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# Influence of the fractal structure of the trapped set on the scattering resonances

Lucas Vacossin

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# Influence of the fractal structure of the trapped set on the scattering resonances

*Influence de la structure fractale de l'ensemble capté  
sur les résonances de scattering*

**Thèse de doctorat de l'université Paris-Saclay**

École doctorale n°574 mathématiques Hadamard (EDMH)  
Spécialité de doctorat : Mathématiques Fondamentales  
Graduate School : Mathématiques. Référent : Faculté des sciences d'Orsay

Thèse préparée dans l'unité de recherche **Laboratoire de mathématiques d'Orsay**  
(Université Paris-Saclay, CNRS),  
sous la direction de **Stéphane NONNENMACHER**, Professeur

**Thèse soutenue à Paris-Saclay, le 5 avril 2023, par**

**Lucas VACOSSIN**

## Composition du jury

Membres du jury avec voix délibérative

<b>Nicolas BURQ</b> Professeur, Université Paris-Saclay	Président
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<b>Maciej ZWORSKI</b> Professeur, Université de Californie (Berkeley)	Rapporteur & Examineur
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**Titre :** Influence de la structure fractale de l'ensemble capté sur les résonances de scattering

**Mots clés :** Résonances, systèmes hyperboliques ouverts, chaos quantique, scattering par des obstacles, analyse semiclassique

**Résumé :** Dans cette thèse, nous nous intéressons aux résonances dites de scattering, qui jouent un rôle fondamental dans l'étude de l'équation des ondes à l'extérieur d'obstacles. Nous tâcherons de comprendre certains aspects de la répartition de ces résonances, plus particulièrement en dimension  $d = 2$  et l'on s'intéresse à des obstacles à frontière lisse, strictement convexes qui vérifient une condition de non-eclipse. Nous établissons l'existence d'une région sans résonance sous l'axe réel - on parle de trou spectral - et améliorons des bornes supérieures pour le comptage de résonances dans des boîtes. Ces résultats font intervenir de façon cruciale la structure fractale de l'ensemble capté du flot du billard. Ces travaux sont dans la lignée des récentes avancées

obtenues dans l'étude des résonances des surfaces hyperboliques convexes co-compactes. L'outil fondamental pour l'obtention du trou spectral est un Principe d'Incertitude Fractal, récemment développé par S. Dyatlov et J. Bourgain, notamment.

Nous obtenons aussi des résultats similaires dans l'étude des résonances de l'opérateur semiclassique  $-\hbar^2\Delta + V$  où  $V$  est un potentiel lisse à support compact. En fait, l'étude du problème des obstacles se fait naturellement à haute fréquence et permet d'utiliser les outils de l'analyse semiclassique. Les méthodes utilisées dans le cas des obstacles s'appliquent alors aux opérateurs semiclassiques  $-\hbar^2\Delta + V$  grâce à deux articles de S. Nonnenmacher, J. Sjöstrand et M. Zworski.

**Title :** Influence of the fractal structure of the trapped set on the scattering resonances

**Keywords :** Resonances, open hyperbolic systems, quantum chaos, obstacle scattering, semiclassical analysis

**Abstract :** In this thesis, we are interested in the study of the scattering resonances, which play a crucial role in the study of the wave equation outside obstacles. We aim at understanding some aspects of the distribution of these resonances, more particularly in dimension  $d = 2$  and for strictly convex obstacles with smooth boundary, satisfying a no-eclipse condition. We prove the existence of a band without resonance below the real axis - called a spectral gap - and improve upper bounds for counting functions of resonances in boxes. These results involve the fractal structure of the trapped set of the billiard flow. These works follow the recent results obtained in the study of convex co-compact hyperbolic surfaces. The

crucial tool in the proof of the spectral gap is a Fractal Uncertainty Principle, recently developed by S. Dyatlov and J. Bourgain, among others.

We also prove similar results in the study of semiclassical scattering resonances for operators  $-\hbar^2\Delta + V$  where  $V$  is a smooth compactly supported potential. In fact, the study of the obstacle problem is naturally turned into a high-frequency problem which allows to use the methods of semiclassical analysis. The strategy used in the case of the obstacles can be applied to the semiclassical operators  $-\hbar^2\Delta + V$  thanks to two papers of S. Nonnenmacher, J. Sjöstrand and M. Zworski.

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 **Mathématiques**  
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**Département de  
mathématiques et  
applications**



# Résumé / Abstract

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*En Français :*

La notion de résonance apparaît dans diverses branches des mathématiques et de la physique. Dans cette thèse, nous nous intéressons aux résonances dites de scattering, qui jouent un rôle fondamental dans l'étude de l'équation des ondes à l'extérieur d'obstacles. Elles sont déjà présentes dans les travaux de Lax-Phillips ([LP16]) sur la théorie du scattering où elles apparaissent comme pôles de la matrice de scattering. L'approche utilisée dans cette thèse est différente et repose sur l'étude de la résolvante du Laplacien  $\Delta$  sur  $L^2(\mathbb{R}^d \setminus \mathcal{O})$ . Nous tâcherons de comprendre certains aspects de la répartition de ces résonances dans le plan complexe, plus particulièrement en dimension  $d = 2$ . Nous établissons l'existence d'une région sans résonance sous l'axe réel - on parle de trou spectral - et améliorons des bornes supérieures pour le comptage de résonances dans des boîtes. Ces résultats font intervenir de façon cruciale la structure fractale de l'ensemble capté du flot du billard. Ces travaux sont dans la lignée des récentes avancées obtenues dans la compréhension du spectre des surfaces hyperboliques convexes co-compactes. L'outil fondamental pour l'obtention du trou spectral est un Principe d'Incertitude Fractal, récemment développé par S. Dyatlov et J. Bourgain, notamment.

Nous obtenons aussi des résultats similaires dans l'étude des résonances de l'opérateur semi-classique  $-h^2\Delta + V$  où  $V$  est un potentiel lisse à support compact. En fait, l'étude du problème des obstacles se fait naturellement à haute fréquence et permet d'utiliser les outils de l'analyse semi-classique. Les méthodes utilisées se généralisent alors sans difficulté aux opérateurs semiclassiques  $-h^2\Delta + V$  grâce aux travaux de [NSZ11] et [NSZ14].

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*In English:*

The notion of resonance appears in several branches of mathematics and physics. In this thesis, we are interested in the scattering resonances, which play a crucial role in the study of the wave equation outside obstacles. They are already present in the works of Lax-Phillips ([LP16]) on scattering theory and appear as the pole of the scattering matrix. The approach in this thesis is slightly different and relies on the study of the resolvent of the Laplace operator  $\Delta$  on  $L^2(\mathbb{R}^d \setminus \mathcal{O})$ . We aim at understanding some aspects of the distribution of these resonances in the complex plane, more particularly in dimension  $d = 2$ . We prove the existence of a band without resonance below the real axis - called a spectral gap - and improve upper bounds for counting functions of resonances in boxes. These results involve the fractal structure of the trapped set of the billiard flow. These works follow naturally the recent results obtained in the study of convex co-compact hyperbolic surfaces. The crucial tool in the proof of the spectral gap is a Fractal Uncertainty Principle, recently developed by S. Dyatlov and J. Bourgain, among others.

We also prove similar results in the study of semiclassical scattering resonances for operators  $-h^2\Delta + V$  where  $V$  is a smooth compactly supported potential. In fact, the study of the obstacle problem is naturally turned into a high-frequency problem which allows to use the methods of semiclassical analysis. The strategy can be generalized without pain to the semiclassical operators  $-h^2\Delta + V$  thanks to the works of [NSZ11] and [NSZ14].

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Ce manuscrit et la soutenance qui l'accompagne clôturent trois années de doctorat. Les pages qui vont suivre regroupent les travaux que j'ai produits durant ces trois années et leur lecture requiert du temps et une certaine expertise. C'est tout le contraire des quelques lignes qui vont suivre et qui sont l'occasion de remercier les personnes qui ont rendu possible ce doctorat, ou qui ont rendu plus agréables ces trois dernières années. Il n'est pas exclu que certaines personnes aient été oubliées et je m'en excuse.



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Part I  
Introduction





# Chapter 1

## Introduction (en français)

Cette introduction contient deux parties. Dans la première, nous présentons les principaux résultats démontrés dans ce manuscrit et nous les contextualisons. Dans la seconde partie, nous donnons un aperçu de quelques outils essentiels qui ont été utilisés dans cette thèse.

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## 1-1 Cadre.

**Motivations.** La principale motivation de cette thèse est l'étude du problème de *scattering par des obstacles strictement convexes*. Nous nous intéressons notamment aux *résonances*, dites de scattering, de tels systèmes, que l'on peut présenter comme des *systèmes hyperboliques ouverts*. D'autres systèmes de ce genre seront exposés en 1-1.5.. Nos principaux résultats, à savoir les Théorèmes A et B présentés dans cette introduction, concernent la répartition de ces résonances dans le cas d'obstacles en dimension 2.

### 1-1.1. Résonances dans le problème de scattering par des obstacles.

Commençons par introduire la notion de résonance, dans le cas qui nous intéresse. Pour ne pas perdre en généralité, nous présentons le problème en dimension quelconque  $d$ . On considère un ouvert borné  $O$  - l'obstacle (ou l'ensemble d'obstacles) - et à ce stade, la seule hypothèse sur  $O$  que nous faisons est qu'il possède une frontière lisse. Nous rajouterons plus loin des hypothèses géométriques importantes - notamment la stricte convexité- mais elles ne sont pas encore nécessaires pour la définition générale des résonances.

Dans ce cadre, le problème principal qui nous intéresse concerne l'étude de la propagation des ondes en dehors de l'obstacle. Autrement dit, nous nous intéressons à l'étude de l'équation des ondes

$$\partial_{tt}^2 u(t, x) - \Delta u(t, x) = 0 ; t \in \mathbb{R}, x \in \mathbb{R}^d \setminus \overline{O}$$

avec conditions initiales  $u(0, x) = u_0(x)$ ,  $\partial_t u(0, x) = u_1(x)$ . Pour que ce problème soit bien posé, il est nécessaire d'imposer des conditions de bord. Dans cette thèse, nous utiliserons la condition de Dirichlet, c'est-à-dire,

$$u(t, x) = 0, x \in \partial O$$

La théorie des semi-groupes assure que le problème ci-dessus est bien posé (voir [Tay10], Chapitre 9, Section 4). En effet, posons  $U = \mathbb{R}^d \setminus \overline{O}$  ainsi que  $H$  l'espace de Hilbert  $\mathcal{H}(U) \oplus L^2(U)$ , où  $\mathcal{H}(U)$  désigne le complété de  $C_c^\infty(U)$  par rapport à la norme  $\|f\|_{\mathcal{H}} = \|\nabla f\|_{L^2(U)}$ . Soit alors  $A$  l'opérateur

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta & 0 \end{pmatrix}$$

de domaine  $D(A) = (H \cap H^2(U)) \oplus H_0^1(U)$ .  $A$  est maximal dissipatif, de sorte que la théorie de Hille-Yosida permet de définir le propagateur  $e^{tA}$  et si  $(u_0, u_1) \in H$ , la première composante  $u(t)$  de  $t \mapsto e^{tA}(u_0, u_1)$  est l'unique solution du problème de Cauchy suivant

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \text{ dans } U \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \\ u(t, x) = 0 \text{ pour } x \in \partial O \end{cases} \quad (1-1.1)$$

Dans le cas d'un ouvert borné  $\Omega$ , (ce qui n'est évidemment pas le cas ici), la théorie spectrale de  $\Delta$  sur  $L^2(\Omega)$  permet de résoudre complètement l'équation des ondes dans  $\Omega$  (avec condition de Dirichlet). En effet, puisque l'injection  $H_0^1(\Omega) \rightarrow L^2(\Omega)$  est compacte,  $-\Delta$  possède une résolvante compacte et donc un spectre discret  $0 < \lambda_0(\Omega) \leq \lambda_1(\Omega) \leq \dots$ , associé à une base Hilbertienne de fonctions propres  $(e_j)_{j \in \mathbb{N}}$ . Dès lors, les solutions de (1-1.1) sont de la forme :

$$u(t, x) = \sum_{j=0}^{\infty} (a_j e^{it\lambda_j} + b_j e^{-it\lambda_j}) e_j(x)$$

où  $a_j, b_j$  s'obtiennent en décomposant la donnée initiale dans la base  $(e_j)$ .

Cela n'est plus possible dans le cas qui nous intéresse d'un ouvert non borné. En fait, le spectre de  $\Delta$  n'est plus discret et nous avons

$$\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, +\infty[$$

Néanmoins, il est toujours possible de définir une notion de valeurs propres généralisées, appelées *résonances*, qui jouent un rôle analogue à celui joué par les valeurs propres dans le cas borné.

Celles-ci permettent d'obtenir des informations sur le comportement des solutions de l'équation des ondes. En particulier, il est possible d'obtenir un taux de décroissance pour l'énergie locale des solutions de l'équation des ondes, comme nous le verrons dans cette thèse (voir Chapitre 6). L'énergie globale d'une solution est définie comme

$$E(t) = \frac{1}{2} \|(u(t), \partial_t u(t))\|_H^2 = \frac{1}{2} \int_U |\nabla u(t)|^2 + |\partial_t u(t)|^2$$

Sans amortissement, elle est bien entendu conservée. En revanche, si  $K \Subset \mathbb{R}^2$ , nous pouvons également définir l'énergie locale dans  $K$  de la même façon

$$E_K(t) = \frac{1}{2} \int_{K \cap U} |\nabla u(t)|^2 + |\partial_t u(t)|^2$$

On sait alors que pour tout compact  $K$ ,  $E_K(t) \rightarrow 0$  quand  $t \rightarrow +\infty$  dès lors que les données initiales sont à support compact (voir par exemple [Tay10], Chapitre 9, Proposition 4.1). Une bonne compréhension de la résolvante de  $-\Delta$  permet d'avoir un contrôle sur le taux de décroissance de l'énergie locale.

**Définition rigoureuse des résonances.** Afin d'introduire rigoureusement les résonances, nous allons suivre l'approche du livre [DZ19].

**Remarque.**

Quand il s'agit du problème des obstacles, la dimension 1 n'est pas très intéressante. Néanmoins, en ce qui concerne le scattering par un potentiel, c'est tout à fait pertinent et pour cette raison, nous ne l'excluons pas de la présentation ci-dessous.

**La résolvante libre.** Il paraît naturel de commencer par comprendre le cas de la résolvante libre, à savoir le cas où  $O = \emptyset$  avant de s'aventurer plus loin dans l'étude des problèmes de scattering. En fait, il s'avère que le cas  $U = \mathbb{R}^d$  est fondamental pour développer la théorie du scattering par des obstacles, ou même par un potentiel. Présentons donc les principales propriétés de cette résolvante libre. Dans ce cas, rappelons que l'on souhaite étudier  $-\Delta$  sur  $L^2(\mathbb{R}^d)$  de domaine  $H^2(\mathbb{R}^d)$  et que le spectre est alors donné par  $\sigma(-\Delta) = [0, +\infty[$ . Il est alors possible de définir la résolvante

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^d) \rightarrow H^2(\mathbb{R}^d) ; \operatorname{Im} \lambda > 0 \quad (1-1.2)$$

**Remarque.**

L'une des raisons de choisir d'écrire le paramètre spectral  $z \in \mathbb{C} \setminus [0, +\infty[$  de la forme  $z = \lambda^2$ ,  $\operatorname{Im} \lambda > 0$  est que les fonctions  $e_{\lambda, \omega}(x) = e^{i\lambda x \cdot \omega}$  où  $\omega \in \mathbb{S}^{d-1}$  sont des solutions évidentes de l'équation  $-\Delta u - \lambda^2 u = 0$ . Le signe de  $\operatorname{Im} \lambda$  est purement arbitraire et c'est celui qui est choisi dans ce manuscrit.

Il est possible de décrire analytiquement le noyau de Schwartz de la résolvante libre  $R_0(\lambda)$  : il s'agit d'un noyau de convolution  $R_0(\lambda, x, y) = G(\lambda, |x - y|)$  où  $G$  dépend de la dimension. Ci-dessous, une expression de  $G$  dans le cas des dimensions 1, 2 et 3.

Dimension	1	2	3
$G(\lambda, r)$	$\frac{i}{2\lambda} e^{i\lambda r}$	$\frac{i}{4} H_0^{(1)}(\lambda r)$	$\frac{1}{4\pi r} e^{i\lambda r}$

$H_0^{(1)}$  est une fonction de Hankel du premier type. Quand  $\operatorname{Im} \lambda > 0$ , son comportement asymptotique dans la limite  $r \rightarrow +\infty$  est donné par

$$H_0^{(1)}(\lambda r) = \sqrt{\frac{2}{\pi \lambda r}} e^{i\lambda r - i\pi/4} (1 + O(1/r))$$

En particulier, on voit aisément que pour les 3 premières dimensions, quand  $\operatorname{Im} \lambda > 0$ , le noyau décroît exponentiellement vite quand  $r \rightarrow +\infty$ , ce qui assure que l'opérateur  $R_0(\lambda) : L^2 \rightarrow H^2$  est borné.

En dimension 3, et de façon plus générale, en dimension impaire plus grande que 3, le noyau peut se prolonger analytiquement à tout le plan complexe  $\mathbb{C}$ . En dimension 1, il peut s'étendre de façon méromorphe avec un unique pôle simple en 0. La situation est différente en dimension paire. Cela se voit bien dans le cas  $d = 2$  présenté plus haut. En effet,  $H_0^{(1)}$  ne peut pas être prolongée analytiquement à tout  $\mathbb{C}$ , mais au **recouvrement logarithmique** de  $\mathbb{C}$ , que l'on notera  $\Lambda$ . Nous avons finalement plus généralement le résultat suivant:

**Théorème 1-1.1.** Soit  $\chi \in C_c^\infty(\mathbb{R}^d)$ . Alors, l'opérateur

$$\chi R_0(\lambda)\chi : L^2(\mathbb{R}^d) \rightarrow H^2(\mathbb{R}^d)$$

défini pour  $\text{Im } \lambda > 0$  se prolonge

- méromorphiquement à  $\mathbb{C}$  avec un pôle simple en  $\lambda = 0$  quand  $d = 1$ ;
- analytiquement à  $\Lambda$  quand  $d$  est pair ;
- analytiquement à  $\mathbb{C}$  quand  $d$  est impair de  $d \geq 3$ .

### Remarque.

- Il y a une différence majeure entre les dimensions paires et impaires. Cette différence a aussi des conséquences fondamentales pour l'équation des ondes : le principe de Huyghens fort est valable en dimension impaire, mais pas en dimension paire. Quand on ajoute des obstacles, cela a des conséquences sur le taux de décroissance de l'énergie locale de l'équation des ondes, comme nous le verrons dans cette thèse.
- Une façon d'obtenir les expressions de  $G$ , qui sont données dans le tableau ci-dessus, est de résoudre l'équation  $-\Delta u - \lambda^2 u = 0$  avec des fonctions  $u$  radiales. Cela conduit à résoudre l'EDO donnée dans la table ci-dessous pour les dimensions 1,2 et 3. Chacune a une base de solutions qui sont données ci-dessous également.

Dimension	1	2	3
EDO	$f'' + \lambda^2 f = 0$	$r^2 f'' + r f' + r^2 \lambda^2 f = 0$	$r^2 f'' + 2r f' + r^2 \lambda^2 f = 0$
Solution du 1er type	$\frac{i}{2\lambda} e^{i\lambda r}$	$\frac{i}{4} H_0^{(1)}(\lambda r)$	$\frac{1}{4\pi r} e^{i\lambda r}$
Solution du 2nd type	$\frac{i}{2\lambda} e^{-i\lambda r}$	$\frac{i}{4} H_0^{(2)}(\lambda r)$	$\frac{1}{4\pi r} e^{-i\lambda r}$

$H_0^{(2)}$  est une fonction de Hankel du second type. Quand  $\text{Im } \lambda > 0$ , le comportement asymptotique quand  $r \rightarrow +\infty$  est donné par

$$H_0^{(2)}(\lambda r) = \sqrt{\frac{2}{\pi \lambda r}} e^{-(i\lambda r - i\pi/4)} (1 + O(1/r))$$

Les solutions du premier type sont dites *sortantes*, quand les solutions du second type sont dites *entrantes*. Quand on calcule le noyau de la résolvante libre, on veut que la résolvante  $R_0(\lambda)$  soit bornée de  $L^2$  vers  $L^2$  quand  $\text{Im } \lambda > 0$ . Cette condition "sélectionne" les solutions sortantes. Pour cette raison, on dit que la résolvante  $R_0(\lambda)$  est sortante.

**Prolongement méromorphe.** Nous aurons besoin de la définition suivante.

**Définition 1-1.1.** On dit qu'une famille d'opérateurs  $T(\lambda) : H_1 \rightarrow H_2$  où  $H_1, H_2$  sont des espaces de Hilbert et  $\lambda$  un paramètre complexe appartenant à un ouvert  $U \subset \mathbb{C}$  (ou bien  $U \subset \Lambda$ ) est *meromorphe avec des pôles de rang fini* si pour tout  $\lambda_0 \in U$ , il existe  $r > 0$ ,  $m \in \mathbb{N}$ ,  $T_1, \dots, T_m : H_1 \rightarrow H_2$  de rang fini et  $T_h(\lambda) : H_1 \rightarrow H_2$  une famille holomorphe d'opérateurs

dépendant de  $\lambda \in B(\lambda_0, r)$  tels que  $B(\lambda_0, r) \subset U$  et pour tout  $\lambda \in B(\lambda_0, r)$ ,

$$T(\lambda) = T_h(\lambda) + \sum_{j=1}^m \frac{T_j}{(\lambda - \lambda_0)^j}$$

Les pôles de cette famille sont les points où l'on ne peut pas prendre  $m = 0$ . L'ensemble des pôles est discret.

On peut alors définir précisément les résonances. Leur définition repose sur le résultat suivant (voir [DZ19], Théorème 4.4):

**Théorème 1-1.2.** *Soit  $O$  un ouvert borné ayant une frontière lisse et notons  $U = \mathbb{R}^d \setminus \overline{O}$ . Soit  $\Delta$  le laplacien sur  $L^2(U)$  de domaine  $H^2(U) \cap H_0^1(U)$  (i.e. avec condition de Dirichlet). Alors,*

$$R(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(U) \rightarrow L^2(U)$$

*bien définie pour  $\text{Im } \lambda > 0$ , se prolonge à  $\mathbb{C}$  (resp.  $\Lambda$ ) quand  $d$  est impair (resp. quand  $d$  est pair) en tant que famille méromorphe d'opérateurs avec pôles de rang fini allant de  $L_{comp}^2(U)$  vers  $H_{loc}^2(U)$ . En d'autres termes, pour tout  $\chi \in C_c^\infty(\mathbb{R}^d)$  tel que  $\chi \equiv 1$  dans un voisinage de  $\overline{O}$ ,*

$$\chi R(\lambda) \chi : L^2(U) \rightarrow H^2(U)$$

*est une famille méromorphe d'opérateurs avec pôles de rang fini.*

On peut alors définir rigoureusement les résonances :

**Définition 1-1.2.** On dit que  $\lambda \in \mathbb{C}$  ( $d$  impair) ou  $\Lambda$  ( $d$  pair) est une résonance de  $-\Delta$  si c'est un pôle du prolongement méromorphe de  $R(\lambda)$ . La multiplicité d'une résonance  $\lambda$  est définie comme

$$m_R(\lambda) := \text{rank} \int_{\gamma_\lambda} R(\zeta) d\zeta$$

où le contour  $\gamma_\lambda$  est un cercle centré en  $\lambda$  n'entourant aucun autre pôle.

### Remarque.

- La preuve de ce résultat repose sur la théorie de Fredholm analytique. L'objectif est de construire un bon parametrix  $\tilde{R}(\lambda)$  tel  $(-\Delta - \lambda^2)\tilde{R}(\lambda) = \text{Id} + K(\lambda)$  avec  $K(\lambda)$  une famille analytique (ou éventuellement méromorphe en dimension 1) d'opérateurs compacts. La résolvante libre  $R_0(\lambda)$  fournit un bon parametrix à l'infini et c'est pourquoi il est crucial de bien comprendre  $R_0(\lambda)$ .
- Ce résultat peut s'étendre à une classe plus large de perturbations du laplacien. Tout d'abord, il est possible de l'adapter au cas du scattering par un potentiel, c'est-à-dire, l'étude de  $-\Delta + V$  où  $V \in C_c^\infty(\mathbb{R}^d)$ . Plus généralement, il est possible de généraliser ce résultat au formalisme abstrait de "black-box" de [SZ91] pour des opérateurs  $P$  agissant sur des espaces de Hilbert de la forme  $\mathcal{H} = \mathcal{H}_0 \oplus^\perp L^2(\mathbb{R}^d \setminus B(0, R))$  où  $P$  coïncide avec  $-\Delta$  sur  $L^2(\mathbb{R}^d \setminus B(0, R))$  (voir par exemple [DZ19], Chapter 4).
- Comme expliqué dans [DZ19], ces résonances peuvent aussi être décrites comme les pôles de la matrice de scattering, ce qui correspond directement aux résonances de scattering introduites par [LP16] en dimension impaire (voir par exemple [SZ94]).

La structure de la résolvante sortante au voisinage d'une résonance est donnée par le théorème suivant ([DZ19], Théorème 4.7) :

**Théorème 1-1.3.** Soit  $\lambda \neq 0$ . Alors, il existe  $M_\lambda \leq m_R(\lambda)$  et  $\zeta \mapsto A(\zeta, \lambda)$  une famille analytique d'opérateurs définie dans un voisinage de  $\lambda$ , telle que, dans ce même voisinage,

$$R(\zeta) = A(\zeta, \lambda) - \sum_{k=1}^{M_\lambda} \frac{(-\Delta - \lambda^2)^{k-1}}{(\zeta^2 - \lambda^2)^k} \Pi_\lambda$$

où  $\Pi_\lambda$  est le projecteur spectral défini par

$$\Pi_\lambda = -\frac{1}{2i\pi} \int_{\gamma_\lambda} R(\zeta) 2\zeta d\zeta$$

où le contour  $\gamma_\lambda$  ne contient aucun autre pôle que  $\lambda$ .  $\Pi_\lambda$  vérifie :

$$\text{rang } \Pi_\lambda = m_R(\lambda) ; (-\Delta - \lambda^2)^{M_\lambda} = 0$$

Cela permet de définir la notion d'états résonants, qui jouent le rôle de fonctions propres généralisées de  $-\Delta$ :

**Définition 1-1.3.** On dit que  $u$  est un état résonant (resp. un état résonant généralisé) associé à la résonance  $\lambda \neq 0$  si  $u \in \text{Im } \Pi_\lambda$  et  $(-\Delta - \lambda^2)u = 0$  (resp. si  $v \in \text{Im } \Pi_\lambda$ ).

La caractérisation qui suit des états résonants peut s'avérer utile et montre que ceux-ci vérifient une propriété sortante ([DZ19], Théorème 4.9):

**Théorème 1-1.4.** Soit  $\lambda \neq 0$ . Un état  $u \in H_{loc}^2(U) \cap H_0^1(U)$  est un état résonant associé à  $\lambda$  si et seulement si  $(-\Delta - \lambda^2)u = 0$  et s'il existe  $g \in L_{comp}^2(\mathbb{R}^n)$  et  $R > 0$  tels que  $\bar{O} \subset B(0, R)$  et

$$u|_{\mathbb{R}^d \setminus B(0, R)} = (R_0(\lambda)g)|_{\mathbb{R}^d \setminus B(0, R)}$$

## 1-1.2. Répartition des résonances.

Dans cette thèse, nous nous intéressons à la répartition des résonances dans le plan complexe. Nous étudierons plus particulièrement le cas de la dimension  $d = 2$ . On s'intéressera dans ce cas aux résonances du premier feuillet de  $\Lambda$ , à savoir  $\mathbb{C} \setminus (i\mathbb{R}^-)$ . Deux questions ont tout particulièrement motivé nos travaux. La première question est la suivante :

**Question 1.1** (Trou spectral). Existe-t-il  $\gamma > 0$  tel qu'il n'y a qu'un nombre fini de résonances dans  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > 1, \text{Im } \lambda > -\gamma\}$  ?

### Remarque.

- On sait par ailleurs qu'il n'y a pas de résonances sur l'axe réel. Ce résultat est souvent connu sous le nom de théorème de Rellich ([Rel43]). Il est par exemple prouvé dans [Bur98] en toute dimension.
- En dimension impaire, on peut remplacer de manière équivalente  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > 1, \text{Im } \lambda > -\gamma\}$  par  $\{\lambda \in \mathbb{C}, \text{Im } \lambda > -\gamma\}$ .
- L'existence d'un nombre fini de résonances dans  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > 1, \text{Im } \lambda > -\gamma\}$  revient à dire qu'il existe  $\lambda_0 > 0$  tel qu'il n'y a pas de résonances dans  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > \lambda_0, \text{Im } \lambda > -\gamma\}$ . On dira que  $\gamma$  est la profondeur ou la taille du trou spectral.
- Il est tout à fait envisageable de s'intéresser à d'autres formes de régions sans résonances. On pourrait regarder - mais nous le ferons pas dans cette thèse - des profils  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

et considérer la région

$$\{\lambda \in \mathbb{C}, |\operatorname{Re} \lambda| > \lambda_0, \operatorname{Im} \lambda \geq -g(|\operatorname{Re} \lambda|)\}$$

Comme nous l'expliquerons plus en détail dans cette thèse, la géométrie des obstacles joue un rôle crucial dans l'obtention d'un trou spectral. Avec l'unique hypothèse que les obstacles ont une frontière lisse, le résultat de région sans résonance le plus général donne un profil de la forme  $g(\lambda) = c_1 e^{-c_2 \lambda}$  :

**Théorème 1-1.5** ([Bur98]). *Soit  $O$  un obstacle borné de frontière lisse. Alors, il existe  $\lambda_0, c_1, c_2$  tel qu'il n'y a pas de résonance dans*

$$\{|\operatorname{Re} \lambda| > \lambda_0\} \cap \left\{ \operatorname{Im} \lambda \geq -c_1 e^{-c_2 |\operatorname{Re} \lambda|} \right\}$$

La Question 1.1 est très fortement reliée à la décroissance de l'énergie locale de l'équation des ondes. En fait, afin de contrôler le taux de décroissance, il est aussi nécessaire que le trou spectral soit accompagné d'estimées pour la résolvante tronquée, dans des bandes sous l'axe réel. Autrement dit, pour  $\chi \in C_c^\infty(\mathbb{R}^d)$ , il est nécessaire de trouver une fonction assez explicite  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  telle que pour tout  $\lambda \in \mathbb{C}$

$$\operatorname{Re} \lambda \gg 1; \operatorname{Im} \lambda > -\gamma \implies \|\chi R(\lambda) \chi\|_{L^2 \rightarrow L^2} \leq f(|\operatorname{Re} \lambda|)$$

Nous verrons cela plus en détail dans le Chapitre 6 (voir aussi [Ika82], [Ika88], [Leb96], [Bur98], [Zwo12] Chapter 5).

L'autre question principale concerne des estimées pour des fonctions de comptages de résonances dans des boîtes. Pour  $\gamma > 0$  et  $\delta > 0$ , notons

$$\Omega_{\gamma, \delta}(r) = \{\lambda \in \mathbb{C}, \operatorname{Im} \lambda \in [-\gamma, 0], \operatorname{Re} \lambda \in [r, r + \delta]\}$$

(en dimension 2, on considère  $\Omega_{\gamma, \delta}$  comme un sous-ensemble du premier feuillet de  $\Lambda$ ) et soit  $N_{\gamma, \delta}(r)$  le nombre de résonances dans  $\Omega_{\gamma, \delta}(r)$ , comptées avec multiplicité.

**Question 1.2** (Comptage de résonances). *Peut-on trouver un équivalent asymptotique simple de  $N_{\gamma, \delta}(r)$  quand  $r \rightarrow +\infty$  ?*

**Remarque.**

- En général, un asymptotique exact est une question très complexe. De plus, l'obtention de bornes supérieures se révèle souvent plus facile que pour les bornes inférieures. Dans cette thèse, nous prouvons uniquement des bornes supérieures.
- On parle en général de borne ou asymptotique de Weyl, par analogie avec la loi de Weyl dans le cas des valeurs propres.
- Il est possible de compter les résonances dans d'autres régions - pas nécessairement des boîtes - tant que ces régions bougent avec  $\operatorname{Re} \lambda \rightarrow +\infty$  (ou bien  $|\lambda|$ ).

La borne la plus générale (et sans hypothèse particulière) est donnée par :

**Théorème 1-1.6.** *Soit  $O$  un obstacle borné de frontière lisse. Quand  $d$  est impair (resp. pair), on définit  $N(r), r > 0$  comme le nombre de résonances comptées avec multiplicité dans  $B(0, r)$  (resp.  $B(0, r) \setminus i\mathbb{R}^-$ ). Alors, il existe  $C > 0$  tel que pour tout  $r > 1$ ,*

$$N(r) \leq Cr^d$$

**Remarque.**



- Ce théorème est démontré dans [Mel84] pour le cas de la dimension impaire.
- Dans le cas de la dimension paire, c'est un corollaire des estimées plus générales obtenues dans [Vod94b] pour  $d = 2$  et [Vod94a] pour  $d \geq 4$ . En fait, Vodev donne des bornes supérieures pour des régions de  $\Lambda$  de la forme  $\{|\lambda| < r, |\arg \lambda| \leq a\}$ .

### 1-1.3. Un système hyperbolique ouvert : le cadre géométrique.

Nous en venons à présent aux hypothèses géométriques sur les obstacles, qui sont fondamentales dans cette thèse, et qui sont nécessaires pour énoncer nos résultats. Considérons un nombre fini  $J$  d'obstacles (i.e. des ouverts connexes bornés)  $\mathcal{O}_j \subset \mathbb{R}^d$ ,  $j = 1, \dots, J$  de **frontière lisse** et supposons que ces obstacles sont tous **strictement convexes**. On note

$$\mathcal{O} = \bigcup_{j=1}^J \mathcal{O}_j ; \Omega = \mathbb{R}^d \setminus \mathcal{O}$$

Pour des raisons techniques et pratiques, on supposera de plus que les obstacles vérifient la **condition de non-éclipse** ci-dessous, due à Ikawa : pour tous  $i \neq j \neq k$ ,

$$\overline{\text{conv}}(\mathcal{O}_j \cup \mathcal{O}_i) \cap \overline{\mathcal{O}_k} = \emptyset \quad (1-1.3)$$

où  $\overline{\text{conv}}(A)$  est l'adhérence de l'enveloppe convexe de  $A$ .

Que ce soit le problème du trou spectral ou le problème de comptage de résonances, les deux sont des problèmes à haute-fréquence. Il est alors possible d'introduire un petit paramètre  $h$  où  $\frac{1}{h} = \text{Re}(\lambda)$ . Avec cette nouvelle paramétrisation, on s'intéresse alors à l'opérateur semiclassique

$$P(h) = -h^2 \Delta - 1 \quad , \quad 0 < h \leq h_0$$

dans la limite  $h \rightarrow 0$ . Le nouveau paramètre spectral est  $z \in D(0, Ch)$  pour un certain  $C > 0$  fixé. On dit que  $h$  est un *paramètre semiclassique*. Ce point de vue permet d'utiliser les méthodes de l'analyse semiclassique, qui permettent une étude uniforme des opérateurs  $P(h)$  dans la limite  $h \rightarrow 0$ . Il est notamment possible de relier rigoureusement le comportement à haute fréquence de  $-\Delta$  et le flot classique associé au système : on parle de la *correspondance classique/quantique*. En ce qui concerne ce système ouvert, le flot classique est le flot du billard dans  $\Omega \times \mathbb{S}^1$ , c'est-à-dire, le mouvement libre d'une particule qui se déplace en ligne droite, avec des rebonds élastiques sur le bord des obstacles (voir Figure 1.1).

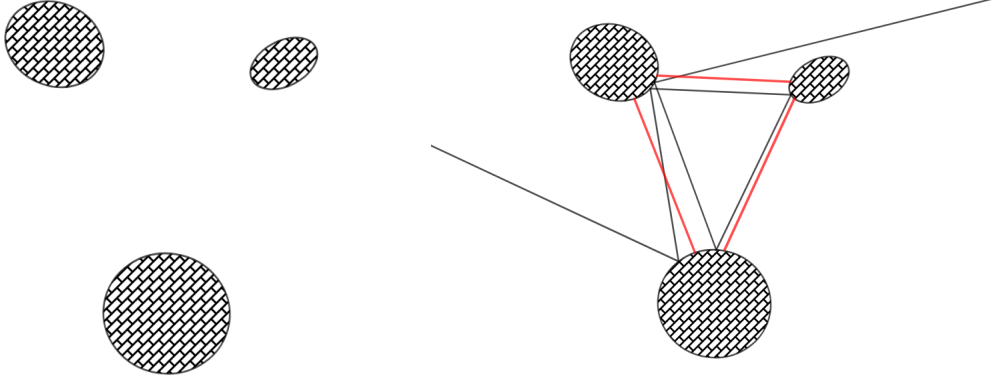
L'ensemble capté, à savoir, l'ensemble des points  $(x, \xi) \in \Omega \times \mathbb{S}^1$  dont la trajectoire reste bornée, à la fois dans le passé et dans le futur, est un objet particulièrement important dans l'étude de ces problèmes. Dans les cas  $J = 0$  et  $J = 1$ , l'ensemble capté est vide. Pour le cas de deux obstacles, il s'agit d'une unique trajectoire périodique. Dès que plus de trois obstacles rentrent en jeu, la structure de l'ensemble capté se complexifie et dévoile un aspect fractal. La Figure 1.2 montre différentes trajectoires captées correspondant à diverses trajectoires périodiques dans le cas de trois disques situés sur les sommets d'un triangle équilatéral. La Figure 1.3 montre une représentation de l'ensemble capté pour un modèle à 4 obstacles. La structure de l'ensemble capté joue un rôle fondamental dans les deux questions qui nous intéressent. Afin d'illustrer mathématiquement la complexité de cet ensemble, il peut être intéressant de citer le théorème suivant :

**Théorème 1-1.7** ([Mor91]). *Supposons que les  $J$  obstacles soient strictement convexes, de frontière lisse et vérifient la condition de non-éclipse. Alors, pour tout  $\alpha \in \Sigma$ , où  $\Sigma$  est l'ensemble des suites d'obstacles admissibles*

$$\Sigma := \{\alpha \in \{1, \dots, J\}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \alpha_j \neq \alpha_{j+1}\}$$

*il existe une unique trajectoire captée  $\Phi : \mathbb{R} \rightarrow \Omega \times \mathbb{S}^1$  telle que  $\Phi(0) \in \partial \mathcal{O}_{\alpha_0} \times \mathbb{S}^1$  et la suite des obstacles rencontrés par la trajectoire  $\Phi$  est exactement  $\alpha$ .*

**Remarque.**



(a) Trois obstacles convexes qui vérifient la condition de non-éclipse. (b) Une trajectoire non captée du flot du billard (en noir) et des trajectoires captées (en rouge).

Figure 1.1: On étudie l'équation des ondes - le problème "quantique" - en dehors d'obstacles strictement convexes qui vérifient la condition non-éclipse (a) . Le flot classique est le flot du billard (b).

L'existence d'une trajectoire périodique avec une suite périodique de  $\Sigma$  prescrite n'est pas difficile à voir. En effet, soit  $(\alpha_1, \dots, \alpha_N)$  une telle suite, il suffit alors de minimiser,

$$(x_1, \dots, x_N) \in \partial\mathcal{O}_{\alpha_1} \times \dots \times \partial\mathcal{O}_{\alpha_N} \mapsto \|x_1 - x_2\|^2 + \dots + \|x_{N-1} - x_N\|^2 + \|x_N - x_1\|^2$$

Pour des suites non périodiques, on les approche par des suites périodiques de période  $N$  et on fait  $N \rightarrow +\infty$ . Pour démontrer l'unicité, il faut utiliser l'hyperbolicité du flot du billard.

La complexité de l'ensemble capté est en fait relié à l'hyperbolicité du flot du billard. Des définitions précises seront rappelées au besoin dans cette thèse, et notamment dans le Chapitre 3. Pour le moment, ce qui est important de garder à l'esprit est que l'hyperbolicité crée de l'instabilité par rapport aux conditions initiales : ceci est illustré par les figures 1.4 et 1.5. Par exemple, une particule peut démarrer sa trajectoire très proche d'une trajectoire captée mais pour autant finir sa course vers l'infini. On dit que ce système est un *système hyperbolique ouvert*.

**Quelques notions de systèmes dynamiques.** Afin de conclure cette présentation succincte du flot du billard, il nous paraît nécessaire d'introduire quelques notions de systèmes dynamiques qui seront importantes pour comprendre certains résultats que nous souhaitons présenter ci-après.

- La **transformation du billard** (voir Figure 1.6). Pour  $j \in \{1, \dots, J\}$ , notons  $B^*\partial\mathcal{O}_j$  le fibré suivant de  $\partial\mathcal{O}_j$

$$B^*\partial\mathcal{O}_j = \{(x, \eta) \in T^*\partial\mathcal{O}_j, |\eta| < 1\}$$

Soit également  $S_{\partial\mathcal{O}_j}^*$  la restriction de  $S^*\Omega$  à  $\partial\mathcal{O}_j$ ,  $\pi_j : S_{\partial\mathcal{O}_j}^* \rightarrow B^*\partial\mathcal{O}_j$  la projection orthogonale naturelle ainsi que  $\nu_j(x)$  la normale extérieure en  $x \in \partial\mathcal{O}_j$ . Pour  $i \neq j$ , on définit l'application  $\mathcal{B}_{ij}$  de la façon suivante

$$\begin{aligned} \mathcal{B}_{ij}(\rho') = \rho &\iff (\rho, \rho') \in B^*\partial\mathcal{O}_i \times B^*\partial\mathcal{O}_j, \\ &\exists t > 0, \exists \xi \in \mathbb{S}^1, \exists x \in \partial\mathcal{O}_j \\ x + t\xi &\in \partial\mathcal{O}_i, \langle \nu_j(x), \xi \rangle > 0, -\langle \nu_i(x + t\xi), \xi \rangle > 0, \\ \pi_j(x, \xi) &= \rho', \pi_i(x + t\xi, \xi) = \rho \end{aligned}$$

- Le **Jacobien instable**. La métrique Riemannienne sur  $T^*\partial\mathcal{O}$  provient de celle de  $T^*\mathbb{R}^d$ . Considérons une trajectoire périodique  $\gamma$  du flot du billard et notons  $\alpha \in \Sigma$  la suite associée,

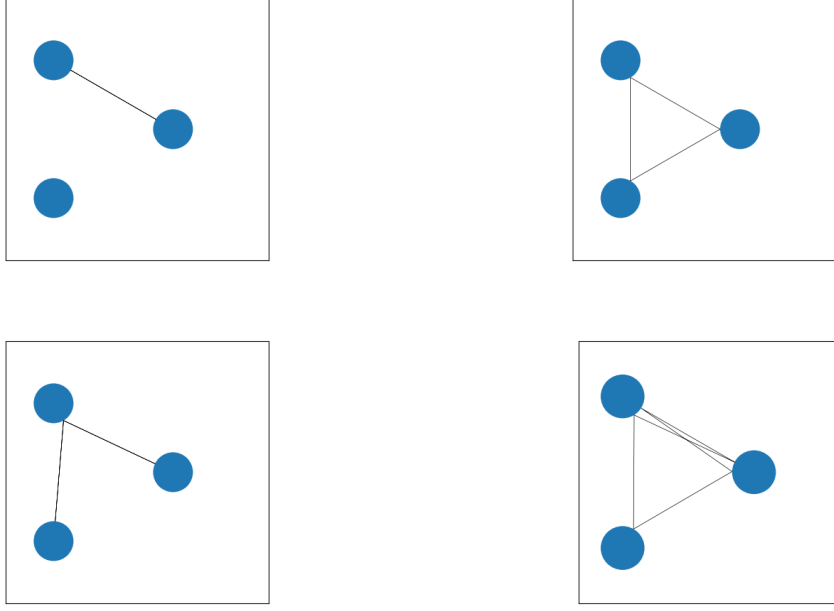


Figure 1.2: Exemples de trajectoires périodiques dans le cas de trois disques situés sur les sommets d'un triangle équilatéral.

comme le permet le Théorème 1-1.7. On notera  $(\alpha_1, \dots, \alpha_N)$  un motif élémentaire de la suite périodique. Disons que  $\gamma$  part d'un point  $\rho_1 \in B^* \partial \mathcal{O}_{\alpha_1}$ , qui est donc un point fixe de

$$\mathcal{B}_\alpha := \mathcal{B}_{\alpha_1 \alpha_N} \circ \dots \circ \mathcal{B}_{\alpha_2 \alpha_1}$$

Dans un voisinage de  $\rho_1$ , l'application  $\mathcal{B}_\alpha$  est bien définie et lisse. Cette affirmation est vraie grâce à la condition de non-éclipse. En effet, celle-ci assure que les trajectoires rasantes, lieux de singularité de l'application du billard, ne sont jamais captées. Nous affirmons que  $\rho_1$  est un point fixe hyperbolique de  $\mathcal{B}_\alpha$  (voir [CM00]), c'est-à-dire, il existe une décomposition

$$T_{\rho_1}^* (T^* \partial \mathcal{O}_{\alpha_1}) = E_u(\rho_1) \oplus E_s(\rho_1)$$

telle que

- (i)  $d_{\rho_1} \mathcal{B}_\alpha (E_{u/s}(\rho_1)) \subset E_{u/s}(\rho_1)$ ;
- (ii)  $\dim E_u(\rho_1) = \dim E_s(\rho_1) = d - 1$
- (iii) il existe  $C > 0$  et  $\lambda < 1$  tels que pour tout  $v \in E_u(\rho_1)$  (resp.  $E_s(\rho_1)$ ) et  $n \in \mathbb{N}$ ,

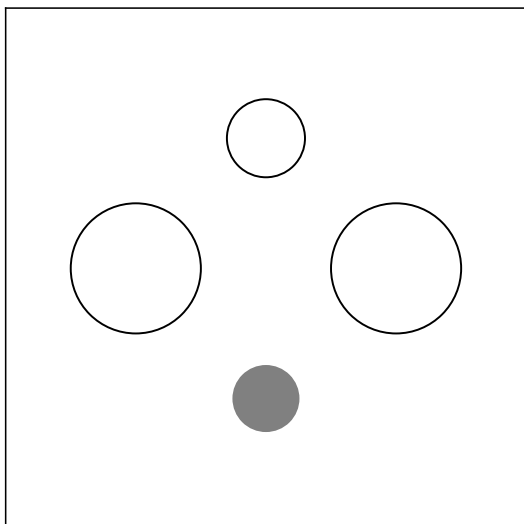
$$\| (d_{\rho_1} \mathcal{B}_\alpha)^{-n} v \| \leq C \lambda^n \| v \| \quad (\text{resp. } \| (d_{\rho_1} \mathcal{B}_\alpha)^n v \| \leq C \lambda^n \| v \| )$$

On définit alors le Jacobien instable associé à la trajectoire périodique  $\gamma$  :

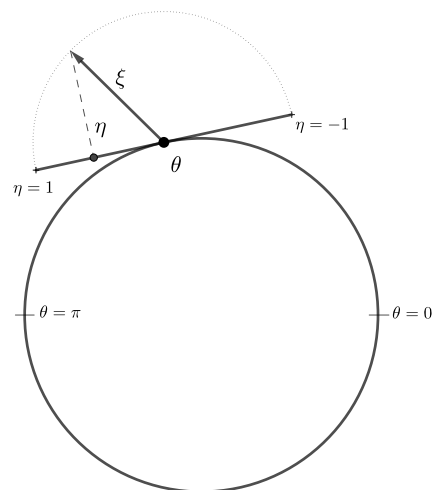
$$J^u(\gamma) = \det d_{\rho_1} (\mathcal{B}_\alpha : E_u(\rho_1) \rightarrow E_u(\rho_1)) \quad (1-1.4)$$

- **La pression topologique.** Cette notion vient de la théorie du Formalisme Thermodynamique (voir par exemple [Rue04]). A ce stade, nous donnons simplement sa définition dans ce contexte pour le cas particulier du flot du billard et du Jacobien instable mais elle peut être généralisée à d'autres systèmes et d'autres applications continues définies sur l'ensemble capté (voir Définition 3-5.1 au Chapitre 3). Soit  $s \in \mathbb{R}$ , on définit la pression topologique  $P(s)$  par

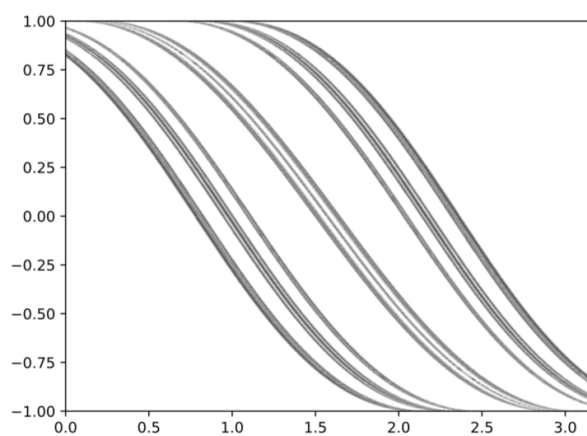
$$P(s) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \left( \sum_{\gamma, T-1 \leq |\gamma| < T} J^u(\gamma)^{-s} \right) \quad (1-1.5)$$



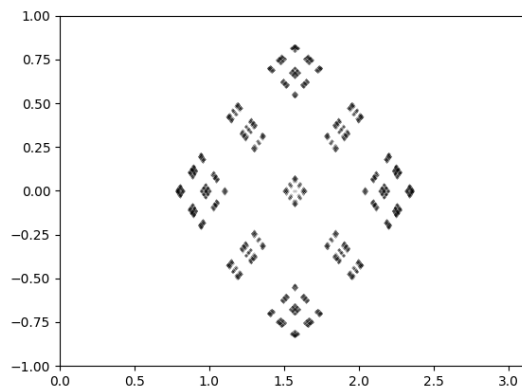
(a) La disposition des obstacles que l'on considère. Nous nous concentrons sur le disque gris pour le tracé de l'ensemble capté.



(b) Les coordonnées  $(\theta, \eta)$  sur le disque gris.



(c) L'ensemble des points captés dans le futur pour le disque gris (*incoming tail*).



(d) L'ensemble capté pour le disque gris.

Figure 1.3: On montre l'ensemble des points captés dans le futur (1.3c) et captés (futur et passé)(1.3d) pour l'exemple de (1.3a) en se restreignant à l'obstacle gris. Les coordonnées utilisées pour le décrire sont données par (1.3b) : la variable en abscisse est  $\theta \in [0, \pi]$  et la variable en ordonnée est  $\eta \in [-1, 1]$ , associé à un vecteur sortant unitaire  $\xi$ . Pour ce qui est de l'ensemble des points captés dans le futur, on voit apparaître une réunion de variétés stables, disposés de manière fractale le long de variété instables.

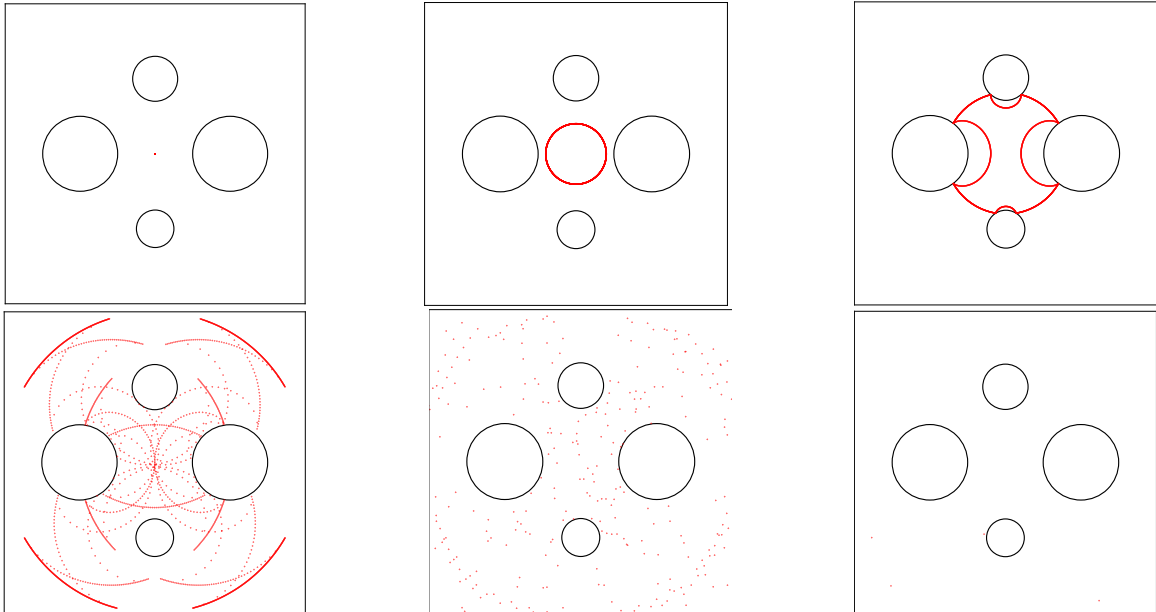


Figure 1.4: Observons la dynamique de nombreuses particules (il y en a 2000) avec la même position initiale et avec des vitesses initiales réparties uniformément sur le cercle unité. Dans cet exemple, les angles des vitesses sont donnés par  $\theta_k = 0.01 + 2\pi \frac{k}{2000}$ . Après un temps plutôt court, la plupart (et en fait, toutes dans cet exemple) les particules, s'échappent de la fenêtre. Voir [la vidéo en ligne](#).

la somme portant sur l'ensemble des trajectoires périodiques  $\gamma$  de longueur  $|\gamma| \in [T - 1, T]$ . La longueur d'une trajectoire  $|\gamma|$  est la somme des longueurs des rayons qui constituent  $\gamma$ . Quand  $s = 0$ ,  $P(0)$  estime simplement le nombre de trajectoire périodiques et est en fait l'entropie topologique  $h_{top}$ . Quand  $s = 1$ , on peut en fait montrer que  $P(1) < 0$  et  $-P(1)$  est souvent appelé taux de décroissance classique du système, noté  $\gamma_{cl}$ .  $P(s)$  est une façon de mesurer la compétition entre la complexité (nombre de trajectoires périodiques) et l'instabilité (due à l'hyperbolicité). Il est connu que  $s \mapsto P(s)$  est strictement décroissante.

- La **dimension** de l'ensemble capté. En raison de l'hyperbolicité, l'ensemble capté possède une structure fractale (voir par exemple le livre [Bar08]). Sa dimension boîte (ou de Minkowski) et sa dimension de Hausdorff coïncident en dimension 2. De plus, pour des raisons liées à la structure bien particulièrement de l'ensemble capté, on note souvent cette dimension  $2d_H + 1$ . En dimension 2,  $P(s)$  et  $d_H$  sont reliés par une formule, connue sous le nom de la formule de Bowen

$$P(d_H) = 0$$

#### Remarque.

Dans le cadre du scattering par des obstacles, il semble plus facile de travailler avec l'application du billard, plutôt que le flot du billard, qui, tel quel, possède des singularités au niveau des points de réflexion. Cependant, comme expliqué dans [DSW21], il est possible de construire un modèle lisse dans lequel le flot du billard est lisse (dans un voisinage de l'ensemble capté) et préserve les propriétés dynamiques (hyperbolicité, longueur des trajectoires périodiques, etc.). Dans ce même article, les auteurs travaillent avec une condition plus faible interdisant les trajectoires captées d'avoir des rayons rasants. Comme expliqué, cette condition est automatiquement vérifiée si l'on impose l'hypothèse de non-éclipse. En fait, il serait envisageable de travailler également avec cette hypothèse plus faible. Néanmoins, cette thèse utilise de façon cruciale [NSZ14], où les auteurs font l'hypothèse de non-éclipse. Pour cette raison, nous faisons le choix de travailler avec cette hypothèse plus forte.

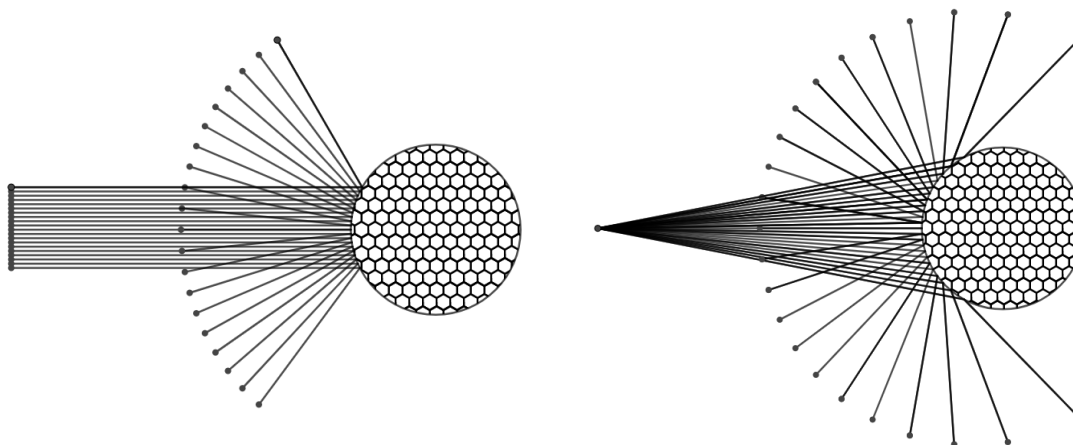


Figure 1.5: Ces figures montrent deux familles de trajectoires qui dépendent continument d'un paramètre. Après une réflexion sur le bord de l'obstacle strictement convexe, on observe distinctement l'instabilité par rapport à la condition initiale.

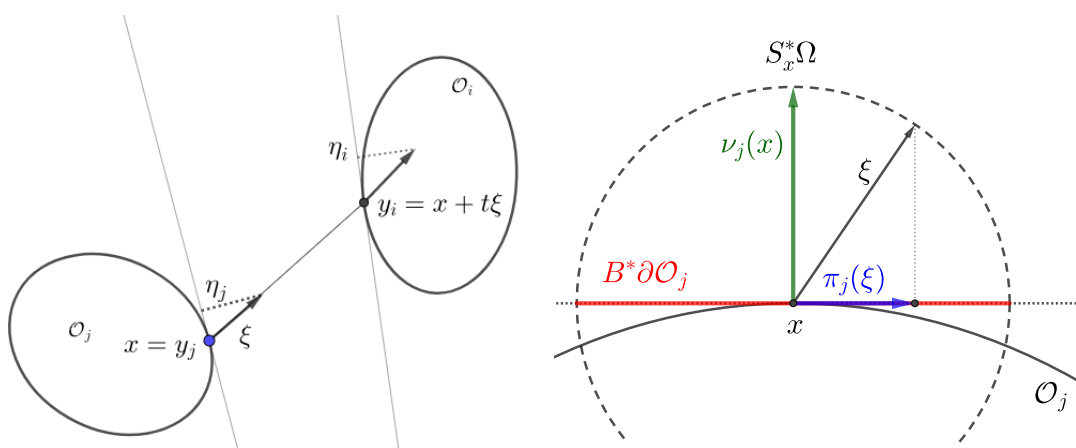


Figure 1.6: La transformation du billard.

#### 1-1.4. Résultats existants et apports principaux de cette thèse.

Nous allons maintenant présenter, de façon non exhaustive, certains résultats existants concernant la répartition des résonances. Nous commençons par le cas d'un seul obstacle, qui est une situation dans laquelle l'ensemble capté est vide (on parle de situation de "non-trapping"). Ensuite, nous détaillerons les résultats dans le cas de deux obstacles : il s'agit d'une situation dans laquelle la répartition des obstacles est particulièrement bien comprise. Finalement, nous explorerons le cas plus complexe de  $J \geq 3$  obstacles et présentons les deux principaux résultats de ce manuscrit.

##### 1-1.4.1. Le cas d'un seul obstacle.

Avant de donner des résultats généraux pour le scattering par un obstacle convexe, intéressons-nous succinctement au cas du scattering par une sphère dans  $\mathbb{R}^3$ . Il s'agit d'une des seules situations où des calculs explicites peuvent être menés, alors profitons-en !

*Exemple.* Scattering par une sphère de  $\mathbb{R}^3$ . (voir aussi [Tay10], Chapitre 9, Section 9)  
Soit  $\mathcal{O} = B(0, R) \subset \mathbb{R}^3$  la boule de centre 0 et de rayon  $R$ . On note  $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$ . On cherche à calculer les résonances de scattering pour cet obstacle. Bien entendu, on va exploiter la symétrie radiale du problème. Toute fonction  $u : \Omega \rightarrow \mathbb{C}$  peut être décomposée dans la base Hilbertienne de  $L^2(\mathbb{S}^2)$  composée des polynômes harmoniques : pour  $l \in \mathbb{N}$  et  $-l \leq m \leq l$ , on note  $\phi_{l,m}$  un élément de cette base associé à la valeur propre  $l(l+1)$  du laplacien sur  $\mathbb{S}^2$ .

*Etats résonants.* Dans ce contexte, il est connu qu'il n'y a pas de résonance réelle, et l'on peut donc

se concentrer sur le cas  $\lambda \neq 0$  afin d'utiliser la caractérisation des états résonants donnée plus haut. On cherche donc  $u \in H_{loc}^2(\Omega) \cap H_0^1(\Omega)$  telle que  $u$  est une solution sortant de  $(-\Delta - \lambda^2)u = 0$ . On peut écrire

$$u(r, \omega) = \sum_{l,m} u_{l,m}(r) \phi_{l,m}(\omega), \quad r > R$$

où  $u_{l,m}(R) = 0$  et  $u_{l,m}$  est solution de l'EDO

$$r^2 u'' + 2r u' + (r^2 \lambda^2 - l(l+1)) u = 0$$

Cette EDO a deux solutions fondamentales  $r \mapsto h_l^{(1)}(\lambda r)$  et  $r \mapsto h_l^{(2)}(\lambda r)$  où  $h_l^{(i)}$  sont les fonctions de Bessel Sphériques du troisième type (voir par exemple [AS72]). Elles peuvent s'écrire

$$h_l^{(1)}(z) = \frac{e^{iz}}{z} p_l(z^{-1}); \quad h_l^{(2)}(z) = \frac{e^{-iz}}{z} q_l(z^{-1})$$

où  $p_l$  et  $q_l$  sont des fonctions polynomiales de degré  $l$ . Plus précisément,  $p_l$  est donné par

$$p_l(X) = (-i)^{l+1} \sum_{k=0}^l \frac{(l+k)!}{k!(l-k)!} \left(\frac{iX}{2}\right)^k$$

*Propriété sortante.* Afin que  $u$  soit sortante, le seul choix possible parmi les solutions de cette EDO est de prendre

$$u_{l,m}(r) = a_{l,m} h_l^{(1)}(\lambda r)$$

En effet, si l'on cherche à résoudre le problème libre en coordonnées sphériques, on est amené à résoudre les mêmes EDOs et quand  $\text{Im } \lambda > 0$ ,  $h_l^{(1)}(\lambda r)$  est la seule solution qui permet de garder le caractère borné sur  $L^2(\mathbb{R}^3)$  de l'opérateur, puisque  $h_l^{(2)}(\lambda r)$  croît exponentiellement vite quand  $r \rightarrow +\infty$  si  $\text{Im } \lambda > 0$ .

*Condition de Dirichlet.* Si l'on veut que  $u$  s'annule sur le bord de la boule sans que  $u$  ne soit identiquement nulle,  $\lambda$  doit vérifier

$$h_l^{(1)}(\lambda R) = 0 \tag{H_l}$$

*Conclusion.* Ainsi, si  $\lambda$  ne vérifie aucune des conditions  $H_l$ , il n'existe aucune solution sortante non triviale de  $-\Delta u - \lambda^2 u = 0$ . Au contraire, si l'une des conditions  $H_l$  est vérifiée, les fonctions  $h_l^{(1)}(r) \phi_{l,m}$  donnent  $2l+1$  solutions indépendantes, de sorte que  $\lambda$  est une résonance et sa multiplicité se calcule en sommant les  $2l+1$  pour  $l$  parcourant l'ensemble des entiers telles que  $h_l^{(1)}(\lambda R) = 0$ . Ces résonances sont représentées sur la Figure 1.7.

Venons-en désormais au cas général d'un obstacle convexe. Le flot du billard est évidemment "non-trapping". Le résultat qui suit est alors standard dans le cas des situations de "non-trapping" et s'appuie sur des résultats de propagations de singularités (voir [DZ19] Section 4.6 et 6.4 et les références indiquées, voir aussi [MRS77], [Mel79], [MS78], [RS82]).

**Théorème 1-1.8.** *Soit  $\mathcal{O}$  un obstacle strictement convexe de frontière lisse. Alors, il existe  $\lambda_0 > 0$  et  $A > 0$  tels qu'il n'y a pas de résonance dans*

$$\{\lambda \in \mathbb{C} \setminus (i\mathbb{R}^-), |\text{Re } \lambda| > \lambda_0, \text{Im } \lambda \geq -A \log |\text{Re } \lambda|\}$$

**Remarque.**

- Ce résultat fournit donc une réponse claire aux deux questions qui nous intéressent. Il y a effectivement un trou spectral et chaque  $N_{\gamma,\delta}(r)$  est nulle pour  $r \geq r_\gamma$ .
- Ce résultat est déjà démontré dans [LP71] en dimension 3.

Cette région logarithmique sans résonance a été améliorée :

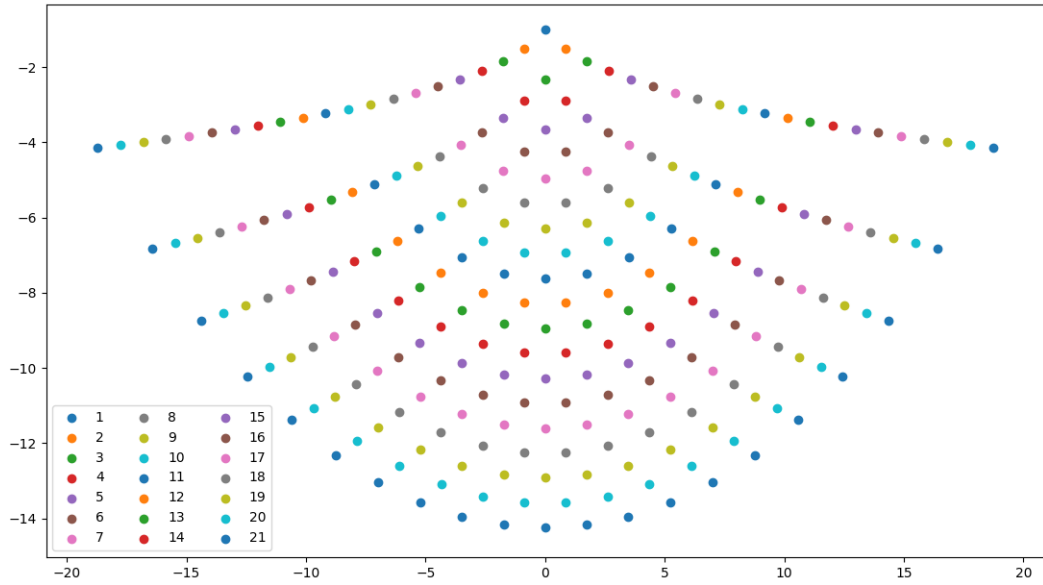


Figure 1.7: Les premières résonances de scattering pour la sphère de rayon 1 sont représentées. Chaque couleur correspond à une famille de pôles, associé à un entier  $l = 1, \dots, 21$ . Dans une telle famille, chaque pôle a en fait une multiplicité  $2l + 1$ .

**Théorème 1-1.9** ([HL94], [SZ95]). *Soit  $\mathcal{O}$  un obstacle strictement convexe de frontière lisse. Alors, il existe  $\lambda_0 > 0$  et  $c_1 > 0$  tel qu'il n'existe pas de résonance  $\lambda$  vérifiant*

$$\text{Im } \lambda \geq -c_0(\text{Re } \lambda)^{1/3} + c_1 |\text{Re } \lambda| \geq \lambda_0$$

*où  $c_0$  est une constante qui se calcule explicitement et qui dépend de  $\mathcal{O}$ .*

Ce résultat a d'abord été établi pour des obstacles avec des frontières de régularité analytique dans [BLR87], où les auteurs ont également démontré que cette borne est optimale dans un certain sens, pour des obstacles génériques. Dans [SZ95], les auteurs fournissent des estimées plus générales pour des fonctions de comptages dans différentes régions sous l'axe réel et obtiennent cette région sans résonance comme un corollaire immédiat. Les deux auteurs ont aussi amélioré ce résultat dans [SZ99] où ils prouvent l'existence de bandes cubiques dans lesquelles les résonances doivent se situer, sous certaines conditions de pincement de la courbure. En un certain sens, plus l'obstacle ressemble à une sphère, plus la répartition des résonances ressemble à celle observée dans la Figure 1.7.

#### 1-1.4.2. Le cas de deux obstacles.

En ce qui concerne le cas de deux obstacles, l'ensemble capté n'est plus vide. Il est néanmoins loin d'être compliqué : il est en effet donné par une unique trajectoire périodique, qui correspond au rayon qui minimise la distance entre les deux obstacles.

En un certain sens, l'ensemble capté est trop fin pour permettre aux résonances d'approcher l'axe réel quand  $\text{Re } \lambda \rightarrow +\infty$ . Cette explication heuristique est en fait l'une des motivations principales de notre travail sur le trou spectral pour des situations plus complexes. Dans cette direction, la première preuve d'un trou spectral pour deux obstacles strictement convexes apparaît dans le travail d'Ikawa [Ika82] en dimension 3. Remarquons au passage que ce résultat implique que la conjecture énoncée par Lax-Phillips pour des obstacles captants est fautive. En effet, ils ont conjecturé que dans une telle situation de "trapping", il devrait exister une suite de résonances  $(\lambda_n)$



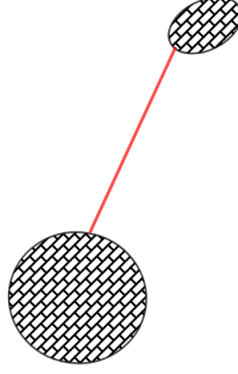


Figure 1.8: Deux obstacles et l'ensemble capté associé.

telle que  $\text{Im } \lambda_n \rightarrow 0$  et  $\text{Re } \lambda_n \rightarrow +\infty$ . Ikawa a alors été amené à formuler une nouvelle conjecture ([Ika06]), connue sous le nom de *conjecture de Lax-Phillips modifiée*, et toujours ouverte à ce jour.

**Conjecture.** *Dans le cas d'un nombre arbitraire d'obstacles  $J \geq 2$ , il existe  $\gamma > 0$  tel que la bande  $\{\text{Re } \lambda \geq 1, \text{Im } \lambda \in [-\gamma, 0]\}$  contient une infinité de résonances.*

Dans le cas de deux obstacles, cette conjecture a été résolue et l'on sait désormais qu'elle est vraie. En fait, dans la limite des hautes fréquences,  $|\text{Re } \lambda| \rightarrow +\infty$ , il est possible de décrire asymptotiquement toutes les résonances dans la bande  $\{\text{Re } \lambda \geq 1, \text{Im } \lambda \in [-\gamma, 0]\}$ . Si Ikawa est le premier à avoir obtenu une bande avec une infinité de résonances ([Ika83], [Ika87]) en dimension 3, son résultat a été significativement amélioré par [Gér88] (en dimension impaire quelconque), qui démontre l'existence d'un pseudo-réseau de résonances. Ce résultat énoncé et prouvé en dimension impaire par [Gér88] pourrait en fait s'adapter en dimension paire (voir [NSZ14], Section 6). Afin d'énoncer ce résultat précisément, il nous faut introduire quelques notations.

Notons  $\mathcal{O}_1$  et  $\mathcal{O}_2$  les deux obstacles strictement convexes et soit  $d_{\text{obs}} = d(\mathcal{O}_1, \mathcal{O}_2)$ . Cette distance est atteinte pour un unique rayon  $\gamma_0$  qui donne une trajectoire périodique du flot du billard : on peut associer au rayon  $\gamma_0$  un point fixe périodique  $\rho_0$  de  $\mathcal{B}^2 := \mathcal{B}_{21} \circ \mathcal{B}_{12}$  et en raison de la stricte convexité des obstacles,  $\rho_0$  est un point fixe hyperbolique, c'est-à-dire que les valeurs propres de  $d_{\rho_0} \mathcal{B}^2$  sont toutes de modules différent de 1. Notons  $\nu_1, \dots, \nu_{d-1}$  les valeurs propres de module plus petit que 1 (éventuellement répétées avec multiplicité). En fait,  $\mathcal{B}^2$  est symplectique, de sorte que les valeurs propres de modules plus grand que 1 sont les  $\nu_i^{-1}$ . Notons alors  $b_0 = (\nu_1 \dots \nu_{d-1})^{1/2} = J^u(\gamma_0)^{-1/2}$  et  $c_0 = \log \frac{2d_{\text{obs}}}{b_0}$ . Pour  $\alpha \in \mathbb{N}^{d-1}$ , on note  $\nu^\alpha = \nu_1^{\alpha_1} \dots \nu_{d-1}^{\alpha_{d-1}}$ . Posons  $I = \{\nu^\alpha, \alpha \in \mathbb{N}^{d-1}\}$  et pour  $m \in I$ , notons

$$N_m = \#\{\alpha \in \mathbb{N}^{d-1}, \nu^\alpha = m\}$$

Pour chaque  $m \in I$ , on peut alors définir une ligne de pseudo-pôles :

$$\lambda_m(j) = j \frac{\pi}{d_{\text{obs}}} + i(c_0 - \log m), \quad j \in \mathbb{Z}$$

**Théorème 1-1.10** ([Gér88]). *Notons  $\text{Res}$  l'ensemble des résonances pour les deux obstacles  $\mathcal{O}_1$  et  $\mathcal{O}_2$ . Fixons  $\gamma > 0$ . Alors, il existe  $\lambda_0 > 0$  et  $C > 0$  tels que, en notant  $\Omega(\lambda_0, \gamma) = \{\lambda, \text{Re } \lambda > \lambda_0, \text{Im } \lambda > -\gamma\}$ , on a*

$$\text{Res} \cap \Omega(\lambda_0, \gamma) \subset \bigcup_{\substack{m \in I, j \in \mathbb{Z} \\ \lambda_m(j) \in \Omega(\lambda_0, \gamma)}} D\left(\lambda_m(j), C|\lambda_m(j)|^{1/2}\right)$$

*et chaque  $D(\lambda_m(j), C|\lambda_m(j)|^{1/2})$  contient exactement  $N_m$  résonances (comptés avec multiplicité).*

**Remarque.**

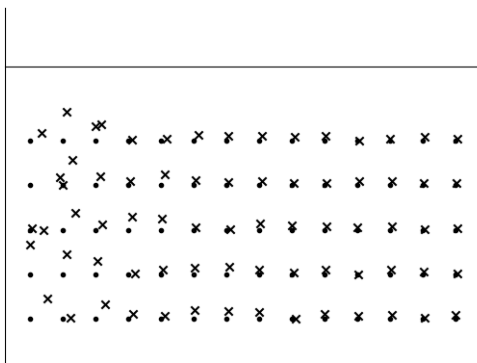


Figure 1.9: Asymptotiquement, les résonances (les croix) pour deux obstacles convexes, convergent vers un pseudo-réseau (les points).

En fait, le résultat de [Gér88] est beaucoup plus précis puisqu'il donne un développement asymptotique en puissance de  $\lambda$  de chacun des  $N_m$  pôles dans  $D(\lambda_m(j), C|\lambda_m(j)|^{1/2})$ .

### 1-1.4.3. Cas général.

Passons finalement au cas général d'un nombre arbitraire d'obstacles. Nous présentons les principaux résultats existants autour des questions 1.1 et 1.2 et exposons les **principaux résultats nouveaux démontrés dans de cette thèse**.

**Trous spectraux.** Dans le cas général de  $J \geq 3$  obstacles, l'existence d'un trou spectral reste une question ouverte. Il existe néanmoins un certains nombres de travaux expérimentaux (voir par exemple [BWP<sup>+</sup>13]) qui ont conduit à formuler la conjecture suivante ([Zwo17]) :

**Conjecture.** *Pour des obstacles strictement convexes avec un frontière lisse et vérifiant la condition de non-éclipse, il existe toujours un trou spectral.*

Cette conjecture, telle qu'énoncée dans [Zwo17] concerne en réalité une plus large classes de systèmes hyperboliques ouverts. Nous y reviendrons très rapidement.

Il semblerait que l'un des premiers résultats positifs qui prouve l'existence d'un trou spectral pour plus de 3 obstacles strictement convexes a été obtenu par Ikawa [Ika88] dans le cas de la dimension 3, et sous certaines hypothèses. Son résultat a été revisité par [NZ09] et généralisé en dimension quelconque :

**Théorème 1-1.11** ([Ika88], [NZ09]). *Supposons que  $P(1/2) < 0$ . Alors il existe un trou spectral. Plus précisément, pour tout  $\varepsilon > 0$ , il existe  $\lambda_\varepsilon > 0$  tel qu'il n'a y pas de résonances dans*

$$\{|\operatorname{Re} \lambda| > \lambda_\varepsilon, \operatorname{Im} \lambda \geq (P(1/2) + \varepsilon)\}$$

### Remarque.

- Le résultat de [NZ09] concerne a priori une situation différente. Néanmoins, il peut aussi s'adapter pour donner un trou spectral dans le problème des obstacles: il est nécessaire pour cela d'utiliser les résultats de [NSZ14]. Cela est expliqué plus en détail dans [Non11] (Section 8).
- Le trou spectral a également été amélioré dans [PS10] sous certaines hypothèses dynamiques supplémentaires. Entre autres, pour obtenir un meilleur trou spectral, les auteurs ont besoin qu'un certain opérateur de transfert, associé au flot du billard, vérifie une *estimée à la Dolgopyat*.

- Ce type de condition de pression faisant intervenir  $\frac{1}{2} \log J^u$  apparaît dans d'autres travaux, en lien avec des problèmes différents. Dans [Bur93], l'auteur démontre un résultat de contrôle pour l'équation des plaques dans un ouvert borné contenant des obstacles strictement convexes qui vérifient la condition de non-éclipse. Ce résultat est valable sous la condition  $P(1/2) < 0$ . C'est aussi ce genre de condition qui apparaît dans les travaux [Sch10] et [Riv14], en lien avec l'étude de l'équation des ondes amortie sur des variétés compactes sans bord.

Dans cette thèse, nous montrons qu'en dimension 2, la conjecture est vraie sans condition sur la pression topologique :

**Théorème A** (Trou spectral, [Vac22b]). *En dimension  $d = 2$ , pour des obstacles strictement convexes avec une frontière lisse et vérifiant la condition de non-éclipse, il existe toujours un trou spectral.*

Ce résultat s'accompagne d'estimations pour la résolvante tronquée dans la bande sans résonance que donne le trou spectral. En particulier, de telles estimées ont des applications pour le contrôle de la décroissance de l'énergie locale pour l'équation des ondes (voir Chapitre 6, [Vac22a]).

**Bornes supérieures pour le comptage des résonances.** Commençons par rappeler que l'on note  $N_{\gamma,\delta}(r)$  le nombre de résonances (avec multiplicité) dans la boîte  $\{\operatorname{Re} \lambda \in [r, r + \delta], \operatorname{Im} \lambda \in [-\gamma, 0]\}$ . Pour simplifier - et parce que cela a peu d'importance tant que  $\delta$  est indépendant de  $r$  - on fixe  $\delta = 1$  et on pose  $N_\gamma = N_{\gamma,1}$ . Plusieurs résultats numériques et expérimentaux ([PWB<sup>+</sup>12], [EMW10]) suggèrent que  $N_\gamma(r)$  possède le comportement suivant :

$$N_\gamma(r) \sim C_\gamma r^{\alpha_\gamma}$$

Cependant, la démonstration d'un tel comportement est aujourd'hui hors de portée. En effet, il est souvent particulièrement difficile d'obtenir des bornes inférieures. Même des formes plus faibles d'un tel résultat, i.e., des résultats comme

$$N_\gamma(r) \geq C_\gamma r^{\alpha_\gamma}$$

constitueraient des avancées remarquables dans ce domaine. A ce jour, des bornes supérieures sont connues, et sont souvent appelées bornes de Weyl fractales supérieures :

**Théorème 1-1.12** ([NSZ14]). *Pour des obstacles strictement convexes avec une frontière lisse et vérifiant la condition de non-éclipse, la borne supérieure suivante est valable : pour tout  $\gamma > 0$  et pour tout  $\varepsilon > 0$ , il existe  $C_{\gamma,\varepsilon} > 0$  tel que pour tout  $r > 1$ ,*

$$N_\gamma(r) \leq C_{\gamma,\varepsilon} r^{d_H + \varepsilon}$$

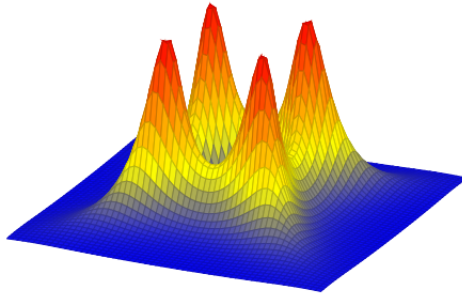
#### Remarque.

- Ce résultat est consistant avec ce qui est connu dans le cas de deux obstacles puisque  $d_H = 0$  dans ce cas et que l'on connaît la répartition asymptotique des résonances.
- Un trou spectral de profondeur  $\gamma$  est équivalent à  $N_\gamma(r) = 0$  pour  $r$  suffisamment grand. En particulier s'il y a un trou spectral, comme conjecturé, cette borne ne peut pas être optimal pour toutes les valeurs de  $\gamma$ .

Dans les travaux d'investigations expérimentaux comme numériques qui ont été menés, la valeur  $\gamma = \gamma_{cl}/2$  semble avoir un rôle important. On observe notamment deux phénomènes importants :

- Le nombre de résonances semble être plus petit que la borne donnée par le théorème précédent, quand  $\gamma < \gamma_{cl}/2$  ;
- On observe une concentration des résonances au voisinage de la droite  $\gamma = \gamma_{cl}/2$ .

Cette première observation peut être reliée à la conjecture dite de Jakobson-Naud ([JN12]) qui concerne d'abord les surfaces hyperboliques convexes co-compactes (voir ci-après). Dans ce contexte, l'équivalent de la conjecture pourrait être : il y a un trou spectral de profondeur  $\gamma_{cl}/2$ . Dans cette thèse, nous donnons plutôt une réponse à une conjecture qui paraît plus accessible [Zwo17] :

Figure 1.10: Exemple d'un potentiel dans  $\mathbb{R}^2$  avec 4 pics.

**Conjecture.** Pour tout  $\gamma < \gamma_{cl}$ ,  $N_\gamma(r)r^{-d_H} \rightarrow 0$  quand  $r \rightarrow +\infty$ .

Nous démontrons cette conjecture dans le cas de la dimension 2, en prouvant une borne supérieure meilleure pour  $\gamma < \gamma_{cl}$ .

**Théorème B** (Borne supérieure de Weyl fractale améliorée). *En dimension 2, pour des obstacles strictement convexes avec une frontière lisse et vérifiant la condition de non-éclipse, il existe une fonction  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  vérifiant*

- $\sigma(\gamma) > 0$  for  $0 \leq \gamma < \gamma_{cl}/2$  ;
- $\sigma(\gamma) = 0$  for  $\gamma \geq \gamma_{cl}/2$

et telle que pour tout  $\gamma > 0$  et pour tout  $\varepsilon > 0$ , il existe  $C_{\gamma,\varepsilon}$  tel que

$$\forall r \geq 1, N(r, \gamma) \leq C_{\gamma,\varepsilon} r^{d_H - \max(\sigma(\gamma) - \varepsilon, 0)}$$

## 1-1.5. Résultats analogues dans d'autres systèmes hyperboliques ouverts.

### 1-1.5.1. Scattering semiclassique par un potentiel.

Le scattering par un potentiel est une façon naturelle d'étendre le scattering par des obstacles, en remplaçant les obstacles, vus comme des barrières de potentiel infini infranchissables, par des potentiels  $V \in C_c^\infty(\mathbb{R}^d)$ . Dans ces systèmes, on cherche à étudier les résonances de l'opérateur  $-\Delta + V$ , qui sont définies, comme précédemment, en tant que pôles du prolongement méromorphe de  $\chi(-\Delta + V - \lambda^2)^{-1}\chi$  (voir par exemple [DZ19]). En fait, pour garder l'analogie avec le problème des obstacles, il est plus pertinent d'étudier le problème semiclassique, à savoir l'étude de l'opérateur semiclassique  $P_0(h) = -h^2\Delta + V$ . Si, dans le cas des obstacles, le paramètre semiclassique  $h$  est avant tout un moyen pratique d'utiliser les outils de l'analyse semiclassique, quand il s'agit du potentiel, l'étude de la famille  $P_0(h)$  dans la limite  $h \rightarrow 0$  est un problème en soi. L'opérateur  $P_0(h)$  est un opérateur semiclassique associé à l'hamiltonien  $p(x, \xi) = |\xi|^2 + V(x)$ . Pour étudier  $P_0(h)$ , il est nécessaire de comprendre la dynamique classique hamiltonienne associée. Contrairement au cas des obstacles, l'étude semiclassique doit avoir lieu autour d'une énergie fixée  $E$  : pour tout  $C > 0$  fixé, on cherche à comprendre la distribution des résonances de  $P_0(h)$  dans  $D(E, Ch)$ , qui est reliée au flot hamiltonien sur la couche d'énergie  $p^{-1}(E)$ . Les questions 1.1 et 1.2 peuvent alors aisément se reformuler dans ce contexte :

- Existe-t-il  $\gamma > 0$  tel qu'il n'y aucune résonance dans  $D(E, Ch) \cap \{\text{Im } z \geq -\gamma\}$  pour  $h$  assez petit ?
- Peut-on trouver un équivalent/borne supérieure/borne inférieure pour le nombre de résonances  $N_{\gamma,E,C}(h)$  dans  $D(E, Ch) \cap \{\text{Im } z \geq -\gamma\}$  dans la limite  $h \rightarrow 0$ ?

Tout comme dans le cas du scattering par des obstacles, ces questions sont reliées aux propriétés de l'ensemble capté  $K_E$  à énergie  $E$  : si l'on  $\varphi^t : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$  le flot hamiltonien associé à  $p$ ,

$$K_E = \{\rho \in p^{-1}(E), \varphi^t(\rho) \not\rightarrow \infty \text{ quand } t \rightarrow +\infty \text{ et } -\infty\}$$

Quand l'ensemble capté est un ensemble hyperbolique pour  $\varphi^t$ , la situation est semblable à celle rencontrée pour des obstacles strictement convexes puisque les flots classiques partagent les mêmes propriétés. Il devient alors possible de prouver les mêmes résultats :

- Si  $K_E = \emptyset$ , des régions logarithmiques sans résonances sont connues ([SZ07], [DZ19], Théorème 6.21 et les références indiquées)
- Quand  $K_E$  est réduit à une unique orbite périodique hyperbolique, la même répartition asymptotique au voisinage d'un pseudo-réseau a lieu ([GS87]).
- Pour des situations plus générales, et donc plus complexes, le trou spectral est connu sous la condition de pression  $P(1/2) < 0$  ([NZ09]) et dans [SZ07], les auteurs démontrent une borne supérieure de Weyl fractale

$$N_{\gamma,C,E}(h) = O(h^{-d_H - \varepsilon})$$

En fait, sous des hypothèses supplémentaires plutôt raisonnables sur l'ensemble capté - notamment si  $K_E$  possède une dimension topologique égale à 1 - les travaux de S. Nonnenmacher, J. Sjöstrand et M. Zworski dans [NSZ11] et [NSZ14] montrent qu'il est possible de traiter uniformément les deux modèles (obstacles et potentiels), en utilisant une réduction du problème de résonances à l'étude *d'applications quantiques hyperboliques ouvertes*. Cette réduction est un des outils fondamentaux exploités dans cette thèse. Dans [NSZ14], les auteurs démontrent la borne fractale obtenue par [SZ07] en utilisant cette réduction, qui amène à une preuve différente (et en un certain sens, plus simple). En utilisant cette réduction, les Théorèmes A et B sont démontrés dans cette thèse également dans le cas du scattering semiclassique par un potentiel.

Il est par ailleurs possible de traiter des potentiels  $V$  plus généraux, et même des opérateurs pseudodifférentiels  $P(x, hD_x)$  (se référer pour cela aux hypothèses de [NSZ11]). Il est également possible de considérer des variétés avec divers types d'infinis. Dans [NSZ11], les auteurs considèrent des variétés  $(X, g)$  qui possèdent un nombre fini d'infinis euclidiens  $\mathbb{R}^d \setminus B(0, R)$ . En outre, les méthodes de Vasy, qui permettent de traiter des infinis asymptotiquement hyperboliques ([Vas13]), combinées aux méthodes de recollement développées avec Datchev permettent de traiter séparément l'infini et la région d'interaction où vit l'ensemble capté ([DV12]), sous des hypothèses géométriques et microlocales raisonnables et sous réserve d'avoir certaines estimées de résolvante dans chacune des régions. En particulier, puisque des estimées de résolvante sont obtenues dans cette thèse (Chapitre 6), ces méthodes de recollement peuvent conduire à des trous spectraux pour des infinis différents que ceux euclidiens et notamment asymptotiquement hyperboliques. Parmi des variétés asymptotiquement hyperboliques, les surfaces hyperboliques convexes co-compactes ont une place particulière et leur étude spectrale partage un certain nombre de propriétés avec le scattering par des obstacles. La prochaine sous-section consiste à présenter les aspects principaux de cette théorie et de mettre en avant l'analogie avec le cas des obstacles.

### 1-1.5.2. Surfaces hyperboliques convexes co-compactes.

Dans cette sous-section, nous proposons une courte présentation de la théorie spectrale des surfaces hyperboliques convexes co-compactes. L'auteur a appris une grande partie de cette théorie dans le livre de D. Borthwick [Bor16].

**Géométrie.** Les surfaces hyperboliques convexes co-compactes sont des quotients du demi-plan de Poincaré

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2, y > 0\},$$

muni la métrique hyperbolique  $y^{-2}(dx^2 + dy^2)$ , par certains sous-groupes particuliers de  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$  (eux mêmes appelés convexes co-compactes). Ce sont des surfaces de courbure -1 et elles possèdent des infinis hyperboliques bien particuliers. Il existe en effet deux sortes d'infinis hyperboliques :

- Les pointes hyperboliques : elles sont construites à partir du modèle du cylindre parabolique. Il s'agit du quotient  $\mathbb{H}^2$  par le groupe  $\Gamma_\infty$  engendré par la transformation  $z \mapsto z + 1$ . Nous n'aurons pas à nous intéresser à ces objets.
- Les entonnoirs hyperboliques : ce sont des demi-cylindres hyperboliques. Un cylindre hyperbolique avec une géodésique centrale de longueur  $l$  est le quotient de  $\mathbb{H}^2$  par le groupe  $\Gamma_l$



(a) Un entonnoir hyperbolique

(b) Une pointe hyperbolique

Figure 1.11: Les deux modèles d'infinis hyperboliques.

engendré par  $z \mapsto e^l z$ . Il est possible de paramétrer le cylindre hyperbolique par les variables  $(r, \theta) \in \mathbb{R} \times \mathbb{S}^1$  avec la métrique

$$dr^2 + \frac{l^2}{4\pi^2} \cosh^2(r) d\theta^2$$

La courbe  $\{r = 0\}$  est une géodésique de longueur  $l$ . Dans ce système de coordonnées, l'entonnoir hyperbolique  $F_l$  est la région  $\{r > 0\} \times \mathbb{S}^1$ .

Une surface hyperbolique  $M = \mathbb{H}^2/\Gamma$  d'aire infini est dite convexe co-compacte si elle est géométriquement finie et si elle ne possède pas de pointe hyperbolique. Elle peut en fait se décomposer de la façon suivante :

$$M = N \sqcup \bigsqcup_{i=1}^L F_i$$

où chaque  $F_i$ ,  $1 \leq i \leq L$  est un entonnoir et  $N$  est une partie compacte (appelée le cœur compact). Ces surfaces sont particulièrement bien adaptées pour la théorie du scattering : chaque entonnoir donne un infini avec lequel il est possible de définir des notions d'ondes sortantes et entrantes.

**Théorie spectrale.** Pour de telles surfaces  $(M, g)$ , on s'intéresse aux propriétés spectrales de l'opérateur de Laplace-Beltrami  $-\Delta_g$ . En ce qui concerne ces surfaces, le spectre de  $-\Delta_g$  vérifie

**Théorème 1-1.13** ([LP82], [LP84], voir également [Bor16], Chapitre 7). *Soit  $(M, g)$  une surface hyperbolique convexe co-compacte. Le spectre  $\sigma(-\Delta_g)$  est constitué du spectre essentiel, égal à  $[1/4, +\infty[$  et éventuellement d'un nombre fini de valeurs propres dans  $]0, 1/4[$ . Il n'y a par ailleurs pas de valeur propre plongée dans  $[1/4, +\infty[$ .*

Puisque le spectre essentiel est  $[1/4, +\infty[$ , on considère généralement la résolvante

$$R(\lambda) := (-\Delta_g - 1/4 - \lambda^2)^{-1}, \operatorname{Im} \lambda > 0, 1/4 + \lambda^2 \notin \sigma(-\Delta_g)$$

Pour  $\operatorname{Im} \lambda > 0$ , c'est une famille méromorphe d'opérateurs avec des pôles de rang fini, les pôles étant en bijection avec l'ensemble des valeurs propres de  $]0, 1/4[$ . En fait, une autre convention est en général utilisée pour décrire le paramètre spectral dans ce cas : la variable spectrale est  $s \in \{\operatorname{Re} s > 1/2\}$  reliée à  $\lambda$  par la relation  $s(1-s) = 1/4 + \lambda^2$ , et l'on étudie alors  $R_M(s) = (-\Delta_g - s(1-s))^{-1}$ . La principale raison pour laquelle ce choix est fait provient du fait que  $y \in \mathbb{H}^2 \mapsto y^s$  and  $y \in \mathbb{H}^2 \mapsto y^{1-s}$  sont des solutions de

$$-\Delta_g u = s(1-s)u$$

En effet,  $-\Delta_g = y^2(\partial_x^2 + \partial_y^2)$  pour  $\mathbb{H}^2$ . En particulier, la fonction  $y^s$  (resp.  $y^{1-s}$ ) joue un rôle similaire à la fonction  $e^{i\lambda|x|}$  (resp.  $e^{-i\lambda|x|}$ ) en tant que fonction sortante (resp. entrante). Tout comme dans le cas euclidien, il est possible de prolonger méromorphiquement la résolvante :

**Théorème 1-1.14** ([GZ95], voir aussi [Bor16], Chapitre 6). *La résolvante*

$$R_M(s) : L^2(M) \rightarrow H^2(M)$$

*bien définie pour  $\operatorname{Re} s > 1/2$ ,  $s(1-s) \notin \sigma(-\Delta)$ , se prolonge à tout le plan complexe en une famille méromorphe d'opérateurs avec des pôles de rang fini*

$$R_M(s) : L_{\text{comp}}^2(M) \rightarrow H_{\text{loc}}^2(M)$$

*Ses pôles sont appelés les résonances de la surface  $M$ .*

**Remarque.**

Ce théorème peut également être démontré en utilisant les méthodes de Vasy ([Vas13]), qui peuvent être utilisées pour une plus large classe de surfaces asymptotiquement hyperboliques.

Dans ce contexte, il est possible de poser les mêmes questions 1.1 et 1.2. Et encore une fois, il est important de comprendre la dynamique classique associée à cet objet "quantique".

**Dynamique.** La dynamique classique sous-jacente n'est autre que le flot géodésique sur  $M = \mathbb{H}^2/\Gamma$ . De façon équivalente, c'est le flot hamiltonien associé à  $p(x, \xi) = |\xi|_g^2$ . Ses propriétés sont héritées du flot géodésique sur le demi-plan de Poincaré et de l'action du sous-groupe  $\Gamma$  sur  $\mathbb{H}^2$ . En particulier, le flot géodésique est hyperbolique puisque la courbure est strictement négative. L'ensemble capté  $K \subset S^*M$  est défini comme dans les autres systèmes hyperboliques ouverts : c'est l'ensemble des points  $\rho \in S^*M$  telle que l'unique géodésique  $\gamma$  qui passe par  $\rho$  est contenue dans une région compacte de  $S^*M$ . En outre, dans ce modèle bien particulier, les résonances peuvent être égales décrites comme les zéros d'une certaine *fonction zeta dynamique*. La fonction zeta de Selberg est définie par

$$Z_M(s) = \prod_{\gamma \in \mathcal{P}} \prod_{l=0}^{\infty} \left(1 - e^{-(s+l)|\gamma|}\right)$$

où  $\mathcal{P}$  est l'ensemble des géodésiques fermées primitives et  $|\gamma|$  est la longueur de la géodésique  $\gamma$ . Ce produit converge absolument pour  $\operatorname{Re} s > \delta$  où  $\delta$  est appelé *l'exposant de convergence* de  $\Gamma$ .  $\delta$  est en fait défini rigoureusement comme étant

$$\delta = \inf \left\{ s \geq 0, \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma \cdot w)} \right\}, \quad z, w \in \mathbb{H}^2$$

où  $d$  désigne la distance hyperbolique sur  $\mathbb{H}^2$  et la définition de  $\delta$  ne dépend ni de  $z$  ni de  $w$ .

$Z_M$  peut se prolonger méromorphiquement à tout  $\mathbb{C}$  avec des zéros triviaux, dit topologiques, aux points  $s = -n$ ,  $n \in \mathbb{N}$  de multiplicité  $2n + 1$  et ses autres zéros sont donnés avec multiplicité par les résonances de  $M$  ([PP01]). Dans le cas des obstacles, il est possible de définir des fonctions zeta dynamiques en s'inspirant de la formule de  $Z_M$ . Néanmoins, s'il existe bien des liens entre résonances de scattering et fonctions zeta (voir par exemple [PS10]), les liens ne sont pas aussi clairs, à part dans le cas de deux obstacles ([Gér88]).

En ce qui concerne les surfaces hyperboliques, il est en fait possible de décrire l'ensemble capté à partir de l'ensemble limite  $\Lambda_\Gamma \subset \partial\mathbb{H}^2$ . Pour cette description, il est plus commode d'utiliser le modèle du disque de Poincaré pour  $\mathbb{H}^2$ . Dans ce modèle,  $\partial\mathbb{H}^2 = \mathbb{S}^1$  et  $\Lambda_\Gamma$  est défini comme

$$\Lambda_\Gamma := \{z \in \partial\mathbb{H}^2, \exists y \in \mathbb{H}^2, \exists (\gamma_n) \in \Gamma^{\mathbb{N}}, \gamma_n \cdot y \rightarrow z\}$$

(on note  $\gamma \cdot z$  l'action d'un élément  $\gamma \in \Gamma$  sur un élément  $z \in \mathbb{H}^2$ ).  $\Lambda_\Gamma$  est un sous-ensemble fractal de  $\mathbb{S}^1$ . Il est par ailleurs possible de donner une description de  $\Lambda_\Gamma$  qui fait apparaître clairement

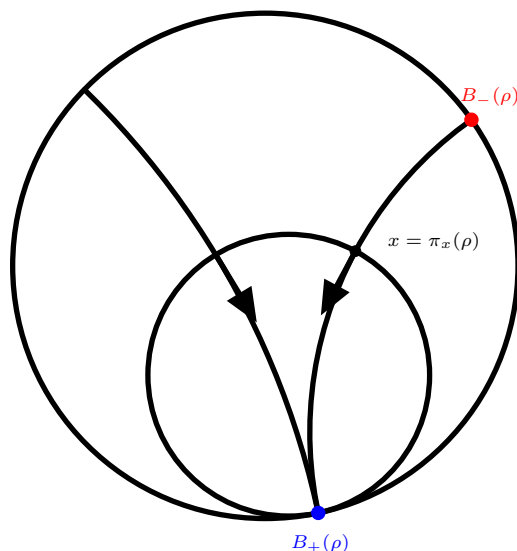


Figure 1.12: Description de  $B_{\pm}(\rho)$  sur  $\mathbb{S}^1$ . Nous avons représenté aussi l'horocycle passant par  $x$  et  $B_+$  ainsi qu'une autre géodésique qui a pour point final  $B_+(\rho)$  en  $+\infty$ .

une structure d'ensemble de Cantor, en utilisant le modèle de surface de Schottky des surfaces hyperboliques convexes co-compactes (voir [Bor16], Chapitre 15). Il y a un résultat remarquable de [Sul79], [Pat76] qui énonce que:

$$\delta := \dim_H \Lambda_{\Gamma} \quad (1-1.6)$$

où  $\dim_H$  est la dimension de Hausdorff de  $\Lambda_{\Gamma}$ .

Pour relier l'ensemble capté  $K$  à l'ensemble limite, il faut encore introduire les applications suivantes  $B_{\pm}$  (voire Figure 1.12) définies sur  $T^*\mathbb{H}^2 \setminus 0$ . Pour tout  $\rho = (x, \xi) \in T^*\mathbb{H}^2 \setminus 0$ , il existe une unique géodésique  $\gamma : \mathbb{R} \rightarrow T^*\mathbb{H}^2$  qui passe par ce point. Le projeté de  $\gamma$  sur  $M$ ,  $\pi_x(\gamma)$  est un arc de cercle qui coupe  $\mathbb{S}^1$  en deux points :

$$B_{\pm}(\rho) = \lim_{t \rightarrow \pm\infty} \pi_x(\gamma(t))$$

Remarquons que  $B_{\pm}$  est constant sur une géodésique. Enfin, notons  $\pi_{\Gamma} : T^*\mathbb{H}^2 \rightarrow T^*M$  la projection naturelle sur le quotient, relevée à l'espace cotangent. On a finalement, pour tout  $\rho = (x, \xi) \in S^*\mathbb{H}^2$ :

$$\pi_{\Gamma}(\rho) \in K \iff B_-(\rho) \in \Lambda_{\Gamma} \text{ and } B_+(\rho) \in \Lambda_{\Gamma}$$

En particulier, cela permet de comprendre pourquoi l'on a

$$\dim_H K = 2\delta + 1$$

Ainsi,  $\delta$  joue le même rôle que  $d_H$  dans les modèles précédents de scattering par des obstacles ou par un potentiel. De plus, la pression topologique associée au flot géodésique sur l'ensemble capté prend la forme simple suivant :

$$P(s) = \delta - s$$

**Trou spectral.** Dans ce cadre, le problème de trou spectral consiste à montrer qu'il n'existe qu'un nombre fini de résonances dans  $\{\operatorname{Re} s \geq 1/2 - \gamma\}$  pour  $\gamma > 0$ . En fait, la première résonance est bien connue : c'est une des conséquences de la théorie de Patterson-Sullivan (voir [Bor16], Chapitre 14).

**Théorème 1-1.15** ([Pat88]).  $\delta$  est une résonance simple et il n'y a aucune autre résonance dans  $\{\operatorname{Re} s \geq \delta\}$ .



Quand  $\delta \geq 1/2$ , il est connu qu'il n'y a qu'un nombre fini de résonances  $s$  dans  $\{\operatorname{Re} s \geq 1/2\}$ , qui sont associées aux valeurs propres de la forme  $s(1-s)$  dans  $[0, 1/4[$ . En conséquence, le résultat précédent permet de dire que  $\delta(1-\delta)$  est la première valeur propre de  $-\Delta_g$ . Quand  $\delta < 1/2$ , ce théorème donne un trou spectral explicite de taille  $\gamma = 1/2 - \delta = |P(1/2)|$ . On peut donc voir ce résultat comme l'analogie du trou spectral donné par le Théorème 1-1.11. Quand  $\delta < 1/2$ , la Question 1.1 peut alors être reprécisée de la façon suivante : peut-on améliorer le trou spectral de taille  $1/2 - \delta$  ? Une réponse positive a été apportée à cette question par F. Naud ([Nau05]) puis dans [BD17].

**Théorème 1-1.16.** *Pour toute surface hyperbolique convexe co-compacte  $M = \mathbb{H}^2/\Gamma$  d'exposant de convergence  $\delta$ , il existe  $\varepsilon > 0$  tel qu'il n'y a qu'un nombre fini de résonances dans  $\{\operatorname{Re} s > \delta - \varepsilon\}$ .*

Contrairement au résultat de F. Naud, où  $\varepsilon$  dépend *a priori* de la surface, le travail de [BD17] fournit un  $\varepsilon$  qui ne dépend plus que de  $\delta$ . Ce travail fait partie d'une série de papiers, commencée en 2016 avec [DZ16], qui reposent sur l'utilisation d'un nouvel outil, appelé Principe d'Incertitude Fractal. Il s'agira également d'un outil clé dans cette thèse, et nous nous y attarderons plus en détail un peu plus loin dans cette introduction. En ce qui concerne la Question 1.1, cet outil a permis d'obtenir le résultat suivant :

**Théorème 1-1.17** ([BD18]). *Pour toute surface hyperbolique convexe co-compacte, il existe  $\varepsilon > 0$  tel qu'il n'y a qu'un nombre fini de résonances dans  $\{\operatorname{Re} s > 1/2 - \varepsilon\}$ .*

En particulier, la condition de pression  $\delta < 1/2$  n'est plus nécessaire. Ce résultat, tout comme la stratégie utilisée pour le prouver, a largement motivé notre travail afin de démontrer le Théorème A.

**Borne supérieure de Weyl fractale.** Dans ce cadre, la reformulation de la question 1.2 concerne la fonction de comptage :

$$N_\gamma(r) = \#\{s \text{ résonances, } \operatorname{Re} s \geq 1/2 - \gamma, |\operatorname{Im} s| \in [r, r+1]\} \text{ comptées avec multiplicité}$$

Des bornes supérieures de la forme  $N_\gamma(r) \leq C_\gamma r^\delta$  pour les surfaces hyperboliques convexes co-compactes sont apparues dans le travail de M. Zworski [Zwo99]. Elles ont pu être étendues au cas des variétés asymptotiquement hyperboliques dans [DD13]. Il a été conjecturé dans [Zwo17], (Conjecture 5) que cette borne devrait être optimale pour des valeurs de  $\gamma$  assez grande. Par ailleurs, des travaux numériques sur ces résonances (voir par exemple l'annexe dans [DBW19]) ont laissé penser qu'il devait être possible d'améliorer cette borne pour des valeurs de  $\gamma$  plus petites. Un premier résultat dans cette direction a été obtenu par F. Naud dans [Nau12], où l'auteur démontre une borne analogue à celle du Théorème B (et sans perte de  $\varepsilon$ ), avec une fonction  $\sigma$  possédant les mêmes propriétés. A l'heure d'écrire ce manuscrit, la meilleure borne connue, à notre connaissance, est celle obtenue par [DBW19] :

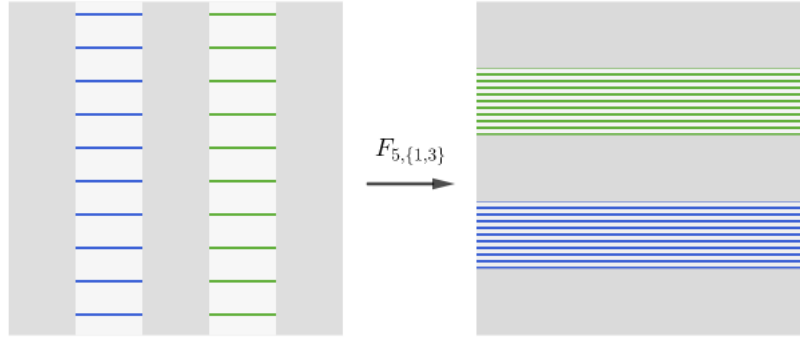
**Théorème 1-1.18** ([DBW19]). *Pour tout  $\gamma \geq 0$  et  $\varepsilon > 0$ , il existe  $C > 0$  tel que pour tout  $r \geq 1$ ,*

$$N_\gamma(r) \leq Cr^{\delta - \sigma(\gamma) + \varepsilon}$$

où  $\sigma(\gamma) = \max(0, \delta + 1 - 2\gamma)$

Remarquons que la fonction  $\sigma$ , très explicite et agréable à écrire, vérifie les propriétés de la fonction  $\sigma$  du Théorème B puisque  $\gamma_{cl} = 1 - \delta$  dans ce cadre. Ces résultats sont donc des analogues du Théorème B en ce qui concerne les surfaces hyperboliques convexes co-compactes. Cependant, la stratégie que nous développons pour le démontrer est très différente de celle de [DBW19] (et même de [Nau12]). En effet, dans [DBW19], la stratégie repose sur l'utilisation d'un calcul semiclassique anisotrope, associé à certaines variétés Lagrangiennes bien particulières, dont la description analytique est possible grâce aux propriétés du flot géodésique sur le disque de Poincaré.

En ce qui concerne des bornes inférieures, il reste difficile d'en obtenir et malgré des résultats numériques convaincants ([Bor14], [BW16], [LSZ03], [DBW19]), un équivalent simple et exacte de

Figure 1.13: Un exemple : l'application du boulanger ouverte  $F_{5,\{1,3\}}$ 

la fonction de comptage dans les bandes reste un problème ouvert. A ce jour, quelques bornes inférieures sont connues, dont celle de [GZ99] : pour tout  $0 < \varepsilon < 1/2$  et  $0 > \sigma > -\frac{1}{2\varepsilon}$ , le nombre de résonances  $\tilde{N}_\sigma(r)$  (avec multiplicité) dans  $\{|s - 1/2| \leq r, \text{Im } s \geq \sigma\}$  vérifie

$$\tilde{N}_\sigma(r) = \Omega(r^{1-\varepsilon})$$

c'est-à-dire que  $\tilde{N}_\sigma(r)r^{-1+\varepsilon}$  prend des valeurs arbitrairement grandes quand  $r \rightarrow +\infty$ . La preuve de ce résultat repose, comme la plupart des estimées inférieures, sur l'analyse d'une formule de trace.

Les surfaces hyperboliques convexes co-compactes sont un parfait exemple de systèmes hyperboliques ouverts pour lequel l'étude des résonances partagent de nombreux points communs avec le scattering par des obstacles (et le scattering par un potentiel). Un des intérêts de ce modèle est que d'autres méthodes d'analyse sont possibles, fondées sur les propriétés particulières du flot géodésiques et les propriétés algébriques du sous-groupe  $\Gamma$ . Cependant, certains résultats restent à prouver et semblent parfois encore difficiles à obtenir. Pour tester certaines conjectures qui restent encore inaccessibles sur ce modèle, il est souvent utile d'introduire des modèles jouets plus simples. C'est ce que nous allons désormais présenter.

### 1-1.5.3. Quantification de l'application du boulanger ouverte.

Les transformations dites du boulanger sont des modèles jouet pour les systèmes dynamiques chaotiques, définie sur l'espace des phases  $\mathbb{T}^2$ . On les qualifie d'ouvertes quand elles ne sont définies que sur un sous-ensemble de  $\mathbb{T}^2$ . Les applications ouvertes du boulanger sont des modèles jouet pour comprendre des systèmes chaotiques ouverts, comme l'application du billard pour des obstacles strictement convexes.

Pour définir de telles applications, on fixe un entier  $M \in \mathbb{N}$  et un alphabet  $\mathcal{A} \subset \{0, \dots, M-1\}$ . On identifie  $\mathbb{T}^2$  avec  $[0, 1]^2$ . On définit l'ensemble de départ

$$D_{M,\mathcal{A}} = \bigcup_{a \in \mathcal{A}} [aM^{-1}, (a+1)M^{-1}] \times [0, 1[$$

L'application  $F = F_{M,\mathcal{A}}$  est définie comme

$$F_{M,\mathcal{A}}(x, \xi) = (Mx - a, M^{-1}(\xi + a)), \quad \frac{a}{M} \leq x < \frac{a+1}{M}, \quad a \in \mathcal{A}$$

Afin que l'application soit effectivement ouverte, on suppose que  $\mathcal{A} \neq \{0, \dots, M-1\}$ .

L'ensemble d'arrivée est alors  $A_{M,\mathcal{A}} = F_{M,\mathcal{A}}(D_{M,\mathcal{A}})$  et l'on peut définir un inverse  $F_{M,\mathcal{A}}^{-1}$ , bien défini sur  $A_{M,\mathcal{A}}$ . L'ensemble capté  $K$  est défini comme l'ensemble des points  $\rho = (x, \xi) \in \mathbb{T}^2$  tels que  $F^n(\rho)$  est bien défini pour tout  $n \in \mathbb{Z}$ . On peut facilement le décrire en utilisant la dynamique symbolique associée à l'application et il prend la forme d'un ensemble de Cantor :

$$K = \mathcal{C} \times \mathcal{C}$$

où

$$\mathcal{C} = \bigcap_{k \in \mathbb{N}} \bigcup_{a \in \mathcal{C}_k} \left[ \frac{a}{M^k}, \frac{a+1}{M^k} \right]; \mathcal{C}_k = \{a_0 + a_1 M + \dots + a_{k-1} M^{k-1}, (a_0, \dots, a_{k-1}) \in \mathcal{A}^k\}$$

est un ensemble de Cantor de dimension  $\delta = \frac{\log |\mathcal{A}|}{\log M}$ . On peut définir une notion de pression topologique pour cette application :

$$P(s) = \lim_{N \rightarrow +\infty} \frac{1}{N \log M} \log \left( \sum_{\gamma, |\gamma|=N} M^{-sN} \right)$$

où la somme parcourt l'ensemble des trajectoires périodiques de période  $N$ . Puisqu'il y en a exactement  $|\mathcal{A}|^N$ , on voit que

$$P(s) = \frac{1}{\log M} (\log 2 - s \log M) = \delta - s$$

Voilà pour ce qui est de l'application classique. [BV89] en a initié la quantification, et leur étude a été poursuivie par d'autres ([SV96] [NZ05], [NZ07]). Ces quantifications fournissent des modèles en dimension finie : il s'agit de matrices de taille  $N$ , où  $N$  joue le rôle de l'inverse d'un paramètre semiclassique  $h$  (plus précisément  $N = (2\pi h)^{-1}$ ). On cherche à les étudier dans la limite  $N \rightarrow +\infty$ . Nous choisissons de présenter le modèle de [DJ17], en introduisant des fonctions cut-off  $\chi \in C_c^\infty([0, 1])$ , qui permettent d'éviter d'avoir à traiter les singularités des applications du boulanger (voir par exemple [NZ07]) et pour  $N \in M\mathbb{Z}$ , on définit

$$B_N = B_{N,M,\mathcal{A},\chi} = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/M} \mathcal{F}_{N/M} \chi_{N/M} & & \\ & \ddots & \\ & & \chi_{N/M} \mathcal{F}_{N/M} \chi_{N/M} \end{pmatrix} I_{M,\mathcal{A}} \quad (1-1.7)$$

où  $\mathcal{F}_N$  est la matrice unitaire de la transformée de Fourier discrète en dimension  $N$  et

$$\chi_{N/M} = \begin{pmatrix} \chi(0) & & & \\ & \chi\left(\frac{M}{N}\right) & & \\ & & \ddots & \\ & & & \chi\left(\left(\frac{N}{M}-1\right)\left(\frac{M}{N}\right)\right) \end{pmatrix}; I_{M,\mathcal{A}} = \begin{pmatrix} \mathbb{1}_{0 \in \mathcal{A}} I_{N/M} & & \\ & \ddots & \\ & & \mathbb{1}_{M-1 \in \mathcal{A}} I_{N/M} \end{pmatrix}$$

Ces applications quantiques ouvertes sont pensées pour modéliser le propagateur au temps  $t = \log M$  de systèmes quantiques ouverts et les valeurs propres de  $B_N$  sont reliées, de façon heuristique, aux résonances de scattering d'un tel système ouvert. La formule heuristique peut s'énoncer comme suit : si  $\lambda$  est une résonance,  $z = e^{-it\lambda}$  est une valeur propre de  $B_N$ . En fait, les travaux de [NSZ11] et [NSZ14] prouvent de façon rigoureuse la relation entre les résonances de scattering (respectivement pour le potentiel et les obstacles) et les valeurs propres d'applications quantiques ouvertes plus générales. Cette relation est un ingrédient clé de cette thèse, à la fois pour le problème de trou spectral et celui de comptage de résonances. Avec cette analogie, le problème du trou spectral (Question 1.1) et le comptage de résonances (Question 1.2) ont un équivalent :

- L'application  $B_N$  vérifie  $\|B_N\|_{\ell^2 \rightarrow \ell^2} \leq 1$ , de sorte que 1.1 devient : peut-on trouver  $\gamma > 0$  et  $N_0 \geq 0$  tel que pour tout  $N \geq N_0$ ,  $\sigma(B_N) \cap \{|z| \geq M^{-\gamma}\} = \emptyset$  ?
- La Question 1.2 concerne le comptage de valeurs propres dans des anneaux. On veut des équivalents/bornes supérieures/bornes inférieures pour  $N_\gamma(N)$  dans la limite  $N \rightarrow +\infty$  où  $N_\gamma(N)$  est le nombre de valeurs propres de  $B_N$ , avec multiplicité, dans  $\{|z| \geq M^{-\gamma}\}$ . Des calculs numériques ont été faits dans [NZ05], et suggèrent des équivalents de la forme

$$N_\gamma(N) \sim C_\gamma N^{\mu_\gamma}$$

**Trou spectral.** Comme dans les modèles déjà présentés, la valeur  $\delta = 1/2$  est une valeur importante. En effet, les résultats de [NZ09] pourraient s'adapter dans ce modèle et les méthodes de [DJ17], utilisée pour étudier le cas des entiers  $N$  de la forme  $M^k$ , permettent de montrer que si  $\delta < 1/2$ , il y a un trou spectral de taille  $1/2 - \delta + \varepsilon$  pour tout  $\varepsilon > 0$ . Dans ce travail, les auteurs montrent que le trou spectral peut en fait être amélioré et quand  $\delta \geq 1/2$ , il y a toujours un trou spectral :

**Théorème 1-1.19** ([DJ17]). *Il existe  $\beta = \beta(M, \mathcal{A}) > \max(0, 1/2 - \delta)$  et  $k_0 \in \mathbb{N}$  tel que pour tout  $k \geq k_0$ ,*

$$\sigma(B_{M^k}) \cap \{|z| \geq M^{-\beta}\} = \emptyset$$

**Bornes supérieures de Weyl fractales.** Le comportement asymptotique de  $N_\gamma(N)$  a été testé numériquement dans plusieurs travaux ([Non11], Section 6, [NR07], [NZ05]). Il n'y a toujours pas de bornes inférieures connues dans ce modèle mais des bornes semblables à celle du Théorème 1-1.18 ont été obtenues dans le cas  $N = M^k$  dans la limite  $k \rightarrow \infty$  :

**Théorème 1-1.20** ([DJ17]). *Soit  $N = M^k$  pour  $k \in \mathbb{N}$ , et notons  $B_N$  la matrice définie par 1-1.7. Soit  $N_\gamma(k)$  le nombre de valeurs propres de  $B_N$ , avec multiplicité, dans  $\{|z| \geq M^{-\gamma}\}$ . Alors, pour tout  $\gamma > 0$  et  $\varepsilon > 0$ , il existe  $C > 0$  tel que pour tout  $k \in \mathbb{N}$ ,*

$$N_\gamma(k) \leq CN^{\delta - \sigma(\gamma) + \varepsilon}$$

où  $\sigma(\gamma) = \max(0, 1 - \delta - 2\gamma)$ .

En fait, la stratégie utilisée dans [DJ17] pour démontrer ce résultat est inspirée de celle de [DBW19]. Remarquons une fois de plus que  $\sigma$  se comporte comme celle du Théorème B. Dans un récent travail de [Zhe22], la borne  $N_\gamma(\delta) = O(N^\delta)$  a été prouvée pour tous les entiers  $N \in M\mathbb{Z}$  (et sans la perte en  $\varepsilon$ ).

Ceci conclut la première partie de l'introduction, dédiée à la présentation et à la mise en perspective des résultats de cette thèse.

## 1-2 Boîte à outils.

Nous en venons à présent à la deuxième partie de cette introduction. Nous avons voulu présenter quelques outils importants utilisés dans cette thèse. Nous avons fait des choix, guidés à la fois par l'importance des outils et les goûts mathématique de l'auteur. Cette liste est donc, bien entendu, non exhaustive et entièrement subjective.

### Remarque.

Il aurait pu être pertinent d'évoquer la machinerie de l'analyse semiclassique, qui est largement utilisée dans cette thèse. Nous avons décidé de ne pas le faire, et ceci pour deux principales raisons. D'une part, il y aurait eu beaucoup trop à dire et nous préférons y dédier un chapitre. D'autre part, il nous paraissait très réducteur de dire que l'analyse semiclassique est un simple outil. C'est en effet une branche à part entière de la théorie des EDPs linéaires ! Le point de départ de toute la théorie est la construction d'une procédure dite de quantification qui permet de quantifier des symboles lisses  $a \in \mathcal{S}(\mathbb{R}^{2d})$  en un opérateur sur  $L^2(\mathbb{R}^d)$ . Par exemple, la quantification de Weyl repose sur la formule :

$$a^W(x, hD_x)u(x) = \text{Op}_h(a)u(x) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}(x-y)\cdot\xi} u(y) dy d\xi$$

Une telle formule permet de construire un calcul semiclassique consistant sur  $\mathbb{R}^d$  (composition, caractère borné sur  $L^2$ , inégalités de Gårding, théorème d'Egorov, etc. ) et peut tout aussi bien se développer sur des variétés différentielles. Des ouvrages de référence sont disponibles dans

la littérature, comme [Zwo12], dans lequel l'auteur a pu apprendre une grande partie de cette théorie.<sup>a</sup>. Le Chapitre 4 rappelle certains objets et résultats importants en analyse semiclassique, qui seront utilisés dans cette thèse.

<sup>a</sup>L'auteur a également été initié à l'analyse semiclassique par son directeur de thèse, lors d'un cours de master. Cette première rencontre - avec l'analyse semiclassique comme avec S.Nonnenmacher - est sûrement le point de départ de tout ce manuscrit !

### 1-2.1. Complex scaling.

Le premier outil que nous voulions évoquer est la méthode dite de *complex scaling*. Pour être tout à fait exact, cette méthode a été plutôt implicitement utilisée dans ce manuscrit, à travers les travaux de [NSZ11] et [NSZ14] où c'est le point de départ qui permet de passer d'un problème de résonances à un problème de valeurs propres, nécessaire pour établir un problème de Grushin. Le *complex scaling* est notamment utile quand il s'agit d'obtenir des estimées pour la résolvante tronquée - voir Chapitre 6. Nous commençons par rappeler les tenants de cette méthode et comment elle permet de passer d'un problème de résonances à un problème de valeurs propres pour des opérateurs non auto-adjoints. Dans un second temps, nous avons voulu profiter de cette mention aux opérateurs non auto-adjoints pour évoquer succinctement quelques effets pseudospectraux qui peuvent apparaître quand on traite ce genre d'opérateurs. En particulier, cela permet d'expliquer pourquoi il est souvent difficile d'obtenir des bornes inférieures pour le nombre de résonances.

#### 1-2.1.1. Des résonances aux valeurs propres.

La méthode de *complex scaling* dans l'étude des résonances de scattering apparaît dans les travaux d'Aguilar, Combes et Balslev [AC71], [BC71]. Elle a ensuite été très largement développée (et utilisée!) par de nombreux autres auteurs. Dans le cas du formalisme de boîte noire de Sjöstrand-Zworski, qui permet de traiter une large classe de perturbations du laplacien, elle a d'abord été utilisée dans [SZ91]. Cette méthode est devenue quasiment indispensable dans l'étude des problèmes de résonances (voir par exemple [Sjö97], [SZ95]). Le principal intérêt de cette méthode est qu'elle permet de passer d'un problème de résonances - il faut trouver les pôles d'un prolongement méromorphe - à un problème de valeurs propres d'un opérateur non auto-adjoint. Cela en fait une méthode d'une importance capitale, tant d'un point de vue théorique que numérique. L'auteur a appris les rouages de cette machinerie dans la présentation détaillée faite dans [DZ19] (Section 4.5).

Par soucis de clarté et de concision, nous présenterons les détails du *complex scaling* dans le cas du scattering par des obstacles: on fixe un obstacle  $\mathcal{O} \subset \mathbb{R}^d$  et l'on considère  $-\Delta_\Omega$  sur  $L^2(\Omega)$ ,  $\Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$ , de domaine  $H_0^1(\Omega) \cap H^2(\Omega)$ . On fixe  $R > 0$  tel que  $\overline{\mathcal{O}} \subset B(0, R)$ . Pour un paramètre  $\theta \in ]0, \pi/2[$ , on considère une déformation complexe totalement réelle  $\Gamma_\theta \subset \mathbb{C}^d$  de  $\mathbb{R}^d$  - on dit que  $\Gamma_\theta$  est totalement réelle si  $T_z \Gamma_\theta \cap i\Gamma_\theta = \emptyset$  pour tout  $z \in \Gamma_\theta$  - telle que pour un certain  $R' > R$ ,

$$\begin{aligned} \Gamma_\theta \cap B_{\mathbb{C}^d}(0, R) &= \mathbb{R}^d \cap B_{\mathbb{C}^d}(0, R) \\ \Gamma_\theta \cap \mathbb{C}^d \setminus B_{\mathbb{C}^d}(0, R') &= e^{i\theta} \mathbb{R}^d \cap \mathbb{C}^d \setminus B_{\mathbb{C}^d}(0, R') \end{aligned}$$

Comme expliqué dans [DZ19], il est possible de construire une telle déformation : considérons  $g \in C^\infty(\mathbb{R}_+, \mathbb{R})$  telle que

$$g(t) = 0 \text{ for } t \leq R, \quad g(t) = \frac{1}{2}t^2 \text{ for } t \geq 2R, \quad g''(t) \geq 0$$

et pour  $0 < \theta < \pi/2$ , posons  $F_\theta(x) = \tan(\theta)g(|x|)$ ,  $f_\theta(x) = x + i\partial_x F_\theta(x)$  et  $\Gamma_\theta = f_\theta(\mathbb{R}^d)$ .

Pour définir le laplacien déformé, on introduit les opérateurs différentiels sur  $\mathbb{C}^d$ ,

$$\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j}); \quad \Delta_z = \sum_{j=1}^d \partial_{z_j}^2.$$

Alors, pour  $u \in C^\infty(\Gamma_\theta)$ , il existe un prolongement presque analytique<sup>1</sup>  $\tilde{u} \in C^\infty(\mathbb{C}^d)$  et l'on peut montrer que la valeur de  $\Delta_z \tilde{u}$  on  $\Gamma_\theta$  ne dépend pas du choix du prolongement presque analytique.

<sup>1</sup>c'est-à-dire que pour tout  $1 \leq j \leq d$ ,  $\bar{\partial}_{z_j} \tilde{u}(z) = O(d(z, \Gamma_\theta)^{-\infty})$

Cela permet de définir la déformation complexe du laplacien libre,  $\Delta_{\theta,0}$  sur  $L^2(\Gamma_\theta)$  de domaine  $H^2(\Gamma_\theta) : \Delta_\theta u = (-\Delta_z \tilde{u})|_{\Gamma_\theta}$ . En fait, en utilisant la déformation  $f_\theta : \mathbb{R}^d \rightarrow \Gamma_\theta$ , on peut montrer que  $-\Delta_{\theta,0}$  est équivalent à

$$\left( (\text{Id} + i\partial_x^2 F_\theta(x))^{-1} \partial_x \right) \cdot \left( (\text{Id} + i\partial_x^2 F_\theta(x))^{-1} \partial_x \right) \text{ sur } L^2(\mathbb{R}^d)$$

Il est alors possible de définir la version déformée de  $-\Delta_\Omega$  avec domaine  $H_0^1(\Omega) \cap H^2(\Omega)$  : fixons  $\chi \in C_c^\infty(B(0, R))$  tel que  $\chi = 1$  près de  $\overline{\mathcal{O}}$  ; la fonction  $1 - \chi$  vaut 1 en dehors de  $B(0, R)$  de sorte que l'on peut la prolonger en une fonction lisse sur  $\Gamma_\theta$  ; puisque  $\Gamma_\theta \cap B(0, R) = \mathbb{R}^d \cap B(0, R)$ , l'ensemble  $\Gamma_\theta \setminus \mathcal{O}$  s'identifie à  $\Omega$  près de  $\partial\mathcal{O}$  et l'on peut définir le domaine  $\mathcal{D}_\theta = H^2(\Gamma_\theta \setminus \mathcal{O}) \cap H_0^1(\Gamma_\theta \setminus \mathcal{O})$ . Pour  $u \in \mathcal{D}_\theta$ , on définit finalement

$$-\Delta_\theta u = -\Delta_\Omega(\chi u) - \Delta_{\theta,0}((1 - \chi)u)$$

La relation fondamentale entre  $-\Delta_\theta$  et  $-\Delta_\Omega$  est donnée par le théorème suivant :

**Théorème 1-2.1** ([DZ19] Section 4.5, [SZ91]). *Supposons que  $0 \leq \theta < \pi/2$ . Soit  $\lambda \in \mathbb{C}^*$  tel que  $\text{Im}(e^{i\theta}\lambda) > 0$ , alors  $-\Delta_\theta - \lambda^2$  est un opérateur de Fredholm d'indice 0 et  $\lambda^2$  est une valeur propre de  $-\Delta_\theta$  si et seulement si c'est une résonance de  $-\Delta_\Omega$ . De plus, les multiplicités coïncident. Enfin, si  $\psi \in C_c^\infty(B(0, R))$ ,  $\psi = 1$  suffisamment près de  $\overline{\mathcal{O}}$  et si  $\lambda$  n'est pas une résonance,*

$$\psi(-\Delta_\theta - \lambda^2)^{-1}\psi = \psi(-\Delta_\Omega - \lambda^2)^{-1}\psi$$

### 1-2.1.2. Excursion : quelques phénomènes spectraux remarquables pour des opérateurs non auto-adjoints.

L'objectif de cette sous-section, quelque peu déconnectée du reste de cette thèse, est de présenter quelques phénomènes de pseudospectres qui peuvent apparaître dans l'étude d'opérateurs non auto-adjoints. Les exemples choisis ont été rencontrés pendant cette thèse et ont particulièrement intéressé l'auteur. Ils permettent d'illustrer l'instabilité du spectre pour ces opérateurs et de comprendre qu'il peut être difficile d'obtenir des bornes inférieures dans le comptage de résonances. Les deux derniers exemples ont été étudiés par l'auteur pour la préparation d'un exposé donné en Juin 2022 pour un Workshop de l'ANR Adyct. Bien entendu, cette sous-section n'a *pas* pour objectif de donner une présentation de la théorie du pseudospectre (à ce propos, on peut consulter [TE05]).

**L'exemple de la matrice de Jordan.** Il est assez courant, pour débiter dans l'étude des phénomènes pseudospectraux, de s'intéresser à un petit exemple simple qui concerne des perturbations de la matrice de Jordan  $J_N \in \mathcal{M}_N(\mathbb{C})$  :

$$J_N = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & & 0 \end{pmatrix}$$

Le spectre de  $J_N$  est réduit à  $\{0\}$  mais quand  $z \in D(0, 1)$  et quand  $N$  est grand, sa résolvante  $(J_N - z)^{-1}$  a une norme  $\ell^2 \rightarrow \ell^2$  plutôt grande comparée à  $z$ . En effet, si  $u = (1, z, \dots, z^{N-1})^t$  alors

$$(z - J_N)u = (0, \dots, 0, z^N)^t$$

de sorte que

$$\|(z - J_N)^{-1}\|_{\ell^2 \rightarrow \ell^2} \geq \frac{1}{|z|^N} \|u\|_{\ell^2} \geq |z|^{-N}$$

Ces comportements mènent à des phénomènes d'instabilité du spectre. Une première manifestation déterministe est donnée par l'exemple suivant :

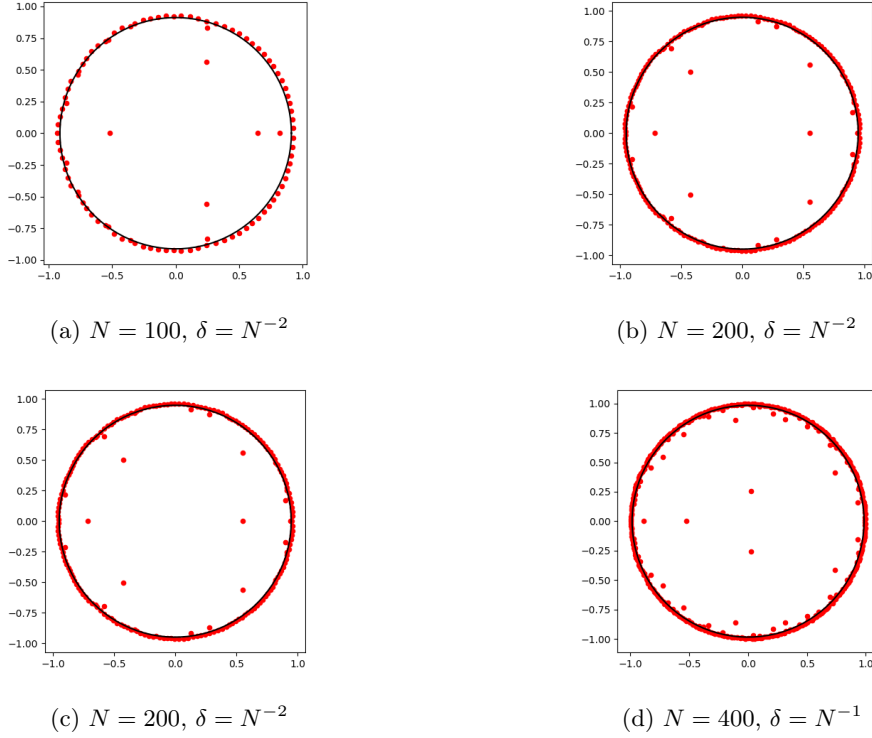


Figure 1.14: Tracé du spectre  $J_N + \delta G_N$  pour différentes valeurs de  $\delta$  et différentes valeurs de  $N$ , où  $G_N$  est une perturbation Gaussienne aléatoire. Le cercle noir est le cercle de rayon  $\delta^{1/N}$ .

*Exemple.* Soit  $0 < \varepsilon < 1$  un petit paramètre et soit  $R_N = (\delta_{iN} \delta_{j1})_{1 \leq i, j \leq N}$ . Il n'est pas difficile de voir que le spectre de  $J_N + \varepsilon R_N$  est donné par

$$\{\varepsilon^{1/N} e^{\frac{2ik\pi}{N}}, 0 \leq k \leq N-1\}$$

qui sont  $N$  points répartis de façon régulière sur le cercle de rayon  $\varepsilon^{1/N}$ . Dans la limite  $N \rightarrow +\infty$ , on peut dire que le spectre  $J_N + \varepsilon R_N$  converge vers  $\mathbb{S}^1$ .

*Exemple.* Il est aussi intéressant de regarder des perturbations aléatoires de  $J_N$  comme dans [DH09]. Par exemple, mentionnons le corollaire suivant de l'article [GWZ14] : Pour  $A \in \mathcal{M}_N(\mathbb{C})$ , notons  $L^A$  la mesure  $N^{-1} \sum_{\mu \in \sigma(A)} \delta_\mu$  où les valeurs propres sont répétées avec multiplicité. Soit  $(G_{ij})_{i, j \in \mathbb{N}}$  une suite i.i.d. de variables Gaussiennes de moyenne nulle et de variance 1. Notons  $G_N = (G_{ij})_{1 \leq i, j \leq N}$ . Soit  $(\delta_N) \in \mathbb{R}^{\mathbb{N}}$  une suite telle que pour certains  $\kappa', \kappa > 0$ ,

$$N^{-\kappa'} \leq \delta_N \leq N^{-1/2-\kappa}$$

Alors,  $L^{J_N + \delta_N G_N}$  converge faiblement en probabilité vers la mesure uniforme sur  $\mathbb{S}^1$  (voir la Figure 1.14).

**Pathologie spectrale pour un opérateur différentiel.** L'exemple suivant est inspiré de [See86]. Considérons la fonction suivante :

$$f_\alpha(x) = e^{ix} + \alpha e^{2ix}, x \in \mathbb{R}, |\alpha| < 1$$

ainsi que l'opérateur différentiel défini sur  $L^2(\mathbb{R}/2\pi\mathbb{Z})$ :

$$d(\alpha) = f_\alpha(x) \partial_x$$

Un petit calcul permet de montrer qu'une primitive de  $1/f_\alpha : \mathbb{R} \rightarrow \mathbb{C}$  est donnée par

$$g_\alpha(x) = e^{-ix} - \alpha x - i\alpha \log(1 + \alpha e^{ix})$$

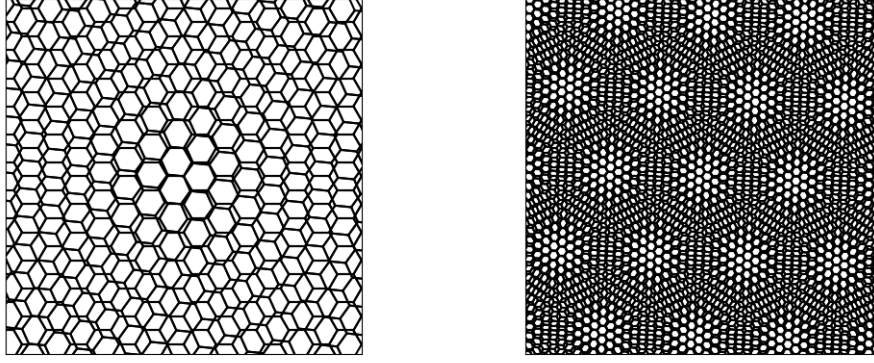


Figure 1.15: Deux couches de graphène décalées d'un petit angle forment les fameux motifs de Moiré .

Pour calculer le spectre de  $d(\alpha)$ , il suffit de résoudre le problème aux valeurs propres

$$\begin{cases} d(\alpha)u = \lambda u \\ u(0) = u(2\pi) \end{cases}$$

Une solution de ce problème a la forme  $u(x) = u_0 \exp(\lambda g_\alpha(x))$ , qui fournit effectivement une solution si et seulement si  $\lambda(g_\alpha(2\pi) - g_\alpha(0)) \in 2i\pi\mathbb{Z}$ . Mais,  $g_\alpha(2\pi) - g_\alpha(0) = -2\pi\alpha$ . Donc:

$$\begin{cases} \alpha = 0 \implies \text{Sp}(d(\alpha)) = \mathbb{C} \\ \alpha \neq 0 \implies \text{Sp}(d(\alpha)) = \frac{1}{\alpha}\mathbb{Z} \end{cases}$$

Ainsi, malgré le fait que  $d(\alpha)$  dépende analytiquement de  $\alpha$  (c'est une perturbation vraiment simple de  $d(0)$ ), le comportement de  $d(\alpha)$  change brutalement en  $\alpha = 0$ , comparé avec le cas  $\alpha \neq 0$ .

**Angles magiques pour des bicouches de graphène décalées.** Le dernier exemple que nous voulions présenter apparaît dans le travail récent [BEWZ22], qui étudie le modèle physique introduit dans [TKV19] pour des bicouches de graphène décalées d'un certain angle (TBG en anglais pour *twisted bilayer graphene*). L'étude de ce modèle conduit à s'intéresser au spectre d'une certaine famille d'opérateurs différentiels  $D(\alpha)$  sur  $L^2(\mathbb{C}/\Gamma, \mathbb{C}^2)$ , où  $\Gamma = 4\pi(i\omega\mathbb{Z} \oplus i\omega^2\mathbb{Z})$  est relié au réseau hexagonal qui apparaît dans l'étude du graphène et  $\alpha \in \mathbb{C}$  est un paramètre relié à l'angle de décalage entre les deux couches de graphène. (voir la Figure 1.15).

$D(\alpha)$  a pour domaine  $H^1(\mathbb{C}/\Gamma, \mathbb{C}^2)$  et est explicitement donné par

$$D(\alpha) = \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix}$$

où  $D_{\bar{z}} = \frac{1}{i}\partial_{\bar{z}} = \frac{1}{2i}(\partial_{x_1} + i\partial_{x_2})$  et  $U(z)$  est relié à l'interaction entre les deux réseaux de graphène et a pour expression exacte

$$U(z) = \sum_{k=0}^2 e^{\frac{1}{2}(z\omega^{-k} - \bar{z}\omega^k)} \omega^k, \quad \omega = e^{2i\pi/3}$$

Dans [BEWZ22], les auteurs démontrent, entre autres, qu'il existe un ensemble discret non vide  $\mathcal{A}$  tel que  $D(\alpha)$  vérifie:

$$\text{Sp}(D(\alpha)) = \begin{cases} \Gamma^* & \text{si } \alpha \notin \mathcal{A} \\ \mathbb{C} & \text{if } \alpha \in \mathcal{A} \end{cases}$$

$\Gamma^*$  est le réseau dual de  $\Gamma$ , donné explicitement par  $\Gamma^* = \frac{1}{\sqrt{3}}(\omega\mathbb{Z} \oplus \omega^2\mathbb{Z})$ . Cet exemple fournit une autre illustration d'effets pseudospectraux : ce qui est frappant dans cette situation c'est qu'ils jouent un rôle important dans une situation physique pertinente. L'ensemble  $\mathcal{A}$  est appelé l'ensemble des angles magiques et possède des applications remarquables pour le modèle physique.



## 1-2.2. Principe d'Incertitude Fractal.

Le Principe d'Incertitude Fractal est un outil d'analyse harmonique récemment développé et qui a permis des progrès remarquables dans le domaine du Chaos Quantique. Sa première utilisation remonte au travail [DZ16], puis il a été développé notamment par S. Dyatlov en collaboration avec J. Bourgain ([BD17],[BD18]). Pour l'expliquer très grossièrement, on peut le présenter ainsi : il permet de dire qu'une fonction  $f : \mathbb{R} \rightarrow \mathbb{R}$  ne peut pas être bien localisée à la fois en position et en fréquence près d'un ensemble fractal. C'est l'outil principal dans la preuve du Théorème A (voir Chapitre 5). S'il a des applications pour les trous spectraux, il donne également des résultats intéressants dans d'autres problèmes de Chaos Quantique.

### 1-2.2.1. Énoncé.

En mécanique quantique, le principe d'incertitude énonce qu'aucun état ne peut-être parfaitement localisé dans l'espace des phases, i.e. en position et en moment. Cet énoncé est généralement quantifié par le principe d'incertitude d'Heisenberg, connu sous la forme :

$$\Delta x \Delta p \geq h/2$$

Cette inégalité dit grossièrement qu'un état quantique occupe toujours une boîte de volume plus grand que  $h$  dans l'espace des phases. Pour énoncer le principe d'incertitude fractal, nous allons plutôt utiliser une autre façon de quantifier le principe d'incertitude : si  $\mathcal{F}_h : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  désigne la  $h$ -transformée de Fourier unitaire :

$$\mathcal{F}_h u(\xi) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi/h} u(x) dx$$

on peut facilement vérifier que

$$\|\mathbb{1}_{[0,h]} \mathcal{F}_h \mathbb{1}_{[0,h]}\|_{L^2 \rightarrow L^2} \leq h^{1/2}.$$

Il faut lire cette inégalité de la façon suivante : si un état  $u$ , de norme 1, est localisé en position dans  $[0, h]$ , alors la masse de  $\mathcal{F}_h u$  dans  $[0, h]$  est petite - en fait plus petite que  $h^{1/2}$ . C'est donc bien une autre façon de quantifier l'énoncé : "aucun état ne peut être totalement localisé en position et en moment près d'un point". Remarquons au passage qu'il est possible de généraliser légèrement cette inégalité d'incertitude en jouant avec la saturation d'un volume quantique élémentaire d'ordre  $h$  : si  $\gamma_1, \gamma_2 > 0$  vérifient  $\gamma_1 + \gamma_2 > 1$ , alors

$$\|\mathbb{1}_{[0,h^{\gamma_1}]} \mathcal{F}_h \mathbb{1}_{[0,h^{\gamma_2}]}\|_{L^2 \rightarrow L^2} \leq h^\beta ; \beta = \frac{1}{2}(\gamma_1 + \gamma_2 - 1) > 0. \quad (1-2.1)$$

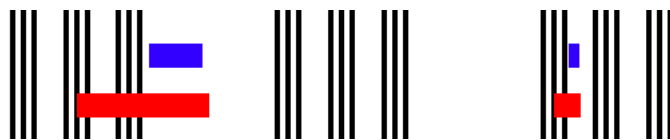
En effet, on a,

$$\begin{aligned} \|\mathbb{1}_{[0,h^{\gamma_1}]} \mathcal{F}_h \mathbb{1}_{[0,h^{\gamma_2}]}\|_{L^2 \rightarrow L^2} &\leq \|\mathbb{1}_{[0,h^{\gamma_1}]}\|_{L^\infty \rightarrow L^2} \times \|\mathcal{F}_h\|_{L^1 \rightarrow L^\infty} \times \|\mathbb{1}_{[0,h^{\gamma_2}]}\|_{L^2 \rightarrow L^1} \\ &\leq h^{\gamma_1/2} \times (2\pi h)^{-1/2} \times h^{\gamma_2/2} \\ &\leq h^\beta. \end{aligned}$$

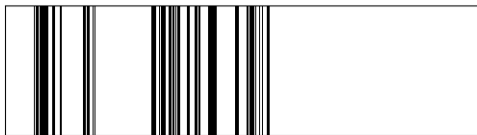
Le principe d'incertitude fractal quantifie l'énoncé plus général "aucun état ne peut être totalement localisé en position et en moment près d'un ensemble fractal". Autrement dit, il permet d'obtenir des bornes du type

$$\|\mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)}\|_{L^2 \rightarrow L^2} = O(h^\beta) \quad (1-2.2)$$

où les ensembles  $X(h)$  et  $Y(h)$  sont "fractals". A ce stade, nous sommes restés très évasifs sur la notion de "fractal". Il devient nécessaire de préciser les notions de fractalité que nous allons utiliser.



(a) Un ensemble de Cantor est poreux. Les intervalles rouges correspondent à des choix de  $I$  tandis que les bleus correspondent à  $J$ .



(b) Un ensemble poreux aléatoire.

Figure 1.16: Exemples d'ensembles poreux.

**Ensembles fractals.** Commençons par rappeler une définition assez standard de dimension fractale, en la personne de la dimension boîte supérieure d'un espace métrique compact  $(X, d)$ . Si l'on note  $N_X(\varepsilon)$  le nombre minimal de boules ouvertes de rayon  $\varepsilon$  nécessaire pour recouvrir  $X$ , alors l'on définit la dimension boîte supérieure de  $X$  par

$$\overline{\dim} X := \limsup_{\varepsilon \rightarrow 0} \frac{\log N_X(\varepsilon)}{-\log \varepsilon} \quad (1-2.3)$$

En particulier, remarquons que si  $\delta > \overline{\dim} X$ , il existe  $\varepsilon_0 > 0$  tel que pour tout  $\varepsilon \leq \varepsilon_0$ ,  $N_X(\varepsilon) \leq \varepsilon^{-\delta}$ . Ce qui va en réalité s'avérer fondamental pour les ensembles fractals qui vont nous intéresser, c'est plus la présence de trous à toutes les échelles (ou au moins jusqu'à un échelle très petite). On parle dans ce cas de porosité, et il est possible de définir cette notion (cf. Figure 1.16) :

**Définition 1-2.1.** Soit  $\nu \in (0, 1)$  et  $0 \leq \alpha_0 \leq \alpha_1$ . On dit qu'un ensemble  $\Omega \subset \mathbb{R}$  est  $\nu$ -poreux de l'échelle  $\alpha_0$  à  $\alpha_1$  si pour tout intervalle  $I \subset \mathbb{R}$  de taille  $|I| \in [\alpha_0, \alpha_1]$ , il existe un sous-intervalle  $J \subset I$  de taille  $|J| = \nu|I|$  tel que  $J \cap \Omega = \emptyset$ .

La notion de porosité peut être reliée à la dimension boîte supérieure. En effet, on a le lemme suivant, démontré dans le chapitre 5, Lemme 5-5.2 :

**Lemme 1-2.1.** Soit  $\Omega \subset \mathbb{R}$ . Supposons qu'il existe  $0 < \delta < 1$ ,  $C > 0$  et  $\varepsilon_0 > 0$  tel que

$$\forall \varepsilon \leq \varepsilon_0, N_\Omega(\varepsilon) \leq C\varepsilon^{-\delta}$$

Alors, il existe  $\nu = \nu(\delta, \varepsilon_0, C)$  tel que  $\Omega$  est  $\nu$ -poreux de l'échelle 0 à 1.

Il est également possible d'utiliser la notion de  $\delta$ -régularité :

**Définition 1-2.2.** Soit  $X \subset \mathbb{R}$  un ensemble fermé non vide et soit  $0 \leq \delta \leq 1$ ,  $C_R \geq 1$  et  $0 \leq \alpha_{min} \leq \alpha_{max} \leq +\infty$ . On dit que  $X$  est  $\delta$ -régulier avec constante  $C_R$  de l'échelle  $\alpha_{min}$  à  $\alpha_{max}$  s'il existe une mesure localement finie  $\mu_X$  supportée dans  $X$  tel que pour tout intervalle  $I$  centré en un point de  $X$  et tel que  $\alpha_{min} \leq |I| \leq \alpha_{max}$ , on a

$$C_R^{-1}|I|^\delta \leq \mu_X(I) \leq C_R|I|^\delta$$

Cette notion est en fait très proche à la notion de porosité. A ce titre, mentionnons un résultat qui montre qu'un ensemble poreux peut toujours être inclus dans un ensemble  $\delta$ -régulier :

**Lemme 1-2.2** ([DJ18b], Lemme 5.4). Soit  $X \subset \mathbb{R}$  un ensemble  $\nu$ -poreux de l'échelle  $\alpha_{min}$  à  $\alpha_{max}$ . Alors il existe  $Y \subset \mathbb{R}$  qui est  $\delta$ -régulier avec constante  $C_R = C_R(\nu)$  de l'échelle 0 à  $\alpha_{max}$  et tel que  $X \subset Y(\alpha_{min})$ .

**Énoncé du principe d'incertitude fractal.** Le principe d'incertitude, tel qu'énoncé dans [BD18], n'utilise pas directement la  $h$ -transformée de Fourier mais plutôt la transformée de Fourier usuelle  $f \mapsto \hat{f}$ . L'échelle  $h$  y apparaît indirectement à travers une constante  $N$ , avec  $N \sim h^{-1}$ .

**Théorème 1-2.2** ([BD18], Theorem 4). *Soit  $0 \leq \delta < 1$ ,  $C_R \geq 1$  et  $N \geq 1$ . Supposons que*

- $X \subset [-1, 1]$  est  $\delta$ -régulier avec constante  $C_R$  de l'échelle  $N^{-1}$  à 1 ;
- $Y \subset [-N, N]$  est  $\delta$ -régulier avec constante  $C_R$  de l'échelle 1 à  $N$  ;

*Alors, il existe  $\beta = \beta(C_R, \delta)$  et  $C = C(C_R, \delta) > 0$  tels que pour tout  $f \in L^2(\mathbb{R})$ ,*

$$\text{supp } \hat{f} \subset Y \implies \|f\|_{L^2(X)} \leq CN^{-\beta} \|f\|_{L^2(\mathbb{R})}$$

### Remarque.

Dans [JZ20], les auteurs fournissent une expression explicite de  $\beta$  en fonction de  $C_R$  et  $\delta$ . Ils prouvent que l'on peut prendre :

$$\beta = \exp \left[ -\exp \left( K(C_R \delta^{-1} (1 - \delta)^{-1})^{K(1-\delta)^{-2}} \right) \right]$$

où  $K$  est une constante universelle.

Il est également possible d'énoncer un principe d'incertitude fractal sous la forme 1-2.2. En effet, il n'est pas difficile de voir que le théorème précédent implique :

**Lemma 1-2.3.** Soit  $0 \leq \delta < 1$  et  $C_R \geq 1$ . Il existe  $\beta, C > 0$  (qui dépendent uniquement de  $\delta, C_R$ ) tel que pour tout  $0 < h \leq 1$ , l'inégalité

$$\left\| \mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)} \right\|_{L^2 \rightarrow L^2} \leq Ch^\beta$$

est vérifiée pour tous les ensembles  $X(h), Y(h) \subset [-1, 1]$  qui sont  $\delta$ -réguliers avec constante  $C_R$  de l'échelle  $h$  à 1.

### Remarque.

- C'est en fait sous cette forme qu'il est d'abord apparu dans [DZ16], dans le cadre de l'étude de surfaces hyperboliques convexes co-compactes, pour des ensembles  $\delta$ -réguliers bien spécifiques, reliés à l'ensemble limite de la surface. Dans cet article,  $\mathcal{F}_h$  est remplacé par un opérateur intégral de Fourier particulier.
- On peut montrer que si  $X \subset [-1, 1]$  est  $\delta$ -régulier avec constante  $C_R$  de l'échelle  $h$  à 1, alors la mesure de Lebesgue de  $X$  vérifie  $\text{Vol}(X) \leq 24C_R h^{1-\delta}$  (voir [BD18], Lemme 2.9). On voit alors, en raisonnant comme pour la preuve de 1-2.1, que

$$\left\| \mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)} \right\|_{L^2 \rightarrow L^2} \leq Ch^{1/2-\delta}$$

est valable pour tous ensembles  $X(h), Y(h) \subset [-1, 1]$   $\delta$ -réguliers avec constante  $C_R$  de l'échelle  $h$  à 1, où  $C$  ne dépendant que de  $C_R$ .

- La démonstration de [BD18] utilise des outils d'analyse harmonique. Dans [DJ18a], les auteurs utilisent une autre stratégie, qui repose sur la méthode dite de Dolgopyat, mais leur résultat est différent car ils obtiennent un principe d'incertitude fractal avec un exposant de la forme  $\beta = 1/2 - \delta + \varepsilon$ , qui n'améliore l'exposant  $1/2 - \delta$  que quand  $\delta \leq 1/2$ .

Mentionnons également une version légèrement différente, sous une condition de porosité à différentes échelles. Cette version est énoncée dans [DJN21] et utilisée dans ce manuscrit de thèse :

**Théorème 1-2.3.** ([DJN21], Proposition 2.11) Soit  $\gamma_j^\pm \in \mathbb{R}$ ,  $j = 0, 1$  tel que

$$0 \leq \gamma_1^\pm \leq \gamma_0^\pm \leq 1; \gamma_1^+ + \gamma_1^- < 1 < \gamma_0^+ + \gamma_0^-$$

et définissons

$$\gamma = \min(\gamma_0^+, 1 - \gamma_1^-) - \max(\gamma_1^+, 1 - \gamma_0^-)$$

Alors, pour tout  $\nu > 0$ , il existe  $\beta = \beta(\nu)$  et  $C = C(\nu)$  telle que l'inégalité

$$\|\mathbb{1}_{\Omega_-(h)} \mathcal{F}_h \mathbb{1}_{\Omega_+(h)}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq Ch^{\gamma\beta}$$

est vérifiée pour tout  $0 < h \leq 1$  et pour tous sous-ensembles  $\Omega_\pm(h) \subset \mathbb{R}$  qui sont  $\nu$ -poreux de l'échelle  $h^{\gamma_0^\pm}$  à  $h^{\gamma_1^\pm}$ .

**Une version discrète.** Il existe également une versions discrète de ce principe d'incertitude fractal. Ce résultat discret a été introduit dans [DJ17] pour l'étude des applications ouvertes du boulanger quantifiées et a permis l'obtention d'un trou spectral. Ce principe d'incertitude concerne une classe plus restreinte d'ensembles fractals : il se concentre en réalité sur des ensembles de Cantor. Pour le présenter, il nous faut réintroduire le même bagage que précédemment pour les applications du boulanger, à savoir, un entier  $M \geq 3$  ainsi qu'un alphabet  $\mathcal{A} \subset \{0, \dots, M-1\}$ . Pour  $k \in \mathbb{N}^*$ , on note

$$\mathcal{C}_k = \{a_0 + a_1M + \dots + a_{k-1}M^{k-1}, (a_0, \dots, a_{k-1}) \in \mathcal{A}^k\} \subset \{0, \dots, M^k - 1\}$$

qui est un ensemble de Cantor discret. Il est naturellement relié à l'ensemble de Cantor

$$\mathcal{C}_\infty = \bigcap_{k=0}^{\infty} \bigcup_{a \in \mathcal{C}_k} \left[ \frac{a}{M^k}, \frac{a+1}{M^k} \right]$$

dont la dimension de Hausdorff est

$$\delta = \frac{\log |\mathcal{A}|}{\log M}$$

Au passage, notons que  $\mathcal{C}_k$  contient exactement  $|\mathcal{A}|^k = M^{\delta k}$  éléments. Enfin, pour énoncer le principe d'incertitude fractal de [DJ17], on note  $\mathcal{F}_N : \ell^2(\mathbb{C}^N) \rightarrow \ell^2(\mathbb{C}^{\mathbb{N}})$  la transformée de Fourier discrète (unitaire).

**Théorème 1-2.4.** Soit  $M \geq 3$  et  $\mathcal{A}$  un alphabet vérifiant  $\delta \in ]0, 1[$ . Alors, il existe  $\beta > \max(0, 1/2 - \delta)$  et  $C > 0$  (qui dépendent uniquement de  $M, \mathcal{A}$ ) tels que pour tout  $k \in \mathbb{N}^*$ ,

$$\|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_{M^k} \mathbb{1}_{\mathcal{C}_k}\|_{\ell^2 \rightarrow \ell^2} \leq CM^{-k\beta}$$

On dit dans ce cas que  $\mathcal{C}_k$  vérifie un Principe d'Incertainité Fractal d'exposant  $\beta$ .

**Remarque.**

Posons  $r_k = \|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_{M^k} \mathbb{1}_{\mathcal{C}_k}\|_{\ell^2 \rightarrow \ell^2}$ .

- Pour y voir un parallèle avec le cas continu qui fait intervenir un paramètre semiclassique, il faut interpréter  $M^k$  comme étant l'inverse de  $2\pi h$ .
- La borne  $r_k \leq 1$  est triviale puisque  $\mathcal{F}_{M^k}$  est unitaire.
- La borne  $r_k \leq M^{-k(1/2-\delta)}$  n'est pas non plus très difficile à obtenir. Il s'agit en effet d'une estimée de volume :  $r_k$  est majoré par la norme Hilbert-Schmidt de la matrice  $M_k = \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_{M^k} \mathbb{1}_{\mathcal{C}_k}$ , qui n'est autre que  $\sqrt{\sum_{i,j} |(M_k)_{i,j}|^2}$ . Puisque  $M_k$  possède exactement  $|\mathcal{C}_k|^2 = M^{2k\delta}$  coefficients non nuls et puisque chacun de ces coefficients est en fait un coefficient de  $\mathcal{F}_{M^k}$ , de norme exactement  $M^{-k/2}$ , on trouve finalement que  $r_k \leq M^{k(\delta-1/2)}$ .
- La démonstration d'une telle version est en fait bien plus simple que pour le cas général

du Principe d'Incertitude Fractal (voir par exemple [Dya19]). En effet, on peut utiliser la sous-multiplicativité de la suite  $(r_k)_k$ , ce qui donne  $\lim_{k \rightarrow +\infty} \frac{1}{k} \log r_k = \inf_{k \in \mathbb{N}^*} \frac{1}{k} \log r_k$ . Il suffit alors de prouver qu'il existe  $k \in \mathbb{N}^*$  tel que

$$\frac{1}{k} \log r_k < -\max(0, 1/2 - \delta) \log M$$

**Généralisations en dimension supérieure.** À ce jour, l'unique version connue de ce principe d'incertitude fractal n'est valable qu'en dimension 1 et permet d'obtenir des trous spectraux pour des objets en dimension 2 (surfaces hyperboliques, scattering par des obstacles ou par un potentiel en dimension 2). Une généralisation en dimension supérieure nécessiterait des hypothèses supplémentaires : comme expliqué dans [Dya19] (Exemple 6.1), le principe d'incertitude fractal ne pourrait pas s'appliquer tel quel à n'importe quelle paire d'ensembles  $\delta$ -réguliers. Par exemple, dans [HS20], les auteurs démontrent une version du Théorème 1-2.2 en dimension quelconque  $d$  pour des ensembles  $X$   $\delta$ -réguliers généraux ( $\delta \in ]0, d[$ ) mais pour des ensembles  $Y$  d'une forme bien précise : les  $Y$  pour qui leur théorème s'applique sont essentiellement des produits d'ensembles  $\delta_1$ -réguliers 1-dimensionnels avec  $\delta_1 \in ]0, 1[$  (voir Figure 1.17). En ce qui concerne la version discrète, dans une récente prépublication, [Coh22] apporte une condition nécessaire et suffisante pour que le principe d'incertitude fractal discret appliqué à des ensembles de Cantor en 2D soit vérifié.

### 1-2.2.2. Applications.

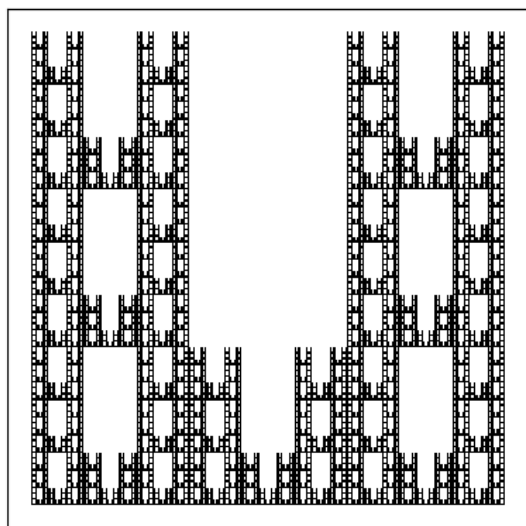
Si le principe d'incertitude fractal est à lui seul un résultat élégant d'analyse harmonique, il s'avère également être d'une très grande utilité en chaos quantique. Dans cette partie, nous présentons diverses applications de cet outil qui ont été obtenues depuis son introduction en 2016.

