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THÈSE PRÉSENTÉE
POUR OBTENIR LE GRADE DE
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DE L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUE ET INFORMATIQUE

SPÉCIALITÉ : Mathématiques appliquées et calcul scientifique

Par **AARACH Nacer**

Titre de la thèse : Étude de quelques approximations hydrostatiques des
modèles de la mécanique des fluides

Sous la direction de : **Marius-Gheorghe PAICU**

Soutenue le 20 mars 2023

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Résumé : Dans la première partie de cette thèse, nous étudions trois modèles décrivant une approximation hydrostatique des équations issues de la mécanique des fluides telles que les équations de la magnétohydrodynamique, les équations primitives et les équations hyperboliques de Navier-Stokes. Dans le cadre de ce projet, nous avons déterminé l'impact que la méthode des estimations analytiques pourrait avoir sur le problème de la limite hydrostatique des équations de la mécanique des fluides avec une faible viscosité dans un domaine mince. Les équations de la magnéto-hydrodynamique et hyperbolique de Navier-Stokes avec une viscosité évanescence et conditions Dirichlet au bord du domaine, font intervenir des équations de type Prandtl qui régissent le comportement du fluide dans une couche limite proche du bord. Ces équations semblent être mal posées dans des espaces de Sobolev, mais elles sont bien posées pour des données initiales analytiques. Nous avons démontré par la méthode des estimations analytique qu'il est possible d'obtenir un résultat d'existence globale en temps dans l'espace des fonctions analytiques, à donnée initiale petite.

Dans la deuxième partie de cette thèse, nous étudions le système magnétohydrodynamique dans tout l'espace de dimension trois \mathbb{R}^3 . Dans ce cadre, on a obtenu deux résultats d'existence de solutions fortes globales pour ce système magnéto-hydrodynamique homogène et inhomogène. En effet, on a étudié ces équations dans un cadre homogène et inhomogène où on a pu montrer que nos équations sont globalement bien posées lorsque la dérivée verticale de nos données initiales satisfait une condition de petitesse.

Mots-clés : Hydrostatique approximation, Existence globale, Couche limite, viscosité évanescence

Abstract: In the first part of this thesis, we study three models describing a hydrostatic approximation of equations derived from equations of fluid mechanics such as the equations of magnetohydrodynamic, the primitive equations and the hyperbolic equations of Navier-Stokes. In this project, we determined the impact that the method of analytical estimates could have on the problem of the hydrostatic limit of the equations of fluid mechanics with low viscosity in a thin domain. The equations of the magnetohydrodynamic and hyperbolic Navier-Stokes with evanescent viscosity and Dirichlet conditions at the edge, involve the Prandtl equations that govern the behavior of the fluid in a boundary layer close to the edge. These equations seem to be poorly posed in Sobolev spaces, but they are well posed for initial analytical data. We have demonstrated by the method of analytical estimates that it is possible to obtain a global existence result in time in the space of analytical functions, with a small initial data.

In the second part of this thesis, we study the magneto-hydrodynamic system in all the space of dimension three \mathbb{R}^3 . In this context, we obtained two results of existence of strong global solutions for this homogeneous and inhomogeneous magneto-hydrodynamic system. Indeed, we studied these equations in a homogeneous and an inhomogeneous framework where we were able to show that our equations are generally well-posed when the vertical derivation of our initial data satisfies a smallness condition.

Keywords: Hydrostatic approximation, global existence, Boundary layers, evanescent viscosity

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Chapter 1

Introduction (Version Française)

La physique est une science qui nous permet de comprendre les différents phénomènes liés à notre planète. Dans la théorie de la physique on observe trois branches principales qui sont: La physique classique, la physique quantique et la relativité générale. Dans notre thèse on s'intéresse à la physique classique, plus précisément sur "La mécanique des fluides" qui a pour principe d'étudier le comportement des fluides (liquides, gaz, plasmas, métaux liquides, eaux salées, électrolytes, ...) et des forces internes associées. La mécanique des fluides se divise en deux parties: statique et la dynamique des fluides. La statique des fluides représente l'étude des fluides au repos c'est à dire les fluides qui ne se déplacent pas et la dynamique correspondant l'étude des fluides en mouvement c'est à dire les fluides qui se déplacent au cours du temps.

Dans cette thèse, nous étudierons des équations de la dynamique des fluides. L'histoire de ces équations a commencé avec **L. Euler** et **J. d'Alembert** et leur travail sur des fluides parfaits, et par la suite avec **L. Navier** pour les fluides visqueux. Nous étudierons en outre les propriétés magnétiques des fluides conducteurs.

Partant du paradoxe de D'Alembert et de Leonhard Euler, l'étude de la dynamique des fluides a fourni de nombreuses raisons pour regarder de plus près la frontière qui confine l'écoulement, au 19e siècle. Les expériences physiques, ainsi que les résultats mathématiques, suggèrent que la couche mince proche de la frontière est une partie importante de la dynamique. Pour comprendre le mouvement du fluide réel, Ludwig Prandtl a introduit en 1904 le concept de couche limite et a proposé un modèle mathématique décrivant ce phénomène.

L'idée de L. Prandtl est de diviser en deux zones le champ d'écoulement : une à l'intérieur de la couche limite, dominée par la viscosité et créant la majorité de la traînée subie par la couche limite, et l'autre à l'extérieur de la couche limite, où nous pouvons négliger la viscosité, sans effets significatifs sur la solution. Ces deux zones permettent de considérer que le fluide est proche à l'intérieur de l'écoulement parfait incompressible décrit par les équations d'Euler incompressibles perturbé autour des bords par des équations des couches limites, ce qui simplifie considérablement les équations de Navier-Stokes associées. Ces équations de la couche limite, complétées par la condition d'incompressibilité et les conditions aux limites et conditions initiales appropriées, constitue le système de Prandtl, dont l'étude a permis de mieux comprendre les écoulements visqueux.

1.1 Concepts principaux en mécanique des fluides

1.1.1 Compressibilité et incompressibilité

Un fluide est dit compressible si son volume peut changer au cours du temps, c'est-à-dire qu'on peut par exemple le comprimer dans un espace plus restreint en lui appliquant une pression. Tous les gaz sont des fluides compressibles (air, oxygène, hydrogène, azote, etc). En revanche, un fluide incompressible est un fluide qui ne peut être ni comprimé ni dilaté, et son volume reste constant. Cette hypothèse est vérifiée pour l'eau liquide à température fixe et les métaux fondus. Elle est aussi vérifiée pour les gaz quand le nombre de Mach Ma est très faible convergeant vers zéro. En réalité, un fluide rigoureusement incompressible n'existe pas.

La principale différence entre fluides compressibles et fluides incompressibles est qu'une force appliquée à un fluide compressible modifie la densité du fluide alors qu'une force appliquée à un fluide incompressible n'en modifie pas la densité dans une mesure considérable.

1.1.2 Viscosité

La viscosité d'un fluide est une mesure de sa résistance à la déformation à une vitesse donnée. Dans un fluide, on distingue deux types de viscosité : "cinématique" et "dynamique". Dans les équations de Navier-Stokes, la viscosité cinématique joue un rôle important, car elle est de taille comparable au nombre de Rossby (ce nombre représente le rapport entre les forces d'inerties et les forces dues à la rotation qui caractérisent le mouvement d'un fluide dans un repère tournant). En réalité, le fluide est turbulent cela signifie que la viscosité fréquemment notée par le paramètre ν n'est pas une viscosité cinématique, mais plutôt une viscosité turbulente (c'est-à-dire une viscosité différente dans chaque direction). Le fait d'avoir une viscosité turbulente impact l'apparition de l'anisotropie dans le comportement des fluides : le mouvement dans la direction horizontale est bien plus important que dans la direction verticale.

Un fluide qui n'a pas de résistance au cisaillement est appelé fluide idéal ou inviscide. Une viscosité nulle n'est observée qu'à très basses températures dans les fluides parfaits. Autrement, la deuxième loi de la thermodynamique exige que tous les fluides aient une viscosité positive; ces fluides sont techniquement dits visqueux. Un fluide à forte viscosité comme le brai peut même sembler solide.

1.1.3 Nombre de Reynolds

Le nombre de Reynolds Re permet de prédire les modèles dans différentes situations d'écoulement du fluide. Ce nombre correspond au rapport entre les forces d'inertie et les forces visqueuses à l'intérieur d'un fluide qui a subi un mouvement interne relatif en raison des différentes vitesses du fluide. Une zone où ces forces changent leur comportement est appelée couche limite, comme la surface d'enclenchement à l'intérieur d'un tube. Un effet similaire est créé par l'introduction d'un flux de fluide à grande vitesse dans un fluide à faible vitesse, tel que les gaz chauds émis

par une flamme dans l'air. Le nombre de Reynolds est donné par

$$Re = \frac{\text{forces d'inertie}}{\text{forces visqueuses}} = \rho UL/\nu$$

où U et L sont les échelles de vitesse et de longueur du mouvement. Lorsque le nombre de Reynolds est très faible ($Re \ll 1$), les forces d'inertie ($\partial_t u + u \cdot \nabla u$) sont très faibles par rapport aux forces visqueuses et peuvent être négligées lors de la résolution des équations (NS). En revanche, lorsque ce nombre est très élevé ($Re \gg 1$), les forces inertielles sont plus grandes que les forces visqueuses. De tels problèmes d'écoulement turbulent sont de nature transitoire, un maillage suffisamment fin adapté à la taille des petits tourbillons de l'écoulement doit être utilisé.

1.2 Équations du mouvement en mécanique des fluides

Dans ce qui suit, nous nous intéressons aux équations différentielles de la mécanique des fluides incompressibles.

1.2.1 Fluide Incompressible

Dans la mécanique des fluides ou plus généralement dans la mécanique continue, l'écoulement incompressible désigne un écoulement dans lequel la densité du matériau est constante au sein du flux, dans un volume infinitésimal qui se déplace avec la vitesse d'écoulement. Une affirmation équivalente qui implique l'incompressibilité est que la divergence de la vitesse d'écoulement est nulle. (La dérivation ci-dessous illustre en quoi ces conditions sont équivalentes).

Nous commençons en déduisant du principe fondamental de la dynamique, les équations **d'Euler** et de **Navier-Stokes**, pour un fluide incompressible. Considérons maintenant l'ouvert Ω comme un élément du fluide dans \mathbb{R}^n . Les particules de fluide évoluent dans le temps selon des trajectoires $\psi(t, x)$, qui suivent l'équation suivante :

$$\begin{cases} \partial_t \psi(t, x) = u(t, \psi(t, x)), \\ u(0, x) = x. \end{cases}$$

où $u(t, x)$ est le champ de vitesse du fluide au point $\psi(t, x)$.

La condition d'incompressibilité du fluide aboutit mathématiquement à la relation

$$\forall t \in [0, T], \quad \text{Volume}(\Omega) = \text{Volume}(\psi(t, \Omega)).$$

En effectuant un changement de variable et en supposant que la fonction $\psi(t, x)$ est un difféomorphisme de classe C^1 , nous avons :

$$\int_{\Omega} 1 dx = \int_{\Omega} |\det J\psi(t, y)| dy.$$

Nous choisissons comme domaine $\Omega = \Omega_\epsilon = B(x, \epsilon)$ et en prenant la limite de ϵ à zéro, nous obtenons :

$$\det J\psi(t, x) = 1.$$

En outre, le théorème de Liouville nous affirme que toute solution matricielle de l'équation $X'(t) = A(t)X(t)$, vérifie :

$$(\det X)'(t) = (\operatorname{tr} A(t))\det X(t).$$

Nous constatons que

$$\partial_t(\det J\psi(t, x)) = \operatorname{tr}(\nabla u)(t, \psi(t, x)) \times \det J(\psi)(t, x).$$

Comme $\det J\psi(t, x) = 1$, alors $\partial_t(\det J\psi(t, x)) = 0$. En conséquence, nous obtenons le résultat suivant :

$$\operatorname{tr}(\nabla u)(t, \psi(t, x)) = \operatorname{div} u = 0.$$

Ainsi, la condition d'incompressibilité se traduit par l'équation $\operatorname{div} u = 0$.

1.2.2 Conservation de la masse

En physique, la loi de conservation de la masse ou le principe de conservation de la masse, stipule que pour tout système fermé à tout transfert de matière et d'énergie, la masse du système doit rester constante au cours du temps, car la masse du système ne peut pas changer, de sorte que la quantité ne peut être ni ajoutée ni supprimée. Par conséquent, la quantité de masse est conservée au fil du temps.

Mathématiquement, la conservation de la masse peut être formulée dans les domaines de la mécanique des fluides et de la mécanique des milieux continus. La conservation de la masse est généralement exprimée à l'aide de l'équation de continuité, donnée sous forme différentielle par

$$\partial_t \rho + \operatorname{div}(u\rho) = 0,$$

où ρ est la densité (masse par unité de volume), t est le temps, div est l'opérateur divergence (pour un champ de dimension trois, l'opérateur divergence est défini par $\operatorname{div} u = \partial_x u^1 + \partial_y u^2 + \partial_z u^3$), et $u = (u^1, u^2, u^3)$ est le champ de vitesse.

En particulier, si u est un champ à divergences nulle (cela signifie que $\operatorname{div} u = 0$) et que notre fluide est homogène c'est à dire $\rho(t, x) = \tilde{\rho}(t)$, alors la densité reste constante dans le temps, c'est-à-dire

$$\frac{\partial \rho}{\partial t} = 0.$$

Nous avons alors $\rho(t, x) = \rho_0$ pour tous t, x .

1.2.3 Expression des grandeurs cinématiques

Trajectoire : C'est un ensemble de points occupés successivement par la même particule de fluide. Une particule de fluide qui passe à l'instant t au point $\psi(t, x)$ a pour vitesse u définie par :

$$u(t, \psi(t, x)) = \partial_t \psi(t, x).$$

Accélération : On note l'accélération du fluide par le paramètre γ . Elle est définie comme la dérivée en temps de la vitesse du fluide u , notée $\frac{du}{dt}$, est donnée par la formule

$$\begin{aligned} \gamma &= \partial_t [u(t, \psi(t, x))] \\ &= \partial_t u(t, \psi(t, x)) + \partial_t \psi(t, x) \cdot \nabla u(t, \psi(t, x)) \\ (1.2.1) \quad &= (\partial_t u + u \cdot \nabla u)(t, \psi(t, x)). \end{aligned}$$

Loi de Newton: La loi de Newton nous donne

$$\rho \gamma(t, x) = F(t, x)$$

où γ est l'accélération et F désigne les forces extérieures qui s'exercent sur le fluide.

Par (1.2.1), on a

$$\rho(\partial_t u + u \cdot \nabla u) = F.$$

Equation d'Euler : Considérons l'écoulement incompressible d'un fluide parfait défini dans un champ de force massique f . En tout point du fluide $\psi(t, x)$, les champs de pression $p(t, \psi(t, x))$ et de vitesse $u(t, \psi(t, x))$ vérifiant l'état d'incompressibilité implique

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + f, \text{ l'incompressibilité implique } \operatorname{div} u = 0.$$

Equation de Navier-Stokes : La deuxième loi de Newton s'écrit :

$$F = \rho(\partial_t u + u \cdot \nabla u),$$

avec $F = -\operatorname{div} \sigma$, où σ est le tenseur des contraintes. Pour un fluide général du type différentiel d'ordre n , $\sigma = -pId + G(A_1, A_2, \dots, A_n)$ où G est le polynôme homogène dans ses variables

$$A_n = \frac{dA_{n-1}}{dt} + (\nabla u)^t A_{n-1} + A_{n-1}(\nabla u).$$

Pour un fluide Newtonien, nous avons :

$$-\operatorname{div} (pId + \nu \nabla u) = -\nabla p + \nu \Delta u = F,$$

donc,

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + \nu \Delta u,$$

où $\Delta = \sum_{i=1}^n \partial_i^2$. Les équations de Navier-Stokes sont alors données par le système d'équations suivant :

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) - \nu \Delta u = -\nabla p, \\ \operatorname{div} u = 0, \\ u(0, x) = u_0(x). \end{cases}$$

Pour les fluides homogène on peut considérer que $\rho = \rho_0 = 1$.

Si le fluide est inhomogène, le système d'équations prend la forme suivante

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \nu \Delta u = -\nabla p, \\ \operatorname{div} u = 0, \\ u(0, x) = u_0(x). \end{cases}$$

1.3 Équations magnéto-hydrodynamiques

Dans cette thèse, nous avons étudié les équations couplées de Navier-Stokes avec une approximation des équations de Maxwell pour le champ électromagnétique, qui décrivent les interactions du champ électrique et du champ magnétique. Ce modèle est appelé modèle magnétohydrodynamique (**MHD**). L'équation décrivant l'évolution du champ magnétique noté \mathbf{B} , est déduite des équations de Maxwell (Maxwell 1865, Jackson 1975)

$$(1.3.1) \quad \begin{cases} \eta \lambda \partial_t E = \operatorname{curl} B - \lambda j, \\ \partial_t B = -\operatorname{curl} E, \\ \operatorname{div} B = 0, \end{cases}$$

où E et B obéissent à l'équation de Maxwell de l'électromagnétique et désignent respectivement le champ électrique et magnétique du fluide, η et λ sont respectivement la permittivité et le coefficient de perméabilité magnétique. Nous supposons que le courant électrique j suit la loi d'Ohm :

$$j = \sigma(E + u \times B)$$

où σ représente la conductivité électrique du fluide. Dans le système magnétohydrodynamique usuel, nous supposons que les vitesses typiques du fluide sont faibles par rapport à la vitesse de la lumière c (on note que $c^2 = \frac{1}{\lambda \eta}$); l'équation de Maxwell-Ampère se simplifie alors en :

$$\operatorname{curl} B = \lambda j.$$

On combinant ces équations et en définissant la diffusivité magnétique par $\mu = 1/\lambda\sigma$, on obtient l'équation d'induction

$$\begin{aligned} \partial_t B &= -\operatorname{curl} E \\ &= -\operatorname{curl} \left(\frac{1}{\sigma} j - u \times B \right) \\ &= -\operatorname{curl} \left(\frac{1}{\sigma} \left(\frac{1}{\lambda} \operatorname{curl} B \right) - u \times B \right) \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\sigma\lambda} \operatorname{curl} \operatorname{curl} B + \operatorname{curl}(u \times B) \\
&= \mu\Delta B - u \cdot \nabla B + B \cdot \nabla u.
\end{aligned}$$

Le champ électromagnétique E agit sur le fluide par la force de Lorentz $F = j \times B + \rho E$, où ρ désigne la densité électrique. Cependant, dans les fluides denses, la charge électrostatique est neutralisée sur les distances macroscopiques, ce qui est connu sous le nom de quasi-neutralité. Par conséquent, nous pouvons supposer que $\rho = 0$, ce qui donne la force

$$(1.3.2) \quad F = j \times B.$$

Selon l'hypothèse qui dit que les vitesses typiques du fluide sont petites par rapport à la vitesse de la lumière c , la première équation de (1.3.1) et (1.3.2) se réécrit.

$$F = \frac{1}{\lambda} \operatorname{curl} B \times B = \frac{1}{\lambda} \left((B \cdot \nabla) B - \frac{1}{2} \nabla |B|^2 \right).$$

Nous obtenons donc les équations de Navier-Stokes couplées avec la force de Lorentz et les équations de Maxwell pour le champ électromagnétique.

$$(1.3.3) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla \left(p + \frac{|B|^2}{2} \right) + B \cdot \nabla B \\ \partial_t B + u \cdot \nabla B - \mu \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ u|_{t=0} = u_0, \quad B|_{t=0} = B_0. \end{cases}$$

Si le fluide est inhomogène, notre système d'équations prend la forme suivante

$$(1.3.4) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \nu \Delta u = -\nabla \left(p + \frac{|B|^2}{2} \right) + B \cdot \nabla B \\ \partial_t B + u \cdot \nabla B - \mu \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ u|_{t=0} = u_0, \quad B|_{t=0} = B_0. \end{cases}$$

Remarque 1.3.1. *Dans cette thèse, nous cherchons à obtenir l'existence globale de solutions du système homogène et inhomogène incompressible **MHD** en dimension trois d'espace, avec seulement une condition de petitesse sur la dérivée verticale de la donnée initiale dans certains espaces possédant des invariances d'échelle.*

1.4 La version hyperbolique des équations Navier-Stokes

Dans cette thèse, nous nous intéressons à la version hyperbolique des équations Navier-Stokes. Dans cet esprit, nous approchons les solutions de (NS) par des solutions d'équations hyper-

boliques, afin de corriger le paradoxe de la vitesse infinie de propagation dans les équations paraboliques.

Nous commençons d'abord par l'approximation proposée par Cattaneo en 1949 (et d'autres comme Chester, Vernotte, etc.) pour l'étude de l'équation de la chaleur comme limite des équations hyperboliques (voir [24,25]). Ils ont proposé le modèle hyperbolique suivant

$$\frac{1}{c^2}\partial_t^2\theta + \frac{1}{\beta}\partial_t\theta - \Delta\theta = 0.$$

Cette équation est appelée **équation du télégraphe**. Elle a une vitesse de propagation finie et est compatible à la fois avec le principe de relativité et la deuxième loi de la thermodynamique, de sorte qu'elle est un modèle physique satisfaisant.