**Trous spectraux.** Comme déjà expliqué, la première utilisation du principe d'incertitude fractal remonte à [DZ16], où les auteurs l'ont développé pour obtenir un trou spectral dans le cadre des surfaces hyperboliques convexes co-compactes. Il leur a permis d'obtenir un meilleur trou spectral, comparé aux précédents résultats, pour des valeurs de  $\delta$  proches de la valeur critique  $\delta = 1/2$ . Il a ensuite été développé par S. Dyatlov et J. Bourgain dans [BD17] et [BD18], ce qui leur a permis d'améliorer encore ce trou spectral. Afin d'utiliser cet outil pour l'obtention d'un trou spectral, les auteurs ont décrit précisément l'évolution du propagateur de  $-\Delta$  jusqu'à deux fois le temps d'Ehrenfest, ce qui n'est normalement pas permis par le calcul semiclassique standard. Pour contourner cet écueil, les auteurs ont développé un calcul semiclassique anisotrope qui leur permet d'appliquer le théorème d'Egorov à un temps de l'ordre de deux fois le temps d'Ehrenfest. L'existence de ce calcul anisotrope repose sur la structure géométrique bien particulière des surfaces hyperboliques qui permet d'avoir :

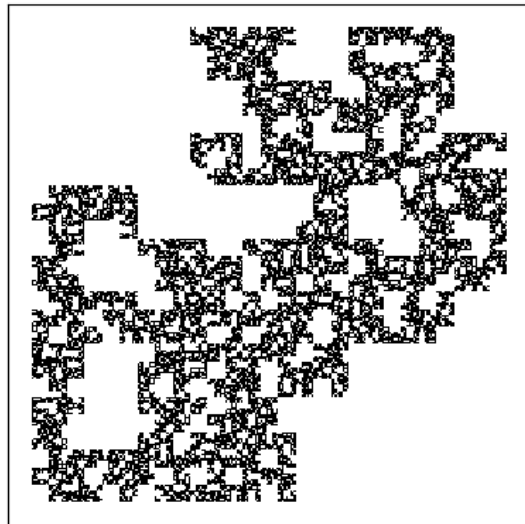
- Une description analytique des espaces stables et instables, quand on les relève au disque de Poincaré. En particulier, les distributions de ces espaces sont lisses.
- Un Jacobien instable constant.

Ces deux points, qui s'avèrent en fait cruciaux dans les travaux mentionnés plus haut, ne sont plus vérifiés dès lors que l'on s'intéresse à des situations chaotiques (i.e. hyperboliques) plus générales. Ce fut l'une des principales difficultés à dépasser dans [Vac22b] (et au Chapitre 5), mais en réalité, cela avait déjà été accompli dans [DJN21] pour une application différente (voir plus bas). Cette approche est bien entendu impossible à reproduire pour des systèmes hyperboliques ouverts généraux. En ce qui concerne la version discrète, elle est apparue dans [DJ17] et leur a permis d'obtenir un trou spectral pour des applications du boulanger ouvertes quantiques. Étant donné que nous nous sommes déjà attardés sur les trous spectraux, nous nous arrêterons ici en ce qui concerne cette partie et nous nous concentrons à présent sur d'autres applications en chaos quantique.

**Ergodicité quantique.** Le but de ce paragraphe n'est *pas* de présenter de façon détaillée et exhaustive l'ergodicité quantique, mais plutôt d'expliquer quelques problèmes et les récentes avancées qu'a permis le principe d'incertitude fractal. Pour une plus large présentation, nous conseillons par exemple [Non13], [Dya21a], [Dya21b].



(a) Un ensemble poreux (auto-similaire) en 2D.



(b) Un autre ensemble poreux en 2D (généralé aléatoirement).

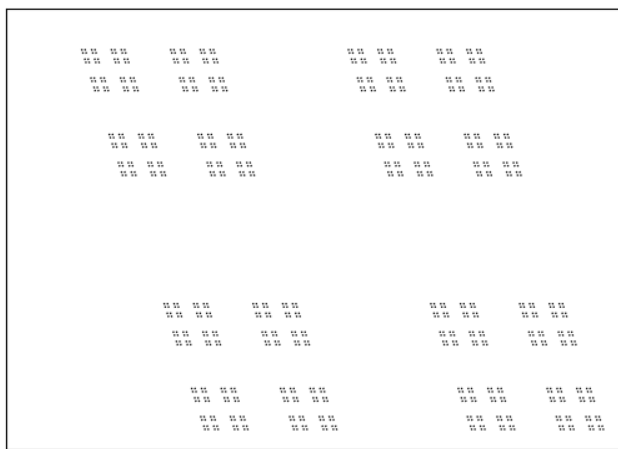
(c) Un produit d'ensembles  $\delta$ -réguliers.

Figure 1.17: Le principe d'incertitude fractal de [HS20] s'applique pour des ensembles quelconques  $X$   $\delta$ -réguliers, comme ceux des figures (a) et (b). En revanche, leur résultat impose des restrictions à l'ensemble  $Y$ , qui doit avoir la forme de celui présenté sur la figure (c).

L'Ergodicité Quantique concerne plusieurs situations chaotiques dans des domaines bornés. Pour simplifier notre propos, on s'intéressera uniquement aux surfaces compactes de courbure strictement négative. Soit donc  $(M, g)$  une telle surface (connexe), par exemple, une surface hyperbolique. Il est connu que le flot géodésique (sur  $S^*M$ ) est ergodique. L'ergodicité quantique s'intéresse au comportement haute-fréquence des fonctions propres de l'opérateur de Laplace-Beltrami  $-\Delta_g$ . Cet opérateur possède un spectre discret  $\lambda_0 = 0 < \lambda_1 \leq \dots \lambda_n \rightarrow +\infty$  dont les fonctions propres sont notées  $(u_j)$ :

$$-\Delta u_j = \lambda_j u_j ; \|u_j\|_{L^2(M)} = 1$$

Parmi les questions auxquelles tente de répondre l'ergodicité quantique :

- Que peut-on dire du comportement de  $(u_j)$  quand  $j \rightarrow +\infty$  ?
- Comment l'ergodicité du flot géodésique influence-t-elle leur comportement ?

Pour répondre à ces questions, il est pertinent d'étudier les limites faibles des mesures  $|u_j(x)|^2 dx$  sur  $M$ . En fait, on peut étudier un objet plus général, appelé mesure semiclassique :

**Définition 1-2.3.** Soit  $\text{Op}_h$  une procédure de quantification semiclassique sur  $M$ .

- Soit  $v \in L^2(M)$  et  $0 < h \leq 1$ . La *distribution de Wigner* associée à  $v$  est la mesure  $\mu_v^h$  sur  $T^*M$  définie par

$$\int_{T^*M} a(x, \xi) d\mu_v^h(x, \xi) = (\text{Op}_h(a)v, v)_{L^2} ; a \in C_c^\infty(T^*M)$$

- On dit qu'une mesure  $\mu$  sur  $T^*M$  est une *mesure semiclassique* associée à la famille de fonctions propres  $(u_j)$  s'il existe une sous-suite  $(u_{j_k})_k$  telle que, en notant  $h_k = \lambda_{j_k}^{-1}$ ,  $\mu_{u_{j_k}}^{h_k}$  converge faible-\* vers  $\mu$ , c'est-à-dire, pour tout  $a \in C_c^\infty(T^*M)$ ,

$$\lim_{k \rightarrow +\infty} (\text{Op}_{h_k}(a)u_{j_k}, u_{j_k})_{L^2} = \int_{T^*M} a(x, \xi) d\mu(x, \xi)$$

**Remarque.**

- Quand on choisit  $a(x, \xi) = a(x)$ , on retrouve le cas des mesures  $|u_j(x)|^2 dx$ .
- Le calcul semiclassique permet de montrer que la mesure  $\mu$  vérifie :
  - (i)  $\mu$  est supportée dans  $S^*M$  ;
  - (ii)  $\mu$  est invariante par le flot géodésique.
- Les mesures semiclassiques ont été introduites dans [Gér91] et [LP93].

La question principale de l'ergodicité quantique est la suivante :

**Question 1.3.** *Quelles sont les mesures semiclassiques possibles pour  $M$  ?*

En fait, en l'état actuel des connaissances, on peut préciser cette question et demander : la mesure de Liouville  $\mu_L$  sur  $S^*M$  est-elle la seule mesure semiclassique sur  $M$  ? En effet, il est désormais bien connu que la plupart des fonctions propres s'équidistribuent au sens suivant :

**Théorème 1-2.5** (QE [dV85] dans le cas général, [Zel87] pour les surfaces hyperboliques, [Sch74] pour un premier énoncé). *Il existe une suite  $(j_k) \in \mathbb{N}^{\mathbb{N}}$  de densité 1 telle que  $\mu_{u_{j_k}}^{h_k}$  converge faible-\* vers la mesure de Liouville  $\mu_L$  sur  $S^*M$ .*

Le mesure de Liouville  $\mu_L$  sur  $T^*M$  est  $\frac{\omega^n}{n!}$  où  $\omega$  est la forme symplectique canonique sur  $T^*M$ . Ce résultat amène alors naturellement à la conjecture, connue sous le nom d'Unique Ergodicité Quantique (QUE) :

**Conjecture (QUE).** *La mesure de Liouville  $\mu_L$  est la seule mesure semiclassique.*

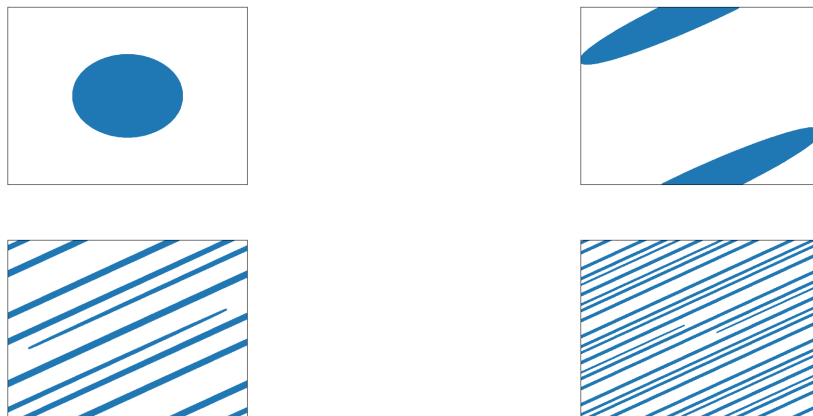


Figure 1.18: Évolution d'un paquet d'onde dans une situation hyperbolique. Il s'étire dans la direction instable. Ces images ont été générées en utilisant la dynamique de  $A \in \mathrm{SL}_2(\mathbb{Z})$ ,  $|\mathrm{tr}(A)| > 2$ , sur le tore. Un ensemble initial de points  $U$  (en haut à gauche) est itéré. Les 3 autres figures montrent  $A^k(U)$  pour différentes valeurs de  $k \in \mathbb{N}$ .

Cette conjecture est encore ouverte (à l'heure d'écrire ces lignes). Certaines restrictions sur les mesures semiclassiques ont été obtenues dans [AN07b], [AN07a], qui imposent des bornes à l'entropie des mesures. Récemment, l'utilisation du principe d'incertitude fractal a permis d'obtenir une nouvelle restriction importante :

**Théorème 1-2.6** ([DJ18b] pour les surfaces hyperboliques, [DJN21] dans le cas général).  
*Soit  $\mu$  une mesure semiclassique. Alors  $\mu$ , vue comme mesure sur  $S^*M$ , a un support total i.e. pour tout ouvert  $U \subset S^*M$ ,  $\mu(U) > 0$ .*

La preuve de [DJ18b] utilise le calcul anisotrope développé dans [DZ16]. Une fois de plus, s'il est possible de l'utiliser dans le cas des surfaces hyperboliques, ce n'est pas le cas dans le cas des surfaces de courbure négative générales. Néanmoins, dans [DJN21], les auteurs ont réussi à adapter la stratégie avec une approche différente. La régularité  $C^\infty$  des espaces stables et instables n'est plus requise dans leur travail, même si une régularité  $C^{2-\varepsilon}$  reste nécessaire. Ils ont aussi pu gérer le cas d'un jacobien instable variable. Leur approche a été adaptée dans notre travail ([Vac22b] et Chapitre 5) pour obtenir un trou spectral dans le problème de scattering par des obstacles.

Tout comme dans le cas des systèmes hyperboliques ouverts, il existe un modèle jouet bien connu en ergodicité quantique, la célèbre application du chat d'Arnold. Le modèle correspond à la quantification d'une application chaotique classique sur le tore  $\mathbb{T}^2$ . L'objet classique est une matrice  $A \in \mathrm{SL}_2(\mathbb{Z})$  telle que  $|\mathrm{tr}(A)| > 2$ , autrement dit, telle que  $A$  a des valeurs propres  $\lambda_u, \lambda_u^{-1}$  avec  $|\lambda_u| > 1$ .  $A$  agit sur  $\mathbb{T}^2$  et la dynamique est chaotique (voir Figure 1.18). La quantification d'une telle application est réalisée, par exemple, dans [BB96]. Sans entrer dans les détails du modèle, il est possible de construire un espace de Hilbert  $\mathcal{H}_N$  de dimension finie  $N$ , où  $N = (2\pi h)^{-1}$  correspond à l'inverse d'un paramètre semiclassique, et une matrice  $\mathcal{M}_N(A)$  qui agit sur  $\mathcal{H}_N$ . De façon analogue aux cas de l'opérateur de Laplace-Beltrami, cette matrice possède une base d'états propres et il est possible de construire une notion analogue de mesure semiclassique sur le tore. Dans un récent travail, [Sch21] utilise le principe d'incertitude fractal pour démontrer un théorème analogue à 1-2.6 dans ce contexte. Plus récemment, dans [DJ21], les auteurs généralisent ce résultat en dimension supérieure, mais avec des hypothèses supplémentaires sur la matrice  $A \in \mathrm{SL}_2(\mathbb{Z}^d)$ , qui leur permet *in fine* d'appliquer le principe d'incertitude fractal 1-dimensionnel.

**Applications aux EDPs.** La stratégie utilisée dans [DJ18b] et [DJN21], qui repose sur le principe d'incertitude fractal, permet également d'obtenir des applications pour des EDPs sur les surfaces de courbure négative.

*Contrôle et observabilité pour l'équation de Schrödinger.* On parle de contrôle et d'observabilité, deux notions qui sont en fait fortement reliées, quand on souhaite étudier les deux questions suiv-



antes. Soit  $\Omega \subset M$  un ouvert et soit  $T > 0$  :

- *Observabilité* : Notons  $u(t) = e^{it\Delta}u_0$ ,  $u_0 \in L^2(M)$ , c'est-à-dire, l'unique solution de

$$\begin{cases} (i\partial_t + \Delta)u(t, x) = 0, & t \in \mathbb{R}, x \in M \\ u(t = 0, x) = u_0(x), & x \in M \end{cases}$$

Si l'on observe que  $u(t, x) = 0$  pour tout  $x \in \Omega$  et pour tout  $t \in [0, T]$ , peut-on dire que  $u_0 = 0$  ?

- *Contrôle* : étant donné  $u_0 \in L^2(M)$ , peut-on trouver un terme de forçage (le contrôle)  $f \in L^2(]0, T[ \times \Omega)$  telle que l'unique solution  $u$  de

$$\begin{cases} (i\partial_t + \Delta)u(t, x) = f(t, x)\mathbb{1}_{]0, T[ \times \Omega}, & t \in \mathbb{R}, x \in M \\ u(t = 0, x) = u_0(x), & x \in M \end{cases}$$

vérifie  $u(T, x) = 0$  pour tout  $x \in M$ .

Dans le contexte des surfaces de courbure négative, le résultat suivant a été démontré :

**Théorème 1-2.7** ([Jin18] (pour les surfaces hyperboliques) et [DJN21] (pour le cas général)). *Soit  $(M, g)$  une surface de courbure strictement négative,  $\Omega \subset M$  un ouvert et  $T > 0$ .*

- (*Observabilité*) *Il existe une constante  $K > 0$  telle que pour tout  $u_0 \in L^2(M)$ ,*

$$\|u\|_{L^2(M)}^2 \leq K \int_0^T \|e^{it\Delta}u_0\|^2 dt.$$

- (*Contrôle*) *Pour tout  $u_0 \in L^2(M)$ , il existe  $f \in L^2(]0, T[ \times \Omega)$  telle que l'unique solution de*

$$(i\partial_t + \Delta)u(t, x) = f(t, x)\mathbb{1}_{]0, T[ \times \Omega} u(0, x) = u_0(x)$$

*vérifie  $u(T, x) = 0$  pour tout  $x \in M$ .*

*Équation des ondes amortie.* En ce qui concerne l'équation des ondes amortie,

$$(\partial_t^2 - \Delta + b(x)\partial_t)u(t, x) = 0; \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \quad (1-2.4)$$

où  $b \in C^\infty(M)$ ,  $b \geq 0$ , est le terme d'amortissement, il est bien connu qu'il y a une relation entre la décroissance de l'énergie et les interactions entre le flot géodésique et la région d'amortissement  $\{b > 0\}$ . Pour tout  $(u_0, u_1) \in H^1(M) \times L^2(M)$ , (1-2.4) possède une unique solution  $u(t)$ , définie par la théorie des semigroupes. L'énergie d'une solution est donnée par

$$E(u(t)) = \frac{1}{2} \int_M |\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 dx$$

Quand  $b$  n'est pas identiquement nulle, on sait que  $E(u(t)) \rightarrow 0$  quand  $t \rightarrow +\infty$ . Il est bien connu que pour que le taux de décroissance de l'énergie soit exponentiel, il suffit que l'amortissement vérifie la condition de contrôle géométrique, à savoir, que toute géodésique traverse la région d'amortissement (voir [Leb96]). Dans ce cas, on a même que  $E(u(t)) \leq e^{-\alpha t} E(u_0)$  pour un certain  $\alpha > 0$ . Dans le cas des surfaces de courbure négative, l'hyperbolicité empêche les paquets d'onde de rester concentrés dans un voisinage d'un point de l'espace des phases et les étire le long des variétés instables (voir Figure 1.18). L'utilisation du principe d'incertitude fractal a permis d'obtenir un résultat frappant qui montre qu'indépendamment de la taille de la région d'amortissement, dès lors que  $\{b > 0\} = \emptyset$ , l'énergie décroît exponentiellement vite :

**Théorème 1-2.8** ([Jin20] pour les surfaces hyperboliques, [DJN21] pour le cas général). *Soit  $(M, g)$  une surface de courbure strictement négative et  $b \in C^\infty(M)$ . Supposons  $b \geq 0$ ,  $b \not\equiv 0$ . Pour tout  $s > 0$ , il existe  $C > 0$  et  $\gamma > 0$  tel que pour tout  $(u_0, u_1) \in H^{s+1}(M) \times H^s(M)$ , l'énergie de la solution de l'équation des ondes amortie 1-2.4 décroît exponentiellement vite et l'on a :*

$$E(u(t)) \leq e^{-\gamma t} (\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s})$$

### 1-2.3. Propagation d'états cohérents.

Les états cohérents, qui portent bien d'autres noms dans la littérature (états gaussiens, paquets d'onde gaussiens, etc.) s'avèrent être un outil efficace pour relier les dynamiques quantique et classique. C'est en fait assez naturel de les utiliser quand il s'agit d'utiliser la correspondance classique/quantique. L'auteur s'est initié aux états cohérents et leurs applications en analyse semiclassique dans le livre [CR12].

**États cohérents.** L'état cohérent semiclassique centré en zéro est

$$\varphi_0(x) = \frac{1}{(\pi h)^{1/4}} e^{-\frac{|x|^2}{2h}} \quad (1-2.5)$$

et l'état cohérent centré en  $\rho \in \mathbb{R}^{2d}$  est

$$\varphi_\rho := T_\rho \varphi_0 \quad (1-2.6)$$

où  $T_\rho$  est l'opérateur de translation de Weyl-Heisenberg :

$$T_{(x_0, \xi_0)} u(x) = e^{-\frac{i}{2h} \xi_0 \cdot x_0} e^{\frac{i}{h} \xi_0 \cdot x} u(x - x_0)$$

On rappelle que  $\varphi_0$  est par ailleurs l'état fondamental de l'oscillateur harmonique  $-\hbar^2 \Delta + |x|^2$ . Les autres états propres de l'oscillateur harmonique, appelés états excités, sont obtenus à partir de  $\varphi_0$  en appliquant l'opérateur de création  $\mathbf{a}_i^\dagger = \frac{1}{\sqrt{2h}} (-\hbar \partial_{x_i} + x_i)$ . Pour  $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$ , on notera par exemple

$$\varphi_0^\nu = \frac{1}{\nu_1! \dots \nu_d!} \left( \mathbf{a}_1^\dagger \right)^{\nu_1} \dots \left( \mathbf{a}_d^\dagger \right)^{\nu_d} \varphi_0$$

pour l'état normalisé excité. C'est une fonction propre de  $-\hbar^2 \Delta + x^2$  pour la valeur propre  $\hbar(\nu_1 + \dots + \nu_d + d/2)$ . La famille  $(\varphi_0^\nu)_\nu$  constitue une base Hilbertienne de  $L^2(\mathbb{R}^d)$ . On rappelle au passage que l'opérateur d'annihilation est

$$\mathbf{a}_i = \frac{1}{\sqrt{2h}} (\hbar \partial_{x_i} + x_i)$$

**États cohérents déformés.** Il peut aussi être intéressant de définir les états cohérents déformés, parfois appelés états cohérents de Hagedorn.

**Définition 1-2.4.** On note  $\Sigma_d$  le demi plan supérieur de Siegel, formé des matrices symétriques  $\Gamma \in \mathcal{M}_d(\mathbb{C})$  telles que  $\text{Im}(\Gamma)$  est définie positive.

Soit  $\Gamma \in \Sigma_d$ . On définit un état cohérent centré en 0 déformé par la formule

$$\varphi_0^{(\Gamma)}(x) = (a_\Gamma \pi h)^{-1/4} \exp\left(i \frac{\Gamma x \cdot x}{2h}\right)$$

où  $a_\Gamma > 0$  est choisi pour que cet état soit normalisé. De façon analogue, on définit l'état cohérent déformé en  $\rho \in \mathbb{R}^{2d}$  par

$$\varphi_\rho^{(\Gamma)} = T_\rho \varphi_0^{(\Gamma)}$$

Les états cohérents déformés ont un lien avec les repères Lagrangiens (voir par exemple [LST18] qui en donnent une interprétation géométrique). Pour expliquer cette interprétation, notons, pour  $\Gamma \in \Sigma_d$ ,

$$L_\Gamma = \{(\Gamma q, q), q \in \mathbb{C}^d\}$$

On dit que  $L_\Gamma$  est un repère Lagrangien positif, dans le sens suivant : si

$$J = \begin{pmatrix} 0 & \text{I}_d \\ -\text{I}_d & 0 \end{pmatrix}$$

- (Lagrangien) pour tous  $l, l' \in L_\Gamma$ ,  $l \cdot J l' = 0$  ;
- (Positif) pour tout  $l \in L_\Gamma$ ,  $\frac{1}{i} (J l) \cdot l > 0$ .

Il est par ailleurs possible de définir des opérateurs de création et d'annihilation plus généraux : pour tout  $l \in \mathbb{C}^{2d} = (l_x, l_\xi) \in \mathbb{C}^d \oplus \mathbb{C}^d$ , on note

$$A[l] = \frac{i}{\sqrt{2h}} \sum_{j=1}^d \frac{h}{i} (l_\xi)_j \partial_{x_j} - (l_x)_j x_j ; \quad A^\dagger[l] = -\frac{i}{\sqrt{2h}} \sum_{j=1}^d \frac{h}{i} \overline{(l_\xi)_j} \partial_{x_j} - \overline{(l_x)_j} x_j$$

Remarquons les opérateurs de création et d'annihilation usuels  $\mathbf{a}_j^\dagger$  et  $\mathbf{a}_j$  sont obtenus avec  $l_j = (ie_j, e_j)$  où  $e_j$  est le  $j$ -ème élément de la base canonique, et notons que  $l_j \in L_{i\mathbb{1}_d}$ . Il est alors possible de montrer que les états cohérents déformés sont caractérisés par ([LST18], Proposition 3.5) : soit  $u \in \mathcal{D}'(\mathbb{R}^d)$ , alors

$$\forall l \in L_\Gamma, A(l)u = 0 \iff \exists c \in \mathbb{C}, u = c\varphi_0^{(\Gamma)}$$

Les états cohérents déformés apparaissent également de façon naturelle quand on étudie la propagation d'états cohérents par des hamiltoniens quadratiques. Soit  $S(t) \in \mathcal{S}_{2d}(\mathbb{R})$  une famille de matrices symétriques réelles dépendant continument de  $t$  et considérons le hamiltonien associé, à savoir

$$H(t, x, \xi) = \frac{1}{2} \begin{pmatrix} x \\ \xi \end{pmatrix} \cdot S(t) \begin{pmatrix} x \\ \xi \end{pmatrix}.$$

$H(t)$  peut être quantifiée en  $H^W(t, x, hD_x) = \widehat{H}(t)$ . Par exemple, pour  $S(t) = I_{2d}$ ,  $\widehat{H}$  n'est autre que l'oscillateur harmonique  $\frac{1}{2}(-h^2\Delta + x^2)$ . On considère alors le propagateur quantique  $U(t)$  associé à  $H$ , qui résout l'équation de Schrödinger

$$ih\partial_t U(t) = \widehat{H}(t)U(t) ; \quad U(0) = \text{Id}.$$

On peut calculer explicitement la propagation des états cohérents  $\varphi_\rho$  dès lors que l'on connaît le flot classique associé à  $H(t, x, \xi)$ . En effet, soit  $F(t)$  la matrice symplectique qui permet de résoudre l'EDO

$$\frac{d}{dt} F(t) = JS(t)F(t).$$

Écrivons-la par blocs

$$F(t) = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}.$$

On peut montrer que  $A_t + iB_t$  est toujours inversible et que si  $\Gamma_t = (C_t + iD_t)(A_t + iB_t)^{-1}$ , alors, on a ([CR12], Théorème 16, Chapitre 3) : il existe  $c_t \in \{z \in \mathbb{C}, |z| = 1\}$  tel que

$$U(t)\varphi_\rho = c_t\varphi_{\rho_t}^{(\Gamma_t)} ; \quad \rho_t = F(t)\rho.$$

En d'autres termes, le centre de l'état cohérent suit les lignes du flot hamiltonien classique et l'état subit une déformation. S'il l'on souhaite s'attaquer à des Hamiltoniens plus généraux  $\widehat{H}(t)$ , les mêmes idées s'appliquent, mais les calculs ne sont plus exactes. Toutefois, il est possible de produire un développement asymptotique en puissances de  $h^{1/2}$ . Pour cela, il faut écrire un développement de Taylor de  $H(t)$  au voisinage du centre de l'état cohérent, qui se déplace encore selon le flot classique. La partie quadratique de l'hamiltonien contrôle l'approximation au premier ordre, tandis que les termes supérieurs fournissent des corrections qui font apparaître les états excités (voir par exemple [CR97]).

*Exemple.* (cf. Figure 1.19 et Figure 1.20). Voyons ce qu'il se passe dans une situation très simple en dimension  $d = 1$ , avec un flot possédant un unique point fixe hyperbolique. C'est une situation jouet qui peut modéliser ce qui se passe pour le scattering par deux obstacles. Dans notre exemple, on veut donc que la matrice symplectique  $F_t$  soit

$$F_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

ce qui est possible en considérant  $H(t, x, \xi) = H(x, \xi) = x\xi$ . 0 est bien évidemment un point fixe hyperbolique : l'espace et la variété instables (resp. stables) sont donnés par l'axe des  $x$  (resp.

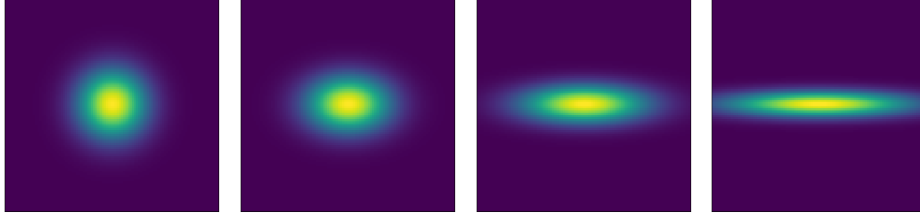


Figure 1.19: Évolution d'un état cohérent centré initialement en 0, selon la dynamique de l'exemple 1-2.7. La couleur dépend du module de la distribution de Wigner de l'état évolué.

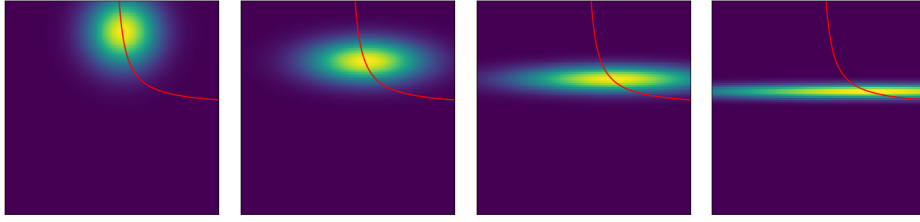


Figure 1.20: Évolution d'un état cohérent centré initialement en un point  $\rho_0$ , selon la dynamique de l'exemple 1-2.7. Le centre  $\rho_0 = (x_0, \xi_0)$  de cet exemple vérifie à la fois  $x_0 \neq 0$  et  $\xi_0 \neq 0$ . La courbe rouge représente la courbe sur laquelle évolue le centre de l'état gaussien. La couleur dépend du module de la distribution de Wigner de l'état évolué.

l'axe des  $\xi$ ). On a donc  $\widehat{H} = xhD_x + \frac{h}{2i}$  et le propagateur quantique n'est autre que l'opérateur de dilatation :

$$U(t)u(x) = e^{-t/2}u(e^{-t}x) \quad (1-2.7)$$

Il est alors très aisé de calculer explicitement l'évolution d'un état cohérent  $\varphi_\rho$  et de vérifier notamment que le résultat général présenté plus haut est vérifié. On voit bien que l'état cohérent devient de plus en plus fin et s'étire le long de la direction instable quand  $t \rightarrow +\infty$ . C'est une propriété que nous utiliserons dans cette thèse (voir Chapitre 7).

**Décomposition en états cohérents.** Les états cohérents ne sont pas orthogonaux, et l'on peut même calculer explicitement : pour tout  $\rho, \rho' \in \mathbb{R}^{2d}$ ,

$$(\varphi_\rho, \varphi_{\rho'})_{L^2} = \exp\left(-\frac{|\rho - \rho'|^2}{4h}\right) \exp\left(i\frac{\sigma(\rho, \rho')}{2h}\right) ; \sigma((x_0, \xi_0), (x_1, \xi_1)) = \xi_1 \cdot x_0 - \xi_0 \cdot x_1.$$

Néanmoins, ils forment un système complet qui permet malgré tout de décrire de façon précise n'importe quel état  $u \in L^2(\mathbb{R}^d)$ , voir les opérateurs agissant sur  $L^2(\mathbb{R}^d)$ . Pour comprendre cela, on peut introduire la transformée de Bargman :

$$u \in L^2(\mathbb{R}^d) \mapsto \mathcal{B}_h u \in L^2(\mathbb{R}^{2d}) ; \mathcal{B}_h u(\rho) = (2\pi h)^{-d/2} (\varphi_\rho, u)_{L^2} = (2\pi h)^{-d/2} \int_{\mathbb{R}^d} \overline{\varphi_\rho(x)} u(x) dx.$$

On peut montrer que  $\mathcal{B}_h$  est en fait une isométrie de  $L^2(\mathbb{R}^d)$  vers  $L^2(\mathbb{R}^{2d})$ . En particulier, son adjoint formel  $\mathcal{B}_h^*$  est aussi bien défini et donné par

$$\mathcal{B}_h^* v(x) = (2\pi h)^{-d/2} \int_{\mathbb{R}^{2d}} \varphi_\rho(x) v(\rho) d\rho, \quad v \in L^2(\mathbb{R}^{2d}).$$

Cette formule, tout comme le fait que  $\mathcal{B}_h^* \mathcal{B}_h = \text{Id}_{L^2(\mathbb{R}^d)}$  permet de décomposer un état  $u \in L^2(\mathbb{R}^d)$  en combinaison linéaire d'états cohérents, selon la formule bien pratique suivante, connue sous le nom de résolution de l'identité,

$$u(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} (\varphi_\rho, u)_{L^2} \varphi_\rho(x) d\rho. \quad (1-2.8)$$

Parmi les conséquences intéressantes de cette formule, on peut citer la possibilité de calculer la norme Hilbert-Schmidt ou bien la trace d'un opérateur en ne connaissant son action que sur les états cohérents :

**Lemma 1-2.4.** Soit  $B : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  un opérateur Hilbert-Schmidt. Alors,

$$\|A\|_{HS} = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} \|B\varphi_\rho\|^2 d\rho.$$

Soit  $A : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  un opérateur à trace. Alors,

$$\text{tr}(A) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} (A\varphi_\rho, \varphi_\rho) d\rho.$$

Cette propriété a été utilisée dans le Chapitre 7 pour estimer la norme Hilbert-Schmidt d'un opérateur, ce qui, à son tour, s'est avéré efficace pour estimer le nombre de résonance dans des boîtes et obtenir le théorème B.

**Remarque.**

- On peut définir une procédure de quantification, appelée quantification anti-Wick, qui repose sur la formule (1-2.8) : pour  $a \in \mathcal{S}(\mathbb{R}^{2d})$  (et en fait, on pourrait très bien prendre  $a \in L^\infty(\mathbb{R}^{2d})$ ) :

$$\text{Op}_\hbar^{AW}(a)u(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} a(\rho) (\varphi_\rho, u)_{L^2} \varphi_\rho(x) d\rho.$$

Par rapport aux autres procédures de quantifications (et notamment la quantification de Weyl ou les quantifications à gauche/droite), le principal intérêt de la quantification anti-Wick est qu'elle est positive :

$$a \geq 0 \implies \forall u \in L^2(\mathbb{R}^d), \left( \text{Op}_\hbar^{AW}(a)u, u \right) \geq 0.$$

- Il existe aussi une approche légèrement différente de celle de la transformée de Bargman, qui envoie un état  $u \in L^2(\mathbb{R}^d)$  sur

$$u^\#(z) = (\mathcal{B}_\hbar u(x, \xi)) e^{\frac{x^2 + \xi^2}{4\hbar}} ; z = x - i\xi.$$

Il s'agit en fait de la fameuse transformation FBI (FBI pour Fourier-Bros-Iagolnitzer, voir par exemple [CR12], Section 1.3, [Zwo12], Chapitre 13) : c'est une isométrie de  $L^2(\mathbb{R}^d)$  vers l'espace dit de Fock  $\mathcal{F}(\mathbb{C}^d)$  des fonctions entières  $f : \mathbb{C}^d \rightarrow \mathbb{C}$  telles que

$$\|f\|_{\mathcal{F}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} |f(z)|^2 e^{-\frac{z \cdot \bar{z}}{\hbar}} |dz \wedge d\bar{z}| < +\infty.$$

**Applications de la propagation des états cohérents.** On peut retrouver l'évolution de n'importe quel état en connaissant uniquement l'évolution des états cohérents, ce qui en fait donc un outil de travail particulièrement utile. Les formules du lemme 1-2.4 ont par exemple été utilisées dans [CRR99] pour démontrer la formule de trace semiclassique de Gutzwiller, en utilisant la description précise de l'évolution des états cohérents obtenue dans [CR97]. Les paquets d'onde sont aussi souvent utilisés pour la construction de quasi-modes, i.e. des approximations de fonctions propres pour des opérateurs pseudodifférentiels. C'est par exemple le cas dans [PU93]. Ces idées restent fructueuses encore aujourd'hui, comme en témoignent les récentes travaux, parmi d'autres, de [LST18] ou de [Arn20]. Il n'est bien entendu pas possible de donner une liste exhaustive de l'application des états gaussiens et de leur propagation, qui sont un outil important en physique mathématique. [CR12] propose plusieurs applications d'un point de vue semiclassique et les auteurs y donnent des références vers une littérature plus large.

Ceci conclut l'introduction de ce manuscrit de thèse. Encore une fois, les outils présentés ne forment pas une liste exhaustive. Nous serons bien entendu amenés à présenter d'autres outils importants dans cette thèse et nous reviendrons plus en détail sur ceux déjà évoqués quand nous en aurons besoin. Les prochains chapitres 3 et 4 sont dédiés à des résultats préliminaires concernant les systèmes dynamiques hyperboliques et l'analyse semiclassique, tout en présentant les objets d'étude principaux de cette thèse : les applications hyperboliques ouvertes et leurs équivalents quantiques. Nous rentrons dans le vif du sujet au Chapitre 5 et nous démontrons le théorème A dans le Chapitre 6, ainsi que d'autres applications dans le problème de scattering par des obstacles. La démonstration du Théorème B sera faite au Chapitre 7.



# Chapter 2

## Introduction (in english)

This introduction is organized into two parts. In the first one, we present the main results proved in this thesis and contextualize them. In the second part, we give an overview of a few tools used in this thesis.

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## 2-1 Framework.

**Motivations.** The main motivation of this thesis is the study of the problem of *scattering by strictly convex obstacles*. We will be interested in the scattering resonances of such systems, which can be presented as *open hyperbolic systems*. Other such systems will be displayed below in 2-1.5.. Our main results, namely Theorem A and B, presented in this introduction, concern the repartition of these resonances, when dealing with obstacles in dimension 2.

### 2-1.1. Resonances in obstacle scattering.

For the purpose of introducing resonances, we present the general problem in dimension  $d$ . We consider an bounded open set  $O$ , considered as an obstacle (or a set of obstacles), and at this stage, the unique assumption we make is that  $O$  has smooth boundary. Further, we will add important geometric conditions - in particular, the strict convexity assumption - but this is not necessary to present the notion of resonances.

In this setting, the main problem we are interested in is the study of the propagation of waves outside the obstacle, that is, we want to study the wave equation

$$\partial_{tt}^2 u(t, x) - \Delta u(t, x) = 0 ; t \in \mathbb{R}, x \in \mathbb{R}^d \setminus \overline{O}$$

with initial conditions  $u(0, x) = u_0(x)$ ,  $\partial_t u(0, x) = u_1(x)$ . For this problem being well-posed, we need to impose boundary conditions on  $\partial O$ . In this thesis, we will consider Dirichlet boundary conditions, that is,

$$u(t, x) = 0, x \in \partial O$$

Semi-group theory ensures that the above problem is well-posed ([Tay10], Chapter 9). Indeed, let  $U = \mathbb{R}^d \setminus \overline{O}$  and let  $H$  be the Hilbert space  $\mathcal{H}(U) \oplus L^2(U)$ , where  $\mathcal{H}$  is the completion of  $C_c^\infty(U)$  with respect to the norm  $\|f\|_{\mathcal{H}} = \|\nabla f\|_{L^2(U)}$ . Let also  $A$  be the operator

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta & 0 \end{pmatrix}$$

with domain  $D(A) = (H \cap H^2(U)) \oplus H_0^1(U)$ .  $A$  is maximal dissipative, so that Hille-Yosida theory allows to define the propagator  $e^{tA}$  and for  $(u_0, u_1) \in H$ , the first component  $u(t)$  of  $t \mapsto e^{tA}(u_0, u_1)$  is the unique solution of the following Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \text{ in } U \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \\ u(t, x) = 0 \text{ for } x \in \partial O \end{cases} \quad (2-1.1)$$

In the case of a bounded open set  $\Omega$  (it is obviously not the case here), the spectral theory of  $\Delta$  allows to solve completely the wave equation inside  $\Omega$  (with Dirichlet boundary condition). Indeed, since the injection  $H_0^1(\Omega) \rightarrow L^2(\Omega)$  is compact,  $-\Delta$  has compact resolvent and hence, discrete spectrum  $0 < \lambda_0(\Omega) \leq \lambda_1(\Omega) \leq \dots$ , associated to an Hilbert basis of functions  $(e_j)_{j \in \mathbb{N}}$ . Hence, the solutions of (2-1.1) have the form :

$$u(t, x) = \sum_{j=0}^{\infty} (a_j e^{it\lambda_j} + b_j e^{-it\lambda_j}) e_j(x)$$

where  $a_j, b_j$  can be computed by decomposing the initial data in the basis  $(e_j)$ .

This is no more possible in the open case we are interested in. In fact, the spectrum of  $\Delta$  is no more discrete but we have

$$\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, +\infty[$$

Nevertheless, it is still possible to define a set of generalized eigenvalues, called *resonances*, playing a role similar to the eigenvalues in the compact case, and allowing to obtain information on the behavior of solutions of the wave equation. In particular, we can obtain estimates for the decay of

the local energy for solutions of the wave equation, as it will be shown in this thesis (see Chapter 6). The global energy of the solution is defined as

$$E(t) = \frac{1}{2} \|(u(t), \partial_t u(t))\|_H^2 = \frac{1}{2} \int_U |\nabla u(t)|^2 + |\partial_t u(t)|^2$$

When there is no damping, it is conserved. If  $K \Subset \mathbb{R}^2$ , we also define the local energy in  $K$  as

$$E_K(t) = \frac{1}{2} \int_{K \cap U} |\nabla u(t)|^2 + |\partial_t u(t)|^2$$

It is known that for every  $K$ ,  $E_K(t) \rightarrow 0$  as  $t \rightarrow +\infty$  (see [Tay10], Chapter 9, Proposition 4.1). A good understanding of the resolvent of  $-\Delta$  allows to control the decay rate of the local energy.

**Rigorous definition of resonances.** To properly introduce resonances, we follow the approach presented in the recent book [DZ19].

**Remark.**

When dealing with obstacles, the case  $d = 1$  is not interesting. It becomes interesting in the study of scattering by a potential.

**The free resolvent.** It is natural to understand the case  $O = \emptyset$  that is  $U = \mathbb{R}^d$  before going further in the study. In fact, it turns out that the free case is crucial to develop the theory of scattering by obstacles (and potential). So let's present the basic facts for the resolvent of the operator  $-\Delta$  on  $L^2(\mathbb{R}^d)$  with domain  $H^2(\mathbb{R}^d)$ . The spectrum is given by  $\sigma(-\Delta) = [0, +\infty[$ , so that the following resolvent

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^d) \rightarrow H^2(\mathbb{R}^d) ; \operatorname{Im} \lambda > 0 \quad (2-1.2)$$

is well defined.

**Remark.**

The choice of writing the spectral parameter  $z \in \mathbb{C} \setminus [0, +\infty[$  of the form  $z = \lambda^2$ ,  $\operatorname{Im} \lambda > 0$  essentially comes from the fact that the functions  $e_{\lambda, \omega}(x) = e^{i\lambda x \cdot \omega}$  where  $\omega \in \mathbb{S}^{d-1}$  are trivial solutions of the equation  $-\Delta u - \lambda^2 u = 0$ . The sign of  $\operatorname{Im} \lambda$  is an arbitrary choice.

It is possible to give an analytic description of the Schwartz kernel of the free resolvent  $R_0(\lambda)$ . It is in fact a convolution kernel  $R_0(\lambda, x, y) = G(\lambda, |x - y|)$  where  $G$  depends on the dimension. We give the expression of  $G$  for  $d = 1, 2, 3$  in the following table.

Dimension	1	2	3
$G(\lambda, r)$	$\frac{i}{2\lambda} e^{i\lambda r}$	$\frac{i}{4} H_0^{(1)}(\lambda r)$	$\frac{1}{4\pi r} e^{i\lambda r}$

Here,  $H_0^{(1)}$  is a Hankel function of the first kind. When  $\operatorname{Im} \lambda > 0$ , the leading asymptotic as  $r \rightarrow +\infty$  is given by

$$H_0^{(1)}(\lambda r) = \sqrt{\frac{2}{\pi \lambda r}} e^{i\lambda r - i\pi/4} (1 + O(1/r))$$

In particular, when  $\operatorname{Im} \lambda > 0$ , the kernel has an exponential decay when  $r \rightarrow +\infty$ , ensuring that the operator  $R_0(\lambda) : L^2 \rightarrow H^2$  is bounded.

In dimension 3, and in fact more generally in odd dimensions bigger than 3, the kernel can be extended from  $\operatorname{Im} \lambda > 0$  to  $\mathbb{C}$ , as a holomorphic function. In dimension 1, it can be extended meromorphically with a unique pole at 0. The situation is different in even dimensions. This is clear in the case  $d = 2$  since the function  $H_0^{(1)}$  can be analytically extended to the **logarithmic cover**  $\Lambda$  of  $\mathbb{C}$ , but not to  $\mathbb{C}$ . In fact, we have the following crucial result :

**Theorem 2-1.1.** Let  $\chi \in C_c^\infty(\mathbb{R}^d)$ . Then, the operator

$$\chi R_0(\lambda) \chi : L^2(\mathbb{R}^d) \rightarrow H^2(\mathbb{R}^d)$$

well defined for  $\text{Im } \lambda > 0$  extends

- meromorphically to  $\mathbb{C}$  with a simple pole at  $\lambda = 0$  when  $d = 1$ ;
- analytically to  $\Lambda$  when  $d$  is even ;
- analytically to  $\mathbb{C}$  when  $d$  is odd.

**Remark.**

- There is a significant difference between odd and even dimensions. This has deep implications for the free wave equation : the strong Huyghens principle holds only in odd dimensions. When adding obstacles, as we will see in this thesis, it has implications for the decay rate of the local energy.
- One way to obtain the expressions of  $G$  given in the table is to solve the equation  $-\Delta u - \lambda^2 u = 0$  with a radial solution. It leads to solving the following ODEs. Each one has a basis of solutions given in the following table.

Dimension	1	2	3
ODE	$f'' + \lambda^2 f = 0$	$r^2 f'' + r f' + r^2 \lambda^2 f = 0$	$r^2 f'' + 2r f' + r^2 \lambda^2 f = 0$
Solution of first kind	$\frac{i}{2\lambda} e^{i\lambda r}$	$\frac{i}{4} H_0^{(1)}(\lambda r)$	$\frac{1}{4\pi r} e^{i\lambda r}$
Solution of second kind	$\frac{i}{2\lambda} e^{-i\lambda r}$	$\frac{i}{4} H_0^{(2)}(\lambda r)$	$\frac{1}{4\pi r} e^{-i\lambda r}$

Here,  $H_0^{(2)}$  is a Hankel function of the second kind. When  $\text{Im } \lambda > 0$ , the leading asymptotic as  $r \rightarrow +\infty$  is given by

$$H_0^{(2)}(\lambda r) = \sqrt{\frac{2}{\pi \lambda r}} e^{-(i\lambda r - i\pi/4)} (1 + O(1/r))$$

The solutions of the first kind are called *outgoing*, while the solutions of the second kind are *incoming*. In some sense, when computing the kernel of the free resolvent, the condition that  $R_0(\lambda)$  is bounded from  $L^2$  to  $L^2$  when  $\text{Im } \lambda > 0$  selects the outgoing part. For this reason, it is said that  $R_0(\lambda)$  is the *outgoing resolvent*.

**Meromorphic continuation.** We need the following definition.

**Definition 2-1.1.** We say that a family of operators  $T(\lambda) : H_1 \rightarrow H_2$  where  $H_1, H_2$  are Hilbert spaces and  $\lambda$  a parameter lying in an open set  $U \subset \mathbb{C}$  (or  $U \subset \Lambda$ ) is *meromorphic with poles of finite rank* if for every  $\lambda_0 \in U$ , there exist  $r > 0$ ,  $m \in \mathbb{N}$ ,  $T_1, \dots, T_m : H_1 \rightarrow H_2$  of finite rank and  $T_h(\lambda) : H_1 \rightarrow H_2$  a family of operators depending analytically on  $\lambda \in B(\lambda_0, r)$  such that  $B(\lambda_0, r) \subset U$  and for all  $\lambda \in B(\lambda_0, r)$ ,

$$T(\lambda) = T_h(\lambda) + \sum_{j=1}^m \frac{T_j}{(\lambda - \lambda_0)^j}$$

The poles of this family are the points  $\lambda \in U$  where we cannot take  $m = 0$  and they form a discrete set.

We can now turn to the precise definition of resonances. It relies on the following (see [DZ19], Theorem 4.4):

**Theorem 2-1.2.** Assume that  $O$  is a bounded open set with smooth boundary and let us note  $U = \mathbb{R}^d \setminus \overline{O}$ . Let  $-\Delta$  be the Laplacian on  $L^2(U)$  with domain  $H^2(U) \cap H_0^1(U)$  (that is, with Dirichlet boundary condition). Then,

$$R(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(U) \rightarrow L^2(U)$$

well defined for  $\text{Im } \lambda > 0$ , continues to  $\mathbb{C}$  (resp.  $\Lambda$ ) when  $d$  is odd (resp. when  $d$  is even) as a meromorphic family of operators with poles of finite rank as operators from  $L_{comp}^2(U)$  to  $H_{loc}^2(U)$ . In other words, for any  $\chi \in C_c^\infty(\mathbb{R}^d)$  such that  $\chi \equiv 1$  in a neighborhood of  $\overline{O}$ ,

$$\chi R(\lambda) \chi : L^2(U) \rightarrow H^2(U)$$

is a meromorphic family of operators with poles of finite rank.

We can now define rigorously the resonances :

**Definition 2-1.2.** We say that  $\lambda \in \mathbb{C}$  ( $d$  odd) or  $\Lambda$  ( $d$  even) is a resonance of  $-\Delta$  if it is a pole of the meromorphic continuation of  $R(\lambda)$ . The multiplicity of a resonance  $\lambda$  is defined as

$$m_R(\lambda) := \text{rank} \int_{\gamma_\lambda} R(\zeta) d\zeta$$

where the contour  $\gamma_\lambda$  is a circle centered at  $\lambda$  containing no other pole than  $\lambda$ .

**Remark.**

- The proof of this result relies on analytic Fredholm theory. The point is to build a good parametrix  $\tilde{R}(\lambda)$  such that  $(-\Delta - \lambda^2)\tilde{R}(\lambda) = \text{Id} + K(\lambda)$  where  $K(\lambda)$  is an analytic (or meromorphic in dimension 1) family of compact operators. The free resolvent  $R_0(\lambda)$  gives a parametrix for  $R(\lambda)$  near infinity.
- This result can be extended to a wider class of perturbations of the Laplace operator. First, it is possible to adapt this result to treat the case of scattering by a potential, that is, the study of  $-\Delta + V$  where  $V \in C_c^\infty(\mathbb{R}^d)$ . More generally, it can be adapted to the abstract black-box setting of [SZ91] for operators  $P$  acting on an Hilbert space of the form  $\mathcal{H} = \mathcal{H}_0 \oplus^\perp L^2(\mathbb{R}^d \setminus B(0, R))$  where  $P$  coincides with  $-\Delta$  on  $L^2(\mathbb{R}^d \setminus B(0, R))$  (see [DZ19], Chapter 4).
- As explained in [DZ19], these resonances can also be described as the poles of the scattering matrix, and correspond to the scattering resonances introduced in [LP16] in odd dimension (see for instance [SZ94]).

The structure of the outgoing resolvent near a resonance is given by the following (see [DZ19], Theorem 4.7).

**Theorem 2-1.3.** Suppose that  $\lambda \neq 0$  is a resonance. Then, there exists  $M_\lambda \leq m_R(\lambda)$  and  $\zeta \mapsto A(\zeta, \lambda)$  a holomorphic family of operators defined in a neighborhood of  $\lambda$ , such that in this neighborhood,

$$R(\zeta) = A(\zeta, \lambda) - \sum_{k=1}^{M_\lambda} \frac{(-\Delta - \lambda^2)^{k-1}}{(\zeta^2 - \lambda^2)^k} \Pi_\lambda$$

where  $\Pi_\lambda$  is a spectral projector defined as

$$\Pi_\lambda = -\frac{1}{2i\pi} \int_{\gamma_\lambda} R(\zeta) 2\zeta d\zeta$$

where the contour  $\gamma_\lambda$  contains no other pole than  $\lambda$ .  $\Pi_\lambda$  satisfies:

$$\text{rank } \Pi_\lambda = m_R(\lambda) ; (-\Delta - \lambda^2)^{M_\lambda} = 0$$

This allows to define resonant states, which are generalized eigenvalues of  $-\Delta$  :

**Definition 2-1.3.** We say that  $u$  is a resonant state (resp. generalized resonant state) associated with the resonance  $\lambda \neq 0$  if  $u \in \text{Im } \Pi_\lambda$  and  $(-\Delta - \lambda^2)u = 0$  (resp. if  $u \in \text{Im } \Pi_\lambda$ ).

The following useful characterization shows that resonant states satisfy outgoing properties (see [DZ19], Theorem 4.9):

**Theorem 2-1.4.** Let  $\lambda \neq 0$ . A state  $u \in H_{loc}^2(U) \cap H_0^1(U)$  is a resonant state associated with  $\lambda$  if and only if  $(-\Delta - \lambda^2)u = 0$  and there exists  $g \in L_{comp}^2(\mathbb{R}^n)$  and  $R > 0$  such that  $\bar{O} \subset B(0, R)$  and

$$u|_{\mathbb{R}^d \setminus B(0, R)} = (R_0(\lambda)g)|_{\mathbb{R}^d \setminus B(0, R)}$$

## 2-1.2. Distribution of resonances.

In this thesis, we study the distribution of resonances in the complex plane, and especially in 2D scattering problems. In dimension 2, we focus only on the resonances on the first sheet of  $\Lambda$ , that is on  $\mathbb{C} \setminus (i\mathbb{R}^-)$ . Two main questions interested us. The first one concerns the existence of a spectral gap :

**Question 2.1** (Spectral gap). Does there exist  $\gamma > 0$  such that there is only a finite number of resonances in  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > 1, \text{Im } \lambda > -\gamma\}$  ?

### Remark.

- It is known that there is no real resonance. This result is often referred as Rellich's theorem (see [Rel43] and see for instance [Bur98] for an elementary proof in any dimension).
- In odd dimensions, the question is equivalent if one changes  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > 1, \text{Im } \lambda > -\gamma\}$  into  $\{\lambda \in \mathbb{C}, \text{Im } \lambda > -\gamma\}$ .
- We say that  $\gamma$  is the size of the spectral gap: indeed, the existence of a finite number of resonances in  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > 1, \text{Im } \lambda > -\gamma\}$  is equivalent to the fact that there exists  $\lambda_0 > 0$  such that there is no resonances in  $\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > \lambda_0, \text{Im } \lambda > -\gamma\}$ .
- It is also possible to consider different kind of resonance free region. More generally, we can consider profile  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and consider the region

$$\{\lambda \in \mathbb{C}, |\text{Re } \lambda| > \lambda_0, \text{Im } \lambda \geq -g(|\text{Re } \lambda|)\}$$

As we will explain in this thesis, the geometry of the obstacles plays a crucial role in obtaining such spectral gaps. With the unique assumption that the obstacles are bounded open sets with smooth boundary, the most general resonance-free region is given by a profile of the form  $g(\lambda) = c_1 e^{-c_2 \lambda}$  :

**Theorem 2-1.5** ([Bur98]). Assume that  $O$  is a bounded open set with smooth boundary. Then, there exists  $\lambda_0, c_1, c_2 > 0$  such that there is no resonance in the region

$$\{|\text{Re } \lambda| > \lambda_0\} \cap \left\{ \text{Im } \lambda \geq -c_1 e^{-c_2 |\text{Re } \lambda|} \right\}$$

Question 2.1 is highly related with the decay of the local energy for the wave equation outside the obstacles. In fact, to obtain such decay, it is also important that spectral gaps come with estimates in strips for the cut-off resolvent, that is, for  $\chi \in C_c^\infty(\mathbb{R}^d)$  equal to 1 in a neighborhood of  $\bar{O}$ , we need to find  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that for all  $\lambda \in \mathbb{C}$

$$\text{Re } \lambda \gg 1; \text{Im } \lambda > -\gamma \implies \|\chi R(\lambda)\chi\|_{L^2 \rightarrow L^2} \leq f(|\text{Re } \lambda|)$$

This will be explained in much detail in Chapter 6 (see for instance [Ika82], [Ika88], [Leb96] [Bur98], [Zwo12] Chapter 5).

The other main question concerns estimates for counting functions of resonances in boxes. For  $\gamma > 0$  and  $\delta > 0$ , let's note

$$\Omega_{\gamma,\delta}(r) = \{\lambda \in \mathbb{C}, \text{Im } \lambda \in [-\gamma, 0], \text{Re } \lambda \in [r, r + \delta]\}$$

and let  $N_{\gamma,\delta}(r)$  be the number of resonances in  $\Omega_{\gamma,\delta}(r)$ , counted with multiplicity.

**Question 2.2** (Resonance counting). *Can we find an asymptotic for  $N_{\gamma,\delta}(r)$  as  $r \rightarrow +\infty$  ?*

**Remark.**

- An exact asymptotic is a very delicate question. Finding upper bounds is often easier than finding lower bounds. In this thesis, we have only studied upper bounds.
- This question is often presented as an analogue of the Weyl's law for resonances.
- It is also possible to consider different regions, not necessarily boxes, as soon as they move with  $\text{Re } \lambda \rightarrow +\infty$ .

The most general bound concerns the counting function in balls :

**Theorem 2-1.6.** *Assume that  $O$  is a bounded obstacle with smooth boundary. When  $d$  is odd (resp. even), we define  $N(r), r > 0$  as the number of resonances in  $B(0, r)$  ( resp.  $B(0, r) \setminus i\mathbb{R}^-$  ), counted with multiplicity. Then, there exists  $C > 0$  such that for all  $r > 1$ ,*

$$N(r) \leq Cr^d$$

**Remark.**

- This theorem is proved by [Mel84] in the odd-dimensional case.
- In the case where  $d$  is even, this is a corollary of the main estimates obtained in [Vod94b] when  $d = 2$  and [Vod94a] for  $d \geq 4$ . In fact, Vodev gave more general upper bounds for regions in the logarithmic cover of the form  $\{|\lambda| < r, |\arg \lambda| \leq a\}$ .

### 2-1.3. An open hyperbolic system : the geometric framework.

We now precise the geometric assumptions on the obstacles for the results obtained in this thesis. We consider a finite number of open sets (the obstacles)  $\mathcal{O}_j \subset \mathbb{R}^d$ ,  $j = 1, \dots, J$  with smooth boundary and we assume that they are strictly convex. We note

$$\mathcal{O} = \bigcup_{j=1}^J \mathcal{O}_j ; \Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$$

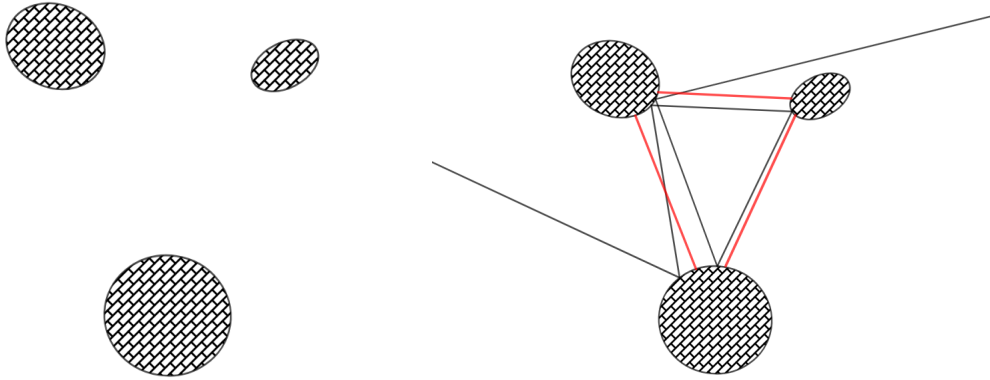
For technical reasons, we will also assume that they satisfy the following no-eclipse condition, due to Ikawa: for all  $i \neq j \neq k$ ,

$$\overline{\text{conv}(\mathcal{O}_j \cup \mathcal{O}_i)} \cap \overline{\mathcal{O}_k} = \emptyset \tag{2-1.3}$$

where  $\text{conv}(A)$  denotes the convex hull of a subset  $A \subset \mathbb{R}^d$ .

Both the spectral gap problem and resonance counting are high-frequency problems and justify the introduction of a small parameter  $h$ , where  $\frac{1}{h} = \text{Re}(\lambda)$ . Under this rescaling, we are interested in the semiclassical operator

$$P(h) = -h^2 \Delta - 1 \quad , \quad 0 < h \leq h_0$$



(a) Three convex obstacles satisfying the no-eclipse condition.

(b) Some billiard trajectories.

Figure 2.1: We study the wave equation - the quantum problem - outside strictly convex obstacles satisfying the no-eclipse condition (a). The classical flow is the billiard flow (b).

with spectral parameter  $z \in D(0, Ch)$  for some  $C > 0$ . We say that  $h$  is a *semiclassical parameter*. This point of view allows us to use the techniques of semiclassical analysis (see for instance [Zwo12]) to uniformly study the operators  $P(h)$  in the limit  $h \rightarrow 0$ . In particular, we are able to rigorously relate the high-frequency behavior of  $-\Delta$  and the classical flow associated with this system : this is the *quantum/classical correspondence*. For this open system, the classical dynamics is the billiard flow in  $\Omega \times \mathbb{S}^1$ , that is to say, the free motion outside the obstacles with normal reflection on their boundaries (see Figure 2.1).

A relevant dynamical object is the trapped set corresponding to the points  $(x, \xi) \in \Omega \times \mathbb{S}^1$  that do not escape to infinity in the backward and forward direction of the flow. This trapped set is obviously empty when  $J = 0$  or  $J = 1$ . In the case of two obstacles, it is a single closed trajectory. As soon as more obstacles are involved, the structure of the trapped set becomes complex and exhibits a fractal structure. Figure 2.2 shows different trapped trajectories corresponding to periodic ones in the case of three discs at the vertices of an equilateral triangle. Figure 2.3 shows the trapped set for a model with 4 obstacles. The structure of the trapped set plays a crucial role in the two questions above. It seems interesting to mention the following result, which illustrates the complexity of the trapped set :

**Theorem 2-1.7** ([Mor91]). *Assume that the  $J$  obstacles are strictly convex, have smooth boundary and satisfy the no-eclipse condition. Then, for each admissible sequence  $\alpha \in \Sigma$  where*

$$\Sigma := \{\alpha \in \{1, \dots, J\}^{\mathbb{Z}}, \forall j \in \mathbb{Z}, \alpha_j \neq \alpha_{j+1}\}$$

*is the set of admissible sequences, there exists a unique trapped trajectory of the billiard flow  $\Phi : \mathbb{R} \rightarrow \Omega \times \mathbb{S}^1$  such that  $\Phi(0) \in \partial\mathcal{O}_{\alpha_0} \times \mathbb{S}^1$  and the sequence of obstacles hit by  $\Phi$  is exactly  $\alpha$ .*

**Remark.**

The existence of periodic trajectories with prescribed sequence in  $\Sigma$  is not difficult to see. Indeed, let  $(\alpha_1, \dots, \alpha_N)$  be such a sequence, it suffices to minimize the function

$$(x_1, \dots, x_N) \in \partial\mathcal{O}_{\alpha_1} \times \dots \times \partial\mathcal{O}_{\alpha_N} \mapsto \|x_1 - x_2\|^2 + \dots + \|x_{N-1} - x_N\|^2 + \|x_N - x_1\|^2$$

For non periodic sequences, one can approximate them by periodic trajectories of length  $N$  and let  $N \rightarrow +\infty$ . The proof of uniqueness strongly relies on the hyperbolicity of the flow.

The complexity of the trapped set is in fact related to the hyperbolicity of the billiard flow. Precise definitions of hyperbolicity will be recalled in this thesis (see Chapter 3). Here, what is

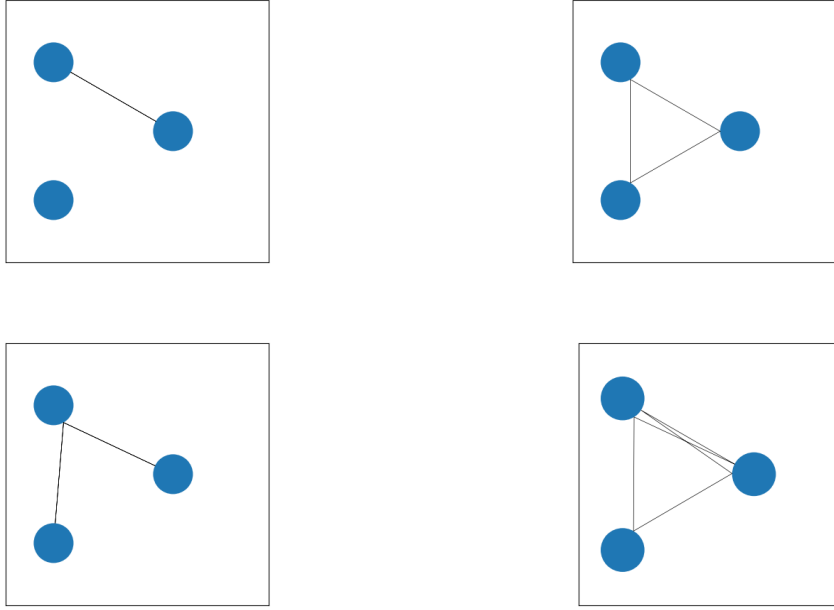


Figure 2.2: Examples of periodic trajectories for three discs at the vertices of an equilateral triangle.

relevant to have in mind in that hyperbolicity creates instability with respect to initial conditions : this is illustrated by the figure 2.4 and 2.5. For instance, a particle can start with initial conditions very close to a trapped trajectory but escape toward infinity. This system is said to be in the class of *open hyperbolic systems*.

**A little dynamical background.** To finish this presentation of the billiard flow, it seems necessary to introduce some important notions to describe its dynamics and to understand some results we want to present below.

- The **billiard map** (see Figure 2.6). For  $j \in \{1, \dots, J\}$ , let  $B^*\partial\mathcal{O}_j$  be the co-ball bundle of  $\partial\mathcal{O}_j$  that is

$$B^*\partial\mathcal{O}_j = \{(x, \eta) \in T^*\partial\mathcal{O}_j, |\eta| < 1\}$$

Let also  $S_{\partial\mathcal{O}_j}^*$  be the restriction of  $S^*\Omega$  to  $\partial\mathcal{O}_j$ ,  $\pi_j : S_{\partial\mathcal{O}_j}^* \rightarrow B^*\partial\mathcal{O}_j$  the natural orthogonal projection and  $\nu_j(x)$  be the outward normal vector at  $x \in \partial\mathcal{O}_j$ . For  $i \neq j$ , let  $\mathcal{B}_{ij}$  be the map defined by

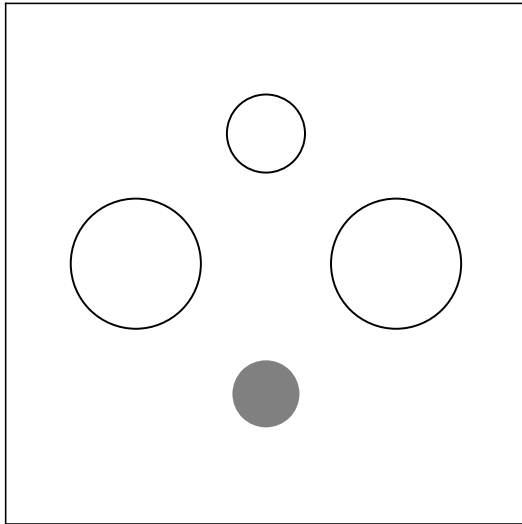
$$\begin{aligned} \mathcal{B}_{ij}(\rho) = \rho' &\iff (\rho', \rho) \in B^*\partial\mathcal{O}_i \times B^*\partial\mathcal{O}_j, \\ &\exists t > 0, \exists \xi \in \mathbb{S}^1, \exists x \in \partial\mathcal{O}_j \\ &x + t\xi \in \partial\mathcal{O}_i, \langle \nu_j(x), \xi \rangle > 0, -\langle \nu_i(x + t\xi), \xi \rangle > 0, \\ &\pi_j(x, \xi) = \rho, \pi_i(x + t\xi, \xi) = \rho' \end{aligned}$$

- The **unstable Jacobian**. The Riemannian metric on  $T^*\partial\mathcal{O}$  is inherited from the one of  $T^*\mathbb{R}^d$ . Let's consider a periodic trajectory  $\gamma$  for the billiard flow, associated with the sequence  $\alpha \in \Sigma$ , as permitted by Theorem 2-1.7. Let's note  $(\alpha_1, \dots, \alpha_N)$  an elementary pattern of  $\alpha$ . Let's say that  $\gamma$  starts from a point  $\rho_1 \in B^*\partial\mathcal{O}_{\alpha_1}$ , which is a fixed point of

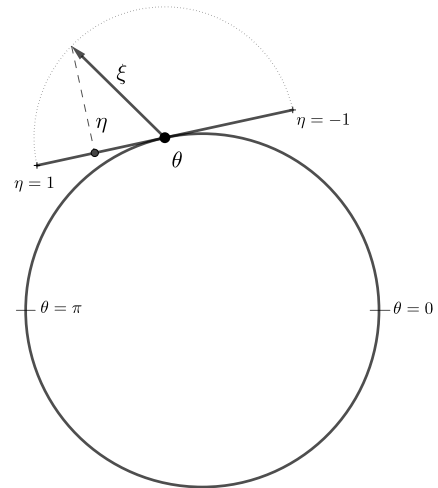
$$\mathcal{B}_\alpha := \mathcal{B}_{\alpha_1\alpha_N} \circ \dots \circ \mathcal{B}_{\alpha_2\alpha_1}$$

Locally near  $\rho_1$ , the map  $\mathcal{B}_\alpha$  is well defined and smooth. This holds due to the no-eclipse condition which ensures that the trapped set contains no glancing points, which are singularities of the billiard map. We claim that  $\rho_1$  is a hyperbolic fixed point of  $\mathcal{B}_\alpha$  (see [CM00]),

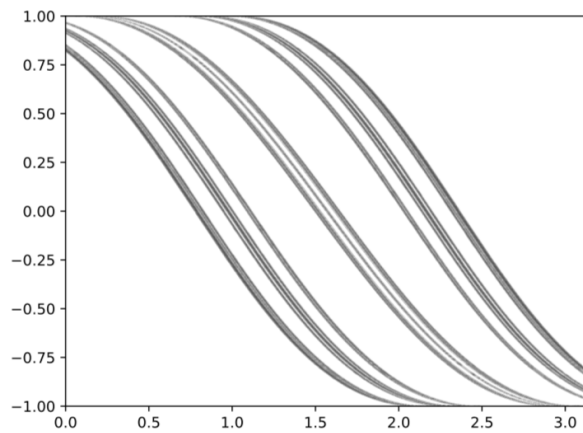




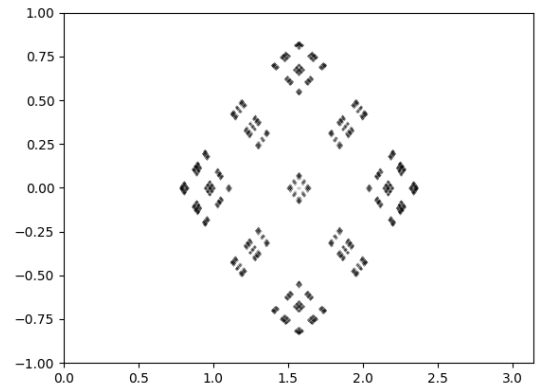
(a) The disposition of the obstacles we consider. We focus on the gray disc to plot the trapped set and the incoming tail.



(b) The coordinate  $(\theta, \eta)$  on the gray circle.



(c) The incoming tail for the gray disc.



(d) The trapped set for the gray disc.

Figure 2.3: We show the incoming tail (2.3c) and the trapped set (2.3d) in the example of (2.3a) when restricted to the gray disc. The coordinates used are described in (2.3b) : the variable in the horizontal axis is  $\theta \in [0, \pi]$  while the variable on the vertical axis is  $\eta \in [-1, 1]$ , associated with a unit outward vector  $\xi$ . The incoming tail appears as a union of stable manifolds, distributed in a fractal way along the unstable manifolds.

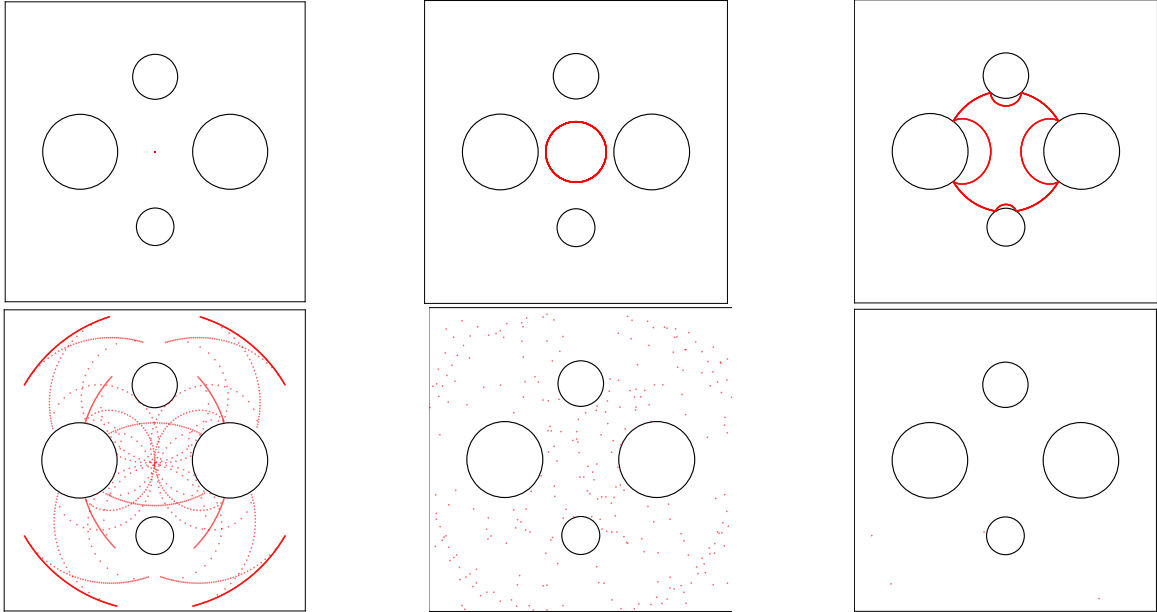


Figure 2.4: Let's observe the dynamics of many (2000) particles with the same initial position and initial velocities uniformly distributed on the unit circle : in this example, the angles of the velocities are  $\theta_k = 0.01 + 2\pi \frac{k}{2000}$ . After a rather short time, most of (and in fact all in this example) the particles end outside the window. See [the online video](#).

that is, there exists a decomposition

$$T_{\rho_1}^* (T^* \partial \mathcal{O}_{\alpha_1}) = E_u(\rho_1) \oplus E_s(\rho_1)$$

such that

- (i)  $d_{\rho_1} \mathcal{B}^\alpha(E_{u/s}(\rho_1)) \subset E_{u/s}(\rho_1)$ ;
- (ii)  $\dim E_u(\rho_1) = \dim E_s(\rho_1) = d - 1$
- (iii) there exists  $C > 0$  and  $\lambda < 1$  such that for each  $v \in E_u(\rho_1)$  (resp.  $E_s(\rho_1)$ ) and  $n \in \mathbb{N}$ ,

$$\|(d_{\rho_1} \mathcal{B}_\alpha)^{-n} v\| \leq C \lambda^n \|v\| \quad (\text{resp. } \|(d_{\rho_1} \mathcal{B}_\alpha)^n v\| \leq C \lambda^n \|v\|)$$

We then define the unstable Jacobian associated with the periodic trajectory  $\gamma$  as

$$J^u(\gamma) = \det d_{\rho_1} (\mathcal{B}^\alpha : E_u(\rho_1) \rightarrow E_u(\rho_1)) \quad (2-1.4)$$

- The **topological pressure**. This notion comes from the Thermodynamic formalism (see for instance [Rue04]). At this stage, we present only its definition in this context for the particular case of the unstable Jacobian and the billiard map, but it can be generalized to continuous functions defined on the trapped set (see Definition 3-5.1 in Chapter 3). Let  $s \in \mathbb{R}$ , we define the topological pressure  $P(s)$  as

$$P(s) = \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \left( \sum_{\gamma, T-1 \leq |\gamma| < T} J^u(\gamma)^{-s} \right) \quad (2-1.5)$$

Here, the sum runs over all the periodic trajectories  $\gamma$  of length  $|\gamma| \in [T - 1, T[$ . The length of a trajectory  $|\gamma|$  is the sum of the lengths of the rays forming  $\gamma$ . When  $s = 0$ ,  $P(0)$  simply estimates the number of periodic rays and is in fact the topological entropy  $h_{top}$ . When  $s = 1$ , it can be shown that  $P(1) < 0$  and  $-P(1)$  is sometimes called the classical escape (or decay) rate of the system, written  $\gamma_{cl}$ .  $P(s)$  measures the competition between the complexity of the system (number of periodic rays) and the instability (due to the hyperbolicity). It is known that  $s \mapsto P(s)$  is strictly decreasing.

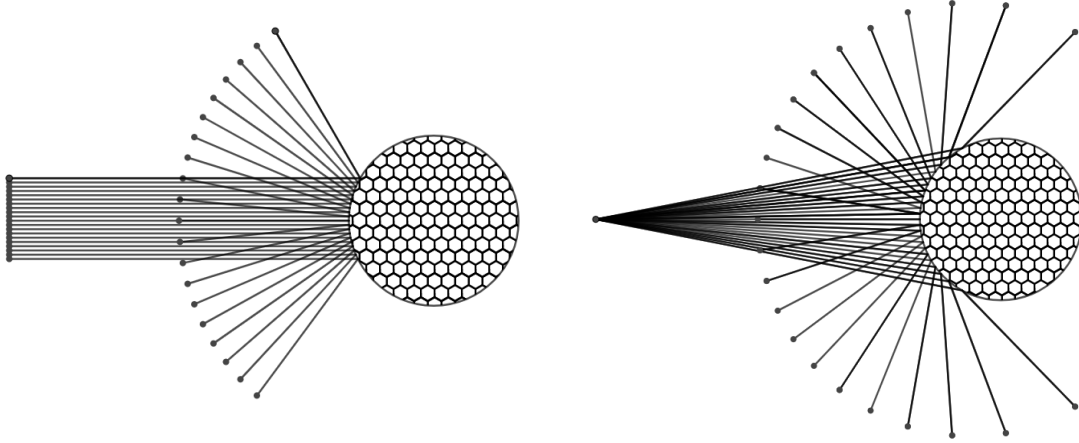


Figure 2.5: These figures show two families of trajectories depending continuously on a parameter. After one reflexion on the boundary of a strictly convex obstacle, we clearly observe unstability with respect to the initial condition.

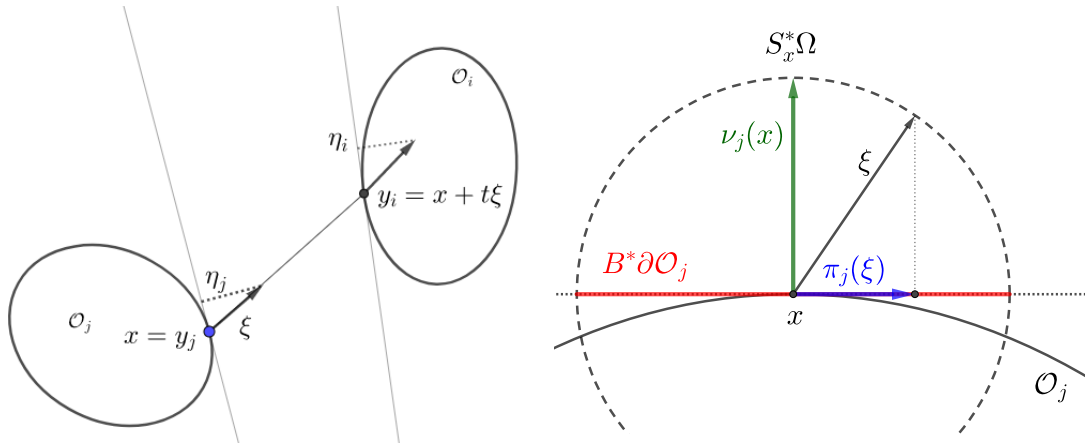


Figure 2.6: The billiard map.

- The **dimension** of the trapped set. In presence of such hyperbolicity, the trapped set is known to exhibit a fractal structure (see for instance the book [Bar08]). Its box (or Minkowski) and Hausdorff dimensions coincide in dimension 2 and for some reason related to the special structure of the trapped set, we write it  $2d_H + 1$ . In dimension 2,  $P(s)$  and  $d_H$  are related by a formula, known as Bowen's formula

$$P(d_H) = 0$$

**Remark.**

In the context of obstacle scattering, it seems easier to deal with the billiard map, instead of the billiard flow since the later has singularities. Nevertheless, as explained in [DSW21], it is possible to build a smooth model in which the billiard flow is smooth and which preserves the properties of the dynamics (hyperbolicity, length of the periodic trajectories, etc.). In this paper, the authors also work with a weaker "no-grazing" assumption, implied by the no-eclipse condition. In fact, it should be possible to work with this weaker condition. Nevertheless, a significant part of this thesis relies on the paper [NSZ14], in which the no-eclipse condition is made. For this reason, we choose to keep this assumption in our work.

### 2-1.4. Existing results and main contributions of this thesis.

We now present a non exhaustive list of existing results concerning the distribution of resonances. We start with the case of 1 obstacle, which is a non-trapping situation. We then give the result for the case of two obstacles, situation in which the distribution of resonances is well understood. Finally, we explore some results in the general case of  $J \geq 3$  obstacles and we present the two main results of this manuscript.

#### 2-1.4.1. The case of 1 obstacle

Before giving general results for scattering by one strictly convex obstacles, let's present the case of the sphere in  $\mathbb{R}^3$ , one of the only situation where analytic computations are possible.

*Example.* Scattering by a sphere in  $\mathbb{R}^3$ . (see also [Tay10], Chapter 9, Section 9)

Let  $\mathcal{O} = B(0, R) \subset \mathbb{R}^3$  be the ball of center 0 and radius  $R$ . We note  $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$ . We aim at computing the scattering resonances of this obstacle. To do so, we will exploit the radial symmetry. Every function  $u : \Omega \rightarrow \mathbb{C}$  can be decomposed with the help of the Hilbert basis of  $L^2(\mathbb{S}^2)$  made of harmonic polynomials : for  $l \in \mathbb{N}$  and  $-l \leq m \leq l$ , we note  $\phi_{l,m}$  an element of this basis associated with the eigenvalue  $l(l+1)$ .

*Resonant states.* In fact, in this context, it is known that there is no real resonance. So we can focus on the case  $\lambda \neq 0$  and use the characterization given above. We are looking for  $u \in H_{loc}^2(\Omega) \cap H_0^1(\Omega)$  such that  $(-\Delta - \lambda^2)u = 0$  with  $u$  outgoing. We can write

$$u(r, \omega) = \sum_{l,m} u_{l,m}(r) \phi_{l,m}(\omega), \quad r > R$$

where  $u_{l,m}(R) = 0$  and  $u_{l,m}$  satisfies the following ODE

$$r^2 u'' + 2ru' + (r^2 \lambda^2 - l(l+1))u = 0$$

This ODE has 2 fundamental solutions  $r \mapsto h_l^{(1)}(\lambda r)$  and  $r \mapsto h_l^{(2)}(\lambda r)$  where  $h_l^{(i)}$  are the Spherical Bessel functions of the third kind (see for instance [AS72]). They can be written on the form

$$h_l^{(1)}(z) = \frac{e^{iz}}{z} p_l(z^{-1}); \quad h_l^{(2)}(z) = \frac{e^{-iz}}{z} q_l(z^{-1})$$

where  $p_l$  and  $q_l$  are polynomials of degree  $l$ . More precisely,  $p_l$  is given by

$$p_l(X) = (-i)^{l+1} \sum_{k=0}^l \frac{(l+k)!}{k!(l-k)!} \left( \frac{iX}{2} \right)^k$$

*Outgoing property.* The only possible choice for the outgoing property to hold is to take

$$u_{l,m}(r) = a_{l,m} h_l^{(1)}(\lambda r)$$

Indeed, one can try to compute the free resolvent by using this decomposition and has to solve the same ODE. When  $\text{Im } \lambda > 0$ , the only possible choice to have  $L^2$  boundedness is to choose the solution  $h_l^{(1)}(\lambda r)$  since  $h_l^{(2)}(\lambda r)$  grows exponentially fast as  $r \rightarrow +\infty$  when  $\text{Im } \lambda > 0$ .

*Boundary condition.* If we want the boundary conditions to hold without  $u_{l,m}$  being identically zero,  $\lambda$  must satisfy

$$h_l^{(1)}(\lambda R) = 0 \tag{H_l}$$

*Conclusion.* As a consequence, if  $\lambda$  does not satisfy any of the condition  $H_l$ , there is no non trivial outgoing solution of  $-\Delta u - \lambda^2 u = 0$ . On the contrary, if one of the  $H_l$  is satisfied, the functions  $h_l^{(1)}(r) \phi_{l,m}$  give  $2l+1$  independent outgoing solutions, so  $\lambda$  is a resonance and its multiplicity is given by the sum of the  $2l+1$  where  $h_l^{(1)}(\lambda R) = 0$  (see Figure 2.7).

Let's now turn to the general case of one strictly convex obstacle. The billiard flow is obviously non trapping, that is, the trapped set is empty. The following result is now rather standard for non trapping situations and relies on propagation of singularities (see [DZ19] Section 4.6 and 6.4 and the references given there, see also [MRS77], [Mel79], [MS78], [RS82]).

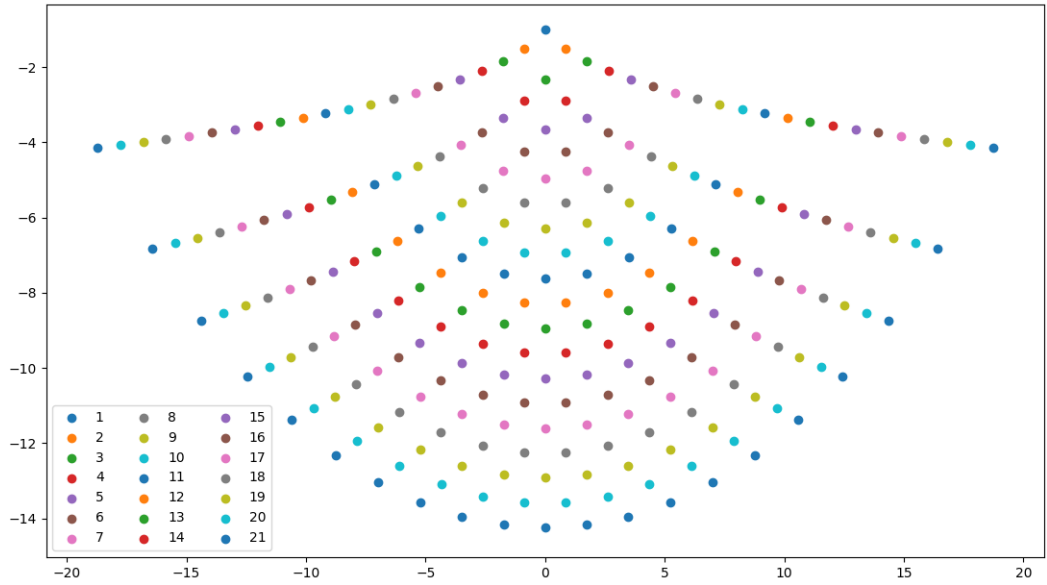


Figure 2.7: Locations of the first scattering resonances for the exterior of a sphere of radius 1. Each color is associated to a family of poles, labeled by the integer  $l = 1, \dots, 21$ . In such a family, each pole has multiplicity  $2l + 1$ .

**Theorem 2-1.8.** *Assume that  $\mathcal{O}$  is a single strictly convex obstacle with smooth boundary. Then, there exists  $\lambda_0 > 0$  and  $A > 0$  such that there is no resonance in*

$$\{\lambda \in \mathbb{C} \setminus (i\mathbb{R}^-), |\operatorname{Re} \lambda| > \lambda_0, \operatorname{Im} \lambda \geq -A \log |\operatorname{Re} \lambda|\}$$

**Remark.**

- This result gives a clear answer to the two questions mentioned above. There is indeed a spectral gap and each  $N_{\gamma, \delta}(r)$  is zero for  $r \geq r_\gamma$ .
- This result was already proved in dimension 3 in [LP71].

This logarithmic resonance free region has been improved to an inverse cubic region :

**Theorem 2-1.9** ([HL94], [SZ95]). *Assume that  $\mathcal{O}$  is a single strictly convex obstacle with smooth boundary. Then, there exists  $\lambda_0 > 0$  and  $c_1 > 0$  such that there is no resonance  $\lambda$  satisfying*

$$\operatorname{Im} \lambda \geq -c_0(\operatorname{Re} \lambda)^{1/3} + c_1 |\operatorname{Re} \lambda| \geq \lambda_0$$

*where  $c_0$  is an explicit computable constant depending only on  $\mathcal{O}$ .*

This result has first been proved for obstacles with analytic boundaries in [BLR87], where the authors also show that this bound is optimal in some sense, for generic analytic obstacles. In [SZ95], the author give more general estimates for counting functions in different neighborhoods of the real axis and obtained this resonance free region as a direct corollary. The two authors improved this analysis in [SZ99] where they establish the existence of cubic bands where the resonances live, under some pinching condition for the curvature. The heuristics is that the more the obstacle looks like a sphere, the more the picture looks like the one in Figure 2.7.

### 2-1.4.2. The case of 2 obstacles.

In this situation, the trapped set is no more empty, but is far from being complicated ! Indeed, it is given by a unique periodic trajectory which is the ray minimizing the distance between the two obstacles.

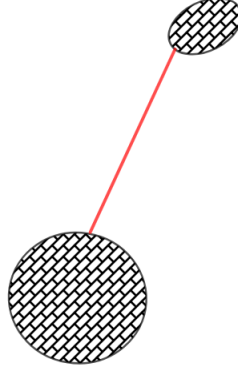


Figure 2.8: Two obstacles and the trapped set.

In some sense, this trapped set is too small to allow resonance to approach the real axis when  $\text{Re } \lambda \rightarrow +\infty$ . As we will see, this heuristic explanation is still relevant for more complicated trapped set. The first proof of the existence of a spectral gap for 2 strictly convex obstacles has been given by Ikawa [Ika82] in the case of  $d = 3$ . Let's note that his result shows that the conjecture made by Lax-Phillips for trapping obstacles doesn't hold. Indeed, they conjectured that in such a trapping situation, there should exist a sequence  $(\lambda_n)$  of resonances with  $\text{Im } \lambda_n \rightarrow 0$  and  $\text{Re } \lambda_n \rightarrow +\infty$ . This lead Ikawa to formulate the following conjecture ([Ika06]), known as the *modified Lax-Phillips conjecture* and still open :

**Conjecture.** *In the case of an arbitrary number of obstacles  $J \geq 2$ , there exists  $\gamma > 0$  such that the band  $\{\text{Re } \lambda \geq 1, \text{Im } \lambda \in [-\gamma, 0]\}$  contains an infinite number of resonances.*

In the case of 2 obstacles, this conjecture has been solved and is known to hold. In fact, in the limit  $|\text{Re } \lambda| \rightarrow +\infty$ , it is possible to describe asymptotically all the resonances in any band  $\{\text{Re } \lambda \geq 1, \text{Im } \lambda \in [-\gamma, 0]\}$ . Ikawa is the first one to have constructed a band of resonances ([Ika83], [Ika87]) in dimension 3. His result has been significantly improved by [Gér88] in any odd dimension, by proving the existence of a pseudo-lattice of resonances. This result stated and proved in odd dimensions in [Gér88] could be adapted for even dimensions (see [NSZ14]). To state this result precisely, let us introduce the following notations. Let's note  $\mathcal{O}_1$  and  $\mathcal{O}_2$  the two strictly convex obstacles and set  $d_{obs} = d(\mathcal{O}_1, \mathcal{O}_2)$ . The distance is reached for a unique ray  $\gamma_0$  corresponding to a periodic billiard trajectory. The ray  $\gamma_0$  is associated with a periodic point  $\rho_0$  of  $\mathcal{B}^2 := \mathcal{B}_{21} \circ \mathcal{B}_{12}$  and due to the strict convexity of the obstacles,  $\rho_0$  is a hyperbolic fixed point, that is  $d_{\rho_0} \mathcal{B}^2$  has all its eigenvalues with modulus different from 1. Let's note  $\nu_1, \dots, \nu_{d-1}$  its eigenvalues with modulus smaller than one (repeated with multiplicity). In fact,  $\mathcal{B}^2$  is symplectic, so that the eigenvalues of modulus bigger than 1 are the  $\nu_i^{-1}$ . Let's note  $b_0 = (\nu_1 \dots \nu_{d-1})^{1/2} = J^u(\gamma_0)^{-1/2}$  and  $c_0 = \log \frac{2d_{obs}}{b_0}$  and for  $\alpha \in \mathbb{N}^{d-1}$ ,  $\nu^\alpha = \nu_1^{\alpha_1} \dots \nu_{d-1}^{\alpha_{d-1}}$ . Set  $I = \{\nu^\alpha, \alpha \in \mathbb{N}^{d-1}\}$  and for  $m \in I$ , set

$$N_m = \#\{\alpha \in \mathbb{N}^{d-1}, \nu^\alpha = m\}$$

For each  $m \in I$ , we can now define a line of pseudo-poles :

$$\lambda_m(j) = j \frac{\pi}{d_{obs}} + i(c_0 - \log m), j \in \mathbb{Z}$$

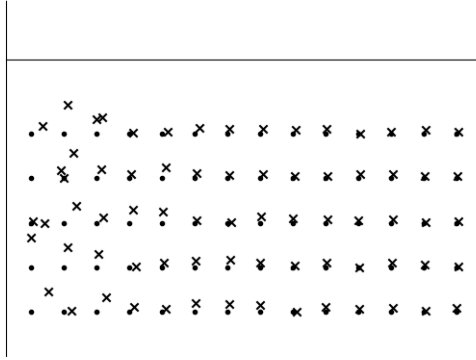


Figure 2.9: Asymptotically, the resonances (crosses) for two strictly convex obstacles converge to a pseudo-lattice (points).

**Theorem 2-1.10** ([Gér88]). *Let's note  $\text{Res}$  the set of resonances for the two obstacles  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Fix  $\gamma > 0$ . Then, there exists  $\lambda_0 > 0$  and  $C > 0$  such that, by denoting  $\Omega(\lambda_0, \gamma) = \{\lambda, \text{Re } \lambda > \lambda_0, \text{Im } \lambda > -\gamma\}$ , we have*

$$\text{Res} \cap \Omega(\lambda_0, \gamma) \subset \bigcup_{\substack{m \in I, j \in \mathbb{Z} \\ \lambda_m(j) \in \Omega(\lambda_0, \gamma)}} D\left(\lambda_m(j), C|\lambda_m(j)|^{1/2}\right)$$

*and each  $D(\lambda_m(j), C|\lambda_m(j)|^{1/2})$  contains exactly  $N_m$  resonances (with multiplicity).*

**Remark.**

In fact, the result of [Gér88] is much more precise since it gives an entire asymptotic expansion of the  $N_m$  poles in  $D(\lambda_m(j), C|\lambda_m(j)|^{1/2})$ .

**2-1.4.3. General case.**

We now come to the general case of an arbitrary number of obstacles. We present the existing results in this context and give the main results of this thesis.

**Spectral gaps.** In the general case of  $J \geq 3$ , the existence of a spectral gap remains open. Based on experimental results (see for instance, [BWP<sup>+</sup>13]) the following conjecture has been made in [Zwo17] :

**Conjecture.** *For strictly convex obstacles with smooth boundary and satisfying the no-eclipse condition, there is always a spectral gap.*

This conjecture concerns in fact much more general open hyperbolic systems. We will come to this point later.

It seems that the first positive result for the existence of a spectral gap for more than 3 obstacles has been given by Ikawa [Ika88] for the case of dimension 3. His result has been revisited by [NZ09] and generalized in any dimension :

**Theorem 2-1.11** ([Ika88], [NZ09]). *Assume that  $P(1/2) < 0$ . Then, there exists a spectral gap. More precisely, for every  $\varepsilon > 0$ , there exists  $\lambda_\varepsilon > 0$  such that there is no resonance in*

$$\{|\text{Re } \lambda| > \lambda_\varepsilon, \text{Im } \lambda \geq (P(1/2) + \varepsilon)\}$$

**Remark.**

- The result of [NZ09] rather concerns a different situation of scattering by a potential. To see why it implies the spectral gap for obstacle scattering, it is necessary to use the result of [NSZ14]. This is explained in [Non11] (Section 8).
- The spectral gap has been improved in [PS10] under additional dynamical assumptions. In particular, the authors required a Dolgopyat-type estimate for some transfer operator related to the billiard flow.
- This kind of pressure condition involving  $\frac{1}{2} \log J^u$  also appears in other works, for different problems. In [Bur93], the author proves a controllability result for the plate equation inside a bounded open set, containing strictly convex obstacles satisfying Ikawa condition. His result holds under the pressure condition  $P(1/2) < 0$ . This is also this kind of pressure condition which appears in [Sch10] and [Riv14], concerning the study of the damped wave equation on compact manifolds without boundary.

In this thesis, we show that in dimension 2, the conjecture holds without the pressure condition :

**Theorem A** (Spectral gap, [Vac22b]). *For strictly convex obstacles with smooth boundary and satisfying the no-eclipse condition in dimension 2, there is always a spectral gap.*

This result comes with polynomial estimates for the cut-off resolvent in strips, which have applications for the decay of the local energy for the wave equation (see Chapter 6, [Vac22a]).

**Estimates for the counting function.** Recall the definition of  $N_{\gamma,\delta}(r)$  as the number of resonances (with multiplicity) in the box  $\{\operatorname{Re} \lambda \in [r, r + \delta], \operatorname{Im} \lambda \in [-\gamma, 0]\}$ . For simplicity, we fix  $\delta = 1$  and set  $N_\gamma = N_{\gamma,1}$ . Numerical computations and experiments (see [PWB<sup>+</sup>12], [EMW10]) suggest a behavior of the form

$$N_\gamma(r) \sim C_\gamma r^{\alpha_\gamma}$$

Nevertheless, it is a very challenging open problem. Indeed, it is often very difficult to find lower bounds. Even weak form such

$$N_\gamma(r) \geq C_\gamma r^{\alpha_\gamma}$$

would be a great progress in understanding the behavior of this quantity. Up to now, some upper bounds are known, referred as fractal Weyl upper bounds :

**Theorem 2-1.12** ([NSZ14]). *For strictly convex obstacles with smooth boundary and satisfying the no-eclipse condition, the following upper bounds for the counting function  $N_\gamma(r)$  holds : for all  $\gamma > 0$  and for all  $\varepsilon > 0$ , there exists  $C_{\gamma,\varepsilon} > 0$  such that for all  $r > 1$ ,*

$$N_\gamma(r) \leq C_{\gamma,\varepsilon} r^{d_H + \varepsilon}$$

**Remark.**

- In dimension 2, the results holds with  $\varepsilon = 0$ .
- This is consistent with what is known for 2 obstacles since in that case  $d_H = 0$  and we explicitly know the asymptotic distribution of the resonances.
- Spectral gap of size  $\gamma$  are equivalent with  $N_\gamma(r) = 0$  for  $r$  large enough. In particular, if there is a spectral gap - as conjectured - this bound cannot be optimal for all values of  $\gamma$ .

In experiments, the value  $\gamma = \gamma_{cl}/2$  seems to play an important role. It is possible to observe two phenomena :

- The number of resonances seems to be smaller than the bound given by the previous theorem when  $\gamma < \gamma_{cl}/2$  ;
- A concentration of resonances near the line  $\gamma = \gamma_{cl}/2$  is observed.

The first observation can be related to the so-called Jakobson-Naud conjecture ([JN12]) for convex co-compact hyperbolic surfaces (see below), which, in this context, would translate in : there is a spectral gap of size  $\gamma_{cl}/2$ . In this thesis, we tackle a much more accessible conjecture, made in [Zwo17] :



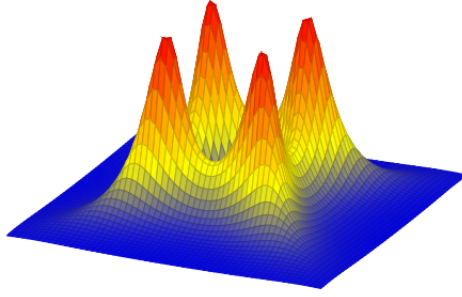


Figure 2.10: Example of potential in  $\mathbb{R}^2$  with 4 bumps.

**Conjecture.** For every  $\gamma < \gamma_{cl}$ ,  $N_\gamma(r)r^{-d_H} \rightarrow 0$  as  $r \rightarrow +\infty$ .

We succeed to prove this conjecture in the case of dimension 2, giving an improved upper bound for  $\gamma \leq \gamma_{cl}/2$ .

**Theorem B** (Improved fractal Weyl upper bound). For strictly convex obstacles with smooth boundary and satisfying the no-eclipse condition in dimension 2, there exists a non increasing function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying

- $\sigma(\gamma) > 0$  for  $0 \leq \gamma < \gamma_{cl}/2$  ;
- $\sigma(\gamma) = 0$  for  $\gamma \geq \gamma_{cl}/2$

and such that for all  $\gamma > 0$  and for all  $\varepsilon > 0$  there exists  $C_{\gamma,\varepsilon}$  such that  $\forall r \geq 1$ ,

$$N(r, \gamma) \leq C_{\gamma,\varepsilon} r^{d_H - \max(\sigma(\gamma) - \varepsilon, 0)}$$

## 2-1.5. Similar results in other models of open hyperbolic systems.

### 2-1.5.1. Semiclassical scattering by a potential.

Scattering by a potential is a natural way to extend obstacle scattering, by replacing the obstacles, seen as infinite potential barriers, by smooth compactly supported potentials  $V \in C_c^\infty(\mathbb{R}^d)$ . In other words, we want to study the resonances of the operator  $-\Delta + V$ , which are well defined as poles of the meromorphic continuation of the cut-off resolvent  $\chi(-\Delta + V - \lambda^2)^{-1}\chi$  (see for instance [DZ19]). In fact, for the parallel with obstacle scattering to hold, it is necessary to study semiclassical scattering by a potential, that is of the form,  $P_0(h) = -h^2\Delta + V$ . If, in the case of obstacle scattering, the semiclassical parameter  $h$  is a convenient way to apply semiclassical analysis, in scattering by a potential, the uniform study of  $P_0(h)$  in the limit  $h \rightarrow 0$  has its own interest. The operator  $P_0(h)$  is a semiclassical pseudodifferential operator associated with the Hamiltonian  $p(x, \xi) = |\xi|^2 + V(x)$ . The study of  $P_0(h)$  requires to understand the classical Hamiltonian dynamics of this Hamiltonian. Contrary to the case of obstacle scattering, it is necessary to study the resonances around a fixed frequency  $E$  : for any  $C > 0$  fixed, the study of the resonances of  $P_0(h)$  in the ball  $D(E, Ch)$  is related with the Hamiltonian flow on the energy shell  $p^{-1}(E)$ . The analogues of Questions 2.1 and 2.2 in this context are now :

- Is there some  $\gamma > 0$  such that there is no resonance in  $D(E, Ch) \cap \{\text{Im } z \geq -\gamma\}$  for  $h$  small enough ?
- Can we find asymptotics/lower bounds/upper bounds for the number of resonances  $N_{\gamma,E,C}(h)$  in  $D(E, Ch) \cap \{\text{Im } z \geq -\gamma\}$  as  $h \rightarrow 0$ ?

As in the case of obstacle scattering, such questions are related with the trapped set  $K_E$  at energy  $E$  : if we note  $\varphi^t : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$  the Hamiltonian flow associated with  $p$ ,

$$K_E = \{\rho \in p^{-1}(E), \varphi^t(\rho) \not\rightarrow \infty \text{ when } t \rightarrow +\infty \text{ and } -\infty\}$$

When the trapped set is a hyperbolic set for the flow  $\varphi^t$ , the situation is very similar to the case of strictly convex obstacles since the classical flows shares the same properties. It is then possible to prove analogues results :

- When  $K_E = \emptyset$ , logarithmic resonance free regions are known ([SZ07], [DZ19], Theorem 6.21 and the references cited there).
- When  $K_E$  is reduced to a single hyperbolic periodic orbit, the same distribution on a pseudo-lattice occurs ([GS87]).
- For more general (and necessarily more complicated) trapped set, the spectral gap under the pressure condition  $P(1/2) < 0$  has been proved in [NZ09]. In [SZ07], the authors prove the fractal upper bound

$$N_{\gamma,C,E}(h) = O(h^{-d_H - \varepsilon})$$

In fact, under some additional, but reasonable, assumptions on the trapped set of the Hamiltonian flow - essentially, one requires that  $K_E$  is topologically 1 dimensional - the works of S. Nonnenmacher, J. Sjöstrand and M. Zworski in [NSZ11] and [NSZ14] show that it is possible to lead a uniform study of the two models (obstacle scattering and semiclassical scattering by a potential), by use of a reduction to *hyperbolic open quantum maps*. This is one of the main tools of this thesis. In [NSZ14], the authors prove the fractal upper bound of [SZ07] in a different (and in some sense easier) way. By use of this reduction, which is one of the main tool of this thesis, Theorems A and B are obtained also for semiclassical scattering by a potential.

It is possible to treat more general potential  $V$ , and in fact more general pseudodifferential operators  $P(x, hD_x)$  (see for instance the assumptions in [NSZ11] Section 2). It is also possible to consider different manifolds with different types of infinite ends. In [NSZ11], the authors consider manifolds  $(X, g)$  with a finite number of euclidean ends  $\mathbb{R}^d \setminus B(0, R)$ . The methods of Vasy allows to treat asymptotically hyperbolic ends ([Vas13]) and the gluing method developed by Datchev-Vasy allows to treat separately the interaction region (where lives the trapped set) and the infinite ends ([DV12]), under reasonable assumptions on the geometry and the resolvent in each region. In particular, since resolvent estimates are obtained in our work (see Chapter 6), this gluing method can lead to spectral gaps for different types of infinite ends too. Among asymptotically hyperbolic manifolds, convex co-compact hyperbolic surfaces have a very particular place and their spectral study share many properties with obstacle scattering. The next subsection aims at presenting the main aspects of this theory and at showing the analogy with obstacle scattering.

### 2-1.5.2. Convex co-compact hyperbolic surfaces.

In this section, we give a quick presentation of the spectral theory of convex co-compact hyperbolic surfaces. The author learned most of this theory in the book of D. Borthwich ([Bor16]).

**Geometry.** Convex co-compact hyperbolic surfaces are quotient of the Poincaré half-plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2, y > 0\}$$

(with usual metric  $y^{-2}(dx^2 + dy^2)$ ) by some specific subgroups  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$  (called themselves convex co-compact). They are indeed of constant curvature  $-1$  and have funnels ends. There indeed exist only two type of infinite hyperbolic ends :

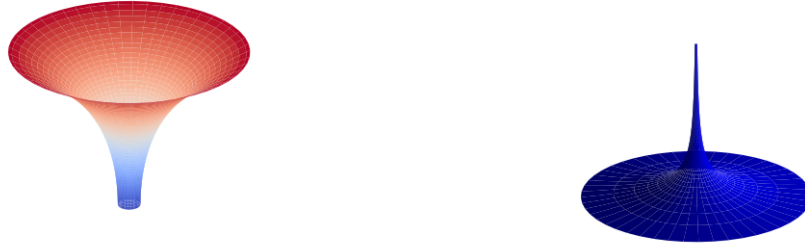
- The cusps, which are given by the model of the parabolic cylinder, that is the quotient of  $\mathbb{H}^2$  by the group  $\Gamma_\infty$  generated by the transformation  $z \mapsto z + 1$ . It will not interest us in this thesis.
- The funnels, which are half hyperbolic cylinders. A hyperbolic cylinder with central geodesics of length  $l$  is the quotient of  $\mathbb{H}^2$  by the group  $\Gamma_l$  generated by  $z \mapsto e^l z$ . It is possible to parametrize the hyperbolic cylinder with variables  $(r, \theta) \in \mathbb{R} \times \mathbb{S}^1$  with metric

$$dr^2 + \frac{l^2}{4\pi^2} \cosh^2 r d\theta^2$$

The curve  $\{r = 0\}$  is a geodesic of length  $l$ . In this system of coordinates, the funnel  $F_l$  is  $\{r > 0\} \times \mathbb{S}^1$ .

A hyperbolic surface  $M = \mathbb{H}^2/\Gamma$  (of infinite area) is said to be convex co-compact if it is geometrically finite with no cusps. It can be decomposed as

$$M = N \sqcup \bigsqcup_{i=1}^L F_i$$



(a) A funnel

(b) A cusp

Figure 2.11: The two types of hyperbolic end.

where each  $F_i$ ,  $1 \leq i \leq L$  is a funnel and  $N$  is a compact subset (called the compact core). Such surfaces are well adapted for scattering theory : each funnel furnishes an infinity, in which it is possible to define incoming and outgoing waves.

**Spectral theory.** For such surfaces  $(M, g)$ , we are interested in the spectral property of the Laplace operator  $-\Delta_g$ . On such surfaces, the spectrum of  $-\Delta_g$  satisfies the properties listed in the following theorem :

**Theorem 2-1.13** ([LP82], [LP84], see also [Bor16], Chapter 7). *Let  $(M, g)$  be a convex co-compact hyperbolic surface. The spectrum  $\sigma(-\Delta_g)$  consists of the essential spectrum, equal to  $[1/4, +\infty[$  and possibly a finite number of eigenvalues in  $]0, 1/4[$ . There is no embedded eigenvalues in  $[1/4, +\infty[$ .*

Since the essential spectrum is  $[1/4, +\infty[$ , it is natural to consider the resolvent

$$R(\lambda) := (-\Delta_g - 1/4 - \lambda^2)^{-1}, \text{Im } \lambda > 0, 1/4 + \lambda^2 \notin \sigma(-\Delta_g)$$

For  $\text{Im } \lambda > 0$ , it is a meromorphic family of operators with poles of finite rank, the poles being in bijection with the eigenvalues in  $]0, 1/4[$ . In fact, the following convention is preferred for hyperbolic surfaces : the spectral variable is  $s \in \{\text{Re } s > 1/2\}$ , related to  $\lambda$  by the relation  $s(1-s) = 1/4 + \lambda^2$ , and we study  $R_M(s) = (-\Delta_g - s(1-s))^{-1}$ . The main reason for this writing is the fact that the functions  $y \in \mathbb{H}^2 \mapsto y^s$  and  $y \in \mathbb{H}^2 \mapsto y^{1-s}$  are solutions of the equation

$$-\Delta_g u = s(1-s)u$$

Indeed,  $-\Delta_g = y^2(\partial_x^2 + \partial_y^2)$  for  $\mathbb{H}^2$ . In particular, the function  $y^s$  (resp.  $y^{1-s}$ ) plays a role similar to the function  $e^{i\lambda|x|}$  (resp.  $e^{-i\lambda|x|}$ ) as outgoing (resp. incoming) functions. As in the Euclidian case for obstacle scattering, it is possible to meromorphically extend the resolvent :

**Theorem 2-1.14** ([GZ95], see also [Bor16], Chapter 6). *The resolvent*

$$R_M(s) : L^2(M) \rightarrow H^2(M)$$

*well defined for  $\text{Re } s > 1/2$ ,  $s(1-s) \notin \sigma(-\Delta)$ , extends to all  $\mathbb{C}$  as a meromorphic family of operators with poles of finite rank*

$$R_M(s) : L^2_{\text{comp}}(M) \rightarrow H^2_{\text{loc}}(M)$$

*Its poles are called the resonances of the surface  $M$ .*

This could also be proved using the methods of [Vas13], which allow to treat a larger class of surfaces with asymptotically hyperbolic ends.

It is now possible to ask the same questions 2.1 and 2.2 in this context. Again, to be able to answer these questions, it is useful to understand the classical dynamics associated with this "quantum" object.

**Dynamics.** This dynamics is the geodesic flow on  $M = \mathbb{H}^2/\Gamma$ . Equivalently, this is the Hamiltonian flow associated with  $p(x, \xi) = |\xi|_g^2$ . Its properties are inherited from the geodesic flow on the Poincaré half-plane and the action of  $\Gamma$  on  $\mathbb{H}^2$ . In particular, the geodesic flow is hyperbolic since the curvature is negative. The trapped set  $K$  in the cosphere bundle  $S^*M$  is defined similarly as before in the other open systems : it is the set of points  $\rho \in S^*M$  such that the unique geodesic  $\gamma$  passing through  $\rho$  stays in a compact subset of  $S^*M$ . In fact, in this special model, the dynamics allows to obtain the resonances in a different way since the resonances can be described as the zeros of a certain *dynamical zeta function*. Selberg's zeta function is defined as

$$Z_M(s) = \prod_{\gamma \in \mathcal{P}} \prod_{l=0}^{\infty} \left(1 - e^{-(s+l)|\gamma|}\right)$$

where  $\mathcal{P}$  is the set of primitive closed geodesics and  $|\gamma|$  denotes the length of a closed geodesic  $\gamma$ . This product converges absolutely of  $\text{Re } s > \delta$  where  $\delta$  is called the *exponent of convergence* of  $\Gamma$ .  $\delta$  is in fact defined rigorously as

$$\delta = \inf \left\{ s \geq 0, \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma \cdot w)} \right\}, \quad z, w \in \mathbb{H}^2$$

where  $d$  denotes the hyperbolic distance in  $\mathbb{H}^2$  and the definition of  $\delta$  does not depend on  $z$  and  $w$ .

$Z_M$  can be continued meromorphically to all  $\mathbb{C}$  with trivial topological zeros at  $s = -n, n \in \mathbb{N}$  of multiplicity  $2n+1$  and its other zeros are given, with multiplicity, by the resonances of  $M$  ([PP01]). In obstacle scattering, it is possible to define such dynamical zeta functions. There exists relations between their zeros and the scattering resonances (see for instance [PS10]) but the link is not as clear as in the case of hyperbolic surfaces, except in the case of two obstacles ([Gér88]).

Due to the special algebraic structure of hyperbolic surfaces, it is possible to give a description of the trapped set in term on the limit set  $\Lambda_\Gamma \subset \partial\mathbb{H}^2$ . For this description, let us adopt the Poincaré disc model for the hyperbolic plane. In this model,  $\partial\mathbb{H}^2 = \mathbb{S}^1$  and  $\Lambda_\Gamma$  is defined as

$$\Lambda_\Gamma := \{z \in \partial\mathbb{H}^2, \exists y \in \mathbb{H}^2, \exists(\gamma_n) \in \Gamma^{\mathbb{N}}, \gamma_n \cdot y \rightarrow z\}$$

(we note  $\gamma \cdot z$  the action of an element  $\gamma \in \Gamma$  on an element  $z \in \mathbb{H}^2$ ).  $\Lambda_\Gamma$  is a fractal subset of  $\mathbb{S}^1$ . It is also possible to give a Cantor-like description of  $\Lambda_\Gamma$  using the Schottky surface model for convex co-compact surfaces (see [Bor16], Chapter 15). There is a remarkable result of Patterson ([Pat76]) and Sullivan ([Sul79]) which states :

$$\delta := \dim_H \Lambda_\Gamma \tag{2-1.6}$$

where  $\dim_H$  denotes the Hausdorff dimension of  $\Lambda_\Gamma$ .

To relate the trapped set  $K$  to the limit set, it is necessary to introduce the following maps  $B_\pm$  defined on  $T^*\mathbb{H}^2 \setminus 0$  (see Figure 2.12). For any  $\rho = (x, \xi) \in T^*\mathbb{H}^2 \setminus 0$ , there exists a unique geodesic  $\gamma : \mathbb{R} \rightarrow T^*\mathbb{H}^2$  passing through this point. The projected geodesic  $\pi_x(\gamma)$  is an arc of circle intersecting  $\mathbb{S}^1$  into two points :

$$B_\pm(\rho) = \lim_{t \rightarrow \pm\infty} \pi_x(\gamma(t))$$

Note that  $B_\pm$  is invariant on a geodesic. Finally, let  $\pi_\Gamma : T^*\mathbb{H}^2 \rightarrow T^*M$  be the natural projection. We then have, for any  $\rho = (x, \xi) \in S^*\mathbb{H}^2$ :

$$\pi_\Gamma(\rho) \in K \iff B_-(\rho) \in \Lambda_\Gamma \text{ and } B_+(\rho) \in \Lambda_\Gamma$$

In particular, this allows to understand why we have

$$\dim_H K = 2\delta + 1$$

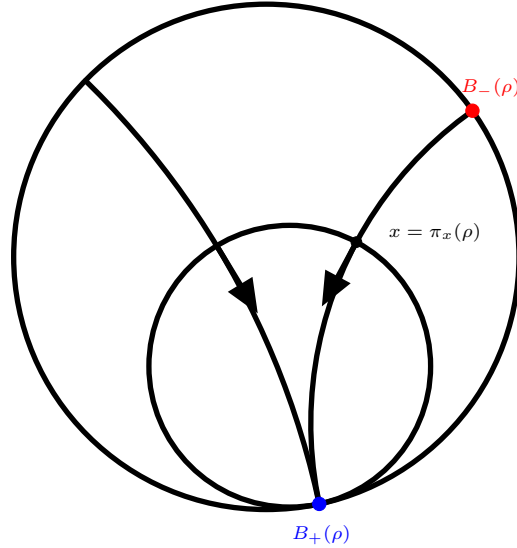


Figure 2.12: Description of the points  $B_{\pm}(\rho)$  on  $\mathbb{S}^1$ . We have also represented the horocycle passing through  $x$  and  $B_+$  and another geodesic joining  $B_+(\rho)$  at  $+\infty$ .

Hence,  $\delta$  plays the same role as  $d_H$  in the previous model of obstacle scattering and scattering by a potential. Moreover, the topological pressure associated to the geodesic flow on the trapped set is:

$$P(s) = \delta - s$$

**Spectral gap.** In this context, the spectral gap consists in showing that there is a finite number of resonances in  $\{\operatorname{Re} s \geq 1/2 - \gamma\}$  for some  $\gamma > 0$ . In fact, the first resonance is perfectly known due to the so-called Patterson-Sullivan theory (see [Bor16], Chapter 14):

**Theorem 2-1.15** ([Pat88]). *There is a simple resonance at  $\delta$  and no other resonance in  $\{\operatorname{Re} s \geq \delta\}$ .*

When  $\delta \geq 1/2$ , it is known that there is a finite number of resonance in  $\{\operatorname{Re} s \geq 1/2\}$  corresponding to the eigenvalues  $s(1-s)$  in  $[0, 1/4[$ . Hence, this result tells that  $\delta(1-\delta)$  is the first eigenvalue of  $-\Delta_g$ . When  $\delta < 1/2$ , this theorem gives a quantitative spectral gap of size  $\gamma = 1/2 - \delta = |P(1/2)|$ . This result is the analogue of the spectral gap of Theorem 2-1.11. When  $\delta < 1/2$ , Question 2.1 can be refined in : Can the gap of size  $1/2 - \delta$  be improved ? This question has been positively answered by Naud ([Nau05]) and then by Bourgain-Dyatlov ([BD17]) :

**Theorem 2-1.16.** *For every convex co-compact surface  $M = \mathbb{H}^2/\Gamma$  of exponent of convergence  $\delta$ , there exists  $\varepsilon > 0$  such that there is a finite number of resonances in  $\{\operatorname{Re} s > \delta - \varepsilon\}$ .*

Contrary to the work of Naud where  $\varepsilon = \varepsilon(M)$ , the method of [BD17] allows to show that one can take  $\varepsilon = \varepsilon(\delta)$ . It is part of a series of recent results, initiated by [DZ16], based on the use of a new tool, called a Fractal Uncertainty Principle. It will be a crucial tool of our analysis and we will present this tool in detail below in the second part of the introduction. For Question 2.1, the use of this tool lead to

**Theorem 2-1.17** ([BD18]). *For every convex co-compact surface  $M$ , there exists  $\varepsilon > 0$  such that there is a finite number of resonances in  $\{\operatorname{Re} s > 1/2 - \varepsilon\}$ .*

In particular, the pressure condition  $\delta < 1/2$  is no more a necessary assumption. This result has inspired our work to prove Theorem A, so as their method which uses the Fractal Uncertainty Principle.

**Fractal Weyl upper bound.** In this situation, the reformulation of Question 2.2 concerns :

$$N_\gamma(r) = \#\{s \text{ resonances, } \operatorname{Re} s \geq 1/2 - \gamma, |\operatorname{Im} s| \in [r, r + 1]\} \text{ counted with multiplicity}$$

Upper bounds of the form  $N_\gamma(r) \leq C_\gamma r^\delta$  appeared for convex co-compact surfaces in the work of Zworski [Zwo99]. It has been generalized for asymptotically hyperbolic manifolds in [DD13]. It has been conjectured (see [Zwo17], Conjecture 5) that this bound is optimal when the strip is sufficiently large. However, numerical experiments (see for instance the appendix of [DBW19] for the case of convex co-compact surfaces) show that it should be possible to improve this bound for small values of  $\gamma$ . A first result in that direction has been obtained in [Nau12] where the author showed a bound similar to the one in Theorem B (without the loss of  $\varepsilon$ ), with a function  $\sigma$  having the same properties. At the time of writing this manuscript, the most explicit bound, to the author knowledge, is given by the following theorem.

**Theorem 2-1.18** ([DBW19]). *For each  $\gamma \geq 0$  and  $\varepsilon > 0$ , there exists a constant  $C > 0$  such that for all  $r \geq 1$ ,*

$$N_\gamma(r) \leq C r^{\delta - \sigma(\gamma) + \varepsilon}$$

where  $\sigma(\gamma) = \max(0, \delta + 1 - 2\gamma)$

Note that the function  $\sigma$  satisfies the same properties as the one in Theorem B since  $\gamma_{cl} = 1 - \delta$  in this context. This result is the analogue of Theorem B for convex co-compact hyperbolic surfaces, but the function  $\sigma$  is simpler. Nevertheless, the methods used to prove Theorem B are completely different : in [DBW19], the strategy uses anisotropic semiclassical calculus associated with special Lagrangian submanifolds of  $T^*M$ , whose analytic description relies on the particular properties of the geodesic flow.

Lower bounds are still difficult to obtain and despite convincing numerical results ([Bor14], [BW16], [LSZ03], [DBW19]), an exact asymptotic for the counting function in strips is still an open problem. At the time of writing this manuscript, some lower bounds are known such as the one of [GZ99] : for any  $0 < \varepsilon < 1/2$  and  $0 > \sigma > -\frac{1}{2\varepsilon}$ , the number of resonances  $\tilde{N}_\sigma(r)$  (with multiplicity) in  $\{|s - 1/2| \leq r, |\operatorname{Im} s| \geq \sigma\}$  satisfies

$$\tilde{N}_\sigma(r) = \Omega(r^{1-\varepsilon})$$

that is  $\tilde{N}_\sigma(r)r^{-1+\varepsilon}$  takes arbitrarily large values as  $r \rightarrow +\infty$ . The proof of this result, like most lower bounds in this context, relies on the analysis of a trace formula.

Convex co-compact surfaces furnish an example of an open hyperbolic systems in which the study of resonances share many aspects with obstacle scattering (or scattering by a potential). The main interest of this model is that other methods of analysis are available, based on the particular properties of the geodesic flow and the algebraic structure of the subgroup  $\Gamma$ . However, some results are still not proven and sometimes seem very difficult. To test some conjectures which are still inaccessible in this model, it is often useful to study much easier toy model. We now present such a model.

### 2-1.5.3. Quantization of open baker's map.

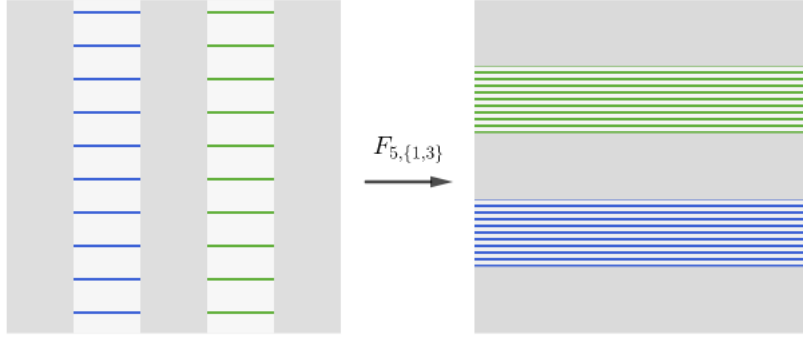
Baker's transformations are toy models for chaotic dynamics. They are said to be open when they are defined on a subset of  $\mathbb{T}^2$ . Open baker's maps are toy models to understand open chaotic dynamics, such as the dynamics of the billiard map with strictly convex obstacles.

To define an open baker's map, we fix an integer  $M \in \mathbb{N}^*$  and an alphabet  $\mathcal{A} \subset \{0, \dots, M-1\}$  and we identify  $\mathbb{T}^2$  with  $[0, 1]^2$ . We define the departure set

$$D_{M,\mathcal{A}} = \bigcup_{a \in \mathcal{A}} [aM^{-1}, (a+1)M^{-1}[ \times [0, 1[$$

The map  $F = F_{M,\mathcal{A}}$  is defined as

$$F_{M,\mathcal{A}}(x, \xi) = (Mx - a, M^{-1}(\xi + a)), \quad \frac{a}{M} \leq x < \frac{a+1}{M}, \quad a \in \mathcal{A}$$

Figure 2.13: Example of the baker's map  $F_{5,\{1,3\}}$ 

So that the map is effectively open, we assume that  $\mathcal{A} \neq \{0, \dots, M-1\}$ .

The arrival set is  $A_{M,\mathcal{A}} = F_{M,\mathcal{A}}(D_{M,\mathcal{A}})$  and we can define an inverse  $F_{M,\mathcal{A}}^{-1}$ , well defined on  $A_{M,\mathcal{A}}$ . The trapped set  $K$  is defined as the set of points  $\rho = (x, \xi) \in \mathbb{T}^2$  such that  $F^n(\rho)$  is well defined for all  $n \in \mathbb{Z}$ . It can easily be described using the symbolic dynamics associated with the map and has the form

$$K = \mathcal{C} \times \mathcal{C}$$

where

$$\mathcal{C} = \bigcap_{k \in \mathbb{N}} \bigcup_{a \in \mathcal{C}_k} \left[ \frac{a}{M^k}, \frac{a+1}{M^k} \right[ ; \mathcal{C}_k = \{a_0 + a_1 M + \dots + a_{k-1} M^{k-1}, (a_0, \dots, a_{k-1}) \in \mathcal{A}^k\}$$

is a Cantor set of Hausdorff dimension  $\delta = \frac{\log |\mathcal{A}|}{\log M}$ . It is possible to define a topological pressure for such a map

$$P(s) = \lim_{N \rightarrow +\infty} \frac{1}{N \log M} \log \left( \sum_{\gamma, |\gamma|=N} M^{-sN} \right)$$

where the sum runs over the periodic trajectories of period  $N$ . Since there is exactly  $|\mathcal{A}|^N$  such trajectories, we find that

$$P(s) = \frac{1}{\log M} (\log 2 - s \log M) = \delta - s$$

Quantization of such classical maps was initiated by [BV89], then followed by others ([SV96], [NZ05], [NZ07]). They are finite dimensional matrices of dimension  $N$ , which play the role of the inverse of the semiclassical parameter  $h$  (more precisely  $N = (2\pi h)^{-1}$ ). The goal is to study the behavior of these matrices in the limit  $N \rightarrow +\infty$ . Following [DJ17], to define quantized open baker's map, we introduce a cut-off function  $\chi \in C_c^\infty(]0, 1[)$ , which allows to avoid the singularities of the baker's map (see for instance [NZ07]) and we define for  $N \in M\mathbb{Z}$ ,

$$B_N = B_{N,M,\mathcal{A},\chi} = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/M} \mathcal{F}_{N/M} \chi_{N/M} & & \\ & \ddots & \\ & & \chi_{N/M} \mathcal{F}_{N/M} \chi_{N/M} \end{pmatrix} I_{M,\mathcal{A}} \quad (2-1.7)$$

where  $\mathcal{F}_N$  is the unitary discrete Fourier transform in dimension  $N$  and

$$\chi_{N/M} = \begin{pmatrix} \chi(0) & & & \\ & \chi\left(\frac{M}{N}\right) & & \\ & & \ddots & \\ & & & \chi\left(\left(\frac{N}{M}-1\right)\left(\frac{M}{N}\right)\right) \end{pmatrix} ; I_{M,\mathcal{A}} = \begin{pmatrix} \mathbb{1}_{0 \in \mathcal{A}} I_{N/M} & & \\ & \ddots & \\ & & \mathbb{1}_{M-1 \in \mathcal{A}} I_{N/M} \end{pmatrix}$$

Quantized open maps are built to be the propagators at time  $t = \log M$  of an open quantum system and eigenvalues of  $B_N$  are heuristically related to scattering resonances of such a system.

The heuristic formula reads as follows : if  $\lambda$  is a resonance,  $z = e^{-it\lambda}$  is an eigenvalue of  $B_N$ . In fact, the works [NSZ11] and [NSZ14] give a rigorous proof of such a correspondence between scattering resonances and eigenvalues of more general open hyperbolic quantum maps. This correspondence is a crucial ingredient in our works. With this analogy, the spectral gap problem (Question 2.1) and resonance counting (Question 2.2) have their counterpart :

- The map  $B_N$  satisfies  $\|B_N\|_{\ell^2 \rightarrow \ell^2} \leq 1$ , so that Question 2.1 becomes : can we find  $\gamma > 0$  and  $N_0 \geq 0$  such that for all  $N \geq N_0$ ,  $\sigma(B_N) \cap \{|z| \geq M^{-\gamma}\} = \emptyset$  ?
- Question 2.2 concerns counting eigenvalues in annulus. We want asymptotics/upper bound/lower bounds for  $N_\gamma(N)$  as  $N \rightarrow +\infty$  where  $N_\gamma(N)$  is the number of eigenvalues of  $B_N$ , counted with multiplicity, in  $\{|z| \geq M^{-\gamma}\}$ . Numerical results, as the one obtained in [NZ05], suggest asymptotics of the form

$$N_\gamma(N) \sim C_\gamma N^{\mu_\gamma}$$

**Spectral gap.** As in the previous model, the threshold  $\delta = 1/2$  is still important. Indeed, the method of [NZ09] applied to this context or the method of [DJ17] can show that if  $\delta < 1/2$ , there is a spectral gap of gap  $1/2 - \delta + \varepsilon$  for any  $\varepsilon > 0$ . When restricting to integers  $N$  of the form  $M^k$ , this spectral gap can in fact be improved and when  $\delta \geq 1/2$ , there is also a spectral gap :

**Theorem 2-1.19** ([DJ17]). *There exists  $\beta = \beta(M, \mathcal{A}) > \max(0, 1/2 - \delta)$  and  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$ ,*

$$\sigma(B_{M^k}) \cap \{|z| \geq M^{-\beta}\} = \emptyset$$

**Fractal Weyl upper bounds.** The asymptotic behavior of  $N_\gamma(N)$  has been numerically tested in different works (see [Non11], Section 6, [NR07], [NZ05]). Lower bounds are still not known but similar upper bounds as in Theorem 2-1.18 have been obtained in the case  $N = M^k$  in the limit  $k \rightarrow \infty$  :

**Theorem 2-1.20** ([DJ17]). *Let  $N = M^k$  for  $k \in \mathbb{N}$  and  $B_N$  be the matrix defined in 2-1.7. Let  $N_\gamma(k)$  be the number of eigenvalues of  $B_N$ , counted with multiplicity, in  $\{|z| \geq M^{-\gamma}\}$ . Then, for any  $\gamma > 0$  and  $\varepsilon > 0$ , there exists  $C > 0$  such that for all  $k \in \mathbb{N}$ ,*

$$N_\gamma(k) \leq CN^{\delta - \sigma(\gamma) + \varepsilon}$$

where  $\sigma(\gamma) = \max(0, 1 - \delta - 2\gamma)$ .

In fact, the strategy used in [DJ17] to prove this result is inspired by the one in [DBW19]. Note again that the function  $\sigma$  behaves like the one in Theorem B. In the recent work [Zhe22], the bound  $N_\gamma(\delta) = O(N^\delta)$  has been proved for all integer  $N \in M\mathbb{Z}$  (and without the loss of  $\varepsilon$ ).

This ends the first part of the introduction, dedicated to the presentation of the main results of this thesis and their context.

## 2-2 Main tools.

We now turn to the second part of the introduction. We present a few important tools that are used in this thesis. This list is - of course - non exhaustive and rather subjective.

### Remark.

It would be worth mentioning that this thesis uses the whole machinery of semiclassical analysis. We won't do it in this introduction since it would be too long. Moreover, mentioning semiclassical analysis as a tool is probably reductive since it can be described as a whole branch of the theory of linear partial differential equations. For these reasons, we won't give a presentation of the whole theory of semiclassical analysis here and we devote Chapter 4 to detail the main results of



semiclassical analysis used in this thesis.

The starting point of the theory is a formula for quantizing smooth symbols  $a \in \mathcal{S}(\mathbb{R}^{2d})$ , which gives a quantization procedure on  $L^2(\mathbb{R}^d)$ . For instance, the Weyl quantization reads as follows :

$$a^W(x, hD_x)u(x) = \text{Op}_h(a)u(x) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}(x-y)\cdot u(y)} dy d\xi$$

Such a formula allows to build a consistent semiclassical calculus on  $\mathbb{R}^d$  (composition,  $L^2$ -boundedness, Gårding inequalities, Egorov's theorem, etc.) and can be developed also on manifolds. Reference textbooks are now available in the literature such as [Zwo12], in which the author has learned most of this theory.

### 2-2.1. Complex scaling.

We start with the method of complex scaling. To be precise, this method is not directly used in this thesis, but rather indirectly through the works [NSZ11] and [NSZ14] where it is the crucial starting point to change the resonance problem into an eigenvalue problem, necessary to construct a Grushin problem. Complex scaling also turns out to be useful when dealing with cut-off resolvent estimates - see Chapter 6. We first recall how complex scaling works and allows to treat the scattering resonances as eigenvalues of non self-adjoint operators. Then, we wanted to focus on a few pseudo-spectral effects that can occur when dealing with non self-adjoint operators. In particular, this explains why lower bounds for resonance counting can be difficult to obtain.

#### 2-2.1.1. From resonances to eigenvalues.

The method of complex scaling in the study of scattering resonances originates in the works of Aguilar, Combes and Balslev [AC71], [BC71]. It has then been widely developed by many others. In the black-box setting of Sjöstrand-Zworski, allowing to treat a large variety of perturbations of the Laplace operator, it has been first used in [SZ91]. The method has become almost unavoidable in the study of scattering resonances (see for instance [Sjö97], [SZ95]). The main advantage of this method is that it changes the resonance problem - finding poles of a meromorphic continuation - to an eigenvalue problem of a non self-adjoint operator. It makes it both theoretically and numerically important. The author learned this machinery in the detailed presentation of [DZ19] (Section 4.5).

To make our presentation clear and concise, we present complex scaling in the case of obstacle scattering : we fix an open set  $\mathcal{O} \subset \mathbb{R}^d$  and consider  $-\Delta_\Omega$  on  $L^2(\Omega)$ ,  $\Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$ , with domain  $H_0^1(\Omega) \cap H^2(\Omega)$ . We fix  $R > 0$  such that  $\overline{\mathcal{O}} \subset B(0, R)$ . For a parameter  $\theta \in ]0, \pi/2[$ , we consider a totally real complex deformation  $\Gamma_\theta \subset \mathbb{C}^d$  of  $\mathbb{R}^d$  -  $\Gamma_\theta$  being totally real means that  $T_z \Gamma_\theta \cap i\Gamma_\theta = \emptyset$  for all  $z \in \Gamma_\theta$  - such that for some  $R' > R$ ,

$$\begin{aligned} \Gamma_\theta \cap B_{\mathbb{C}^d}(0, R) &= \mathbb{R}^d \cap B_{\mathbb{C}^d}(0, R) \\ \Gamma_\theta \cap \mathbb{C}^d \setminus B_{\mathbb{C}^d}(0, R') &= e^{i\theta} \mathbb{R}^d \cap \mathbb{C}^d \setminus B_{\mathbb{C}^d}(0, R') \end{aligned}$$

As explained in [DZ19], it is possible to construction such a deformation : consider  $g \in C^\infty(\mathbb{R}_+, \mathbb{R})$  such that

$$g(t) = 0 \text{ for } t \leq R, \quad g(t) = \frac{1}{2}t^2 \text{ for } t \geq 2R, \quad g''(t) \geq 0$$

and for  $0 < \theta < \pi/2$ , set  $F_\theta(x) = \tan(\theta)g(|x|)$ ,  $f_\theta(x) = x + i\partial_x F_\theta(x)$  and  $\Gamma_\theta = f_\theta(\mathbb{R}^d)$ .

To define the complex-scaled Laplace operator, we introduce the differential operators on  $\mathbb{C}$ ,  $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$  and  $\Delta_z = \sum_{j=1}^d \partial_{z_j}^2$ . Then, for  $u \in C^\infty(\Gamma_\theta)$ , there exists an almost-analytic extension<sup>1</sup>  $\tilde{u} \in C^\infty(\mathbb{C}^d)$  and it can be shown that the value of  $\Delta_z \tilde{u}$  on  $\Gamma_\theta$  does not depend on the choice of the almost analytic extension. This allows to define a free complex-scaled Laplace operator  $\Delta_{\theta,0}$  on  $L^2(\Gamma_\theta)$  with domain  $H^2(\Gamma_\theta) : \Delta_{\theta,0}u = (-\Delta_z \tilde{u})|_{\Gamma_\theta}$ . In fact, using the identification  $f_\theta : \mathbb{R}^d \rightarrow \Gamma_\theta$ , it can be shown that  $-\Delta_{\theta,0}$  is equivalent with

$$\left( (\text{Id} + i\partial_x^2 F_\theta(x))^{-1} \partial_x \right) \cdot \left( (\text{Id} + i\partial_x^2 F_\theta(x))^{-1} \partial_x \right) \text{ on } L^2(\mathbb{R}^d)$$

<sup>1</sup>that is for  $1 \leq j \leq d$ ,  $\bar{\partial}_{z_j} \tilde{u}(z) = O(d(z, \Gamma_\theta)^{-\infty})$

It is now possible to define a complex-scaled version of  $-\Delta_\Omega$  with domain  $H_0^1(\Omega) \cap H^2(\Omega)$  : fix  $\chi \in C_c^\infty(B(0, R))$  such that  $\chi = 1$  near  $\bar{\mathcal{O}}$  ; the function  $1 - \chi$  is equal to one outside  $B(0, R)$  so that it can be extended to a smooth function on  $\Gamma_\theta$  ; since  $\Gamma_\theta \cap B_{\mathbb{C}^d}(0, R) = \mathbb{R}^d \cap B_{\mathbb{C}^d}(0, R)$ , the set  $\Gamma_\theta \setminus \bar{\mathcal{O}}$  can be identified with  $\Omega$  near  $\partial\mathcal{O}$  and we define the domain  $\mathcal{D}_\theta = H^2(\Gamma_\theta \setminus \bar{\mathcal{O}}) \cap H_0^1(\Gamma_\theta \setminus \bar{\mathcal{O}})$ . For  $u \in \mathcal{D}_\theta$ , we define

$$-\Delta_\theta u = -\Delta_\Omega(\chi u) - \Delta_{\theta,0}((1 - \chi)u)$$

The relation between  $-\Delta_\theta$  and  $-\Delta_\Omega$  is given by the following theorem :

**Theorem 2-2.1** ([DZ19] Section 4.5, [SZ91]). *Assume that  $0 \leq \theta < \pi/2$ . Let  $\lambda \in \mathbb{C}^*$  be such that  $\text{Im}(e^{i\theta}\lambda) > 0$ , then  $-\Delta_\theta - \lambda^2$  is a Freedom operator of index 0 and  $\lambda^2$  is an eigenvalue of  $-\Delta_\theta$  if and only if it is a resonance of  $-\Delta_\Omega$ . Moreover, the multiplicities coincide. Finally, if  $\psi \in C_c^\infty(B(0, R))$ ,  $\psi = 1$  near  $\bar{\mathcal{O}}$  and if  $\lambda$  is not a resonance,*

$$\psi(-\Delta_\theta - \lambda^2)^{-1}\psi = \psi(-\Delta_\Omega - \lambda^2)^{-1}\psi$$

### 2-2.1.2. Excursion : striking phenomena for non self-adjoint operators.

This subsection is more or less disconnected with the rest of the manuscript. Its goal is *not* to give a complete presentation of the theory of pseudospectrum (for this purpose, see for instance [TE05]). We rather aim at presenting a few striking phenomena that the author has been lead to encounter during his PhD for different reason, and which had aroused his curiosity. The last 2 examples have been studied by the author for the preparation of a talk given in June 2022 for a Workshop of the ANR Adyct.

**The example of the Jordan matrix.** A very standard example to present pseudospectral effects are perturbations of the Jordan matrix  $J_N \in \mathcal{M}_N(\mathbb{C})$  :

$$J_N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

The spectrum of  $J_N$  is reduced to  $\{0\}$  but when  $z \in D(0, 1)$  and  $N$  is large, its resolvent  $(J_N - z)^{-1}$  has large  $\ell^2 \rightarrow \ell^2$  norm compared to  $z$ . Indeed, if  $u = (1, z, \dots, z^{N-1})^t$  then

$$(z - J_N)u = (0, \dots, 0, z^N)^t$$

so that

$$\|(z - J_N)^{-1}\|_{\ell^2 \rightarrow \ell^2} \geq \frac{1}{|z|^N} \|u\|_{\ell^2} \geq |z|^{-N}$$

This kind of behavior leads to instability of the spectrum. A first deterministic manifestation is given by the following :

*Example.* Let  $0 < \varepsilon < 1$  be a small fixed parameter and  $R_N = (\delta_{iN}\delta_{j1})_{1 \leq i, j \leq N}$ . It is not difficult to see that the spectrum of  $J_N + \varepsilon R_N$  is given by

$$\{\varepsilon^{1/N} e^{\frac{2ik\pi}{N}}, 0 \leq k \leq N-1\}$$

which are  $N$  points distributed on the circle of radius  $\varepsilon^{1/N}$ . In the limit  $N \rightarrow +\infty$ , one could say that the spectrum of  $J_N + \varepsilon R_N$  converges to  $\mathbb{S}^1$ .

*Example.* It is also relevant to study random perturbations of  $J_N$  as in [DH09]. For instance, let us mention the following corollary of the paper [GWZ14]. For  $A \in \mathcal{M}_N(\mathbb{C})$ , let  $L^A$  be the measure  $N^{-1} \sum_{\mu \in \sigma(A)} \delta_\mu$  where the eigenvalues are repeated with multiplicity. Let  $(G_{ij})_{i, j \in \mathbb{N}}$  be a sequence of i.i.d Gaussian variables of mean 0 and variance 1 and let's note  $G_N = (G_{ij})_{1 \leq i, j \leq N}$ . Let  $(\delta_N) \in \mathbb{R}^{\mathbb{N}}$  be a sequence such that for some  $\kappa', \kappa > 0$ ,

$$N^{-\kappa'} \leq \delta_N \leq N^{-1/2-\kappa}$$

Then,  $L^{J_N + \delta_N G_N}$  converges weakly in probability to the uniform measure of  $\mathbb{S}^1$  (see Figure 2.14).

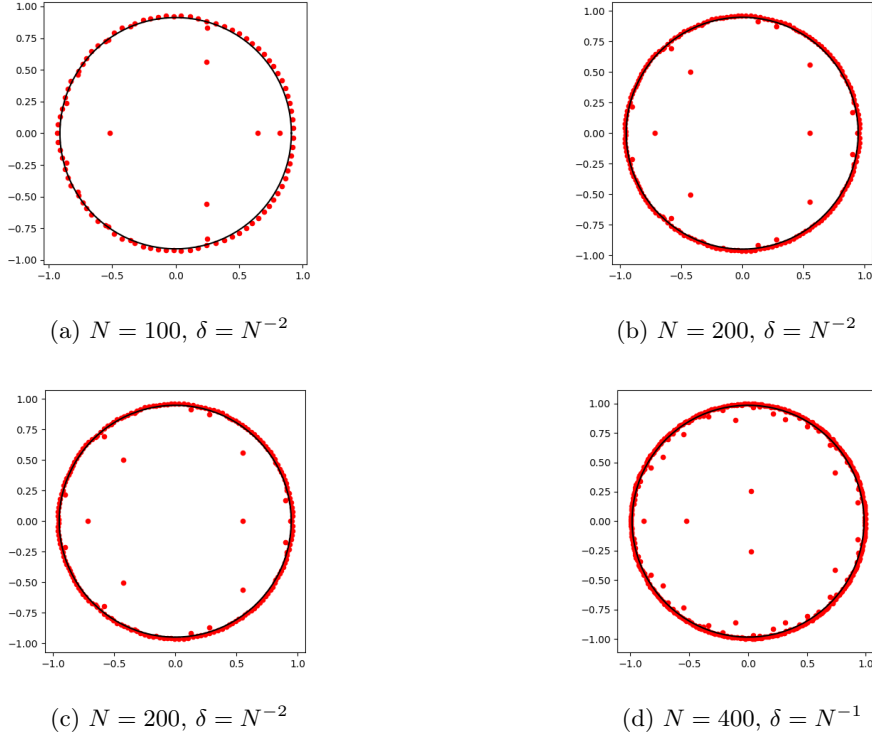


Figure 2.14: Plot of the spectrum of  $J_N + \delta G_N$  for different values of  $\delta$  and different values of  $N$ , where  $G_N$  is a Gaussian random perturbation. The black circle is the circle of radius  $\delta^{1/N}$ .

**A spectral pathology for a differential operator.** The following example is inspired by [See86]. Let's consider the following function :

$$f_\alpha(x) = e^{ix} + \alpha e^{2ix}, \quad x \in \mathbb{R}, \quad |\alpha| < 1$$

and the following differential operator on  $L^2(\mathbb{S}^1)$ :

$$d(\alpha) = f_\alpha(x) \partial_x$$

A simple computation shows that a primitive of  $1/f_\alpha : \mathbb{R} \rightarrow \mathbb{C}$  is given by

$$g_\alpha(x) = e^{-ix} - \alpha x - i\alpha \log(1 + \alpha e^{ix})$$

To compute the spectrum of  $d(\alpha)$ , it is enough to solve the eigenvalue problem

$$\begin{cases} d(\alpha)u = \lambda u \\ u(0) = u(2\pi) \end{cases}$$

A solution of this problem has the form  $u(x) = u_0 \exp(\lambda g_\alpha(x))$ , which gives a solution if and only if  $\lambda(g_\alpha(2\pi) - g_\alpha(0)) \in 2i\pi\mathbb{Z}$ . But,  $g_\alpha(2\pi) - g_\alpha(0) = -2\pi\alpha$ . As a consequence:

$$\begin{cases} \alpha = 0 \implies \text{Sp}(d(\alpha)) = \mathbb{C} \\ \alpha \neq 0 \implies \text{Sp}(d(\alpha)) = \frac{1}{\alpha}\mathbb{Z} \end{cases}$$

As a consequence, despite the fact that  $d(\alpha)$  depends holomorphically on  $\alpha$  (it's a very simple perturbation), the behavior of  $d(\alpha)$  drastically changes when  $\alpha = 0$ , compared with  $\alpha \neq 0$ .

**Magic angles in twisted bilayer graphene.** The last example we wanted to present appears in the recent work [BEWZ22], based on the analysis of the physical model of [TKV19] for twisted bilayer graphene. The study of this model leads to explore the spectrum of a certain holomorphic family of differential operators  $D(\alpha)$  on  $L^2(\mathbb{C}/\Gamma, \mathbb{C}^2)$ , where  $\Gamma = 4\pi(i\omega\mathbb{Z} \oplus i\omega^2\mathbb{Z})$  is related to the

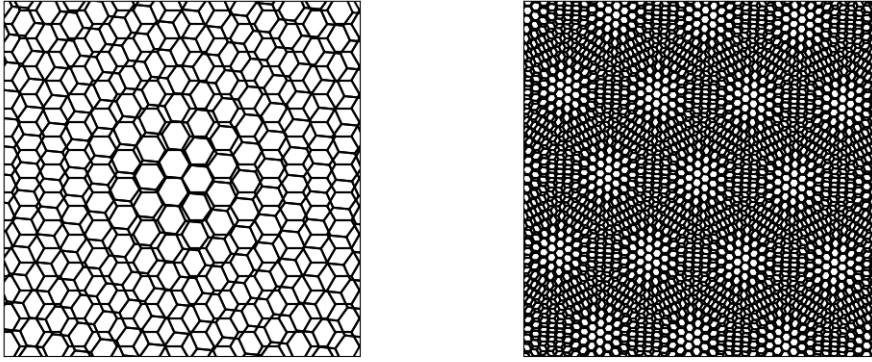


Figure 2.15: Two layers of graphene with one twisted by a small angle lead to the so-called Moiré pattern in twisted bilayer graphene.

honeycomb lattice which appears in the study of graphene and  $\alpha \in \mathbb{C}$  is a parameter related to the angle of twisting between the two sheets of graphene (see Figure 2.15).

$D(\alpha)$  has domain  $H^1(\mathbb{C}/\Gamma, \mathbb{C}^2)$  and is given by

$$D(\alpha) = \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix}$$

where  $D_{\bar{z}} = \frac{1}{i}\partial_{\bar{z}} = \frac{1}{2i}(\partial_{x_1} + i\partial_{x_2})$  and  $U(z)$  is related to the interactions between the two lattices and is given exactly by

$$U(z) = \sum_{k=0}^2 e^{\frac{1}{2}(z\omega^{-k} - \bar{z}\omega^k)} \omega^k, \quad \omega = e^{2i\pi/3}$$

In [BEWZ22], the authors prove, inter alia, that there exists a non-empty discrete set  $\mathcal{A}$  such that the spectrum  $\sigma(D(\alpha))$  of  $D(\alpha)$  satisfies :

$$\sigma(D(\alpha)) = \begin{cases} \Gamma^* & \text{if } \alpha \notin \mathcal{A} \\ \mathbb{C} & \text{if } \alpha \in \mathcal{A} \end{cases}$$

Here  $\Gamma^*$  is the dual lattice of  $\Gamma$ , explicitly given by  $\Gamma^* = \frac{1}{\sqrt{3}}(\omega\mathbb{Z} \oplus \omega^2\mathbb{Z})$ . This example shows another kind of pseudo-spectral effect : what is striking here is that it appears in a physically relevant situation. The set  $\mathcal{A}$  is called the set of *magic angles* and has striking implications on the physical model.

## 2-2.2. Fractal Uncertainty Principle.

The Fractal Uncertainty Principle (FUP) is a recent tool of harmonic analysis which has been successfully applied in Quantum Chaos. It first appeared in [DZ16], and has further been developed, among others, by S. Dyatlov in collaboration with J. Bourgain ([BD17],[BD18]). Essentially, it says that no function  $f : \mathbb{R} \rightarrow \mathbb{R}$  can be localized both in position and frequency near a fractal set. It is the main tool in the proof of the spectral gap (Theorem A, see Chapter 5). It has applications not only in obtaining spectral gaps but also in other problems of quantum chaos.

### 2-2.2.1. Statement.

In quantum mechanics, the uncertainty principle states that no state can be perfectly localized both in position and momentum. It is usually quantified with the famous Heisenberg's formula :

$$\Delta x \Delta p \geq \hbar/2$$

This inequality roughly says that a quantum state occupies at least a box of volume  $h$ . Here, we will use a different way to write an uncertainty principle : if  $\mathcal{F}_h : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is the (unitary)  $h$ -Fourier transform :

$$\mathcal{F}_h u(\xi) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi/h} u(x)$$

then one can check that

$$\left\| \mathbb{1}_{[0,h]} \mathcal{F}_h \mathbb{1}_{[0,h]} \right\|_{L^2 \rightarrow L^2} \leq h^{1/2}.$$

This has to be read as : " if a state  $u$  of norm 1 is localized in position in  $[0, h]$ , then  $\mathcal{F}_h u$  has little mass - in fact less than  $h^{1/2}$  - in  $[0, h]$ ". This is another way to quantify the statement : "no state can be fully localized both in position and momentum near a point". Note that it is possible to slightly generalize this simple uncertainty principle by playing with the saturation of the quantum box of volume of order  $h$  : if  $\gamma_1, \gamma_2 > 0$  satisfy  $\gamma_1 + \gamma_2 > 1$ , then

$$\left\| \mathbb{1}_{[0,h^{\gamma_1}]} \mathcal{F}_h \mathbb{1}_{[0,h^{\gamma_2}]} \right\|_{L^2 \rightarrow L^2} \leq h^\beta ; \beta = \frac{1}{2}(\gamma_1 + \gamma_2 - 1) > 0. \quad (2-2.1)$$

Indeed,

$$\begin{aligned} \left\| \mathbb{1}_{[0,h^{\gamma_1}]} \mathcal{F}_h \mathbb{1}_{[0,h^{\gamma_2}]} \right\|_{L^2 \rightarrow L^2} &\leq \left\| \mathbb{1}_{[0,h^{\gamma_1}]} \right\|_{L^\infty \rightarrow L^2} \times \left\| \mathcal{F}_h \right\|_{L^1 \rightarrow L^\infty} \times \left\| \mathbb{1}_{[0,h^{\gamma_2}]} \right\|_{L^2 \rightarrow L^1} \\ &\leq h^{\gamma_1/2} \times (2\pi h)^{-1/2} \times h^{\gamma_2/2} \\ &\leq h^\beta. \end{aligned}$$

The Fractal Uncertainty Principle aims at quantifying the most general statement : "no state can be fully localized both in position and momentum near a fractal set". In other words, we want bounds on the form

$$\left\| \mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)} \right\|_{L^2 \rightarrow L^2} = O(h^\beta) \quad (2-2.2)$$

for fractal sets  $X(h), Y(h)$ . At this point, we have been vague on the word "fractal". It is necessary to present some notions of fractal sets that can be used.

**Fractal sets.** Let's start with a standard definition of fractal dimension. Let us recall the definition of the upper box dimension of a compact metric space  $(X, d)$ . We denote by  $N_X(\varepsilon)$  the minimal number of open balls of radius  $\varepsilon$  needed to cover  $X$ . Then, the upper box dimension of  $X$  is defined by :

$$\overline{\dim} X := \limsup_{\varepsilon \rightarrow 0} \frac{\log N_X(\varepsilon)}{-\log \varepsilon} \quad (2-2.3)$$

In particular, if  $\delta > \overline{\dim} X$ , there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \leq \varepsilon_0$ ,  $N_X(\varepsilon) \leq \varepsilon^{-\delta}$ . What is in fact important in the fractal sets we want to focus on, is the fact that there are holes at every scale (or at least down to a very small scale). This is possible to define such a notion of porosity (see Figure 2.16) :

**Definition 2-2.1.** Let  $\nu \in (0, 1)$  and  $0 \leq \alpha_0 \leq \alpha_1$ . We say that a subset  $\Omega \subset \mathbb{R}$  is  $\nu$ -porous on scales  $\alpha_0$  to  $\alpha_1$  if for every interval  $I \subset \mathbb{R}$  of size  $|I| \in [\alpha_0, \alpha_1]$ , there exists a subinterval  $J \subset I$  of size  $|J| = \nu|I|$  such that  $J \cap \Omega = \emptyset$ .

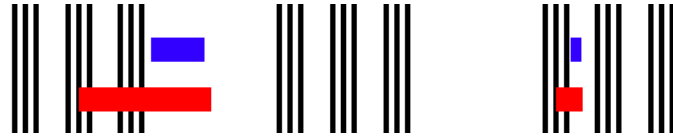
The notion of porosity can be related with the upper-box dimension. Indeed, we have the following result, proved in Chapter 5, Lemma 5-5.2 :

**Lemma 2-2.1.** Let  $\Omega \subset \mathbb{R}$ . Suppose that there exist  $0 < \delta < 1$ ,  $C > 0$  and  $\varepsilon_0 > 0$  such that

$$\forall \varepsilon \leq \varepsilon_0, N_\Omega(\varepsilon) \leq C\varepsilon^{-\delta}$$

Then, there exists  $\nu = \nu(\delta, \varepsilon_0, C)$  such that  $\Omega$  is  $\nu$ -porous on scale 0 to 1.

It is also possible to use the notion of  $\delta$ -regularity :



(a) A Cantor set is porous. Red intervals correspond to choices of  $I$ , blue ones correspond to  $J$ .



(b) A porous set randomly constructed.

Figure 2.16: Examples of porous sets.

**Definition 2-2.2.** Let  $X \subset \mathbb{R}$  be a non-empty closed set,  $0 \leq \delta \leq 1$ ,  $C_R \geq 1$  and  $0 \leq \alpha_{min} \leq \alpha_{max} \leq +\infty$ . We say that  $X$  is  $\delta$ -regular with constant  $C_R$  on scales  $\alpha_{min}$  to  $\alpha_{max}$  if there exists a locally finite measure  $\mu_X$  supported on  $X$  such that for every interval  $I$  centered at a point in  $X$  and  $\alpha_{min} \leq |I| \leq \alpha_{max}$ , we have

$$C_R^{-1}|I|^\delta \leq \mu_X(I) \leq C_R|I|^\delta$$

In fact, this notion is essentially equivalent with the notion of porosity. Let us mention a result in that sense which says that porous sets can be included in  $\delta$ -regular set :

**Lemma 2-2.2** ([DJ18b], Lemma 5.4). Assume that  $X$  is  $\nu$ -porous on scale  $\alpha_{min}$  to  $\alpha_{max}$ . Then, there exists  $Y \subset \mathbb{R}$  which is  $\delta$ -regular with constant  $C_R = C_R(\nu)$  on scale 0 to  $\alpha_{max}$  and such that  $X \subset Y(\alpha_{min})$ .

**Statement of the FUP.** As stated in [BD18], the Fractal Uncertainty Principle deals with the standard Fourier transform  $f \mapsto \hat{f}$  and the scale  $h$  appears through a large constant  $N \sim h^{-1}$  :

**Theorem 2-2.2** ([BD18], Theorem 4). Let  $0 \leq \delta < 1$ ,  $C_R \geq 1$  and  $N \geq 1$ . Assume that

- $X \subset [-1, 1]$  is  $\delta$ -regular with constant  $C_R$  on scales  $N^{-1}$  to 1 ;
- $Y \subset [-N, N]$  is  $\delta$ -regular with constant  $C_R$  on scales 1 to  $N$  ;

Then, there exists  $\beta = \beta(C_R, \delta)$  and  $C = C(C_R, \delta) > 0$  such that for all  $f \in L^2(\mathbb{R})$ ,

$$\text{supp } \hat{f} \subset Y \implies \|f\|_{L^2(X)} \leq CN^{-\beta} \|f\|_{L^2(\mathbb{R})}$$

**Remark.**

In [JZ20], the authors give an explicit dependence of  $\beta$  on  $C_R$  and  $\delta$ . They prove that in the previous theorem, one can take :

$$\beta = \exp \left[ -\exp \left( K(C_R \delta^{-1} (1 - \delta)^{-1})^{K(1-\delta)^{-2}} \right) \right]$$

where  $K$  is a universal constant.

It is possible to state a FUP under the form 2-2.2. Indeed, it is no difficult to see that the previous theorem implies :

**Lemma 2-2.3.** Let  $0 \leq \delta < 1$  and  $C_R \geq 1$ . There exists  $\beta, C > 0$  (depending only on  $\delta, C_R$ ) such that for all  $0 < h \leq 1$ , the uncertainty principle

$$\| \mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)} \|_{L^2 \rightarrow L^2} \leq h^\beta$$

holds for all subsets  $X(h), Y(h) \subset [-1, 1]$  which are  $\delta$ -regular with constant  $C_R$  on scales  $h$  to 1.

**Remark.**

- It appeared under this form in [DZ16] in the context of convex co-compact hyperbolic surfaces, for specific  $\delta$ -regular sets related to the limit set of the surface, with  $\mathcal{F}_h$  replaced by a specific Fourier integral operator.
- It is possible to show that if  $X \subset [-1, 1]$  is  $\delta$ -regular with constant  $C_R$  on scale  $h$  to 1, then the Lebesgue measure of  $X$  satisfies  $\text{Vol}(X) \leq 24C_R h^{1-\delta}$  (see [BD18], Lemma 2.9). Hence, arguing as in the proof of 2-2.1, we see that

$$\|\mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)}\|_{L^2 \rightarrow L^2} \leq C h^{1/2-\delta}$$

holds for all subsets  $X(h), Y(h) \subset [-1, 1]$  which are  $\delta$ -regular with constant  $C_R$  on scales  $h$  to 1, with a constant  $C$  only depending on  $C_R$ .

- The proof of [BD18] uses tools of harmonic analysis. In [DJ18a], the authors use a different strategy to prove a FUP, based on Dolgopyat's method, but their result is different since they prove a FUP with exponent  $\beta = 1/2 - \delta + \varepsilon$ , which improves the exponent  $1/2 - \delta$  when  $\delta \leq 1/2$ .

Let us mention a slightly different version, concerning a FUP under a porosity assumption on different scales. This version is stated in [DJN21] and is used in this thesis :

**Theorem 2-2.3.** ([DJN21], Proposition 2.11) Fix  $\gamma_j^\pm \in \mathbb{R}$ ,  $j = 0, 1$  such that

$$0 \leq \gamma_1^\pm \leq \gamma_0^\pm \leq 1; \gamma_1^+ + \gamma_1^- < 1 < \gamma_0^+ + \gamma_0^-$$

and define

$$\gamma = \min(\gamma_0^+, 1 - \gamma_1^-) - \max(\gamma_1^+, 1 - \gamma_0^-)$$

Then, for each  $\nu > 0$ , there exists  $\beta = \beta(\nu)$  and  $C = C(\nu)$  such that the estimate

$$\|\mathbb{1}_{\Omega_-(h)} \mathcal{F}_h \mathbb{1}_{\Omega_+(h)}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C h^{\gamma\beta}$$

holds for all  $0 < h \leq 1$  and all sets  $\Omega_\pm(h) \subset \mathbb{R}$  which are  $\nu$ -porous on scale  $h^{\gamma_0^\pm}$  to  $h^{\gamma_1^\pm}$ .

**A discrete version.** There is also a discrete version of the Fractal Uncertainty Principle. Such a version has been first introduced in [DJ17] for the study of quantized open baker's map and has led to a spectral gap. It concerns a more restrictive class of fractal sets since it has been stated for Cantor sets. To present it, we reintroduce the same material as before for quantized baker's map, namely, an integer  $M \geq 3$  and an alphabet  $\mathcal{A} \subset \{0, \dots, M-1\}$ . For  $k \in \mathbb{N}^*$ , we note

$$\mathcal{C}_k = \{a_0 + a_1 M + \dots + a_{k-1} M^{k-1}, (a_0, \dots, a_{k-1}) \in \mathcal{A}^k\} \subset \{0, \dots, M^k - 1\}$$

which is a discrete Cantor set. It is related with the Cantor set

$$\mathcal{C}_\infty = \bigcap_{k=0}^{\infty} \bigcup_{a \in \mathcal{C}_k} \left[ \frac{a}{M^k}, \frac{a+1}{M^k} \right]$$

which has Hausdorff dimension

$$\delta = \frac{\log |\mathcal{A}|}{\log M}$$

Note that  $\mathcal{C}_k$  contains exactly  $|\mathcal{A}|^k = M^{\delta k}$  elements. To state the Fractal Uncertainty Principle of [DJ17], we note  $\mathcal{F}_N : \ell^2(\mathbb{C}^N) \rightarrow \ell^2(\mathbb{C}^N)$  the unitary discrete Fourier transform :

**Theorem 2-2.4.** Let  $M \geq 3$  and  $\mathcal{A}$  be an alphabet with  $\delta \in ]0, 1[$ . Then, there exists  $\beta > \max(0, 1/2 - \delta)$  and  $C > 0$  depending only on  $M, \mathcal{A}$  such that for all  $k \in \mathbb{N}^*$ ,

$$\|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_{M^k} \mathbb{1}_{\mathcal{C}_k}\|_{\ell^2 \rightarrow \ell^2} \leq C M^{-k\beta}$$

It is said that  $\mathcal{C}_k$  satisfies a discrete Fractal Uncertainty Principle with exponent  $\beta$ .

**Remark.**

Let's note  $r_k = \|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_{M^k} \mathbb{1}_{\mathcal{C}_k}\|_{\ell^2 \rightarrow \ell^2}$ .

- To see a parallel with the continuous version of the FUP, it is necessary to interpret  $M^k$  as  $(2\pi h)^{-1}$ .
- The bound  $r_k \leq 1$  is trivial since  $\mathcal{F}_{M^k}$  is unitary.
- The bound  $r_k \leq M^{-k(1/2-\delta)}$  is not difficult to obtain and is a volume bound. Indeed,  $r_k$  is smaller than the Hilbert-Schmidt norm of the matrix  $M_k = \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_{M^k} \mathbb{1}_{\mathcal{C}_k}$ , which is nothing but  $\sqrt{\sum_{i,j} |(M_k)_{ij}|^2}$ . Since  $M_k$  has exactly  $|\mathcal{C}_k|^2 = M^{2k\delta}$  non-zero coefficients and since each non zero coefficient of  $M_k$ , which is a coefficient of  $\mathcal{F}_{M^k}$ , has modulus  $M^{-k/2}$ , we find  $r_k \leq M^{k(\delta-1/2)}$ .
- The proof of such a version is much easier than the one of the general FUP (see for instance [Dya19]) since it uses the sub-multiplicativity of the sequence  $(r_k)_k$ . Hence, since  $\lim_{k \rightarrow +\infty} \frac{1}{k} \log r_k = \inf_{k \in \mathbb{N}^*} \frac{1}{k} \log r_k$ , it is enough to prove that there exists  $k \in \mathbb{N}^*$  such that

$$\frac{1}{k} \log r_k < -\max(0, 1/2 - \delta) \log M$$

**Generalization to higher dimensions.** The currently known FUP only holds in dimension 1 and allows to obtain spectral gaps for 2D objects (hyperbolic surfaces, obstacle scattering and scattering by a potential in 2D). A generalization to higher dimensions requires care since (2-2.2) cannot hold for any pair of  $\delta$ -regular sets, as explained in [Dya19] (Example 6.1). Some additional assumptions seem to be required to state uncertainty principles for fractal (e.g. porous or  $\delta$ -regular) sets. For instance, in [HS20], the authors prove a version of Theorem 2-2.2 in higher dimensions  $d$ , with  $X$  a general  $\delta$ -regular set ( $\delta \in ]0, d[$ ) but with a particular choice of  $Y$ , which is essentially a product of 1-dimensional  $\delta_1$ -regular sets with  $\delta_1 \in ]0, 1[$  (see Figure 2.17). Concerning the discrete FUP, a recent progress has been made in [Coh22], where the author gives a necessary and sufficient condition on Cantor-like sets in 2D so that a FUP hold.

### 2-2.2.2. Applications.

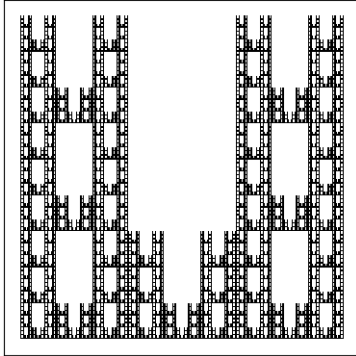
The Fractal Uncertainty Principle is an elegant result in itself but it is also a powerful tool in quantum chaos. We present different applications of this tool which have been obtained since it has been introduced in [DZ16].

**Spectral gaps.** As already said, the first use of a FUP was in [DZ16], where the authors developed it to obtain a spectral gap for hyperbolic surfaces. They obtained an improved gap - compared with the previous known results - for  $\delta$  close to the critical value of  $\delta = 1/2$ . It has been further developed by S. Dyatlov and J. Bourgain in [BD17] and [BD18] in this same context. To reduce the problem of the spectral gap to a FUP, the authors give a precise description of the evolution of the propagator of  $-\Delta$  up to twice the Ehrenfest time, which is generally not possible with the usual semiclassical calculus. To do so, they introduced an anisotropic semiclassical calculus which allows to apply Egorov's theorem up to twice the Ehrenfest time. The existence of this anisotropic calculus is made possible by the algebraic structure of the hyperbolic surfaces which allows to have:

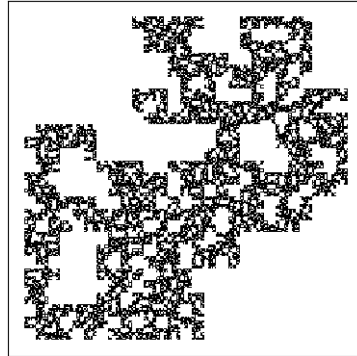
- An analytic description of the unstable and stable spaces when lifted to the Poincaré disk model. In particular, these spaces depend smoothly on the base point.
- A constant unstable Jacobian.

These two crucial points, necessary in the cited works, fail when dealing with more general chaotic (i.e. hyperbolic) situations. It has been one of the main difficulties to overpass in [Vac22b] (see also Chapter 5), but it was already accomplished in [DJN21] for a different application of the FUP (see below). Let us also mention the different approach of [DZ18] which exploits the FUP with transfer operators for hyperbolic surfaces. This approach is indeed not reproducible in more general open hyperbolic systems. Concerning the discrete FUP, it has been first developed in [DJ17] to obtain a spectral gap for quantized open baker's maps. Since we have already said a lot on spectral gaps





(a) A (self-similar) porous set in 2D.



(b) Another (random) porous set in 2D.

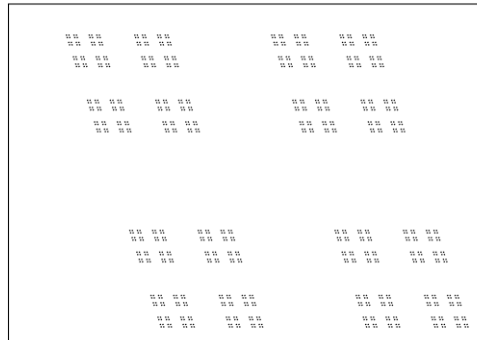
(c) A product of  $\delta$ -regular sets.

Figure 2.17: The FUP of [HS20] can be applied to  $\delta$ -regular sets  $X$  such as the ones in subfigures (a) and (b). Nevertheless, they have to restrict to subsets  $Y$  like in subfigure (c) .

in the first part of the Introduction of this thesis, we stop here concerning spectral gaps and focus on other applications in quantum chaos.

**Quantum Ergodicity** . The purpose of this paragraph is *not* to give a complete presentation of this area, but rather to explain the problem and the recent progress made thanks to the FUP. For a broader introduction, see for instance [Non13], [Dya21a], [Dya21b]. Quantum Ergodicity concerns different *closed* chaotic situations. For simplicity, we focus on negatively curved compact surfaces. Let  $(M, g)$  be a compact (connected) surface with negative curvature (for instance, a hyperbolic surface). It is known that the geodesic flow is ergodic. The question raised by Quantum Ergodicity concerns the high-frequency behavior of the eigenfunctions of the Laplace-Beltrami operator  $-\Delta$ . This operator has spectrum  $\lambda_0 = 0 < \lambda_1 \leq \dots \lambda_n \rightarrow +\infty$  with eigenfunctions  $(u_j)$ :

$$-\Delta u_j = \lambda_j u_j ; \|u_j\|_{L^2(M)} = 1$$

What can we say about the behavior of  $(u_j)$  as  $j \rightarrow +\infty$  ? How does the ergodicity of the geodesic flow influence its behavior ? To answer these questions, it is relevant to study the weak limits of the measures  $|u_j(x)|^2 dx$  on  $M$ . In fact, it is possible to study a more general object : the semiclassical measures.

**Definition 2-2.3.** Let  $\text{Op}_h$  be a quantization procedure on  $M$ .

- Let  $v \in L^2(M)$  and  $0 < h \leq 1$ . The *Wigner distribution* associated with  $v$  is the measure  $\mu_v^h$  on  $T^*M$  defined by

$$\int_{T^*M} a(x, \xi) d\mu_v^h(x, \xi) = (\text{Op}_h(a)v, v)_{L^2} ; a \in C_c^\infty(T^*M)$$

- We say that a measure  $\mu$  on  $T^*M$  is a *semiclassical measure* associated with the family of eigenfunctions  $(u_j)$  if there exists a subsequence  $(u_{j_k})_k$  such that, with  $h_k = \lambda_{j_k}^{-1}$ ,  $\mu_{u_{j_k}}^{h_k}$  converges weakly- $\star$  to  $\mu$ , that is, for all  $a \in C_c^\infty(T^*M)$ ,

$$\lim_{k \rightarrow +\infty} (\text{Op}_{h_k}(a)u_{j_k}, u_{j_k})_{L^2} = \int_{T^*M} a(x, \xi) d\mu(x, \xi)$$

**Remark.**

- When we take  $a(x, \xi) = a(x)$ , we recover the case  $|u_j(x)|^2 dx$ .
- The semiclassical calculus allows to show that a semiclassical measure  $\mu$  satisfies :
  - (i)  $\mu$  is supported on  $S^*M$  ;
  - (ii)  $\mu$  is invariant by the geodesics flow.
- Semiclassical measures were introduced in [Gér91] and [LP93].

The main question in Quantum Ergodicity is the following :

**Question 2.3.** *What are the possible semiclassical measures on  $M$  ?*

In fact, it could be refined as " Is the Liouville measure  $\mu_L$  the only semiclassical measure on  $M$  ?". Indeed, it is now well known that most eigenfunctions equidistribute in the following sense.

**Theorem 2-2.5** (Quantum Ergodicity, [dV85] for the general case, [Zel87] for hyperbolic surface, [Sch74] for a first statement ). *There exists a density-1 sequence  $(u_{j_k})$  such that  $\mu_{u_{j_k}}^{h_k}$  converges weakly- $\star$  to the Liouville measure  $\mu_L$*

This naturally leads to the conjecture, known as the Quantum Unique Ergodicity (QUE) conjecture.

**Conjecture (QUE).** *The Liouville measure  $\mu_L$  is the only semiclassical measure.*

This conjecture is still open. Restrictions on potential semiclassical measures have been obtained in [AN07b], [AN07a], concerning bounds on the entropy. More recently, the use of the FUP has permitted to obtain a new kind of restrictions :

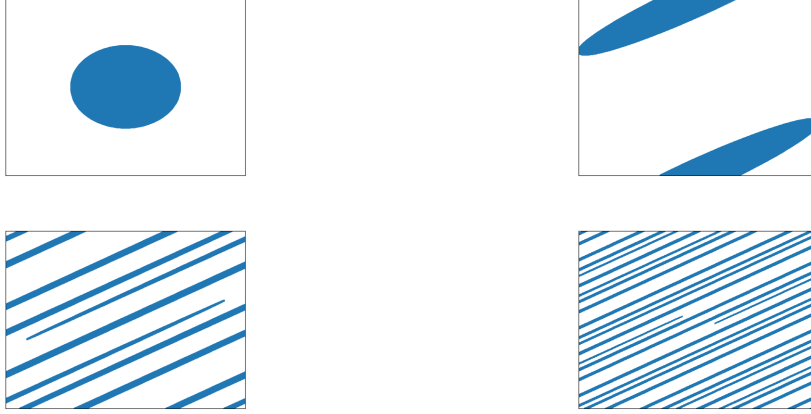


Figure 2.18: Evolution of a wave packet in a hyperbolic situation. It spreads along unstable manifolds. The pictures has been made using the dynamics of  $A \in \text{SL}_2(\mathbb{Z})$ ,  $|\text{tr}(A)| > 2$ , on the torus. A set of points  $U$  (upper left corner) is iterated. The three other figures show  $A^k(U)$  for different values of  $k \in \mathbb{N}$ .

**Theorem 2-2.6** ([DJ18b] for hyperbolic surfaces, [DJN21] for the general case). *Let  $\mu$  be a semiclassical measure. Then,  $\mu$ , seen as a measure on  $S^*M$ , has full support i.e. for any open set  $U \subset S^*M$ ,  $\mu(U) > 0$ .*

The proof of [DJ18b] uses the anisotropic calculus developed in [DZ16]. Again, this is possible due to the special properties of hyperbolic surfaces. This was no more possible to use this calculus for general negatively curved surfaces. However, in [DJN21], the authors succeeded to adapt the strategy with a different approach. The smoothness of the unstable and stable spaces is no more needed, but their  $C^{2-\varepsilon}$  regularity was required. They also made it possible to deal with varying unstable Jacobian. Their approach is the one that has been adapted in [Vac22b] (see Chapter 5) to obtain a spectral gap in obstacle scattering.

As in the case of open quantum systems, there is a famous toy model to test conjectures in quantum ergodicity, the so-called cat map. The model concerns the action of a classical chaotic map on the torus  $\mathbb{T}^2$  and its quantization. The classical object is a matrix  $A \in \text{SL}_2(\mathbb{Z})$  with  $|\text{tr}(A)| > 2$ , that is,  $A$  has eigenvalues  $\lambda_u, \lambda_u^{-1}$  with  $|\lambda_u| > 1$ .  $A$  acts on  $\mathbb{T}^2$  and the dynamics is chaotic (see Figure 2.18). Its quantization has been performed for instance in [BB96]. Without entering into the details of the model, it is possible to construct a finite dimensional Hilbert space  $\mathcal{H}_N$ , with  $N = (2\pi\hbar)^{-1}$  and an operator  $\mathcal{M}_N(A)$  on  $\mathcal{H}_N$ . As in the case of the Laplace-Beltrami operator, it has an eigenbasis of eigenfunctions and it is possible to define the analogue notion of semiclassical measure on the torus. In the recent work of [Sch21], the FUP has been successfully used to prove an analogue of Theorem 2-2.6 in this context. More recently, [DJ21] generalized this result in higher dimensions with additional assumptions on  $A$ , allowing them to apply the one-dimensional FUP.

**Applications to PDEs.** The strategy used in [DJ18b] and [DJN21], based on the use of the Fractal Uncertainty Principle, also has applications to PDEs on negatively curved manifolds  $(M, g)$ .

*Control and observability for Schrödinger equation.* Control and observability, which are in fact closely related, concern the following questions which can be formulated as follows. Let  $\Omega \subset M$  be an open set and  $T > 0$  :

- *Observability* : Let  $u(t) = e^{it\Delta}u_0$ ,  $u_0 \in L^2(M)$ , that is the solution of

$$\begin{cases} (i\partial_t + \Delta)u(t, x) = 0, & t \in \mathbb{R}, x \in M \\ u(t = 0, x) = u_0(x), & x \in M \end{cases}$$

If we observe that  $u(t, x) = 0$  for all  $x \in \Omega$  and for all  $t \in [0, T]$ , can we say that  $u_0 = 0$  ?

- *Control* : given  $u_0 \in L^2(M)$ , can we find a control term  $f \in L^2(]0, T[ \times \Omega)$  such that the unique solution of

$$\begin{cases} (i\partial_t + \Delta)u(t, x) = f(t, x)\mathbb{1}_{]0, T[ \times \Omega}, & t \in \mathbb{R}, x \in M \\ u(t = 0, x) = u_0(x), & x \in M \end{cases}$$

satisfies  $u(T, x) = 0$  for all  $x \in M$ .

In the context of negatively curved surfaces, the following result has been proved :

**Theorem 2-2.7** ([Jin18] for hyperbolic surfaces, [DJN21] for general negatively curved surfaces). *Let  $(M, g)$  be a negatively curved surface,  $\Omega \subset M$  be an open set and  $T > 0$ .*

- (*Observability*) *There exists a constant  $K > 0$  such that for all  $u_0 \in L^2(M)$ ,*

$$\|u\|_{L^2(M)}^2 \leq K \int_0^T \|e^{it\Delta}u_0\|^2 dt.$$

- (*Control*) *For any  $u_0 \in L^2(M)$ , there exists  $f \in L^2(]0, T[ \times \Omega)$  such that the solution of the equation*

$$(i\partial_t + \Delta)u(t, x) = f(t, x)\mathbb{1}_{]0, T[ \times \Omega} u(0, x) = u_0(x)$$

*satisfies  $u(T, x) = 0$  for all  $x \in M$ .*

*Damped wave equation.* Concerning the damped wave equation,

$$(\partial_t^2 - \Delta + b(x)\partial_t)u(t, x) = 0; \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \quad (2-2.4)$$

where  $b \in C^\infty(M)$ ,  $b \geq 0$ , is the damping function, it is known that there are relations between the decay of the energy and the interactions between the geodesic flow and the damping region  $\{b > 0\}$ . For each,  $(u_0, u_1) \in H^1(M) \times L^2(M)$ , (2-2.4) has a unique solution  $u(t)$ , defined with semigroup theory. The energy of the solution is defined as

$$E(u(t)) = \frac{1}{2} \int_M |\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 dx$$

When  $b$  is not zero, it is known that  $E(u(t)) \rightarrow 0$  when  $t \rightarrow +\infty$ . It is known that to obtain an exponential decay rate for the energy, it is sufficient that the damping satisfies the geometric control condition, that is, all the geodesics meet the damping region (see [Leb96]). In fact, in that case, we have  $E(u(t)) \leq e^{-\alpha t} E(u_0)$  for some  $\alpha > 0$ . In the case of surfaces of negative curvature, hyperbolicity prevent wave packets to stay concentrated near a point in phase space and it spreads wave packets along unstable manifolds (see Figure 2.18). The use of the FUP has lead to a striking result showing that no matter how thin is the damping region, as soon as  $\{b > 0\} = \emptyset$ , exponential decay holds:

**Theorem 2-2.8** ([Jin20] for hyperbolic surfaces, [DJN21] for general negatively curved surface). *Let  $(M, g)$  be a negatively curved surface. Assume that  $b \geq 0$  but  $b \not\equiv 0$ . For every  $s > 0$ , there exists  $C > 0$  and  $\gamma > 0$  such that for any  $(u_0, u_1) \in H^{s+1}(M) \times H^s(M)$ , the energy of the solution decays exponentially and we have:*

$$E(u(t)) \leq e^{-\gamma t} (\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s})$$

### 2-2.3. Propagation of coherent states.

Coherent states, which can have different names in the literature (Gaussian states, Gaussian beams or Gaussian wave packets) turn out to be an efficient tool to relate the quantum dynamics and the classical flow dynamics. This is a very natural tool when dealing with quantum/classical correspondence. The author has learned on coherent states in the book [CR12].

**Coherent states.** The semiclassical coherent state centered at zero is

$$\varphi_0(x) = \frac{1}{(\pi\hbar)^{1/4}} e^{-\frac{|x|^2}{2\hbar}} \quad (2-2.5)$$

and the coherent state centered at  $\rho \in \mathbb{R}^{2d}$  is

$$\varphi_\rho := T_\rho \varphi_0 \quad (2-2.6)$$

where  $T_\rho$  is the Weyl-Heisenberg translation operator:

$$T_{(x_0, \xi_0)} u(x) = e^{-\frac{i}{2\hbar} \xi_0 \cdot x_0} e^{\frac{i}{\hbar} \xi_0 \cdot x} u(x - x_0)$$

We recall that  $\varphi_0$  is the ground state of the harmonic oscillator  $-h^2\Delta + |x|^2$ . The other eigenfunctions of this harmonic oscillator, called excited states, are obtained from  $\varphi_0$  by applying the creation (or raising) operator  $\mathbf{a}_i^\dagger = \frac{1}{\sqrt{2\hbar}}(-h\partial_{x_i} + x_i)$ . For  $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$ , we can note for instance

$$\varphi_0^\nu = \frac{1}{\nu_1! \dots \nu_d!} (\mathbf{a}_1^\dagger)^{\nu_1} \dots (\mathbf{a}_d^\dagger)^{\nu_d} \varphi_0$$

for the normalized excited states. It is an eigenfunction of  $-h^2\Delta + x^2$  with eigenvalue  $h(\nu_1 + \dots + \nu_d + d/2)$ . The family  $(\varphi_0^\nu)_\nu$  forms an Hilbert basis of  $L^2(\mathbb{R}^d)$ . Recall also that the annihilator (or lowering) operators are

$$\mathbf{a}_i = \frac{1}{\sqrt{2\hbar}}(h\partial_{x_i} + x_i)$$

**Squeezed coherent states.** It can also be interesting to define squeezed coherent states, sometimes called Hagedorn coherent states.

**Definition 2-2.4.** We note  $\Sigma_d$  for the Siegel upper half plane, consisting of symmetric matrices  $\Gamma \in \mathcal{M}_d(\mathbb{C})$  such that  $\text{Im}(\Gamma)$  is positive definite.

Let  $\Gamma \in \Sigma_d$ . The squeezed coherent state centered at zero is

$$\varphi_0^{(\Gamma)}(x) = (a_\Gamma \pi \hbar)^{-1/4} \exp\left(i \frac{\Gamma x \cdot x}{2\hbar}\right)$$

where  $a_\Gamma > 0$  makes the norm of this state equal to one. We also define the squeezed coherent state centered at  $\rho \in \mathbb{R}^{2d}$  by

$$\varphi_\rho^{(\Gamma)} = T_\rho \varphi_0^{(\Gamma)}$$

Squeezed coherent states have relations with positive Lagrangian frames (see for instance [LST18] which gives a geometric interpretation of squeezed coherent states). To see that, for  $\Gamma \in \Sigma_d$ , let's note

$$L_\Gamma = \{(\Gamma q, q), q \in \mathbb{C}^d\}$$

$L_\Gamma$  is a positive Lagrangian frame in the sense that, if

$$J = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$$

- (Lagrangian) for all  $l, l' \in L_\Gamma$ ,  $l \cdot J l' = 0$  ;
- (Positive) for all  $l \in L_\Gamma$ ,  $\frac{1}{i}(J l) \cdot l > 0$ .

It is then possible to define more general raising and lowering operators : for each  $l \in \mathbb{C}^{2d} = (l_x, l_\xi) \in \mathbb{C}^d \oplus \mathbb{C}^d$ , we set

$$A[l] = \frac{i}{\sqrt{2\hbar}} \sum_{j=1}^d \frac{\hbar}{i} (l_\xi)_j \partial_{x_j} - (l_x)_j x_j ; A^\dagger[l] = -\frac{i}{\sqrt{2\hbar}} \sum_{j=1}^d \frac{\hbar}{i} \overline{(l_\xi)_j} \partial_{x_j} - \overline{(l_x)_j} x_j$$

The creation and annihilation operators  $\mathbf{a}_j^\dagger$  and  $\mathbf{a}_j$  are obtained with  $l_j = (ie_j, e_j)$  where  $e_j$  is the  $j$ -th element of the canonical basis. Let's note that  $l_j \in L_{i\text{Id}}$ . It is then possible to show the following characterization of the squeezed coherent states ([LST18], Proposition 3.5) : let  $u \in \mathcal{D}'(\mathbb{R}^d)$ , then

$$\forall l \in L_\Gamma, A(l)u = 0 \iff \exists c \in \mathbb{C}, u = c \varphi_0^{(\Gamma)}$$

Squeezed coherent states also naturally appear when propagating coherent states by quadratic Hamiltonian. Let  $S(t) \in \mathcal{S}_{2d}(\mathbb{R})$  be time-dependent family of real symmetric matrices, depending continuously on  $t$ , and consider the associated time-dependent quadratic Hamiltonian

$$H(t, x, \xi) = \frac{1}{2} \begin{pmatrix} x \\ \xi \end{pmatrix} \cdot S(t) \begin{pmatrix} x \\ \xi \end{pmatrix}$$

$H(t)$  is quantized into  $H^W(t, x, hD_x) = \widehat{H}(t)$ . For instance, when  $S(t) = I_{2d}$ ,  $\widehat{H}$  is the harmonic oscillator  $\frac{1}{2}(-h^2\Delta + x^2)$ . We consider the quantum propagator  $U(t)$  associated with  $H$ , which is the solution of the Schrödinger equation

$$ih\partial_t U(t) = \widehat{H}(t)U(t); U(0) = \text{Id}.$$

The propagation of coherent states  $\varphi_\rho$  can be explicitly computed, once the classical flow of  $H(t, x, \xi)$  is known. Indeed, let  $F(t)$  be the symplectic matrix solving the ODE

$$\frac{d}{dt}F(t) = JS(t)F(t).$$

Let's write  $F(t)$  by blocks

$$F(t) = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}.$$

It can be shown  $A_t + iB_t$  is always invertible and if  $\Gamma_t = (C_t + iD_t)(A_t + iB_t)^{-1}$ , then, one has ([CR12], Theorem 16 in Chapter 3) : there exist  $c_t \in \{z \in \mathbb{C}, |z| = 1\}$  such that

$$U(t)\varphi_\rho = c_t\varphi_{\rho_t}^{\Gamma_t}; \rho_t = F(t)\rho.$$

Namely, the center of the coherent state moves along the classical Hamiltonian flow and the state undergoes a squeezing. When dealing with the propagation of general Hamiltonian  $\widehat{H}(t)$ , the same ideas apply but the computations are no more exact. Nevertheless, it is still possible to give an asymptotic expansion in powers of  $h^{1/2}$ . One has to perform a Taylor expansion of  $H(t)$  near the moving center of the coherent state, which follows again the classical flow. The quadratic part of the expansion gives the first term of the approximation and the higher-order terms lead to corrections which involve the excited states (see for instance [CR97]).

*Example.* (see Figure 2.19 and Figure 2.20). Let's see what happens in a very simple situation in dimension  $d = 1$ , where the classical flow has a unique fixed point, which is hyperbolic. This is a model situation which shares similarities with scattering by 2 obstacles. The symplectic matrix  $F_t$  is in our example

$$F_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

that is, we consider  $H(t, x, \xi) = H(x, \xi) = x\xi$ . 0 is a hyperbolic fixed point: the unstable (resp. stable) space is given by the  $x$ -axis (resp.  $\xi$ -axis). Simple computations shows that  $\widehat{H} = xhD_x + \frac{h}{2i}$  and that the quantum propagator is the following scaling operator :

$$U(t)u(x) = e^{-t/2}u(e^{-t}x)$$

It is then very easy to compute the evolution of a coherent state  $\varphi_\rho$  and check that it is consistent with the general case above. The coherent state becomes thinner and thinner while  $t \rightarrow +\infty$  : it is stretched along the unstable direction. Such a property will be used in this thesis (see Chapter 7).

**Decomposition into coherent states.** Coherent states are not orthogonal, since for all  $\rho, \rho' \in \mathbb{R}^{2d}$ , one has

$$(\varphi_\rho, \varphi_{\rho'})_{L^2} = \exp\left(-\frac{|\rho - \rho'|^2}{4h}\right) \exp\left(i\frac{\sigma(\rho, \rho')}{2h}\right); \sigma((x_0, \xi_0), (x_1, \xi_1)) = \xi_1 \cdot x_0 - \xi_0 \cdot x_1.$$

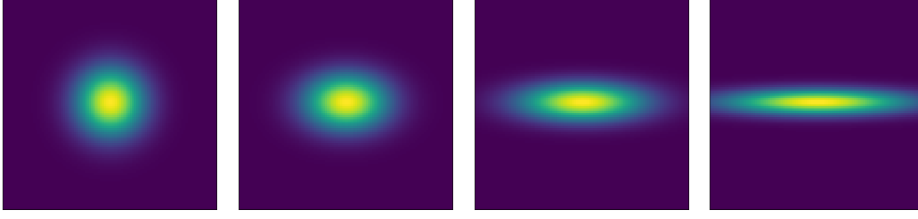


Figure 2.19: Evolution of a coherent state initially centered at 0, through the dynamics of Example 2-2.3.. The color represents the modulus of the Wigner distribution of the evolved Gaussian state.

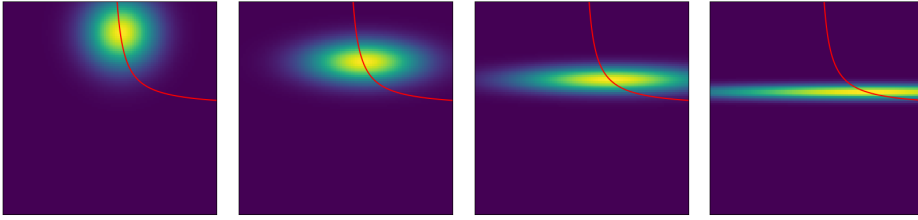


Figure 2.20: Evolution of a coherent state through the dynamics of Example 2-2.3.. The state is initially centered at a point  $(x_0, \xi_0)$  with both  $x_0 \neq 0$  and  $\xi_0 \neq 0$ . The red curve represents the trajectory of the center of the state. The color represents the modulus of the Wigner distribution of the evolved Gaussian state.

Nevertheless, they form an over-complete system which allows to describe all the states  $u \in L^2(\mathbb{R}^d)$  and operators on  $L^2(\mathbb{R}^d)$ . To see that, let's introduce the Bargman transform :

$$u \in L^2(\mathbb{R}^d) \mapsto \mathcal{B}_h u \in L^2(\mathbb{R}^{2d}) ; \mathcal{B}_h u(\rho) = (2\pi h)^{-d/2} (\varphi_\rho, u)_{L^2} = (2\pi h)^{-d/2} \int_{\mathbb{R}^d} \overline{\varphi_\rho(x)} u(x) dx.$$

It can be shown that  $\mathcal{B}_h$  is in fact an isometry from  $L^2(\mathbb{R}^d)$  to  $L^2(\mathbb{R}^{2d})$ . In particular, the formal adjoint  $\mathcal{B}_h^*$  is also well-defined and is given by

$$\mathcal{B}_h^* v(x) = (2\pi h)^{-d/2} \int_{\mathbb{R}^{2d}} \varphi_\rho(x) v(\rho) d\rho, \quad v \in L^2(\mathbb{R}^{2d}).$$

This formula and the fact that  $\mathcal{B}_h^* \mathcal{B}_h = \text{Id}_{L^2(\mathbb{R}^d)}$  show that each state  $u \in L^2(\mathbb{R}^d)$  admits the following fundamental decomposition into coherent states :

$$u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} (\varphi_\rho, u)_{L^2} \varphi_\rho(x) d\rho. \quad (2-2.7)$$

An interesting consequence of this formula, which is known as a resolution of the identity, concerns the possibility of computing the Hilbert-Schmidt norm or the trace of operators by simply computing their action on coherent states :

**Lemma 2-2.4.** Let  $B : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  be an Hilbert-Schmidt operator. Then,

$$\|A\|_{HS} = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} \|B\varphi_\rho\|^2 d\rho.$$

Let  $A : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  be a trace class operator. Then,

$$\text{tr}(A) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} (A\varphi_\rho, \varphi_\rho)_{L^2} d\rho.$$

This fact has been exploited in Chapter 7 to estimate the Hilbert-Schmidt norm of an operator, which in turn, turns out to be useful to estimate the number of resonances in boxes and obtain Theorem B.

**Remark.**

- It is also possible to define a quantization procedure, called the anti-Wick quantization, inspired by (2-2.7) : for  $a \in \mathcal{S}(\mathbb{R}^{2d})$  (and in fact, for  $a \in L^\infty(\mathbb{R}^{2d})$  it would also make sense):

$$\text{Op}_h^{AW}(a)u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} a(\rho)(\varphi_\rho, u)_{L^2} \varphi_\rho(x).$$

Compared with other quantizations (such as the Weyl quantization, or the left/right quantizations), the main interest of anti-Wick quantization is its positivity :

$$a \geq 0 \implies \forall u \in L^2(\mathbb{R}^d), \left( \text{Op}_h^{AW}(a)u, u \right)_{L^2} \geq 0.$$

- There is also a slightly different approach which transforms a state  $u \in L^2(\mathbb{R}^d)$  to

$$u^\#(z) = (\mathcal{B}_h u(x, \xi)) e^{\frac{x^2 + \xi^2}{4h}} ; z = x - i\xi.$$

This is the so-called FBI transform (FBI for Fourier-Bros-Iagolnitzer, see for instance [CR12], Section 1.3, [Zwo12], Chapter 13) : it is an isometry from  $L^2(\mathbb{R}^d)$  to the Fock space  $\mathcal{F}(\mathbb{C}^d)$ , the space of entire functions  $f : \mathbb{C}^d \rightarrow \mathbb{C}$  such that

$$\|f\|_{\mathcal{F}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} |f(z)|^2 e^{-\frac{z \cdot \bar{z}}{h}} |dz \wedge d\bar{z}| < +\infty.$$

**Applications of the propagation of coherent states.** The fact that it is possible to recover the evolution of any state by knowing the evolution of the coherent states makes them particularly useful. The formulas of Lemma 2-2.4 have been for instance used in [CRR99] to give a proof of the Gutzwiller Semiclassical Trace Formula, using the precise description of the propagation of coherent states of [CR97]. Wave packets are also often used for the construction of quasi-modes, that is, approximate eigenstates of pseudodifferential operators. This is for instance the case in [PU93]. These ideas are still fruitful and alive, as evidenced by the recent works of, among others, [LST18] or [Arn20]. Of course, this is impossible to give an exhaustive list of applications of wave packets and their propagation. Several applications from a semiclassical point of view are listed in [CR12] and the authors give a references to a broader literature on the subject.

This ends the introduction of this manuscript. We again insist on the fact that the tools presented above do not form an exhaustive list. We will be lead to introduce other tools in the core of this manuscript, and we will come back to the tools presented here, when needed. The next two chapters 3 and 4 are dedicated to the presentation of preliminary results on hyperbolic dynamical systems and semiclassical analysis. They are also used to introduce the main objects studied in this thesis : open hyperbolic maps and their quantum counterpart. We get to the heart of the matter in Chapter 5 and we prove Theorem A in Chapter 6, so as other applications in the problem of scattering by obstacles. We prove Theorem B in Chapter 7.





## Part II

# Preliminary results



The core of this thesis is the study of open quantum hyperbolic systems. This field is a place where meet two main areas of mathematics : hyperbolic dynamical systems and semiclassical analysis. Most of the tools we use in this thesis are issued from these two fields and most of them are rather standard (with some notorious textbooks which were widely used for the redaction of this manuscript : [HK95] for hyperbolic dynamical systems and [Zwo12] for semiclassical analysis). The following two chapters are dedicated to the presentation of main notions from hyperbolic dynamical systems (Chapter 3) and semiclassical analysis (Chapter 4). We recall some definitions and important properties. When quantitative versions of rather standard results are needed, we provide some proofs. Most of this material is stated and proved in [Vac22b].

The major personal contribution of this part is the proof of Theorem 3-3.1, concerning the regularity of stable and unstable distributions for 2-dimensional hyperbolic systems. Theorem 3-3.1 is the main ingredient to prove the existence of special adapted charts (see Section 3-4), which turns out to be crucial in the proof of the spectral gap (Theorem A).

We also use these preliminary chapters to introduce the main objects that are studied in this thesis :

- **Open hyperbolic maps**, defined on particular phase space (see 3-1.1.). It encompasses the case of the billiard map, whose definition is recalled.
- **Open hyperbolic quantum maps**, which are the quantum counterpart of the above maps. These are Fourier integral operators. (see 4-3.2).

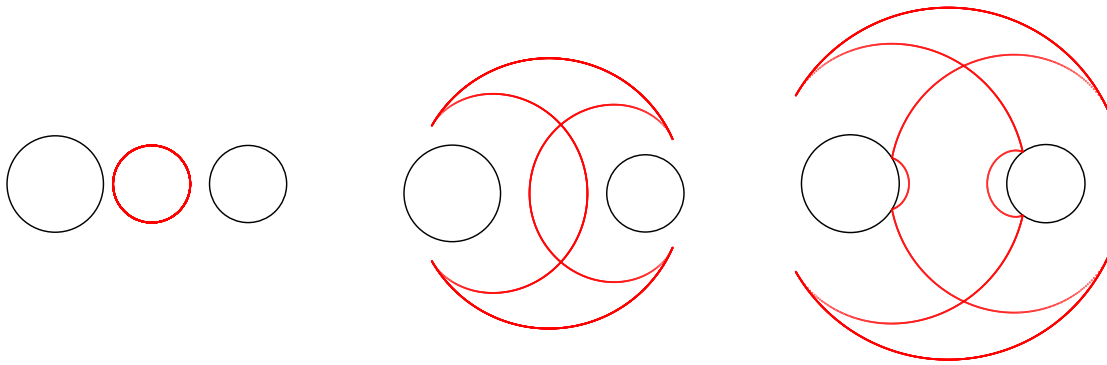


Figure 2.21: The billiard flow is a hyperbolic dynamical system. These 3 pictures show the evolution of many particles with the same initial position and velocities uniformly distributed in  $\mathbb{S}^1$ .



# Chapter 3

## Hyperbolic dynamics

In this chapter, we present the general classical systems we work with, that is, open hyperbolic maps. We present a general notion of such map, keeping in mind that the billiard map belongs to this category. We then gather rather standard properties of hyperbolic dynamical systems. We also prove a result concerning the regularity of the unstable and stable distribution for 2D hyperbolic maps. This is the content of Theorem 3-3.1. We provide our own proof, since it turned out to be delicate to find a version of such regularity in the literature, for our particular open situation. Finally, we apply this result to state the existence of "adapted charts" in which the geometry of the unstable and stable manifolds is straightened. This is a crucial point in the proof of the spectral gap (Theorem A).

### 3-1 Open hyperbolic maps.

#### 3-1.1. Definitions and assumptions.

The following long definition is based on the definitions in the works of Nonnenmacher, Sjöstrand and Zworski in [NSZ11] and [NSZ14] specialized to the 2-dimensional phase space. Consider open intervals  $Y_1, \dots, Y_J$  of  $J$  copies of  $\mathbb{R}$ . Let us set :

$$Y = \bigsqcup_{j=1}^J Y_j \subset \bigsqcup_{j=1}^J \mathbb{R}$$

and let us consider

$$U = \bigsqcup_{j=1}^J U_j \subset \bigsqcup_{j=1}^J T^*\mathbb{R} \quad ; \quad \text{where } U_j \in T^*Y_j \text{ are open sets.}$$

The Hilbert space  $L^2(Y)$  is the orthogonal sum  $\bigoplus_{i=1}^J L^2(Y_i)$ .

For  $j = 1, \dots, J$ , we consider open disjoint subsets  $\tilde{D}_{ij} \in U_j$ ,  $1 \leq i \leq J$ , the *departure sets*, and similarly, for  $i = 1, \dots, J$ , we consider open disjoint subsets  $\tilde{A}_{ij} \in U_i$ ,  $1 \leq j \leq J$ , the *arrival sets* (see Figure 3.1). We assume that there exist smooth symplectomorphisms

$$F_{ij} : \tilde{D}_{ij} \rightarrow F_{ij}(\tilde{D}_{ij}) = \tilde{A}_{ij} \tag{3-1.1}$$

We note  $F$  for the global smooth map  $F : \tilde{D} \rightarrow \tilde{A}$  where  $\tilde{A}$  and  $\tilde{D}$  are the full arrival and departure sets, defined as

$$\begin{aligned} \tilde{A} &= \bigsqcup_{i=1}^J \bigsqcup_{j=1}^J \tilde{A}_{ij} \subset \bigsqcup_{i=1}^J U_i \\ \tilde{D} &= \bigsqcup_{j=1}^J \bigsqcup_{i=1}^J \tilde{D}_{ij} \subset \bigsqcup_{j=1}^J U_j \end{aligned}$$

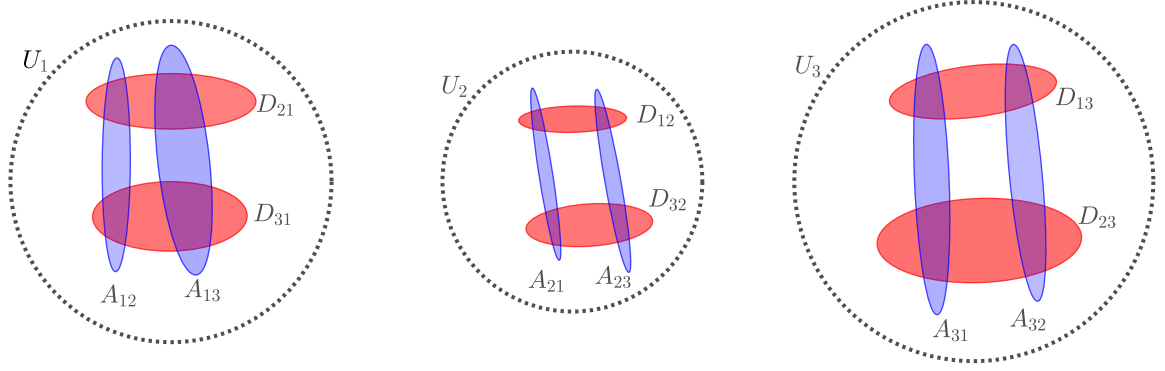


Figure 3.1: A schematic example of an open hyperbolic map with  $J = 3$  in a case where  $D_{ii} = \emptyset$  for  $i = 1, 2, 3$ .

We define the outgoing (resp. incoming) tail by  $\mathcal{T}_+ := \{\rho \in U; F^{-n}(\rho) \in U, \forall n \in \mathbb{N}\}$  (resp.  $\mathcal{T}_- := \{\rho \in U; F^n(\rho) \in U, \forall n \in \mathbb{N}\}$ ). We assume that they are closed subsets of  $U$  and that the *trapped set*

$$\mathcal{T} = \mathcal{T}_+ \cap \mathcal{T}_- \quad (3-1.2)$$

is compact.

In this thesis, we will make the following assumption :

$\mathcal{T}$  is *totally disconnected*.

**Remark.**

It is possible that for some values of  $i$  and  $j$ ,  $\widetilde{D}_{ij} = \emptyset$ . For instance, when dealing with the billiard map (see 3-1.1.1.), the sets  $\widetilde{D}_{ii}$  are all empty.

We then make the following crucial assumption.

$\mathcal{T}$  is a hyperbolic set for  $F$  (Hyperbolicity)

Namely, for every  $\rho \in \mathcal{T}$ , we assume that there exist stable/unstable tangent spaces  $E^s(\rho)$  and  $E^u(\rho)$  such that :

- $\dim E^s(\rho) = \dim E^u(\rho) = 1$
- $T_\rho U = E^s(\rho) \oplus E^u(\rho)$
- there exists  $\lambda > 0$ ,  $C > 0$  such that for all  $\rho \in \mathcal{T}$  and for all  $v \in T_\rho U$  and  $n \in \mathbb{N}$ ,

$$v \in E^s(\rho) \implies \|d_\rho F^n(v)\| \leq C e^{-n\lambda} \|v\| \quad (3-1.3)$$

$$v \in E^u(\rho) \implies \|d_\rho F^{-n}(v)\| \leq C e^{-n\lambda} \|v\| \quad (3-1.4)$$

where  $\|\cdot\|$  is a fixed Riemannian metric on  $U$ .

The decomposition of  $T_\rho U$  into stable and unstable spaces is assumed to be continuous. It is possible to fix a Riemannian metric on  $U$  such that (3-1.3) and (3-1.4) hold with  $C = 1$  ([Mat68]).

The last assumption we make on  $\mathcal{T}$  is a "fractal" assumption. To state it, we introduce the map

$$\phi_u : \rho \in \mathcal{T} \mapsto -\log \|d_\rho F|_{E_u(\rho)}\| \quad (3-1.5)$$

We suppose that

$$-P(\phi_u, F) > 0 \quad (\text{Fractal})$$

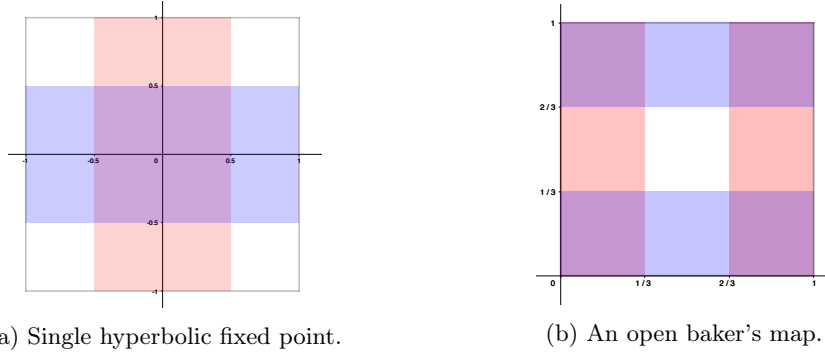


Figure 3.2: Examples when  $J = 1$ . The departure sets are in blue, the arrival sets in red. In the first example,  $U_1 = ]-1, 1[^2 \subset T^*\mathbb{R}$ ,  $D_{11} = ]-1, 1[\times ]-1/2, 1/2[$ ,  $A_{11} = ]-1/2, 1/2[\times ]-1, 1[$  with  $F(x, \xi) = (x/2, 2\xi)$ . The trapped set is reduced to a single hyperbolic fixed point. The second example is built on the model of an open baker's map. We have  $U_1 = ]0, 1[^2 \subset T^*\mathbb{R}$ ,  $D_{11} = ]0, 1[\times ]0, 1/3[\cup ]0, 1[\times ]2/3, 1[$ ,  $A_{11} = ]0, 1/3[\times ]0, 1[\cup ]2/3, 1[\times ]0, 1[$ . In such a model, the map  $F$  is piecewise affine and given by  $F(x + a, \xi) = (3x, a + \xi/3)$  for  $a \in \{0, 2\}$ ,  $(x, \xi) \in ]0, 1[^2$ .

Here, in terms of thermodynamics formalism,  $P$  denotes the topological pressure of the map  $\phi_u$  (see Definition 3-5.1). The norm  $\|\cdot\|$  is associated with any Riemannian metric on  $U$ . For instance, a possible formula for the definition of the pressure of a map  $\phi$ , for the dynamical system  $F : \mathcal{T} \rightarrow \mathcal{T}$  is

$$P(\phi, F) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \sup_{E \subset \mathcal{T}} \sum_{\rho \in E} \exp^{\sum_{k=0}^{n-1} \phi(F^k \rho)}$$

where the supremum ranges over all the  $(n, \varepsilon)$  separated subsets  $E \subset \mathcal{T}$  ( $E$  is said to be  $(n, \varepsilon)$  separated if for every  $\rho, \rho' \in E$ , there exists  $k \in \{0, \dots, n-1\}$ ,  $d(F^k(\rho), F^k(\rho')) > \varepsilon$ ).

**Remark.**

- $\gamma_1 := -P((\phi_u, F))$  is the classical decay rate of the dynamical system. It has the following physical interpretation : fix a point  $\rho_0 \in \mathcal{T}$  and consider the set  $B_m(\rho_0, \varepsilon)$  of points  $\rho \in U$  such that  $|F^k(\rho) - F^k(\rho_0)| < \varepsilon$  for  $0 \leq k \leq m-1$ . Then, its Lebesgue measure roughly behaves like  $e^{-m\gamma_1}$ .
- This condition implies that  $\mathcal{T}$  is indeed "fractal". More precisely, the trace of  $\mathcal{T}$  along the unstable and stable manifolds (see Lemma 3-2.1 for the definitions of these manifolds) have upper-box dimension strictly smaller than one. A quantitative version of this fact is stated in Proposition 5-5.2. In fact, Bowen's formula (see for instance [Bar08], the references given there and Proposition 3-6.1) gives that this upper-box dimension corresponds to the Hausdorff dimension  $d_H$  and it is the unique solution of the equation

$$P(s\phi_u, F) = 0, s \in \mathbb{R}$$

The Hausdorff dimension of the trapped set is then  $2d_H$ .

Here ends the description of the classical map. We present in more detail the case of the billiard map, for obstacles satisfying a no-eclipse assumption. Such a map satisfies all the assumptions above.

**3-1.1.1. The billiard map.**

Let  $\mathcal{O} = \bigcup_{i=1}^J \mathcal{O}_i$  where  $\mathcal{O}_i$  are open, strictly convex obstacles in  $\mathbb{R}^2$  having smooth boundary and satisfying the *Ikawa condition* : for  $i \neq j \neq k$ ,  $\overline{\mathcal{O}_i}$  does not intersect the convex hull of  $\mathcal{O}_j \cup \mathcal{O}_k$ .

For  $j \in \{1, \dots, J\}$ , let  $B^*\partial\mathcal{O}_j$  be the co-ball bundle of  $\partial\mathcal{O}_j$ ,  $S^*_{\partial\mathcal{O}_j}$  be the restriction of  $S^*\Omega$  to  $\partial\mathcal{O}_j$ ,  $\pi_j : S^*_{\partial\mathcal{O}_j} \rightarrow B^*\partial\mathcal{O}_j$  the natural projection and  $\nu_j(x)$  be the outward normal vector at  $x \in \partial\mathcal{O}_j$  (see Figure 3.3).



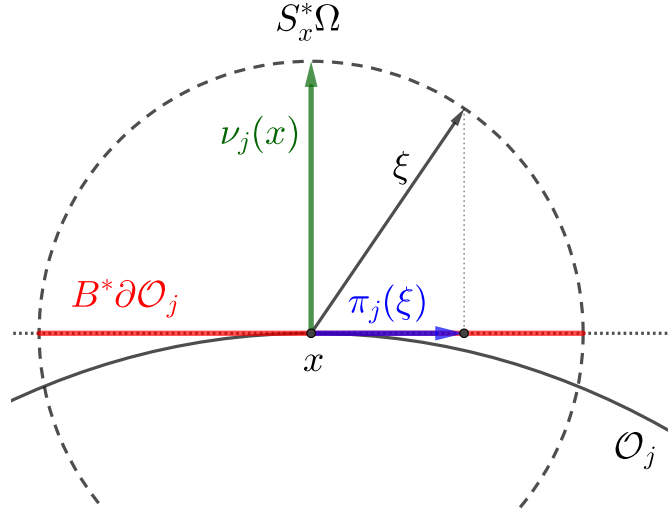


Figure 3.3: The notations used to define the billiard map.

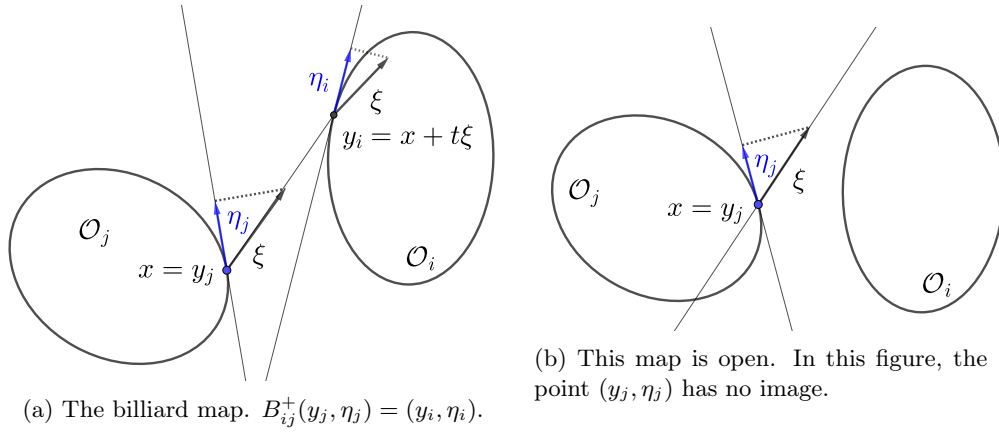


Figure 3.4: Description of billiard map.

For  $i \neq j$ , let  $\mathcal{B}_{ij}^+ : B^*\partial\mathcal{O}_j \rightarrow B^*\partial\mathcal{O}_i$  be the symplectic open map defined by

$$\begin{aligned} \rho = \mathcal{B}_{ij}^+(\rho') &\iff \exists t > 0, \exists \xi \in \mathbb{S}^1, \exists x \in \partial\mathcal{O}_j \\ x + t\xi &\in \partial\mathcal{O}_i, \langle \nu_j(x), \xi \rangle > 0, \langle \nu_i(x + t\xi), \xi \rangle < 0, \\ \pi_j(x, \xi) &= \rho', \pi_i(x + t\xi, \xi) = \rho \end{aligned}$$

$\mathcal{B}_{ij}^+$  is the billiard map. This map is open. (see Figure 3.4). Note that due to our definition, the glancing rays (that is the rays associated with a point  $\rho = (y, \eta) \in B^*\partial\mathcal{O}$  with  $|\eta| = 1$ ) are not in the set of definition of  $\mathcal{B}_{ij}^+$ .

It is a standard fact in the study of chaotic billiards (see for instance [CM00]) that the billiard map is hyperbolic due to the strict convexity assumption. Ikawa's condition ensures that the restriction of the dynamical system to the trapped set has a symbolic representation ([Mor91]). In particular, the trapped set is totally disconnected. The pressure condition (*Fractal*) is known to hold since the trapped set is not an attractor ([BR75]).

Strictly speaking this open map is defined in  $\bigcup_{j=1}^J B^*\partial\mathcal{O}_j$ , which has not the required form. Nevertheless, there exist  $Y_j \subset \partial\mathcal{O}_j$ ,  $1 \leq j \leq J$ , open subsets, not equal to all  $\partial\mathcal{O}_j$ , such that the departure set of the billiard map lies inside a compact subset of  $\bigcup_{j=1}^J T^*Y_j$ . As a consequence, the formalism of 3-1.1. applies.

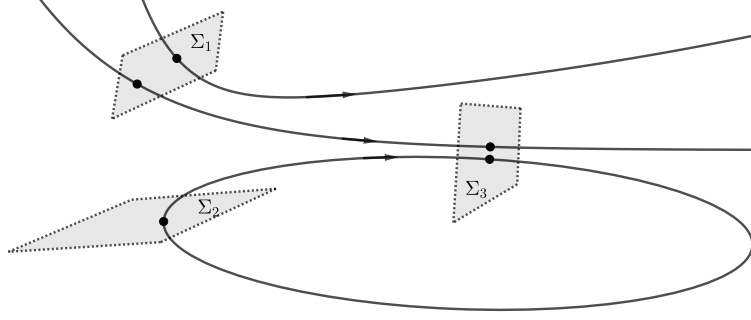


Figure 3.5: Schematic representation of Poincaré sections for the flow  $\Phi_t$  on an energy shell. The energy shell has dimension 3, so that the Poincaré section are 2-dimensional.

**3-1.1.2. Poincaré return map.**

Let us present another situation in which this formalism can be used. We will consider a simple example of Hamiltonian flow in  $\mathbb{R}^2$ . Other manifolds with several euclidian ends can be treated (see [NSZ11] -Section 2.1- for more general assumptions). Here, we simply consider a smooth compactly supported potential  $V \in C_c^\infty(\mathbb{R}^2)$ . We fix an energy  $E_0 > 0$  and consider  $p(x, \xi) = \xi^2 + V - E_0$ . We assume that 0 is not a critical energy of  $p$ , that is

$$dp \neq 0 \text{ on } p^{-1}(0)$$

Let's note  $H_p$  the Hamiltonian vector field associated with  $p$  and  $\Phi_t = \exp(tH_p)$  the corresponding Hamiltonian flow. The trapped set at energy 0 is the set

$$K_0 = \{(x, \xi) \in p^{-1}(0), \exists R > 0, \forall t \in \mathbb{R}, \Phi_t(x, \xi) \in B(0, R)\}$$

It is a compact subset of  $p^{-1}(0)$ . Here are the two crucial assumptions :

- (i)  $\Phi_t$  is hyperbolic on  $K_0$  ;
- (ii)  $K_0$  is topologically one dimensional.

**Poincaré sections.** A consequence of this second assumption (see [NSZ11], Section 2.2), is that there exist finitely many smooth contractible hypersurfaces  $\Sigma_i \subset p^{-1}(0)$ ,  $i = 1, \dots, J$  with smooth boundary and such that (see Figure 3.5)

$$\partial\Sigma_i \cap K_0 = \emptyset ; \Sigma_i \cap \Sigma_k = \emptyset, k \neq i$$

$$H_p \text{ is transversal to } \Sigma_i \text{ uniformly up to the boundary}$$

Moreover, for  $1 \leq i \leq J$  and for every  $\rho \in K_0$ , there exists  $t_-(\rho)$  and  $i_-(\rho)$  (resp.  $t_+(\rho)$  and  $i_+(\rho)$ ) such that

$$\begin{aligned} \Phi_{t_\pm(\rho)}(\rho) &\in K_0 \cap \Sigma_{i_\pm(\rho)} \\ \Sigma \cap \{\Phi_t(\rho), t_-(\rho) < t < t_+(\rho), t \in \mathbb{R}^*\} &= \emptyset \end{aligned}$$

where we note

$$\Sigma = \bigsqcup_{i=1}^J \Sigma_i$$

The maps  $t_\pm(\rho)$  are uniformly bounded on  $K_0$  and can be smoothly extended in a neighborhood of  $K_0$ . Finally, there exist  $\tilde{\Sigma}_i$  and symplectic diffeomorphisms

$$\kappa_i : \Sigma_i \rightarrow \tilde{\Sigma}_i$$

smooth up to the boundary.

**Poincaré return map.** For  $1 \leq i, j \leq J$ , the map  $\rho \mapsto \Phi_{t_+(\rho)}(\rho)$  extends smoothly to a symplectic diffeomorphism

$$F_{ij} : D_{ij} \rightarrow A_{ij}$$

where  $D_{ij}$  (resp.  $A_{ij}$ ) is a neighborhood of

$$\{\rho \in \mathcal{T} \cap \Sigma_j, i_+(\rho) = i\} \text{ (resp. } \{\rho \in \mathcal{T} \cap \Sigma_i, i_-(\rho) = j\})$$

The map  $F_{ij}$  is called the Poincaré return map. By writing it in the charts  $\kappa_i$  and  $\kappa_j$ , we can consider the following map between open sets of  $T^*\mathbb{R}$

$$\tilde{F}_{ij} = \kappa_i \circ F_{ij} \circ \kappa_j^{-1} : \tilde{D}_{ij} \subset \tilde{\Sigma}_j \rightarrow \tilde{A}_{ij} \subset \tilde{\Sigma}_i$$

The hyperbolicity of the flow implies the hyperbolicity of these open maps and  $K_0$  being topologically one dimensional implies that  $K_0 \cap \Sigma$  is totally disconnected. Finally, since  $K_0$  is not an attractor, the condition (Fractal) is satisfied ([BR75]). Hence, the formalism of 3-1.1. applies.

## 3-2 Standard results for hyperbolic dynamical systems.

We assumed that  $F$  is hyperbolic on the trapped set  $\mathcal{T}$ . As already mentioned, we can fix an adapted Riemannian metric on  $U$  such that the following stronger version of the hyperbolic estimates are satisfied for some  $\lambda_0 > 0$  : for every  $\rho \in \mathcal{T}$ ,  $n \in \mathbb{N}$ ,

$$v \in E_u(\rho) \implies \|d_\rho F^{-n}(v)\| \leq e^{-\lambda_0 n} \|v\| \quad (3-2.1)$$

$$v \in E_s(\rho) \implies \|d_\rho F^n(v)\| \leq e^{-\lambda_0 n} \|v\| \quad (3-2.2)$$

**Notations.** We now use the induced Riemannian distance on  $U$  and denote it  $d$ .

We also use the same notation  $\|\cdot\|$  to denote the subordinate norm on the space of linear maps between tangent spaces of  $U$ , namely, if  $F(\rho_1) = \rho_2$ ,

$$\|d_{\rho_1} F\| = \sup_{v \in T_{\rho_1} U, \|v\|_{\rho_1} = 1} \|d_{\rho_1} F(v)\|_{\rho_2}$$

If  $\rho \in \mathcal{T}$ ,  $n \in \mathbb{Z}$ , we use this Riemannian metric to define the unstable Jacobian  $J_n^u(\rho)$  and stable Jacobian  $J_n^s(\rho)$  at  $\rho$  by :

$$v \in E_u(\rho) \implies \|d_\rho F^n(v)\| = J_n^u(\rho) \|v\| \quad (3-2.3)$$

$$v \in E_s(\rho) \implies \|d_\rho F^n(v)\| = J_n^s(\rho) \|v\| \quad (3-2.4)$$

These Jacobians quantify the local hyperbolicity of the map.

**Notations.** Suppose that  $f$  and  $g$  are some real-valued functions depending on the same family of parameters  $\mathcal{P}$ . For instance, for  $J_n^u(\rho)$ ,  $\mathcal{P} = \{n, \rho\}$ . We will note  $f \sim g$  to mean that there exist constant a  $C \geq 1$  depending only on  $(U, F)$ , but not on  $\mathcal{P}$ , such that  $C^{-1}g \leq f \leq Cg$ .

For instance, if we define unstable and stable Jacobian  $\tilde{J}_n^u$  and  $\tilde{J}_n^s$  using another Riemannian metric, then, for every  $n \in \mathbb{Z}$  and  $\rho \in \mathcal{T}$ ,

$$\tilde{J}_n^u(\rho) \sim J_n^u(\rho) \quad ; \quad \tilde{J}_n^s(\rho) \sim J_n^s(\rho)$$

From the compactness of  $\mathcal{T}$ , there exists  $\lambda_1 \geq \lambda_0$  which satisfies

$$e^{n\lambda_0} \leq J_n^u(\rho) \leq e^{n\lambda_1} \quad \text{and} \quad e^{-n\lambda_1} \leq J_n^s(\rho) \leq e^{-n\lambda_0} \quad ; \quad n \in \mathbb{N}, \rho \in \mathcal{T} \quad (3-2.5)$$

$$e^{n\lambda_0} \leq J_{-n}^s(\rho) \leq e^{n\lambda_1} \quad \text{and} \quad e^{-n\lambda_1} \leq J_{-n}^u(\rho) \leq e^{-n\lambda_0} \quad ; \quad n \in \mathbb{N}, \rho \in \mathcal{T} \quad (3-2.6)$$

We record here standard facts about the stable and unstable manifolds (see for instance [HK95], Chapter 6).

**Lemma 3-2.1.** For any  $\rho \in \mathcal{T}$ , there exist local stable and unstable manifolds  $W_s(\rho), W_u(\rho) \subset U$  satisfying, for some  $\varepsilon_1 > 0$  (only depending on  $F$ ) :

- (1)  $W_s(\rho), W_u(\rho)$  are  $C^\infty$ -embedded curves, with the  $C^\infty$  norms of the embedding uniformly bounded in  $\rho$ .
- (2) the boundaries of  $W_u(\rho)$  and  $W_s(\rho)$  do not intersect  $\overline{B(\rho, \varepsilon_1)}$  and  $W_{u/s}(\rho) \subset B(\rho, 2\varepsilon_1)$  (these are local unstable/stable manifolds).
- (3)  $W_s(\rho) \cap W_u(\rho) = \{\rho\}$ ,  $T_\rho W_{u/s}(\rho) = E_{u/s}(\rho)$
- (4)  $F(W_s(\rho)) \subset W_s(F(\rho))$  and  $F^{-1}(W_u(\rho)) \subset W_u(F^{-1}(\rho))$
- (5) (a) For each  $\rho' \in W_s(\rho)$ ,  $d(F^n(\rho), F^n(\rho')) \rightarrow 0$ .  
(b) For each  $\rho' \in W_u(\rho)$ ,  $d(F^{-n}(\rho), F^{-n}(\rho')) \rightarrow 0$ .
- (6) Let  $\theta > 0$  satisfying  $e^{-\lambda_0} < \theta < 1$ . There exists  $C > 0$  (independent of  $\varepsilon_1$ ) such that the following holds :
  - (a) If  $\rho' \in U$  satisfies  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$  for all  $i = 0, \dots, n$  then  $d(\rho', W_s(\rho)) \leq C\theta^n \varepsilon_1$  and for  $0 \leq i \leq n$ ,  $d(F^i(\rho), F^i(\rho')) \leq C\varepsilon_1 \theta^{\min(i, n-i)}$ .
  - (b) If  $\rho' \in U$  satisfies  $d(F^{-i}(\rho), F^{-i}(\rho')) \leq \varepsilon_1$  for all  $i = 0, \dots, n$  then  $d(\rho', W_u(\rho)) \leq C\theta^n \varepsilon_1$  and for  $0 \leq i \leq n$ ,  $d(F^{-i}(\rho), F^{-i}(\rho')) \leq C\varepsilon_1 \theta^{\min(i, n-i)}$ .
- (7) If  $\rho, \rho' \in \mathcal{T}$  satisfy  $d(\rho, \rho') \leq \varepsilon_1$ , then  $W_u(\rho) \cap W_s(\rho')$  consists of exactly one point of  $\mathcal{T}$ .

Below, we will require that  $C\varepsilon_1 < 1$ . Up to making  $\varepsilon_1$  smaller, we assume that this condition holds.

For our purpose, we will need a more precise version of these results. The following lemmas are an adaptation of Lemma 2.1 in [DJN21] to our setting.

**Lemma 3-2.2.** There exist constants  $\varepsilon_1 > 0$  and  $C > 0$  depending only on  $(U, F)$ , such that for all  $\rho, \rho' \in U$ ,

- (1) if  $\rho \in \mathcal{T}$  and  $\rho' \in W_s(\rho)$  satisfy  $d(\rho, \rho') \leq \varepsilon_1$ , then

$$C^{-1}J_n^s(\rho)d(\rho, \rho') \leq d(F^n(\rho), F^n(\rho')) \leq CJ_n^s(\rho)d(\rho, \rho') \quad , \quad \forall n \in \mathbb{N} \quad (3-2.7)$$

- (2) if  $\rho \in \mathcal{T}$  and  $\rho' \in W_u(\rho)$  satisfy  $d(\rho, \rho') \leq \varepsilon_1$ , then

$$C^{-1}J_{-n}^u(\rho)d(\rho, \rho') \leq d(F^{-n}(\rho), F^{-n}(\rho')) \leq CJ_{-n}^u(\rho)d(\rho, \rho') \quad , \quad \forall n \in \mathbb{N} \quad (3-2.8)$$

*Proof.* We prove (1). (2) is proved in a similar way by inverting the time direction. Let  $\rho \in \mathcal{T}, \rho' \in W_s(\rho)$ . Since  $T_\rho(W_s(\rho)) = E_s(\rho)$  and  $d_\rho F(E_s(\rho)) = E_s(F(\rho))$ , the Taylor development of  $F$  along  $W_s(\rho)$  gives :

$$d(F(\rho), F(\rho')) = J_1^s(\rho)d(\rho, \rho') + O(d(\rho, \rho')^2) = J_1^s(\rho)d(\rho, \rho')(1 + O(d(\rho, \rho'))) \quad (3-2.9)$$

since  $J_1^s \geq e^{-\lambda_1}$ . Applying this equality with  $F^k(\rho)$  and  $F^k(\rho')$  instead of  $\rho$  and  $\rho'$ , and recalling that, by lemma 3-2.1,  $d(F^k(\rho), F^k(\rho')) \leq C\theta^k d(\rho, \rho')$ , we can write,

$$d(F^{k+1}(\rho), F^{k+1}(\rho')) = J_1^s(F^k(\rho))d(F^k(\rho), F^k(\rho'))(1 + O(\theta^k \varepsilon_1)) \quad (3-2.10)$$

By this last inequality and the chain rule, we have

$$J_n^s(\rho)d(\rho, \rho') \prod_{k=0}^{n-1} (1 - C\theta^k \varepsilon_1) \leq d(F^n(\rho), F^n(\rho')) \leq J_n^s(\rho)d(\rho, \rho') \prod_{k=0}^{n-1} (1 + C\theta^k \varepsilon_1) \quad (3-2.11)$$

We conclude by noting that

$$\prod_{k=0}^{n-1} (1 + C\theta^k \varepsilon_1) \leq \prod_{k=0}^{+\infty} (1 + C\theta^k \varepsilon_1) < +\infty ; \quad \prod_{k=0}^{n-1} (1 - C\theta^k \varepsilon_1) \geq \prod_{k=0}^{+\infty} (1 - C\theta^k \varepsilon_1) \geq C^{-1}$$

(note that in the last inequality and in (3-2.11), we need to ensure that  $\varepsilon_1 C < 1$  so that the product is effectively non zero).  $\square$

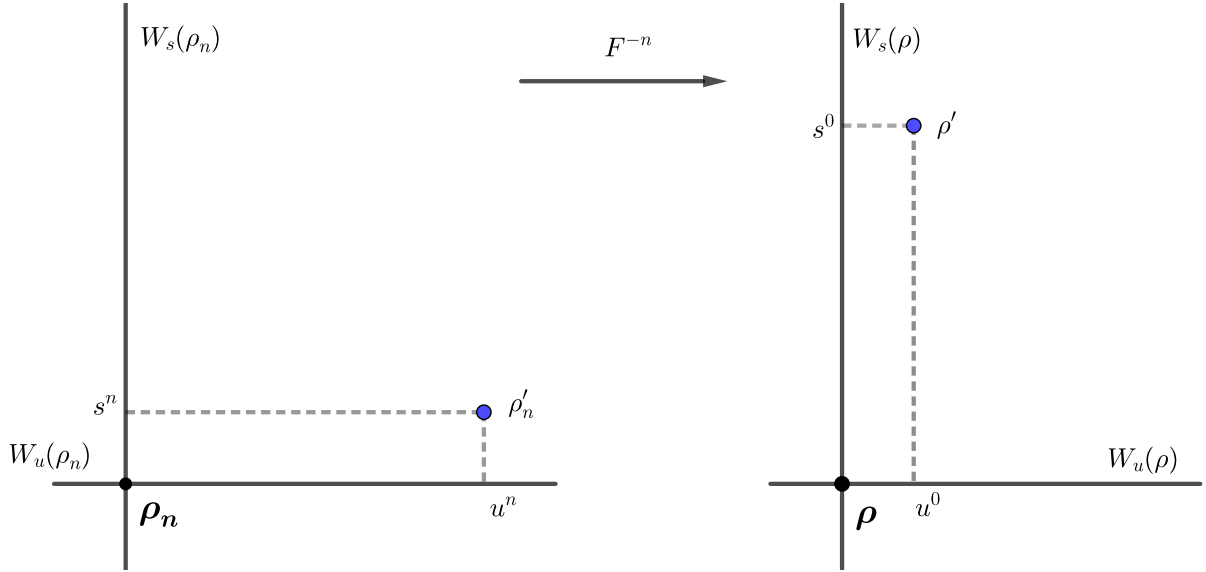


Figure 3.6: Framework for the proof of Lemma 3-2.3

The following lemma gives a stronger version of (6) in Lemma 3-2.1.

**Lemma 3-2.3.** There exist  $C > 0$  and  $\varepsilon_1 > 0$ , depending only on  $(U, F)$ , such that for all  $\rho, \rho' \in U$  and  $n \in \mathbb{N}$  :

- (1) if  $\rho \in \mathcal{T}$  and  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$  for all  $i \in \{0, \dots, n\}$  then

$$d(\rho', W_s(\rho)) \leq \frac{C}{J_n^u(\rho)} \quad (3-2.12)$$

and

$$\|d_{\rho'} F^n\| \leq C J_n^u(\rho) \quad (3-2.13)$$

- (2) if  $\rho \in \mathcal{T}$  and  $d(F^{-i}(\rho), F^{-i}(\rho')) \leq \varepsilon_1$  for all  $i \in \{0, \dots, n\}$  then

$$d(\rho', W_u(\rho)) \leq \frac{C}{J_{-n}^s(\rho)} \quad (3-2.14)$$

and

$$\|d_{\rho'} F^{-n}\| \leq C J_{-n}^s(\rho) \quad (3-2.15)$$

*Proof.* We prove (1). (2) is proved in a similar way by inverting the time direction. Let  $\rho \in \mathcal{T}$  and  $\rho' \in U$  such that  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$  for  $0 \leq i \leq n$  with  $\varepsilon_1$  to be determined. Denote  $\rho_k = F^k(\rho)$ . The first condition on  $\varepsilon_1$  is that it is smaller than the one of lemma 3-2.1 so that we ensure the following estimates : for  $k \in \{0, \dots, n\}$

$$d(F^k(\rho'), W_s(F^k(\rho))) \leq C\theta^{n-k}\varepsilon_1 \quad (3-2.16)$$

$$d(F^k(\rho'), W_s(F^k(\rho))) \leq C\theta^k\varepsilon_1 \quad (3-2.17)$$

We will use coordinates charts  $\kappa_k : \hat{\rho} \in U_k \mapsto (u^k, s^k) \in V_k$  adapted to the dynamical system (see [HK95], Theorem 6.2.3, the explanations below and Theorem 6.2.8 for the existence of this chart). More precisely, we want these charts to satisfy

- $\kappa_k(\rho_k) = (0, 0)$

<sup>1</sup>in other words, there exists a smooth curve  $\gamma : [-\delta, \delta] \rightarrow U$  such that  $\gamma(0) = \rho$ ,  $\text{ran}\gamma \subset W_{u/s}(\rho)$  and  $\overline{B(\rho, \varepsilon_1)} \cap W_{i/s}(\rho) = \gamma([-\delta/2, \delta/2])$  : it means that the size of the unstable and stable manifolds is bounded from below uniformly.

- $\kappa_k(W_s(\rho_k) \cap U_k) = \{(0, s), s \in \mathbb{R}\} \cap V_k$
- $\kappa_k(W_u(\rho_k) \cap U_k) = \{(u, 0), u \in \mathbb{R}\} \cap V_k$
- For  $\hat{\rho} \in U_k$ ,  $|u^k| \sim d(\hat{\rho}, W_s(\rho_k)); |s^k| \sim d(\hat{\rho}, W_u(\rho_k)); |s^k|^2 + |u^k|^2 \sim d(\rho_k, \hat{\rho})^2$ .
- $(\kappa_k)_{0 \leq k \leq n}$  are uniformly bounded in the  $C^N$  topology for all  $N$ , independently of  $\rho_0$  and  $n$ . In particular, we may assume that  $\varepsilon_1$  is chosen small enough so that  $B(\rho_k, \varepsilon_1) \subset U_k$  for all  $0 \leq k \leq n$ .
- Up to changing the metric we work with (which is not problematic), we may assume that the restrictions of  $d\kappa_k(\rho)$  to  $E_s(\rho)$  and  $E_u(\rho)$  are isometries for the metrics  $|\cdot|_s$  and  $|\cdot|_u$ .

If we note  $\widetilde{F}_k = \kappa_k \circ F \circ \kappa_{k-1}^{-1}$ , we can check that in this pair of coordinates charts, the action of  $F^{-1}$  is given by

$$\widetilde{F}_k^{-1}(u^k, s^k) = (\pm J_{-1}^u(\rho_k)u^k + \alpha_k(u^k, s^k), \pm J_{-1}^s(\rho_k)s^k + \beta_k(u^k, s^k)) \quad (3-2.18)$$

where  $\alpha_k, \beta_k$  are smooth functions, uniformly bounded in  $k$  for the  $C^2$  topology and such that  $\alpha_k(0, s^k) = 0, \beta_k(u^k, 0) = 0, d\alpha_k(0, 0) = 0, d\beta_k(0, 0) = 0$ .

With these properties, one can check that

$$\alpha_k(u^k, s^k) \leq C|u^k| \|(u^k, s^k)\| \quad (3-2.19)$$

Let's now denote  $\rho'_k = F^k(\rho')$  and  $(u^k, s^k) = \kappa_k(\rho'_k)$ . By (3-2.16), (3-2.17), (3-2.18), (3-2.19), we can write

$$\begin{aligned} |u^{k-1}| &\leq J_{-1}^u(\rho_k)|u^k| + C|u^k| \|(u^k, s^k)\| \\ &\leq J_{-1}^u(F^k(\rho))|u^k| (1 + C\varepsilon_1(\theta_1^k + \theta_1^{n-k})) \\ &\leq J_{-1}^u(F^k(\rho))|u^k| \left(1 + C\varepsilon_1\theta^{\min(k, n-k)}\right) \end{aligned}$$

Then, using the chain rule, one has

$$d(\rho', W_s(\rho)) \leq C|u^0| \leq C J_{-n}^u(F^n(\rho)) \prod_{k=0}^{n-1} \left(1 + C\varepsilon_1\theta^{\min(k, n-k)}\right) \quad (3-2.20)$$

Finally, we can estimate

$$\prod_{k=0}^n \left(1 + C\varepsilon_1\theta^{\min(k, n-k)}\right) \leq \prod_{k=0}^{\lceil n/2 \rceil} (1 + C\varepsilon_1\theta^k)^2 \leq C$$

which gives

$$d(\rho', W_s(\rho)) \leq C J_{-n}^u(F^n(\rho)) = \frac{C}{J_n^u(\rho)} \quad (3-2.21)$$

This proves (3-2.12).

To prove (3-2.13), we first construct a metric which simplifies the computations. If  $\rho \in \mathcal{T}$ , we pick  $v_*(\rho) \in E_*(\rho)$ <sup>2</sup> such that  $\|v_*(\rho)\| = 1$ . There exists a Riemannian metric  $|\cdot|$  on  $\mathcal{T}$  such that for every  $\rho \in \mathcal{T}$ ,  $(v_u(\rho), v_s(\rho))$  is an orthonormal basis of  $T_\rho U$ . This metric is  $\gamma$ -Hölder in  $\rho \in \mathcal{T}$  since stable and unstable distributions are  $\gamma$ -Hölder for some  $\gamma \in (0, 1)$ .

If  $\rho \in \mathcal{T}$  and  $n \in \mathbb{Z}$ , we note  $\tilde{J}_n^{u/s}(\rho) \in \mathbb{R}$  the numbers such that

$$d_\rho(F^n)(v_u(\rho)) = \tilde{J}_n^u(\rho)v_u(F^n(\rho)); d_\rho(F^n)(v_s(\rho)) = \tilde{J}_n^s(\rho)v_s(F^n(\rho))$$

As already observed,  $|\tilde{J}_n^u(\rho)| \sim J_n^u(\rho)$ , for all  $n$  (with constants independent of  $n$ ). We can also assume that  $|\tilde{J}_1^u(\rho)| > |\tilde{J}_1^s(\rho)|$  for all  $\rho$ . In the orthonormal basis  $(v_u(\rho), v_s(\rho))$  and  $(v_u(F^n(\rho)), v_s(F^n(\rho)))$ ,  $d_\rho F^n$  has the form

$$\begin{pmatrix} \tilde{J}_n^u(\rho) & 0 \\ 0 & \tilde{J}_n^s(\rho) \end{pmatrix}$$

<sup>2</sup>Here, we are not concerned by the orientation. It is simply a matter of direction.

Due to the orthonormality of these basis, we have that for the subordinate norms,  $\|d_\rho F^n\| = |\tilde{J}_n^u(\rho)|$ . Hence, the chain rule implies the following equality for this particular Riemannian metric defined on  $\mathcal{T}$  :

$$\forall \rho \in \mathcal{T}, \|d_\rho(F^n)\| = |\tilde{J}_n^u(\rho)| = \prod_{i=0}^{n-1} |\tilde{J}_1^u(F^i(\rho))| = \prod_{i=0}^{n-1} \|d_{F^i(\rho)}F\| \quad (3-2.22)$$

We now claim that we can extend  $|\cdot|$  to a relatively compact neighborhood  $V$  of  $\mathcal{T}$  such that  $\rho \in V \mapsto |\cdot|_\rho$  is still  $\gamma$ -Hölder. To do so, it is enough to extend the coefficients of the metric in a coordinate chart in a  $\gamma$ -Hölder way, which is possible (for instance, in virtue of Corollary 1 in [McS34]), which still defines a non-degenerate 2-form in a sufficiently small neighborhood of  $\mathcal{T}$ .

We now aim at proving (3-2.13) for this particular metric. (3-2.13) will hold in the general case since two continuous metric are always uniformly equivalent in a compact neighborhood of  $\mathcal{T}$ .

In the following, we assume that  $\varepsilon_1$  is small enough so that  $\rho$  belongs to the neighborhood of  $\mathcal{T}$  in which  $|\cdot|$  is defined. Since  $\rho \mapsto \|d_\rho F\|_{T_\rho U \rightarrow T_{F(\rho)}U}$  is  $\gamma$ -Hölder (in the following, we will drop the subscript in the norm) we have, for all  $i \in \{0, \dots, n\}$

$$\left| \|d_{F^i(\rho')}F\| - \|d_{F^i(\rho)}F\| \right| \leq Cd(F^i(\rho'), F^i(\rho))^\gamma \leq C\varepsilon_1 \theta^{\gamma \min(i, n-i)} \quad (3-2.23)$$

Using the chain rule and the submultiplicativity of  $\|\cdot\|$ , we have

$$\|d_{\rho'}F^n\| \leq \prod_{i=0}^n \|d_{F^i(\rho')}F\| \leq \prod_{i=0}^n \|d_{F^i(\rho)}F\| \left(1 + C\varepsilon_1 \theta^{\gamma \min(i, n-i)}\right) \quad (3-2.24)$$

Eventually, by (3-2.22) and the fact that  $\prod_{i=0}^n (1 + C\varepsilon_1 \theta^{\gamma \min(i, n-i)}) \leq \prod_{i=0}^\infty (1 + C\varepsilon_1 \theta^i)^2 < +\infty$ , (3-2.13) holds.  $\square$

As an immediate consequence of this lemma, we get :

**Corollary 3-2.1.** There exist  $C > 0$  and  $\varepsilon_1 > 0$  (depending only on  $(U, F)$ ) such that for all  $\rho, \rho' \in \mathcal{T}$  and  $n \in \mathbb{N}$  :

(1) if  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$  for all  $i \in \{0, \dots, n\}$  then

$$C^{-1}J_n^u(\rho) \leq J_n^u(\rho') \leq CJ_n^u(\rho) \quad (3-2.25)$$

(2) if  $d(F^{-i}(\rho), F^{-i}(\rho')) \leq \varepsilon_1$  for all  $i \in \{0, \dots, n\}$  then

$$C^{-1}J_{-n}^s(\rho) \leq J_{-n}^s(\rho') \leq CJ_{-n}^s(\rho) \quad (3-2.26)$$

*Proof.* This is a consequence of the previous lemma and of the fact that uniformly in  $\rho$  and  $n \in \mathbb{N}$ ,

$$\begin{aligned} \|d_\rho F^n\| &\sim J_n^u(\rho) \\ \|d_\rho F^{-n}\| &\sim J_{-n}^s(\rho) \end{aligned}$$

$\square$

### 3-3 Regularity of the invariant splitting.

It is known for Anosov diffeomorphisms that stable and unstable distributions are in fact  $C^{2-\varepsilon}$  in dimension 2 (see [HK90]). For our purpose, we need to extend this result to our setting, where the hyperbolic invariant set  $\mathcal{T}$  is not the full phase space, but a fractal subset of it. In fact, we will show that one can extend the stable and unstable distributions to an open neighborhood of  $\mathcal{T}$  and that these extensions are  $C^{1,\beta}$  for some  $\beta > 0$ . Actually, since what happens outside a fixed neighborhood of  $\mathcal{T}$  is irrelevant (one can always use cut-offs), we will prove the following theorem which might be of independent interest.

**Theorem 3-3.1.** *Let us denote  $\mathcal{G}_1(U)$  the Grassmanian bundle of 1-plane in  $TU$ . There exists  $\beta > 0$  and sections  $E_u, E_s : U \rightarrow \mathcal{G}_1(U)$  such that :*

- For every  $\rho \in \mathcal{T}$ ,  $E_u(\rho)$  (resp.  $E_s(\rho)$ ) is the unstable (resp. stable) distribution at  $\rho$  ;
- $E_u$  and  $E_s$  have regularity  $C^{1,\beta}$

**Remark.**

Our proof relies on the techniques of [HP69]. In fact, in [HK95] 19.1.d, the authors show how one can obtain  $C^1$  regularity of the map  $\rho \in \mathcal{T} \mapsto E_u(\rho)$  and explains how to prove  $C^{1,\beta}$  regularity. Their notion of differentiability on the set  $\mathcal{T}$  (which is clearly not open in our case) relies on the existence of linear approximations. Here, we choose to show a slightly different version of this regularity by proving that  $\rho \in \mathcal{T} \mapsto E_u(\rho)$  can be obtained as the restriction of a  $C^{1,\beta}$  map defined in an open neighborhood of  $\mathcal{T}$ .

### 3-3.1. Proof of the $C^{1,\beta}$ regularity.

**Preliminaries.** We recall that  $\mathcal{T}$  is an invariant hyperbolic set for  $F$ . Hence, there exists a continuous splitting of  $T_{\mathcal{T}}U$ , into stable and unstable spaces  $\rho \in \mathcal{T} \mapsto E_s(\rho), \rho \in \mathcal{T} \mapsto E_u(\rho)$ . We use a continuous Riemannian metric on  $T_{\mathcal{T}}U$  such that  $d_{\rho}F$  is a contraction from  $E_s(\rho) \rightarrow E_s(F(\rho))$  and expanding from  $E_u(\rho) \rightarrow E_u(F(\rho))$ , and making  $E_u(\rho)$  and  $E_s(\rho)$  orthogonal.

Let  $\rho \in \mathcal{T} \mapsto e_u(\rho) \in TU$  and  $\rho \in \mathcal{T} \mapsto e_s(\rho) \in TU$  be two continuous sections<sup>3</sup> such that, for every  $\rho \in \mathcal{T}$ ,

- $e_u(\rho)$  spans  $E_u(\rho)$ ,
- $e_s(\rho)$  spans  $E_s(\rho)$ ,
- $\|e_u(\rho)\| = 1, \|e_s(\rho)\| = 1$

The matrix representation of  $d_{\rho}F$ <sup>4</sup> in these basis is

$$d_{\rho}F = \begin{pmatrix} \tilde{J}^u(\rho) & 0 \\ 0 & \tilde{J}^s(\rho) \end{pmatrix}$$

with  $\nu := \sup_{\rho \in \mathcal{T}} \max \left[ \left( |\tilde{J}^u(\rho)| \right)^{-1}, |\tilde{J}^s(\rho)| \right] < 1$ .

We can extend  $e_u$  and  $e_s$  to  $U$  to continuous functions, still denoted  $e_u$  and  $e_s$ . Let us consider smooth vector fields  $v_u$  and  $v_s$  on  $U$  approximating  $e_u$  and  $e_s$  and a smooth Riemannian metric approximating the one considered above. By slightly modifying this vector fields, we can assume that for this new metric,  $(v_u(\rho), v_s(\rho))$  is an orthonormal basis for all  $\rho \in U$ . In these new basis, we now write

$$d_{\rho}F = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix}$$

We assume that  $v_u$  and  $v_s$  are sufficiently close to  $e_u$  and  $e_s$  to ensure that, for some  $\eta > 0$  small enough,

$$\begin{aligned} \sup_{\rho \in \mathcal{T}} \max (|b(\rho)|, |c(\rho)|) &\leq \eta \\ \sup_{\rho \in \mathcal{T}} |d(\rho)| &\leq \nu + \eta \leq 1 - 4\eta \\ \inf_{\rho \in \mathcal{T}} |a(\rho)| &\geq \nu^{-1} - \eta \geq 1 + 4\eta \end{aligned}$$

<sup>3</sup>Note that there is no problem of orientation to construct such global sections. Indeed,  $\mathcal{T}$  is totally disconnected and hence, one can cover  $\mathcal{T}$  by a disjoint union of open sets small enough so that it is possible to construct local sections in each such sets. Since these open sets are disjoint, these local sections allow us to build a global continuous section.

<sup>4</sup>The definition of  $\tilde{J}^{u/s}$  may differ from the one of  $J_1^{u/s}$  above since we don't work *a priori* with the same metric.



We consider an open neighborhood  $\Omega$  of  $\mathcal{T}$  such that the following holds :

$$\begin{aligned} \sup_{\rho \in \Omega} \max(|b(\rho)|, |c(\rho)|) &\leq 2\eta \\ \sup_{\rho \in \Omega} |d(\rho)| &\leq \nu + 2\eta \leq 1 - 3\eta \\ \inf_{\rho \in \Omega} |a(\rho)| &\geq \nu^{-1} - 2\eta \geq 1 + 3\eta \end{aligned}$$

Our method relies on different uses of the Contraction Map Theorem. We state the Fiber Contraction Theorem of [HP69] (Section 1), which will be used further. We recall that a fixed point  $x_0$  of a continuous map  $f : X \rightarrow X$  is said to be *attractive* if for every  $x \in X$ ,  $f^n(x) \rightarrow x_0$ .

**Theorem 3-3.2. Fiber Contraction Theorem**

Let  $(X, d)$  be a metric space and  $h : X \rightarrow X$  a map having an attractive fixed point  $x_0$ . Let us consider  $Y$  another metric space and a family of maps  $(g_x : Y \rightarrow Y)_{x \in X}$  and denote by  $H$  the map

$$H : (x, y) \in X \times Y \mapsto (h(x), g_x(y)) \in X \times Y$$

Assume that

- $H$  is continuous ;
- For all  $x \in X$ ,  $\limsup_{n \rightarrow +\infty} L(g_{h^n(x)}) < 1$  where  $L(g_{h^n(x)})$  denotes the best Lipschitz constant for  $g_{h^n(x)}$  ;
- $y_0$  is an attractive fixed point for  $g_{x_0}$ .

Then  $(x_0, y_0)$  is an attractive fixed point for  $H$ .

In the following, we study the regularity of the unstable distribution. The same holds for the stable distribution by changing the roles of  $F^{-1}$  and  $F$ .

**$E_u$  is a fixed point of a contraction.** By our assumption on  $v_u$  and  $v_s$ , there exists a continuous function  $\lambda : U \rightarrow \mathbb{R}$  such that

$$\mathbb{R}e_u(\rho) = \mathbb{R}(v_u(\rho) + \lambda(\rho)v_s(\rho))$$

Hence, we will represent the extension of the unstable distribution by a continuous map  $\lambda : \Omega \rightarrow \mathbb{R}$ . Our aim is to show that we can find  $\lambda$  regular enough such that for  $\rho \in \mathcal{T}$ ,

$$E_u(\rho) = \mathbb{R}(v_u(\rho) + \lambda(\rho)v_s(\rho))$$

To do so, we will start by constructing  $\lambda$  as a fixed point of a contraction in a nice space. This contraction will be related to invariance properties of the unstable distribution.

First of all, if  $\rho' = F(\rho) \in \Omega \cap F(\Omega)$ , and if  $v = v_u(\rho) + \lambda v_s(\rho)$ ,  $d_\rho F$  maps  $v$  to

$$w = (a(\rho) + \lambda b(\rho))v_u(\rho') + (c(\rho) + \lambda d(\rho))v_s(\rho')$$

Hence, the line of  $T_\rho U$  represented by  $\lambda$  is sent to the line represented by  $t(\rho, \lambda)$  in  $T_{\rho'} U$  where

$$t(\rho, \lambda) = \frac{\lambda d(\rho) + c(\rho)}{a(\rho) + \lambda b(\rho)} \tag{3-3.1}$$

Set  $\Omega_1 = \Omega \cap F(\Omega)$  and let us consider a cut-off function  $\chi \in C_c^\infty(\Omega_1)$  such that  $0 \leq \chi \leq 1$  and  $\chi \equiv 1$  in a neighborhood of  $\mathcal{T}$ . Let us introduce the complete metric space

$$X = \{\lambda \in C(\Omega, \mathbb{R}), \|\lambda\|_\infty \leq 1\}$$

and consider the map  $T : X \rightarrow X$  defined, for  $\lambda \in X$  and  $\rho' \in \Omega$ ,

$$(T\lambda)(\rho') = \chi(\rho')t(F^{-1}(\rho'), \lambda(F^{-1}(\rho'))) \tag{3-3.2}$$

To see that this is well defined, first note that  $F^{-1}$  is well defined on  $\text{supp } \chi$  and  $F^{-1}(\text{supp } \chi) \subset \Omega$ . It is clear that if  $\lambda \in X$ ,  $T\lambda$  is continuous. To see that  $\|T\lambda\|_\infty \leq 1$ , it is enough to note that if  $\rho \in \Omega$  and  $|\lambda| \leq 1$ ,

$$|t(\rho, \lambda)| \leq \frac{|d(\rho)| + |c(\rho)|}{|a(\rho)| - |b(\rho)|} \leq \frac{1 - 3\eta + 2\eta}{1 + 3\eta - 2\eta} \leq \frac{1 - \eta}{1 + \eta} < 1$$

Let us now prove the following

**Proposition 3-3.1.**

- $T$  is a contraction.
- If  $\lambda_u$  denotes its unique fixed point, then, for every  $\rho \in \mathcal{T}$ ,  $E_u(\rho) = \mathbb{R}\left(v_u(\rho) + \lambda_u(\rho)v_s(\rho)\right)$

*Proof.* Let  $\lambda, \mu \in X$ . If  $\rho' \in \Omega \setminus \text{supp } \chi$ , we have  $T\mu(\rho') = T\lambda(\rho') = 0$ . Now assume that  $\rho' \in \text{supp } \chi$  and write  $\rho' = F(\rho)$  with  $\rho \in \Omega$ .

$$|T\lambda(\rho') - T\mu(\rho')| = |\chi(\rho')||t(\rho, \lambda(\rho)) - t(\rho, \mu(\rho))| \leq |t(\rho, \lambda(\rho)) - t(\rho, \mu(\rho))|$$

The map  $\lambda \in [-1, 1] \mapsto t(\rho, \lambda)$  is smooth, so that we can write

$$\|T\lambda - T\mu\|_\infty \leq \sup_{\rho' \in \text{supp } \chi} |T\lambda(\rho') - T\mu(\rho')| \leq \sup_{\Omega \times [-1, 1]} |\partial_\lambda t| \times \|\lambda - \mu\|_\infty$$

It is then enough to show that  $\sup_{\Omega \times [-1, 1]} |\partial_\lambda t| < 1$ . For  $(\rho, \lambda) \in \Omega \times [-1, 1]$ , we have

$$\partial_\lambda t(\rho, \lambda) = \frac{d(\rho)}{a(\rho) + \lambda b(\rho)} - b(\rho) \frac{\lambda d(\rho) + c(\rho)}{(a(\rho) + \lambda b(\rho))^2} \quad (3-3.3)$$

Hence, we can control

$$|\partial_\lambda t(\rho, \lambda)| \leq \frac{1 - 3\eta}{1 + \eta} + \eta \frac{1 - \eta}{(1 + \eta)^2} = \kappa_\eta < 1$$

if  $\eta$  is small enough. This demonstrates that  $T$  is a contraction.

As a consequence,  $T$  has a unique fixed point,  $\lambda_u$ . We note  $v(\rho) = v_u(\rho) + \lambda_u(\rho)v_s(\rho)$ . We want to show that  $v(\rho) \in \mathbb{R}e_u(\rho)$  for  $\rho \in \mathcal{T}$  (recall that  $e_u : U \rightarrow TU$  is continuous and that  $e_u(\rho)$  spans  $E_u(\rho)$  if  $\rho \in \mathcal{T}$ ). Since  $\chi = 1$  on  $\mathcal{T}$ , we see by definition of  $T$  that for every  $\rho \in \mathcal{T}$ ,

$$d_\rho F(v(\rho)) \in \mathbb{R}v(F(\rho)) \quad (3-3.4)$$

If  $v_u$  is sufficiently close to  $e_u$ , we can find a continuous and bounded function  $\mu$  such that

$$\mathbb{R}v(x) = \mathbb{R}(e_u(x) + \mu(x)e_s(x))$$

From (3-3.4), if  $\rho' = F(\rho) \in \mathcal{T}$ ,

$$d_\rho F\left(e_u(\rho) + \mu(\rho)e_s(\rho)\right) = \tilde{J}_1^u(\rho) \left( e_u(\rho') + \mu(\rho) \frac{\tilde{J}_1^s(\rho)}{\tilde{J}_1^u(\rho)} e_s(\rho') \right) \in \mathbb{R}\left(e_u(\rho') + \mu(\rho')e_s(\rho')\right)$$

This implies the equality

$$\mu(\rho') = \mu(\rho) \frac{\tilde{J}_1^s(\rho)}{\tilde{J}_1^u(\rho)} \quad (3-3.5)$$

This equality implies that  $\mu = 0$  on  $\mathcal{T}$  and hence,  $v = e_u$  on  $\mathcal{T}$ , as expected.  $\square$

**Remark.**

As long as  $\rho' \in \{\chi = 1\}$ , the vector field  $v(\rho') = v_u(\rho') + \lambda(\rho')v_s(\rho')$  is invariant by  $dF$ . When  $\rho' \in W_u(\rho) \cap \{\chi = 1\}$  for some  $\rho \in \mathcal{T}$ , we will see below that the direction given by  $v(\rho')$  coincides with the tangent space to  $W_u(\rho)$ , namely  $T_{\rho'}W_u(\rho) = \mathbb{R}v(\rho')$ . When  $\rho' \notin \bigcup_{\rho \in \mathcal{T}} W_u(\rho)$ , there exists  $n \in \mathbb{N}$  such that  $F^{-n}(\rho') \notin \text{supp } \chi$ . Hence,  $\lambda_u(\rho')$  is given by an explicit expression obtained by iterating the fixed point formula.

**Differentiability of  $\lambda_u$ .** We go on by showing that  $\lambda$  is  $C^1$  by adapting the method of [HP69]. We now introduce the Banach space  $Y$  of bounded continuous sections  $\alpha : \Omega \rightarrow T^*\Omega$ . We will use the norm on  $T^*\Omega$  adapted to the metric on  $T\Omega$ , namely if  $\alpha \in Y$ ,

$$\|\alpha\|_Y = \sup_{\rho \in \Omega} \sup_{v \in T_\rho\Omega, v \neq 0} \frac{|\alpha(\rho)(v)|}{\|v\|_{T_\rho\Omega}}$$

For  $\lambda \in X$ , let us introduce the map  $G_\lambda : Y \rightarrow Y$ , defined as follows. For  $\alpha \in Y$  and  $\rho' \in \Omega$ ,

$$(G_\lambda \alpha)(\rho') = \chi(\rho') \left[ d_\rho t(\rho, \lambda(\rho)) + \partial_\lambda t(\rho, \lambda(\rho)) \alpha(\rho) \right] \circ (d_\rho F)^{-1} + t(\rho, \lambda(\rho)) d_{\rho'} \chi \quad (3-3.6)$$

with  $\rho = F^{-1}(\rho')$ , which is well defined since  $\rho \in \Omega$  if  $\rho' \in \text{supp}(\chi)$ .  $G_\lambda$  is constructed to satisfy : for  $\lambda \in X$ , if  $\lambda$  is  $C^1$ , then the following relation holds :

$$G_\lambda(d\lambda) = d(T\lambda) \quad (3-3.7)$$

Let us first state the key tool to show the differentiability of  $\lambda_u$ .

**Proposition 3-3.2.** For every  $\lambda \in X$ ,  $G_\lambda$  is a contraction with Lipschitz constant  $L_\lambda$  satisfying

$$\sup_{\lambda \in X} L_\lambda < 1$$

Before proving it, let us show how it leads us to

**Proposition 3-3.3.**  $\lambda_u$  is  $C^1$ .

*Proof.* We use the Contraction Fiber Theorem. Let  $\alpha_u$  be the unique fixed point of  $G_{\lambda_u}$ . The map

$$H : (\lambda, \alpha) \in X \times Y \mapsto (T\lambda, G_\lambda \alpha) \in X \times Y$$

is continuous and the previous proposition shows that for every  $\lambda \in X$ ,  $\sup_n L(G_{T^n \lambda}) < 1$ . The Contraction Fiber Theorem implies that  $(\lambda_u, \alpha_u)$  is an attractive fixed point for  $H$ .

Let  $\lambda \in X$  be  $C^1$ . Hence,  $H^n(\lambda, d\lambda) \rightarrow (\lambda_u, \alpha_u)$ . But  $H^n(\lambda, d\lambda) = (T^n \lambda, \alpha_n)$  with

$$\alpha_n = G_{T^{n-1} \lambda} \circ \dots \circ G_\lambda d\lambda$$

It is clear that if  $\lambda \in C^1$ , so is  $T\lambda$  and an iterative use of (3-3.7) implies that  $\alpha_n = d(T^n \lambda)$ . This shows that  $\lambda_u$  is  $C^1$  and  $d\lambda_u = \alpha_u$ .  $\square$

Let us now prove Proposition 3-3.2.

*Proof.* Let  $\lambda \in X$  and fix  $\alpha, \beta \in Y$ . It is of course enough to control  $\|G_\lambda \alpha(\rho') - G_\lambda \beta(\rho')\|$  for  $\rho' \in \text{supp}(\chi)$  since both  $G_\lambda \alpha$  and  $G_\lambda \beta$  vanishes outside. Let us fix  $\rho' = F(\rho) \in \text{supp}(\chi)$ .  $G_\lambda \alpha(\rho') - G_\lambda \beta(\rho')$  is given by

$$\chi(\rho') \partial_\lambda t(\rho, \lambda(\rho)) [\alpha(\rho) - \beta(\rho)] \circ (d_\rho F)^{-1}$$

so it is enough to control  $\partial_\lambda t(\rho, \lambda(\rho)) \gamma(\rho) \circ (d_\rho F)^{-1}$  for  $\gamma = \alpha - \beta$ . With the precise expression of  $\partial_\lambda t(\rho, \lambda(\rho))$  given by (3-3.3), we can estimate

$$|\partial_\lambda t(\rho, \lambda(\rho))| = \frac{|d(\rho)|}{|a(\rho) + \lambda(\rho)b(\rho)|} + O_\nu(\eta) = \frac{|d(\rho)|}{|a(\rho)|} + O_\nu(\eta)$$

(By the notation  $O_\nu(\eta)$ , we mean that this term is bounded by  $C\eta$  where  $C$  is a constant depending only on  $\nu$  and  $(F, U)$ ).

Moreover, we have  $\|(d_\rho F)^{-1}\| = \max\left(\frac{1}{a(\rho)}, \frac{1}{d(\rho)}\right) + O_\nu(\eta) = \frac{1}{d(\rho)} + O_\nu(\eta)$ . Hence,

$$\|\partial_\lambda t(\rho, \lambda(\rho))\gamma(\rho) \circ (d_\rho F)^{-1}\| \leq \left(\frac{1}{a(\rho)} + O_\nu(\eta)\right) \|\gamma(\rho)\| \leq (\nu + O_\nu(\eta)) \|\gamma\|_Y$$

Hence, if  $\eta$  is small enough, the proposition is proved.  $\square$

**Hölder regularity of  $\alpha_u$ .** In fact, as explained at the end of 19.1.d in [HK95], we can improve the  $C^1$  regularity.

To deal with Hölder regularity of sections  $\alpha : \Omega \rightarrow T^*\Omega$ , we will simply evaluate the distance between  $\alpha(\rho_1)$  and  $\alpha(\rho_2)$  for  $\rho_1, \rho_2 \in \Omega$  using the natural identification  $T^*\Omega = \Omega \times (\mathbb{R}^2)^*$ , where we see  $\alpha(\rho_1)$  as an element of  $(\mathbb{R}^2)^*$ . This allows us to write  $\alpha(\rho_1) - \alpha(\rho_2)$  and compute  $\|\alpha(\rho_1) - \alpha(\rho_2)\|$  where  $\|\cdot\|$  is a norm on  $(\mathbb{R}^2)^*$ . There exists  $C > 0$  such that for every  $\alpha \in Y$ ,  $\sup_{\rho \in \Omega} \|\alpha(\rho)\| \leq C\|\alpha\|_Y$ .

Let us introduce  $\mu$  a Lipschitz constant for  $F^{-1}$  on  $\Omega$  and an exponent  $\beta > 0$  such that

$$\nu\mu^\beta < 1 \tag{3-3.8}$$

This condition is called a *bunching condition* in [HK95] (19.1.d). Such a  $\beta$  exists. We will then show the following, which finally concludes the proof of Theorem 3-3.1.

**Proposition 3-3.4.**  $\alpha_u$  is  $\beta$ -Hölder, that is to say,  $\lambda_u$  is  $C^{1,\beta}$ .

*Proof.* Let us introduce

$$Y^\beta := \{\alpha \in Y; \alpha \text{ is } \beta\text{-Hölder}\}$$

Let us consider some  $\varepsilon > 0$  to be determined later and we equip  $Y^\beta$  with the norm

$$\|\alpha\|_{Y^\beta} = \|\alpha\|_Y + \varepsilon\|\alpha\|_\beta; \quad \|\alpha\|_\beta = \sup_{\rho_1 \neq \rho_2} \frac{\|\alpha(\rho_1) - \alpha(\rho_2)\|}{d(\rho_1, \rho_2)^\beta}$$

The map  $T : X \rightarrow X$  defined by (3-3.2) actually maps  $X \cap C^1(\Omega, \mathbb{R})$  to  $X \cap C^1(\Omega, \mathbb{R})$ . Moreover, our previous results have proved that  $\lambda_u$  is an attractive fixed point for  $T$  in  $X \cap C^1(\Omega, \mathbb{R})$ , where  $X \cap C^1(\Omega, \mathbb{R})$  is now equipped with the  $C^1$  norm. For  $\lambda \in X$  and  $\alpha \in Y$ , we can write,

$$G_\lambda \alpha = \gamma_\lambda + \tilde{G}_\lambda \alpha$$

where for  $\rho' = F(\rho) \in \text{supp } \chi$ ,

$$\begin{aligned} \gamma_\lambda(\rho') &= \chi(\rho')d_\rho t(\rho, \lambda(\rho)) + t(\rho, \lambda(\rho))d_{\rho'} \chi \\ \tilde{G}_\lambda \alpha(\rho') &= \chi(\rho')\partial_\lambda t(\rho, \lambda(\rho))\alpha(\rho) \circ (d_\rho F)^{-1} \end{aligned}$$

We state here some obvious facts on  $\gamma_\lambda$  and  $\tilde{G}_\lambda$

- $C_1 := \sup_{\lambda \in X} \|\gamma_\lambda\|_\infty < +\infty$ ;
- if  $\lambda \in X \cap C^1(\Omega, \mathbb{R})$ ,  $\gamma_\lambda$  is also  $C^1$ ;
- According to Proposition 3-3.2;  $\tilde{G}_\lambda : Y \rightarrow Y$  is a contraction with Lipschitz constant  $L_\lambda$  and  $\nu_1 := \sup_{\lambda \in X} L_\lambda < 1$ ;
- if  $\lambda \in X \cap C^1(\Omega, \mathbb{R})$  and  $\alpha$  is  $\beta$ -Hölder,  $\tilde{G}_\lambda \alpha$  is  $\beta$ -Hölder.

If  $M > \frac{C_1}{1-\nu_1}$  and  $\lambda \in X \cap C^1(\Omega, \mathbb{R})$ , then  $\|d\lambda\|_Y \leq M \implies \|d(T\lambda)\|_Y \leq M$ . Indeed, we have

$$\|d(T\lambda)\|_Y = \|G_\lambda(d\lambda)\|_Y = \|\gamma_\lambda + \tilde{G}_\lambda d\lambda\|_Y \leq C_1 + \nu_1 M \leq M$$

Hence, we introduce the complete metric space

$$X' = \{\lambda \in X \cap C^1(\Omega, \mathbb{R}), \|d\lambda\|_Y \leq M\} \quad (3-3.9)$$

$T(X') \subset X'$  and  $\lambda_u$  is an attractive fixed point for  $(X', T)$ .

We now wish to apply the Fiber Contraction Theorem to

$$H_\beta : (\lambda, \alpha) \in X' \times Y^\beta \mapsto (T\lambda, G_\lambda \alpha) \in X' \times Y^\beta$$

To do so, we need to show that for every  $\lambda \in X'$ ,  $G_\lambda : Y^\beta \rightarrow Y^\beta$  is a contraction and find a uniform estimate for the Lipschitz constants.

Let's consider  $\alpha_1, \alpha_2 \in Y^\beta$  and set  $\gamma = \alpha_1 - \alpha_2$ . We want to estimate the  $Y^\beta$  norm of  $\tilde{G}_\lambda \gamma$ . We already know that  $\|\tilde{G}_\lambda \gamma\|_Y \leq \nu_1 \|\gamma\|_Y$ . Take  $\rho'_1, \rho'_2 \in \Omega$  and let's estimate  $\|\tilde{G}_\lambda \gamma(\rho'_1) - \tilde{G}_\lambda \gamma(\rho'_2)\|$ . We distinguish 3 cases :

- $\rho'_1, \rho'_2 \notin \text{supp } \chi$  : there is nothing to write.
- $\rho'_1 \in \text{supp } \chi, \rho'_2 \notin \Omega \cap F(\Omega)$ . In this case,  $d(\rho'_1, \rho'_2) \geq \delta > 0$  where  $\delta$  is the distance between  $\text{supp } \chi$  and  $(\Omega \cap F(\Omega))^c$ . Hence,

$$\frac{\|\tilde{G}_\lambda \gamma(\rho'_1) - \tilde{G}_\lambda \gamma(\rho'_2)\|}{d(\rho'_1, \rho'_2)^\beta} \leq \delta^{-\beta} \|\tilde{G}_\lambda \gamma(\rho'_1)\| \leq \delta^{-\beta} C \|\tilde{G}_\lambda \gamma\|_Y \leq \nu_1 \delta^{-\beta} C \|\gamma\|_Y$$

- $\rho'_1, \rho'_2 \in \Omega \cap F(\Omega)$ . Let's write  $\rho'_1 = F(\rho_1), \rho'_2 = F(\rho_2)$  and note that  $d(\rho_1, \rho_2) \leq \mu d(\rho'_1, \rho'_2)$ .

$$\begin{aligned} \tilde{G}_\lambda \gamma(\rho'_1) - \tilde{G}_\lambda \gamma(\rho'_2) &= \chi(\rho'_1) \partial_\lambda t(\rho_1, \lambda(\rho_1)) [\gamma(\rho_1) - \gamma(\rho_2)] \circ (d_{\rho_1} F)^{-1} \} (1) \\ &\quad + [\chi(\rho'_1) \partial_\lambda t(\rho_1, \lambda(\rho_1)) - \chi(\rho'_2) \partial_\lambda t(\rho_2, \lambda(\rho_2))] \gamma(\rho_2) \circ (d_{\rho_1} F)^{-1} \} (2) \\ &\quad + \chi(\rho'_2) \partial_\lambda t(\rho_2, \lambda(\rho_2)) \gamma(\rho_2) \circ [(d_{\rho_1} F)^{-1} - (d_{\rho_2} F)^{-1}] \} (3) \end{aligned}$$

To handle the last two terms (2) and (3), we notice that  $\rho' \in \Omega \cap F(\Omega) \mapsto \chi(\rho') \partial_\lambda t(\rho, \lambda(\rho))$  is Lipschitz since  $\lambda$  is  $C^1$ , with Lipschitz constant which can be chosen uniform for  $\lambda \in X'$ . The same is true for  $\rho \mapsto d_\rho F^{-1}$ . Hence, there exists a uniform constant  $C > 0$  such that

$$\|(2) + (3)\| \leq C d(\rho'_1, \rho'_2)^\beta \|\gamma\|_Y$$

To deal with the first term (1), we recall that by previous computations,

$$|\chi(\rho') \partial_\lambda t(\rho, \lambda(\rho))| \cdot \|(d_\rho F)^{-1}\| \leq \nu + O_\nu(\eta)$$

As consequence, we have

$$\|(1)\| \leq (\nu + O_\nu(\eta)) \|\gamma\|_\beta d(\rho_1, \rho_2)^\beta \leq (\nu + O_\nu(\eta)) \mu^\beta \|\gamma\|_\beta d(\rho'_1, \rho'_2)^\beta$$

Henceforth, if  $\eta$  is small enough, so that  $\nu_2 := (\nu + O_\nu(\eta)) \mu^\beta < 1$ ,

$$\|H_\lambda \gamma\|_\beta \leq \nu_2 \|\gamma\|_\beta + C \|\gamma\|_Y$$

Eventually,

$$\begin{aligned} \|\tilde{G}_\lambda \gamma\|_{Y^\beta} &\leq \nu_1 \|\gamma\|_Y + \varepsilon (\nu_2 \|\gamma\|_\beta + C \|\gamma\|_Y) \\ &\leq (\nu_1 + \varepsilon C) \|\gamma\|_Y + \nu_2 \varepsilon \|\gamma\|_\beta \\ &\leq \nu_3 \|\gamma\|_{Y^\beta} \end{aligned}$$

where  $\nu_3 = \max(\nu_1 + \varepsilon C, \nu_2) < 1$  if  $\varepsilon$  is small enough.

The Fiber Contraction Theorem applies and says that  $(\lambda_u, \alpha_u)$  is an attractive fixed point for  $H_\beta$ . We conclude as previously : consider  $\lambda \in C^{1,\beta}(\Omega, \mathbb{R}) \cap X'$  so that  $(\lambda, d\lambda) \in X' \times Y^\beta$ .  $H_\beta^n(\lambda, d\lambda) = (T^n \lambda, dT^n \lambda) \rightarrow (\lambda_u, \alpha_u)$  in  $X' \times Y^\beta$ . That ensures that  $\alpha_u$  is  $\beta$ -Hölder.  $\square$

### 3-3.2. Regularity of the stable and unstable leaves.

Once we've extended the unstable distribution to an open neighborhood of  $\mathcal{T}$ , we take advantage of the fact that these distributions are 1-dimensional to integrate the vector field defined by their unit vector.

We set  $E_u(\rho) = \mathbb{R}(v_u(\rho) + \lambda_u(\rho)v_s(\rho))$ . Recall that in a compact neighborhood of  $\mathcal{T}$ , the relation  $d_\rho F(E_u(\rho)) = E_u(F(\rho))$  is valid due to the definition of  $\lambda_u$  as the fixed point of  $T$  defined in (3-3.2).  $T^*U$  is equipped with a smooth Riemannian metric such that  $dF^{-1}$  is a contraction on  $E_u(\rho)$  for  $\rho \in \mathcal{T}$  and hence, in a compact neighborhood of  $\mathcal{T}$ , this is also true. We can consider the vector field

$$\rho \in U \mapsto e_u(\rho)$$

where  $e_u(\rho)$  is a unit vector spanning  $E_u(\rho)$ . By our previous result, this vector field is  $C^{1,\beta}$  and if  $\rho$  lies in a sufficiently small neighborhood of  $\mathcal{T}$ ,  $d_\rho(F^{-1})(e_u(\rho)) = \tilde{J}^u(\rho)e_u(F^{-1}(\rho))$  where  $|\tilde{J}^u(\rho)| \leq \nu < 1$ .

We denote by  $\varphi_u^t(\rho)$  the flow generated by  $e_u(\rho)$  and we will show that one can identify the unstable manifolds and the flow lines of  $e_u$  in a small neighborhood of  $\mathcal{T}$ .

**Proposition 3-3.5.** There exists  $t_0$  such that for every  $\rho \in \mathcal{T}$ ,  $\{\varphi_u^t(\rho), |t| \leq t_0\} \subset W_u(\rho)$

*Proof.* Consider  $t_0$  is sufficiently small such that  $|\tilde{J}^u(\varphi_u^t(\rho))| \leq \nu < 1$  for  $\rho \in \mathcal{T}$ ,  $t \in [-t_0, t_0]$ . For  $(t, \rho) \in \mathbb{R} \times U$ , set  $\mu(t, \rho) = \int_0^t \tilde{J}^u(\varphi_u^s(\rho)) ds$  and we claim that for  $t_0$  small enough, if  $|t| \leq t_0$ ,

$$F^{-1}(\varphi_u^t(\rho)) = \varphi_u^{\mu(t,\rho)}(F^{-1}(\rho))$$

Indeed, in  $t = 0$ , both are equal to  $F^{-1}(\rho)$  and a quick computation shows that both satisfy the ODE

$$\frac{d}{dt}Y(t) = J^u(\varphi_u^t(\rho))e_u(Y(t))$$

As a consequence, by induction, we see that one can write for  $n \in \mathbb{N}$ ,

$$F^{-n}(\varphi_u^t(\rho)) = \varphi_u^{\mu_n(t,\rho)}(F^{-n}(\rho))$$

where  $\mu_n$  is defined by induction by  $\mu_{n+1}(t, \rho) = \mu(\mu_n(t, \rho), F^{-n}(\rho))$ . Hence, if  $|t| \leq t_0$  and  $\rho \in \mathcal{T}$ , we see that  $\mu_n(t, \rho)$  stays in  $[-t_0, t_0]$  and moreover  $|\mu_n(t, \rho)| \leq \nu^n |t|$ . We then see that if  $|t| \leq t_0$  and  $\rho \in \mathcal{T}$ ,

$$d(F^{-n}(\varphi_u^t(\rho)), F^{-n}(\rho)) = d(\varphi_u^{\mu_n(t,\rho)}(F^{-n}(\rho)), F^{-n}(\rho)) \leq C|\mu_n(t, \rho)| \leq C\nu^n$$

This shows that  $\varphi_u^t(\rho)$  belongs to the global unstable manifold at  $\rho$ , and hence, if  $t_0$  is small enough,  $\varphi_u^t(\rho)$  belongs to the local manifold  $W_u(\rho)$  and  $t_0$  can be chosen uniformly with respect to  $\rho \in \mathcal{T}$ .  $\square$

The regularity of the unstable distributions implies the same regularity for the flow  $\varphi_u^t$ . Indeed, this is a simple consequence of

**Lemma 3-3.1.** Let  $U \subset \mathbb{R}^n$  be open and  $Y : U \rightarrow \mathbb{R}^n$  be a complete  $C^{1+\beta}$  vector field. We note  $\phi^t(x)$  the flow generated by  $Y$ . Then, for any  $T \in \mathbb{R}$  and  $K \subset U$  compact, the map

$$(t, x) \in [-T, T] \times K \mapsto \phi^t(x)$$

is  $C^{1+\beta}$ .

*Proof.* We fix  $T, K$  as in the statement. We'll use the same constants  $C, C'$  at different places, with different meaning. In addition to  $Y$ , they will depend on  $T, K$ .

Since  $Y$  is  $C^1$ , Cauchy-Lipschitz theorem gives the local existence and uniqueness of the flow. It is standard that the flow is also  $C^1$  and satisfies

$$\partial_t d\phi^t(x) = dY(\phi^t(x)) \circ d\phi^t(x) \tag{3-3.10}$$

Let's note  $A^t(x) = d\phi^t(x)$  and  $\Xi(t, x) = dY(\phi^t(x))$ . The assumption on  $Y$  implies that  $\Xi$  is  $\beta$ -Hölder.

Fix  $(t_0, x_0), (t_1, x_1) \in [-T, T] \times K$  and let's estimate  $\|A^{t_1}(x_1) - A^{t_0}(x_0)\|$ . We split it into two pieces and control it with the triangle inequality :

$$\|A^{t_1}(x_1) - A^{t_0}(x_0)\| \leq \|A^{t_1}(x_1) - A^{t_0}(x_1)\| + \|A^{t_0}(x_1) - A^{t_0}(x_0)\|$$

It is not hard to control the first term of the right hand side using (3-3.10) since

$$\|A^{t_1}(x_1) - A^{t_0}(x_1)\| = \left\| \int_{t_0}^{t_1} \Xi(s, x_1) \circ A^s(x_1) ds \right\| \leq C|t_1 - t_0|$$

To estimate the second term, we estimate

$$\begin{aligned} \|\partial_t(A^t(x_1) - A^t(x_0))\| &\leq \|(\Xi(t, x_1) - \Xi(t, x_0)) \circ A^t(x_1) + \Xi(t, x_0) \circ (A^t(x_1) - A^t(x_0))\| \\ &\leq Cd(x_0, x_1)^\beta + C'\|A^t(x_1) - A^t(x_0)\| \end{aligned}$$

By Gronwall's lemma,

$$\|A^{t_0}(x_1) - A^{t_0}(x_0)\| \leq Cd(x_0, x_1)^\beta e^{C't_0} \leq Cd(x_0, x_1)^\beta$$

This concludes the proof.  $\square$

Since the flow is  $C^{1,\beta}$ , we deduce that, up to reducing the size of the local unstable manifolds, these local unstable manifolds  $W_u(\rho)$  depend  $C^{1,\beta}$  on the base point  $\rho \in \mathcal{T}$ . We'll also use this proposition to show the same regularity for holonomy maps. Suppose that  $\varepsilon_0$  is small enough. We know that if  $\rho_1, \rho_2 \in \mathcal{T}$  satisfy  $d(\rho_1, \rho_2) \leq \varepsilon_0$ , then  $W_u(\rho_2) \cap W_s(\rho_1)$  consists of exactly one point. Let's note it  $H_{\rho_1}^u(\rho_2)$ .

Finally, we define the holonomy map

$$H_{\rho_1, \rho_2}^u : \rho_3 \in W_s(\rho_2) \cap \mathcal{T} \mapsto H_{\rho_1}^u(\rho_3) \in W_s(\rho_1) \cap \mathcal{T}$$

**Lemma 3-3.2.** If  $\varepsilon_0$  is small enough, for every  $\rho_1 \in \mathcal{T}$ , the map

$$H_{\rho_1}^u : \mathcal{T} \cap B(\rho_1, \varepsilon_0) \rightarrow W_s(\rho_1) \cap \mathcal{T}$$

is the restriction of a map  $\tilde{H}_{\rho_1}^u : B(\rho_1, \varepsilon_0) \rightarrow W_s(\rho_1)$  which is  $C^{1,\beta}$ .

*Proof.* Let  $\rho_1 \in \mathcal{T}$ . As in the proof of Lemma 3-2.3, consider a smooth chart  $\kappa : U_1 \rightarrow V_1 \subset \mathbb{R}^2$ ,  $\rho_1 \in U_1, 0 \in V_1$  such that :

- $\kappa(\rho_1) = (0, 0)$
- $\kappa(W_s(\rho_1) \cap U_1) = \{(0, s), s \in \mathbb{R}\} \cap V_1$
- $\kappa(W_u(\rho_1) \cap U_1) = \{(u, 0), u \in \mathbb{R}\} \cap V_1$
- $d_{\rho_1} \kappa(e_u(\rho_1)) = (1, 0)$ .

We now work in this chart  $V_1$  and note  $\Phi^t = \kappa \circ \varphi_u^t \circ \kappa^{-1}$  the flow in this chart, well defined for  $t$  small enough. Consider the map

$$\psi(u, s) = \Phi^u(0, s)$$

$\psi$  is  $C^{1,\beta}$  and  $d_0\psi = I_2$ . By the Inverse Function Theorem,  $\psi$  is a local diffeomorphism between neighborhoods of 0 :

$$\psi : V_2 \rightarrow V_2'$$

Since  $d_{(u,s)}(\psi^{-1}) = (d_{\psi^{-1}(u,s)}\psi)^{-1}$ ,  $\psi^{-1}$  is  $C^{1,\beta}$ . We now consider

$$\kappa_0 = \psi^{-1} \circ \kappa : \kappa^{-1}(V_2) := U_2 \rightarrow V_2'$$

and observe that :

- $\kappa_0(W_s(\rho_1) \cap U_2) = \{(0, s), s \in \mathbb{R}\} \cap V_2'$  ;
- $\kappa_0 \circ \varphi_u^t \circ \kappa_0^{-1}(u, s) = (u + t, s)$ . In other words  $\kappa_0$  rectifies the unstable manifolds.

Armed with these facts, we define

$$\tilde{H}_{\rho_1}^u : U_2 \rightarrow W_s(\rho_1) \quad ; \quad \tilde{H}_{\rho_1}^u = \kappa_0^{-1} \circ \pi_s \circ \kappa_0$$

where  $\pi_s(u, s) = (0, s)$ .  $\tilde{H}_{\rho_1}^u$  is  $C^{1,\beta}$ . We assume that  $B(0, \varepsilon_0) \subset U_1$ . Let us check that  $\tilde{H}_{\rho_1}^u$  extends the holonomy map in  $B(\rho_1, \varepsilon_0)$  (if  $\varepsilon_0$  is small enough). Let  $\rho_2 \in \mathcal{T} \cap B(\rho_1, \varepsilon_0)$  and note  $\rho_2' = \tilde{H}_{\rho_1}^u(\rho_2)$ . By definition of  $\tilde{H}_{\rho_1}^u$ ,  $\rho_2'$  can be written  $\rho_2' = \varphi_u^t(\rho_1)$  and hence, if  $\varepsilon_0$  is small enough,  $\rho_2' \in W_u(\rho_1)$ . Since,  $\rho_2' \in W_s(\rho_2)$ , we see that  $\rho_2' = H_{\rho_1}^u(\rho_2)$ .  $\square$

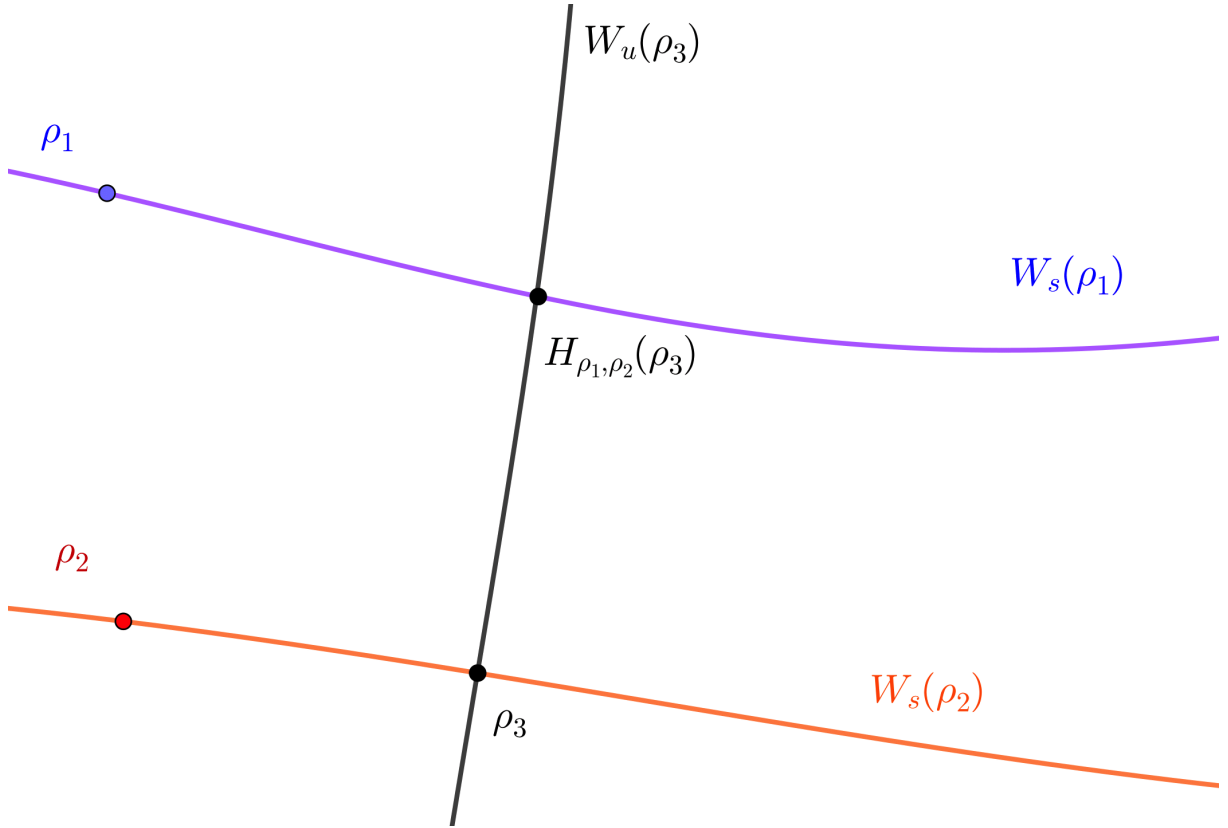


Figure 3.7: The holonomy map.

Note that by compactness,  $\varepsilon_0$  can be chosen uniformly in  $\rho_1 \in \mathcal{T}$  and the  $C^{1,\beta}$  norms of  $\tilde{H}_{\rho_1}^u$  are uniform. As a corollary, we get the following :

**Corollary 3-3.1.** Suppose that  $\varepsilon_0$  is small enough. Then, the holonomy maps, defined for  $\rho_1, \rho_2 \in \mathcal{T}$  with  $d(\rho_1, \rho_2) \leq \varepsilon_0$ ,

$$H_{\rho_1, \rho_2}^u : W_s(\rho_2) \cap \mathcal{T} \rightarrow W_s(\rho_1) \cap \mathcal{T}$$

are the restrictions of  $C^{1,\beta} : \tilde{H}_{\rho_1, \rho_2}^u : W_s(\rho_1) \rightarrow W_s(\rho_2)$  with  $C^{1,\beta}$  norms uniform in  $\rho_1, \rho_2$ .

### 3-4 Adapted charts.

We construct charts in which the unstable manifolds are close to horizontal lines. These charts will be used at different places and their existence relies on the  $C^{1+\beta}$  regularity of the unstable distribution.

**Weak version.** We start with a weak version of these charts.

**Lemma 3-4.1.** Suppose that  $C > 0$  is a fixed global constant and  $\varepsilon_0$  is chosen small enough. For every  $\rho_0 \in \mathcal{T}$ , there exists a canonical transformation

$$\kappa_0 : U'_{\rho_0} \rightarrow V'_{\rho_0} \subset \mathbb{R}^2$$

satisfying (we note  $(y, \eta)$  the variable in  $\mathbb{R}^2$ ) :

- (1)  $B(\rho_0, C\varepsilon_0) \subset U'_{\rho_0}$ ;
- (2)  $\kappa_0(\rho_0) = 0$ ,  $d_{\rho_0}\kappa_0(E_u(\rho_0)) = \mathbb{R} \times \{0\}$ ;  $d_{\rho_0}\kappa_0(E_s(x)) = \{0\} \times \mathbb{R}$ ;
- (3) The image of the unstable manifold  $W_u(\rho_0) \cap U'_{\rho_0}$  is exactly  $\{(y, 0), y \in \mathbb{R}\} \cap V'_{\rho_0}$ .

Moreover, for every  $N$ , the  $C^N$  norms of  $\kappa_0$  are bounded uniformly with respect to  $\rho_0 \in \mathcal{T}$ .

**Remark.**



The difference with the charts used in the proof of Lemma 3-2.3 is that we require  $\kappa_0$  to be a smooth canonical transformation.

*Proof.*  $W_u(\rho_0)$  is a  $C^\infty$  manifold, hence there exists a  $C^\infty$  defining function  $\eta$  defined in a neighborhood  $\rho_0$  : namely  $d_{\rho_0}\eta \neq 0$  and  $W_u(\rho_0) = \{\eta = 0\}$  locally near  $\rho_0$ . Darboux's theorem gives a function  $y$  defined in a neighborhood of  $\rho_0$  such that  $(y, \eta)$  forms a system of symplectic coordinates. We can assume that  $y(\rho_0) = 0$ . If  $\kappa(\rho) = (y, \eta)$ , the third point is satisfied by assumption on  $\eta$  and we need to ensure that  $d_{\rho_0}\kappa(E_s(\rho_0)) = \{0\} \times \mathbb{R}$  by modifying  $\eta$  in a symplectic way. Assume that  $d_{\rho_0}\kappa(E_s(\rho_0)) = \mathbb{R}^t(a, 1)$ . The symplectic matrix

$$A = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

maps the basis  $({}^t(1, 0), {}^t(a, 1))$  to the canonical basis of  $\mathbb{R}^2$  and we can set  $\kappa_0 := A \circ \kappa$  which is the required canonical transformation, defined in a small neighborhood  $U'_{\rho_0}$  of  $\rho_0$ .

We can ensure that  $B(\rho_0, C\varepsilon_0) \subset U'_{\rho_0}$  for  $\varepsilon_0$  small enough and the uniformity of the  $C^N$  norms of  $\kappa$  thanks to the compactness of  $\mathcal{T}$  and the fact that the unstable distribution depends continuously on  $\rho_0 \in \mathcal{T}$ .  $\square$

**Straightened version.** We now straighten the unstable manifolds in a stronger version of the previous charts. The construction and the use of these charts is similar to [DJN21] (Lemma 2.3).

**Lemma 3-4.2.** Suppose that  $\varepsilon_0$  is chosen small enough. For every  $\rho_0 \in \mathcal{T}$  there exists a canonical transformation

$$\kappa = \kappa_{\rho_0} : U_{\rho_0} \subset U \rightarrow V_{\rho_0} \subset \mathbb{R}^2$$

satisfying (we note  $(y, \eta)$  the variable in  $\mathbb{R}^2$ ) :

- (1)  $B(\rho_0, 2\varepsilon_0) \subset U_{\rho_0}$  ;
- (2)  $\kappa(\rho_0) = 0$  ,  $d_{\rho_0}\kappa(E_u(\rho_0)) = \mathbb{R} \times \{0\}$ ;  $d_{\rho_0}\kappa(E_s(\rho_0)) = \{0\} \times \mathbb{R}$
- (3) The images of the unstable manifolds  $W_u(\rho)$ ,  $\rho \in U_{\rho_0} \cap \mathcal{T}$ , are described by

$$\kappa(W_u(\rho) \cap U_{\rho_0}) = \left\{ \left( y, g(y, \zeta(\rho)) \right), y \in \Omega \right\} \quad (3-4.1)$$

where  $\Omega \subset \mathbb{R}$  is an open set,  $\zeta : U_{\rho_0} \rightarrow \mathbb{R}$  is  $C^{1+\beta}$ ,  $g : \Omega \times I \rightarrow \mathbb{R}$  is  $C^{1+\beta}$  (where  $I$  is a neighborhood of  $\zeta(U_{\rho_0})$ ) and they satisfy

- (i)  $\zeta$  is constant on the unstable manifolds ;
- (ii)  $\zeta(\rho_0) = 0$ ,  $g(y, 0) = 0$  ;
- (iii)  $g(0, \zeta) = \zeta$  ;
- (iv)  $\partial_{\zeta}g(y, 0) = 1$

The derivatives of  $\kappa_{\rho_0}$  and the  $C^{1+\beta}$  norms of  $g, \zeta$  are bounded uniformly in  $\rho_0$ .

**Remark.**

The most important condition, which will be used later on, is the last one : it makes the unstable manifolds very close to horizontal lines. The model situation we expect is when the unstable distribution is constant and horizontal.

*Proof.* Around a point  $\rho_0 \in \mathcal{T}$ , we work in the charts given by Lemma 3-4.1 :  $\kappa_0 : U'_{\rho_0} \rightarrow V'_{\rho_0}$ . We recall that the unstable distribution is given by the restriction of a  $C^{1+\beta}$  vector field  $e_u$ . If  $U'_{\rho_0}$  is a sufficiently small neighborhood of  $\rho_0$ , we can write, for  $\rho \in U'_{\rho_0}$ ,

$$d_{\rho}\kappa_0(e_u(\rho)) \in \mathbb{R}\tilde{e}_u(\rho) \quad \text{with } \tilde{e}_u(\rho) = {}^t(1, f_0(\rho)) \quad (3-4.2)$$

where  $f_0 : U'_{\rho_0} \rightarrow \mathbb{R}$  is a  $C^{1+\beta}$  function which is nothing but the slope of the unstable direction in the chart  $\kappa_0$ . In the  $(y, \eta)$  variable, we still note  $f_0(\rho) = f_0(y, \eta)$  and we observe that due to the assumption on  $\kappa_0$ , we have

$$f_0(y, 0) = 0 \quad , \quad (y, 0) \in V'_{\rho_0}$$

We consider  $\Phi^t(y, \eta)$  the flow generated by the vector field  $\tilde{e}_u$ . Due to the form of  $\tilde{e}_u$ , we can write,

$$\Phi^t(y, \eta) = (y + t, Z^t(y, \eta))$$

The reparametrization made in (3-4.2) does not change the flow lines of the vector field  $(\kappa_0)_* e_u$ . In particular, in virtue of Proposition 3-3.5, they coincide locally with the unstable manifolds. More precisely, if we set,

$$g_0(y, \eta) := Z^y(0, \eta)$$

then, for  $(0, \eta) = \kappa_0(\rho) \in \kappa_0(\mathcal{T} \cap W_s(\rho_0))$ ,

$$\kappa_0(W_u(\rho)) \cap \{|y| < y_0\} = \left\{ (y, g_0(y, \eta)), |y| < y_0 \right\}$$

for some  $y_0$  small enough (which can be chosen uniformly in  $\rho_0$ ). To define  $\zeta$ , we go back up the flow : suppose that  $\rho \in U'_{\rho_0}$  and write  $\kappa_0(\rho) = (y, \eta)$  and assume  $|y| < y_0$ . We set

$$\zeta(\rho) := Z^{-y}(y, \eta)$$

To say it differently,  $\kappa_0(W_u(\rho))$  intersects the axis  $\{y = 0\}$  at  $(0, \zeta(\rho))$ .

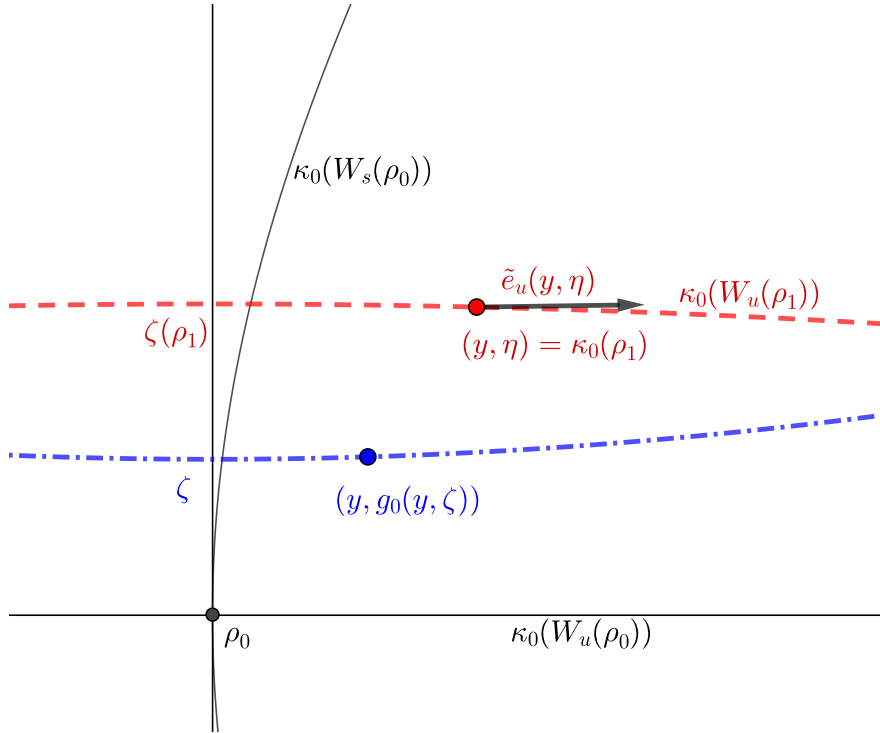


Figure 3.8: The definitions of  $g_0$  and  $\zeta$  use the flow generated by  $\tilde{e}_u$ .

$\zeta$  and  $g_0$  are  $C^{1+\beta}$ , their  $C^{1+\beta}$  norms depend uniformly on  $\rho_0$  and they satisfy :

- By definition,  $\zeta$  is constant on the flow lines, and hence, on the unstable manifolds  $W_u(\rho)$  if  $\rho \in \mathcal{T} \cap U'_{\rho_0} \cap \{|y| < y_0\}$  ;
- $\zeta(\rho_0) = 0$  ;
- Since  $f_0(y, 0) = 0$ ,  $Z^y(0, 0) = 0$  and hence  $g_0(y, 0) = 0$  ;
- Since  $Z^0(0, \eta) = \eta$ ,  $g_0(0, \eta) = \eta$ .

However, at this stage, the last condition ( $\partial_\zeta g_0(y, 0) = 1$ ) is not satisfied by  $g_0$  and we need to modify the chart. To do so, we'll make use of the following lemma.

**Lemma 3-4.3.** The map  $y \in \{|y| < y_0\} \mapsto \partial_\eta f_0(y, 0)$  is smooth, with  $C^N$  norms bounded uniformly in  $\rho_0$ .

*Proof.* The proof uses the construction of  $e_u$  in the proof of Theorem 3-3.1. It is inspired by techniques usually used to show the unstable manifold's theorem (see for instance [Dya18]). In fact, the smoothness of  $y \mapsto f_0(y, 0)$  is a direct consequence of the smoothness of the unstable manifold  $W_u(\rho_0)$ . It was not clear for us if it was possible to easily deduce from this the required smoothness of  $y \mapsto \partial_\eta f_0(y, 0)$ . This is why we decided to give a proof of this proposition. It uses the fact that  $e_u$  has been constructed to satisfy  $\mathbb{R}d_\rho F(e_u(\rho)) = \mathbb{R}e_u(F(\rho))$  for  $\rho$  in a small neighborhood of  $\mathcal{T}$ . To show the lemma, we need information along all the orbit of  $\rho_0$ . For this purpose, we introduce the following, for  $m \in \mathbb{Z}$ ,

- $\rho_m = F^m(\rho_0)$  ;
- $\kappa_m : U_m \rightarrow V_m \subset \mathbb{R}^2$  the chart given by Lemma 3-4.1 centered at  $\rho_m$  and we assume that the relation  $\mathbb{R}d_\rho F(e_u(\rho)) = \mathbb{R}e_u(F(\rho))$  holds for  $\rho \in U_m$ . We will note  $(y_m, \eta_m)$  the variable in  $V_m$  ;
- $G_m = \kappa_{m+1} \circ F \circ \kappa_m^{-1} : V_m \rightarrow V_{m+1}$  ;
- A reparametrization of the vector field  $(\kappa_m)_* e_u : \mathbb{R}(\kappa_m)_* e_u = \mathbb{R}e_m$  where  $e_m(y_m, \eta_m) = {}^t(1, s_m(y_m, \eta_m))$  where  $s_m$  is a slope function which is known to be  $C^{1+\beta}$ .

Note that  $s_m(y_m, 0) = 0$  due to the fact that  $\kappa_m(W_u(\rho_m)) \subset \mathbb{R} \times \{0\}$ . The hyperbolicity assumption on  $F$  and the properties of  $\kappa_m$  allow us to write

$$G_m(y_m, \eta_m) = \left( \lambda_m y_m + \alpha_m(y_m, \eta_m), \mu_m \eta_m + \beta_m(y_m, \eta_m) \right)$$

where

- For some  $\nu < 1$ ,  $0 \leq |\mu_m| \leq \nu$ ,  $|\lambda_m| \geq \nu^{-1}$  for all  $m \in \mathbb{N}$  ;
- $\alpha_m(0, 0) = \beta_m(0, 0) = 0$  ;
- $\beta_m(y_m, 0) = 0$  for  $(y_m, 0) \in V_m$
- $d\alpha_m(0, 0) = d\beta_m(0, 0) = 0$  ;
- We can assume that  $U_m$  are sufficiently small neighborhoods of  $\rho_m$  so that  $\beta_m, \alpha_m = O(\delta_0)_{C^1(U_m)}$  for some small  $\delta_0 > 0$ .

The property  $d_\rho F(e_u(\rho)) \in \mathbb{R}e_u(F(\rho))$  implies that  $d_{(y_m, \eta_m)} G_m(e_m(y_m, \eta_m)) \in \mathbb{R}e_{m+1}(G_m(y_m, \eta_m))$ . As a consequence, the transformation of the slopes gives an equation satisfied by the family of slopes  $(s_m)_{m \in \mathbb{Z}}$  :

$$s_{m+1}(G_m(y_m, \eta_m)) = Q_m(y_m, \eta_m, s_m(y_m, \eta_m)) \quad (3-4.3)$$

where  $Q_m$  is the smooth function

$$Q_m(y_m, \eta_m, s) = \frac{s \times (\mu_m + \partial_{\eta_m} \beta_m(y_m, \eta_m)) + \partial_{y_m} \beta_m(y_m, \eta_m)}{\lambda_m + \partial_{y_m} \alpha_m(y_m, \eta_m) + s \times \partial_{\eta_m} \alpha_m(y_m, \eta_m)}$$

Writing  $G_m(y_m, \eta_m) = (y_{m+1}, \eta_{m+1})$ , we deduce by differentiation of (3-4.3) with respect to  $\eta_{m+1}$ : (we omit the point of evaluation of the maps involved in the right hand side to alleviate the line)

$$\begin{aligned} \partial_{\eta_{m+1}} s_{m+1}(y_{m+1}, \eta_{m+1}) &= \partial_{y_m} Q_m \times \partial_{\eta_{m+1}} y_m + \partial_{\eta_m} Q_m \times \partial_{\eta_{m+1}} \eta_m \\ &\quad + \partial_s Q_m \times (\partial_{y_m} s_m \times \partial_{\eta_{m+1}} y_m + \partial_{\eta_m} s_m \times \partial_{\eta_{m+1}} \eta_m) \end{aligned} \quad (3-4.4)$$

This last equation gives the transformation of vertical derivative of the slope. We now evaluate this identity at the point  $(y_{m+1}, 0)$ . In the following lines, when the variable  $y_m$  and  $y_{m+1}$  appear in the same equation, we implicitly assume that they are related by  $(y_{m+1}, 0) = G_m(y_m, 0)$ , namely  $y_{m+1} = \lambda_m y_m + \alpha_m(y_m, 0)$ . We remark that due to the fact that  $\beta_m(y_m, 0) = 0$ ,  $Q_m(y_m, 0, 0) = 0$  and the first term of the right hand side vanishes. The term  $\partial_{y_m} s_m$  also vanishes at  $(y_m, 0)$ . We will note

$$\begin{aligned} \sigma_m(y_m) &= \partial_{\eta_m} s_m(y_m, 0) \\ h_m(y_m) &= \partial_{\eta_m} Q_m(y_m, 0, 0) \times \partial_{\eta_{m+1}} \eta_m(y_{m+1}, 0) \\ c_m(y_m) &= \partial_s Q_m(y_m, 0, 0) \times \partial_{\eta_{m+1}} \eta_m(y_{m+1}, 0) \end{aligned}$$

These notations allow us to rewrite (3-4.4) at  $(y_{m+1}, 0)$  :

$$\sigma_{m+1}(y_{m+1}) = h_m(y_m) + c_m(y_m) \times \sigma_m(y_m) \quad (3-4.5)$$

We observe that  $|\partial_{\eta_{m+1}}\eta_m(y_m, 0)| = |\mu_m^{-1} + O(\delta_0)_{C^0}|$  and after some computations, we see that

$$\partial_s Q_m(y_m, 0, 0) = \frac{\mu_m}{\lambda_m} + O(\delta_0)_{C^0}$$

As a consequence,

$$|c_m(y_m)| = |\lambda_m^{-1}| + O(\delta_0)_{C^0} \leq \nu_1 \quad (3-4.6)$$

where, if  $\delta_0$  is small enough, we can fix  $\nu_1 < 1$ . Moreover,  $c_m$  and  $h_m$  are smooth functions and their  $C^N$  norms are bounded uniformly in  $m$ , and actually by global constants depending only on  $F$ . Furthermore,  $y_m \mapsto y_{m+1}$  is given by  $y_m \mapsto \lambda_m y + \alpha_m(y_m, 0)$  and is an expanding diffeomorphism provided  $\delta_0$  is small enough.

We fix some small  $\varepsilon$  such that  $(-\varepsilon, \varepsilon) \times \{0\} \subset U_m$  for all  $m$ . Let's note  $I = (-\varepsilon, \varepsilon)$ . We will make use of the Fiber Contraction Theorem to show that  $y_m \in I \mapsto \sigma_m(y_m)$  is smooth for every  $m$ , with uniform  $C^N$  norms. For this purpose, let us introduce the following notations :

- $C_0 \leq C_1 \leq \dots C_N \leq \dots$  a family of constant which will be specified in the sequel ;
- The complete metric space  $X_N = \{\gamma \in C^N(I); \|\gamma\|_{C^k} \leq C_k, 0 \leq k \leq N\}$  equipped with the  $C^N$  norm ;
- The auxiliary metric space  $X_N^{aux} = \{\gamma \in C^0(I); \|\gamma\|_\infty \leq C_N\}$  equipped with the  $C^0$  norm ;
- The complete metric space  $E_N = (X_N)^{\mathbb{Z}}$  equipped with the metric

$$d(\gamma_1, \gamma_2) = \sup_{m \in \mathbb{Z}} \|(\gamma_1)_m - (\gamma_2)_m\|_{C^N}$$

- Its auxiliary counterpart  $E_N^{aux} = (X_N^{aux})^{\mathbb{Z}}$  equipped with the metric

$$d(\gamma_1, \gamma_2) = \sup_{m \in \mathbb{Z}} \|(\gamma_1)_m - (\gamma_2)_m\|_{C^0}$$

For  $\gamma \in E_N$ , let's define  $T\gamma$  with the formula (3-4.5) :

$$(T\gamma)_{m+1}(y_{m+1}) = (h_m + c_m \gamma_m)(y_m)$$

Since  $y_m \mapsto y_{m+1}$  is expanding, we see that  $y_{m+1} \in I \implies y_m \in I$ . Hence,  $(T\gamma)_{m+1}$  is well defined on  $I$ . Our aim is to show by induction on  $N$  that for every  $N \in \mathbb{N}$ ,  $\sigma := (\sigma_m)_{m \in \mathbb{Z}}$  is in  $E_N$  and is an attractive fixed point of  $T : E_N \rightarrow E_N$ .

We start with the case  $N = 0$ . We need to check that  $T(E_0) \subset E_0$ . It will be the case as soon as

$$C_0 \nu_1 + \sup_m \|h_m\|_\infty \leq C_0$$

For instance, take  $C_0 = \frac{2 \sup_m \|h_m\|_\infty}{1 - \nu_1}$ . Due to the fact that  $\|c_m\|_{C^0(I)} \leq \nu_1$ ,  $T$  is a contraction with contraction rate  $\nu_1$  and hence  $T : E_0 \rightarrow E_0$  has a unique attractive fixed point. This fixed point is necessarily  $\sigma$  since  $\sigma$  satisfies (3-4.5).

Arguing by induction, we assume that  $\sigma \in E_N$ ,  $T(E_N) \subset E_N$  and  $\sigma$  is an attractive fixed point for  $T$  and we want to show that the same is true for  $N + 1$ . For this purpose, suppose that  $\gamma \in E_N$  is of class  $C^{N+1}$ . Analyzing the formula defining  $T$ , we see that can can write, for  $m \in \mathbb{Z}$ ,

$$(T\gamma)_m^{(N+1)}(y_{m+1}) = h_m^{(N+1)}(y_m) + c_m(y_m) \times \left( \frac{\partial y_{m+1}}{\partial y_m}(y_m) \right)^{-N-1} \times \gamma_m^{(N+1)}(y_m) + R_{N,m} \left( y_m, \gamma_m(y_m), \dots, \gamma_m^{(N)}(y_m) \right) \quad (3-4.7)$$

where  $R_{N,m} : I \times [-C_0, C_0] \times \dots \times [-C_N, C_N] \rightarrow \mathbb{R}$  is a polynomial in the last  $N + 1$  variables with smooth coefficients in  $y_m$ , uniformly bounded in  $m$ . As a consequence, there exists a global constant  $C'_{N+1}$  such that

$$\sup_m \sup_{I \times [-C_0, C_0] \times \dots \times [-C_N, C_N]} |R_{N,m}(y_m, \tau_0, \dots, \tau_N)| \leq C'_{N+1}$$

We can then choose  $C_{N+1} \geq C_N$  such that

$$\sup_m \|h_m\|_{C^{N+1}} + C'_{N+1} + \nu_1 C_{N+1} \leq C_{N+1}$$

which ensures that  $T : E_{N+1} \rightarrow E_{N+1}$ . We now wish to use the Fiber Contraction Theorem (Theorem 3-3.2). If  $\gamma \in E_N$ , we define the map  $S_\gamma : E_{N+1}^{aux} \rightarrow E_{N+1}^{aux}$  by

$$(S_\gamma \theta)_{m+1}(y_{m+1}) = h_m^{(N+1)}(y_m) + c_m(y_m) \times \left( \frac{\partial y_{m+1}}{\partial y_m}(y_m) \right)^{-N-1} \times \theta_m(y_m) + R_{N,m}(y_m, \gamma_m(y_m), \dots, \gamma_m^N(y_m))$$

Due to the choice of  $C_{N+1}$ , we see that  $S_\gamma$  is well defined and since we have

$$\left| \frac{\partial y_{m+1}}{\partial y_m}(y_m) \right| \geq 1$$

and  $\|c_m\|_{C^0(I)} \leq \nu_1$ ,  $S_\gamma$  is a contraction with contraction rate  $\nu_1$ , for every  $\gamma \in E_N$ . In particular, the map  $S_\sigma$  has a unique fixed point  $\sigma_{N+1} \in E_{N+1}^{aux}$ .

The Fiber Contraction Theorem (Theorem 3-3.2) applies to the continuous map

$$T_N : (\gamma, \theta) \in E_N \times E_{N+1}^{aux} \mapsto (T\gamma, S_\gamma \theta) \in E_N \times E_{N+1}^{aux}$$

and  $(\sigma, \sigma_{N+1})$  is an attractive fixed point of  $T_N$  in  $E_N \times E_{N+1}^{aux}$ .

In particular, if  $\gamma \in E_{N+1}$ , then  $\tilde{\gamma} := (\gamma, \gamma^{(N+1)}) \in E_N \times E_{N+1}^{aux}$  and

$$\lim_{p \rightarrow +\infty} T_N^p \tilde{\gamma} = (\sigma, \sigma_{N+1}) \text{ in } E_N \times E_{N+1}^{aux}$$

However, by definition of  $S_\gamma$ ,

$$T_N^p \tilde{\gamma} = \left( T^p \gamma, (T^p \gamma)^{(N+1)} \right)$$

Hence, for every fixed  $m$ ,  $(T^p \gamma)_m$  converges to  $\sigma_m$  in  $X_N$  and  $(T^p \gamma)_m^{(N+1)}$  converges uniformly on  $I$  to  $\sigma_{N+1}$ . This proves that  $\sigma$  is  $C^{N+1}$  and  $\sigma^{(N+1)} = \sigma_{N+1}$ . We conclude that  $\sigma \in E_{N+1}$  is then an attractive fixed point of  $T : E_{N+1} \rightarrow E_{N+1}$ , which proves the induction and concludes the proof of Lemma 3-4.3.  $\square$

We first show that this lemma implies that  $y \in \{|y| < y_0\} \mapsto \partial_\eta g_0(y, 0)$  is smooth. Indeed, due to the  $C^{1+\beta}$  regularity of  $E_u$ ,  $(t, y, \eta) \mapsto Z^t(y, \eta)$  is  $C^1$  and satisfies :

$$\frac{d}{dt} \partial_\eta Z^t(y, \eta) = \partial_\eta f_0(y + t, Z^t(y, \eta))$$

Specifying in  $(y, \eta) = (0, 0)$ , we have

$$\frac{d}{dt} \partial_\eta Z^t(0, 0) = \partial_\eta f_0(t, 0)$$

This exactly says that  $y \mapsto \partial_\eta g_0(y, 0)$  is  $C^1$  and has  $\partial_\eta f_0(y, 0)$  as derivative with respect to  $y$  and hence  $y \mapsto \partial_\eta g_0(y, 0)$  is smooth, as required.

Due to the relation  $g_0(0, \eta) = \eta$ , we have  $\partial_\eta g_0(0, 0) = 1$ . As a consequence, if  $y_0$  is small enough, we can assume that  $\partial_\eta g_0(y, 0) > 0$  for  $|y| < y_0$  and consider the smooth diffeomorphism defined in  $\{|y| < y_0\}$

$$\psi : y \mapsto \int_0^y \partial_\eta g_0(s, 0) ds$$

We then use the canonical transformation

$$\Psi : (y, \eta) \mapsto \left( \psi(y), \frac{\eta}{\psi'(y)} \right)$$

We finally consider the chart  $\kappa_{\rho_0} = \Psi \circ \kappa_0$  defined in  $U_{\rho_0} = U'_{\rho_0} \cap \{|y| < y_0\}$  and if  $\varepsilon_0$  is small enough, we can ensure that  $B(\rho_0, 2\varepsilon_0) \subset U_{\rho_0}$ . In this chart, the graph of  $g_0(\cdot, \zeta)$  is sent to the graph of the function

$$g : y \in \Omega := \psi((-y_0, y_0)) \mapsto \frac{g_0(\psi^{-1}(y), \zeta)}{\psi'(\psi^{-1}(y))}$$

We eventually check that

- $g(y, 0) = 0$  since  $g_0(y, 0) = 0$  ;
- $g(0, \zeta) = \zeta$  since  $\psi(0) = 0$ ,  $\psi'(0) = 1$  and  $g(0, \zeta) = \zeta$  ;
- $\partial_\eta g(y, 0) = 1$ ;
- The  $C^{1+\beta}$  norm of  $g$  is bounded uniformly in  $\rho_0$  ;
- The  $C^N$  norms of  $\kappa_{\rho_0}$  are bounded uniformly in  $\rho_0$ .

□

### 3-5 Topological pressure.

We recall the definition and some formulas for the topological pressure associated with a continuous function  $\varphi : \mathcal{T} \rightarrow \mathbb{R}$ . The dynamical system we consider is the restriction of  $F$  on  $\mathcal{T}$ . We consider a distance function  $d$  on  $\mathcal{T}$ . For  $n \in \mathbb{N}$  and  $\epsilon > 0$ , we say that a subset  $E \subset \mathcal{T}$  is  $(n, \epsilon)$  separated if for every  $\rho, \rho' \in E$ ,  $\rho \neq \rho'$ , there exists  $0 \leq i \leq n-1$ ,  $d(F^i(\rho), F^i(\rho')) > \epsilon$ .

**Definition 3-5.1.** If  $\varphi$  is a continuous function on  $\mathcal{T}$ , the topological pressure  $P(\varphi, F)$  (or simply  $P(\varphi)$  if there is no ambiguity) associated with  $\varphi \in C(\mathcal{T}, \mathbb{R})$  is defined as

$$P(\varphi) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log P_0(\varphi, n, \epsilon)$$

where

$$P_0(\varphi, n, \epsilon) = \sup \left\{ \sum_{\rho \in E} \exp \left( \sum_{i=0}^{n-1} \varphi(F^i(\rho)) \right) ; E \text{ is } (n, \epsilon) \text{ separated} \right\}$$

In this thesis, we will use another formula for the pressure. To state it, let us introduce a few notations : if  $\mathcal{Q}$  is a finite open cover of  $\mathcal{T}$ , we note  $\text{diam } \mathcal{Q} = \sup_{A \in \mathcal{Q}} \text{diam } A$  and for  $n \in \mathbb{N}$ ,  $\mathcal{Q}^{\wedge n}$  is the open cover of  $\mathcal{T}$  by the sets  $\bigcap_{i=0}^{n-1} f^{-i}(A_i)$  where  $A_0, \dots, A_{n-1} \in \mathcal{Q}$ . For  $\varphi : \mathcal{T} \rightarrow \mathbb{R}$  continuous,  $n \in \mathbb{N}$  and an open cover  $\mathcal{Q}$  of  $\mathcal{T}$ , we define

$$P_1(\varphi, n, \mathcal{Q}) = \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A} \exp \left( \sum_{i=0}^{n-1} \varphi(F^i(x)) \right) ; \alpha \subset \mathcal{Q}^{\wedge n}, \mathcal{T} \subset \bigcup_{A \in \alpha} A \right\}$$

**Proposition 3-5.1.** [Wal75] (Theorem 1.6). The following formula holds: for any  $\varphi \in C(\mathcal{T}, \mathbb{R})$ ,

$$P(\varphi) = \lim_{\text{diam } \mathcal{Q} \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log P_1(\varphi, n, \mathcal{Q}) \quad (3-5.1)$$

Note that in particular, it asserts that the limit in  $n$  exists for all open cover  $\mathcal{Q}$ .

**Relation with the flow.** This formalism applies for the trapped set of the two situations presented above, namely the billiard map (3-1.1.1.) and a Poincaré return map (3-1.1.2.). The billiard map is naturally inherited from the billiard flow and the Poincaré return map from a Hamiltonian flow. It is in fact possible to define pressures for the corresponding flow. Relations exist between these notions and we aim at explaining them in the following lines, as done in [BR75] (Section 3). It will be useful in particular in Chapter 7.