Ces équations peuvent également être considérées comme une relaxation des équations d'Euler, ces équations ont été considérées par Brenier, Natalini and Puel dans [16]. Considérons les équations d'Euler incompressibles, à savoir

$$(1.4.1) \quad \begin{cases} \partial_t U + U \cdot \nabla U = -\nabla p, \\ \operatorname{div} U = 0, \\ U(0, x) = U_0(x). \end{cases}$$

Pour approcher ces équations, nous introduisons leur version relaxée, qui est obtenue par une perturbation singulière du terme non linéaire $U \cdot \nabla U = \nabla \cdot (U \otimes U)$, au moyen d'une variable à valeur matricielle supplémentaire V . Cela mène au système suivant :

$$(1.4.2) \quad \begin{cases} \partial_t U + \nabla \cdot (V) = -\nabla p, \\ \partial_t V + \nu \nabla U = -\frac{1}{\tau}(V - U \otimes U), \\ \operatorname{div} U = 0, \\ U(0, x) = U_0(x), \quad V(0, x) = V_0(x). \end{cases}$$

Notons que si τ converge vers zéro, nous récupérons formellement le système d'Euler incompressible (1.4.1). Considérons maintenant une échelle diffusive. Notons pour $\epsilon > 0$

$$(1.4.3) \quad \begin{cases} U^\epsilon(t, x) := \frac{1}{\sqrt{\epsilon}} U \left(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}} \right), \\ V^\epsilon(t, x) := \frac{1}{\epsilon} V \left(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}} \right), \\ p^\epsilon(t, x) := \frac{1}{\epsilon} p \left(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}} \right). \end{cases}$$

En prenant $\tau = 1$, le système (1.4.2) devient avec les inconnues de (1.4.3)

$$(1.4.4) \quad \begin{cases} \partial_t U^\epsilon + \nabla \cdot (V^\epsilon) = -\nabla p^\epsilon, \\ \sqrt{\epsilon} \partial_t V^\epsilon + \frac{\nu}{\sqrt{\epsilon}} \nabla U^\epsilon = -\frac{1}{\sqrt{\epsilon}} (V^\epsilon - U^\epsilon \otimes U^\epsilon), \\ \operatorname{div} U^\epsilon = 0, \\ U^\epsilon(0, x) = U_0^\epsilon(x), \quad V^\epsilon(0, x) = V_0^\epsilon(x). \end{cases}$$

Remarquons que notre mise à l'échelle peut être considérée comme une perturbation hyperbolique des équations de Navier-Stokes, qui est semblable à celle proposée par Cattaneo pour les équations de la chaleur [24], En éliminant l'inconnue V dans les équations (1.4.4), et en dénotant par $q^\epsilon = p^\epsilon + \epsilon \partial_t p^\epsilon$, nous obtenons

$$(1.4.5) \quad \begin{cases} \epsilon \partial_t^2 U^\epsilon + \partial_t U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \nu \Delta U^\epsilon = -\nabla q^\epsilon, \\ \operatorname{div} U^\epsilon = 0, \\ U^\epsilon(0, x) = U_0^\epsilon(x), \quad \partial_t U^\epsilon(0, x) = U_1^\epsilon(x). \end{cases}$$

Cette perturbation, considérée comme une relaxation des équations d'Euler, a été examinée pour la première fois par Brenier, Natalini et Puel dans [16]. Plus tard, cette équation a été considérée par Paicu et Raugel dans [108, 109].

Ce modèle admet plusieurs justifications dans de nombreux articles avec des méthodes différentes. Citons par exemple [74] où les auteurs ont considéré ces équations comme un modèle de formation de glace dans les lacs. D'un point de vue numérique, citons [80, 81] où cette approximation est utilisée pour calculer les solutions de Navier-Stokes en utilisant des schémas numériques tel que schémas d'éléments finis. Dans le deuxième chapitre, nous présenterons le résultat obtenu par Brenier, Natalini et Puel dans [16], ainsi que le résultat obtenu par Paicu et Raugel dans [108, 109] et le résultat de Coulaud, Hachicha et Raugel dans [38] de la solution globale en 2D et 3D. Dans cette thèse, nous obtenons un résultat d'existence et unicité globale des solutions de l'équation (1.4.5) dans un domaine mince $\mathbb{R} \times (0, 1)$, pour certaines données initiales analytiquement petites dans la variable tangentielle. Nous allons brièvement les énoncer dans le deuxième chapitre.

Nous allons maintenant expliciter la version hyperbolique des équations magnéto-hydrodynamiques. Nous commençons par rappeler la forme répandue du système bidimensionnel Navier-Stokes-Maxwell avec la loi de Cattaneo:

$$(1.4.6) \quad \begin{cases} \frac{\rho \nu \mathcal{J}}{\epsilon^2} \partial_\tau^2 \vec{U} + \rho (\partial_t \vec{U} + \vec{U} \cdot \nabla \vec{U}) - \rho \nu \Delta \vec{U} + \nabla P = \vec{J} \times \vec{B} & \text{La quantité de mouvement,} \\ \operatorname{div} \vec{U} = 0 & \text{conservation de la masse,} \\ \partial_t \vec{B} + \operatorname{curl} \vec{E} = 0 & \text{Loi de Faraday,} \\ \vec{J} = \sigma (\vec{E} + \vec{U} \times \vec{B}) & \text{Loi d'Ohm,} \\ \frac{1}{2} \partial_t \vec{E} + \mu_0 \vec{J} = \operatorname{curl} \vec{B} & \text{Loi d'Ampère,} \\ \operatorname{div} \vec{B} = 0 & \text{Loi de Gauss pour le magnétisme,} \\ \operatorname{div} \vec{E} = 0 & \text{Loi de Gauss.} \end{cases}$$

Le système et les variables d'état correspondantes dépendent de $(t, X, Y) \in (0, T) \times \mathbb{R} \times \mathbb{R}_+$, pour un temps positif $T > 0$.

Les constantes $\rho > 0$ et $\nu > 0$ sont respectivement la densité du fluide et la viscosité cinématique, tandis que $c > 0$ représente la vitesse de la lumière. Le premier terme $(\rho\nu\mathcal{J}/c^2)\partial_t^2\vec{U}$ dans l'équilibre de la quantité de mouvement linéaire est dû à la loi de Cattaneo [6, 16, 38, 108, 111, 122] et dépend d'une constante d'inertie générale $\mathcal{J} > 0$. Cette loi se développe autour d'un développement de Taylor du premier ordre d'une relation retardée sur le tenseur des contraintes de Cauchy

$$\mathbb{S}(t + t_{\text{rel}}, \cdot) = \nu \frac{\nabla u + \nabla u^T}{2}(t, \cdot)$$

où pour nous le temps de relaxation est donné par $t_{\text{rel}} = \rho\nu\mathcal{J}/c^2$. Cette forme particulière, sera en effet importante lors du redimensionnement de notre système pour les couches limites.

On note $\sigma > 0$ la conductivité électrique, et $\mu_0 > 0$ la perméabilité magnétique. On note aussi $\vec{U}(t, X, Y) = (U_1(t, X, Y), U_2(t, X, Y))^T \in \mathbb{R}^2$ et $\vec{B}(t, X, Y) = (B_1(t, X, Y), B_2(t, X, Y))^T \in \mathbb{R}^2$ le champ de vitesse et le champ magnétique du média, respectivement. La pression scalaire $P(t, X, Y) \in \mathbb{R}$ est le multiplicateur de Lagrange qui assure l'incompressibilité du champ de vitesse. La densité de courant $\vec{J} = (0, 0, J(t, X, Y))^T$ et le champ électrique $\vec{E} = (0, 0, E(t, X, Y))^T$ sont considérés comme des champs vectoriels tridimensionnels, étant perpendiculaires au plan dans lequel le mouvement du fluide se produit. Puisque nous avons affaire à la version bidimensionnelle des équations, nous allons clarifier la notation employée:

$$\begin{aligned} \vec{J} \times \vec{B} &= J\vec{B}^\perp = J \begin{pmatrix} -B_2 \\ B_1 \end{pmatrix}, \quad \text{curl}\vec{E} = \begin{pmatrix} \partial_Y E \\ -\partial_X E \end{pmatrix}, \\ \vec{U} \times \vec{B} &= \begin{pmatrix} 0 \\ 0 \\ U_1 B_2 - U_2 B_1 \end{pmatrix}, \quad \text{curl}\vec{B} = \begin{pmatrix} 0 \\ 0 \\ \partial_X B_2 - \partial_Y B_1 \end{pmatrix}. \end{aligned}$$

Les paramètres positifs ν , μ_0 et ε_0 correspondent respectivement à la viscosité cinématique, à la perméabilité magnétique et à la permittivité de l'espace libre. De plus, le paramètre σ représente la conductivité électrique du milieu.

Certains des termes de (1.4.6) sont redondants, en effet nous pouvons obtenir le système global en cinq équations en fonction de \vec{U} , \vec{B} et \vec{E} . Tout d'abord, nous formulons la loi de Faraday dans (1.4.6) uniquement en termes de champ magnétique \vec{B} , en utilisant la loi d'Ohm (1.4.6):

$$(1.4.7) \quad \partial_\tau \vec{B} = -\text{curl}\vec{E} = \text{curl}(\vec{U} \times \vec{B}) - \frac{1}{\sigma} \text{curl}\vec{J}.$$

De plus, pour se débarrasser de la densité du courant dans $\text{curl}\vec{J}$, nous appliquons l'opérateur curl à la loi d'Ampère:

$$\frac{1}{c^2} \partial_\tau (\text{curl}\vec{E}) + \mu_0 \text{curl}\vec{J} = \text{curlcurl}B,$$

ce qui implique

$$\text{curl}\vec{J} = \frac{1}{\mu_0 c^2} \partial_\tau^2 \vec{B} + \frac{1}{\mu_0} (\nabla \text{div}\vec{B} - \Delta B) = \frac{1}{\mu_0 c^2} \partial_\tau^2 \vec{B} + \frac{1}{\mu_0} (\nabla \text{div}\vec{B} - \Delta B)$$

Ainsi, on peut brancher cette dernière relation dans l'équation (1.4.7), pour finalement obtenir une forme hyperbolique de la loi d'Ampère en termes de champ magnétique \vec{B} :

$$(1.4.8) \quad \frac{1}{\sigma\mu_0c^2}\partial_\tau^2\vec{B} + \partial_\tau\vec{B} - \frac{1}{\sigma\mu_0}\Delta\vec{B} = \text{curl}(\vec{U} \times \vec{B}) = \vec{B} \cdot \nabla\vec{U} - \vec{U} \cdot \nabla\vec{B}.$$

De même, nous pouvons nous débarrasser de \vec{J} également dans l'équilibre de la quantité de mouvement linéaire dans (1.4.6) grâce à

$$\frac{\rho\nu\mathcal{J}}{c^2}\partial_t^2\vec{U} + \rho(\partial_t\vec{U} + \vec{U} \cdot \nabla\vec{U}) - \nu\Delta\vec{U} + \nabla P = \sigma(\vec{E} + \vec{U} \times \vec{B}) \times \vec{B} = \sigma\vec{E} \times \vec{B} + \sigma(\vec{B}(\vec{U} \cdot \vec{B}) - \vec{U}|\vec{B}|^2).$$

Nous sommes maintenant en état de réduire le nombre d'équations dans (1.4.6). En considérant le champ électrique $\vec{E}(t, X, Y) = (0, 0, E(t, X, Y))^T$ (dont la divergence est toujours nulle) et en rappelant la définition du champ vectoriel $\vec{B}^T = (-B_2, B_1)^T$, on rassemble finalement

$$(1.4.9) \quad \begin{cases} \frac{\rho\nu\mathcal{J}}{c^2}\partial_t^2\vec{U} + \rho(\partial_t\vec{U} + \vec{U} \cdot \nabla\vec{U}) - \rho\nu\Delta\vec{U} + \nabla P = \sigma(\vec{B}(\vec{U} \cdot \vec{B}) - \vec{U}|\vec{B}|^2) + \sigma E\vec{B}^T, \\ \text{div}\vec{U} = 0, \\ \frac{1}{\sigma\mu_0c^2}\partial_\tau^2\vec{B} + \partial_t\vec{B} - \frac{1}{\sigma\mu_0}\Delta\vec{B} = \vec{B} \cdot \nabla\vec{U} - \vec{U} \cdot \nabla\vec{B}, \\ \partial_t\vec{B} + \text{curl}\vec{E} = 0, \\ \text{div}\vec{B} = 0, \end{cases}$$

1.4.1 Couches limites

En mécanique des fluides, une couche limite est une partie du fluide à proximité immédiate d'une surface limitante où les effets visqueux sont au moins aussi importants que les effets inertiels. Le fluide dans la couche limite est caractérisé par de grandes variations en variable verticale à l'approche de la surface. Les couches limites apparaissent dans plusieurs phénomènes par exemple **autour des ailes d'un avion** et dans **l'atmosphère de la Terre**. Sur les ailes d'un avion, la couche limite est la partie de l'écoulement proche des ailes de l'avion, où les forces visqueuses déforment l'écoulement non visqueux environnant. D'autre part, dans le cas de l'atmosphère de la Terre, nous voyons la couche limite atmosphérique comme la couche d'air à proximité du sol.

Le concept fondamental de couche limite a été défini par L. Prandtl (1904), comme une couche de fluide se développant dans un écoulement avec un nombre de Reynolds \mathbf{Re} très élevé, c'est-à-dire avec une viscosité relativement faible par rapport aux forces inertielles.

La déduction des équations de couche limite a été l'un des progrès majeurs en dynamique des fluides. En utilisant l'analyse de l'ordre de grandeur, les équations bien connues de Navier-Stokes d'écoulement de fluides visqueux peuvent être grandement simplifiées à l'intérieur de la couche limite. En faisant l'approximation de la couche limite, notre flux peut être alors divisé en une partie non-visqueuse et une couche limite, qui est régie par une équation plus facile à résoudre. Les équations de continuité et de Navier-Stokes pour un flux continu incompressible en deux dimensions dans les coordonnées cartésiennes sont données par

$$(1.4.10) \quad \begin{cases} \partial_t u^\nu + u^\nu \partial_x u^\nu + v^\nu \partial_y u^\nu - \nu(\partial_x^2 u^\nu + \partial_y^2 u^\nu) = -\partial_x p^\nu, \\ \partial_t v^\nu + u^\nu \partial_x v^\nu + v^\nu \partial_y v^\nu - \nu(\partial_x^2 v^\nu + \partial_y^2 v^\nu) = -\partial_y p^\nu, \\ \partial_x u^\nu + \partial_y v^\nu = 0, \\ u^\nu(t, x, 0) = 0, \quad v^\nu(t, x, 0) = 0. \end{cases}$$

L'idée de Prandtl est d'injecter un ansatz pour que les deux composantes de la vitesse s'expriment sous une forme plus utile près du bord. En exploitant le fait que la partie la plus importante de l'écoulement se produit près du bord, cette idée conduit à un redimensionnement sur la variable verticale et les composantes de vitesse.

L'hypothèse suggérée par Prandtl est de transformer la variable verticale en $Y = \frac{y}{\sqrt{\nu}}$. Nous obtenons les redimensionnements de nos vitesses et de notre pression suivants

$$(1.4.11) \quad \begin{cases} u^\nu(t, x, y) = u\left(t, x, \frac{y}{\sqrt{\nu}}\right), \\ v^\nu(t, x, y) = \sqrt{\nu}v\left(t, x, \frac{y}{\sqrt{\nu}}\right), \\ p^\nu(t, x, y) = p\left(t, x, \frac{y}{\sqrt{\nu}}\right). \end{cases}$$

En insérant l'expression ci-dessus dans (1.4.10) et en conservant les termes d'ordre principaux, nous obtenons le système de Prandtl (en écrivant y au lieu de Y) :

$$(1.4.12) \quad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = -\partial_x p, \\ \partial_y p = 0, \\ \partial_x u + \partial_y v = 0, \\ u(t, x, 0) = 0, \quad v(t, x, 0) = 0. \end{cases}$$

Si on se place dans le domaine $\Omega = \mathbb{R} \times (0, \sqrt{\nu})$, cela signifie que nous sommes dans un domaine mince. Le système (1.4.12) sera complété donc par les conditions de Dirichlet homogène

$$u_{/y=0} = u_{/y=1} = 0 \text{ and } v_{/y=0} = v_{/y=1} = 0.$$

Remarque 1.4.1. *L'une des conséquences de ce travail formel est que nous avons éliminé du système l'équation d'évolution de la composante verticale de la vitesse. Cette composante verticale n'est récupérée qu'à partir de la condition d'incompressibilité. Nous appliquons cette élimination dans plusieurs modèles (par exemple système MHD et version hyperbolique du système Navier-Stokes) pour obtenir l'existence globale des solutions de ces modèles ainsi que leur stabilité.*

1.4.2 Équations primitives

Les équations primitives sont un ensemble d'équations différentielles non linéaires utilisées pour estimer le flux atmosphérique global. Les équations primitives sont une version simplifiée des équations de Navier-Stokes. Elles sont applicables dans le cas d'un fluide sur la surface d'une sphère en supposant que la composante verticale du mouvement est beaucoup moins importante que la composante horizontale du mouvement et que la couche du fluide est très mince par rapport au rayon de la sphère.

Si nous regardons la modélisation faite pour les fluides dans la section 1.2, nous avons négligé un effet important qui est la gravité terrestre qui a une grande influence dans l'étude des fluides géophysiques. Dans la nature, l'apparition de la stratification due à la densité variable du fluide, provoque le couplage entre les équations de Navier-Stokes et les équations de transport de la chaleur (ou stratification de la densité).

Nous pouvons également mentionner les phénomènes dus à la stratification, qui ne peuvent pas s'expliquer simplement par l'équation du mouvement rapide du fluide en rotation, et qui sont observés par de nombreux océanographes. C'est le phénomène "des eaux mortes" (pour une description, nous pouvons suggérer au lecteur la référence [39]). Il arrive qu'un bateau subisse une forte résistance dans son mouvement alors que l'eau à la surface est apparemment très calme. La cause est la présence de vagues internes. Le bateau navigue sur une mince couche d'eau relativement douce située sur une couche d'eau très salée. Il crée des vagues internes à l'interface, invisibles à la surface mais de grande énergie, provoquant ainsi la résistance au mouvement du bateau.

Pour décrire le mouvement des fluides géophysiques à l'échelle, nous considérons deux phénomènes très différents : la rotation tendant à une distribution en colonnes verticale et la stratification, tendant à maintenir une distribution en couches horizontales de même densité. En utilisant la conservation de masse, la quantité de mouvement et l'approximation de Boussinesq, nous obtenons le système des équations primitives

$$(1.4.13) \quad \begin{cases} \partial_t U + U \cdot \nabla U - \nu \Delta U + \frac{T}{Fr} = -\nabla p, \\ \partial_t T + U \cdot \nabla T - \nu' \Delta T = 0, \\ \operatorname{div} U = 0, \\ U|_{t=0} = U_0, \quad T|_{t=0} = T_0, \end{cases}$$

où U est le champ de vitesse, T est la température potentielle et p est la pression. Les paramètres ν et ν' sont respectivement la viscosité cinématique et la diffusivité thermique, Fr est le nombre Froude.

Chapter 2

Historique des résultats et contributions de la thèse (Version Française)

Dans le cadre de cette thèse, nous nous intéressons à des systèmes issus de la mécanique des fluides géophysiques newtoniens. Dans une première partie, on considère des approximations hydrostatiques des fluides géophysiques newtoniens, à savoir les équations de Navier-Stokes dans leur version hyperbolique, les équations de la magnéto-hydrodynamique et les équations primitives, dans des domaines minces. Dans une deuxième partie, nous allons étudier les systèmes magnéto-hydrodynamique dans un cadre anisotrope dans tout l'espace de dimension 3.

On commencera notre étude par des modèles de fluides géophysiques dans un domaine mince complété par la condition de Dirichlet au bord, de sorte que l'étendue verticale du domaine est négligeable devant son homologue horizontale. Dans ce cas, la viscosité remise à l'échelle n'est pas isotrope. On utilisera donc des viscosités anisotropes (viscosités turbulentes). L'idée principale est de prendre des espaces invariants par changement d'échelle tels que les espaces de Besov et Sobolev. L'objectif est d'effectuer un changement d'échelle afin que nous soyons proches de la limite de notre domaine d'étude. Le changement considéré pour nos données est $U(t, x, y) = \left(u^\epsilon(t, x, \frac{y}{\epsilon}), v^\epsilon(t, x, \frac{y}{\epsilon}) \right)$ où ϵ est la largeur du domaine. La disparition de la viscosité dans la limite combinée aux conditions de Dirichlet aux bords du domaine engendre l'apparition de couches limites. Les équations qui décrivent l'évolution du fluide dans des couches limites minces ont été la découverte de Prandtl [119] en 1904 pour expliquer l'écart entre les conditions aux limites vérifiées par un fluide idéal et un fluide visqueux à viscosité évanescence.

Nos modèles sont étudiés dans la première partie, dans un cadre de couches limites ayant une faible viscosité et diffusivité magnétique ($\nu = \epsilon^2$ et $\sigma = \epsilon^2$) et sont aussi dits à viscosité évanescence. En utilisant un changement d'échelle sur nos solutions, on obtient des systèmes anisotropes. Par un passage à la limite ($\epsilon \rightarrow 0$), ces systèmes anisotropes tendent vers des équations hydrostatiques de type Prandtl.

Dans un cadre anisotrope, où la viscosité ainsi que la diffusivité magnétique horizontale sont faibles par rapport à leurs homologues verticales, l'intérêt est d'obtenir des preuves d'existence des solutions fortes dans des espaces qui sont presque optimaux. On rappelle qu'un espace est dit optimal ou critique s'il est invariant par changement d'échelle pour les systèmes considérés, c'est à dire que si $u(t, x)$ est une solution de notre équation Navier-Stokes hyperbolique, alors $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ et $p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x)$ est aussi une solution. Dans notre cas, les espaces critiques dans lesquels on travaillera seront l'espace de Besov $\mathcal{B}^{\frac{1}{2}, 0}$ défini par:

$$\mathcal{B}^{\frac{1}{2}, 0} \triangleq \left\{ f \in \mathcal{S}'_h : \|f\|_{\mathcal{B}^{\frac{1}{2}, 0}} \triangleq \|(2^{\frac{q}{2}} \|\Delta_q^h f\|_{L^2})_{q \in \mathbb{Z}}\|_{\ell^1(\mathbb{Z})} < +\infty \right\}.$$

Dans la deuxième partie de cette thèse, nous étudions le système magnéto-hydrodynamique dans tout l'espace \mathbb{R}^3 . Dans ce cadre, on a obtenu deux résultats d'existence de solutions fortes globales pour ce système magnéto-hydrodynamique. En effet, il s'agissait d'étudier le cas où la dérivée verticale de nos données initiales satisfait une condition non-linéaire de petitesse.

2.1 La version hyperbolique des équations de Navier-Stokes

Avant de commencer à présenter les résultats de cette thèse sur le modèle Navier-Stokes hyperbolique, on commence par rappeler quelques résultats fondamentaux sur l'existence, la régularité et l'unicité connus pour ce type d'équations.