In the two situations, we have a bijection  $F : \mathcal{T} \rightarrow \mathcal{T}$  and a return time function  $t_{ret} : \mathcal{T} \rightarrow [t_{\min}, +\infty[$  for some  $t_{\min} > 0$ . In an abstract setting, we can introduce the suspension flow on

$$K = \mathcal{T} \times \mathbb{R} / \sim$$

where  $\sim$  is a relation identifying  $(\rho, 0)$  and  $(F(\rho), t_{ret}(\rho))$ , and the flow  $f^t$  on  $K$  is defined as

$$f^t(\rho, s) = (\rho, s + t) \text{ for } 0 \leq s + t \leq t_{ret}(\rho).$$

Concretely, for the Poincaré return map,  $K$  is realized by the trapped set of the Hamiltonian flow  $\Phi^t$  and  $f^t$  is nothing but its restriction on  $K$ . This is not rigorously the case for the billiard flow since there is a difference between the coordinates used for the billiard map and the global phase space in which the billiard flow lives. Nevertheless, using the smooth model of [DSW21], one could

realize  $K$  as the trapped set of a smooth open hyperbolic flow. As a consequence, in the two models, the map  $F$  is inherited from a smooth open hyperbolic flow  $\Phi^t$ .

For this flow  $(f^t)$  on  $K$ , one can define an analogue notion of pressure. We fix a metric on  $K$  (the pressure will not depend on this choice) and for  $T > 0$  and  $\varepsilon > 0$ , we say that a set  $E \subset K$  is  $(T, \varepsilon)$ -separated if for all  $\rho_1, \rho_2 \in E$ ,

$$\rho_1 \neq \rho_2 \implies \forall t \in [0, T], d(f^t(\rho_1), f^t(\rho_2)) > \varepsilon.$$

Then, we can define the topological pressure associated with a continuous function  $\psi : K \rightarrow \mathbb{R}$

$$P(\psi, (f^t)) = \lim_{\varepsilon \rightarrow 0} \limsup_{T \rightarrow +\infty} \frac{1}{T} \log Z(\psi, T, \varepsilon)$$

where

$$Z(\psi, T, \varepsilon) = \sup \left\{ \sum_{\rho \in E} \exp \left( \int_0^T \psi(f^t(\rho)) dt \right) ; E \text{ is } (T, \varepsilon)\text{-separated} \right\}$$

ow, if  $\psi : K \rightarrow \mathbb{R}$  is a continuous function on  $K$ , one can define the following function  $\varphi$  on  $\mathcal{T}$  :

$$\varphi(\rho) = \int_0^{t_{ret}(\rho)} \psi(f^t(\rho)) dt$$

Then, if  $\varphi$  is Hölder-continuous,  $P(\psi, (f^t))$  satisfies the following formula (see [BR75], Section 3): let  $\Phi = \varphi - P(\psi, (f^t))t_{ret}$ , then

$$P(\Phi, F) = 0 \tag{3-5.2}$$

By the way, since  $t_{ret} \geq t_{\min} > 0$ , the map  $s \mapsto \varphi - st_{ret}$  is strictly decreasing, so that  $P(\psi, (f^t))$  is the unique root of the equation  $P(\varphi - st_{ret}, F) = 0$ .

We conclude this section by specializing these formulas in the particular case of the unstable Jacobian. As explained, in the two models we consider, the map  $F$  and the trapped set  $K$  are inherited from a smooth open hyperbolic flow  $\Phi^t$ . By fixing a Riemannian metric on  $K$ , one can define a local unstable Jacobian  $\lambda_u^t(\rho)$ , which is the determinant of the differential  $D\Phi^t(\rho) : E_u(\rho) \rightarrow E_u(\rho)$ , where  $E_u(\rho)$  is the unstable space at  $\rho \in K$ . We can now consider the following map

$$\psi^u(\rho) = - \left. \frac{d \log \lambda_u^t(\rho)}{dt} \right|_{t=0} = - \left. \frac{d \lambda_u^t(\rho)}{dt} \right|_{t=0}$$

and the following important parameter

$$\gamma_{cl} = -P(\psi^u, (f^t)) \tag{3-5.3}$$

$\gamma_{cl}$  is called the classical decay rate of the system. For the billiard flow, it coincides with the one defined in Chapter 2, in the paragraph after (2-1.5). It is not hard to see that for this special choice

$$\int_0^{t_{ret}} \psi^u(f^t(\rho)) dt = \phi_u(\rho)$$

where  $\phi_u$  is defined in (3-1.5) (and provided the Riemannian metric on  $\mathcal{T}$  used in this definition coincides with the global one on  $K$ ). (3-5.2) now reads

$$P(\phi_u + \gamma_{cl} t_{ret}, F) = 0 \tag{3-5.4}$$

### Remark.

In the case of the Hamiltonian flow, there is not a canonical choice of Poincaré section, and both the return map and the return time function depend on this choice. Nevertheless, the classical decay rate depends on the flow, and not on this choice. In particular, (3-5.4) holds no matter the choice of Poincaré sections.

### 3-6 Fractal dimension.

Let us recall the definition of the upper box dimension of a compact metric space  $(X, d)$ . We denote by  $N_X(\varepsilon)$  the minimal number of open balls of radius  $\varepsilon$  needed to cover  $X$ . Then, the upper box dimension of  $X$  is defined as :

$$\overline{\dim} X := \limsup_{\varepsilon \rightarrow 0} \frac{\log N_X(\varepsilon)}{-\log \varepsilon} \quad (3-6.1)$$

In particular, if  $\delta > \overline{\dim} X$ , there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \leq \varepsilon_0$ ,  $N_X(\varepsilon) \leq \varepsilon^{-\delta}$ .

We record the following well known result (see for instance [Bar08], Theorem 4.3.2). Recall the definition of  $\phi_u$  in (3-1.5).

**Proposition 3-6.1.** There exists a unique  $s_0 \in \mathbb{R}$  such that  $P(s_0 \phi_u) = 0$ . Moreover, for every  $\rho \in \mathcal{T}$ ,  $\overline{\dim}(\mathcal{T} \cap W_u(\rho)) = \overline{\dim}(\mathcal{T} \cap W_s(\rho)) = s_0$ . Finally,  $\overline{\dim} \mathcal{T} = 2s_0$ .

**Remark.**

| In fact, this holds also for the Hausdorff dimension but we will mainly use the upper-box dimension for practical and technical reasons. In the following, we note  $s_0 = d_H$ .

We will need the slightly more precise following result, which allows to control  $N_{W_{u/s}(\rho) \cap \mathcal{T}}$  uniformly in  $\rho$  :

**Proposition 3-6.2.** There exists  $\varepsilon_1 > 0$  such that the following holds. For every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that for every  $\rho \in \mathcal{T}$ , if  $X_\rho = W_{u/s}(\rho) \cap \mathcal{T}$ ,

$$N_{X_\rho}(r) \leq C_\varepsilon r^{-(d_H + \varepsilon)}; \quad \forall r \leq \varepsilon_1$$

*Proof.* Obviously, this holds at every  $\rho \in \mathcal{T}$  with  $C_\varepsilon$  a priori depending on  $\rho$ . The uniformity is a consequence of the fact that the holonomy maps are Lipschitz, with uniform Lipschitz norm due to the compactness of  $\mathcal{T}$  in virtue of Corollary 3-3.1. Then, due to the compactness of  $\mathcal{T}$ , one can estimate  $N_{W_{u/s}(\rho) \cap \mathcal{T}}(\varepsilon)$  by considering only a finite number of (un)stable leaves as references and apply : *Assume that  $(X, d)$ ,  $(Y, d')$  are compact metric spaces and  $f : X \rightarrow Y$  is  $C$ -Lipschitz. Then, for every  $\varepsilon > 0$ ,*

$$N_{f(X)}(\varepsilon) \leq N_X(\varepsilon/C)$$

□

We finish by a lemma estimating the number of balls of size  $\delta$  needed to cover  $\mathcal{T} \cap W_u(\rho_0) \cap J$  where  $J \subset W_u(\rho_0)$  is an interval of size  $l$ . The difference with Proposition 3-6.2 is that the size of  $J$  can be much small than  $\varepsilon_1$ .

**Lemma 3-6.1.** Let  $\rho_0 \in \mathcal{T}$ . Let  $\kappa : U_0 \rightarrow V_0 \subset \mathbb{R}$  be a smooth chart such that the image of the local unstable manifold passing through  $\rho_0$  is given by a graph

$$\kappa(W_u(\rho_0) \cap U_0) = \{(x, g(x)), x \in I\}$$

for some open interval  $I$ . For  $J \subset I$ , let's note

$$X(J) = \{x \in J, \kappa^{-1}(x, g(x)) \in \mathcal{T}\}$$

Then, for every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  depending only on  $\varepsilon$ ,  $F$  and  $\kappa$  such that : for all  $J \subset I$  interval of length  $l$  and for all  $\delta \in ]0, l]$ ,

$$N_{X(J)}(\delta) \leq C_\varepsilon \left( \frac{\delta}{l} \right)^{-(d_H + \varepsilon)}$$

*Proof.* Let's note  $N = N_{X(J)}(\delta)$ . If  $N = 0$ , there is obviously nothing to prove. So let's assume that  $N \neq 0$  and let's fix a reference point  $x_0 \in X(J)$  : to  $x_0$  corresponds a point  $\rho_0 = \kappa^{-1}(x_0, g(x_0)) \in \mathcal{T}$  and we are interested in a piece of unstable manifold of  $\rho_0$  of size  $l$ .



We know that the upper-box dimension of each  $\mathcal{T} \cap W_u(\rho)$  is equal to  $d_H$ . However, since here we are interested by a small piece of an unstable manifold of size  $l$ , we will expand this piece to reach a size of order 1. We note  $J_0 = \kappa^{-1}(\{(x, g(x)), x \in J\})$  and for  $m \in \mathbb{N}$ , we note  $\rho_m = F^m(\rho_0)$  and introduce

$$T := \max\{m \in \mathbb{N}, F^m(J) \subset W_u(\rho_m) \text{ and } \text{diam} F^m(J_0) \leq \varepsilon_1\}$$

In particular, the definition of  $T$  implies that for all  $\rho \in J_0$ ,  $F^m(\rho)$  is well-defined for  $0 \leq m \leq T$  and satisfies  $d(F^m(\rho), F^m(\rho_0)) \leq \varepsilon_1$ .

**Claim :** We first claim that if  $J' \subset J$  is a subinterval with  $X(J') \neq \emptyset$ , then

$$\text{diam } J'_T \sim \frac{\text{diam } J'}{J_{-T}^u(\rho_T)}$$

where  $J'_T = F^T(J'_0)$  for  $J'_0 = \kappa^{-1}(\{(x, g(x)), x \in J'\})$ . In particular, it holds for  $J' = J$ .

**Proof of the claim :** Let's prove this claim and suppose that  $J' \subset J$  is an interval of length  $l'$  and consider  $x' \in X(J')$ . Let's note  $\rho' = \kappa^{-1}(x', g(x')) \in \mathcal{T}$ . If  $\hat{x} \in J'$  and  $\hat{\rho} = \kappa^{-1}(\hat{x}, g(\hat{x})) \in W_u(\rho')$ , we have

$$d(F^T(\rho'), F^T(\hat{\rho})) \sim \frac{d(\rho', \hat{\rho})}{J_{-T}^u(F^T(\rho'))} \sim \frac{|x' - \hat{x}|}{J_{-T}^u(F^T(\rho'))}$$

Since  $d(F^m(\rho'), F^m(\rho_0)) \leq \varepsilon_1$  for  $0 \leq m \leq T$ , we have

$$J_{-T}^u(F^T(\rho')) \sim J_{-T}^u(\rho_T)$$

In particular, if we choose  $\hat{x}$  such that  $|x' - \hat{x}| \geq \text{diam } J'/3$ , we have

$$\text{diam } J'_T \geq C^{-1} \frac{|x' - \hat{x}|}{J_{-T}^u(\rho_T)} \geq C^{-1} \frac{\text{diam } J'}{J_{-T}^u(\rho_T)}$$

For the converse inequality, assume that  $\rho_1, \rho_2 \in J'_0$ .

$$d(F^T(\rho_1), F^T(\rho_2)) \leq d(F^T(\rho_1), F^T(\rho')) + d(F^T(\rho'), F^T(\rho_2)) \leq C \frac{|x_1 - x'| + |x' - x_2|}{J_{-T}^u(\rho_T)} \leq C \frac{\text{diam } J'}{J_{-T}^u(\rho_T)}$$

which finally gives the required inequality by taking the supremum over  $\rho_1$  and  $\rho_2$ .

**End of proof.** We have,  $J_{-T}^u(\rho_T) \text{diam } J_T \sim \text{diam } J \sim l$ . By definition of  $T$ ,

$$\text{diam } F^{T+1}(J_0) \geq \varepsilon_1$$

so that

$$\text{diam } F^T(J_0) \geq C^{-1} \varepsilon_1$$

and hence,  $J_{-T}^u(\rho_T) \leq Cl$  (this  $C$  also depends on  $\varepsilon_1$ , which is not a problem since  $\varepsilon_1$  depends only on  $F$ ). Let us fix  $k > 0$ , to be determined later. By Proposition 3-5.1, we can cover  $J_T \cap \mathcal{T}$  by  $N$  balls of diameter at most  $k\delta$  with  $N \leq C_\varepsilon(k\delta)^{-d_H - \varepsilon}$ . Let's choose  $\rho_1, \dots, \rho_N \in \mathcal{T} \cap J_T$  such that

$$\mathcal{T} \cap J_T \subset \bigcup_{i=1}^N B(\rho_i, k\delta).$$

We note  $x_i$  the point in  $J$  such that  $\rho_i = F^T(\rho_i)$  with  $\kappa(\rho_i) = (x_i, g(x_i))$ . If  $x \in X(J)$ , then  $\rho := F^T(\kappa^{-1}(x, g(x))) \in \mathcal{T} \cap J_T$  and there exists  $i \in \{1, \dots, N\}$  such that  $d(\rho_i, \rho) \leq k\delta$ . As a consequence,  $|x - x_i| \leq C J_{-T}^u(\rho_T) d(\rho_i, \rho) \leq C' l k \delta$  for some constant  $C'$  depending on  $F$  and  $\kappa$ . We now fix  $k = (2C'l)^{-1}$ , so that  $X(J)$  can be covered by  $N$  intervals of length  $\delta$ . As a consequence,

$$N_{X(J)}(\delta) \leq N \leq C_\varepsilon \left( \frac{\delta}{2C'l} \right)^{-d_H - \varepsilon} = C'_\varepsilon \left( \frac{\delta}{l} \right)^{-d_H - \varepsilon}$$

□

# Chapter 4

## Semiclassical analysis : Pseudodifferential calculus and Fourier integral operators

In this Chapter, we gather important definitions and results of semiclassical analysis. In particular, in Section 4-3, we define a central object of this thesis : **open hyperbolic quantum maps**. They are Fourier Integral operators (see Section 4-2) associated with the open hyperbolic maps introduced in the previous chapter (Section 3-1).

### 4-1 Pseudodifferential operators and Weyl quantization

We recall some basic notions and properties of the Weyl quantization on  $\mathbb{R}^n$ . We refer the reader to [Zwo12] for the proofs of the statements and further considerations on semiclassical analysis and quantizations. We start by defining classes of  $h$ -dependent symbols.

**Definition 4-1.1.** Let  $0 \leq \delta \leq \frac{1}{2}$ . We say that an  $h$ -dependent family  $a := (a(\cdot; h))_{0 < h \leq 1}$  is in the class  $S_\delta(T^*\mathbb{R}^n)$  (or simply  $S_\delta$  if there is no ambiguity) if for every  $\alpha \in \mathbb{N}^{2n}$ , there exists  $C_\alpha > 0$  such that :

$$\forall 0 < h \leq 1, \sup_{(x, \xi) \in \mathbb{R}^{2n}} |\partial^\alpha a(x, \xi; h)| \leq C_\alpha h^{-\delta|\alpha|}$$

In this paper, we will mostly be concerned with  $\delta < 1/2$ . We will also use the notation  $S_{0+} = \bigcap_{\delta > 0} S_\delta$ .

We write  $a = O(h^N)_{S_\delta}$  to mean that for every  $\alpha \in \mathbb{N}^{2n}$ , there exists  $C_{\alpha, N}$  such that

$$\forall 0 < h \leq 1, \sup_{(x, \xi) \in \mathbb{R}^{2n}} |\partial^\alpha a(x, \xi; h)| \leq C_{\alpha, N} h^{-\delta|\alpha|} h^N$$

If  $a = O(h^N)_{S_\delta}$  for all  $N \in \mathbb{N}$ , we'll write  $a = O(h^\infty)_{S_\delta}$ .

For a given symbol  $a \in S_\delta(T^*\mathbb{R}^n)$ , we say that  $a$  has a compact essential support if there exists a compact set  $K$  such :

$$\forall \chi \in C_c^\infty(\mathbb{R}^n), \text{supp } \chi \cap K = \emptyset \implies \chi a = O(h^\infty)_{S(T^*\mathbb{R}^n)}$$

(here  $\mathcal{S}$  stands for the Schwartz space). We note  $\text{ess supp } a \subset K$  and say that  $a$  belongs to the class  $S_\delta^{\text{comp}}(T^*\mathbb{R}^n)$ . The essential support of  $a$  is then the intersection of all such compact  $K$ 's. In particular, the class  $S_\delta^{\text{comp}}$  contains all the symbols in  $S_\delta$  supported in a  $h$ -independent compact set and these symbols correspond, modulo  $O(h^\infty)_{S(T^*\mathbb{R}^n)}$ , to all symbols of  $S_\delta^{\text{comp}}$ . For this reason, we will adopt the following notation : for an open set  $\Omega \subset \mathbb{R}^n$ ,  $a \in S_\delta^{\text{comp}}(\Omega) \iff \text{ess supp } a \Subset \Omega$ .

For a symbol  $a \in S_\delta(T^*\mathbb{R}^n)$ , we will quantize it using Weyl's quantization procedure. It is written as :

$$(\text{Op}_h(a)u)(x) = (a^W(x, hD_x)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} a\left(\frac{x+y}{2}, \xi\right) u(y) e^{i\frac{(x-y)\cdot\xi}{h}} dy d\xi$$

We will note  $\Psi_\delta(\mathbb{R}^n)$  the corresponding classes of pseudodifferential operators. By definition, the wavefront set of  $A = \text{Op}_h(a)$  is  $\text{WF}_h(A) = \text{ess supp } a$ .

We say that a family  $u = u(h) \in \mathcal{D}'(\mathbb{R}^n)$  is  $h$ -tempered if for every  $\chi \in C_c^\infty(\mathbb{R}^n)$ , there exist  $C > 0$  and  $N \in \mathbb{N}$  such that  $\|\chi u\|_{H_h^{-N}} \leq Ch^{-N}$ . For a  $h$ -tempered family  $u$ , we say that a point  $\rho \in T^*\mathbb{R}^n$  does *not* belong to the wavefront set of  $u$  if there exists  $a \in S^{comp}(T^*\mathbb{R}^n)$  such that  $a(\rho) \neq 0$  and  $\text{Op}_h(a)u = O(h^\infty)_S$ . We note  $\text{WF}_h(u)$  the wavefront set of  $u$ .

We say that a family of operators  $B = B(h) : C_c^\infty(\mathbb{R}^{n_2}) \rightarrow \mathcal{D}'(\mathbb{R}^{n_1})$  is  $h$ -tempered if its Schwartz kernel  $\mathcal{K}_B \in \mathcal{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  is  $h$ -tempered. The wavefront set of  $B$ , denoted  $\text{WF}_h'(B)$  is defined as

$$\text{WF}_h'(B) = \{(x, \xi, y, -\eta) \in T^*\mathbb{R}^{n_1} \times T^*\mathbb{R}^{n_2}, (x, \xi, y, \eta) \in \text{WF}_h(\mathcal{K}_B)\}$$

Let us now recall standard results in semi-classical analysis concerning the  $L^2$ -boundedness of pseudodifferential operators and their composition. We'll use the following version of Calderon-Vaillancourt Theorem ([Zwo12], Theorem 4.23).

**Theorem 4-1.1.** *There exists  $C_n > 0$  such that the following holds. For every  $0 \leq \delta < \frac{1}{2}$ , and  $a \in S_\delta(T^*\mathbb{R}^n)$ ,  $\text{Op}_h(a)$  is a bounded operator on  $L^2$  and*

$$\|\text{Op}_h(a)\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C_n \sum_{|\alpha| \leq 8n} h^{|\alpha|/2} \|\partial^\alpha a\|_{L^\infty}$$

As a consequence of the sharp Gårding inequality (see [Zwo12], Theorem 4.32), we also have the precise estimate of  $L^2$  norms of pseudodifferential operator,

**Proposition 4-1.1.** Assume that  $a \in S_\delta(\mathbb{R}^{2n})$ . Then, there exists  $C_a$  depending on a finite number of semi-norms of  $a$  such that :

$$\|\text{Op}_h(a)\|_{L^2 \rightarrow L^2} \leq \|a\|_\infty + C_a h^{\frac{1}{2}-\delta}$$

We recall that the Weyl quantizations of real symbols are self-adjoint in  $L^2$ . The composition of two pseudodifferential operators in  $\Psi_\delta$  is still a pseudodifferential operator. More precisely (see [Zwo12], Theorem 4.11 and 4.18), if  $a, b \in S_\delta$ ,  $\text{Op}_h(a) \circ \text{Op}_h(b)$  is given by  $\text{Op}_h(a \# b)$ , where  $a \# b$  is the Moyal product of  $a$  and  $b$ . It is given by

$$a \# b(\rho) = e^{ihA(D)}(a \otimes b)|_{\rho=\rho_1=\rho_2}$$

where  $a \otimes b(\rho_1, \rho_2) = a(\rho_1)b(\rho_2)$ ,  $e^{ihA(D)}$  is a Fourier multiplier acting on functions on  $\mathbb{R}^{4n}$  and, writing  $\rho_i = (x_i, \xi_i)$ ,

$$A(D) = \frac{1}{2}(D_{\xi_1} \circ D_{x_2} - D_{x_1} \circ D_{\xi_2})$$

We can estimate the Moyal product by a quadratic stationary phase and get the following expansion: for all  $N \in \mathbb{N}$ ,

$$a \# b(\rho) = \sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (a \otimes b)|_{\rho_1=\rho_2=\rho} + r_N$$

where for all  $\alpha \in \mathbb{N}^{2n}$ , there exists  $C_\alpha$ , independent of  $a$  and  $b$ , such that

$$\|\partial^\alpha r_N\|_\infty \leq C_\alpha h^N \|a \otimes b\|_{C^{2N+4n+1+|\alpha|}}$$

As a consequence of this asymptotic expansion, we have the precise product formula :

**Lemma 4-1.1.** For every  $N \in \mathbb{N}$ , there exists  $C_N > 0$  such that, for every  $a, b \in S_\delta(\mathbb{R}^n)$ ,

$$\text{Op}_h(a) \circ \text{Op}_h(b) = \text{Op}_h \left( \sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (a \otimes b)|_{\rho=\rho_1=\rho_2} \right) + R_N \quad (4-1.1)$$

where

$$\|R_N\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_N h^N \|a \otimes b\|_{C^{2N+12n+1}} \quad (4-1.2)$$

**Remark.**

It will be important in the sequel to understand the derivatives of  $a$  and  $b$  involved in the  $k$ -th term of the previous expansion. A quick recurrence using the precise form of the operator  $A(D)$  shows that  $A(D)^k(a \otimes b)(\rho_1, \rho_2)$  is of the form

$$\sum_{|\alpha|=k, |\beta|=k} \lambda_{\alpha, \beta} \partial^\alpha a(\rho_1) \partial^\beta b(\rho_2)$$

This can be rewritten  $l_k(d^k a(\rho_1), d^k b(\rho_2))$  where  $l_k$  is a bilinear form on the spaces of  $k$ -symmetric forms on  $\mathbb{R}^{2n}$ . Of, course, we make use of the the identifications  $T_{\rho_1} T^* \mathbb{R}^n \simeq T_{\rho_2} T^* \mathbb{R}^n \simeq \mathbb{R}^{2n}$

As a simple corollary, we get an expression for the commutator of pseudodifferential operators.

**Corollary 4-1.1.** For every  $N \in \mathbb{N}$ , there exists  $C_N > 0$  such that, for every  $a, b \in S_\delta(\mathbb{R}^n)$ ,

$$[\text{Op}_h(a), \text{Op}_h(b)] = \text{Op}_h \left( \frac{h}{i} \{a, b\} + \sum_{k=2}^{N-1} h^k L_k(d^k a, d^k b) \right) + R_N$$

where

$$\|R_N\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_N h^N \|a \otimes b\|_{C^{2N+12n+1}}$$

where the  $L_k$  are bilinear forms on the spaces of  $k$ -symmetric forms on  $\mathbb{R}^{2n}$ .

## 4-2 Fourier Integral Operators and Egorov's theorem

We now review some aspects of the theory of Fourier integral operators. We follow [Zwo12], Chapter 11 and [NSZ14]. We refer the reader to [GS13] for further details. We introduce the material needed to understand the definition 4-3.2 of open hyperbolic quantum maps in Section 4-3. We also provide a quantitative version of Egorov's theorem.

### 4-2.1. Local symplectomorphisms and their quantization

We momentarily work in dimension  $n$ . Let us note  $\mathcal{K}$  the set of symplectomorphisms  $\kappa : T^* \mathbb{R}^n \rightarrow T^* \mathbb{R}^n$  such that the following holds : there exist continuous and piecewise smooth, with respect to  $t$ , families of smooth functions  $(\kappa_t)_{t \in [0,1]}$ ,  $(q_t)_{t \in [0,1]}$  such that :

- $\forall t \in [0, 1]$ ,  $\kappa_t : T^* \mathbb{R}^n \rightarrow T^* \mathbb{R}^n$  is a symplectomorphism ;
- $\kappa_0 = \text{Id}_{T^* \mathbb{R}^n}$ ,  $\kappa_1 = \kappa$  ;
- $\forall t \in [0, 1]$ ,  $\kappa_t(0) = 0$  ;
- there exists  $K \Subset T^* \mathbb{R}^n$  compact such that  $\forall t \in [0, 1]$ ,  $q_t : T^* \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\text{supp } q_t \subset K$  ;
- $\frac{d}{dt} \kappa_t = (\kappa_t)^* H_{q_t}$

If  $\kappa \in \mathcal{K}$ , we note  $G_\kappa = Gr'(\kappa) = \{(x, \xi, y, -\eta), (x, \xi) = \kappa(y, \eta)\}$  the twisted graph of  $\kappa$ , which is Lagrangian in  $T^* \mathbb{R}^n$ . We recall [Zwo12], Lemma 11.4, which asserts that local symplectomorphisms fixing the origin can be seen as elements of  $\mathcal{K}$ , as soon as we have some geometric freedom.

**Lemma 4-2.1.** Let  $U_0, U_1$  be open and precompact subsets of  $T^* \mathbb{R}^n$ . Assume that  $\kappa : U_0 \rightarrow U_1$  is a local symplectomorphism fixing 0 and which extends to  $V_0 \ni U_0$  an open star-shaped neighborhood of 0. Then, there exists  $\tilde{\kappa} \in \mathcal{K}$  such that  $\tilde{\kappa}|_{U_0} = \kappa$ .

If  $\kappa \in \mathcal{K}$  and if  $(q_t)$  denotes the family of smooth functions associated with  $\kappa$  in its definition, we note  $Q(t) = \text{Op}_h(q_t)$ . It is a continuous and piecewise smooth family of operators. Then the Cauchy problem

$$\begin{cases} hD_t U(t) + U(t)Q(t) = 0 \\ U(0) = \text{Id} \end{cases} \quad (4-2.1)$$

is globally well-posed.

Following [NSZ14], Definition 3.9, we adopt the definition (with  $G_\kappa = Gr'(\kappa)$ ) :

**Definition 4-2.1.** Fix  $\delta \in [0, 1/2)$ . We say that  $T \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n; G_\kappa)$  if there exist  $a \in S_\delta(T^* \mathbb{R}^n)$  and a path  $(\kappa_t)$  from  $\text{Id}$  to  $\kappa$  satisfying the above assumptions such that  $T = \text{Op}_h(a)U(1)$ , where

$t \mapsto U(t)$  is the solution of the Cauchy problem (4-2.1).

The class  $I_{0^+}(\mathbb{R} \times \mathbb{R}, G_\kappa)$  is by definition  $\bigcap_{\delta > 0} I_\delta(\mathbb{R} \times \mathbb{R}, G_\kappa)$ .

It is a standard result, known as Egorov's theorem (see [Zwo12], Theorem 11.1) that if  $U(t)$  solves the Cauchy problem (4-2.1) and if  $b_0 \in S_\delta$ , then  $U(1)^{-1} \text{Op}_h(b_0)U(1)$  is a pseudodifferential operator in  $\Psi_\delta$  and if  $b_1 = b_0 \circ \kappa$ , then  $U(t)^{-1} \text{Op}_h(b_0)U(t) - \text{Op}_h(b_1) \in h^{1-2\delta}\Psi_\delta$ .

**Remark.**

Applying Egorov's theorem and Beal's theorem, it is possible to show that if  $(\kappa_t)$  is a closed path from Id to Id, and  $U(t)$  solves (4-2.1), then  $U(1) \in \Psi_0(\mathbb{R}^n)$ . In other words,  $I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}'(\text{Id})) \subset \Psi_\delta(\mathbb{R}^n)$ . But the other inclusion is trivial. Hence, this in an equality :

$$I_\delta(\mathbb{R}^n \times \mathbb{R}^n, \text{Gr}'(\text{Id})) = \Psi_\delta(\mathbb{R}^n)$$

The notation  $I(\mathbb{R}^n \times \mathbb{R}^n, G_\kappa)$  comes from the fact that the Schwartz kernels of such operators are Lagrangian distributions associated with  $G_\kappa$ , and in particular have wavefront sets included in  $C$ . As a consequence, if  $T \in I_\delta(\mathbb{R}^n \times \mathbb{R}^n, G_\kappa)$ ,  $\text{WF}'_h(T) \subset \text{Gr}(\kappa)$ .

Let us state a simple proposition concerning the composition of Fourier integral operators :

**Proposition 4-2.1.** Let  $\kappa_1, \kappa_2 \in \mathcal{K}$  and  $T_1 \in I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}'(\kappa_1)), T_2 \in I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}'(\kappa_1))$ . Then,

$$T_1 \circ T_2 \in I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}'(\kappa_1 \circ \kappa_2))$$

*Proof.* Let's write  $T_1 = \text{Op}_h(a_1)T_1(1)$ ,  $T_2 = \text{Op}_h(a_2)T_2(1)$  with the obvious notations associated with the Cauchy problems (4-2.1) for  $\kappa_1$  and  $\kappa_2$ . Egorov's theorem asserts that  $T_1(1) \text{Op}_h(a_2)T_1(1)^{-1} = \text{Op}_h(b_2)$  for some  $b_2 \in S_\delta$  and  $\text{Op}_h(a_1) \text{Op}_h(b_2) = \text{Op}_h(a_1 \# b_2)$ . It is then enough to focus on the case  $a_1 = a_2 = 1$ . We set

$$T_3(t) := \begin{cases} T_1(2t) & \text{for } 0 \leq t \leq 1/2 \\ T_1(1) \circ T_2(2t-1) & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

It solves the Cauchy problem

$$\begin{cases} hD_t T_3(t) + T_3(t)Q_3(t) = 0 \\ T_3(0) = \text{Id} \end{cases}$$

with

$$Q_3(t) := \begin{cases} 2Q_1(2t) & \text{for } 0 \leq t \leq 1/2 \\ 2Q_2(2t-1) & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

To conclude the proof, it is enough to notice that this Cauchy problem is associated with the path  $\kappa_3(t)$  between  $\kappa(0) = \text{Id}$  and  $\kappa_3(1) = \kappa_1 \circ \kappa_2$  where

$$\kappa_3(t) := \begin{cases} \kappa_1(2t) & \text{for } 0 \leq t \leq 1/2 \\ \kappa_1 \circ \kappa_2(2t-1) & \text{for } 1/2 \leq t \leq 1 \end{cases}$$

□

#### 4-2.2. Precise version of Egorov's theorem

We will need a more quantitative version of Egorov's theorem, similar to the one in [DJN21] (Lemma A.7). The result does not show that  $U(1)^{-1} \text{Op}_h(a)U(1)$  is a pseudodifferential operator (one would need Beal's theorem to say that) but it gives a precise estimate on the remainder, depending on the semi-norms of  $a$ . We now specialize to the case of dimension  $n = 1$  but the following result holds in any dimension by changing the constant 15 in something of the form  $Mn$ .

**Proposition 4-2.2.** Consider  $\kappa \in \mathcal{K}$  and note  $U(t)$  the solution of (4-2.1). There exists a family of differential operators  $(D_j)_{j \in \mathbb{N}}$  of order  $j$  such that for all  $a \in S_\delta$  and all  $N \in \mathbb{N}$ ,

$$U(1)^{-1} \text{Op}_h(a)U(1) = \text{Op}_h \left( a \circ \kappa + \sum_{j=1}^{N-1} h^j (D_{j+1} a) \circ \kappa \right) + O_\kappa(h^N \|a\|_{C^{2N+15}}) \quad (4-2.2)$$

*Proof.* We keep the notations introduced previously. Let us first note

$$A_0(t) = U(t) \text{Op}_h(a \circ \kappa_t) U(t)^{-1}$$

and compute

$$\begin{aligned} U(t)^{-1} \partial_t A_0(t) U(t) &= -\frac{i}{h} [Q(t), \text{Op}_h(a \circ \kappa_t)] + \text{Op}_h(\{q_t, a \circ \kappa_t\}) \\ &= \text{Op}_h(\{q_t, a \circ \kappa_t\}) - \frac{i}{h} \left( \text{Op}_h \left( \frac{h}{i} \{q_t, a \circ \kappa_t\} + \sum_{j=2}^N h^j L_j(d^j q_t, d^j(a \circ \kappa_t)) \right) \right) \\ &\quad + O(h^N \|q_t \otimes (a \circ \kappa_t)\|_{C^{2(N+1)+13}}) \\ &= \text{Op}_h \left( \sum_{j=1}^{N-1} -i h^j L_{j+1}(d^{j+1} q_t, d^{j+1}(a \circ \kappa_t)) \right) + O_{\kappa_t}(h^N \|a\|_{C^{2N+15}}) \end{aligned}$$

We now define by induction a family of functions  $a_j(t), j = 0, \dots, N-1$  by

$$a_0(t) = a; \quad a_k(t) = \sum_{m=0}^{k-1} \int_0^t i L_{k+1-m}(d^{k+1-m} q_s, d^{k+1-m}(a_m(s) \circ \kappa_s)) \circ \kappa_s^{-1} ds$$

and set  $A_k(t) = U(t) \text{Op}_h \left( \sum_{j=0}^k h^j a_j(t) \circ \kappa_t \right) U(t)^{-1}$ . We first remark by an easy induction on  $k$ , that  $a_k(t)$  is of the form  $D_{k+1}(t)a$  where  $D_{k+1}(t)$  is a differential operator of order at most  $k+1$ , with coefficients depending continuously on  $t$  and on  $(\kappa_t)_t$ . We now check by induction the following :

$$U(t)^{-1} \partial_t A_k(t) U(t) = -i \text{Op}_h \left( \sum_{j=k+1}^{N-1} \sum_{m=0}^k h^j L_{j+1-m}(d^{j+1-m} q_t, d^{j+1-m}(a_m(t) \circ \kappa_t)) \right) + O_{\kappa_t}(h^N \|a\|_{C^{2N+15}})$$

We've already done it for  $k=0$ . Let's assume that the equality holds for  $k-1$  and let's prove it for  $k \geq 1$ .

$$U(t)^{-1} \partial_t A_k(t) U(t) = U(t)^{-1} \partial_t A_{k-1}(t) U(t) + h^k U(t)^{-1} \partial_t \text{Op}_h(a_k(t) \circ \kappa_t) U(t)$$

Let's compute the second part of the right hand side.

$$\begin{aligned} &U(t)^{-1} \partial_t \text{Op}_h(a_k(t) \circ \kappa_t) U(t) \\ &= -\frac{i}{h} [Q(t), \text{Op}_h(a_k(t) \circ \kappa_t)] + \text{Op}_h(\{q_t, a_k(t) \circ \kappa_t\}) + \text{Op}_h(\partial_t a_k(t) \circ \kappa_t) \\ &= -i \text{Op}_h \left( \sum_{l=1}^{N-1-k} h^l L_{l+1}(d^{l+1} q_t, d^{l+1}(a_k(t) \circ \kappa_t)) \right) + O_{\kappa_t}(h^{N-k} \|a_k(t)\|_{C^{2(N+1-k)+13}}) + \text{Op}_h(\partial_t a_k(t) \circ \kappa_t) \end{aligned}$$

We can estimate the remainder by

$$O_{\kappa_t}(h^{N-k} \|a_k(t)\|_{C^{2(N+1-k)+13}}) = O_{\kappa_t}(h^{N-k} \|a\|_{C^{2(N+1-k)+13+k+1}}) = O_{\kappa_t}(h^{N-k} \|a\|_{C^{2N+15}})$$

We now combine this with the value of

$$U(t)^{-1} \partial_t A_{k-1}(t) U(t) = -i \text{Op}_h \left( \sum_{j=k}^{N-1} \sum_{m=0}^{k-1} h^j L_{j+1-m}(d^{j+1-m} q_t, d^{j+1-m}(a_m(t) \circ \kappa_t)) \right) + O_{\kappa_t}(h^N \|a\|_{C^{2N+15}})$$

By definition of  $a_k(t)$ , the term  $h^k \text{Op}_h(\partial_t a_k(t) \circ \kappa_t)$  cancels the term corresponding to  $j=k$  in the sum. Moreover, for every  $j > k$ , writing  $j = k+l, l \in \{1, \dots, N-1-k\}$ , the term  $h^{k+l} L_{l+1}(d^{l+1} q_t, d^{l+1}(a_k(t) \circ \kappa_t))$ , gives the missing term  $h^j L_{j+1-k}(d^{j+1-k} q_t, d^{j+1-k}(a_k(t) \circ \kappa_t))$ . This gives the required equality for  $A_k(t)$ .

In particular,  $\partial_t A_{N-1}(t) = O_\kappa(h^N \|a\|_{C^{2N+15}})$ . We now use the fact that at  $t = 0$ ,  $a_0(0) = a$ ,  $a_k(0) = 0$ ,  $k = 1, \dots, N-1$ ,  $U(0) = \text{Id}$ ,  $\kappa_0 = \text{Id}$ , and hence  $A_{N-1}(0) = \text{Op}_h(a)$ . Integrating between 0 and 1, we hence have

$$A_{N-1}(t) - \text{Op}_h(a) = O_\kappa(h^N \|a\|_{C^{2N+15}})$$

Conjugating by  $U(1)$ , we finally have

$$U(1)^{-1} \text{Op}_h(a) U(1) = \text{Op}_h \left( a \circ \kappa + \sum_{k=1}^{N-1} h^k a_k(1) \circ \kappa \right) + O_\kappa(h^N \|a\|_{C^{2N+15}})$$

which is the what we wanted, since  $a_k(1) = D_{k+1}(1)a$ .  $\square$

By using local charts and composition results (see for instance [DZ19], Proposition E.10), it is possible to build local Fourier integral operators, which, combined with the last proposition, gives

**Proposition 4-2.3.** Let  $V \subset \mathbb{R}^2 = T^*\mathbb{R}$  an open set and  $\kappa : V \rightarrow U \subset \mathbb{R}^2$  a symplectic map. Fix  $\rho_V \in V$ . There exists  $W \subset V' \subset V$ , neighborhoods of  $\rho_V$  and a pair  $(B, B')$  of Fourier integral operators in  $I_0(\kappa(V') \times V', \text{Gr}'(\kappa)) \times I_0(V' \times \kappa(V'), \text{Gr}'(\kappa^{-1}))$  which satisfy : there exists differential operator  $(L_j)_{j \geq 1}$  of order  $2j$  and supported in  $V'$  such that for all  $a \in S_\delta(\mathbb{R}^2)$  with  $\text{supp } a \subset W$  and for all  $N \in \mathbb{N}$ ,

$$B \text{Op}_h(a) B' = \text{Op}_h(a \circ \kappa^{-1}) + \sum_{j=1}^{N-1} h^j \text{Op}_h((L_j a) \circ \kappa^{-1}) + O(h^N \|a\|_{C^{2N+M}})$$

for some universal integer  $M$ .

*Proof.* It is enough to treat the case  $\rho_V = 0 = \kappa(\rho_V)$ . It suffices to consider sufficiently small neighborhoods of 0 so that the restriction of  $\kappa$  can be seen as the restriction of an element of  $\mathcal{K}$ . Then, one uses Proposition 4-2.2.  $\square$

### 4-2.3. An important example

Let us focus on a particular case of canonical transformations. Suppose that  $\kappa : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$  is a canonical transformation such that

$$(x, \xi, y, \eta) \in \text{Gr}(\kappa) \mapsto (x, \eta)$$

is a local diffeomorphism near  $(x_0, \xi_0, y_0, \eta_0)$ . Then, there exists a phase function  $\psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ ,  $\Omega_x, \Omega_\eta$  open sets of  $\mathbb{R}^n$  and  $\Omega$  a neighborhood of  $(x_0, \xi_0, y_0, -\eta_0)$ , such that

$$\text{Gr}'(\kappa) \cap \Omega = \{(x, \partial_x \psi(x, \eta), \partial_\eta \psi(x, \eta), -\eta), x \in \Omega_x, \eta \in \Omega_\eta\}$$

One says that  $\psi$  generates  $\text{Gr}'(\kappa)$ . Suppose that that  $a \in S_\delta^{\text{comp}}(\Omega_x \times \Omega_\eta)$ . Then, the following operator  $T$  is an element of  $I_\delta^{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n, \text{Gr}'(\kappa))$  :

$$Tu(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\psi(x, \eta) - y \cdot \eta)} a(x, \eta) u(y) dy d\eta \quad (4-2.3)$$

and if  $T^*T = \text{Id}$  microlocally near  $(y_0, \eta_0)$  then  $|a(x, \eta)|^2 = |\det D_{x\eta}^2 \psi(x, \eta)| + O(h^{1-2\delta})_{S_\delta}$  near  $(x_0, \xi_0, y_0, \eta_0)$ . The converse statement holds : microlocally near  $(x_0, \xi_0, y_0, \eta_0)$  and modulo a smoothing operator which is  $O(h^\infty)$ , the elements of  $I_\delta(\mathbb{R}^n \times \mathbb{R}^n, \text{Gr}'(\kappa))$  can be written under this form.

## 4-3 Open quantum hyperbolic maps.

Let us consider an open hyperbolic map  $F$ , as described by the formalism in 3-1. We recall that this formalism relies on :

- open intervals  $Y_1, \dots, Y_J$  of  $\mathbb{R}$  and  $Y = \bigsqcup_{j=1}^J Y_j \subset \bigsqcup_{j=1}^J \mathbb{R}$  ;

- $U = \bigsqcup_{j=1}^J U_j \subset \bigsqcup_{j=1}^J T^*\mathbb{R}^d$  where  $U_j \Subset T^*Y_j$  are open sets;
- For  $j = 1, \dots, J$ , open disjoint subsets  $\widetilde{D}_{ij} \Subset U_j$ ,  $1 \leq i \leq J$ , the departure sets, and for  $i = 1, \dots, J$  open disjoint subsets  $\widetilde{A}_{ij} \Subset U_i$ ,  $1 \leq j \leq J$ , the arrival sets ;
- Smooth symplectomorphisms  $F_{ij} : \widetilde{D}_{ij} \rightarrow F_{ij}(\widetilde{D}_{ij}) = \widetilde{A}_{ij}$

Then,  $F$  is the global smooth map  $F : \widetilde{D} \rightarrow \widetilde{A}$  where  $\widetilde{A}$  and  $\widetilde{D}$  are the full arrival and departure sets, defined as

$$\widetilde{A} = \bigsqcup_{i=1}^J \bigsqcup_{j=1}^J \widetilde{A}_{ij} \subset \bigsqcup_{i=1}^J U_i$$

$$\widetilde{D} = \bigsqcup_{j=1}^J \bigsqcup_{i=1}^J \widetilde{D}_{ij} \subset \bigsqcup_{j=1}^J U_j$$

Finally, we recall that we note  $\mathcal{T} \subset U$  the trapped set of  $F$ .

Our aim is to define open quantum maps associated with  $F$ . We fix a compact set  $W \subset \widetilde{A}$  containing some neighborhood of  $\mathcal{T}$ . Our definition will depend on  $W$ . Following [NSZ14] (Section 3.4.2), we now focus on the definition of the elements of  $I_\delta(Y \times Y; \text{Gr}(F)')$ . An element  $T \in I_\delta(Y \times Y; \text{Gr}(F)')$  is a matrix of operators

$$T = (T_{ij})_{1 \leq i, j \leq J} : \bigoplus_{j=1}^J L^2(Y_j) \rightarrow \bigoplus_{i=1}^J L^2(Y_i)$$

Each  $T_{ij}$  is an element of  $I_\delta(Y_i \times Y_j, \text{Gr}(F_{ij})')$ . Let's now describe the recipe to construct elements of  $I_\delta(Y_i \times Y_j, \text{Gr}(F_{ij})')$ .

We fix  $i, j \in \{1, \dots, J\}$ .

- Fix some small  $\varepsilon > 0$  and two open covers of  $U_j$ ,  $U_j \subset \bigcup_{l=1}^L \Omega_l$ ,  $\Omega_l \Subset \widetilde{\Omega}_l$ , with  $\widetilde{\Omega}_l$  star-shaped and having diameter smaller than  $\varepsilon$ . We note  $\mathcal{L}$  the sets of indices  $l$  such that  $\widetilde{\Omega}_l \subset \widetilde{D}_{ij} \subset U_j$  and we require (this is possible if  $\varepsilon$  is small enough)

$$F^{-1}(W) \cap U_j \subset \bigcup_{l \in \mathcal{L}} \Omega_l$$

- Introduce a smooth partition of unity associated with the cover  $(\Omega_l)$ ,  $(\chi_l)_{1 \leq l \leq L} \in C_c^\infty(\Omega_l, [0, 1])$ ,  $\text{supp } \chi_l \subset \Omega_l$ ,  $\sum_l \chi_l = 1$  in a neighborhood of  $\overline{U_j}$ .
- For each  $l \in \mathcal{L}$ , we denote  $F_l$  the restriction to  $\widetilde{\Omega}_l$  of  $F_{ij}$ . By Lemma 4-2.1, there exists  $\kappa_l \in \mathcal{K}$  which coincides with  $F_l$  on  $\Omega_l$ .
- We consider  $T_l = \text{Op}_h(\alpha_i)U_l(1)$  where  $U_l(t)$  is the solution of the Cauchy problem (4-2.1) associated with  $\kappa_l$  and  $\alpha_i \in S_\delta^{\text{comp}}(T^*\mathbb{R})$ .
- We set

$$T^{\mathbb{R}} = \sum_{l \in \mathcal{L}} T_l \text{Op}_h(\chi_l) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \quad (4-3.1)$$

$T^{\mathbb{R}}$  is a globally defined Fourier integral operator. We will note  $T^{\mathbb{R}} \in I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}(F_{ij})')$ . Its wavefront set is included in  $\widetilde{A}_{ij} \times \widetilde{D}_{ij}$ .

- Finally, we fix cut-off functions  $(\Psi_i, \Psi_j) \in C_c^\infty(Y_i, [0, 1]) \times C_c^\infty(Y_j, [0, 1])$  such that  $\Psi_i \equiv 1$  on  $\pi(U_i)$  and  $\Psi_j \equiv 1$  on  $\pi(U_j)$  (here,  $\pi : (x, \xi) \in T^*Y_\bullet \mapsto x \in Y_\bullet$  is the natural projection) and we adopt the following definitions :



**Definition 4-3.1.** We say that  $T : \mathcal{D}'(Y_j) \rightarrow C^\infty(\overline{Y_i})$  is a Fourier integral operator in the class  $I_\delta(Y_i \times Y_j, \text{Gr}(F_{ij})')$  if there exists  $T^{\mathbb{R}} \in I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}(F)')$  as constructed above such that

- $T - \Psi_i T \Psi_j = O(h^\infty)_{\mathcal{D}'(Y_j) \rightarrow C^\infty(\overline{Y_i})}$ ;
- $\Psi_i T \Psi_j = \Psi_i T^{\mathbb{R}} \Psi_j$

**Remark.**

The definition of this class is not canonical since it depends in fact on the compact set  $W$  through the partition of unity.

For  $U'_j \subset U_j$  and  $U'_i = F(U'_j) \subset U_i$ , we say that  $T$  (or  $T^{\mathbb{R}}$ ) is microlocally unitary in  $U'_i \times U'_j$  if  $TT^* = \text{Id}$  microlocally in  $U'_i$  and  $T^*T = \text{Id}$  microlocally in  $U'_j$ .

We can now state our definition for open quantum hyperbolic maps associated with  $F$  :

**Definition 4-3.2.** Fix  $\delta \in [0, 1/2[$ . We say that  $T = T(h)$  is an *open quantum hyperbolic map* associated with  $F$ , and we note  $T = T(h) \in I_\delta(Y \times Y, \text{Gr}(F)')$  if : for each couple  $(i, j) \in \{1, \dots, J\}^2$ , there exists a semi-classical Fourier integral operator  $T_{ij} = T_{ij}(h) \in I_\delta(Y_j \times Y_i, \text{Gr}(F_{ij})')$  associated with  $F_{ij}$  in the sense of definition 4-3.1, such that

$$T = (T_{ij})_{1 \leq i, j \leq J} : \bigoplus_{i=1}^J L^2(Y_i) \rightarrow \bigoplus_{i=1}^J L^2(Y_i)$$

In particular  $\text{WF}_h'(T) \subset \tilde{A} \times \tilde{D}$ . We note  $I_{0+}(Y \times Y, \text{Gr}(F)') = \bigcap_{\delta > 0} I_\delta(Y \times Y, \text{Gr}(F)')$ .

We will say that  $T \in I_{0+}(Y \times Y, \text{Gr}(F)')$  is microlocally invertible near  $\mathcal{T}$  if there exists a neighborhood  $U' \subset U$  of  $\mathcal{T}$  and an operator  $T' \in I_{0+}(Y \times Y, \text{Gr}(F^{-1})')$  such that, for every  $u = (u_1, \dots, u_J) \in L^2(Y)$

$$\forall j \in \{1, \dots, J\}, \text{WF}_h(u_j) \subset U' \cap U_j \implies TT'u = u + O(h^\infty)\|u\|_{L^2}, T'Tu = u + O(h^\infty)\|u\|_{L^2}$$

Suppose that  $T$  is microlocally invertible near  $\mathcal{T}$  and recall that  $T^*T \in \Psi_{0+}(Y)$ . Then we can write

$$T^*T = \text{Op}_h(a_h)$$

where  $a_h$  is a smooth symbol in the class  $S_{0+}(U)$ . We note  $\alpha_h = \sqrt{|a_h|}$  and call it the *amplitude* of  $T$ . Since  $T$  is microlocally invertible near  $\mathcal{T}$ ,  $|a_h| > c^2$  near  $\mathcal{T}$ , for some  $h$ -independent constant  $c > 0$ , showing that  $\alpha_h$  is smooth and larger than  $c$  in a neighborhood of  $\mathcal{T}$ .

**Remark.**

If  $T$  has amplitude  $\alpha$ , at first approximation,  $T$  transforms a wave packet  $u_{\rho_0}$  of norm 1 centered at a point  $\rho_0$  lying in a small neighborhood of  $\mathcal{T}$  into a (squeezed) wave packet of norm  $\alpha(\rho_0)$  centered at the point  $F(\rho_0)$ .

We will say that  $T$  is *microlocally unitary near  $\mathcal{T}$*  if the two following conditions hold :

- $\|TT^*\| \leq 1 + O(h^\varepsilon)$  for some  $\varepsilon > 0$
- there exists a neighborhood  $\Omega \subset U$  of  $\mathcal{T}$  such that, for every  $u = (u_1, \dots, u_J) \in \bigoplus_{j=1}^J L^2(Y_j)$ ,
 
$$\forall j \in \{1, \dots, J\}, \text{WF}_h(u_j) \subset \Omega \cap U_j \implies TT^*u = u + O(h^\infty)\|u\|_{L^2}, T^*Tu = u + O(h^\infty)\|u\|_{L^2}$$

The first condition is not microlocal but global. Nevertheless, we choose to include it for convenience (as we will see, these operators are in fact relevant near  $\mathcal{T}$ , where this condition is implied by the second one). It is clear that the second condition implies that the amplitude is equal to one in a neighborhood of  $\mathcal{T}$ . Let us now briefly see what it implies for the components of  $T^*T$ . First focus on the off-diagonal entries.

$$(T^*T)_{ij} = \sum_{k=1}^J (T^*)_{ik} T_{kj} = \sum_{k=1}^J (T_{ki})^* T_{kj}$$

If  $k \in \{1, \dots, J\}$  and  $i \neq j$ ,  $(T_{ki})^* T_{kj} = O(h^\infty)$  since

$$\text{WF}_h'(T_{ki}^*) \subset \widetilde{D}_{ki} \times \widetilde{A}_{ki} \quad ; \quad \text{WF}_h'(T_{kj}) \subset \widetilde{A}_{kj} \times \widetilde{D}_{kj} \quad \text{and} \quad \widetilde{A}_{kj} \cap \widetilde{A}_{ki} = \emptyset$$

As a consequence, the off-diagonal terms are always  $O(h^\infty)$ . For the diagonal entries,

$$(T^*T)_{ii} = \sum_{k=1}^J (T_{ki})^* T_{ki}$$

Each term of this sum is a pseudodifferential operator with wavefront set

$$\text{WF}_h(T_{ki}^* T_{ki}) \subset \widetilde{D}_{ki}$$

Since the  $\widetilde{D}_{ki}$  are pairwise disjoint,  $T^*T = \text{Id}_{L^2(Y)} + O(h^\infty)$  microlocally near  $\mathcal{T}$  if and only if for all  $k, i$ ,  $T_{ki}^* T_{ki} = \text{Id}_{L^2(Y_i)} + O(h^\infty)$  microlocally near  $D_{ki}$ . The same computations apply to  $TT^*$ . As a consequence,  $T$  is microlocally unitary near  $\mathcal{T}$  if and only if for all  $(k, i)$ ,  $T_{ki}$  is a Fourier integral operator associated with  $F_{ki}$ , microlocally unitary near  $D_{ki} \times A_{ki}$  (see the paragraph below Definition 4-3.1).

**Another version of Egorov's theorem.** The precise version of Egorov's theorem in Proposition 4-2.2 is only stated for globally unitary Fourier integral operator defined using the Cauchy problem 4-2.1. We extend it here to microlocally unitary and globally defined Fourier integral operators. We fix  $i, j \in \{1, \dots, J\}$ .

**Lemma 4-3.1.** Let  $T \in I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}(F_{ij})')$ . Suppose that  $B(\rho, 4\varepsilon) \subset U_j$  and that  $T$  is microlocally unitary in  $F_{ij}(B(\rho, 4\varepsilon)) \times B(\rho, 4\varepsilon)$ . Then, there exists a family  $(D_k)_{k \in \mathbb{N}}$  of differential operators of order  $k$ , compactly supported in  $B(\rho, 3\varepsilon)$  such that the following holds : For every  $N \in \mathbb{N}$  and for all  $b \in C_c^\infty(B(\rho, 2\varepsilon))$ ,

$$T \text{Op}_h(b) = \text{Op}_h \left( b \circ \kappa^{-1} + \sum_{k=1}^{N-1} h^k (D_{k+1} b) \circ \kappa^{-1} \right) T + O(h^N \|b\|_{C^{2N+15}})_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}$$

The constants in  $O$  depend on  $T$  and  $F$ .

*Proof.* First, introduce some cut-off function  $\chi$  such that  $\chi \equiv 1$  in a neighborhood of  $B(\rho, 2\varepsilon)$  and  $\text{supp } \chi \subset B(\rho, 3\varepsilon)$ . Due to these properties and Proposition 4-1.1, we have

$$\text{Op}_h(b) = \text{Op}_h(\chi) \text{Op}_h(b) \text{Op}_h(\chi) \text{Op}_h(\chi) + O(h^N \|b\|_{C^{2N+13}})_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}$$

Moreover,  $\text{Op}_h(\chi) T^* T = \text{Op}_h(\chi) + O(h^\infty)$  and hence,

$$T \text{Op}_h(b) = T \text{Op}_h(\chi) \text{Op}_h(b) \text{Op}_h(\chi) \text{Op}_h(\chi) T^* T + O(h^N \|b\|_{C^{2N+13}})_{L^2 \rightarrow L^2} + O(h^\infty) \| \text{Op}_h(b) \|_{L^2 \rightarrow L^2}$$

The term  $O(h^\infty) \| \text{Op}_h(b) \|_{L^2 \rightarrow L^2}$  can be absorbed in  $O(h^N \|b\|_{C^{2N+13}})_{L^2 \rightarrow L^2}$ . Consider  $\tilde{\kappa} \in \mathcal{K}$  extending  $\kappa|_{B(\rho, 3\varepsilon)}$  and construct  $U = U(1)$  by solving the Cauchy problem (4-2.1) associated with  $\tilde{\kappa}$ . Due to the properties on composition of Fourier integral operators (Proposition 4-2.1),  $T \text{Op}_h(\chi) U^{-1}$  and  $U \text{Op}_h(\chi) T^*$  are pseudodifferential operators, and we note them  $\text{Op}_h(a_1), \text{Op}_h(a_2)$ . Now write

$$\begin{aligned} T \text{Op}_h(b) &= [T \text{Op}_h(\chi) U^{-1}] U \text{Op}_h(b) \text{Op}_h(\chi) U^{-1} [U \text{Op}_h(\chi) T^*] T + O(h^N \|b\|_{C^{2N+13}})_{L^2 \rightarrow L^2} \\ &= \text{Op}_h(a_1) [U \text{Op}_h(b) \text{Op}_h(\chi) U^{-1}] \text{Op}_h(a_2) T + O(h^N \|b\|_{C^{2N+13}})_{L^2 \rightarrow L^2} \end{aligned}$$

By using the precise version in Proposition 4-2.2, one can write

$$U \text{Op}_h(b) \text{Op}_h(\chi) U^{-1} = \text{Op}_h \left( b \circ \kappa^{-1} + \sum_{k=1}^{N-1} (L_{k+1} b) \circ \kappa^{-1} \right) + O(h^N \|b\|_{C^{2N+15}})_{L^2 \rightarrow L^2}$$

Applying Lemma 4-1.1, we see that we can write

$$T \text{Op}_h(b) = \text{Op}_h \left( b_0 \circ \kappa^{-1} + \sum_{k=1}^{N-1} (D_{k+1} b) \circ \kappa^{-1} \right) T + O(h^N \|b\|_{C^{2N+15}})_{L^2 \rightarrow L^2}$$

where  $b_0 = a_1 \times b \circ \kappa^{-1} \times a_2$ .  $T$  being microlocally unitary in  $B(\rho, 4\varepsilon)$ , the product  $a_1 a_2$  is equal to 1 in  $B(\rho, 2\varepsilon)$ , and hence, the lemma is proved.  $\square$



## Part III

# Spectral gaps



# Chapter 5

## Spectral gap for open quantum maps

In this chapter, we prove a spectral gap for open hyperbolic quantum maps (as defined in 4-3.2) associated with open hyperbolic maps satisfying the formalism of 3-1.1.. The material of this chapter comes from the paper [Vac22b], to appear in Analysis & PDE.

### 5-1 Introduction.

#### 5-1.1. Motivations : scattering by convex obstacles and spectral gap.

To introduce the theorem we prove in this chapter, let us recall our main motivation : obstacle scattering. We refer the reader to Part I for a broader introduction to the related problems. Assume that

$$\mathcal{O} = \bigcup_{j=1}^J \mathcal{O}_j$$

where  $\mathcal{O}_j$  are open, strictly convex connected obstacles in  $\mathbb{R}^2$  having smooth boundary and satisfying the *Ikawa condition* : for  $i \neq j \neq k$ ,  $\overline{\mathcal{O}_i}$  does not intersect the convex hull of  $\overline{\mathcal{O}_j} \cup \overline{\mathcal{O}_k}$ . Let

$$\Omega = \mathbb{R}^2 \setminus \overline{\mathcal{O}}$$

The resolvent of the Dirichlet Laplacian in  $\Omega$  continues meromorphically to the logarithmic cover of  $\mathbb{C}$ . More precisely, suppose that  $\chi \in C_c^\infty(\mathbb{R}^2)$  is equal to one in a neighborhood of  $\overline{\mathcal{O}}$ .

$$\chi(-\Delta_\Omega - \lambda^2)^{-1} \chi : L^2(\Omega) \rightarrow L^2(\Omega)$$

is holomorphic in the region  $\{\text{Im } \lambda > 0\}$  and it continues meromorphically to the logarithmic cover of  $\mathbb{C}$ . Its poles are the *scattering resonances* and as explained in Part I, we are interested in the problem of the existence of a spectral gap in the first sheet of the logarithmic cover (i.e.  $\mathbb{C} \setminus i\mathbb{R}^-$ ).

The spectral gap has for instance been studied by [Ika88] in dimension 3. It has been experimentally investigated in [BWP+13] for three disks and five disks systems. In this study, the author bring an experimental evidence of the presence of a spectral gap, no matter how thin the trapped set is. We will also widely use the presentation and the arguments of [NSZ14] concerning the study of the scattering resonances in obstacle scattering.

The spectral gap problem is a high-frequency problem and justifies the introduction of a small parameter  $h$ , where  $\frac{1}{h}$  corresponds to a large frequency scale. Under this rescaling, we are interested in the semiclassical operator

$$P(h) = -h^2 \Delta - 1 \quad , \quad 0 < h \leq 1$$

and spectral parameter  $z \in D(0, Ch)$  for some  $C > 0$ . In the semiclassical limit, the classical dynamics associated to this quantum problem is the billiard flow in  $\Omega \times \mathbb{S}^1$  and the trapped set plays a crucial role in the spectral gap problem.

A good dynamical object to study this structure is the topological pressure associated with the unstable Jacobian  $\phi_u$ . This dynamical quantity is a strictly decreasing function  $s \mapsto P(s)$  which measures the instability of the flow (see Section 3-5.1). In dimension 2, Bowen's formula shows that the Hausdorff and upper box dimensions of the trapped set are  $2s_0$  where  $s_0$  is the unique

root of the equation  $P(s) = 0$  (see Proposition 3-6.1). In [NZ09], the existence of a spectral gap for such systems has been proved under the pressure condition

$$P\left(\frac{1}{2}\right) < 0$$

Their result holds in any dimension, with a quantitative spectral gap. Our result doesn't need this assumption anymore. In fact, it relies on the weaker pressure condition :

$$P(1) < 0$$

It is known that this condition is always satisfied in the scattering problem we consider since the trapped set is not an attractor ([BR75]). Due to Bowen's formula, this condition can be interpreted as a fractal condition. This is this fractal property that will be crucial in the analysis.

**Reduction to open hyperbolic quantum maps.** An important aspect of our analysis to prove Theorem A relies on previous results of [NSZ14]. Their Theorem 5 (Section 6) reduces the study of the scattering poles to the study of the cancellation of

$$z \mapsto \det(I - M(z))$$

where

$$M(z) : L^2(\partial\mathcal{O}) \rightarrow L^2(\partial\mathcal{O}) \tag{5-1.1}$$

is a family of *hyperbolic open quantum map* (see Section 4-3). The family  $z \mapsto M(z)$  depends holomorphically on  $z \in D(0, Ch)$  for some  $C > 0$  and is sometimes called a *hyperbolic quantum monodromy operator*. The construction of this operator relies on the study of the operators  $M_0(z)$  defined as follows : for  $1 \leq j \leq J$ , let  $H_j(z) : C^\infty(\partial\mathcal{O}_j) \rightarrow C^\infty(\mathbb{R}^2 \setminus \mathcal{O}_j)$  be the resolvent of the problem

$$\begin{cases} (-h^2\Delta - 1 - z)(H_j(z)v) = 0 \\ H_j(z)v \text{ is outgoing} \\ H_j(z)v = v \text{ on } \partial\mathcal{O}_j. \end{cases}$$

Let  $\gamma_j$  be the restriction of a smooth function  $u \in C^\infty(\mathbb{R}^2)$  to  $C^\infty(\partial\mathcal{O}_j)$  and define  $M_0(z)$  by :

$$M_0(z) = \begin{cases} 0 & \text{if } i = j \\ -\gamma_i H_j(z) & \text{otherwise.} \end{cases}$$

Due to results of Gérard ([Gér88], Appendix II), this matrix is a Fourier integral operator associated with a Lagrangian relation related to the billiard flow. *A priori*, it does exclude neither the glancing rays nor the shadow region. Ikawa condition ensures that they do not play a role when considering the trapped set and allows the author to neglect the effects of these regions (see Section 6 in [NSZ14]). A consequence of their analysis is that  $M(z)$  is associated with a simpler Lagrangian relation  $\mathcal{B}^+$ , which is the billiard map presented in 3-1.1.1.. This reduction will be presented in more details in the following Chapter (see Chapter 6).

**Spectral gap for hyperbolic open quantum maps.** Using this reduction, Theorem A will be proved once we are able to show that the spectral radius of  $M(z)$  is strictly smaller than 1 for  $z \in D(0, Ch) \cap \{\text{Im } z \in [-\delta h, 0]\}$ , for some  $\delta > 0$ . This will be a consequence of the following statement, which will be partly demonstrated in this chapter (see subsection 5-1.2. below for the exact version proved in this chapter).

**Theorem 5-1.1.** *Let  $(M(z))_z$  be the family introduced in (5-1.1), that is a hyperbolic quantum monodromy operator associated with the billiard map  $\mathcal{B}^+$ . Then, there exist  $h_0 > 0$ ,  $\gamma > 0$  and  $\tau_{\max} > 0$  such that the spectral radius of  $M(z)$ ,  $\rho_{\text{spec}}(z)$ , satisfies : for all  $0 < h \leq h_0$  and all  $z \in D(0, Ch)$ ,*

$$\rho_{\text{spec}}(z) \leq e^{-\gamma - \tau_{\max} \text{Im } z}$$

When  $z \in \mathbb{R}$ , the operator  $M(z)$  is microlocally unitary near the trapped set and its  $L^2$  norm is essentially 1. Then, we have the trivial bound

$$\rho_{\text{spec}}(z) \leq 1$$

The bound given by the theorem is a spectral gap since we obtain

$$\rho_{\text{spec}}(z) \leq e^{-\gamma} < 1$$

The dependence of the bound with the parameter  $z$  is related to the amplitude of the open quantum map  $M(z)$ .

The link between open quantum maps and the resonances of open quantum systems has also been established in [NSZ11] for the case of potential scattering and this is why we will also obtain a spectral gap in this context. We review this reduction both in obstacle and potential scattering in the following Chapter 6. We will show how it implies the spectral gap, and even obtain estimates for the cut-off resolvent. The present chapter focuses on the proof of a more general Theorem (see Theorem 5-1.2 in Section 5-1.2. below) concerning spectral gaps for open hyperbolic quantum maps. The proof strongly uses the Fractal Uncertainty Principle.

**On the fractal uncertainty principle.** A broader introduction has been made in 2-2.2.. We will give the precise definitions and statements in Section 5-5. We rather explain here the general idea of this principle in the spirit of our use. Roughly speaking, it says that no function can be concentrated both in frequencies and positions near a fractal set. Suppose that  $X, Y \subset \mathbb{R}$  are fractal sets. To fix the ideas, let's say that  $X$  and  $Y$  have upper box dimension  $\delta_X$  and  $\delta_Y$  strictly smaller than one. For  $c > 0$ , let's note  $X(c) = X + [-c, +c]$  and the same for  $Y$ . Also denote  $\mathcal{F}_h$  the  $h$ -Fourier transform :

$$\mathcal{F}_h u(\xi) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} e^{-i \frac{x\xi}{h}} u(x) dx$$

The fractal uncertainty principle then states that there exists  $\beta > 0$  depending on  $X$  and  $Y$  (See Proposition 5-5.1 for the precise dependence) such that, for  $h$  small enough,

$$\|\mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq h^\beta$$

It will be a key ingredient in the proof of the main theorem of this Chapter. It has been successfully used to show spectral gaps for convex co-compact hyperbolic surfaces ([DZ16], [BD17], [DJ18b], [DZ18]). A discrete version of the fractal uncertainty principle is also the main ingredient of [DJ17] where the author proved a spectral gap for open quantum maps in a toy model case. Their result concerning open baker's map on the torus  $\mathbb{T}^2$  partly motivates our theorem on open quantum maps.

The fractal uncertainty principle has also given new results in quantum chaos on negatively curved compact surfaces. It has first been successfully used for compact hyperbolic surfaces in [DJ17] where the authors proved that semiclassical measures have full support. The hyperbolic case was treated using quantization procedures developed in [DZ16], which allow to have a good semiclassical calculus for symbols very irregular in the stable direction, but smooth in the unstable one (or conversely). In [Sch21], the same ideas lead to a full delocalization of eigenstates for quantum cat maps. The quantization procedures used in these papers rely on the smoothness of the unstable and stable distributions. This strategy fails for general negatively curved surfaces. However, in [DJN21], the authors bypassed this obstacle and succeeded to extend these results to the case of negatively curved surfaces. This is mainly from this paper that we borrow the techniques and we adapt them in our setting. Theorem 3-3.1 on the regularity of the stable and unstable distributions turns out to be crucial for this purpose, thanks to the adapted charts it allows to build (see 3-4).

### 5-1.2. Statement of the main theorem.

Let us consider an open hyperbolic map  $F$ , as described by the formalism in 3-1.1.. We recall that this formalism relies on :

- open intervals  $Y_1, \dots, Y_J$  of  $\mathbb{R}$  and  $Y = \bigsqcup_{j=1}^J Y_j \subset \bigsqcup_{j=1}^J \mathbb{R}$  ;
- $U = \bigsqcup_{j=1}^J U_j \subset \bigsqcup_{j=1}^J T^*\mathbb{R}^d$  where  $U_j \Subset T^*Y_j$  are open sets;
- For  $j = 1, \dots, J$ , open disjoint subsets  $\tilde{D}_{ij} \Subset U_j$ ,  $1 \leq i \leq J$ , the *departure sets*, and similarly, for  $i = 1, \dots, J$  open disjoint subsets  $\tilde{A}_{ij} \Subset U_i$ ,  $1 \leq j \leq J$ , the *arrival sets* ;



- Smooth symplectomorphisms  $F_{ij} : \widetilde{D}_{ij} \rightarrow F_{ij}(\widetilde{D}_{ij}) = \widetilde{A}_{ij}$

Then,  $F$  is the global smooth map  $F : \widetilde{D} \rightarrow \widetilde{A}$  where  $\widetilde{A}$  and  $\widetilde{D}$  are the full arrival and departure sets, defined as

$$\widetilde{A} = \bigsqcup_{i=1}^J \bigsqcup_{j=1}^J \widetilde{A}_{ij} \subset \bigsqcup_{i=1}^J U_i$$

$$\widetilde{D} = \bigsqcup_{j=1}^J \bigsqcup_{i=1}^J \widetilde{D}_{ij} \subset \bigsqcup_{j=1}^J U_j$$

Finally, we recall that we note  $\mathcal{T}$  the trapped set of  $F$  and  $F$  is supposed to satisfy the assumptions (**Hyperbolicity**) and (**Fractal**). We also refer the reader to 4-3 for the definition of open quantum hyperbolic maps (see Definition 4-3.2). We note  $\rho_{spec}$  for the spectral radius of a bounded operator.

The goal of this chapter is to prove the following Theorem :

**Theorem 5-1.2.** *Suppose that  $F$  is an open hyperbolic map defined on  $U \Subset T^*Y$ . Then, there exists  $\gamma > 0$  such that the following holds :*

*Let  $T = T(h) \in I_{0+}(Y \times Y, F')$  be an open quantum hyperbolic map associated with  $F$  in the sense of definition (4-3.2) and  $\alpha \in S_\delta^{comp}(U)$ . Assume that  $T$  is microlocally unitary in a neighborhood of  $\mathcal{T}$ . Then, there exists  $h_0 > 0$  such that*

$$\forall 0 < h \leq h_0 \quad , \quad \rho_{spec}(T(h) \text{Op}_h(\alpha)) \leq e^{-\gamma} \|\alpha\|_\infty$$

*$h_0$  depends on  $(U, F)$ ,  $T$  and semi-norms of  $\alpha$  in  $S_\delta$ .*

**Notations.** An element of  $S_\delta^{comp}(U)$  is a  $J$ -uple  $\alpha = (\alpha_1, \dots, \alpha_J)$  where each  $\alpha_j$  is an element of  $S_{comp}^\delta(\mathbb{R}^2)$  such that  $\text{esssupp } \alpha_j \subset U_j$ .

We fix a smooth function  $\Psi_Y = (\Psi_1, \dots, \Psi_J)$  such that, for  $1 \leq j \leq J$ ,  $\Psi_j \in C_c^\infty(Y_j, [0, 1])$  satisfies  $\Psi_j = 1$  on  $\pi(U_j)$  (recall that  $U_j \Subset T^*Y_j$ ).

For  $\alpha \in S_\delta^{comp}(U)$ , we also note  $\text{Op}_h(\alpha)$  the diagonal operator valued matrix:

$$\text{Op}_h(\alpha) = \text{Diag}(\Psi_1 \text{Op}_h(\alpha_1) \Psi_1, \dots, \Psi_J \text{Op}_h(\alpha_J) \Psi_J) : \bigoplus_{j=1}^J L^2(Y_j) \rightarrow \bigoplus_{j=1}^J L^2(Y_j)$$

Note that as operators on  $L^2(\mathbb{R})$ ,  $\text{Op}_h(\alpha_j)$  and  $\Psi_j \text{Op}_h(\alpha_j) \Psi_j$  are equal modulo a smoothing  $O(h^\infty)$ .

### Remark.

- If the value  $h_0$  depends on  $T$  and  $\alpha$ , this is not the case of  $\gamma$  which depends on the dynamical system  $(U, F)$ .
- This is a spectral gap : it has to be compared with the easy bound we could have by using the  $L^2$ -norm of  $T \text{Op}_h(\alpha)$  which is essentially  $\|\alpha\|_\infty$  :

$$\rho_{spec}(T \text{Op}_h(\alpha)) \leq \|\alpha\|_\infty + o(1)$$

In particular, if  $\alpha \equiv 1$  in a neighborhood of  $\mathcal{T}$  and  $|\alpha| \leq 1$  everywhere,

$$\rho_{spec}(T(h)) \leq e^{-\gamma} < 1$$

- $T \text{Op}_h(\alpha)$  is the way we've chosen to write our Fourier integral operator with amplitude  $\alpha$  in this Chapter.

**A model example.** To explain the main ideas of the proof of Theorem 5-1.2, let us show how it works in an example where the trapped set is the smallest possible : a single point. In this context, we only need a simpler uncertainty principle. We focus on a simplified open quantum map modeling the case of 2 obstacles.

We consider the hyperbolic map

$$F : (x, \xi) \in \mathbb{R}^2 \mapsto (2^{-1}x, 2\xi) \in \mathbb{R}^2$$

It has a unique hyperbolic fixed point  $\rho_0 = 0$  and the stable (resp. unstable) manifold at 0 is given by  $\{\xi = 0\}$  (resp.  $\{x = 0\}$ ). The scaling operator

$$U : v \in L^2(\mathbb{R}) \mapsto \sqrt{2}v(2x)$$

is a quantum map quantizing  $F$ . To open it, we consider a cut-off function  $\chi \in C_c^\infty(\mathbb{R}^2)$  such that  $\chi \equiv 1$  in  $B(0, 1/2)$  and  $\text{supp } \chi \Subset B(0, 1)$  and we consider the open quantum map

$$M = M(h) = \text{Op}_h(\chi)U$$

where  $\text{Op}_h$  is in this example (and only in this example) the left quantization :

$$\text{Op}_h(\chi)u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} \chi(x, \xi) e^{i\frac{(x-y)\xi}{h}} u(y) dy d\xi$$

One easily checks that Egorov's property for  $U$  is true without remainder term :

$$U^* \text{Op}_h(\chi)U = \text{Op}_h(\chi \circ F) \quad , \quad U \text{Op}_h(\chi)U^* = \text{Op}_h(\chi \circ F^{-1})$$

To show a spectral gap for  $M$ , we study  $M^n$  with

$$n = n(h) \sim -\frac{3 \log h}{4 \log 2}$$

This time is longer than the Ehrenfest time  $-\frac{\log h}{\log 2}$ . We write :

$$M^n = U^n \text{Op}_h(\chi \circ F^n) \dots \text{Op}_h(\chi \circ F^1)$$

The formula  $[\text{Op}_h(a), \text{Op}_h(b)] = O(h^{1-2\delta})$  is valid for  $a, b$  symbols in  $S_\delta$  and  $\delta < 1/2$ . The problem here is that for  $1 \leq k \leq n$ ,  $\chi \circ F^k$  are uniformly in  $S_{3/4}$  : this is not a good symbol class. To bypass this difficulty, we observe that the symbols  $\chi \circ F^k$  are uniformly in  $S_{3/8}$  for  $k \in \{-n/2, \dots, n/2\}$ . As a consequence, for  $j \in \{1, \dots, n\}$  we write:

$$\begin{aligned} [\text{Op}_h(\chi \circ F^n), \text{Op}_h(\chi \circ F^j)] &= U^{-n/2} \left[ \text{Op}_h(\chi \circ F^{n/2}), \text{Op}_h(\chi \circ F^{j-n/2}) \right] U^{n/2} \\ &= U^{-n/2} O(h^{1/4}) U^{n/2} \\ &= O(h^{1/4}) \end{aligned}$$

where the constants in  $O$  are uniform with respect to  $j$  and depend only on  $\chi$ . Applying this formula recursively to move the term  $\text{Op}_h(\chi \circ F^n)$  to the right, we get that

$$M^n = U^n \text{Op}_h(\chi \circ F^{n-1}) \dots \text{Op}_h(\chi \circ F^1) \text{Op}_h(\chi \circ F^n) + O(h^{1/4} \log h)$$

Similarly, we can write :

$$M^{n+1} = \text{Op}_h(\chi \circ F^{-n}) \text{Op}_h(\chi) \dots \text{Op}_h(\chi \circ F^{-n+1}) U^{n+1} + O(h^{1/4} \log h)$$

Hence, we have

$$M^{2n+1} = A \text{Op}_h(\chi \circ F^n) \text{Op}_h(\chi \circ F^{-n}) B + O(h^{1/4} \log h)$$

with

$$A = A(h) = U^n \text{Op}_h(\chi \circ F^{n-1}) \dots \text{Op}_h(\chi \circ F^1) = O(1)$$

and

$$B = B(h) = \text{Op}_h(\chi) \dots \text{Op}_h(\chi \circ F^{-n+1}) U^{n+1} = O(1)$$

We have the following properties on the supports

$$\text{supp } \chi \circ F^n \subset \{|\xi| \leq 2^{-n}\} \quad , \quad \text{supp } \chi \circ F^n \subset \{|x| \leq 2^{-n}\}$$

Assuming that  $n(h) \geq -\frac{3}{4} \frac{\log h}{\log 2}$ , we observe that

$$\text{Op}_h(\chi \circ F^n) = \text{Op}_h(\chi \circ F^n) \mathbb{1}_{[-h^{3/4}, h^{3/4}]}(hD_x)$$

$$\text{Op}_h(\chi \circ F^{-n}) = \mathbb{1}_{[h^{-3/4}, h^{3/4}]}(x) \text{Op}_h(\chi \circ F^{-n})$$

Finally, we have

$$M^{2n+1} = A \text{Op}_h(\chi \circ F^n) \mathbb{1}_{[-h^{3/4}, h^{3/4}]}(hD_x) \mathbb{1}_{[h^{-3/4}, h^{3/4}]}(x) \text{Op}_h(\chi \circ F^{-n}) B + O\left(h^{1/4} \log h\right)$$

This is where we need an uncertainty principle :

$$\begin{aligned} \|\mathbb{1}_{[-h^{3/4}, h^{3/4}]}(hD_x) \mathbb{1}_{[h^{-3/4}, h^{3/4}]}(x)\|_{L^2 \rightarrow L^2} &= \|\mathbb{1}_{[-h^{3/4}, h^{3/4}]} \mathcal{F}_h \mathbb{1}_{[-h^{3/4}, h^{3/4}]}\|_{L^2 \rightarrow L^2} \\ &\leq \|\mathbb{1}_{[-h^{3/4}, h^{3/4}]}\|_{L^\infty \rightarrow L^2} \times \|\mathcal{F}_h\|_{L^1 \rightarrow L^\infty} \times \|\mathbb{1}_{[-h^{3/4}, h^{3/4}]}\|_{L^2 \rightarrow L^1} \\ &\leq Ch^{3/8} \times h^{-1/2} \times h^{3/8} = Ch^{1/4} \end{aligned}$$

Here, the bound can be understood as a volume estimate : the box in phase space of size  $h^{3/4}$  is smaller than a "quantum box". Gathering all the computations together, we see that

$$\|M^{2n+1}\|_{L^2 \rightarrow L^2} = O\left(h^{1/4} \log h\right)$$

Elevating this to the power  $\frac{1}{2n+1}$ , we see that for every  $\varepsilon > 0$ , we can find  $h_\varepsilon$  such that for  $h \leq h_\varepsilon$ ,

$$\rho(M) \leq (1 + \varepsilon)2^{-1/6}$$

**Remark.**

What matters in this example is the strategy we use, and not particularly the bound, which is in fact not optimal.

**Sketch of proof.** The strategy presented in this simple model case is the guideline, but its direct application will encounter major pitfalls that we'll have to bypass.

- The trapped set being a more complex fractal set, we'll need the general fractal uncertainty principle developed by S. Dyatlov and his collaborators.
- Even in small coordinate charts, the trapped set cannot be written as a product of fractal sets in the unstable and stable directions. To tackle this difficulty, we use the adapted coordinate charts constructed in Section 3-4 in which we straighten the unstable manifolds. The existence of such coordinate charts is made possible by Theorem 3-3.1, which asserts that the unstable (and stable) distribution can be extended in a neighborhood of the trapped set to a  $C^{1,\beta}$  vector field.
- In the model case, there is only one point and hence one unstable Jacobian to consider which gives the Lyapouov exponent of the map  $\log J_u^1(0) = \log 2$ . Generally, the growth rate of the unstable Jacobian differs from one point to another (see 5-3.2.) and the choice of the integer  $n(h)$  is not as simple. In fact, we prefer to break the symmetry  $2n(h) = n(h) + n(h)$  and split  $2n(h)$  into a small logarithmic time  $N_0(h)$  and a long logarithmic time  $N_1(h)$  (see section 5-2). The first one is supposed to be smaller than the Ehrenfest time and allows us to use semiclassical calculus to handle  $M^{N_0}$ . As a matter of fact, the major technical difficulties concerns the study of  $M^{N_1}$ .

- The study of  $M^{N_1}$  requires fine microlocal techniques. The trick used in the model case to have the commutator estimate is no more possible and we have to use propagation results up to twice the Ehrenfest time. This is what we do in section 5-3.3. but this study has to be made locally and we need to split  $M^{N_1}$  into a sum of many terms  $U_{\mathbf{q}}$ .
- We could use the fractal uncertainty principle to get the decay for single terms  $M^{N_0}U_{\mathbf{q}}$ . However, a simple triangle inequality to handle their sum will no more give a decay for  $M^{N_0+N_1}$  since the number of terms in the sum grows like a negative power of  $h$ . To bypass this problem, we need a more careful analysis and we gather them into clouds (see 5-3.6.). These clouds are supposed to interact with a few other ones, so that a Cotlar-Stein type estimate reduces the study of the norm of the sum, to the norm of each cloud. The elements of a single cloud are supposed to be close to each other, so that the fractal uncertainty principle can be applied to all of them in the same time and gives the required decay for a single cloud.

Our strategy follows the main lines of the proof of [DJN21]. In particular, their strategy allows us to apply the fractal uncertainty principle of [BD18] in a case where the unstable foliation is not smooth (and in fact, a priori defined only in a fractal set). Their strategy relies on the existence of adapted charts based on  $C^{2^-}$  regularity of the unstable foliations in negatively curved surfaces. It is based on results of [HK90] for Anosov flows. In our work, this strategy relies on the adapted charts of 3-4. Another aspect which changes from [DJN21] is the proof of porosity. In their study, the porous sets arise as iteration of artificial "holes" and they had to control the evolution of such holes. In our context, this study is easier since we already know that the trapped set has a fractal structure, characterized by its fractal (Hausdorff or upper-box) dimension.

**Restrictions.** The main restriction of our theorem is that it only applies to quantum maps with two-dimensional phase space. In terms of open systems, it only concerns problems with physical space of dimension 2. Several points explain this restriction :

- The fractal uncertainty principle works in dimension 1. In higher dimension, the result is currently not well understood and the only known cases require strong assumptions on the fractal sets (See [Dya18], Section 6).
- Our proof strongly relies on the regularity of the stable and unstable laminations.
- The growth of the unstable Jacobian controls the contraction (resp. expansion) rate in the unique stable (resp. unstable) direction.

## 5-2 Start of the proof and a key result.

We start the proof of Theorem 5-1.2. The rest of the Chapter is dedicated to this proof.

**Notations.** Throughout this chapter, the meaning of the constants  $C$  can change from line to line but these constants will only depend on our dynamical system  $(U, F)$ . If there is another dependence, it will be specified.

We consider  $T = T(h) \in I_{0^+}(Y \times Y, F')$  an open quantum hyperbolic maps associated with  $F$ , microlocally unitary in a neighborhood of  $\mathcal{T}$ , and a symbol  $\alpha \in S_{0^+}(U)$ . We want to show a bound for the spectral radius of  $M(h) = T(h) \text{Op}_h(\alpha)$ , independent of  $h$ . We'll use the standard fact :

$$\|M^n\|_{L^2 \rightarrow L^2} \leq \rho \implies \rho_{spec}(M) \leq \rho^{1/n}$$

The trivial lemma which follows reduces the theorem to the study of  $\|M^n\|$  with  $n = n(h) \sim \delta |\log h|$ .

**Lemma 5-2.1.** Let  $\delta > 0$  and  $N(h) \in \mathbb{N}$  satisfy  $N(h) \sim \delta |\log h|$ . Suppose that there exists  $h_0 > 0$  and  $\gamma > 0$  such that

$$\forall 0 < h < h_0 \quad , \quad \|M(h)^{N(h)}\| \leq h^\gamma \|\alpha\|_\infty^{N(h)} \quad (5-2.1)$$

Then, for every  $\varepsilon > 0$ , there exists  $h_\varepsilon$  such that, for  $h \leq h_\varepsilon$ ,

$$\rho_{spec}(M(h)) \leq e^{-\frac{\gamma}{\delta} + \varepsilon} \|\alpha\|_\infty$$

*Proof.* It suffices to observe that under the assumption (5-2.1), we have  $\rho_{spec}(M(h)) \leq e^{\frac{\gamma \log h}{N(h)}} \|\alpha\|_\infty$  and use the equivalence for  $N(h)$ .  $\square$

**Remark.**

If we use the bound  $\|M\| \leq \|\alpha\|_\infty + O(h^{1/2-\varepsilon})$ , one get the obvious bound  $\|M^N\| \leq \|\alpha\|_\infty^N (1 + o(1))$ . Hence, (5-2.1) is a decay bound.

The proof of Theorem 5-1.2 is then reduced to the proof of the following key result, which will be exploited in Chapter 6 :

**Theorem 5-2.1.** *There exists  $\delta > 0$ ,  $h_0 > 0$ , a family of integers  $N(h) \sim \delta |\log(h)|$  and  $\gamma > 0$  such that, for  $0 < h \leq h_0$*

$$\|M(h)^{N(h)}\| \leq h^\gamma \|\alpha\|_\infty^{N(h)} \quad (5-2.2)$$

Actually, the precise value of  $N(h)$  we'll use is rather explicit and we now describe it. Recall that the unstable distribution has  $C^{1,\beta}$  regularity for some  $\beta \in ]0, 1[$  (see Theorem 3-3.1). We set

$$\mathfrak{b} = \frac{1}{1 + \beta} \quad (5-2.3)$$

We now choose  $\delta_0 \in (0, \frac{1}{2})$  such that

$$\mathfrak{b} + \delta_0 < 1 \quad (5-2.4)$$

For instance, let us set

$$\delta_0 = \frac{1 - \mathfrak{b}}{2} = \frac{\beta}{2(1 + \beta)}$$

Let us also recall the definitions of the exponent  $\lambda_0 \leq \lambda_1$  in (3-2.5) and (3-2.6) :

$$\begin{aligned} e^{n\lambda_0} \leq J_n^u(\rho) \leq e^{n\lambda_1} & \quad \text{and} \quad e^{-n\lambda_1} \leq J_n^s(\rho) \leq e^{-n\lambda_0} & ; n \in \mathbb{N}, \rho \in \mathcal{T} \\ e^{n\lambda_0} \leq J_{-n}^s(\rho) \leq e^{n\lambda_1} & \quad \text{and} \quad e^{-n\lambda_1} \leq J_{-n}^u(\rho) \leq e^{-n\lambda_0} & ; n \in \mathbb{N}, \rho \in \mathcal{T} \end{aligned}$$

We then introduce the following notations

$$N(h) = N_0(h) + N_1(h) \quad ; \quad N_0(h) = \left\lceil \frac{\delta_0}{\lambda_1} |\log(h)| \right\rceil \quad ; \quad N_1(h) = \left\lceil \frac{1}{\lambda_0} |\log(h)| \right\rceil \quad (5-2.5)$$

$N_0(h)$  (resp.  $N_1(h)$ ) corresponds to a short (resp. long) logarithmic time. We will omit the dependence on  $h$  in the following.

To be complete with the numerology, we introduce another number  $\tau < 1$  such that

$$\mathfrak{b} < \tau < 1 \quad \text{and} \quad \delta_0 \frac{\lambda_0}{\lambda_1} + \tau > 1 \quad (5-2.6)$$

The meaning of these conditions will be clear in the core of the proof and we won't miss to recall where they are used. For instance, we set

$$\tau = 1 - \frac{\lambda_0}{\lambda_1} \frac{1 - \mathfrak{b}}{4} \quad (5-2.7)$$

**An important remark.** If two operators  $M_1(h)$  and  $M_2(h)$  are equal modulo  $O(h^\infty)$ , this is also the case for  $M_1(h)^{N(h)}$  and  $M_2(h)^{N(h)}$  as long as

- $N(h) = O(\log h)$ .
- $M_1(h), M_2(h) = O(h^{-K})$  for some  $K$ .

This will be widely used in the following. In particular, recall that we work with operators acting on  $L^2(Y)$  but these operators take the form  $M_1(h) = \Psi_Y M_2(h) \Psi_Y$  where  $\Psi_Y \in C_c^\infty(Y, [0, 1])$  and  $M_2(h)$  is a bounded operator on  $\bigoplus_{j=1}^J L^2(\mathbb{R})$  such that  $M_2(h) = \Psi_Y M_2(h) \Psi_Y + O(h^\infty)$ . As a consequence, modulo  $O(h^\infty)$ , it is enough to focus on  $M_2(h)^{N(h)}$ . For this reason, from now on and even if we keep the same notation, we work with

$$M(h) = T(h) \text{Op}_h(\alpha) : \bigoplus_{j=1}^J L^2(\mathbb{R}) \rightarrow \bigoplus_{j=1}^J L^2(\mathbb{R})$$

where  $T(h) = (T_{ij}(h))$  with  $T_{ij} \in I_{0+}(\mathbb{R} \times \mathbb{R}, F'_{ij})$  and

$$\text{Op}_h(\alpha) = \text{Diag}(\text{Op}_h(\alpha_1), \dots, \text{Op}_h(\alpha_J))$$

### 5-3 Construction of a refined quantum partition.

#### 5-3.1. Microlocal partition of unity and notations.

We consider some  $\varepsilon_0 > 0$ , which is supposed small enough to satisfy all the assumptions which will appear in the following.

We consider a cover of  $\mathcal{T}$  by a finite number of balls of radius  $\varepsilon_0$  :

$$\mathcal{T} \subset \bigcup_{q=1}^Q B(\rho_q, \varepsilon_0) \quad ; \quad \rho_q \in \mathcal{T}$$

and we assume that for all  $q \in \{1, \dots, Q\}$ , there exists  $j_q, l_q, m_q \in \{1, \dots, J\}$  such that

$$B(\rho_q, 2\varepsilon_0) \subset \tilde{A}_{j_q l_q} \cap \tilde{D}_{m_q j_q} \subset U_{j_q}$$

We also assume that  $T$  is microlocally unitary in  $B(\rho_q, 4\varepsilon_0)$ . We then note

$$\mathcal{V}_q = B(\rho_q, 2\varepsilon_0) \tag{5-3.1}$$

#### Remark.

In the case of obstacle scattering, with obstacles satisfying the non-eclipse condition, it is possible to choose a simple partition of unity, related to the coding of the trapped set according to the sequence of obstacles hit by a trajectory. Indeed, due to a result of [Mor91], there is a homeomorphism between the trapped set and the admissible - that is two consecutive obstacles are different - sequences of obstacles. As a consequence, if the obstacles are numbered from 1 to  $J$ , we can partition the trapped set by open subsets  $U_{\vec{\alpha}}$  indexed by

$$\{\vec{\alpha} = (\alpha_{-N}, \dots, \alpha_N) \in \{1, \dots, J\}^{2N+1}, \alpha_{i+1} \neq \alpha_i\}$$

The diameter of such partition goes to 0 as  $N$  goes to  $+\infty$  and we could get the required partition  $(\mathcal{V}_q)_q$ , with the additional property of being disjoint open subsets of  $U$ . This would simplify the study in this particular setting.

We complete this cover with

$$\mathcal{V}_\infty = U' \setminus \bigcup_{q=1}^Q \overline{B(\rho_q, \varepsilon_0)} \tag{5-3.2}$$

$U' \Subset U$  is an open set such that  $\text{WF}_h(M) \Subset U' \times U'$ . We note  $U'_j$  the component of  $U'$  inside  $U_j$ .

We note  $\mathcal{A} = \{1, \dots, Q\}$  and  $\mathcal{A}_\infty = \mathcal{A} \cup \{\infty\}$ .

We then consider a partition of unity associated with the cover  $\mathcal{V}_1, \dots, \mathcal{V}_Q, \mathcal{V}_\infty$ , namely a family of smooth functions  $\chi_q \in C_c^\infty(U)$ , for  $q \in \mathcal{A}_\infty$  such that :

- $\text{supp } \chi_q \subset \mathcal{V}_q$

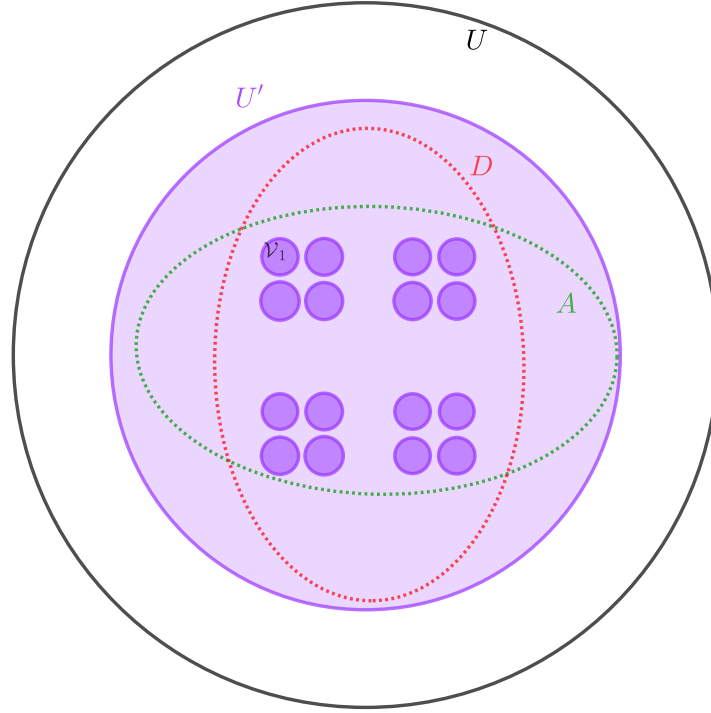


Figure 5.1: The partition  $(\mathcal{V}_q)_{q \in \mathcal{A}_\infty}$  is made by small neighborhoods of  $\mathcal{T}$  (small purple disks) and a big open set included in  $U'$ .

- $0 \leq \chi_q \leq 1$
- $1 = \sum_{q \in \mathcal{A}_\infty} \chi_q$  in  $\bigcup_{q \in \mathcal{A}_\infty} \mathcal{V}_q$

More precisely, if  $q \in \mathcal{A}$ ,  $\chi_q \in C^\infty(U_{j_q})$  and for every  $j \in \{1, \dots, J\}$ , there exists  $b_j \in C_c^\infty(U_j)$  such that on  $U'_j$ ,  $1 = b_j + \sum_{q \in \mathcal{A}, j_q=j} \chi_q$ . Thus,  $\chi_\infty = \sum_{j=1}^J b_j$ .

We can then quantize these symbols so as to get a pseudodifferential partition of unity. More precisely, to respect the matrix structure, we may write this quantization in a diagonal operator valued matrix, still denoted  $\text{Op}_h$  :

- for  $q \in \mathcal{A}$ ,  $A_q = \text{Op}_h(\chi_q)$  is the diagonal matrix  $\text{Diag}(0, \dots, \text{Op}_h(\chi_q), 0, \dots, 0)$  where the block  $\text{Op}_h(\chi_q)$  is in the  $j_q$ -th position ;
- $\text{Op}_h(\chi_\infty) = \text{Diag}(\text{Op}_h(b_1), \dots, \text{Op}_h(b_J))$ .

The family  $(A_q)_{q \in \mathcal{A}_\infty}$  satisfies the following properties :

$$\sum_{q \in \mathcal{A}_\infty} A_q = \text{Id} \text{ microlocally in } U' \quad ; \quad \forall q \in \mathcal{A}_\infty, \|A_q\| \leq 1 + O(h^{1/2}) \quad (5-3.3)$$

Since  $M = \sum_{q \in \mathcal{A}_\infty} MA_q + O(h^\infty)$ , we may write

$$M^n = \sum_{\mathbf{q} \in \mathcal{A}_\infty^n} U_{\mathbf{q}} + O(h^\infty)$$

where for  $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}_\infty^n$ ,

$$U_{\mathbf{q}} := MA_{q_{n-1}} \dots MA_{q_0} \quad (5-3.4)$$

For  $\mathbf{q} = q_0 \dots, q_{n-1} \in \mathcal{A}_\infty^n$ , we also define a family of refined neighborhoods, forming a refined cover of  $\mathcal{T}$ ,

$$\mathcal{V}_{\mathbf{q}}^- = \bigcap_{i=0}^{n-1} F^{-i}(\mathcal{V}_{q_i}) \quad ; \quad \mathcal{V}_{\mathbf{q}}^+ = F^n(\mathcal{V}_{\mathbf{q}}^-) = \bigcap_{i=0}^{n-1} F^{n-i}(\mathcal{V}_{q_i}) \quad (5-3.5)$$

This definition imply that a point  $\rho \in \mathcal{V}_{\mathbf{q}}^-$  lies in  $\mathcal{V}_{q_i}$  at time  $i$  (i.e  $F^i(\rho) \in \mathcal{V}_{q_i}$ ) for  $0 \leq i \leq n-1$  and a point  $\rho \in \mathcal{V}_{\mathbf{q}}^+$  lies in  $\mathcal{V}_{q_{n-i}}$  at time  $-i$ , for  $1 \leq i \leq n$ . Roughly speaking, we expect that each operator  $U_{\mathbf{q}}$  acts from  $\mathcal{V}_{\mathbf{q}}^-$  to  $\mathcal{V}_{\mathbf{q}}^+$  and is negligible (in some sense to be specified later on) elsewhere. Combining (5-3.3) and the bound on  $M$ , the following bound is valid (for any  $\varepsilon > 0$ ):

$$\|U_{\mathbf{q}}\|_{L^2 \rightarrow L^2} \leq \left( \|\alpha\|_{\infty} + O(h^{1/2-\varepsilon}) \right)^n \quad (5-3.6)$$

As soon as  $|n| \leq C_0 |\log h|$ , we have  $\|U_{\mathbf{q}}\|_{L^2 \rightarrow L^2} \leq C \|\alpha\|_{\infty}^n$ , for some  $C$  depending on  $C_0$  and a finite number of semi-norms of  $\alpha$ .

**Reduction to words in  $\mathcal{A}$ .** We can find a uniform  $T_0 \in \mathbb{N}$  such that if  $\rho \in \mathcal{V}_{\infty}$ , there exists  $k \in \{-T_0, \dots, T_0\}$  such that  $F^k(\rho)$  "falls" in the hole. By standard properties of the Fourier integral operators, each component  $(M^{T_0})_{ij}$  of  $M^{T_0}$  is a Fourier integral operator associated with the component  $(F^{T_0})_{ij}$  of  $F^{T_0}$ . In particular,  $\text{WF}_h'(M^{T_0}) \subset \text{Gr}'(F^{T_0})$ .

Let us study  $M^{2T_0+N(h)}$ . If  $\mathbf{q} = q_0 \dots q_{N-1} \in \mathcal{A}_{\infty}^N$  and if there exists an index  $i \in \{0, \dots, N-1\}$  such that  $q_i = \infty$ , one can isolate this index  $i$  and trap  $A_{q_i}$  between two Fourier integral operators  $M_1, M_2$ , belonging to a finite family of FIO associated with  $F^{T_0}$ , so that we can write

$$M^{T_0} U_{\mathbf{q}} M^{T_0} = B_1 M_1 A_{\infty} M_2 B_2$$

where  $B_1, B_2$  satisfy the  $L^2$ -bound :

$$\|B_1\| \times \|B_2\| \leq (\|\alpha\|_{\infty} + O(h^{1/4}))^{N-1} = O(h^{-K})$$

for some integer  $K$ . Since,

$$\text{WF}_h'(M_1 A_{\infty} M_2) \subset \{(F^{T_0}(\rho), F^{-T_0}(\rho)); \rho \in \text{WF}_h(A_{\infty})\} = \emptyset$$

we have  $M_1 A_{\infty} M_2 = O(h^{\infty})$ , with constants that can be chosen independent of  $\mathbf{q}$ . Hence, the same is true for  $M^{T_0} U_{\mathbf{q}} M^{T_0}$ .  $|\mathcal{A}^N|$  is bounded by a negative power of  $h$ . So, we can write :

$$\begin{aligned} M^{N+2T_0} &= \sum_{\mathbf{q} \in \mathcal{A}_{\infty}^N} M^{T_0} U_{\mathbf{q}} M^{T_0} \\ &= \sum_{\mathbf{q} \in \mathcal{A}^N} M^{T_0} U_{\mathbf{q}} M^{T_0} + O(h^{\infty}) \\ &= M^{T_0} \left( \sum_{\mathbf{q} \in \mathcal{A}^N} U_{\mathbf{q}} \right) M^{T_0} + O(h^{\infty}) \end{aligned}$$

We can then replace  $M$  by

$$\mathfrak{M} = \sum_{q \in \mathcal{A}} M A_q = M(\text{Id} - A_{\infty}) + O(h^{\infty})_{L^2 \rightarrow L^2} \quad (5-3.7)$$

A decay bound

$$\|\mathfrak{M}(h)^{N(h)}\| \leq h^{\gamma} \|\alpha\|_{\infty}^{N(h)} \quad (5-3.8)$$

will imply the required decay bound (5-2.2) for  $M$  with  $N(h)$  replaced by  $N(h) + 2T_0$ . We are hence reduced to prove the decay bound (5-3.8).



### 5-3.2. Local Jacobian.

**A first definition.** Following [DJN21], we introduce local unstable and stable Jacobians and we then state several properties. For  $n \in \mathbb{N}^*$  and  $\mathbf{q} \in \mathcal{A}^n$ , let us define its local stable and unstable Jacobian.

$$J_{\mathbf{q}}^- := \inf_{\rho \in \mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^-} J_n^u(\rho), \quad J_{\mathbf{q}}^+ := \inf_{\rho \in \mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^+} J_{-n}^s(\rho) \quad (5-3.9)$$

By the chain rule, we have for  $\rho \in \mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^-$ ,

$$J_n^u(\rho) = \prod_{i=0}^{n-1} J_1^u(F^i(\rho))$$

A similar formula is true for  $\rho \in \mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^+$  :

$$J_{-n}^s(\rho) = \prod_{i=0}^{n-1} (J_1^s(F^{i-n}(\rho)))^{-1} = \prod_{i=0}^{n-1} J_{-1}^s(F^{-i}(\rho))$$

Hence, we've got the basic estimates :

$$\mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^- \neq \emptyset \implies e^{\lambda_0 n} \leq J_{\mathbf{q}}^- \leq e^{\lambda_1 n} \quad (5-3.10)$$

$$\mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^+ \neq \emptyset \implies e^{\lambda_0 n} \leq J_{\mathbf{q}}^+ \leq e^{\lambda_1 n} \quad (5-3.11)$$

If  $\mathbf{q} = q_0 \dots q_{n-1}$  and  $\mathbf{q}_- = q_0 \dots q_{n-2}$ , then  $\mathcal{V}_{\mathbf{q}}^- \subset \mathcal{V}_{\mathbf{q}_-}^-$  and thus

$$J_{\mathbf{q}}^- \geq e^{\lambda_0} J_{\mathbf{q}_-}^- \quad (5-3.12)$$

Similarly, if  $\mathbf{q}_+ = q_1 \dots q_{n-1}$ ,  $\mathcal{V}_{\mathbf{q}}^+ \subset \mathcal{V}_{\mathbf{q}_+}^+$  and

$$J_{\mathbf{q}}^+ \geq e^{\lambda_0} J_{\mathbf{q}_+}^+ \quad (5-3.13)$$

As a consequence of Corollary 3-2.1, if  $\varepsilon_0$  is small enough, the local stable and unstable Jacobians give the expansion rate of the flow at every point of  $\mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^\pm$ . If  $\mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^\pm \neq \emptyset$ ,

$$\forall \rho \in \mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^-, J_n^u(\rho) \sim J_{\mathbf{q}}^- \quad (5-3.14)$$

$$\forall \rho \in \mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^+, J_{-n}^s(\rho) \sim J_{\mathbf{q}}^+ \quad (5-3.15)$$

This definition is slightly not satisfactory since  $J_{\mathbf{q}}^\pm = +\infty$  as soon as  $\mathcal{V}_{\mathbf{q}}^\pm \cap \mathcal{T} = \emptyset$ . However, when  $\mathcal{V}_{\mathbf{q}}^\pm \neq \emptyset$ , this set can still stay relevant. For this purpose, we will give a definition of local stable and unstable Jacobian for such words with help of the Shadowing Lemma ([HK95], Section 18.1).

**Enlarged definition.** Let  $n \in \mathbb{N}$  and  $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}^n$ . We focus on  $\mathcal{V}_{\mathbf{q}}^-$ , with the case of  $\mathcal{V}_{\mathbf{q}}^+$  handled similarly by considering  $F^{-1}$  instead of  $F$ .

If  $\mathcal{V}_{\mathbf{q}}^- \cap \mathcal{T} \neq \emptyset$ , we keep the definition given in 5-3.9. Assume now that  $\mathcal{V}_{\mathbf{q}}^- \neq \emptyset$  but  $\mathcal{V}_{\mathbf{q}}^- \cap \mathcal{T} = \emptyset$ . Fix  $\rho \in \mathcal{V}_{\mathbf{q}}^-$ . By definition of  $\mathcal{V}_{\mathbf{q}_i}$ , for  $i \in \{0, \dots, n-1\}$ , we have  $d(\rho_i, F^i(\rho)) \leq 2\varepsilon_0$ . Hence,

$$d(F(\rho_i), \rho_{i+1}) \leq d(F(\rho_i), F^{i+1}(\rho)) + d(F^{i+1}(\rho), \rho_{i+1}) \leq C\varepsilon_0$$

for a constant  $C$  only depending on  $F$ . That is to say,  $(\rho_0, \dots, \rho_{n-1})$  is a  $C\varepsilon_0$  pseudo orbit. Assume that  $\delta_0 > 0$  is a small fixed parameter. In virtue of the shadowing lemma, if  $\varepsilon_0$  is sufficiently small,  $(\rho_0, \dots, \rho_{n-1})$  is  $\delta_0$  shadowed by an orbit of  $F$  : there exists  $\rho' \in \mathcal{T}$  such that for  $i \in \{0, \dots, n-1\}$ ,  $d(\rho_i, F^i(\rho')) \leq \delta_0$ . Consequently,  $d(F^i(\rho), F^i(\rho')) \leq \delta_0 + C\varepsilon_0$ . If  $\rho_2$  is another point in  $\mathcal{V}_{\mathbf{q}}^-$ , for  $i = 0, \dots, n-1$ ,  $d(F^i(\rho_2), F^i(\rho')) \leq 2\varepsilon_0 + C\varepsilon_0 + \delta_0$ . For convenience, set  $\varepsilon_2 = 2\varepsilon_0 + \delta_0 + C\varepsilon_0$  and note that  $\varepsilon_2$  can be arbitrarily small depending on  $\varepsilon_0$ . As a consequence, we have proven the following

**Lemma 5-3.1.** If  $\mathcal{V}_{\mathbf{q}}^- \neq \emptyset$ , there exists  $\rho' \in \mathcal{T}$  such that  $\forall i \in \{0, \dots, n-1\}$  and  $\rho \in \mathcal{V}_{\mathbf{q}}^-$ ,  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_2$ .

Fix any  $\rho'$  satisfying the conclusions of this lemma and we arbitrarily set

$$J_{\mathbf{q}}^- = J_n^u(\rho') \quad (5-3.16)$$

If  $\rho'_1$  is another point satisfying this conclusion, we have  $d(F^i(\rho'), F^i(\rho'_1)) \leq 2\varepsilon_2$  for  $i \in \{0, \dots, n-1\}$  and in virtue of Corollary (3-2.1),

$$J_n^u(\rho') \sim J_n^u(\rho'_1)$$

Hence, up to global multiplicative constants, the definition of this unstable Jacobian is independent of the choice of  $\rho'$ . Notice that if  $\mathcal{V}_{\mathbf{q}}^- \cap \mathcal{T} \neq \emptyset$ , any  $\rho' \in \mathcal{T} \cap \mathcal{V}_{\mathbf{q}}^-$  satisfies the conclusions of Lemma 5-3.1 and  $J_{\mathbf{q}}^- \sim J_n^u(\rho')$ .

To define  $J_{\mathbf{q}}^+$ , we can argue similarly and show that there exists  $\rho'$  satisfying  $d(F^i(\rho'), F^i(\rho)) \leq \varepsilon_2$  for  $i \in \{-n, \dots, -1\}$  and  $\rho \in \mathcal{V}_{\mathbf{q}}^+$ . We can assume that this is the same  $\varepsilon_2$  as before and we set  $J_{\mathbf{q}}^+ = J_{-n}^s(\rho')$  for any  $\rho'$ .

**Behavior of the local Jacobian.** The following three lemmas are crucial to understand the behavior of the evolution of points in the sets  $\mathcal{V}_{\mathbf{q}}^{\pm}$ . The first one gives estimates to handle these quantities.

**Lemma 5-3.2.** Let  $n \in \mathbb{N}$  and  $\mathbf{q}, \mathbf{p}$  in  $\mathcal{A}^n$ . If  $\varepsilon_0$  is chosen small enough, then the following holds

1)  $\mathcal{V}_{\mathbf{q}}^+ \neq \emptyset \iff \mathcal{V}_{\mathbf{q}}^- \neq \emptyset$  and in that case  $J_{\mathbf{q}}^- \sim J_{\mathbf{q}}^+$ .

2) If two propagated neighborhoods intersects, the local Jacobians are comparable :

$$\mathcal{V}_{\mathbf{q}}^{\pm} \cap \mathcal{V}_{\mathbf{p}}^{\pm} \neq \emptyset \implies J_{\mathbf{q}}^{\pm} \sim J_{\mathbf{p}}^{\pm} \quad (5-3.17)$$

3) If  $\mathbf{q}$  can be written as the concatenation of  $\mathbf{q}_1$  and  $\mathbf{q}_2$  of lengths  $n_1$  and  $n_2$  such that  $n_1 + n_2 = n$  and if  $\mathcal{V}_{\mathbf{q}}^{\pm} \neq \emptyset$ , then

$$J_{\mathbf{q}}^{\pm} \sim J_{\mathbf{q}_1}^{\pm} J_{\mathbf{q}_2}^{\pm} \quad (5-3.18)$$

**Notations.** The constants in  $\sim$  are independent of  $\rho$  and  $n$ . They depend on  $F$  but also on the partition  $(\mathcal{V}_q)_q$ . In the following, we'll be lead to use constants with the same kind of dependence. These constants will be allowed to depend also on the partition of unity  $(\chi_q)_q$  and on  $M$ . Constants with such dependence will be called **global** constants.

*Proof.* 1) The equivalence is obvious. From the fact that  $F$  is a volume-preserving canonical transformation, we have for some  $C > 0$ ,

$$\forall \rho \in \mathcal{T}, \forall n \in \mathbb{N}, C^{-1} \leq J_n^u(\rho) J_n^s(\rho) \leq C$$

and we write  $J_n^u(\rho) \sim J_n^s(\rho)^{-1}$ . From  $F^{-n} \circ F^n(\rho) = \rho$ , we also get  $J_n^s(\rho)^{-1} = J_{-n}^s(F^n(\rho))$ . Eventually, if  $\rho' \in \mathcal{T}$  satisfies  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_2$  for  $i \in \{0, \dots, n-1\}$  and  $\rho \in \mathcal{V}_{\mathbf{q}}^-$ ,  $F^n(\rho') = \rho^+$  satisfies  $d(F^i(\rho), F^i(\rho^+)) \leq \varepsilon_2$  for  $i \in \{-n, \dots, -1\}$  and  $\rho \in \mathcal{V}_{\mathbf{q}}^+$ . Hence

$$J_{\mathbf{q}}^+ \sim J_{-n}^s(\rho^+) \sim J_n^u(\rho') \sim J_{\mathbf{q}}^-$$

Thanks to this first point, it is enough to show the remaining point only for  $-$ .

2) Pick  $\rho_{\mathbf{q}} \in \mathcal{T}$  (resp.  $\rho_{\mathbf{p}}$ ) satisfying the conclusions of lemma 5-3.1 for  $\mathcal{V}_{\mathbf{q}}^-$  (resp.  $\mathcal{V}_{\mathbf{p}}^-$ ).  $d(F^i(\rho_{\mathbf{q}}), F^i(\rho_{\mathbf{p}})) \leq 2\varepsilon_2$  and hence, in virtue of Corollary 3-2.1,  $J_n^u(\rho_{\mathbf{q}}) \sim J_n^u(\rho_{\mathbf{p}})$ . This gives 2).

3) Pick  $\rho \in \mathcal{T}$  satisfying the conclusions of lemma 5-3.1 for  $\mathcal{V}_{\mathbf{q}}^-$ . By the chain rule,  $J_n^u(\rho) = J_{n_2}^u(F^{n_1}(\rho)) J_{n_1}^u(\rho)$ . Remark that

$$\mathcal{V}_{\mathbf{q}}^- = \mathcal{V}_{\mathbf{q}_1}^- \cap F^{-n_1}(\mathcal{V}_{\mathbf{q}_2}^-)$$

Hence,  $\rho$  satisfies the conclusions of Lemma 5-3.1 for  $\mathbf{q}_1$  with  $\varepsilon_2$  and the same is true for  $F^{n_1}(\rho)$  and  $\mathbf{q}_2$ . It follows that  $J_{\mathbf{q}_1}^- \sim J_{n_1}^u(\rho)$  and  $J_{\mathbf{q}_2}^- \sim J_{n_2}^u(F^{n_1}(\rho))$ . This gives 3).  $\square$

**Remark.**

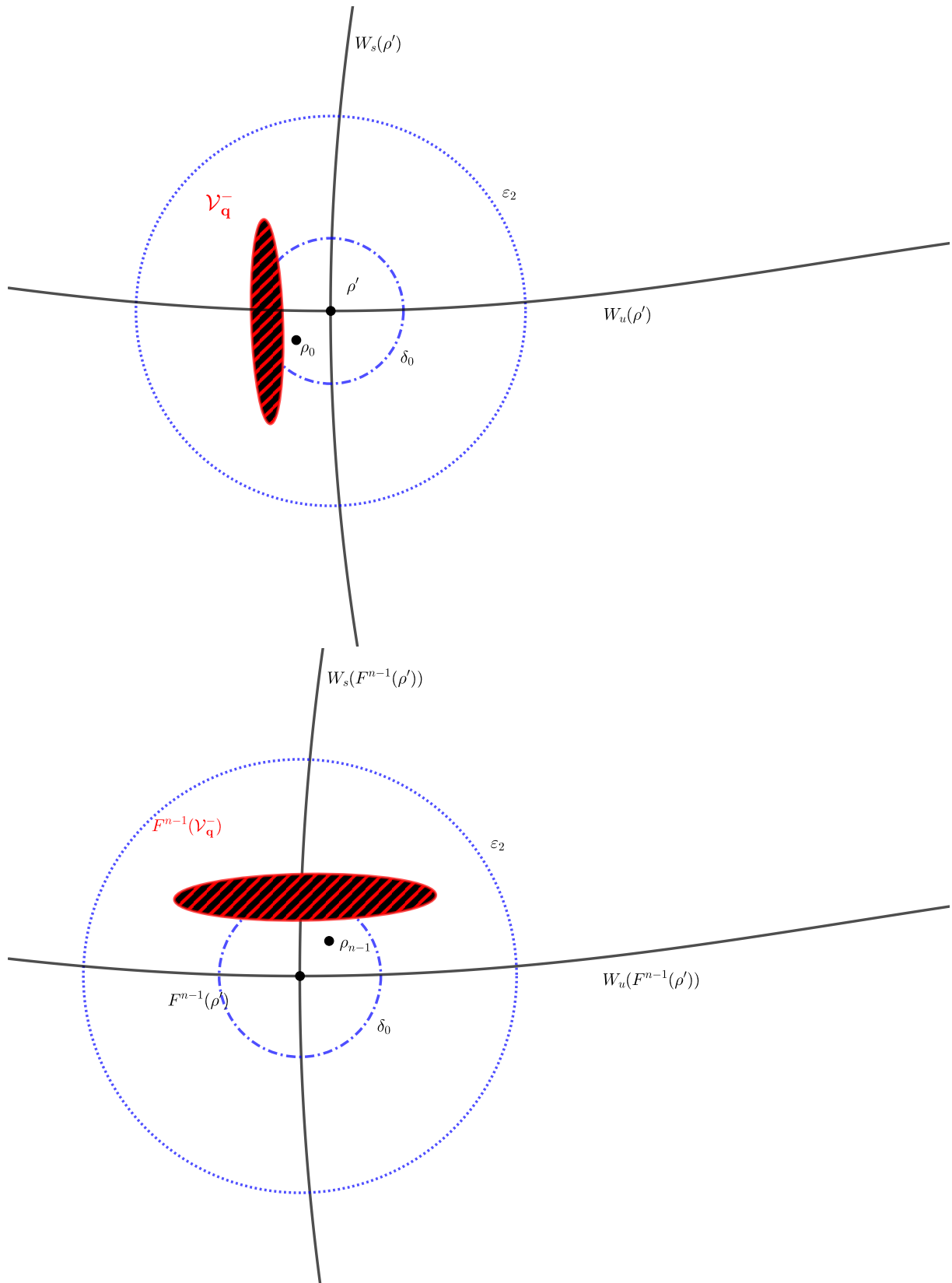


Figure 5.2: Evolution of the set  $\mathcal{V}_q^-$  (the red hatched set) at time 0 and  $n-1$ . The points  $\rho_i, F^i(\rho')$  are represented at these times, so as the balls  $B(F^i(\rho'), \varepsilon_2)$  and  $B(F^i(\rho'), \delta_0)$  (their boundaries are the blue dotted lines). We've also represented the stable (resp. unstable) manifold at  $F^i(\rho')$  to show the directions in which  $F$  contracts (resp. expands).

The first point of the previous lemma shows that we could consider only one of the two quantities. Nevertheless, we prefer keeping trace of it. The reason is that *a priori*  $J^+$  and  $J^-$  support two different kind of information :  $J^+$  controls the growth of  $F^n$  whereas  $J^-$  controls the growth of  $F^{-n}$ . The fact that the two dynamics (in the past and in the future) have similar behaviors is a consequence of the fact that  $F$  is volume-preserving.

The next lemmas relate the local Jacobian to the expansion rates of the flow in the  $\mathcal{V}_\mathbf{q}^\pm$ . It will be important in our semiclassical study of operators microlocally supported in  $\mathcal{V}_\mathbf{q}^\pm$ .

**Lemma 5-3.3. Control of expansion rate by unstable Jacobian.** If  $\varepsilon_0$  is small enough, there exists a global constant  $C > 0$  satisfying the following inequalities. For every  $n \in \mathbb{N}^*$  and  $\mathbf{q} \in \mathcal{A}^n$  such that  $\mathcal{V}_\mathbf{q}^- \neq \emptyset$  we have :

$$\sup_{\rho \in \mathcal{V}_\mathbf{q}^-} \|d_\rho F^n\| \leq C J_\mathbf{q}^- \quad (5-3.19)$$

$$\sup_{\rho \in \mathcal{V}_\mathbf{q}^+} \|d_\rho F^{-n}\| \leq C J_\mathbf{q}^+ \quad (5-3.20)$$

*Proof.* This is a consequence of (3-2.13). Indeed, if  $\mathcal{V}_\mathbf{q}^- \neq \emptyset$  and if  $\rho' \in \mathcal{T}$  satisfies the conclusions of lemma 5-3.1, for every  $\rho \in \mathcal{V}_\mathbf{q}^-$ ,  $\|d_\rho F^n\| \leq C J_n^u(\rho)$  with  $C$  a global constant depending only on  $\varepsilon_2$ .  $\square$

This third lemma emphasizes that  $\mathcal{V}_\mathbf{q}^-$  lies in a small neighborhood of a stable manifold and  $\mathcal{V}_\mathbf{q}^+$  lies in a small neighborhood of an unstable manifold, with the size of this neighborhood controlled by the local Jacobian. It is a direct consequence of Lemma 3-2.3.

**Lemma 5-3.4. Localization of the  $\mathcal{V}_\mathbf{q}^\pm$ .** There exists a global constant  $C > 0$  such that for all  $n \in \mathbb{N}$  and  $\mathbf{q} \in \mathcal{A}^n$ ,

- (1) If  $\mathcal{V}_\mathbf{q}^- \neq \emptyset$  and if  $\rho' \in \mathcal{T}$  satisfies the conclusion of lemma 5-3.1, then, for all  $\rho \in \mathcal{V}_\mathbf{q}^-$ ,

$$d(\rho, W_s(\rho')) \leq \frac{C}{J_\mathbf{q}^-} \quad (5-3.21)$$

- (2) If  $\mathcal{V}_\mathbf{q}^+ \neq \emptyset$  and if  $\rho' \in \mathcal{T}$  satisfies the conclusion of lemma 5-3.1 in the future (namely,  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_2$  for all  $\rho \in \mathcal{V}_\mathbf{q}^+$  and  $i \in \{-n, \dots, -1\}$ ), then for all  $\rho \in \mathcal{V}_\mathbf{q}^+$ ,

$$d(\rho, W_u(\rho')) \leq \frac{C}{J_\mathbf{q}^+} \quad (5-3.22)$$

### 5-3.3. Propagation up to local Ehrenfest time.

In this section, we show that under some control of the local Jacobian defined above, one can handle the operators  $U_\mathbf{q}$  and prove the existence of symbols  $a_\mathbf{q}^\pm$  (in exotic classes  $S_\delta$ ) such that

$$U_\mathbf{q} = \text{Op}_h(a_\mathbf{q}^+) T^{|\mathbf{q}|} + O(h^\infty) \quad (5-3.23)$$

$$U_\mathbf{q} = T^{|\mathbf{q}|} \text{Op}_h(a_\mathbf{q}^-) + O(h^\infty) \quad (5-3.24)$$

with symbols  $a_\mathbf{q}^\pm$  supported in  $\mathcal{V}_\mathbf{q}^\pm$ . We recall that  $U_\mathbf{q} = MA_{q_{n-1}} \dots MA_{q_0}$  with  $M = T \text{Op}_h(\alpha)$ . Let us state the precise statement we will prove.

**Proposition 5-3.1.** Fix  $0 < \delta < \delta_1 < \frac{1}{2}$  and  $C_0 > 0$ .

- (1) For every  $n \in \mathbb{N}$  and for all  $\mathbf{q} \in \mathcal{A}^n$  satisfying

$$J_\mathbf{q}^+ \leq C_0 h^{-\delta} \quad (5-3.25)$$

there exists  $a_\mathbf{q}^+ \in \|\alpha\|_\infty^n S_{\delta_1}^{comp}$  such that

$$U_\mathbf{q} = \text{Op}_h(a_\mathbf{q}^+) T^n + O(h^\infty)_{L^2 \rightarrow L^2} \quad (5-3.26)$$

and

$$\text{supp } a_\mathbf{q}^+ \subset \mathcal{V}_\mathbf{q}^+ \quad (5-3.27)$$

(2) For every  $n \in \mathbb{N}$  and for all  $\mathbf{q} \in \mathcal{A}^n$  satisfying

$$J_{\mathbf{q}}^- \leq C_0 h^{-\delta} \quad (5-3.28)$$

there exists  $a_{\mathbf{q}}^- \in \|\alpha\|_{\infty}^n S_{\delta_1}^{comp}$  such that

$$U_{\mathbf{q}} = T^n \text{Op}_h(a_{\mathbf{q}}^-) + O(h^\infty)_{L^2 \rightarrow L^2} \quad (5-3.29)$$

$$\text{supp } a_{\mathbf{q}}^- \subset \mathcal{V}_{\mathbf{q}}^- \quad (5-3.30)$$

**Remark.**

- The implied constants appearing in the  $O(h^\infty)$  are quasi-global : they have the same dependence as global constants but depend also on  $C_0, \delta, \delta_1$ . What is important is that they are independent of  $n$  and  $\mathbf{q}$  as soon as the assumption (5-3.25) is satisfied.
- (5-3.25) implies that  $\mathcal{V}_{\mathbf{q}}^+ \neq \emptyset$ . In particular, if  $\mathbf{q}$  satisfies this assumption, there exists a sequence  $(i_0, \dots, i_n)$  such that for all  $p \in \{0, \dots, n-1\}$ ,  $\mathcal{V}_{q_p} \subset \widetilde{D}_{i_{p+1}, i_p} \subset U_{i_p}$
- In fact,  $\text{supp } a_{\mathbf{q}}^+ \subset F(\mathcal{V}_{q_{n-1}}) \subset U_{i_n}$ . Hence, the operator  $\text{Op}_h(a_{\mathbf{q}}^+)$  acting on  $\bigoplus_{i=1}^J L^2(\mathbb{R})$  is the diagonal matrix  $\text{Diag}(0, \dots, \text{Op}_h(a_{\mathbf{q}}^+), \dots, 0)$ .
- The symbol  $a_{\mathbf{q}}^+$  has an asymptotic expansion in power of  $h$ . The principal symbol is given by

$$(a_{\mathbf{q}}^+)_0 = \prod_{p=1}^n a_{q_{n-p}} \circ F^{-p} \quad (5-3.31)$$

where  $a_q = \chi_q \times \alpha$ . Note that if the functions  $a_{q_{n-p}} \circ F^{-p}$  are not necessarily well defined, the product is well defined thanks to the assumptions on the supports of  $\chi_q$ , namely  $\text{supp } \chi_q \Subset \mathcal{V}_q$ . Indeed, such a symbol can be constructed inductively as the  $n$ -th term  $b_n$  of the sequence of functions  $b_1 = a_{q_0} \circ F^{-1}$  and  $b_{i+1}$  is obtained from  $a_i$  by the following

$$b_{i+1} = (a_{q_i} \times a_i) \circ F^{-1}$$

If we assume that  $\text{supp } b_i \Subset \mathcal{V}_{q_0 \dots q_{i-1}}^+$ , then  $\text{supp}(a_{q_i} \times b_i) \Subset F^{-1}(\mathcal{V}_{q_0 \dots q_i}^+)$ . This property allows us to define  $b_{i+1}$  and  $\text{supp } b_{i+1} \Subset \mathcal{V}_{q_0 \dots q_i}^+$ .

- The same holds for  $a_{\mathbf{q}}^-$  with principal symbol

$$(a_{\mathbf{q}}^-)_0 = \prod_{p=0}^{n-1} a_{q_p} \circ F^p \quad (5-3.32)$$

- Our proof follows the sketch of proof of [DJN21] (Section 5) and [Riv10] (Section 7).

In the end of this section, we focus on proving this proposition. We only prove the first point. The second point can be proved similarly by using the same techniques.

### 5-3.3.1. Iterative construction of the symbols.

Let us start by a lemma combining the precise versions of the expansion of the Moyal product (Lemma 4-1.1) and of Egorov theorem (Proposition 4-2.2). This lemma is the key ingredient for the iterative formulas below.

**Lemma 5-3.5.** Let  $q \in \mathcal{A}$  and let  $a \in S_{\delta_1}^{comp}$  such that  $\text{supp } a \Subset U_j$  for some  $j \in \{1, \dots, J\}$ . Then, there exists a family of differential operators  $L_{k,q}$  of order  $2k$ , with smooth coefficients compactly supported in  $\mathcal{V}_q$ , such that for every  $N \in \mathbb{N}$ , we have the following expansion

$$MA_q \text{Op}_h(a) = \text{Op}_h \left( \sum_{k=0}^{N-1} h^k (L_{k,q} a) \circ F^{-1} \right) T + O(\|a\|_{C^{2N+15}} h^N)_{L^2 \rightarrow L^2} \quad (5-3.33)$$

Moreover, one has  $L_{0,q} = \chi_q \times \alpha := a_q$ .

**Remark.**

- Again, since  $\text{supp } a \subset U_j$ ,  $\text{Op}_h(a)$  is a diagonal matrix with only one non-zero block equal to  $\text{Op}_h(a)$ .
- Recall that we've supposed that  $\mathcal{V}_q \subset \widetilde{D}_{m_q j_q}$ . As a consequence, the symbols

$$a_1^{(k)} := L_{k,q} a \circ F^{-1}$$

are equal to  $L_{k,q} a \circ (F_{m_q j_q})^{-1}$  and are supported in  $U_{m_q}$  :  $\text{Op}_h(a_1^{(k)})$  is still a diagonal matrix.

*Proof.* Let us first work at the order of operators  $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  and let us study :

$$M_{m_q j_q} \text{Op}_h(\chi_q) \text{Op}_h(a) = T_{m_q j_q} \text{Op}_h(\alpha_{j_q}) \text{Op}_h(\chi_q) \text{Op}_h(a)$$

Using Lemma 4-1.1, we write

$$\text{Op}_h(\chi_q) \text{Op}_h(a) = \text{Op}_h \left( \sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (\chi_q \otimes a)|_{\rho=\rho_1=\rho_2} \right) + O(h^N \|\chi_q \otimes a\|_{C^{2N+13}})$$

the principal term of the expansion being  $\chi_q a$ . Set  $a_{q,k}(\rho) = A(D)^k (\chi_q \otimes a)|_{\rho=\rho_1=\rho_2}$  and use Lemma 4-1.1 to write

$$\text{Op}_h(\alpha_{j_q}) \text{Op}_h(\chi_q) \text{Op}_h(a) = \sum_{k_1+k_2 < N} \frac{i^{k_1+k_2} h^{k_1+k_2}}{k_1! k_2!} \text{Op}_h(A(D)^{k_2} (\alpha_{j_q} \otimes a_{q,k_1})|_{\rho=\rho_1=\rho_2}) + O(h^N \|a\|_{C^{2N+13}})$$

The principal term in the expansion is  $\alpha_{j_q} \chi_q a$ . We note that

$$a \mapsto \sum_{k_1+k_2=k} A(D)^{k_2} (\alpha_{j_q} \otimes a_{q,k_1})|_{\rho=\rho_1=\rho_2}$$

is a differential operator of order  $2k$ . Using the precise version of Egorov theorem in Lemma 4-3.1, we see that for any  $b$  with  $\text{supp}(b) \subset \mathcal{V}_q$ ,

$$T_{m_q j_q} \text{Op}_h(b) = \text{Op}_h \left( b \circ (F_{m_q j_q})^{-1} + \sum_{k=1}^{N-1} h^k (D_k b) \circ (F_{m_q j_q})^{-1} \right) + O(h^N \|b\|_{C^{2N+15}})$$

where  $D_k$  are differential of order  $2k$  compactly supported in  $\mathcal{V}_q$ . Applying this to the previous expansion, we see that we can write :

$$T_{m_q j_q} \text{Op}_h(\alpha_{j_q}) \text{Op}_h(\chi_q) \text{Op}_h(a) = \text{Op}_h \left( (\alpha_{j_q} \chi_q a) \circ F^{-1} + \sum_{k=1}^{N-1} h^k (L_{k,q} a) \circ F^{-1} \right) + O(h^N \|a\|_{C^{2N+15}})$$

We now come to the entire matrix operator. Note that the matrix  $M \text{Op}_h(\chi_q) \text{Op}_h(a)$  is of the form

$$\begin{pmatrix} 0 & \dots & M_{1j_q} \text{Op}_h(\chi_q) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & M_{Jj_q} \text{Op}_h(\chi_q) & \dots & 0 \end{pmatrix} \text{Op}_h(a)$$

Recall that  $\text{WF}_h(\text{Op}_h(\chi_q)) \subset \widetilde{D}_{m_q j_q}$  and  $\text{WF}_h'(M_{m_q j_q}) \subset \text{Gr}'(F_{m_q j_q})$ . Hence, for  $m \neq m_q$ ,  $M_{m j_q} \text{Op}_h(\chi_q) = O(h^\infty)$  and the previous matrix can be written

$$\begin{pmatrix} 0 & \dots & O(h^\infty) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & M_{m_q j_q} \text{Op}_h(\chi_q) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & O(h^\infty) & \dots & 0 \end{pmatrix} \text{Op}_h(a) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & M_{m_q j_q} \text{Op}_h(\chi_q) \text{Op}_h(a) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} + O(h^\infty) \|\text{Op}_h(a)\|_{L^2}$$

With constant in  $O(h^\infty)$  depending on  $\chi_q, M$  and  $\|\text{Op}_h(a)\|_{L^2 \rightarrow L^2} = O(\|a\|_{C^s})$ . Let's note

$$a_1^{(k)} = L_{k,q} a \circ F^{-1}$$

and observe that  $\text{supp}(a_1^{(k)}) \subset F(\text{supp } \chi_q) \Subset \tilde{A}_{m_q j_q}$ . Consider a cut-off function  $\tilde{\chi}_q$  such that  $\tilde{\chi}_q \equiv 1$  in a neighborhood of  $F(\text{supp } \chi_q)$  and  $\text{supp } \tilde{\chi}_q \subset \tilde{A}_{m_q j_q}$ . Using Lemma 4-1.1 and the support properties of  $\tilde{\chi}_q$ , one has

$$\text{Op}_h(a_1^{(k)}) = \text{Op}_h(a_1^{(k)}) \text{Op}_h(\tilde{\chi}_q) + O\left(h^{N-k} \|a_1^{(k)}\|_{C^{2(N-k)+13}}\right) = \text{Op}_h(a_1^{(k)}) \text{Op}_h(\tilde{\chi}_q) + O\left(h^{N-k} \|a\|_{C^{2N+13}}\right)$$

Then, one can write  $\text{Op}_h(a_1^{(k)})T$  on the form

$$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ \text{Op}_h(a_1^{(k)}) \text{Op}_h(\tilde{\chi}_q) T_{m_q 1} & \dots & \text{Op}_h(a_1^{(k)}) \text{Op}_h(\tilde{\chi}_q) T_{m_q J} \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \end{pmatrix} + O\left(h^{N-k} \|a\|_{C^{2N+13}}\right)$$

and for  $j \neq j_q$ ,  $\text{Op}_h(\tilde{\chi}_q) T_{m_q j} = O(h^\infty)$ . We can conclude that

$$\begin{aligned} \text{Op}_h(a_1^{(k)})T &= \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \text{Op}_h(a_1^{(k)}) \text{Op}_h(\tilde{\chi}_q) T_{m_q j_q} & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} + O(h^\infty) \|\text{Op}_h(a_1^{(k)})\|_{L^2 \rightarrow L^2} + O\left(h^{N-k} \|a\|_{C^{2N+13}}\right) \\ &= \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \text{Op}_h(a_1^{(k)}) T_{m_q j_q} & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} + O\left(h^{N-k} \|a\|_{C^{2N+13}}\right) \end{aligned}$$

Combining this with the version obtained with  $M_{m_q j_q}$ , we get (5-3.33).  $\square$

Let us now start the iterative construction of the symbols. Fix  $N \in \mathbb{N}$  which can be taken arbitrarily large. Recall that we want to write

$$U_{\mathbf{q}} = \text{Op}_h(a_{\mathbf{q}}^+) T^{|\mathbf{q}|} + O(h^\infty)_{L^2 \rightarrow L^2} \quad (5-3.34)$$

Note  $U_r = U_{q_0 \dots q_{r-1}}$ . We want to write

$$U_r = \text{Op}_h\left(\sum_{k=0}^{N-1} h^k a_r^{(k)}\right) T^r + R_r^{(N)} \quad (5-3.35)$$

We start by writing

$$U_1 = \text{Op}_h\left(\sum_{k=0}^{N-1} h^k a_1^{(k)}\right) T + R_1^{(N)} \quad (5-3.36)$$

which is possible in virtue of (5-3.33). To pass from  $U_r$  to  $U_{r+1}$ , we have the relation

$$U_{r+1} = M A_{q_r} U_r = \sum_{k=0}^{N-1} h^k M A_{q_r} \text{Op}_h\left(a_r^{(k)}\right) T^r + M A_{q_r} R_r^{(N)}$$

So, we will construct inductively our symbols by setting

$$a_{r+1}^{(k)} = \sum_{p=0}^k \left(L_{p,q_r} a_r^{(k-p)}\right) \circ (F_{i_{r+1}, i_r})^{-1} \quad (5-3.37)$$

and

$$R_{r+1}^{(N)} = MA_{q_r} R_r^{(N)} + \sum_{k=0}^{N-1} O\left(\|a_r^{(k)}\|_{C^{2(N-k)+15}}\right) \quad (5-3.38)$$

The  $O$  encompasses the remainder terms in Lemma 5-3.33. The constants in the  $O$  only depend on  $M$  and the  $\chi_q, q \in \mathcal{A}$ , but not on  $\mathbf{q}$ .

To make this construction work, we will have to prove that the symbols  $a_r^{(k)}$  lie in a good symbol class  $S_{\delta_1}^{comp}$ .

Before reaching this step, let us just note that by induction one sees that :

•

$$\|R_r^{(N)}\| \leq C_N h^N \left(1 + \sum_{k=0}^{N-1} \sum_{l=0}^{r-1} \|a_l^{(k)}\|_{C^{2(N-k)+15}}\right) \quad (5-3.39)$$

with  $C_N$  depending on  $N, M$  and the  $a_q$ , but neither on  $r$  nor  $\mathbf{q}$ .

- Since  $L_{p,q_r}$  has coefficient supported in  $\mathcal{V}_{q_r}$ , we see by induction that  $\text{supp } a_{r+1}^{(k)} \subset \mathcal{V}_{q_0 \dots q_r}^+$  as announced.
- $a_{r+1}^{(0)} = \prod_{p=1}^{r+1} a_{q_{r+1-p}} \circ F^{-p}$

### 5-3.3.2. Control of the symbols.

We aim at estimating the semi-norms  $\|a_r^{(k)}\|_{C^m}$  for  $k < N, 1 \leq r \leq n$  and  $m \in \mathbb{N}$ . We will show the following :

**Proposition 5-3.2.** For every  $r \in \{1, \dots, n\}, k \in \{0, \dots, N-1\}$  and  $m \in \mathbb{N}$ , there exists  $C(k, m)$ , such that with  $\Gamma_{k,m} = (k+1)(m+k+1)$ ,

$$\|a_r^{(k)}\|_{C^m} \leq C(k, m) r^{\Gamma_{k,m}} \left(J_{q_0 \dots q_{r-1}}^+\right)^{2k+m} \|\alpha\|_{\infty}^r \quad (5-3.40)$$

**Remark.**

- What is important in this result is the way in which the bound depends on  $r$  and  $\mathbf{q}$ . Up to the term  $r^{\Gamma_{k,m}}$ , which is supposed to behave like  $O(|\log h|^{\Gamma_{k,m}})$ , the significant part of the estimate is that we can control the symbols by the local Jacobian.
- Since  $\text{supp } a_r^{(k)} \subset \mathcal{V}_{q_0 \dots q_{r-1}}^+$ , we need to focus on points  $\rho \in \mathcal{V}_{q_0 \dots q_{r-1}}^+$ .
- Our method is very close to the ones developed in [Riv10] and [DJN21]. However, we've changed a few things at the cost of being less precise on the exponent  $\Gamma_{k,m}$ . Our aim was to treat our problem as if we wanted to control the product of  $r$  triangular matrices.

Let us pick  $\rho \in \mathcal{V}_{q_0 \dots q_{r-1}}^+$ . With (5-3.37), one sees that if  $k, m \in \mathbb{N}$ ,  $d^m a_{r+1}^{(k)}$  depends on  $d^{m'} a_r^{(k')}(F^{-1}(\rho))$  for several  $m', k'$ . Before going deeper in the analysis of this dependence, let us note two obvious facts :

- This dependence is linear, with coefficients smoothly depending on  $\rho$ .
- If  $d^m a_{r+1}^{(k)}$  depends effectively on  $d^{m'} a_r^{(k')}(F^{-1}(\rho))$ , then  $k' \leq k$  and  $2k' + m' \leq 2k + m$ .

**Precise analysis of the dependence.** That being said, let us pick  $m_0, k_0 \in \mathbb{N}$ . Set  $N_0 = 2k_0 + m_0$  and consider the (column) vector

$$A_r(\rho) := \left(d^m a_r^{(k)}(\rho)\right)_{k \leq k_0, 2k+m \leq N_0} \in \bigoplus_{k \leq k_0, 2k+m \leq N_0} S^m T_{\rho}^* U \quad (5-3.41)$$

Here  $S^m T_{\rho}^* U$  is the spaces of  $m$ -linear symmetric form on  $T_{\rho} U$ . To define a norm on the fibers  $S^m T_{\rho}^* U$ , we can use for  $f \in S^m T_{\rho}^* U$ ,

$$\|f\|_{m,\rho} = \sup_{v_1, \dots, v_m \in T_{\rho} U} \frac{f(v_1, \dots, v_m)}{\|v_1\|_{\rho} \dots \|v_m\|_{\rho}} \quad (5-3.42)$$



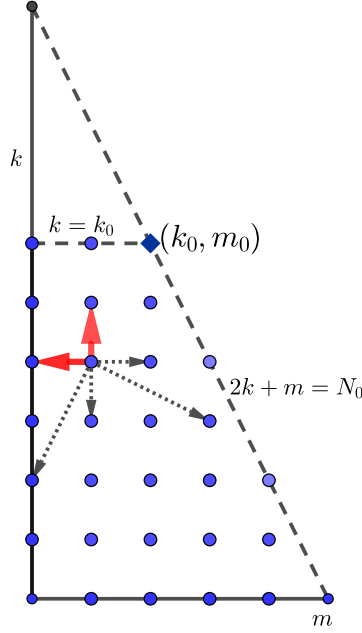


Figure 5.3: The starting point  $(k_0, m_0)$  is represented by a diamond. The set  $\mathcal{I}$  corresponds to the couple  $(k, m) \in \mathbb{N}^2$  in the region under the dotted lines  $k = k_0$  and  $2k + m = N_0$ . We've represented a family of arrows starting from a point  $\gamma_1 \in \mathcal{I}$ . The dotted arrows points toward  $\beta$  such that  $\gamma_2 \prec \gamma_1$ . The big red arrows points toward points  $\gamma_2$  such that  $P_{\gamma_1 \gamma_2}^{(r)} = 0$ .

where  $\|v\|_\rho$  for  $v \in T_\rho U$  is the norm induced by the Riemannian metric used to define  $J_1^u$  in 3-2.3. Note that for any fixed neighborhood of  $\mathcal{T}$ , there exists a global constant  $C > 0$  such that for each  $a \in C_c^\infty(U)$  supported in this neighborhood, one has

$$C^{-1} \|a\|_{C^m} \leq \sup_{m' \leq m} \sup_{\rho \in U} \|d^{m'} a\|_{m', \rho} \leq C \|a\|_{C^m}$$

We will denote by  $\gamma_1, \gamma_2$ , etc. elements of  $\mathcal{I} := \mathcal{I}(k_0, m_0) = \{(k, m) \in \mathbb{N}^2, k \leq k_0, 2k + m \leq N_0\}$ . We equip  $\mathcal{I}$  with the lexicographic order  $\prec$  and note  $\#\mathcal{I} := \Gamma_{k_0, m_0}$  (see Figure 5.3). We order the indices of  $A_r(\rho)$  with  $\prec$ .  $A_r(\rho)$  depends linearly on  $A_{r-1}(F^{-1}(\rho))$  and this dependence can be made explicit by a matrix

$$P^{(r)}(\rho) = \left( P_{\gamma_1 \gamma_2}^{(r)}(\rho) \right)_{\gamma_1, \gamma_2 \in \mathcal{I}}, \text{ where } P_{\gamma_1 \gamma_2}^{(r)}(\rho) \in L\left(S^{m'} T_{F^{-1}(\rho)}^* U, S^m T_\rho^* U\right) \text{ if } \gamma_1 = (k, m), \gamma_2 = (k', m')$$

so that

$$A_r(\rho) = P^{(r)}(\rho) A_{r-1}(F^{-1}(\rho)) \quad (5-3.43)$$

**Notations.** If  $\gamma_1 = (k, m), \gamma_2 = (m', k'), \rho, \rho' \in U$  and if  $A : S^{m'} T_{\rho'}^* U \rightarrow S^m T_\rho^* U$  is a linear operator, we will note

$$\|\cdot\|_{\gamma_1, \rho, \gamma_2, \rho'}$$

its subordinate norm for the norms defined by (5-3.42).

Analyzing (5-3.37), it turns out that if  $\gamma_1 = (k, m), \gamma_2 = (k', m') \in \mathcal{I}$ , then

- if  $k' > k$ ,  $P_{\gamma_1 \gamma_2}^{(r)}(\rho) = 0$  ;
- if  $k = k'$ , the contribution to  $d^m a_r^{(k)}(\rho)$  of  $a_{r-1}^{(k)}$  comes from

$$\begin{aligned} & d^m \left( (a_{q_{r-1}} a_{r-1}^{(k)}) \circ F^{-1} \right) (\rho) \\ &= a_{q_{r-1}}(F^{-1}(\rho)) \times d^m \left( a_{r-1}^{(k)} \circ F^{-1} \right) (\rho) + (\text{derivatives of order strictly less than } m \text{ for } a_{r-1}^{(k)}) \\ &= a_{q_{r-1}}(F^{-1}(\rho)) \times ({}^t dF^{-1}(\rho))^{\otimes m} d^m a_{r-1}^{(k)}(F^{-1}(\rho)) + (\text{idem}) \end{aligned}$$

In particular, if  $\gamma_1 = (k, m) \prec \gamma_2 = (k, m')$  doesn't hold, we see that  $P_{\gamma_1 \gamma_2}^{(r)}(\rho) = 0$ .

- If  $k' < k$ , we can have  $P_{\gamma_1 \gamma_2}^{(r)}(\rho) \neq 0$  with  $m' > m$ . But, the use of the lexicographic order ensures that  $\gamma_1 < \gamma_2$  in that case.

Hence,  $P^{(r)}(\rho)$  is a lower triangular matrix and the diagonal coefficients for the index  $\gamma_1 = (k, m)$  are given by

$$P_{\gamma_1 \gamma_1}^{(r)}(\rho) : f \in S^m T_{F^{-1}(\rho)}^* U \mapsto a_{q_{r-1}}(F^{-1}(\rho)) \times ({}^t dF^{-1}(\rho))^{\otimes m} f \in S^m T_\rho^* U \quad (5-3.44)$$

Iterating (5-3.43), we have

$$A_r(\rho) = P^{(r)}(\rho) P^{(r-1)}(F^{-1}(\rho)) \dots P^{(2)}(F^{-(r-2)}(\rho)) A_1(F^{1-r}(\rho))$$

For  $\gamma \in \mathcal{I}$ , we note

$$\mathcal{E}_r(\gamma) = \{ \vec{\gamma} = (\gamma_1, \dots, \gamma_r) \in \mathcal{I}^r; \quad \gamma_r = \gamma, \gamma_i < \gamma_{i+1} \}$$

The triangular property of  $P$  allows us to write :

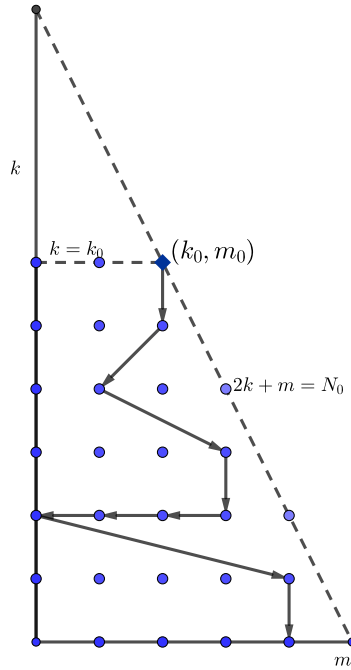


Figure 5.4: We've represented the reduction of an element  $\vec{\gamma} \in \mathcal{E}_r(k_0, m_0)$ , i.e, the arrows between  $\gamma_i$  and  $\gamma_{i+1}$  when  $\gamma_i \neq \gamma_{i+1}$ . During the descent, the value of  $m$  can only increase when  $k$  decreases strictly.

$$(A_r(\rho))_\gamma = \sum_{\vec{\gamma} \in \mathcal{E}_r(\gamma)} P_{\gamma_r \gamma_{r-1}}^{(r)}(\rho) \dots P_{\gamma_2 \gamma_1}^{(2)}(F^{-(r-2)}(\rho)) (A_1(F^{1-r}(\rho)))_{\gamma_1}$$

**Control of individual terms.** Let us fix  $\gamma = (k, m)$  and pick  $\vec{\gamma} \in \mathcal{E}_r(\gamma)$ . We wish to analyze the operator

$$P_{\vec{\gamma}}(\rho) := P_{\gamma_r \gamma_{r-1}}^{(r)}(\rho) \dots P_{\gamma_2 \gamma_1}^{(2)}(F^{-(r-2)}(\rho))$$

First of all,  $\#\{i \in \{1, \dots, r-1\}, \gamma_{i+1} \neq \gamma_i\} \leq \Gamma_{k_0, m_0}$ . So let us write

$$\{i \in \{1, \dots, r-1\}, \gamma_{i+1} \neq \gamma_i\} = \{t_1 < \dots < t_d\}$$

with  $d \leq \Gamma_{k_0, m_0}$ . We can set  $t_{d+1} = r, t_0 = 0$  and we can rewrite

$$\vec{\gamma} = (\underbrace{\beta_1, \dots, \beta_1}_{t_1}, \underbrace{\beta_2, \dots, \beta_2}_{t_2 - t_1}, \dots, \underbrace{\beta_d, \dots, \beta_d}_{t_d - t_{d-1}}, \underbrace{\beta_{d+1}, \dots, \beta_{d+1}}_{t_{d+1} - t_d})$$

For  $p \in \{1, \dots, d+1\}$ , we introduce the operator

$$D_p(\rho) = P_{\beta_p \beta_p}^{(t_p)} \left( F^{-(r-t_p)}(\rho) \right) \dots P_{\beta_p \beta_p}^{(t_{p-1}+2)} \left( F^{-(r-t_{p-1}-2)}(\rho) \right)$$

and for  $p \in \{1, \dots, d\}$

$$T_p(\rho) = P_{\beta_{p+1} \beta_p}^{t_p+1} \left( F^{-(r-t_p-1)}(\rho) \right)$$

so that we can write

$$P_{\vec{\gamma}}(\rho) = D_{d+1}(\rho) T_d(\rho) D_d(\rho) \dots T_1(\rho) D_1(\rho)$$

For  $p \in \{1, \dots, d+1\}$ , if  $\beta_p = (k, m)$ , we can see that

$$\begin{aligned} D_p(\rho) &= \left[ \prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho) \right] \left[ \left( {}^t dF^{-1} \left( F^{-(r-t_p)}(\rho) \right) \right)^{\otimes m} \circ \dots \circ \left( {}^t dF^{-1} \left( F^{-(r-t_{p-1}-2)}(\rho) \right) \right)^{\otimes m} \right] \\ &= \left[ \prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho) \right] \left( {}^t dF^{-(t_p-t_{p-1}-1)} \left( F^{-(r-t_p)}(\rho) \right) \right)^{\otimes m} \end{aligned}$$

We introduce the word

$$\mathbf{q}_p = q_{t_{p-1}} \dots q_{t_p-1}$$

and set  $\rho_p = F^{-(r-t_p)}(\rho)$ ,  $\rho'_p = F^{-(t_p-t_{p-1}-1)}(\rho_p)$ . To estimate the subordinate norm of  $D_p(\rho)$ , we use Lemma 5-3.3. Since  $\rho \in \mathcal{V}_{\mathbf{q}}^+$ ,  $\rho_p \in \mathcal{V}_{\mathbf{q}_p}^+$  and we have

$$\begin{aligned} \|D_p(\rho)\|_{\beta_p, \rho_p, \beta_p, \rho'_p} &\leq \left| \prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho) \right| \sup_{\rho_p \in \mathcal{V}_{\mathbf{q}_p}^+} \|dF^{-(t_p-t_{p-1}-1)}(\rho_p)\|^m \\ &\leq (C J_{\mathbf{q}_p}^+)^m \left| \prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho) \right| \\ &\leq C_{k_0, m_0} \left( J_{\mathbf{q}_p}^+ \right)^{N_0} \left| \prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)}(\rho) \right| \end{aligned}$$

To estimate the norms of  $T_p(\rho)$ , we simply note that they depend smoothly on  $\rho_p$  which lies in a compact set, so we can bound them by a uniform constant  $C_1$ . This is not a problem since they appear  $d$  times in  $P_{\vec{\gamma}}$  with  $d \leq \Gamma_{k_0, m_0}$ . Consequently, we can estimate  $\|P_{\vec{\gamma}}(\rho)\|_{\gamma, \rho, \gamma_1, F^{-(r-1)}(\rho)}$ ,

$$\|P_{\vec{\gamma}}(\rho)\|_{\gamma, \rho, \gamma_1, F^{-(r-1)}(\rho)} \leq C_{k_0, m_0} \left( J_{\mathbf{q}_1}^+ \dots J_{\mathbf{q}_{d+1}}^+ \right)^{N_0} |a_{\mathbf{q}, \vec{\gamma}}(\rho)| \leq C_{k_0, m_0} \left( J_{\mathbf{q}}^+ \right)^{N_0} |a_{\mathbf{q}, \vec{\gamma}}(\rho)| \quad (5-3.45)$$

where

$$a_{\mathbf{q}, \vec{\gamma}} = \prod_{p=1}^{d+1} \prod_{j=t_{p-1}+1}^{t_p-1} a_{q_j} \circ F^{-(r-j)} \quad (5-3.46)$$

Here, the last inequality holds by applying  $d$  times (5-3.18), with  $d \leq \Gamma_{k_0, m_0}$ , once we've noted that

$$\mathbf{q} = \mathbf{q}_1 \dots \mathbf{q}_{d+1}$$

Finally, if  $\gamma_1 = (k_1, m_1)$ , to estimate  $\|(A_1(F^{1-r}(\rho)))_{\gamma_1}\|_{m_1, F^{1-r}(\rho)}$ , we simply note that it depends smoothly on  $F^{1-r}(\rho)$ , so that we can bound it by a uniform constant. Hence, we have

$$\|P_{\vec{\gamma}}(\rho) A_1(F^{1-r}(\rho))\|_{m, \rho} \leq C_{k_0, m_0} \left( J_{\mathbf{q}}^+ \right)^{N_0} |a_{\mathbf{q}, \vec{\gamma}}(\rho)| \quad (5-3.47)$$

**Cardinality of  $\mathcal{E}_r(\gamma)$ .** The bound we will provide is far from being optimal but it will turn out to be enough for our purpose. To count the number of elements in  $\mathcal{E}_r(\gamma)$ , we remark that it is similar than counting the number of decreasing sequences of length  $r$  starting from  $\gamma$ . This number is smaller than the number of increasing sequences of length  $r$  in  $\{1, \dots, \Gamma_{k_0, m_0}\}$ . Recalling that the number of sequences  $u_1 \leq u_2 \leq \dots \leq u_r$  satisfying  $u_1 = 1$  and  $u_r = b$  is equal to  $\binom{b+r-2}{r-2}$ , one can estimate

$$\#\mathcal{E}_r(\gamma) \leq \sum_{b=1}^{\Gamma_{k_0, m_0}} \binom{b+r-2}{r-2} \leq \Gamma_{k_0, m_0} (r-1)^{\Gamma_{k_0, m_0}} \quad (5-3.48)$$

Finally, we can compute explicitly  $\Gamma_{k_0, m_0}$  and we find  $\Gamma_{k_0, m_0} = (k_0 + 1)(m_0 + 1 + k_0)$ .

**Conclusion.** We finally combine (5-3.48) and (5-3.47) to prove Proposition 5-3.2. Recall that  $|a_q| = |\alpha| \chi_q \leq \|\alpha\|_\infty$ .

$$\begin{aligned} \sup_{\rho \in \mathcal{V}_{q_0 \dots q_{r-1}}} \|d^{m_0} a_r^{(k_0)}\|_{m_0, \rho} &= \sup_{\rho \in \mathcal{V}_{q_0 \dots q_{r-1}}} \|(A_r(\rho))_{(k_0, m_0)}\|_{m_0, \rho} \\ &\leq \sum_{\vec{\gamma} \in \mathcal{E}_r(k_0, m_0)} \|P_{\vec{\gamma}}(\rho) A_1(F^{1-r}(\rho))\|_{m_0, \rho} \\ &\leq \Gamma_{k_0, m_0} r^{\Gamma_{k_0, m_0}} C_{k_0, m_0} (J_{\mathbf{q}}^+)^{N_0} |a_{\mathbf{q}, \vec{\gamma}}(\rho)| \\ &\leq C_{k_0, m_0} r^{\Gamma_{k_0, m_0}} (J_{\mathbf{q}}^+)^{N_0} \|\alpha\|_\infty^r \end{aligned}$$

Finally, we get as expected

$$\|a_r^{(k_0)}\|_{C^{m_0}} \leq C_{k_0, m_0} r^{\Gamma_{k_0, m_0}} (J_{\mathbf{q}}^+)^{N_0} \|\alpha\|_\infty^r$$

### 5-3.3.3. End of proof of proposition 5-3.1.

Armed with these estimates, we are now able to conclude the proof of Proposition 5-3.1 under the assumptions (5-3.25). Assume that this assumption is satisfied and construct inductively the symbols  $a_r^{(k)}$  with the formula (5-3.37). Since  $J_{\mathbf{q}}^+ \leq Ch^{-\delta}$ , it implies that  $n = O(\log h)$ . Hence, we have for  $r \leq n$ ,

$$\|a_r^{(k)}\|_{C^m} \leq C_{k, m} h^{-\delta m} h^{-2k\delta} |\log h|^{\Gamma_{k, m}} \|\alpha\|_\infty^r \leq C_{k, m} h^{-\delta_1 m} h^{-2k\delta_1} \|\alpha\|_\infty^r$$

The symbol  $h^{2\delta_1 k} a_r^{(k)}$  lies in  $\|\alpha\|_\infty^r S_{\delta_1}^{comp}(T^*\mathbb{R})$ . Using Borel's theorem with the parameter  $h' = h^{1-2\delta_1}$ , we can construct a symbol

$$a_{q_0 \dots q_{r-1}}^+ \sim \sum_{k=0}^{\infty} (h')^k h^{2\delta_1 k} a_r^{(k)} = \sum_{k=0}^{\infty} h^k a_r^{(k)} \in \|\alpha\|_\infty^r S_{\delta_1}^{comp}$$

that is, for every  $N \in \mathbb{N}$ ,

$$a_{q_0 \dots q_{r-1}}^+ - \sum_{k=0}^{N-1} h^k a_r^{(k)} = O\left(h^{(1-2\delta_1)N} \|\alpha\|_\infty^r\right)$$

By construction of the  $a_r^{(k)}$ , for every  $N \in \mathbb{N}$ , we have

$$U_{\mathbf{q}}^+ - \text{Op}_h(a_{\mathbf{q}}^+) T^{|\mathbf{q}|} = R_n^{(N)} + O\left(h^{(1-2\delta_1)N} \|\alpha\|_\infty^r\right)$$

Fix some  $K \geq 0$  such that  $\min(1, \|\alpha\|_\infty^n) = O(h^{-K})$ , so that  $\|\alpha\|_\infty^r = O(h^{-K})$ . With (5-3.39) and our estimates, we can control

$$\|R_n^{(N)}\| \leq C_N h^N \left(1 + |\log h|^{\Gamma_{k, m} + 1} h^{-\delta(2N+15)} h^{-K}\right) \leq C_N h^{-15\delta_1 + N(1-2\delta_1) - K}$$

Since we can choose  $N$  as large as we want, we have finally proved that

$$U_{\mathbf{q}}^+ - \text{Op}_h(a_{\mathbf{q}}^+) T^{|\mathbf{q}|} = O(h^\infty)$$

□

### 5-3.3.4. Norm of sums over many words.

We'll make use of the tools and notations developed in this subsection to prove the following proposition. To state it, we introduce the notations

$$\mathcal{Q}(n, \tau, C_0) := \{\mathbf{q} \in \mathcal{A}^n; J_{\mathbf{q}}^+ \leq C_0 h^{-\tau}\} \quad (5-3.49)$$

**Proposition 5-3.3.** There exists  $C = C(C_0, \tau)$  such that for every  $\mathcal{Q} \subset \mathcal{Q}(n, \tau, C_0)$ , the following bound holds :

$$\left\| \sum_{\mathbf{q} \in \mathcal{Q}} U_{\mathbf{q}} \right\|_{L^2 \rightarrow L^2} \leq C \|\alpha\|^n |\log h| \quad (5-3.50)$$

*Proof.* Throughout the proof, we'll denote by  $C$  quasi-global constants, i.e. constants depending on  $C_0, \tau$  and the same other parameters as global constants. We will also be lead to use a constant  $C_1$ : it has the same dependence.

*Step 1:* First note that since  $J_{\mathbf{q}}^+ \leq C_0 h^{-\tau}$ ,  $n$  satisfies the bound  $n = O(\log h)$ .

*Step 2 :* If  $\mathbf{q} \in \mathcal{Q}(n, \tau, C_0)$ , denote by  $l(\mathbf{q}) = l$  the largest integer such that

$$J_{q_0 \dots q_{l-1}}^+ \leq h^{-\tau/2}$$

Since  $J_{q_0 \dots q_l} > h^{-\tau/2}$ ,  $J_{q_0 \dots q_{l-1}}^+ > C h^{-\tau/2}$  and hence

$$J_{q_l \dots q_{n-1}}^+ \leq C \frac{h^{-\tau}}{J_{q_0 \dots q_{l-1}}^+} \leq C_1 h^{-\tau/2}$$

We can then write  $\mathbf{q} = \mathbf{s}\mathbf{r}$  with  $\mathbf{s} \in \mathcal{Q}(l, \tau/2, 1)$ ,  $\mathbf{r} \in \mathcal{Q}(n-l, \tau/2, C_1)$ . It follows that we can write

$$\sum_{\mathbf{q} \in \mathcal{Q}} U_{\mathbf{q}} = \sum_{l=1}^n \sum_{\substack{\mathbf{s} \in \mathcal{Q}(l, \tau/2, 1) \\ \mathbf{r} \in \mathcal{Q}(n-l, \tau/2, C_1)}} F_l(\mathbf{s}, \mathbf{r}) U_{\mathbf{r}} U_{\mathbf{s}}$$

with  $F_l(\mathbf{s}, \mathbf{r}) = \mathbb{1}_{\mathbf{s}\mathbf{r} \in \mathcal{Q}}$ . It is then enough to show the bound

$$\max_{1 \leq l \leq n} \left\| \sum_{\substack{\mathbf{s} \in \mathcal{Q}(l, \tau/2, 1) \\ \mathbf{r} \in \mathcal{Q}(n-l, \tau/2, C_1)}} F_l(\mathbf{s}, \mathbf{r}) U_{\mathbf{r}} U_{\mathbf{s}} \right\| \leq C \|\alpha\|^n \quad (5-3.51)$$

In the following, we fix some  $1 \leq l \leq n$  and we'll simply note  $\sum_{\mathbf{s}, \mathbf{r}}$  to alleviate the notations. Note that the number of terms in the sum is bounded by

$$|\mathcal{Q}(l, \tau/2, 1) \times \mathcal{Q}(n-l, \tau/2, C_1)| \leq |\mathcal{A}|^l \times |\mathcal{A}|^{n-l} \leq |\mathcal{A}|^n \leq h^{-Q}$$

where  $Q = C \log |\mathcal{A}|$ .

*Step 3:* We fix some large  $N \in \mathbb{N}$  and  $\delta_1 \in (\tau/2, 1/2)$ . Recall that we can write,

$$\begin{aligned} U_{\mathbf{s}} &= \left( \text{Op}_h \left( \sum_{k=0}^{N-1} h^k a_{\mathbf{s}}^{(k)} \right) + O_{L^2 \rightarrow L^2} \left( h^{(1-2\delta_1)N-15\delta_1} \|\alpha\|_{\infty}^l \right) \right) T^l \\ U_{\mathbf{r}} &= T^{n-l} \left( \text{Op}_h \left( \sum_{k=0}^{N-1} h^k a_{\mathbf{r}}^{(k)} \right) + O_{L^2 \rightarrow L^2} \left( h^{(1-2\delta_1)N-15\delta_1} \|\alpha\|_{\infty}^{n-l} \right) \right) \end{aligned}$$

with bounds on  $a_{\mathbf{s}}^{(k)}$  and  $a_{\mathbf{r}}^{(k)}$  given by Proposition 5-3.1.

We then use the formula for the composition of operators in  $\Psi_{\delta_1}^{comp}(T^*\mathbb{R})$  (Lemma 4-1.1) and for simplicity, we note  $\mathcal{L}_k(a, b)(\rho) = \frac{i^k}{k!}(A(D))^k(a \otimes b)(\rho, \rho)$ . For  $0 \leq k \leq N-1$ , we set

$$a_{\mathbf{s}, \mathbf{r}, k} = \sum_{j+k_-+k_+=k} \mathcal{L}_j \left( a_{\mathbf{r}}^{(k_-)}, a_{\mathbf{s}}^{(k_+)} \right)$$

Note that if  $j + k_- + k_+ \geq N$ ,

$$\begin{aligned} \|a_{\mathbf{r}}^{(k_-)} \otimes a_{\mathbf{s}}^{(k_+)}\|_{C^{2j+13}} &\leq C_j \sup_{m_++m_-=2j+13} \|a_{\mathbf{r}}^{(k_-)}\|_{C^{m_-}} \|a_{\mathbf{s}}^{(k_+)}\|_{C^{m_+}} \\ &\leq C_{j, k_-, k_+} h^{-(2k_-+m_-)\delta_1} h^{-(2k_++m_+)\delta_1} \|\alpha\|_{\infty}^n \\ &\leq C_{j, k_-, k_+} h^{-2\delta_1(j+k_-+k_+)-13\delta_1} \|\alpha\|_{\infty}^n \\ &\leq C_{j, k_-, k_+} h^{-2\delta_1 N - 13\delta_1} \|\alpha\|_{\infty}^n \end{aligned}$$

and henceforth,

$$O \left( h^{j+k_-+k_+} \|a_{\mathbf{r}}^{(k_-)} \otimes a_{\mathbf{s}}^{(k_+)}\|_{C^{2j+13}} \right) = O \left( h^{(1-2\delta_1)N-15\delta_1} \|\alpha\|_{\infty}^n \right)$$

As a consequence, we can write

$$U_{\mathbf{r}} U_{\mathbf{s}} = T^{n-l} \left( \text{Op}_h \left( \sum_{k=0}^{N-1} h^k a_{\mathbf{s}, \mathbf{r}, k} \right) \right) T^l + O_{L^2 \rightarrow L^2} \left( h^{(1-2\delta_1)N-15\delta_1} \|\alpha\|_{\infty}^n \right)$$

It follows that

$$\sum_{\mathbf{s}, \mathbf{r}} U_{\mathbf{r}} U_{\mathbf{s}} = T^{n-l} \left( \text{Op}_h \left( \sum_{k=0}^{N-1} h^k a^{(k)} \right) \right) T^l + O_{L^2 \rightarrow L^2} \left( h^{(1-2\delta_1)N-15\delta_1-Q} \|\alpha\|_{\infty}^n \right)$$

where

$$a^{(k)} = \sum_{\mathbf{s}, \mathbf{r}} F(\mathbf{s}, \mathbf{r}) a_{\mathbf{s}, \mathbf{r}, k} \quad (5-3.52)$$

Suppose that  $N$  has been chosen such that

$$(1-2\delta_1)N > 15\delta_1 + Q$$

The remainder term is thus controlled by the desired bound since it is of order  $O(\|\alpha\|_{\infty}^n)$ .

*Step 4:  $C^0$  norm of  $a^{(0)}$ .*

$$a^{(0)} = \sum_{\mathbf{s}, \mathbf{r}} F(\mathbf{s}, \mathbf{r}) a_{\mathbf{s}}^{(0)} a_{\mathbf{r}}^{(0)}$$

where, in virtue of (5-3.31) and (5-3.32),

$$a_{\mathbf{s}}^{(0)} = \prod_{p=1}^l a_{s_{l-p}} \circ F^{-p}; \quad a_{\mathbf{r}}^{(0)} = \prod_{p=0}^{n-l-1} a_{r_p} \circ F^p$$

As a consequence, we can estimate

$$\begin{aligned} |a^{(0)}| &\leq \sum_{\mathbf{s}, \mathbf{r}} |a_{\mathbf{s}}^{(0)}| |a_{\mathbf{r}}^{(0)}| \\ &\leq \prod_{p=1}^l \left( \sum_{q \in \mathcal{A}} |a_q| \right) \circ F^{-p} \times \prod_{p=0}^{n-l-1} \left( \sum_{q \in \mathcal{A}} |a_q| \right) \circ F^p \\ &\leq \|\alpha\|_{\infty}^n \end{aligned}$$

*Step 5:  $C^m$  norms of  $a^{(k)}$ .* We will show the following : there exists constants  $C_{k,m}$  (depending only on  $C_0, \delta_1, \tau$  and  $m, k$ ) such that for all  $0 \leq k \leq N-1$  and  $m \in \mathbb{N}$ ,

$$\|a^{(k)}\|_{C^m} \leq C_{k,m} h^{-(2k+m)\delta_1} \|\alpha\|_\infty^n \quad (5-3.53)$$

Let's compute :

$$\begin{aligned} \|a^{(k)}\|_{C^m} &\leq \sum_{\mathbf{s}, \mathbf{r}} \|a_{\mathbf{s}, \mathbf{r}, k}\|_{C^m} \\ &\leq \sum_{\mathbf{s}, \mathbf{r}} \sum_{j+k_++k_- = k} \left\| \mathcal{L}_j \left( a_{\mathbf{r}}^{(k_-)}, a_{\mathbf{s}}^{(k_+)} \right) \right\|_{C^m} \\ &\leq \sum_{\mathbf{s}, \mathbf{r}} \sum_{j+k_++k_- = k} \left\| a_{\mathbf{r}}^{(k_-)} \otimes a_{\mathbf{s}}^{(k_+)} \right\|_{C^{2j+m}} \\ &\leq \sum_{\mathbf{s}, \mathbf{r}} \sum_{\substack{j+k_++k_- = k \\ m_+ + m_- \leq m + 2j}} \left\| a_{\mathbf{r}}^{(k_-)} \right\|_{C^{m_-}} \left\| a_{\mathbf{s}}^{(k_+)} \right\|_{C^{m_+}} \end{aligned}$$

and hence

$$\|a^{(k)}\|_{C^m} \leq C_{k,m} \sup_{\substack{j+k_++k_- = k \\ m_+ + m_- \leq m + 2j}} \sum_{\mathbf{s}, \mathbf{r}} \left\| a_{\mathbf{r}}^{(k_-)} \right\|_{C^{m_-}} \left\| a_{\mathbf{s}}^{(k_+)} \right\|_{C^{m_+}} \quad (5-3.54)$$

Let us fix  $j, k_+, k_-, m_+, m_-$  satisfying  $j + k_+ + k_- = k, m_- + m_+ \leq m + 2j$  and let us estimate

$$\sum_{\mathbf{s}} \left\| a_{\mathbf{s}}^{(k_+)} \right\|_{C^{m_+}} \times \sum_{\mathbf{r}} \left\| a_{\mathbf{r}}^{(k_-)} \right\|_{C^{m_-}}$$

We estimate the sum over  $\mathbf{s}$ . The same kind of estimates will hold for  $\mathbf{r}$  with the same methods. We reuse the tools developed in the last subsections. Namely, we set  $N_+ = 2k_+ + m_+, \gamma_+ = (k_+, m_+), \mathcal{I} = \mathcal{I}(\gamma_+)$  and

$$(A_{\mathbf{s}}(\rho)) = \left( d^m a_{\mathbf{s}}^{(k)} \right)_{k \leq k_+, 2k+m \leq N_+}$$

We have shown that there exists a global constant  $C > 0$  such that

$$\begin{aligned} \|a_{\mathbf{s}}^{(k_+)}\|_{C^{m_+}} &\leq \sup_{\rho} \|A_{\mathbf{s}}(\rho)\| \leq C \sum_{\vec{\gamma} \in \mathcal{E}_l(\gamma_+)} \|P_{\vec{\gamma}}(\rho)\| \\ &\leq \sum_{\vec{\gamma} \in \mathcal{E}_l(\gamma_+)} C_{N_+, k_+} (J_{\mathbf{s}}^+)^{N_+} |a_{\mathbf{s}, \vec{\gamma}}(\rho)| \\ &\leq C_{N_+, k_+} h^{-\tau N_+/2} \sum_{\vec{\gamma} \in \mathcal{E}_l(\gamma_+)} |a_{\mathbf{s}, \vec{\gamma}}(\rho)| \end{aligned}$$

where  $C_{N_+, k_+}$  depends on  $C_0, \tau, N_+, k_+$  and global parameters. We hence have to estimate

$$\sum_{\mathbf{s}} \sum_{\vec{\gamma} \in \mathcal{E}_l(\gamma_+)} |a_{\mathbf{s}, \vec{\gamma}}(\rho)|$$

Fix  $\vec{\gamma} \in \mathcal{E}_l(\alpha_+)$  and write it

$$\vec{\gamma} = \underbrace{(\beta_1, \dots, \beta_1)}_{t_1} \underbrace{(\beta_2, \dots, \beta_2)}_{t_2 - t_1} \dots \underbrace{(\beta_d, \dots, \beta_d)}_{t_d - t_{d-1}} \underbrace{(\beta_{d+1}, \dots, \beta_{d+1})}_{t_{d+1} - t_d} \text{ where } d \leq \Gamma_{k_+, m_+}$$

and recall that

$$a_{\mathbf{s}, \vec{\gamma}} = \prod_{p=1}^{d+1} \prod_{j=t_{p-1}+1}^{t_p-1} a_{s_j} \circ F^{-(l-j)}$$

When one sums over  $\mathbf{s} \in \mathcal{A}^l$ , the values of  $\mathbf{s}$  at the indices  $t_i, 1 \leq i \leq d$  do not play a role and we write :

$$\begin{aligned}
\sum_{\mathbf{s}} |a_{\mathbf{s}, \vec{\gamma}}| &= \sum_{s_{t_1} \in \mathcal{A}} \cdots \sum_{s_{t_d} \in \mathcal{A}} \prod_{p=1}^{d+1} \prod_{j=t_{p-1}+1}^{t_p-1} \left( \sum_{s \in \mathcal{A}} |a_s| \right) \circ F^{-(l-j)} \\
&\leq |\mathcal{A}|^d \sup_{\rho} \left( \sum_{s \in \mathcal{A}} |a_s| \right)^l \\
&\leq K^{\Gamma_{k_+, m_+}} \|\alpha\|_{\infty}^l \leq C_{k_+, m_+} \|\alpha\|_{\infty}^l
\end{aligned}$$

As a consequence,

$$\sum_{\mathbf{s}} \sum_{\vec{\gamma} \in \mathcal{E}_l(\gamma_+)} |a_{\mathbf{s}, \vec{\gamma}}| \leq \#\mathcal{E}_l(\gamma_+) C_{k_+, m_+} \|\alpha\|_{\infty}^l \leq C_{k_+, m_+} (l-1)^{\Gamma_{k_+, m_+}} \|\alpha\|_{\infty}^l$$

which gives

$$\sum_{\mathbf{s}} \left\| a_{\mathbf{s}}^{(k_+)} \right\|_{C^{m_+}} \leq C_{k_+, m_+} h^{-\tau N_+/2} (l-1)^{\Gamma_{k_+, m_+}} \|\alpha\|_{\infty}^l \leq C_{k_+, m_+} h^{-\delta_1 N_+} \|\alpha\|_{\infty}^l$$

where the last inequality (with a different value of  $C_{k_+, m_+}$ ) follows from the fact that  $l = O(\log h)$  and  $\delta_1 > \frac{\tau}{2}$ . The same kind of estimates holds for the sum over  $\mathbf{r}$  :

$$\sum_{\mathbf{r}} \left\| a_{\mathbf{r}}^{(k_-)} \right\|_{C^{m_-}} \leq C_{k_-, m_-} h^{-\delta_1 N_-} \|\alpha\|_{\infty}^{n-l}$$

Eventually, using (5-3.54), we get (5-3.53) since

$$N_+ + N_- = 2k_+ + m_+ + 2k_- + m_- \leq 2(k_+ + k_- + j) + m = 2k + m$$

*Step 6 : Conclusion.* We can conclude the proof of the Proposition 5-3.3. The bound (5-3.53) shows that for  $0 \leq k \leq N-1$ ,  $a^{(k)} \in h^{-2k\delta_1} \|\alpha\|_{\infty}^n S_{\delta_1}^{comp}$  and thus  $\sum_{k=0}^{N-1} h^k a^{(k)} \in S_{\delta_1}^{comp} \|\alpha\|_{\infty}^n$ . From the  $L^2$ -boundedness of pseudodifferential operators with symbol in  $S_{\delta_1}$ ,

$$\left\| \text{Op}_h \left( \sum_{k=0}^{N-1} h^k a^{(k)} \right) \right\| \leq \sum_{k=0}^{N-1} \sum_{m \leq M} h^{k+m/2} \|a^{(k)}\|_{C^m} \leq \sum_{k=0}^{N-1} \sum_{m \leq M} C_{k,m} h^{(k+2m)(1/2-\delta_1)} \|\alpha\|_{\infty}^n \leq C \|\alpha\|_{\infty}^n$$

where  $C$  depends only on  $C_0, \tau, \delta_1$ . Since  $\|T\| \leq 1$ , we get

$$\left\| \sum_{\mathbf{s}, \mathbf{r}} F(\mathbf{s}, \mathbf{r}) U_{\mathbf{r}} U_{\mathbf{s}} \right\| \leq C \|\alpha\|_{\infty}^n$$

which concludes the proof of Proposition 5-3.3.  $\square$

### 5-3.4. Manipulations of the $U_{\mathbf{q}}$ .

#### 5-3.4.1. First consequences.

We now make use of Proposition 5-3.1 to deduce several important facts. We go on following [DJN21]. In the whole subsection, we fix  $0 \leq \delta < \delta_1 < \frac{1}{2}$  and  $C_0 > 0$ . We denote  $\mathcal{A}^{\rightarrow} = \bigcup_{n \in \mathbb{N}} \mathcal{A}^n$ .

**Remark.**

The constants in  $O(h^{\infty})$  depend on  $\mathbf{p}$  and  $\mathbf{q}$  only through  $C_0, \delta, \delta_1$ , not on the precise value of  $\mathbf{p}$  and  $\mathbf{q}$ . It will always be the case in the following and we won't precise it anymore. As already done, all the quasi-global constants (i.e. depending on global parameters and  $C_0, \delta, \tau, \delta_1$ ) will be noted by the letter  $C$ .

**Lemma 5-3.6.** Let  $\mathbf{q}, \mathbf{p} \in \mathcal{A}^{\rightarrow}$  satisfying  $\mathcal{V}_{\mathbf{q}}^+ \cap \mathcal{V}_{\mathbf{p}}^- = \emptyset$  and  $\max(J_{\mathbf{q}}^+, J_{\mathbf{p}}^-) \leq C_0 h^{-\delta}$ . Then

$$U_{\mathbf{p}} U_{\mathbf{q}} = O(h^{\infty})_{L^2 \rightarrow L^2}$$



*Proof.* In virtue of Proposition 5-3.1, we can write

$$U_{\mathbf{p}} = T^{|\mathbf{p}|} \text{Op}_h(a_{\mathbf{p}}^-) + O(h^\infty)$$

$$U_{\mathbf{q}} = \text{Op}_h(a_{\mathbf{q}}^+) T^{|\mathbf{q}|} + O(h^\infty)$$

With  $a_{\mathbf{q}}^+ \in \|\alpha\|_\infty^{|\mathbf{q}|} S_{\delta_1}^{\text{comp}}$ ,  $a_{\mathbf{p}}^- \in \|\alpha\|_\infty^{|\mathbf{p}|} S_{\delta_1}^{\text{comp}}$  and  $\text{supp } a_{\mathbf{p}}^- \subset \mathcal{V}_{\mathbf{p}}^-, \text{supp } a_{\mathbf{q}}^+ \subset \mathcal{V}_{\mathbf{q}}^+$ . Since  $\mathcal{V}_{\mathbf{q}}^+ \cap \mathcal{V}_{\mathbf{p}}^- = \emptyset$ ,  $\text{Op}_h(a_{\mathbf{p}}^-) \text{Op}_h(a_{\mathbf{q}}^+) = O(h^\infty)$  as a consequence of the composition of two symbols of  $S_{\delta_1}$ . The constants in  $O(h^\infty)$  depend on semi-norms of these symbols, themselves depending on  $C_0, \tau, \delta_1$ . Since  $T^n = O(1)$ , the result is proved.  $\square$

Lemma 5-3.6 will have interesting consequences, starting with the following lemma which enables use to get rid (that is to say to control by  $O(h^\infty)$ ) of words  $\mathbf{q}$  where  $\mathcal{V}_{\mathbf{q}}^\pm = \emptyset$ , under some assumptions. In particular, it can be applied without trouble to words of "small" lengths  $N \leq \frac{1}{2\lambda_1} |\log h|$ , what could also be deduced from applying Egorov's theorem up to the global Ehrenfest time  $\frac{1}{2\lambda_1} |\log h|$ .

**Lemma 5-3.7.** Let  $\mathbf{q} \in \mathcal{A}^\rightarrow$  such that  $n = |\mathbf{q}| \leq C_0 |\log h|$  and assume that  $\mathcal{V}_{\mathbf{q}}^- = \emptyset$ . We suppose that one of the above assumptions is satisfied :

- (i) If  $m = \max\{k \in \{1, \dots, n\}, \mathcal{V}_{q_0 \dots q_{k-1}}^- \neq \emptyset\}$ ,  $J_{q_0 \dots q_{m-1}}^- \leq C_0 h^{-2\delta}$ .
- (ii) If  $m = \min\{k \in \{0, \dots, n-1\}, \mathcal{V}_{q_m \dots q_{n-1}}^- \neq \emptyset\}$ ,  $J_{q_m \dots q_{n-1}}^- \leq C_0 h^{-2\delta}$ .

Then,  $U_{\mathbf{q}} = O(h^\infty)$ .

*Proof.* We prove this lemma under assumption (i). This is similar under (ii). We note  $m = \max\{k \in \{1, \dots, n\}, \mathcal{V}_{q_0 \dots q_{k-1}}^- \neq \emptyset\}$  and assume  $J_{q_0 \dots q_{m-1}}^- \leq C_0 h^{-2\delta}$ . Due to (5-3.6), it is enough to show that  $U_{q_0 \dots q_m} = O(h^\infty)$ . Let us denote  $l = \max\{k \in \{1, \dots, m\}, J_{q_0 \dots q_{l-1}}^- \leq h^{-\delta}\}$  and notice that  $l < m$  (if  $h$  is small enough). By maximality of  $l$ , it is clear that  $J_{q_0 \dots q_l}^- \geq h^{-\delta}$ . According to the third point of Lemma 5-3.2,

$$J_{q_{l+1} \dots q_{m-1}}^- \sim \frac{J_{q_0 \dots q_{m-1}}^-}{J_{q_0 \dots q_l}^-} \leq C h^{-\delta}$$

Set  $\mathbf{p} = q_l \dots q_m$ . We distinguish now between two cases

- $\mathcal{V}_{\mathbf{p}}^- \neq \emptyset$  : We set  $\mathbf{r} = q_0 \dots q_{l-1}$ . It follows that

$$\max(J_{\mathbf{p}}^-, J_{\mathbf{r}}^-) \leq C h^{-\delta}$$

Moreover,

$$\mathcal{V}_{\mathbf{p}}^- \cap \mathcal{V}_{\mathbf{r}}^+ = F^l (\mathcal{V}_{q_0 \dots q_m}^-) = \emptyset$$

By Lemma 5-3.6,  $U_{\mathbf{p}} U_{\mathbf{r}} = U_{q_0 \dots q_m} = O(h^\infty)$ .

- $\mathcal{V}_{\mathbf{p}}^- = \emptyset$  : This time, we have  $\max(J_{q_l \dots q_{m-1}}^-, J_{q_m}^-) \leq C h^{-\delta}$  and  $\mathcal{V}_{q_m}^- \cap \mathcal{V}_{q_l \dots q_{m-1}}^+ = \emptyset$ . According to Lemma 5-3.6,  $U_{q_l \dots q_m} = U_{q_m} U_{q_l \dots q_{m-1}} = O(h^\infty)$ . It follows that  $U_{q_0 \dots q_m} = O(h^\infty)$ .  $\square$

### 5-3.4.2. Orthogonality of the $U_{\mathbf{q}}$ .

We now focus on terms  $U_{\mathbf{q}} U_{\mathbf{p}}^*$  and  $U_{\mathbf{q}}^* U_{\mathbf{p}}$  when  $\mathcal{V}_{\mathbf{q}}^+$  and  $\mathcal{V}_{\mathbf{p}}^+$  are disjoint, under growth conditions of the Jacobian. The following result shows that the operators  $U_{\mathbf{q}}$  and  $U_{\mathbf{p}}$  are (up to  $O(h^\infty)$ ) orthogonal. These estimates will turn out to be important to apply Cotlar-Stein type estimates.

**Proposition 5-3.4.** Assume that  $\mathbf{q}, \mathbf{p} \in \mathcal{A}^\rightarrow$  are two words of same length  $|\mathbf{q}| = |\mathbf{p}| = n$  satisfying  $\mathcal{V}_{\mathbf{q}}^+ \cap \mathcal{V}_{\mathbf{p}}^+ = \emptyset$  and  $\max(J_{\mathbf{q}}^+, J_{\mathbf{p}}^+) \leq C_0 h^{-2\delta}$ . Then,

$$U_{\mathbf{q}} U_{\mathbf{p}}^* = O(h^\infty)$$

$$U_{\mathbf{q}}^* U_{\mathbf{p}} = O(h^\infty)$$

Before proving it, we need the following lemma, whose proof relies on the iterative construction of the symbols  $a_{\mathbf{q}}^{\pm}$ .

**Lemma 5-3.8.** Assume that  $\mathbf{q}, \mathbf{p} \in \mathcal{A}^{\rightarrow}$  are two words of same length  $|\mathbf{q}| = |\mathbf{p}| = n$  satisfying  $\max(J_{\mathbf{q}}^+, J_{\mathbf{p}}^+) \leq C_0 h^{-\delta}$ . Then,

$$\begin{aligned} U_{\mathbf{q}} U_{\mathbf{p}}^* &= \text{Op}_h(a_{\mathbf{q}}^+) \text{Op}_h(a_{\mathbf{p}}^+)^* + O(h^\infty) \\ U_{\mathbf{q}}^* U_{\mathbf{p}} &= \text{Op}_h(a_{\mathbf{q}}^-)^* \text{Op}_h(a_{\mathbf{p}}^-) + O(h^\infty) \end{aligned}$$

*Proof. (of the lemma)* We prove the first equality. The second one could be treated similarly. Recall the construction procedure of the subsection 5-3.3.. We adopt the same notations. We will show by induction on  $r \in \{0, \dots, n-1\}$  that :

$$V_r := U_{q_0 \dots q_{r-1}} U_{p_0 \dots p_{r-1}}^* = \text{Op}_h(a_{q_0 \dots q_{r-1}}^+) \text{Op}_h(a_{p_0 \dots p_{r-1}}^+)^* + O(h^\infty)$$

The case  $r = 1$  follows from

$$M A_{q_0} A_{p_0}^* M^* = \text{Op}_h(a_{q_0}^+) T T^* \text{Op}_h(a_{p_0}^+)^* + O(h^\infty) = \text{Op}_h(a_{q_0}^+) \text{Op}_h(a_{p_0}^+)^* + O(h^\infty)$$

where we use the fact that  $T T^* = I + O(h^\infty)$  microlocally in  $\mathcal{V}_{p_0}^+$ , Assume that the assumption is satisfied for  $r$ , namely :

$$V_r = \text{Op}_h(a_{q_0 \dots q_{r-1}}^+) \text{Op}_h(a_{p_0 \dots p_{r-1}}^+)^* + O(h^\infty)$$

and let's prove it for  $r + 1$ .

$$\begin{aligned} V_{r+1} &= M A_{q_r} V_r A_{p_r}^* M^* \\ &= M A_{q_r} \text{Op}_h(a_{q_0 \dots q_{r-1}}^+) \text{Op}_h(a_{p_0 \dots p_{r-1}}^+)^* A_{p_r}^* M^* + O(h^\infty) \\ &= \text{Op}_h(a_{q_0 \dots q_r}^+) T T^* \text{Op}_h(a_{p_0 \dots p_r}^+)^* + O(h^\infty) \\ &= \text{Op}_h(a_{q_0 \dots q_r}^+) \text{Op}_h(a_{p_0 \dots p_r}^+)^* + O(h^\infty) \end{aligned}$$

The last equality follows from  $T T^* = I + O(h^\infty)$  microlocally in  $\mathcal{V}_{p_r}^+$  and the one before is due to the recursive construction of the symbols  $a_{q_0 \dots q_r}^+$  in the subsection 5-3.3..  $\square$

*Proof. (of the proposition)* Let us begin with the first equality. Consider the largest integer  $l$  such that

$$\max(J_{q_0 \dots q_{l-1}}^+, J_{p_0 \dots p_{l-1}}^+) \leq h^{-\delta}$$

We set  $\mathbf{q}_{\leftarrow} = q_0 \dots q_{l-1}$  and  $\mathbf{q}_{\rightarrow} = q_l \dots q_{n-1}$ , and the same notations for  $\mathbf{p}$ . We obviously have :

$$U_{\mathbf{q}} U_{\mathbf{p}}^* = U_{\mathbf{q}_{\rightarrow}} U_{\mathbf{q}_{\leftarrow}} U_{\mathbf{p}_{\leftarrow}}^* U_{\mathbf{p}_{\rightarrow}}^*$$

We then consider two cases,

►  $\mathcal{V}_{\mathbf{q}_{\leftarrow}}^+ \cap \mathcal{V}_{\mathbf{p}_{\leftarrow}}^+ = \emptyset$  : we may write

$$U_{\mathbf{q}_{\leftarrow}} U_{\mathbf{p}_{\leftarrow}}^* = T^l \text{Op}_h(a_{\mathbf{q}_{\leftarrow}}^-) \text{Op}_h(a_{\mathbf{q}_{\leftarrow}}^-)^* T^l + O(h^\infty)$$

Since,  $\mathcal{V}_{\mathbf{q}_{\leftarrow}}^- \cap \mathcal{V}_{\mathbf{p}_{\leftarrow}}^- = \emptyset$ , we can use the composition formula in  $S_{\delta_1}^{comp}$  to conclude that  $\text{Op}_h(a_{\mathbf{q}_{\leftarrow}}^-) \text{Op}_h(a_{\mathbf{q}_{\leftarrow}}^-)^* = O(h^\infty)$ , which gives the desire result, recalling that  $U_{\mathbf{q}} = O(1)$ .

►  $\mathcal{V}_{\mathbf{q}_{\leftarrow}}^+ \cap \mathcal{V}_{\mathbf{p}_{\leftarrow}}^+ \neq \emptyset$  : in this case, we use the previous lemma and we can write

$$U_{\mathbf{q}_{\leftarrow}} U_{\mathbf{p}_{\leftarrow}}^* = \text{Op}_h(a_{\mathbf{q}_{\leftarrow}}^+) \text{Op}_h(a_{\mathbf{p}_{\leftarrow}}^+)^* + O(h^\infty)$$

In virtue of the second point of Lemma 5-3.2,  $J_{\mathbf{q}_{\leftarrow}}^+ \sim J_{\mathbf{p}_{\leftarrow}}^+$ . Moreover, by maximality of  $l$ , either  $J_{\mathbf{q}_{\leftarrow} q_l}^+ > h^{-\delta}$  or  $J_{\mathbf{p}_{\leftarrow} p_l}^+ > h^{-\delta}$ . But

$$J_{\mathbf{q}_{\leftarrow} q_l}^+ \sim J_{\mathbf{q}_{\leftarrow}}^+$$

Hence,  $J_{\mathbf{q}_{\leftarrow}}^+ \sim h^{-\delta}$ . Using now the third point of Lemma 5-3.2, we conclude that

$$J_{\mathbf{q}_{\rightarrow}}^+ \sim J_{\mathbf{p}_{\rightarrow}}^+ \sim h^{-\delta}$$

This estimate allows us to write

$$U_{\mathbf{q}} U_{\mathbf{p}}^* = T^{n-l} \text{Op}_h(a_{\mathbf{q}_{\rightarrow}}^-) \text{Op}_h(a_{\mathbf{q}_{\leftarrow}}^+) \text{Op}_h(a_{\mathbf{p}_{\leftarrow}}^+)^* \text{Op}_h(a_{\mathbf{p}_{\rightarrow}}^-)^* (T^*)^{n-l} + O(h^\infty)$$

with all the symbols in  $h^{-M} S_{\delta_1}^{comp}$  for some  $M > 0$ . To conclude, we use the composition formula in this symbol class, noting that

$$\mathcal{V}_{\mathbf{q}_{\leftarrow}}^+ \cap \mathcal{V}_{\mathbf{q}_{\rightarrow}}^- \cap \mathcal{V}_{\mathbf{p}_{\leftarrow}}^+ \cap \mathcal{V}_{\mathbf{p}_{\rightarrow}}^- = F^l (\mathcal{V}_{\mathbf{q}_{\rightarrow}}^- \cap \mathcal{V}_{\mathbf{p}_{\leftarrow}}^-) = \emptyset$$

To deal with the second equality, we consider the smallest integer  $l$  such that :

$$\max(J_{q_l \dots q_{n-1}}^+, J_{p_l \dots p_{n-1}}^+) \leq h^{-\delta}$$

As before, we write  $\mathbf{q}_{\leftarrow} = q_0 \dots q_{l-1}$  and  $\mathbf{q}_{\rightarrow} = q_l \dots q_{n-1}$ , and the same notations for  $\mathbf{p}$ . We obviously have :

$$U_{\mathbf{q}}^* U_{\mathbf{p}} = U_{\mathbf{q}_{\leftarrow}}^* U_{\mathbf{q}_{\rightarrow}}^* U_{\mathbf{p}_{\rightarrow}} U_{\mathbf{p}_{\leftarrow}}$$

We distinguish the cases  $\mathcal{V}_{\mathbf{q}_{\rightarrow}}^+ \cap \mathcal{V}_{\mathbf{p}_{\rightarrow}}^+ = \emptyset$  or not and argue similarly.  $\square$

### 5-3.5. Reduction to sub-words with precise growth of their Jacobian.

Recall that we are interested in a decay bound for  $\|\mathfrak{M}^{N_0+N_1}\|$  where  $\mathfrak{M} = M(\text{Id} - A_\infty) = \sum_{q \in \mathcal{A}} M A_q$ . For this purpose, we decompose  $\mathfrak{M}^{N_1} = \sum_{\mathbf{q} \in \mathcal{A}^{N_1}} U_{\mathbf{q}}$ .

If  $\mathbf{q} \in \mathcal{A}^{N_1}$ , either  $\mathcal{V}_{\mathbf{q}}^+ = \emptyset$ , and in this case  $J_{\mathbf{q}}^+ = +\infty$ , or  $\mathcal{V}_{\mathbf{q}}^+ \neq \emptyset$ , which implies that  $J_{\mathbf{q}}^+ \geq e^{\lambda_1 N_1} \geq h^{-1} \gg h^{-\tau}$ . In both cases, the following integer is well defined :

$$n(\mathbf{q}) = \max\{k \in \{1, N_1\}, J_{q_{N_1-k} \dots q_{N_1-1}}^+ \leq h^{-\tau}\} \quad (5-3.55)$$

We then set  $\mathbf{q}_\tau = q_{N_1-n(q)-1} \dots q_{N_1-1}$ . The case  $\mathcal{V}_{\mathbf{q}_\tau} = \emptyset$  is irrelevant. Indeed, if  $\mathbf{q} \in \mathcal{A}^{N_1}$  and if  $\mathcal{V}_{\mathbf{q}_\tau} = \emptyset$ , then  $U_{\mathbf{q}} = O(h^\infty)$ , as an obvious consequence of Lemma 5-3.7. Then, we set

$$Q = \{\mathbf{q} \in \mathcal{A}^{N_1}, \mathcal{V}_{\mathbf{q}_\tau} \neq \emptyset\} \quad (5-3.56)$$

so that, due to the fact that  $|\mathcal{A}^{N_1}| = O(h^{-M})$ , for some  $M > 0$ , we have

$$\mathfrak{M}^{N_1} = \sum_{\mathbf{q} \in Q} U_{\mathbf{q}} + O(h^\infty)$$

We partition  $Q$  in function of the length of  $\mathbf{q}_\tau$  and the value of  $q_{N_1-1}$ . Namely, we set

$$Q_0(n, a) = \{\mathbf{q} \in Q; |\mathbf{q}_\tau| = n, q_{N_1-1} = a\}$$

We finally set  $Q(n, a) = \{\mathbf{q}_\tau, \mathbf{q} \in Q_0(n, a)\}$  which is simply the set of words  $\mathbf{q} \in \mathcal{A}^n$  such that  $q_{n-1} = a$  and  $J_{q_1 \dots q_{n-1}}^+ \leq h^{-\tau} < J_{\mathbf{q}}^+$ . Note that every word  $\mathbf{q} \in Q_0(n, a)$  can be written in the form  $\mathbf{q} = \mathbf{r}\mathbf{p}$  with  $\mathbf{p} \in Q(n, a)$  and  $\mathbf{r} \in \mathcal{A}^{N_1-n}$ . We deduce that, modulo  $O(h^\infty)$ ,

$$\begin{aligned} \mathfrak{M}^{N_1} &= \sum_{n=1}^{N_1} \sum_{a \in \mathcal{A}} \sum_{\mathbf{q} \in Q_0(n, a)} U_{\mathbf{q}} \\ &= \sum_{n=1}^{N_1} \sum_{a \in \mathcal{A}} \sum_{\substack{\mathbf{p} \in Q(n, a) \\ \mathbf{r} \in \mathcal{A}^{N_1-n}}} U_{\mathbf{p}} U_{\mathbf{r}} \\ &= \sum_{n=1}^{N_1} \sum_{a \in \mathcal{A}} \left( \sum_{\mathbf{q} \in Q(n, a)} U_{\mathbf{q}} \right) \mathfrak{M}^{N_1-n} \end{aligned}$$

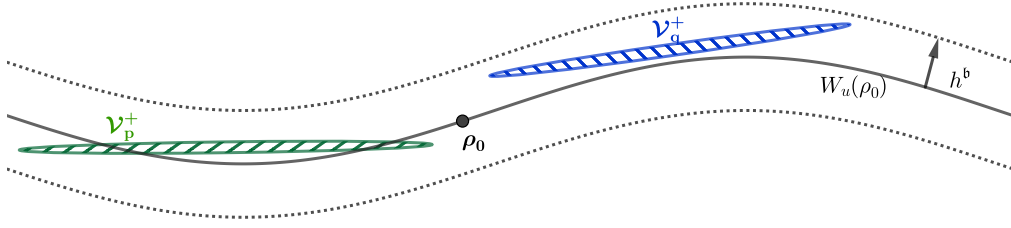


Figure 5.5: Two words  $\mathbf{q}, \mathbf{p} \in Q(n, a)$  are close to each other if  $\mathcal{V}_{\mathbf{q}}^+$  and  $\mathcal{V}_{\mathbf{p}}^+$  lie in the  $h^{\mathfrak{b}}$ -neighborhood of the same unstable leaves, as stated in Definition 5-3.1.

As a consequence, we get

$$\|\mathfrak{M}^{N_0+N_1}\| \leq CN_1|\mathcal{A}| \sup_{\substack{1 \leq n \leq N_1 \\ a \in \mathcal{A}}} \|\mathfrak{M}^{N_0} U_{Q(n,a)}\| (\|\alpha\|_{\infty})^{N_1-n} \quad (5-3.57)$$

where we've noted

$$U_{Q(n,a)} = \sum_{\mathbf{q} \in Q(n,a)} U_{\mathbf{q}} \quad (5-3.58)$$

Since  $N_1 = O(\log h)$ , the proof of (5-3.8) is reduced to prove :

**Proposition 5-3.5.** There exists  $\gamma > 0$  such that, for  $h$  small enough, we have

$$\sup_{\substack{1 \leq n \leq N_1 \\ a \in \mathcal{A}}} \frac{\|\mathfrak{M}^{N_0} U_{Q(n,a)}\|}{\|\alpha\|_{\infty}^{n+N_0}} \leq h^{\gamma} \quad (5-3.59)$$

### 5-3.6. Partition into clouds.

We fix  $1 \leq n \leq N_1$  and  $a \in \mathcal{A}$ . We aim at gathering pieces of  $\mathfrak{M}^{N_0} U_{Q(n,a)}$  into clouds and we want these clouds to interact (with a meaning we will define further) with only a finite and uniform number of other clouds, so that the global norm of  $\|\mathfrak{M}^{N_0} U_{Q(n,a)}\|$  can be deduced from a uniform bound for each cloud.

Recall that  $\delta_0$  and  $\tau$  (see (5-2.3), (5-2.4) and (5-2.6)) have been chosen such that

$$\mathfrak{b} + \delta_0 < 1 ; \mathfrak{b} < \tau$$

We start by defining a notion of closeness between two words  $\mathbf{q}, \mathbf{p} \in Q(n, a)$ . We choose  $\varepsilon_2$  as in Lemma 5-3.1.

**Definition 5-3.1.** Let  $\mathbf{q}, \mathbf{p} \in Q(n, a)$ . We say that these two words are *close to each other* if there exists  $\rho_0 \in \mathcal{T} \cap F(\mathcal{V}_a(\varepsilon_2))$  such that :

$$\forall \rho \in \mathcal{V}_{\mathbf{q}}^+ \cup \mathcal{V}_{\mathbf{p}}^+, d(\rho, W_u(\rho_0)) \leq h^{\mathfrak{b}}$$

Otherwise, we say that  $\mathbf{q}$  and  $\mathbf{p}$  are *far from each other*.

#### Remark.

By definition of  $\mathcal{V}_{\mathbf{q}}^+$ , if  $\mathbf{q} \in Q(n, a)$  and if  $\rho \in \mathcal{V}_{\mathbf{q}}^+$ ,  $\rho$  does not lie in  $\mathcal{V}_a$ , but  $F^{-1}(\rho)$  does. Hence, we work with  $F(\mathcal{V}_a)$  instead of  $\mathcal{V}_a$ . Moreover, the set  $F(\mathcal{V}_a(\varepsilon_2))$  is chosen to fit well in the computations below and in particular in the proof of Lemma 5-3.9. We could replace it by  $\mathcal{V}_a^+(C\varepsilon_2)$ , where  $C$  is any Lipschitz constant for  $F$ .

The important fact on words far from each other is that the associated operator  $\mathfrak{M}^{N_0} U_{\mathbf{q}}$  are almost orthogonal :

**Proposition 5-3.6.** Assume that  $\mathbf{q}, \mathbf{p} \in Q(n, a)$  are far from each other. Then,

$$(\mathfrak{M}^{N_0} U_{\mathbf{q}})^* (\mathfrak{M}^{N_0} U_{\mathbf{p}}) = O(h^\infty) \quad (5-3.60)$$

$$(\mathfrak{M}^{N_0} U_{\mathbf{q}}) (\mathfrak{M}^{N_0} U_{\mathbf{p}})^* = O(h^\infty) \quad (5-3.61)$$

We will need the following lemma.

**Lemma 5-3.9.** If  $\mathbf{q}, \mathbf{p} \in Q(n, a)$  are far from each other, there exist words  $\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2$  such that

- $|\mathbf{p}_1| = |\mathbf{q}_1|, |\mathbf{p}_2| = |\mathbf{q}_2|$  ;
- $\mathbf{q} = \mathbf{q}_1 \mathbf{q}_2, \mathbf{p} = \mathbf{p}_1 \mathbf{p}_2$  ;
- $\mathcal{V}_{\mathbf{q}_2}^+ \cap \mathcal{V}_{\mathbf{p}_2}^+ = \emptyset$ ;
- $\max(J_{\mathbf{q}_2}^+, J_{\mathbf{p}_2}^+) \leq Ch^{-b}$  (for some global constant  $C > 0$ ).

In particular,  $\mathcal{V}_{\mathbf{q}}^+ \cap \mathcal{V}_{\mathbf{p}}^+ = \emptyset$

Let's momentarily admit it and prove the proposition.

*Proof. (of the proposition).* Fix  $\mathbf{q}, \mathbf{p} \in Q(n, a)$  far from each other. Since  $\mathcal{V}_{\mathbf{q}}^+ \cap \mathcal{V}_{\mathbf{p}}^+ = \emptyset, U_{\mathbf{q}} U_{\mathbf{p}}^* = O(h^\infty)$  in virtue of Proposition 5-3.4. Hence, using the polynomial bounds  $\|\mathfrak{M}^{N_0}\| = O(h^{-M})$  (for some  $M > 0$ ), we have

$$(\mathfrak{M}^{N_0} U_{\mathbf{q}}) (\mathfrak{M}^{N_0} U_{\mathbf{p}})^* = O(h^\infty)$$

To prove the first point, we write

$$(\mathfrak{M}^{N_0} U_{\mathbf{q}})^* (\mathfrak{M}^{N_0} U_{\mathbf{p}}) = \sum_{\mathbf{s}, \mathbf{t} \in \mathcal{A}^{N_0}} U_{\mathbf{q}_1}^* U_{\mathbf{q}_2}^* U_{\mathbf{s}}^* U_{\mathbf{t}} U_{\mathbf{p}_2} U_{\mathbf{p}_1}$$

Hence, it is enough to show that  $U_{\mathbf{q}_2}^* U_{\mathbf{s}}^* U_{\mathbf{t}} U_{\mathbf{p}_2} = O(h^\infty)$  uniformly in  $\mathbf{s}, \mathbf{t}$ . To do so, we note that

$$\begin{aligned} \mathcal{V}_{\mathbf{q}_2 \mathbf{s}}^+ \cap \mathcal{V}_{\mathbf{p}_2 \mathbf{t}}^+ &\subset F^{N_0} (\mathcal{V}_{\mathbf{q}_2}^+ \cap \mathcal{V}_{\mathbf{p}_2}^+) = \emptyset \\ J_{\mathbf{q}_2 \mathbf{s}}^+ &\leq C J_{\mathbf{s}}^+ J_{\mathbf{q}_2}^+ \leq C e^{\lambda_1 N_0} h^{-b} \leq C h^{-(\delta_0 + b)} \\ J_{\mathbf{p}_2 \mathbf{t}}^+ &\leq C h^{-(\delta_0 + b)} \end{aligned}$$

and apply Proposition 5-3.4, with  $\delta = \frac{\delta_0 + b}{2} < 1/2$  (here we use the condition (5-2.4)).  $\square$

We now prove the lemma.

*Proof. (of the lemma)* Consider  $\mathbf{q}, \mathbf{p} \in Q(n, a)$  far from each other. Consider the smallest integer  $m$  such that  $\mathcal{V}_{q_m \dots q_{n-1}}^+ \cap \mathcal{V}_{p_m \dots p_{n-1}}^+ \neq \emptyset$ . We will show that  $m > 0$  and set  $\mathbf{p}_2 = p_{m-1} \dots p_{n-1}, \mathbf{q}_2 = q_{m-1} \dots q_{n-1}$ . Pick  $\rho \in \mathcal{V}_{q_m \dots q_{n-1}}^+ \cap \mathcal{V}_{p_m \dots p_{n-1}}^+$ . By choice of  $\varepsilon_2$  after Lemma 5-3.1, there exists  $\rho_0 \in \mathcal{T}$  such that  $d(F^{-i}(\rho), F^{-i}(\rho_0)) \leq \varepsilon_2$  for  $i \in \{1, \dots, n-m\}$ . In particular,  $d(F^{-1}(\rho), F^{-1}(\rho_0)) \leq \varepsilon_2$  and  $F^{-1}(\rho) \in \mathcal{V}_a$ , so that  $\rho_0 \in F(\mathcal{V}_a(\varepsilon_2))$ . Since,  $\mathbf{q}, \mathbf{p}$  are far from each other, there exists  $\rho_1 \in \mathcal{V}_{\mathbf{q}}^+ \cup \mathcal{V}_{\mathbf{p}}^+$  such that  $d(\rho_1, W_u(\rho_0)) > h^b$  (otherwise, it would contradict the definition 5-3.1).

Suppose for instance that  $\rho_1 \in \mathcal{V}_{\mathbf{q}}^+ \subset \mathcal{V}_{q_m \dots q_{n-1}}^+$ . Hence,  $d(F^{-i}(\rho_0), F^{-i}(\rho_1)) \leq 2\varepsilon_0 + \varepsilon_2$  for  $i \in \{1, \dots, n-m\}$ . From (3-2.12),  $d(\rho_1, W_u(\rho_0)) \leq C (J_s^{n-m}(\rho_0))^{-1}$  and hence,  $J_s^{n-m}(\rho_0) \leq Ch^{-b}$ . But,  $J_s^{n-m}(\rho_0) \sim J_{p_m \dots p_{n-1}}^+ \sim J_{q_m \dots q_{n-1}}^+$ , which gives

$$\max \left( J_{p_m \dots p_{n-1}}^+, J_{q_m \dots q_{n-1}}^+ \right) \leq Ch^{-b}$$

Since  $\min(J_{\mathbf{q}}^+, J_{\mathbf{p}}^+) > h^{-\tau} \gg h^{-b}$  (here we use (5-2.6)), we cannot have  $m = 0$  (if  $h$  small enough). Thus, we can set  $\mathbf{p}_2 = p_{m-1} \dots p_{n-1}, \mathbf{q}_2 = q_{m-1} \dots q_{n-1}$  which satisfy the required properties by minimality of  $m$ .  $\square$

We now decompose  $U_{Q(n, a)}$  into a sum of operators, each of them corresponding to a *cloud* of words. In the following, we'll use the term *cloud* to mean a subset  $\mathcal{Q} \subset Q(n, a)$  and we'll adopt the notation

$$\mathcal{V}_{\mathcal{Q}}^+ = \bigcup_{\mathbf{q} \in \mathcal{Q}} \mathcal{V}_{\mathbf{q}}^+$$

and the definition :

**Definition 5-3.2.** We say that two clouds  $\mathcal{Q}_1, \mathcal{Q}_2$  do not interact if for all couples  $(\mathbf{q}_1, \mathbf{q}_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ ,  $\mathbf{q}_1$  and  $\mathbf{q}_2$  are far from each other.

The existence of such a decomposition follows from the key proposition :

**Proposition 5-3.7.** Suppose  $\varepsilon_0$  is small enough.

There exists a partition of  $Q(n, a)$  into clouds  $\mathcal{Q}_1, \dots, \mathcal{Q}_r$  and a global constant  $C > 0$  such that, for  $i = 1, \dots, r$ ,

- i) there exists  $\rho_i \in \mathcal{T}$  such that for all  $\rho \in \mathcal{V}_{\mathcal{Q}_i}^+$ ,  $d(\rho, W_u(\rho_i)) \leq Ch^b$  ;
- ii) if  $\mathcal{Q}_i$  interacts with exactly  $c_i$  clouds, then  $c_i \leq C$ .

**Remark.**

Actually,  $r$  and the clouds  $\mathcal{Q}_i$  depend on  $n$  and  $a$ . We do not write this dependence explicitly here to make the notations lighter. The second point is relevant since *a priori*, the only obvious bound on  $r = r(n, a)$  is  $|r| \leq |\mathcal{A}|^n$ , where  $n = O(\log h)$ .

*Proof.* Keeping in mind that for all  $\mathbf{q} \in Q(n, a)$ ,  $\mathcal{V}_{\mathbf{q}}^+ \subset \mathcal{V}_a^+$ , we fix  $\rho_a \in \mathcal{V}_a^+$ . If  $\varepsilon_0$  is small enough,  $\mathcal{V}_a^+$  do not intersect the boundaries of  $W_s(\rho_a)$  and  $W_u(\rho_a)$ .

For  $\mathbf{q} \in Q(n, a)$ , there exists  $\rho_{\mathbf{q}} \in \mathcal{T}$  such that  $d(F^{-i}(\rho), F^{-i}(\rho_{\mathbf{q}})) \leq \varepsilon_2$  for all  $\rho \in \mathcal{V}_{\mathbf{q}}^+$  and for  $i = 1, \dots, n$ , according to Lemma 5-3.1 and since  $J_{\mathbf{q}}^+ \sim h^\tau$ ,

$$d(\rho, W_u(\rho_{\mathbf{q}})) \leq Ch^{-\tau}$$

$d(\rho_a, \rho_{\mathbf{q}}) \leq C(\varepsilon_2 + \varepsilon_0)$  and hence, if  $\varepsilon_0$  is small enough,  $z_{\mathbf{q}} := H_{\rho_a}^u(\rho_{\mathbf{q}})$  (here,  $H_{\rho_a}^u : B(\rho_a, \varepsilon'_0) \rightarrow W_s(\rho_a)$ ) is the unstable holonomy map defined before Lemma 3-3.2) is well defined, and depends Lipschitz-continuously on  $\rho_{\mathbf{q}}$  (with global Lipschitz constant).

Next, consider a maximal subset  $\{z_1, \dots, z_r\} \subset \{z_{\mathbf{q}}, \mathbf{q} \in Q(n, a)\}$  which is  $h^b$  separated. By maximality, for every  $\mathbf{q} \in Q(n, a)$ , there exists  $i \in \{1, \dots, r\}$  such that  $|z_i - z_{\mathbf{q}}| \leq h^b$  and we use these  $z_i$  to partition  $Q(n, a)$  into clouds  $\mathcal{Q}_i$  where for  $i \in \{1, \dots, r\}$ ,  $|z_i - z_{\mathbf{q}}| \leq h^b$  for all  $\mathbf{q} \in \mathcal{Q}_i$ . We now show that this partition satisfies the required properties.

Let  $i \in \{1, \dots, r\}$ ,  $\mathbf{q} \in \mathcal{Q}_i$  and  $\rho \in \mathcal{V}_{\mathbf{q}}^+$ . By local uniqueness of the unstable leaves, we may assume that  $\varepsilon_0$  is small enough so that  $W_u(\rho_{\mathbf{q}}) \cap \mathcal{V}_a^+ = W_u(z_{\mathbf{q}}) \cap \mathcal{V}_a^+$ . Hence,

$$d(\rho, W_u(z_{\mathbf{q}})) \leq Ch^{-\tau}$$

Since the unstable leaves depend Lipschitz-continuously on  $\rho \in \mathcal{T}$ , we have

$$d(\rho, W_u(z_i)) \leq C|z_i - z_{\mathbf{q}}| + Cd(\rho, W_u(z_{\mathbf{q}})) \leq Ch^b + Ch^\tau \leq Ch^b$$

This gives i).

To show ii), suppose that  $\mathcal{Q}_i$  and  $\mathcal{Q}_j$  interact. Then, there exists  $(\mathbf{q}, \mathbf{p}) \in \mathcal{Q}_i \times \mathcal{Q}_j$  and  $\rho_0 \in \mathcal{T}$  such that for all  $\rho \in \mathcal{V}_{\mathbf{q}}^+ \cup \mathcal{V}_{\mathbf{p}}^+$ ,  $d(\rho, W_u(\rho_0)) \leq h^b$ . It follows that  $d(z_{\mathbf{q}}, W_u(\rho_0)) \leq Ch^\tau + h^b \leq Ch^b$  and if we note  $z_0 = H_{\rho_a}^u(\rho_0)$  the unique point in  $W_u(\rho_0) \cap W_s(\rho_a)$  then  $|z_0 - z_{\mathbf{q}}| \leq Ch^b$ . The same is true for  $\mathbf{p}$  and we have  $|z_{\mathbf{q}} - z_{\mathbf{p}}| \leq Ch^b$  and eventually,  $|z_i - z_j| \leq Ch^b$ . Since  $z_1, \dots, z_r$  are  $h^b$  separated, we see after rescaling that the number of  $j$  such that  $\mathcal{Q}_i$  and  $\mathcal{Q}_j$  interact is smaller than the maximal number of points in  $B(0, C)$  which are 1-separated (one can for instance bound it by  $(2C + 1)^2$ , but what matters is that it is a global constant).  $\square$

This partition into clouds allows us to decompose  $\mathfrak{M}^{N_0} U_{Q(n, a)}$  into a sum of operators

$$B_i = \mathfrak{M}^{N_0} U_{\mathcal{Q}_i} = \sum_{\mathbf{q} \in \mathcal{Q}_i} \mathfrak{M}^{N_0} U_{\mathbf{q}} \quad ; \quad \mathfrak{M}^{N_0} U_{Q(n, a)} = \sum_{i=1}^r B_i \quad (5-3.62)$$

The use of Cotlar-Stein theorem ([Zwo12]), Theorem C.5) reduces the control of the sum by the control of individual clouds :

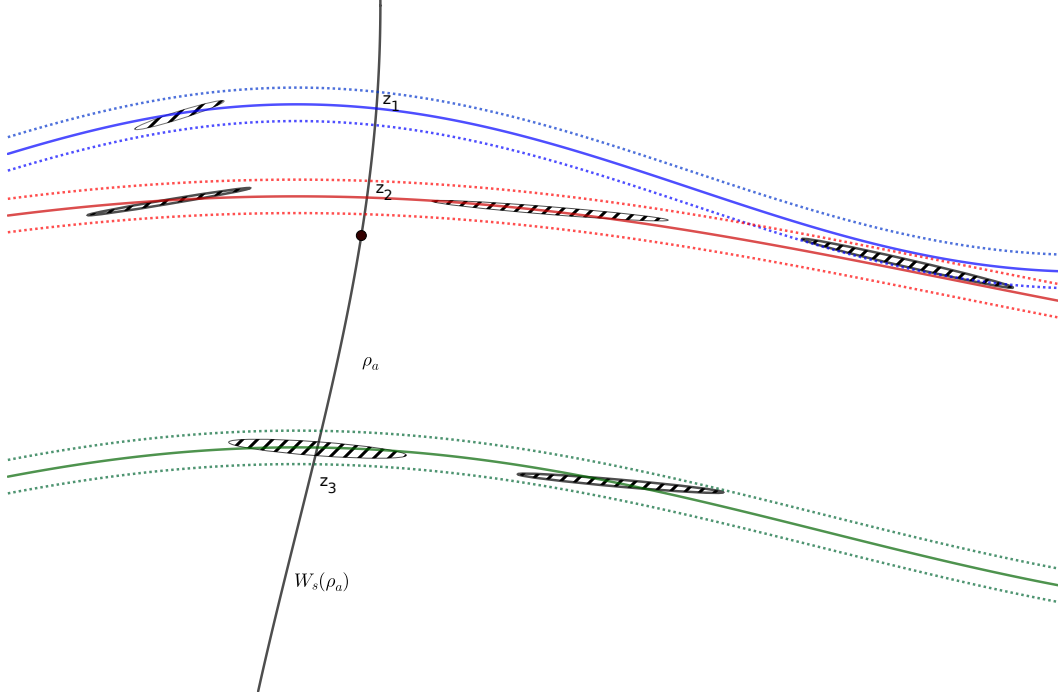


Figure 5.6: We gather the 6 small sets  $\mathcal{V}_{\mathbf{q}}$  into 3 clouds corresponding to  $z_1, z_2$  and  $z_3$ . Here,  $\mathcal{Q}_1 = \{\mathbf{q}_1\}$ ,  $\mathcal{Q}_2 = \{\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ ,  $\mathcal{Q}_3 = \{\mathbf{q}_5, \mathbf{q}_6\}$ . The clouds  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  interact. The dotted lines draw tubes of width  $Ch^b$  around the unstable leaves  $W_u(z_i)$ . The sets  $\mathcal{V}_{\mathbf{q}}$  have width of order  $h^\tau$ .

**Lemma 5-3.10.** With the above notations, there exists a global constant  $C > 0$  such that

$$\|\mathfrak{M}^{N_0} U_{Q(n,a)}\| \leq C \sup_{1 \leq i \leq r} \|B_i\| + O(h^\infty) \quad (5-3.63)$$

*Proof.* Cotlar-Stein theorem reduces to control

$$\begin{aligned} & \max_i \sum_j \|B_i^* B_j\|^{1/2} \\ & \max_i \sum_j \|B_j B_i^*\|^{1/2} \end{aligned}$$

Fix  $i \in \{1, \dots, r\}$ .

If  $\mathcal{Q}_i$  and  $\mathcal{Q}_j$  do not interact,  $\|B_i^* B_j\|^{1/2}$  (resp.  $\|B_j B_i^*\|^{1/2}$ ) is a sum of terms of the form  $(\mathfrak{M}^{N_0} U_{\mathbf{q}})^* (\mathfrak{M}^{N_0} U_{\mathbf{p}})$  (resp.  $(\mathfrak{M}^{N_0} U_{\mathbf{q}}) (\mathfrak{M}^{N_0} U_{\mathbf{p}})^*$ ) where  $\mathbf{p}$  and  $\mathbf{q}$  are far from each other. In virtue of Proposition 5-3.4, these terms are uniformly  $O(h^\infty)$  and since the number of terms in the sum grows at most polynomially with  $h$ , we can gather all these terms in a single uniform  $O(h^\infty)$ . As a consequence, we have

$$\begin{aligned} \sum_j \|B_i^* B_j\|^{1/2} & \leq \sum_{\mathcal{Q}_i \text{ and } \mathcal{Q}_j \text{ interact}} \|B_i^* B_j\|^{1/2} + O(h^\infty) \\ & \leq \sum_{\mathcal{Q}_i \text{ and } \mathcal{Q}_j \text{ interact}} \max_k \|B_k\| + O(h^\infty) \\ & \leq C \max_k \|B_k\| + O(h^\infty) \end{aligned}$$

and the same holds for the second sum. This gives the desired inequalities.  $\square$

The proof of (5-3.8) and, as a consequence, of Proposition 5-2.1 is then reduced to the proof of

**Proposition 5-3.8.** There exists  $\gamma > 0$  such that the following holds for  $h$  small enough. Assume that  $\mathcal{Q} \subset \mathcal{Q}(n, a)$  satisfies, for some global constant  $C > 0$ ,

$$\exists \rho_0 \in \mathcal{T}, \quad \forall \rho \in \mathcal{V}_{\mathcal{Q}}^+, \quad d(\rho, W_u(\rho_0)) \leq Ch^b$$

where  $\mathfrak{b} = \frac{1}{1+\beta}$  is defined in (5-2.3). Then,

$$\frac{\|\mathfrak{M}^{N_0} U_{\mathcal{Q}}\|}{\|\alpha\|_{\infty}^{N_0+n}} \leq h^{\gamma}$$

## 5-4 Reduction to a fractal uncertainty principle via microlocalization properties

In this section, we reduce the proof of Proposition 5-3.8 to a fractal uncertainty principle. To do so, we aim at showing microlocalization properties of the operators involved. The disymmetry between  $N_0$  and  $N_1$  in the decomposition  $N = N_0 + N_1$  will appear clearly in this section. Since  $N_0$  is below the Ehrenfest time, we can actually use semiclassical tools. By contrast, things are more complicated for operators  $U_{\mathbf{q}}$ , with  $\mathbf{q} \in \mathcal{Q}(n, a)$  and we'll use methods of propagation of Lagrangian leaves. These methods are inspired by [AN07b], [AN07a] and [NZ09] and are also used in [DJN21].

### 5-4.1. Microlocalization of $\mathfrak{M}^{N_0}$ .

We first state a microlocalization result for  $\mathfrak{M}^{N_0}$ . This is the easiest one to obtain since  $N_0$  is below the Ehrenfest time. We recall the definition of  $\mathcal{T}_-$  the set of the future trapped points

$$\mathcal{T}_- = \bigcap_{n \in \mathbb{N}} F^{-n}(U)$$

and focus on  $\mathcal{T}_-^{\text{loc}} := \mathcal{T}_- \cap \mathcal{T}(4\varepsilon_0)$ .  $\mathcal{T}_-$  is laminated by the weak global stable leaves. Hence, if  $\varepsilon_0$  is small enough, ensuring that the boundaries of the local stable leaves  $W_s(\rho)$ ,  $\rho \in \mathcal{T}$  do not intersect  $\mathcal{T}(4\varepsilon_0)$ , we have

$$\mathcal{T}_-^{\text{loc}} \subset \bigcup_{\rho \in \mathcal{T}} W_s(\rho)$$

When  $\mathbf{q} \in \mathcal{A}^{N_0}$  and  $\mathcal{V}_{\mathbf{q}}^- \neq \emptyset$ ,  $\mathcal{V}_{\mathbf{q}}^-$  lies in a  $O\left(h^{\delta_0 \frac{\lambda_0}{\lambda_1}}\right)$  neighborhood of a stable leaves, as stated in the following lemma. In the following, we write

$$\delta_2 = \delta_0 \frac{\lambda_0}{\lambda_1} \tag{5-4.1}$$

We recall that we have defined  $\mathfrak{b}$  in (5-2.3) and  $\tau$  in (5-2.7) such that  $\alpha < \tau < 1$  and  $\delta_2 + \tau > 1$  (see 5-2.6). Moreover,  $N_0 = \lceil \frac{\delta_0}{\lambda_1} |\log h| \rceil$ .

**Lemma 5-4.1.** There exists a global constant  $C_2 > 0$  such that for all  $\mathbf{q} \in \mathcal{A}^{N_0}$  satisfying  $\mathcal{V}_{\mathbf{q}}^- \neq \emptyset$ ,

$$d(\mathcal{V}_{\mathbf{q}}^-, \mathcal{T}_-^{\text{loc}}) \leq C_2 h^{\delta_2}$$

#### Remark.

| In the end of this section, the use of  $C_2$  will always refer to the constant appearing in this lemma. | On other places, we keep our convention on global constants, noting them always  $C$ .

*Proof.* We already know by Lemma 5-3.4 that there exists  $C > 0$  such that if  $\mathcal{V}_{\mathbf{q}}^- \neq \emptyset$ , there exists  $\rho_0 \in \mathcal{T}$  such that

$$d(\mathcal{V}_{\mathbf{q}}^-, W_s(\rho_0)) \leq \frac{C}{J_{\mathbf{q}}}$$

But  $J_{\mathbf{q}}^- \geq e^{\lambda_0 N_0} \geq C^{-1} h^{-\delta_0 \frac{\lambda_0}{\lambda_1}}$ . Finally,  $d(\mathcal{V}_{\mathbf{q}}^-, \mathcal{T}_-^{\text{loc}}) \leq C h^{\delta_2}$ , as required.  $\square$

The following lemma allows us to construct symbols in nice symbol classes with supports in  $h^{\delta}$  neighborhood. Its proof can be found in [DZ16] (Lemma 3.3).



**Lemma 5-4.2.** Let  $\varepsilon > 0$  and  $\delta \in [0, \frac{1}{2}[$ . Let  $V_0(h) \subset V_1(h) \subset \mathbb{R}^d$  be sets depending on  $h$  and assume that for  $0 \leq h \leq 1$ ,  $d(V_0(h), V_1(h)^c) > \varepsilon h^\delta$ . Then, there exist a family  $\chi_h \in C_c^\infty(\mathbb{R}^d)$  such that, for all  $h \leq 1$ ,

- $\chi_h = 1$  on  $V_0(h)$  ;
- $\text{supp } \chi \subset V_1(h)$ .
- For every  $\alpha \in \mathbb{N}^d$ , there exists  $C_\alpha$  depending only on  $\varepsilon$  such that for all  $x \in \mathbb{R}^d$  and for all  $0 < h \leq 1$ ,

$$|\partial^\alpha \chi_h(x)| \leq C_\alpha h^{-\delta|\alpha|}$$

Applying this lemma with  $V_0(h) = \mathcal{T}_-^{loc}(2C_2 h^{\delta_2})$ ,  $V_1(h) = \mathcal{T}_-^{loc}(4C_2 h^{\delta_2})$  with  $\varepsilon = 2C_2$ , we consider a family of smooth cut-offs  $\chi_h \in S_{\delta_2}^{comp}$  and we can consider it as an element of  $S_{\delta_2}^{comp}(U)$  since (at least for  $h$  small enough) the support of  $\chi_h$  is included in  $U$ . We are now ready to state the microlocalization property of  $\mathfrak{M}^{N_0}$ .

**Proposition 5-4.1.**

$$\mathfrak{M}^{N_0} = \mathfrak{M}^{N_0} \text{Op}_h(\chi_h) + O(h^\infty)_{L^2(Y) \rightarrow L^2(Y)} \quad (5-4.2)$$

*Proof.* We need to show that  $\mathfrak{M}^{N_0}(\text{Op}_h(1 - \chi_h)) = O(h^\infty)$ . To do so, we decompose  $\mathfrak{M}^{N_0} = \sum_{\mathbf{q} \in \mathcal{A}^{N_0}} U_{\mathbf{q}}$ . Since the number of terms in this sum grows polynomially with  $h$ , it is enough to show that

$$\forall \mathbf{q} \in \mathcal{A}^{N_0}, U_{\mathbf{q}}(\text{Op}_h(1 - \chi_h)) = O(h^\infty)$$

with bounds uniform in  $\mathbf{q}$ . We then consider two cases :

- $\mathcal{V}_{\mathbf{q}}^- = \emptyset$  : Lemma 5-3.7 applies. Indeed, if  $m \leq N_0$  and  $\mathcal{V}_{q_0 \dots q_{m-1}}^- \neq \emptyset$ , we have

$$J_{q_0 \dots q_{m-1}}^- \leq e^{m\lambda_1} \leq e^{N_0\lambda_1} \leq Ch^{-\delta_0}$$

Hence,  $U_{\mathbf{q}} = O(h^\infty)$ , with global constants in the  $O(h^\infty)$ .

- $\mathcal{V}_{\mathbf{q}}^- \neq \emptyset$  : We apply Proposition 5-3.1. Since  $J_{\mathbf{q}}^- \leq Ce^{\lambda_1 N_0} \leq Ch^{-\delta_0}$ , we take some  $\delta_1 \in ]\delta_0, \frac{1}{2}[$  (in particular,  $\delta_2 < \delta_1$ ) and we can write  $U_{\mathbf{q}} = T^{N_0} \text{Op}_h(a_{\mathbf{q}}^-) + O(h^\infty)$  with  $a_{\mathbf{q}}^- \in S_{\delta_1}^{comp}(U)$  and  $\text{supp } a_{\mathbf{q}}^- \subset \mathcal{V}_{\mathbf{q}}^-$ . Noticing that  $\chi_h = 1$  on  $\mathcal{V}_{\mathbf{q}}^- \subset \mathcal{T}_-^{loc}(2C_2 h^{\delta_2})$ , the composition formula in  $S_{\delta_1}^{comp}$  implies that  $\text{Op}_h(a_{\mathbf{q}}^-) \text{Op}_h(1 - \chi_h) = O(h^\infty)$ . Since the seminorms of  $a_{\mathbf{q}}^-$  are uniformly bounded in  $\mathbf{q}$ , the constants appearing in  $O(h^\infty)$  are uniform in  $\mathbf{q}$ .

This concludes the proof.  $\square$

## 5-4.2. Propagation of Lagrangian leaves and Lagrangian states.

So as to study the microlocalization of  $U_{\mathbf{q}}$ , we'll use the same strategy as in [DJN21], themselves inspired by [AN07b], [AN07a] and [NZ09]. We cannot show that  $U_{\mathbf{q}}$  is a Fourier integral operator since the propagation goes behind the Ehrenfest time. Instead, we show a weaker result which will be enough for our purpose. The idea is to decompose a state  $u$  in a sum of Lagrangian states associated with Lagrangian leaves almost parallel to unstable leaves, what we will call horizontal leaves (because we will consider them in charts where the unstable leaves are close to be horizontal). Studying the precise behavior of these states, we can get fine information on the microlocalization of  $U_{\mathbf{q}}u$ . Roughly speaking, we'll show that if  $u$  is a Lagrangian state associated with an original horizontal Lagrangian  $\mathcal{L}_{q_0, \theta} \subset \mathcal{V}_{q_0}$ , then  $U_{\mathbf{q}}u$  is a Lagrangian state associated with the piece of the evolved Lagrangian  $F^n(\mathcal{L}_{q_0, \theta})$  inside  $\mathcal{V}_{\mathbf{q}}^+$ .

To define "horizontal" Lagrangian leaves, we need to work in adapted coordinate charts in which the notion of horizontality (thinking  $W_u(\rho)$  as the reference) makes sens. For this purpose, for  $q \in \mathcal{A}$ , we consider charts centered around the points  $\rho_q$ , associated with the fixed macroscopic partition of  $\mathcal{T}$  by the  $\mathcal{V}_q = B(\rho_q, 2\varepsilon_0)$ . First, we consider symplectic maps

$$\kappa_q : W_q \subset U_{k_q} \rightarrow V_q \subset \mathbb{R}^2$$

satisfying (we note  $(x, \xi)$  the variable in  $U$  and  $(y, \eta)$  in  $\mathbb{R}^2$ ) :

- (1)  $B(\rho_q, C\varepsilon_0) \subset W_q$  for some global constant  $C \gg 2$ ;
- (2)  $\kappa(\rho_q) = 0$ ,  $d\kappa(\rho_q)(E_u(\rho_q)) = \mathbb{R} \times \{0\}$ ;  $d\kappa(\rho_q)(E_s(\rho_q)) = \{0\} \times \mathbb{R}$  ;
- (3) The image of the unstable leaf  $W_u(\rho_q)$  is exactly  $\{(y, 0), y \in \mathbb{R}\} \cap \tilde{V}_q$ .

Theses charts are for instance given by Lemma 3-4.1 (at this stage, the strong straightening property is not necessary). In these adapted charts where  $W_u(\rho_q)$  coincides with  $\mathbb{R} \times \{0\}$ , the horizontal Lagrangian leaves will be the of the form

$$\mathcal{C}_\theta := \{(y, \theta), y \in \mathbb{R}\} \quad (5-4.3)$$

Finally, we fix unit vectors on  $E_u(\rho_q)$  and  $E_s(\rho_q)$ ,  $e_u(\rho_q)$  and  $e_s(\rho_q)$ , used to defined the unstable and stable Jacobians in section 3-2. Let's write

$$d\kappa_q(e_u(\rho_q)) = (\lambda_{q,u}, 0) \quad ; \quad d\kappa_q(e_s(\rho_q)) = (0, \lambda_{q,s})$$

Note  $D_q = \begin{pmatrix} \lambda_{q,u} & 0 \\ 0 & \lambda_{q,s} \end{pmatrix}$ . We dilate the chart  $\tilde{\kappa}_q$  and define

$$\tilde{\kappa}_q : \rho \in W_q \mapsto D_q \kappa_q(\rho) \in \tilde{V}_q := D_q(V_q)$$

#### 5-4.2.1. Horizontal Lagrangian and their evolution.

Let us fix a word  $\mathbf{q} \in \mathcal{A}^n$  and let us define

$$\mathcal{L}_{q_0, \theta} = \kappa_{q_0}^{-1}(\mathcal{C}_\theta \cap V_{q_0}) \cap \mathcal{V}_{q_0} \quad (5-4.4)$$

Then, let's define inductively

$$\mathcal{L}_{q_0 \dots q_j, \theta} = F(\mathcal{L}_{q_0 \dots q_{j-1}, \theta}) \cap \mathcal{V}_{q_j} \quad (5-4.5)$$

which allows to define  $\mathcal{L}_{\mathbf{q}, \theta}$ . One can check that

$$\mathcal{L}_{\mathbf{q}, \theta} = F^{-1}(\mathcal{V}_{\mathbf{q}}^+) \cap F^{n-1}(\mathcal{L}_{q_0, \theta}) \quad (5-4.6)$$

The term  $F^{-1}$  comes from the definition of  $\mathcal{V}_{\mathbf{q}}^+$  :

$$\rho \in \mathcal{V}_{\mathbf{q}}^+ \iff \forall 1 \leq i \leq n, F^{-i}(\rho) \in \mathcal{V}_{q_{n-i}}$$

Finally, let's define

$$\mathcal{C}_{\mathbf{q}, \theta} = \kappa_{q_{n-1}}(\mathcal{L}_{\mathbf{q}, \theta}) \quad (5-4.7)$$

We first focus on one step of the iterative process.

In  $\tilde{V}_q \subset \mathbb{R}^2$ , we use the notations  $\tilde{B}_q(0, r)$  for the cube  $]-r, r[ \times ]-r, r[$ . We keep the subscript  $q$  to keep trace of the chart in which this cube is supposed to live. Finally, we set

$$B_q(0, r) = D_q^{-1}(\tilde{B}_q(0, r)) \subset V_q$$

$B_q(0, r)$  is simply a rectangle centered at zero with size only depending on  $q$  (this is also a ball for some norm in  $\mathbb{R}^2$ ). The advantage of  $\tilde{B}_q$  and  $\tilde{\kappa}_q$  compared with  $B_q$  and  $\kappa_q$  will appear below. However,  $\tilde{\kappa}_q$  is not symplectic, and for further use, it is not possible to use  $\tilde{\kappa}_q$  as a symplectic change of coordinates.

Let  $q, p \in \mathcal{A}$  and suppose that  $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) \neq \emptyset$ . As a consequence there exists a global constant  $C' > 0$  such that  $d(F(\rho_q), \rho_p) \leq C'\varepsilon_0$  and if  $C$  in (1) of Lemma 3-4.1 is large enough, we can assume that for some global constant  $C_1 > 0$ ,

$$\kappa_q(\mathcal{V}_q) \subset B_q(0, C_1\varepsilon_0) \subset V_q \quad \kappa_p \circ F \circ \kappa_q^{-1}(B_q(0, C_1\varepsilon_0)) \subset V_p \quad (5-4.8)$$

The following map is hence well defined

$$\tau_{p,q} := \kappa_p \circ F \circ \kappa_q^{-1} : B_q(0, C_1\varepsilon_0) \rightarrow \tau_{p,q}(B_q(0, C_1\varepsilon_0)) \subset V_p$$

$\tau_{p,q}$  is nothing but the writing of  $F$  between the charts  $V_q$  and  $V_p$ . Note that since the number of possible transitions is finite, we can assume that  $C_1$  is uniform for all  $q, p \in \mathcal{A}$  such that  $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) \neq \emptyset$ .

We also adopt the following definitions and notations :

**Definition 5-4.1.** Let  $G_q : ] - C_1\varepsilon_0, C_1\varepsilon_0[ \rightarrow ] - C_1\varepsilon_0, C_1\varepsilon_0[$  be a smooth map. It represents the horizontal Lagrangian

$$\mathcal{L}_{G_q} := D_q^{-1}(\{(y, G_q(y), y) \in ] - C_1\varepsilon_0, C_1\varepsilon_0[\}) \subset B_q(0, C_1\varepsilon_0) \subset V_q$$

We say that such a Lagrangian lies in the  $\gamma$ -unstable cone if

$$\|G'_q\|_\infty \leq \gamma$$

and we note  $G_q \in \mathcal{C}_q^u(C_1\varepsilon_0, \gamma)$ .

**Remark.**

This is where the use of  $\tilde{\kappa}_q$  and  $\tilde{B}_q$  turns out to be useful : to represent horizontal Lagrangian in  $V_q$ , we use the cube  $\tilde{B}_q(0, C_1\varepsilon_0) \subset \tilde{V}_q$  of fixed size.

With this definition, we show in the following lemma an invariance property of the  $\gamma$ -unstable cones :

**Lemma 5-4.3.** There exist global constants  $C > 0, C_1 > 0$  such that if  $\varepsilon_0$  is sufficiently small, then the following holds.

For every  $G_q \in \mathcal{C}_q^u(C_1\varepsilon_0, C\varepsilon_0)$ , there exists  $G_p \in \mathcal{C}_p^u(C_1\varepsilon_0, C\varepsilon_0)$  such that

- (i)  $\tau_{p,q}(\mathcal{L}_{G_q}) \cap B_p(0, C_1\varepsilon_0) = \mathcal{L}_{G_p}$  ;
- (ii) For some global constants  $C_l, l \geq 2$ ,  $\|G_q\|_{C^l} \leq C_l \implies \|G_p\|_{C^l} \leq C_l$  ;

Moreover, let's define  $\phi_{qp} : ] - C_1\varepsilon_0, C_1\varepsilon_0[ \rightarrow \mathbb{R}$  by

$$y_q = \phi_{qp}(y_p) \iff (y_p, G_p(y_p)) = D_p \circ \tau_{pq} \circ D_q^{-1}((\phi_{qp}(y_p), G_q \circ \phi_{qp}(y_p)))$$

Then,  $\phi_{pq}$  is smooth contracting diffeomorphism onto its image. In particular, there exists a global constant  $\nu < 1$  such that  $\|\phi'_{pq}\|_\infty \leq \nu$ .

*Proof.* Take  $C_1$  large but fixed (with conditions further imposed) and assume that  $\varepsilon_0$  is small enough so that (5-4.8) holds. Let us note  $\lambda_q = J_1^u(\rho_q) > 1$  and  $\mu_q = J_1^s(\rho_q) < 1$  and let us fix some global  $\nu$  satisfying

$$\forall q \in \mathcal{A}, \max(\lambda_q^{-1}, \mu_q) < \nu < 1$$

Recall that  $e_u$  and  $e_s$  are  $C^{1,\varepsilon}$  in  $\rho$ . We write  $\partial_y$  and  $\partial_\eta$  to denote the unit vector of  $\mathbb{R} \times \{0\}$  and  $\{0\} \times \mathbb{R}$  respectively. We fix a constant  $C > 0$  with conditions imposed further and we assume that  $\|G'_p\|_\infty \leq C\varepsilon_0$ . We note  $\tilde{\tau} = D_p \circ \tau_{p,q} \circ D_q^{-1}$  (we drop the subscript for  $\tilde{\tau}$  to alleviate the notations). In the computations below, the implied constants in the  $O$  are global constants (depending also on the choices on  $\kappa_q$ ):

- \*  $\tilde{\tau}(0) = \tilde{\kappa}_p \circ F(\rho_q) = O(\varepsilon_0)$ ;
- \*  $d\tilde{\tau}(0) = d\tilde{\kappa}_p(F(\rho_q)) \circ dF(\rho_q) \circ [d\tilde{\kappa}_q(\rho_q)]^{-1}$  ;
- \*  $d\tilde{\tau}(0)(\partial_y) = d\tilde{\kappa}_p(F(\rho_q))(\lambda_q e_u(F(\rho_q))) = \lambda_q (d\tilde{\kappa}_p(\rho_p) + O(\varepsilon_0))(e_u(\rho_p) + O(\varepsilon_0)) = \lambda_q \partial_y + O(\varepsilon_0)$ , where we use the Lipschitz regularity of  $\rho \mapsto e_u(\rho)$  in the second equality ;
- \* Similarly,  $d\tilde{\tau}(0)(\partial_\eta) = \mu_q \partial_\eta + O(\varepsilon_0)$ ;

(this is here that we use the renormalization of  $\kappa_q$  into  $\tilde{\kappa}_q$ ). Eventually, we use the fact that  $\tilde{\tau} - \tilde{\tau}(0) - d\tilde{\tau}(0) = O(C_1\varepsilon_0)_{C^1(B(0, C_1\varepsilon_0))}$  and we get that

$$\tilde{\tau}(y, \eta) = (\lambda_q y + y_r(y, \eta), \mu_q \eta + \eta_r(y, \eta)), (y, \eta) \in \tilde{B}_q(0, C_1\varepsilon_0) \quad (5-4.9)$$

where  $y_r(y, \eta)$  and  $\eta_r(y, \eta)$  are  $O(C_1\varepsilon_0)_{C^1}$ . Before going further, let us show that we can fix  $C_1$  such that

$$(y, \eta) \in \tilde{B}_q(0, C_1\varepsilon_0) \implies |\mu_q \eta + \eta_r(y, \eta)| \leq C_1\varepsilon_0 \quad (5-4.10)$$

To do so, let us note that in fact  $\tilde{\tau} - \tilde{\tau}(0) - d\tilde{\tau}(0) = O((C_1\varepsilon_0)^2)_{C^0(B(0, C_1\varepsilon_0))}$  and hence if  $(y, \eta) \in \tilde{B}_q(0, C_1\varepsilon_0)$  we have :

$$|\eta_r(y, \eta)| = O(\varepsilon_0) + O((C_1\varepsilon_0)^2)_{C^0(B(0, C_1\varepsilon_0))} \leq C'\varepsilon_0(1 + C_1^2\varepsilon_0)$$

Assume that  $C_1$  is large enough such that  $\nu C_1 + C' < C_1 \frac{\nu+1}{2}$ . If  $(y, \eta) \in \tilde{B}_q(0, C_1\varepsilon_0)$ , we have

$$|\mu_q\eta + \eta_r(y, \eta)| \leq \nu C_1\varepsilon_0 + C'\varepsilon_0(1 + C_1^2\varepsilon_0) \leq \left(C_1 \frac{\nu+1}{2} + C_1^2\varepsilon_0\right)\varepsilon_0$$

This fixes  $C_1$ . Since  $C_1$  is now a global fixed parameter, we can remove it from the  $O$  in the estimates. If  $\varepsilon_0$  is small enough, depending on our choice of  $C_1$ , (5-4.10) holds.

To write the image of the leaf as a graph, we observe that, if  $\varepsilon_0$  is small enough (depending only on global parameters) the map

$$\psi : y \in ] - C_1\varepsilon_0, C_1\varepsilon_0[ \mapsto \lambda_q y + y_r(y, G_q(y))$$

is expanding and we can impose  $|\psi'| \geq \nu^{-1}$ . In particular,  $\text{Im } \psi$  contains an interval of size  $2\nu^{-1}C_1\varepsilon_0$ . Moreover,  $\psi(0) = y_r(0, G_q(0)) \leq \|y_r\|_{C^1} |G_q(0)| = O(\varepsilon_0^2)$ . We claim that if  $\varepsilon_0$  is small enough,  $\text{Im } \psi$  contains  $] - C_1\varepsilon_0, C_1\varepsilon_0[$ . Indeed, it suffices to have

$$\nu^{-1}C_1\varepsilon_0 - |\psi(0)| \geq C_1\varepsilon_0$$

But we have

$$C_1\varepsilon_0 + |\psi(0)| \leq C_1\varepsilon_0(1 + O(\varepsilon_0)) \leq C_1\varepsilon_0\nu^{-1}$$

if  $1 + O(\varepsilon_0) \leq \nu^{-1}$ , condition that can be satisfied if  $\varepsilon_0$  is small enough. Hence,  $\phi := \phi_{pq} = \psi|_{]-C_1\varepsilon_0, C_1\varepsilon_0[}$  is well defined and we set

$$G_p(y) = \mu_q G_q(\phi(y)) + \eta_r(\phi(y), G_q(\phi(y))), y \in ] - C_1\varepsilon_0, C_1\varepsilon_0[ \quad (5-4.11)$$

By definition, it is clear that  $\tau_{p,q}(\mathcal{L}_{G_q}) \cap B_p(0, C_1\varepsilon_0) = \mathcal{L}_{G_p}$  and  $(y, G_p(y)) = \tilde{\tau}(\phi(y), G_q(\phi(y)))$ .  $\phi$  is obviously a smooth contracting diffeomorphism and  $\|\phi'\| \leq \frac{1}{\inf|\psi'(y)|} \leq \nu$ . Moreover, due to (5-4.10),  $|G_p(y)| \leq C_1\varepsilon_0$ . To prove that  $G_p \in \mathcal{C}_p^u(C_1\varepsilon_0, C\varepsilon_0)$ , we compute :

$$\begin{aligned} G'_p(y) &= \mu_q G'_q(\phi(y)) \times \phi'(y) + (\partial_y \eta_r + \partial_\eta \eta_r \times G'_q(\phi(y))) \phi'(y) \\ |G'_p(y)| &\leq \nu^2 C\varepsilon_0 + O(\varepsilon_0(1 + C\varepsilon_0))\nu \leq [\nu^2 C + \nu C'(1 + C\varepsilon_0)]\varepsilon_0 \end{aligned}$$

for some global  $C' > 0$ . If we assume  $\nu^2 + \varepsilon_0 C'\nu < 1$ , which is possible if  $\varepsilon_0$  is small enough, then we can choose  $C$  large enough satisfying

$$C \times (\nu^2 + \nu C'\varepsilon_0) + \nu C' \leq C$$

This ensures that  $\|G'_p\|_\infty \leq C\varepsilon_0$ .

Finally, we prove (ii) by induction on  $l$  : the case  $l = 1$  is done. Assume that there exists a constant  $C_l$  such that  $\|G_q\|_{C^l} \leq C_l \implies \|G_p\|_{C^l} \leq C_l$ . We want to find a constant  $C_{l+1}$  fitting for the  $C^{l+1}$  norm. Using (5-4.11), we see by induction that the  $(l+1)$  derivatives of  $G_p$  has the form

$$G_p^{(l+1)}(y) = \phi'(y)^{l+1} \times G_q^{(l+1)}(y) \times \left(1 + \partial_\eta \eta_r(y, \phi(y))\right) + P_y \left(G_q(y), \dots, G_q^{(l)}(y)\right)$$

where  $P_y(\tau_0, \dots, \tau_l)$  is a polynomial with smooth coefficients in  $y$ . Hence, there exists a constant  $M(C_l)$  such that for  $y \in ] - C_1\varepsilon_0, C_1\varepsilon_0[$ ,  $\left|P_y \left(G_q(y), \dots, G_q^{(l)}(y)\right)\right| \leq M(C_l)$ . Since

$$\left|\phi'(y)^{l+1} \left(1 + \partial_\eta \eta_r(y, \phi(y))\right)\right| \leq \nu(1 + \varepsilon_0 C') := \nu_1$$

if  $\varepsilon_0$  is small enough ensuring that  $\nu_1 < 1$ , we can take

$$C_{l+1} = \max \left( C_l, \frac{M(C_l)}{1 - \nu_1} \right)$$

Indeed, with such a constant, assuming that  $\|G_q\|_{C^{l+1}} \leq C_{l+1}$ , we have

$$|G_p^{(l+1)}(y)| \leq C_{l+1}\nu_1 + M(C_l) \leq C_{l+1}$$

□

Armed with this lemma, we can now iterate the process and get the following proposition describing the evolution of the Lagrangian  $\mathcal{C}_{\mathbf{q},\theta}$ .

**Proposition 5-4.2.** Assume that  $\varepsilon_0$  is small enough. Then, for every  $n \in \mathbb{N}^*$ ,  $\mathbf{q} \in \mathcal{A}^n$ , and  $\theta \in \mathbb{R}$ , there exists an open subset  $I_{\mathbf{q},\theta} \subset \mathbb{R}$  and a smooth map  $G_{\mathbf{q},\theta}$  such that :

- $\mathcal{C}_{\mathbf{q},\theta} = \left\{ (y, G_{\mathbf{q},\theta}(y)), y \in I_{\mathbf{q},\theta} \right\}$  ;
- $\|G'_{\mathbf{q},\theta}\|_{\infty} \leq C\varepsilon_0$  for some global constant  $C$ ;
- For every  $l \geq 2$ ,  $\|G_{\mathbf{q},\theta}\|_{C^l} \leq C_l$  for some global  $C_l$ ;
- If  $\phi_{\mathbf{q},\theta} : I_{\mathbf{q},\theta} \rightarrow \mathbb{R}$  is defined by

$$\kappa_{q_{n-1}} \circ F^{n-1} \circ \kappa_{q_0}^{-1} (\phi_{\mathbf{q},\theta}(y), \theta) = (y, G_{\mathbf{q},\theta}(y))$$

Then, for some global constants  $C > 0$  and  $0 < \nu < 1$ ,  $\|\phi'_{\mathbf{q},\theta}\| \leq C\nu^{n-1}$ .

*Proof.* Assume that  $\mathcal{L}_{\mathbf{q},\theta} \neq \emptyset$ , otherwise, there is nothing to prove. In particular, we can restrict our attention to small  $\theta$ ,  $|\theta| \leq C_1\varepsilon_0$ . As a consequence, for every  $i \in \{1, \dots, n\}$ ,  $F(\mathcal{V}_{q_{i-1}}) \cap \mathcal{V}_{q_i} \neq \emptyset$ . Hence, we can consider the maps  $\tau_i := \tau_{q_i, q_{i-1}}$  and since we assume that  $\kappa_{q_i}(\mathcal{V}_{q_i}) \subset B_{q_i}(0, C_1\varepsilon_0)$ ,

$$C_{q_0 \dots q_i, \theta} = \tau_i (C_{q_0 \dots q_{i-1}, \theta}) \cap \kappa_{q_i}(\mathcal{V}_{q_i})$$

We start with a constant function  $G_0 \in C_0^u(C_1\varepsilon_0, 0)$  such that  $\mathcal{L}_{G_0} = \mathcal{C}_\theta$  (it suffices to take  $G_0 = \lambda_{q_0, s}\theta$ ) and we inductively apply the previous lemma to show the existence of a family  $G_j \in C_j^u(C_1\varepsilon_0, C\varepsilon_0)$ ,  $0 \leq j \leq n-1$ , such that

- (i)  $\tau_i(\mathcal{L}_{G_i}) \cap B_{q_i}(0, C_1\varepsilon_0) = \mathcal{L}_{G_{i+1}}$  ;
- (ii)  $\|G_i\|_{C^l} \leq C_l$ ;
- (iii) If we define  $\phi_i : ]-C_1\varepsilon_0, C_1\varepsilon_0[ \rightarrow ]-C_1\varepsilon_0, C_1\varepsilon_0[$  by

$$(y, G_i(y)) = D_{q_i} \circ \tau_i \circ D_{q_{i-1}}^{-1} (\phi_i(y), G_{i-1} \circ \phi_i(y))$$

then there exists  $\nu < 1$  such that  $\|\phi'_i\|_{\infty} \leq \nu$ .

- (iv)  $C_{q_0 \dots q_i, \theta}$  is an open subset of  $\mathcal{L}_{G_i}$ .

We have

$$\mathcal{L}_{G_{n-1}} = D_{q_{n-1}}^{-1} \left( \{(y, G_{n-1}(y)), y \in ]-C_1\varepsilon_0, C_1\varepsilon_0[\} \right)$$

This can be also written

$$\mathcal{L}_{G_{n-1}} = \left\{ \left( y, \lambda_{q_{n-1}, s}^{-1} G_{n-1}(\lambda_{q_{n-1}, u} y) \right), |y| < \lambda_{q_{n-1}, u}^{-1} C_1\varepsilon_0 \right\}$$

It suffices to consider

$$\begin{aligned} G_{\mathbf{q},\theta}(y) &= \lambda_{q_{n-1}, s}^{-1} G_{n-1}(\lambda_{q_{n-1}, u} y) \\ I_{\mathbf{q},\theta} &= \left\{ y \in ]-\lambda_{q_{n-1}, u}^{-1} C_1\varepsilon_0, \lambda_{q_{n-1}, u}^{-1} C_1\varepsilon_0[, (y, G_{\mathbf{q},\theta}(y)) \in \mathcal{C}_{\mathbf{q},\theta} \right\} \\ \phi_{\mathbf{q},\theta}(y) &= \lambda_{q_1, u}^{-1} \phi_1 \circ \dots \circ \phi_{n-1}(\lambda_{q_{n-1}, u} y) \end{aligned}$$

□

### 5-4.2.2. Evolution of Lagrangian states.

Once we've studied the evolution of the Lagrangian leaves starting from  $\mathcal{C}_\theta$ , we can study the evolution of the corresponding Lagrangian states. In our case, since the leaves stay rather horizontal, the form of the Lagrangian states we'll consider is the simplest :

$$a(x)e^{i\psi(x)/h}$$

where  $a$  is an amplitude and  $\psi$  a generating phase function. It is associated with the Lagrangian,

$$\mathcal{L} = \{(y, \psi'(y)), y \in \text{supp } a\}$$

For  $q \in \mathcal{A}$ , we quantize  $\kappa_q$ . Remind that we denoted  $k_q$  the integer such that  $\mathcal{V}_q \Subset U_{k_q}$ . There exist Fourier integral operators  $B_q, B'_q \in I_0^{\text{comp}}(\kappa_q) \times I_0^{\text{comp}}(\kappa_q^{-1})$ ,

$$\begin{aligned} B_q &: L^2(Y_{k_q}) \rightarrow L^2(\mathbb{R}); \\ B'_q &: L^2(\mathbb{R}) \rightarrow L^2(Y_{k_q}) \end{aligned}$$

such that they quantize  $\kappa_q$  in a neighborhood of  $\kappa_q(\overline{\mathcal{V}_q}) \times \overline{\mathcal{V}_q}$ . Moreover, we impose that  $\text{WF}_h(B_q B'_q)$  is a compact subset of  $\mathbb{R}^2$ . We will still denoted  $B_q$  and  $B'_q$  the operators

$$B_q = (0, \dots, \underbrace{B_q}_{k_q}, \dots, 0) : L^2(Y) \rightarrow L^2(\mathbb{R}) \quad ; \quad B'_q = {}^t(0, \dots, \underbrace{B'_q}_{k_q}, \dots, 0) : L^2(\mathbb{R}) \rightarrow L^2(Y)$$

If  $\text{supp}(c_q) \subset \mathcal{V}_q$  and if  $C$  denotes the operator valued matrix with only one non zero entry  $\text{Op}_h(c_q)$  in position  $(k_q, k_q)$ , then as operators  $L^2(Y) \rightarrow L^2(Y)$ ,

$$B'_q B_q C = C + O(h^\infty) \quad ; \quad C B'_q B_q = C + O(h^\infty)$$

The proposition we aim at proving in the following :

**Proposition 5-4.3.** Fix  $C_0 > 0$ . For every  $n \in \mathbb{N}$ ,  $\mathbf{q} \in \mathcal{A}^n$  and  $\theta \in \mathbb{R}$  satisfying

$$n \leq C_0 |\log h| \quad ; \quad |\theta| \leq C_0 \tag{5-4.12}$$

and for every  $N \in \mathbb{N}$ , there exists a symbol  $a_{\mathbf{q}, \theta, N} \in C_c^\infty(I_{\mathbf{q}, \theta})$  such that :

$$(i) \quad U_{\mathbf{q}} \left( B'_{q_0} e^{i \frac{\theta}{h}} \right) = M A_{q_{n-1}} B'_{q_{n-1}} \left( e^{i \frac{\psi_{\mathbf{q}}}{h}} a_{\mathbf{q}, \theta, N} \right) + O(h^N)_{L^2}$$

$$(ii) \quad \|a_{\mathbf{q}, \theta, N}\|_{C_l} \leq C_{l, N} h^{-C_0 \log B}$$

$$(iii) \quad \text{There exists } \delta > 0 \text{ such that } d(\text{supp}(a_{\mathbf{q}, \theta, N}), \mathbb{R} \setminus I_{\mathbf{q}, N, \theta}) \geq \delta$$

where  $\psi_{\mathbf{q}, \theta}$  is a primitive of  $G_{\mathbf{q}, \theta}$  and  $B > 0$  is a global constant.

#### Remark.

- As usual,  $\delta, C_{l, N}$  and  $C_N$  depend only on  $F, A_q, B_q, B'_q, \kappa_q$  and the indices indicated in their notations.
- In other words, the Lagrangian state  $e^{i \frac{\theta}{h}}$  is changed to a Lagrangian state associated with  $\mathcal{C}_{\mathbf{q}, \theta}$ .

The end of this subsection is devoted to the proof of Proposition 5-4.3. In the rest of this section, we fix a constant  $C_0 > 0$  and we work with a fixed word  $\mathbf{q} \in \mathcal{A}^n$  with length  $n \leq C_0 |\log h|$  and a fixed momentum  $|\theta| \leq C_0$ . From now on and until the end of the proof, the constants below will always be uniform in  $\mathbf{q}, \theta$  satisfying the previous assumption. They will depend on global parameters and on  $C_0$ . If they depend on other parameters, we will specify it with subscripts. This is also the case for implicit constants in  $O$  (such as in  $O(h^\infty)$ ).

**Preparatory work.** We first note the following fact : if  $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) = \emptyset$ ,  $A_p M A_q = O(h^\infty)$ . As a consequence, if  $\mathcal{V}_{q_{i-1}} \cap F^{-1}(\mathcal{V}_{q_i}) = \emptyset$  for some  $i$ , then  $U_{\mathbf{q}} = O(h^\infty)$ . In the sequel, it is enough to consider words  $\mathbf{q}$  for which  $\mathcal{V}_{q_{i-1}} \cap F^{-1}(\mathcal{V}_{q_i}) \neq \emptyset$  for  $1 \leq i \leq n-1$ .

We consider symbols  $\tilde{a}_q$  such that  $\text{supp}(\tilde{a}_q) \subset \mathcal{V}_q$  and  $\tilde{a}_q \equiv 1$  on  $\text{supp}(\chi_q)$ . We denote  $\tilde{A}_q = \text{Op}_h(\tilde{a}_q)$  (as usual thought as a diagonal operator valued matrix). The following computations holds since  $n = O(\log h)$  and  $\|M A_q\| \leq \|\alpha\|_\infty + o(1)$  uniformly in  $q$  :

$$\begin{aligned} U_{\mathbf{q}} B'_{q_0} &= M A_{q_{n-1}} \tilde{A}_{q_{n-1}} M A_{q_{n-2}} \tilde{A}_{q_{n-2}} \cdots M A_{q_1} \tilde{A}_{q_1} M A_{q_0} B'_{q_0} + O(h^\infty) \\ &= M A_{q_{n-1}} B'_{q_{n-1}} B_{q_{n-1}} \tilde{A}_{q_{n-1}} M \cdots M A_{q_1} B'_{q_1} B_{q_1} \tilde{A}_{q_1} M A_{q_0} B'_{q_0} + O(h^\infty) \end{aligned}$$

We set  $T_{p,q} = B_p \tilde{A}_p M A_q B'_q$  and  $M_q = M A_q B'_q$ , which allows us to write

$$U_{\mathbf{q}} B'_{q_0} = M_{q_{n-1}} T_{q_{n-1}, q_{n-2}} \cdots T_{q_1, q_0} + O(h^\infty)$$

For  $p, q \in \mathcal{A}$  with  $\mathcal{V}_q \cap F^{-1}(\mathcal{V}_p) \neq \emptyset$ ,  $T_{q,p} \in I_0^{\text{comp}}(\tau_{p,q})$ . Moreover, the previous computations have shown that  $\tau_{p,q}$  has the form

$$\tau_{p,q}(y, \eta) = (\lambda_{p,q} y + y_r(y, \eta), \mu_{p,q} \eta + \eta_r(y, \eta)), (y, \eta) \in B_q(0, C_1 \varepsilon_0)$$

where  $y_r(y, \eta)$  and  $\eta_r(y, \eta)$  are  $O(\varepsilon_0)_{C^1}$ . This time,  $\lambda_{p,q}, \mu_{p,q}$  are simply constants uniformly bounded from below and from above for  $p, q \in \mathcal{A}$  (recall that  $B_q(0, C_1 \varepsilon_0)$  is a rectangle in  $\mathbb{R}^2$ , built from the cube  $\tilde{B}_q(0, C_1 \varepsilon_0)$  adapted to the definition of the unstable Jacobian). If  $\varepsilon_0$  small enough, the projection  $\pi : (y, \eta, x, \xi) \in \mathcal{L}_{q,p} \mapsto (y, \xi) \in \mathbb{R}^2$  is a diffeomorphism onto its image. where

$$\mathcal{L}_{q,p} = \left\{ (\tau_{q,p}(x, \xi), x, -\xi), (x, \xi) \in B_q(0, C_1 \varepsilon_0) \right\}$$

is the twisted graph of  $\tau_{p,q}$ . As a consequence, there exists a smooth phase function  $S_{p,q}$  defined in an open set  $\Omega_{p,q}$  of  $\mathbb{R}^2$ , generating  $\mathcal{L}_{p,q}$  locally i.e.

$$\mathcal{L}_{p,q} \cap \tau_{p,q}(B_q(0, C_1 \varepsilon_0)) \times B_q(0, C_1 \varepsilon_0) = \left\{ (y, \partial_y S_{p,q}(y, \xi), \partial_\xi S_{p,q}(y, \xi), -\xi), (y, \xi) \in \Omega_{q,p} \right\}$$

Hence,  $T_{p,q}$  can be written in the following form, up to a  $O(h^\infty)$  remainder and for some symbol  $\alpha_{p,q}(\cdot; h) \in C_c^\infty(\Omega_{p,q})$ :

$$T_{p,q} u(y) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(S_{p,q}(y, \xi) - x\xi)} \alpha_{p,q}(y, \xi; h) u(x) dx d\xi \quad (5-4.13)$$

Moreover, due to the operators  $\tilde{A}_p$  and  $A_q$  in the definition of  $T_{p,q}$ , we can assume that

$$(y, \xi) \in \text{supp}(\alpha_{p,q}) \implies (\partial_\xi S_{p,q}(y, \xi), \xi) \in \kappa_q(\text{supp } a_q), (y, \partial_y S_{p,q}(y, \xi)) \in \kappa_p(\text{supp } \tilde{a}_p)$$

In the sequel, we write

$$\mathcal{C}_i = \mathcal{C}_{q_0 \dots q_i, \theta}$$

and we change the subscripts  $(q_{i-1}, q_i)$  to  $i$  in all the objects  $T, \alpha, S, \tau$ . Due to the previous results, we can write  $\mathcal{C}_i = \left\{ (y, G_i(y)), y \in I_i \right\}$  with  $I_i := I_{q_0 \dots q_i, \theta}$  and  $G_i := G_{q_0 \dots q_i, \theta}$ . We also have projection maps  $\Phi_i : I_i \rightarrow \mathbb{R}$  defined by :

$$\tau_i \circ \cdots \circ \tau_1(\Phi_i(y), \theta) = (y, G_i(y))$$

satisfying  $\|\Phi_i'\|_\infty \leq C\nu^i < 1$ . Moreover, if we note the intermediate corresponding projection  $\phi_i := \Phi_i \circ \Phi_{i-1}^{-1} : I_i \rightarrow I_{i-1}$ , we observe that  $\phi_i$  is constructed using the properties of  $F$  and  $G_{i-1}$  (see the proof of Lemma 5-4.2) and hence, for every  $l$ ,  $\|\phi_i\|_{C^l} \leq C_l$  for some  $C_l$  not depending on  $\mathbf{q}, \theta$  nor  $i$ .

For  $0 \leq i \leq n-1$ , we consider a primitive  $\psi_i$  of  $G_i$  so that  $\mathcal{C}_i$  is generated by  $\psi_i$  i.e.

$$\mathcal{C}_i = \left\{ (y, \psi'_i(y), y \in I_i) \right\}$$

The following lemma can be found in [NZ09] (Lemma 4.1). We state it without proof, since it is the reference but it is a direct application of the stationary phase theorem.

**Lemma 5-4.4.** Pick  $i \in \{1, \dots, n-1\}$ .

For any  $a \in C_c^\infty(I_{i-1})$ , the application of  $T_i$  to the Lagrangian state  $ae^{i\frac{\psi_{i-1}}{h}}$  associated with  $\mathcal{C}_{i-1}$  gives a Lagrangian state associated with  $\mathcal{C}_i$  and satisfies

$$T_i \left( ae^{i\frac{\psi_{i-1}}{h}} \right) (y) = e^{i\frac{\beta_i}{h}} e^{i\frac{\psi_i(y)}{h}} \left( \sum_{j=0}^{N-1} b_j(y) h^j + h^N r_N(y; h) \right) \quad (5-4.14)$$

where, if we note  $x = \phi_i(y)$ ,  $b_j(y) = (L_{j,i}(x, D_x)a)(x)$  for some differential operator  $L_{j,i}$  of order  $2j$  with smooth coefficients supported in  $I_{i-1}$  and  $\beta_i \in \mathbb{R}$ . Moreover, one have :

- $b_0(y) = \frac{\alpha_i(y, \xi)}{|\det D_{y, \xi}^2 S_i(y, \xi)|^{1/2}} |\phi'_i(y)|^{1/2} a(x)$  with  $\xi = \psi'_{i-1}(x)$ ;
- $\|b_j\|_{C^l(I_i)} \leq C_{l,j} \|a\|_{C^{l+2j}(I_{i-1})}$ ,  $l \in \mathbb{N}$ ,  $0 \leq j \leq N-1$  ;
- $\|r_N\|_{C^l(I_i)} \leq C_N \|a\|_{C^{l+1+2N}(I_{i-1})}$

The constants  $C_N$  and  $C_{l,j}$  depend on  $\tau_i, \alpha_i, \|\psi_i^{(m)}\|_{\infty, I_i}$ .

**Remark.**

- In particular, in virtue of Proposition 5-4.2, the constant  $C_{l,j}$  and  $C_N$  can be chosen uniform in  $\mathbf{q}, \theta$  as soon as they satisfy the required assumptions.  $|q| \leq C_0 |\log h|, \theta \leq C_0$ .
- Without loss of generality, we can replace  $\psi_i$  by  $\beta_i + \psi_i$  (this actually corresponds to fixing an antiderivative on  $\mathcal{C}_{i+1}$ ) and hence we can assume that  $\beta_i = 0$ .
- The properties on the support of  $\alpha_i$  imply the following ones on the support of the differential operators  $L_{j,i}$  :

$$y \in \text{supp } L_{j,i} \implies (y, \psi'_i(y)) \in \kappa_{q_i}(\text{supp } \tilde{a}_{q_i}) \cap \tau_{i-1} \circ \kappa_{q_{i-1}}(\text{supp } a_{q_{i-1}}) \quad (5-4.15)$$

**Iteration formulas and analysis of the symbols** Then, we iterate this lemma starting from  $\psi_0(x) = x \cdot \theta$ , in the spirit of Proposition 4.1 in [NZ09]. In the sequel, we adopt the following convention : we note  $x_k$  the variable in  $I_k$  and we naturally denote  $(x_k, x_{k-1}, \dots, x_1, x_0)$  the sequence defined by  $x_{i-1} = \phi_i(x_i)$ . We also note

$$\beta_i(x_i) = \frac{\alpha_i(x_i, \xi)}{|\det D_{x_i, \xi}^2 S_i(x_i, \xi)|^{1/2}} \quad ; \quad \xi = \psi'_{i-1}(x_{i-1})$$

$$f_i(x_i) = \beta(x_i) |\phi'_i(x_i)|^{1/2}$$

We fix a constant  $B > 0$  (depending only on  $F, A_q, B_q, B'_q, C_0$ ) satisfying for all  $1 \leq i \leq n-1$ ,

$$\sup_{x_i \in I_i} |\beta_i(x_i)| \leq B$$

$$\|T_i\| \leq B$$

Roughly speaking,  $B$  is of order  $\|\alpha\|_\infty$ , but in this part, the precise value of  $B$  is not relevant. Finally, note that there exists  $\nu < 1$  (again depending only on  $F, A_q, B_q, B'_q$ ) such that  $|\phi'_i(x_i)| \leq \nu$  for  $x_i \in I_i$ . Fix  $N \in \mathbb{N}$  and denote

$$\tilde{N} = 1 + \lceil N + C_0 \log B \rceil \quad (5-4.16)$$

We iteratively define a sequence of symbols  $a_{i,j}$ ,  $0 \leq i \leq n-1$ ,  $0 \leq j \leq \tilde{N}-1$  by  $a_{0,0} = 1, a_{0,j} = 0$  and for  $0 \leq j \leq \tilde{N}-1$



$$a_{i,j}(x_i) = \sum_{p=0}^j L_{j-p,i}(a_{i-1,p})(x_{i-1}) \quad (5-4.17)$$

The following lemma controls the growth of the symbols. The proof is a precise analysis of the iteration formula (5-4.17) and is rather technical. We will first use this lemma to finish the proof of Proposition 5-4.3. We write the detailed proof below in 5-4.2.3.) and refer the reader to [NZ09] (Proposition 4.1), where the author lead the same analysis (but in the case  $B = 1$ ).

**Lemma 5-4.5.** For all  $j \in \{0, \dots, \tilde{N} - 1\}$ ,  $l \in \mathbb{N}$ , there exists  $C_{j,l} > 0$  such that for all  $i \in \{0, \dots, n - 1\}$ , one has

$$\|a_{i,j}\|_{C^l(I_i)} \leq C_{j,l} \left( B\nu^{1/2} \right)^i (i+1)^{l+3j} \quad (5-4.18)$$

**Remark.**

Again, what is important is the fact that  $C_{j,l}$  does *not* depend on  $\mathbf{q}, n, \theta$  nor  $i$ : it depends on  $C_0$  and global parameters.

**Control of the remainder** Let us call  $r_{i,N}(a)$  the remainder appearing in Lemma 5-4.4. Define inductively  $(R_{i,\tilde{N}})$  by  $R_{0,\tilde{N}} = 0$  and

$$R_{i+1,\tilde{N}} = e^{-\frac{i\psi_{i+1}}{h}} T_{i+1} \left( e^{\frac{i\psi_i}{h}} R_{i,\tilde{N}} \right) + \sum_{j=0}^{\tilde{N}-1} r_{i+1,\tilde{N}-j}(a_{i,j}) \quad (5-4.19)$$

This definition ensures that for all  $1 \leq i \leq n$ ,

$$T_i \dots T_1 \left( e^{\frac{i\psi_0}{h}} \right) = e^{i\frac{\psi_i(y)}{h}} \left( \sum_{j=0}^{\tilde{N}-1} h^j a_{i,j} + h^{\tilde{N}} R_{i,\tilde{N}} \right) \quad (5-4.20)$$

**Lemma 5-4.6.** There exists  $C_{\tilde{N}}$  depending only on  $\tilde{N}$ ,  $C_0$  and global parameters such that for all  $1 \leq i \leq n - 1$ ,

$$\|R_{i,\tilde{N}}\|_{L^2(\mathbb{R})} \leq C_{\tilde{N}} B^i$$

*Proof.* Recalling that  $\|T_i\|_{L^2 \rightarrow L^2} \leq B$  and the bound on the remainder in Lemma 5-4.4, the recursive definition of  $R_{i,\tilde{N}}$  gives the following bound:

$$\|R_{i,\tilde{N}}\|_{L^2} \leq B \|R_{i-1,\tilde{N}}\|_{L^2} + \sum_{j=0}^{\tilde{N}-1} C_{\tilde{N}-j} \|a_{i-1,j}\|_{C^{1+2(\tilde{N}-j)}}$$

By induction and using the previous bounds on  $\|a_{i,j}\|_{C^l}$ , we get

$$\begin{aligned} \|R_{\tilde{N},i}\|_{L^2} &\leq \sum_{p=0}^{i-1} B^{i-1-p} \sum_{j=0}^{\tilde{N}-1} C_{\tilde{N}-j} \|a_{p,j}\|_{C^{1+2(\tilde{N}-j)}} \\ &\leq \sum_{p=0}^{i-1} B^{i-1-p} \sum_{j=0}^{N_1-1} C_{\tilde{N}-j} C_{\tilde{N}-j,0} (B\nu^{1/2})^p (p+1)^{1+2\tilde{N}+j} \\ &\leq C_{\tilde{N}} B^i \sum_{p=0}^{i-1} \nu^{p/2} (p+1)^{1+3N_1} \\ &\leq C_{\tilde{N}} B^i \end{aligned}$$

using that the sum is absolutely convergent.  $\square$

**End of proof of Proposition 5-4.3.** We've got now all the elements to conclude the proof. We set

$$a_{\mathbf{q},\theta,N} := \sum_{j=0}^{\tilde{N}-1} h^j a_{n-1,j}$$

We know that

$$U_{\mathbf{q}} B'_{q_0} \left( e^{i\frac{\theta}{h}} \right) = M_{q_{n-1}} \left( e^{i\frac{\psi_{\mathbf{q}}}{h}} a_{\mathbf{q},\theta,N} \right) + M_{q_{n-1}} (h^{\tilde{N}} R_{n-1,\tilde{N}})$$

Since  $M_q$  are uniformly bounded in  $q$  and  $R_{n-1,\tilde{N}} \leq C_{\tilde{N}} B^{n-1} \leq C_{N_1} h^{-C_0 \log B}$ , we have :

$$\|M_{q_{n-1}}(h^{\tilde{N}} R_{n-1,\tilde{N}})\|_{L^2} \leq C_N h^{\tilde{N}-C_0 \log B} \leq C_N h^N$$

Concerning the bounds on  $a_{\mathbf{q},\theta,N}$ , we have

$$\begin{aligned} \|a_{\mathbf{q},\theta,N}\|_{C^l} &\leq \sum_{j=0}^{\tilde{N}-1} h^j \|a_{n-1,j}\|_{C^l} \\ &\leq \sum_{j=0}^{\tilde{N}-1} C_{j,l} \left( B\nu^{1/2} \right)^{n-1} n^{l+3j} h^j \\ &\leq C_{l,N} n^{l+3\tilde{N}} \left( B\nu^{1/2} \right)^{n-1} \\ &\leq C_{l,N} h^{-C_0 \log B} n^{l+3\tilde{N}} \nu^{\frac{n-1}{2}} \\ &\leq C_{l,N} h^{-C_0 \log B} \end{aligned}$$

where we use the fact that  $n \leq C_0 |\log h|$  and bound  $n^{l+3\tilde{N}} \nu^{\frac{n-1}{2}}$  by some  $C_{l,\tilde{N}}$  since  $\nu < 1$ .

Finally, we need to prove the property on the support of  $a_{\mathbf{q},\theta,N}$ . To do so, let us introduce, for  $q \in \mathcal{A}$ , an open set  $\mathcal{W}_q$  satisfying

$$\text{supp } \tilde{a}_q \Subset \mathcal{W}_q \subset \mathcal{V}_q$$

This allows us to define new objects replacing  $\mathcal{V}_q$  by  $\mathcal{W}_q$  in the definitions :

$$\begin{aligned} \mathcal{W}_{\mathbf{q}}^+ &= \bigcap_{i=0}^{n-1} F^{n-i}(\mathcal{W}_{q_i}) \Subset \mathcal{V}_{\mathbf{q}}^+ \\ \mathcal{D}_{\mathbf{q},\theta} &= \kappa_{q_{n-1}} \left( F^{-1}(\mathcal{W}_{\mathbf{q}}^+) \cap F^{n-1}(\mathcal{L}_{q_0,\theta}) \right) \Subset \mathcal{C}_{\mathbf{q},\theta} \end{aligned}$$

and the associated subinterval  $J_{\mathbf{q},\theta} \Subset I_{\mathbf{q},\theta}$  built thanks to Proposition 5-4.2 such that

$$\mathcal{D}_{\mathbf{q},\theta} = \left\{ (y, G_{\mathbf{q},\theta}(y)); y \in J_{\mathbf{q},\theta} \right\}$$

Let us fix  $\delta > 0$  small (with further conditions imposed). We will show the following stronger statement

$$d(\text{supp}(a_{\mathbf{q},\theta,N}), \mathbb{R} \setminus J_{\mathbf{q},\theta}) \geq \delta$$

Suppose that this is not the case. We can find  $x_{n-1} \in \text{supp } a_{\mathbf{q},\theta,N}$ ,  $y_{n-1} \in I_{\mathbf{q},\theta} \setminus J_{\mathbf{q},\theta}$  such that  $|x_{n-1} - y_{n-1}| \leq \delta$ . As already done, we denote by  $x_i$  (resp.  $y_i$ ) the points defined by  $x_{i-1} = \phi_i(x_i)$  (resp.  $y_{i-1} = \phi_i(y_i)$ ). Since  $\phi_i$  are contractions, we have  $|x_i - y_i| \leq \delta$  for  $1 \leq i \leq n-1$ . If we note

$$\rho_i = \kappa_{q_i}^{-1}(x_i, \psi'_i(x_i)) \quad ; \quad \zeta_i = \kappa_{q_i}^{-1}(y_i, \psi'_i(y_i))$$

we have for some  $C > 0$  :  $d(\rho_i, \zeta_i) \leq C\delta$ . By definition, one also has

$$F^{-i}(\rho_{n-1}) = \rho_{n-1-i} \quad ; \quad F^{-i}(\zeta_{n-1}) = \zeta_{n-1-i}$$

By the support property (5-4.15) of the operators  $L_{j,i}$ ,  $\rho_i \in \text{supp } \tilde{a}_{q_i}$  for  $0 \leq i \leq n-1$ . Let's assume that  $\delta$  is small enough so that for all  $q \in \mathcal{A}$ ,

$$d(\text{supp } \tilde{a}_q, (\mathcal{W}_q)^c) \geq 2C\delta$$

Hence,

$$\rho_i \in \text{supp } \tilde{a}_{q_i} \text{ and } d(\rho_i, \zeta_i) \leq C\delta \implies \zeta_i \in \mathcal{W}_{q_i}$$

As a consequence, for all  $0 \leq i \leq n-1$ ,  $F^{i+1-n}(\zeta_{n-1}) \in \mathcal{W}_{q_i}$ , or equivalently  $\zeta_{n-1} \in F^{-1}(\mathcal{W}_{q_i}^+)$ . Hence,

$$(y_{n-1}, \psi'_{n-1}(y_{n-1})) \in \mathcal{C}_{\mathbf{q},\theta} \cap \kappa_{q_{n-1}}(F^{-1}(\mathcal{W}_{q_i}^+)) \subset \mathcal{D}_{\mathbf{q},\theta}$$

showing that  $y_{n-1} \in J_{\mathbf{q},\theta}$ , and giving a contradiction with  $y_{n-1} \in I_{\mathbf{q},\theta} \setminus J_{\mathbf{q},\theta}$ .

### 5-4.2.3. Proof of Lemma 5-4.5.

We give the missing proof of Lemma 5-4.5. The proof is a precise analysis of the iteration formula (5-4.17). We adopt the notations introduced for Lemma 5-4.5. We argue by induction on  $J$  to show the property  $\mathcal{P}_J$ : "the bound (5-4.18) is valid for all  $j \leq J$  and for all  $1 \leq i \leq n-1, l \in \mathbb{N}$  with some constants  $C_{j,l}$ ".

**1. Base case.** Let us start with  $\mathcal{P}_0$ . The iteration formula (5-4.17) implies that

$$a_{i,0}(x_i) = \prod_{l=1}^i f_l(x_l)$$

Hence, the bound  $\|a_{i,0}\|_{C^0} \leq (B\nu^{1/2})^i$  is obvious and we can set  $C_{0,0} = 1$ . We now argue by induction on  $i$  and prove the property  $\mathcal{P}_{0,i}$ : "the bound (5-4.18) is valid for  $j = 0, i$  and for all  $l \in \mathbb{N}$  for some constants  $C_{j,l}$ ". These bounds are trivially true for  $i = 0$  and are direct consequences of Lemma 5-4.4 for  $i = 1$ . Suppose that the property holds for  $i-1$  for some  $i \geq 1$  and let's show it for  $i$ .

**1.1. Case  $l = 1$ .** Let us first deal with  $l = 1$  and compute the derivative of  $a_{i,0}$ , using the formula :  $a_{i,0}(x_i) = f_i(x_i)a_{i-1,0}(x_{i-1})$ .

$$a'_{i,0}(x_i) = f'_i(x_i)a_{i-1,0}(x_{i-1}) + f_i(x_i)a'_{i-1,0}(x_{i-1}) \left( \frac{\partial x_{i-1}}{\partial x_i} \right)$$

We use the (weak) bound  $\left| \frac{\partial x_{i-1}}{\partial x_i} \right| \leq 1$  and the property  $\mathcal{P}_{0,i-1}$  to show that :

$$\|a_{i,0}\|_{C^1} \leq C \left( B\nu^{1/2} \right)^{i-1} + C_{0,1} \left( B\nu^{1/2} \right) \times \left( B\nu^{1/2} \right)^{i-1} i \leq C_{0,1} \left( B\nu^{1/2} \right)^i (i+1)$$

assuming that  $C_{0,1} > C \left( B\nu^{1/2} \right)^{-1}$ .

**1.2. General case for  $l > 0$ .** We now come back to the general case  $l > 0$ . By using the formula  $a_{i,0}(x_i) = f_i(x_i)a_{i-1,0}(x_{i-1})$ , one sees that we can write  $a_{i,0}^{(l)}$  on the form :

$$a_{i,0}^{(l)}(x_i) = f_i(x_i)a_{i-1,0}^{(l)}(x_{i-1}) \left( \frac{\partial x_{i-1}}{\partial x_i} \right)^l + O(\|a_{i-1,0}\|_{C^{l-1}})$$

The constants appearing in the  $O$  depend on  $C^l$  norms of  $f_i$  and  $\phi_i$ , which, by assumption are controlled by some uniform  $C'_l$ . Hence, using the assumption  $\mathcal{P}_{0,i-1}$ ,

$$\begin{aligned} |a_{i,0}^{(l)}(x_i)| &\leq \left( B\nu^{1/2} \right) \|a_{i-1,0}\|_{C^l} \left( \frac{\partial x_{i-1}}{\partial x_i} \right)^l + C'_l \|a_{i-1,0}\|_{C^{l-1}} \\ &\leq C_{0,l} \left( B\nu^{1/2} \right) \left( B\nu^{1/2} \right)^{i-1} i^l + C'_l C_{0,l-1} \left( B\nu^{1/2} \right)^{i-1} i^{l-1} \\ &\leq C_{0,l} \left( B\nu^{1/2} \right)^i (i+1)^l \end{aligned}$$

assuming that  $C_{0,l}$  is chosen bigger than  $\frac{1}{l} C'_l C_{0,l-1} \left( B\nu^{1/2} \right)^{-1}$ . As a consequence, we can build constants satisfying these conditions by defining inductively  $C_{0,l} = \max \left( C_{0,l-1}, \frac{1}{l} C'_l C_{0,l-1} \left( B\nu^{1/2} \right)^{-1} \right)$ . This ends the proof of  $\mathcal{P}_{0,i}$  and hence of  $\mathcal{P}_0$ .

**2. Induction step.** We now assume that  $\mathcal{P}_{j-1}$  is true for some  $j \geq 1$  and aim at proving  $\mathcal{P}_j$ . Again, we do it by induction on  $i$  by proving the properties  $\mathcal{P}_{j,i}$  : "the bound (5-4.18) is true for  $j, i$  and all  $l \in \mathbb{N}$ ". These bounds are trivially true for  $i = 0$  and are direct consequences of Lemma 5-4.4 for  $i = 1$ . Suppose that the property holds for  $i - 1$  for some  $i \geq 2$  and let's show it for  $i$ .

**2.1. Case  $l = 0$ .** Let's start with  $l = 0$ . The iteration formula shows that

$$a_{i,j}(x_i) = f_i(x_i)a_{i-1,j}(x_{i-1}) + \sum_{p=0}^{j-1} L_{j-p,i}(a_{i-1,p})(x_{i-1})$$

By Lemma 5-4.4, there exists constants  $C'_{p,m} > 0$  such that

$$\|L_{p,i}a\|_{C^m(I_i)} \leq C'_{p,m} \|a\|_{C^{2p+m}(I_{i-1})}$$

Hence, assuming that (5-4.18) holds for  $a_{i-1,j}$  with  $l = 0$ .

$$\begin{aligned} \|a_{i,j}\|_{\infty} &\leq C_{j,0} \left(B\nu^{1/2}\right) \left(B\nu^{1/2}\right)^{i-1} i^{3j} + \sum_{p=0}^{j-1} C'_{j-p,0} \|a_{i-1,p}\|_{C^{2(j-p)}} \\ &\leq C_{j,0} \left(B\nu^{1/2}\right)^i i^{3j} + \sum_{p=0}^{j-1} C'_{j-p,0} C_{p,2(j-p)} \left(B\nu^{1/2}\right)^{i-1} i^{2(j-p)+3p} \\ &\leq C_{j,0} \left(B\nu^{1/2}\right)^i i^{3j} + i^{2j} \left(B\nu^{1/2}\right)^{i-1} \sum_{p=0}^{j-1} C'_{j-p,0} C_{p,2(j-p)} i^p \\ &\leq C_{j,0} \left(B\nu^{1/2}\right)^i i^{3j} + i^{2j} \left(B\nu^{1/2}\right)^{i-1} \left[ \sup_{0 \leq p \leq j-1} C'_{j-p,0} C_{p,2(j-p)} \right] \frac{i^j - 1}{i - 1} \\ &\leq C_{j,0} \left(B\nu^{1/2}\right)^i i^{3j} + i^{3j-1} \left(B\nu^{1/2}\right)^{i-1} \left[ \sup_{0 \leq p \leq j-1} C'_{j-p,0} C_{p,2(j-p)} \right] \tilde{C}_j \text{ where } \frac{i^j - 1}{i - 1} \leq \tilde{C}_j i^{j-1} \\ &\leq C_{j,0} \left(B\nu^{1/2}\right)^i (i+1)^{3j} \end{aligned}$$

assuming that  $C_{j,0}$  is chosen bigger than  $K_j := \frac{1}{3j} \left(B\nu^{1/2}\right)^{-1} \left[ \sup_{0 \leq p \leq j-1} C'_{j-p,0} C_{p,2(j-p)} \right] \tilde{C}_j$ . As a consequence, the bounds hold for  $l = 0$  and  $i, j$  if we set  $C_{j,0} = \max(1, K_j)$ .

**2.2. Case  $l > 0$ .** Consider now  $l > 0$ . As already done, one can write

$$a_{i,j}^{(l)}(x_i) = f_i(x_i) a_{i-1,j}^{(l)}(x_{i-1}) \left( \frac{\partial x_{i-1}}{\partial x_i} \right)^l + O(\|a_{i-1,j}\|_{C^{l-1}}) + \sum_{p=0}^{j-1} (L_{j-p,i}(a_{i-1,p}))^{(l)}(x_{i-1})$$

As usual, the constants in  $O$  depend on  $l, j$  but not on  $i$  and we note  $C''_{l,j}$  the constant in this  $O$ . Hence, we can control :

$$\begin{aligned} \|a_{i,j}^{(l)}\|_{\infty} &\leq C_{j,l} \left(B\nu^{1/2}\right) \left(B\nu^{1/2}\right)^{i-1} i^{l+3j} + C''_{l,j} C_{j,l-1} \left(B\nu^{1/2}\right)^{i-1} i^{l+3j-1} + \sum_{p=0}^{j-1} \|L_{j-p,i}(a_{i-1,p})\|_{C^l} \\ &\leq C_{j,l} \left(B\nu^{1/2}\right)^i i^{l+3j} + C''_{l,j} C_{j,l-1} \left(B\nu^{1/2}\right)^{i-1} i^{l+3j-1} + \sum_{p=0}^{j-1} C'_{j-p,l} \|a_{i-1,p}\|_{C^{l+2(j-p)}} \\ &\leq C_{j,l} \left(B\nu^{1/2}\right)^i i^{l+3j} + C''_{l,j} C_{j,l-1} \left(B\nu^{1/2}\right)^{i-1} i^{l+3j-1} + \sum_{p=0}^{j-1} C'_{j-p,l} C_{p,l+2(j-p)} \left(B\nu^{1/2}\right)^{i-1} i^{l+2(j-p)+3p} \\ &\leq C_{j,l} \left(B\nu^{1/2}\right)^i \left( i^{l+3j} + i^{l+3j-1} \frac{1}{C_{j,l}} \underbrace{\left( B\nu^{1/2} \right)^{-1} \left( C''_{l,j} C_{j,l-1} + \sup_{0 \leq p \leq j-1} C'_{j-p,l} C_{p,l+2(j-p)} \tilde{C}_j \right)}_{C_{j,l}} \right) \\ &\leq C_{j,l} \left(B\nu^{1/2}\right)^i (i+1)^{l+3j} \end{aligned}$$

if  $C_{j,l} \geq \tilde{C}_{j,l}$ . Eventually, we define by induction on  $l$  the constants  $C_{j,l}$  by setting  $C_{j,l} = \max(C_{j,l-1}, \tilde{C}_{j,l})$ , achieving the proof of  $\mathcal{P}_j$ . This concludes the proof of the lemma.

### 5-4.3. Microlocalization of $U_{\mathcal{Q}}$ .

We now fix a cloud  $\mathcal{Q} \subset \mathcal{Q}(n, a)$ , centered at a point  $\rho_0 \in \mathcal{T}$ , namely satisfying the condition of Proposition 5-3.8 :

$$\forall \rho \in \bigcup_{\mathbf{q} \in \mathcal{Q}} \mathcal{V}_{\mathbf{q}}^+, d(\rho, W_u(\rho_0)) \leq Ch^b$$

Let us note

$$U_{\mathcal{Q}} = \sum_{\mathbf{q} \in \mathcal{Q}} U_{\mathbf{q}} \quad (5-4.21)$$

and

$$\mathcal{V}_{\mathcal{Q}}^+ = \bigcup_{\mathbf{q} \in \mathcal{Q}} \mathcal{V}_{\mathbf{q}}^+ \quad (5-4.22)$$

We fix an adapted chart  $\kappa := \kappa_{\rho_0} : U_0 \rightarrow V_0$  around  $\rho_0$  as permitted by the Lemma 3-4.2. We can assume that  $\mathcal{V}_a^+ \Subset U_0$  (if  $\varepsilon_0$  is small enough and since the local unstable leaf  $W_u(\rho_0)$  is close to points in  $\mathcal{V}_a^+$ ). We consider a cut-off function  $\tilde{\chi}_a \in C_c^\infty(U_0)$  such that  $\tilde{\chi}_a \equiv 1$  on  $F(\text{supp } \chi_a)$  and  $\text{supp } \tilde{\chi}_a \subset \mathcal{V}_a^+$ . Let us note  $\Xi_a = \text{Op}_h(\tilde{\chi}_a)$ . Since  $\Xi_a M A_a = M A_a + O(h^\infty)$ ,  $|\mathcal{Q}| = O(h^{-K})$  and  $\|U_{\mathbf{q}}\| = O(h^{-K})$  for some  $K > 0$ , we have

$$\mathfrak{M}^{N_0} U_{\mathcal{Q}} = \mathfrak{M}^{N_0} \Xi_a U_{\mathcal{Q}} + O(h^\infty)$$

Let us introduce Fourier integral operators  $B, B'$  quantizing  $\kappa$  in  $\text{supp}(\chi_a)$  :

$$B'B = I + O(h^\infty) \text{ microlocally in } \text{supp}(\chi_a)$$

Hence :

$$\mathfrak{M}^{N_0} U_{\mathcal{Q}} = \mathfrak{M}^{N_0} \Xi_a B' B U_{\mathcal{Q}} + O(h^\infty)$$

We introduce the following sets :

$$\Gamma^+ = \eta(\kappa(\mathcal{V}_{\mathcal{Q}}^+)) ; \Omega^+ = \Gamma^+(h^\tau) \quad (5-4.23)$$

and for  $\mathbf{q} \in \mathcal{Q}$ ,

$$\Gamma_{\mathbf{q}}^+ = \eta(\kappa(\mathcal{V}_{\mathbf{q}}^+)) \quad (5-4.24)$$

We will prove in the following lemma that the pieces  $U_{\mathbf{q}}$  are microlocalized in thin horizontal rectangles (see Figure 5.7).

**Lemma 5-4.7.** For every  $\mathbf{q} \in \mathcal{Q}$ ,

$$\mathbb{1}_{\Gamma_{\mathbf{q}}^+(h^\tau)}(hD_y) B U_{\mathbf{q}} = B U_{\mathbf{q}} + O(h^\infty)_{L^2 \rightarrow L^2} \quad (5-4.25)$$

with uniform bounds in the  $O(h^\infty)$ .

Using the polynomial bounds  $|\mathcal{Q}| = O(h^{-C})$  and  $\|U_{\mathbf{q}}\| = O(h^{-C})$ , we immediately deduce the

**Proposition 5-4.4.**

$$\mathbb{1}_{\Omega^+}(hD_y) B U_{\mathcal{Q}} = B U_{\mathcal{Q}} + O(h^\infty)_{L^2 \rightarrow L^2} \quad (5-4.26)$$

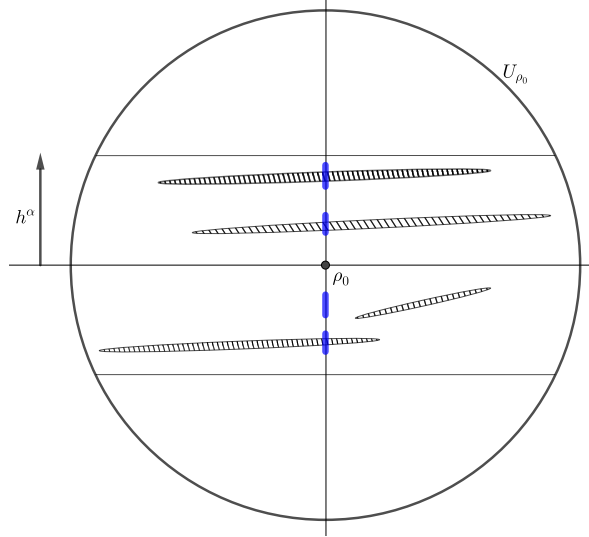


Figure 5.7: The definition of the sets  $\Gamma_{\mathbf{q}}^+$ . They are represented by the blue segments on the  $\eta$ -axis and are the projections on the  $\eta$  variable of the sets  $\mathcal{V}_{\mathbf{q}}^+$  (the hatched sets). They are of width of order  $h^\tau$ .

**5-4.3.1. Proof of Lemma 5-4.7.**

We fix a word  $\mathbf{q} = q_0 \dots q_{n-2}a \in \mathcal{Q}$ . Since  $\text{WF}_h(A_{q_0})$  is compact, we can find  $\chi \in C_c^\infty(\mathbb{R})$  such that

$$A_{q_0} = A_{q_0} B'_{q_0} \chi(hD_y) B_{q_0} + O(h^\infty)$$

Since there is a finite number of symbols in  $\mathcal{A}$ , we can choose one single  $\chi$  for all the possible symbols  $q_0$ . We are hence reduced to prove that

$$\underbrace{\mathbb{1}_{\mathbb{R} \setminus \Gamma_{\mathbf{q}}^+(h^\tau)}(hD_y) B U_{\mathbf{q}} B'_{q_0} \chi(hD_y)}_T = O(h^\infty)_{L^2 \rightarrow L^2} \tag{5-4.27}$$

If  $u \in L^2(\mathbb{R})$ , writing

$$(\chi(hD_y)u)(y) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} \chi(\theta) \mathcal{F}_h u(\theta) e^{i\frac{\theta y}{h}} d\theta$$

we have

$$T(\chi(hD_y)u) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} \chi(\theta) \mathcal{F}_h u(\theta) \left( T e^{i\frac{\theta \cdot}{h}} \right) d\theta$$

Hence,

$$\begin{aligned} \|T(\chi(hD_y)u)\|_{L^2} &\leq \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} |\chi(\theta) \mathcal{F}_h u(\theta)| \left\| T e^{i\frac{\theta \cdot}{h}} \right\|_{L^2} d\theta \\ &\leq \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} |\chi(\theta) \mathcal{F}_h u(\theta)| \sup_{\theta \in \text{supp } \chi} \left\| T e^{i\frac{\theta \cdot}{h}} \right\|_{L^2} \\ &\leq \frac{C_\chi}{h^{1/2}} \|\mathcal{F}_h u\|_{L^2} \sup_{\theta \in \text{supp } \chi} \left\| T e^{i\frac{\theta \cdot}{h}} \right\|_{L^2} \\ &\leq \frac{C_\chi}{h^{1/2}} \|u\|_{L^2} \sup_{\theta \in \text{supp } \chi} \left\| T e^{i\frac{\theta \cdot}{h}} \right\|_{L^2} \end{aligned}$$

As a consequence, we are lead to estimate  $\sup_{\theta \in \text{supp } \chi} \left\| T e^{i\frac{\theta \cdot}{h}} \right\|_{L^2}$ . We fix  $\theta \in \text{supp } \chi$ . Writing that  $\text{supp } \chi \subset [-C_0, C_0]$  and recalling  $|\mathbf{q}| = n \leq C_0 |\log h|$  for some global  $C_0$ , we are in the framework of Proposition 5-4.3.

We fix  $N \in \mathbb{N}$  and we aim at proving that  $Te^{i\frac{\theta}{h}} = O(h^N)$ . By Proposition 5-4.3, there exists  $a_{\mathbf{q},N,\theta} \in C_c^\infty(I_{\mathbf{q},\theta})$  such that

$$U_{\mathbf{q}}B'_{q_0} \left( e^{i\frac{\theta}{h}} \right) = MA_aB'_a \left( a_{\mathbf{q},N,\theta} e^{i\frac{\Phi_{\mathbf{q},\theta}}{h}} \right) + O(h^N)$$

Set  $S := BMA_aB'_a$ .  $S$  is a Fourier integral operator associated with  $s := \kappa \circ F \circ \kappa_a^{-1}$ . Recall that the definitions and the description of the Lagrangian

$$\begin{aligned} C_{\mathbf{q},\theta} &= \kappa_a \left( F^{-1} (\mathcal{V}_{\mathbf{q}}^+) \cap F^{n-1} (\mathcal{L}_{q_0,\theta}) \right) \\ &= \{(y, \Phi'_{\mathbf{q},\theta}(y)), y \in I_{\mathbf{q},\theta}\} \end{aligned}$$

with  $\Phi_{\mathbf{q},\theta} \in C^\infty(I_{\mathbf{q},\theta})$ ;  $\|\Phi_{\mathbf{q},\theta}\|_{C^1} \leq C\varepsilon_0$ ;  $\|\Phi_{\mathbf{q},\theta}\|_{C^l} \leq C_l$ . Assuming that  $\varepsilon_0$  is small enough, we can assume that :

- $s$  is well defined on  $B_a(0, C_1\varepsilon_0)$  and satisfies the conclusion of Lemma 5-4.3. As a consequence, the Lagrangian line

$$s(C_{\mathbf{q},\theta}) = \kappa (\mathcal{V}_{\mathbf{q}}^+) \cap \kappa \circ F^n (\mathcal{L}_{q_0,\theta})$$

can be written  $\{(y, \Psi'(y)), y \in I\}$  for some open  $I \subset \mathbb{R}$  and some function  $\Psi \in C^\infty(I)$  satisfying

$$\|\Psi\|_{C^1} \leq C\varepsilon_0; \|\Psi\|_{C^l} \leq C_l$$

with global constants  $C$  and  $C_l$ .

- $S$  has the form (5-4.13) with a phase function and a symbol having  $C^l$  norms bounded by global constants (depending on  $l$ ).

Hence, we can apply Lemma 5-4.4 to see that there exists  $b \in C_c^\infty(I)$  such that

$$S \left( a_{\mathbf{q},N,\theta} e^{i\frac{\Phi_{\mathbf{q},\theta}}{h}} \right) = b e^{i\frac{\Psi}{h}} + O(h^N)_{L^2}$$

$b$  satisfies the same type of bounds as  $a_{\mathbf{q},N,\theta}$ , namely :

$$\|b\|_{C^l} \leq C_{l,N} h^{-C_0 \log B}$$

Moreover, since  $d(\text{supp } a_{\mathbf{q},N,\theta}, \mathbb{R} \setminus I_{\mathbf{q},\theta}) \geq \delta$ , there exists  $\delta' > 0$  such that  $d(\text{supp } b, \mathbb{R} \setminus I) \geq \delta'$ . The constants  $C_{l,N}$  and  $\delta'$  are global constants. Since  $N$  is arbitrary, to conclude the proof of Lemma 5-4.7, it remains to show that

$$\mathbb{1}_{\mathbb{R} \setminus \Gamma_{\mathbf{q}}^+(h\tau)}(hD_y) \left( b e^{i\frac{\Psi}{h}} \right) = O(h^N) \quad (5-4.28)$$

To do so, we make use of the fine Fourier localization statement from Proposition 2.7 in [DJN21]. We state it for convenience but refer the reader to the quoted paper for the proof.

**Proposition 5-4.5.** Let  $U \subset \mathbb{R}^n$  open,  $K \subset U$  compact,  $\Phi \in C^\infty(U)$  and  $a \in C_c^\infty(U)$  with  $\text{supp } a \subset K$ . Assume that there is a constant  $C_0$  and constants  $C_N, N \in \mathbb{N}^*$  such that :

$$\text{vol}(K) \leq C_0 \quad (5-4.29)$$

$$d(K, \mathbb{R}^n \setminus U) \geq C_0^{-1} \quad (5-4.30)$$

$$\max_{0 < |\alpha| \leq N} \sup_U |\partial^\alpha \Phi| \leq C_N; N \geq 1 \quad (5-4.31)$$

$$\max_{0 \leq |\alpha| \leq N} \sup_U |\partial^\alpha a| \leq C_N; N \geq 1 \quad (5-4.32)$$

$$(5-4.33)$$

Finally, assume that the projection of the Lagrangian  $\{(x, \Phi'(x)), x \in U\}$  on the momentum variable has a diameter of order  $h^\tau$ , namely :

$$\text{diam}(\Omega_\Phi) \leq C_0 h^\tau \text{ where } \Omega_\Phi = \{\Phi'(x), x \in U\} \quad (5-4.34)$$

Define the Lagrangian state

$$u(x) = a(x)e^{i\frac{\Phi(x)}{h}} \in C_c^\infty(U) \subset C_c^\infty(\mathbb{R}^n) \quad (5-4.35)$$

Then, for every  $N \geq 1$ , there exists  $C'_N$  such that

$$\|\mathbb{1}_{\mathbb{R}^n \setminus \Omega_\Phi(h\tau)} u\| \leq C'_N h^N \quad (5-4.36)$$

$C'_N$  depends on  $\tau, n, N, C_0, C_{N'}$  for some  $N'(n, N, \tau)$ .

When  $U = I$ ,  $K = \text{supp } b$ ,  $a = h^{C_0 \log B} b$ ,  $\Phi = \Psi$ , the assumptions (5-4.29) to (5-4.32) are satisfied for some global constants  $C_0, C_N$ . In this case,

$$\Omega_\Psi = \{\Psi'(y), y \in I\} = \eta(\kappa(\mathcal{V}_\mathbf{q}^+) \cap \kappa \circ F^n(\mathcal{L}_{q_0, \theta}))$$

Since  $\Omega_\Psi \subset \Gamma_\mathbf{q}^+$ , to prove (5-4.28), it is enough to prove it with  $\Gamma_\mathbf{q}^+$  replaced by  $\Omega_\Psi$  and to apply the last proposition, it remains to check that the last point (5-4.34) is satisfied. Since who can do more, can do less, we will show that

$$\text{diam}(\Gamma_\mathbf{q}^+) \leq C_0 h^\tau$$

This is where the strong assumption on the adapted charts will play a role. To insist on this role, we state the following lemma :

**Lemma 5-4.8.** Let  $C_0 > 0$ . Assume that  $\rho_1 \in \mathcal{T} \cap U_{\rho_0}$  satisfies  $d(\rho_1, W_u(\rho_0)) \leq C_0 h^\mathbf{b}$ . If  $\rho_2 \in W_u(\rho_1)$ , then for some global constant  $C > 0$ ,

$$|\eta(\kappa(\rho_1)) - \eta(\kappa(\rho_2))| \leq C C_0^{1+\beta} h \quad (5-4.37)$$

*Proof.* Recall that the chart  $(\kappa, U_{\rho_0})$  is the one centered at  $\rho_0$ , given by Lemma 3-4.2. In this chart,  $\kappa(W_u(\rho_1))$  is almost horizontal : we have

$$\kappa(W_u(\rho_1)) = \{y, g(y, \zeta(\rho_1)), y \in \Omega\}$$

where  $\Omega$  is some open bounded set of  $\mathbb{R}$ , with  $g$  and  $\zeta$  satisfying the properties of Lemma 3-4.2. Hence, to prove the lemma, it is enough to estimate  $|g(y, \zeta(\rho_1)) - g(0, \zeta(\rho_1))|, y \in \Omega$ . Since  $\zeta(\rho_0) = 0$  and  $\zeta$  is Lipschitz,  $|\zeta(\rho_1)| \leq C_0 h^\mathbf{b}$ . Indeed, if  $\rho'_0 \in W_u(\rho_0)$  satisfies  $d(\rho'_0, \rho_1) \leq 2C_0 h^\mathbf{b}$ ,

$$|\zeta(\rho_1)| = |\zeta(\rho_1) - \zeta(\rho'_0)| \leq C d(\rho_1, \rho'_0) \leq C C_0 h^\mathbf{b}$$

Then, we have

$$\begin{aligned} |g(y, \zeta(\rho_1)) - g(0, \zeta(\rho_1))| &= |g(y, \zeta(\rho_1)) - g(y, 0) - \partial_\zeta g(y, 0) \zeta(\rho_1)| \\ &= \left| \int_0^{\zeta(\rho_1)} (\partial_\zeta g(y, \zeta) - \partial_\zeta g(y, 0)) d\zeta \right| \\ &\leq \left| \int_0^{\zeta(\rho_1)} C \zeta^\beta d\zeta \right| \\ &\leq C \zeta(\rho_1)^{1+\beta} \leq C C_0^{1+\beta} h^{\mathbf{b}(1+\beta)} \end{aligned}$$

In the first equality, we've used the facts that  $g(0, \zeta) = \zeta$ ,  $\partial_\zeta g(y, 0) = 1$  and  $g(y, 0) = 0$ . This concludes the proof since, by definition (see (5-2.3)),  $\mathbf{b}(1+\beta) = 1$ .  $\square$

**Remark.**

| This lemma explains our definition of  $\mathbf{b}$ .



From this lemma, we can deduce (5-4.34). Indeed, recall that there exists  $\rho_{\mathbf{q}} \in \mathcal{T}$  such that  $\mathcal{V}_{\mathbf{q}}^+ \subset W_u(\rho_{\mathbf{q}})(Ch^\tau)$ . If  $\rho_1, \rho_2 \in \mathcal{V}_{\mathbf{q}}^+$ , there exists  $\rho'_1, \rho'_2 \in W_u(\rho_{\mathbf{q}})$  such that

$$d(\rho_i, \rho'_i) \leq Ch^\tau ; i = 1, 2$$

Hence, one can estimate

$$|\eta(\kappa(\rho_1)) - \eta(\kappa(\rho_2))| \leq \underbrace{|\eta(\kappa(\rho_1)) - \eta(\kappa(\rho'_1))|}_{\leq Ch^\tau} + \underbrace{|\eta(\kappa(\rho'_1)) - \eta(\kappa(\rho'_2))|}_{\leq Ch} + \underbrace{|\eta(\kappa(\rho'_2)) - \eta(\kappa(\rho_2))|}_{\leq Ch^\tau}$$

The inequality in the middle is a consequence of the previous lemma. Indeed,  $\rho'_1, \rho'_2 \in W_u(\rho_{\mathbf{q}})$  where (recall that  $\tau > \mathfrak{b}$ )

$$d(\rho'_1, W_u(\rho_0)) \leq d(\rho_1, \rho'_1) + d(\rho_1, W_u(\rho_0)) \leq Ch^\tau + Ch^{\mathfrak{b}} \leq 2Ch^{\mathfrak{b}}$$

#### 5-4.4. Reduction to a fractal uncertainty principle.

We go on the work started in the last subsection and we keep the same notations. In virtue of Proposition 5-4.1 and Proposition 5-4.4, we can write

$$\mathfrak{M}^{N_0} U_{\mathcal{Q}} = \mathfrak{M}^{N_0} B' B \text{Op}_h(\chi_h) \Xi_a B' \mathbf{1}_{\Omega^+}(hD_y) BU_{\mathcal{Q}} + O(h^\infty)_{L^2 \rightarrow L^2} \quad (5-4.38)$$

where

- $\chi_h \in S_{\delta_2}^{comp}$ ,  $\chi_h \equiv 1$  on  $\mathcal{T}_-^{loc}(2C_2h^{\delta_2})$  and  $\text{supp } \chi_h \in \mathcal{T}_-^{loc}(4C_2h^{\delta_2})$  (see Proposition 5-4.1 and before);
- $\Xi_a = \text{Op}_h(\tilde{\chi}_a)$  where  $\tilde{\chi}_a \in C_c^\infty(U_0)$  is a cut-off function such that  $\tilde{\chi}_a \equiv 1$  on  $F(\text{supp } \chi_a)$  and  $\text{supp } \tilde{\chi}_a \subset \mathcal{V}_a^+$  (see the beginning of subsection 5-4.3.) ;
- $\Omega^+ = \eta(\kappa(\mathcal{V}_{\mathcal{Q}}^+))(h^\tau)$  (see 5-4.23 and Proposition 5-4.4).

In  $V_{\rho_0}$ ,  $U_{\mathcal{Q}}$  is microlocalized in a region  $\{|\eta| \leq Ch^{\mathfrak{b}}\}$ . To work with symbols in usual symbol classes, we will rather consider a bigger region  $\{|\eta| \leq h^{\delta_0}\}$ . For this purpose, let us denote

$$\Gamma^- = y\left(\kappa(\mathcal{V}_a^+ \cap \mathcal{T}_-^{loc}(4C_2h^{\delta_2})) \cap \{|\eta| \leq h^{\delta_0}\}\right) ; \Omega^- = \Gamma^-(h^{\delta_0}) \quad (5-4.39)$$

Since  $\mathcal{V}_{\mathcal{Q}}^+ \subset W_u(\rho_0)(Ch^{\mathfrak{b}})$ ,  $\Omega_+ \subset [-C_0h^{\mathfrak{b}}, C_0h^{\mathfrak{b}}] \subset [-h^{\delta_0}, h^{\delta_0}]$  for  $h$  small enough. By Lemma 5-4.2, there exists  $\chi_+(\eta) := \chi_+(\eta; h) \in C_c^\infty(\mathbb{R})$  such that :

- $\chi_+ \equiv 1$  on  $\Omega^+$ ;
- $\text{supp } \chi_+ \subset [-h^{\delta_0}, h^{\delta_0}]$  ;
- $\forall k \in \mathbb{N}$  and  $\eta \in \mathbb{R}$ ,  $|\chi_+^{(k)}(\eta)| \leq C_k h^{-\delta_0 k}$  for some global constants  $C_k$ .

$\chi_+$  satisfies :

$$\mathbf{1}_{\Omega^+}(hD_y) = \chi_+(hD_y) \mathbf{1}_{\Omega^+}(hD_y)$$

Let's now consider the following subset of  $\Gamma^-$  :

$$\tilde{\Gamma}^- = y\left(\kappa(\mathcal{V}_a^+ \cap \mathcal{T}_-^{loc}(4C_2h^{\delta_2})) \cap \{\eta \in \text{supp } \chi_+\}\right)$$

The inclusion  $\tilde{\Gamma}^- \subset \Gamma^-$  comes from the support property of  $\chi_+$ .

Using again Lemma 5-4.2, we construct a family  $\chi_-(y) := \chi_-(y; h) \in C_c^\infty(\mathbb{R})$  such that :

- $\chi_- \equiv 1$  on  $\tilde{\Gamma}^-$ ;
- $\text{supp } \chi_- \subset \Omega^- = \Gamma^-(h^{\delta_0})$  ;
- $\forall k \in \mathbb{N}$  and  $y \in \mathbb{R}$ ,  $|\chi_-^{(k)}(y)| \leq C_k h^{-\delta_0 k}$ .

and  $\chi_-$  allows to write

$$\chi_-(y) \mathbf{1}_{\Omega^-}(y) = \chi_-(y)$$

We now claim that

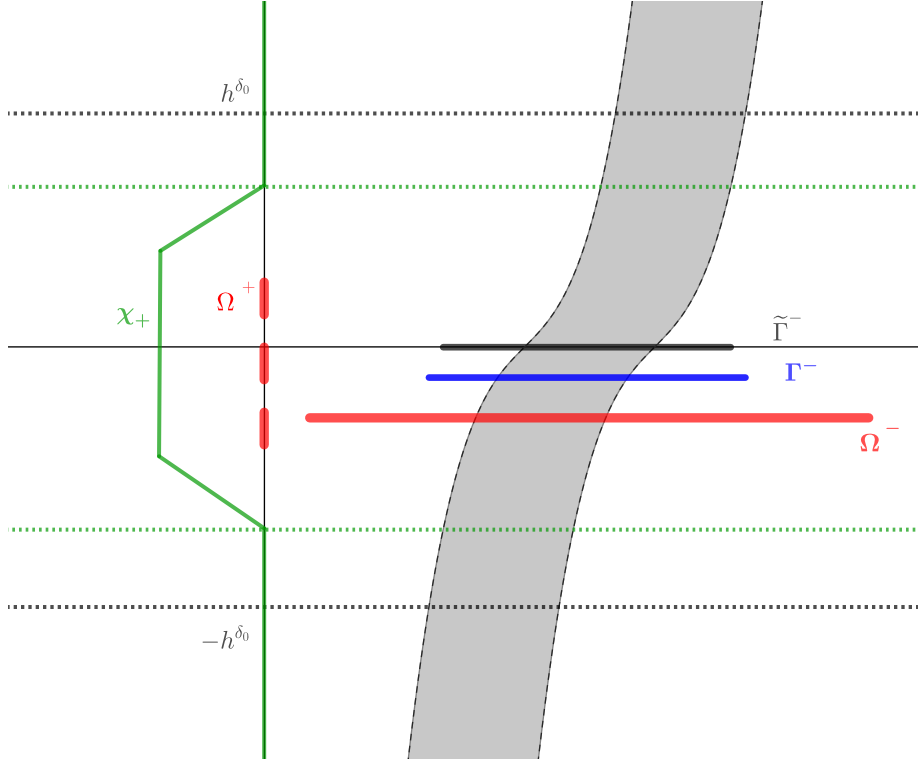


Figure 5.8: The set  $\Omega^+$  is represented on the  $\eta$ -axis, with the support of the function  $\chi_+$ . On the  $y$ -axis, we project the gray set  $\kappa(\mathcal{V}_a^+ \cap \mathcal{T}_-^{loc}(4C_2h^{\delta_2}))$  to obtain both  $\Gamma^-$  and  $\tilde{\Gamma}^-$  depending on the size of the  $\eta$ -window. The larger set  $\Omega^-$  is also represented in red.

$$\mathfrak{M}^{N_0}U_{\mathcal{Q}} = \mathfrak{M}^{N_0} \text{Op}_h(\chi_h)\Xi_a B' \chi_-(y) \mathbb{1}_{\Omega^+}(hD_y) BU_{\mathcal{Q}} + O(h^\infty)_{L^2 \rightarrow L^2} \quad (5-4.40)$$

Due to the polynomial bounds on  $\|\mathfrak{M}^{N_0}\|$  and  $\|U_{\mathcal{Q}}\|$ , it is then enough to show that

$$\text{Op}_h(\chi_h)\Xi_a B'(1 - \chi_-(y))\chi_+(hD_y) = O(h^\infty)$$

Using Egorov's theorem in  $\Psi_{\delta_2}(\mathbb{R})$ , we see that  $\Xi_0 := B \text{Op}_h(\chi_h)\Xi_a B'$  is in  $\Psi_{\delta_2}(\mathbb{R})$  and  $\text{WF}_h(\Xi_0) \subset \kappa(\text{supp } \chi_a \cap \text{supp } \chi_h)$ . We now observe that

$$\begin{aligned} (y, \eta) \in \text{WF}_h(\Xi_0) \cap \text{WF}_h(1 - \chi_-(y)) \cap \text{WF}_h(\chi_+(hD_y)) &\implies \\ (y, \eta) \in \kappa(\text{supp } \chi_a \cap \text{supp } \chi_h), \eta \in \text{supp } \chi_+, y \notin \tilde{\Gamma}^-, & \end{aligned}$$

But the first two conditions imply that  $y \in \tilde{\Gamma}^-$ . Hence,

$$\text{WF}_h(\Xi_0) \cap \text{WF}_h(1 - \chi_-(y)) \cap \text{WF}_h(\chi_+(hD_y)) = \emptyset$$

By the composition formulas in  $\Psi_{\delta_0}(\mathbb{R})$ ,  $\Xi_0(1 - \chi_-(y))\chi_+(hD_y) = O(h^\infty)$ . Note that the constants in  $O(h^\infty)$  depend on the semi-norms of  $\chi_\pm, \chi_h$  and  $\chi_a$ . Due to their construction, the semi-norms of  $\chi_\pm$  and  $\chi_h$  are bounded by global constants. As a consequence, the constants  $O(h^\infty)$  are global constants.

This proves the claim 5-4.40. Recalling the bound

$$\|\mathfrak{M}^{N_0}\|_{L^2 \rightarrow L^2} \leq \|\alpha\|^{N_0}(1 + o(1)) \quad , \quad \|U_{\mathcal{Q}}\|_{L^2 \rightarrow L^2} \leq C |\log h| \|\alpha\|_\infty^{N_1}$$

we see that the proof of Proposition 5-3.8 and hence of Proposition 5-2.1, has been reduced to proving the following proposition.

**Proposition 5-4.6.** With the above notations, There exists  $\gamma > 0$  and  $h_0 > 0$  such that :

$$\forall h \leq h_0, \|\mathbb{1}_{\Omega^-}(y)\mathbb{1}_{\Omega^+}(hD_y)\|_{L^2 \rightarrow L^2} \leq h^\gamma \quad (5-4.41)$$

**Remark.**

$\gamma$  and  $h_0$  are global : they do not depend on the particular  $\mathcal{Q} \subset \mathcal{Q}(n, a)$  satisfying the conditions of Proposition 5-3.8, nor on  $n$ .

The proof of this proposition is the aim of the next section and relies on a fractal uncertainty principle.

## 5-5 Application of the fractal uncertainty principle.

The fractal uncertainty principle, first introduced by Dyatlov-Zahl in [DZ16] and further proved in full generality by Bourgain-Dyatlov in [BD18], is the key tool for our decay estimate. We'll use the slightly more general version proved and used in [DJN21].

### 5-5.1. Porous sets.

We start by recalling the definition of porous sets and then we state the version of the fractal uncertainty principle we'll use.

**Definition 5-5.1.** Let  $\nu \in (0, 1)$  and  $0 \leq \alpha_0 \leq \alpha_1$ . We say that a subset  $\Omega \subset \mathbb{R}$  is  $\nu$ -porous on scales  $\alpha_0$  to  $\alpha_1$  if for every interval  $I \subset \mathbb{R}$  of size  $|I| \in [\alpha_0, \alpha_1]$ , there exists a subinterval  $J \subset I$  of size  $|J| = \nu|I|$  such that  $J \cap \Omega = \emptyset$ .

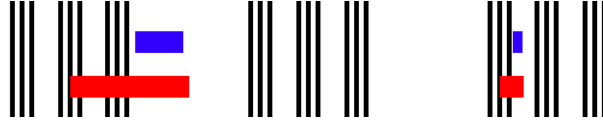


Figure 5.9: Example of a porous set. Its construction is based on a Cantor-like set. Red intervals correspond to choices of  $I$ , blue ones correspond to  $J$ .