2.1.1 Rappel sur les résultats connus d'existence et de régularité des solutions

Brenier, Natalini et Puel dans [16] ont considéré pour la première fois la version hyperbolique de Navier-Stokes (1.4.5) sur $\mathbb{R}^+ \times \mathbb{T}^2$, où \mathbb{T}^2 est le tore bidimensionnel $\mathbb{R}^2/\mathbb{Z}^2$. On rappelle que l'équation est donnée par

$$(2.1.1) \quad \begin{cases} \tau \partial_t^2 u^\tau + \partial_t u^\tau + u^\tau \cdot \nabla u^\tau - \nu \Delta u^\tau = -\nabla p^\tau, \\ \operatorname{div} u^\tau = 0, \\ u^\tau(0, x) = u_0^\tau(x), \quad \partial_t u^\tau(0, x) = u_1^\tau(x). \end{cases}$$

Cette équation, peut être réécrite sous la forme:

$$(2.1.2) \quad \begin{cases} \partial_t u^\tau + \operatorname{div}(V^\tau) = -\nabla p^\tau, \\ \sqrt{\tau} \partial_t V^\tau + \frac{\nu}{\sqrt{\tau}} \nabla u^\tau = -\frac{1}{\sqrt{\tau}} (V^\tau - u^\tau \otimes u^\tau), \\ \operatorname{div} u^\tau = 0, \\ u^\tau(0, x) = u_0^\tau(x), \quad V^\tau(0, x) = V_0^\tau(x). \end{cases}$$

Ils ont démontré que si les données initiales (u_0^τ, V_0^τ) sont petites dans $H^2(\mathbb{T}^2)^2 \times H^1(\mathbb{T}^2)^4$ alors ils arrivent à obtenir un résultat d'existence globale des solutions, ainsi que la convergence de ces solutions vers les solutions de Navier-Stokes avec une donnée initiale régulière. Dans leur résultat, les auteurs montrent que cette convergence n'est pas seulement formelle. Le résultat obtenu dans [16] est le suivant

Théorème 2.1.1. *Soit $T \geq 0$ et u_0 un champ de vecteurs régulier de divergence nulle sur le tore \mathbb{T}^2 . Soit $\tau > 0$ et (u_0^τ, V_0^τ) une suite de données initiales pour le système (2.1.2). On suppose qu'il existe une constante C positive indépendante de τ telle que*

$$\|u_0^\tau\|_{H^1(\mathbb{T}^2)} \leq C, \quad \|V_0^\tau\|_{H^2(\mathbb{T}^2)} \leq \frac{C}{\sqrt{\tau}},$$

et

$$\|u_0^\tau\|_{H^2(\mathbb{T}^2)} \leq \frac{C_0}{K_s \sqrt{\tau}}, \quad \|u_0^\tau - u_0\|_{L^2(\mathbb{T}^2)}^2 \leq C\sqrt{\tau},$$

où $0 < C_0 < 1$ et K_s est la constante intervenant dans l'inégalité de Sobolev sur $H^2(\mathbb{T})$. Si u est la solution de Navier-Stokes avec u_0 donnée initiale, alors il existe une constante C_T telle que

$$\sup_{t \in [0, T]} \|u^\tau - u\|_{L^2(\mathbb{T}^2)}^2 \leq C_T \sqrt{\tau}.$$

On rappelle que l'espace de Sobolev a pour définition dans \mathbb{R}^n

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \widehat{f} \in L^1_{loc}(\mathbb{R}^n), \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < +\infty \right\},$$

où \mathcal{S}' est l'espace des distributions tempérées dans \mathbb{R}^n . On définit la norme dans H^s par

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Pour prouver leur résultat, ils ont utilisé la méthode d'énergie hyperbolique et l'énergie modulée de Dafermos pour la convergence. L'énergie obtenue en multipliant l'équation (2.1.1) par $u^\tau + 2\tau \partial_t u^\tau$ puis en intégrant en espace est

$$E_\tau(t) = \int_{\mathbb{T}^2} \left(\frac{1}{2} |u^\tau(t) + \tau \partial_t u^\tau(t)|^2 + \frac{\tau^2}{2} |\partial_t u^\tau(t)|^2 + \tau |\nabla u^\tau(t)|^2 \right) dx.$$

Afin qu'ils puissent avoir des informations sur la convergence, ils ont modulé cette énergie par une solution u de l'équation Navier-Stokes, obtenant l'énergie modulée de Dafermos

$$E_{\tau, u}(t) = \int_{\mathbb{T}^2} \left(\frac{1}{2} |u^\tau(t) - u(t) + \tau \partial_t u^\tau(t)|^2 + \frac{\tau^2}{2} |\partial_t u^\tau(t)|^2 + \tau |\nabla u^\tau(t)|^2 \right) dx.$$

Cette énergie permet de contrôler la norme $\|u^\tau(t) - u(t)\|_{L^2}$.

La méthode que les auteurs utilisent pour prouver leur résultat reste restreinte au cadre bi-dimensionnel et demande beaucoup de régularité sur les données initiales.

Une amélioration de ce résultat a été faite par Paicu et Raugel dans [108, 109], qui ont approché les équations de Navier-Stokes avec force extérieure f qui peut dépendre du temps par une perturbation hyperbolique

$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p + f.$$

grâce à la même méthode d'énergie introduite dans [16] et une inégalité de type Strichartz sur les hautes fréquences, les auteurs obtiennent des résultats d'existence globale et d'unicité pour ce type d'équation avec des données initiales dans $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$. Pour la convergence, les auteurs obtiennent des résultats qui améliorent ceux de Brenier, Natalini et Puel.

Plus précisément, ils obtiennent les résultats suivants. Pour simplifier, on considère que la force extérieure f est nulle

Théorème 2.1.2. *Soit $\tau_0 > 0$, il existe une constante strictement positive K telle que, si $0 < \tau < \tau_0$ et si les vecteurs à divergence nulle $u_0^\tau \in H^1(\mathbb{R}^2)^2$, $u_1^\tau \in L^2(\mathbb{R}^2)^2$ satisfont*

$$\|u_0^\tau\|_{L^2} + \|u_1^\tau\|_{L^2} + \|\nabla u_0^\tau\|_{L^2} \leq K,$$

alors l'équation (2.1.1) admet une solution unique intégrale globale

$$(u^\tau, \partial_t u^\tau) \in C^0(\mathbb{R}^+, H^1(\mathbb{R}^2)^2) \times C^0(\mathbb{R}^+, L^2(\mathbb{R}^2)^2).$$

Lorsqu'on fait converger τ vers 0, la limite formelle du système (2.1.1) est donnée par le système de Navier-Stokes. Dans le théorème suivant, Paicu et Raugel montrent que cette limite n'est pas seulement formelle.

Théorème 2.1.3. *Pour tous nombres strictement positifs R et T , il existe un nombre strictement positif $\tau_1 = \tau_1(R, T)$ tel que, pour $0 < \tau < \tau_1$, si le couple de vecteurs à divergence nulle $(u^\tau, \partial_t u^\tau) \in H^1(\mathbb{R}^2)^2$, $u_0^\tau \in L^2(\mathbb{R}^2)^2$ vérifie*

$$\|u_0^\tau\|_{L^2} + \sqrt{\tau}\|u_1^\tau\|_{L^2} \leq R,$$

alors le système (2.1.1) admet une solution unique intégrale

$$(u^\tau, \partial_t u^\tau) \in C^0(\mathbb{R}^+, H^1(\mathbb{R}^2)^2) \times C^0(\mathbb{R}^+, L^2(\mathbb{R}^2)^2).$$

Pour tout $0 \leq t \leq T$,

$$\sqrt{\tau}\|\partial_t(t(u^\tau(t) - u(t)))\|_{L^2} + \|t(u^\tau(t) - u(t))\|_{H^1} \leq \tau e^K,$$

où K est une constante qui ne dépend que de R et T (voir théorème 3 [108]).

Dans le cadre de deux dimensions et trois dimensions, on peut aussi citer les travaux de Hachicha dans [68], qui a obtenu un résultat d'existence globale et d'unicité des solutions pour le système (2.1.1) avec des données initiales bien choisies. Le résultat en dimension 2 est le suivant

Théorème 2.1.4. *Soient $0 < s, \delta < 1$ et $u_0 \in H^s(\mathbb{R}^2)^2$ un champ de vecteur à divergence nulle. Soit $(u_0^\tau, u_1^\tau)|_{\tau>0} \in H^{1+\delta}(\mathbb{R}^2)^2 \times H^\delta(\mathbb{R}^2)^2$ une suite de données initiales à divergence nulle du système (2.1.1). Supposons que les conditions suivantes soient vérifiées :*

$$\left\{ \begin{array}{l} \|u_0^\tau - u_0\|_{L^2} + \tau \|u_1^\tau\|_{L^2} + \tau^{\frac{1}{2}} \|u_0^\tau\|_{\dot{H}^1} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{\frac{1+\delta}{2}} \|u_0^\tau\|_{\dot{H}^{1+\delta}} + \tau^{\frac{\delta}{2}} \|u_0^\tau\|_{\dot{H}^\delta} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{1+\frac{\delta}{2}} \|u_1^\tau\|_{\dot{H}^\delta} = \mathcal{O}(1). \end{array} \right.$$

Alors, si τ est assez petit, l'équation (2.1.1) admet une unique solution globale u^τ qui converge, en norme $L_{loc}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)^2)$, lorsque τ tend vers 0, vers l'unique solution forte u de Navier-Stokes, avec donnée initiale u_0 . De plus, pour tout $T > 0$, il existe une constante C_T ; qui ne dépend que de T et u , telle que

$$\sup_{t \in [0; T]} \int_{\mathbb{R}^2} |u^\tau - u|^2 dx \leq C_T \tau^{(\frac{s}{2})^-}.$$

En dimension 3, l'auteur a imposé une condition de petitesse supplémentaire sur la donnée initiale u_0^τ . Elle obtient un résultat analogue à celui de la dimension 2.

Théorème 2.1.5. *Soient $0 < s, \delta < 1$ et $u_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ un champ de vecteur à divergence nulle. Soit $(u_0^\tau, u_1^\tau)|_{\tau>0} \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$ une suite de données initiales à divergence nulle du système (2.1.1) telle que $\|u_0^\tau\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{16}$. Supposons que les conditions suivantes soient vérifiées*

$$\left\{ \begin{array}{l} \|u_0^\tau - u_0\|_{\dot{H}^{\frac{1}{2}}} + \tau \|u_1^\tau\|_{\dot{H}^{\frac{1}{2}}} + \tau^{\frac{1}{2}} \|u_0^\tau\|_{\dot{H}^{\frac{3}{2}}} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{\frac{1+\delta}{2}} \|u_0^\tau\|_{\dot{H}^{\frac{3}{2}+\delta}} + \tau^{\frac{\delta}{2}} \|u_0^\tau\|_{\dot{H}^{\frac{1}{2}+\delta}} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{1+\frac{\delta}{2}} \|u_1^\tau\|_{\dot{H}^{\frac{1}{2}+\delta}} = \mathcal{O}(1). \end{array} \right.$$

Alors, si τ est assez petit, l'équation (2.1.1) admet une unique solution globale u^τ qui converge, en norme $L_{loc}^\infty(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3)$, lorsque τ tend vers 0, vers l'unique solution forte u de Navier-Stokes, avec donnée initiale u_0 . De plus, pour tout $T > 0$, il existe une constante C_T qui ne dépend que de T et de u , telle que

$$\sup_{t \in [0; T]} \int_{\mathbb{R}^3} |\Lambda^{\frac{1}{2}}(u^\tau - u)|^2 dx \leq C_T \tau^{(\frac{s}{2})^-}.$$

L'idée principale de la preuve est d'utiliser la méthode d'énergie hyperbolique inspirée du papier de Brenier, Natalini et Puel dans [16]. On rappelle que l'espace de Sobolev homogène est défini pour tous $s \in \mathbb{R}$ par

$$\dot{H}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \widehat{f} \in L_{loc}^1(\mathbb{R}^n), \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi < +\infty \right\},$$

où \mathcal{S}' est l'espace des distributions tempérées dans \mathbb{R}^n . On définit la semi-norme dans H^s par

$$\|f\|_{\dot{H}^s} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

2.1.2 Contribution de la thèse

Dans le chapitre 3 de cette thèse, pour $\epsilon > 0$ et $\tau > 0$ on considère le système suivant :

$$(HNS) \quad \begin{cases} \tau \partial_t^2 U^\tau + \partial_t U^\tau + U^\tau \cdot \nabla U^\tau - \epsilon^2 \Delta U^\tau + \nabla P^\tau = 0, \\ \operatorname{div} U^\tau = 0, \\ U^\tau|_{t=0} = U_0^\tau, \quad \partial_t U^\tau|_{t=0} = U_1^\tau, \end{cases}$$

Tous les résultats qu'on a cité précédemment, sont obtenus lorsque leur domaine Ω est \mathbb{R}^n ($n = 2, 3$) ou \mathbb{T}^n . Dans notre cas, on a voulu étudier le système lorsque le fluide évolue dans un domaine mince avec des conditions de Dirichlet au bord. On note $\mathbf{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\}$, tel que ϵ est la largeur du domaine. L'avantage de travailler dans un domaine mince complété par les conditions de Dirichlet au bord est de faire apparaître dans notre étude les équations de type Prandtl dans leur version hyperbolique, c'est à dire qu'on va avoir un terme de type ∂_t^2 dans l'équation de Prandtl classique. L'idée principale du résultat est d'obtenir l'existence et l'unicité des solutions pour le système (HNS) dans un domaine mince. Pour simplifier notre système, nous éliminons la dépendance τ , pour cela nous effectuons la remise à l'échelle

$$(2.1.3) \quad U^\tau(t, X) = \tau^{-\frac{1}{2}} U(\tau^{-1}t, \tau^{-\frac{1}{2}}X), \quad P^\tau(t, X) = \tau^{-1} P(\tau^{-1}t, \tau^{-\frac{1}{2}}X).$$

On remplace dans le système (HNS) pour obtenir

$$(2.1.4) \quad \begin{cases} \partial_t^2 U + \partial_t U + U \cdot \nabla U - \epsilon^2 \Delta U + \nabla P = 0, \\ \operatorname{div} U = 0, \\ U|_{t=0} = \sqrt{\tau} U_0^\tau(\sqrt{\tau}X) = U_0, \\ \partial_t U|_{t=0} = \tau^{\frac{3}{2}} U_1^\tau(\sqrt{\tau}X) = U_1 \end{cases}$$

ce système est complété par les conditions de Dirichlet au bord

$$U|_{y=0} = 0 \quad \text{et} \quad U|_{y=\epsilon} = 0$$

et par la donnée initiale

$$U|_{t=0} = U_0^\epsilon = \left(u_0 \left(x, \frac{y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{y}{\epsilon} \right) \right).$$

On va chercher des solution à notre système de la forme

$$(2.1.5) \quad \begin{cases} U(t, x, y) = \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ P(t, x, y) = p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

Soit $\mathbf{S} := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$. Après un changement naturel d'échelle le système (4.1.4)

devient le système de Navier-Stokes hyperbolique anisotrope mis à l'échelle suivant:

$$(2.1.6) \quad \begin{cases} \partial_t^2 u^\epsilon + \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon + \partial_x p^\epsilon = 0, & \text{dans }]0, \infty[\times \mathbf{S} \\ \epsilon^2 (\partial_t^2 v^\epsilon + \partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) + \partial_y p^\epsilon = 0, & \text{dans }]0, \infty[\times \mathbf{S} \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, & \text{dans }]0, \infty[\times \mathbf{S} \\ (u^\epsilon, v^\epsilon)|_{t=0} = (u_0, v_0) \text{ and } \partial_t (u^\epsilon, v^\epsilon)|_{t=0} = (u_1, v_1), & \text{dans } \mathbf{S} \\ (u^\epsilon, v^\epsilon)|_{y=0} = (u^\epsilon, v^\epsilon)|_{y=1} = 0. & \end{cases}$$

Prenant $\epsilon \rightarrow 0$ dans le système (2.1.6), on obtient l'équation de Prandtl hyperbolique

$$(2.1.7) \quad \begin{cases} \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = 0, & \text{dans }]0, \infty[\times \mathbf{S} \\ \partial_y p = 0, & \text{dans }]0, \infty[\times \mathbf{S} \\ \partial_x u + \partial_y v = 0, & \text{dans }]0, \infty[\times \mathbf{S} \\ u|_{t=0} = u_0, & \text{dans } \mathbf{S} \\ \partial_t u|_{t=0} = u_1, & \text{dans } \mathbf{S}, \end{cases}$$

où la vitesse $U = (u, v)$ satisfait la condition aux limites de non-glissement de Dirichlet

$$(u, v)|_{y=0} = (u, v)|_{y=1} = 0.$$

Le but est de justifier le passage à la limite du système (2.1.6) vers le système (2.1.7) lorsqu'on considère des données initiales analytiques. Travailler avec des données analytiques permet de surmonter la difficulté du terme non-linéaire $v \partial_y u$ dans (2.1.7), dans le cas général ou on n'impose pas d'hypothèses structurelles sur les données initiales.

Notre premier résultat consiste à prouver que le système de Prandtl (2.1.7) avec des données initiales analytiques et petites est globalement bien posé.

Théorème 2.1.6. *Soient $a > 0$ et $s \in]0, 1[$. Il existe une constante $c_0 > 0$ suffisamment petite, telle que pour (u_0, u_1) de données initiales à divergence nulle vérifiant la condition de compatibilité $\int_0^1 u_0 dy = 0$, si les données initiales satisfont la condition de petitesse*

$$(2.1.8) \quad \left\| e^{a|D_x|} (u_0 + u_1) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} \partial_y u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \| e^{a|D_x|} u_1 \|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a,$$

alors le système (2.1.7) admet une solution globale unique u satisfaisant l'estimation

$$(2.1.9) \quad \begin{aligned} & \frac{1}{2} \| e^{\mathcal{R}t} (u + \partial_t u)_\phi \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \| e^{\mathcal{R}t} \partial_y u_\phi \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \| e^{\mathcal{R}t} (\partial_t u)_\phi \|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & + \frac{1}{2} \| e^{\mathcal{R}t} (\partial_t u)_\phi \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \leq C \left(\| e^{a|D_x|} \partial_y u_0 \|_{\mathcal{B}^s} + \| e^{a|D_x|} (u_0 + u_1) \|_{\mathcal{B}^s} + \| e^{a|D_x|} u_1 \|_{\mathcal{B}^s} \right), \end{aligned}$$

où $C = C(s)$ et on définit w_ϕ par

$$(2.1.10) \quad \begin{cases} w_\phi(t, x, y) = e^{\phi(t, D_x)} w(t, x, y) \triangleq \mathcal{F}_h^{-1} (e^{\phi(t, \xi)} \hat{w}(t, \xi, y)), \\ \phi(t, \xi) = (a - \lambda \theta(t)) |\xi|, \end{cases}$$

La quantité $\theta(t)$ décrit l'évolution de la bande d'analyticité au cours du temps, cette quantité

est positive pour tout $t \in \mathbb{R}_+^*$ satisfait l'équation $\dot{\theta}(t) = \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$ et lorsque $t = 0$ on a $\theta(0) = 0$.

\mathcal{R} est une constante déterminée par l'inégalité de Poincaré pour la bande \mathbf{S} . L'espace fonctionnel utilisé pour obtenir ce résultat est l'espace de Besov du type Chemin-Lerner.

$$\|u\|_{\tilde{L}_T^p(\mathcal{B}^s(\mathbf{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \left\| \Delta_q^h u(t) \right\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

Remarque 2.1.1. L'idée principale pour prouver les deux théorèmes ci-dessus est de contrôler la nouvelle inconnue u_ϕ définie par (2.1.10), où u est la vitesse horizontale et u_ϕ est une fonction pondérée de u dans la variable de Fourier duale avec une fonction exponentielle de $(a - \lambda\theta(t))|\xi|$.

Le deuxième résultat démontré est que le système de Navier-Stokes perturbé (2.1.6) avec de petites données analytiques dans la variable tangentielle est globalement bien posé.

Théorème 2.1.7. Soit $a > 0$ et $s \in]0, 1[$. il existe une constante $c_1 > 0$ suffisamment petite, telle que pour (u_0, v_0) de données initiales à divergence nulle, si les données initiales satisfont la condition de petitesse

$$(2.1.11) \quad \begin{aligned} & \|e^{a|D_x|} \partial_y(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|e^{a|D_x|} \partial_x(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} \\ & + \|e^{a|D_x|}(u_0 + u_1, \epsilon(v_0 + v_1))\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(u_1, \epsilon v_1)\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a, \end{aligned}$$

alors le système (2.1.6) admet une solution globale unique (u, v) , telle que

$$(2.1.12) \quad \begin{aligned} & \frac{1}{2} \|e^{\mathcal{R}t}(u + \partial_t u, \epsilon(v + \partial_t v))_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \epsilon \|e^{\mathcal{R}t} \partial_x(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\ & + \frac{1}{2} \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & \leq C \left(\|e^{a|D_x|} \partial_y(u_0, \epsilon v_0)\|_{\mathcal{B}^s} + \epsilon \|e^{a|D_x|} \partial_x(u_0, \epsilon v_0)\|_{\mathcal{B}^s} \right. \\ & \quad \left. + \|e^{a|D_x|}(u_1, \epsilon v_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}(u_0 + u_1, \epsilon(v_0 + v_1))\|_{\mathcal{B}^s} \right), \end{aligned}$$

où $C = C(s)$ et (u_Θ, v_Θ) est donné par (2.1.10) avec $\phi = \Theta$.

Le troisième résultat concerne la convergence des solutions du système Navier-Stokes hyperbolique transformé par le changement d'échelle anisotrope (2.1.6) vers celles du système hyperbolique de Prandtl (2.1.7).

Théorème 2.1.8. Soit $a > 0$, et $(u_0^\epsilon, v_0^\epsilon)$ satisfaisant (2.1.11). Soit (u_0, u_1) satisfaisant $e^{a|D_x|}(u_0, u_1) \in (\mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{7}{2}})^2$, $e^{a|D_x|} \partial_y(u_0, u_1) \in (\mathcal{B}^{\frac{3}{2}})^2$, et la condition de compatibilité $\int_0^1 u_0 dy = 0$. Si nos données initiales satisfont la condition de petitesse

$$\left\| e^{a|D_x|}(u_0 + u_1) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} \partial_y u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} u_1\|_{\mathcal{B}^{\frac{1}{2}}}$$

$$(2.1.13) \quad \leq \frac{c_2 a}{2 + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{3}{2}}}},$$

pour c_2 une constante suffisamment petite, alors on a

$$(2.1.14) \quad \begin{aligned} & \frac{1}{2} \|(R^1 + \partial_t R^1, \epsilon(R^2 + \partial_t R^2))_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|\partial_x(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \\ & + \frac{1}{2} \|(\partial_t R^1)_\varphi, \epsilon(\partial_t R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|(\partial_t R^1, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ & \leq \|e^{a|D_x|}((u_1^\epsilon - u_1), \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}} \\ & + C \left(\|e^{a|D_x|}\partial_y(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|e^{a|D_x|}\partial_x(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} \right. \\ & \quad \left. + \|e^{a|D_x|}((u_0^\epsilon - u_0) + (u_1^\epsilon - u_1), \epsilon(v_0^\epsilon - v_0) + \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}} + M\epsilon \right). \end{aligned}$$

où $C = C(s)$ et

$$(2.1.15) \quad \begin{cases} R^1 = u^\epsilon - u, \\ R^2 = v^\epsilon - v, \end{cases}$$

et v_0 est déterminé d'après u_0 passant par $\partial_x u + \partial_y v = 0$ et $v_0|_{y=0} = v_0|_{y=1} = 0$, et $(R_\varphi^1, \epsilon R_\varphi^2)$ est donné par (2.1.10).

Remarque 2.1.2. *L'idée principale de la preuve de ces théorèmes ci-dessus est d'utiliser des estimations d'énergie analytiques, qui sont motivées par [112] et qui proviennent de [27].*

2.2 Approximation hydrostatique des Équations de la magnéto-hydrodynamique

Le but de cette section est de présenter les résultats obtenus lors de cette thèse sur une approximation hydrostatique des équations de la magnéto-hydrodynamique dans un domaine mince complété avec des conditions de Dirichlet au bord. Avant d'exposer les résultats, on commence par rappeler les résultats fondamentaux sur l'existence ainsi que la régularité et l'unicité des solutions de ces équations.

2.2.1 Résultats connus

On commence par l'origine de ces approximations, qui ont été obtenues à l'issue d'une dérivation systématique des modèles de couche limite en magnéto-hydrodynamique par D. Gérard-Varet et M. Prestipino dans [63]. En effet, ces derniers ont considéré le système magnéto-hydrodynamique dans un sous-espace Ω de \mathbb{R}^3 donné par

$$(2.2.1) \quad \begin{cases} \partial_t U + U \nabla U - \frac{1}{Re} \Delta U = -\nabla P + SB \cdot \nabla B, \\ \partial_t B - \text{curl}(U \times B) - \frac{1}{Rm} \Delta B = 0, \\ \text{div } U = \text{div } B = 0. \end{cases}$$

Les paramètres Re et Rm représentent les nombres de Reynolds hydrodynamique et magnétique respectivement. On suppose que ces paramètres sont très grands ($Re \gg 1$, and $Rm \sim Re$). Ce choix de régime des paramètres est nécessaire pour la création d'une couche limite. Le paramètre S est un paramètre de couplage donné par

$$S = \frac{B_0^2}{\mu \rho V^2} = \frac{Ha}{Re Rm}, \quad \text{où } Ha = B_0 L \left(\frac{\sigma}{\eta} \right)^{\frac{1}{2}},$$

où le paramètre Ha est le nombre de Hartmann, B_0 et V sont les amplitudes typiques pour les champs magnétiques et de vitesse, L est une échelle de longueur typique de l'écoulement, ρ correspond à la densité du fluide, μ est sa perméabilité magnétique et η est le coefficient de viscosité.

Dans notre étude, on va s'intéresser à des fluides qui évoluent dans des domaines minces à deux dimensions par exemple $\Omega_1 = \mathbb{R} \times (0, 1)$. On considère les solutions du système (2.2.1)

$$(2.2.2) \quad \begin{cases} U(t, x, y) \sim \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ B(t, x, y) \sim \left(b^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon c^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ P(t, x, y) \sim p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

Le paramètre ϵ désigne la taille de la couche limite, c'est un paramètre très petit ($\epsilon \ll 1$).

Remarque 2.2.1. *Notons que la condition de divergence nulle sur U et B , reste toujours valide par la remise à l'échelle des composantes verticales par le facteur ϵ . En effet*

$$\begin{aligned} \text{div } U &= \partial_x u^\epsilon + \epsilon \frac{1}{\epsilon} \partial_y v^\epsilon = \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \\ \text{div } B &= \partial_x b^\epsilon + \epsilon \frac{1}{\epsilon} \partial_y c^\epsilon = \partial_x b^\epsilon + \partial_y c^\epsilon = 0. \end{aligned}$$

Si on remplace U , B et P par les solutions proposées dans (2.2.2) dans le système (2.2.1), on obtient

$$\partial_t U + U \cdot \nabla U - \frac{1}{Re} \Delta U + \nabla P - SB \cdot \nabla B$$

$$\begin{aligned}
&= \left(\partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y u^\epsilon - \frac{1}{Re} \partial_x^2 u^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 u^\epsilon + \partial_x p^\epsilon - S b^\epsilon \partial_x b^\epsilon - S \epsilon c^\epsilon \frac{1}{\epsilon} \partial_y b^\epsilon, \right. \\
&\partial_t \epsilon v^\epsilon + u^\epsilon \partial_x \epsilon v^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y \epsilon v^\epsilon - \frac{1}{Re} \partial_x^2 \epsilon v^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 \epsilon v^\epsilon + \frac{1}{\epsilon} \partial_y p^\epsilon - S b^\epsilon \partial_x \epsilon c^\epsilon - S \epsilon c^\epsilon \frac{1}{\epsilon} \partial_y \epsilon c^\epsilon \left. \right) \\
&= (0, 0),
\end{aligned}$$

et

$$\begin{aligned}
&\partial_t B + U \cdot \nabla B - B \cdot \nabla U - \frac{1}{Rm} \Delta B \\
&= \left(\partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y b^\epsilon - \frac{1}{Re} \partial_x^2 b^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 b^\epsilon - b^\epsilon \partial_x u^\epsilon - \epsilon c^\epsilon \frac{1}{\epsilon} \partial_y u^\epsilon, \right. \\
&\partial_t \epsilon c^\epsilon + u^\epsilon \partial_x \epsilon c^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y \epsilon c^\epsilon - \frac{1}{Re} \partial_x^2 \epsilon c^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 \epsilon c^\epsilon - b^\epsilon \partial_x \epsilon v^\epsilon - \epsilon c^\epsilon \frac{1}{\epsilon} \partial_y \epsilon v^\epsilon \left. \right) \\
&= (0, 0).
\end{aligned}$$

Donc le nouveau système est de la forme:

$$(2.2.3) \quad \left\{ \begin{array}{l} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \frac{1}{Re} \partial_x^2 u^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 u^\epsilon = -\partial_x p^\epsilon + S b^\epsilon \partial_x b^\epsilon + S \epsilon c^\epsilon \partial_y b^\epsilon, \\ \partial_t \epsilon v^\epsilon + \epsilon u^\epsilon \partial_x v^\epsilon + \epsilon v^\epsilon \partial_y v^\epsilon - \frac{\epsilon}{Re} \partial_x^2 v^\epsilon - \frac{1}{Re} \frac{1}{\epsilon} \partial_y^2 v^\epsilon = -\frac{1}{\epsilon} \partial_y p^\epsilon + \epsilon S b^\epsilon \partial_x c^\epsilon + S \epsilon c^\epsilon \partial_y c^\epsilon, \\ \partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + v^\epsilon \partial_y b^\epsilon - \frac{1}{Re} \partial_x^2 b^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 b^\epsilon = b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y u^\epsilon, \\ \epsilon \partial_t c^\epsilon + \epsilon u^\epsilon \partial_x c^\epsilon + \epsilon v^\epsilon \partial_y c^\epsilon - \frac{1}{Re} \epsilon \partial_x^2 c^\epsilon - \frac{1}{Re} \frac{1}{\epsilon} \partial_y^2 c^\epsilon = \epsilon b^\epsilon \partial_x v^\epsilon + \epsilon c^\epsilon \partial_y v^\epsilon, \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \text{ et } \partial_x b^\epsilon + \partial_y c^\epsilon = 0. \end{array} \right.$$

ce système est complété par les conditions de Dirichlet au bord:

$$(u^\epsilon, v^\epsilon)_{/y=0,1} = (b^\epsilon, c^\epsilon)_{/y=0,1} = 0.$$

Dans ce qui suit, on suppose que la constante $S = 1$. On définit la taille de notre couche par $\epsilon = \frac{1}{\sqrt{Re}} = \frac{1}{\sqrt{Rm}}$, cela signifie que nos nombres de Reynolds hydrodynamique et magnétique sont identiques. Si on remplace dans le système (2.2.3), on obtient

$$(2.2.4) \quad \left\{ \begin{array}{l} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon = -\partial_x p^\epsilon + b^\epsilon \partial_x b^\epsilon + c^\epsilon \partial_y b^\epsilon, \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) = -\partial_y p^\epsilon + \epsilon^2 (b^\epsilon \partial_x c^\epsilon + c^\epsilon \partial_y c^\epsilon), \\ \partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + v^\epsilon \partial_y b^\epsilon - \epsilon \partial_x^2 b^\epsilon - \partial_y^2 b^\epsilon = b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y u^\epsilon, \\ \epsilon (\partial_t c^\epsilon + u^\epsilon \partial_x c^\epsilon + v^\epsilon \partial_y c^\epsilon - \epsilon^2 \partial_x^2 c^\epsilon - \partial_y^2 c^\epsilon) = \epsilon (b^\epsilon \partial_x v^\epsilon + c^\epsilon \partial_y v^\epsilon) \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \text{ et } \partial_x b^\epsilon + \partial_y c^\epsilon = 0. \end{array} \right.$$

Prenant la limite formelle de ϵ vers 0 dans le système (2.2.4), on obtient le système de type Prandtl suivant:

$$(2.2.5) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u = -\partial_x p + b \partial_x b + c \partial_y b, \\ \partial_y p = 0, \\ \partial_t b + u \partial_x b + v \partial_y b - \partial_y^2 b = b \partial_x u + c \partial_y u, \\ \partial_x u + \partial_y v = 0, \text{ and } \partial_x b + \partial_y c = 0. \end{cases}$$

Le système (2.2.5) a fait l'objet de plusieurs études récentes, l'existence et la régularité des solutions globales et locales sont des questions difficiles que les chercheurs ont essayé de prouver pour ce système. Cependant, Liu, Xie et Yang dans [95] ont réussi à prouver que ce système est bien posé localement dans l'espace de Sobolev sans demander aucune condition de monotocité sur la vitesse tangentielle. La seule hypothèse essentielle dans leur travail est que la composante horizontale du champ magnétique admet une borne inférieure positive. Ce résultat donne une bonne compréhension de la physique générale selon laquelle le champ magnétique stabilise la couche limite.

D'autre part, lorsque les données initiales sont analytiques, N. Liu and P. Zhang dans [99] ont prouvé un résultat d'existence et d'unicité globale des solutions pour le système magnétohydrodynamique en dimension deux dans l'espace supérieur $\mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}^+\}$. Avant d'énoncer leur résultat on rappelle que leur système est de la forme:

$$(2.2.6) \quad \begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - \partial_y^2 u_1 = -\partial_x p + b_1 \partial_x b_1 + b_2 \partial_y b_1, \\ \partial_t b_1 + u_1 \partial_x b_1 + u_2 \partial_y b_1 - \kappa \partial_y^2 b_1 = b_1 \partial_x u_1 + b_2 \partial_y u_1, \\ \partial_x u_1 + \partial_y u_2 = 0, \text{ et } \partial_x b_1 + \partial_y b_2 = 0, \\ (u_1, u_2)_{/y=0} = 0 \text{ et } (\partial_y b_1, b_2)_{/y=0} = 0, \\ \lim_{y \rightarrow +\infty} u_1 = U_1 \text{ et } \lim_{y \rightarrow +\infty} b_1 = B_1, \\ u_1/t=0 = u_{1,0}, \quad b_1/t=0 = b_{1,0}, \end{cases}$$

où (u_1, u_2) and (b_1, b_2) représentent respectivement la vitesse du fluide et le champ magnétique, le paramètre κ est une constante positive représentant le rapport entre le nombre de Reynolds hydrodynamique et le nombre de Reynolds magnétique, $(U_1, B_1, p)(t, x)$ sont les traces des champs tangentiels et de la pression de l'écoulement sur la frontière et satisfont la loi de Bernoulli :

$$(2.2.7) \quad \begin{cases} \partial_t U_1 + U_1 \partial_x U_1 + \partial_x p = B_1 \partial_x B_1, \\ \partial_t B_1 + U_1 \partial_x B_1 = B_1 \partial_x U_1. \end{cases}$$

Le but de leur travail est d'obtenir un résultat d'existence globale des solutions pour ce problème. Pour cela, ils ont défini pour toute constante \bar{B}_κ , une fonction plateau $\chi \in C^\infty[0, \infty)$ définie par $\chi(y) = \begin{cases} y & \text{si } y \geq 2, \\ 0 & \text{si } y \leq 1, \end{cases}$ et ont fait le changement de variable suivant :

$$u = u_1 - \chi'(y)U \quad \text{et} \quad v = u_2 + \chi(y)\partial_x U,$$

$$(2.2.8) \quad b = b_1 - \chi'(y)B - \bar{B}_\kappa \quad \text{et} \quad c = b_2 + \chi(y)\partial_x B,$$

où $U = U_1$ et $B = B_1 - \bar{B}_\kappa$. Comme (u_1, u_2, b_1, b_2) résout le système (2.2.6), alors (u, v, b, c) résout

$$(2.2.9) \quad \begin{cases} \partial_t u - \partial_y^2 u - \bar{B}_\kappa \partial_x b + u \partial_x u - b \partial_x b + v \partial_y u - c \partial_y b + \chi'(U \partial_x u - B \partial_x b) \\ \quad + \chi'(\partial_x U u - \partial_x B b) + \chi(-\partial_x U \partial_y u + \partial_x B \partial_y b) + \chi''(U v - B c) = M_U, \\ \partial_t b - \kappa \partial_y^2 b - \bar{B}_\kappa \partial_x u + u \partial_x b - b \partial_x u + v \partial_y b - c \partial_y u + \chi'(U \partial_x b - B \partial_x u) \\ \quad + \chi'(\partial_x B u - \partial_x U b) + \chi(-\partial_x U \partial_y b + \partial_x B \partial_y u) + \chi''(B v - U c) = M_B, \\ \partial_x u + \partial_y v = 0, \quad \text{and} \quad \partial_x b + \partial_y c = 0 \\ (u, v)_{/y=0} = 0 \quad \text{and} \quad (\partial_y b, c)_{/y=0} = 0, \\ \lim_{y \rightarrow +\infty} (u, v) = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} (b, c) = 0, \\ u_{/t=0} = u_0 = u_{1,0} - \chi' U_0, \quad b_{/t=0} = b_0 = b_{1,0} - \chi' B_0 - \bar{B}_\kappa, \end{cases}$$

Les termes M_U et M_B sont supportés dans $y \in [0, 2]$ pour tout $t > 0$. Ces termes sont définis par

$$(2.2.10) \quad \begin{aligned} M_U &= (1 - \chi')(\partial_t U - \bar{B}_\kappa \partial_x B) + \chi''' U + (1 - (\chi')^2 + \chi \chi'')(U \partial_x U - B \partial_x B), \\ M_B &= (1 - \chi')(\partial_t B - \bar{B}_\kappa \partial_x U) + \chi''' B + (1 - (\chi')^2 - \chi \chi'')(U \partial_x B - B \partial_x U). \end{aligned}$$

D'autre part, comme $\partial_x u + \partial_y v = 0$ et $\partial_x b + \partial_y c = 0$, alors il existe deux fonctions potentielles (φ, ψ) telle que

$$(u, b) = \partial_y(\varphi, \psi) \quad \text{et} \quad (v, c) = -\partial_x(\varphi, \psi).$$

Ces fonctions primitives peuvent avoir les conditions aux limites suivantes

$$(\varphi, \psi)_{/y=0} = 0 \quad \text{et} \quad \lim_{y \rightarrow +\infty} (\varphi, \psi) = 0.$$

En intégrant les équations (2.2.9) satisfaites par (u, b) par rapport à la variable verticale y sur l'intervalle $[y, \infty[$, on obtient

$$(2.2.11) \quad \begin{cases} \partial_t \varphi - \partial_y^2 \varphi - \bar{B}_\kappa \partial_x \psi + u \partial_x \varphi - b \partial_x \psi + 2 \int_y^\infty (\partial_x \varphi \partial_y u - \partial_x \psi \partial_y b) dy' \\ \quad + \chi'(U \partial_x \varphi - B \partial_x \psi) + 2 \int_y^\infty \chi''(U \partial_x \varphi - B \partial_x \psi) dy' + \chi(-\partial_x U u + \partial_x B b) \\ \quad + 2 \chi'(\partial_x U \varphi - \partial_x B \psi) + 2 \int_y^\infty \chi''(\partial_x U \varphi - \partial_x B \psi) = m_U, \\ \partial_t \psi - \kappa \partial_y^2 \psi - \bar{B}_\kappa \partial_x \varphi + u \partial_x \psi - b \partial_x \varphi + \chi'(U \partial_x \psi - B \partial_x \varphi) \\ \quad + \chi(-\partial_x U b + \partial_x B u) = m_B, \\ \varphi_{/y=0} = \psi_{/y=0} = 0, \quad \text{and} \quad \lim_{y \rightarrow +\infty} \varphi = \lim_{y \rightarrow +\infty} \psi = 0, \\ \varphi_{/t=0} = \varphi_0 = - \int_y^\infty u_0 dy', \quad \psi_{/t=0} = \psi_0 = - \int_y^\infty b_0 dy', \end{cases}$$

où $(m_U, m_B) = - \int_y^\infty (M_U, M_B) dy'$ aussi supporté dans $y \in [0, 2]$ pour tout $t > 0$. Le résultat

obtenu dans leur travail est le suivant:

Théorème 2.2.1 (N. Liu et P. Zhang). *Soit $\kappa \in]0, 2[$, $\bar{B}_\kappa = \begin{cases} 1 & \text{si } \kappa = 1, \\ 0 & \text{sinon,} \end{cases}$ et $\epsilon, \delta > 0$. Nous supposons que les traces des champs tangentiels (U, B) satisfont*

$$(2.2.12) \quad \|\langle t \rangle^{\frac{9}{4}} e^{\delta|D_x|}(U, B)\|_{\tilde{L}^\infty(\mathbb{R}^+; B_h^{\frac{3}{2}})} + \|\langle t \rangle^{\frac{7}{4}} e^{\delta|D_x|}(\partial_t U, \partial_t B, U, B)\|_{\tilde{L}^\infty(\mathbb{R}^+; B_h^{\frac{1}{2}})} \leq \epsilon,$$

$$(2.2.13) \quad \int_0^\infty \langle t \rangle^{\frac{5}{4}} \|e^{\delta|D_x|}(U, B)\|_{B_h^{\frac{1}{2}}} dt \leq \epsilon.$$

Soient $(u_0, b_0, \varphi_0, \psi_0)$ les données initiales, satisfaisant la condition de compatibilité $\int_0^\infty u_0 dy = \int_0^\infty b_0 dy = 0$ et

$$(2.2.14) \quad \|e^{\frac{y^2}{8}} e^{\delta|D_x|}(u_0, b_0, \varphi_0, \psi_0)\|_{B_{\frac{1}{2}, 0}} < \infty \quad \text{et} \quad \|e^{\frac{y^2}{8}} e^{\delta|D_x|}(G_0, H_0)\|_{B_{\frac{1}{2}, 0}} \leq \sqrt{\epsilon},$$

où $G_0 = u_0 + \frac{y}{2\langle t \rangle} \varphi_0$ et $H_0 = b_0 + \frac{y}{2\kappa\langle t \rangle} \psi_0$. Alors il existe des constantes positives λ, a et $\epsilon_0(\lambda, a, \kappa, \delta)$ telle que pour $\epsilon \leq \epsilon_0$ et $l_\kappa = \frac{\kappa(2-\kappa)}{4} \in]0, \frac{1}{4}[$, le système (2.2.9) admet une solution unique globale (u, b) satisfaisant

$$\sup_{t \in [0, \infty[} \theta(t) \leq \frac{\delta}{2\lambda}.$$

La quantité $\theta(t)$ est une quantité clé qui nous permet de décrire l'évolution de la bande d'analyticité de (u, b) . Dans ce cas, cette quantité est définie par

$$(2.2.15) \quad \begin{cases} \dot{\theta}(t) = \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_y(G, H)_\phi(t)\|_{B_{\frac{1}{2}, 0}} + \frac{1}{\sqrt{\epsilon}} \langle t \rangle^{\frac{5}{4}} \|(U, B)_\phi(t)\|_{B_h^{\frac{1}{2}}} \\ \theta|_{t=0} = 0. \end{cases}$$

Ici la fonction de phase ϕ est définie par

$$\phi(t, \xi) = (\delta - \lambda\theta(t))|\xi|,$$

et la fonction pondérée Ψ est déterminée par

$$\Psi(t, y) = \frac{y^2}{8\langle t \rangle} \quad \text{avec} \quad \langle t \rangle = 1 + t.$$

Remarque 2.2.2. *L'idée principale de la preuve du théorème ci-dessus consiste à utiliser les estimations d'énergie analytiques, qui sont motivées par [112] et qui proviennent de [27].*

Ce résultat est obtenu dans l'espace fonctionnel de Besov de type Chemin-Lerner

$$(2.2.16) \quad \|u\|_{\tilde{L}^p(T_0, T; B^{s, 0})} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_{T_0}^T \|\Delta_q^h u(t)\|_{L^2_+}^p dt \right)^{\frac{1}{p}},$$

where $L_+^2 = L^2(\mathbb{R}_+^2)$. En particulier, quand $T_0 = 0$, on simplifie la notation de $\|u\|_{\tilde{L}^p(0,T;B^{s,0})}$ en $\|u\|_{\tilde{L}_T^p(B^{s,0})}$.

Remarque 2.2.3. L'espace $B_h^{\frac{3}{2}}$ désigne l'espace de Besov $B_{2,1}^{\frac{3}{2}}(\mathbb{R}_h)$ et l'espace $B_h^{\frac{1}{2}}$ désigne l'espace de Besov $B_{2,1}^{\frac{1}{2}}(\mathbb{R}_h)$.