The following simple lemma shows that when one fattens a porous set, one gets another porous set. For its (very elementary) proof, see [DJN21] (Lemma 2.12).

**Lemma 5-5.1.** Let  $\nu \in (0, 1)$  and  $0 \leq \alpha_0 < \alpha_1$ . Assume that  $\alpha_2 \in (0, \frac{\nu}{3}\alpha_1]$  and  $\Omega \subset \mathbb{R}$  is  $\nu$ -porous on scales  $\alpha_0$  to  $\alpha_1$ . Then, the neighborhood  $\Omega(\alpha_2) = \Omega + [-\alpha_2, \alpha_2]$  is  $\frac{\nu}{3}$ -porous on scale  $\max(\alpha_0, \frac{3}{\nu}\alpha_2)$  to  $\alpha_1$ .

The notion of porosity can be related to the different notions of fractal dimensions. Let us recall the definition of the upper box dimension of a metric space  $(X, d)$ . We denote by  $N_X(\varepsilon)$  the minimal number of open balls of radius  $\varepsilon$  needed to cover  $X$ . Then, the upper box dimension of  $X$  is defined by :

$$\overline{\dim} X := \limsup_{\varepsilon \rightarrow 0} \frac{\log N_X(\varepsilon)}{-\log \varepsilon} \quad (5-5.1)$$

In particular, if  $\delta > \overline{\dim} X$ , there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \leq \varepsilon_0$ ,  $N_X(\varepsilon) \leq \varepsilon^{-\delta}$ . This observation motivates the following lemma :

**Lemma 5-5.2.** Let  $\Omega \subset \mathbb{R}$ . Suppose that there exist  $0 < \delta < 1$ ,  $C > 0$  and  $\varepsilon_0 > 0$  such that

$$\forall \varepsilon \leq \varepsilon_0, N_\Omega(\varepsilon) \leq C\varepsilon^{-\delta}$$

Then, there exists  $\nu = \nu(\delta, \varepsilon_0, C)$  such that  $\Omega$  is  $\nu$ -porous on scale 0 to 1.

**Remark.**

The proof will give an explicit value of  $\nu$ . This quantitative statement will be important in the sequel to ensure the same porosity for all the sets  $W_{u/s}(\rho_0) \cap \mathcal{T}$ .

*Proof.* Let us set  $T = \lfloor \max\left((6\varepsilon_0)^{-1}, (6^\delta C)^{\frac{1}{1-\delta}}\right) \rfloor + 1$  and  $\nu = (3T)^{-1}$ . We will show that  $\Omega$  is  $\nu$ -porous on scale 0 to 1. Let  $I \subset \mathbb{R}$  be an interval of size  $|I| \in (0, 1]$ . Cut  $I$  into  $3T$  consecutive closed intervals of size  $\nu$ :  $J_0, \dots, J_{3T-1}$ . We argue by contradiction and assume that each of these intervals does intersect  $\Omega$ . Let us show that

$$N_\Omega(\nu/2) \geq T \tag{5-5.2}$$

Assume that  $U_1, \dots, U_k$  is a family of open intervals of size  $\nu$  covering  $\Omega$ . For  $i = 0, \dots, T-1$ , there exists  $x_i \in J_{3i+1}$  and  $j_i \in \{1, \dots, k\}$  such that  $x_i \in U_{j_i}$ . It follows that  $U_{j_i} \subset J_{3i} \cup J_{3i+1} \cup J_{3i+2}$  and hence  $i \neq l \implies U_{j_i} \cap U_{j_l} = \emptyset$ . The map  $i \in \{0, \dots, T-1\} \mapsto j_i \in \{1, \dots, k\}$  is one-to-one, and it gives (5-5.2). Since  $T \geq \frac{1}{6\varepsilon_0}$ ,  $\nu/2 \leq \varepsilon_0$ . As a consequence ,

$$T \leq N(\nu/2) \leq C(6T)^\delta$$

which implies  $T^{1-\delta} \leq C6^\delta$ . This contradicts the definition of  $T$ . □

In the extra section 5-5.4., we give a result in the other way, namely, porous sets down to scale 0 have an upper box dimension strictly smaller than one.

For further use, we also record the easy lemma :

**Lemma 5-5.3.** Assume that  $(X, d), (Y, d')$  are metric spaces and  $f : X \rightarrow Y$  is  $C$ -Lipschitz. Then, for every  $\varepsilon > 0$ ,

$$N_{f(X)}(\varepsilon) \leq N_X(\varepsilon/C)$$

In particular, if  $N_X(\varepsilon) \leq C_1^\delta \varepsilon^\delta$  for  $\varepsilon \leq \varepsilon_0$ , then for  $\varepsilon \leq C\varepsilon_0$ ,  $N_{f(X)}(\varepsilon) \leq (C_1 C)^\delta \varepsilon^{-\delta}$ .

### 5-5.2. Fractal uncertainty principle.

We state here the version of the fractal uncertainty principle we'll use. This version is stated in Proposition 2.11 in [DJN21]. The difference with the original version in [BD18] is that it relaxes the assumption regarding the scales on which the sets are porous. We refer the reader to the review of Dyatlov [Dya19] to an overview on the fractal uncertainty principle with other references and applications.

**Proposition 5-5.1. Fractal uncertainty principle.** Fix numbers  $\gamma_0^\pm, \gamma_1^\pm$  such that

$$0 \leq \gamma_1^\pm < \gamma_0^\pm \leq 1, \gamma_1^+ + \gamma_1^- < 1 < \gamma_0^+ + \gamma_0^-$$

and define

$$\gamma := \min(\gamma_0^+, 1 - \gamma_1^-) - \max(\gamma_1^+, 1 - \gamma_0^-)$$

Then for each  $\nu > 0$  , there exists  $\beta = \beta(\nu) > 0$  and  $C = C(\nu)$  such that the estimate

$$\|\mathbb{1}_{\Omega_-} \mathcal{F}_h \mathbb{1}_{\Omega_+}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq Ch^{\gamma\beta} \tag{5-5.3}$$

holds for all  $0 < h \leq 1$  and all  $h$ -dependent sets  $\Omega_\pm \subset \mathbb{R}$  which are  $\nu$ -porous on scale  $h^{\gamma_0^\pm}$  to  $h^{\gamma_1^\pm}$ .

**Remark.**

In the sequel, we will use this result with  $\gamma_1^\pm = 0$ . In this case, the condition on  $\gamma_0^\pm$  becomes  $\gamma_0^- + \gamma_0^+ > 1$  and the exponent  $\gamma$  is  $\gamma_0^- + \gamma_0^+ - 1$ . This condition can be interpreted as a condition of saturation of the standard uncertainty principle : a rectangle of size  $h^{\gamma_0^+} \times h^{\gamma_0^-}$  will be subplanckian.

### 5-5.3. Porosity of $\Omega^+$ and $\Omega^-$ .

Since we want to apply Proposition 5-5.1 to prove Proposition 5-4.6, we need to show the porosity of the sets  $\Omega^\pm$  defined in (5-4.23) and (5-4.39). The main tool is the following proposition.

**Proposition 5-5.2.** There exist  $\delta \in [0, 1[$ ,  $C > 0$  and  $\varepsilon_0 > 0$  such that for every  $\rho_0 \in \mathcal{T}$ , if  $X = W_{u/s}(\rho_0) \cap \mathcal{T} \cap U_{\rho_0}$ ,

$$N_X(\varepsilon) \leq C\varepsilon^{-\delta}; \forall \varepsilon \leq \varepsilon_0$$

**Remark.**

Recall that  $W_{u/s}(\rho_0)$  is a local unstable (resp. stable) manifold at  $\rho_0$ , and in particular a single smooth curve.  $U_{\rho_0}$  is the domain of the chart adapted  $\kappa_{\rho_0}$  (see 3-4.2).

Roughly speaking, this proposition says that the upper box dimension of the sets  $W_{u/s}(\rho) \cap \mathcal{T}$ , the trace of  $\mathcal{T}$  along the stable and unstable manifolds, is strictly smaller than one. This condition on the upper box dimension is a fractal condition. In our case, we need uniform estimates on the numbers  $N_X(\varepsilon)$  for  $X = W_{u/s}(\rho) \cap \mathcal{T}$ . This uniformity is a consequence of the fact that the holonomy maps are  $C^1$  with uniform  $C^1$  bounds (and thus Lipschitz, which is enough to conclude). This result is clearly linked with Bowen's formula which has been proved in different contexts and links the dimension of  $X$  with the topological pressure of the map  $\phi_u = -\log |J_u^1|$ . This is where the assumption (Fractal) is used.

*Proof.* We will simply recall some arguments which lead to give an upper bound to the upper box dimension. We borrow this arguments from [Bar08] (Section 4.3) and refer the reader to this book for the definitions and properties of topological pressure (definition 2.3.1), Markov partition (definition 4.2.6) and other references on this theory.

We'll show that the pressure condition (Fractal) implies Proposition 5-5.2. We prove it for the unstable manifolds. The proof is similar in the case of stable manifolds by changing  $F$  into  $F^{-1}$ . We first begin by fixing a Markov partition for  $\mathcal{T}$  with diameter at most  $\eta_0$ . This is possible in virtue of Theorem 18.7.3 in [HK95]. We note  $R_1, \dots, R_p \subset \mathcal{T}$  this Markov partition. Here,  $\eta_0$  is smaller than the diameter of the local stable and unstable manifolds and the holonomy maps  $H_{\rho, \rho'}^{u/s}$  are well defined for  $d(\rho, \rho') \leq \eta_0$  :

$$H_{\rho, \rho'}^{u/s} : W_{s/u}(\rho) \rightarrow W_{s/u}(\rho'), \zeta \mapsto \text{the unique point in } W_u(\zeta) \cap W_s(\rho')$$

Due to our results on the regularity of the stable and unstable distributions, these maps are Lipschitz with global Lipschitz constants. In particular, if an inequality of the kind

$$N_{W_u(\rho) \cap \mathcal{T}}(\varepsilon) \leq C\varepsilon^{-\delta}$$

holds for some  $\rho$ , it holds for  $\rho'$  if  $d(\rho, \rho') \leq \eta_0$  with  $C$  replaced by  $K^\delta C$  where  $K$  is a Lipschitz constant for the holonomy maps. We fix  $(\rho_1, \dots, \rho_p)$  in  $(R_1, \dots, R_p)$  and we set  $V = \bigcup_{i=1}^p W_u(\rho_i) \cap R_i$ . It is then enough to show that

$$\overline{\dim} V < 1$$

Indeed, if  $\overline{\dim} V < 1$ , for  $\delta \in (\overline{\dim} V, 1)$ , there exists  $\varepsilon_0 > 0$  such that

$$\forall \varepsilon \leq \varepsilon_0, N_V(\varepsilon) \leq \varepsilon^{-\delta}$$

and we conclude the proof of Proposition 5-5.2 with the above considerations on the holonomy maps.

$\delta := \overline{\dim} V$  satisfies the equation  $P(\delta\phi_u) = 0$ . We will actually show that  $P(\delta\phi_u) \geq 0$ . Since  $s \mapsto P(s\phi_u)$  is strictly decreasing and has a unique root, the assumption  $P(\phi_u) < 0$  will give  $\delta < 1$ . We will note

$$R_{i_0, \dots, i_n} = \bigcap_{k=0}^n F^{-i}(R_{i_k}); V_{i_0, \dots, i_n} = R_{i_0, \dots, i_n} \cap V$$

the elements of the refined partition at time  $n$ . Similarly to the definitions of  $J_{\mathbf{q}}^+$ , we will note

$$J_{i_0, \dots, i_n} = \inf\{J_u^n(\rho), \rho \in R_{i_0, \dots, i_n}\}$$

and write

$$c_n(s) = \sum_{i_0, \dots, i_n} J_{i_0, \dots, i_n}^{-s} = \sum_{i_0, \dots, i_n} \exp \max_{R_{i_0, \dots, i_n}} \left( s \sum_{k=0}^{n-1} \phi_u \circ F^k \right)$$

(the last equality follows from the chain rule). Properties of Markov partitions ensure that

$$P(s\phi_u) = \lim_{n \rightarrow \infty} \frac{1}{n} \log c_n(s)$$

Fix  $s > \delta$ . Hence, there exists  $\varepsilon_1$  such that  $\forall \varepsilon \leq \varepsilon_1, N_V(\varepsilon) \leq \varepsilon^{-s}$ .

Fix  $n \in \mathbb{N}^*$ . By writing  $V = \bigcup_{i_0, \dots, i_n} V_{i_0, \dots, i_n}$  we have

$$N_V(\varepsilon) \leq \sum_{i_0, \dots, i_n} N_{V_{i_0, \dots, i_n}}(\varepsilon)$$

Note that

$$F^n(V_{i_0, \dots, i_n}) \subset W_u(F^n(\rho_{i_0})) \cap R_{i_n}$$

and

$$H_{F^n(\rho_{i_0}), \rho_{i_n}}^s(F^n(V_{i_0, \dots, i_n})) \subset V_{i_n}$$

Hence, if we cover  $V_{i_n}$  by  $N$  sets of diameter at most  $\varepsilon, U_1, \dots, U_N$ , the sets  $F^{-n} \circ H_{\rho_{i_n}, F^n(\rho_{i_0})}^s(U_i), 1 \leq i \leq N$  cover  $V_{i_0, \dots, i_n}$  and have diameters at most  $K\varepsilon J_{i_0, \dots, i_n}^{-1}$ . Hence,

$$N_{V_{i_n}}(\varepsilon) \geq N_{V_{i_0, \dots, i_n}}(K\varepsilon J_{i_0, \dots, i_n}^{-1})$$

which gives

$$N_V(\varepsilon) \leq \sum_{i_0, \dots, i_n} N_{V_{i_n}}(\varepsilon K^{-1} J_{i_0, \dots, i_n})$$

As a consequence, if  $\varepsilon < \varepsilon_1 K J_n^{-1}$ , where  $J_n = \sup_{i_0, \dots, i_n} J_{i_0, \dots, i_n}$ , we have

$$N_V(\varepsilon) \leq \sum_{i_0, \dots, i_n} K^s J_{i_0, \dots, i_n}^{-s} \varepsilon^{-s} = K^s \varepsilon^{-s} c_n(s)$$

By iterating this process, we see that for all  $m \in \mathbb{N}$ , if  $\varepsilon < \varepsilon_1 (K J_n^{-1})^m$ ,

$$N_V(\varepsilon) \leq \varepsilon^{-s} K^{ms} c_n(s)^m$$

Hence,

$$\frac{\log N_V(\varepsilon)}{-\log \varepsilon} \leq s + m \frac{\log(K^s c_n(s))}{-\log \varepsilon} \leq s + m \frac{\log(K^s c_n(s))}{-\log(\varepsilon_1 (K J_n^{-1})^m)}$$

We then take the lim sup as  $\varepsilon \rightarrow 0$  first and then pass to the limit as  $m \rightarrow +\infty$  and find that

$$\overline{\dim} V \leq s + \frac{\log K^s c_n(s)}{-\log K J_n^{-1}}$$

Then, we pass to the limit  $s \rightarrow \delta$  and find that  $\log(K^\delta c_n(\delta)) \geq 0$ . Hence,

$$P(\delta\phi_u) = \lim_{n \rightarrow \infty} \frac{1}{n} \log c_n(\delta) \geq \lim_{n \rightarrow \infty} \frac{-\delta \log K}{n} = 0$$

This ends the proof of the required inequality and gives that  $\overline{\dim} V < 1$ . □

From the Proposition 5-5.2, we get

**Corollary 5-5.1.** There exists  $\nu > 0$  such that for every  $\rho_0 \in \mathcal{T}$ , the sets  $y \circ \kappa(W_u(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})$  and  $\zeta(W_s(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})$  are  $\nu$ -porous on scale 0 to 1.

*Proof.* The maps  $y \circ \kappa$  and  $\zeta$  are  $C$ -Lipschitz for a global constant  $C$ . As a consequence, the previous lemma and Lemma 5-5.3 give

$$\forall \varepsilon \leq \varepsilon_0/C, N_\Omega(\varepsilon) \leq C^\delta \varepsilon^{-\delta}, \text{ where } \Omega = y \circ \kappa(W_u(\rho_0) \cap \mathcal{T} \cap U_{\rho_0}) \text{ or } \zeta(W_s(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})$$

Applying Lemma 5-5.2, the  $\nu$ -porosity is proved for some  $\nu = \nu(\delta, C, \varepsilon_0)$ .  $\square$

To conclude, we use this corollary to show the porosity of  $\Omega^\pm$ . We start by studying  $\Omega^+$ .

**Lemma 5-5.4.** There exists a global constant  $C > 0$  such that

$$\Omega^+ \subset \zeta(W_s(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})(Ch^\tau)$$

*Proof.* Since  $\Omega^+ = \Gamma^+(h^\tau)$ , it is enough to show the same statement for  $\Gamma^+ = \eta \circ \kappa_{\rho_0}(\mathcal{V}_\mathcal{Q}^+)$ . Let  $\rho \in \mathcal{V}_\mathcal{Q}^+$ . By assumption on  $\mathcal{Q}$  and  $\rho_0$ ,  $d(\rho, W_u(\rho_0)) \leq Ch^b$ . Since  $\rho \in \mathcal{V}_\mathbf{q}$  for some  $\mathbf{q} \in \mathcal{Q}$ , there exists  $\rho_1 \in \mathcal{T}$  such that  $d(\rho, W_u(\rho_1)) \leq \frac{C}{J_\mathbf{q}^+(\rho_1)} \leq Ch^\tau$ . Fix  $\rho_2 \in W_u(\rho_1)$  such that  $d(\rho, \rho_2) \leq Ch^\tau$ .

$$|\eta \circ \kappa(\rho) - \zeta(\rho_1)| = |\eta \circ \kappa(\rho) - \zeta(\rho_2)| \leq |\eta \circ \kappa(\rho) - \eta \circ \kappa(\rho_2)| + |\eta \circ \kappa(\rho_2) - \zeta(\rho_2)|$$

Since  $\eta \circ \kappa$  is Lipschitz, we can control the first term by

$$|\eta \circ \kappa(\rho) - \eta \circ \kappa(\rho_2)| \leq Cd(\rho, \rho_2) \leq Ch^\tau$$

To estimate the second term, the same arguments used after Lemma 5-4.8 show that

$$|\eta \circ \kappa(\rho_2) - \zeta(\rho_2)| \leq \text{diam}[\eta \circ \kappa(W_u(\rho_2) \cap U_{\rho_0})] \leq Ch$$

It gives  $|\eta \circ \kappa(\rho) - \zeta(\rho_1)| \leq Ch^\tau$ . To conclude, note that there exists a unique point  $\rho'_1 \in W_s(\rho_0) \cap W_u(\rho_1)$  and  $\zeta(\rho_1) = \zeta(\rho'_1)$ .  $\square$

As a simple corollary of this lemma and of Lemma 5-5.1, we get

**Corollary 5-5.2.**  $\Omega^+$  is  $\nu/3$ -porous on scale  $\frac{3}{\nu}Ch^\tau$  to 1.

We now turn to the study of  $\Omega^-$ . We can state and prove similar results with different scales of porosity. Recall that  $\delta_2 = \frac{\lambda_0}{\lambda_1}\delta_0$ .

**Lemma 5-5.5.** There exists a global constant  $C > 0$  such that

$$\Omega^- \subset y \circ \kappa(W_u(\rho_0) \cap \mathcal{T} \cap U_{\rho_0})(Ch^{\delta_2})$$

*Proof.* Since  $\Omega^- = \Gamma^-(h^{\delta_0})$  with  $\delta_0 > \delta_2$ , it is enough to prove if for

$$\Gamma^- = y \circ \kappa(\mathcal{V}_a^+ \cap \mathcal{T}_-^{loc}(4C_2h^{\delta_2}) \cap \{|\eta| \leq h^{\delta_0}\})$$

Recall that  $\mathcal{T}_-^{loc} \subset \bigcup_{\rho \in \mathcal{T}} W_s(\rho)$ . Since in  $\mathcal{V}_a^+$ , all the local stable leaves intersect  $W_u(\rho_0)$ , we have

$$\mathcal{V}_a^+ \cap \mathcal{T}_-^{loc}(4C_2h^{\delta_2}) \subset \bigcup_{\rho \in W_u(\rho_0) \cap \mathcal{T}} W_s(\rho)(4C_2h^{\delta_2})$$

Fix  $\rho \in W_u(\rho_0) \cap \mathcal{T}$ . Since  $d\kappa(E_s(\rho_0)) = \mathbb{R}\partial_\eta$ , if  $\varepsilon_0$  is small enough, we can write  $\kappa(W_s(\rho)) = \{(G_\rho(\eta), \eta), \eta \in O\}$  where  $O$  is some open subset of  $\mathbb{R}$  and  $G_\rho : O \rightarrow \mathbb{R}$  is  $C^\infty$ . In particular, it is Lipschitz with a global Lipschitz constant  $C$ . If  $|\eta| \leq h^{\delta_0}$ ,  $|G_\rho(\eta) - G_\rho(0)| \leq Ch^{\delta_0}$ . Recall that  $\kappa(W_u(\rho_0) \cap U_{\rho_0}) \subset \mathbb{R} \times \{0\}$  and hence,  $G_\rho(0) = y \circ \kappa(\rho)$ . As a consequence, if  $\rho_1 \in W_s(\rho) \cap \{|\eta| \leq h^{\delta_0}\}$ , writing  $\kappa(\rho_1) = (G_\rho(\eta), \eta)$ , we have

$$|y \circ \kappa(\rho_1) - y \circ \kappa(\rho)| = |G_\rho(\eta) - G_\rho(0)| \leq Ch^{\delta_0}$$

Then, if  $\rho_2 \in W_s(\rho)(4C_2h^{\delta_2})$ , since  $\kappa$  is Lipschitz with global Lipschitz constant ,

$$|y \circ \kappa(\rho_2) - y \circ \kappa(\rho)| \leq Ch^{\delta_2} + Ch^{\delta_0} \leq Ch^{\delta_2}$$

This shows that  $y \circ \kappa(\rho_2) \in y \circ \kappa(W_u(\rho_0) \cap \mathcal{T})(Ch^{\delta_2})$  and concludes the proof.  $\square$

As a corollary, using Lemma 5-5.1, we get

**Corollary 5-5.3.**  $\Omega^-$  is  $\nu/3$ -porous on scale  $\frac{3}{\nu}Ch^{\delta_2}$  to 1.

We can now prove the last Proposition 5-4.6 needed to end the proof of the main Theorem 5-2.1. This is a consequence of the porosity of  $\Omega^\pm$  and the fractal uncertainty principle. To apply Proposition 5-5.1, we need to ensure that the scale condition is satisfied, that is to say

$$\delta_2 + \tau > 1$$

which has been supposed when defining  $\tau$  in (5-2.6) and (5-2.7). Theorem 5-2.1 then comes with any  $0 < \gamma < (\delta_2 + \tau - 1)\beta(\nu/3)$ . This ends the proof of Theorem 5-2.1.

#### 5-5.4. From porosity to upper box dimension.

Let us conclude this Chapter with an extra subsection related with the porous sets. We have shown in Lemma 5-5.2 that sets with upper box dimension strictly smaller than one are porous. In this extra section, we show a result in the other way, namely, porous sets down to scale 0 have an upper box dimension strictly smaller than one. The following lemma gives a quantitative version of this statement. This is not useful for our use (we only needed the first implication) but we found that it could be of independent interest. Our proof is based on the proof of Lemma 5.4 in [DJ18b]. We adopt the same notations as in 5-5.1.

**Lemma 5-5.6.** Let  $M \in \mathbb{N}, \nu > 0, \alpha_1 > 0$ . Let  $X \subset [-M, M]$  be a closed set and assume that  $X$  is  $\nu$ -porous on scale 0 to  $\alpha_1$ . Then, there exists  $C = C(\nu, \alpha_1, M) > 0$ ,  $\varepsilon_0 = \varepsilon_0(\nu, \alpha_1, M)$  and  $\delta = \delta(\nu) \in [0, 1)$  such that

$$\forall \varepsilon \leq \varepsilon_0 ; N_X(\varepsilon) \leq C\varepsilon^{-\delta}$$

In particular,

$$\overline{\dim} X \leq \delta$$

*Proof.* We note  $L = \lceil \frac{2}{\nu} \rceil$  and  $k_0$  the unique integer such that

$$L^{-k_0} \leq \alpha_1 < L^{-k_0+1}$$

We will note  $I_{m,k} = [mL^{-k}, (m+1)L^{-k}]$  for  $k \in \mathbb{N}, m \in \mathbb{Z}$ .

We now show by induction on  $k \geq k_0$  that there exists  $Y_k \subset \mathbb{Z}$  such that :

$$\#Y_k \leq 2ML^{k_0}(L-1)^{k-k_0} ; \Omega \subset \bigcup_{m \in Y_k} I_{m,k} \quad (5-5.4)$$

namely, at each level  $k \geq k_0$ , one new interval  $I_{m,k}$  does not intersect  $\Omega$ .

The case  $k = k_0$  is trivial since we simply cover  $\Omega$  by the intervals  $I_{m,k_0}$ , for  $ML^{k_0} \leq m < ML^{k_0}$ . We now assume that the result is proved for  $k \geq k_0$  and we prove it for  $k+1$ . Fix  $m \in Y_k$ . We write  $I = \bigcup_{j=0}^{L-1} I_{mL+j,k+1}$ . We claim that among the intervals  $I_{mL+j,k+1}$ , at least one does not intersect  $\Omega$ . Indeed, since  $|I| \leq L^{-k_0} \leq \alpha_1$ , the porosity of  $\Omega$  implies the existence of an interval  $J \subset I$  of size  $\nu|I| = \nu L^{-k} \geq 2L^{-k-1}$  such that  $J \cap \Omega = \emptyset$ . Since  $|J| \geq 2L^{-k-1}$ ,  $J$  contains at least one of the intervals  $I_{mL+j,k+1}$ . We note this index  $j_m$ . We now set

$$Y_{k+1} = \bigcup_{m \in Y_k} \{mL+j, j \in \{0, \dots, L-1\} \setminus \{j_m\}\}$$

By the property of  $j_m$ ,  $\Omega \subset \bigcup_{m \in Y_{k+1}} I_{m,k+1}$  and  $\#Y_{k+1} \leq (L-1)\#Y_k \leq (L-1)^{k+1-k_0} 2ML^{k_0}$ .

We now consider  $\varepsilon \leq \frac{1}{2}L^{-k_0}$  and write  $k$  the unique integer such that

$$L^{-k} \leq 2\varepsilon < L^{-k+1} \quad i.e. \quad k = \left\lceil \frac{-\log(2\varepsilon)}{\log L} \right\rceil$$

Since we can cover  $\Omega$  by  $2ML^{k_0}(L-1)^{k-k_0}$  closed intervals of size  $L^{-k}$ , we can cover  $\Omega$  by  $4ML^{k_0}(L-1)^{k-k_0}$  open intervals of size  $2\varepsilon$ . Hence,

$$N_\Omega(\varepsilon) \leq 4ML^{k_0}(L-1)^{k-k_0} \leq 4M \left( \frac{L}{L-1} \right)^{k_0} (L-1)^{-\frac{\log(2\varepsilon)}{\log L} + 1} \leq C\varepsilon^{-\delta}$$

with  $\delta = \frac{\log(L-1)}{\log L} \in [0, 1)$  and  $C = 4M \left( \frac{L}{L-1} \right)^{k_0} (L-1)^{1-\frac{\log 2}{\log L}}$ . □



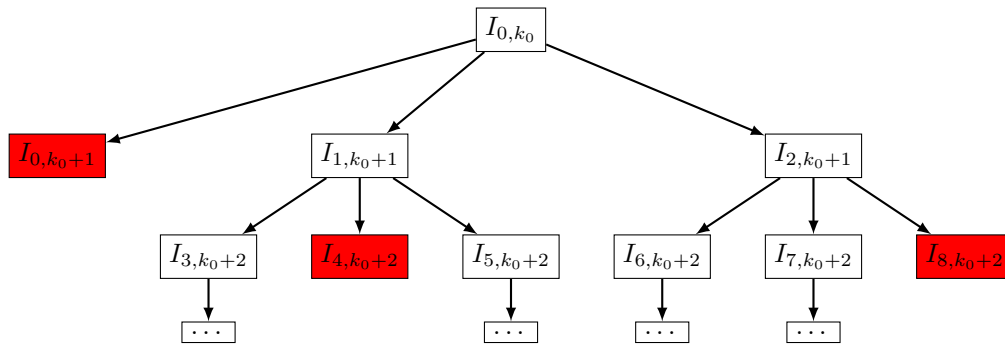


Figure 5.10: It illustrates the tree structure of the family of intervals  $I_{k,m}$  with  $L = 3$ . The porosity allows us to withdraw at least one child to any parent. The missing children are drawn in red.

# Chapter 6

## Applications

In this chapter, we use the crucial result of the previous Chapter 5 given by Theorem 5-2.1 to prove a spectral gap in obstacle scattering and to obtain polynomial resolvent estimates in strips below the real axis. We deduce estimates in  $O(|\lambda| \log |\lambda|)$  for the truncated resolvent  $\chi R(\lambda) \chi$  on the real line and give an application to the decay of the local energy for the wave equation. The material of this chapter is also in the preprint [Vac22a].

### 6-1 Introduction.

#### 6-1.1. Spectral gap and resolvent estimates.

Let  $(\mathcal{O}_j)_{1 \leq j \leq J}$  be open, strictly convex obstacles in  $\mathbb{R}^2$  having smooth boundary and satisfying the *Ikawa condition* of no-eclipse: for  $i \neq j \neq k$ ,  $\overline{\mathcal{O}_i}$  does not intersect the convex hull of  $\overline{\mathcal{O}_j} \cup \overline{\mathcal{O}_k}$ . Let

$$\mathcal{O} = \bigcup_{j=1}^J \mathcal{O}_j ; \quad \Omega = \mathbb{R}^2 \setminus \overline{\mathcal{O}}.$$

Let us note  $R(\lambda)$  the resolvent of the Dirichlet Laplacian  $-\Delta_\Omega$  in  $\Omega$ , continued meromorphically to the logarithmic cover  $\Lambda$  of  $\mathbb{C}$ . In this chapter, we detail the arguments of [NSZ14] and use the main estimate of Chapter 5 in Theorem 5-2.1 to prove the spectral gap announced in the introduction (Theorem A) and to obtain estimates for the cut-off resolvent in this region. We will rather state these resolvent estimates in a semiclassical form, so that it can also be applied to more general semiclassical problems such as the scattering by a smooth compactly potential (see [NSZ11], Section 2, for precise assumptions). We are interested in the semiclassical operator

$$P(h) = -h^2 \Delta_\Omega - 1 \quad , \quad 0 < h \leq h_0$$

and spectral parameter  $z \in [-\delta, \delta] + i[-Kh, Kh]$  for some fixed  $K > 0$  and some  $\delta > 0$ . We note

$$R_h(z) = (P(h) - z)^{-1} \tag{6-1.1}$$

continued meromorphically from  $\text{Im } z > 0$  to  $z \in [-\delta, \delta] + i[-Kh, Kh]$ . We prove :

**Theorem 6-1.1.** *Suppose that  $P(h) = -h^2 \Delta_\Omega - 1$  where  $-\Delta_\Omega$  is the Dirichlet Laplacian in  $\Omega$ , or  $P(h) = -h^2 \Delta + V - E_0$  where  $V \in C_c^\infty(\mathbb{R}^2)$  and  $E_0 \in \mathbb{R}_+^*$  satisfying the assumptions of [NSZ11], recalled in 6-2.3.1.. Let  $\chi \in C_c^\infty(\mathbb{R}^2)$  be equal to one in a neighborhood of  $\overline{\mathcal{O}}$  (in the case of obstacle scattering) or  $\text{supp } V$  (in the case of scattering by a potential). Fix  $K > 0$ . There exists  $\delta_0 > 0$ ,  $\gamma > 0$ ,  $C > 0$ ,  $h_0 > 0$  and  $\beta \geq 0$  such that for all  $0 < h \leq h_0$ ,  $P(h)$  has no resonance in*

$$\mathcal{D}_h := \{z \in \mathbb{C}, \text{Re } z \in [-\delta_0, \delta_0], -\gamma h \leq \text{Im } z \leq Kh\} \tag{6-1.2}$$

and for all  $z \in \mathcal{D}_h$ ,

$$\|\chi R_h(z) \chi\|_{L^2 \rightarrow L^2} \leq Ch^{-\beta} \tag{6-1.3}$$

In particular, Theorem A on the spectral gap is proved.

**Remark.**

In the case of the obstacles, with these notations, for  $\delta$  small enough and  $h$  small enough,  $z$  is related to the spectral parameter  $\lambda_h(z)$  by the relation  $\lambda_h(z)^2 = h^{-2}(1+z)$ . As a consequence,  $\lambda_h(z)$  lies in a neighborhood of  $1/h$  in  $\Lambda$ . In particular, it lives in the first sheet of  $\Lambda$ , that is  $\arg \lambda_h(z) \in ]-\pi/2, \pi/2[$ .

### 6-1.2. Applications.

**Decay of the local energy for the wave equation.** As a first application, we obtain a decay rate  $O(t^{-2})$  for the local energy of the wave equation outside the obstacles. The link between resolvent estimates and energy decay is quite standard now (see for instance [Zwo12], Chapter 5, [Leb96]). In the particular case of obstacle scattering, Ikawa showed exponential decay *in dimension 3*, for the case of two obstacles ([Ika82]) and for more obstacles under a dynamical assumption ([Ika88]) involving the topological pressure  $P(s)$  of the billiard flow. This assumption requires the pressure to be strictly negative at  $s = 1/2$  (see also [NZ09]). In the case of dimension 2 (and more generally, of even dimensions), one cannot expect such an exponential decay due to the logarithmic singularity at 0 for the free resolvent and the fact that the strong Huygens principle does not hold. Even for the free case, the bound for the local energy is  $O(t^{-2})$ . This is the bound we obtain here, assuming that the initial data are sufficiently regular :

**Theorem 6-1.2.** *There exists  $k \in \mathbb{N}$  such that for all  $R > 0$ , there exists  $C_R > 0$  such that the following holds: let  $(u_0, u_1) \in (H^{k+1}(\Omega) \cap H_0^1(\Omega)) \times H^k(\Omega)$  be initial data supported in  $B(0, R) \cap \Omega$  and consider the unique solution of the Cauchy problem*

$$\begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0 & \text{in } \Omega \\ u(t, x) = 0 & \text{on } \partial\Omega \\ u_{t=0} = u_0 \\ \partial_t u_{t=0} = u_1 \end{cases}$$

*Then, for  $t \geq 1$ , the local energy in the ball  $B(0, R)$ ,  $E_R(t)$ , satisfies the bound*

$$E_R(t) := \int_{B(0, R) \cap \Omega} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \leq \frac{C_R}{t^2} (\|u_0\|_{H^{k+1}}^2 + \|u_1\|_{H^k}^2)$$

Theorem 6-1.2 is a consequence of Theorem 6-1.1. This fact is proved in Section 6-3 and the proof uses the strategy of [Bur98].

**Resolvent estimates on the real line.** Polynomial resolvent bounds in strips are known to imply better bounds on the real line, by using a semiclassical maximum principle (see for instance [Bur04], Lemma 4.7, or [Ing18]). As a consequence, we deduce the following estimates on the real line :

**Corollary 6-1.1.** Let  $P(h)$  be one of the operators described in Theorem 6-1.1 and let  $\chi \in C_c^\infty(\mathbb{R}^2)$  as in this Theorem. There exist  $C_0 > 0$ ,  $\delta_0 > 0$  and  $h_0 > 0$  such that for all  $0 < h \leq h_0$  and for all  $z \in [-\delta_0, \delta_0]$ ,

$$\|\chi(P(h) - z)^{-1}\chi\|_{L^2 \rightarrow L^2} \leq C_0 \frac{|\log h|}{h}$$

**Remark.**

As a direct corollary of the proof of Lemma 4.7 in [Bur04], we can obtain a more general bound. For  $h > 0$  small enough,

$$\|\chi(P(h) - z)^{-1}\chi\|_{L^2 \rightarrow L^2} \leq C_0 |\log h| h^{-1+\sigma} |\operatorname{Im} z|^{1/h}, \quad z \in [-\delta_0, \delta_0] + i[-\gamma h, 0] \quad (6-1.4)$$

where  $\sigma > 0$ . With this method, based on the maximum principle for analytic functions, the value of  $\sigma$  is not explicit. In fact, our proof gives a bound of the form

$$\|\chi(P(h) - z)^{-1}\chi\|_{L^2 \rightarrow L^2} \leq C |\log h| h^{-1-M_1-M_2} |\operatorname{Im} z|/h, \quad z \in [-\delta_0, \delta_0] + i[-\gamma h, 0]$$

where  $M_2$  only depends on constants related to the billiard map (see (6-2.15)). The extra  $M_1$  is a consequence of the method we use, based on the use of an escape function. It is possible that a more careful analysis could allow to get rid of this extra  $M_1$  and we could straightly obtain a bound of the form (6-1.4).

This kind of estimates is known to be useful to prove smoothing effects for the Schrödinger equation and to obtain Strichartz estimates, which turns out to be crucial for the local-well posedness of the non-linear Schrödinger equation (see for instance [Bur04], [BGT04]). Let's for instance mention the following smoothing estimates (see the references above for the proof and for pointers to the literature concerning these estimates) :

**Corollary 6-1.2.** Let  $\Omega$  be as in Theorem 6-1.1 and let  $e^{-it\Delta_\Omega}$  be the Schrödinger propagator of the Dirichlet Laplacian  $-\Delta_\Omega$  in  $\Omega$ . Then, for any  $\varepsilon > 0$  and for any  $\chi \in C_c^\infty(\mathbb{R}^2)$  equal to one in a neighborhood of  $\mathcal{O}$ , there exists  $C > 0$  such that for any  $u_0 \in L^2(\Omega)$ ,

$$\|\chi e^{-it\Delta_\Omega} \chi u_0\|_{L^2(\mathbb{R}_t, H^{1/2-\varepsilon}(\Omega))} \leq C \|\chi u_0\|_{L^2}$$

## 6-2 Proof of Theorem 6-1.1

In this section, we prove the main resolvent estimate of this Chapter. The central point, concerning a resolvent bound for open hyperbolic quantum maps, is common to the case of obstacle scattering and scattering by a potential. However, the reduction to open quantum maps differs in the two above situations, this is why we distinguish the two cases.

### 6-2.1. Crucial resolvent bound.

Let  $F : \tilde{D} \rightarrow \tilde{A}$  be an open hyperbolic map, as presented in 3-1.1.. We recall that this formalism relies on :

- open intervals  $Y_1, \dots, Y_J$  of  $\mathbb{R}$  and  $Y = \bigsqcup_{j=1}^J Y_j \subset \bigsqcup_{j=1}^J \mathbb{R}$  ;
- $U = \bigsqcup_{j=1}^J U_j \subset \bigsqcup_{j=1}^J T^*\mathbb{R}^d$  where  $U_j \Subset T^*Y_j$  are open sets;
- For  $j = 1, \dots, J$ , open disjoint subsets  $\tilde{D}_{ij} \Subset U_j$ ,  $1 \leq i \leq J$ , the *departure sets*, and, for  $i = 1, \dots, J$  open disjoint subsets  $\tilde{A}_{ij} \Subset U_i$ ,  $1 \leq j \leq J$ , the *arrival sets* ;
- Smooth symplectomorphisms  $F_{ij} : \tilde{D}_{ij} \rightarrow F_{ij}(\tilde{D}_{ij}) = \tilde{A}_{ij}$

Then,  $F$  is the global smooth map  $F : \tilde{D} \rightarrow \tilde{A}$  where  $\tilde{A}$  and  $\tilde{D}$  are the full arrival and departure sets. Finally, we recall that we note  $\mathcal{T}$  the trapped set of  $F$ , which is supposed to be totally disconnected and  $F$  is supposed to satisfy the assumptions (Hyperbolicity) and (Fractal). We also refer the reader to 4-3 for the definition of open quantum hyperbolic maps (see Definition 4-3.2).

We now consider  $M(h)$  an open quantum hyperbolic map, associated with  $F$ . We suppose that  $M(h)$  is microlocally invertible near  $\mathcal{T}$ . Additionally, we make the following assumption : there exists  $L > 0$  and  $\phi_0 \in C_c^\infty(T^*Y, [0, 1])$  such that  $\operatorname{supp}(\phi_0)$  is contained in a compact neighborhood  $\mathcal{W}$  of  $\mathcal{T}$ ,  $\mathcal{W} \subset \tilde{D}$ ,  $\phi_0 = 1$  in a neighborhood of  $\mathcal{T}$  and

$$M(h)(1 - \operatorname{Op}_h(\phi_0)) = O(h^L) \tag{6-2.1}$$

Let us note  $\alpha_h$  the amplitude of  $M(h)$ , as defined after Definition 4-3.2 and  $\|\alpha_h\|_\infty$  its sup norm in  $\mathcal{W}$ . It is a priori  $h$ -dependent, but it is uniformly bounded in  $h$ . A direct consequence of Theorem 5-2.1 is the following :

**Proposition 6-2.1.** Suppose that  $M(h)$  satisfies the above assumptions. There exists  $\delta > 0$ ,  $\gamma > 0$ ,  $h_0 > 0$  and a family of integer  $N(h) \sim \delta |\log h|$ , defined for  $0 < h \leq h_0$ , such that for all  $0 < h \leq h_0$ ,

$$\|M(h)^{N(h)}\|_{L^2 \rightarrow L^2} \leq h^\gamma \|\alpha_h\|_\infty^{N(h)} \quad (6-2.2)$$

**Remark.**

- Strictly speaking, the result of Chapter 5 applies to operators of the form  $T(h) \text{Op}_h(\alpha)$  where  $T$  is microlocally unitary near  $\mathcal{T}$ . We can reduce (6-2.2) to this case. Indeed, locally near every point  $\rho_0 \in \mathcal{T}$ ,  $M(h)$  takes this form, and  $\mathcal{T}$  is totally disconnected, so that  $M(h)$  takes this form in a small neighborhood of  $\mathcal{T}$ . Finally, as showed in Section 5-3.1., the behavior of  $M(h)$  outside any neighborhood of  $\mathcal{T}$  contributes as a  $O(h^\infty)$  in (6-2.2), as soon as  $N(h)$  is bigger than a fixed  $N_0$  depending on this neighborhood.
- Note that the constant  $\delta$  and  $\gamma$  are purely dynamical, that is, depend only on the dynamics of  $F$  near  $\mathcal{T}$ . Indeed,  $\delta$  is defined in Section 5-2 using only dynamical parameters, such as the Jacobian of  $F$ . Concerning  $\gamma$ , it is implicitly defined using the porosity of the trapped set (see Section 5-5).  $h_0$  depends on  $\alpha$  (through a finite number of semi-norms). This remark will turn out to be important when dealing with scattering by a potential.

This estimate, which is the crucial point in [Vac22b] to prove the spectral gap naturally leads to a resolvent bound for  $(\text{Id} - M(h))^{-1}$  :

**Proposition 6-2.2.** Suppose that  $M(h)$  satisfies the above assumptions. Let  $\gamma$  and  $\delta$  be given in Proposition 6-2.1 and assume that for some  $h_1 > 0$ , for all  $0 < h \leq h_1$ ,

$$\|\alpha_h\|_\infty < \exp\left(\frac{\gamma}{\delta}\right) \quad (6-2.3)$$

Let us consider  $A \geq 1$  such that for all  $0 < h \leq h_1$ ,  $\|\alpha_h\|_\infty \leq A$ . Then, there exists  $h_0 \in ]0, h_1]$  such that for all  $0 < h \leq h_0$ ,

$$\left\| (\text{Id} - M(h))^{-1} \right\|_{L^2 \rightarrow L^2} \leq 2\delta |\log h| h^{-\delta \log A} \quad (6-2.4)$$

*Proof.* First recall that  $M^*M \in \Psi_{0+}$  with  $\sigma_0(M^*M) = \alpha_h^2$  and  $M = O(h^L)$  microlocally outside  $\mathcal{W}$ . Hence, we can estimate the operator norm of  $M(h)$  (see [Zwo12], Theorem 13.13),

$$\|M(h)\|_{L^2 \rightarrow L^2} \leq \|\alpha_h\|_\infty + O(h^\eta)$$

where  $\eta$  is any fixed number in  $]0, 1[$ .

Let  $N(h) \sim \delta |\log h|$  be the family of integers given by Proposition 6-2.1. Without loss of generality, we may assume that  $N(h) \leq \delta |\log h|$ . We use the fact that  $h^\gamma \|\alpha_h\|_\infty^{N(h)} = o(1)$  when  $h \rightarrow 0$  if  $\|\alpha_h\|_\infty < e^{\frac{\gamma}{\delta}}$ . As a consequence,  $\text{Id} - M^{N(h)}$  is invertible for  $h$  small enough with

$$\|(\text{Id} - M^{N(h)})^{-1}\| \leq \frac{3}{2}, \quad 0 \leq h \ll 1 \quad (6-2.5)$$

This implies that  $I - M$  is invertible with inverse

$$(\text{Id} - M)^{-1} = (\text{Id} + M + \dots + M^{N(h)-1})(\text{Id} - M^{N(h)})^{-1} \quad (6-2.6)$$

We hence estimate

$$\begin{aligned} \|\text{Id} + M + \dots + M^{N(h)-1}\| &\leq N(h)(\|\alpha_h\|_\infty + O(h^\eta))^{N(h)} \leq N(h)(A + O(h^\eta))^{N(h)} \\ &\leq \delta |\log h| h^{-\delta \log A} (1 + o(1)) \\ &\leq \frac{4}{3} \delta |\log h| h^{-\delta \log A} \end{aligned}$$

if  $h$  is small enough. Using (6-2.6), we multiply with (6-2.5) and find the required inequality.  $\square$

**Remark.**

The constant 2 can be changed into any  $1 + \varepsilon$  by changing  $h_0$  into  $h_0(\varepsilon)$ . If  $\liminf_{h \rightarrow 0} \|\alpha_h\|_\infty > 1$  we can get rid of the  $\log h$  term by changing it into a constant depending on  $\|\alpha_h\|_\infty$ . More precisely, a better estimate of the sum can show that

$$\left\| (\text{Id} - M(h))^{-1} \right\|_{L^2 \rightarrow L^2} \leq \frac{2}{\|\alpha_h\|_\infty - 1} h^{-\delta \log \|\alpha_h\|_\infty}$$

The main interest of the estimate in Proposition 6-2.2 is that it gives a uniform estimate in the limit  $\|\alpha_h\|_\infty \rightarrow 1$ .

### 6-2.2. Proof in the case of obstacle scattering.

In this subsection, we recall the main ingredients of [NSZ14] and prove the resolvent estimate of Theorem 6-1.1 in obstacle scattering.<sup>1</sup>

Let  $\mathcal{O} = \bigcup_{j=1}^J \mathcal{O}_j$  where  $\mathcal{O}_j$  are open, strictly convex obstacles in  $\mathbb{R}^2$  having smooth boundary and satisfying the *Ikawa's no-eclipse condition* : for  $i \neq j \neq k$ ,  $\overline{\mathcal{O}_i}$  does not intersect the convex hull of  $\overline{\mathcal{O}_j} \cup \overline{\mathcal{O}_k}$ . Let  $P(h) = -h^2 \Delta_\Omega - 1$  and fix a cut-off function  $\chi \in C_c^\infty(\mathbb{R}^2)$  equal to one in a neighborhood of  $\overline{\mathcal{O}}$ . First note that by a simple scaling argument, it is enough to prove (6-1.3) for  $z \in \{z \in D(0, Kh), \text{Im } z \geq -\gamma h\}$  for any  $K > 0$  fixed.

**Complex scaling.** We fix  $R_\chi > 0$  such that  $\text{supp } \chi \subset B(0, R_\chi)$ . For a parameter  $\theta \in ]0, \pi/2[$ , we consider a complex deformation  $\Gamma_\theta \subset \mathbb{C}^2$  of  $\mathbb{R}^2$  such that for some  $R' > R_\chi$ ,

$$\begin{aligned} \Gamma_\theta \cap B_{\mathbb{C}^2}(0, R_\chi) &= \mathbb{R}^2 \cap B_{\mathbb{R}^2}(0, R_\chi) \\ \Gamma_\theta \cap \mathbb{C}^2 \setminus B_{\mathbb{C}^2}(0, R') &= e^{i\theta} \mathbb{R}^2 \cap \mathbb{C}^2 \setminus B_{\mathbb{C}^2}(0, R') \\ \Gamma_\theta &= f_\theta(\mathbb{R}^2) ; f_\theta : \mathbb{R}^2 \rightarrow \mathbb{C}^2 \text{ injective} \end{aligned}$$

By identifying  $\mathbb{R}^2$  and  $\Gamma_\theta$  through  $f_\theta$ , we note  $\Delta_\theta$  the corresponding complex-scaled free Laplacian, and  $\Delta_{\Omega, \theta}$  the complex scaled Laplacian on  $H^2(\Omega) \cap H_0^1(\Omega)$ . We fix  $K > 0$  (which can be chosen arbitrarily large) and for  $z \in D(0, Kh)$ , we note

$$P_\bullet(z) = -h^2 \Delta_\bullet - 1 - z \tag{6-2.7}$$

with either  $\bullet = \theta$  or  $\bullet = \Omega, \theta$ . We note the associated resolvent, when they are defined,

$$R_{\Omega, \theta}(z) : L^2(\Omega) \rightarrow H^2(\Omega) ; R_\theta(z) : L^2(\mathbb{R}^2) \rightarrow H^2(\mathbb{R}^2)$$

#### Remark.

With these notations, the parameter  $\lambda$  of the usual resolvent  $R(\lambda)$  takes the form  $\lambda = \lambda_h(z) = h^{-1}(1+z)^{1/2}$  with  $z \in D(0, Kh) \subset D(0, 1)$  if  $h$  small enough, so that the square root is well defined and gives a holomorphic function of  $z$ .

Thanks to the usual properties of the complex scaling method (see for instance [DZ19], Section 4.5 in Chapter 4 and the references given there), we have :

- The operators  $P_\theta(z)$  and  $P_{\Omega, \theta}(z)$  are Fredholm operators of index 0;
- $z$  is a pole of  $R_{\Omega, \theta}(z)$  if and only if  $\lambda_h(z)$  is a scattering resonance ;
- For  $z$  not a pole of  $R_{\Omega, \theta}(z)$ , in virtue of the properties of  $\chi$  and  $\Gamma_\theta$ , we have :

$$\chi R_{\Omega, \theta}(z) \chi = \chi R_h(z) \chi = h^{-2} \chi R(\lambda_h(z)) \chi$$

- Finally, we recall that we have the following standard estimate for  $R_\theta(z)$  (see for instance [DZ19], Theorem 6.10)

$$\|R_\theta(z)\|_{L^2(\mathbb{R}^2) \rightarrow H_h^2(\mathbb{R}^2)} \leq Ch^{-1} ; z \in D(0, Kh) \tag{6-2.8}$$

In particular, it tells that  $R_\theta$  is holomorphic in  $D(0, Kh)$ . Here,  $H_h^2(\mathbb{R}^2)$  is a semiclassical Sobolev space i.e.  $H^2(\mathbb{R}^2)$  with the norm  $\|u\|_{H_h^2(\mathbb{R}^2)} = \|(1 - h^2 \Delta)u\|_{L^2}$ .

<sup>1</sup>We use notations similar to the ones in [NSZ14] but beware that we do not use the exact same conventions.

To prove Theorem 6-1.1, it is then enough to give a bound for  $\chi R_{\Omega, \theta}(z)\chi : L^2(\Omega) \rightarrow L^2(\Omega)$  in the corresponding region.

**Reduction to the boundary of the obstacles.** Following [NSZ14] (Section 6), we introduce the following operators to obtain a reduction to the boundary. For  $j = 1, \dots, J$ , let

$$\gamma_j : u \in H^2(\Omega) \mapsto u|_{\partial\mathcal{O}_j} \in H^{3/2}(\partial\mathcal{O}_j) \quad (6-2.9)$$

be the (bounded) trace operator and  $\gamma u = (\gamma_j u)_j \in H^{3/2}(\partial\mathcal{O}) := H^{3/2}(\partial\mathcal{O}_1) \times \dots \times H^{3/2}(\partial\mathcal{O}_J)$ , and let

$$H_j(z) : H^{3/2}(\partial\mathcal{O}_j) \rightarrow H^2(\mathbb{R}^2 \setminus \mathcal{O}_j) \xrightarrow{\text{extension by } 0} L^2(\mathbb{R}^2) \quad (6-2.10)$$

be the Poisson operator, defined, for  $v \in H^{3/2}(\partial\mathcal{O}_j)$ , as the solution to the problem

$$\begin{cases} P_\theta(z)H_j(z)v = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\mathcal{O}_j} \\ \gamma_j H_j(z)v = v. \end{cases}$$

$u = H_j(z)v$  is a solution of the problem  $P_\theta(z)u = 0$  with outgoing properties. So as  $P_\theta(z)$ ,  $H_j(z)$  implicitly depends on  $h$ . For  $\vec{v} = (v_1, \dots, v_J) \in H^{3/2}(\partial\mathcal{O})$ , we set

$$H(z)\vec{v} = \sum_{j=1}^J H_j(z)v_j$$

Let us define the following operator-valued matrix  $\mathcal{M}(z) : H^{3/2}(\partial\mathcal{O}) \rightarrow H^{3/2}(\partial\mathcal{O})$  by the relation

$$\text{Id} - \mathcal{M}(z) = \gamma H(z) \quad (6-2.11)$$

We state a few facts concerning these operators. In the following lemma, we give estimates involving the semiclassical version of the Sobolev spaces  $H^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j})$  and  $H^{3/2}(\partial\mathcal{O}_j)$ , denoted  $H_h^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j})$  and  $H_h^{3/2}(\partial\mathcal{O}_j)$  respectively.

**Lemma 6-2.1.** For  $j = 1, \dots, J$ , there exists  $C > 0$  such that for all  $0 < h \leq 1$ , the norm of the bounded operator  $\gamma_j$  from  $H_h^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j})$  to  $H_h^{3/2}(\partial\mathcal{O}_j)$  satisfies

$$\|\gamma_j\|_{H_h^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j}) \rightarrow H_h^{3/2}(\partial\mathcal{O}_j)} \leq Ch^{-1/2}$$

*Proof.* Using a partition of unity argument and local charts, it is sufficient to prove that the above result holds with  $\mathbb{R}^2 \setminus \overline{\mathcal{O}_j}$  replaced by  $\mathbb{R} \times \mathbb{R}_+^*$  and  $\partial\mathcal{O}_j$  replaced by  $\mathbb{R}$ . In this setting, we note  $\gamma$  the associated trace operator. First, we extend an element  $u \in H_h^2(\mathbb{R} \times \mathbb{R}_+^*)$  to an element  $\tilde{u} \in H_h^2(\mathbb{R} \times \mathbb{R})$  such that  $\|\tilde{u}\|_{H_h^2} \leq C\|u\|_{H_h^2}$  (see for instance [Eva10], Chapter 5, Section 4 : in the proof of Theorem 1, one can extend  $u \in H_h^2(\mathbb{R} \times \mathbb{R}_+^*)$  with the formula : for  $y > 0$ ,  $u(x, -y) = -3u(x, y) + 4u(x, -y/2)$ ). Then we observe that, with  $\mathcal{F}_h^1$  (resp.  $\mathcal{F}_h$ ) the semiclassical unitary Fourier transform in 1D (resp. 2D),

$$\mathcal{F}_h^1(\gamma u)(\xi) = \frac{1}{(2\pi h)^{1/2}} \int_{\mathbb{R}} \mathcal{F}_h(\tilde{u})(\xi, \eta) d\eta$$

From, this we get

$$\|\gamma u\|_{H_h^{3/2}} \leq Ch^{-1/2} \|\tilde{u}\|_{H_h^2} \quad (6-2.12)$$

Indeed, by Cauchy-Schwarz, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{F}_h(\tilde{u})(\xi, \eta) d\eta \right|^2 &\leq \left( \int_{\mathbb{R}} |\mathcal{F}_h(\tilde{u})(\xi, \eta)|^2 (1 + \xi^2 + \eta^2)^2 d\eta \right) \left( \int_{\mathbb{R}} (1 + \xi^2 + \eta^2)^{-2} d\eta \right) \\ &\leq \left( \int_{\mathbb{R}} |\mathcal{F}_h(\tilde{u})(\xi, \eta)|^2 (1 + \xi^2 + \eta^2)^2 d\eta \right) (1 + \xi^2)^{-3/2} \int_{\mathbb{R}} (1 + \eta^2)^{-2} d\eta \end{aligned}$$

We find (6-2.12) by multiplying by  $(1 + \xi^2)^{3/2}$  and integrating over  $\xi$ . This concludes the proof.  $\square$

**Lemma 6-2.2.** For  $j = 1, \dots, J$ , for any  $K > 0$ , there exists  $h_0 > 0$  such that for all  $0 < h \leq h_0$ ,  $\chi H_j(z)$  is holomorphic in  $D(0, Kh)$  and satisfies for some  $C > 0$  independent of  $h$ , and for  $z \in D(0, Kh)$ ,

$$\|\chi H_j(z)\|_{H_h^{3/2}(\partial\mathcal{O}_j) \rightarrow L^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j})} \leq Ch^{-1/2}$$

*Proof.* We follow the main lines of the proof of Lemma 6.1 in [NSZ14].

First, let us introduce an extension operator  $T_j^h : H_h^{3/2}(\partial\mathcal{O}_j) \rightarrow H_h^2(\mathbb{R}^2)$  such that for  $v \in H_h^{3/2}(\partial\mathcal{O}_j)$ ,  $T_j^h v$  is supported in a small neighborhood of  $\partial\mathcal{O}_j$  and

$$T_j^h = O(h^{1/2}) : H_h^{3/2}(\partial\mathcal{O}_j) \rightarrow H_h^2(\mathbb{R}^2)$$

This is possible, for instance by taking the extension operator given in the proof of Lemma 6.1 in [NSZ14]. Another approach consists in using a partition of unity and local charts to replace  $\partial\mathcal{O}_j$  by  $\mathbb{R}$ , as in the proof of Lemma 6-2.1. Then, one can consider the following operator

$$T_h : v \in H_h^{3/2}(\mathbb{R}) \mapsto T_h v \in H_h^2(\mathbb{R}^2) ; T_h v(x, y) = \left( \chi \left( \frac{x}{h} \langle hD_y \rangle \right) v \right) (y)$$

where  $\chi \in C_c^\infty(\mathbb{R})$ ,  $\chi(0) = 1$ . Then,  $T_h v(0, y) = v$  and one has

$$\|T_h v\|_{H_h^2(\mathbb{R}^2)} \leq Ch^{1/2} \|v\|_{H_h^{3/2}(\mathbb{R})}$$

Indeed, one has

$$\mathcal{F}_h(T_h v)(\xi, \eta) = h^{1/2} \langle \eta \rangle^{-1} \mathcal{F}_1^1(\chi) \left( \frac{\xi}{\langle \eta \rangle} \right) \times \mathcal{F}_h^1(v)(\eta)$$

and hence

$$\begin{aligned} \|T_h v\|_{H_h^2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} |\mathcal{F}_h(T_h v)(\xi, \eta)|^2 (1 + \xi^2 + \eta^2)^2 d\xi d\eta \\ &\leq h \int \langle \eta \rangle^{-2} \left| \mathcal{F}_1^1(\chi) \left( \frac{\xi}{\langle \eta \rangle} \right) \right|^2 |\mathcal{F}_h^1(v)(\eta)|^2 (1 + \xi^2)^2 (1 + \eta^2)^2 d\eta d\xi \\ &\leq h \left( \int_{\mathbb{R}} |\mathcal{F}_1^1(\chi)(\xi)|^2 (1 + \xi^2)^2 d\xi \right) \left( \int_{\mathbb{R}} |\mathcal{F}_h^1(v)(\eta)|^2 (1 + \eta^2)^{3/2} d\eta \right) \\ &\leq Ch \|v\|_{H_h^{3/2}(\mathbb{R})}^2 \end{aligned}$$

We then assume that for all  $v \in H_h^{3/2}(\partial\mathcal{O}_j)$ ,  $\text{supp}(T_j^h v) \subset \text{supp} \chi$ . Then, we claim that

$$H_j(z) = \mathbf{1}_{\mathbb{R}^2 \setminus \overline{\mathcal{O}_j}} T_j^h - R_{j,\theta}(z) \mathbf{1}_{\mathbb{R}^2 \setminus \overline{\mathcal{O}_j}} P_\theta(z) T_j^h$$

where  $R_{j,\theta}(z)$  is the resolvent of the complex scaled Dirichlet realization of  $-h^2\Delta - 1$  on  $\mathbb{R}^2 \setminus \overline{\mathcal{O}_j}$ . Indeed, the boundary condition on  $\partial\mathcal{O}_j$  is satisfied since  $\text{Ran}(R_{j,\theta}) \subset H_0^1(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j})$ , and by definition,  $P_\theta R_{j,\theta} w = w$  in  $\mathbb{R}^2 \setminus \overline{\mathcal{O}_j}$ , for  $w \in L^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j})$ .

As a consequence, it suffices to show that  $\chi R_{j,\theta}(z) \chi$  is holomorphic in  $D(0, Kh)$  with the bound

$$\chi R_{j,\theta}(z) \chi = O(h^{-1}) : L^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j}) \rightarrow L^2(\mathbb{R}^2 \setminus \overline{\mathcal{O}_j})$$

This is a rather standard non-trapping estimates (here, when there is a single obstacle, the billiard flow is non-trapping). As explained in [NSZ14] in the proof of Lemma 6.1, such an estimate relies on propagation of singularities concerning the wave propagator : one can check that an abstract non-trapping condition for black box Hamiltonian is satisfied (see for instance [DZ19], Definition 4.42). This implies that the required statement ([DZ19], Theorem 4.43) holds.  $\square$

Finally, we recall the crucial relation between  $R_{\Omega,\theta}(z)$  and  $\mathcal{M}(z)$  (see [NSZ14], formula 6.11 and the references given there). Assume that  $z \in D(0, Kh)$  and that  $\text{Id} - \mathcal{M}(z)$  is invertible. Then, so is  $R_{\Omega,\theta}(z)$  and we have

$$R_{\Omega,\theta}(z) = \mathbf{1}_\Omega R_\theta(z) - \mathbf{1}_\Omega H(z) (\text{Id} - \mathcal{M}(z))^{-1} \gamma R_\theta(z) \quad (6-2.13)$$



In particular, we see that if we have a bound for

$$\|(\text{Id} - \mathcal{M}(z))^{-1}\|_{H_h^{3/2} \rightarrow H_h^{3/2}}$$

we find a resolvent bound for  $R_{\Omega, \theta}(z)$ . In fact, as explained in [NSZ14], it is sufficient to work on  $L^2(\partial\mathcal{O})$  in virtue of the following result :

**Lemma 6-2.3.** ([NSZ14], Lemma 6.5) For  $j = 1, \dots, J$ , let  $B^*\partial\mathcal{O}_j := \{(y, \eta), y \in \partial\mathcal{O}_j, |\eta| \leq 1\}$  and consider  $\chi_j \in C_c^\infty(T^*\partial\mathcal{O}_j)$  such that  $\chi_j = 1$  near  $B^*\partial\mathcal{O}_j$ . Then, by denoting  $\text{Op}_h(\chi_j)$  a quantization of  $\chi_j$  and by  $D$  the diagonal operator-valued matrix  $\text{Diag}(\text{Op}_h(\chi_1), \dots, \text{Op}_h(\chi_J))$ , we have

$$\begin{aligned} (\text{Id} - D)(\text{Id} - \mathcal{M}(z)) &= O(h^\infty)_{L^2(\partial\mathcal{O}) \rightarrow C^\infty(\partial\mathcal{O})} \\ (\text{Id} - \mathcal{M}(z))(\text{Id} - D) &= O(h^\infty)_{L^2(\partial\mathcal{O}) \rightarrow C^\infty(\partial\mathcal{O})} \end{aligned}$$

As a consequence of this lemma,  $\text{Id} - \mathcal{M}(z)$  extends to an operator  $L^2(\partial\mathcal{O}) \rightarrow L^2(\partial\mathcal{O})$  and as soon as  $\text{Id} - \mathcal{M}(z)$  is invertible and  $z \in D(0, Kh)$

$$\|(\text{Id} - \mathcal{M}(z))^{-1}\|_{H_h^{3/2} \rightarrow H_h^{3/2}} \leq C_1 \|(\text{Id} - \mathcal{M}(z))^{-1}\|_{L^2 \rightarrow L^2}$$

(with a constant  $C_1$  independent of  $z$ ).

**Microlocal properties of  $\mathcal{M}(z)$  and reduction to a simpler problem.** We recall the main microlocal properties of  $\mathcal{M}(z)$  and reduce the invertibility of  $\text{Id} - \mathcal{M}(z)$  to a nicer Fourier integral operator, as explained in [NSZ14] (Section 6). To do so, let us introduce the following notations.

For  $j \in \{1, \dots, J\}$ , let  $B^*\partial\mathcal{O}_j$  be the co-ball bundle of  $\partial\mathcal{O}_j$ ,  $S_{\partial\mathcal{O}_j}^*$  be the restriction of  $S^*\Omega$  to  $\partial\mathcal{O}_j$ ,  $\pi_j : S_{\partial\mathcal{O}_j}^* \rightarrow B^*\partial\mathcal{O}_j$  the natural projection and  $\nu_j(x)$  be the outward normal vector at  $x \in \partial\mathcal{O}_j$  (see Figure 6.1a).

For  $i \neq j$ , let  $\mathcal{B}_{ij}^\pm : B^*\partial\mathcal{O}_j \rightarrow B^*\partial\mathcal{O}_i$  be the symplectic open maps defined by

$$\begin{aligned} \rho &= \mathcal{B}_{ij}^\pm(\rho') \iff \exists t > 0, \exists \xi \in \mathbb{S}^1, \exists x \in \partial\mathcal{O}_j \\ x + t\xi &\in \partial\mathcal{O}_i, \langle \nu_j(x), \xi \rangle > 0, \pm \langle \nu_i(x + t\xi), \xi \rangle < 0, \\ \pi_j(x, \xi) &= \rho', \pi_i(x + t\xi, \xi) = \rho \end{aligned}$$

$\mathcal{B}_{ij}^+$  is the billiard map as defined in 3-1.1.1., whereas  $\mathcal{B}_{ij}^-$  is a shadow map (see Figure 6.1b and 6.1c). These maps are open. (see Figure 6.1d). Note that due to our definition of these maps, the glancing rays (that is the rays associated with a point  $\rho = (y, \eta) \in B^*\partial\mathcal{O}$  with  $|\eta| = 1$ ) are not in the set of definition of  $\mathcal{B}_{ij}^\pm$ . Moreover, due to Ikawa's condition, if a point  $\rho \in B^*\partial\mathcal{O}_j$  has an image by  $\mathcal{B}_{ij}^\pm$ , it cannot have one by  $\mathcal{B}_{kj}^\pm$  for  $k \neq i$ .

Let  $\bar{A}_{ij}$  be the closure of the arrival set of the billiard map, that is

$$\bar{A}_{ij} = \overline{\{\rho \in B^*\partial\mathcal{O}_i, \exists \rho' \in B^*\partial\mathcal{O}_j, \rho = \mathcal{B}_{ij}^+(\rho')\}}$$

Similarly, let  $\bar{D}_{ij}$  be the closure of the departure set of the billiard map, that is

$$\bar{D}_{ij} = \overline{\{\rho' \in B^*\partial\mathcal{O}_j, \exists \rho \in B^*\partial\mathcal{O}_i, \rho = \mathcal{B}_{ij}^+(\rho')\}}$$

We also note

$$\bar{A}_i = \bigcup_{j \neq i} \bar{A}_{ij}; \quad \bar{D}_i = \bigcup_{j \neq i} \bar{D}_{ji}$$

Finally, we introduce the arrival and departure glancing regions :

$$\tilde{A}_i^{\mathcal{G}} = \bar{A}_i \cap S^*\partial\mathcal{O}_i; \quad \tilde{D}_i^{\mathcal{G}} = \bar{D}_i \cap S^*\partial\mathcal{O}_i$$

We recall the main facts proved in [NSZ14] concerning these relations and their link with  $\mathcal{M}(z)$  :

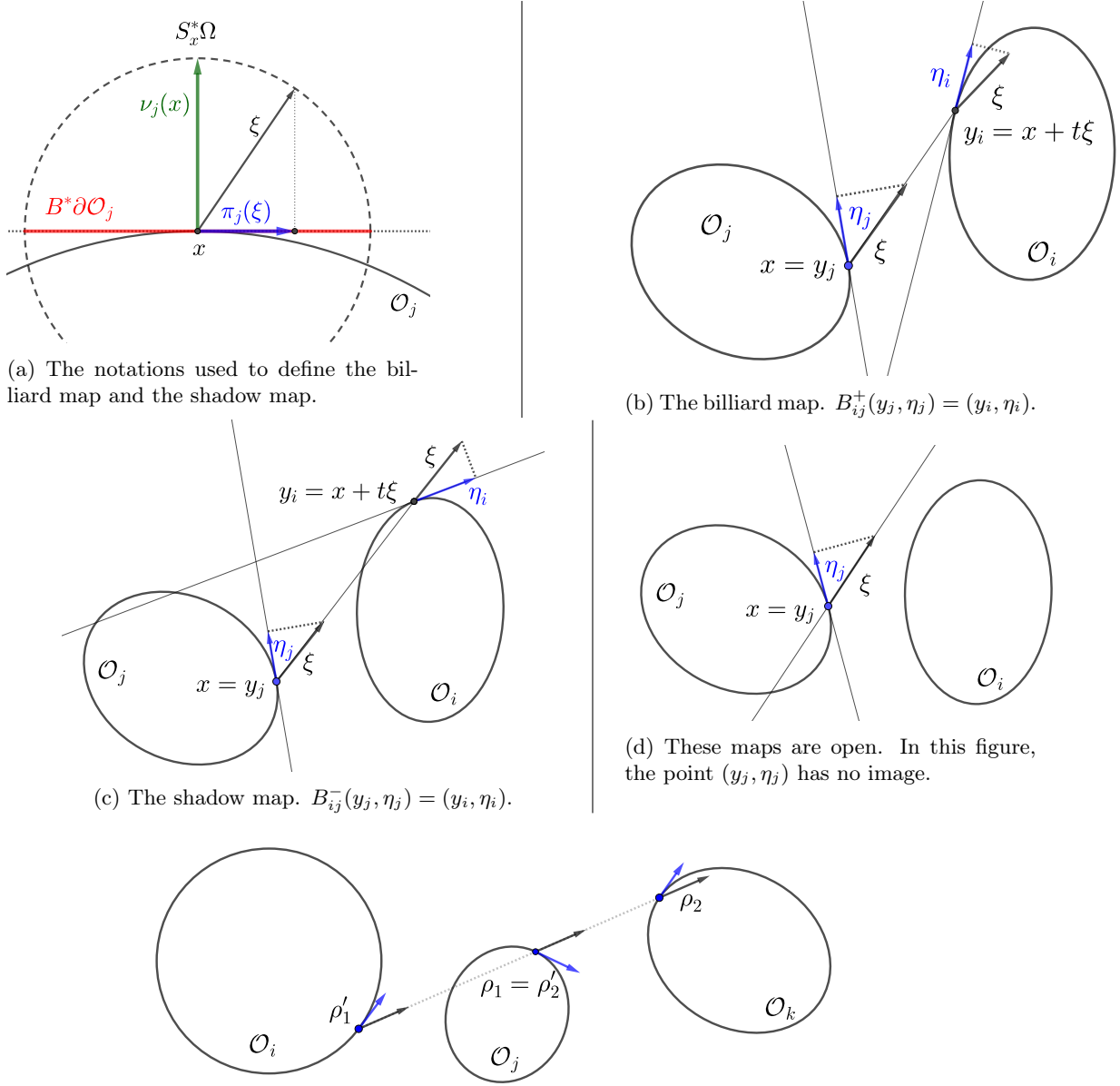


Figure 6.1: Description of the billiard map (6.1b) and the shadow map (6.1d), with the notations recalled in (6.1a). These maps are open, as shown in (6.1d) and the no-eclipse condition prevents some situations in the composition of these maps (6.1e).

**Lemma 6-2.4.** (See Figure 6.1e). Assuming that the obstacles satisfy the no-eclipse condition, the following holds : let  $i \neq j \neq k$  and let  $(\rho'_1, \rho_1) \in \overline{\text{Gr}(\mathcal{B}_{ji}^-)}$  and  $(\rho'_2, \rho_2) \in \overline{\text{Gr}(\mathcal{B}_{kj}^\pm)}$ . Then,

$$\rho_1 \neq \rho'_2$$

In particular, it is possible to consider open neighborhoods  $U_i^A$  and  $U_i^D$  of  $\tilde{A}_i^{\mathcal{G}}$  and  $\tilde{D}_i^{\mathcal{G}}$  respectively, such that (see Figure 6.2), by noting  $\pi_R$  (resp.  $\pi_L$ ) the projection  $(\rho', \rho) \mapsto \rho$  (resp.  $(\rho', \rho) \mapsto \rho'$ )

$$\rho' \in U_i^A \implies \rho' \notin \bigcup_{k \neq i} \pi_R(\text{Gr}(\mathcal{B}_{ki}^\pm)) \quad (\text{see Figure 6.2a : it means that the ray continues to infinity})$$

$$\rho \in U_i^D \implies \rho \notin \bigcup_{k \neq i} \pi_L(\text{Gr}(\mathcal{B}_{ik}^\pm)) \quad (\text{see Figure 6.2b : it means that the ray comes from infinity})$$

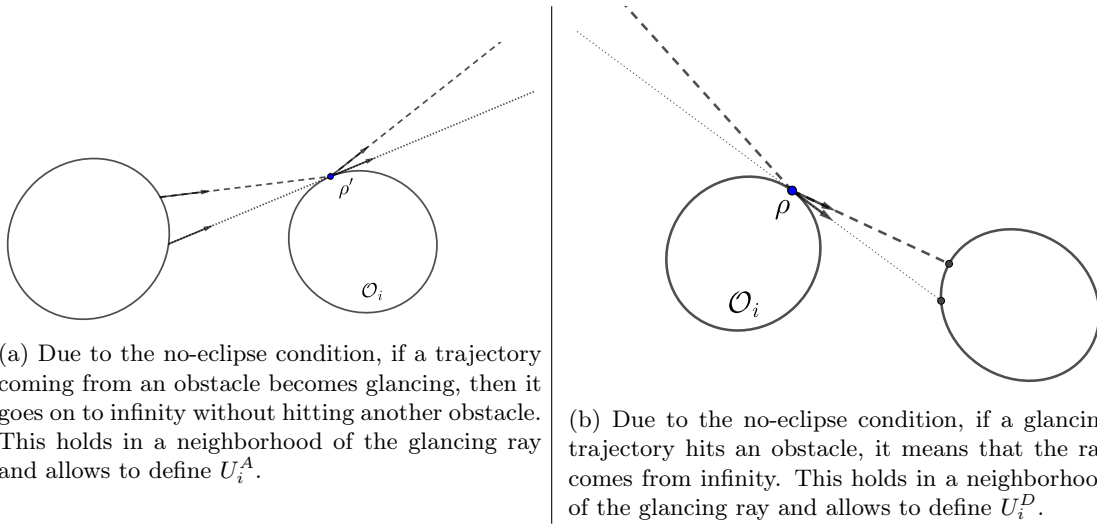


Figure 6.2: The sets  $U_i^A$  and  $U_i^D$  are built by using the properties of the glancing rays. The dotted lines correspond to glancing rays, the broken lines represent trajectories close to the glancing region.

Let us fix cut-off functions  $\chi_i^D$  (resp.  $\chi_i^A$ ) such that  $\chi_i^D = 1$  near  $D_i^{\mathcal{G}}$  (resp.  $\chi_i^A = 1$  near  $A_i^{\mathcal{G}}$ ) and  $\text{supp } \chi_i^D \subset U_i^D$  (resp.  $\text{supp } \chi_i^A \subset U_i^A$ ).

We gather the results of Proposition 6.7 in [NSZ14] and some of its consequence in the following proposition. It is based on the microlocal analysis of the operators involved.

**Proposition 6-2.3.** For  $i \neq j$ ,

- uniformly in  $z \in D(0, Kh)$ ,

$$\mathcal{M}_{ij}(z) \text{Op}_h(\chi_j^A) = O(h^\infty)_{L^2 \rightarrow C^\infty} ; \text{Op}_h(\chi_i^D) \mathcal{M}_{ij}(z) = O(h^\infty)_{L^2 \rightarrow C^\infty}$$

- By excluding the glancing region on the left and on the right, we have

$$(1 - \text{Op}_h(\chi_i^D)) \mathcal{M}_{ij}(z) (1 - \text{Op}_h(\chi_j^A)) \in I_0(\partial \mathcal{O}_i \times \partial \mathcal{O}_j, \text{Gr}(\mathcal{B}_{ij}^+))' + I_0(\partial \mathcal{O}_i \times \partial \mathcal{O}_j, \text{Gr}(\mathcal{B}_{ij}^-))'$$

so let us write

$$(1 - \text{Op}_h(\chi_i^D)) \mathcal{M}_{ij}(z) (1 - \text{Op}_h(\chi_j^A)) = \mathcal{M}_{ij}^+(z) + \mathcal{M}_{ij}^-(z)$$

with  $\mathcal{M}_{ij}^\pm(z) \in I_0(\partial \mathcal{O}_i \times \partial \mathcal{O}_j, \text{Gr}(\mathcal{B}_{ij}^\pm))'$ . Only compact parts of the interior of the graphs of  $\mathcal{B}_{ij}^\pm$  are involved in the definition of the class  $I_0(\partial \mathcal{O}_i \times \partial \mathcal{O}_j, \text{Gr}(\mathcal{B}_{ij}^\pm))'$ , depending on the support of  $\chi_i^D$  and  $\chi_j^A$ .

- The operators  $\mathcal{M}_{ij}^\pm(z)$  have amplitude  $\alpha_{ij}^\pm(z)$  satisfying, for  $z \in D(0, Kh)$  and for some  $C_1, \tau > 0$ ,

$$\alpha_{ij}^\pm(z) \leq C_1 e^{-\tau \frac{\text{Im} z}{h}}$$

- Finally, in virtue of Lemma 6-2.4,  $\mathcal{M}_{ij}^\pm(z) \circ \mathcal{M}_{jk}^\mp(z) = O(h^\infty)_{L^2 \rightarrow C^\infty}$  uniformly for  $z \in D(0, Kh)$ .

Let us note  $\mathcal{M}^\pm(z)$  the matrix of operators with

$$(\mathcal{M}^\pm(z))_{ij} = \begin{cases} \mathcal{M}_{ij}^\pm(z) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

Then, we observe that

$$(\text{Id} - \mathcal{M}(z)) (\text{Id} + \mathcal{M}^-(z)) = \text{Id} - \mathcal{M}^+(z) + O(h^\infty)_{L^2 \rightarrow C^\infty} \tag{6-2.14}$$

Since we are interested in invertibility in strips, let's note :

$$\Omega(\gamma, K, h) = D(0, Kh) \cap \{\text{Im } z \geq -\gamma h\}$$

We have the rather obvious lemma :

**Lemma 6-2.5.** Assume that for  $z \in \Omega(\gamma, K, h)$ ,  $\text{Id} - \mathcal{M}^+(z)$  is invertible and satisfies the bound

$$\|(\text{Id} - \mathcal{M}^+(z))^{-1}\|_{L^2 \rightarrow L^2} \leq a(z, h)$$

with  $a(z, h) \leq h^{-N}$  for some  $N$  independent of  $z \in \Omega(\gamma, K, h)$ . Then, there exists  $h_0 > 0$  and  $C > 0$ , such that for  $0 < h \leq h_0$ , and for all  $z \in \Omega(\gamma, K, h)$ ,  $\text{Id} - \mathcal{M}(z)$  is invertible and satisfies

$$\|(\text{Id} - \mathcal{M}(z))^{-1}\|_{L^2 \rightarrow L^2} \leq Ca(z, h)$$

*Proof.* Assuming the invertibility of  $\text{Id} - \mathcal{M}^+(z)$ , it suffices to write

$$(\text{Id} - \mathcal{M}(z)) (\text{Id} + \mathcal{M}^-(z)) (\text{Id} - \mathcal{M}^+(z))^{-1} = \text{Id} + R(z, h)$$

with  $R(z, h) = (\text{Id} - \mathcal{M}^+(z))^{-1} O(h^\infty)_{L^2 \rightarrow C^\infty} = O(h^\infty)_{L^2 \rightarrow L^2}$  uniformly in  $z$ . We conclude by a Neumann series argument to invert the right hand side  $\text{Id} + R(z, h)$  and use the bound on the amplitude of  $\mathcal{M}^-(z)$  given in Proposition 6-2.3, which gives a uniform bound for  $\mathcal{M}^-(z)$  in  $D(0, Kh)$ .  $\square$

It is then enough to prove the invertibility of  $\text{Id} - \mathcal{M}^+(z)$  with polynomial resolvent bounds.

**Conjugation by an escape function.** The operator  $\mathcal{M}^+(z)$  satisfies almost all the assumptions of Proposition 6-2.1 for the relation  $F = \mathcal{B}^+$ , except that it is not very small outside a fixed compact neighborhood of  $\mathcal{T}$ <sup>2</sup>. To fix this problem, following [NSZ14] (Section 6.3), we can introduce a smooth escape function  $g_0$ . Recall that  $\mathcal{T}$  is the trapped set for  $F = \mathcal{B}^+$  and let  $\mathcal{W}_1 \Subset \mathcal{W}_2 \Subset \mathcal{W}_3$  be subsets of  $\bigcup_i \widetilde{D}_i$  such that  $\mathcal{T} \subset \mathcal{W}_1$  and such that  $\mathcal{W}_3$  is large enough so that  $\mathcal{M}^+(z) \text{Op}_h(\phi) = O(h^\infty)_{L^2 \rightarrow C^\infty}$  for any smooth function  $\phi$  such that  $\text{supp}(\phi) \cap \mathcal{W}_3 = \emptyset$ . This is possible in virtue of the third point in Proposition 6-2.3. Concerning  $\mathcal{W}_2$ , it can be an arbitrarily small neighborhood of  $\mathcal{T}$ . Then, one can construct  $g_0$  such that (see Lemma 4.5 in [NSZ14]),

$$\begin{aligned} g_0 &= 0 \text{ in } \mathcal{W}_1 \\ g_0 \circ F - g_0 &\geq 0 \text{ in } \mathcal{W}_3 \\ g_0 \circ F - g_0 &\geq 1 \text{ in } \mathcal{W}_3 \setminus \mathcal{W}_2 \end{aligned}$$

Then, we set  $g = T \log(1/h) g_0$  for some  $T > 0$  fixed and large enough, so that  $e^{\pm \text{Op}_h(g)}$  are pseudodifferential operators and satisfies

$$e^{\pm \text{Op}_h(g)} = O(h^{-TC_0}) ; C_0 = \|g_0\|_\infty$$

---

<sup>2</sup>strictly speaking,  $\partial\mathcal{O}$  is not a disjoint union of intervals, but since we work with the relation  $\mathcal{B}^+$ , we can use microlocal cut-offs to restrict to the relevant part of the obstacles, which is included in a disjoint union of open intervals

(Note that  $\text{Op}_h(g)$  is a diagonal matrix-valued operator on  $L^2(\partial\mathcal{O}) = \bigoplus L^2(\partial\mathcal{O}_i)$ ), and in virtue of Egorov's theorem, the operator

$$\mathcal{M}_g^+ := e^{-\text{Op}_h(g)} \mathcal{M}^+(z) e^{\text{Op}_h(g)}$$

is  $O(h^L)$  for some  $L > 0$ , microlocally outside a neighborhood of  $\mathcal{T}$ , which can be made as small as necessary if  $\mathcal{W}_2$  is small enough.

**End of proof.** We can now apply Proposition 6-2.1 and then, Proposition 6-2.2, to  $M(h) = \mathcal{M}_g^+(z)$  for  $z \in D(0, Kh)$  with  $\text{Im } z \geq -\gamma h$ . To control the amplitude  $\alpha_h(z)$  of  $\mathcal{M}_g^+(z)$ , we simply need a bound in a small neighborhood of  $\mathcal{T}$  in which  $\mathcal{M}_g^+$  is not  $O(h^L)$ . In virtue of Egorov's theorem, the amplitude of  $\mathcal{M}_g^+$  is smaller than the amplitude of  $\mathcal{M}^+$ . We now claim that there exists  $\tau > 0$  such that the amplitude satisfies :

$$\|\alpha_h(z)\|_\infty \leq e^{-\tau \frac{\text{Im } z}{h}}$$

In fact, as explained in [Non11] (Theorem 6), microlocally near the trapped set, it is possible to write

$$M^+(z) = M^+(0) \text{Op}_h(a_{h,z}) + O(h^{1-\varepsilon}); \quad a_{h,z}(\rho) = \exp\left(\frac{izt(\rho)}{h}\right)$$

where  $t(\rho)$  is the time needed for a ray emanating from  $\rho$  to hit another obstacle. This fact is a consequence of the microlocal analysis performed in [Gér88] (see Appendix II for the construction of a parametrix and III.2 for precise computations near the unique trapped ray for two obstacles, see also [SV95]). In particular,  $\tau$  in the estimate above is a maximal return time for the billiard flow, in a small neighborhood of  $\mathcal{T}$ .

Now, let  $\gamma, \delta$  be the constants given by Proposition 6-2.1, depending, in this context, on the dynamics of the billiard map. Let us introduce the following threshold  $\gamma_{\text{lim}} = \frac{1}{2\tau} \frac{\gamma}{\delta}$  so that

$$z \in \Omega(\gamma_{\text{lim}}, K, h) \implies \|\alpha_h(z)\|_\infty \leq e^{\gamma/2\delta} < e^{\gamma/\delta}$$

Proposition 6-2.2 now gives for  $z \in \Omega(\gamma_{\text{lim}}, K, h)$ ,

$$\left\| (\text{Id} - \mathcal{M}_g^+(z))^{-1} \right\| \leq 2\delta |\log h| h^{-\delta \log A}$$

where

$$A := \max(1, \tau \gamma_{\text{lim}})$$

Indeed,  $A \geq 1$  and it allows to have  $\|\alpha_h(z)\|_\infty \leq A$  for  $z \in \Omega(\gamma_{\text{lim}}, K, h)$ . Going back to  $\mathcal{M}^+(z)$ , we get that

$$\left\| (\text{Id} - \mathcal{M}^+(z))^{-1} \right\| \leq C |\log h| h^{-\delta \log A - 2C_0 T}$$

where the extra  $h^{-2C_0 T}$  comes from the norm of  $e^{\pm \text{Op}_h(g)}$ . We conclude with Lemma 6-2.5 and the formula (6-2.13), using the estimates of Lemma 6-2.1 and 6-2.2. This gives for  $h$  small enough and  $z \in \Omega(\gamma_{\text{lim}}, K, h)$ ,

$$\|R_\theta(z)\|_{L^2 \rightarrow L^2} \leq C |\log h| h^{-1-2C_0 T - \delta \log A} \quad (6-2.15)$$

### 6-2.3. Proof in the case of scattering by a potential.

The treatment of scattering by a potential is different and relies on a reduction to Poincaré sections of the Hamiltonian flow, under the assumption that the trapped set is totally disconnected.

#### 6-2.3.1. Assumptions.

We refer the reader to [NSZ11] (Section 2.1) for more general assumptions. Here, we simply consider a smooth compactly supported potential  $V \in C_c^\infty(\mathbb{R}^2)$  and work with the semiclassical differential operator  $-h^2 \Delta + V$ . We fix an energy  $E_0 > 0$  and consider

$$P_h = -h^2 \Delta + V - E_0$$

We note  $p(x, \xi) = \xi^2 + V - E_0$  and we assume that 0 is not a critical energy of  $p$ , that is

$$dp \neq 0 \text{ on } p^{-1}(0)$$

Let's note  $H_p$  the Hamiltonian vector field associated with  $p$  and  $\Phi_t = \exp(tH_p)$  the corresponding Hamiltonian flow. The trapped set at energy 0 is the set

$$K_0 = \{(x, \xi) \in p^{-1}(0), \exists R > 0, \forall t \in \mathbb{R}, \Phi_t(x, \xi) \in B(0, R)\}$$

It is a compact subset of  $p^{-1}(0)$ . Here are the two crucial assumptions :

- (i)  $\Phi_t$  is hyperbolic on  $K_0$  ;
- (ii)  $K_0$  is topologically one dimensional.

### 6-2.3.2. The reduction of [NSZ11].

We recall the main ingredients of the reduction to open quantum maps performed in [NSZ11]. The aim of the following lines is to explain their crucial Theorem 5.

Let us note

$$\mathcal{R}(\eta, M_0, h) = \{z \in \mathbb{C}, |\operatorname{Re} z| \leq \eta, |\operatorname{Im} z| \leq M_0 h\}$$

Here,  $M_0$  is fixed (but large). As in the case of obstacle scattering, we fix once and for all the cut-off function  $\chi \in C_c^\infty(\mathbb{R}^2)$  (with  $\chi = 1$  in a neighborhood of  $\operatorname{supp}(V)$ ) and we consider a complex scaled version of  $P(h)$ ,  $P_\theta(h)$  whose eigenvalues coincide with the resonances in  $\mathcal{R}(\eta, M_0, h)$  and such that  $\chi(P_\theta - z)^{-1}\chi = \chi(P - z)^{-1}\chi$  for  $z \in \mathcal{R}(\delta, M_0, h)$ . Note that the parameter  $\theta$  chosen in [NSZ11] depends on  $h$ .

Here are now the crucial ingredients of the reduction.

- **Poincaré sections.** There exist finitely many smooth contractible hypersurfaces  $\Sigma_i \subset p^{-1}(0)$ ,  $i = 1, \dots, J$  with smooth boundary and such that

$$\partial\Sigma_i \cap K_0 = \emptyset ; \Sigma_i \cap \Sigma_k = \emptyset, k \neq i$$

$H_p$  is transversal to  $\Sigma_i$  uniformly up to the boundary

Moreover, for  $1 \leq i \leq J$  and for every  $\rho \in K_0$ , there exists  $t_-(\rho) < 0$  and  $i_-(\rho)$  (resp.  $t_+(\rho) > 0$  and  $i_+(\rho)$ ) such that

$$\begin{aligned} \Phi_{t_\pm(\rho)}(\rho) &\in K_0 \cap \Sigma_{i_\pm(\rho)} \\ \Sigma \cap \{\Phi_t(\rho), t_-(\rho) < t < t_+(\rho), t \in \mathbb{R}^*\} &= \emptyset \end{aligned}$$

where we note

$$\Sigma = \bigsqcup_{i=1}^J \Sigma_i$$

The maps  $t_\pm(\rho)$  are uniformly bounded on  $K_0$  and can be smoothly extended in a neighborhood of  $K_0$ . For convenience, it is also assumed that  $i_+(\rho) \neq i$  for all  $\rho \in K_0 \cap \Sigma_i$ . This can be achieved by taking smaller and more Poincaré sections. Finally, there exist  $\tilde{\Sigma}_i \Subset \mathbb{R}^2$  and symplectic diffeomorphisms

$$\kappa_i : \Sigma_i \rightarrow \tilde{\Sigma}_i$$

smooth up to the boundary.

- **Poincaré return map.** For  $1 \leq i, j \leq J$ , the map  $\rho \mapsto \Phi_{t_+(\rho)}(\rho)$ , initially defined on  $K_0$ , extends smoothly to a symplectic diffeomorphism

$$F_{ij} : D_{ij} \rightarrow A_{ij}$$

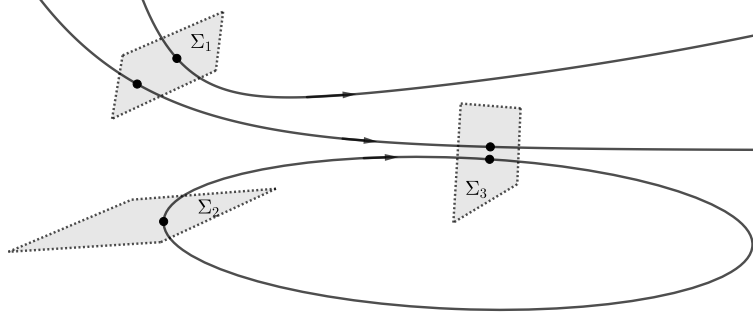


Figure 6.3: Schematic representation of Poincaré sections for the flow  $\Phi_t$  on an energy shell. The energy shell has dimension 3, so that the Poincaré sections are 2-dimensional.

by taking the intersection of the flow of a point  $\rho \in D_{ij}$  with  $\Sigma_i$  where  $D_{ij}$  (resp.  $A_{ij}$ ) is a neighborhood of

$$\{\rho \in \mathcal{T} \cap \Sigma_j, i_+(\rho) = i\} \text{ (resp. } \{\rho \in \mathcal{T} \cap \Sigma_i, i_-(\rho) = j\})$$

in  $\Sigma_j$  (resp.  $\Sigma_i$ ). The map  $F_{ij}$  is called the Poincaré return map. By writing it in the charts  $\kappa_i$  and  $\kappa_j$ , we can consider the following map between open sets of  $T^*\mathbb{R}$

$$\tilde{F}_{ij} = \kappa_i \circ F_{ij} \circ \kappa_j^{-1} : \tilde{D}_{ij} \subset \tilde{\Sigma}_j \rightarrow \tilde{A}_{ij} \subset \tilde{\Sigma}_i$$

Using the continuity of the flow, the same objects can be defined on energy shells  $p^{-1}(z)$  for  $z \in [-\delta, \delta]$  with  $\delta$  small enough and we will note these objects  $\Sigma_i^{(z)}$ ,  $F_{ij}^{(z)}$ , etc. In fact, it is possible to use the same open sets  $\tilde{\Sigma}_i$  and define,

$$\tilde{F}_{ij}^{(z)} = \kappa_i \circ F_{ij}^{(z)} \circ \kappa_j^{-1} : \tilde{D}_{ij}^{(z)} \subset \tilde{\Sigma}_j \rightarrow \tilde{A}_{ij}^{(z)} \subset \tilde{\Sigma}_i$$

The hyperbolicity of the flow implies the hyperbolicity of these open maps.

- **Open quantum maps.** The notion of open quantum hyperbolic map associated with  $\tilde{F}$  has been given in Definition 4-3.2. Since  $\tilde{\Sigma}_j \subset T^*\mathbb{R}$ , we will simply say that it is an operator-valued matrix  $T = (T_{ij})_{1 \leq i, j \leq J} : L^2(\mathbb{R})^J \rightarrow L^2(\mathbb{R})^J$  with  $T_{ij} \in I_\delta(\mathbb{R} \times \mathbb{R}, \text{Gr}(\tilde{F}_{ij})')$ . In [NSZ11], the authors construct a particular family of open quantum hyperbolic maps, called  $\mathcal{M}(z)$ , where  $\mathcal{M}(z)$  is associated with  $\tilde{F}^{(\text{Re } z)}$ . This family is first microlocally defined near the trapped set and satisfies uniformly in  $\mathcal{R}(\eta, M_0, h)$  and microlocally in a fixed neighborhood of the trapped set :

$$\mathcal{M}_{ij}(z) = \mathcal{M}_{ij}(\text{Re } z) \text{Op}_h(a_z) + O(h \log h) ; a_z = \exp\left(i \frac{(\text{Im } z)t_+^{(z)}}{h}\right)$$

This particular family is built to solve a microlocal Grushin problem (see section 4 in [NSZ11]).

- **Escape functions.** To perform a global study (i.e. no more microlocal) and to make the amplitude of  $\mathcal{M}(z)$  very small outside a fixed neighborhood of the trapped set, the authors introduce an escape function for the flow  $\Phi_t$ , denoted  $G_0 \in C_c^\infty(T^*\mathbb{R}^2)$  (see Lemma 5.3 in [NSZ11], where it is chosen independent of the energy variable near  $\Sigma$ ). Let us note  $G = Mh \log(1/h)G_0$ ,  $g_j = G|_{\Sigma_j} \circ \kappa_j^{-1}$  and

$$g : \bigsqcup_{j=1}^J \tilde{\Sigma}_j \rightarrow \mathbb{R}, \rho \in \tilde{\Sigma}_j \mapsto g_j(\rho)$$

$g$  is an escape function for the map  $\tilde{F}$ . Each  $g_j$  can be extended as an element of  $C_c^\infty(\mathbb{R}^2)$  (see equation 5.2 and below in [NSZ11]).

- **Conjugated operators.** As in the case of obstacle scattering, we can consider the operators

$$e^{\pm \text{Op}_h(G)} : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2) ; e^{\pm \text{Op}_h(g)} : L^2(\mathbb{R})^J \rightarrow L^2(\mathbb{R})^J$$

Again, their norm is bounded by  $O(h^{-K_G})$  for some  $K_G > 0$  depending on  $G_0$  and  $M$ . We now introduce the following conjugated operators :

$$P_{\theta,G} = e^{-\text{Op}_h(G)} P_{\theta} e^{\text{Op}_h(G)} ; \mathcal{M}_{ij,g}(z) = e^{-\text{Op}_h(g_i)} \mathcal{M}_{ij}(z) e^{\text{Op}_h(g_j)}$$

$$\mathcal{M}_g(z) = (\mathcal{M}_{ij,g}(z))_{1 \leq i,j \leq J}$$

The escape function  $G_0$  is built so that  $\mathcal{M}_g(z) : L^2(\mathbb{R})^J \rightarrow L^2(\mathbb{R})^J$  is  $O(h^{K_0})$  for some (large)  $K_0$  microlocally outside a small neighborhood of the trapped set  $\mathcal{T}$ . In particular,  $\mathcal{M}_g(z)$  satisfies the assumptions of the propositions 6-2.1 and 6-2.2.

- **A finite dimensional space.** For practical and technical reasons,<sup>3</sup> the authors choose to work with a finite dimensional version of the open quantum map  $\mathcal{M}_g(z)$ . To do so, they introduce finite rank projections  $\Pi_j : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  and the finite dimensional subspace of  $L^2(\mathbb{R})^J$

$$\mathcal{H} = \text{Ran } \Pi_1 \times \cdots \times \text{Ran } \Pi_J$$

The  $\Pi_j$ 's are built so that the projector  $\Pi = \text{Diag}(\Pi_1, \dots, \Pi_J)$  satisfies the very important relation

$$\Pi \mathcal{M}(z) \Pi = \mathcal{M}(z) \Pi + O(h^{K_1}) \quad (6-2.16)$$

for some large  $K_1$  (in particular  $K_1 \gg K_G$  so that the same relation holds after conjugation by  $e^{\text{Op}_h g}$  with  $K_1$  replaced by  $K_1 - 2K_G$ ). We will note  $\Pi_g = e^{-\text{Op}_h(g)} \Pi e^{\text{Op}_h(g)}$ .

- **The Grushin problem.** To obtain a global Grushin problem (see section 5 in [NSZ11]) the authors construct global operators  $R_+(z) : H_h^2(\mathbb{R}^2) \rightarrow \mathcal{H}$ ,  $R_-(z) : \mathcal{H} \rightarrow H_h^2(\mathbb{R}^2)$  which depend holomorphically on  $z \in \mathcal{R}(\eta, M_0, h)$ . The Grushin problem concerns

$$\mathcal{P}_g(z) = \begin{pmatrix} P_{\theta,G} - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : H_h^2(\mathbb{R}^2) \times \mathcal{H} \rightarrow L^2(\mathbb{R}^2) \times \mathcal{H} \quad (6-2.17)$$

The goal of such a Grushin problem is to transform the eigenvalue equation  $P_{\theta,G} u = zu$  into an equation on a simpler operator  $E_{\pm}(z)$ . This transformation is possible when  $\mathcal{P}_g(z)$  is invertible. Indeed, in virtue of the so-called Schur complement formula, if  $\mathcal{P}_g(z)$  is invertible with inverse

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{\pm}(z) \end{pmatrix} : L^2(\mathbb{R}^2) \times \mathcal{H} \rightarrow H_h^2(\mathbb{R}^2) \times \mathcal{H},$$

then  $P_{\theta,G} - z$  is invertible if and only if  $E_{\pm}(z)$  is and we have

$$(P_{\theta,G} - z)^{-1} = E(z) - E_+(z) E_{\pm}(z)^{-1} E_-(z)$$

The authors prove the following result :

**Theorem 6-2.1** ([NSZ11], Theorem 5). *The Grushin problem (6-2.17) is invertible for all  $z \in \mathcal{R}(\eta, M_0, h)$ . If we note*

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{\pm}(z) \end{pmatrix} : L^2(\mathbb{R}^2) \times \mathcal{H} \rightarrow H_h^2(\mathbb{R}^2) \times \mathcal{H}$$

the inverse of  $\mathcal{P}_g(z)$ , then

- <sup>a</sup>  $\|E\|, \|E_+\|, \|E_-\|, \|E_{\pm}\| = O(h^{-1})$  uniformly in  $\mathcal{R}(\eta, M_0, h)$ .
- The operator  $E_{\pm}(z)$  takes the form, for some  $L_2 > 0$

$$E_{\pm}(z) = \text{Id} - M_g(z, h) + O(h^{L_2}) ; M_g(z, h) := \Pi_g \mathcal{M}_g(z) \Pi_g$$

<sup>a</sup>the norms are associated with the spaces mentionned above. For instance,  $\|E_-\|_{\mathcal{H} \rightarrow H_h^2(\mathbb{R}^2)}$ .

### Remark.

<sup>3</sup>Mainly, to ensure the existence of determinants without discussion



As explained after Theorem 2 in [NSZ11],  $L_2 = c'M$  for some  $c'$ , where  $M$  is the one in the definition of the escape function  $G$ . In particular,  $M$  can be chosen arbitrarily large, independently of  $c'$ , so that  $L_2$  can be made as large as necessary.

### 6-2.3.3. End of proof.

To rigorously apply Proposition 6-2.1 to  $\mathcal{M}_g(z)$ , we fix  $\eta_0 \in [-\eta, \eta]$  for  $\eta$  small enough and consider  $z \in D(\eta_0, Kh)$  for some fixed  $0 < K < M_0$ . For such  $z$ ,  $\mathcal{M}(z)$  is an open quantum map associated with the Poincaré return map between the Poincaré sections  $\Sigma(\eta_0) = \bigsqcup_{1 \leq j \leq J} \Sigma_j(\eta_0)$  inside the energy shell  $p^{-1}(\eta_0)$ .

Since  $\mathcal{M}_g(z)$  satisfies the assumption of Proposition 6-2.1 for  $z \in D(\eta_0, Kh) \cap \mathcal{R}(\eta, M_0, h)$ , it also satisfies its conclusion :

$$\exists h_0 > 0, \delta, \gamma > 0 ; \forall 0 < h \leq h_0, \|\mathcal{M}(z)_g^{N(h)}\|_{L^2 \rightarrow L^2} \leq h^\gamma \|\alpha_h(z)\|_\infty^{N(h)} \quad (6-2.18)$$

with  $N(h) \sim \delta |\log h|$ . *A priori*,  $h_0, \delta, \gamma$  depend on  $\eta_0$ . Nevertheless, as explained after Proposition 6-2.1,  $\delta$  and  $\gamma$  depend only on the properties of the Poincaré return map  $F^{(\eta_0)}$  and  $h_0$  depends on semi-norms of  $\alpha_h(z) = e^{i\frac{z}{h}t_+^{(\eta_0)}}$ . As explained in Section 4.1.1 in [NSZ11], this dynamics depends continuously on  $\eta_0$  in a neighborhood of 0 : that is, the departure sets, the arrival sets, the Poincaré maps and the return time function depend continuously on  $\eta_0$ . As a consequence, we can find  $\eta$  and constants  $\delta, \gamma, h_0$  such that (6-2.18) holds for  $z \in \Omega(\eta, \gamma, h) := \{|\operatorname{Re} z| \leq \eta, -\gamma h \leq \operatorname{Im} z \leq 0\}$ . From this, we see that for  $0 < h \leq h_0$  and for  $z \in \Omega(\eta, \gamma, h)$ ,

$$\|\mathcal{M}(z)_g^{N(h)}\|_{L^2 \rightarrow L^2} \leq h^\gamma e^{-N(h)\tau\frac{\operatorname{Im} z}{h}} ; \tau = \sup_{|\eta_0| \leq \eta} \|t_\infty^{(\eta_0)}\|_\infty$$

From (6-2.16), we see that for  $N(h) \sim \delta |\log h|$ ,

$$M_g(z)^{N(h)} = \Pi_g \mathcal{M}_g^{N(h)} \Pi_g + O((\log h)h^{K_1})$$

so that we deduce that  $M_g(z)$  satisfies also the conclusion of Proposition 6-2.1 and hence, of Proposition 6-2.2. As a direct consequence, we obtain that for  $h$  small enough,  $E_\pm(z)$  is invertible for all  $z \in \Omega(\eta, \gamma, h)$  and it satisfies for some  $\beta > 0$  :

$$\|E_\pm(z)^{-1}\|_{\mathcal{H} \rightarrow \mathcal{H}} \leq h^{-\beta}$$

We now conclude the proof as in the case of obstacle scattering, essentially replacing the formula (6-2.13) by the standard Schur complement formula for the Grushin problem above :  $E_\pm(z)$  is invertible if and only if  $P_\theta - z$  is and

$$(P_{\theta, G} - z)^{-1} = E(z) - E_+(z)E_\pm(z)^{-1}E_-(z)$$

Then, for  $h$  small enough and for  $z \in \Omega(\eta, \gamma, h)$ ,

$$\|(P_{\theta, G} - z)^{-1}\|_{L^2 \rightarrow H_h^2} = O(h^{-1}) + O(h^{-\beta-2}) = O(h^{-\beta-2})$$

which gives

$$\|(P_\theta - z)^{-1}\|_{L^2 \rightarrow H_h^2} = O(h^{-\beta-2-2K_G s})$$

where  $c_0$  depends on  $G_0$ .

## 6-3 Application to the local energy decay for the wave equation.

We present an application of the resolvent estimate obtained in the case of obstacle scattering to the decay of the local energy for the wave equation outside the obstacles. In this chapter, we follow the main arguments of [Bur98] to prove Theorem 6-1.2.

### 6-3.1. Resolvent estimates.

Let us rewrite the resolvent estimate of Theorem 6-1.1 in term of  $\lambda$  : there exists  $\gamma > 0$ ,  $\lambda_0 > 0$  and  $\beta > 0$  such that for any  $\chi \in C_c^\infty(\mathbb{R}^2)$  equal to one in a neighborhood of  $\overline{\mathcal{O}}$ , there exists  $C_\chi > 0$  such that for all  $\lambda \in \mathbb{C}$ ,

$$|\lambda| \geq \lambda_0, \operatorname{Im} \lambda \geq -\gamma \implies \|\chi R(\lambda)\chi\|_{L^2 \rightarrow L^2} \leq C_\chi |\lambda|^\beta \quad (6-3.1)$$

Recalling that for  $f \in L_{comp}^2$ , with  $g = R(\lambda)f$ , it holds that  $\chi g \in H_0^1(\Omega)$  and  $g$  satisfies  $-\Delta g + \lambda^2 g = f$ , it is not hard to see that the above estimate implies that

$$\|\chi R(\lambda)\chi\|_{L^2 \rightarrow H_0^1} \leq C'_\chi |\lambda|^{\beta+1} \quad (6-3.2)$$

for  $|\lambda| \geq \lambda_0$  and  $\operatorname{Im} \lambda \geq -\gamma$  (see for instance the proof of Proposition 2.5 in [BGT04]).

This gives resolvent estimates for large  $\lambda$ . We will also need to control the resolvent for small  $\lambda$ , in angular neighborhoods of the logarithmic singularity at 0. For this purpose, we state a consequence of a result proved in [Bur98] (Appendix B.2) :

**Lemma 6-3.1.** For  $\varepsilon > 0$ , let  $\mathcal{S}_\varepsilon = \{z \in \mathbb{C}^*, |z| \leq \varepsilon, \arg z \in [-\pi/4, 5\pi/4]\}$ .

There exists  $\varepsilon_0 > 0$  such that there is no resonance in  $\mathcal{S}_{\varepsilon_0}$  and for any  $\chi \in C_c^\infty(\mathbb{R}^2)$  equal to one in a neighborhood of  $\overline{\mathcal{O}}$ , there exists  $C_\chi > 0$  such that for all  $\lambda \in \mathcal{S}_{\varepsilon_0}$ ,

$$\|\chi R(\lambda)\chi\|_{L^2 \rightarrow H_0^1} \leq C_\chi \quad (6-3.3)$$

Finally, to relate what happens between the two regions, we let us also mention the useful result, proved in [Bur98] (Appendix B.1) :

**Lemma 6-3.2.** There are no real resonances (that is with  $\arg(z) = 0$  or  $\pi$ ).

### 6-3.2. The wave equation generator.

Let  $H$  be the Hilbert space  $\mathcal{H}(\Omega) \oplus L^2(\Omega)$ , where  $\mathcal{H}$  is the completion of  $C_c^\infty(\Omega)$  with respect to the norm  $\|f\|_{\mathcal{H}} = \|\nabla f\|_{L^2(\Omega)}$ <sup>4</sup> and let  $A$  be the operator

$$A = \begin{pmatrix} 0 & \operatorname{Id} \\ \Delta & 0 \end{pmatrix}$$

with domain  $D(A) = (\mathcal{H} \cap H^2(\Omega)) \oplus H_0^1(\Omega)$ .  $A$  is maximal dissipative, so that Hille-Yosida theory allows to define the propagator  $e^{tA}$  and for  $(u_0, u_1) \in H$ , the first component  $u(t)$  of  $t \mapsto e^{tA}(u_0, u_1)$  is the unique solution of the following Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

Note also that since  $A$  is maximal dissipative, for  $\xi$  with  $\operatorname{Re}(\xi) > 0$ ,  $A - \xi$  is invertible and

$$\|(A - \xi)^{-1}\|_{H \rightarrow H} \leq |\operatorname{Re} \xi|^{-1} \quad (6-3.4)$$

The global energy of the solution is defined as

$$E(t) = \frac{1}{2} \|(u(t), \partial_t u(t))\|_H^2 = \frac{1}{2} \int_\Omega |\nabla u(t)|^2 + |\partial_t u(t)|^2$$

It is conserved. If  $K \Subset \mathbb{R}^2$ , we also define the local energy in  $K$  as

$$E_K(t) = \frac{1}{2} \int_{K \cap \Omega} |\nabla u(t)|^2 + |\partial_t u(t)|^2$$

<sup>4</sup>This choice of Hilbert space makes the wave propagator unitary on  $H$ , since the energy of a solution of the wave equation is its norm in  $H$ ; see [Tay10], Chapter 9, Section 4

Note that, by Poincaré inequality, if  $B \subset \bar{\Omega}$  is bounded and if  $\chi \in C_c^\infty(\mathbb{R}^2)$  is equal to one in a neighborhood of  $\bar{\mathcal{O}}$  and is supported in  $B$ , then for  $f \in \mathcal{H}(\Omega)$ ,

$$\|\chi f\|_{\mathcal{H}} \sim \|\chi f\|_{H^1(B)} \sim \|\chi f\|_{L^2(B)} + \|\nabla(\chi f)\|_{L^2(B)}$$

If  $\chi \in C_c^\infty(\mathbb{R}^2)$  is equal to one in a neighborhood of  $\bar{\mathcal{O}}$ , by abuse we note  $\chi$  the bounded operator of  $(u, v) \in H \mapsto (\chi u, \chi v) \in H$ .

A short computation shows that for  $\lambda \in \mathbb{C}$ ,  $(u_0, u_1) \in D(A)$  and  $(v_0, v_1) \in H$ ,

$$(A + i\lambda) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \iff \begin{cases} (-\Delta - \lambda^2)u_0 = i\lambda v_0 - v_1 \\ u_1 = v_0 - i\lambda u_0 \end{cases}$$

This relation and the remark above for bounded sets  $B$  show that for any  $\chi \in C_c^\infty(\mathbb{R}^2)$ , the cut-off resolvent  $\chi(A + i\lambda)^{-1}\chi$ , well defined for  $\text{Im } \lambda > 0$  extends to the logarithmic cover  $\Lambda$  of  $\mathbb{C}$  and we have for  $\lambda \in \Lambda$

$$\chi(A + i\lambda)^{-1}\chi = \begin{pmatrix} i\lambda\chi R(\lambda)\chi & -\chi R(\lambda)\chi \\ \chi^2 + \lambda^2\chi R(\lambda)\chi & i\lambda\chi R(\lambda)\chi \end{pmatrix} \quad (6-3.5)$$

We deduce that  $\chi(A + i\lambda)^{-1}\chi$  has no real resonance and satisfies the following resolvent estimates, for some constant  $C_\chi$ ,

$$\lambda \in \mathcal{S}_{\varepsilon_0} \implies \|\chi(A + i\lambda)^{-1}\chi\|_{H \rightarrow H} \leq C_\chi \quad (6-3.6)$$

$$|\lambda| \geq \lambda_0, \text{Im } \lambda \geq -\gamma \implies \|\chi(A + i\lambda)^{-1}\chi\|_{H \rightarrow H} \leq C_\chi |\lambda|^{\beta+2} \quad (6-3.7)$$

### 6-3.3. Proof of the local energy decay.

Let us fix  $R > 0$  such that  $\mathcal{O} \Subset B(0, R)$ . We want to estimate the local energy in  $B(0, R)$  for solutions with initial data supported in  $B(0, R)$ , and sufficiently regular, that is in  $D(A^k)$  for a sufficiently large  $k$ . As we will see, the decay will hold for data in  $D(A^k)$  with  $k \geq \beta + 4$ , where  $\beta$  is the one appearing in (6-3.7). For this purpose, let us fix  $\chi \in C_c^\infty(\mathbb{R}^2)$  such that  $\chi = 1$  in  $B(0, R)$ .

Let  $U_0 \in D(A^k)$  with  $\text{supp}(U_0) \Subset B(0, R)$ . We want to estimate the energy of  $\chi e^{tA}U_0$ , or equivalently, we want to control  $\|\chi e^{tA}U_0\|_H$ . Let us write  $U = (I - A)^k U_0 \in H$ , so that  $\|U_0\|_{D(A^k)} = \|U\|_H$ . It is clear that we have  $\text{supp}(U) \subset B(0, R)$ , so that  $U = \chi U$ . With this notation, we want to show that there exists  $C_R > 0$  such that for all  $t \geq 1$ ,

$$I(t) := \|\chi e^{tA}(I - A)^{-k}\chi U\|_H \leq \frac{C}{t} \|U\|_H$$

The starting point of the proof is the following formula :

**Lemma 6-3.3.** Assume that  $k \geq 2$ . For  $t \geq 0$  and for  $U \in H$ , we have

$$e^{tA}(I - A)^{-k}U = \frac{-1}{2\pi} \int_{\lambda \in \frac{i}{2} + \mathbb{R}} e^{-it\lambda} \frac{1}{(1 + i\lambda)^k} (A + i\lambda)^{-1}U d\lambda \quad (6-3.8)$$

*Proof.* First remark that the integral  $I(t)$  in the right hand side is absolutely convergent in virtue of (6-3.4) and since  $k \geq 2$ .

Differentiating the right hand side with respect to  $t$ , we find that

$$\begin{aligned} (\partial_t - A)I(t) &= \frac{-1}{2\pi} \int_{\lambda \in \frac{i}{2} + \mathbb{R}} e^{-it\lambda} \frac{1}{(1 + i\lambda)^k} (-i\lambda - A)(A + i\lambda)^{-1}U d\lambda \\ &= \frac{1}{2\pi} \int_{\lambda \in \frac{i}{2} + \mathbb{R}} e^{-it\lambda} \frac{1}{(1 + i\lambda)^k} U d\lambda = 0 \end{aligned}$$

(To see that the last integral is equal to zero, one can for instance perform a contour deformation from  $\text{Im}(\lambda) = 1/2$  to  $\text{Im}(\lambda) = -\rho$  and let  $\rho$  tend to  $+\infty$ .)

Finally, we need to check that  $I(0) = (\text{Id} - A)^{-k}U$ . We have

$$I(0) = \frac{-1}{2\pi} \int_{\lambda \in \frac{i}{2} + \mathbb{R}} \frac{1}{(1 + i\lambda)^k} (A + i\lambda)^{-1}U d\lambda$$

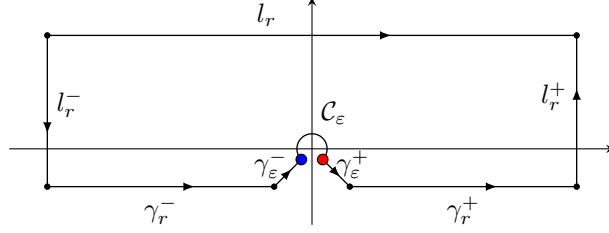


Figure 6.4: The contour used to deform the integral.  $z^-$  (resp.  $z^+$ ) is the blue (resp. red) point on the figure.

We perform a contour deformation. Let  $r > 1$  and let  $\Gamma_r$  be rectangle joining the points  $i/2 + r, r(1+i), r(i-1), i/2 - r$ . We also note  $\gamma_r = \Gamma_r \setminus [-r + i/2, r + i/2]$ . The function  $g_k : z \mapsto -(1+iz)^{-k}(iz \text{Id} + A)^{-1}U$  is meromorphic in  $\text{Im } z > 0$ , with a unique pole at  $z = i$ . As a consequence, we find that

$$\frac{1}{2i\pi} \int_{\Gamma_r} g_k(z) dz = \text{Res}_{z=i} g_k = \frac{-1}{i^k(k-1)!} \partial_z^{k-1} ((iz \text{Id} + A)^{-1}U)|_{z=i} = i(\text{Id} - A)^{-k}U$$

Hence, we have

$$I(0) = \lim_{r \rightarrow +\infty} \frac{-i}{2i\pi} \int_{i/2-r}^{i/2+r} g_k(\lambda) d\lambda = \lim_{r \rightarrow +\infty} -i \left( i(\text{Id} - A)^{-k}U - \int_{\gamma_r} g_k(z) dz \right) = (\text{Id} - A)^{-k}U$$

Indeed, it is not hard to see that the contribution on  $\gamma_r$  tends to 0 as  $r \rightarrow +\infty$ .  $\square$

*End of proof of Theorem 6-1.2.* The proof relies on a contour deformation below the real axis in the integral of (6-3.8) where the cut-off  $\chi$  is inserted, but we need to get around the logarithmic singularity at 0 : it is possible due to (6-3.6).

We fix  $t > 0$ . In the estimates below, the constants denoted by  $C$  (or  $C_k$ ) do not depend on  $t$ . We know that the map  $\lambda \mapsto \chi(A + i\lambda)^{-1}\chi U$  is meromorphic in  $\mathbb{C} \setminus i\mathbb{R}^-$ , with no poles in  $\{\text{Im } \lambda > 0\} \cup \mathcal{S}_{\varepsilon_0} \cup \{\text{Im } \lambda \geq -\gamma, |\lambda| \geq \lambda_0\}$ . By taking  $\varepsilon_0$  smaller if necessary, we may assume that  $2^{-1/2}\varepsilon_0 \leq \gamma$ . Let  $K$  be the union of the rectangles  $K_+$  and  $K_-$  where

$$K_{\pm} = \{\lambda \in \mathbb{C}, \text{Re } \lambda \in [\pm 2^{-1/2}\varepsilon_0, \pm \lambda_0], \text{Im } \lambda \in [-2^{-1/2}\varepsilon_0, 0]\}$$

Since there is only a finite number of resonance in  $K$  and since there are no resonances on  $K \cap \{\text{Im } \lambda = 0\}$ , we can find  $\delta > 0$  such that there is no resonance in  $K \cap \{\text{Im } \lambda \geq -\delta\}$  and since this region is compact, we can find  $C$  such that for  $\lambda \in K \cap \{\text{Im } \lambda \geq -\delta\}$ ,

$$\|\chi(A + i\lambda)^{-1}\chi\|_{H \rightarrow H} \leq C$$

Let's note  $z^+$  (resp.  $z^-$ ) the unique point in  $\{|z| = \varepsilon_0\} \cap \{\text{Im } z = -\delta\} \cap \{\pm \text{Re } z > 0\}$ . Fix  $r \gg 1$  and  $0 < \varepsilon < \varepsilon_0$  and let's note  $z_{\varepsilon}^{\pm}$  the point of the segment  $[0, z^{\pm}]$  with norm  $\varepsilon$  and let's introduce the following paths, oriented from the left point to the right point :

$$\begin{aligned} \gamma_r^+ &= [z^+, r - i\delta]; \gamma_r^- = [-r - i\delta, z^-] \\ \gamma_{\varepsilon}^+ &= [z_{\varepsilon}^+, z^+]; \gamma_{\varepsilon}^- = [z^-, z_{\varepsilon}^-] \\ l_r^+ &= [r - i\delta, r + i/2]; l_r^- = [-r - i\delta, -r + i/2] \\ l_r &= [-r + i/2, r + i/2] \end{aligned}$$

and  $\mathcal{C}_{\varepsilon}$  be the arc of the circle  $\{|z| = \varepsilon\}$  from  $z_{\varepsilon}^-$  to  $z_{\varepsilon}^+$ . (See Figure 6.4).

With  $f_k(z) = -\frac{1}{2\pi} e^{-itz} (1+iz)^{-k} \chi(A+iz)^{-1} \chi U$ , we have  $\chi e^{tA} (\text{Id} - A)^{-k} \chi U = \lim_{r \rightarrow +\infty} \int_{l_r} f_k(z) dz$  and since  $f_k(z)$  is holomorphic in a neighborhood of the compact set surrounded by the above contours, we have

$$\int_{l_r} f_k(z) dz = \int_{l_r^+} f_k(z) dz + \int_{l_r^-} f_k(z) dz + \int_{\gamma_r^+} f_k(z) dz + \int_{\gamma_r^-} f_k(z) dz + \int_{\mathcal{C}_{\varepsilon}} f_k(z) dz + \int_{\gamma_{\varepsilon}^+} f_k(z) dz + \int_{\gamma_{\varepsilon}^-} f_k(z) dz$$

Note that  $\mathcal{C}_\varepsilon \cup \gamma_\varepsilon^+ \cup \gamma_\varepsilon^- \subset \mathcal{S}_{\varepsilon_0}$ . As a consequence, we have

$$\left\| \int_{\mathcal{C}_\varepsilon} f_k(z) dz \right\|_H \leq C_k \varepsilon \|U\|_H \xrightarrow{\varepsilon \rightarrow 0} 0$$

and, with  $\theta = \arg z^+$ ,

$$\left\| \int_{\gamma_\varepsilon^+} f_k(z) \right\|_H \leq C_k \int_\varepsilon^{\varepsilon_0} e^{ts \sin \theta} \|U\|_H ds \leq C_k \frac{\|U\|_H}{t |\sin \theta|} (e^{t\varepsilon \sin \theta} - e^{t\varepsilon_0 \sin \theta}) \leq \frac{C_k}{t} \|U\|_H$$

The case of  $\gamma_\varepsilon^-$  is treated similarly.

On  $\gamma_r^\pm$ , the following holds :

$$\|\chi(A + i\lambda)^{-1} \chi\|_{H \rightarrow H} \leq C |\lambda|^{\beta+2}$$

Indeed, this is true for  $|\lambda| \geq \lambda_0$  and there is no resonance on  $\gamma_r^\pm$ . As a consequence, for  $\lambda = -i\delta + \xi$ ,  $|\xi| \geq \operatorname{Re}(z_+)$ , we have

$$\|\chi e^{-it\lambda} (1 + i\lambda)^{-k} (A + i\lambda)^{-1} \chi\|_{H \rightarrow H} \leq C e^{-t\delta} |\xi|^{\beta+2-k}$$

Hence, we assume here that  $\boxed{k \geq \beta + 4}$  so that

$$\left\| \int_{\gamma_r^\pm} f_k(z) dz \right\|_H \leq C \int_{\operatorname{Re}(z^+)}^r e^{-t\delta} |\xi|^{\beta+2-k} \|U\|_H d\xi \leq C e^{-t\delta} \int_{\operatorname{Re}(z^+)}^{+\infty} |\xi|^{-2} \|U\|_H d\xi \leq C e^{-t\delta} \|U\|_H$$

Finally, we treat the vertical segments  $l_r^\pm$ .

$$\left\| \int_{l_r^\pm} f_k(z) dz \right\|_H \leq C \sup_{y \in [-\delta, 1/2]} \|f_k(\pm r + iy)\|_H$$

For  $y \in [-\delta, 1/2]$ , we have

$$\begin{aligned} \|f_k(\pm r + iy)\|_H &\leq C e^{ty} r^{-k} \|\chi(A + i(r + iy))^{-1} \chi\|_{H \rightarrow H} \times \|U\|_H \\ &\leq C e^{t/2} r^{-k} \|\chi(A + i(r + iy))^{-1} \chi\|_{H \rightarrow H} \times \|U\|_H \end{aligned}$$

Using (6-3.7), we find that for  $y \in [-\delta, 1/2]$ ,  $\|\chi(A + i(r + iy))^{-1} \chi\|_{H \rightarrow H} \leq C r^{\beta+2}$ . As a consequence, one finds that for  $y \in [-\delta, 1/2]$ ,

$$\|f_k(\pm r + iy)\|_H \leq C e^{t/2} r^{\beta+2-k} \|U\|_H \leq C e^{t/2} r^{-2} \|U\|_H$$

As a consequence,

$$\left\| \int_{l_r} f_k(z) dz \right\|_H \leq \left( \frac{C_k}{t} + C e^{-t\delta} + C_k \varepsilon + C e^{t/2} r^{-2} \right) \|U\|_H$$

By letting  $\varepsilon$  tending to 0 and  $r$  to  $+\infty$ , we conclude that

$$\|\chi e^{tA} (\operatorname{Id} - A)^{-k} \chi U\|_H \leq \left( \frac{C_k}{t} + C e^{-t\delta} \right) \|U\|_H \leq \frac{C_k}{t} \|U\|_H$$

which gives the required result.

## Part IV

# Coupling resonances in strips



# Chapter 7

## Improved fractal Weyl upper bound

In this chapter, we prove Theorem B which provides an upper bound for the number  $N(r, \gamma)$  of resonances in obstacle scattering in the box  $\{r \leq \operatorname{Re}(\lambda) \leq r + 1; -\gamma \leq \operatorname{Im}(\lambda) \leq 0\}$ . It was proved in the work of [NSZ14] that  $N(r, \gamma) = O_\gamma(r^{d_H})$  where  $2d_H + 1$  is the Hausdorff dimension of the trapped set of the billiard flow. In this chapter, we provide an improved upper bound in the band  $0 \leq \gamma < \gamma_{cl}/2$ , where  $\gamma_{cl}$  is the classical decay rate of the flow. This improved Weyl upper bound is in the spirit of the ones of [Nau12] and [DBW19] in the case of convex co-compact surfaces, and of [DJ17] in the case of open quantum baker's maps.

### 7-1 Introduction.

#### 7-1.1. An improved fractal upper bound for open quantum hyperbolic maps.

**Motivation : scattering by convex obstacles.** In this chapter, we will obtain upper bounds for the counting functions of resonances in the problem of scattering by strictly convex obstacles in the plane and provide some elements to answer Question 2.2 asked in Part I. We recall that we are interested in the scattering by

$$\mathcal{O} = \bigcup_{j=1}^J \mathcal{O}_j$$

where  $\mathcal{O}_j$  are open, strictly convex connected obstacles in  $\mathbb{R}^2$  having smooth boundary and satisfying the *Ikawa condition* : for  $i \neq j \neq k$ ,  $\overline{\mathcal{O}_i}$  does not intersect the convex hull of  $\overline{\mathcal{O}_j} \cup \overline{\mathcal{O}_k}$ .

We are interested in counting resonances in strips and in this chapter we focus on an upper bound for the quantities

$$N(r, \gamma) = \sum_{\substack{r \leq \operatorname{Re} \lambda \leq r+1 \\ \operatorname{Im} \lambda \geq -\gamma}} m(\lambda)$$

The depth  $\gamma$  of the strip being fixed, we are interested in upper bounds as  $r \rightarrow +\infty$ . We recall that the multiplicity of a (non-zero) resonance  $\lambda_0$  is given by

$$m(\lambda_0) = \operatorname{rank} \frac{1}{2i\pi} \int_\gamma R_\chi(\lambda) d\lambda, \quad \gamma(t) = \lambda_0 + \epsilon^{2i\pi t}, \quad 0 \leq t \leq 1, \quad 0 < \epsilon \ll 1$$

**Fractal Weyl bounds.** In a previous result, the Hausdorff dimension of the trapped set of the billiard flow (see Chapter 3, 3-1.1.1.) plays a crucial role when estimating  $N(r, \gamma)$ . In [NSZ14], the authors proved a Fractal Weyl upper bound involving this fractal dimension.



**Theorem 7-1.1. Fractal Weyl upper bound [NSZ14]**

Assume that the obstacles  $\mathcal{O}_j \subset \mathbb{R}^2$  are strictly convex open sets having smooth boundary and satisfying the Ikawa condition. Assume that the trapped set of the billiard flow has Hausdorff dimension  $2d_H + 1$ . Then, for every  $\gamma > 0$ , there exists  $C_\gamma > 0$  such that for all  $r \geq 1$ ,

$$N(r, \gamma) \leq C_\gamma r^{d_H}.$$

**Remark.**

Their result holds in any dimension, but in dimension  $d > 2$  one has to add an extra loss of  $\varepsilon$  : for every  $\varepsilon > 0$ , for every  $\gamma > 0$ , there exists  $C_{\varepsilon, \gamma} > 0$  such that for all  $r \geq 1$ ,

$$N(r, \gamma) \leq C_{\varepsilon, \gamma} r^{d_H + \varepsilon}.$$

This bound is conjectured to be optimal for large values of  $\gamma$  (see [Zwo17], Conjecture 5). However, as soon as a spectral gap exists, the exponent  $d_H$  cannot be optimal for any  $\gamma$ . It always exists in dimension 2, as proved in Chapters 5 and 6 (and in the paper [Vac22b]) and it holds also in higher dimensions under some pressure condition (see Theorem 2-1.11). Our Theorem 7-1.2 below gives a better bound in dimension 2 for

$$\gamma < \gamma_{cl}/2$$

where  $\gamma_{cl}$  is the classical decay rate of the billiard flow (see 3-5.3). It is also given by the following formula (see [BR75], Proposition 4.4), which explains its name :

$$-\gamma_{cl} = \lim_{\varepsilon \rightarrow 0} \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \text{Leb} \left( \bigcup_{\rho \in K} B_\rho(\varepsilon, T) \right) \quad (7-1.1)$$

where

$$B_\rho(\varepsilon, T) = \{(x, \xi) \in \Omega \times \mathbb{S}^1, \forall t \in [0, T], d(\varphi^t(x, \xi), \varphi^t(\rho)) \leq \varepsilon\}$$

are Bowen balls.

In this chapter, we will obtain as a consequence of a more general result concerning open quantum hyperbolic maps, the following upper bound, which is the content of Theorem B in Part I :

**Theorem 7-1.2.** Assume that the obstacles  $\mathcal{O}_j \subset \mathbb{R}^2$  are strictly convex open sets, having smooth boundary and satisfying the Ikawa condition. Then, there exists a non increasing function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying

- $\sigma(\gamma) > 0$  for  $0 \leq \gamma < \gamma_{cl}/2$  ;
- $\sigma(\gamma) = 0$  for  $\gamma \geq \gamma_{cl}/2$

and such that for all  $\gamma > 0$  and for all  $\varepsilon > 0$  there exists  $C_{\gamma, \varepsilon} > 0$  such that

$$\forall r \geq 1, N(r, \gamma) \leq C_{\gamma, \varepsilon} r^{d_H - (\sigma(\gamma) - \varepsilon)_+}$$

**Remark.**

A rather explicit value of  $\sigma$  in term of topological pressure is given by the formula (7-2.2). Here, we can take  $(\sigma(\gamma) - \varepsilon)_+ = \max(\sigma(\gamma) - \varepsilon, 0)$  due to the result of [NSZ14]. When  $\gamma \geq \gamma_{cl}/2$ , we always have  $(\sigma(\gamma) - \varepsilon)_+ = 0$ .

**Fractal Weyl upper bounds in open quantum systems.** As already explained in Introduction (Part I), the problem of wave scattering by obstacles has a long history in the physics and mathematics literature and the case of two obstacles is particularly well-understood (see [Gér88], [Ika82]), so is the diffraction by one convex obstacle (see for instance [BLR87], [HL94]). As soon as 3 or more obstacles are involved, the underlying classical flow - in this case, the billiard flow - becomes highly chaotic. A particularly interesting model is the  $n$ -disk system, which has been intensively studied both numerically and experimentally (see for instance [GR89a],[GR89b], [BWP<sup>+</sup>13]) and the fractal upper bound has been successfully tested in [PWB<sup>+</sup>12], [LSZ03]. The question of estimating the number of resonances also appears in other open hyperbolic systems (see 2-1.5.), such as scattering by a potential or the case of convex co-compact hyperbolic surfaces.

The first Fractal Weyl upper bound for the counting function in strips appeared in the work of Sjöstrand [Sjö90] (see Section 5, Theorem 5.7) for Schrödinger operators  $-h^2\Delta + V$  in the analytic case. The author estimated the number of resonances in larger boxes  $\{| \operatorname{Re} z | \leq \delta, -\gamma h \leq \operatorname{Im} z \leq 0\}$  in the limit  $h \rightarrow 0$ . More precise upper bounds  $O(h^{-d_H})$  for smaller boxes  $\{| \operatorname{Re} z | \leq Ch, -\gamma h \leq \operatorname{Im} z \leq 0\}$ , which correspond, under the rescaling  $r = h^{-1}$ , to the boxes we consider in obstacle scattering, were obtained in different smooth situations : for convex co-compact hyperbolic surfaces ([Zwo99]), in scattering by a potential ([SZ07]), in obstacle scattering ([NSZ14]), for asymptotically hyperbolic manifold ([DD13]). It has been conjectured (see [Zwo17], Conjecture 5) that the bound  $N(r, \gamma) = O(r^{d_H})$  is optimal when the strip is sufficiently large. However, numerical experiments (see for instance the appendix of [DBW19] for the case of convex co-compact surfaces) show that it should be possible to improve this bound for strips of width smaller than some threshold. These numerical results lead [Zwo17] to conjecture that

$$\lim_{r \rightarrow +\infty} N(r, \gamma)r^{-d_H} = 0 \text{ when } 0 \leq \gamma < \frac{\gamma_{cl}}{2}$$

First results in this direction were obtained in the case of convex co-compact hyperbolic surfaces :

- In [Nau12], the author showed a bound similar to the one in Theorem 7-1.2 (without the loss of  $\varepsilon$ ), with a function  $\sigma$  having the same properties ;
- In [DBW19], the author obtained the same result with an explicit function  $\sigma$  given by  $\sigma(\gamma) = \max(0, 1 - d_H - 2\gamma)$ , which satisfies the same properties as the one in Theorem 7-1.2 (since in this context  $\gamma_{cl} = 1 - d_H$ ). His result can be generalized to higher dimensional convex co-compact hyperbolic manifold.

Theorem 7-1.2 gives a positive answer to this conjecture in obstacle scattering in dimension 2. There is also a stronger conjecture, due to Jakobson-Naud ([JN12]) in the case of convex co-compact surfaces, which states that for every  $\gamma < \gamma_{cl}/2$ ,  $N(r, \gamma) = 0$  for  $r \gg 1$ . Our work is still far from this strong result, which would give a quantitative (and pleasant!) spectral gap.

**Toy models and open quantum maps.** To test these conjectures, it is useful to work on toy models where numerical and theoretical computations are sometimes easier. A very appreciated toy model in the study of open hyperbolic systems is the open baker's map (see 2-1.5.3.). We recall that the open baker's map is a piecewise affine open map  $F_{M, \mathcal{A}}$  on the torus  $\mathbb{T}^2$ , associated with an alphabet  $\mathcal{A} \subset \{0, \dots, M - 1\}$  and its quantization is given by a matrix  $B_N \in \mathcal{M}_N(\mathbb{R})$  where  $N$  plays the role of  $(2\pi h)^{-1}$  (see Definition 2-1.7). In this context, one wants to count the number of eigenvalues of the family of matrices  $B_N \in \mathcal{M}_N(\mathbb{R})$  in the annulus  $\{|z| \geq \nu\}$  in the limit  $N \rightarrow +\infty$ . There are convincing numerical and theoretical results. In [NR07], the author gave numerical evidence of Weyl upper bounds of the type

$$\#(\operatorname{Spec}(B_N) \cap \{|z| \geq \nu\}) \leq C_\nu N^{d_H}$$

In [DJ17], the author proved an even more precise upper bound, when  $N = M^k$  :

$$\#(\operatorname{Spec}(B_{M^k}) \cap \{|z| \geq \nu\}) \leq C_\nu (M^k)^{d_H + \varepsilon - \Sigma(\nu)}, \forall k \in \mathbb{N} \quad ; \quad \Sigma(\nu) = \sigma\left(-\frac{\log \nu}{\log J^u}\right)$$

where  $J^u = M$  is the unstable Jacobian of the system and  $\sigma(\gamma) = \max(1 - d_H - 2\gamma, 0)$ . In particular,  $\sigma$  shares the same properties as the one in Theorem 7-1.2, since the classical decay rate of the baker's map is  $1 - d_H$ . The link between  $\Sigma(\nu)$  and  $\sigma(\gamma)$  comes from the heuristics interpretation of the eigenvalues of  $B_N$  : they are thought to have the form  $e^{-it\frac{z}{h}}$  with  $t = \log M$  and  $z$  is a resonance in a hypothetical open quantum system.

### 7-1.2. Main theorem and applications.

Our proof of Theorem 7-1.2 relies on previous results of [NSZ14], recalled in detail in Chapter 6 (see 6-2.2.). Indeed, their Theorem 5 reduces the study of the scattering poles  $\lambda \in ]1/h - R, 1/h + R[ + i] - R, R[$  to the study of the cancellation of

$$z \in ] - R, R[ + i] - R, R[ \mapsto \det(I - M(z; h))$$

where

$$M(z) = M(z; h) : L^2(\partial\mathcal{O}) \rightarrow L^2(\partial\mathcal{O}) \quad (7-1.2)$$

is a family of *open quantum hyperbolic maps*. The zeros  $z$  and the resonances  $\lambda$  are related by the relation  $h\lambda = 1 + z$ .

We will now state the main theorem 7-1.3 of this chapter. We show how Theorem 7-1.3 implies Theorem 7-1.2 using the results of [NSZ14] in 7-2.1..

#### 7-1.2.1. Statement of the main theorem.

Let  $F : \tilde{D} \rightarrow \tilde{A}$  be an open hyperbolic map, as presented in 3-1.1.. We recall that this formalism relies on :

- open intervals  $Y_1, \dots, Y_J$  of  $\mathbb{R}$  and  $Y = \bigsqcup_{j=1}^J Y_j \subset \bigsqcup_{j=1}^J \mathbb{R}$  ;
- $U = \bigsqcup_{j=1}^J U_j \subset \bigsqcup_{j=1}^J T^*\mathbb{R}^d$  where  $U_j \Subset T^*Y_j$  are open sets;
- For  $j = 1, \dots, J$ , open disjoint subsets  $\tilde{D}_{ij} \Subset U_j$ ,  $1 \leq i \leq J$ , the *departure sets*, and, for  $i = 1, \dots, J$  open disjoint subsets  $\tilde{A}_{ij} \Subset U_i$ ,  $1 \leq j \leq J$ , the *arrival sets* ;
- Smooth symplectomorphisms  $F_{ij} : \tilde{D}_{ij} \rightarrow F_{ij}(\tilde{D}_{ij}) = \tilde{A}_{ij}$

Then,  $F$  is the global smooth map  $F : \tilde{D} \rightarrow \tilde{A}$  where  $\tilde{A}$  and  $\tilde{D}$  are the full arrival and departure sets. Finally, we recall that we note  $\mathcal{T} \subset U$  the trapped set of  $F$ , which is supposed to be totally disconnected and  $F$  is supposed to satisfy the assumptions ([Hyperbolicity](#)) and ([Fractal](#)). In particular, the hyperbolicity of  $F$  allows us to define stable and unstable Jacobians at  $\rho \in \mathcal{T}$ , denoted respectively  $J_n^u(\rho)$  and  $J_n^s(\rho)$  (see Definition 7-4.3). We will note  $\varphi_u(\rho) = \log J_1^u(\rho)$ . We also define the maximal Lyapounov exponent  $\lambda_{\max}$  :

$$\lambda_{\max} = \sup_{\rho \in \mathcal{T}} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log J_n^u(\rho) \quad (7-1.3)$$

We refer the reader to 4-3 for the definition of open quantum hyperbolic maps (see Definition 4-3.2).

**Assumptions of Theorem 7-1.3.** We consider a family  $(M_h(z))_z = (M(z; h))_z$  of open hyperbolic quantum maps, associated with  $F$ , as defined in Definition 4-3.2, and depending holomorphically on a parameter  $z \in \Omega = \Omega_R = ] - R, R[ + i] - R, R[$  with  $R$  fixed (but in practice, it can be chosen arbitrarily large). We suppose that there exists  $L > 0$  and  $a \in C_c^\infty(T^*Y)$  such that  $\text{supp}(a)$  is contained in a compact neighborhood  $\mathcal{W}$  of  $\mathcal{T}$ ,  $\mathcal{W} \subset \tilde{D}$ ,  $a = 1$  in a neighborhood of  $\mathcal{T}$  and uniformly in  $\Omega$ ,

$$M_h(z)(1 - \text{Op}_h(a)) = O(h^L)_{L^2 \rightarrow L^2}$$

Let's note  $\alpha_h(z)$  the amplitude of  $M_h(z)$  (as defined after definition 4-3.2). We make the following assumption on  $\alpha_h$  : there exists a neighborhood  $V \subset U$  of  $\mathcal{T}$  and a smooth function  $t_{ret} : V \rightarrow \mathbb{R}_+^1$  such that  $\inf_V t_{ret} > 0$ ,  $\sup_V t_{ret} < +\infty$  and for all  $z \in \Omega$  and  $\rho \in V$ ,

$$\alpha_h(z)(\rho) = \exp(-(\text{Im } z)t_{ret}(\rho)) + O(h^{1-})_{S_{0+}}$$

that is, there exists  $\chi \in C_c^\infty(U)$  such that  $\chi \equiv 1$  on  $V$  and supported in a larger neighborhood of  $\mathcal{T}$  and such that for every  $\eta > 0$ ,  $\chi(\alpha_h(z) - \exp(-(\text{Im } z)t_{ret}))$  is in  $h^{1-\eta}S_\eta$  uniformly for  $z \in \Omega$ .

**Remark.**

<sup>1</sup>In the applications,  $t_{ret}$  is a return time function.

In particular, the principal part  $\alpha_z(\rho) = \exp(-(\text{Im } z)t_{ret}(\rho))$  of  $\alpha_h$  is independent of  $h$  in  $V$ . This assumption, which may look strong at first glance, is satisfied in the two applications we consider (see 7-2.1. and 7-2.2.). This assumption is in fact not necessary for a more general study, but it makes the formula (7-1.4) and the Theorem 7-1.3 below more readable. The works of [NSZ11] and [NSZ14] allow to work up to  $\text{Im } z = C \log h$ . For such  $z$ ,  $\alpha_z$  is clearly  $h$  dependent and lives in the symbol class  $S_{0+}$ .

We also assume that  $M_h(z)$  is uniformly bounded for  $z \in \Omega$  and for all  $h$  small enough

$$\|M_h(z)\|_{L^2 \rightarrow L^2} \leq C.$$

Let us now define the following quantity :

$$p(\beta) = -\frac{1}{6\lambda_{\max}} P(-\varphi_u + 2\beta t_{ret}) \tag{7-1.4}$$

where  $P$  denotes the topological pressure of  $\varphi : \rho \in \mathcal{T} \mapsto -\varphi_u + 2\beta t_{ret}$  with respect to the dynamics of  $F$  on  $\mathcal{T}$  (see 3-5.1).

For  $\Omega' \subset \Omega$ , we note  $m_M(\Omega') = \sum_{z \in \Omega', f_h(z)=0} m(z)$  where  $m(z)$  stands for the multiplicity of  $z$  as a zero of  $f_h(z) = \det(1 - M_h(z))$ . Note that this determinant is well-defined since these operators are constructed trace-class (see 4-3).

**Theorem 7-1.3.** *For every  $\varepsilon > 0$ ,  $\gamma > 0$  and  $0 < R' < R$ , there exist  $C = C_{\varepsilon, \gamma, R'}$  and  $h_0 > 0$  such that*

$$m_M(\{| \text{Re } z | < R', \text{Im } z \in [-\gamma, 0]\}) \leq Ch^{-d_H + \max(p(\gamma + \varepsilon) - \varepsilon, 0)}, \forall 0 < h \leq h_0$$

where  $2d_H$  is the Hausdorff dimension of  $\mathcal{T}$ .

**7-1.2.2. Application in semiclassical scattering by a potential.**

The reduction from an open quantum system to an open quantum hyperbolic map, proved in [NSZ14] for the case of obstacle scattering, is also proved in the case of potential scattering in [NSZ11], as recalled in 6-2.3.. As a consequence, we can prove a bound similar to the one given by Theorem 7-1.2 in potential scattering. The following theorem is proved in 7-2.2. using Theorem 7-1.3.

**Theorem 7-1.4.** *Let  $V \in C_c^\infty(\mathbb{R}^2)$ ,  $E_0 > 0$  and consider the semiclassical pseudodifferential operator  $P_h = -h^2\Delta + V - E_0$ . Let's note  $p(x, \xi) = \xi^2 + V - E_0$  and assume that*

$$dp \neq 0 \text{ on } p^{-1}(0)$$

*Let's note  $H_p$  the Hamiltonian vector field associated with  $p$  and  $\Phi_t = \exp(tH_p)$  the corresponding Hamiltonian flow. Let's note  $K_0$  the trapped set of  $p$  at energy 0 and let's assume that*

- (i)  $\Phi_t$  is hyperbolic on  $K_0$  ;
- (ii)  $K_0$  is topologically one dimensional.

*Let  $\gamma_{cl}$  be the classical escape rate of the system at energy 0 and  $2d_H + 1$  be the Hausdorff dimension of  $K_0$ . Let  $N(R, \gamma; h)$  be the number of resonances of  $P_h/h$  in  $\{| \text{Re } z | < R, \text{Im } z \in [-\gamma, 0]\}$ , counted with multiplicity. Then, there exists a non increasing function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying*

- $\sigma(\gamma) > 0$  for  $0 \leq \gamma < \gamma_{cl}/2$  ;
- $\sigma(\gamma) = 0$  for  $\gamma \geq \gamma_{cl}/2$

*and such that for all  $R, \gamma > 0$  and for all  $\varepsilon > 0$  there exists  $C_{R, \gamma, \varepsilon}$  and  $h_0 > 0$  such that*

$$\forall 0 < h \leq h_0, N(R, \gamma; h) \leq C_{R, \gamma, \varepsilon} h^{-d_H + (\sigma(\gamma) - \varepsilon)_+}$$

**Remark.**

We are interested in resonances of a Schrödinger operator  $P_h = -h^2\Delta + V - E_0$  in a neighborhood of 0 of size  $h$ . To keep notations consistent with the spectral parameter  $z$  appearing in Theorem 7-1.3, we renormalize to study the resonances of  $P_h/h$  in a fixed neighborhood of 0.

**Remark.**

The theorem could be extended to a wider class a perturbations of the Laplacian in manifolds with Euclidean ends. We refer the reader to [NSZ11] (Section 2.1) for more general assumptions

**7-1.2.3. Skecth of proof of Theorem 7-1.3.**

In [NSZ14], to prove the Fractal Weyl upper bound, the author modify the monodromy operator  $M(z; h)$  and replace it by

$$M_{tG}(z; h) = e^{-tG} M(z; h) e^{tG}$$

where  $G = \text{Op}_h(g)$  with  $g$  an escape function constructed such that the Fourier integral operator  $M_{tG}$  has a small amplitude outside a neighborhood  $\mathcal{T}(h^{1/2-})$  of  $\mathcal{T}$ , and  $t$  is a fixed parameter. Here, to avoid the critical symbol class  $\tilde{S}_{1/2}$ , we will work in the symbol class  $S_\delta$  for some  $\delta = 1/2 - \varepsilon$ , so that the interesting neighborhood of  $\mathcal{T}$  becomes  $\mathcal{T}(h^\delta)$ , which has a volume comparable to  $h^{\delta(2-2d_H)}$ .

Since the zeros of  $z \mapsto \det(1 - M_{tG}(z; h))$  coincide (with multiplicity) with the zeros of  $\det(1 - M(z; h))$ , we wish to count the zeros of  $\det(1 - M_{tG}(z; h))$ . Jensen’s formula and standard spectral inequalities on spectral determinants reduce the estimates on the zeros of  $\det(1 - M_{tG}(z; h))$  to a control on the Hilbert-Schmidt norm of  $M_{tG}(z; h)$ . In [NSZ14], the author show that  $M_{tG}$  is close to an operator having a rank comparable to  $h^{-d_H}$ , which lead them to a bound of the form

$$\|M_{tG}(z; h)\|_{HS}^2 \leq \|M_{tG}(z; h)\|_{L^2 \rightarrow L^2}^2 \times \text{rank} \leq Ch^{-d_H}$$

To improve the fractal upper bound of [NSZ14] and prove Theorem 7-1.3, we start with the simple observation that the zeros of  $\det(1 - M_{tG}(z; h))$  are among the zeros of  $\det(1 - M_{tG}^n(z; h))$ , for any  $n \in \mathbb{N}^*$ . We use this fact with an exponent  $n = n(h)$  depending on  $h : n(h) \sim \nu \log 1/h$  for some  $\nu > 0$ . *A priori*, when  $n(h)$  grows logarithmically,  $M_{tG}^n$  becomes "nasty" (i.e. no more a Fourier Integral Operator in a suitable class; recall that essentially  $g \in S_\delta$ ), and in particular, it becomes impossible to use Egorov’s theorem as soon as  $n \geq \varepsilon \log 1/h$ , for some small  $\varepsilon$  (essentially  $\frac{1/2-\delta}{\lambda_{\max}}$ ). However, the action of the operator  $M_{tG}(z)$  on coherent states  $\varphi_\rho$  will remain under control for a sufficiently long logarithmic time. We will be able to obtain good estimates up to

$$n(h) \sim \frac{1}{6\lambda_{\max}} \log 1/h$$

To use these estimates, we use the representation of the trace in terms of coherent states :

$$\|M_{tG}(z; h)^n\|_{HS}^2 = \frac{1}{2\pi h} \int_U \|M_{tG}(z; h)^n \varphi_\rho\|^2 d\rho \tag{7-1.5}$$

The main new ingredient in the present chapter will consist in controlling precisely the evolved states  $M_{tG}^n \varphi_\rho$  for such logarithmic times. The behavior of this state will depend on the initial point  $\rho$  (see Proposition 7-5.2 for a precise and rigorous statement)

- If  $\rho$  is not in an  $h^\delta$  neighborhood of  $\mathcal{T}$ , we will show that for any  $L > 0$ , we can find  $t = t(L)$  such that the norm of  $M_{tG}^n \varphi_\rho$  is  $O(h^L)$ . As a consequence, the mass in the integral in (7-1.5) is essentially contained in an  $h^\delta$  neighborhood of  $\mathcal{T}$ . In particular, by simply estimating  $\|M_{tG}(z; h)^n\|^2 \leq C$  in a  $h^\delta$  neighborhood of  $\mathcal{T}$ , we find that

$$\|M_{tG}(z; h)^n\|_{HS}^2 \leq Ch^{-1} h^{\delta(2-2d_H)} \leq Ch^{-d_H + O(\varepsilon)}$$

This gives the previous upper bound of [NSZ14].

- For states sufficiently close to  $\mathcal{T}$ ,  $\varphi_\rho$  will evolve into a squeezed coherent state, aligned along the unstable leaves of  $\mathcal{T}_+$ . This phenomenon can be understood as a deformation of the coherent state. In the unstable direction, the components of this squeezed state far from  $\mathcal{T}_-$  (that it at distance bigger than  $h^\delta$ ) will experience a strong damping due to the escape function. For such state, we are able to control the squared  $L^2$ -norm

$$w_z(\rho) := \|M_{tG}(z; h)^n \varphi_\rho\|^2$$

by (again, see Proposition 7-5.2 for the rigorous statement)

$$w_z(\rho) \leq C \left( \prod_{i=0}^{n-1} \alpha_z(F^i(\rho)) \right)^2 J_n^u(\rho)^{d_H-1} \tag{7-1.6}$$

with  $\alpha_z = \exp(-(\text{Im } z)t_{ret})$ . This is the crucial estimate of this chapter.

Plugging this bound into the integral in (7-1.5), we are able to prove the following upper bound

$$\|M_{tG}(z; h)^n\|_{HS}^2 \leq C_\varepsilon h^{-d_H + \tilde{\sigma}(z) - O(\varepsilon)}; \quad \tilde{\sigma}(z) = -\frac{1}{6\lambda_{\max}} P(-\varphi_u - 2(\text{Im } z)t_{ret})$$

(see Proposition 7-5.1). The link between the pressure and (7-1.6) appears when one writes

$$w_z(\rho) \leq C J_n^u(\rho)^{d_H} \exp \left( \sum_{i=0}^{n-1} (-2(\text{Im } z)t_{ret} - \varphi_u) \circ F^i(\rho) \right)$$

The factor  $J_n^u(\rho)^{d_H}$  disappears after integrating and the sum is the Birkoff sum appearing in the definition of the pressure (see the proof of Proposition 7-5.1). It finally gives Theorem 7-1.3 (see Section 7-5).

The crucial estimate (7-1.6) is the main novelty of this chapter. It relies on propagation of coherent states and a subtle interaction of the evolved state with the escape function. The proof of (7-1.6) relies on the following ideas :

- The term

$$\pi_{\alpha,n}(\rho) := \prod_{i=0}^{n-1} \alpha_z(F^i(\rho)) \tag{7-1.7}$$

comes from the repeated action of  $M(z)$  on  $\varphi_\rho$ .

- The initial state  $\varphi_\rho$  is a wavepacket of size  $h^{1/2}$ .  $M(z)^n \varphi_\rho$  is a squeezed coherent state, microlocalized near  $F^n(\rho)$ . This is due to the fact that we will work with  $n = n(h) \leq \frac{1-\eta}{6\lambda_{\max}} \log(1/h)$  for some  $\eta > 0$ . Nevertheless, it is no more microlocalized in a  $h^{1/2}$  neighborhood of this point. It will be more convenient to write it as a Lagrangian state, associated with a local unstable leaf  $W_u(\rho_n)$ , for some  $\rho_n \in \mathcal{T}$  close to  $F^n(\rho)$  : if  $\psi_u(x)$  is a generating function for  $W_u(\rho_n)$ , that is, if we can write  $W_u(\rho_n) = \{(x, \psi'_u(x)), x \in I\}$ , the state will be written

$$a_h(x) e^{\frac{i}{h} \psi_u(x)}$$

The size of this Lagrangian state along the unstable manifold is controlled by the local Jacobian near  $\rho$  and is  $O(h^{1/2} J_n^u(\rho))$  : we will see that

$$|x| \gg h^{1/2} J_n^u(\rho) \implies a_h(x) = O(h^\infty)$$

- Finally, we need to understand the interaction of the escape function with this evolved state (see Figure 7.1). The action of the escape function damps the part of the state at distance larger than  $h^\delta$  of  $\mathcal{T}$ . Since such a state is very close to an unstable manifold, the only relevant damping on this state comes from the components at distance larger than  $h^\delta$  from  $\mathcal{T}_-$ . Roughly speaking, to obtain the bound we want, we prove that if  $d((x, \psi'_u(x)), \mathcal{T}_-) \leq h^\delta$ , then

$$a_h(x) \leq C \pi_{\alpha,n}(\rho) (J_n^u(\rho))^{-1/2} h^{-1/4} \tag{7-1.8}$$

and we prove that we can neglect the remaining points  $x$  such that  $d((x, \psi'_u(x)), \mathcal{T}_-) \geq h^\delta$  (see Proposition 7-6.6). It gives

$$\|M_{tG}(z)^n \varphi_\rho\|_{L^2}^2 \leq C \pi_{\alpha,n}(\rho)^2 J_n^u(\rho)^{-1} h^{-1/2} \text{Len}(X_-(\rho, \rho_n))$$

where

$$X_-(\rho, \rho_n) = \{x \in \mathbb{R}, |x| \leq C J_n^u(\rho) h^{1/2}, d((x, \psi'_u(x)), \mathcal{T}_-) \leq h^\delta\}$$

- It remains to control the length of  $X_-(\rho, \rho_n)$ . We use the fact that  $\mathcal{T}_- \cap W_u(\rho_n)$  has box dimension  $d_H$ . In fact, we are interested by a piece of  $W_u(\rho_n)$  of size  $h^{1/2} J_n^u(\rho)$  and we show that such a piece can be covered by  $N_h$  balls of radius  $h^\delta$  with (see Lemma 3-6.1)

$$N_h \leq C \left( \frac{h^\delta}{h^{1/2} J_n^u(\rho)} \right)^{-d_H}$$

so that

$$\text{Len}(X_-(\rho, \rho_n)) \leq C N_h h^\delta \leq C h^{1/2} J_n^u(\rho)^{d_H} h^{-O(\varepsilon)}$$

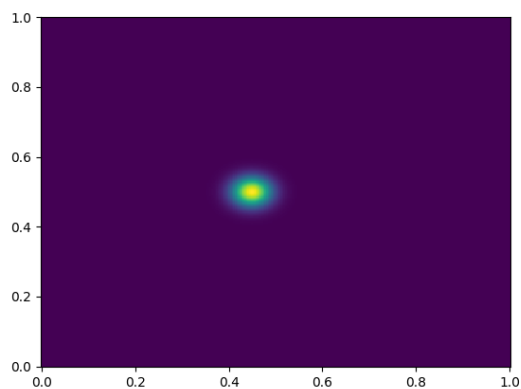
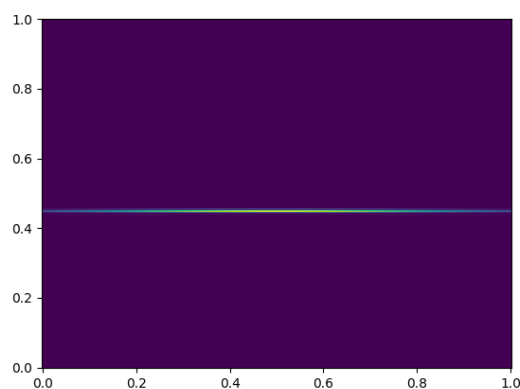
- Putting the pieces together, we obtain (7-1.6).

**Comment on the proof and the exponent.** The proof relies on propagation of coherent states in hyperbolic systems. In such systems, the Ehrenfest time is known to be  $T_E = \frac{1}{2\lambda_{\max}} \log 1/h$ . Here, we only propagate up to the time  $\frac{1}{3} T_E$ . It corresponds to the time at which the approximation in terms of squeezed coherent states fails, which is a well-known fact in the study of the semiclassical propagation of Gaussian coherent states (see [CR12] and the references given there). In fact, using an idea of [SVT12] and techniques developed in [AN07b], [NZ09] concerning the propagation of Lagrangian states, it should be possible to go up to the Ehrenfest time by writing a squeezed coherent states as a highly-oscillating Lagrangian states in the unstable direction. Indeed, the propagation of Lagrangian states aligned along unstable leaves lead to smoothing the amplitude of such state. If such a method work, it would lead to have an exponent  $3\sigma(\gamma)$  in Theorem 7-1.2, instead of  $\sigma(\gamma)$ : this would be a better estimate, but the threshold  $\gamma_{cl}$  does not change. For this reason and because it would require much more material, the author has decided to stop at  $T_E/3$ , where the first approximation stays valid.

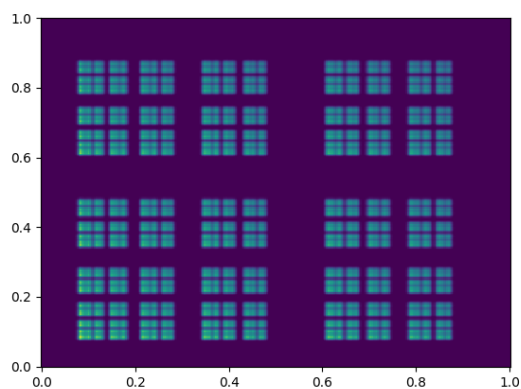
**Plan of the chapter.** The chapter is organized as follows :

- In Section 7-2, we explain how to apply Theorem 7-1.3 in obstacle scattering and scattering by a potential.
- We deal with preliminaries in Section 7-3 and Section 7-4. Section 7-3 is devoted to semi-classical results concerning metaplectic operators and coherent states. Section 7-4 focuses on properties of hyperbolic dynamical systems and we introduce a crucial escape function.
- Section 7-5 reduces the proof of Theorem 7-1.3 to the key proposition 7-5.2, concerning the behavior of the propagated coherent states.
- Section 7-6 is devoted to the proof of Proposition 7-5.2.

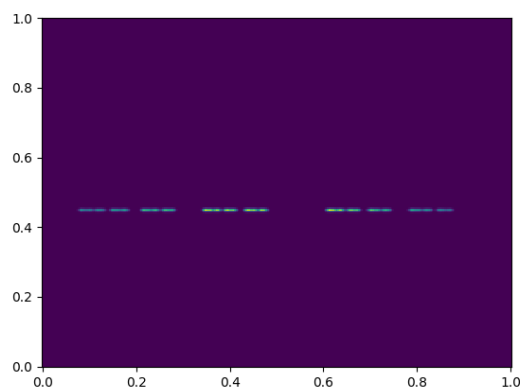
**Notations.** Throughout the chapter, we will use the same constant  $C$  at different places, with different meaning. However, it will always have the same dependence on the dynamical system and the family of operators  $M_h(z)$  we work with. That is, we write  $f \leq Cg$  instead of : there exists  $C > 0$  depending on  $F$  and  $M_h(z)$  such that  $f \leq Cg$ . At some point, we will fix a partition of unity of  $U$ , associated with local charts, depending on parameters  $\varepsilon$  and  $\varepsilon_0$ . The constants  $C$  will also depend on these objects. If the constant  $C$  has other dependencies, we will make it precise or write it in subscript if necessary. Finally, we write  $f \sim g$  to mean  $C^{-1}f \leq g \leq Cf$ .

(a) A coherent state of size  $h^{1/2}$  ...

(b) ...evolved into a squeezed coherent state.



(c) The escape function damps the region far from the trapped set...



(d) ... and is responsible of a damping for the evolved coherent state.

Figure 7.1: We show the evolution in phase space of a coherent state in an open hyperbolic system, associated with an open baker's map. The color is related to the intensity of the Wigner distribution of the state. The damping due to the escape function is shown in (7.1c). The initial coherent state is shown in (7.1a), and the evolved state, without damping in (7.1b). When we apply the damping, the evolved state loses part of its mass (7.1d)



## 7-2 Proofs of the fractal Weyl upper bounds in obstacle scattering and scattering by a potential.

### 7-2.1. Proof of Theorem 7-1.2.

Let us show how Theorem 7-1.3 implies Theorem 7-1.2. Suppose that the obstacles  $\mathcal{O}_j$  are strictly convex, have smooth boundary and satisfy the no-eclipse condition. We refer the reader to 3-1.1.1. for the definition of the billiard map  $\mathcal{B}^+$ . As already explained in 6-2.2.,  $\mathcal{B}^+$  satisfies the assumption of Theorem 7-1.3 and the reduction performed in [NSZ14] leads to the existence of a family  $M_h(z)$  of open hyperbolic quantum maps associated with  $\mathcal{B}^+$  which satisfies :

- $M_h(z)$  depends holomorphically on  $z \in \Omega(h) = ]-R, R[ + i] - C \log 1/h, R[$  for some fixed  $R > 1$  and  $C > 1$  ;
- there exists  $h_0 = h_0(R, C)$  such that for  $0 < h \leq h_0$  and for  $z \in \Omega(h)$ ,  $\frac{1}{h} + z$  is a resonance if and only if  $\det(1 - M_h(z)) = 0$ . The multiplicity of the resonance coincides with the multiplicity of  $z$  as a zero of  $\det(1 - M_h(z))$ .

The construction of this operator is recalled in 6-2.2.. Moreover, we recall that the amplitude of  $M_h(z)$  is related, via the solutions of the eikonal equation, to the distance between two collisions. In particular, near the trapped set, it is given by

$$\alpha_h(z)(\rho) = \exp(-t_{ret}(\rho) \operatorname{Im} z) + O(h^{1-S_0+}) \quad (7-2.1)$$

For  $\rho \in \mathcal{T}$ ,  $t_{ret}(\rho)$  is described as follows : assume that  $\rho = (x, \xi)$  and  $(y, \eta) = \mathcal{B}^+(x, \xi)$ , then  $t_{ret}(\rho) = |x - y|$ .  $t_{ret}$  continues smoothly in a neighborhood of  $\mathcal{T}$  and is called a *return time function*.

We can apply Theorem 7-1.3 to this family of open quantum maps and we find that, for any fixed  $\varepsilon > 0$  and for  $r \gg 1$  (with  $h = r^{-1}$ , recalling that the resonances are given by  $1/h + z$  where  $z$  is a pole of  $\det(1 - M_h(z))$ ), the number  $N(r, \gamma)$  of resonances, counted with multiplicity, in  $[r, r + 1] - i[0, \gamma]$  satisfies

$$N(r, \gamma) \leq m_M \{ |\operatorname{Re} z| < 2, \operatorname{Im} z \geq -\gamma \} \leq C_{\varepsilon, \gamma} r^{d_H - (p(\gamma + \varepsilon) - \varepsilon)_+}$$

Where  $p(\beta)$  is given by

$$p(\beta) = -\frac{1}{6\lambda_{\max}} P(2\beta t_{ret} - \varphi_u)$$

Using the continuity of the pressure, we can choose  $\varepsilon' > 0$  to ensure that

$$P(2(\gamma + \varepsilon') t_{ret} - \varphi_u) \geq P(2\gamma t_{ret} - \varphi_u) + \varepsilon$$

and we may assume that  $\varepsilon' \leq \varepsilon$ . Applying the above formula with  $\varepsilon'$ , we find that

$$N(r, \gamma) \leq C_{\varepsilon, \gamma} r^{d_H - \sigma(\gamma) + \varepsilon}$$

with

$$\sigma(\gamma) = \max \left( 0, -\frac{1}{6\lambda_{\max}} P(-\varphi_u + 2\gamma t_{ret}) \right) \quad (7-2.2)$$

To check that  $\sigma$  satisfies the properties listed in Theorem 7-1.3, we invoke the theory of Axiom A flows (see (3-5.4, see also [BR75], Proposition 3.1) : the map  $s \mapsto P(-\varphi_u + s t_{ret})$  is strictly increasing and has a unique root given by  $\gamma_{cl}$ . In particular, we deduce that  $\sigma(\gamma) > 0$  for  $\gamma < \gamma_{cl}/2$  and  $\sigma(\gamma) = 0$  for  $\gamma \geq \gamma_{cl}/2$ , as expected. Finally, since the bound  $N(r, \gamma) = O(r^{d_H})$  holds for any  $\gamma$ , we can change  $\sigma(\gamma) - \varepsilon$  into  $(\sigma(\gamma) - \varepsilon)_+ = \max(\sigma(\gamma) - \varepsilon, 0)$ . This concludes the proof of Theorem 7-1.2.

### 7-2.2. Proof of Theorem 7-1.4.

Let us show how Theorem 7-1.3 implies Theorem 7-1.4. The ideas are the same as for the case of obstacle scattering and rely on the reduction performed in [NSZ11], presented in 6-2.3..

We consider  $V \in C_c^\infty(\mathbb{R}^2)$ ,  $E_0 > 0$  and the semiclassical pseudodifferential operator  $P_h = -h^2 \Delta + V - E_0$ . We note  $p(x, \xi) = \xi^2 + V - E_0$  and we assume that

$$dp \neq 0 \text{ on } p^{-1}(0)$$

Let's note  $H_p$  the Hamiltonian vector field associated with  $p$  and  $\Phi_t = \exp(tH_p)$  the corresponding Hamiltonian flow. Let's note  $K_0$  the trapped set of  $\Phi_t$  at energy 0 and we assume that  $\Phi_t$  is hyperbolic on  $K_0$  and  $K_0$  is topologically one dimensional. More generally, we could work with more general Schrödinger operators in manifolds with Euclidean ends. We refer the reader to [NSZ11] (Section 2.1) for more general assumptions.

To apply Theorem 7-1.3, we use the result of [NSZ11] (Theorem 1 and 2). Under the assumptions above, there exists a smooth Poincaré hypersurface  $\Sigma$  for the flow  $\Phi_t$  on the energy shell  $p^{-1}(0)$  near  $K_0$ .  $\Sigma$  is made of several disjoint pieces  $\Sigma_j$ ,  $1 \leq j \leq J$ . The reduced trapped set is now  $\mathcal{T} := K_0 \cap \Sigma$  and, if we write  $2d_H + 1$  for the dimension of  $K_0$ ,  $\mathcal{T}$  has dimension

$$\dim \mathcal{T} = \dim K_0 - 1 = 2d_H$$

The assumption that  $\Sigma$  is a smooth Poincaré hypersurface ensures that there exists  $\varepsilon_{\min} > 0$  such that the map

$$(\rho, t) \in \Sigma \times ]-\varepsilon_{\min}, \varepsilon_{\min}[ \mapsto \Phi_t(\rho)$$

is a smooth diffeomorphism onto its image. We note  $t_{ret}$  the return time function on  $\Sigma$  : for  $\rho \in \Sigma$ ,

$$t_{ret}(\rho) = \inf\{t > \varepsilon_{\min}, \Phi_t(\rho) \in \Sigma\} \in [\varepsilon_{\min}, +\infty]$$

$t_{ret} < +\infty$  in a neighborhood  $U \subset \Sigma$  of  $\mathcal{T}$ . We then define the Poincaré return map  $F$ , which is an open hyperbolic map defined on an open subset of  $\Sigma$  :

$$F : \rho \in \Sigma \mapsto \Phi_{t_{ret}}(\rho) \in \Sigma$$

In [NSZ11] (see also 6-2.3.), the authors construct a family of finite-dimensional matrices  $(\mathbf{M}(z; h))$  for  $z \in \Omega(h) = ]-R, R[ + i] - C \log 1/h, R[$  (with  $R$  fixed but large) such that for  $h$  small enough and for all  $z \in \Omega(h)$ ,

$$\det(I - \mathbf{M}(z; h)) = 0 \iff hz \text{ is a resonance of } P_h$$

The matrices  $\mathbf{M}(z; h)$  satisfy uniformly for  $z \in \Omega(h)$  and for  $h$  small enough,

$$\mathbf{M}(z; h) = \Pi_h M(z; h) \Pi_h + O(h^L) \tag{7-2.3}$$

where  $L > 0$  can be chosen as large as necessary,  $\Pi_h$  is a finite rank projector and  $M(z; h)$  is a family of open hyperbolic quantum maps associated with  $F$  (in the sense of Definition 4-3.2). The amplitude of  $M(z; h)$  satisfies

$$\alpha_h(z)(\rho) = \exp(-t_{ret}(\rho) \operatorname{Im} z) + O(h^{1-} S_{0+}).$$

By their construction,  $M(z; h)$  and  $\Pi_h$  satisfy, for some  $L > 0$  as large as necessary, uniformly for  $z \in \Omega(h)$  and for  $h$  small enough,

$$\Pi_h M(z; h) \Pi_h = M(z; h) + O(h^L) \tag{7-2.4}$$

We can apply Theorem 7-1.3 to the family  $M(z; h)$  of open quantum maps and we find that, for any fixed  $\varepsilon > 0$  and  $r > 0$  (with  $r < R$ ) and for  $h \ll 1$ , the number  $N_M(r, \gamma; h)$  of zeros of  $\det(\operatorname{Id} - M(z; h))$  in  $\{|\operatorname{Re} z| < r, \operatorname{Im} z \in [-\gamma, 0]\}$  satisfies

$$N_M(r, \gamma; h) \leq C_{r, \varepsilon, \gamma} h^{-d_H + (p(\gamma + \varepsilon) - \varepsilon)_+}$$

$p(\beta)$  is given by

$$p(\beta) = -\frac{1}{6\lambda_{\max}} P(2\beta t_{ret} - \varphi_u)$$

where  $\varphi_u$  is the unstable Jacobian associated with  $F$ . Here, it can also be obtained by differentiating the flow  $\Phi_t$ . In fact, by inspecting the proof of Theorem 7-1.3 and by using (7-2.3) and (7-2.4), we see that the same conclusion holds for  $\mathbf{M}$  instead of  $M$ . Indeed, in the formula (7-5.2) in Proposition 7-5.1, one can replace  $M(z; h)$  by  $\mathbf{M}(z; h)$  since  $M(z; h)^{N(h)} = \mathbf{M}(z; h)^{N(h)} + O(|\log h| h^L)$  as soon as  $N(h) = O(\log h)$ . We now conclude as for the case of obstacle scattering in 7-2.1. and find that

$$N(r, \gamma; h) \leq C_{r, \gamma, \varepsilon} h^{-d_H + \sigma(\gamma) - \varepsilon}$$

where

$$\sigma(\gamma) = \max\left(0, -\frac{1}{6\lambda_{\max}} P(-\varphi_u + 2\gamma t_{ret})\right) \tag{7-2.5}$$

### 7-3 Semiclassical preliminaries.

We recall that Chapter 4 has already provided the main definitions and properties concerning pseudodifferential operators and Fourier Integral operators. In particular, we refer to 4-3 for the definition of open quantum hyperbolic maps (see Definition 4-3.2) needed to understand the main theorem of this chapter.

#### 7-3.1. Pseudodifferential operators and Weyl quantization.

For the purpose of this chapter, we extend the definitions given in Chapter 4 by considering slightly more general symbol classes, based on particular order functions.

**Definitions.** We start by defining classes of  $h$ -dependent symbols. In the following definitions,  $m$  is a positive functions defined on  $T^*\mathbb{R}^n$  of the form  $\langle \rho \rangle^N$ , for some  $N \in \mathbb{Z}$ , where  $\langle \rho \rangle = \sqrt{1 + |\rho|^2}$  and  $\rho = (x, \xi)$  is a point in phase space  $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ .  $m$  is called an order function (in the sense of [Zwo12], 4.4.1)

**Definition 7-3.1.** Let  $0 \leq \delta < \frac{1}{2}$ . We say that an  $h$ -dependent family  $a := (a(\cdot; h))_{0 < h \leq 1} \in C^\infty(\mathbb{R}^{2n})$  is in the class  $S_\delta(m)$  (and simply  $S_\delta$  if  $m = 1$ ) if for every  $\alpha \in \mathbb{N}^{2n}$ , there exists  $C_\alpha > 0$  such that :

$$\forall 0 < h \leq 1, \sup_{\rho \in \mathbb{R}^{2n}} |\partial^\alpha a(\rho; h)| \leq C_\alpha h^{-\delta|\alpha|} m(\rho)$$

We will use the notation  $S_{0+}(m) = \bigcap_{\delta > 0} S_\delta(m)$ .

We write  $a = O(h^N)_{S_\delta(m)}$  to mean that for every  $\alpha \in \mathbb{N}^{2n}$ , there exists  $C_{\alpha, N}$  such that

$$\forall 0 < h \leq 1, \sup_{\rho \in \mathbb{R}^{2n}} h^{\delta|\alpha|} |\partial^\alpha a(\rho; h)| \leq C_{\alpha, N} h^N m(\rho)$$

If  $a = O(h^N)_{S_\delta(m)}$  for all  $N \in \mathbb{N}$ , we'll write  $a = O(h^\infty)_{S_\delta(m)}$ . For a given symbol  $a \in S_\delta$ , we say that  $a$  has a compact essential support if there exists a compact set  $K$  such that<sup>2</sup> :

$$\forall \chi \in C_c^\infty(\mathbb{R}^{2n}), \text{supp } \chi \cap K = \emptyset \implies \chi a = O(h^\infty)_{S(\mathbb{R}^{2n})}$$

We say that  $a$  belongs to the class  $S_\delta^{comp}$  and its essential support is then the intersection of all such compact  $K$ 's. We denote it  $\text{ess supp } a \subset K$ . In particular, the class  $S_\delta^{comp}$  contains all the symbols supported in a  $h$ -independent compact set and these symbols correspond, modulo  $O(h^\infty)_{S(T^*\mathbb{R})}$ , to all symbols of  $S_\delta^{comp}$ . For this reason, we will adopt the following notation: if  $V \Subset \mathbb{R}^{2n}$  is an open set, we say that  $a \in S_\delta^{comp}(V)$  if  $a \in S_\delta^{comp}(\mathbb{R}^{2n})$  and  $\text{ess supp } a \Subset V$ .

For a symbol  $a \in S_\delta(m)$ , we'll quantize it using Weyl's quantization procedure. It is informally written as :

$$(\text{Op}_h(a)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} a\left(\frac{x+y}{2}, \xi; h\right) u(y) e^{i\frac{(x-y)\cdot\xi}{h}} dy d\xi$$

We will note  $\Psi_\delta(m)$  the corresponding classes of pseudodifferential operators. By definition, the wavefront set of  $A = \text{Op}_h(a)$  is  $\text{WF}_h(A) = \text{ess supp } a$ .

If  $m_1$  and  $m_2$  are two order functions of the form  $\langle \rho \rangle^{N_i}$ ,  $i = 1, 2$ , the composition of two pseudodifferential operators in  $\Psi_\delta(m_1)$  and  $\Psi_\delta(m_2)$  is a pseudodifferential operator in the class  $\Psi_\delta(m_1 m_2)$ . More precisely (see [Zwo12], Theorem 4.11 and 4.18), if  $(a, b) \in S_\delta(m_1) \times S_\delta(m_2)$ ,  $\text{Op}_h(a) \circ \text{Op}_h(b)$  is given by  $\text{Op}_h(a \# b)$ , where  $a \# b$  is the Moyal product of  $a$  and  $b$ . It is given by

$$a \# b(\rho) = e^{ihA(D)}(a \otimes b)|_{\rho=\rho_1=\rho_2}$$

where  $a \otimes b(\rho_1, \rho_2) = a(\rho_1)b(\rho_2)$ ,  $e^{ihA(D)}$  is a Fourier multiplier acting on functions on  $\mathbb{R}^{4n}$  and, writing  $\rho_i = (x_i, \xi_i)$ ,

$$A(D) = \frac{1}{2} (D_{\xi_1} \circ D_{x_2} - D_{x_1} \circ D_{\xi_2})$$

We can estimate the Moyal product by a quadratic stationary phase and get the following expansion which holds in  $S_\delta(m_1 m_2)$  for all  $N \in \mathbb{N}$ ,

<sup>2</sup>Here  $\mathcal{S}$  denotes the Schwartz space and the notation  $O(h^\infty)_{S(\mathbb{R}^{2n})}$  means that every semi-norm is  $O(h^\infty)$ .

$$a \# b(\rho) = \sum_{k=0}^{N-1} \frac{i^k h^k}{k!} A(D)^k (a \otimes b)|_{\rho=\rho_1=\rho_2} + r_N$$

where for all  $\alpha \in \mathbb{N}^{2n}$ , there exists  $C_\alpha$ , independent of  $a$  and  $b$ , such that

$$\|\partial^\alpha r_N\|_\infty \leq C_\alpha h^N \|a \otimes b\|_{C^{2N+4n+1+|\alpha|}}$$

**Weighted Sobolev spaces.** We can also define the weighted Sobolev spaces  $H_h(m)$ . In the case  $m = \langle \rho \rangle^N$ , we have

$$H_h(\langle \rho \rangle^N) = \text{Op}_h(\langle \rho \rangle^{-N})(L^2(\mathbb{R}^n)) \subset \mathcal{S}'(\mathbb{R}^n)$$

When  $N \geq 0$ ,  $H_h(\langle \rho \rangle^N)$  coincides with the space of functions  $u \in \mathcal{S}'(\mathbb{R}^n)$  such that

$$\forall \alpha, \beta \in \mathbb{N}^n \text{ with } |\alpha| + |\beta| \leq N, \quad x^\alpha (h\partial^\beta)u \in L^2(\mathbb{R}^n)$$

and we have the following equivalence of norms :

$$\|u\|_{H_h(\langle \rho \rangle^N)}^2 \sim \sup_{|\alpha|+|\beta| \leq N} \|x^\alpha (h\partial^\beta)u\|_{L^2}^2$$

As a consequence of Calderon-Vaillancourt theorem, we have for symbols  $a \in S_\delta(m)$  :

**Proposition 7-3.1.** Let  $N \in \mathbb{Z}$ . There exists  $M \in \mathbb{N}$  and  $C > 0$  such that the following holds : For all  $a \in S_\delta(\langle \rho \rangle^N)$ ,  $\text{Op}_h(a) : H_h(\langle \rho \rangle^N) \rightarrow L^2$  is uniformly bounded and

$$\|\text{Op}_h(a)\|_{H_h(\langle \rho \rangle^N) \rightarrow L^2} \leq C \sup_{|\alpha| \leq M} h^{|\alpha|/2} \|\langle \rho \rangle^{-N} \partial^\alpha a\|_\infty$$

### 7-3.2. Metaplectic operators.

Among the class of Fourier integral operators acting on  $L^2(\mathbb{R})$ , metaplectic operators are the one quantizing the linear symplectic transformations of  $T^*\mathbb{R} = \mathbb{R}^2$ . The main advantage of metaplectic operators compared with general Fourier Integral operators is that the Egorov property is exact (see definition 7-3.2 below). We recall here a few standard facts on metaplectic operators. We refer the reader to [Zwo12] (Section 11.3) and [CR12] (Chapter 3) for a more precise presentation and other references.

**Definition 7-3.2.** For  $\rho = (x_0, \xi_0) \in \mathbb{R}^2 = T^*\mathbb{R}$ , the phase space translation operator  $T_h(\rho)$  is defined as :

$$T_h(\rho)u(x) = e^{-i\frac{x_0\xi_0}{2h}} e^{i\frac{x\xi_0}{h}} u(x - x_0)$$

It is a unitary on  $L^2(\mathbb{R})$  and  $T_h(\rho)^* = T_h(-\rho)$ . Moreover,  $T_h(\rho)^* \text{Op}_h(a) T_h(\rho) = \text{Op}_h(a(\cdot - \rho))$  for any classical observable  $a \in \mathcal{S}'(\mathbb{R})$ .

**Proposition 7-3.2. (and Definition)** Let  $\kappa : T^*\mathbb{R} \rightarrow T^*\mathbb{R}$  be a symplectic linear map. There exists a unitary operator  $\mathcal{M}_h(\kappa) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  such that one of the two following equivalent conditions hold :

- (i) For every  $\rho \in T^*\mathbb{R}$ ,  $\mathcal{M}_h(\kappa) T_h(\rho) \mathcal{M}_h(\kappa)^* = T_h(\kappa(\rho))$  ;
- (ii) For all  $a \in \mathcal{S}'(\mathbb{R})$ ,  $\mathcal{M}_h(\kappa) \text{Op}_h(a) \mathcal{M}_h(\kappa)^* = \text{Op}_h(a \circ \kappa^{-1})$ .

The operator  $\mathcal{M}_h(\kappa)$  is unique up to multiplication by an element of  $\mathbb{U} = \{z \in \mathbb{C}, |z| = 1\}$ .

Most of the time we won't precise that  $T_h(\rho)$  and  $\mathcal{M}_h(\kappa)$  depend on  $h$  and we will simply write  $T(\rho)$  and  $\mathcal{M}(\kappa)$ . We will write the index  $h$  (or  $h = 1$ ) when needed. In fact, we can relate  $\mathcal{M}_h(\kappa)$  and  $\mathcal{M}_1(\kappa)$  by the relation :

$$\mathcal{M}_h(\kappa) \Lambda_h = \Lambda_h \mathcal{M}_1(\kappa) \tag{7-3.1}$$

where  $\Lambda_h$  is the unitary scaling operator :

$$\Lambda_h u(x) = h^{-1/4} u(h^{-1/2}x) \tag{7-3.2}$$

A way to obtain metaplectic operators is by solving the Schrödinger equation associated with quadratic Hamiltonians.

**Proposition 7-3.3.** Let  $S_2(\mathbb{R})$  be the spaces of symmetric matrices of  $\mathcal{M}_2(\mathbb{R})$ . Let  $t \in [0, 1] \mapsto S(t) \in S_2(\mathbb{R})$  be a  $C^1$  curve. We note

- the quadratic time dependent Hamiltonian  $H(t, \rho) = \frac{1}{2}(\rho, S(t)\rho)$  ;
- $t \in [0, 1] \mapsto \kappa(t)$  the classical flow for the Hamiltonian  $H(t)$ , which solves the equation

$$\dot{\kappa}(t) = JS(t)\kappa(t)$$

where  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .  $\kappa(t)$  is a symplectic linear map for all  $t \in [0, 1]$

- $U(t)$  the propagator of the Schrödinger equation

$$\frac{\hbar}{i} \frac{d}{dt} u(t) + \text{Op}_h(H(t))u = 0$$

$U(t)$  is a unitary operator on  $L^2(\mathbb{R})$  for all  $t \in [0, 1]$ .

Then, for all  $t \in [0, 1]$ ,  $U(t)$  is a metaplectic operator associated with the linear symplectic map  $\kappa(t)$ .

Note that for every  $\kappa_1 \in \text{Sp}_2(\mathbb{R})$ , there always exists a (non unique)  $C^1$  curve  $\kappa : t \in [0, 1] \rightarrow \text{Sp}_2(\mathbb{R})$  such that  $\kappa(0) = I_2$  and  $\kappa(1) = \kappa_1$ . (see for instance [CR12], Proposition 31 in Chapter 3). As a consequence, we can always construct  $\mathcal{M}(\kappa_1)$  by use of the previous proposition.

*Example.* The unitary  $h$ -Fourier transform  $\mathcal{F}_h$ , where

$$\mathcal{F}_h u(\xi) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{\mathbb{R}} u(y) e^{-\frac{iy\xi}{\hbar}} dy$$

is a metaplectic operator associated with  $J$ .

*Example.* Suppose that

$$\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with  $a \neq 0$ . Then, the following operator is a metaplectic operator associated with  $\kappa$  :

$$\mathcal{M}(\kappa)u(x) = \left( \frac{1}{2\pi\hbar|a|} \right)^{1/2} \int_{\mathbb{R}} e^{\frac{i}{2\hbar}(ca^{-1}x^2 + 2a^{-1}x\xi - a^{-1}b\xi^2)} \mathcal{F}_h u(\xi) d\xi \quad (7-3.3)$$

### 7-3.3. Coherent states.

#### 7-3.3.1. Definitions and notations.

In this subsection, we introduce the notations and definitions we will use for studying coherent states. We refer the reader to [CR12]. The semiclassical coherent state (or Gaussian state) centered at zero will be denoted by

$$\varphi_0(x) = \frac{1}{(\pi\hbar)^{1/4}} e^{-\frac{x^2}{2\hbar}} \quad (7-3.4)$$

and the coherent state centered at  $\rho$  is simply

$$\varphi_\rho := T(\rho)\varphi_0 \quad (7-3.5)$$

We also write

$$\varphi_0 = \Lambda_h \Psi_0$$

where  $\Lambda_h$  is defined in (7-3.2) and  $\Psi_0$  is the renormalized coherent state

$$\Psi_0(x) = \frac{1}{\pi^{1/4}} e^{-\frac{x^2}{2}} \quad (7-3.6)$$

We recall that  $\varphi_0$  (resp.  $\Psi_0$ ) is the ground state of the harmonic oscillator  $-\hbar^2 \partial_x^2 + x^2$  (resp.  $-\partial_x^2 + x^2$ ). The other eigenfunctions of this harmonic oscillator, called excited states, are obtained

from  $\varphi_0$  (resp.  $\Psi_0$ ) by applying the creation operator  $\mathbf{a} = \frac{1}{\sqrt{2\hbar}}(-\hbar\partial_x + x)$  (resp.  $\Lambda_h^*\mathbf{a}\Lambda_h = \frac{1}{\sqrt{2}}(-\partial_x + x)$ ). For  $n \in \mathbb{N}$ , we can note for instance

$$\varphi_{0,n} = \mathbf{a}^n \varphi_0; \Psi_n = \Lambda_h^* \mathbf{a}^n \Lambda_h \Psi_0$$

We recall that  $\Psi_n = h_n(x)\Psi_0$  where  $h_n$  is a hermite polynomial of degree  $n$ . In particular, if  $P \in \mathbb{C}[X]$ , it is possible to decompose  $P(x)\Psi_0(x)$  into a linear combination of excited states up to order  $\deg(P)$ .

We can also define *squeezed coherent states* :

**Definition 7-3.3.** Let  $\gamma \in \mathbb{C}$  with  $\text{Im } \gamma > 0$ . The squeezed coherent state, deformed by  $\gamma$  and centered at zero is

$$\varphi_0^{(\gamma)}(x) = (a_\gamma \pi \hbar)^{-1/4} e^{i\gamma \frac{x^2}{2\hbar}}$$

where  $a_\gamma = \text{Im}(\gamma)^{-1}$  makes the norm of this state equal to one. We also define the squeezed coherent state centered at  $\rho \in T^*\mathbb{R}$  by

$$\varphi_\rho^{(\gamma)} = T(\rho)\varphi_0^{(\gamma)}$$

and the squeezed renormalized coherent state at 0

$$\Psi_0^{(\gamma)}(x) = (a_\gamma \pi)^{-1/4} e^{i\gamma \frac{x^2}{2}}$$

We conclude this section by recalling a useful formula - a resolution of the identity - which is the starting point of our analysis (see [CR12], Proposition 6 in Section 1.2)

**Lemma 7-3.1.** Let  $A : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  be a trace class operator. Then,

$$\text{tr}(A) = \frac{1}{2\pi\hbar} \int_{T^*\mathbb{R}} \langle A\varphi_\rho, \varphi_\rho \rangle d\rho$$

where  $d\rho$  denotes the Lebesgue measure of  $\mathbb{R}^2$ .

**7-3.3.2. Action of metaplectic operators on coherent states.**

We recall here how metaplectic operators act on coherent states. We refer the reader to [CR12] (Section 3.2) for a complete proof and a general version in any dimension :

**Proposition 7-3.4.** Let  $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a symplectic linear map. Let  $\mathcal{M}(\kappa)$  be a metaplectic operator associated with  $\kappa$ , constructed by use of Proposition 7-3.3, following a path  $\kappa_t$  from  $I_2$  to  $\kappa$ . Then, we have :

$$\mathcal{M}(\kappa)\varphi_0(x) = (\pi\hbar)^{-1/4}(a + ib)^{-1/2} e^{i\gamma_\kappa \frac{x^2}{2\hbar}}$$

where  $\gamma_\kappa = (c + id)(a + ib)^{-1}$ .

**Remark.**

The square root  $(a + ib)^{1/2}$  is determined by the path  $\kappa_t$  ( $(a_t + ib_t)^{1/2}$  has to be continuous). Since  $\text{Im } \gamma_\kappa = |a + ib|^{-2}$ , this proposition shows that for some  $\lambda \in \mathbb{U}$ ,

$$\mathcal{M}(\kappa)\varphi_0 = \lambda\varphi_0^{(\gamma)}$$

Since the metaplectic operators are defined modulo  $\mathbb{U}$ , in the rest of this chapter, we will sometimes omit to write the factor  $\lambda$  and and by abuse, we could write  $\mathcal{M}(\kappa)\varphi_0 = \varphi_0^{(\gamma)}$ . It won't be specified anymore. Anyway, we are concerned by the norm of such states.

We also give the following formula concerning the action of metaplectic operators on excited coherent states (see [Hag98], Section 2) :

**Proposition 7-3.5.** Let  $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a symplectic linear map. Let  $\mathcal{M}(\kappa)$  be a metaplectic operator associated with  $\kappa$ , constructed by use of Proposition 7-3.3, following a path  $\kappa_t$  from  $I_2$  to  $\kappa$ . Then,

$$\mathcal{M}(\kappa)\varphi_{0,n} = (|a+ib|^2\pi h)^{-1/4} \left( \frac{a-ib}{a+ib} \right)^{n/2} h_n \left( \frac{x}{h^{1/2}|a+ib|} \right) e^{i\gamma_\kappa \frac{x^2}{2h}}$$

where  $\gamma_\kappa = (c+id)(a+ib)^{-1}$ .

In the sequel, we will need to estimate the  $H_h(\langle \rho \rangle^N)$ -norm of squeezed coherent states in terms of the squeezing parameter. Equivalently, we need to control this norm for a state of the form  $\mathcal{M}(\kappa)\varphi_\rho$  in terms of  $\kappa$ . To do so, we start by fixing a norm  $\|\cdot\|$  on  $\mathcal{M}_2(\mathbb{R})$ . For convenience, let's assume that for all linear symplectic map, we have

$$\|\kappa\| \geq 1 \tag{7-3.7}$$

For instance, let's say that  $\|\kappa\| = \sqrt{2} \max(|\kappa|_{11}, |\kappa|_{12}, |\kappa|_{21}, |\kappa|_{22})$ . It is not hard to check that this norm satisfies 7-3.7 since  $\det(\kappa) = 1$ . The main interest of (7-3.7) is that  $\|\kappa\|^a \leq \|\kappa\|^b$  if  $a \leq b$ .

We have :

**Lemma 7-3.2.** There exists a family of universal constants  $(K_{N,k})_{(N,k) \in \mathbb{N}^2}$  such that the following holds : let  $N \in \mathbb{N}$ ,  $k \in \mathbb{N}$  and  $\kappa$  be a symplectic linear map. Then, for all  $0 < h \leq 1$ ,

$$\|\mathcal{M}(\kappa)(x^k \varphi_0)\|_{H_h(\langle \rho \rangle^N)} \leq K_{N,k} \sum_{l=0}^N h^{(l+k)/2} \|\kappa\|^l$$

*Proof.* Let's write  $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . For a state  $u \in H_h(\langle \rho \rangle^N)$ , we have

$$\|u\|_{H_h(\langle \rho \rangle^N)}^2 \sim \sum_{\alpha+\beta \leq N} \|\text{Op}_h(x^\alpha \xi^\beta)u\|_{L^2}^2$$

Let  $\alpha, \beta \in \mathbb{N}$  such that  $\alpha + \beta \leq N$ . We want to estimate  $\|\text{Op}_h(x^\alpha \xi^\beta) \mathcal{M}(\kappa)(x^k \varphi_0)\|_{L^2}^2$ . We have

$$\text{Op}_h(x^\alpha \xi^\beta) \mathcal{M}(\kappa)(x^k \varphi_0) = \mathcal{M}(\kappa) \text{Op}_h((ax + b\xi)^\alpha (cx + d\xi)^\beta)(x^k \varphi_0)$$

Since  $\mathcal{M}(\kappa)$  is unitary on  $L^2$ , it is enough to estimate the  $L^2$  norm of

$$\text{Op}_h((ax + b\xi)^\alpha (cx + d\xi)^\beta)(x^k \varphi_0) = \text{Op}_h \left( \sum_{l=(l_1, l_2) \in \mathbb{N}^2, l_1+l_2=\alpha+\beta} B_l(\kappa) x^{l_1} \xi^{l_2} \right) (x^k \varphi_0)$$

where  $B_l$  is some  $l_1 + l_2$  multilinear form in  $\kappa$ , whose coefficients depend on  $\alpha$  and  $\beta$ . In particular,  $|B_l(\kappa)| \leq C_l \|\kappa\|^{l_1+l_2}$  for some universal  $C_l$ . Finally, we observe that  $\|\text{Op}_h(x^{l_1} \xi^{l_2})(x^k \varphi_0)\|_{L^2} := h^{(l_1+l_2+k)/2} C_{(l_1, l_2, k)}$ , for some  $C_{(l_1, l_2, k)}$  depending only on  $(l_1, l_2, k)$ , and we find that

$$\|\text{Op}_h((ax + b\xi)^\alpha (cx + d\xi)^\beta)(x^k \varphi_0)\| \leq C_{(\alpha, \beta, k)} \sum_{p=0}^{\alpha+\beta} \|\kappa\|^p h^{(p+k)/2}$$

we find the required inequality with  $K_{N,k}$  depending on the the  $C_{\alpha, \beta, k}$  with  $\alpha + \beta \leq N$ .  $\square$

As a corollary, by specializing at  $h = 1$ , we obtain the following :

**Corollary 7-3.1.** There exists a family of constants  $K_{N,d}, d \in \mathbb{N}, N \in \mathbb{N}$  such that : for all  $P \in \mathbb{C}[X]$ , for all symplectic linear map  $\kappa$  and for all  $N \in \mathbb{N}$ ,

$$\|\mathcal{M}_1(\kappa)(P\Psi_0)\|_{H_1(\langle \rho \rangle^N)} \leq K_{N, \deg P} N_\infty(P) \|\kappa\|^N$$

where  $N_\infty(P)$  is the sup norm of the coefficients of  $P$ .

**7-3.3.3. Action of pseudodifferential operators on coherent states.**

In this subsection, we give precise results for the actions of semiclassical pseudodifferential operators on coherent states, when the symbol of the pseudodifferential operator belongs to the class  $S_\delta$ .

**Lemma 7-3.3.** Suppose that  $a \in S_\delta(T^*\mathbb{R})$  with  $0 \leq \delta < 1/2$ . Assume that  $\rho_0 = (x_0, \xi_0) \in T^*\mathbb{R}$ . Then, for every  $N \in \mathbb{N}$ , there exists  $\rho_N(a, \rho_0) \in L^2$  such that

$$\text{Op}_h(a)\varphi_{\rho_0} = \sum_{k=0}^{N-1} h^{k/2}\psi_k(a, \rho_0) + h^{N/2}\rho_N(a, \rho_0)$$

where

$$\psi_k(a, \rho_0) = T(\rho_0)\Lambda_h \text{Op}_1 \left( \sum_{\alpha+\beta=k} \frac{\partial_x^\alpha \partial_\xi^\beta a(\rho_0)}{\alpha!\beta!} x^\alpha \xi^\beta \right) \Psi_0$$

and

$$\|\rho_N(a, \rho_0)\|_{L^2} \leq C_N h^{-\delta N} \sup_{|\gamma| \leq N+M} \|h^{|\gamma|} \partial^\gamma a\|_\infty$$

**Remark.**

- $M$  is a universal constant.
- The first term of the expansions is  $a(\rho_0)\varphi_{\rho_0}$ .
- It is effectively an expansion in power of  $h^{1/2-\delta}$  since  $a \in S_\delta$ .
- We could also write  $\text{Op}_1(x^\alpha \xi^\beta)\Psi_0$  in the form  $P(x)\Psi_0$  where  $P$  is a polynomial of degree  $\alpha + \beta$ , or equivalently, it is a linear combination of the first  $|\alpha| + |\beta|$  excited states.

*Proof.* Let's write  $\varphi_{\rho_0} = T(\rho_0)\Lambda_h\Psi_0$ . We have

$$\begin{aligned} \text{Op}_h(a)\varphi_{\rho_0} &= \text{Op}_h(a)T(\rho_0)\Lambda_h\Psi_0 \\ &= T(\rho_0)\Lambda_h \text{Op}_1(b_h)\Psi_0 \end{aligned}$$

where  $b_h(\rho) = a(\rho_0 + h^{1/2}\rho)$ . Let's write the Taylor expansion of  $a$  around  $\rho_0$  :

$$b_h(x, \xi) = \sum_{\alpha+\beta \leq N-1} h^{(\alpha+\beta)/2} \frac{\partial_x^\alpha \partial_\xi^\beta a(\rho_0)}{\alpha!\beta!} x^\alpha \xi^\beta + h^{N/2} R_N(x, \xi)$$

where

$$R_N(\rho) = \frac{1}{(N-1)!} \int_0^1 \frac{d^N}{dt^N} a(\rho_0 + th^{1/2}\rho) (1-t)^{N-1} dt$$

Applying  $\text{Op}_1$  to this expansion, we get the required asymptotic with

$$\rho_N(a, \rho_0) = T(\rho_0)\Lambda_h \text{Op}_1(R_N)\Psi_0$$

It remains to estimate the  $L^2$  norm of  $\rho_N$ . Since  $T(\rho_0)$  is unitary, it is enough to evaluate

$$\Lambda_h \text{Op}_1(R_N)\Psi_0 = \text{Op}_h(\tilde{R}_N)\varphi_0$$

where  $\tilde{R}_N(\rho) = R_N(h^{-1/2}\rho) = \frac{1}{(N-1)!} \int_0^1 (1-t)^{N-1} d^N a(\rho_0 + t\rho)(\rho^{\otimes N}) dt$ <sup>3</sup>. Using that  $a \in S_\delta$ , it is not hard to see, after derivation under the integral that, for any  $\gamma \in \mathbb{N}^2$  and  $\rho \in T^*\mathbb{R}$ ,

$$|\partial^\gamma \tilde{R}_N(\rho)| \leq C_N \sup_{\gamma_1 \leq N+|\gamma|} \|\partial^{\gamma_1} a\|_\infty \langle \rho \rangle^N \leq h^{-\delta(N+|\gamma|)} \|h^{|\gamma|} \partial^\gamma a\|_\infty \langle \rho \rangle^N$$

This shows that  $\tilde{R}_N \in h^{-\delta N} S_\delta(\langle \rho \rangle^N)$  in the sense of [Zwo12] (Definition 4.4.3). Then, we find that

$$h^{\delta N} \text{Op}_h(\tilde{R}_N) : H_h(\langle \rho \rangle^N) \rightarrow L^2(\mathbb{R})$$

<sup>3</sup>Here, if  $f \in C^N(\mathbb{R}^2, \mathbb{R})$ , we note  $d^N f(\rho)(h^{\otimes N}) = \frac{d^N}{dt^N} \Big|_{t=0} f(\rho + th)$ .



is a uniformly bounded family of operators, with norm depending on a finite number of semi-norms of  $\tilde{R}_N$  in  $S_\delta(\langle \rho \rangle^N)$ . We conclude by noting that for any  $N \in \mathbb{N}$ ,  $\varphi_0$  is in  $H_h(\langle \rho \rangle^N)$ , with a norm bounded uniformly in  $h$ . Hence

$$\|\rho_N\|_{L^2} \leq \| \text{Op}_h(\tilde{R}_N) \|_{H_h(\langle \rho \rangle^N) \rightarrow L^2(\mathbb{R})} \times \|\varphi_0\|_{H_h(\langle \rho \rangle^N)} \leq h^{-\delta N} C_N \sup_{|\gamma| \leq N+M} \|h^{\delta|\gamma|} \partial^\gamma b\|_\infty$$

□

As a simple corollary, we get :

**Corollary 7-3.2.** Assume that  $a$  vanishes at order  $k$  at  $\rho_0$ . Then,

$$\text{Op}_h(a)\varphi_{\rho_0} = O_{L^2} \left( h^{k(1/2-\delta)} \right)$$

In particular, if  $a$  vanishes in a neighborhood of  $\rho_0$ , we recover that  $\text{Op}_h(a)\varphi_{\rho_0} = O(h^\infty)$ . This is something well known since  $\text{WF}_h(\varphi_{\rho_0}) = \{\rho_0\}$ .

**7-3.3.4. Action of Fourier integral operators on coherent states.**

In [CR12] (Chapter 4), the authors study the quantum evolution of coherent states by the propagator of a Schrödinger equation with a time-dependent Hamiltonian. We refer the reader to their work, and in particular to Theorem 21 in this book for this very general version of the evolution of coherent states. Here, we will simply study the action of the particular type of Fourier integral operator of the form given in equation (4-2.3) on states of the form  $T(\rho_0)\mathcal{M}(\kappa)\varphi_0$ . In other words, we want to study the action of a Fourier Integral Operator on these squeezed and translated states. More generally, we will consider also squeezed excited states of the form  $T(\rho_0)\mathcal{M}(\kappa)\Lambda_h(P(x)\Psi_0(x))$ . We will give an asymptotic expansion of these evolved states with a controlled remainder. The dependence of this remainder on  $\kappa$  will be crucial to use recursively the expansion.

Let's describe the framework in which we want to work : we suppose that  $\Omega_x, \Omega_\eta$  are open intervals of  $\mathbb{R}$ ,  $\psi \in C^\infty(\Omega_x \times \Omega_\eta)$  is a phase function that generates the twisted graph of some symplectic map  $F$  in some open set  $\Omega_0 \subset \mathbb{R}^4$ , that is

$$\text{Gr}'(F) \cap \Omega_0 = \{ (x, \partial_x \psi(x, \eta), \partial_\eta \psi(x, \eta), -\eta), x \in \Omega_x, \eta \in \Omega_\eta \}$$

We suppose that  $a \in S_{0+}^{\text{comp}}(\Omega_x \times \Omega_\eta)$  and we consider the Fourier integral operator :

$$Su(x) = \frac{1}{(2\pi h)} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi(x, \eta) - y \cdot \eta)} a(x, \eta) u(y) dy d\eta$$

We do not necessarily assume that  $S$  is microlocally unitary, but if it were the case,  $a$  would satisfy  $|a(x, \eta)|^2 = |\partial_{x\eta}^2 \psi(x, \eta)| + O(h^{1-\varepsilon})$  for any  $\varepsilon > 0$ . More generally, the amplitude  $\alpha$  of  $S$  as a Fourier integral operator is given, modulo  $O(h^{1-})S_{0+}$ , by

$$\alpha_S(y, \eta) = \frac{a(x, \eta)}{|\partial_{x\eta}^2 \psi(x, \eta)|^{1/2}}, F(y, \eta) = (x, \xi)$$

**Proposition 7-3.6.** Assume that  $S$  satisfies the above assumptions. Let  $\kappa \in \mathcal{M}_2(\mathbb{R})$  be a symplectic linear map and  $\rho_0 \in T^*\mathbb{R}$ . Let's note  $\rho_1 = F(\rho_0)$ . Let  $P \in \mathbb{C}[X]$ . Then, there exists a family of polynomials  $Q_k(P)_{k \in \mathbb{N}}$  such that

- $Q_0(P) = \alpha_S(\rho_0)P$  (up to multiplication by an element of  $\mathbb{U}$ ) ;
- $Q_k(P)$  is a polynomial of degree  $\deg P + 3k$  and the map  $P \mapsto Q_k(P)$  is linear, with coefficients depending on  $\kappa$  and the derivatives of  $\psi$  and  $a$  at  $(x_1, \xi_0)$  up to the  $3k$ -th order, and we have

$$N_\infty(Q_k(P)) \leq C_{3k}(\psi) \|a\|_{C^k} \|\kappa\|^{3k} N_\infty(P)$$

Moreover, if  $(x_1, \xi_0) \notin \text{supp } a$ , then  $Q_k = 0$ .

- for every  $N \in \mathbb{N}$ ,

$$S\left(T(\rho_0)\mathcal{M}(\kappa)\Lambda_h[P\Psi_0]\right) = T(\rho_1)\mathcal{M}(d_{\rho_0}F \circ \kappa)\Lambda_h \left[ \sum_{k=0}^{N-1} h^{k/2} Q_k(P)\Psi_0 \right] + R_N \quad (7-3.8)$$

with

$$\|R_N\|_{L^2} \leq h^{N/2} C_{3N+M}(\psi) \|a\|_{C^{N+M}} \|\kappa\|^{3N} K_{N, \deg P} N_\infty(P)$$

Here,

- $C_k(\psi)$  depends on the  $C^k$  norm of  $\psi$  ;
- $M$  is a universal constant ;
- $N_\infty(P)$  is the sup norm on the coefficients of  $P$  ;
- $(K_{N,d})_{(N,d) \in \mathbb{N}^2}$  is a family of universal constants ;
- For every  $\varepsilon > 0$  and  $k \in \mathbb{N}$ , there exists  $C_{\varepsilon,k}$  such that  $\|a\|_{C^k} \leq C_{\varepsilon,k} h^{-\varepsilon}$ .

**Remark.**

This proposition shows that a Fourier Integral operator transforms a wave packet centered at  $\rho_0$  into a wave packet centered at  $F(\rho_0)$ . However, this transformation squeezes the wave packet according to the linearization of  $F$  at  $\rho_0$ : this is the effect of  $\mathcal{M}(d_{\rho_0}F)$ . The control of the error is important if we want to iterate this formula and apply it to squeezed coherent states  $\mathcal{M}(\kappa_h)\varphi_0$ , with a symplectic linear map  $\kappa_h$  potentially depending of  $h$ . As soon as

$$\|\kappa_h\|^3 \ll h^{-1/2},$$

the remainder stays smaller than the terms in the expansion. In particular, suppose that  $\kappa_h = \kappa_{n(h)} \dots \kappa_0$  with  $\|\kappa_i\| \leq e^\lambda$  and  $n(h) \sim \nu \log(1/h)$ . Then, the approximation is valid as soon as

$$\nu \leq \frac{1-\varepsilon}{6\lambda}$$

*Proof.* The following computations are performed modulo multiplication by an element of  $\mathbb{U}$ .

Let's note  $\rho_0 = (x_0, \xi_0)$  and  $\rho_1 = (x_1, \xi_1)$ . Recall that, by definition of  $\psi$ ,

$$\xi_1 = \partial_x \psi(x_1, \xi_0) \quad ; \quad x_0 = \partial_\eta \psi(x_1, \xi_0) \quad (7-3.9)$$

We have, for  $u \in L^2(\mathbb{R})$ ,

$$\begin{aligned} (\Lambda_h^* T(\rho_1)^* S T(\rho_0) \Lambda_h u)(x) &= h^{1/4} e^{-\frac{ix\xi_1}{h^{1/2}}} (S T(\rho_0) \Lambda_h u)(h^{1/2}x + x_1) \\ &= e^{-\frac{ix\xi_1}{h^{1/2}}} \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi(h^{1/2}x+x_1, \eta) - y\eta)} a(h^{1/2}x + x_1, \eta) e^{\frac{iy\xi_0}{h}} u(h^{-1/2}y - x_0) dy d\eta \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\tilde{\psi}_h(x, \eta, y)} a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) u(y) dy d\eta \end{aligned}$$

after a change of variable, with

$$\tilde{\psi}_h(x, \eta, y) = \frac{1}{h} \psi(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) - y\eta - h^{-1/2}(x\xi_1 + x_0\eta) \quad (7-3.10)$$

Let us write the Taylor expansion of  $\psi(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)$  at order  $N+1 \in \mathbb{N}$  :

$$\begin{aligned} \psi(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) &= \psi(x_1, \xi_0) + h^{1/2}(x\partial_x \psi(x_1, \xi_0) + \eta\partial_\eta \psi(x_1, \xi_0)) \\ &\quad + \frac{h}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) + \sum_{k=3}^{N+1} h^{k/2} \psi_k(x, \eta) + h^{(N+2)/2} r_{N+2}^\psi(x, \eta; h) \end{aligned} \quad (7-3.11)$$

where  $\psi_k$  is  $k$ -multilinear in  $(x, \eta)$  with coefficients depending on the derivatives of  $\psi$  of order  $k$  at  $(x_1, \xi_0)$  and for  $\alpha \in \mathbb{N}^2$ ,

$$h^{(N+2)/2} r_{N+2}^\psi(x, \eta; h) = \frac{1}{(N+1)!} \int_0^1 (1-t)^{N+1} \frac{d^{N+2}}{dt^{N+2}} (\psi(x_1 + th^{1/2}x, \xi_0 + th^{1/2}\eta)) dt$$

In particular, we have the estimates

$$|\partial^\alpha r_{N+2}^\psi(x, \eta; h)| \leq C_N \sup_{N+2 \leq |\beta| \leq N+2+|\alpha|} h^{(|\beta|-N-2)/2} \|\partial^\beta \psi\|_\infty \langle (x, \eta) \rangle^{N+2} \quad (7-3.12)$$

Recalling (7-3.9), we can write :

$$\begin{aligned} & (\Lambda_h^* T(\rho_1)^* ST(\rho_0) \Lambda_h u)(x) = \\ & \frac{1}{2\pi} \int_{T^*\mathbb{R}} e^{i(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) - y\eta)} e^{ih^{1/2}r_3^\psi(x, \eta; h)} a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) u(y) dy d\eta \end{aligned} \quad (7-3.13)$$

Then, we write the Taylor expansion of  $e^{ih^{1/2}r_3^\psi(x, \eta; h)}$  at order  $N$ :

$$e^{ih^{1/2}r_3^\psi(x, \eta; h)} = \sum_{k=0}^{N-1} \frac{i^k h^{k/2}}{k!} r_3^\psi(x, \eta; h)^k + \underbrace{\frac{i^N h^{N/2}}{(N-1)!} (r_3^\psi(x, \eta; h))^N \int_0^1 e^{ih^{1/2}sr_3^\psi(x, \eta; h)} (1-s)^{N-1} ds}_{\tilde{r}_N} \quad (7-3.14)$$

Using (7-3.11), we write  $r_3^\psi = \sum_{j=0}^{N-k-1} h^{j/2} \psi_{3+j} + h^{(N-k)/2} r_{3+N-k}^\psi$  and we can expand

$$\left(r_3^\psi\right)^k = \sum_{\alpha_1 + \dots + \alpha_k < N-k} h^{(\alpha_1 + \dots + \alpha_k)/2} \psi_{3+\alpha_1} \dots \psi_{3+\alpha_k} + h^{(N-k)/2} \text{Remainder}$$

where the remainder is a linear combination, with universal coefficients, of terms of the form

$$r_{3+\alpha_1}^\psi \dots r_{3+\alpha_j}^\psi \psi_{3+\alpha_{j+1}} \dots \psi_{3+\alpha_k} \quad ; \quad 0 \leq j \leq k, \alpha_1 + \dots + \alpha_k = N-k \quad (7-3.15)$$

Gathering all the terms of order  $h^{k/2}$  for  $k \leq N-1$ , together and gathering all the terms of order  $h^{N/2}$  in a single remainder term, we have

$$e^{ih^{1/2}r_3^\psi(x, \eta; h)} = \sum_{k=0}^{N-1} h^{k/2} \tilde{P}_k(x, \eta; \psi) + h^{N/2} r_{N,1} + \tilde{r}_N$$

where

- $\tilde{P}_k(\cdot; \psi)$  is a polynomial of order  $3k$  in  $(x, \eta)$  with coefficients of the form  $q((\partial^\alpha \psi(x_1, \xi_0))_{|\alpha| \leq 3+k})$ , where  $q$  is a universal polynomial of degree  $k$ ;
- $r_{N,1}$  is a linear combination of terms of the form (7-3.15) with  $0 \leq k \leq N-1, 0 \leq j \leq k$  and  $\alpha_1, \dots, \alpha_k$ , with  $\alpha_1 + \dots + \alpha_k = N-k$ ;
- $\tilde{r}_N$  is defined in (7-3.14).

Similarly, we can Taylor expand  $a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)$  to find that

$$\begin{aligned} & e^{ih^{1/2}r_3^\psi(x, \eta; h)} a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0) = \sum_{k=0}^{N-1} h^{k/2} P_k(x, \eta; \psi, a) \\ & + h^{N/2} \underbrace{\sum_{k=0}^{N-1} \tilde{P}_k(x, \eta; \psi) r_{N-k}^\alpha(x, \eta; h) + h^{N/2} r_{N,1} \times a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)}_{\text{first remainder term}} + \underbrace{\tilde{r}_N \times a(h^{1/2}x + x_1, h^{1/2}\eta + \xi_0)}_{\text{second remainder term}} \end{aligned} \quad (7-3.16)$$

where  $P_k(\cdot; \psi, a)$  is a polynomial of degree  $3k$  in  $(x, \eta)$ , given by

$$P_k(x, \eta; \psi, a) = \sum_{k_1 + k_2 = k} \tilde{P}_{k_1}(x, \eta; \psi) \times \left( \frac{1}{k_2!} d^{k_2} a(x_1, \xi_0) ((x, \eta)^{\otimes k_2}) \right)$$

and for  $p \in \mathbb{N}$ ,

$$r_p^\alpha(x, \eta; h) = \frac{h^{-p/2}}{p!} \int_0^1 (1-t)^{p-1} \frac{d^p}{dt^p} a(x_1 + th^{1/2}x, \xi_0 + th^{1/2}\eta) dt$$

Plugging (7-3.16) in (7-3.13) with  $u = \mathcal{M}_1(\kappa)(P\Psi_0)$ , we find an expansion in powers of  $h^{1/2}$  for  $\Lambda_h^* T(\rho_1)^* ST(\rho_0) \Lambda_h u$ .

**Identification of the first term.** The first term of the expansion is

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) - y\eta)} a(x_1, \xi_0) u(y) dy d\eta$$

Differentiating the relation

$$F(\partial_\eta \psi(x, \eta), \eta) = (x, \partial_x \psi(x, \eta))$$

it not hard to see that

$$d_{\rho_0} F = \frac{1}{\partial_{x\eta}^2 \psi(x_1, \xi_0)} \begin{pmatrix} 1 & -\partial_{\eta\eta}^2 \psi(x_1, \xi_0) \\ \partial_{xx}^2 \psi(x_1, \xi_0) & (\partial_{x\eta}^2 \psi(x_1, \xi_0))^2 - \partial_{\eta\eta}^2 \psi(x_1, \xi_0) \partial_{xx}^2 \psi(x_1, \xi_0) \end{pmatrix}$$

As a consequence, comparing with (7-3.3), we observe that

$$v \mapsto \frac{1}{2\pi} |\partial_{x\eta} \psi(x_1, \xi_0)|^{1/2} \int_{T^*\mathbb{R}} e^{i(\frac{1}{2} D^2 \psi(x_1, \xi_0)(x, \eta) - y\eta)} v(y) dy d\eta$$

is a metaplectic operator associated with  $d_{\rho_0} F$ , that we note  $\mathcal{M}_1(d_{\rho_0} F)$ , and hence, we see that

$$ST(\rho_0) \mathcal{M}(\kappa) \Lambda_h [P\Psi_0] = T(\rho_1) \mathcal{M}(d_{\rho_0} F \circ \kappa) \Lambda_h \left[ \frac{a(x_1, \xi_0)}{|\partial_{x\eta} \psi(x_1, \xi_0)|^{1/2}} P\Psi_0 \right] + (\text{smaller terms})$$

This gives the required form for  $Q_0(P)$ .

**Identification of higher order terms.** For the term of order  $k$  in the expansion of (7-3.13), based on (7-3.16), we have to understand

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) - y\eta)} P_k(x, \eta; \psi, a) u(y) dy d\eta$$

Hence, we focus on terms of the form

$$S_{l,m}(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) - y\eta)} x^l \eta^m u(y) dy d\eta$$

with  $l + m \leq 3k$ . The term  $x^l$  can be put in front of the integral. Concerning, the  $\eta$  term, repeated integrations by part (or equivalently, using the usual properties of the Fourier transform), we find that

$$S_{l,m}(u) = x^l \mathcal{M}_1(d_{\rho_0} F)((i\partial_y)^m u)$$

Now, combining this with the standard commutations properties of metaplectic operators we write

$$\begin{aligned} S_{l,m}(\mathcal{M}_1(\kappa)[P\Psi_0]) &= \text{Op}_1(x^l) \mathcal{M}_1(d_{\rho_0} F) \text{Op}_1((-\xi)^m) \mathcal{M}_1(\kappa)[P\Psi_0] \\ &= \mathcal{M}_1(d_{\rho_0} F \circ \kappa) \text{Op}_1((x \circ (dF(\rho_0) \circ \kappa)^l) \text{Op}_1((-\xi \circ \kappa)^m) [P\Psi_0] \end{aligned}$$

Finally, the action of  $\text{Op}_1((x \circ (d_{\rho_0} F \circ \kappa)^l) \text{Op}_1((-\xi \circ \kappa)^m)$  transforms  $P\Psi_0$  into another state of the form  $Q\Psi_0$  where  $Q$  is of degree  $\deg P + l + m$ , where the coefficients of  $Q$  depend linearly on those of  $P$ , with coefficients in the linear combination depending on  $\kappa$  and on  $d_{\rho_0} F$ . By developing the powers  $(x \circ (d_{\rho_0} F \circ \kappa)^l$  and  $(-\xi \circ \kappa)^m$ , we see that the coefficients of  $Q$  are bounded by  $C_{l,m} \|d_{\rho_0} F\|^l \|\kappa\|^{l+m}$  for some constant  $C_{l,m}$ .

As a consequence, we can write the entire term of order  $k$  in the form :  $T(\rho_1) \mathcal{M}(d_{\rho_0} F \circ \kappa) \Lambda_h(Q_k(P)\Psi_0)$  where  $Q_k(P)$  is a polynomial of order  $\deg P + 3k$ , the map  $P \mapsto Q_k(P)$  is linear and its coefficients depend on  $\kappa$ , the derivatives of  $\psi$  and  $a$  at  $(x_1, \xi_0)$  up to the  $3k$ -th order. This gives the required polynomial. By putting the terms  $\|d_{\rho_0} F\|^l$  into  $C_{3k}(\psi)$  and using the special form of  $P_k$ , we obtain the required estimate

$$N_\infty(Q_k(P)) \leq C_{3k}(\psi) \|a\|_{C^k} \|\kappa\|^{3k} N_\infty(P).$$

**Control of the remainders.** The last step of the proof consists in proving that the remainder term has the required bound. As already written with the underbrace in (7-3.16), this remainder can be decomposed in two terms: they have different properties. Let us start with the first term, and call it  $\tilde{r}_{N,1}$ .

In the products of the form given by (7-3.15), gathering the factors  $r_{3+\alpha}^\psi$  into a single term and the polynomials  $\psi_k$  into a single polynomial, we see that the term  $r_{N,1}$ , appearing in  $\tilde{r}_{N,1}$ , is a sum of terms of the form  $Q_j^\psi(x, \eta)R_j^\psi(x, \eta; h)$ , for  $0 \leq j \leq k$ , where  $Q_j^\psi$  is a polynomial of degree  $3j$  and  $R_j^\psi(x, \eta; h)$  satisfies for  $\alpha \in \mathbb{N}^2$ ,

$$|\partial^\alpha R_j^\psi(x, \eta; h)| \leq C_{3N-3j+|\alpha|}(\psi) \langle (x, \eta) \rangle^{3N-3j}$$

where  $C_{3N-3j+|\alpha|}(\psi)$  depends on the derivatives of  $\psi$  up to the order  $3N - 3j + |\alpha|$ .<sup>4</sup> Using the same kind of estimates for  $r_{N-k}^a(x, \eta; a; h)$ , we see that  $\tilde{r}_{N,1}$  satisfies :

$$\forall \alpha \in \mathbb{N}^2, (x, \eta) \in \mathbb{R}^2, |\partial^\alpha \tilde{r}_{N,1}(x, \eta)| \leq C_{3N+|\alpha|}(\psi) \|a\|_{C^{N+|\alpha|}} \langle (x, \eta) \rangle^{3N} \quad (7-3.17)$$

We are now interested in controlling

$$\tilde{R}_{N,1}u(x) := \frac{1}{2\pi} \int_{T^*\mathbb{R}} e^{i(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) - y\eta)} \tilde{r}_{N,1}(x, \eta) u(y) dy d\eta$$

We will use the following lemma, proved in the appendix 7-7.1. :

**Lemma 7-3.4.** Let  $\tilde{b}$  be a symbol in  $S(\langle \rho \rangle^N)$ . Then, there exists a symbol  $b \in S(\langle \rho \rangle^N)$  such that for all  $0 < h \leq 1$ ,

$$\frac{1}{2\pi h} \int_{T^*\mathbb{R}} e^{i\frac{1}{h}(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) - y\eta)} \tilde{b}(x, \eta) u(y) dy d\eta = \mathcal{M}(d_{\rho_0}F) \text{Op}_h(b)u(x)$$

Moreover, there exists a universal integer  $M' \in \mathbb{N}$  such that  $b$  satisfies : for all  $\alpha \in \mathbb{N}^2$ ,

$$\langle \rho \rangle^N |\partial^\alpha b(\rho)| \leq C_\alpha \sup_{|\beta| \leq |\alpha| + M'} \sup_{\rho \in T^*\mathbb{R}} \left( |\partial^\beta \tilde{b}(\rho)| \langle \rho \rangle^N \right)$$

where  $C_\alpha$  depends on  $d_{\rho_0}F$ .

By applying lemma 7-3.4 (in the case  $h = 1$  in the lemma), we can find a symbol  $r_{N,h}$  such that

$$\tilde{R}_{N,1} = \mathcal{M}_1(d_{\rho_0}F) \text{Op}_1(r_{N,h})$$

To conclude the treatment of the first part of the remainder, we compute :

$$\begin{aligned} \|\tilde{R}_{N,1} \mathcal{M}_1(\kappa)[P\Psi_0]\|_{L^2} &= \|\mathcal{M}_1(d_{\rho_0}F) \text{Op}_1(r_{N,h}) \mathcal{M}_1(\kappa)[P\Psi_0]\|_{L^2} \\ &\leq \|\text{Op}_1(r_{N,h})\|_{H_1(\langle \rho \rangle^{3N}) \rightarrow L^2} \times \|\mathcal{M}_1(\kappa)[P\Psi_0]\|_{H_1(\langle \rho \rangle^{3N})} \\ &\leq C_M(r_{N,h}) \|\kappa\|^{3N} K_{N,degP} N_\infty(P) \end{aligned}$$

by using Corollary 7-3.1, where  $C_M(r_{N,h})$  depends on the first  $M$  semi-norms of  $r_{N,h}$  in  $S(\langle \rho \rangle^{3N})$ , which, in turn depends on the first  $M + M'$  semi-norms of  $\tilde{r}_{N,h}$  in  $S(\langle \rho \rangle^{3N})$  according to Lemma 7-3.4. By (7-3.17), this can be controlled by some constant  $C_{3N+M+M'}(\psi) \|a\|_{C^{N+M+M'}}$ .

Let's turn to the second remainder in (7-3.16). We want to control

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) - y\eta)} \tilde{r}_N(x, \eta) u(y) dy d\eta$$

<sup>4</sup>These estimates comes from (7-3.12) and in fact, we can take

$$C_{3N-3j+|\alpha|}(\psi) = \sup_{3N-3j \leq |\beta| \leq 3N-3j+|\alpha|} h^{(|\beta|-3N+3j)/2} \|\partial^\beta \psi\|_\infty$$

Recalling the precise description of  $\tilde{r}_N$  in (7-3.14), we set, for  $s \in [0, 1]$  :

$$\tilde{R}_s u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\frac{1}{2}(D^2\psi(x_1, \xi_0)(x, \eta), (x, \eta)) + ish^{1/2}r_3^\psi(x, \eta; h) - y\eta)} r_3^\psi(x, \eta; h)^N a(x_1 + h^{1/2}x, \xi_0 + h^{1/2}\eta) u(y) dy d\eta$$

and we want to estimate  $\|\tilde{R}_s \mathcal{M}_1(\kappa)[P\Psi_0]\|_{L^2}$  uniformly in  $s \in [0, 1]$ . The symbol

$$b_N(x, \eta) := r_3^\psi(x, \eta; h)^N a(x_1 + h^{1/2}x, \xi_0 + h^{1/2}\eta)$$

lies in the symbol class  $S(\langle \rho \rangle^{3N})$ , with a control on its semi-norms due to (7-3.12). Let's admit the following lemma, whose proof is also put in the appendix 7-7.1..

**Lemma 7-3.5.** For every  $s \in [0, 1]$ , there exists  $B_s(\cdot) \in S(\langle \rho \rangle^{6N})$  such that :

- $\tilde{R}_s^* \tilde{R}_s = \text{Op}_1(B_s)$  ;
- There exists a universal  $M \in \mathbb{N}$  such that for all  $\alpha \in \mathbb{N}^2$ , for all  $s \in [0, 1]$ , with some universal constants  $C_\alpha$ ,

$$\sup_{\rho} |\partial^\alpha B_s(\rho)| \leq C_\alpha \left( \sup_{\rho, |\beta| \leq |\alpha| + M} d^\beta b_N(\rho) \langle \rho \rangle^{-3N} \right)^2 \langle \rho \rangle^{6N}.$$

This lemma allows us to control

$$\begin{aligned} \|\tilde{R}_s\|_{H_1(\langle \rho \rangle^{3N}) \rightarrow L^2}^2 &\leq \|\tilde{R}_s^* \tilde{R}_s\|_{H_1(\langle \rho \rangle^{3N}) \rightarrow H_1(\langle \rho \rangle^{-3N})} \\ &\leq \|\text{Op}_1(B_s)\|_{H_1(\langle \rho \rangle^{3N}) \rightarrow H_1(\langle \rho \rangle^{-3N})} \\ &\leq C_N \sup_{|\alpha| \leq M} \sup_{\rho} |\partial^\alpha B_s(\rho)| \langle \rho \rangle^{-6N} \\ &\leq C_N \left( \sup_{|\beta| \leq 2M} \sup_{\rho} d^\beta b_N(\rho) \langle \rho \rangle^{-3N} \right)^2 \\ &\leq (C_{3N+M'}(\psi) \|a\|_{C^{N+M'}})^2 \end{aligned}$$

We finally conclude as before for  $\tilde{R}_{N,1}$  by using Corollary 7-3.1. This concludes the proof of Proposition 7-3.6.  $\square$

## 7-4 Dynamical preliminaries.

### 7-4.1. Hyperbolic dynamics.

We recall that the hyperbolicity of  $F$  on the trapped set  $\mathcal{T}$  allows to define unstable/stable Jacobian (see Chapter 3). As already mentioned, we can fix an adapted Riemannian metric on  $U$  such that the following stronger version of the hyperbolic estimates are satisfied for some  $\lambda_0 > 0$  : for every  $\rho \in \mathcal{T}$ ,  $n \in \mathbb{N}$ ,

$$v \in E_u(\rho) \implies \|d_\rho F^{-n}(v)\| \leq e^{-\lambda_0 n} \|v\| \quad (7-4.1)$$

$$v \in E_s(\rho) \implies \|d_\rho F^n(v)\| \leq e^{-\lambda_0 n} \|v\| \quad (7-4.2)$$

If  $\rho \in \mathcal{T}$ ,  $n \in \mathbb{Z}$ , we use this Riemannian metric to define the unstable Jacobian  $J_n^u(\rho)$  and stable Jacobian  $J_n^s(\rho)$  at  $\rho$  by :

$$v \in E_u(\rho) \implies \|d_\rho F^n(v)\| = J_n^u(\rho) \|v\| \quad (7-4.3)$$

$$v \in E_s(\rho) \implies \|d_\rho F^n(v)\| = J_n^s(\rho) \|v\| \quad (7-4.4)$$

These Jacobians quantify the local hyperbolicity of the map. Since  $F$  is volume preserving,  $J_n^u(\rho) J_n^s(\rho) \sim 1$ .

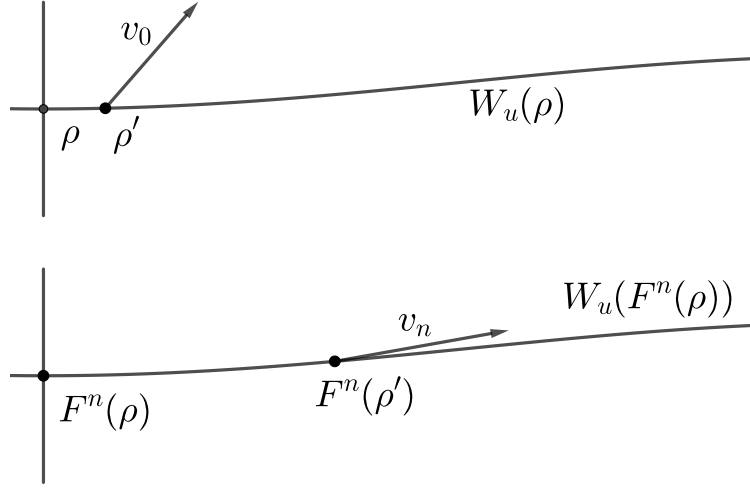


Figure 7.2: The linearized dynamics makes the vector closer and closer to the tangent space of the unstable manifold. See Lemma 7-4.1. The vertical direction corresponds to the stable direction, in which the dynamics contracts.

From the compactness of  $\mathcal{T}$ , the minimal and maximal Lyapounov exponents of are well-defined

$$\lambda_{max} = \sup_{\rho \in \mathcal{T}} \limsup_n \frac{1}{n} \log J_n^u(\rho) < +\infty$$

$$\lambda_{min} = \inf_{\rho \in \mathcal{T}} \liminf_n \frac{1}{n} \log J_n^u(\rho) > 0$$

Standard facts on hyperbolic dynamical systems have been recorded and proved in Chapter 3 and the existence of unstable/stable manifolds have been recorded in Lemma 3-2.1. More quantitative results have been proved in Lemma 3-2.2, Lemma 3-2.3 and Corollary 3-2.1. We will make use of these results in this chapter.

In this chapter, we will also use the following lemma concerning the linearized dynamics. If  $\rho \in \mathcal{T}$  and  $\rho' \in W_u(\rho)$ , the tangent space  $T_{\rho'}W_u(\rho)$  will naturally be denoted  $E_u(\rho')$  and if  $v \in T_{\rho'}U$ , we note  $d(v, E_u(\rho'))$  the distance between  $v$  and  $E_u(\rho')$  computed using the Riemannian metric on  $T_{\rho'}U$ .

**Lemma 7-4.1.** There exist  $\varepsilon_1 > 0$ ,  $\gamma \in (0, 1)$  and  $C > 0$  such that the following holds. Assume that  $\rho \in \mathcal{T}$ ,  $\rho' \in W_u(\rho)$ ,  $v_0 \in T_{\rho'}U$  and  $n \in \mathbb{N}$  satisfy :  $\forall i \in \{0, \dots, n\}$ ,  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_1$ ,  $\|v_0\| = 1$  and  $d(v_0, E_u(\rho')) \leq \gamma$ . Let's note

$$v_n = \frac{d_{\rho'} F^n(v_0)}{\|d_{\rho'} F^n(v_0)\|} \in T_{F^n(\rho')}U$$

Then (see Figure 7.2),

$$d(v_n, E_u(F^n(\rho'))) \leq C J_n^u(\rho)^{-2} d(v_0, E_u(\rho'))$$

**Remark.**

This is a form of inclination lemma : the tangent vectors are attracted toward the unstable direction upon the evolution. We provide a quantitative and local version of this statement. The assumption  $d(v_0, E_u(\rho')) \leq \gamma$  is a transversality assumption : it says that  $v_0$  has to be sufficiently transverse to the stable direction.

*Proof.* First note that due to the assumption on  $\rho$  and  $\rho'$  and Lemma 3-2.2,

$$d(F^i(\rho), F^i(\rho')) \leq C \theta^{n-i} d(F^n(\rho), F^n(\rho')) \leq C \theta^{n-i} \varepsilon_1$$

for some  $0 < \theta < 1$  and for  $0 \leq i \leq n$ . We use coordinates charts  $\kappa_i$  centered at  $F^i(\rho)$  (for  $0 \leq i \leq n$ ), given by Lemma 3-4.1. Let's note  $(u^i, s^i)$  the coordinates in  $\kappa_i$ . Since  $\kappa_i(W_u(F^i(\rho))) = \{(u^i, 0)\}$ , the map  $F$  between the charts  $\kappa_{i-1}$  and  $\kappa_i$  is given by :

$$\kappa_i \circ F \circ \kappa_{i-1}^{-1}(u^{i-1}, s^{i-1}) = (\nu_i u^{i-1} + \alpha_i(u^{i-1}, s^{i-1}), \mu_i s^{i-1} + \beta_i(u^{i-1}, s^{i-1}))$$

with  $\beta_i(u^{i-1}, 0) = 0$ ,  $d\alpha_i(0, 0) = 0$  and  $d\beta_i(0, 0) = 0$ . Remark that  $\nu_1 \dots \nu_i \sim J_i^u(\rho) \sim (\mu_1 \dots \mu_i)^{-1}$  for  $1 \leq i \leq n$ .

Let us note  $w_0 = d_{\rho'} \kappa_0(v_0)$  and  $\tilde{w}_n = d_{F^n(\rho')} \kappa_n(v_n)$ . Hence, we want to show that  $d(\tilde{w}_n, \mathbb{R}e_u) \leq C J_n^u(\rho)^{-1} d(w_0, \mathbb{R}e_u)$  where  $\mathbb{R}e_u = \{(u, 0)\}$ . Here, to compute the distance, both between points and tangent vectors, we can simply use the usual euclidean distance in  $\mathbb{R}^2$ . Let us also introduce  $w_i = d_{\rho'}(\kappa_i \circ F^i)(v_0)$  and write  $w_i = (w_i^u, w_i^s)$ . With these notations, we have  $w_n = \|d_{\rho'} F^n(v_0)\| \tilde{w}_n$  and

$$d(\tilde{w}_n, \mathbb{R}e_u) = \frac{w_n^s}{\|d_{\rho'} F^n(v_0)\|}$$

Since  $\|d_{\rho'} F^n(v_0)\| \sim \|w_n\|$ , we are reduced to prove that

$$\frac{|w_n^s|}{\|w_n\|} \leq C J_n^u(\rho)^{-2} |w_0^s| \quad (7-4.5)$$

If  $\gamma$  is small enough, we can deduce from the transversality assumption on  $v_0$  that  $|w_0^u| \geq \frac{1}{2}|w_0^s|$ . In particular,  $\|w_0\|^2 \geq \frac{4}{3}|w_0^s|^2$ . Let us note  $(u^i, 0)$  the coordinates of  $F^i(\rho')$  in the charts  $\kappa_i$  and recall that  $|u^i| \leq C\theta^{n-i}\varepsilon_1$ . We have the relations

$$\begin{aligned} w_i^u &= \nu_i w_{i-1}^u + d\alpha_i(u^{i-1}, 0) \cdot w_{i-1} \\ w_i^s &= \mu_i w_{i-1}^s + d\beta_i(u^{i-1}, 0) \cdot w_{i-1} \end{aligned}$$

Since  $\beta_i(u, 0) = 0$ ,  $d\beta_i(u^{i-1}, 0) \cdot w_{i-1} = \partial_{s^i} \beta_i(u^{i-1}, 0) w_{i-1}^s$ . Moreover,  $d\beta_i(0, 0) = 0$ , and hence,  $|\partial_{s^i} \beta_i(u^{i-1}, 0) w_{i-1}^s| \leq C|u^{i-1}| |w_{i-1}^s| \leq C\theta^{n+1-i}\varepsilon_1 |w_{i-1}^s|$ . this gives,

$$|w_i^s| \leq (\mu_i + C\theta^{n+1-i}\varepsilon_1) \dots (\mu_1 + C\theta^n) |w_0^s|$$

$$|w_i^s| \leq \mu_1 \dots \mu_i \prod_{k=1}^i \left(1 + \frac{C\varepsilon_1 \theta^{n+1-k}}{\mu_k}\right) |w_0^s|$$

Since  $\mu_k \geq c$  for some  $c > 0$  and for all  $1 \leq k \leq n$ , we can estimate

$$\prod_{k=1}^i \left(1 + \frac{C\varepsilon_1 \theta^{n+1-k}}{\mu_k}\right) \leq \prod_{k=0}^{i-1} (1 + C\varepsilon_1 c^{-1} \theta^k) \leq \prod_{i=0}^{\infty} (1 + C\varepsilon_1 c^{-1} \theta^i) < +\infty$$

As a consequence,  $|w_i^s| \leq C J_i^u(\rho)^{-1} |w_0^s|$ . We now turn to a lower bound for  $\|w_n\|$ . From  $w_i^u = \nu_i w_{i-1}^u + O(|u^{i-1}| |w_{i-1}|)$ , we find that

$$|w_i^u| \geq \nu_i |w_{i-1}^u| - C|u^{i-1}| |w_{i-1}| \geq \nu_i |w_{i-1}^u| - C\theta^{n+1-i} |w_{i-1}|$$

We observe that  $\|w_{i-1}\| \leq |w_{i-1}^u| + |w_{i-1}^s| \leq |w_{i-1}^u| + C J_{i-1}^u(\rho)^{-1} |w_0^s|$ , which gives that

$$|w_i^u| \geq |w_{i-1}^u| (\nu_i - C\theta^{n+1-i}) - C\theta^{n+1-i} J_{i-1}^u(\rho)^{-1} |w_0^s|$$

Recall that for  $\theta_1 = e^{-\lambda_1}$ , we have  $J_i^u(\rho)^{-1} \leq \theta_1^i$ , so that for some  $\theta_2 \in ]\max(\theta, \theta_1), 1[$ ,  $\theta^{n+1-i} J_{i-1}^u(\rho)^{-1} \leq C\theta_2^n$ . Iterating this formula, we find that

$$|w_n^u| \geq (\nu_1 - C\theta^n) \dots (\nu_n - C\theta) |w_0^u| - \sum_{i=1}^{n-1} (\nu_n - C\theta) \dots (\nu_i - C\theta^{n+1-i}) \theta_2^n |w_0^s|$$

By similar arguments as in the case of  $|w_n^s|$ , we can show that  $(\nu_n - C\theta) \dots (\nu_1 - C\theta^n) \geq C^{-1} J_n^u(\rho)$  and  $|(\nu_n - C\theta) \dots (\nu_1 - C\theta^n)| \leq \nu_1 \dots \nu_n \leq C J_n^u(\rho)$ . As a consequence, and using the fact that  $|w_0^u| \geq 1/4 \|w_0\|$  (by the transversality assumption), we find that

$$|w_n^u| \geq C^{-1} (1 - \theta_2^n) J_n^u(\rho) \|w_0\| \geq C^{-1} J_n^u(\rho) \|w_0\|.$$

We conclude that  $\|w_n\| \geq |w_n^u| \geq C^{-1} J_n^u(\rho) \|w_0\|$ , which gives (7-4.5).  $\square$



### 7-4.2. Escape function.

In this subsection, we record the construction of escape functions of [NSZ14], specialized to our open map  $F : \widetilde{D} \subset U \rightarrow \widetilde{A} \subset U$ . We do not give the proof, since it is entirely contained in [NSZ14] (Lemmata 4.1 - 4.4).

**Lemma 7-4.2.** Assume that  $\mathcal{V}_2$  is a small neighborhood of  $\mathcal{T}$  in which  $F$  is well defined. Then, there exist  $C_0 > 0$  and a neighborhood  $\mathcal{V}_1 \subset \mathcal{V}_2$  of  $\mathcal{T}$  such that the following holds : For every  $\epsilon > 0$ , there exist functions  $\hat{\varphi}_\pm = \hat{\varphi}_{\pm, \epsilon} \in C^\infty(\mathcal{V}_1 \cup F(\mathcal{V}_1), [\epsilon, +\infty[)$  such that

$$\begin{aligned} \hat{\varphi}_\pm(\rho) &\sim d(\rho, \mathcal{T}_\pm)^2 + \epsilon; \\ \pm(\hat{\varphi}_\pm(\rho) - \hat{\varphi}_\pm(F(\rho)) + C_0\epsilon) &\sim \hat{\varphi}_\pm(\rho); \\ \hat{\varphi}_+(\rho) + \hat{\varphi}_-(\rho) &\sim \epsilon; \\ \partial^\alpha \hat{\varphi}_\pm(\rho) &= O\left(\hat{\varphi}_\pm(\rho)^{1-|\alpha|/2}\right). \end{aligned}$$

The constants in the  $\sim$  and  $O$  are independent of  $\rho \in \mathcal{V}_1 \cup F(\mathcal{V}_1)$  and  $\epsilon$ .

Armed with these two functions, we construct the following escape function

$$\hat{g}_\epsilon = \log(M\epsilon + \hat{\varphi}_-) - \log(M\epsilon + \hat{\varphi}_+) \quad (7-4.6)$$

where  $M \gg 1$  is a constant independent of  $\epsilon$  and sufficiently large so that the following lemma holds :

**Lemma 7-4.3.** For  $M$  large enough, there exists  $C_1 > 0$  such that, uniformly with respect to  $\epsilon$ ,

$$\rho \in \mathcal{V}_1 \cup F(\mathcal{V}_1), d(\rho, \mathcal{T}) \geq C_1\epsilon \implies \hat{g}_\epsilon(F(\rho)) - \hat{g}_\epsilon \geq 1/C_1.$$

Since we will be interested in the dynamics in a neighborhood of  $\mathcal{T}$ , we fix a smooth cut-off function  $\hat{\chi} \in C_c^\infty(\mathcal{V}_1 \cup F(\mathcal{V}_1))$ , independent of  $\epsilon$ , such that  $\hat{\chi} = 1$  in a neighborhood of  $\mathcal{T}$  and we set

$$g_\epsilon = \hat{\chi} \hat{g}_\epsilon \quad (7-4.7)$$

As a consequence of the construction of  $\hat{\varphi}_\pm$ , it is also possible to check that

**Lemma 7-4.4.**  $g_\epsilon$  satisfies the following estimates : there exist  $C > 0$ ,  $\mu > 0$  and a family of constants  $C_\alpha > 0$ ,  $\alpha \in \mathbb{N}^2$ , independent of  $\epsilon$  such that for all  $\rho, \zeta \in U$ ,

$$\begin{aligned} |g_\epsilon(\rho)| &\leq C |\log \epsilon| \\ |\partial^\alpha g_\epsilon(\rho)| &\leq C_\alpha \left( \epsilon^{-|\alpha|/2} \right) \\ \frac{\exp(g_\epsilon(\rho))}{\exp(g_\epsilon(\zeta))} &\leq C \left\langle \frac{\rho - \zeta}{\sqrt{\epsilon}} \right\rangle^\mu \end{aligned}$$

This last inequality makes  $e^{g_\epsilon}$  an order function in the rescaled variable  $\rho/\sqrt{\epsilon}$ .

#### Remark.

We will specialize to  $\epsilon = h^{2\delta}$  where  $\delta = \frac{1}{2} - \varepsilon$ . For this reason, it is important that the constants do not depend on  $\epsilon$ .

- In this particular case, this lemma will ensure that  $g$  lives in a suitable symbol class. More precisely, for this choice of  $\epsilon = h^{2\delta}$ , we will have  $g/\log h \in S_\delta$  and for  $|\alpha| \geq 1$ ,  $h^{|\alpha|} \partial^\alpha g \in S_\delta$ .
- The properties of  $g$  and  $\hat{\varphi}_\pm$  will be used to estimate terms of the form  $\exp(tg(\rho) - tg(F^n(\rho)))$ . When  $\rho$  is sufficiently far from the trapped set and  $n \sim \alpha \log h$ , we will be able to show that this term is  $O(h^{L(t)})$  where  $L(t)$  can be made as large as necessary by choosing  $t$  large enough. The properties of  $g$  are widely used in 7-6.5.3. and 7-6.5.4..

## 7-5 Proof of Theorem 7-1.3.

From now on,  $M_h(z) = M(z; h)$  is an open hyperbolic quantum map satisfying the assumptions of Theorem 7-1.3. Recall that we note  $\alpha_h(z)$  the amplitude of  $M_h(z)$ . Our aim is to understand the zeros of the Fredholm determinants

$$\det(\text{Id} - M_h(z))$$

Since the spectrum of  $M_h(z)$  doesn't change by conjugation, we will instead study

$$M_t(z; h) := e^{-tG} M_h(z) e^{tG} \tag{7-5.1}$$

where  $t$  will be chosen below and  $G = \text{Op}_h(g)$  where  $g = g_{h^{2\delta}}$  is the escape function constructed (7-4.7), specialized in the case  $\epsilon = h^{2\delta}$  where  $\delta = 1/2 - \epsilon$ , for some fixed  $\epsilon$ . To alleviate the notations, we now omit to write that  $M_t(z)$  depends on  $h$ . The role of this conjugation is to damp the quantum map outside a small neighborhood of the trapped set so that it confers to the new operator nicer microlocal properties. To exploit the hyperbolicity of  $F$  and the special structure of the trapped set, we note that the zeros (repeated with multiplicity) of  $\det(\text{Id} - M_t(z))$  are among the zeros of

$$\det(\text{Id} - M_t(z)^{2N})$$

We will use this remark with an exponent  $N(h)$  depending on  $h$  in a controlled way and we will assume that  $N(h) \leq C \log \frac{1}{h}$  for some  $C > 0$ . A precise value of  $N(h)$  will be given later.

### 7-5.1. Application of a Jensen formula.

The proof of Theorem 7-1.3 relies on the following proposition, whose proof will occupy the end of this section in 7-5.2.. We recall that  $\Omega = ] - R, R[ + i ] - R, R[$  with  $R$  fixed but large (in particular,  $R \geq 4$ ).

**Proposition 7-5.1.** Let  $\epsilon > 0$ . Let  $g = g_{h^{2\delta}}$  be the escape function defined in (7-4.7) with  $1/2 - \delta = \epsilon > 0$ . Let's note  $M_t(z) = e^{-t \text{Op}_h(g)} M_h(z) e^{t \text{Op}_h(g)}$ . Let us fix  $\beta \in ]0, R[$ . Then, there exist  $t = t_\epsilon > 0$ ,  $C = C_\epsilon > 0$ ,  $\nu_\epsilon > 0$ ,  $\vartheta_\epsilon > 0$  and  $N = N_\epsilon(h) \in \mathbb{N}$  such that

- When  $\epsilon \rightarrow 0$ ,

$$\nu_\epsilon = d_H + O(\epsilon) ; \vartheta_\epsilon = \frac{1 - O(\epsilon)}{6\lambda_{\max}}$$

- at fixed  $\epsilon$ , when  $h \rightarrow 0$ ,  $N_\epsilon(h) \sim \vartheta_\epsilon \log(1/h)$
- for all  $h$  sufficiently small and for all  $z \in \Omega$  with  $\text{Im}(z) \in [-\beta, 4]$ ,

$$\text{tr} \left( (M_t(z)^N)^* M_t(z)^N \right) \leq C h^{-\nu_\epsilon} h^{-\vartheta_\epsilon P(-2 \text{Im} z t_{ret} - \varphi_u)} \tag{7-5.2}$$

**Remark.**

Since  $\text{Im} z \geq -\beta$  and since the function  $s \mapsto P(-2st_{ret} - \varphi_u)$  is non increasing, the right hand side can be estimated by  $h^{-\nu_\epsilon} h^{-\vartheta_\epsilon P(2\beta t_{ret} - \varphi_u)} = h^{-d_H + p(\beta) - O(\epsilon)}$ . This is where the function  $p(\beta) = -\frac{1}{6\lambda_{\max}} P(2\beta t_{ret} - \varphi_u)$  appears.

Armed with this proposition, we can conclude the proof of Theorem 7-1.3 by using standard arguments of spectral theory and complex analysis (we mainly borrow the arguments from [DD13], [DBW19]).

*Proof of Theorem 7-1.3.* In Theorem 7-1.3, we want to prove the bound

$$m_M(\{| \text{Re} z | < r, \text{Im} z \geq -\gamma\}) + O \left( h^{-d_H + \max(p(\gamma + \epsilon) - \epsilon, 0)} \right).$$

The exponent  $d_H$  is known from [NSZ14] in Theorem 4. We focus on the potential improvement given by  $p(\gamma + \epsilon) - \epsilon$ .

We fix  $0 < r < R$  and  $\gamma > 0$  and note  $\Omega_0 = \{|\operatorname{Re} z| \leq r, \operatorname{Im}(z) \in [-\gamma, 2]\}$ . For  $\eta > 0$ , we also note  $\Omega_\eta = \{|\operatorname{Re} z| < R, \operatorname{Im} z \in ]-\gamma - \eta, 4[ \}$ . Since  $\det(\operatorname{Id} - M(z; h)) = \det(\operatorname{Id} - M_t(z))$  and due to the relation:  $\operatorname{Id} - A^{2N} = (\operatorname{Id} - A)(\operatorname{Id} + A + \dots + A^{2N-1})$ , we have (we note  $m_T(\Omega_1)$  the numbers of zeros of  $\det(\operatorname{Id} - T)$  in  $\Omega_1$ , counted with multiplicity),

$$m_M(\Omega_0) = m_{M_t}(\Omega_0) \leq m_{M_t^{2N}}(\Omega_0)$$

that is, it is enough to estimate the number of zeros of  $f(z) = \det(\operatorname{Id} - M_t(z)^{2N})$ .

We claim that if  $H$  is some Hilbert space and if  $A : H \rightarrow H$  is a trace-class operator, then  $\log |\det(I - A^2)| \leq \|A\|_{HS}^2 = \operatorname{tr}(A^*A)$ . Indeed, if we denote  $\lambda_j(A)$  (resp.  $\sigma_j(A)$ ) the eigenvalues (resp. singular values) of  $A$  repeated with multiplicity, one has,

$$\log |\det(I - A^2)| = \sum_j \log(|1 - \lambda_j(A^2)|) = \sum_j \log|1 - \lambda_j(A)^2| \leq \sum_j \log(1 + \lambda_j(A)^2) \leq \sum_j \lambda_j(A)^2$$

Weyl's inequalities imply that (see for instance [DZ19], Appendix B.5.1)

$$\sum_j \lambda_j(A)^2 \leq \sum_j \sigma_j(A)^2 = \|A\|_{HS}^2 = \operatorname{tr}(A^*A)$$

which gives the desired result. Hence, we have

$$\log |\det(\operatorname{Id} - M_t(z)^{2N})| \leq \operatorname{tr}((M_t^N(z))^* M_t^N(z)) \quad (7-5.3)$$

which is known to be controlled by Proposition 7-5.1. Let's note  $z_0 = i \in \Omega_0$ . By the Riemann mapping theorem, for any  $\eta > 0$ , there exists a conformal map  $c : \Omega_\eta \rightarrow \{|z| < 1\}$  such that  $c(z_0) = 0$ .  $c(\Omega_0) \Subset c(\Omega_\eta)$ , so that there exists  $\delta > 0$  such that  $c(\Omega_0) \subset \{|z| < 1 - \delta\}$ . We now apply Jensen's formula to the function  $f \circ c$ . Let  $n(t)$  denote the number of zeros of  $f \circ c$  (counted with multiplicities), in the disc of radius  $t$ . We have, by Jensen's formula,

$$\int_0^{1-\delta/2} \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f \circ c((1 - \delta/2)e^{i\theta})| d\theta - \log |f(z_0)|$$

Therefore,

$$\begin{aligned} m_M(\Omega_0) &\leq m_{M_t^{2N}}(\Omega_0) \leq n(1 - \delta) \\ &\leq \frac{2}{\delta(1 - \delta)} \int_{1-\delta}^{1-\delta/2} \frac{n(t)}{t} dt \\ &\leq \frac{2}{\delta(1 - \delta)} \int_0^{1-\delta/2} \frac{n(t)}{t} dt \\ &\leq \frac{2}{\delta(1 - \delta)} \left( \frac{1}{2\pi} \int_0^{2\pi} \log |f \circ c((1 - \delta/2)e^{i\theta})| d\theta - \log |f(z_0)| \right) \\ &\leq \frac{2}{\delta(1 - \delta)} \left( \sup_{z \in \Omega_\eta} \log |f(z)| - \log |f(z_0)| \right) \end{aligned}$$

We apply Proposition 7-5.1 with a small parameter  $\varepsilon'$ , depending on  $\varepsilon$ , giving exponents  $\nu_{\varepsilon'}$ ,  $\vartheta_{\varepsilon'}$ . Since  $\nu_{\varepsilon'} = d_H + O(\varepsilon')$  and  $\vartheta_{\varepsilon'} = \frac{1}{6\lambda_{\max}} + O(\varepsilon')$ , we can choose  $\varepsilon'$  small enough so that

$$\nu_{\varepsilon'} - 6\lambda_{\max} \vartheta_{\varepsilon'} p(\gamma + \eta) \leq d_H - p(\gamma + \eta) + \varepsilon.$$

Hence, we have

$$\sup_{z \in \Omega_\eta} \log |f(z)| \leq \sup_{z \in \Omega_\eta} h^{-\nu_{\varepsilon'}} h^{-\vartheta_{\varepsilon'} P(-2 \operatorname{Im} z t_{ret} - \varphi_u)} \leq h^{-d_H - \varepsilon + p(\gamma + \eta)}$$

since the map  $\beta \mapsto p(\beta)$  is non increasing (recall the definition of  $p(\beta)$  in (7-1.4)). To handle the term  $-\log |f(z_0)|$ , we observe that since  $\limsup_{h \rightarrow 0} \|\alpha_h(z_0)\|_\infty < 1$ , there exists  $\rho \in [0, 1[$  such

that for  $h$  small enough,  $\|M_h(z_0)\| \leq \rho$ . As a consequence,  $\|M_t^{2N(h)}(z_0)\| \leq C_t \rho^{2N}$ , so that for  $h$  small enough,  $\|M_t^{2N}\| < 1/2$ . In particular, for such  $h$ ,  $\text{Id} - M_t(z_0)^{2N}$  is invertible and

$$\left\| (\text{Id} - M_t(z_0)^{2N})^{-1} \right\| \leq 2.$$

As a consequence, one has

$$\begin{aligned} -\log |\det (\text{Id} - M_t(z_0)^{2N})| &= \log \left| \det (\text{Id} - M_t(z_0)^{2N})^{-1} \right| \\ &= \log \left| \det \left( \text{Id} + M_t(z_0)^{2N} (\text{Id} - M_t(z_0)^{2N})^{-1} \right) \right| \\ &\leq \left\| M_t(z_0)^{2N} (\text{Id} - M_t(z_0)^{2N})^{-1} \right\|_{tr} \\ &\leq \|M_t(z_0)^{2N}\|_{tr} \left\| (\text{Id} - M_t(z_0)^{2N})^{-1} \right\| \\ &\leq 2 \|M_t(z_0)^{2N}\|_{HS} \leq C h^{-d_H - \varepsilon + p(\gamma + \eta)} \end{aligned}$$

This concludes the proof. □

### 7-5.2. Proof of Proposition 7-5.1.

We start the proof of Proposition 7-5.1. We fix some  $\varepsilon > 0$  and we freeze the complex variable  $z$  and note  $M_h$  and  $\alpha_h$  instead of  $M_h(z)$  and  $\alpha_h(z)$ : we momentarily forget this dependence but keep in mind that  $\text{Im}(z) \in [-\beta, 4]$  for some  $\beta > 0$ . In particular,  $\alpha_h(z) = e^{-\text{Im} z t_{ret}} + O(h^{-1} S_{0+})$  in a neighborhood of  $\mathcal{T}$  and the constant in the estimates below can be chosen independent of  $z$ .

**Reduction to FIO acting on  $\mathbb{R}$ .** We will note  $\mathbb{R}_J = \bigsqcup_{j=1}^J \mathbb{R}$  and  $L^2(\mathbb{R}_J) = \bigoplus_{j=1}^J L^2(\mathbb{R})$ . Recall that by construction (see 4-3),  $M_h$  is an operator of the form  $(M_{ij}(h))$  where  $M_{ij}(h) : L^2(Y_j) \rightarrow L^2(Y_i)$ . It will be more convenient for us to work on  $L^2(\mathbb{R})$ . For this purpose, recall that, for all  $i, j$ , there exists  $\tilde{M}_{ij}(h) \in I_{0+}(\mathbb{R} \times \mathbb{R}, \text{Gr}(F_{ij})')$  and cut-off functions  $\Psi_i, \Psi_j$  such that as operators  $L^2(Y_j) \rightarrow L^2(Y_i)$

$$M_{ij}(h) = \Psi_i \tilde{M}_{ij}(h) \Psi_j + O(h^\infty)$$

and as operator  $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ ,

$$\tilde{M}_{ij}(h) = \Psi_i \tilde{M}_{ij}(h) \Psi_j + O(h^\infty)$$

where, in the two equalities above, the  $O(h^\infty)$  hold for the trace norm. Let's note  $M_\psi(h) = (\Psi_i M_{ij}(h) \Psi_j)_{ij}$ . As soon as  $N \leq C \log \frac{1}{h}$ ,  $M(h)^N = M_\Psi(h)^N + O(h^\infty)$  as operators  $L^2(Y) \rightarrow L^2(Y)$  and  $M_\Psi(h)^N = \tilde{M}(h)^N + O(h^\infty)$  as operators  $L^2(\mathbb{R}_J) \rightarrow L^2(\mathbb{R}_J)$ . The same holds after conjugation by  $e^{tG}$ . In particular, this sows that

$$\text{tr}_{L^2(Y)} ((M_t^N)^* M_t^N) = \text{tr}_{L^2(\mathbb{R}_J)} ((\tilde{M}_t^N)^* \tilde{M}_t^N) + O(h^\infty)$$

Since the  $O(h^\infty)$  will finally be adsorbed in our required inequality, it is enough to work with  $\tilde{M}(h)$  instead of  $M_h$ .

From now on, we will write  $M_h$  for the operator  $\tilde{M}_h : L^2(\mathbb{R}_J) \rightarrow L^2(\mathbb{R}_J)$ . There exists  $\Psi_A, \Psi_D$  such that

$$\text{supp } \Psi_A \Subset \tilde{A} \quad ; \quad \text{supp } \Psi_D \Subset \tilde{D}$$

and

$$\Psi_A M_h = M_h + O(h^\infty); \quad M_h \Psi_D = M_h + O(h^\infty)$$

Moreover, we will now omit the  $h$ -dependence of the semiclassical operators in the notations when this dependence is obvious. In particular, we will simply write  $M$  or  $M_t$  instead of  $M_h$  or  $M_t(h)$  respectively.

**Notations.** A function  $a$  on  $T^*\mathbb{R}_J = \bigsqcup_{j=1}^J T^*\mathbb{R}$  is a  $J$ -uple of functions  $(a_1, \dots, a_J)$ . The quantization  $\text{Op}_h(a)$  is the diagonal matrix with diagonal entries  $\text{Op}_h(a_j)$ . The support of  $a$  is the disjoint union of the supports of the  $a_j$ 's, so as the wavefront set of  $\text{Op}_h(a)$ .

### 7-5.2.1. Refined quantum partition.

In virtue of Proposition 3-5.1, applied with  $\varphi = -2 \operatorname{Im} z t_{ret} - \varphi_u$ , there exists  $\eta > 0$  such that for any open cover  $\mathcal{Q}$  of  $\mathcal{T}$  of diameter smaller than  $\eta$ , one has

$$\left| \lim_{n \rightarrow +\infty} \frac{1}{n} \log P_1(\varphi, n, \mathcal{Q}) - P(\varphi) \right| \leq \varepsilon/3 \quad (7-5.4)$$

We consider some  $\varepsilon_0 > 0$ , which is supposed to be small enough to satisfy all the assumptions which will appear in the following and which will follow us throughout the end of the chapter. In particular, we first impose  $\varepsilon_0 < \eta$ .

Since  $\mathcal{T}$  is totally disconnected, there exists an open cover of  $\mathcal{T}$  by a finite number of *disjoint* open sets (of  $U$ ) of diameter smaller than  $\varepsilon_0$  :

$$\mathcal{T} \subset \bigcup_{\mathfrak{A} \in \mathcal{Q}} \mathfrak{A}$$

We fix some  $\rho_{\mathfrak{A}} \in \mathcal{T} \cap \mathfrak{A}$  and we assume that for all  $\mathfrak{A} \in \mathcal{Q}$ , there exists  $j_{\mathfrak{A}}, l_{\mathfrak{A}}, m_{\mathfrak{A}} \in \{1, \dots, J\}$  such that

$$\mathfrak{A} \subset B(\rho_{\mathfrak{A}}, 2\varepsilon_0) \subset \tilde{A}_{j_{\mathfrak{A}} l_{\mathfrak{A}}} \cap \tilde{D}_{m_{\mathfrak{A}} j_{\mathfrak{A}}} \subset U_{j_{\mathfrak{A}}}$$

$\varepsilon_0$  is supposed to be small enough so that :

- $e^{-\tau_m} \leq \alpha_h \leq e^{\tau_M}$  in  $B(\rho_{\mathfrak{A}}, 2\varepsilon_0)$  for some  $\tau_m, \tau_M > 0$ , for all  $h$  small enough.
- According to Lemma 3-4.1, there exists a chart  $\kappa_{\mathfrak{A}} : B(\rho_{\mathfrak{A}}, 2\varepsilon_0) \rightarrow W_{\mathfrak{A}} = \kappa_{\mathfrak{A}}(B(\rho_{\mathfrak{A}}, 2\varepsilon_0))$ , adapted to the dynamics, where  $W_{\mathfrak{A}}$  is a subset of  $T^*\mathbb{R}$  centered at 0. .
- There exist Fourier integral operators  $B_{\mathfrak{A}}, B'_{\mathfrak{A}} \in I_0(\mathbb{R} \times \mathbb{R}, \operatorname{Gr}'(\kappa_{\mathfrak{A}})) \times I_0(\mathbb{R} \times \mathbb{R}, \operatorname{Gr}'(\kappa_{\mathfrak{A}}^{-1}))$ , quantizing  $\kappa_{\mathfrak{A}}$  in a neighborhood of  $\kappa_{\mathfrak{A}}(\overline{\mathfrak{A}}) \times \overline{\mathfrak{A}}$ .

**Notations.** We will still denote  $B_{\mathfrak{A}}$  and  $B'_{\mathfrak{A}}$  the operators

$$B_{\mathfrak{A}} = \operatorname{Diag}(0, \dots, B_{\mathfrak{A}}, \dots, 0) : L^2(\mathbb{R}_J) \rightarrow L^2(\mathbb{R}_J) \quad ; \quad B'_{\mathfrak{A}} = \operatorname{Diag}(0, \dots, B'_{\mathfrak{A}}, \dots, 0) : L^2(\mathbb{R}_J) \rightarrow L^2(\mathbb{R}_J)$$

with the non zero entry in position  $j_{\mathfrak{A}}$ . When we say that  $(B_{\mathfrak{A}}, B'_{\mathfrak{A}})$  quantize  $\kappa_{\mathfrak{A}}$  in a neighborhood of  $\kappa_{\mathfrak{A}}(\overline{\mathfrak{A}}) \times \overline{\mathfrak{A}}$ , we mean that  $B'_{\mathfrak{A}} B_{\mathfrak{A}} = I + O(h^\infty)$  microlocally in a neighborhood of  $\overline{\mathfrak{A}}$  (in the sense that if  $\operatorname{supp}(c)$  is included in this neighborhood of  $\overline{\mathfrak{A}}$  and if  $C = \operatorname{Op}_h(c)$ , then  $B'_{\mathfrak{A}} B_{\mathfrak{A}} C = C + O(h^\infty)$ ;  $C B'_{\mathfrak{A}} B_{\mathfrak{A}} = C + O(h^\infty)$ ) and  $B_{\mathfrak{A}} B'_{\mathfrak{A}} = I + O(h^\infty)$  microlocally in a neighborhood of  $\kappa_{\mathfrak{A}}(\overline{\mathfrak{A}})$ .

In virtue of the equation (7-5.4), there exists  $n_0 \in \mathbb{N}$  such that

$$\left| \frac{1}{n_0} \log P_1(\varphi, n_0, \mathcal{Q}) - P(\varphi) \right| \leq 2\varepsilon/3$$

As a consequence, there exists a subpartition  $(\mathcal{W}_q)_{q \in \mathcal{A}} \subset \mathcal{Q}^{n_0}$  such that  $\mathcal{T} \subset \bigcup_{q \in \mathcal{A}} \mathcal{W}_q$  and

$$\sum_{q \in \mathcal{A}} \sup_{\rho \in \mathcal{W}_q \cap \mathcal{T}} \exp \left( \sum_{i=0}^{n_0-1} \varphi(F^i(\rho)) \right) \leq e^{n_0(P(\varphi) + \varepsilon)} \quad (7-5.5)$$

For  $q \in \mathcal{A}$ , we can find an open set  $\mathcal{V}_q \Subset \mathcal{W}_q$  such that  $\mathcal{T} \cap \mathcal{W}_q \subset \mathcal{V}_q$ .  $(\mathcal{V}_q)_{q \in \mathcal{A}}$  is still a cover of  $\mathcal{T}$ . We complete this cover with

$$\mathcal{V}_\infty = \mathbb{R}_J \setminus \bigcup_{q \in \mathcal{A}} \mathcal{V}_q \quad (7-5.6)$$

We note  $\mathcal{A}_\infty = \mathcal{A} \cup \{\infty\}$ . Note also that for  $q \in \mathcal{A}$ ,  $\mathcal{W}_q$  is of the form

$$\mathfrak{A}_0 \cap F^{-1}(\mathfrak{A}_1) \cap \dots \cap F^{-(n_0-1)}(\mathfrak{A}_{n_0-1})$$

and in particular  $\mathcal{W}_q \subset \mathfrak{A}_0$  : we note  $j_q, l_q, m_q, \rho_q, \kappa_q, B_q, B'_q, W_q$ , instead of  $j_{\mathfrak{A}_0}, l_{\mathfrak{A}_0}, m_{\mathfrak{A}_0}, \rho_{\mathfrak{A}_0}, \kappa_{\mathfrak{A}_0}, B_{\mathfrak{A}_0}, B'_{\mathfrak{A}_0}, W_{\mathfrak{A}_0}$ . Then, for  $q \in \mathcal{A}$ , we consider a cut-off function  $\chi_q \in C_c^\infty(\mathbb{R}_J, [0, 1])$  such that

$\text{supp}(\chi_q) \subset \mathcal{W}_q$  and  $\chi_q \equiv 1$  in a neighborhood of  $\overline{\mathcal{V}_q}$ . Finally, we note  $\chi_\infty = 1 - \sum_{q \in \mathcal{A}} \chi_q$ . We note that  $\chi_q$  is supported in only one copy of  $\mathbb{R}$  in  $\mathbb{R}_J$  when  $q \in \mathcal{A}$  and  $\chi_\infty$  has non-zero components in all the copies of  $\mathbb{R}$  in  $\mathbb{R}_J$ . Moreover,  $\text{supp}(\chi_\infty) \subset \mathcal{V}_\infty$ .

We then quantize the symbols  $\chi_q, q \in \mathcal{A}_\infty$  :

$$A_q = \text{Op}_h(\chi_q) \tag{7-5.7}$$

Note that for  $q \in \mathcal{A}$ ,  $A_q$  is a diagonal matrix with a single non zero coefficient. The family  $(A_q)_{q \in \mathcal{A}_\infty}$  satisfies the following properties :

$$\sum_{q \in \mathcal{A}_\infty} A_q = \text{Id} \quad ; \quad \forall q \in \mathcal{A}_\infty, \|A_q\| \leq 1 + O(h) \tag{7-5.8}$$

Since  $M^{n_0} = \sum_{q \in \mathcal{A}_\infty} M^{n_0} A_q$ , we may write

$$M^{n_0} = \sum_{\mathbf{q} \in \mathcal{A}_\infty^n} M_{\mathbf{q}}$$

where for  $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}_\infty^n$ ,

$$M_{\mathbf{q}} := M^{n_0} A_{q_{n-1}} \dots M^{n_0} A_{q_0} \tag{7-5.9}$$

For  $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}_\infty^n$ , we also define a family of refined neighborhoods, forming a refined cover of  $\mathcal{T}$ ,

$$\mathcal{V}_{\mathbf{q}}^- = \bigcap_{i=0}^{n-1} F^{-in_0}(\mathcal{V}_{q_i}) \quad ; \quad \mathcal{V}_{\mathbf{q}}^+ = F^{nn_0}(\mathcal{V}_{\mathbf{q}}^-) = \bigcap_{i=0}^{n-1} F^{(n-i)n_0}(\mathcal{V}_{q_i}) \tag{7-5.10}$$

and we adopt the same definitions by changing  $\mathcal{V}$  into  $\mathcal{W}$ . Roughly speaking, we expect that each operator  $M_{\mathbf{q}}$  acts from  $\mathcal{W}_{\mathbf{q}}^-$  to  $\mathcal{W}_{\mathbf{q}}^+$  and is negligible elsewhere. Combining (7-5.8), the fact that  $\alpha_h \leq e^{\tau M}$  in  $B(\rho_{\mathfrak{A}}, 2\varepsilon_0)$  and the bound on  $M$ , the following bound is valid :

$$\|M_{\mathbf{q}}\|_{L^2 \rightarrow L^2} \leq (e^{\tau M} + O(h^{1-}))^{nn_0} \tag{7-5.11}$$

As soon as  $|n| \leq C_0 |\log h|$ , we have  $\|M_{\mathbf{q}}\|_{L^2 \rightarrow L^2} \leq C e^{nn_0 \tau M}$ , for some  $C$  depending on  $C_0$  and a finite number of semi-norms of  $\alpha_h$  and then

$$\|M_{\mathbf{q}}\| \leq C h^{-M}$$

for some  $C > 0$  and  $M > 0$  depending on  $C_0$  and  $\alpha_h$ .

### 7-5.2.2. Local unstable Jacobian.

We want to define unstable Jacobians associated with this refined partition. Let's fix a word  $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}^n$  and assume that  $\mathcal{W}_{\mathbf{q}}^- \neq \emptyset$ . Fix  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ . By definition of  $\mathcal{W}_{q_i}$ , there exists  $\mathfrak{A}_{0,i}, \dots, \mathfrak{A}_{n_0-1,i} \in \mathcal{Q}$  such that

$$\mathcal{W}_{q_i} = \bigcap_{j=0}^{n_0-1} F^{-j}(\mathfrak{A}_{j,i})$$

Hence, for  $0 \leq l \leq n' = n \times n_0 - 1$ , there exists  $\rho_l \in \mathcal{T}$  such that  $d(\rho_l, F^l(\rho)) \leq 2\varepsilon_0$ . Hence,

$$d(F(\rho_l), \rho_{l+1}) \leq d(F(\rho_l), F^{l+1}(\rho)) + d(F^{l+1}(\rho), \rho_{l+1}) \leq C\varepsilon_0$$

That is to say,  $(\rho_0, \dots, \rho_{n'})$  is a  $C\varepsilon_0$  pseudo orbit. Assume that  $\delta_0 > 0$  is a small fixed parameter. In virtue of the shadowing lemma (see [HK95], Section 18.1), if  $\varepsilon_0$  is sufficiently small,  $(\rho_0, \dots, \rho_{n'})$  is  $\delta_0$  shadowed by an orbit of  $F$  : there exists  $\rho' \in \mathcal{T}$  such that for  $i \in \{0, \dots, n'\}$ ,  $d(\rho_i, F^i(\rho')) \leq \delta_0$ . Consequently,  $d(F^i(\rho), F^i(\rho')) \leq \delta_0 + C\varepsilon_0$ . If  $\rho_2$  is another point in  $\mathcal{W}_{\mathbf{q}}^-$ , for  $i = 0, \dots, n'$ ,  $d(F^i(\rho_2), F^i(\rho')) \leq 2\varepsilon_0 + C\varepsilon_0 + \delta_0$ . For convenience, set  $\varepsilon_2 = 2\varepsilon_0 + \delta_0 + C\varepsilon_0$  and note that  $\varepsilon_2$  can be arbitrarily small depending on  $\varepsilon_0$ . As a consequence, we have proven the following

**Lemma 7-5.1.** If  $\mathcal{W}_{\mathbf{q}}^- \neq \emptyset$ , there exists  $\rho' \in \mathcal{T}$  such that  $\forall l \in \{0, \dots, nn_0 - 1\}$  and for any  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ ,  $d(F^l(\rho), F^l(\rho')) \leq \varepsilon_2$ .

We fix any  $\rho'$  satisfying the conclusions of this lemma and we arbitrarily set (recall also the definition of  $J_n^u(\rho)$  in (7-4.3) for  $\rho \in \mathcal{T}$ )

$$J_{\mathbf{q}}^u := J_{nn_0}^u(\rho') = \prod_{j=0}^{n-1} J_{n_0}^u(F^{jn_0}(\rho')) \quad (7-5.12)$$

If  $\rho'_1$  is another point satisfying this conclusion, we have  $d(F^i(\rho'), F^i(\rho'_1)) \leq 2\varepsilon_2$  for  $i \in \{0, \dots, n'\}$  and in virtue of Corollary (3-2.1),

$$J_{nn_0}^u(\rho') \sim J_{nn_0}^u(\rho'_1)$$

Hence, up to global multiplicative constant, the definition of this unstable Jacobian is independent of the choice of  $\rho'$ . Notice that if  $\mathcal{W}_{\mathbf{q}}^- \cap \mathcal{T} \neq \emptyset$ , any  $\rho' \in \mathcal{T} \cap \mathcal{W}_{\mathbf{q}}^-$  satisfies the conclusions of Lemma 7-5.1 and  $J_{\mathbf{q}}^u \sim J_{nn_0}^u(\rho')$ .

We have the following facts concerning these local unstable Jacobians :

**Lemma 7-5.2.** If  $\varepsilon_0$  is small enough, the following holds. There exists  $C > 0$  such that for all  $\rho \in \mathcal{A}^n$  and for all  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ , we have

- $\|d_{\rho}F^{nn_0}\| \leq CJ_{\mathbf{q}}^u$
- $d(F^{nn_0-1}(\rho), \mathcal{T}_+) \leq C (J_{\mathbf{q}}^u)^{-1} d(\rho, \mathcal{T}_+)$
- $d(\rho, \mathcal{T}_-) \leq C (J_{\mathbf{q}}^u)^{-1} d(F^{nn_0-1}(\rho), \mathcal{T}_-)$

*Proof.* The three points are consequences of Lemma 3-2.3. The first point is an easy one. Concerning the other two, first recall that  $\mathcal{T}_+$  (resp.  $\mathcal{T}_-$ ) is, in a neighborhood of  $\mathcal{T}$ , equal to the union of local unstable (resp. stable manifolds). Let's consider the second inequality. The proof of the third one is similar, by inverting the time direction. We fix  $\zeta \in \mathcal{T}$  such that  $d(\rho, \mathcal{T}_+) = d(\rho, W_u(\zeta))$  and  $d(\zeta, \rho) \leq 2\varepsilon_0$ . Recall that by Lemma 7-5.1, there exists  $\rho'$  such that  $\forall i \in \{0, \dots, nn_0 - 1\}$  and  $d(F^i(\rho), F^i(\rho')) \leq \varepsilon_2$ . We hence consider the unique point  $\zeta' \in W_u(\zeta) \cap W_s(\rho')$ . Since  $\zeta' \in W_s(\rho')$ ,  $d(F^i(\zeta'), F^i(\rho')) \leq CJ_i^s(\rho)d(\rho', \zeta')$  for all  $0 \leq i \leq nn_0 - 1$ . If  $\varepsilon_0$  is small enough, we may assume that  $CJ_i^s(\rho)d(\rho', \zeta') \leq \frac{1}{2}\varepsilon_1$  for  $0 \leq i \leq n-1$  (where  $\varepsilon_1$  appears in Lemma 3-2.3). As a consequence,  $J_{nn_0}^u(\zeta') \sim J_{nn_0}^u(\rho') \sim J_{\mathbf{q}}^u$ . Moreover,  $d(F^i(\rho), F^i(\zeta')) \leq \frac{1}{2}\varepsilon_1 + \varepsilon_2$  for all  $0 \leq i \leq nn_0 - 1$ . Hence, if  $\varepsilon_2 \leq \frac{1}{2}\varepsilon_1$ ,

$$d(F^{nn_0-1}(\rho), \mathcal{T}_+) \leq d(F^{nn_0-1}(\rho), W_u(F^n(\zeta'))) \leq CJ_{nn_0}^u(\zeta')d(\rho, W_u(\zeta')) \leq CJ_{\mathbf{q}}^u d(\rho, \mathcal{T}_+) \quad \square$$

### 7-5.2.3. Numerology.

In this subsection, we introduce the parameters we will work with. Recall that  $\varepsilon$  has been fixed. We set  $\delta = 1/2 - \varepsilon$  : it is related to the regularity of the escape function  $g$ . For technical reasons, we also introduce

$$\delta_0 = \frac{1}{2} - \frac{\varepsilon}{2}; \quad \delta_1 = \delta - \frac{\varepsilon}{2} = \frac{1}{2} - \frac{3\varepsilon}{2} \quad (7-5.13)$$

satisfying  $\delta_1 < \delta < \delta_0 < 1/2$ . Recall that  $n_0$  has been chosen in (7-5.5) and that

$$\lambda_{\max} = \sup_{\rho \in \mathcal{T}} \limsup_{n \rightarrow +\infty} \frac{1}{n} \log J_n^u(\rho).$$

We define precisely the parameter  $\vartheta_{\varepsilon}$  appearing in Proposition 7-5.1 as

$$\vartheta_{\varepsilon} = \frac{1 - 4\varepsilon}{6\lambda_{\max}(1 + \varepsilon)^2}. \quad (7-5.14)$$

The precise value of  $\vartheta_{\varepsilon}$  will be used in the following : what is important is that  $\vartheta_{\varepsilon} = \frac{1}{6\lambda_{\max}} - O(\varepsilon) < 1/6\lambda_{\max}$ . Finally, we set

$$n = n(h) = \left\lceil \frac{\vartheta_{\varepsilon}}{n_0} \log \frac{1}{h} \right\rceil$$

which satisfies

$$e^{\lambda_{\max}(1+\varepsilon)nm_0} \leq h^{-\frac{1-4\varepsilon}{6(1+\varepsilon)}}.$$

In particular, we assume that  $\varepsilon$  is small enough to ensure that

$$h^{\delta_0} h^{-\frac{1-4\varepsilon}{6(1+\varepsilon)}} \leq h^{1/3}.$$

This will constraint the width of the evolved coherent states.

#### 7-5.2.4. Reduction to $L^2$ -bounds of an evolved coherent state.

We can find a uniform  $T_0 \in \mathbb{N}$  such that if  $\rho \in \mathcal{V}_\infty$ , there exists  $k \in \{-T_0, \dots, T_0\}$  such that  $F^k(\rho)$  "falls" in the hole - that is, either there exists  $k \in \{1, \dots, T_0\}$  such that  $F^i(\rho) \in \tilde{D}$  for  $1 \leq i \leq k-1$  and  $F^k(\rho) \in U \setminus \tilde{D}$  or there exists  $k \in \{1, \dots, T_0\}$  such that  $F^{-i}(\rho) \in \tilde{A}$  for  $1 \leq i \leq k-1$  and  $F^{-k}(\rho) \in U \setminus \tilde{A}$ . By standard properties of the Fourier integral operators, each component  $(M^{T_0})_{ij}$  of  $M^{T_0}$  is a Fourier integral operator associated with the component  $(F^{T_0})_{ij}$  of  $F^{T_0}$ . In particular,  $\text{WF}_h'(M^{T_0}) \subset \text{Gr}(F^{T_0})$ .

Let us study  $M^{2T_0+nn_0} = M^{T_0}M^{nn_0}M^{T_0}$ , and let's decompose  $M^{nn_0} = \sum_{\mathbf{q} \in \mathcal{A}_\infty^n} M_{\mathbf{q}}$ . If  $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}_\infty^n$  and if there exists an index  $i \in \{0, \dots, n-1\}$  such that  $q_i = \infty$ , one can isolate this index  $i$  and trap  $A_{q_i}$  between two Fourier integral operators  $M_1, M_2$ , belonging to a finite family of FIO associated to  $F^{T_0}$ , so that we can write

$$M^{T_0}M_{\mathbf{q}}M^{T_0} = B_1M_1A_\infty M_2B_2$$

where  $B_1, B_2$  satisfy the  $L^2$ -bound :

$$\|B_1\| \times \|B_2\| \leq C(\|\alpha_h\|_\infty)^{nn_0-1} = O(h^{-K})$$

for some integer  $K > 0$ , and we have  $M_1A_\infty M_2 = O(h^\infty)$ , with constants that can be chosen independent of  $\mathbf{q}$ . Hence, the same is true for  $M^{T_0}M_{\mathbf{q}}M^{T_0}$ . So, we can write, keeping in mind that  $|\mathcal{A}|^n = O(h^{-K})$  for some  $K > 0$  :

$$\begin{aligned} M^{nn_0+2T_0} &= \sum_{\mathbf{q} \in \mathcal{A}_\infty^n} M^{T_0}M_{\mathbf{q}}M^{T_0} \\ &= \sum_{\mathbf{q} \in \mathcal{A}^n} M^{T_0}M_{\mathbf{q}}M^{T_0} + O(h^\infty) \\ &= M^{T_0} \left( \sum_{\mathbf{q} \in \mathcal{A}^n} M_{\mathbf{q}} \right) M^{T_0} + O(h^\infty) \end{aligned}$$

Let us note

$$\mathfrak{M} = M^{nn_0}(\text{Id} - A_\infty) = M^{nn_0} \sum_{q \in \mathcal{A}} A_q \quad (7-5.15)$$

We have shown the following lemma :

**Lemma 7-5.3.** There exists  $T_0 \in \mathbb{N}$  such that

$$M^{2T_0+nn_0} = M^{T_0}\mathfrak{M}^n M^{T_0} + O(h^\infty)$$

Let us now look at what this equation implies on the trace of  $M^{2T_0+nn_0}$ . In the following computations, we use : If  $A$  is an Hilbert-Schmidt operator and  $B$  a bounded operator,

- (i)  $\text{tr}(A^*A) = \|A\|_{HS}^2$  ;
- (ii)  $\|AB\|_{HS} \leq \|B\| \times \|A\|_{HS}$  ;  $\|BA\|_{HS} \leq \|B\| \times \|A\|_{HS}$



$$\begin{aligned}
\operatorname{tr} \left( \left( M_t^{2T_0+nn_0} \right)^* M_t^{2T_0+nn_0} \right) &= \|M_t^{2T_0+nn_0}\|_{HS}^2 \\
&= \left\| M_t^{T_0} \mathfrak{M}_t^n M_t^{T_0} \right\|_{HS}^2 + O(h^\infty) \\
&\leq \|M_t^{T_0}\|^4 \|\mathfrak{M}_t^n\|_{HS}^2 + O(h^\infty) \\
&\leq \|M_t^{T_0}\|^4 \operatorname{tr} \left( (\mathfrak{M}_t^n)^* \mathfrak{M}_t^n \right) + O(h^\infty)
\end{aligned}$$

Hence, it is enough to find the expected upper bound (7-5.2) for  $\operatorname{tr} \left( (\mathfrak{M}_t^n)^* \mathfrak{M}_t^n \right)$  to obtain the same kind of upper bounds for  $\operatorname{tr} \left( \left( M_t(z)^N \right)^* M_t(z)^N \right)$ .

**Evolution in local adapted charts.** We will be interested in the evolution of coherent states through the action of  $\mathfrak{M}$ . It will be more convenient to work in the charts  $\kappa_q$  in which the action of  $F$  is well adapted to the position-momentum coordinate  $(x, \xi)$ . For this purpose, we start by writing,

$$\mathfrak{M}_t^n = e^{-tG} \mathfrak{M}^{n-1} \sum_{q \in \mathcal{A}} M^{n_0} A_q e^{tG}$$

Recall that  $B'_q B_q = I + O(h^\infty)$  microlocally near  $\operatorname{supp}(a_q)$ , hence,

$$\mathfrak{M}_t^n = e^{-tG} \mathfrak{M}^{n-1} \sum_{q \in \mathcal{A}} M^{n_0} A_q B'_q B_q e^{tG} B'_q B_q + O(h^\infty)$$

Let's note

$$\tilde{E}_t = B_q e^{tG} B'_q \quad (7-5.16)$$

We also fix  $\tilde{A}_q = \operatorname{Op}_h(\tilde{a}_q)$  such that  $\operatorname{WF}_h(\tilde{A}_q) \subset \mathcal{W}_q$  and  $\tilde{a}_q = 1$  near  $\operatorname{supp}(\chi_q)$ . This gives :

$$\begin{aligned}
\operatorname{tr} \left( (\mathfrak{M}_t^n)^* \mathfrak{M}_t^n \right) &= \sum_{q,p \in \mathcal{A}} \operatorname{tr} \left( \left( e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_p B'_p \tilde{E}_t B_p \tilde{A}_p \right)^* e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t B_q \tilde{A}_q \right) + O(h^\infty) \\
&= \sum_{q,p \in \mathcal{A}} \operatorname{tr} \left( B_p^* \left( e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_p B'_p \tilde{E}_t \right)^* e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t B_q \tilde{A}_q \tilde{A}_p^* \right) + O(h^\infty) \\
&= \sum_{q \in \mathcal{A}} \operatorname{tr} \left( B_q^* \left( e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right)^* e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t B_q \tilde{A}_q \tilde{A}_q^* \right) + O(h^\infty) \\
&\leq C |\mathcal{A}| \sup_{q \in \mathcal{A}} \operatorname{tr} \left( \left( e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right)^* e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right) + O(h^\infty)
\end{aligned}$$

where  $C$  is such that  $C_0 \|B_q\| \times \|B_q \tilde{A}_q \tilde{A}_q^*\| \leq C$  for all  $q \in \mathcal{A}$  (and  $0 < h \leq 1$ ). The passage from the second to the third line holds since  $\tilde{A}_q \tilde{A}_p^* = O(h^\infty)$  when  $q \neq p$ , in virtue of the fact that  $\mathcal{W}_p \cap \mathcal{W}_q \neq \emptyset$ . This computations show that it is enough to control, uniformly in  $q$ , the trace

$$\operatorname{tr} \left( \left( e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right)^* e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right) \quad (7-5.17)$$

since we now have :

$$\operatorname{tr} \left( \left( M_t^{2T_0+nn_0} \right)^* M_t^{2T_0+nn_0} \right) \leq C |\mathcal{A}| \sup_{q \in \mathcal{A}} \operatorname{tr} \left( \left( e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right)^* e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right) + O(h^\infty) \quad (7-5.18)$$

From now on, we will note  $\rho, \zeta$ , etc. points in  $U$  and  $\hat{\rho}, \hat{\zeta}$ , etc. their images in the local charts  $\kappa_q$ . The resolution of identity of Lemma 7-3.1, valid at the level of operators on  $L^2(\mathbb{R})$ , extends to the case of matrix operator acting on  $L^2(\mathbb{R}_J)$ , in the following sense :

$$\operatorname{tr}(A) = \sum_{j=1}^J \frac{1}{2\pi h} \int_{T^*\mathbb{R}} \langle A_{jj} \varphi_{\hat{\rho}}, \varphi_{\hat{\rho}} \rangle d\hat{\rho}$$

Hence, if  $K = e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t$ , we have

$$\begin{aligned} \text{tr}(K^*K) &= \sum_{j=1}^J \frac{1}{2\pi h} \int_{T^*\mathbb{R}} \langle (K^*K)_{jj} \varphi_{\hat{\rho}}, \varphi_{\hat{\rho}} \rangle d\hat{\rho} \\ &= \sum_{1 \leq i, j \leq J} \frac{1}{2\pi h} \int_{T^*\mathbb{R}} \langle K_{ij} \varphi_{\hat{\rho}}, K_{ij} \varphi_{\hat{\rho}} \rangle d\hat{\rho} \end{aligned}$$

Since  $A_q B'_q$  is diagonal with only one non-zero diagonal entry in position  $j_q$ ,  $B_{ij} = 0$  except when  $j = j_q$ . We can write :

$$\text{tr} \left( (e^{-tG} \mathfrak{M}^{n-1} M A_q B'_q)^* e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \right) = \frac{1}{2\pi h} \int_{T^*\mathbb{R}} \left\| e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \tilde{\varphi}_{\hat{\rho}} \right\|^2 d\hat{\rho} \tag{7-5.19}$$

where  $\tilde{\varphi}_{\hat{\rho}}$  is the column vector with only one non-zero entry equal to  $\varphi_{\hat{\rho}}$  in position  $j_q$ .

**7-5.2.5. End of the proof.**

The main ingredient for the proof of the improved fractal Weyl law, which is also the main novelty of this chapter is a good control for

$$w(\hat{\rho}) := \left\| e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t \tilde{\varphi}_{\hat{\rho}} \right\|^2 \tag{7-5.20}$$

This weight  $w$  depends on the parameter  $t$  which governs the weight of the escape function. We omit to write this dependence explicitly : indeed, what is important is that once  $t$  is fixed sufficiently large,  $w$  will satisfy the expected decay in Proposition 7-5.2. To state this bound, let's introduce, for  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ ,

$$\Pi_{\alpha, \mathbf{q}}(\rho) = \prod_{i=0}^{nn_0-1} \alpha(F^i(\rho))$$

where

$$\alpha(\rho) = \exp(-\text{Im} z_{\text{ret}}(\rho)) ; \rho \in \mathcal{W}_{\mathbf{q}}^- \tag{7-5.21}$$

so that, for  $\rho \in \mathcal{W}_{\mathbf{q}}$ , we have  $\alpha_h(\rho) = \alpha(\rho) + O(h^{1-} S_{0+})$ . We also introduce the following neighborhood of  $\mathcal{T}$

$$\mathcal{T}_{\delta, \delta_1} = \{ \rho ; d(\rho, \mathcal{T}_-) \leq h^\delta, d(\rho, \mathcal{T}_+) \leq h^{\delta_1} \} \subset U \subset T^*\mathbb{R}_J \tag{7-5.22}$$

**Proposition 7-5.2.** For any  $L > 0$ , there exists  $t = t(\varepsilon, L)$  such that the following holds. Let  $\hat{\rho} \in \mathbb{R}^2$ . If  $\hat{\rho} \notin \kappa_q(\mathcal{W}_q)$ , then  $w(\hat{\rho}) = O\left(\left(\frac{h}{\langle \hat{\rho} \rangle}\right)^\infty\right)$  with uniform constants. Otherwise, assume that  $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$ . We have

1. If, for all  $\mathbf{q} \in \mathcal{A}^{n+1}$ ,  $\rho \notin \mathcal{W}_{\mathbf{q}}^-$ , then  $w(\hat{\rho}) = O(h^\infty)$  with uniform constants.
2. Otherwise, there exists a unique  $\mathbf{q} \in \mathcal{A}^{n+1}$  such that  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ . In that case, for some uniform constants  $C > 0$  and  $h_0 > 0$ , one has, for  $0 < h \leq h_0$ ,
  - (i) If  $\rho \notin \mathcal{T}_{\delta, \delta_1}$ ,  $w(\hat{\rho}) \leq h^L$  ;
  - (ii) If  $\rho \in \mathcal{T}_{\delta, \delta_1}$ ,

$$w(\hat{\rho}) \leq C (\Pi_{\alpha, \mathbf{q}}(\rho))^2 (J_{\mathbf{q}}^u)^{d_H-1+\varepsilon} h^{(\delta_0-\delta)(d_H+\varepsilon)+\delta-1/2}$$

This key proposition is proved in Section 7-6. We will also require the following proposition :

**Proposition 7-5.3.** Let  $\mathbf{q} = q_0 \dots q_n \in \mathcal{A}^{n+1}$  with  $n = n(h)$  and assume  $\mathcal{W}_{\mathbf{q}}^- \neq \emptyset$ . Then, for some uniform constant  $C > 0$ , and for  $h$  small enough, the following estimate holds :

$$\text{Vol}(\mathcal{T}_{\delta, \delta_1} \cap \mathcal{W}_{\mathbf{q}}^-) \leq C h^{2\delta_1} h^{-(\delta+\delta_1)(d_H+\varepsilon)} (J_{\mathbf{q}}^u)^{-(d_H+\varepsilon)}$$

*Proof.* We assume that  $\mathcal{W}_{\mathbf{q}}^- \neq \emptyset$ . According to Lemma 7-5.2, there exists  $\rho_- \in \mathcal{T}$  such that for all  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ ,

$$d(\rho, W_s(\rho_-)) \leq C (J_{\mathbf{q}}^u)^{-1} \varepsilon_0 \quad (7-5.23)$$

We also consider  $\rho_+ \in \mathcal{T}$  such that  $d(\rho, \rho_+) \leq 2h^{\delta_1}$ . In particular,  $d(\rho_-, \rho_+) \ll \varepsilon_0$  and we may consider a point  $\rho_O \in W_s(\rho_-) \cap W_u(\rho_+)$  and we decide to work in an adapted chart  $\kappa$  centered at  $\rho_O$ . We want to estimate the volume of  $\kappa(\mathcal{T}_{\delta, \delta_1} \cap \mathcal{W}_{\mathbf{q}}^-)$ . We assume that  $\mathcal{W}_{\mathbf{q}}$  is included in the domain of this chart (and so is  $\mathcal{W}_{\mathbf{q}}^-$ ) and we choose this chart such that the image of  $W_s(\rho_-) = W_s(\rho_O)$  is given by  $\{(0, \xi), \xi \in V\}$ : this is possible in virtue of Lemma 3-4.1 (by considering  $F^{-1}$  instead of  $F$  to change the unstable manifold into the stable one)<sup>5</sup>. In virtue of (7-5.23), we have for some uniform constant  $C' > 0$

$$(x, \xi) \in \kappa_{\mathbf{q}}(\mathcal{W}_{\mathbf{q}}^-) \implies |x| \leq C' (J_{\mathbf{q}}^u)^{-1} \varepsilon_0$$

Let's consider  $\Xi(\mathcal{T}) = \{\xi \in V, \kappa_{\mathbf{q}}^{-1}(0, \xi) \in \mathcal{T}\}$  and let's cover it by  $N_s$  intervals of size  $2h^{\delta_1}$  centered at point  $\xi_1, \dots, \xi_{N_s} \in \Xi(\mathcal{T})$ . Since  $\overline{\dim} \mathcal{T} \cap W_s(\rho_-) = d_H$  and in virtue of Proposition 3-6.2, we may choose  $N_s$  such that

$$N_s \leq Ch^{-\delta_1(d_H + \varepsilon)} \quad (7-5.24)$$

for some uniform constant  $C > 0$ . For  $1 \leq i \leq N_s$ , let's note  $\sigma_i = \kappa^{-1}(0, \xi_i)$ . The local unstable manifold passing through  $\sigma_i$  can be written, in the chart  $\kappa$ , as a graph  $\{(x, g_i(x)), x \in U_i\}$ . We note

$$X_i(\mathcal{T}) = \{x \in U_i, \kappa^{-1}(x, g_i(x)) \in \mathcal{T}, |x| \leq 2C' (J_{\mathbf{q}}^u)^{-1} \varepsilon_0\}$$

and we cover  $X_i(\mathcal{T})$  by  $N_{i,u}$  intervals of size  $2h^{\delta}$ , centered at points  $x_{i,j}$ ,  $1 \leq j \leq N_{i,u}$ . Lemma 3-6.1 shows that we can take  $N_{i,u}$  such that for all  $1 \leq i \leq N_s$ ,

$$N_{i,u} \leq C (h^{\delta} J_{\mathbf{q}}^u)^{-d_H - \varepsilon} \quad (7-5.25)$$

for some uniform constant  $C$ .

For  $1 \leq i \leq N_s$  and  $1 \leq j \leq N_{i,u}$ , let's also note  $\xi_{i,j} = g_i(x_{i,j})$ . We claim that there exists a uniform constant  $C > 0$  such that

$$\kappa(\mathcal{T}_{\delta, \delta_1} \cap \mathcal{W}_{\mathbf{q}}^-) \subset \bigcup_{i=1}^{N_s} \bigcup_{j=1}^{N_{i,u}} [x_{i,j} - Ch^{\delta_1}, x_{i,j} + Ch^{\delta_1}] \times [\xi_{i,j} - Ch^{\delta_1}, \xi_{i,j} + Ch^{\delta_1}] \quad (7-5.26)$$

This claim obviously implies the proposition, by combining it with the bounds on  $N_s$  (7-5.24) and the  $N_{i,u}$  (7-5.25). We now turn to the proof of this claim.

Let's consider  $(x, \xi) = \kappa(\sigma) \in \kappa(\mathcal{T}_{\delta, \delta_1} \cap \mathcal{W}_{\mathbf{q}}^-)$ . We introduce different points (and encourage the reader to use Figure 7.3 to follow the different steps):

- Since  $d(\sigma, \mathcal{T}_+) \leq h^{\delta_1}$ , there exists  $\sigma_+ \in \mathcal{T}$  such that  $d(\sigma, W_u(\sigma_+)) \leq h^{\delta_1}$ . We can replace  $\sigma_+$  by the unique point in the intersection  $W_u(\sigma_+) \cap W_s(\rho_O)$  and we can note  $\kappa(\sigma_+) = (0, \xi_+)$ .
- Since  $\xi_+ \in \Xi(\mathcal{T})$ , there exists  $i \in \{1, \dots, N_s\}$  such that  $|\xi_i - \xi_+| \leq h^{\delta_1}$ . In particular,  $d(\sigma_i, \sigma_+) \leq Ch^{\delta_1}$ .
- Since  $d(\sigma, \mathcal{T}_-) \leq h^{\delta}$ , there exists  $\sigma_- \in \mathcal{T}$  such that  $d(\sigma, W_s(\sigma_-)) \leq h^{\delta}$ . We note  $\sigma_O$  the unique point in  $W_s(\sigma_-) \cap W_u(\sigma_+)$ .
- We also note  $\sigma'_i$  the unique point in  $W_s(\sigma_-) \cap W_u(\sigma_i)$ . Due to the Lipschitzness of the holonomy maps (with uniform Lipschitz constant),

$$d(\sigma_O, \sigma'_i) \leq Cd(\sigma_+, \sigma_i) \leq Ch^{\delta_1}$$

- Due to the local product structure near  $\sigma_O$ , we have  $d(\sigma, \sigma_O)^2 \sim d(\sigma, W_s(\sigma_O))^2 + d(\sigma, W_u(\sigma_O))^2 \sim h^{2\delta_1} + h^{2\delta}$ . It gives  $d(\sigma, \sigma_O) \leq Ch^{\delta_1}$  and hence,  $d(\sigma'_i, \sigma) \leq Ch^{\delta_1}$ .

<sup>5</sup>In fact, without the assumption on  $\kappa$  being symplectic, we may assume that both  $W_s(\rho_O)$  and  $W_u(\rho_O)$  are rectified

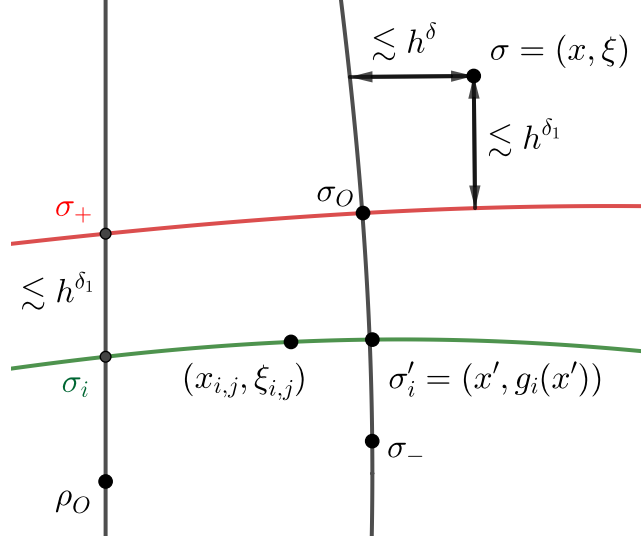


Figure 7.3: The different points introduced in the proof of the claim 7-5.26.

- Let's note  $\sigma'_i = (x', g_i(x'))$ . Since  $x' \in X_i(\mathcal{T})$ , there exists  $j \in \{1, \dots, N_{i,u}\}$  such that  $|x_{i,j} - x'| \leq h^\delta$ . Then we have

$$d(\sigma'_i, \kappa((x_{i,j}, \xi_{i,j})) \leq C|x'_i - x_{i,j}| \leq Ch^\delta$$

- We conclude that  $d(\sigma, \kappa((x_{i,j}, \xi_{i,j})) \leq Ch^{\delta_1}$ , which gives  $|x - x_{i,j}| \leq Ch^{\delta_1}, |\xi - \xi_{i,j}| \leq Ch^{\delta_1}$ .

□

We can now conclude the proof of the main trace estimate. Set  $N = 2T_0 + n(h)$ . We want to plug the estimates of Proposition 7-5.2 into (7-5.18) and (7-5.19). For  $q \in \mathcal{A}$ , let's note

$$O_q = \kappa_q \left( \mathcal{T}_{\delta, \delta_1} \cap \bigcup_{\mathbf{q} \in \mathcal{A}^{n+1}} \mathcal{W}_{\mathbf{q}}^- \right)$$

and let's write

$$\begin{aligned} \text{tr}((M_i^N)^* M_i^N) &\leq C \sup_{q \in \mathcal{A}} \frac{1}{2\pi h} \int_{\mathbb{R}^2} w(\hat{\rho}) d\hat{\rho} \\ &\leq \frac{C}{h} \sup_{q \in \mathcal{A}} \left( \int_{O_q} w(\hat{\rho}) d\hat{\rho} + \int_{\mathbb{R}^2 \setminus O_q} w(\hat{\rho}) d\hat{\rho} \right) \\ &\leq \sup_{q \in \mathcal{A}} \left( Ch^{(\delta - \delta_0)(d_H + \varepsilon) + \delta - 3/2} \sum_{\mathbf{q} \in \mathcal{A}^{n+1}} \int_{\kappa_q(\mathcal{T}_{\delta, \delta_1} \cap \mathcal{W}_{\mathbf{q}}^-)} (\Pi_{\alpha, \mathbf{q}}(\rho))^2 (J_{\mathbf{q}}^u)^{d_H - 1 + \varepsilon} d\hat{\rho} \right) + O(h^L) \end{aligned}$$

The last inequality holds since the integral outside  $O_q$  can be made  $O(h^L)$  by choosing  $t$  with  $L$  arbitrarily large. Indeed, using Proposition 7-5.2, the part outside  $\kappa_q(\mathcal{W}_q)$  is  $O(h^\infty)$  and the integral on  $\kappa_q(\mathcal{W}_q) \setminus O_q$  is  $O(h^L)$ .

Let  $\mathbf{q} = q_0 \dots q_{n-1} \in \mathcal{A}^n$ . We write  $\mathcal{W}_{q_i} = \bigcap_{j=0}^{n_0-1} F^{-1}(\mathfrak{A}_{i,j})$  with  $\mathfrak{A}_{i,j} \in \mathcal{Q}$  and for  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ . Let's note  $\varphi = -2 \text{Im } z t_{ret} - \varphi_u$  and recall that  $\alpha = \exp(-\text{Im } z t_{ret})$ . We have uniformly with

respect to  $\mathbf{q} \in \mathcal{A}^n$  and  $\rho \in \mathcal{W}_{\mathbf{q}}^-$ ,

$$\begin{aligned} (\Pi_{\alpha, \mathbf{q}}(\rho))^2 (J_{\mathbf{q}}^u)^{-1} &\leq C (\Pi_{\alpha, \mathbf{q}}(\rho))^2 (J_{nn_0}^u(\rho))^{-1} \\ &\leq C \prod_{i=0}^{n-1} \left( \exp \left( \sum_{j=0}^{n_0-1} \varphi(F^{in_0+j}(\rho)) \right) \right) \\ &\leq C \prod_{i=0}^{n-1} \left( \sup_{\rho_i \in \mathcal{W}_{q_i}} \exp \left( \sum_{j=0}^{n_0-1} \varphi(F^j(\rho_i)) \right) \right) \\ &\leq C \prod_{i=0}^{n-1} \left( C_0 \sup_{\rho_i \in \mathcal{W}_{q_i} \cap \mathcal{T}} \exp \left( \sum_{j=0}^{n_0-1} \varphi(F^j(\rho_i)) \right) \right) \end{aligned}$$

The last inequality holds for some  $C_0 > 0$  independent of  $n_0$ , since  $\varphi$  is Hölder continuous. Indeed, if  $\varepsilon_0$  is small enough, in virtue of Lemma 3-2.2, there exists  $\theta \in [0, 1)$  and  $C > 0$  such that if  $\rho_1 \in \mathcal{W}_{q_i}$  and if  $\rho_2 \in \mathcal{W}_{q_i} \cap \mathcal{T}$  then  $d(F^j(\rho_1), F^j(\rho_2)) \leq C\theta^{n-j}$ . As a consequence,  $|\varphi(F^j(\rho_1)) - \varphi(F^j(\rho_2))| \leq C\theta_1^{n-j}$  (with  $\theta_1 = \theta^\beta$  for some  $0 < \beta \leq 1$ ). Since  $\sum_{j=0}^{n-1} \theta_1^{n-j} \leq \sum_{j=0}^{\infty} \theta_1^j < +\infty$ , we find that

$$\exp \left( \sum_{j=0}^{n_0-1} \varphi(F^j(\rho_1)) \right) \sim \exp \left( \sum_{j=0}^{n_0-1} \varphi(F^j(\rho_2)) \right)$$

For  $q \in \mathcal{A}$ , let's call

$$p_q = \sup_{\rho \in \mathcal{W}_q \cap \mathcal{T}} \exp \left( \sum_{j=0}^{n_0-1} \varphi(F^j(\rho)) \right)$$

and recall that, due to our special choice of the partition  $(\mathcal{W}_q)_q$  (see (7-5.5)), we have  $\sum_{q \in \mathcal{A}} p_q \leq e^{n_0(P(\varphi)+\varepsilon)}$ . We may assume that  $n_0$  is big enough so that  $C_0 \leq e^{n_0\varepsilon}$ , and hence,  $\sum_{q \in \mathcal{A}} C_0 p_q \leq e^{n_0(P(\varphi)+2\varepsilon)}$ . As a consequence, we find that

$$\begin{aligned} \text{tr}((M_t^N)^* M_t^N) &\leq Ch^{(\delta-\delta_0)(d_H+\varepsilon)+\delta-3/2} \sum_{\mathbf{q} \in \mathcal{A}^n} \text{Vol}(\mathcal{T}_{\delta, \delta_1} \cap \mathcal{W}_{\mathbf{q}}^-) (J_{\mathbf{q}}^u)^{d_H+\varepsilon} \prod_{i=0}^{n-1} C_0 p_{q_i} \\ &\leq Ch^{(\delta-\delta_0)(d_H+\varepsilon)+\delta-3/2} \sum_{\mathbf{q} \in \mathcal{A}^n} h^{2\delta_1} h^{-(\delta+\delta_1)(d_H+\varepsilon)} (J_{\mathbf{q}}^u)^{-(d_H+\varepsilon)} (J_{\mathbf{q}}^u)^{d_H+\varepsilon} \prod_{i=0}^{n-1} C_0 p_{q_i} \\ &\leq Ch^{-\nu_\varepsilon} \sum_{\mathbf{q} \in \mathcal{A}^n} \prod_{i=0}^{n-1} C_0 p_{q_i} = Ch^{-\nu_\varepsilon} \left( \sum_{q \in \mathcal{A}} C_0 p_q \right)^n \leq Ch^{-\nu_\varepsilon} e^{nn_0(P(\varphi)+2\varepsilon)} \end{aligned}$$

where

$$\nu_\varepsilon = d_H + (\delta_0 - \delta)(d_H + \varepsilon) + (1/2 - \delta) + (\delta_1 + \delta)\varepsilon + (2\delta_1 - 1) - d_H(\delta + \delta_1 - 1) = d_H + O(\varepsilon)$$

(see the definitions of these exponents in (7-5.13)). Recalling that  $nn_0 \leq \vartheta_\varepsilon \log 1/h$ , we find that

$$\text{tr}((M_t^N)^* M_t^N) \leq Ch^{-\nu_\varepsilon} h^{-\vartheta_\varepsilon(P(-2\text{Im } zt_{ret} - \varphi_u) + 2\varepsilon)}$$

We can finally insert the term  $2\vartheta_\varepsilon\varepsilon$  into the  $\nu_\varepsilon$  and we find that

$$\text{tr}((M_t^N)^* M_t^N) \leq Ch^{-\nu_\varepsilon} h^{-\vartheta_\varepsilon P(-2\text{Im } zt_{ret} - \varphi_u)}$$

This concludes the proof of Proposition 7-5.1.

## 7-6 Proof of Proposition 7-5.2.

In this section we fix some  $q = q_0$  and we aim at proving Proposition 7-5.2. If  $\hat{\rho} \notin \kappa_q(\mathcal{W}_q)$ , as we will explain, the estimate in  $O\left(\left(\frac{h}{\langle \hat{\rho} \rangle}\right)^\infty\right)$  is nothing but a consequence of the fact that  $\text{WF}_h(B_q A_q B'_q) \Subset \kappa_q(\mathcal{W}_q)$  and one can for instance apply Lemma 15 in [CR12], Chapter 2, Section 3.

The main part of the Proposition 7-5.2 concerns points  $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$ . To prove this proposition, we study separately the actions of the different operators in  $e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \tilde{E}_t$ .

- First, we analyze the action of  $\tilde{E}_t$ . We show that it is essentially given by the multiplication by  $e^{tg(\rho)}$ .
- We go on studying the propagation of Gaussian coherent state through the iterated actions of  $\mathfrak{M}$ . The hyperbolicity of the trajectories leads to a deformation of the Gaussian state. The results we obtain are related to the results of [CR12] with Hamiltonian flow. In particular, this is where we use the fact that  $\vartheta_\varepsilon < (6\lambda_{\max})^{-1}$ . The approximation we use fails for longer logarithmic times.
- Finally, we analyze the action of  $e^{-tG}$  on the evolved coherent states. In a way, we treat this evolved state as a Lagrangian state with rapidly oscillating amplitude, of the form  $a(x)e^{i\frac{\phi(x)}{h}}$ . The scale of oscillation of  $a$  is larger than  $h^\delta$ , scale on which  $g$  oscillates. We show that, at leading order, the action of  $e^{-tG}$  is well approximated by the multiplication by  $e^{-tg(x, \phi'(x))}$ .

**Notations.** In the following, we will be lead to consider states  $u \in L^2(\mathbb{R}_J)$  such that all the components of  $u$  are  $O(h^\infty)$ , except one equal to some  $v \in L^2(\mathbb{R})$ . By abuse, we will note  $v$  instead of  $u$  as soon as the component where  $u$  is non zero is explicit in the context. For instance, we can simply note  $\varphi_{\hat{\rho}}$  instead of  $\tilde{\varphi}_{\hat{\rho}}$  as soon as we specify that  $\hat{\rho} \in \kappa_q(\mathcal{W}_q)$ . Another example : for any  $u \in L^2(\mathbb{R}_J)$  and  $q \in \mathcal{A}$ ,  $B_q A_q u$  has only one non zero component at  $j_q$  and we can use this component to write  $u$ . This will be widely used in the sequel since most of the time we will consider this type of elements.

### 7-6.1. Preparatory work.

Due to standard properties of Fourier integral operators, we can consider a pseudodifferential operator  $\Xi_q$  such that  $\text{WF}_h(\Xi_q) \subset W_q$ ,  $T^*\mathbb{R} \setminus \text{WF}_h(1 - \Xi_q) \subset \kappa_q(\mathcal{W}_q)$  and  $\Xi_q B'_q B_q = \Xi_q + O(h^\infty)$  (recall that  $W_q = \kappa_q(B(\rho_q, 2\varepsilon_0))$  and that  $\mathcal{W}_q \Subset B(\rho_q, 2\varepsilon_0)$  by construction). With these properties, we have in particular  $A_q B'_q \tilde{E}_t = A_q B'_q \tilde{E}_t \Xi_q + O(h^\infty)$ . This allows us to change harmlessly  $\tilde{E}_t$  into  $E_t := \tilde{E}_t \Xi_q$  in all the computations below. We first write

$$\begin{aligned} e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q E_t &= \sum_{(q_1, \dots, q_{n-1}) \in \mathcal{A}^{n-1}} e^{-tG} M^{n_0} A_{q_{n-1}} \dots M^{n_0} A_{q_1} M^{n_0} A_q B'_q E_t \\ &= \sum_{(q_1, \dots, q_n) \in \mathcal{A}^n} A_{q_n} e^{-tG} M^{n_0} A_{q_{n-1}} \dots M^{n_0} A_{q_1} M^{n_0} A_q B'_q E_t + O(h^\infty) \\ &= \sum_{(q_1, \dots, q_n) \in \mathcal{A}^n} A_{q_n} e^{-tG} B'_{q_n} M_{q_n, q_{n-1}} \dots M_{q_2, q_1} M_{q_1, q} E_t + O(h^\infty) \end{aligned}$$

where

$$M_{p,q} = B_p M^{n_0} A_q B'_q \quad (7-6.1)$$

We say that a pair  $(p, q)$  is admissible if  $F^{n_0}(\mathcal{W}_q) \cap \mathcal{W}_p \neq \emptyset$ . By standard properties of Fourier integral operators, if  $(p, q)$  is not admissible,  $M_{p,q} = O(h^\infty)$ . We say that a word  $(q_1, \dots, q_n) \in \mathcal{A}^n$  is admissible if all the pairs  $(q_i, q_{i-1})$  are admissible (with  $q_0 = q$ ). Hence, since  $n = O(\log \frac{1}{h})$ , we can restrict the indices in the above sum to the admissible words.

Suppose that  $(p, q)$  is an admissible pair. By composition of Fourier integral operators,  $M_{p,q}$  is a Fourier integral operator associated with the symplectic map  $F_{p,q} := \kappa_p \circ F^{n_0} \circ \kappa_q^{-1}$ . Since  $\text{diam}(\mathcal{W}_q) \leq \varepsilon_0$ , by taking  $\varepsilon_0$  sufficiently small, we can assume that  $F^{n_0}(\mathcal{W}_q)$  is included in the domain of  $\kappa_p$ . Indeed, there exists  $\tilde{\rho}_q \in \mathcal{W}_q$  such that  $F^{n_0}(\tilde{\rho}_q) \in \mathcal{W}_p$  and hence if  $\rho \in \mathcal{W}_q$ ,

$$d(F^{n_0}(\rho), \rho_p) \leq d(F^{n_0}(\rho), F^{n_0}(\tilde{\rho}_q)) + d(F^{n_0}(\tilde{\rho}_q), \rho_p) \leq C\varepsilon_0$$

We note  $(y, \eta)$  the variables in the charts and  $(\partial_y, \partial_\eta)$  the canonical basis of  $\mathbb{R}^2$  and we have

- $F_{p,q}(0) = \kappa_p \circ F^{n_0}(\rho_q) = O(\varepsilon_0)$ ;
- $d_0 F_{p,q} = d_{F(\rho_q)} \kappa_p \circ d_{\rho_q} F^{n_0} \circ [d_{\rho_q} \kappa_q]^{-1}$  ;
- $d_{\rho_q} F^{n_0}(E_u(\rho_q)) = E_u(F^{n_0}(\rho_q))$  and  $\rho \mapsto E_u(\rho)$  is Lipschitz. Hence, if we note  $e_u(\rho_q) = (d_{\rho_q} \kappa_q)^{-1}(\partial_y) \in E_u(\rho_q)$ , due to the definitions of the adapted charts in Lemma 3-4.1, there exists  $\lambda_{p,q} \in \mathbb{R}^*$  such that

$$d_{\rho_q}(\kappa_p \circ F^{n_0})(e_u(\rho_q)) = \lambda_{p,q} \partial_y + O(\varepsilon_0),$$

that is  $d_0 F_{p,q}(\partial_y) = \lambda_{p,q} \partial_y + O(\varepsilon_0)$ .

- Similarly,  $d_0 F_{p,q}(\partial_\eta) = \mu_{p,q} \partial_\eta + O(\varepsilon_0)$  for some  $\mu_{p,q} \in \mathbb{R}^*$

Eventually, we use the fact that  $F_{p,q} - F_{p,q}(0) - d_0 F_{p,q} = O(\varepsilon_0)_{C^1(W_q)}$  and we get that

$$F_{p,q}(y, \eta) = (\lambda_{p,q} y + y_r(y, \eta), \mu_{p,q} \eta + \eta_r(y, \eta)), (y, \eta) \in W_q \quad (7-6.2)$$

where  $y_r(y, \eta)$  and  $\eta_r(y, \eta)$  are  $O(\varepsilon_0)_{C^1}$ . In particular, if  $\varepsilon_0$  is small enough,  $(x, \xi, y, \eta) \in \text{Gr}(F_{p,q}) \mapsto (x, \eta)$  is a local diffeomorphism near  $(0, 0, 0, 0)$ . Then, there exists a phase function  $\psi_{p,q}$  which generates  $F_{p,q}$  in a neighborhood  $\Omega$  of  $(0, 0, 0, 0)$ . Assuming  $\varepsilon_0$  small enough, we can assume that  $F_{p,q}(W_q) \times W_q \subset \Omega$ .

As a consequence (see for instance [Ale08], [Zwo12] Chapter 10), the Fourier integral operator  $M_{p,q}$  can be written under the form (4-2.3), up to  $O(h^\infty)$ , that is,

$$M_{p,q} u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi_{p,q}(x, \eta) - y\eta)} \alpha_{p,q}(x, \eta) u(y) dy d\eta \quad (7-6.3)$$

where  $\alpha_{p,q}$  is a symbol in  $S_{0+}(\mathbb{R}^2)$ . It has an asymptotic expansion

$$\alpha_{p,q} \sim \sum_{j \geq 0} h^j \alpha_{p,q}^{(j)} \quad (7-6.4)$$

where  $\alpha_{p,q}^{(j)} \in h^{0-} S_{0+}$ , for all  $j \geq 1$  (that is,  $\alpha_{p,q}^{(j)} \in \bigcap_{\eta > 0} h^{-\eta} S_{0+}$ ) and we have

$$|\alpha_{p,q}^{(0)}(x, \eta)| = |\partial_{x,\eta}^2 \psi(x, \eta)|^{1/2} \chi_q(\rho) \times \left( \prod_{i=0}^{n_0-1} \alpha \circ F^i(\rho) \right) \quad ; \quad \rho = \kappa_q^{-1}(y, \eta) ; (x, \xi) = F_{p,q}(y, \eta) \quad (7-6.5)$$

Here, we use the fact that in  $W_q$ ,  $\alpha_h = \alpha + O(h^{1-} S_{0+})$  to put the  $O(h^{1-} S_{0+})$  in  $\alpha_{p,q}^{(1)}$ . Moreover, we have the following support properties : for  $j \in \mathbb{N}$ ,

$$(x, \eta) \in \text{supp}(\alpha_{p,q}^{(j)}) \implies (y, \eta) \in \kappa_q(\text{supp}(\chi_q)) \quad ; \quad (x, \xi) = F_{p,q}(y, \eta) \quad (7-6.6)$$

We now pick an admissible word  $(q_1, \dots, q_n)$  and for  $\hat{\rho} \in \mathbb{R}^2$ , we aim at studying

$$\|A_{q_n} e^{-tG} B'_{q_n} M_{q_n, q_{n-1}} \dots M_{q_2, q_1} M_{q_1, q} E_t \tilde{\varphi}_{\hat{\rho}}\|$$

We have  $M_{q_1, q} E_t = B_{q_1} M^{n_0} A_q B'_q B_q e^{tG} B'_q$ . Since  $\text{WF}_h(A_q B'_q B_q e^{tG}) \subset \text{supp}(\chi_q)$  and  $B'_q$  is a Fourier integral operator associated with  $\kappa_q^{-1}$ , we can find  $\tilde{\chi}_q$  such that  $\text{supp}(\tilde{\chi}_q) \subset \kappa_q(\mathcal{W}_q) \subset W_q$  and

$$A B'_q B_q e^{tG} B'_q = A B'_q B_q e^{tG} B'_q \text{Op}_h(\tilde{\chi}_q) + O(h^\infty)$$

To prove the estimate in  $O\left(\left(\frac{h}{\langle \hat{\rho} \rangle}\right)^\infty\right)$ , we invoke Lemma 15 in [CR12], Chapter 2, Section 3, which allows us to say that, if  $\hat{\rho} \notin \kappa_q(\mathcal{W}_q)$ ,

$$\text{Op}_h(\tilde{\chi}_q) \varphi_{\hat{\rho}} = O\left(\left(\frac{h}{\langle \hat{\rho} \rangle}\right)^\infty\right).$$

Since both  $M_{q_n, q_{n-1}} \dots M_{q_1, q}$  and the number of terms in the sum are  $O(h^{-K})$  for some  $K > 0$ , we deduce the first part of Proposition 7-5.2 :

**Lemma 7-6.1.** Uniformly for  $\hat{\rho} \notin \kappa_q(\mathcal{W}_q)$ ,  $w(\hat{\rho}) = O\left(\left(\frac{h}{\langle \hat{\rho} \rangle}\right)^\infty\right)$ .