Un deuxième résultat dans la direction des données analytiques pour le système magnétohydrodynamique en couche limite est le résultat de S. Li et F. Xie dans [88]. Leur résultat concerne l'étude du caractère bien-posé global des solutions du système magnétohydrodynamique (2.2.6) dans l'espace des fonctions analytiques. Dans ce travail, les auteurs s'intéressent au cas où $B_1 = 1$, $\kappa = 1$ et $U_1(t, x) = f(t)$ dans le système (2.2.6). Par la loi de Bernoulli, leur terme de pression satisfait $\partial_x p = -f'(t)$. Soit $\chi \in C^\infty[0, \infty)$ une fonction plateau définie par $\chi(y) = \begin{cases} 1 & \text{si } y \geq 2, \\ 0 & \text{si } y \leq 1, \end{cases}$, on note $U = u_1 - f(t)\chi(y)$. Alors (U, u_2, b_1, b_2) résout le système

$$(2.2.17) \quad \begin{cases} \partial_t U + (U + f(t)\chi(y))\partial_x U + u_2\partial_y(U + f(t)\chi(y)) \\ \quad - \partial_y^2 U - b_1\partial_x b_1 - b_2\partial_y b_1 = m(t, y), \\ \partial_t b_1 + \partial_y(u_2 b_1 - (U + f(t)\chi(y))b_2) = \partial_y^2 b_1, \\ \partial_x U + \partial_y u_2 = 0, \text{ and } \partial_x b_1 + \partial_y b_2 = 0, \end{cases}$$

où $m(t, y) = (1 - \chi(y))f'(t) + f(t)\chi''(y)$. Si on se place dans le cadre d'un écoulement de cisaillement on a que $(u^s(t, s), 0, 1, 0)$ est une solution triviale du système (2.2.17) avec $u^s(t, y)$ résolvant

$$(2.2.18) \quad \begin{cases} \partial_t u^s - \partial_y^2 u^s = m(t, y), \\ u^s(t, y)/_{y=0} = 0 \text{ et } \lim_{y \rightarrow +\infty} u^s(t, y) = 0, \\ u^s(t, y)/_{t=0} = 0. \end{cases}$$

Cela nous permet de voir la solution du système (2.2.17) comme une perturbation (u, v, b, c) de $(u^s(t, y), 0, 1, 0)$. Cette solution est donnée par

$$\begin{aligned} U &= u^s + u \text{ et } u_2 = v, \\ b_1 &= 1 + b \text{ et } b_2 = c. \end{aligned}$$

Par suite, le système (2.2.17) devient

$$(2.2.19) \quad \begin{cases} \partial_t u + (u + u^s + f(t)\chi(y))\partial_x u + v\partial_y(u + u^s + f(t)\chi(y)) \\ \quad - \partial_y^2 u - (1 + b)\partial_x b - c\partial_y b = 0, \\ \partial_t b + (u + u^s + f(t)\chi(y))\partial_x b - c\partial_y(u + u^s + f(t)\chi(y)) \\ \quad - \partial_y^2 b - (1 + b)\partial_x u + v\partial_y b = 0, \\ \partial_x u + \partial_y v = 0, \text{ and } \partial_x b + \partial_y c = 0, \\ (u, v, \partial_y b, c)/_{y=0} = 0 \text{ et } \lim_{y \rightarrow +\infty} (u, b) = 0, \\ u(0, x, y) = u_0 = u_{1,0} - f(0)\chi(y) \text{ et } b(0, x, y) = b_0 = b_{1,0} - 1. \end{cases}$$

De la même manière, comme $\partial_x u + \partial_y v = 0$ et $\partial_x b + \partial_y c = 0$ il existe deux fonctions potentielles (φ, ψ) telles que $(u, b) = \partial_y(\varphi, \psi)$ et $(v, c) = -\partial_x(\varphi, \psi)$. Intégrant les équations (2.2.19) satisfaites par (u, b) par rapport à la variable verticale y sur l'intervalle $[y, \infty[$, on obtient

$$(2.2.20) \quad \begin{cases} \partial_t \varphi + (u + u^s + f(t)\chi(y))\partial_x \varphi + 2 \int_y^\infty \partial_y(u + u^s + f(t)\chi(y)) \cdot \partial_x \varphi dy \\ \quad - (1 + b)\partial_x \psi - 2 \int_y^\infty \partial_y b \cdot \partial_x \psi dy - \partial_y^2 \varphi = 0, \\ \partial_t \psi + (u + u^s + f(t)\chi(y))\partial_x \psi - (1 + b)\partial_x \varphi - \partial_y^2 \psi = 0, \\ \varphi|_{y=0} = \psi|_{y=0} = 0, \quad \text{and} \quad \lim_{y \rightarrow +\infty} \varphi = \lim_{y \rightarrow +\infty} \psi = 0, \\ \varphi|_{t=0} = \varphi_0 = - \int_y^\infty u_0 dy', \quad \psi|_{t=0} = \psi_0 = - \int_y^\infty b_0 dy'. \end{cases}$$

Le résultat obtenu dans leur papier est le suivant

Théorème 2.2.2 (S.Li et F.Xie). *Soit $\delta > 0$, et $f \in H^1(\mathbb{R}_+)$ qui satisfait:*

$$(2.2.21) \quad C_f = \int_0^\infty \langle t \rangle (|f(t)| + |f'(t)|) dt + \left(\int_0^\infty \langle t \rangle^3 (f^2(t) + (f'(t))^2) dt \right)^{\frac{1}{2}} < \infty,$$

avec $\langle t \rangle = 1 + t$ et $u_0 = \partial_y \varphi_0$, $b_0 = \partial_y \psi_0$ satisfaisant la condition de compatibilité $\int_0^\infty u_0 dy = \int_0^\infty b_0 dy = 0$. Supposons en outre que $G_0 = u_0 + \frac{y}{2\langle t \rangle} \varphi_0$ et $H_0 = b_0 + \frac{y}{2\langle t \rangle} \psi_0$ satisfont

$$(2.2.22) \quad \|e^{\frac{y^2}{8}} e^{2\delta|D_x|}(u_0, b_0)\|_{B^{\frac{1}{2},0}} < \epsilon \quad \text{et} \quad \|e^{\frac{y^2}{8}} e^{\delta|D_x|}(G_0, H_0)\|_{B^{\frac{1}{2},0}} \leq \infty,$$

pour un ϵ suffisamment petit. Alors le système (2.2.19) admet une solution globale unique (u, b) , qui satisfait:

$$(2.2.23) \quad \|e^{\frac{y^2}{8\langle t \rangle}} e^{\delta/2|D_x|}(u, b)\|_{L^\infty(\mathbb{R}_+; B^{\frac{1}{2},0})} + \|e^{\frac{y^2}{8\langle t \rangle}} e^{\delta/2|D_x|} \partial_y(u, b)\|_{L^2(\mathbb{R}_+; B^{\frac{1}{2},0})} \leq C \|e^{\frac{y^2}{8}} e^{2\delta|D_x|}(u_0, b_0)\|_{B^{\frac{1}{2},0}},$$

où C est une constante indépendante de C_f .

Remarque 2.2.4. *L'espace utilisé pour obtenir ce résultat est l'espace de Besov défini par*

$$(2.2.24) \quad \|u\|_{\tilde{L}^p(T_0, T; B^{s,0})} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_{T_0}^T \left\| \Delta_q^h u(t) \right\|_{L^2_+}^p dt \right)^{\frac{1}{p}}.$$

Remarque 2.2.5. *L'idée principale de la preuve du théorème ci-dessus consiste à utiliser les estimations d'énergie analytiques, qui sont motivés par [112] et à définir la fonction de phase*

$$\phi(t, \xi) = (\delta - \lambda\theta(t))|\xi|.$$

La quantité $\theta(t)$ est une quantité clé qui nous permet de décrire l'évolution de la bande d'analyticité de (u, b) .

2.2.2 Énoncé des résultats sur le système magnétohydrodynamique

Dans le chapitre 4 de cette thèse, pour $\epsilon > 0$, on considère le système suivant

$$(MHD) \quad \begin{cases} \partial_t U + U \cdot \nabla U - \epsilon^2 \Delta U + \nabla P = B \cdot \nabla B, \\ \partial_t B + U \cdot \nabla B - \epsilon^2 \Delta B = B \cdot \nabla U, \\ \operatorname{div} U = \operatorname{div} B = 0, \\ U|_{t=0} = U_0, \quad B|_{t=0} = B_0, \end{cases}$$

Tous les résultats qu'on a cité, sont obtenus lorsque le domaine Ω est

$$\mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}^+\}.$$

Dans notre cas, on a étudié le système magnétohydrodynamique pour un fluide qui évolue dans un domaine mince avec des conditions de Dirichlet au bord, notre domaine d'étude est le suivant

$$\mathbf{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\},$$

où ϵ représente la largeur du domaine. L'intérêt d'être dans un domaine mince complété par les conditions de Dirichlet au bord est l'apparition d'équations de type Prandtl pour les champs de vitesse et magnétique. L'idée principale du résultat est d'obtenir l'existence et l'unicité des solutions pour le système (MHD) dans un domaine mince.

Le système (MHD) est complété par les conditions de Dirichlet au bord

$$(U, B)|_{y=0} = 0 \quad \text{et} \quad (U, B)|_{y=\epsilon} = 0$$

Nos données initiales sont de la forme suivante

$$U|_{t=0}^\epsilon = U_0^\epsilon = \left(u_0 \left(x, \frac{y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{y}{\epsilon} \right) \right) \quad \text{dans} \quad \mathbf{S}^\epsilon,$$

et

$$B|_{t=0}^\epsilon = B_0^\epsilon = \left(b_0 \left(x, \frac{y}{\epsilon} \right), \epsilon c_0 \left(x, \frac{y}{\epsilon} \right) \right) \quad \text{dans} \quad \mathbf{S}^\epsilon.$$

On va chercher des solutions pour notre système de la forme

$$(2.2.25) \quad \begin{cases} U(t, x, y) = \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ B(t, x, y) = \left(b^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon c^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ P(t, x, y) = p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

Soit $\mathbf{S} := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$. Après un changement d'échelle naturel, le système (MHD)

devient un système anisotrope

$$(2.2.26) \quad \left\{ \begin{array}{ll} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon + \partial_x p^\epsilon = b^\epsilon \partial_x b^\epsilon + c^\epsilon \partial_y b^\epsilon, & \text{in }]0, \infty[\times \mathbf{S} \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) + \partial_y p^\epsilon = \epsilon^2 (b^\epsilon \partial_x c^\epsilon + c^\epsilon \partial_y c^\epsilon), & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + v^\epsilon \partial_y b^\epsilon - \epsilon^2 \partial_x^2 b^\epsilon - \partial_y^2 b^\epsilon = b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y u^\epsilon, & \text{in }]0, \infty[\times \mathbf{S} \\ \epsilon (\partial_t c^\epsilon + u^\epsilon \partial_x c^\epsilon + v^\epsilon \partial_y c^\epsilon - \epsilon^2 \partial_x^2 c^\epsilon - \partial_y^2 c^\epsilon) = \epsilon (b^\epsilon \partial_x v^\epsilon + c^\epsilon \partial_y v^\epsilon), & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x b^\epsilon + \partial_y c^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{t=0} = (u_0, v_0, b_0, c_0), & \text{in } \mathbf{S} \\ (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{y=0} = (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{y=1} = 0. & \end{array} \right.$$

Par un passage à la limite formelle de $\epsilon \rightarrow 0$ dans le système (2.2.26), on obtient l'équation de Prandtl magnétique

$$(2.2.27) \quad \left\{ \begin{array}{ll} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = b \partial_x b + c \partial_y b, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_y p = 0, & \text{in } \mathbf{S} \times]0, \infty[\\ \partial_t b + u \partial_x b + v \partial_y b - \partial_y^2 b = b \partial_x u + c \partial_y u, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x u + \partial_y v = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x b + \partial_y c = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ u|_{t=0} = u_0, & \text{in } \mathcal{S} \\ b|_{t=0} = b_0, & \text{in } \mathcal{S}, \end{array} \right.$$

où le champ vitesse $U = (u, v)$ et le champ magnétique $B = (b, c)$ satisfont la condition aux limites de non-glissement de Dirichlet

$$(u, v, b, c)|_{y=0} = (u, v, b, c)|_{y=1} = 0.$$

Notre objectif est d'obtenir l'existence globale des solutions pour les systèmes (2.2.26) et (2.2.27) lorsque nos données initiales sont analytiques, puis nous voulons montrer la convergence du système MHD anisotrope mis à l'échelle (2.2.26) au système limite (2.2.27) lorsque ϵ tend vers zéro.

Le premier résultat est que le système MHD limite (2.2.27), avec des petites données analytiques dans la variable horizontale, est globalement bien posé.

Théorème 2.2.3. *Soit $a > 0$, on suppose que pour une constante c_0 suffisamment petite, indépendante de ϵ , et pour toute donnée initiale $(u_0, v_0, b_0, c_0) = (U_0, B_0)$ satisfaisant*

$$(2.2.28) \quad \|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} b_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a,$$

et la condition de compatibilité $\int_0^1 (u_0, b_0) dy = 0$ est satisfaite. Alors le système limite (2.2.27) admet une unique solution globale $(U, B) = (u, v, b, c)$ satisfaisant

$$(2.2.29) \quad \|e^{\mathcal{R}t} (u_\phi, b_\phi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_y (u_\phi, b_\phi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \leq C \|e^{a|D_x|} (u_0, b_0)\|_{\mathcal{B}^{\frac{1}{2}}},$$

où $C = C(s)$ et (u_ϕ, b_ϕ) sont donnés par

$$(2.2.30) \quad \begin{cases} (u_\phi, b_\phi)(t, x, y) = e^{\phi(t, D_x)}(u(t, x, y), b(t, x, y)) \triangleq \mathcal{F}_h^{-1}(e^{\phi(t, \xi)}(\widehat{u}(t, \xi, y), \widehat{b}(t, \xi, y))) \\ \phi(t, \xi) = (a - \lambda\theta(t))|\xi|. \end{cases}$$

La quantité $\theta(t)$ décrit l'évolution de la diminution de la bande d'analyticité au cours du temps, cette quantité est positive pour tout $t \in \mathbb{R}_+^*$ satisfait l'équation $\dot{\theta}(t) = \|\partial_y(u_\phi, b_\phi)\|_{\mathcal{B}^{\frac{1}{2}}}$ et lorsque $t = 0$ on a $\theta(0) = 0$.

\mathcal{R} est une constante déterminée par l'inégalité de Poincaré pour la bande \mathcal{S} , l'espace fonctionnel utilisé pour obtenir ce résultat est l'espace de Besov du type Chemin-Lerner.

$$\|u\|_{\tilde{L}_T^p(\mathcal{B}^s(\mathcal{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \left\| \Delta_q^h u(t) \right\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

Le deuxième résultat démontré est que le système de MHD anisotrope (2.2.26) avec des données initiales petites analytiques dans la variable tangentielle est globalement bien posé.

Théorème 2.2.4. *Soit $a > 0$, il existe une constante c_1 suffisamment petite indépendante de ϵ , telle que pour toute donnée initiale $(u_0^\epsilon, v_0^\epsilon, b_0^\epsilon, c_0^\epsilon) = (U_0^\epsilon, B_0^\epsilon)$ satisfaisant*

$$(2.2.31) \quad \|e^{a|D_x|}(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0, \epsilon c_0)\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a,$$

alors le système anisotrope (2.2.26) admet une unique solution globale (U^ϵ, B^ϵ) satisfaisant

$$(2.2.32) \quad \begin{aligned} & \|e^{\mathcal{R}t}(u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}(b_\varphi, \epsilon c_\varphi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y(u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \\ & + \|e^{\mathcal{R}t}\partial_y(b_\varphi, \epsilon c_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \epsilon \|e^{\mathcal{R}t}\partial_x(u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \\ & + \epsilon \|e^{\mathcal{R}t}\partial_x(b_\varphi, \epsilon c_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \leq C \left(\|e^{a|D_x|}(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0, \epsilon c_0)\|_{\mathcal{B}^{\frac{1}{2}}} \right), \end{aligned}$$

où $C = C(s)$ et (u_φ, v_φ) and (b_φ, c_φ) sont aussi donnés par (2.2.30). La fonction φ est donnée par

$$\varphi(t, D_x) = (a - \lambda\tau(t))|D_x|,$$

tel que $\tau(t)$ est l'évolution de la bande d'analyticité.

Le troisième résultat concerne la convergence des solutions de l'anisotrope mis à l'échelle du système MHD (2.2.26) vers le système limite du type Prandtl magnétique (2.2.27).

Théorème 2.2.5. *Soit $a > 0$, et $(u_0^\epsilon, v_0^\epsilon, b_0^\epsilon, c_0^\epsilon)$, satisfaisant (2.2.31). Soit u_0, b_0 satisfaisant $e^{a|D_x|}(u_0, b_0) \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}}$, $e^{a|D_x|}\partial_y(u_0, b_0) \in \mathcal{B}^{\frac{3}{2}}$, la condition de compatibilité $\int_0^1 (u_0, b_0) dy = 0$ et*

$$\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_2 a}{1 + \|e^{a|D_x|}(u_0, b_0)\|_{\mathcal{B}^{\frac{3}{2}}}},$$

Pour une certaine constante $c_2 > 0$ suffisamment petit indépendamment de ϵ , on a

$$(2.2.33) \quad \begin{aligned} & \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(t)\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_y(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ & + \epsilon \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ & \leq C \left(\|e^{a|D_x|}(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0^\epsilon - b_0, \epsilon(c_0^\epsilon - c_0))\|_{\mathcal{B}^{\frac{1}{2}}} + M\epsilon \right), \end{aligned}$$

où $C = C(s)$ et

$$(2.2.34) \quad \begin{cases} (\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, q^\epsilon) = (u^\epsilon - u, v^\epsilon - v, p^\epsilon - p), \\ (\Phi^{1,\epsilon}, \Phi^{2,\epsilon}) = (b^\epsilon - b, c^\epsilon - c), \end{cases}$$

et v_0 est déterminé à partir de u_0 via $\partial_x u + \partial_y v = 0$ et $v_0|_{y=0} = v_0|_{y=1} = 0$, et $(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)$, $(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)$ sont donnés par (2.2.30). La fonction φ est donnée par

$$\Theta(t, D_x) = (a - \mu\eta(t)) |D_x|,$$

tel que $\eta(t)$ est l'évolution de la bande d'analyticité, et $\mu \geq \lambda$ est une constante.

Remarque 2.2.6. L'idée principale de la preuve de ces théorèmes ci-dessus basés sur l'utilisation des estimations d'énergie analytiques, qui sont motivées par [112] et qui proviennent de [27].

2.2.3 Énoncé des résultats sur la version hyperbolique du système magnétohydrodynamique

Dans le chapitre 5 de cette thèse, on s'intéresse à la version hyperbolique du système magnétohydrodynamique en deux dimension suivant:

$$(2.2.35) \quad \begin{cases} \mathbb{J}\partial_t^2 u + \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = \mathbb{H}^2(b_1 b_2 v - u b_2^2 - b_2 e) & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_y p = \mathbb{H}^2(b_1 b_2 u - b_1^2 v + b_1 e) & (0, T) \times \mathbb{R} \times (0, 1), \\ \frac{\kappa}{\text{Pr}_m} \partial_t^2 b_1 + \partial_t b_1 + u\partial_x b_1 + v\partial_y b_1 - \frac{1}{\text{Pr}_m} \partial_y^2 b_1 = b_1 \partial_x u + b_2 \partial_y u & (0, T) \times \mathbb{R} \times (0, 1), \\ \frac{\kappa}{\text{Pr}_m} \partial_t^2 b_2 + \partial_t b_2 + u\partial_x b_2 + v\partial_y b_2 - \frac{1}{\text{Pr}_m} \partial_y^2 b_2 = b_1 \partial_x v + b_2 \partial_y v & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_t b_1 + \partial_y e = 0 \quad \text{and} \quad \partial_t b_2 - \partial_x e = 0 & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_x u + \partial_y v = 0 \quad \text{and} \quad \partial_x b_1 + \partial_y b_2 = 0 & (0, T) \times \mathbb{R} \times (0, 1), \end{cases}$$

couplé avec les conditions initiales et aux limites suivantes:

$$(2.2.36) \quad (\text{IC}) \quad \begin{cases} (u, b_1, b_2)|_{t=0} = (\bar{u}, \bar{b}_1, \bar{b}_2) & \mathbb{R} \times (0, 1), \\ (\partial_t u, \partial_t b_1, \partial_t b_2)|_{t=0} = (\tilde{u}, \tilde{b}_1, \tilde{b}_2) & \mathbb{R} \times (0, 1), \end{cases}$$

$$(2.2.37) \quad (\text{BC}) \quad \begin{cases} (u, b_1, b_2, e)|_{y=1} = (0, 0, 0, 0) & (0, T) \times \mathbb{R}, \\ (u, b_1, b_2, e)|_{y=1} = (0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{e}) & (0, T) \times \mathbb{R}. \end{cases}$$

Toutes les variables d'état (u, v, b_1, b_2, e) dans (2.2.35) dépendent du temps $t \in (0, T)$ et de l'espace $(x, y) \in \mathbb{R} \times (0, 1)$. Les champs vectoriels $(u, v)^T \in \mathbb{R}^2$ et $(b_1, b_2)^T \in \mathbb{R}^2$ ont une divergence nulle et représentent respectivement la vitesse et les champs magnétiques du plasma. Le champ électrique prend une taille $e \in \mathbb{R}$ perpendiculaire au plan contenant le plasma. Toutes les constantes \mathbb{H} , κ , Pr_m et \mathbb{J} sont positives et dépendent des paramètres standards.

Si on suppose que nos données initiales sont analytiques et suffisamment petites dans la variable horizontale, alors le résultat obtenu dans le chapitre 5 représente l'existence et l'unicité des solutions globales du système (2.2.35).

Théorème 2.2.6. *Supposons que les conditions aux limites homogènes dans (2.2.36) et (2.2.37): $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{e}) = (0, 0, 0)$. Pour tout $s > 2$, il existe une constante positive suffisamment petite $\varepsilon_s \in [0, 1)$ (qui dépend uniquement de s), telle que le résultat suivant est vrai. Soit \bar{u} , \bar{b}_1 et \tilde{b}_1 les données initiales de (2.2.35) qui sont analytiques dans la variable $x \in \mathbb{R}$ avec un rayon d'analyticité $\tau_0 > 0$:*

$$(2.2.38) \quad \begin{aligned} e^{\tau_0(1+|D_x|)}\bar{u} \quad \text{et} \quad e^{\tau_0(1+|D_x|)}\bar{b}_1 & \text{ appartiennent à } H^{s+1,1}(\mathbb{R} \times (0, 1)), \\ e^{\tau_0(1+|D_x|)}\tilde{u} \quad \text{et} \quad e^{\tau_0(1+|D_x|)}\tilde{b}_1 & \text{ appartiennent à } H^{s,0}(\mathbb{R} \times (0, 1)). \end{aligned}$$

Si la condition de petitesse suivante sur les données initiales est vraie

$$(2.2.39) \quad \begin{aligned} & \|e^{\tau_0(1+|D_x|)}\bar{u}\|_{H^{s+1,0}} + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\tilde{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\bar{b}_1\|_{H^{s+1,0}} + \\ & + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{b}_1\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\tilde{b}_1\|_{H^{s,0}} \\ & \leq \left(\frac{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}}}{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{5}{2}} \max\{1, \mathbb{H}^2\} \max\{\text{Pr}_m^{-1}, \text{Pr}_m\}^{\frac{1}{2}}} \min\{\tau_0, \tau_0^{-1}\}^{\frac{3}{2}} \right) \varepsilon_s, \end{aligned}$$

alors il existe une solution analytique globale dans le temps (u, b_1) de (2.2.35), qui a un rayon d'analyticité décroissant $\tau : \mathbb{R}_+ \rightarrow (0, \tau_0]$ donné par:

$$(2.2.40) \quad \tau(t) := \tau_0 \exp \left\{ - \frac{t}{16 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\} > 0.$$

De plus, les normes analytiques de la solution se tend vers zero exponentiellement dans le temps $t \in \mathbb{R}_+$ comme suit:

$$(2.2.41) \quad \begin{aligned} & \|e^{\tau(t)(1+|D_x|)}u(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_t u(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_y u(t)\|_{H^{s,0}}^2 + \\ & + \|e^{\tau(t)(1+|D_x|)}b_1(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_t b_1(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_y b_1(t)\|_{H^{s,0}}^2 \\ & \leq C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) \left\{ \|e^{\tau_0(1+|D_x|)}\bar{u}\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)}\tilde{u}\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{u}\|_{H^{s,0}}^2 + \right. \\ & \left. + \|e^{\tau_0(1+|D_x|)}\bar{b}\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)}\tilde{b}\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{b}\|_{H^{s,0}}^2 \right\} \exp \left\{ - \frac{t}{8 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\}. \end{aligned}$$

où la constante $C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0)$ est définie par

$$C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) = 4^3 \frac{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3} \max\{\text{Pr}_m, \text{Pr}_m^{-1}\} \max\{\tau_0, \tau_0^{-1}\}^2.$$

2.3 Approximation hydrostatique des Équations Primitives

Dans cette section, nous cherchons à donner les résultats obtenus au cours des travaux de cette thèse sur la modélisation des équations primitives dans les couches limites bidimensionnelles. Le modèle étudié représente les fluides géophysiques à grande échelle. Dans notre cas, on va prendre en compte la forte stratification due à la gravité de la Terre. Notre modèle est de la forme suivante

$$(2.3.1) \quad \begin{cases} \partial_t u + U \cdot \nabla u - \nu_1 \partial_x^2 u - \nu_2 \partial_y^2 u + \partial_x p = 0, \\ \partial_t v + U \cdot \nabla v - \nu_1 \partial_x^2 v - \nu_2 \partial_y^2 v + \partial_y p = \frac{T}{F_r}, \\ \partial_t T + U \cdot \nabla T - \kappa_1 \partial_x^2 T - \kappa_2 \partial_y^2 T = 0 \\ \operatorname{div} U = 0, \\ (U, T)_{/t=0} = (U_0, T_0), \end{cases}$$

2.3.1 Rappel des résultats existants

Les résultats de cette thèse concernent l'étude des équations primitives dans un domaine mince complété par des conditions de Dirichlet au bord. La structure du domaine mince ainsi que le choix des données initiales font disparaître les viscosités horizontales à la limite, lorsque l'épaisseur du domaine converge vers zéro. Nous allons donc être amenés à étudier l'approximation hydrostatique du système de Navier-Stokes qui est une équation de type Prandtl, couplée à une équation du type transport-diffusion sur la température. Avant de discuter des résultats obtenus pour ce type de système nous rappelons brièvement certains résultats obtenus antérieurement.

On commence en premier lieu par rappeler quelques résultats obtenus lorsqu'on considère toutes les viscosités et la diffusivité non nulles. Les premiers à traiter mathématiquement ce modèle sont Lions-Temam-Wang dans [91–93] en 1990. Leurs travaux concernent l'étude de l'existence globale des solutions faibles pour les équations primitives en prenant compte de la viscosité et la diffusivité. En ce qui concerne l'unicité de la solution, Guillén-Gonzalez, Masmoudi et Rodriguez-Bellido dans [67] ont prouvé l'unicité de la solution faible pour ces équations dans un cadre de dimension 2. Ils ont aussi obtenu un résultat d'existence locale dans [101]. L'existence globale bidimensionnelle pour ces équations a été obtenue par Bresch, Kazhikhov et Lemoine en [17].

Lorsqu'on considère seulement une viscosité verticale qui apparaît dans le système et toute la diffusivité, C. Cao, Q. Lin and E.S.Titi ont obtenu un résultat d'existence locale et globale avec de petites données initiales de la solution forte du modèle d'ajustement géostrophique primitif 3D réduit avec une faible dissipation dans [21]. Plus spécifiquement, ils se sont penchés sur le

modèle suivant

$$(2.3.2) \quad \begin{cases} \partial_t u + u\partial_x u + w\partial_z u + \epsilon_1 u - f_0 v - \nu\partial_z^2 u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + w\partial_z v + \epsilon_1 v + f_0 u - \nu\partial_z^2 v = 0, \\ \epsilon_2 w + \partial_z p + T = 0, \\ \partial_t T + u\partial_x T + w\partial_z T - \kappa\Delta T = 0 \\ \partial_x u + \partial_z w = 0 \\ (u, v, T)|_{t=0} = (u_0, v_0, T_0), \end{cases}$$

Ce modèle est étudié dans le domaine $\{(x, z) \in \mathbb{R}^2 : 0 \leq z \leq H\}$, complété par les conditions au bord

$$(\partial_z u, \partial_z v, w, T)|_{z=0, H=0}, \text{ et } u, v, w, T \text{ sont périodiques en } x \text{ et de période } 1,$$

par $\partial_x u + \partial_z w = 0$, on déduit

$$w(x, z) = - \int_0^z \partial_x u(x, s) ds.$$

Dans le modèle (2.3.2), ϵ_1 et ϵ_2 sont des constantes positives, ν est une constante positive qui représente la viscosité verticale des équations du moment horizontal et f_0 correspond au paramètre de Coriolis. Ces travaux ont abouti par C. Cao, Q. Lin and E.S.Titi dans [21] aux résultats suivants:

Théorème 2.3.1. *On suppose que $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ vérifient les conditions de symétrie suivantes:*

$$\begin{aligned} &u, v \text{ et } T \text{ sont périodiques en } x \text{ et } z \text{ de période } 1; \\ &u, v \text{ sont pairs en } z, \text{ et } T \text{ est impair en } z, \end{aligned}$$

avec la condition de compatibilité $\int_0^1 \partial_x u_0 dz = 0$. Alors il existe un temps \mathcal{T} positif tel qu'existe une unique solution forte (u, v, T) du système (2.3.2), sur l'intervalle $[0, \mathcal{T}]$. En outre, la seule solution forte (u, v, T) dépend continûment des données initiales.

Remarque 2.3.1. *La preuve de ce théorème se fonde sur l'approximation standard de Galerkin. En effet, les auteurs établissent d'abord les estimations formelles a priori pour les solutions du système (2.3.2). Ces estimations peuvent être rigoureusement justifiées en les dérivant du système d'approximation Galerkin et en passant ensuite à la limite en utilisant le théorème de compacité Aubin-Lions.*

Le second résultat concerne l'existence et l'unicité globale de la solution forte du système (2.3.2), sous réserve que la donnée initiale soit suffisamment petite.

Théorème 2.3.2. *On suppose que $u_0, v_0, T_0, \partial_x u_0, \partial_y v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ vérifient les conditions de symétrie suivantes:*

$$u, v \text{ et } T \text{ sont périodiques en } x \text{ et } z \text{ de période } 1;$$

u, v sont pairs en z , et T est impair en z ,

avec la condition de compatibilité $\int_0^1 \partial_x u_0 dz = 0$. De plus, supposons que

$$\|u_0\|_{H^1} + \|v_0\|_{H^1} + C_0 \|T_0\|_{H^1} + \|\partial_x u_0\|_{H^1} + \|\partial_x v_0\|_{H^1} + C_0 \|\partial_x T_0\|_{H^1} \ll 1$$

pour certains $C_0 > 0$. Alors pour tout \mathcal{T} positif, il existe une seule solution forte (u, v, T) du système (2.3.2) sur l'intervalle $[0, \mathcal{T}]$. De plus, la solution forte unique (u, v, T) dépend continûment de la donnée initiale.

2.3.2 Contribution de la thèse

Dans le chapitre 8 de cette thèse, pour $\epsilon > 0$, on considère le système

$$(PE) \quad \begin{cases} \partial_t U + U \cdot \nabla U - \epsilon^2 \Delta U + \nabla P = \begin{pmatrix} 0 \\ \frac{T}{Fr} \end{pmatrix}, \\ \partial_t T + U \cdot \nabla T - \kappa_1 \partial_x^2 T - \kappa_2 \partial_y^2 T = 0, \\ \operatorname{div} U = 0, \\ U|_{t=0} = U_0, \quad T|_{t=0} = T_0, \end{cases}$$

Dans notre étude, nous nous sommes penchés sur les équations primitives, sur un domaine mince avec des conditions de Dirichlet au bord. Notre domaine d'étude est

$$\mathcal{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\},$$

où ϵ représente la largeur du domaine. L'idée générale pour traiter ce type d'équations est d'imposer l'analyticité suivant la variable horizontale. Cela nous permet de contrôler les termes non linéaires qui posent une difficulté due aux pertes de dérivée dans la variable horizontale. Cette méthode nous a conduit à trouver un résultat d'existence et d'unicité globale des solutions pour le système (PE) dans un domaine mince.

Dans le système (PE) les constantes κ_1 et κ_2 représentent la diffusivité thermique horizontale et verticale. Au long de nos résultats, ces constantes sont données par $\kappa_1 = 1$ et $\kappa_2 = \epsilon^2$. Fr représente le nombre de Froude mesurant l'importance de la stratification, qui est censée être $Fr = \epsilon$.

Le système (PE) est complété par les conditions de Dirichlet au bord

$$(U, T)|_{y=0} = 0 \quad \text{et} \quad (U, T)|_{y=\epsilon} = 0.$$

Nos données initiales ont la forme

$$U|_{t=0} = U_0^\epsilon = \left(u_0 \left(x, \frac{y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{y}{\epsilon} \right) \right) \quad \text{dans} \quad \mathcal{S}^\epsilon,$$

et

$$T|_{t=0} = T_0 \left(x, \frac{y}{\epsilon} \right) \quad \text{dans } \mathcal{S}^\epsilon.$$

On cherche des solutions pour notre système de la forme

$$(2.3.3) \quad \begin{cases} U(t, x, y) = \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ T(t, x, y) = T^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \\ P(t, x, y) = p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

Effectuant la modification de mise à l'échelle $\bar{y} = \frac{y}{\epsilon}$, soit $\mathcal{S} := \{(x, \bar{y}) \in \mathbb{R}^2 : 0 < \bar{y} < 1\}$, on obtient le système primitive anisotrope

$$(2.3.4) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_{\bar{y}} u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_{\bar{y}}^2 u^\epsilon + \partial_x p^\epsilon = 0, \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_{\bar{y}} v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_{\bar{y}}^2 v^\epsilon) + \partial_{\bar{y}} p^\epsilon = T^\epsilon, \\ \partial_t T^\epsilon + u^\epsilon \partial_x T^\epsilon + v^\epsilon \partial_{\bar{y}} T^\epsilon - \Delta T^\epsilon = 0, \\ \partial_x u^\epsilon + \partial_{\bar{y}} v^\epsilon = 0, \\ (u^\epsilon, v^\epsilon, T^\epsilon)|_{t=0} = (u_0, v_0, T_0), \\ (u^\epsilon, v^\epsilon, T^\epsilon)|_{\bar{y}=0} = (u^\epsilon, v^\epsilon, T^\epsilon)|_{\bar{y}=1} = 0. \end{cases}$$

Par un passage à la limite formelle de $\epsilon \rightarrow 0$ dans le système (2.3.4), on obtient l'équation hydrostatique suivante:

$$(2.3.5) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = 0, \\ \partial_y p = T, \\ \partial_t T + u \partial_x T + v \partial_y T - \Delta T = 0, \\ \partial_x u + \partial_y v = 0, \\ (u, T)|_{t=0} = (u_0, T_0), \\ (u, v, T)|_{y=0,1} = 0, \end{cases}$$

Notre objectif est d'obtenir l'existence globale des solutions pour les systèmes (2.3.4) et (4.3.5) lorsque nos données initiales sont analytiques, puis nous voulons montrer la convergence des équations primitives anisotropes mises à l'échelle (2.3.4) vers le système limite (2.3.5) lorsque ϵ tend vers zéro.

Le premier résultat obtenu concerne l'existence de solutions globales du système hydrostatique (2.3.5), avec de petites données analytiques selon la variable horizontale.

Théorème 2.3.3. *Soit $a > 0$, on suppose que pour une constante c_0 suffisamment petite, indépendante de ϵ , et pour toute donnée initiale (u_0, T_0) satisfaisant*

$$(2.3.6) \quad \|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} T_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a,$$

et la condition de compatibilité $\int_0^1 u_0 dy = 0$ est satisfaite, alors le système limite (2.3.5) admet

une unique solution globale (u, T) satisfaisant

$$(2.3.7) \quad \|e^{\mathcal{R}t} e^{\frac{a}{2}|D_x|}(u, T)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t} e^{\frac{a}{2}|D_x|} \partial_y(u, T)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \leq C \|e^{a|D_x|}(u_0, T_0)\|_{\mathcal{B}^{\frac{1}{2}}},$$

De plus, si $e^{a|D_x|}u_0 \in \mathcal{B}^{\frac{5}{2}}$, $e^{a|D_x|}T_0 \in \mathcal{B}^{\frac{3}{2}}$, $e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$ et

$$(2.3.8) \quad \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_1 a}{1 + \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{3}{2}}}}$$

pour un certain c_1 suffisamment petit, alors il existe C une constante positive telle que pour $1 \leq s \leq \frac{5}{2}$, on a

$$(2.3.9) \quad \begin{aligned} & \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \\ & \lesssim C \left(\|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{5}{2}}} + \|e^{a|D_x|} \partial_y T_0\|_{\mathcal{B}^{\frac{3}{2}}} \right). \end{aligned}$$

où $C = C(s)$ et \mathcal{R} est une constante déterminée par l'inégalité de Poincaré pour la bande \mathcal{S} , l'espace fonctionnel utilisé pour obtenir ce résultat est l'espace de Besov Chemin-Lerner donné par la norme

$$\|u\|_{\tilde{L}_T^p(\mathcal{B}^s(\mathcal{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \left\| \Delta_q^h u(t) \right\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

Le deuxième résultat démontré est que les équations primitives anisotropes (2.3.4) avec de faibles données analytiques initiales dans la variable tangentielle sont globalement bien posées.

Théorème 2.3.4. *Soit $a > 0$, il existe une constante c_1 suffisamment petite indépendante de ϵ , telle que pour toute donnée initiale $(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon)$ satisfaisant*

$$(2.3.10) \quad \|e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a.$$

Alors le système anisotrope (2.3.4) admet une unique solution globale $(u^\epsilon, v^\epsilon, T^\epsilon)$ satisfaisant

$$(2.3.11) \quad \begin{aligned} & \|e^{\mathcal{R}t} e^{\frac{a}{2}|D_x|}(u^\epsilon, \epsilon v^\epsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t} e^{\frac{a}{2}|D_x|} T^\epsilon\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t} e^{\frac{a}{2}|D_x|} \partial_y(u^\epsilon, \epsilon v^\epsilon)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \\ & + \|e^{\mathcal{R}t} e^{\frac{a}{2}|D_x|} \nabla T^\epsilon\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \epsilon \|e^{\mathcal{R}t} e^{\frac{a}{2}|D_x|} \partial_x(u^\epsilon, \epsilon v^\epsilon)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \\ & \leq C \left(\|e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \right), \end{aligned}$$

où $C = C(s)$ et \mathcal{R} est une constante déterminée par l'inégalité de Poincaré pour la bande \mathcal{S} .

Le troisième résultat concerne la convergence des solutions de l'équation anisotrope mise à l'échelle des équations primitive (2.3.4) au système limite (2.3.5).

Théorème 2.3.5. *Soit $a > 0$, et $(u_0^\epsilon, v_0^\epsilon, b_0^\epsilon, c_0^\epsilon)$, satisfaisant (2.3.10). Soit u_0, T_0 satisfaisant $e^{a|D_x|}u_0 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}}$, $e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$, $e^{a|D_x|}T_0 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}}$, et la condition de compatibilité $\int_0^1 u_0 dy = 0$ est satisfaite et*

$$\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_2 a}{1 + \|e^{a|D_x|}(u_0, T_0)\|_{\mathcal{B}^{\frac{3}{2}}}},$$

pour un certain c_2 suffisamment petit indépendamment de ϵ , on a

$$(2.3.12) \quad \begin{aligned} & \| (w_\varphi^1, \epsilon w_\varphi^2) \|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \| \partial_y (w_\varphi^1, \epsilon w_\varphi^2) \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon \| (w_\varphi^1, \epsilon w_\varphi^2) \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ & \leq C \left(\| e^{a|D_x|} (u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0)) \|_{\mathcal{B}^{\frac{1}{2}}} + C \| e^{a|D_x|} (T_0^\epsilon - T_0) \|_{\mathcal{B}^{\frac{1}{2}}} + M\epsilon \right). \end{aligned}$$

où $C = C(s)$ et $w^1 = u^\epsilon - u$, $w^2 = v^\epsilon - v$, $\theta = T^\epsilon - T$ et v_0 est déterminé à partir de u_0 via $\partial_x u + \partial_y v = 0$ ainsi que les conditions au bords pour les données initiales suivant la variable verticale, et $M \geq 1$ est une constante indépendante de ϵ .

$(w_\varphi^1, w_\varphi^2, \theta_\varphi)$ sont données par

$$(2.3.13) \quad \left\{ \begin{array}{l} (w_\varphi^1, w_\varphi^2, \theta_\varphi)(t, x, y) = e^{\varphi(t, D_x)} \left(w^1(t, x, y), w^2(t, x, y), \theta(t, x, y) \right) \\ \quad \triangleq \mathcal{F}_h^{-1} \left(e^{\varphi(t, \xi)} \left(\widehat{w^1}(t, \xi, y), \widehat{w^2}(t, \xi, y), \widehat{\theta}(t, \xi, y) \right) \right) \\ \varphi(t, \xi) = (a - \lambda\eta(t))|\xi|. \end{array} \right.$$

La quantité $\eta(t)$ décrit l'évolution de la bande d'analyticité au cours du temps, cette quantité est positive pour tout $t \in \mathbb{R}_+^*$ satisfait l'équation

$$\dot{\eta}(t) = \| (\partial_y u_\Theta^\epsilon, \epsilon \partial_x u_\Theta^\epsilon, \partial_y T_\Theta^\epsilon)(t) \|_{\mathcal{B}^{\frac{1}{2}}} + \| \partial_y (u_\phi, T_\phi)(t) \|_{\mathcal{B}^{\frac{1}{2}}}$$

et lorsque $t = 0$ on a $\eta(0) = 0$.

Remarque 2.3.2. *L'idée principale de la preuve de ces théorèmes ci-dessus est d'utiliser des estimations d'énergie analytiques, qui sont motivées par [112] et qui proviennent de [27].*

2.4 Étude des équations de la magnétohydrodynamique homogène et inhomogène

Dans la deuxième partie de cette thèse, nous nous sommes intéressés à l'étude de l'existence et la régularité ainsi qu'à l'unicité des solutions de l'équation de la magnétohydrodynamique, lorsqu'on a une condition de petitesse de la dérivée unidirectionnelle de la vitesse initiale dans certains espaces invariants d'échelle.

2.4.1 Rappel des résultats

Les résultats de cette thèse sont motivés par le résultat obtenu par Y. Liu, M. Paicu et P. Zhang dans [94]. Les auteurs ont prouvé que tant que la dérivée unidirectionnelle de la vitesse initiale est suffisamment petite dans certains espaces invariants d'échelle, alors le système de Navier-Stokes admet une unique solution globale. En effet, les auteurs ont étudié le système suivant :

$$(2.4.1) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta_h u + \nabla p = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0}(t, x) = u_0(x), \end{cases}$$

où $\Delta_h = \partial_x^2 + \partial_y^2$. Il est globalement bien posé si les données initiales u_0 satisfont une condition de petitesse sur sa dérivée verticale $\partial_3 u_0$ dans certains espaces critiques. Le résultat principal de leur article s'énonce comme suit

Théorème 2.4.1. *Soit Λ_h^{-1} un multiplicateur de Fourier dont le symbole est $|\xi_h|^{-1}$, soit $u_0 \in \mathcal{B}^{0, \frac{1}{2}}$ un champ vectoriel solénoïdal avec $\Lambda_h^{-1} \partial_3 u_0 \in \mathcal{B}^{0, \frac{1}{2}}$. Alors il existe une constante positive suffisamment petite ϵ_0 et quelques constantes positives universelles L, M, N , tel que si*

$$(2.4.2) \quad \|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(L(1 + \|u_0^3\|_{\mathcal{B}^{-\frac{1}{2}, \frac{1}{2}}}^4) \exp(M \mathcal{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}))\right) \leq \epsilon_0,$$

avec

$$\begin{aligned} \mathcal{A}_N(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}) &= N^{\frac{1}{2}} \|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp(C \|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2) \\ &\quad + \|u_{0,N}^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(N^2 \exp(C \|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2)\right) \end{aligned}$$

et N est pris suffisamment grand que $\|u_{0,N}^h\|_{\mathcal{B}^{0, \frac{1}{2}}}$ soit suffisamment petit, alors le système (2.4.1)

admet une solution globale unique $u = v + e^{t\Delta_h} \begin{pmatrix} 0 \\ u_{0,hh}^3 \end{pmatrix}$ avec

$$v \in C([0, \infty[; \mathcal{B}^{0, \frac{1}{2}}), \quad \nabla_h v \in L^2([0, \infty[; \mathcal{B}^{0, \frac{1}{2}})$$

et $u_{0,hh}^3 = \sum_{k \geq l-1} \Delta_k^h \Delta_l^v u^3$.