We now focus on  $\hat{\rho} \in \kappa_q(\mathcal{W}_q) \Subset W_q$ , for which  $F_{p,q}(\hat{\rho})$  is well defined. We finish this preparatory subsection with an important computation. First note that the neighborhood  $\bigcup_{q \in \mathcal{A}} \mathcal{W}_q$  has been fixed by dynamical considerations. We may assume that the cut-off function  $\hat{\chi}$  used in (7-4.7) for the construction of  $g$ , is chosen such that  $\text{supp } \hat{\chi} \Subset \bigcup_{q \in \mathcal{A}} \mathcal{W}_q$ . As a consequence, we can apply Proposition 4-2.3 and we have :

**Lemma 7-6.2.** For all  $q \in \mathcal{A}$ , there exists  $(g_{j,q})_{j \geq 1} \in S_\delta$  such that for all  $N \in \mathbb{N}$ , the following holds:

$$B_q G B'_q = \text{Op}_h(g \circ \kappa_q^{-1}) + \sum_{j=1}^{N-1} h^{j(1-2\delta)} \text{Op}_h(g_{j,q}) + R_N$$

where  $\|R_N\|_{L^2 \rightarrow L^2} \leq C_{2N+M,g} h^{N(1-2\delta)}$ , for some constants  $C_{2N+M,g}$  depending on semi-norms of  $g$  in  $S_\delta$  up to order  $2N+M$ .

**Remark.**

Even if  $g \notin S_\delta$ , it still satisfies  $|\partial^\alpha g| \leq C_{|\alpha|,g} h^{-\delta|\alpha|}$  as soon as  $\alpha \neq 0$ . This allows us to fairly define these semi-norms.

*Proof.* Let's note that  $g \in \log \frac{1}{h} S_\delta$  so that we can apply Proposition 4-2.3 to  $(\log \frac{1}{h})^{-1} g$ . We can find differential operators  $L_{j,q}$  such that

$$B_q G B'_q = \text{Op}_h(g \circ \kappa_q^{-1}) + \sum_{j=1}^{N-1} h^j \text{Op}_h((L_{j,q} g) \circ \kappa_q^{-1}) + O_{L^2 \rightarrow L^2}(h^N \|g\|_{C^{2N+M}})$$

In fact, due to the properties of  $\hat{g}$ ,  $\partial^\alpha g \in h^{-\delta|\alpha|} S_\delta$  as soon as  $\alpha \neq 0$  and the terms  $g_{j,q} := h^{2\delta j} (L_{j,q} g) \circ \kappa_q^{-1} \in S_\delta$  for  $j \geq 1$ . Moreover, the  $O$  is in fact an  $O(C_{2N+M,g} h^{N(1-2\delta)})$ , where  $C_{2N+M,g}$  depends on semi-norms of  $g$  in  $S_\delta$  up to order  $2N+M$ .  $\square$

## 7-6.2. Action of $E_t$ .

We begin with the action of  $E_t$ . Recall the definition of  $E_t = B_q e^{tG} B'_q \Xi_q$  in (7-5.16).

**Lemma 7-6.3.** For any  $N \in \mathbb{N}$  and  $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$ , there exists  $\psi_0, \dots, \psi_{2N-1}$  and  $r_N$  such that

$$E_t \varphi_{\hat{\rho}} = \sum_{j=0}^{2N-1} h^{j(1/2-\delta)} \psi_j + r_N$$

satisfying :

- $\psi_0 = e^{tg(\rho)} \varphi_{\hat{\rho}}$  ;
- For  $0 \leq j \leq 2N-1$ ,  $\psi_j$  is of the form

$$\psi_j = e^{tg(\rho)} T(\hat{\rho}) \Lambda_h \left( P_{t,h}^{(j)} \Psi_0 \right)$$

where  $P_{t,h}^{(j)}$  is a polynomial of degree at most  $2j$ , with coefficients depending on  $t, g$  (and hence,  $h$ ) and  $\hat{\rho}$ . It satisfies,  $N_\infty(P_{t,h}^{(j)}) \leq C_{j,t}$  with  $h$ -independent constants, depending on derivatives of  $g$ .

- $\|r_N\|_{L^2} \leq C_N (1+|t|)^{2N+1} h^{N(1-2\delta)} h^{-K_0 t}$  for some  $K_0 > 0$  depending on  $g$ .

**Remark.**



- $t$  is supposed to be fixed, so that the only meaningful term involving  $t$  is  $h^{-Bt}$ . The other mentions of  $t$  can be put into the constants  $C_N$ . All the polynomials depend also on  $h$ , we will omit to mention it in the subscripts, but we keep in mind that in the following, all the polynomials potentially depend on  $h$ . Nevertheless, their  $N_\infty$ -norm can be controlled in an  $h$ -independent way.
- This lemma shows that the action of  $e^{tG}$  on  $\varphi_\rho$  is well approximated by the multiplication by  $e^{tg(\rho)}$ . A priori, this value strongly depends on  $h$  since  $g$  oscillates at scale  $h^\delta$ . Nevertheless, at the scale of a coherent state (that is  $h^{1/2}$ ), the variations of  $g$  are insignificant.

*Proof.* Let's fix  $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$ ,  $N \in \mathbb{N}$  and set  $\phi(t) = E_t \varphi_{\hat{\rho}}$ .  $\phi$  solves the equation

$$\phi'(t) = B_q G e^{tG} B_q' \Xi_q \varphi_{\hat{\rho}}$$

Since  $B_q B_q' = I$  microlocally near  $\text{WF}_h(\Xi_q)$ , we have  $e^{tG} B_q' \Xi_q = B_q' B_q e^{tG} B_q' \Xi_q + O(h^\infty)$ . Hence, up to  $O(h^\infty)$ ,  $\psi(t)$  solves  $\phi'(t) = \tilde{G} \phi(t)$  with  $\tilde{G} = B_q G B_q'$ . It is enough to find an expansion for the solution of this equation. By Lemma 7-6.2, there exists  $C_N$  (depending on  $g$ ) such that, with  $G_j = \text{Op}_h(g_{j,q})$  and  $G_0 = \text{Op}_h(g \circ \kappa_q^{-1})$ ,

$$\left\| \tilde{G} - \sum_{j=0}^{N-1} h^{j(1-2\delta)} G_j \right\| \leq C_N h^{N(1-2\delta)}$$

Set  $\psi(t) = T(\hat{\rho})^* \phi(t)$ . It solves :  $\psi'(t) = T(\hat{\rho})^* \tilde{G} T(\hat{\rho}) \psi(t)$ . We also set  $u(t) = \Lambda_h^* \psi(t)$ , which solves  $u'(t) = A u(t)$  where  $A = \Lambda_h^* T(\hat{\rho})^* \tilde{G} T(\hat{\rho}) \Lambda_h$ . Let's also note  $\tilde{A}_j = \Lambda_h^* T(\hat{\rho})^* G_j T(\hat{\rho}) \Lambda_h = \text{Op}_1(a_j)$  where  $a_j(\hat{\zeta}) = g_{j,q}(\hat{\rho} + h^{1/2} \hat{\zeta})$ . We wish to apply the formalism of Appendix 7-7.2. with  $H = L^2(\mathbb{R})$ , the operator  $A : H \rightarrow H$ ,  $\mathcal{C} = \{P\Psi_0, P \in \mathbb{C}[X]\}$  with initial state  $u(0) = \Psi_0$ . The parameter  $h$  in Appendix 7-7.2. is replaced by  $\tilde{h} = h^{1/2-\delta}$ . If  $P \in \mathbb{C}[X]$ , we approximate the action of  $A$  by

$$\begin{aligned} A(P\Psi_0) &= \sum_{j=0}^{N-1} \tilde{h}^{2j} \tilde{A}_j(P\Psi_0) + O_N \left( \tilde{h}^{2N} \|P\Psi_0\| \right) \\ &= \sum_{j=0}^{N-1} \tilde{h}^{2j} \left( \sum_{k=0}^{2N-1-2j} \tilde{h}^k A_{j,k}(P\Psi_0) + R_{N,j}(P\Psi_0) \right) + O_N \left( \tilde{h}^{2N} \|P\Psi_0\| \right) \end{aligned}$$

where, according to Lemma 7-3.3,

$$A_{j,k} = \text{Op}_1 \left( \sum_{\alpha+\beta=k} h^{-\delta k} \frac{\partial_x^\alpha \partial_\xi^\beta h_j}{\alpha! \beta!} (0) x^\alpha \xi^\beta \right) \quad ; \quad R_{N,j}(P\Psi_0) = O_{N,j,\text{deg } P} \left( \tilde{h}^{2N-2j} N_\infty(P) \right)$$

where the constant in  $O_{N,j,\text{deg } P}$  depend on  $g$  through its semi-norms, but are  $h$ -independent. Gathering the term of same order together, we can write

$$A(P\Psi_0) = \sum_{l=0}^{2N-1} \tilde{h}^l A_l(P\Psi_0) + O_{N,\text{deg } P} \left( \tilde{h}^{2N} N_\infty(P) \right)$$

here  $A_l = \sum_{2j+k=l} A_{j,k}$ . It is not hard to see that  $A_l(P\Psi_0) = P_l \Psi_0$  where  $P \mapsto P_l$  is linear and  $\text{deg } P_l \leq \text{deg } P + 2l$ . Since  $g_{j,q} \in S_\delta$  if  $j \geq 1$  and since  $h^{|\gamma|\delta} \partial^\gamma g = O(1)$ , we observe that as soon as  $l \geq 1$ , there exists  $C_l$  depending on  $g$  (through a finite number of semi-norms), but independent of  $h$ , such that

$$N_\infty(P_l) \leq C_l N_\infty(P)$$

Concerning  $A_0$ , it is clear that it is in fact  $g(\rho) \text{Id}$ . We now apply the formulas given in Appendix 7-7.2. and use the notations introduced in this appendix, that is

$$R_{2N-1}(t) = e^{tA} \Psi_0 - e^{tA_0} \sum_{l=0}^{2N-1} \tilde{h}^l v_l(t)$$

with  $v_l$  constructed inductively by (7-7.3) and  $v_0 = \Psi_0$ . Since  $A_0$  is a multiplication,  $A_k(s) = A_k$  for all  $s \in \mathbb{R}$  and we see by induction that  $v_k(t)$  is of the form

$$v_k(t) = \sum_{l=0}^k t^l P_{l,k} \Psi_0 = P_k(t) \Psi_0$$

where  $P_{l,k} \in \mathbb{C}[X]$  has degree at most  $2k$ . In particular,  $N_\infty(P_k(t)) \leq c_k(1 + |t|)^k$  for some  $h$ -independent  $c_k$  depending on  $g$ . Concerning the remainder, we have

$$\begin{aligned} \|\tilde{r}_{k,2N}(t)\| &= e^{tg(\rho)} \left\| \left( A - \sum_{j=0}^{2N-k-1} \tilde{h}^j A_j \right) v_k(t) \right\| \\ &= O_{N,k} \left( e^{tg(\rho)} \tilde{h}^{2N-k} N_\infty(P_k(t)) \right) \\ &\leq C_{N,k} (1 + |t|)^k e^{tg(\rho)} \tilde{h}^{2N-k} \end{aligned}$$

Finally, we recall that  $\dot{R}_{2N-1}(t) = AR_{2N-1}(t) + \sum_{j=0}^{2N-1} \tilde{h}^j \tilde{r}_{j,2N}(t)$ . Hence, integrating this inequality, we find that

$$\|R_{2N-1}(t)\| \leq \int_0^t \|A\| \times \|R_{2N-1}(s)\| ds + C_N \tilde{h}^N e^{tg(\rho)} (1 + |t|)^{2N-1}$$

By a version of Gronwall's lemma, we can find a constant  $C_N$  such that

$$\|R_{2N-1}(t)\| \leq C_N \tilde{h}^{2N} e^{t \max(|g(\rho)|, \|A\|)} t^{2N+1}$$

(where  $C_N$  depends on finitely many semi-norms of  $g$ ). Since  $g \in \log(1/h)S_\delta$ , we can find  $K_0 > 0$  such that  $\max(|g(\rho)|, \|A\|) \leq K_0 \log(1/h)$ . Going back to  $\phi(t)$ , we have proved the Lemma.  $\square$

### 7-6.3. Repeated actions of $M_{q_i, q_{i-1}}$ .

We fix some  $\mathbf{q} = q_1 \dots q_n \in \mathcal{A}^{n+1}$ . Each term in the development of  $E_t \varphi_{\hat{\rho}}$  is a sum of term of the form

$$e^{tg(\rho)} T(\hat{\rho}) \Lambda_h(P_0 \Psi_0)$$

with some  $P_0 \in \mathbb{C}[X]$  depending on  $h$ . We now focus on the evolution of each of these terms under the repeated actions of  $M_{q_i, q_{i-1}}$ . We recall that this operator has the form

$$M_{p,q} u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi_{p,q}(x,\eta) - y\eta)} \alpha_{p,q}(x, \eta) u(y) dy d\eta \quad (7-6.7)$$

with

$$\alpha_{p,q} \sim \sum_{j \geq 0} h^j \alpha_{p,q}^{(j)} \quad (7-6.8)$$

This will allow us to use Proposition 7-3.6, but we will have to deal with two different scales of asymptotic expansion :  $h$  and  $h^{1/2}$ . To simplify the notations in this context, we note for  $1 \leq i \leq n$ ,

$$\begin{aligned} M_{q_i, q_{i-1}} &= M_i \\ \psi_{q_i, q_{i-1}} &= \psi_i \\ F_{q_i, q_{i-1}} &= F_i \\ F^{(i)} &= F_i \circ \dots \circ F_1 = \kappa_{q_i} \circ F^{n_0 i} \circ \kappa_q^{-1} \\ \alpha_{q_i, q_{i-1}}^{(j)} &= \alpha_i^{(j)} \end{aligned}$$

For  $0 \leq i \leq n$ , we also note  $\hat{\rho}_i = F_i \circ \dots \circ F_1(\hat{\rho})$  (with  $\hat{\rho}_0 = \hat{\rho}$ ) and set  $\hat{\rho}_i = (x_i, \xi_i)$ .

Momentarily, we do not write the  $e^{tg(\rho)}$  since it is not changed by the actions of the  $M_i$ 's. It will be restored later in Corollary 7-6.2. We fix a parameter  $N$  and we start with an initial state

$$u_0 = T(\hat{\rho}_0)\Lambda_h(P_0\Psi_0) \quad (7-6.9)$$

with  $P_0$  a polynomial of degree  $d_0$ . The coefficients of  $P_0$  may depend on  $h$ , but in a controlled way (see Lemma 7-6.3). Our aim is to show that we have an asymptotic expansion for  $u_i = M_i \dots M_1 u_0$  of the form

$$u_i = \sum_{2j+k < 2N} h^j h^{k/2} u_i^{(j,k)} + r_i^{(N)}$$

where  $u_i^{(j,k)}$  has the form

$$T(\hat{\rho}_i)\mathcal{M}\left(d_{\hat{\rho}}F^{(i)}\right)\Lambda_h\left(P_i^{(j,k)}\Psi_0\right)$$

with  $P_i^{(j,k)}$  polynomial and with a good control on  $r_i^{(N)}$ . For  $1 \leq i \leq n$  and  $0 \leq j \leq N-1$ , we apply Proposition 7-3.6 to the operator

$$(M_i^{(j)}u)(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\psi_i(x,\eta) - y\eta)} \alpha_i^{(j)}(x,\eta) u(y) dy d\eta$$

and for a state of the form

$$u = T(\hat{\rho}_{i-1})\mathcal{M}\left(d_{\hat{\rho}}F^{(i-1)}\right)\Lambda_h(P\Psi_0)$$

For each such polynomial  $P$ , we can find a family  $Q_i^{(j,k)}(P)$  of polynomials such that

- $Q_i^{(j,0)}(P) = \frac{\alpha_i^{(j)}(x_i, \xi_{i-1})}{|\partial_{x\eta}^2 \psi_i(x_i, \xi_{i-1})|^{1/2}} P$  (up to a multiplicative factor of norm 1 that we omit in the proof) ;
- $Q_i^{(j,k)}(P)$  is a polynomial of degree  $\deg P + 3k$  and the map  $P \mapsto Q_i^{(j,k)}$  is linear, with coefficients depending on  $F^{(i)}$  and the derivatives of  $\psi_i$  and  $\alpha_i^{(j)}$  at  $(x_i, \xi_{i-1})$  up to the  $3k$ -th order and we have

$$N_\infty(Q_i^{(j,k)}) \leq C_{3k}(\psi_i) \|\alpha_i^{(j)}\|_{C^k} \|d_{\hat{\rho}}F^{(i)}\|^{3k} N_\infty(P)$$

Moreover, if  $(x_i, \xi_{i-1}) \notin \text{supp } \alpha_i^{(j)}$ , then  $Q_i^{(j,k)} = 0$ .

- for every  $N \in \mathbb{N}$ ,

$$M_i^{(j)}\left(T(\hat{\rho}_{i-1})\mathcal{M}(d_{\hat{\rho}}F^{(i-1)})\Lambda_h[P\Psi_0]\right) = T(\hat{\rho}_i)\mathcal{M}(d_{\hat{\rho}}F^{(i)})\Lambda_h\left[\sum_{k=0}^{N-1} h^{k/2} Q_i^{(j,k)}(P)\Psi_0\right] + R_i^{(j,N)}(P) \quad (7-6.10)$$

with

$$\|R_i^{(j,N)}(P)\|_{L^2} \leq h^{N/2} C_{3N+M}(\psi_i) \|\alpha_i^{(j)}\|_{C^{N+M}} \|d_{\hat{\rho}}F^{(i)}\|^{3N} K_{N,\deg P} N_\infty(P)$$

**Remark.**

In virtue of the properties of  $\alpha_i^{(j)}$ , the condition  $(x_i, \xi_{i-1}) \in \text{supp } \alpha_i^{(j)} \iff F^{in_0}(\rho) \in \text{supp}(\chi_q \alpha)$ .

We also write the expansion of  $M_i$  on the form, for every  $N$ ,

$$M_i = \sum_{j=0}^{N-1} h^j M_i^{(j)} + \tilde{S}_i^{(N)} \quad (7-6.11)$$

with

$$\|\tilde{S}_i^{(N)}\| \leq \tilde{C}_{i,N,\varepsilon} h^{N(1-\varepsilon)}$$

Since  $M_i$  belongs to the finite family  $(M_{p,q})$ , we can replace  $\tilde{C}_{i,N,\varepsilon}$  by  $\tilde{C}_{N,\varepsilon} = \sup_i \tilde{C}_{i,N,\varepsilon}$ .

We now give the iteration formulas for the required expansion. We state  $P_0^{(0,0)} = P_0$  and  $P_0^{(j,k)} = 0$  for the other values of  $(j, k)$ . For  $2j + k < 2N$ , we define inductively  $P_i^{(j,k)} = P_{i,P_0}^{(j,k)}$  by the formula (to alleviate the notations, we will omit to specify the dependence in  $P_0$  when this is not necessary) :

$$P_i^{(j,k)} = \sum_{j_1+j_2=j} \sum_{k_1+k_2=k} Q_i^{(j_2,k_2)} \left( P_{i-1}^{(j_1,k_1)} \right) \tag{7-6.12}$$

Concerning the remainder term, we set

$$\begin{aligned} r_i^{(N)} = r_{i,P_0}^{(N)} = M_i \left( r_{i-1}^{(N)} \right) &+ \sum_{2j+k < 2N} h^{j+k/2} \tilde{S}_i^{(N-j-\lceil k/2 \rceil)} \left( u_{i-1}^{(j,k)} \right) \\ &+ \sum_{2j_1+2j_2+k_1 < 2N} h^{N-j_1-j_2-k_1/2} R_i^{(j_1, 2(N-j_1-j_2)-k_1)} \left( P_{i-1}^{(j_1,k_1)} \right) \end{aligned} \tag{7-6.13}$$

**Lemma 7-6.4.** With the above notations, we have for  $1 \leq i \leq n$ ,

$$u_i = M_i \dots M_1 u_0 = \sum_{2j+k < 2N} h^j h^{k/2} u_{i,P_0}^{(j,k)} + r_{i,P_0}^{(N)} \quad ; \quad u_i^{(j,k)} = T(\hat{\rho}_i) \mathcal{M} \left( d_{\hat{\rho}} F^{(i)} \right) \Lambda_h \left( P_{i,P_0}^{(j,k)} \Psi_0 \right)$$

We now analyze these formulas to understand more precisely these terms and obtain a good control of the remainder. In particular, concerning the polynomial  $P_i^{(j,k)}$ , we want to control their degree and the norms of their coefficients.

**Leading term.** First note that the leading term (that is the term  $(0, 0)$ ) has a nice form. Indeed, up to a factor of norm 1, it is given by

$$\begin{aligned} P_i^{(0,0)} &= P_0 \times \prod_{l=1}^i \frac{\alpha_l^{(0)}(x_l, \xi_{l-1})}{|\partial_{x_l}^2 \psi_l(x_l, \xi_{l-1})|^{1/2}} \\ &= P_0 \times \prod_{l=0}^{i-1} \left[ \left( \chi_{q_l} \prod_{j=0}^{n_0-1} \alpha \circ F^j \right) (F^{ln_0}(\rho)) \right] \end{aligned}$$

The product on the right plays a crucial role in the analysis. Let's note

$$p_{\alpha,l}(\rho) = \left( \chi_{q_l} \prod_{j=0}^{n_0-1} \alpha \circ F^j \right) (F^{ln_0}(\rho)) \quad ; \quad \pi_{\alpha,i}(\rho) = \prod_{l=0}^{i-1} p_{\alpha,l}(\rho)$$

We remark that

$$\pi_{\alpha,i}(\rho) \leq \Pi_{\alpha,q_0 \dots q_{i-1}}(\rho)$$

Recall that  $\Pi_{\alpha,\mathbf{q}}(\rho) = \prod_{i=0}^{n_0-1} \alpha(F^i(\rho))$ . To simplify the notations, let's note  $\Pi_{\alpha,i} = \Pi_{\alpha,q_0 \dots q_{i-1}}$ .

Moreover, combining the support property (7-6.6) of  $\alpha_i^{(j)}$ , Remark 7-6.3. and the properties of  $Q_i^{(j,k)}$  given by Proposition 7-3.6, we see that for  $\mathbf{q} = q_0 \dots q_{i-1}$ ,

$$\rho \notin \mathcal{W}_{\mathbf{q}}^- \implies \forall j, k \in \mathbb{N}, \quad Q_i^{(j,k)} = 0 \tag{7-6.14}$$

**Analysis of the polynomial  $P_{i,P_0}^{(j,k)}$ .** According to (7-6.14), we assume that  $\rho = \kappa_q^{-1}(\hat{\rho}) \in \mathcal{W}_{q_0 \dots q_{i-1}}^-$ . Otherwise, there is nothing more to say. We start by the easiest part of the analysis :

**Lemma 7-6.5.** For all  $0 \leq i \leq n$  and all  $(j, k)$  with  $2j + k < 2N$ ,  $P_{i,P_0}^{(j,k)}$  is of degree at most  $3k + \deg P_0$ .

*Proof.* We argue by induction on  $i$ . This is obvious for the case  $i = 0$ . To pass from  $i - 1$  to  $i$ , we use (7-6.12) which shows that

$$\begin{aligned} \deg P_i^{(j,k)} &\leq \max_{j_1+j_2=j, k_1+k_2=k} \deg Q_i^{(j_2, k_2)} \left( P_{i-1}^{(j_1, k_1)} \right) \\ &\leq \max_{j_1+j_2=j, k_1+k_2=k} 3k_2 + \deg P_{i-1}^{(j_1, k_1)} \\ &\leq \max_{j_1+j_2=j, k_1+k_2=k} 3k_2 + 3k_1 + d_0 \\ &\leq 3k + d_0 \end{aligned}$$

□

The analysis of  $N_\infty \left( P_i^{(j,k)} \right)$  is a bit more tedious.

**Lemma 7-6.6.** For every  $\varepsilon$ , there exists a family of constants  $C_{j,k,\varepsilon}$  depending on the dynamical system and on  $M$  such that: For all  $0 \leq i \leq n$  and all  $(j, k)$  with  $2j + k < 2N$ , we have

$$N_\infty \left( P_{i, P_0}^{(j,k)} \right) \leq C_{j,k,\varepsilon} h^{-k\varepsilon} i^{2j+k} \Pi_{\alpha,i}(\rho) \left( J_{q_0 \dots q_{i-1}}^u \right)^{3k} N_\infty(P_0)$$

**Remark.**

The dependence on  $i$  is of major importance. Here,  $i \leq n = O(\log 1/h)$ . Hence, the term  $i^{2j+k}$  is essentially harmless compared to the second part  $\Pi_{\alpha,i}(\rho) \left( J_{q_0 \dots q_{i-1}}^u \right)^{3k} h^{-k\varepsilon}$ . The factor  $\Pi_{\alpha,i}(\rho)$  does not depend on  $k$  and is common to all the terms. It can be put in front of the all expansion. On the contrary, the growth of  $J_{q_0 \dots q_{i-1}}^u$  influences the precision and the validity of the expansion. So that the expansion holds, we need to require

$$\left( J_{q_0 \dots q_{i-1}}^u \right)^3 h^{-\varepsilon} \ll h^{-1/2}$$

As a consequence, this is where the assumption

$$\vartheta_\varepsilon < \frac{1 - 4\varepsilon}{6\lambda_{\max}}$$

(see its definition in (7-5.14)) is important and lead to a valid expansion.

**Remark.**

The constant  $C_{j,k,\varepsilon}$  depends on  $M_h$  through its amplitude  $\alpha_h$  as a Fourier integral operator in the class  $I_\varepsilon(\mathbb{R} \times \mathbb{R}, \text{Gr}(F)')$  and it depends only a finite number  $N_{j,k}$  of derivatives.

*Proof.* To alleviate the notations, we renormalize  $P_0$  so that  $N_\infty(P_0) = 1$ . We fix  $(j, k)$  such that  $2j + k < 2N$ . By iterating (7-6.12), we find that

$$P_i^{(j,k)} = \sum_{\substack{j_1 + \dots + j_i = j \\ k_1 + \dots + k_i = k}} Q_i^{(j_i, k_i)} \circ \dots \circ Q_1^{(j_1, k_1)}(P_0)$$

We now use the simple following idea : when  $i$  is large that is when  $i \gg 2j + k$ , and when  $j_1 + \dots + j_i = j$  and  $k_1 + \dots + k_i = k$ , most of the couples  $(j_l, k_l)$  are equal to  $(0, 0)$ . From a more quantitative point of view, we have

$$\#\{1 \leq l \leq i, (j_l, k_l) \neq (0, 0)\} \leq 2j + k$$

Indeed,

$$2j + k = 2(j_1 + \dots + j_i) + (k_1 + \dots + k_i) \geq \#\{1 \leq l \leq i, (j_l, k_l) \neq (0, 0)\}$$

Let's note  $\mathcal{P}(i, 2j+k)$  the set of subsets of  $\{1, \dots, i\}$  of cardinals smaller than  $2j+k$ . For  $\mathcal{L} \in \mathcal{P}(i, 2j+k)$  we define the set of indices  $\mathcal{I}_{\mathcal{L}} \subset \mathbb{N}^i \times \mathbb{N}^i$  by

$$(\vec{j}, \vec{k}) = ((j_1, \dots, j_i), (k_1, \dots, k_i)) \in \mathcal{I}_{\mathcal{L}} \iff \begin{cases} j_1 + \dots + j_i = j \\ k_1 + \dots + k_i = k \\ \forall 1 \leq l \leq i, (j_l, k_l) \neq (0, 0) \iff l \in \mathcal{L} \end{cases}$$

With these notations, we have

$$P_i^{(j,k)} = \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} \sum_{(\vec{j}, \vec{k}) \in \mathcal{I}_{\mathcal{L}}} Q_i^{(j_i, k_i)} \circ \dots \circ Q_1^{(j_1, k_1)}(P_0)$$

Let's fix  $\mathcal{L} \in \mathcal{P}(i, 2j+k)$  and  $(\vec{j}, \vec{k}) \in \mathcal{I}_{\mathcal{L}}$ . Let's write  $\mathcal{L} = \{l_1 < \dots < l_m\}$ . Since  $Q_l^{(0,0)}$  is simply a multiplication by  $p_{\alpha, l}(\rho)$ , we have :

$$Q_i^{(j_i, k_i)} \circ \dots \circ Q_1^{(j_1, k_1)}(P_0) = \left( \prod_{l \notin \mathcal{L}} p_{\alpha, l} \right) \times Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \dots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0)$$

Using Proposition 7-3.6, we can estimate

$$N_{\infty} \left( Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \dots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0) \right) \leq N_{\infty}(P_0) \times \prod_{p=1}^m C_{3k_{l_p}}(\psi_{l_p}) \|\alpha_{l_p}^{(j_{l_p})}\|_{C^{k_{l_p}}} \|d_{\hat{\rho}} F^{(l_p)}\|^{3k_{l_p}}$$

For  $1 \leq l \leq i$ ,  $\psi_l$  and  $\alpha_l^{(\cdot)}$  belong to a finite family of functions (corresponding to the finite number of admissible transitions). Hence, recalling that  $\alpha_l^{(j)} \in S_{0+}$  if  $j=0$  and  $h^{0-} S_{0+}$  if  $j \geq 1$ , we can find a global uniform constant depending on the dynamical system, and on a certain number  $N_{j,k}$  of derivatives of  $\alpha$  such that for all  $j' \leq j$ ,  $k' \leq k$  and for all  $l$ ,

$$C_{3k'}(\psi_l) \|\alpha_l^{(j')}\|_{C^{k'}} \leq C_{j,k,\varepsilon} \begin{cases} h^{-k'\varepsilon/2} & \text{if } j' = 0 \\ h^{-k'\varepsilon/2} h^{-\eta_{k,j}} & \text{if } j' \geq 1 \end{cases} .$$

where we artificially choose  $\eta_{k,j} = \frac{k\varepsilon}{2j}$  and use the fact that  $\alpha_l^{(j)} \in S_{\varepsilon/2}$  (resp.  $h^{-\eta_{j,k}} S_{\varepsilon/2}$ ) if  $j' = 0$  (resp.  $j' \geq 1$ ). As a consequence, since  $j_{l_1} + \dots + j_{l_m} = j$  and  $k_{l_1} + \dots + k_{l_p} = k$ , we have

$$N_{\infty} \left( Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \dots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0) \right) \leq C_{j,k,\varepsilon}^m N_{\infty}(P_0) h^{-k\varepsilon/2} h^{-j\eta_{k,j}} \left( \sup_{1 \leq l \leq i} \|d_{\hat{\rho}} F^{(l)}\| \right)^{3k}$$

Since  $m \leq 2j+k$ , there exists a global constant, still denoted  $C_{j,k,\varepsilon}$ , such that, uniformly in  $\mathcal{I}_{\mathcal{L}}$ ,

$$N_{\infty} \left( Q_{l_m}^{(j_{l_m}, k_{l_m})} \circ \dots \circ Q_{l_1}^{(j_{l_1}, k_{l_1})}(P_0) \right) \leq C_{j,k,\varepsilon} N_{\infty}(P_0) h^{-k\varepsilon} \left( \sup_{1 \leq l \leq i} \|d_{\hat{\rho}} F^{(l)}\| \right)^{3k}$$

We remark that for  $1 \leq l \leq n$ ,  $\|d_{\hat{\rho}} F^{(l)}\| \leq C \|d_{\hat{\rho}} F^{n_0 l}\| \leq C J_{q_0 \dots q_{l-1}}^u$ . Finally, since  $|\alpha| \geq e^{-4\tau_m}$  in the neighborhood  $\bigcup_{q \in \mathcal{A}} \mathcal{W}_q$  of  $\mathcal{T}$ , we see that for every  $(\vec{j}, \vec{k}) \in \mathcal{I}_{\mathcal{L}}$  we have

$$N_{\infty} \left( Q_i^{(j_i, k_i)} \circ \dots \circ Q_1^{(j_1, k_1)}(P_0) \right) \leq C_{j,k,\varepsilon} N_{\infty}(P_0) h^{-k\varepsilon} \Pi_{\alpha, i}(\rho) \left( J_{q_0 \dots q_{i-1}}^u \right)^{3k}$$

We can now conclude the proof. Indeed, we have

$$N_{\infty} \left( P_i^{(j,k)} \right) \leq \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} \#\mathcal{I}_{\mathcal{L}} \times \left( C_{j,k,\varepsilon} N_{\infty}(P_0) h^{-k\varepsilon} \Pi_{\alpha, i}(\rho) \left( J_{q_0 \dots q_{i-1}}^u \right)^{3k} \right)$$

If  $\mathcal{L} \in \mathcal{P}(i, 2j+k)$ , we estimate (crudely) the cardinal of  $\mathcal{I}_{\mathcal{L}}$  by

$$\#\mathcal{I}_{\mathcal{L}} \leq (j+1)^{\#\mathcal{L}} (k+1)^{\#\mathcal{L}} \leq (j+1)^{2j+k} (k+1)^{2j+k}$$

Finally,

$$i \mapsto \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} 1 = \sum_{m=0}^{2j+k} \binom{i}{m}$$

is a polynomial function of  $i$ , of degree  $2j+k$ . Hence, there exists  $C_{j,k}$  such that

$$\left| \sum_{\mathcal{L} \in \mathcal{P}(i, 2j+k)} \binom{i}{\#\mathcal{L}} \right| \leq C_{j,k} i^{2j+k}$$

This concludes the proof.  $\square$

**Control of the remainder.** Armed with Lemma 7-6.6 and the iterative formula (7-6.13), we can deduce a control for the remainder term. Let's consider  $B \geq 1$  such that  $\|M_{q,p}\| \leq B$  for all admissible pair  $(q,p)$  (it is possible to take  $B \leq (1+\varepsilon)\|\alpha\|_\infty$ , or even with  $\varepsilon$  going to 0 as  $h \rightarrow 0$ , but the precise value of  $B$  is not relevant for this term). For this reason, we will also get rid of the precise value of  $\Pi_{\alpha,i}$  and assume that  $\|\alpha\|_\infty \leq B$  so that  $\|\Pi_{\alpha,i}\| \leq B^i$ .

Plugging the previous estimates into (7-6.13), we get

$$\begin{aligned} \|r_i^{(N)}\| &\leq B \|r_{i-1}^{(N)}\| + \sum_{2j+k < 2N} h^{j+k/2} \widetilde{C}_{N-j-\lceil k/2 \rceil, \varepsilon} h^{(N-j-\lceil k/2 \rceil)(1-\varepsilon)} \|u_{i-1}^{(j,k)}\| \\ &+ \sum_{2j_1+2j_2+k_1 < 2N} h^N C_{3(2N-2j_1-2j_2-k_1)+M}(\psi_{i-1}) h^{-\varepsilon M} \left( h^{-\varepsilon} \|d_{\hat{\rho}} F^{(i)}\| \right)^3 K_{3k_1, 3k_1+\deg P_0} N_\infty(P_{i-1}^{(j_1, k_1)}) \end{aligned}$$

Recall that  $\|P\Psi_0\|_{L^2} \leq K_{\deg P} N_\infty(P)$  for some family of constants  $K_n$  depending only on  $n$ . By the expression of  $u_i^{(j,k)}$ , we have

$$\|u_i^{(j,k)}\|_{L^2} = \|P_i^{(j,k)}\Psi_0\|_{L^2} \leq K_{3k+\deg P_0} N_\infty(P_i^{(j,k)})$$

We also recall that we can bound  $\|d_{\hat{\rho}} F^{(i)}\|$  by  $C J_{q_0 \dots q_{i-1}}$  for some global constant  $C$ .

$$\begin{aligned} \|r_i^{(N)}\| &\leq B \|r_{i-1}^{(N)}\| + \sum_{2j+k < 2N} h^{N(1-\varepsilon)} \widetilde{C}_{N-j-\lceil k/2 \rceil, \varepsilon} K_{3k} C_{j,k, \varepsilon} B^i i^{2j+k} \left( J_{q_0 \dots q_{i-1}}^u \right)^{3k} \\ &+ \sum_{2j_1+2j_2+k_1 < 2N} h^N C_{N, j, \varepsilon} h^{-M\varepsilon} K_{3k_1} C_{j_1, k_1, \varepsilon} B^i i^{2j_1+k_1} \left( h^{-\varepsilon} \left( J_{q_0 \dots q_{i-1}}^u \right)^3 \right)^{2(N-j_1-j_2)} \end{aligned}$$

Finally, we plug the bound  $J_{q_0 \dots q_{i-1}}^u \leq C_\varepsilon e^{i\lambda_{\max}(1+\varepsilon)}$  into the previous inequality. We can find a constant  $C_{N, \deg P_0, \varepsilon}$  such that

$$\|r_i^{(N)}\| \leq B \|r_{i-1}^{(N)}\| + C_{N, \deg P_0, \varepsilon} B^i i^N e^{6Ni\lambda_{\max}(1+\varepsilon)} h^{N(1-2\varepsilon)} h^{-M\varepsilon}$$

This being valid for all  $1 \leq i \leq n$ , by induction on  $i$ , we find that

$$\begin{aligned} \|r_i^{(N)}\| &\leq \sum_{l=0}^i B^{i-l} \times C_{N, \deg P_0, \varepsilon} B^l l^N e^{6Nl\lambda_{\max}(1+\varepsilon)} h^{N(1-2\varepsilon)} h^{-M\varepsilon} \\ &\leq C_{N, \deg P_0, \varepsilon} B^i \sum_{l=0}^i l^N e^{6Nl\lambda_{\max}(1+\varepsilon)} h^{N(1-2\varepsilon)} h^{-M\varepsilon} \end{aligned}$$

Let  $c_{N, \varepsilon} > 0$  be such that  $\sum_{l=0}^i l^N e^{6Nl\lambda_{\max}(1+\varepsilon)} \leq c_{N, \varepsilon} e^{6Ni\lambda_{\max}(1+\varepsilon)^2}$  for all  $i \in \mathbb{N}$ . This gives, for a constant  $C_{N, \deg P_0, \varepsilon}$ ,

$$\|r_i^{(N)}\| \leq C_{N, \deg P_0, \varepsilon} B^i e^{6Ni\lambda_{\max}(1+\varepsilon)^2} h^{N(1-2\varepsilon)} h^{-M\varepsilon}$$

To conclude, recall that  $n(h) \leq \vartheta_\varepsilon \log 1/h$  with  $\vartheta_\varepsilon = \frac{1-4\varepsilon}{6\lambda_{\max}(1+\varepsilon)^2}$ . Hence, as soon as  $i \leq n(h)$ ,  $e^{6Ni\lambda_{\max}(1+\varepsilon)^2} \leq h^{-N(1-4\varepsilon)}$  and this shows the following lemma

**Lemma 7-6.7.** There exists constants  $C_{N,d,\varepsilon}$  such that for all  $N \in \mathbb{N}$  and for all  $P_0 \in \mathbb{C}[X]$ , we have for all  $1 \leq i \leq n(h)$ ,

$$\|r_{i,P_0}^{(N)}\| \leq C_{N,\deg P_0,\varepsilon} h^{2N\varepsilon} h^{-K} N_\infty(P_0)$$

with  $K = \vartheta_\varepsilon \log B + M\varepsilon$ .

**First consequences.** Since  $N$  can be taken arbitrarily large, we recover the known fact that a wave packet centered at  $\hat{\rho}$  is changed after  $n$  steps into an excited squeezed state centered at  $F^{(n)}(\hat{\rho})$ . The squeezing is governed by the unstable Jacobian along the orbit of  $\rho$ . In particular, we obtain the expected following corollary, which gives the first point in Proposition 7-5.2.

**Corollary 7-6.1.** Let's note  $\mathbf{q} = q_0 \dots q_n \in \mathcal{A}^{n+1}$ . Let  $\hat{\rho} \in \kappa_q(\mathcal{W}_q)$  and let us note  $\rho = \kappa_q^{-1}(\hat{\rho})$ .

- If  $\rho \notin \mathcal{W}_q^-$ , then  $A_{q_n} e^{-tG} B'_{q_n} M_{q_n,q_{n-1}} \dots M_{q_1,q} E_t \varphi_{\hat{\rho}} = O(h^\infty)$ .
- If  $\rho \in \mathcal{W}_q^-$ ,

$$e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q E_t \varphi_\rho = A_{q_n} e^{-tG} B'_{q_n} M_{q_n,q_{n-1}} \dots M_{q_1,q_0} E_t \varphi_\rho + O(h^\infty)$$

with constants independent of  $\mathbf{q}$  and  $\rho$ .

*Proof.* This is a consequence of the previous results and the fact that  $\text{WF}_h(B_{q_n} A_{q_n} e^{-tG} B'_{q_n}) \in \kappa_{q_n}(\mathcal{W}_{q_n})$ .  $\square$

Moreover, we can combine Lemma 7-6.3 (the running index in the formula of Lemma 7-6.3 was  $j$ , it becomes  $l$  in the sum below) and Lemma 7-6.4 to get :

**Proposition 7-6.1.** Assume that  $\hat{\rho} = \kappa_q(\rho) \in \kappa_q(\mathcal{W}_q)$  with  $\rho \in \mathcal{W}_{q_0 \dots q_n}^-$ . Then, for any  $N \in \mathbb{N}$ , we have the following expansion (with  $n = n(h)$ )

$$M_{q_n,q_{n-1}} \dots M_{q_1,q_0} E_t \varphi_{\hat{\rho}} = \sum_{2j+k+l < 2N} h^{j+k/2} h^{l(1/2-\delta)} u_n^{(j,k,l)} + R_n^{(N)}$$

where

$$u_n^{(j,k,l)} = e^{tg(\rho)} T(\hat{\rho}_n) \mathcal{M}(d_{\hat{\rho}} F^{(n)}) \Lambda_h \left( P_n^{(j,k,l)} \Psi_0 \right)$$

$P_n^{(j,k,l)}$  is a polynomial of degree at most  $3k + 2l$  and

$$N_\infty \left( P_n^{(j,k,l)} \right) \leq C_{j,k,l,\varepsilon} n^{2j+k} \Pi_{\alpha,n}(\rho) \left( J_{q_0 \dots q_{n-1}}^u \right)^{3k} h^{-k\varepsilon}$$

Concerning the leading term,  $P_n^{(0,0,0)} = \pi_{\alpha,n}(\rho)$ . Concerning the remainder  $R_n^{(N)}$  we have

$$\|R_n^{(N)}\|_{L^2} \leq C_{N,\varepsilon} h^{-(K+K_0 t)} h^{2N\varepsilon}$$

*Proof.* We simply state  $P_n^{(j,k,l)} = P_{n,P_{t,h}^{(l)}}^{(j,k)}$  which satisfies the required bound for the degree and

$N_\infty$ . Here,  $P_{t,h}^{(l)}$  appears in the expansion of Lemma 7-6.3. Lemma 7-6.3 and Lemma 7-6.4 show that

$$M_{q_n,q_{n-1}} \dots M_{q_1,q_0} E_t \varphi_\rho = \sum_{2j+k+l < 2N} h^{j+k/2} h^{l(1/2-\delta)} u_n^{(j,k,l)} + R_n^{(N)}$$

with  $u_n^{(j,k,l)}$  given by the required formula and

$$R_n^{(N)} = M_{q_n,q_{n-1}} \dots M_{q_1,q_0} r_{2N} + \sum_{l=0}^{2N-1} h^{l(1/2-\delta)} r_{n,P_{t,h}^{(l)}}^{(N-\lfloor l/2 \rfloor)}$$

We can use the bound  $\|r_{2N}\| \leq C_N h^{N(1-2\delta)}$  and the bound for  $r_{n,P}^{(N)}$  in Lemma 7-6.7. Since the degrees of the polynomial  $P_{t,h}^{(l)}$  are bounded by  $4N$ , we can forget the dependence in  $\deg P$  in the estimates of Lemma 7-6.7, so that we find

$$\|R_n^{(N)}\| \leq C_{N,\varepsilon} h^{-K} h^{-K_0 t} \sum_0^{2N} h^{l(1/2-\delta)} h^{2(N-l/2)\varepsilon} \leq C_{N,\varepsilon} h^{-(K+K_0 t)} h^{2N\varepsilon}$$

where the last inequality follows from  $\varepsilon = 1/2 - \delta$ .  $\square$



**Remark.**

This expansion mixes up different scales :

- the scale  $h^{1-2\delta} = h^{2\varepsilon}$  : it comes from the symbol class in which  $g$  lives ;
- a second scale which is the scale  $h^{1/2}$  when  $n$  is independent of  $h$ . In our context, it is better to think this second scale to be  $h^{1/2}(J_{q_0\dots q_n}^u)^3 h^{-\varepsilon}$ . This scale depends on the starting point  $\rho$ . The definition of  $\vartheta_\varepsilon$  ensures that the higher order terms are smaller than the leading term.

Since we can choose  $N$  as large as we want, we can ensure that the remainder decays in  $h$  and that the leading term controls the whole expansion. Note also that the constants  $C_{j,k,l,\varepsilon}$  and  $C_{N,\varepsilon}$  depends on  $g$  and  $M = M_h(z)$  and they can be chosen uniform in  $z \in \Omega(h) \cap \{\text{Im } z \in [-\beta, 4]\}$ .

**7-6.4. Final action of  $A_{q_n} e^{-tG} B'_{q_n}$ .**

From now on, and until the end of the section, we assume that  $\rho \in \mathcal{W}_{\mathbf{q}}^-$  and we prove the missing items of Proposition 7-5.2.

We need to understand the action of  $e^{-tG} B'_{q_n}$  on the terms  $u_n^{(j,k,l)}$  of the last expansion. Since all these terms have the same form, we consider a general polynomial  $P$  of degree  $d$  and want to understand

$$e^{-tG} B'_{q_n} \left( T(\hat{\rho}_n) \mathcal{M} \left( d_{\hat{\rho}} F^{(n)} \right) \Lambda_h(P\Psi_0) \right)$$

It is no more possible to reuse the strategy of Lemma 7-6.3. Indeed, if  $g$  still oscillates on scale  $h^\delta$ ,  $\mathcal{M} \left( d_{\hat{\rho}} F^{(n)} \right) \Lambda_h(P\Psi_0)$  is no more a wave packet in a box of size  $h^{1/2}$ . To see that in a model case, assume that  $d_{\hat{\rho}} F^{(n)}$  is given by the diagonal matrix

$$\begin{pmatrix} \lambda_h & 0 \\ 0 & \lambda_h^{-1} \end{pmatrix}$$

with  $\lambda_h \sim J_{q_0\dots q_{n-1}}^u \sim h^{-\alpha}$  where

$$\lambda_{\min} \vartheta_\varepsilon \leq \alpha \leq \lambda_{\max} \vartheta_\varepsilon = \frac{1-4\varepsilon}{6(1+\varepsilon)^2}$$

Then  $\mathcal{M} \left( d_{\hat{\rho}} F^{(n)} \right)$  is nothing but  $\Lambda_{\lambda_h^2}$  and hence,  $\mathcal{M} \left( d_{\hat{\rho}} F^{(n)} \right) \Lambda_h(P\Psi_0) = \Lambda_{h\lambda_h^2}(P\Psi_0)$ . This states oscillate in the  $x$ -direction on a scale  $h^{1/2-\alpha} \gg h^\delta$ .

**7-6.4.1. Precise description of  $d_{\hat{\rho}} F^{(n)}$ .**

It is not possible to write  $d_{\hat{\rho}} F^{(n)}$  exactly as a diagonal matrix in the standard position/momentum variable. However, the following lemma shows that  $d_{\hat{\rho}} F^{(n)}$  stays close to a diagonal matrix :

**Lemma 7-6.8.** There exists  $\varepsilon_2$  which can be made arbitrarily small depending on  $\varepsilon_0$  such that the following holds. There exists  $\lambda_{n,\mathbf{q}}, \mu_{n,\mathbf{q}} \in \mathbb{R}^+$  such that for all  $n, \mathbf{q} = q_0 \dots q_n$  and  $\hat{\rho} \in \kappa_{\mathbf{q}}(\mathcal{W}_{\mathbf{q}}^-)$ , we have for some global constant  $C > 0$  :

- $C^{-1} J_{\mathbf{q}}^u \leq \lambda_{n,\mathbf{q}} \leq C J_{\mathbf{q}}^u$  ;
- $C^{-1} \leq \mu_{n,\mathbf{q}} \lambda_{n,\mathbf{q}} \leq C$  ;
- $d_{\hat{\rho}} F^{(n)}$  is close to a diagonal matrix :

$$\left\| d_{\hat{\rho}} F^{(n)} - \begin{pmatrix} \lambda_{n,\mathbf{q}} & 0 \\ 0 & \mu_{n,\mathbf{q}} \end{pmatrix} \right\| \leq \varepsilon_2 J_{\mathbf{q}}^u$$

*Proof.* We note  $\rho_i = F^{in_0}(\rho) = \kappa_{q_i}^{-1} \circ F^{(i)}(\hat{\rho})$ . Recall also that  $F^{(i)} = \kappa_{q_i} \circ F^{in_0} \circ \kappa_{q_0}^{-1}$ .

**Step 1 : Reduction to  $\rho \in \mathcal{T}$ .** By definition of  $\mathcal{W}_{q_i}$ , for  $i \in \{0, \dots, n\}$ , we have  $d(\rho_{q_i}, \rho_i) \leq 2\varepsilon_0$ . Hence,

$$d(F^{n_0}(\rho_{q_i}), \rho_{i+1}) \leq d(F^{n_0}(\rho_{q_i}), F^{n_0(i+1)}(\rho)) + d(F^{n_0(i+1)}(\rho), \rho_{i+1}) \leq C\varepsilon_0$$

for a constant  $C$  only depending on  $F$ . That is to say,  $(\rho_0, \dots, \rho_n)$  is a  $C\varepsilon_0$  pseudo orbit for  $F^{n_0}$ . Assume that  $\delta_0 > 0$  is a small fixed parameter. In virtue of the Shadowing Lemma ([HK95], Section 18.1), if  $\varepsilon_0$  is sufficiently small,  $(\rho_0, \dots, \rho_n)$  is  $\delta_0$  shadowed by an orbit of  $F^{n_0}$  i.e. there exists  $\rho' \in \mathcal{T}$  such that for  $i \in \{0, \dots, n\}$ ,  $d(\rho_i, F^{i n_0}(\rho')) \leq \delta_0$ . Consequently,  $d(F^{i n_0}(\rho), F^{i n_0}(\rho')) \leq \delta_0 + C\varepsilon_0$ . For convenience, set  $\varepsilon_2 = \delta_0 + C\varepsilon_0$  and note that  $\varepsilon_2$  can be arbitrarily small depending on  $\varepsilon_0$ . By Lemma 3-2.3, for  $1 \leq i \leq n$ ,

$$\|d_\rho F^{i n_0}\| \leq C J_{i n_0}^u(\rho') \quad ; \quad C^{-1} J_{q_0 \dots q_{i-1}}^u \leq J_{i n_0}^u(\rho') \leq C J_{q_0 \dots q_{i-1}}^u$$

Hence, using the relation

$$d_\rho F^{n n_0} - d_{\rho'} F^{n n_0} = \sum_{k=0}^{n-1} d_{F^{n_0(k+1)}(\rho')} F^{n_0(n-k-1)} \circ (d_{F^{n_0 k}(\rho)} F^{n_0} - d_{F^{n_0 k}(\rho')} F^{n_0}) \circ d_\rho F^{n_0 k}$$

we find that

$$\begin{aligned} \|d_\rho F^{n n_0} - d_{\rho'} F^{n n_0}\| &\leq C \sum_{k=0}^{n-1} J_{n_0(n-k-1)}^u \left( F^{n_0(k+1)}(\rho') \right) \|d_{F^{n_0 k}(\rho)} F^{n_0} - d_{F^{n_0 k}(\rho')} F^{n_0}\| J_{n_0 k}^u(\rho') \\ &\leq C \sum_{k=0}^{n-1} d(F^{n_0 k}(\rho), F^{n_0 k}(\rho')) J_{n n_0}^u(\rho') \\ &\leq C J_{\mathbf{q}}^u \sum_{k=0}^{n-1} \theta^{\min(k, n-k)} \varepsilon_0 \\ &\leq C J_{\mathbf{q}}^u \varepsilon_0 \end{aligned}$$

where we use the Lemma 3-2.1 in the third equality and the last one follows from  $\sum_{k=0}^{n-1} \theta^{\min(k, n-k)} \leq 2 \sum_{k=0, \lceil n/2 \rceil} \theta^k \leq 2 \sum_{k=0}^{\infty} \theta^k < +\infty$ . It is not hard to deduce from this that

$$\|d_{\hat{\rho}} F^{(n)} - d_{\kappa_q(\rho')} F^{(n)}\| \leq C J_{\mathbf{q}}^u \varepsilon_0$$

Hence, it is enough to prove the Lemma for  $d_{\kappa_q(\rho')} F^{(n)}$ .

**Step 2 : The case  $\rho \in \mathcal{T}$ .** We assume that  $\rho \in \mathcal{T}$ . The spaces  $E_u(\rho), E_s(\rho), E_u(F^{n n_0}(\rho))$  and  $E_s(F^{n n_0}(\rho))$  are well-defined. For  $q \in \mathcal{A}$  and  $\bullet = s, u$ , the maps  $\zeta \in \mathcal{W}_q \cap \mathcal{T} \mapsto d_\zeta \kappa_q(E_\bullet(\zeta))$  are Lipschitz. Since  $d_\rho \kappa_q(E_u(\rho_q)) = \mathbb{R} \partial_y$ ,  $d_\rho \kappa_q(E_s(\rho_q)) = \mathbb{R} \partial_\eta$  and  $d(\rho_{q_0}, \rho) \leq C\varepsilon_2$ ,  $d(\rho_{q_n}, F^{n n_0}(\rho)) \leq C\varepsilon_2$ , we can fix unit vectors

$$e_0^u \in d_\rho \kappa_{q_0}(E_u(\rho)) \quad , \quad e_0^s \in d_\rho \kappa_{q_0}(E_s(\rho))$$

$$e_n^u \in d_{F^{n n_0}(\rho)} \kappa_{q_n}(E_u(F^{n n_0}(\rho))) \quad e_n^s \in d_{F^{n n_0}(\rho)} \kappa_{q_n}(E_s(F^{n n_0}(\rho)))$$

such that  $e_0^u, e_n^u = \partial_y + O(\varepsilon_2)$  and  $e_0^s, e_n^s = \partial_\eta + O(\varepsilon_2)$ . If we note  $P_0$  (resp.  $P_n$ ) the change-of-basis matrix from the natural basis of  $\mathbb{R}^2$  to  $(e_0^u, e_0^s)$  (resp.  $(e_n^u, e_n^s)$ ), then  $P_0, P_n = I_2 + O(\varepsilon_2)$  (with global constants in  $O$  not depending on  $n$ ). Moreover, since  $d_{\hat{\rho}} F^{(n)}(e_0^u) \in \mathbb{R} e_n^u$  and  $d_{\hat{\rho}} F^{(n)}(e_0^s) \in \mathbb{R} e_n^s$ , the matrix  $P_n^{-1} d_{\hat{\rho}} F^{(n)} P_0$  is diagonal. Let's write it

$$\begin{pmatrix} \lambda_{n, \mathbf{q}} & 0 \\ 0 & \mu_{n, \mathbf{q}} \end{pmatrix}$$

$\lambda_{n, \mathbf{q}}$  (resp.  $\mu_{n, \mathbf{q}}$ ) is nothing but an unstable (resp. stable) Jacobian for  $\rho$ , and hence  $\lambda_{n, \mathbf{q}} \sim J_{\mathbf{q}}^u$ . Since  $\det d_\rho F^{n n_0} = 1$ ,  $\lambda_{n, \mathbf{q}} \mu_{n, \mathbf{q}} = \det(P_0)^{-1} \det(P_n) = 1 + O(\varepsilon_2)$ . Finally,

$$P_n^{-1} d_{\hat{\rho}} F^{(n)} P_0 = (I_2 + O(\varepsilon_2)) d_{\hat{\rho}} F^{(n)} (I_2 + O(\varepsilon_2)) = d_{\hat{\rho}} F^{(n)} + O\left(\|d_{\hat{\rho}} F^{(n)}\| \varepsilon_2\right) = d_{\hat{\rho}} F^{(n)} + O(\varepsilon_2 J_{\mathbf{q}}^u)$$

This concludes the proof.  $\square$

As a consequence of this lemma, in the standard position/momentum coordinates, we can write

$$d_{\hat{\rho}}F^{(n)} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}; \quad a_n \sim J_{\mathbf{q}}^u; \quad b_n, c_n, d_n = O(\varepsilon_2 J_{\mathbf{q}}^u) \quad (7-6.15)$$

Here,  $a_n, b_n, c_n, d_n$  depend on  $\rho$ , but we won't make this dependence precise since  $\rho$  is fixed until the end of the section. Since we want to understand the action of  $\mathcal{M}(d_{\hat{\rho}}F^{(n)})$  on excited coherent states, we also introduce

$$\gamma_n = (c_n + id_n)(a_n + ib_n)^{-1}; \quad \beta_n = \operatorname{Re}(\gamma_n); \quad \alpha_n = \operatorname{Im}(\gamma_n)^{-1} = |a_n + ib_n|^2 \quad (7-6.16)$$

We've got the basic estimates

$$\alpha_n \sim (J_{\mathbf{q}}^u)^2; \quad \beta_n = O(\varepsilon_2) \quad (7-6.17)$$

Now assume that  $P \in \mathbb{C}[X]$  and decompose  $P$  into the basis of the renormalized hermite polynomials  $(h_n) : P = \sum_{k=0}^{\deg P} a_k(P)h_k$ . By Proposition 7-3.5,

$$\mathcal{M}(d_{\hat{\rho}}F^{(n)}) \Lambda_h(h_k \Psi_0)(x) = (\alpha_n \pi h)^{-1/4} \left( \frac{a_n - ib_n}{a_n + ib_n} \right)^{k/2} h_k \left( \frac{x}{(\alpha_n h)^{1/2}} \right) e^{i\gamma_n \frac{x^2}{2h}} = c_{n,k} \Lambda_{\alpha_n h}(h_k \Psi_0)(x) e^{i\beta_n \frac{x^2}{2h}}$$

with  $|c_{n,k}| = 1$ . As a consequence, there exist linear maps  $\Phi_n : \mathbb{C}[X] \rightarrow \mathbb{C}[X]$  such that for all  $n \in \mathbb{N}$  and  $P \in \mathbb{C}[X]$ ,

- $\deg \Phi_n(P) = \deg P$  for all  $P \in \mathbb{C}[X]$ ;
- $N_{\infty}(\Phi_n(P)) \leq K_{\deg P} N_{\infty}(P)$  where  $K_{\deg P}$  depends only on  $\deg P$ ;
- and the following relation holds

$$\mathcal{M}(d_{\hat{\rho}}F^{(n)}) \Lambda_h(P \Psi_0) = \Lambda_{\alpha_n h}(\Phi_n(P) \Psi_0) e^{i\beta_n \frac{x^2}{2h}} \quad (7-6.18)$$

**Remark.**

This state is written as a Lagrangian state associated with the Lagrangian manifold  $\{(x, \beta_n x)\}$ , with amplitude  $a(x) = \Lambda_{\alpha_n h}(\Phi_n(P) \Psi_0)(x)$ . Since  $\alpha_n \sim (J_{\mathbf{q}}^u)^2$ ,  $\alpha_n \sim h^{-\alpha}$  for some  $\alpha \geq 2\lambda_{\min} \vartheta_{\varepsilon}$ , the amplitude  $a$  oscillates on a scale  $h^{1/2-\alpha/2}$  and this is indeed not a true Lagrangian state. Nevertheless, this scale is larger than the scale  $h^{\delta}$  on which the symbol  $g$  oscillates and when considering the final action of  $e^{tG}$ , we will interpret this state as a Lagrangian state seen from the scale  $h^{\delta}$ . Note that compared with the initial state  $\varphi_0$ , localized in position in an interval of size  $h^{1/2}$ , this expression shows a stretching in position.

**7-6.4.2. Asymptotic expansion for the exponential.**

We now aim at understanding the state  $A_{q_n} e^{-tG} B'_{q_n} u$  where  $u$  is of the form

$$u(x) = T(\hat{\rho}_n) (\Lambda_{\alpha_n h} f)(x) e^{i\beta_n \frac{x^2}{2h}}$$

where  $f = P\Psi$  for some  $P \in \mathbb{C}[X]$ . We first claim that

$$A_{q_n} e^{-tG} B'_{q_n} = A_{q_n} B'_{q_n} e^{-tB_{q_n} G B'_{q_n}} + O(h^{\infty})$$

*Proof.* Set  $A(t) = A_{q_n} e^{-tG} B'_{q_n} e^{tB_{q_n} G B'_{q_n}}$ . At  $t = 0$ ,  $A(0) = A_{q_n} B'_{q_n}$ . We differentiate:

$$\dot{A}(t) = A_{q_n} e^{-tG} [B'_{q_n} B_{q_n} G B'_{q_n} - G B'_{q_n}] e^{tB_{q_n} G B'_{q_n}}$$

The operator  $A_{q_n} e^{-tG}$  is bounded on  $L^2$  and has its semiclassical wavefront set included in  $\operatorname{supp} \chi_{q_n}$ . In particular,  $A_{q_n} e^{-tG} (B'_{q_n} B_{q_n} - \operatorname{Id}) = O(h^{\infty})$  (uniformly for  $t$  in a bounded interval). This shows that  $A'(t) = O(h^{\infty})$ . We conclude that  $A(t) = A_{q_n} B'_{q_n} + O(h^{\infty})$ .  $\square$

Hence we aim at understanding the action of  $e^{-tB_{q_n}GB'_{q_n}}$ . We make use of Lemma 7-6.2 and we write for all  $N \in \mathbb{N}$ ,

$$G_{q_n} := B_{q_n}GB'_{q_n} = \text{Op}_h(g \circ \kappa_{q_n}^{-1}) + \sum_{j=1}^{N-1} h^{j(1-2\delta)} \text{Op}_h(g_{j,q_n}) + R_N$$

with  $\|R_N\| \leq C_N h^{N(1-2\delta)}$ . Let's write  $g_{0,q_n} = g \circ \kappa_{q_n}^{-1}$ . Similarly, we have

$$A_{q_n}B'_{q_n}e^{-tG_{q_n}}T(\hat{\rho}_n) = A_{q_n}B'_{q_n}T(\hat{\rho}_n)e^{-tT(\hat{\rho}_n)^*G_{q_n}T(\hat{\rho}_n)}$$

and we recall that  $T(\hat{\rho}_n)^*\text{Op}_h(a)T(\hat{\rho}_n) = \text{Op}_h(a(\cdot + \hat{\rho}_n))$  for any  $a \in \mathcal{S}'$ . Let's note  $h_j(\hat{\zeta}) = g_{j,q_n}(\hat{\rho}_n + \hat{\zeta})$ , so that

$$A := T(\hat{\rho}_n)^*G_{q_n}T(\hat{\rho}_n) = \sum_{j=0}^{\infty} h^{j(1-2\delta)} \text{Op}_h(h_j) + O_N(h^{N(1-2\delta)})$$

Recall that in virtue of Lemma 7-6.2,  $h_0 \in \log(1/h)S_\delta$  and  $h_j \in S_\delta$  for  $j \geq 1$ .

Finally, we need to understand the action of  $e^{-tA}$  on states  $u(x) = \Lambda_{\alpha_n h} f(x) e^{i\beta_n \frac{x^2}{2h}}$ . We want to apply the formalism of Appendix 7-7.2. with  $H = L^2(\mathbb{R})$  and  $A$ . The class of elements which will interest us is defined as follows : we say that a  $h$ -dependent family of states  $u = u_h \in L^2(\mathbb{R})$  belongs to the class  $\mathcal{C}$  if  $u$  has the form :

$$u(x) = a(x) e^{i\beta_n \frac{x^2}{2h}}$$

where  $a = a_h \in C^\infty(\mathbb{R})$  satisfies : for all  $p \in \mathbb{N}$ , there exists  $C_p$  such that

$$|a^{(p)}(x)| \leq C_p h^{-\delta p} (\alpha_n h)^{-1/4} \left(1 + \frac{x^2}{\alpha_n h}\right)^{-1} \quad (7-6.19)$$

This class depends on  $h$  (and  $n$ , which himself depends on  $h$ ). For such a state  $u$ , we define the natural semi-norms on  $\mathcal{C}$  :

$$q_p(u) = \sup_{k \leq p} \sup_{x \in \mathbb{R}} \left( |a^{(k)}(x)| h^{\delta k} (\alpha_n h)^{1/4} \left(1 + \frac{x^2}{\alpha_n h}\right) \right) \quad (7-6.20)$$

In particular, one has  $\|u\| \leq C q_0(u)$ .

**Remark.**

In fact, the introduction of the semi-norms  $q_j$  with the factor  $(1 + \frac{x^2}{\alpha_n h})^{-1}$  is purely technical : it allows to work in a symbol class depending on this order function (see the proof of Lemma 7-6.10 in the appendix 7-7.1.3.). In the end, we will simply need to estimate the semi-norm  $q_0$  of each term of the expansion of an evolved state  $e^{-tA}u$ , but this will require to control (a finite number of) semi-norms  $q_j$  of the initial state  $u$ . This reason has motivated the introduction of the  $q_j$ 's. We will mainly consider states  $u$  with exponential decay and what is important is that  $\partial^k \Psi_0 \leq C_{k,p} (1+x^2)^{-p/2}$  for all  $k, p \in \mathbb{N}$ .

The following lemma ensures that the states we work with are indeed in  $\mathcal{C}$ , as soon as  $h^{2\delta} \ll \alpha_n h$ . Recall that  $\alpha_n \geq Ch^{-\alpha_{\min}}$  where  $\alpha_{\min} = 2\lambda_{\min} \vartheta_\varepsilon$ . Then, it suffices to require

$$\varepsilon = 1/2 - \delta \leq \alpha_{\min}/2$$

This is clearly not a problem since we want to work with  $\delta = 1/2 - \varepsilon$  very close to  $1/2$  and we assume that this is true, that is, we assume that

$$\varepsilon \leq \alpha_{\min}/2.$$

**Lemma 7-6.9.** Assume that  $u(x) = \Lambda_{\alpha_n h}(P\Psi_0) e^{i\beta_n \frac{x^2}{2h}}$  where  $P \in \mathbb{C}[X]$  has degree  $d$ . Then  $u \in \mathcal{C}$  and for all  $j \in \mathbb{N}$ , there exist constants  $C_{d,j}$  depending only on  $d$  and  $j$  such that  $q_j(u) \leq C_{d,j} N_\infty(P)$

*Proof.*

$$\begin{aligned} \left| (\Lambda_{\alpha_n h} (P\Psi_0))^{(j)}(x) \right| &= \left| (\alpha_n h)^{-j/2} \Lambda_{\alpha_n h} ((P\Psi_0)^{(j)})(x) \right| \\ &\leq h^{-\delta j} (\pi h \alpha_n)^{-1/4} \left| (P_j \Psi_0)((\alpha_n h)^{-1/2} x) \right| \end{aligned}$$

Here, we use that  $\alpha_n h \gg h^{2\delta}$ . and  $P_j$  is a polynomial which depends linearly on  $P$ , with  $\deg P_j = \deg P + j$  and  $N_\infty(P_j) \leq C_{d,j} N_\infty(P)$ . Hence, we have

$$\begin{aligned} q_k(u) &\leq \sup_{j \leq k} \sup_{x \in \mathbb{R}} \left| (P_j \Psi_0)((\alpha_n h)^{-1/2} x) \right| \left( 1 + \frac{x^2}{\alpha_n h} \right) \\ &\leq \sup_{j \leq k} \sup_{x \in \mathbb{R}} \left| (P_j \Psi_0)(x) \right| (1 + x^2) \\ &\leq \sup_{j \leq k} C_{d,j} N_\infty(P_j) \leq C_{d,k} N_\infty(P) \end{aligned}$$

□

To apply the formalism of Appendix 7-7.2., we will require the following lemma. This a more or less direct application of the stationary phase theorem in the quadratic case. We write its proof in appendix 7-7.1.3.. This lemma explains how to compute  $\text{Op}_h(m)u$  for  $u \in \mathcal{C}$  and  $m \in S_\delta$ .

**Lemma 7-6.10.** There exists  $M > 0$  such that the following holds. Assume that  $m \in S_\delta$  or  $m = h_0$ . Then, for all  $k \in \mathbb{N}$ , there exists  $A_k(m) : \mathcal{C} \rightarrow \mathcal{C}$  such that for  $u \in \mathcal{C}$ , written under the form  $u(x) = a(x)e^{i\beta_n \frac{x^2}{2k}}$

- $A_0(m)u(x) = m(x, \beta_n x)u(x)$ ;
- For  $k \geq 1$ ,  $A_k(m)$  is of the form

$$A_k(m)u(x) = \sum_{l \leq 2k} c_l(x) \partial_x^l a(x) e^{i\beta_n \frac{x^2}{2k}} \quad (7-6.21)$$

where  $|c_l^{(p)}(x)| \leq C_{l,k,p} h^{(l-p)\delta}$ .

- For all  $(j, k) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ , there exists  $c_{j,k} > 0$  such that for all  $u \in \mathcal{C}$ ,  $q_j(A_k(m)u) \leq c_{j,k} q_{2k+j}(u)$ ;
- For all  $N \in \mathbb{N}^*$  and for all  $j \in \mathbb{N}$ , there exists  $C_{j,N} > 0$  such that

$$q_j \left( \text{Op}_h(m)u - \sum_{k=0}^{N-1} h^{k(1-2\delta)} A_k(m)u \right) \leq C_{j,N} q_{j+2N+M}(u) h^{N(1-2\delta)}$$

**Remark.**

We need to distinguish the cases  $m = h_0$  and  $m \in S_\delta$  because  $h_0$  is not in  $S_\delta$  (recall that we only have  $h_0 = O(\log(1/h))$ ). However,  $h_0$  satisfies  $|\partial^\alpha h_0| \leq C_\alpha h^{-|\alpha|\delta}$  as soon as  $|\alpha| \geq 1$ . This explains why we restrict on  $(j, k) \neq (0, 0)$  in the third item but in the case  $m \in S_\delta$ , the expression given in the first item shows that it also holds for  $(j, k) = (0, 0)$ .

Gathering the terms of same order in the expansions of each  $\text{Op}_h(h_k)$  given by Lemma 7-6.10, we can build the family of operators

$$A_k : \mathcal{C} \rightarrow \mathcal{C} ; A_k = \sum_{j+l=k} A_j(h_l).$$

Each  $A_k$  has the same form as (7-6.21) and they satisfy, for all  $u \in \mathcal{C}$ ,

- $A_0 u(x) = h_0(x, \beta_n x)u(x)$ .

- For all  $(j, k) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ , there exists  $c_{j,k} > 0$  such that for all  $u \in \mathcal{C}$ ,  $q_j(A_k u) \leq c_{j,k} q_{2k+j}(u)$ ;
- For all  $N \in \mathbb{N}^*$  and for all  $j \in \mathbb{N}$ , there exists  $C_{j,N} > 0$  such that

$$q_j \left( Au - \sum_{k=0}^{N-1} h^{k(1-2\delta)} A_k u \right) \leq C_{j,N} q_{j+2N+M}(u) h^{N(1-2\delta)}$$

We now use the formulas and notations of Appendix 7-7.2. to show :

**Proposition 7-6.2.** Assume that  $P \in \mathbb{C}[X]$  is of degree  $d$  and consider the state  $u = \Lambda_{\alpha_n h}(P\Psi_0) e^{\frac{i\beta_n x^2}{2h}}$ . Then,  $t$  being fixed, there exists a family of functions  $(f_k)$  and  $K_1 > 0$  such that,

- $v_0(x) = u(x)$  ;
- For all  $N \in \mathbb{N}^*$ , there exists  $C_{N,d}$  such that

$$\left\| A_{q_n} \left( e^{-tG} B'_{q_n} T(\hat{\rho}_n) u - \sum_{k=0}^{N-1} h^{k(1-2\delta)} B'_{q_n} T(\hat{\rho}_n) u_k \right) \right\| \leq C_{N,d} h^{N(1-2\delta)} h^{-tK_1} N_\infty(P)$$

where

$$u_k(x) = \exp(-th_0(x, \beta_n x)) v_k(x) ; v_k(x) = f_k(x) (\Lambda_{\alpha_n h} \Psi_0)(x) e^{\frac{i\beta_n x^2}{2h}}$$

- For all  $k \in \mathbb{N}$ , there exists  $c_{k,d} > 0$  such that for all  $x \in \mathbb{R}$ ,

$$|f_k(x)| \leq c_{k,d} \left( 1 + \frac{x^2}{\alpha_n h} \right)^{k/2} N_\infty(P)$$

**Remark.**

| In particular, these last estimates imply that  $v_k \in \mathcal{C}$ .

*Proof.* We use the notations and formulas of Appendix 7-7.2., with parameter  $\tilde{h} = h^{1-2\delta}$ . We define a family  $(v_k(t))$  by the iterative formula 7-7.3. The operator  $A_0$  is nothing but the multiplication by

$$a_0(x) = h_0(x, \beta_n x)$$

and hence,  $e^{sA_0}$  is the multiplication by  $\exp(sa_0)$ .

Let us note  $A_k(s) = e^{-sA_0} A_k e^{sA_0}$  and let us show that  $A_k(s)u(x)$  has the same form as (7-6.21), with the functions  $c_l(x)$  replaced by functions  $\tilde{c}_l(s, x)$ . We have

$$\begin{aligned} e^{-sa_0} c_l(x) \partial_x^l (e^{sa_0} a(x)) &= c_l(x) \sum_{m=0}^l \binom{l}{m} a_m(s, x) a^{(l-m)}(x) \\ &= \sum_{m=0}^l \tilde{c}_{l,m}(s, x) \partial_x^m a(x) \end{aligned}$$

where  $a_m(s, x) = e^{-sa_0} \partial_x^m (e^{sa_0})$  is a sum of terms of the form

$$s^i \prod_{j=1}^i a_0^{(k_j)} \text{ with } (k_1, \dots, k_i) \in (\mathbb{N}^*)^i \text{ and } k_1 + \dots + k_i = m$$

and  $\tilde{c}_{l,m}(s, x) = \binom{m}{l} c_l(x) a_{l-m}(s, x)$ . It is not hard to see that  $|\partial_x^p a_m(s, x)| \leq C_{m,p} (1+|s|)^m h^{-\delta(m+p)}$  so that we have

$$\begin{aligned} |\partial_x^p \tilde{c}_{l,m}(x)| &= \binom{m}{l} \left| \sum_{p_1+p_2=p} \binom{p}{p_1} c_l^{(p_1)}(x) a_{l-m}^{(p_2)}(s, x) \right| \\ &\leq C_{p,l,m} \sup_{p_1+p_2=p} h^{\delta(l-p_1)\delta} h^{-(l-m+p_2)\delta} \leq C_{p,l,m} h^{(m-p)\delta} \end{aligned}$$

which shows that the term in front of  $\partial_x^m$  has the correct behavior to be of the form (7-6.21) and we can set  $\tilde{c}_m(s, x) = \sum_{l \leq 2k} \tilde{c}_{l,m}(s, x)$  so that

$$A_k(s)u(x) = \sum_{m \leq 2k} \tilde{c}_m(s, x) \partial_x^m \left( u e^{-i\beta_n x^2/2h} \right) e^{i\beta_n x^2/2h}$$

Let us now analyze the action of  $A_k(s)$  on states of the form  $c(x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n \frac{x^2}{2h}}$ . We claim that we can write

$$A_k(s) \left( c(x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n \frac{x^2}{2h}} \right) = d_k(s, x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n \frac{x^2}{2h}} \quad (7-6.22)$$

where

$$|\partial_x^p d_k(s, x)| \leq C_{k,p} h^{-p\delta} (1 + |s|)^k \sup_{y \in \mathbb{R}, m \leq 2k+p} |c^{(m)}(y)| \left( 1 + \frac{x^2}{\alpha_n h} \right)^k. \quad (7-6.23)$$

To see that, let us write

$$\begin{aligned} \tilde{c}_m(s, x) \partial_x^m (c(x)\Lambda_{\alpha_n h}(\Psi_0)) &= \tilde{c}_m(s, x) \sum_{l=0}^m \binom{m}{l} c^{(m-l)}(x) (\Lambda_{\alpha_n h}(\Psi_0))^{(l)}(x) \\ &= \underbrace{\left( \sum_{l=0}^m \binom{m}{l} \tilde{c}_m(s, x) c^{(m-l)}(x) \frac{Q_l((\alpha_n h)^{-1/2} x)}{(\alpha_n h)^{l/2}} \right)}_{C_m(s, x)} \Lambda_{\alpha_n h}(\Psi_0)(x) \end{aligned}$$

where  $Q_l \in \mathbb{R}[X]$  are some polynomials of degree  $l$ . We hence have,

$$\begin{aligned} \left| \partial_x^{(p)} C_m(s, x) \right| &= \left| \partial_x^p \left( \sum_{m_1+m_2=m} \binom{m}{m_1} \tilde{c}_m(s, x) c^{(m_1)}(x) \frac{Q_{m_2}((\alpha_n h)^{-1/2} x)}{(\alpha_n h)^{m_2/2}} \right) \right| \\ &\leq C_{m,p} \sup_{\substack{m_1+m_2=m \\ p_1+p_2+p_3=p}} |\partial_x^{p_1} \tilde{c}_m(s, x)| |c^{(m_1+p_2)}(x)| (\alpha_n h)^{-m_2/2-p_3/2} |Q_{m_2}^{(p_3)}((\alpha_n h)^{-1/2} x)| (1 + |s|)^m \\ &\leq C_{m,p} \sup_{\substack{m_1+m_2=m \\ p_1+p_2+p_3=p}} h^{(m-p_1)\delta} |c^{(m_1+p_2)}(x)| (\alpha_n h)^{-m_2/2-p_3/2} (1 + |s|)^m \left( 1 + \frac{x^2}{\alpha_n h} \right)^{(m_2-p_3)/2} \\ &\leq C_{m,p} \sup_{y \in \mathbb{R}, l \leq m+p} |c^{(l)}(y)| \sup_{\substack{m_1+m_2=m \\ p_1+p_2+p_3=p}} h^{(l-p_1)\delta} h^{-(m_2+p_3)\delta} (1 + |s|)^m \left( 1 + \frac{x^2}{\alpha_n h} \right)^{m/2} \\ &\leq C_{m,p} \sup_{y \in \mathbb{R}, l \leq m+p} |c^{(l)}(y)| h^{-\delta p} (1 + |s|)^m \left( 1 + \frac{x^2}{\alpha_n h} \right)^{m/2} \end{aligned}$$

and the claim is proved, with  $d_k(s, x) = \sum_{m \leq 2k} C_m(s, x)$ .

We now analyze precisely the iteration formula (7-7.3) in Appendix 7-7.2.. We use the notations of this appendix (in particular, for the remainders  $\tilde{r}_{j,N}$  and  $R_N$ ). Let  $K_0 > 0$  be such that  $|h_0| \leq K_0 \log 1/h$  so that  $|e^{ta_0}| \leq h^{-tK_0}$ . For  $j \geq 0$ , we have

$$q_j(e^{ta_0} u) \leq h^{-|t|K_0} c_{0,j} q_j(u)$$

This is obvious for  $j = 0$ . For  $j \geq 1$ , it comes from the fact that the derivatives of  $h_0$  satisfy  $|\partial^\alpha h_0| \leq C_\alpha h^{-|\alpha|\delta}$  for  $\alpha \neq 0$  and the definition of  $q_j$  in (7-6.20).

**Leading term.** For our leading term in the expansion we want in Proposition 7-6.2, we simply have

$$u_0(t) = e^{ta_0} u$$

As a consequence,

$$q_j(\tilde{r}_{0,N}(t)) \leq C_{j,N} \tilde{h}^N q_{j+2N+M}(e^{ta_0} u) \leq C_{j,N} \tilde{h}^N c_{0,j+2N+M} h^{-|t|K_0} q_{j+2N+M}(u) \leq C_{d,j,N} \tilde{h}^N h^{-|t|K_0} N_\infty(P)$$

**Iteration.** By induction, using the formulas (7-7.3) and (7-6.22), we see that if the initial state is  $u = \Lambda_{\alpha_n h}(P\Psi_0)e^{i\beta_n \frac{x^2}{2h}}$  then

$$v_k(t) = f_k(t, x)\Lambda_{\alpha_n h}(\Psi_0)(x)e^{i\beta_n \frac{x^2}{2h}}$$

where  $|\partial_x^p f_k(s, x)| \leq C_{k,p} h^{-p\delta} (1 + |s|)^k \left(1 + \frac{x^2}{\alpha_n h}\right)^{k/2} N_\infty(P)$ . When  $p = 0$ , it gives the required estimate for  $|f_k(x)|$  in Proposition 7-6.2. It follows that

$$q_j(e^{tA_0} v_k(t)) \leq h^{-|t|K_0} c_{0,j} q_j(v_k(t)) \leq h^{-|t|K_0} c_{0,j} (1 + |t|)^k h^{-|t|K_0} \tilde{C}_{d,j,k} (1 + |t|)^k N_\infty(P)$$

Moreover, we can estimate

$$q_j(\tilde{r}_{k,N}(t)) \leq C_{j,N-k} \tilde{h}^{N-k} q_{j+2(N-k)+M}(e^{tA_0} v_k(t)) \leq C_{d,N,j,k} \tilde{h}^{N-k} h^{-|t|K_0} (1 + |t|)^k N_\infty(P)$$

**Conclusion** We find that for  $j \in \mathbb{N}$ ,

$$\begin{aligned} q_j \left( \sum_{k=0}^{N-1} \tilde{h}^k \tilde{r}_{k,N}(t) \right) &\leq \tilde{h}^N h^{-|t|K_0} \sum_{k=0}^{N-1} C_{d,N,j,k} (1 + |t|)^k N_\infty(P) \\ &\leq C_{d,j,N} \tilde{h}^N h^{-|t|K_0} (1 + |t|)^{N-1} N_\infty(P) \end{aligned}$$

Integrating (7-7.4), and recalling that  $\|\cdot\| \leq Cq_0$  in  $\mathcal{C}$ , we have

$$\|R_{N-1}(t)\| \leq \int_0^{|t|} \|A\| \|R_{N-1}(s)\| ds + C_{d,0,N} \tilde{h}^N h^{-|t|K_0} (1 + t^2)^{N-1} N_\infty(P)$$

By a version of Gronwall's lemma, we can find a constant  $C_{N,d}$  such that

$$\|R_{N-1}(t)\| \leq C_{N,d} \tilde{h}^N e^{|t| \max(K_0 \log h, \|A\|)} (1 + t^2)^N N_\infty(P)$$

Since,  $\|A\| = O(\log h)$ , there exists  $K_1 > 0$  such that  $e^{|t| \max(K_0 \log h, \|A\|)} \leq h^{-|t|K_1}$  and it concludes the proof of Proposition 7-6.2.  $\square$

Combining Proposition 7-6.1, (7-6.18) and Proposition 7-6.2, we deduce the following expansion.

**Corollary 7-6.2.**

$$\begin{aligned} A_{q_n} e^{-tG} \mathfrak{M}^{n-1} M^{n_0} A_q B'_q \varphi_{\hat{\rho}} &= A_{q_n} \sum_{2j+k+l+2m < 2N} h^{j+k/2+l\varepsilon+m\varepsilon} u_n^{(j,k,l,m)} \\ &\quad + O \left( h^{K-t(K_0+K_1)} h^{2N\varepsilon} (\log h)^N \right) \end{aligned} \quad (7-6.24)$$

where

$$T(\hat{\rho}_n)^* u_n^{(j,k,l,m)}(x) = e^{tg(\rho) - th_0(x, \beta_n x)} f_n^{(j,k,l,m)}(x) \Lambda_{\alpha_n h}(\Psi_0)(x) e^{\frac{i\beta_n x^2}{2h}}$$

where we have, for all  $x \in \mathbb{R}$ ,

$$\left| f_n^{(j,k,l,m)}(x) \right| \leq C_{j,k,l,m} n^{2j+k} \Pi_{\alpha,n}(\rho) (J_{\mathbf{q}}^u)^{3k} h^{-k\varepsilon} \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2}$$

Concerning the leading term,  $f_n^{(0,0,0,0)}$  is constant equal to  $\pi_{\alpha,n}(\rho)$ .

*Proof.* In the expansion of Proposition 7-6.1, we transform the states  $u_n^{(j,k,l)}$  using formula (7-6.18). Finally we use Proposition 7-6.2 on each such state. For  $u_n^{(j,k,l)}$ , we keep the  $N_{j,k,l}$  first terms of the expansion, where  $N_{j,k,l} = N - j - \lceil (k+l)/2 \rceil$ . It gives a remainder term  $r_n^{(j,k,l)}$  satisfying

$$\begin{aligned} \|r_n^{(j,k,l)}\|_{L^2} &\leq C_{N,j,k,l} h^{j+k/2+l\varepsilon} h^{2\varepsilon N_{j,k,l}} N_\infty \left( \Phi_n \left( P_n^{(j,k,l)} \right) \right) h^{-t(K_0+K_1)} \\ &\leq C_{N,j,k,l} h^{j+k/2+l\varepsilon} h^{2\varepsilon N_{j,k,l}} h^{-k\varepsilon} n^{2j+k} \Pi_{\alpha,n}(\rho) (J_{\mathbf{q}}^u)^{3k} h^{-t(K_0+K_1)} \\ &\leq C_{N,j,k,l} h^{j+k/2+l\varepsilon} h^{2\varepsilon N_{j,k,l}} n^{2j+k} h^{-k\varepsilon} h^{-k \frac{1-4\varepsilon}{2(1+\varepsilon)}} h^{-K-t(K_0+K_1)} \\ &\leq C_{N,j,k,l} (\log 1/h)^N h^{2\delta j} h^{2\varepsilon N} h^{k(1/2-2\varepsilon-\frac{1-4\varepsilon}{2(1+\varepsilon)})} h^{-K-t(K_0+K_1)} \end{aligned}$$



But we have  $\frac{1}{2} - 2\varepsilon - \frac{1-4\varepsilon}{2(1+\varepsilon)} \geq 0$  (assuming that  $\varepsilon \leq 1/4$ , which is not a problem since we work with  $\varepsilon$  small). Hence,  $\|r_n^{(j,k,l)}\| \leq C_{N,j,k,l} (\log 1/h)^N h^{-K-t(K_0+K_1)}$ . As a consequence, gathering all the remainders  $r_n^{(j,k,l)}$  together and adding them to  $e^{-tG} B'_{q_n} R_n^{(N)}$ , we obtain a remainder term controlled by  $C_N h^{-K-t(K_0+K_1)} h^{2N\varepsilon} (\log 1/h)^N$  as expected.  $\square$

### 7-6.5. Crucial estimates for the terms of the expansion.

In the expansion of Corollary 7-6.2, the leading term is given by

$$u_n^0 := T^*(\hat{\rho}_n) u_n^{(0,0,0)} = \exp(tg(\rho) - th_0(x, \beta_n x)) \Lambda_{\alpha_n h}(\Psi_0)(x) e^{\frac{i\beta_n x^2}{2h}} \pi_{\alpha_n}(\rho)$$

As a consequence of the Corollary 7-6.2, the other terms have the form

$$T^*(\hat{\rho}_n) = u_n^{(j,k,l,m)}(x) = f_n^{(j,k,l,m)}(x) \frac{u_n^0(x)}{\pi_{\alpha_n}(\rho)}$$

with

$$\left| f_n^{(j,k,l,m)}(x) \right| \leq C_{j,k,l,m} n^{2j+k} (J_{q_0 \dots q_n}^u)^{3k} \Pi_{\alpha_n}(\rho) h^{-k\varepsilon} \left( 1 + \frac{x^2}{\alpha_n h} \right)^{m/2}$$

so that, denoting

$$v_n = \frac{u_n^0}{\pi_{\alpha_n}} = \exp(tg(\rho) - th_0(x, \beta_n x)) \Lambda_{\alpha_n h}(\Psi_0)(x) e^{\frac{i\beta_n x^2}{2h}}$$

we have

$$\frac{\left\| h^{j+k/2+l\varepsilon+m\varepsilon} u_n^{(j,k,l,m)} \right\|_{L^2}}{\Pi_{\alpha_n}(\rho) \times \left\| \left( 1 + \frac{x^2}{\alpha_n h} \right)^{m/2} v_n \right\|_{L^2}} \leq C_{j,k,l,m} h^{j+k/2+l\varepsilon+m\varepsilon} n^{2j+k} (J_{q_0 \dots q_n}^u)^{3k} h^{-k\varepsilon} \quad (7-6.25)$$

Recalling that  $(J_{q_0 \dots q_n}^u)^{3k} h^{-k\varepsilon} \leq C_k e^{3kn\lambda_{\max}(1+\varepsilon)} h^{-k\varepsilon} \leq C_k h^{-k\frac{1-4\varepsilon}{2(1+\varepsilon)} - \varepsilon} \ll C_k h^{-k/2}$ , we see that the right hand side in the above inequality tends to 0 when  $h \rightarrow 0$ . As a consequence, it is enough to control

$$\left\| \left( 1 + \frac{x^2}{\alpha_n h} \right)^{m/2} v_n \right\|_{L^2}$$

This is what we do in the rest of this subsection.

#### 7-6.5.1. Reduction to a compact interval.

Note that since  $\text{WF}_h(A_q)$  is compact, there exists  $\chi \in C_c^\infty(\mathbb{R})$  such that

$$A_q B'_q = A_q B'_q \chi(x) + O(h^\infty)$$

and it is possible to choose a single  $\chi$  for all the  $A_q$ . Indeed, recall that  $\text{WF}_h(A_q) = \text{supp}(\chi_q) \Subset \mathcal{W}_q \subset B(\rho_q, 2\varepsilon_0)$  and that  $\kappa_q$  is well-defined in a neighborhood of  $\rho_q$  of fixed size  $\varepsilon_1$  bigger than  $\varepsilon_0$ . There exists a  $\Xi_q \in \Psi_0(\mathbb{R}^2)$  such that

$$A_q B'_q \Xi_q = A_q B'_q + O(h^\infty) \quad ; \quad \text{WF}_h(\Xi_q) \Subset \kappa_q(\mathcal{W}_q)$$

In particular,  $\text{diam}(\text{WF}_h(\Xi_q)) = O(\varepsilon_0)$ . Hence, it is enough to fix  $\chi \in C_c^\infty(\mathbb{R})$  such that  $\chi = 1$  in a neighborhood of  $\pi_x(\text{WF}_h(\Xi_q))$  for all  $q \in \mathcal{A}$  and such that  $\text{supp} \chi \subset [-C\varepsilon_0, C\varepsilon_0]$  for some large constant  $C$  independent of  $\varepsilon_0$ . As a consequence, we focus on  $\chi v_n$ .

We set

$$\zeta_n(x) = \kappa_{q_n}^{-1}(\hat{\rho}_n + (x, \beta_n x)) \in \mathcal{W}_{q_n} \quad (7-6.26)$$

It describes a curve, preimage by  $\kappa_{q_n}$  of the line  $\hat{\rho}_n + (x, \beta_n x)$ . To ensure that  $\zeta_n(x)$  is well defined,  $\hat{\rho}_n + (x, \beta_n x)$  has to be at distance at most  $\varepsilon_1$  of  $\kappa_{q_n}(\rho_{q_n}) = 0$ . We claim that we may choose  $\varepsilon_0$  small enough so that

$$x \in \text{supp}(\chi) \implies \zeta_n(x) \text{ is well defined} \quad (7-6.27)$$

Indeed,  $x \in \text{supp} \chi \implies |x| \leq C\varepsilon_0$ , so that  $d(\hat{\rho}_n, \kappa_{q_n}(\rho_{q_n})) = O(\varepsilon_0)$  and we choose  $\varepsilon_0$ , ensuring the good definition of  $\zeta_n(x)$ .

**7-6.5.2. Control of the norm of  $v_n$ .**

Our goal is to control the norm of  $\chi v_n$ , which allows to control the leading term. In fact, as already explained, to control the higher order terms, it is also necessary to control the norm of  $\chi \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} v_n$ . Let us note  $\widetilde{\Psi}_m(x) = \pi^{-1/4}(1+x^2)^{m/2}e^{-x^2/2}$ .

$$\begin{aligned} \left\| \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n \right\|_{L^2}^2 &= \int_{\mathbb{R}} |\chi(x)|^2 e^{2tg(\rho) - 2tg(\zeta_n(x))} |\Lambda_{\alpha_n h}(\widetilde{\Psi}_m)(x)|^2 dx \\ &= \int_{\mathbb{R}} \left| \chi((\alpha_n h)^{1/2} x) \right|^2 e^{2tg(\rho) - 2tg(\zeta_n((\alpha_n h)^{1/2} x))} (1+x^2)^m e^{-x^2} dx \end{aligned}$$

We have

**Lemma 7-6.11.**

$$\mathbb{1}_{|x| \geq \alpha_n^{1/2} h^{\delta_0}} \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi(x) v_n(x) = O(h^\infty)_{L^2}$$

The constants in  $O(h^\infty)$  depend on  $m$  and  $\varepsilon$ , but neither on  $\rho$  nor  $n$  as soon as  $n \sim \vartheta_\varepsilon \log 1/h$ .

*Proof.* Since  $g = O(\log h)$ , we have

$$e^{2tg(\rho) - 2tg(\zeta_n((\alpha_n h)^{1/2} x))} \leq \exp(O(\log(h))) = O(h^{-C})$$

for some  $C$  depending on  $t$  and  $g$ . We also have  $|\chi| = O(1)$ . Hence, after a change of variable  $x = \alpha_n^{1/2} h^{1/2} y$ , it suffices to estimate

$$\int_{\mathbb{R}} \mathbb{1}_{[-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]} (\alpha_n^{1/2} h^{1/2} y) (1+y^2)^m e^{-y^2} dy = \int_{|y| \geq h^{\delta_0 - 1/2}} (1+y^2)^m e^{-y^2} dy$$

Since  $\delta_0 < 1/2$ , we conclude by using the standard estimate

$$\int_{|y| > \lambda} (1+y^2)^m e^{-y^2} dy = O_m(\lambda^{-\infty})$$

□

A very important consequence of this lemma is that we only need to focus on  $\zeta_n(x)$  where  $|x| \leq \alpha_n^{1/2} h^{\delta_0}$ . In particular,

$$|x| \leq \alpha_n^{1/2} h^{\delta_0} \implies d(\zeta_n(x), \rho_n) \leq C \alpha_n^{1/2} h^{\delta_0}$$

Recall that  $\alpha_n^{1/2} \leq C e^{n \lambda_{\max}(1+\varepsilon)} \leq C h^{-\frac{1-4\varepsilon}{6(1+\varepsilon)}}$  and recall that  $\delta_0$  is such that

$$\delta_0 - \frac{1-4\varepsilon}{6(1+\varepsilon)} \geq \frac{1}{3} \quad (7-6.28)$$

ensuring that  $\alpha_n^{1/2} h^{\delta_0} \leq C h^{1/3}$ . It will turn out to be important.

**7-6.5.3. Control outside an  $h^\delta$ -neighborhood of  $\mathcal{T}$ .**

We first treat the case where  $\rho$  lies outside an  $h^\delta$ -neighborhood of  $\mathcal{T}$  (in fact, we will be slightly less precise in the unstable direction, see the Lemmata below). The following estimate strongly relies on the structure of the escape function  $g$ . The escape property of  $g$  has been used in [NSZ14] to damp the symbol of the Fourier integral operator  $M(h)$  and they prove that the norm of  $M_t(h)$  outside an  $h^\delta$ -neighborhood of  $\mathcal{T}$  can be smaller than any arbitrary  $\varepsilon$  as soon as  $t$  is well chosen. Here, we want to obtain strong polynomial decay of the form  $h^L$  for some  $L = L(t)$  as large as we want if  $t$  is sufficiently large. This will be possible since we propagate on logarithmic times  $n(h)$ .

We are interested in controlling the term

$$d(x) := \exp(tg(\rho) - tg(\zeta_n(x)))$$

which controls the norm of  $\chi v_n$ . Indeed, since  $\|\widetilde{\Psi}_m\|_\infty < +\infty$ , we have

$$\begin{aligned} \left\| \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n \right\|_{L^2}^2 &\leq \int_{|x| \leq \alpha_n^{1/2} h^{\delta_0}} d(x)^2 |\Lambda_{\alpha_n h} \widetilde{\Psi}_m(x)|^2 dx + O(h^\infty) \\ &\leq C_m (\alpha_n h)^{-1/2} \int_{|x| \leq \alpha_n^{1/2} h^{\delta_0}} d(x)^2 dx + O(h^\infty) \end{aligned}$$

In virtue of the construction of  $g$  in (7-4.6), we have

$$d(x) = (R_-(x)R_+(x))^t, \quad R_-(x) := \frac{Mh^{2\delta} + \hat{\varphi}_-(\rho)}{Mh^{2\delta} + \hat{\varphi}_-(\zeta_n(x))}, \quad R_+(x) := \frac{Mh^{2\delta} + \hat{\varphi}_+(\zeta_n(x))}{Mh^{2\delta} + \hat{\varphi}_+(\rho)} \quad (7-6.29)$$

(These terms depend on  $\rho$  and  $h$ , but we voluntarily omit this dependence to alleviate the notations). Recall also that  $\rho_n = F^{n\alpha_0}(\rho) = \kappa_{q_n}^{-1}(\hat{\rho}_n) = \zeta_n(0)$ .

**Proposition 7-6.3. Estimates for  $R_-$ .** There exists a global constant  $C > 0$  (i.e. depending only  $F$  and  $\varepsilon$  through the choice of the partition of unity, but independent of  $\rho$ ,  $h$  and  $\mathbf{q}$ ) such that for all  $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$ , we have

- If  $d(\rho, \mathcal{T}_-) \geq h^\delta$ ,  $R_-(x) \leq C (J_{\mathbf{q}}^u)^{-2}$  ;
- If  $d(\rho, \mathcal{T}_-) \leq h^\delta$ ,  $R_-(x) \leq C$ .

*Proof.* We pick  $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$ .

- We assume that  $d(\rho, \mathcal{T}_-) \geq h^\delta$ . By Lemma 7-5.2 and (7-6.17),  $d(\rho_n, \mathcal{T}_-) \geq C^{-1} J_{\mathbf{q}}^u d(\rho, \mathcal{T}_-) \geq C^{-1} \alpha_n^{1/2} h^\delta$ . Then, we have

$$d(\zeta_n(x), \mathcal{T}_-) \geq d(\rho_n, \mathcal{T}_-) - d(\zeta_n(x), \rho_n) \geq C^{-1} \alpha_n^{1/2} d(\rho, \mathcal{T}_-) - C \alpha_n^{1/2} h^{\delta_0} \geq \tilde{C}^{-1} \alpha_n^{1/2} d(\rho, \mathcal{T}_-)$$

since  $\delta < \delta_0$  and  $h^\delta \leq d(\rho, \mathcal{T}_-)$ . Recall that  $\hat{\varphi}_-(\zeta) \sim h^{2\delta} + d(\zeta, \mathcal{T}_-)^2$ . Hence,

$$R_-(x) \leq \frac{C_1 d(\rho, \mathcal{T}_-)^2}{C_2 \alpha_n d(\rho, \mathcal{T}_-)^2} \leq C \alpha_n^{-1} \leq C (J_{\mathbf{q}}^u)^{-2}$$

- The second point is much easier (and in fact very crude at this stage) : if  $d(\rho, \mathcal{T}_-) \leq h^\delta$ , the numerator  $Mh^{2\delta} + \hat{\varphi}_-(\rho)$  is smaller than  $Ch^{2\delta}$ . Concerning the denominator, we simply use the fact that  $\hat{\varphi}_- \geq 0$  to bound it from below by  $Mh^{2\delta}$ , and we deduce that  $R_-(x) \leq C$ .

□

We now come to the case of  $R_+$ . Before, we need to understand more precisely the Lagrangian space  $\{(x, \beta_n x)\}$ . We expect it to be a good first order approximation of an unstable manifold. This is the content of the following lemma :

**Lemma 7-6.12.** There exists a global constant  $C > 0$  such that the following holds : there exists  $\zeta_n^* \in \mathcal{T}$  such that for all  $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$ , we have

$$d(\zeta_n(x), \mathcal{T}_+) \leq d(\zeta_n(x), W_u(\zeta_n^*)) \leq C \left( h^{1/3} + (J_{\mathbf{q}}^u)^{-1} \right) d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

**Remark.**

The different terms which compose the error above appear at different places in the proof. One of this term is due to the fact that it is a first order approximation of an unstable manifold : we need to control the error term in this approximation. It turns out that as soon as  $|x| \leq \alpha_n^{1/2} h^{\delta_0} = O(h^{1/3})$ , this error is  $O(|x|^2) = O(h^{2/3})$ . Depending on  $\rho$  (and  $\mathbf{q}$ ), the main term of the error can differ. As we will see, when  $d(\rho, \mathcal{T}) \geq h^{\delta_1}$ , the term  $h^{\delta_0}$  is negligible.

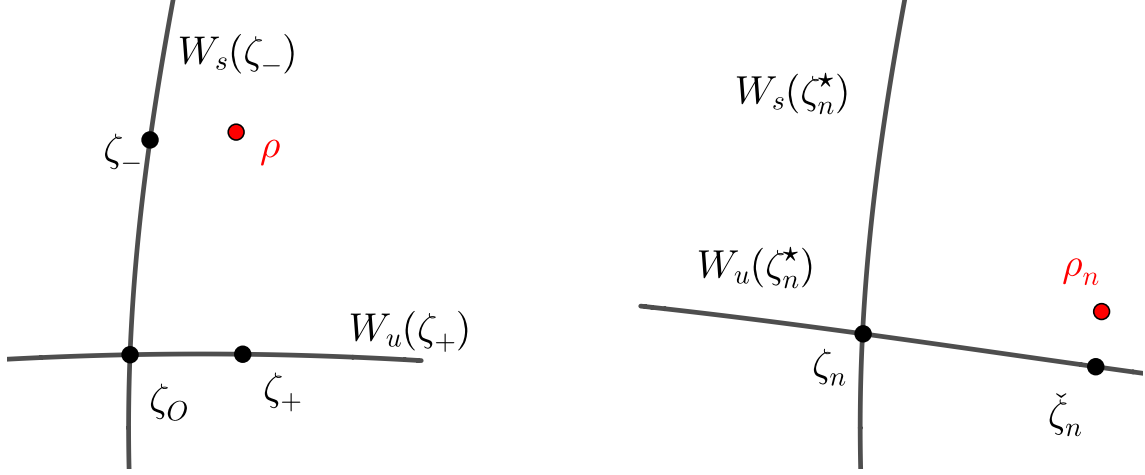


Figure 7.4: The points introduced in Step 1 of the proof Lemma 7-6.12.

*Proof. Step 1 :  $\rho_n$  is close to a reference unstable manifold  $W_u(\zeta_n^*)$ .* (See Figure 7.4)

In this first step, we want to show that  $\rho_n$  is close to an unstable manifold  $W_u(\zeta_n^*)$ . As in the proof of Lemma 7-5.2, we consider a point  $\zeta_+ \in \mathcal{T}$  such that  $d(\rho, \mathcal{T}_+) = d(\rho, W_u(\zeta_+))$  and such that for all  $0 \leq i \leq n$ ,  $d(F^{ino}(\zeta_+), F^{ino}(\rho)) \leq \varepsilon_2$  for some small  $\varepsilon_2$  depending on  $\varepsilon_0$ . Note also that, by the third point of Lemma 7-5.2,  $d(\rho, \mathcal{T}_-) \leq C_2 (J_{\mathbf{q}}^u)^{-1} \varepsilon_0$ . Let's fix a point  $\zeta_-$  such that  $d(\rho, \mathcal{T}_-) = d(\rho, W_s(\zeta_-))$  and let's consider  $\zeta_O$  the unique point in  $W_u(\zeta_+) \cap W_s(\zeta_-)$ . Then, we still have  $d(\rho, \mathcal{T}_+) = d(\rho, W_u(\zeta_O))$  and we also have

$$d(\rho, \zeta_O)^2 \sim d(\rho, \mathcal{T}_-)^2 + d(\rho, \mathcal{T}_+)^2 \leq C \left( (J_{\mathbf{q}}^u)^{-1} \varepsilon_0 \right)^2 + d(\rho, \mathcal{T}_+)^2$$

For  $0 \leq i \leq n$ , set  $\zeta_i^* = F^{ino}(\zeta_O)$ . We have

$$d(\rho_n, W_u(\zeta_n^*)) \leq C (J_{\mathbf{q}}^u)^{-1} d(\rho, W_u(\zeta_O)) \leq C (J_{\mathbf{q}}^u)^{-1} d(\rho, \mathcal{T}_+)$$

and

$$d(\rho_n, \zeta_n^*)^2 \leq C (J_{\mathbf{q}}^u)^2 d(\rho, \mathcal{T}_-)^2 + C (J_{\mathbf{q}}^u)^{-2} d(\rho, \mathcal{T}_+)^2 \leq C \varepsilon_0$$

Let us fix  $\check{\zeta}_n \in W_u(\zeta_n^*)$  such that

$$d(\rho_n, W_u(\zeta_n^*)) = d(\rho_n, \check{\zeta}_n)$$

**Step 2 : The curve  $\zeta_n(x)$  is close to the (unstable) tangent space  $E_u(\check{\zeta}_n)$ .**

*Step 2-a : First approximation.* (See Figure 7.5).

We now want to show that the curve is a rather good approximation of the tangent space of  $W_u(\zeta_n^*)$  at  $\check{\zeta}_n$ . To do so we make the following observation (recall the notations of (7-6.15) and the definition of  $\beta_n$  in (7-6.16)).

$$\mathbf{v}_n := \begin{pmatrix} 1 \\ \beta_n \end{pmatrix} = \alpha_n^{-1/2} d_{\hat{\rho}} F^{(n)}(\mathbf{v}'_n); \quad \mathbf{v}'_n = \alpha_n^{-1/2} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

and note that  $\|\mathbf{v}'_n\| = 1$  (since  $\alpha_n^2 = a_n^2 + b_n^2$ ). We compare this vector  $\mathbf{v}_n$  to  $\mathbf{w}_n := \alpha_n^{-1/2} d_{\hat{\zeta}} F^{(n)}(\mathbf{v}'_n)$  where  $\hat{\zeta} = \kappa_{q_0}(\check{\zeta}_0)$  with  $\check{\zeta}_0 = F^{-nn_0}(\check{\zeta}_n)$ . Arguing as in the proof of Lemma 7-6.8, we can show that

$$\|d_{\hat{\rho}} F^{(n)} - d_{\hat{\zeta}} F^{(n)}\| \leq C J_{\mathbf{q}}^u d(\rho, \check{\zeta}_0)$$

By the triangular inequality,  $d(\rho, \check{\zeta}_0) \leq d(\rho, \zeta_O) + d(\zeta_O, \check{\zeta}_0)$  where the first term is controlled by  $C \left( (J_{\mathbf{q}}^u)^{-1} \varepsilon_0 + d(\rho, \mathcal{T}_+) \right)$ . For the second term, we use the fact that  $\check{\zeta}_0 \in W_u(\zeta_O)$  and  $d(\check{\zeta}_n, F^{nn_0}(\zeta_O)) \leq C \varepsilon_0$ , this gives  $d(\zeta_O, \check{\zeta}_0) \leq C (J_{\mathbf{q}}^u)^{-1} \varepsilon_0$ . As a consequence, we find that

$$\|d_{\hat{\rho}} F^{(n)} - d_{\hat{\zeta}} F^{(n)}\| \leq C J_{\mathbf{q}}^u d(\rho, \mathcal{T}_+) + C \varepsilon_0$$

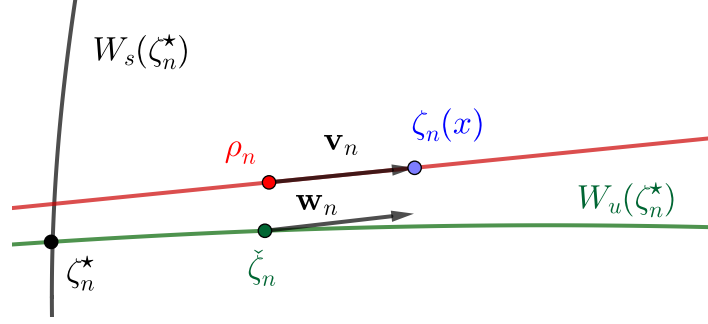


Figure 7.5: The curve  $\zeta_n(x)$  passing through  $\rho_n$  (in red) is close to an unstable manifold  $W_u(\zeta_n^*)$  (in green).  $W_u(\zeta_n^*)$  is well approximated, near  $\check{\zeta}_n$ , by its tangent space at  $\check{\zeta}_n$ , spanned by a vector close to  $\mathbf{w}_n$ .

Finally, recalling that  $\alpha_n^{1/2} \sim J_{\mathbf{q}}^u$  - so that  $\mathbf{v}_n$  and  $\mathbf{w}_n$  are close from being normalized -, we get that  $\|\mathbf{v}_n - \mathbf{w}_n\| \leq Cd(\rho, \mathcal{T}_+) + C\varepsilon_0 (J_{\mathbf{q}}^u)^{-1}$ .

Let's now define  $\tilde{\zeta}_n(x)$  by

$$\kappa_{q_n}(\tilde{\zeta}_n(x)) = \kappa_{q_n}(\check{\zeta}_n) + x\mathbf{w}_n$$

We have (recall that  $\zeta_n(x) = \kappa_{q_n}^{-1}(\hat{\rho}_n + x\mathbf{v}_n)$ )

$$\begin{aligned} d(\tilde{\zeta}_n(x), \zeta_n(x)) &\leq Cd(\kappa_{q_n}(\check{\zeta}_n) + x\mathbf{w}_n, \hat{\rho}_n + x\mathbf{v}_n) \\ &\leq Cd(\check{\zeta}_n, \rho_n) + |x|\|\mathbf{w}_n - \mathbf{v}_n\| \\ &\leq C (J_{\mathbf{q}}^u)^{-1} d(\rho, \mathcal{T}_+) + |x| \left( Cd(\rho, \mathcal{T}_+) + \varepsilon_0 (J_{\mathbf{q}}^u)^{-1} \right) \\ &\leq C (J_{\mathbf{q}}^u)^{-1} d(\rho, \mathcal{T}_+) + CJ_{\mathbf{q}}^u h^{\delta_0} \left( d(\rho, \mathcal{T}_+) + C\varepsilon_0 (J_{\mathbf{q}}^u)^{-1} \right) \text{ if } |x| \leq \alpha_n^{1/2} h^{\delta_0} \\ &\leq C \left( (J_{\mathbf{q}}^u)^{-1} + h^{1/3} \right) d(\rho, \mathcal{T}_+) + Ch^{\delta_0} \end{aligned}$$

where we use the fact that  $J_{\mathbf{q}}^u h^{\delta_0} \leq Ch^{1/3}$ . We will now control the distance of  $\tilde{\zeta}_n(x)$  to  $\mathcal{T}_+$ .

*Step 2-b : Comparison with the tangent space.* (See Figure 7.5).

In this step, we want to show that  $\mathbf{w}_n$  is close to a vector spanning  $d\kappa_{q_n}(T_{\check{\zeta}_n} W_u(\zeta_n^*))$ . To do so, we use Lemma 7-4.1. If  $\varepsilon_0$  is small enough (depending on the parameter  $\varepsilon_1$  appearing in Lemma 7-4.1), we can ensure that the vector  $\mathbf{v}_n$  is sufficiently close to  $\mathbb{R} \times \{0\}$  and hence,  $(d_{\check{\zeta}_0} \kappa_0)^{-1} \mathbf{v}'_n$  is sufficiently close to  $T_{\check{\zeta}_0} W_u(\zeta_0)$ , so that we can apply this lemma with initial vector  $(d_{\check{\zeta}_0} \kappa_0)^{-1} \mathbf{v}'_n$ . To alleviate the notations, let's note  $\mathcal{L} = \kappa_{q_n}(W_u(\zeta_n^*))$ ,  $m = \kappa_{q_n}(\check{\zeta}_n)$ . By applying Lemma 7-4.1 and sending the result in the chart  $\kappa_{q_n}$ , we obtain that

$$d\left(\frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}, T_m \mathcal{L}\right) \leq C (J_{\mathbf{q}}^u)^{-2}$$

since  $\|\mathbf{w}_n\| \leq C$ , the same is true for  $\mathbf{w}_n$ . Let's pick  $\mathbf{w}'_n \in T_m \mathcal{L}$  such that  $\|\mathbf{w}_n - \mathbf{w}'_n\| \leq C (J_{\mathbf{q}}^u)^{-2}$ . We now define  $Z_n(x)$  by the relation

$$\kappa_{q_n}(Z_n(x)) = \kappa_{q_n}(\tilde{\zeta}_n(x)) + \mathbf{w}'_n x$$

If  $|x| \leq \alpha_n^{1/2} h^{\delta_0}$ , it is clear that

$$d(Z_n(x), \tilde{\zeta}_n(x)) \leq |x|\|\mathbf{w}_n - \mathbf{w}'_n\| \leq C (J_{\mathbf{q}}^u)^{-2} \alpha_n^{1/2} h^{\delta_0} \leq C \alpha_n^{-1/2} h^{1/2} \ll h^{\delta_0}$$

Gathering the steps 2-a and 2-b, we see that

$$d(\zeta_n(x), \mathcal{T}_+) \leq d(Z_n(x), \mathcal{T}_+) + C \left( (J_{\mathbf{q}}^u)^{-1} + h^{1/3} \right) d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

**Step 3 : The tangent space is a good approximation.** The only remaining point is to control  $d(Z_n(x), \mathcal{T}_+)$ . We observe that  $\mathbf{w}'_n \in T_m \mathcal{L}$ . Hence, by standard results of differential geometry,  $d(m + x\mathbf{w}'_n, \mathcal{L}) \leq Cx^2$  where  $C$  depends on  $\|\mathbf{w}'_n\|$  and on the curvature of  $\mathcal{L}$  - which can be controlled independently of the base point  $\tilde{\zeta}_n$  of this unstable manifold. As a consequence, if  $|x| \leq \alpha_n^{1/2} h^{\delta_0} \leq h^{1/3}$ ,  $d(m + x\mathbf{w}'_n, \mathcal{L}) \leq Ch^{2/3} \ll Ch^{\delta_0}$ . This shows that  $d(Z_n(x), \mathcal{T}_+) \leq Cd(m + x\mathbf{w}'_n, \mathcal{L}) \leq Ch^{\delta_0}$  and concludes the proof of the lemma.  $\square$

This Lagrangian being well understood, we can now come to the estimates for  $R_+$  :

**Proposition 7-6.4. Estimates for  $R_+$ .** There exists a global constant  $C > 0$  such that for all  $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$ , we have

- If  $d(\rho, \mathcal{T}_+) \geq h^{\delta_1}$ ,  $R_+(x) \leq Ch^{2\varepsilon}$  ;
- If  $d(\rho, \mathcal{T}_+) \leq h^{\delta_1}$ ,  $R_+(x) \leq C$

(for some constant  $C > 0$ ).

*Proof.* Recall that  $\delta_1 = \delta - \varepsilon$ . We pick  $x \in [-\alpha_n^{1/2} h^{\delta_0}, \alpha_n^{1/2} h^{\delta_0}]$ . Here, we will use the inequality  $d(\rho_n, \mathcal{T}_+) \leq C (J_{\mathbf{q}}^u)^{-1} d(\rho, \mathcal{T}_+)$  and the result of the previous lemma, namely,

$$d(\zeta_n(x), \mathcal{T}_+) \leq C \left( h^{1/3} + (J_{\mathbf{q}}^u)^{-1} \right) d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

Recall that  $J_{\mathbf{q}}^u \geq C_\varepsilon e^{n(h)\lambda_{\min}(1-\varepsilon)} \geq C_\varepsilon h^{-\vartheta_\varepsilon \lambda_{\min}(1-\varepsilon)}$ .

We choose some  $0 < \beta < \min(1/3, \vartheta_\varepsilon \lambda_{\min}(1-\varepsilon))$ , which ensures that

$$d(\zeta_n(x), \mathcal{T}_+) \leq Ch^\beta d(\rho, \mathcal{T}_+) + Ch^{\delta_0}$$

Note that since we work with  $\varepsilon$  small, it is harmless to assume that  $\varepsilon < \beta$ . We treat the two points separately :

- For this first point, we distinguish two cases :

**First case :**  $h^{\delta_1} \leq d(\rho, \mathcal{T}_+) \leq h^{\delta-\beta}$  In this context, one has  $d(\zeta_n(x), \mathcal{T}_+) \leq Ch^\beta h^{\delta-\beta} + Ch^{\delta_0} \leq Ch^\delta$ . As a consequence,  $\hat{\varphi}_+(\zeta_n(x)) \leq Ch^{2\delta}$ . We also have  $\hat{\varphi}_+(\rho) \geq C^{-1}(h^{2\delta} + h^{2\delta_1}) \geq C^{-1}h^{2\delta_1}$  which gives

$$R_+(x) \leq \frac{(M+C)h^{2\delta}}{Mh^{2\delta} + C^{-1}h^{2\delta_1}} \leq Ch^{2(\delta-\delta_1)} = Ch^{2\varepsilon}$$

**Second case:**  $d(\rho, \mathcal{T}_+) \geq h^{\delta-\beta}$ . In this context, we have  $d(\rho, \mathcal{T}_+)^2 \gg h^{2\delta}$  so that we can bound the denominator  $Mh^{2\delta} + \hat{\varphi}_+(\rho)$  from below by  $C^{-1}d(\rho, \mathcal{T}_+)^2$ . Concerning the numerator, we have

$$d(\zeta_n(x), \mathcal{T}_+) \leq Ch^\beta d(\rho, \mathcal{T}_+) + Ch^{\delta_0} \leq Ch^\beta d(\rho, \mathcal{T}_+)$$

since  $h^\beta d(\rho, \mathcal{T}_+) \geq h^\beta h^{\delta-\beta} \geq h^\delta \gg h^{\delta_0}$ . We deduce also that  $\hat{\varphi}_+(\zeta_n(x)) \leq Ch^{2\beta} d(\rho, \mathcal{T}_+)^2$ . As a consequence,

$$R_+(x) \leq \frac{Ch^{2\beta} d(\rho, \mathcal{T}_+)^2}{C^{-1} d(\rho, \mathcal{T}_+)^2} \leq Ch^{2\beta} \ll h^{2\varepsilon}$$

- We now assume that  $d(\rho, \mathcal{T}_+) \leq h^{\delta_1}$ . As in the first case above, we can bound the numerator by  $Ch^{2\delta}$ . Concerning the denominator, we simply use the fact that  $\hat{\varphi}_+ \geq 0$  to bound it from below by  $Mh^{2\delta}$ , and this gives, as expected

$$R_+(x) \leq C$$

$\square$

Let's recap these two estimates and their implications concerning  $d(x)$  (and recall that by definition,  $\beta > \varepsilon$  and  $J_{\mathbf{q}}^u \geq C^{-1}h^{-\beta}$ )

$$\begin{aligned} d(\rho, \mathcal{T}_-) \geq h^\delta &\implies d(x) \leq (Ch^{2\beta})^t \ll h^{2t\varepsilon}, \forall x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}] \\ d(\rho, \mathcal{T}_-) \leq h^\delta \text{ and } d(\rho, \mathcal{T}_+) \geq h^{\delta_1} &\implies d(x) \leq (Ch^{2\varepsilon})^t, \forall x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}] \end{aligned}$$

As a consequence, the  $L^2$  norm of  $\chi v_n$  is very small when  $\rho$  lies outside the neighborhood of  $\mathcal{T}$  defined before Proposition 7-5.2 :

$$\mathcal{T}_{\delta, \delta_1} = \{\rho, d(\rho, \mathcal{T}_-) \leq h^\delta, d(\rho, \mathcal{T}_+) \leq h^{\delta_1}\} \quad (7-6.30)$$

Indeed, we obviously have

**Proposition 7-6.5.** For all  $L > 0$ , there exists  $t = t(\varepsilon, L)$  such that the following holds. Assume that  $\rho \notin \mathcal{T}_{\delta, \delta_1}$ . Then,

$$\int_{|x| \leq \alpha_n^{1/2}h^{\delta_0}} d(x)^2 dx \leq Ch^L$$

#### 7-6.5.4. Crucial estimates in $\mathcal{T}_{\delta, \delta_1}$ .

We now turn to the crucial estimate which helps to control the  $L^2$  norm of  $\chi v_n$  when  $\rho \in \mathcal{T}_{\delta, \delta_1}$ .

**Proposition 7-6.6.** Assume that  $\rho \in \mathcal{T}_{\delta, \delta_1}$ . Then,

$$\int_{|x| \leq \alpha_n^{1/2}h^{\delta_0}} d(x)^2 dx \leq C (J_{\mathbf{q}}^u)^{d_H + \varepsilon} h^{(\delta_0 - \delta)(d_H + \varepsilon)} h^\delta$$

*Proof. Step 0 : A simple estimates for  $d(x)$ .* First recall from Proposition 7-6.4,  $\rho \in \mathcal{T}_{\delta, \delta_1} \implies d(x) \leq CR_-(x)^t$ . Moreover, the numerator in  $R_-(x)$  is bounded by  $Ch^{2\delta}$  and since  $\hat{\varphi}_-(\zeta_n(x)) \geq Ch^{2\delta} + Cd(\zeta_n(x), \mathcal{T}_-)^2$ , we find that

$$d(x) \leq C \left( 1 + \left( \frac{d(\zeta_n(x), \mathcal{T}_-)}{h^\delta} \right)^2 \right)^{-t}$$

**Step 1 : The mass is supported in an  $h^\delta$ -neighborhood of  $\mathcal{T}$ .** We use Lemma 7-6.12 which asserts that there exists  $\zeta_n^*$  such that

$$d(\zeta_n(x), W_u(\zeta_n^*)) \leq Ch^\beta d(\rho, \mathcal{T}_+) + Ch^{\delta_0} \ll h^\delta \quad (7-6.31)$$

with  $\beta$  defined in the proof of Proposition 7-6.4. Recall that in the chart  $\kappa_{q_n}, \kappa_{q_n}(\zeta_n(x)) = \hat{\rho}_n + (x, \beta_n x)$ . Moreover, if  $\varepsilon_0$  is small enough, we may assume that  $\kappa_{q_n}(W_u(\zeta_n^*))$  can be written as the graph of a function :

$$\kappa_{q_n}(W_u(\zeta_n^*)) = \{(x, G_u(x)), x \in I_u\}$$

where  $I_u$  is a small interval of size  $\sim \varepsilon_0$  and  $G_u$  a smooth function with bounded  $C^\infty$  norms (with bounds depending only on the  $F$  and the charts). Since  $d(\rho_n, W_u(\zeta_n^*)) \ll h^\delta$ , up to translating, we may assume that  $\hat{\rho}_n = (0, \xi_n)$  and  $|G_u(0) - \xi_n| \ll h^\delta$ . In particular, if  $h$  is small enough, we may assume that  $[-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}] \subset I_u$ . Finally, if  $\varepsilon_0$  is small enough, we can also assume that  $|G'_u(x)| \leq 1/4$  if  $|x| \leq 2\alpha_n^{1/2}h^{\delta_0} \ll 1$  (recall that the chart  $\kappa_q$  is centered at a point  $\rho_q$  such that  $\kappa_q(E_u(\rho_q)) = \mathbb{R} \times \{0\}$ ). We now set

$$X(\mathcal{T}) = \{x \in [-2\alpha_n^{1/2}h^{\delta_0}, 2\alpha_n^{1/2}h^{\delta_0}], \kappa_{q_n}^{-1}(x, G_u(x)) \in \mathcal{T}\}$$

Let's cover  $X(\mathcal{T})$  by  $N$  intervals of size  $2h^\delta$ , centered at points  $x_1, \dots, x_N \in X(\mathcal{T})$ . In virtue of Lemma 3-6.1, we can choose  $N$  such that

$$N \leq C (J_{\mathbf{q}}^u h^{\delta_0 - \delta})^{d_H + \varepsilon}$$

Each interval around  $x_i$  of size  $O(h^\delta)$  supports a mass of order  $O(h^\delta)$ . Our aim in the following lines is to show that the weight of the integral supported at distance larger than  $h^\delta$  of the  $x_i$  is also

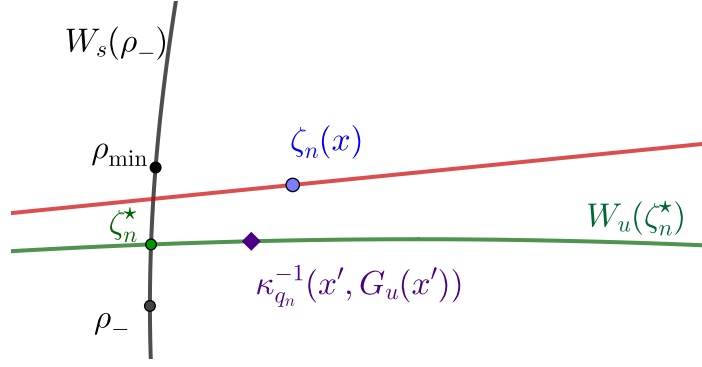


Figure 7.6: The points appearing in the proof of the claim 7-6.32

$O(h^\delta)$ , so that we will be able to estimate the whole integral by  $Nh^\delta$ , which would conclude the proof. Let us consider  $x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}]$  and assume that for all  $1 \leq i \leq N$ ,  $|x - x_i| \geq 2h^\delta$ . Let us choose  $i$  such that  $|x - x_i| = \min_{1 \leq k \leq N} |x - x_k|$ . We claim that there exists  $\nu > 0$ , uniform with respect to  $\rho$ ,  $h$  and  $x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}]$  such that

$$d(\zeta_n(x), \mathcal{T}_-) \geq \nu|x - x_i| \quad (7-6.32)$$

Let's admit it for a while. For  $i \in \{1, \dots, N\}$ , let's note

$$J_i = \{x \in [-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}], |x - x_i| = \min_{1 \leq k \leq N} |x - x_k|\}$$

These intervals form a partition of  $[-\alpha_n^{1/2}h^{\delta_0}, \alpha_n^{1/2}h^{\delta_0}]$ .

$$\begin{aligned} \int_{|x| \leq \alpha_n^{1/2}h^{\delta_0}} d(x)^2 dx &\leq \int_{|x| \leq \alpha_n^{1/2}h^{\delta_0}} C \left( 1 + \left( \frac{d(\zeta_n(x), \mathcal{T}_-)}{h^\delta} \right)^2 \right)^{-2t} dx \\ &\leq C \sum_{i=1}^N \int_{J_i} \left( 1 + \left( \frac{d(\zeta_n(x), \mathcal{T}_-)}{h^\delta} \right)^2 \right)^{-2t} dx \\ &\leq C \sum_{i=1}^N \left( \int_{|x_i - x| \leq 2h^\delta} 1 dx + \int_{|x_i - x| > 2h^\delta} \left( 1 + \left( \frac{\nu|x - x_i|}{h^\delta} \right)^2 \right)^{-2t} dx \right) \\ &\leq C \sum_{i=1}^N \left( 4h^\delta + \int_{|y| > 2} \left( \frac{1}{1 + (\nu y)^2} \right)^{2t} h^\delta dy \right) \\ &\leq CNh^\delta \end{aligned}$$

Here,  $t$  is large enough and in particular, we may ensure that  $t \geq 1$  so that the integral converges.

**Step 2 : Proof of the claim (7-6.32).** We argue by contradiction and assume that  $d(\zeta_n(x), \mathcal{T}_-) \leq \nu|x - x_i|$  for some sufficiently small  $\nu$  (with conditions specified below). Since  $\mathcal{T}_-$  is made of local stable leaves near  $\mathcal{T}$  (and  $\zeta_n(x)$  lies in a small neighborhood of  $\mathcal{T}$ ), we may choose  $\rho_- \in \mathcal{T}$  such that  $d(\zeta_n(x), \mathcal{T}_-) = d(\zeta_n(x), W_s(\rho_-))$ . Let's still note  $\rho_- \in \mathcal{T}$  the unique point of  $W_u(\zeta_n^*) \cap W_s(\rho_-)$  and let's write  $\kappa_{q_n}(\rho_-) = (x_-, G_u(x_-))$ . Again if  $\varepsilon_0$  is small enough, all the stable leaves in  $\kappa_{q_n}$  can be written as graphs in the vertical variable : let us write

$$\kappa_{q_n}(W_s(\rho_-)) = \{(H_s(\xi), \xi), \xi \in I_s\}$$

where  $I_s$  is a small interval of size  $O(\varepsilon_0)$  and  $H_s$  a smooth function with  $C^\infty$  norms bounded by constants only depending on the dynamics and the chart. Up to translating, we may assume that  $(H_s(0), 0) = \kappa_{q_n}(\rho_-) = (x_-, 0)$ . As for  $G_u$ , if  $\varepsilon_0$  is small enough, we can assume that  $|H'_s(\xi)| \leq 1$  for all  $\xi \in I_s$ . Finally, let us note  $\rho_{\min} = \kappa_{q_n}^{-1}(H_s(\xi_{\min}), \xi_{\min})$  a point in  $W_s(\rho_-)$  such that  $d(\zeta_n(x), \rho_{\min}) = d(\zeta_n(x), W_s(\rho_-))$  (see Figure 7.6).



Since by (7-6.31),  $d(\zeta_n(x), W_u(\zeta_n^*)) \ll h^\delta$ , we can find  $x' \in I_u$  such that  $\|(x, \beta_n x) - (x', G_u(x'))\| \ll h^\delta$ . This inequality implies

$$\begin{aligned} |x - x'| &\ll h^\delta, \\ |\beta_n x - G_u(x')| &\ll h^\delta, \\ |G_u(x) - G_u(x')| &\ll h^\delta, \\ |G_u(x) - \beta_n x| &\ll h^\delta. \end{aligned}$$

Since by assumption,  $|x - x_i| \geq 2h^\delta$ , when  $h$  is small enough, the inequality  $|G_u(x) - \beta_n x| \leq \nu|x - x_i|$  holds. We have

$$\|(H_s(\xi_{\min}), \xi_{\min}) - (x, \beta_n x)\| \leq Cd(\zeta_n(x), \rho_{\min}) \leq Cd(\zeta_n(x), \mathcal{T}_-) \leq C\nu|x - x_i|$$

From this we deduce that

$$\begin{aligned} |\xi_{\min}| &\leq |\xi_{\min} - G_u(x)| + |G_u(x)| = |\xi_{\min} - G_u(x)| + |G_u(x) - G_u(x_-)| \\ &\leq |\xi_{\min} - \beta_n x| + |\beta_n x - G_u(x)| + \frac{1}{4}|x - x_-| \\ &\leq \|(H(\xi_{\min}), \xi_{\min}) - (x, \beta_n x)\| + \nu|x - x_i| + \frac{1}{4}|x - x_-| \\ &\leq C\nu|x - x_i| + \frac{1}{4}|x - x_-| \end{aligned}$$

Finally, we find that,

$$\begin{aligned} |x_- - x| &\leq |x_- - H_s(\xi_{\min})| + |H_s(\xi_{\min}) - x| \\ &\leq |H(0) - H(\xi_{\min})| + \|(H(\xi_{\min}), \xi_{\min}) - (x, \beta_n x)\| \\ &\leq |\xi_{\min}| + C\nu|x - x_i| \text{ (recall that } |H'| \leq 1) \\ &\leq \frac{1}{4}|x - x_-| + C\nu|x - x_i| \end{aligned}$$

From this, we deduce that

$$|x - x_-| \leq \frac{4}{3}C\nu|x - x_i| \quad (7-6.33)$$

A first consequence of this inequality is that if  $\nu$  is small enough so that  $\frac{4\nu C}{3} \leq \frac{1}{4}$ , we have

$$|x_-| \leq |x| + \frac{1}{4}|x - x_i| \leq \frac{5}{4}|x| + |x_i| \leq \frac{5}{4}\alpha_n^{1/2}h^{\delta_0} + \frac{2}{4}\alpha_n^{1/2}h^{\delta_0} \leq 2\alpha_n^{1/2}h^{\delta_0}$$

Since  $\kappa_{q_n}^{-1}(x_-, G_u(x_-)) = \rho_- \in \mathcal{T}$ , we deduce that  $x_- \in X(\mathcal{T})$ . In particular, there exists  $j \in \{1, \dots, N\}$  such that  $|x_- - x_j| \leq h^\delta$ . But then, we would have

$$|x_i - x| \leq |x_j - x| \leq |x_- - x_j| + |x_- - x| \leq h^\delta + \frac{1}{4}|x - x_i| \leq \frac{1}{2}|x - x_i| + \frac{1}{4}|x - x_i| < |x - x_i|$$

(recall that  $|x - x_i| \geq 2h^\delta$ ). This gives the required contradiction and concludes the proof of the claim (7-6.32).  $\square$

### 7-6.6. End of the proof.

We can use Lemma 7-6.11, Proposition 7-6.5 and Proposition 7-6.6 to conclude the proof of Proposition 7-5.2. Indeed, since  $\|\widetilde{\Psi}_m\|_\infty < +\infty$ , we have

$$\begin{aligned} \left\| \left(1 + \frac{x^2}{\alpha_n h}\right)^{m/2} \chi v_n \right\|_{L^2}^2 &\leq \int_{|x| \leq \alpha_n^{1/2} h^{\delta_0}} d(x)^2 |\Lambda_{\alpha_n h} \widetilde{\Psi}_m(x)|^2 dx + O(h^\infty) \\ &\leq C_m (\alpha_n h)^{-1/2} \int_{|x| \leq \alpha_n^{1/2} h^{\delta_0}} d(x)^2 dx + O(h^\infty) \end{aligned}$$

It gives a bound  $C_m h^L$  when  $\rho \notin \mathcal{T}_{\delta, \delta_1}$  (with  $L$  as large as necessary by choosing  $t$  large enough) and when  $\rho \in \mathcal{T}_{\delta, \delta_1}$ , we find that

$$\left\| \left( 1 + \frac{x^2}{\alpha_n h} \right)^{m/2} \chi v_n \right\|_{L^2}^2 \leq C_m (J_{\mathbf{q}}^u)^{d_H - 1 + \varepsilon} h^{(\delta_0 - \delta)(d_H + \varepsilon) + \delta - 1/2}$$

When  $m = 0$ , it gives a control of the leading term, since we have

$$\|u_n^0\|_{L^2}^2 \leq \Pi_{\alpha, n}(\rho)^2 \|\chi v_n\|_{L^2}^2$$

and since  $\Pi_{\alpha, n}(\rho)^2 = O(h^{-L_2})$  for some  $L_2 > 0$ , so that for  $\rho \notin \mathcal{T}_{\delta, \delta_1}$  we can have  $\|u_n^0\|_{L^2}^2 = O(h^L)$  for any  $L$  by choosing  $t$  large enough.

This controls the first term of the expansion given by Corollary 7-6.2. We recall that the number of terms in the expansion and the remainder are controlled by a integer  $N \in \mathbb{N}$ . For the other terms in the expansion given by Corollary 7-6.2, as already explained with (7-6.25), they all have their  $L^2$  norms controlled by some

$$\varepsilon(h) \Pi_{\alpha, n}(\rho) \left\| \left( 1 + \frac{x^2}{\alpha_n h} \right)^{m/2} \chi v_n \right\|_{L^2}$$

with  $m \leq N$  and  $\varepsilon(h) \rightarrow 0$  when  $h \rightarrow 0$ . Finally, we can choose  $N = N(\varepsilon)$  such that the remainder has an  $L^2$  norm  $O(h^{2L})$ . This concludes the proof of Proposition 7-5.2, and eventually of Theorem 7-1.3.

## 7-7 Appendix of the Chapter.

### 7-7.1. Missing proofs of the Lemmata involving stationary phase expansions.

In this appendix, we give the missing proofs of Lemmas 7-3.4, 7-3.5 and 7-6.10. It relies on different uses of stationary phase theorems.

#### 7-7.1.1. Proof of of Lemma 7-3.4.

To alleviate the notations, let's note  $q(x, \eta) = \langle D^2 \psi(x_1, \xi_0)(x, \eta), (x, \eta) \rangle$  and write it  $q(x, \eta) = ux^2 + 2vx\eta + w\eta^2$ . The metaplectic operator  $\mathcal{M}(d_{\rho_0} F)$  admits the kernel

$$k(x, y) := \frac{|v|^{1/2}}{2\pi h} \int_{\mathbb{R}} e^{\frac{i}{h}(\frac{1}{2}q(x, \eta) - y\eta)} d\eta$$

and  $\bar{k}(y, x)$  is the kernel of  $\mathcal{M}(d_{\rho_0} F)^*$ . We also note

$$\mathcal{M}_b u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\frac{1}{2}q(x, \eta) - y\eta)} \tilde{b}(x, \eta) u(y) dy d\eta$$

We have

$$\begin{aligned} (\mathcal{M}(d_{\rho_0} F)^* \mathcal{M}_b u)(x) &= \frac{|v|}{(2\pi h)^2} \int_{\mathbb{R}^4} e^{\frac{i}{h}(-\frac{1}{2}q(y, \eta) + x\eta + \frac{1}{2}q(y, \xi) - z\xi)} \tilde{b}(y, \xi) u(z) dy d\eta dz d\xi \\ &= \frac{1}{2\pi h} \int_{\mathbb{R}^2} u(z) e^{\frac{i}{h}(x-z)\xi} \underbrace{\left( \frac{|v|}{2\pi h} \int_{\mathbb{R}^2} e^{\frac{i}{h}(\frac{1}{2}q(y, \xi) - \frac{1}{2}q(y, \eta) + x(\eta - \xi))} \tilde{b}(y, \xi) dy d\eta \right)}_{\tilde{b}(x, \xi)} dz d\xi \\ &= \text{Op}_h^R(\tilde{b})u(x) \\ &= \text{Op}_h(b)u(x) \end{aligned}$$

where  $\text{Op}_h^R$  denotes the right quantization, and by [Zwo12] (Theorem 4.13),  $b(x, \xi) = e^{-\frac{i\hbar}{2}\langle D_x, D_\xi \rangle} \check{b}(x, \xi)$ . Let's analyze  $\check{b}$ :

$$\begin{aligned} \check{b}(x, \xi) &= \frac{|v|}{2\pi\hbar} \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}(\frac{1}{2}q(y, \xi) - \frac{1}{2}q(y, \eta + \xi) + x\eta)} \tilde{b}(y, \xi) dy d\eta \\ &= \frac{|v|}{2\pi\hbar} \int_{\mathbb{R}^2} e^{\frac{i}{\hbar}(\frac{1}{2}w\eta^2 - w\eta\xi - vy\eta + x\eta)} \tilde{b}(y, \xi) dy d\eta \\ &= \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} e^{\frac{i}{2\hbar}w\eta^2} e^{\frac{i}{\hbar}(x-y)\eta} \tilde{b}(v^{-1}(y - w\xi), \xi) dy d\eta \quad (\text{change of variable } vy + w\xi \mapsto y) \\ &= e^{\frac{i\hbar}{2}wD_x^2} \tilde{b}(v^{-1}(x - w\xi), \xi) \end{aligned}$$

In particular, if  $w = 0$ , we directly find that  $\check{b}(x, \xi) = \tilde{b}(v^{-1}x, \xi)$ . Otherwise, it is represented by the formula ([Zwo12], Theorem 4.8):

$$\check{b}(x, \xi) = \frac{e^{\frac{i\pi}{4}\frac{w}{|w|}}}{2\pi\hbar|w|} \int_{\mathbb{R}} e^{-\frac{i\hbar}{2w}y^2} \tilde{b}(v^{-1}(y + x - w\xi), \xi) dy$$

As a consequence, we see that  $b$  is obtained from  $\tilde{b}$  by composing 3 actions : the one of  $e^{-\frac{i\hbar}{2}\langle D_x, D_\xi \rangle}$ , the change of variable  $(x, \xi) \mapsto (v^{-1}(x - w\xi), \xi)$  and  $e^{\frac{i\hbar}{2}wD_x^2}$ . The second one is obviously continuous from  $S(\langle \rho \rangle^{3N})$  to  $S(\langle \rho \rangle^{3N})$ . We can now use [Zwo12] Theorem 4.17 (or more precisely, the estimates given in the proof) : both the action of  $e^{\frac{i\hbar}{2}wD_x^2}$  and  $e^{-\frac{i\hbar}{2}\langle D_x, D_\xi \rangle}$  are continuous from  $S(\langle \rho \rangle^{3N})$  to  $S(\langle \rho \rangle^{3N})$ , and more precisely, there exists a universal integer  $M$  and universal constants  $C_\alpha$  such that, for every  $\alpha \in \mathbb{N}^2$ ,  $(x, \xi) \in T^*\mathbb{R}$ ,

$$|\partial^\alpha(L\tilde{b})(x, \xi)| \leq C_\alpha \sup_{|\beta| \leq |\alpha| + M} \|\langle \rho \rangle^{-3N} \partial^\beta \tilde{b}\| \|\langle \rho \rangle^{3N}$$

with  $L$  being either  $e^{\frac{i\hbar}{2}wD_x^2}$  or  $e^{-\frac{i\hbar}{2}\langle D_x, D_\xi \rangle}$ . The same holds for the change of variable. This gives the required estimates for the symbol  $b$  and concludes the proof of the Lemma.  $\square$

### 7-7.1.2. Proof of Lemma 7-3.5.

Fix  $s \in [0, 1]$  and recall that, with the notation  $q$  introduced above

$$\tilde{R}_s u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\frac{1}{2}q(x, \eta) + sh^{1/2}r_3^\psi(x, \eta; h) - y\eta)} b_N(x, \eta) u(y) dy d\eta$$

Let's introduce

$$R_s = \Lambda_h \tilde{R}_s \Lambda_h^*$$

and observe that the Schwartz kernel of  $R_s$  is given by

$$k_s(x, y) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} e^{\frac{i}{\hbar}(\frac{1}{2}q(x, \eta) + s\rho_3^\psi(x, \eta; h) - y\eta)} \tilde{b}_N(x, \eta) u(y) d\eta$$

where

$$\rho_3^\psi(x, \eta) = h^{3/2} r_3^\psi(h^{-1/2}x, h^{-1/2}\eta) = \psi(x_1 + x, \xi_0 + \eta) - \psi(x_1, \xi_0) - x\partial_x \psi(x_1, \xi_0) - \eta\partial_\eta \psi(x_1, \xi_0) - \frac{1}{2}q(x, \eta)$$

and  $\tilde{b}_N(x, \eta) = b_N(h^{-1/2}x, h^{-1/2}\eta)$  which lies in  $S_{0+}(\langle \rho \rangle^{3N})$ . Let's note  $\psi_s(x, \eta) = \frac{1}{2}q(x, \eta) + s\rho_3^\psi$  and remark that

$$\partial_{x\eta}^2 \psi_s = (1-s)\partial_{x\eta}^2 \psi(x_1, \xi_0) + s\partial_{x\eta}^2 \psi(x_1 + x, \xi_0 + \eta)$$

Since  $\partial_{x\eta}\psi$  does not vanish on  $\Omega_x \times \Omega_\eta$ , it has constant sign and hence,  $\partial_{x\eta}^2\psi_s(x, \eta) \neq 0$  on  $\Omega_x \times \Omega_\eta$ . We now analyze the kernel  $K_s$  of  $R_s^*R_s$  and find that this kernel is

$$\begin{aligned} K_s(x, y) &= \int_{\mathbb{R}} \overline{k_s(z, x)} k_s(z, y) dz \\ &= \int_{\mathbb{R}^3} \exp\left(\frac{i}{h}(\psi_s(z, \eta) - \psi_s(z, \xi) - y\eta + x\xi)\right) \overline{\tilde{b}_N(z, \xi)} \tilde{b}_N(z, \eta) d\eta d\xi dz \\ &= \int_{\mathbb{R}} d\xi e^{\frac{i}{h}(x-y)\xi} \underbrace{\int_{\mathbb{R}^2} \exp\left(\frac{i}{h}(\psi_s(z, \eta) - \psi_s(z, \xi) - y(\eta - \xi))\right) \overline{\tilde{b}_N(z, \xi)} \tilde{b}_N(z, \eta) d\eta dz}_{B_s(y, \xi)} \end{aligned}$$

which is the kernel of  $\text{Op}_h^R(B_s)$ . To analyze  $B_s$ , we want to apply a stationary phase theorem and we need to know the stationary points in the variable  $(z, \eta)$ , of the phase

$$\Phi_s(z, \eta, y, \xi) = \psi_s(z, \eta) - \psi_s(z, \xi) - y(\eta - \xi)$$

We have

$$\partial_z \Phi_s(z, \eta, y, \xi) = \partial_x \psi_s(z, \eta) - \partial_x \psi_s(z, \xi) = \partial_{x\eta}^2 \psi_s(z, \eta(z, \xi, \eta))(\eta - \xi)$$

for some  $\eta(z, \xi, \eta) \in [\eta, \xi]$ . Hence, since  $\partial_{x\eta}^2 \psi$  does not vanish,

$$\partial_z \Phi_s(z, \eta, y, \xi) = 0 \iff \xi = \eta$$

We also have

$$\partial_\eta \Phi_s(z, \eta, y, \xi) = \partial_\eta \psi_s(z, \eta) - y$$

so that the equation  $\partial_\eta \Phi_s(z, \xi, y, \xi) = 0$  has at most one solution, using again the fact that  $\partial_{x\eta}^2 \psi_s$  does not vanish. When there is no stationary point, a non stationary phase argument gives that  $|B_s(y, \xi)| \leq O(h^\infty) \langle \rho \rangle^{6N}$ . If there is a stationary point, it is given by a smooth function  $z_s(y, \xi)$  locally around  $(y, \xi)$  and a stationary phase argument shows that  $|B_s(y, \xi)| \leq C_M \langle \rho \rangle^{6N}$  where  $C_M$  depends on the first  $M$  semi-norms (for some universal integer  $M$ ) of  $b_N$ . We can treat the derivatives of  $B_s$  by differentiating under the integral and integration by part to obtain the same estimates for  $\partial^\alpha B_s$ , involving derivatives of  $\tilde{b}_N$  up to order  $|\alpha| + M$ . This shows that  $B_s \in S(\langle \rho \rangle^{3N})$ . We conclude the proof by passing from  $\text{Op}_h^R$  to  $\text{Op}_h$  as in the proof of Lemma 7-3.4 and we come back to  $h = 1$  by standard scaling arguments.  $\square$

### 7-7.1.3. Proof of Lemma 7-6.10.

Let's write  $u(x) = a(x)e^{i\beta_n \frac{x^2}{2h}}$  with  $a$  satisfying (7-6.19).

$$\begin{aligned} \text{Op}_h(m)u(x) &= \frac{1}{2\pi h} \int_{\mathbb{R}^2} m\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}(x-y)\xi} a(y) e^{i\beta_n \frac{y^2}{2h}} dy d\xi \\ &= \frac{1}{2\pi h} \int_{\mathbb{R}^2} m\left(x + \frac{y}{2}, \beta_n x + \xi\right) a(x+y) e^{-\frac{i}{h}y(\xi + \beta_n x)} e^{i\beta_n \frac{(x+y)^2}{2h}} dy d\xi \\ &= e^{i\beta_n \frac{x^2}{2h}} \underbrace{\frac{1}{2\pi h} \int_{\mathbb{R}^2} m\left(x + \frac{y}{2}, \beta_n x + \xi\right) a(x+y) e^{\frac{i}{2h}(\beta_n y^2 - 2y\xi)} dy d\xi}_{B(x)} \end{aligned}$$

To analyze  $B(x)$ , we invoke the stationary phase theorem in the quadratic case (see [Zwo12], Theorem 3.13) with the non singular symmetric matrix  $Q_n = \begin{pmatrix} \beta_n & -1 \\ -1 & 0 \end{pmatrix}$  and we follow the proof of [Zwo12], Theorem 4.17. We fix a cut-off function  $\chi \in C_c^\infty(\mathbb{R}^2)$  with  $\text{supp } \chi \subset B(0, 1)$  and  $\chi = 1$  in a neighborhood of 0. We write (with  $\chi_1 = \chi, \chi_2 = 1 - \chi$ )

$$B(x) = B_1(x) + B_2(x) \quad ; \quad B_i(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} \chi_i(y, \xi) m\left(x + \frac{y}{2}, \beta_n x + \xi\right) a(x+y) e^{\frac{i}{2h}(\beta_n y^2 - 2y\xi)} dy d\xi$$

We also set  $v_i(x) = B_i(x)e^{i\beta_n \frac{x^2}{2h}}$ . By the stationary phase expansion, we can expand  $B_1$  : for every  $N \in \mathbb{N}$ ,

$$B_1(x) = \sum_{k=0}^{N-1} \frac{h^k}{k!} \left( \frac{(Q_n^{-1}D, D)}{2i} \right)^k c(x, 0, 0) + R_N(x)$$

$$c(x, y, \xi) = \chi(y, \xi)m \left( x + \frac{y}{2}, \beta_n x + \xi \right) a(x + y) \quad ; \quad D = \begin{pmatrix} D_y \\ D_\xi \end{pmatrix}$$

$$R_N(x) = O \left( h^N \sup_{y, \xi} \sup_{k+l \leq 2N+2} |\partial_y^k \partial_\xi^l c(x, y, \xi)| \right)$$

We observe that :

- The first term of the expansion of  $B_1$  is given by  $m(x, \beta_n x)a(x)$  ;
- $B_1$  is smooth since we can derive under the integral and obtain the same kind of expansion;
- The  $k$ -th term, that is  $c_k(x) = \frac{1}{k!} \left( \frac{(Q_n^{-1}D, D)}{2i} \right)^k c(x, 0, 0)$  is a sum of terms of the form  $c_\alpha \partial^\alpha m(x, \beta_n x)a^{(l)}(x)$  with  $\alpha \in \mathbb{N}^2$ ,  $l \in \mathbb{N}$ ,  $|\alpha| + l \leq 2k$  and  $c_\alpha \in \mathbb{R}$ . The coefficients  $c_\alpha$  of this sum depend on  $Q_n$ . Since  $\beta_n = O(\varepsilon_0)$ , these coefficients are bounded uniformly in  $n$ . As a consequence, there exists  $c_{k,p} = c_{k,p}(m)$  such that for  $p \in \mathbb{N}$  with  $k + p > 0$ ,

$$q_p(c_k) \leq c_{k,p} h^{-2k\delta} q_{2k+p}(u)$$

Hence we set  $A_k u(x) = h^{2k\delta} c_k(x) i\beta_n \frac{x^2}{2h}$ , which has the required form in virtue of the expression of  $c_k(x)$ .

Concerning the remainder term, we have

$$R_N(x) \leq C_N(m) h^N h^{-(2N+2)\delta} \sup_{|y| \leq 1} \left( 1 + \frac{(x+y)^2}{\alpha_n h} \right)^{-2}$$

It is not hard to see that

$$\sup_{|y| \leq 1} \left( 1 + \frac{(x+y)^2}{\alpha_n h} \right)^{-2} \leq C \left( 1 + \frac{x^2}{\alpha_n h} \right)^{-2}$$

We choose  $M > 0$  such that  $M(1-2\delta) - 2\delta > 0$ , so that  $R_{N+M}(x) \leq C_N(m) h^{N(1-2\delta)} \sup_{|y| \leq 1} \left( 1 + \frac{(x+y)^2}{\alpha_n h} \right)^{-2}$ .

By writing,  $B_1(x) = \sum_{k=0}^{N-1} c_k(x) + \sum_{k=N}^{N+M-1} c_k(x) + R_{N+M}(x)$ , we see that

$$q_0 \left( v_1 - \sum_{k=0}^{N-1} h^k A_k u \right) \leq C_N h^{N(1-2\delta)} q_{2N+M}(u)$$

By differentiating under the integral, we can show similarly that

$$q_j \left( v_1 - \sum_{k=0}^{N-1} h^k A_k u \right) \leq C_N h^{N(1-2\delta)} q_{j+2N+M}(u)$$

It remains to analyze  $B_2$ . Since there is no stationary point in the integral defining  $B_2$ , we do repeated integration by part using the differential operator  $L(y, \xi) = \frac{(Q_n(y, \xi), D)}{|Q_n(y, \xi)|^2}$  which satisfies  $L \left( e^{\frac{i}{h} (Q_n(y, \xi), (y, \xi))} \right) = e^{\frac{i}{h} (Q_n(y, \xi), (y, \xi))}$ . Set  $c_2(x, y, \xi) = (1 - \chi(y, \xi))m(x + y/2, \beta_n + y/2)a(x + y)$ . Since  $|Q(y, \xi)| \geq c(y^2 + \xi^2)^{1/2}$  on  $\text{supp}(1 - \chi)$ , we observe that for  $M \in \mathbb{N}$ .

$$\begin{aligned} (L^*)^{2M} c_2(x, y, \xi) &\leq C_M (1 + y^2 + \xi^2)^{-M/2} h^{2(1-\delta)M} q_{2M}(u) (\alpha_n h)^{1/4} (1 + (x+y)^2 / \alpha_n h)^{-2} \\ &\leq C_M (1 + y^2 + \xi^2)^{-M/2} h^{2(1-\delta)M} q_{2M}(u) (\alpha_n h)^{1/4} (1 + x^2 / \alpha_n h)^{-2} (1 + y^2 / \alpha_n h)^{-2} \\ &\leq C_M (1 + y^2 + \xi^2)^{-M/2} h^{2(1-\delta)M} q_{2M}(u) (\alpha_n h)^{1/4} (1 + x^2 / \alpha_n h)^{-2} \end{aligned}$$

Integrating over  $\mathbb{R}^2$ , we find that  $|B_2(x)| \leq C_M h^{2(1-\delta)M} q_{2M}(u) (\alpha_n h)^{1/4} (1 + x^2 / \alpha_n h)^{-2}$ . In particular, with  $M = N$ ,  $q_0(v_2) \leq C_N h^{N(1-2\delta)} q_{2N}(u)$ . Similarly, we can show that  $q_j(v_2) \leq C_{j,N} h^{N(1-2\delta)} q_{j+2N}(u)$ . Since  $\text{Op}_h(m)u = v_1 + v_2$ , this concludes the proof of the Lemma 7-6.10.

□

### 7-7.2. Formulas for approximation of exponential.

We consider

- a Hilbert space  $H$  ( $H = L^2(\mathbb{R})$  for applications in this chapter) ;
- a bounded operator  $A : H \rightarrow H$  ;
- a parameter  $h$ ;
- a "class"  $\mathcal{C}$  of elements of  $H$ , that is a subspace of  $H$ .

We assume that for each  $j \in \mathbb{N}$ , there exists  $A_j : \mathcal{C} \rightarrow \mathcal{C}$  such that, in some sense to be specified in applications,  $Au \sim \sum_{j=0}^{\infty} h^j A_j u$ . More precisely, we assume that for all  $N \in \mathbb{N}$  and all  $u \in \mathcal{C}$ , we can write

$$Au = \sum_{j=0}^{N-1} h^j A_j u + h^N R_N(u)$$

We are interested in understanding the action of the operator  $e^{tA}$  on elements of  $\mathcal{C}$ . Recall that if  $u_0 \in H$ ,  $t \mapsto e^{tA} u_0$  is the solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt} u(t) = Au(t) \\ u(0) = u_0 \end{cases}$$

Moreover, we assume that  $A_0$  extends to a bounded operator on  $H$ , so that  $e^{tA_0}$  is a well-defined operator and we assume also that  $e^{tA_0}(\mathcal{C}) \subset \mathcal{C}$  for all  $t \in \mathbb{R}$ . We introduce in this appendix formulas and notations to give an approximation of  $e^{tA} u$ . Of course, the interesting mathematical work lies in controlling the following terms and the accuracy of the expansion, which is done in applications. Let us fix an integer  $N \in \mathbb{N}$  and an initial state  $u \in \mathcal{C}$ .

**Leading term.** For our leading term, we simply state  $u_0(t) = e^{tA_0} u$ . Then, we set  $R_0(t) = e^{tA} u - e^{tA_0} u$ . We have  $\dot{R}_0(t) = Ae^{tA} u - A_0 e^{tA_0} u$ . Hence, we have

$$\dot{R}_0(t) = AR_0(t) + \sum_{j=1}^{N-1} h^j A_j e^{tA_0} u + \tilde{r}_{0,N}(t) \quad ; \quad \tilde{r}_{0,N}(t) = h^N R_N(u_0(t)) \quad (7-7.1)$$

**First correction.** When  $N = 1$ , we stop. Otherwise, we can correct this first approximation by a term of order  $h$ . Of course, it is possible to write down directly a general formula for every  $j$ , but it seems to the author that the case  $j = 1$  helps to understand the general case. Let's try the Ansatz  $u_1(t) = e^{tA_0} v_1(t)$  and set

$$R_1(t) = e^{tA} u - e^{tA_0} (u + h v_1(t))$$

Then we have,

$$\begin{aligned} \dot{R}_1(t) &= \dot{R}_0(t) - h e^{tA_0} (A_0 v_1(t) + v_1'(t)) \\ &= AR_0(t) + \sum_{j=1}^{N-1} h^j A_j e^{tA_0} u + \tilde{r}_{0,N}(t) - h A e^{tA_0} v_1(t) + h(A - A_0) e^{tA_0} v_1(t) - h e^{tA_0} v_1'(t) \\ &= AR_1(t) + \sum_{j=1}^{N-1} h^j A_j e^{tA_0} u + \tilde{r}_{0,N}(t) + h(A - A_0) e^{tA_0} v_1(t) - h e^{tA_0} v_1'(t) \end{aligned}$$

To cancel the term of order  $h$  in the sum, we set

$$v_1(t) = \int_0^t e^{-sA_0} A_1 e^{sA_0} u ds \quad (7-7.2)$$

To proceed with our expansion, we need to assume that  $v_1(t) \in \mathcal{C}$  for all  $t \in \mathbb{R}$ . This will be the case in the applications, with precise control on  $v_1(t)$ .

**Higher order terms.** For convenience, let's note  $A_j(s) = e^{-sA_0} A_j e^{sA_0}$ . We can construct by induction a family of functions  $v_k(t)$  by setting  $v_0(t) = u$  and for  $1 \leq k \leq N-1$ ,

$$v_k(t) = \sum_{l=0}^{k-1} \int_0^t A_{k-l}(s) v_l(s) ds \quad (7-7.3)$$

For these formulas to hold, we assume that this construction ensures that  $v_k(t) \in \mathcal{C}$  for all  $t \in \mathbb{R}$ . It will be satisfied in applications. We also set

$$\tilde{r}_{k,N}(t) = (A - A_0) e^{tA_0} v_k(t) - \sum_{j=1}^{N-k-1} h^j A_j e^{tA_0} v_k(t)$$

and

$$R_k(t) = e^{tA} u - e^{tA_0} \sum_{l=0}^k h^l v_l(t)$$

$\tilde{r}_{k,N}(t)$  has to be seen as a term of order  $h^{N-k}$ . These formulas ensure that

$$\dot{R}_k(t) = AR_k(t) + \sum_{j=k+1}^{N-1} h^j \left( \sum_{l=0}^k A_{k-l} e^{tA_0} v_l(t) \right) + \sum_{j=0}^k h^j \tilde{r}_{j,N}(t)$$

In particular, when  $k = N-1$ ,

$$\dot{R}_{N-1}(t) = AR_{N-1}(t) + \sum_{j=0}^{N-1} h^j \tilde{r}_{j,N} \quad (7-7.4)$$

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**Titre:** Influence de la structure fractale de l'ensemble capté sur les résonances de scattering

**Mots clés:** Résonances, systèmes hyperboliques ouverts, chaos quantique, scattering par des obstacles

**Résumé:** Dans cette thèse, nous nous intéressons aux résonances dites de scattering, qui jouent un rôle fondamental dans l'étude de l'équation des ondes à l'extérieur d'obstacles. Nous tâcherons de comprendre certains aspects de la répartition de ces résonances, plus particulièrement en dimension  $d = 2$  et l'on s'intéresse à des obstacles à frontière lisse, strictement convexes qui vérifient une condition de non-eclipse. Nous établissons l'existence d'une région sans résonance sous l'axe réel - on parle de trou spectral - et améliorons des bornes supérieures pour le comptage de résonances dans des boîtes. Ces résultats font intervenir de façon cruciale la structure fractale de l'ensemble capté du flot du billard. Ces travaux sont dans la lignée des récentes avancées

obtenues dans l'étude des résonances des surfaces hyperboliques convexes co-compactes. L'outil fondamental pour l'obtention du trou spectral est un Principe d'Incertitude Fractal, récemment développé par S. Dyatlov et J. Bourgain, notamment.

Nous obtenons aussi des résultats similaires dans l'étude des résonances de l'opérateur semiclassique  $-h^2\Delta + V$  où  $V$  est un potentiel lisse à support compact. En fait, l'étude du problème des obstacles se fait naturellement à haute fréquence et permet d'utiliser les outils de l'analyse semiclassique. Les méthodes utilisées dans le cas des obstacles s'appliquent alors aux opérateurs semiclassiques  $-h^2\Delta + V$  grâce à deux articles de S. Nonnenmacher, J. Sjöstrand et M. Zworski.

**Title:** Influence of the fractal structure of the trapped set on the scattering resonances

**Keywords:** Resonances, open hyperbolic systems, quantum chaos, obstacle scattering

**Abstract:** In this thesis, we are interested in the study of the scattering resonances, which play a crucial role in the study of the wave equation outside obstacles. We aim at understanding some aspects of the distribution of these resonances, more particularly in dimension  $d = 2$  and for strictly convex obstacles with smooth boundary, satisfying a no-eclipse condition. We prove the existence of a band without resonance below the real axis - called a spectral gap - and improve upper bounds for counting functions of resonances in boxes. These results involve the fractal structure of the trapped set of the billiard flow. These works follow the recent results obtained in the study of convex co-compact hyperbolic surfaces. The

crucial tool in the proof of the spectral gap is a Fractal Uncertainty Principle, recently developed by S. Dyatlov and J. Bourgain, among others.

We also prove similar results in the study of semiclassical scattering resonances for operators  $-h^2\Delta + V$  where  $V$  is a smooth compactly supported potential. In fact, the study of the obstacle problem is naturally turned into a high-frequency problem which allows to use the methods of semiclassical analysis. The strategy used in the case of the obstacles can be applied to the semiclassical operators  $-h^2\Delta + V$  thanks to two papers of S. Nonnenmacher, J. Sjöstrand and M. Zworski.