Remarque 2.4.1. *La preuve de ce théorème consiste à décomposer la solution du système (2.4.1) en somme d'une solution du système de Navier-Stokes en deux dimensions avec un paramètre et une solution du système de Navier-Stokes anisotrope tridimensionnel perturbé.*

Remarque 2.4.2. *Dans le théorème 2.4.1, les auteurs obtiennent leur résultat dans l'espace de Besov homogène suivant:*

$$\mathcal{B}^{0, \frac{1}{2}} \triangleq \left\{ f \in \mathcal{S}'_h(\mathbb{R}^3) : \|u\|_{\mathcal{B}^{0, \frac{1}{2}}} \triangleq \sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q^v u\|_{L^2(\mathbb{R}^3)} < +\infty \right\}.$$

2.4.2 Contribution de la thèse

Le travail du dernier chapitre de cette thèse porte sur l'étude du système magnétohydrodynamique inhomogène. Le but est de supposer que la dérivée verticale des données initiales est suffisamment petite et d'arriver alors à prouver l'existence et l'unicité globale des solutions du

système magnétohydrodynamique inhomogène 3D. On considère donc le système

$$(2.4.3) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \rho(\partial_t u + u \cdot \nabla u) - \mu_1 \Delta u + \nabla p = b \cdot \nabla b, \\ \partial_t b - \mu_2 \Delta b - \operatorname{curl}(u \times b) = 0, \\ \operatorname{div} u = 0, \operatorname{div} b = 0, \\ \rho|_{t=0} = \rho_0(x), u|_{t=0} = u_0(x), b|_{t=0} = u_0(x), \end{cases}$$

où ρ est un scalaire représentant la densité du fluide, $u = (u_1, u_2, u_3)$ et $b = (b_1, b_2, b_3)$ représentent respectivement le champ de vitesse et le champ magnétique du fluide, p est une fonction scalaire indiquant la pression. μ_1 est un paramètre représentant le coefficient de viscosité cinématique et μ_2 est un paramètre représentant le coefficient de diffusion magnétique. Comme μ_1 et μ_2 ne jouent aucun rôle dans l'existence et l'unicité de la solution globale, donc on suppose que ces paramètres valent 1 dans le reste des calculs.

Quand la densité est constante dans le système (2.4.3), il devient le système MHD standard homogène suivant

$$(2.4.4) \quad \begin{cases} \partial_t u^H + u^H \cdot \nabla u^H - \Delta u^H = -\nabla p^H + b^H \cdot \nabla b^H, \\ \partial_t b^H - \Delta b^H + u^H \cdot \nabla b^H - b^H \cdot \nabla u^H = 0, \\ \operatorname{div} u^H = \operatorname{div} b^H = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u^H|_{t=0} = u_0^H(x), b^H|_{t=0} = b_0^H(x). \end{cases}$$

En utilisant la théorie de Littlewood-Paley et en adaptant la méthode développée dans le travail de Y. Liu, M. Paicu et P. Zhang dans [94] pour les équations de magnétohydrodynamique homogènes, nous obtenons un résultat de l'existence et l'unicité globale des solutions dans des espaces de Besov. Le résultat est le suivant:

Théorème 2.4.2. *Soient $u_0^H = (u_0^{H,h}, u_0^{H,3}) \in \mathcal{B}_p^0(\mathbb{R}^3)$, $b_0^H = (b_0^{H,h}, b_0^{H,3}) \in \mathcal{B}_p^0(\mathbb{R}^3)$ deux champs de vecteurs de divergence nulle avec $(u_0^{H,h}, b_0^{H,h}) \in H^1(\mathbb{R}^3)$, $\Lambda_h^{-1} \partial_3(u_0^H, b_0^H) \in \mathcal{B}_p^0(\mathbb{R}^3)$ ($1 < p < 4$), où Λ_h^{-1} est le multiplicateur de Fourier de symbole $|\xi_h|^{-1}$. Alors il existe une constante positive suffisamment petite ε_0 et une constante positive universelle C telle que si (u_0^H, b_0^H) satisfont la condition de petitesse*

$$(2.4.5) \quad \begin{aligned} & \|\Lambda_h^{-1} \partial_3(u_0^H, b_0^H)\|_{\mathcal{B}_p^0} B(u_0^{H,h}, b_0^{H,h}) \times \exp \left(C \left(\|(u_0^{H,3}, b_0^{H,3})\|_{\mathcal{B}_p^0} \right. \right. \\ & \left. \left. + \|(u_0^{H,h}, b_0^{H,h})\|_{\mathcal{B}_p^0}^2 + \|(u_0^{H,h}, b_0^{H,h})\|_{H^1}^4 \right) B(u_0^{H,h}, u_0^{H,h}) \right) \leq \varepsilon_0. \end{aligned}$$

où

$$B(u_0^{H,h}, b_0^{H,h}) = \exp \left(C \left(\|(u_0^{H,h}, b_0^{H,h})\|_{\mathcal{B}_p^0} + \|(u_0^{H,h}, b_0^{H,h})\|_{H^1}^2 \exp(\|(u_0^{H,h}, b_0^{H,h})\|_{L^2}) \right) \right),$$

alors le système (4.4.4) a une solution globale unique (u^{MHD}, b^{MHD}) qui satisfait

$$u^H \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2), \quad b^H \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2).$$

Passons maintenant au cas inhomogène du système MHD (2.4.3). Dans la suite, on suppose que la densité initiale vérifie $\inf \rho_0 > 0$. Par le principe du maximum pour l'équation de transport, on a $\inf \rho(t, x) > 0$. On peut alors utiliser la transformation $a \triangleq \frac{1}{\rho} - 1$ qui nous permet d'obtenir le système

$$(2.4.6) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1+a)(-\Delta u + \nabla p) = (1+a)(b \cdot \nabla b), \\ \partial_t b + u \cdot \nabla b - \Delta b = b \cdot \nabla u, \\ \operatorname{div} u = 0, \operatorname{div} b = 0, \\ a|_{t=0} = a_0(x), u|_{t=0} = u_0(x), b|_{t=0} = b_0(x). \end{cases}$$

Le théorème suivant montre le résultat d'existence et d'unicité globale des solutions associées au système (2.4.6).

Théorème 2.4.3. *Soient $u_0 = (u_0^h, u_0^3) \in \mathcal{B}_p^0(\mathbb{R}^3)$, $b_0 = (b_0^h, b_0^3) \in \mathcal{B}_p^0(\mathbb{R}^3)$ deux champs de vecteurs de divergence nul avec $(u_0^h, b_0^h) \in H^1(\mathbb{R}^3)$, $\Lambda_h^{-1} \partial_3(u_0, b_0) \in \mathcal{B}_p^0$ ($1 < p < 4$), $a_0 \in B_p^{\frac{3}{p}}$, où Λ_h^{-1} est le multiplicateur de Fourier de symbole $|\xi_h|^{-1}$. Alors il existe une constante positive suffisamment petite ε_1 et une constante positive universelle C_1 tel que si (a_0, u_0, b_0) satisfont la condition de petitesse suivante*

$$(2.4.7) \quad \left(\|a_0\|_{B_p^{\frac{3}{p}}} + \|\Lambda_h^{-1} \partial_3(u_0, b_0)\|_{\mathcal{B}_p^0} \right) L(u_0, b_0) \leq \varepsilon_1,$$

avec

$$L(u_0, b_0) = B(u_0^h, b_0^h) \exp(C_1(\|(u_0^3, b_0^3)\|_{\mathcal{B}_p^0} + \|(u_0^h, b_0^h)\|_{\mathcal{B}_p^0}^2 + \|(u_0^h, b_0^h)\|_{H^1}^4)) B(u_0^h, b_0^h)$$

et

$$B(u_0^h, b_0^h) = \exp(C_1(\|(u_0^h, b_0^h)\|_{\mathcal{B}_p^0} + \|(u_0^h, b_0^h)\|_{H^1}^2) \exp(\|(u_0^h, b_0^h)\|_{L^2}))$$

Alors le système (2.4.6) admet une solution globale unique

$$a \in \tilde{L}^\infty(\mathbb{R}_+; B_p^{\frac{3}{p}}), \quad u \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2) \quad \text{et} \quad b \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2).$$

Remarque 2.4.3. *La preuve de ce théorème consiste à décomposer la solution inhomogène en solution homogène plus un terme de reste, qu'on note dans le dernier chapitre par R .*

Remarque 2.4.4. *Dans les théorèmes 2.4.2-2.4.2, nous utilisons l'espace de Besov homogène suivant:*

$$B_{p,r}^{s_1, s_2} \triangleq \{u \in \mathcal{S}'_h(\mathbb{R}^3) : \|u\|_{B_{p,r}^{s_1, s_2}} \triangleq (2^{s_1 j} 2^{s_2 k} \|\Delta_j^h \Delta_k^v u\|_{L^p})_{\ell^1(\mathbb{Z})} < \infty\}.$$

Pour plus de simplicité, nous utilisons la notation suivante

$$\mathcal{B}_p^0 \triangleq \mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}, \quad \mathcal{B}_p^1 \triangleq \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}} \quad \text{and} \quad \mathcal{B}_p^2 \triangleq \mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}, 2+\frac{1}{p}},$$

nous mentionnons que l'espace \mathcal{B}_p^0 est critique. Rappelons que l'espace $\mathcal{B}_p^{\frac{3}{p}}$ est défini par:

$$B_p^{\frac{3}{p}} \triangleq \{u \in \mathcal{S}'_h(\mathbb{R}^3) : \|u\|_{B_p^{\frac{3}{p}}} \triangleq (2^{\frac{3j}{p}} \|\Delta_j u\|_{L^p(\mathbb{R}^3)})_{\ell^1(\mathbb{Z})} < \infty\}.$$

Chapter 3

Introduction (English version)

Physics is a science that allows us to understand the different phenomena related to our planet. In the theory of physics there are three main branches which are: classical physics, quantum physics and general relativity. In our thesis we are interested in classical physics, more precisely on "Fluid mechanics" which has for principle to study the behavior of fluids (liquids, gases, plasmas, liquid metals, salt water, electrolytes, ...) and the associated internal forces. Fluid mechanics is divided into two parts: static and fluid dynamics. The statics of fluids represents the study of fluids at rest, i.e. fluids that do not move and the dynamics corresponding to the study of fluids in motion, i.e. fluids that move over time.

During this thesis, we study the equations of fluid mechanics in motion, the history of these equations began with **L. Euler** and **J. d'Alembert**, when they were working on perfect fluids, after with **L. Navier** for a viscous fluid. We study also the magnetic properties of electrically conducting fluid.

Starting from the D'Alembert paradox and Leonhard Euler, the study of fluid dynamics provided many reasons to take a closer look near the border that confines the flow in the 19th century. Physical experiments, as well as mathematics results, suggest that in the thin layer near to the boundary lies an important part of crucial dynamics. To understand the motion of real fluid, Ludwig Prandtl in 1904 introduced the concept of the boundary layer and proposed a mathematical model for it.

The idea of L.Prandtl is to divide the flow field areas: one inside the boundary layer, dominated by the viscosity and creating the majority of drag experienced by the boundary body, and one outside the boundary layer, where we can neglect the viscosity, without significant effects on the solution. Those two areas allow a closed-form solution for the flow, thus a significant simplification of the Navier-Stokes equations. This equation, completed with the incompressibility condition and the proper boundary and initial conditions, constitutes the Prandtl system, whose study brought a deeper understanding of viscous flows.

3.1 Presentation of main actor in fluid mechanics

3.1.1 Compressible and incompressible

The property of volume change is called compressibility and a fluid whose volume changes is called compressible fluid. On the other hand, incompressible fluid is a fluid that is not compressed or expanded, and its volume is always constant, this hypothesis is verified for liquid water at a fixed temperature and molten metals. It is also checked for gases when the Mach number Ma is low. In reality, a rigorous incompressible fluid does not exist.

The main Difference Compressible vs Incompressible Fluids The main difference between compressible and incompressible fluid is that a force applied to a compressible fluid changes the density of fluid whereas a force applied to an incompressible fluid does not change the density to a considerable degree.

3.1.2 Viscosity

The viscosity of a fluid is a measure of its resistance to deformation at a given rate. There are two types of viscosity "cinematic" and "dynamic", but in the Navier-Stokes equations, it is the kinematic viscosity that has a large role, because it is of comparable size with the Rossby number. In reality, the fluid is turbulent and ν is not the cinematic viscosity it is rather a turbulent viscosity. The Coriolis force creates an asymmetry between horizontal motion and motion vertical, vertical movement being limited when rotation is fast. This induces an anisotropy in fluid behavior: movement in the horizontal direction is much more important than the one in the vertical direction.

A fluid that has no resistance to shear stress is known as an ideal or inviscid fluid. Zero viscosity is observed only at very low temperatures in super-fluids. Otherwise, the second law of thermodynamics requires all fluids to have positive viscosity; such fluids are technically said to be viscous or viscid. A fluid with a high viscosity, such as pitch, may appear to be solid.

3.1.3 Reynolds number

The Reynolds number Re helps predict patterns in different fluid flow situation. This number is the ratio of the inertial forces to viscous forces within a fluid which subjected to relative internal movement due to different fluid velocities. A region where these forces change behavior is known as a boundary layer, such as the bounding surface in the interior of a pipe. A similar effect is created by the introduction of a stream of high-velocity fluid into a low-velocity fluid, such as the hot gases emitted from a flame in air. The Reynolds number is given by

$$Re = \frac{\text{forces d'inertie}}{\text{forces visqueuses}} = \rho UL/\nu$$

where U and L is the velocity and length scales of the motion, and we denote them respectively. When the Reynolds number is very small ($Re \ll 1$), the inertial forces ($\partial_t u + u \cdot \nabla u$) are very

small compared to the viscous forces and they can be neglected when solving (NS) equations. On the other hand when this number is very high ($Re \gg 1$), in this case the initial forces are very larger than the viscous forces. such turbulent flow problems are transient in nature, a mesh that is fine enough to resolve the size of the smallest eddies in the flow needs to be used.

3.2 Equations of motion in fluid mechanics

In the following, we are interested in the differential equations of the mechanics of incompressible fluids.

3.2.1 Incompressible fluid

In fluid mechanics or more generally continuum mechanics, incompressible flow (isochoric flow) refers to a flow in which the material density is constant within a fluid parcel an infinitesimal volume that moves with the flow velocity. An equivalent statement that implies incompressibility is that the divergence of the flow velocity is zero (see the derivation below, which illustrates why these conditions are equivalent).

We start by deduce from the principle of physics, the **d'Euler** and **Navier-Stocks** equations, for an incompressible fluid. Let now consider Ω as an element of the fluid in \mathbb{R}^n , the fluid evolves over time by trajectories $\psi(t, x)$, who satisfies the following equation:

$$\begin{cases} \partial_t \psi(t, x) = u(t, \psi(t, x)), \\ u(0, x) = x. \end{cases}$$

where $u(t, x)$ is the velocity field that is the tangent to the point $\psi(t, x)$ to the trajectory.

The condition of incompressibility of the fluid translates mathematically into the following relation :

$$\forall t \in [0, T], \quad Volume(\Omega) = Volume(\psi(t, \Omega)).$$

By changing the variable and assuming that the function $\psi(t, x)$ is a C^1 -diffeomorphism we have:

$$\int_{\Omega} 1 dx = \int_{\Omega} |\det J\psi(t, y)| dy.$$

We choose our domain $\Omega = \Omega_\epsilon = B(x, \epsilon)$ and by taking the limit of ϵ to zero, we obtain:

$$\det J\psi(t, x) = 1.$$

Otherwise, Liouville's theorem tells us that for any matrix solution of the equation $X'(t) = A(t)X(t)$, we have the following equality::

$$(\det X)'(t) = (\text{tr} A(t)) \det X(t).$$

We find that

$$\partial_t (\det J\psi(t, x)) = \text{tr}(\nabla u)(t, \psi(t, x)) \times \det J(\psi)(t, x).$$

As $\det J\psi(t, x) = 1$, then $\partial_t(\det J\psi(t, x)) = 0$, as a result we obtain the following result:

$$\text{tr}(\nabla u)(t, \psi(t, x)) = \text{div } u = 0.$$

So the condition of incompressibility translates into the equation $\text{div } u = 0$.

3.2.2 Conservation of mass

In physics, the law of conservation of mass or principle of mass conservation states that for any system closed to all transfers of matter and energy, the mass of the system must remain constant over time, as the system's mass cannot change, so quantity can neither be added nor be removed. Therefore, the quantity of mass is conserved over time.

Mathematically, the conservation of mass law can be formulated in the fields of fluid mechanics and continuum mechanics, where the conservation of mass is usually expressed using the continuity equation, given in the differential form in three dimension as:

$$\partial_t \rho + \text{div}(u\rho) = 0,$$

where ρ is the density (mass per unit volume), t is the time, ∇ is the divergence (for example in three dimension $\text{div } u = \partial_x u^1 + \partial_y u^2 + \partial_z u^3$), and $u = (u^1, u^2, u^3)$ is the velocity field.

In particular, if u is a particle of zero divergence (it mean that $\text{div } u = 0$) and our fluid is homogeneous (it mean that $\rho(t, x) = \tilde{\rho}(t)$), then the density remains constant over time, i.e.

$$\frac{\partial \rho}{\partial t} = 0,$$

and so we have $\rho = \rho_0$ for all t, x .

3.2.3 Expression of kinematic quantities

Trajectory : It is a set of points occupied successively by the same fluid. A parcel that passes at the moment t to the point $\psi(t, x)$ at speed u whose expression is of the following form :

$$u(t, \psi(t, x)) = \partial_t \psi(t, x).$$

Acceleration : noted by γ . By definition, it is the derivative concerning the time of the velocity u , which will be denoted $\frac{du}{dt}$, and given by the following formula :

$$\begin{aligned} \gamma &= \partial_t [u(t, \psi(t, x))] \\ &= \partial_t u(t, \psi(t, x)) + \partial_t \psi(t, x) \nabla u(t, \psi(t, x)) \\ (3.2.1) \quad &= (\partial_t u + u \cdot \nabla u)(t, \psi(t, x)). \end{aligned}$$

Newton law: Newton's law gives us:

$$\rho\gamma(t, x) = F(t, x)$$

where γ is the acceleration and F denotes the external forces exerted on the fluid.

By (3.2.1), we have:

$$\rho(\partial_t u + u\nabla u) = F.$$

Euler equation: Let consider the incompressible flow of a perfect fluid in a mass force field f . At any point of the fluid $\psi(t, x)$, the pressure fields $p(t, \psi(t, x))$ and velocities $u(t, \psi(t, x))$ verifying the incompressible condition, we obtain the following equation:

$$\frac{\partial u}{\partial t} + u\nabla u = -\nabla p + f \quad \text{and} \quad \text{div } u = 0.$$

Navier-stocks equations: The second Newton law give us :

$$F = \rho(\partial_t u + u\nabla u),$$

here the force $F = -\text{div}\sigma$, where σ is the stress tensor. For a general fluid of the differential type of order n , $\sigma = -pId + G(A_1, A_2, \dots, A_n)$ where G is the polynomial homogeneous in its variables and

$$A_n = \frac{dA_{n-1}}{dt} + (\nabla u)^t A_{n-1} + A_{n-1}(\nabla u).$$

Note that for a Newtonian fluid, we have:

$$-\text{div } (pId + \nu\nabla u) = -\nabla p + \nu\Delta u = F,$$

so,

$$\rho(\partial_t u + u\nabla u) = -\nabla p + \nu\Delta u,$$

where $\Delta = \sum_{i=1}^n \partial_i^2$, then the Navier-stocks equations are given by the following system:

$$\begin{cases} \rho(\partial_t u + u\nabla u) - \nu\Delta u = -\nabla p, \\ \text{div } u = 0, \\ u(0, x) = u_0(x). \end{cases}$$

For homogeneous fluids it can be considered that $\rho = \rho_0 = 1$.

In addition if the fluid is inhomogeneous, so our equation have the following form:

$$\begin{cases} \partial_t \rho + u\nabla \rho = 0, \\ \rho(\partial_t u + u\nabla u) - \nu\Delta u = -\nabla p, \\ \text{div } u = 0, \\ u(0, x) = u_0(x). \end{cases}$$

3.3 Magneto-hydrodynamics equations

In this thesis, we studied the coupled Navier-Stokes equations with an approximation of the Maxwell equations for the electromagnetic field, which describes the behavior of magma currents in the Earth's core. This model is called the Magneto-hydrodynamique system (**MHD**). The equation describing the evolution of the magnetic field noted \mathbf{B} , derives from the Maxwell equations (Maxwell 1865, Jackson 1975)

$$(3.3.1) \quad \begin{cases} \eta\lambda\partial_t E = \text{curl } B - \lambda j, \\ \partial_t B = -\text{curl } E, \\ \text{div } B = 0, \end{cases}$$

where E and B obey the Maxwell equation of electromagnetic, η and λ are the permittivity and the magnetic permeability coefficient, respectively. We assume that the electric current j follow the Ohm law :

$$j = \sigma(E + u \times B)$$

where σ represent the electric conductivity of the fluid. In classical magneto-hydrodynamique system, we assume that typical speeds are low compared to the speed of light c (we note that $c^2 = \frac{1}{\lambda\eta}$); the equation of Maxwell-Ampere is simplified into:

$$\text{curl } B = \lambda j.$$

By combining these equations and defining the magnetic diffusivity $\mu = 1/\lambda\sigma$, we get the induction equation

$$\begin{aligned} \partial_t B &= -\text{curl } E \\ &= -\text{curl} \left(\frac{1}{\sigma} j - u \times B \right) \\ &= -\text{curl} \left(\frac{1}{\sigma} \left(\frac{1}{\lambda} \text{curl } B \right) - u \times B \right) \\ &= -\frac{1}{\sigma\lambda} \text{curl } \text{curl } B + \text{curl}(u \times B) \\ &= \mu\Delta B - u \cdot \nabla B + B \cdot \nabla u. \end{aligned}$$

The electromagnetic field act back on the fluid through the action of the Lorentz force $F = j \times B + \rho E$, where ρ denote the electric density. However, in dense fluids, electrostatic enforce charge neutrally over macroscopic distances, which is known as quasi-neutrality. Therefore, we may assume that $\rho = 0$, which yields the force

$$(3.3.2) \quad F = j \times B.$$

Under the assumption of the speed of light c , the first equation of (3.3.1) and (3.3.2) yield

$$F = \frac{1}{\lambda} \text{curl} B \times B = \frac{1}{\lambda} \left((B \cdot \nabla) B - \frac{1}{2} \nabla |B|^2 \right).$$

Thus, we obtain the coupled Navier-Stokes equations with the Lorentz force and the Maxwell equations for the electromagnetic field

$$(3.3.3) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla \left(p + \frac{|B|^2}{2} \right) + B \cdot \nabla B \\ \partial_t B + u \cdot \nabla B - \mu \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ u|_{t=0} = u_0, \quad B|_{t=0} = B_0. \end{cases}$$

In addition if the fluid is in-homogeneous, so our equation have the following form:

$$(3.3.4) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \nu \Delta u = -\nabla \left(p + \frac{|B|^2}{2} \right) + B \cdot \nabla B \\ \partial_t B + u \cdot \nabla B - \mu \Delta B = B \cdot \nabla u, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ u|_{t=0} = u_0, \quad B|_{t=0} = B_0. \end{cases}$$

Remark 3.3.1. *In this thesis we investigate to obtain a global well-posedness of the homogeneous and in-homogeneous incompressible **MHD** system in three dimension with only one-directional derivative of the initial velocity and magnetic is sufficiently small in some scaling invariant spaces.*

3.4 Presentation of the hyperbolic version of the Navier-Stokes and Magneto-hydrodynamic equations

In this thesis, we interested by the hyperbolic version of Navier-Stokes and Magneto-hydrodynamic equations. With this in mind, we approach the solutions of (NS) and (MHD) by solutions of better known hyperbolic equations. In particular, we consider an equation of damped waves, whose solutions will propagate instantly, and an equation of weakly compressible waves that will have a finite propagation speed.

We start first by the Cattaneo approximation in 1949 for the study of the heat equation (see [24,25]) and others (Chester, Vernotte, etc.) proposed the following hyperbolic model.

$$\frac{1}{c^2} \partial_t^2 \theta + \frac{1}{\beta} \partial_t \theta - \Delta \theta = 0.$$

This equation is called **The Telegraph equation**. It has a finite propagation speed and is compatible with both the principle of relativity and the second law of thermodynamics, so it is a satisfactory physical model.

This equations can be also seen as a relaxation of Euler equations, these equations were considered by Brenier, Natalini and Puel in [16]. Let consider the incompressible Euler equations,

namely

$$(3.4.1) \quad \begin{cases} \partial_t U + U \cdot \nabla U = -\nabla p, \\ \operatorname{div} U = 0, \\ U(0, x) = U_0(x). \end{cases}$$

To approximate these equations, we introduce its relaxed version, which is obtained by a singular perturbation of the nonlinear term $U \cdot \nabla U = \nabla \cdot (U \otimes U)$, through a supplementary matrix valued variable noted $\nabla \cdot (V)$. This leads to the following system:

$$(3.4.2) \quad \begin{cases} \partial_t U + \nabla \cdot (V) = -\nabla p, \\ \partial_t V + \nu \nabla U = -\frac{1}{\tau} (V - U \otimes U), \\ \operatorname{div} U = 0, \\ U(0, x) = U_0(x), \quad V(0, x) = V_0(x). \end{cases}$$

We note that if τ goes to zero, we formally recover the Euler system (3.4.1). Let us now consider a diffusive scaling, for $\epsilon > 0$ we have

$$(3.4.3) \quad \begin{cases} U^\epsilon(t, x) := \frac{1}{\sqrt{\epsilon}} U\left(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}}\right), \\ V^\epsilon(t, x) := \frac{1}{\epsilon} V\left(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}}\right), \\ p^\epsilon(t, x) := \frac{1}{\epsilon} p\left(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}}\right). \end{cases}$$

Taking $\tau = 1$, therefore the system (3.4.2) becomes

$$(3.4.4) \quad \begin{cases} \partial_t U^\epsilon + \nabla \cdot (V^\epsilon) = -\nabla p^\epsilon, \\ \sqrt{\epsilon} \partial_t V^\epsilon + \frac{\nu}{\sqrt{\epsilon}} \nabla U^\epsilon = -\frac{1}{\sqrt{\epsilon}} (V^\epsilon - U^\epsilon \otimes U^\epsilon), \\ \operatorname{div} U^\epsilon = 0, \\ U^\epsilon(0, x) = U_0^\epsilon(x), \quad V^\epsilon(0, x) = V_0^\epsilon(x). \end{cases}$$

Finally let us remark that our scaling can be considered as a hyperbolic perturbation of the Navier-Stokes equations, which is similar to the Cattaneo hyperbolic heat equations [24], we eliminate just the unknown V in the equations (3.4.4), then we find

$$(3.4.5) \quad \begin{cases} \epsilon \partial_t^2 U^\epsilon + \partial_t U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \nu \Delta U^\epsilon = -\nabla p^\epsilon, \\ \operatorname{div} U^\epsilon = 0, \\ U^\epsilon(0, x) = U_0^\epsilon(x), \quad \partial_t U^\epsilon(0, x) = U_1^\epsilon(x). \end{cases}$$

This perturbation, is seen as a relaxation of Euler equations, was considered by Brenier, Natalini, and Puel in [16]. Later this equation was considered by Paicu and Raugel in [108, 109].

This model has a various justification in many articles with different methods. We cite for example [74] where the authors considered these equations as a model of ice formation in lakes. In a numerical and analytic point of view we cites [80, 81] where this approximation used to compute the solutions of Navier-Stokes. In the second chapter we will introduce the result obtained by Brenier, Natalini, and Puel in [16], also after the result obtained by Marius and Raugel in [108, 109] and the result of O. Coulaud, I. Hachicha and G. Raugel in [38] of the global solution in 2D and 3D. In this thesis, we obtain a result of existence and uniqueness of solutions to (3.4.5) in a striped domain $\mathbb{R} \times (0, 1)$, for some analytically small initial data in the tangential variable, We will briefly state them in the second chapter.

We will now explain the hyperbolic version of the magneto-hydrodynamic equations. We begin by recalling the most widespread form of the two-dimensional Navier-Stokes-Maxwell system with Cattaneo's law:

$$(3.4.6) \quad \begin{cases} \frac{\rho\nu\mathcal{J}}{c^2}\partial_\tau^2\vec{U} + \rho(\partial_t\vec{U} + \vec{U} \cdot \nabla\vec{U}) - \rho\nu\Delta\vec{U} + \nabla P = \vec{J} \times \vec{B} & \text{balance of linear momentum,} \\ \operatorname{div}\vec{U} = 0 & \text{conservation of mass,} \\ \partial_t\vec{B} + \operatorname{curl}\vec{E} = 0 & \text{Faraday's law,} \\ \vec{J} = \sigma(\vec{E} + \vec{U} \times \vec{B}) & \text{Ohm's law,} \\ \frac{1}{c^2}\partial_t\vec{E} + \mu_0\vec{J} = \operatorname{curl}\vec{B} & \text{Ampere's law,} \\ \operatorname{div}\vec{B} = 0 & \text{Gauss's law for magnetism,} \\ \operatorname{div}\vec{E} = 0 & \text{Gauss's law for electric field.} \end{cases}$$

The system and the corresponding state variables depend on $(t, X, Y) \in (0, T) \times \mathbb{R} \times \mathbb{R}_+$, for a positive time $T > 0$.

The constants $\rho > 0$ and $\nu > 0$ are fluid density and kinematic viscosity, respectively, while $c > 0$ represents the speed of light. The first term $(\rho\nu\mathcal{J}/c^2)\partial_t^2\vec{U}$ in the balance of linear momentum is due to Cattaneo's law [6, 16, 38, 108, 111, 122] and depends on a general inertia constant $\mathcal{J} > 0$. This law develops around a first-order Taylor expansion of a delayed relation on the Cauchy stress tensor

$$\mathbb{S}(t + t_{\text{rel}}, \cdot) = \nu \frac{\nabla u + \nabla u^T}{2}(t, \cdot)$$

where for us the relaxation time is given by $\tau_{\text{rel}} = \rho\nu\mathcal{J}/c^2$. This particular form (in terms of $\mathcal{J} > 0$ and not directly in $\tau_{\text{rel}} > 0$), will be important indeed when rescaling our system for the boundary layers.

We have denoted by $\sigma > 0$ the electrical conductivity, by $\mu_0 > 0$ the magnetic permeability. We have further denoted $\vec{U}(\tau, X, Y) = (U_1(\tau, X, Y), U_2(\tau, X, Y))^T \in \mathbb{R}^2$ and $\vec{B}(\tau, X, Y) = (B_1(\tau, X, Y), B_2(\tau, X, Y))^T \in \mathbb{R}^2$ the velocity field and magnetic field of the media, respectively. The scalar pressure $P(\tau, X, Y) \in \mathbb{R}$ is the Lagrangian multiplier that ensures the incompressibility of the velocity field. The current density $\vec{J} = (0, 0, J(\tau, X, Y))^T$ and the electric field $\vec{E} = (0, 0, E(\tau, X, Y))^T$ are considered as three dimensional vector fields, being perpendicular

to the plane in which the fluid motion occurs. Since we are dealing with the two dimensional version of the equations, we shall clarify the employed notation:

$$\vec{J} \times \vec{B} = J\vec{B}^\perp = J \begin{pmatrix} -B_2 \\ B_1 \end{pmatrix}, \quad \text{curl} \vec{E} = \begin{pmatrix} \partial_Y E \\ -\partial_X E \end{pmatrix},$$

$$\vec{U} \times \vec{B} = \begin{pmatrix} 0 \\ 0 \\ U_1 B_2 - U_2 B_1 \end{pmatrix}, \quad \text{curl} \vec{B} = \begin{pmatrix} 0 \\ 0 \\ \partial_X B_2 - \partial_Y B_1 \end{pmatrix}.$$

The positive parameters ν , μ_0 and ε_0 correspond to the kinematic viscosity, magnetic permeability and permittivity of free space, respectively. In addition, the parameter σ represents the electrical conductivity of the medium.

Some of the terms in (3.4.6) are redundant, indeed we can recast the overall system as five equations depending on \vec{U} , \vec{B} and \vec{E} . First, we formulate the Faraday's law in (3.4.6) only in terms of the magnetic field \vec{B} , making use of the Ohm's law (3.4.6):

$$(3.4.7) \quad \partial_\tau \vec{B} = -\text{curl} \vec{E} = \text{curl}(\vec{U} \times \vec{B}) - \frac{1}{\sigma} \text{curl} \vec{J}.$$

Also, to get rid of the density of the current in $\text{curl} \vec{J}$, we apply the operator curl to Ampère's law:

$$\frac{1}{c^2} \partial_\tau (\text{curl} \vec{E}) + \mu_0 \text{curl} \vec{J} = \text{curl} \text{curl} B,$$

which implies that

$$\text{curl} \vec{J} = \frac{1}{\mu_0 c^2} \partial_\tau^2 \vec{B} + \frac{1}{\mu_0} (\nabla \text{div} \vec{B} - \Delta B) = \frac{1}{\mu_0 c^2} \partial_\tau^2 \vec{B} + \frac{1}{\mu_0} (\nabla \text{div} \vec{B} - \Delta B)$$

Thus, we can plug this last relation into the equation (3.4.7), to finally obtain a hyperbolic form of Ampère's law in terms of magnetic field \vec{B} :

$$(3.4.8) \quad \frac{1}{\sigma \mu_0 c^2} \partial_\tau^2 \vec{B} + \partial_\tau \vec{B} - \frac{1}{\sigma \mu_0} \Delta \vec{B} = \text{curl}(\vec{U} \times \vec{B}) = \vec{B} \cdot \nabla \vec{U} - \vec{U} \cdot \nabla \vec{B}.$$

Similarly, we can get rid of \vec{J} also in the linear momentum equilibrium in (3.4.6) thanks to

$$\frac{\rho \nu \mathcal{J}}{c^2} \partial_t^2 \vec{U} \rho (\partial_t \vec{U} + \vec{U} \cdot \nabla \vec{U}) - \nu \Delta \vec{U} + \nabla P = \sigma (\vec{E} + \vec{U} \times \vec{B}) \times \vec{B} = \sigma \vec{E} \times \vec{B} + \sigma (\vec{B} (\vec{U} \cdot \vec{B}) - \vec{U} |\vec{B}|^2).$$

We are now able to reduce the number of equations in (3.4.6). Considering the electric field $\vec{E}(t, X, Y) = (0, 0, E(t, X, Y))^T$ (whose divergence is always zero) and recalling the definition

of the vector field $\vec{B}^T = (-B_2, B_1)^T$, we finally gather

$$(3.4.9) \quad \begin{cases} \frac{\rho\nu\mathcal{J}}{c^2}\partial_t^2\vec{U} + \rho(\partial_t\vec{U} + \vec{U}\cdot\nabla\vec{U}) - \rho\nu\Delta\vec{U} + \nabla P = \sigma(\vec{B}(\vec{U}\cdot\vec{B}) - \vec{U}|\vec{B}|^2) + \sigma E\vec{B}^T, \\ \operatorname{div}\vec{U} = 0, \\ \frac{1}{\sigma\mu_0c^2}\partial_t^2\vec{B} + \partial_t\vec{B} - \frac{1}{\sigma\mu_0}\Delta\vec{B} = \vec{B}\cdot\nabla\vec{U} - \vec{U}\cdot\nabla\vec{B}, \\ \partial_t\vec{B} + \operatorname{curl}\vec{E} = 0, \\ \operatorname{div}\vec{B} = 0, \end{cases}$$

3.4.1 Boundary layer

In physics and fluids mechanics, a boundary layer is layer of the fluid in the immediate vicinity of a bounding surface where the effect are significant. The liquid or gaz in the boundary layer is characterized by an approach to the surface. The boundary layer can be seen in several phenomena for example **an aircraft** and **Earth's atmosphere**. If we stand on an aircraft wings, we observe that the boundary layer is the part of the flow close to the wing of the aircraft, where viscous forces distort the surrounding non-viscous flow. On the other hand, the case of Earth's atmosphere, we see the atmospheric boundary layer as the air layer near to the ground.

A boundary layer is a thin layer of viscous fluid close to the surface of a wall in contact with a moving stream in which the flow of velocity varies from zero at the wall. The fundamental concept of boundary layer was defined by L.Pradtl (1904), it defines the boundary layer as a layer of fluid developing in flow with a very high Reynolds number \mathbf{Re} , that is with relatively low viscosity as compared with inertial forces.

The deduction of the boundary layer equations was one of the most important advances in fluid dynamic. Using the order of magnitude analysis, the well-known governing Navier-Stokes equations of viscous fluid flow can be greatly simplified within the boundary layer. By making the boundary layer approximation, our flow can be divided into an inviscid portion and boundary layer, which is governed by an easier to solve PDE. The continuity and Navier-Stokes equations for a two-dimensional steady incompressible flow in Cartesian coordinates are given by

$$(3.4.10) \quad \begin{cases} \partial_t u^\nu + u^\nu \partial_x u^\nu + v^\nu \partial_y u^\nu - \nu(\partial_x^2 u^\nu + \partial_y^2 u^\nu) = -\partial_x p^\nu. \\ \partial_t v^\nu + u^\nu \partial_x v^\nu + v^\nu \partial_y v^\nu - \nu(\partial_x^2 v^\nu + \partial_y^2 v^\nu) = -\partial_y p^\nu \\ \partial_x u^\nu + \partial_y v^\nu = 0 \\ u^\nu(t, x, 0) = 0, \quad v^\nu(t, x, 0) = 0. \end{cases}$$

The idea of Prandtl is to inject an argument for the two components of the velocity to express them in a more useful form. Exploiting the fact that the most meaningful part of the flow takes place near the boundary, the argument provides a re-scaling of the vertical variable and of the velocity components. On the system (3.4.10), we note that the limit of $\nu \rightarrow 0$ would give just

Euler equation for perfect fluids, whose the solution will not satisfy the non-slip condition at the wall .

The assumption suggested by Prandtl is to transform the vertical motion by $Y = \frac{y}{\sqrt{\nu}}$, so we obtain the re-scaling of our velocities and pressure

$$(3.4.11) \quad \begin{cases} u^\nu(t, x, y) = u\left(t, x, \frac{y}{\sqrt{\nu}}\right) \\ v^\nu(t, x, y) = v\left(t, x, \frac{y}{\sqrt{\nu}}\right) \\ p^\nu(t, x, y) = p\left(t, x, \frac{y}{\sqrt{\nu}}\right). \end{cases}$$

Plugging the expression above in (3.4.10) and keeping the leading order terms we derive the Prandtl system (denoting y instead of Y):

$$(3.4.12) \quad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = -\partial_x p^\nu. \\ \partial_y p = 0 \\ \partial_x u + \partial_y v = 0 \\ u(t, x, 0) = 0, \quad v(t, x, 0) = 0. \end{cases}$$

If we consider the domain $\Omega = \mathbb{R} \times (0, 1)$ it mean that we are in a thin striped domain, then the system (3.4.12) completed by the Dirichlet boundary condition

$$u_{/y=0} = u_{/y=1} = 0 \text{ and } v_{/y=0} = v_{/y=1} = 0.$$

Remark 3.4.1. *One of the consequences of this formal work is that we eliminated from the system the evolution equation on the vertical component. This vertical component is recovered only through the divergence-free condition. we apply this elimination in several models (for example MHD system and Hyperbolic version of Navier-Stokes system) to obtain the overall existence of the solutions of these models and also their stability.*

3.4.2 Primitive equations

The primitive equations are a set of nonlinear partial differential equations, those equations used to approximate the global atmospheric flow. The primitive atmospheric equations are a simplified version of the Navier-Stokes equations. They are applicable in the case of a fluid on the surface of a sphere by assuming that the vertical component of the motion is much weaker than the horizontal component and that the fluid layer is very thin relative to the radius of the sphere.

If we look at the modeling done for fluids in section 3.2, we have neglected an important effect which is Earth's gravity that in situations this gravity has a great influence on geophysical fluids. In nature, the appearance of stratification due to fluid variable density, causes the coupling

between the equations of Navier-stokes and the equations of transport of heat (or stratification of density) and salinity.

We can also mention the phenomena due to stratification, which can not be explained simply by the equation of rapidly rotating fluid motion, and which are observed by many oceanographers. This is the phenomenon of "dead water" (for a description, we can suggest [39]). It happens that a boat undergoes a strong resistance in its movement while the water on the surface is apparently very calm. The cause is the presence of internal waves. The boat sails on a thin layer of relatively fresh water located on a layer of very salty water. The boat creates waves internal on the interface, invisible on the surface but of great energy, thus causing the resistance to the movement of the boat.

To describe the movement of geophysical fluids at the scale, we consider two very different phenomena : the rotation tending to a distribution in columns vertical, and stratification, tending to maintain a distribution in horizontal layers of the same density. Using the mass conservation of the movement quantity and the Boussinesq approximation, we obtain the system of primitives equations

$$(3.4.13) \quad \left\{ \begin{array}{l} \partial_t U + U \cdot \nabla U - \nu \Delta U + \frac{T}{Fr} = -\nabla p, \\ \partial_t T + U \cdot \nabla T - \nu' \Delta T = 0, \\ \operatorname{div} U = 0, \\ U|_{t=0} = U_0, \quad T|_{t=0} = T_0, \end{array} \right.$$

where U is the velocity field, T is the potential temperature and p is the pressure. The Parameters are the kinematic viscosity ν , the thermal diffusivity ν' , Fr is the Froude number.

Chapter 4

History of the results and contributions of the thesis (English version)

As part of this thesis, we are interested in systems derived from Newtonian geophysical fluids mechanics. In a first part, we consider Newtonian geophysical fluids hydrostatic approximations, to namely the Navier-Stokes equations in their hyperbolic version, the magneto-hydrodynamic equations and primitive equations, in thin domains. In a second part, we will study magneto-hydrodynamic systems in an anisotropic framework through three-dimensional space.

We begin our study with models of geophysical fluids in a thin domain, supplemented by the Dirichlet condition at the edge, so that the vertical extent of the domain is negligible compared to its horizontal counterpart. In this case, the rescaled viscosity is not isotropic. Anisotropic viscosities (turbulent viscosities) will therefore be used. The main idea is to take scale-invariant spaces such as Besov and Sobolev spaces. The goal is to make a change of scale so that we are close to the limit of our field of study. The change considered for our data is $U(t, x, y) = \left(u^\epsilon(t, x, \frac{y}{\epsilon}), \epsilon v^\epsilon(t, x, \frac{y}{\epsilon}) \right)$ where ϵ is the width of the domain. The disappearance of viscosity within the boundary combined with Dirichlet conditions at the edges of the domain causes the appearance of boundary layers. The equations that describe the evolution of the fluid in the thin boundary domain were the discovery of Prandtl [119] in 1904 to explain the discrepancy between the boundary conditions verified by an ideal fluid and a viscous fluid with evanescent viscosity.

Our models are studied in the first part, in a framework of boundary layers having low viscosity and magnetic diffusivity ($\nu = \epsilon^2$ and $\sigma = \epsilon^2$) and are also called evanescent viscosity. By using a change of scale on our solutions, we obtain anisotropic systems. By crossing the boundary ($\epsilon \rightarrow 0$), these anisotropic systems tend towards hydrostatic equations of the Prandtl type.

In an anisotropic framework, where the viscosity as well as horizontal magnetic diffusivity are low compared to their vertical counterparts, the interest is to obtain evidence of the existence of

strong solutions in spaces that are almost optimal. We recall that a space is said to be optimal or critical if it is invariant by change of scale for the systems considered, i.e. if $u(t, x)$ is a solution of our hyperbolic NS equation, then $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ and $p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x)$ is also a solution. In our case, the critical spaces in which we work will be the Besov space $\mathcal{B}^{\frac{1}{2}, 0}$ defined by:

$$\mathcal{B}^{\frac{1}{2}, 0} \triangleq \left\{ f \in \mathcal{S}'_h : \|f\|_{\mathcal{B}^{\frac{1}{2}, 0}} \triangleq \|(2^{\frac{q}{2}} \|\Delta_q^h f\|_{L^2})_{q \in \mathbb{Z}}\|_{\ell^1(\mathbb{Z})} < +\infty \right\}.$$

In the second part of this thesis, we study the magneto-hydrodynamic system in all space \mathbb{R}^3 . In this context, two results were obtained of the existence of strong global solutions for this magneto-hydrodynamic system. Indeed, it was a question of studying the case where the vertical derivative of our initial data satisfies a non-linear condition of smallness.

4.1 The hyperbolic version of the Navier-Stokes equations

Before beginning to present the results of this thesis on the hyperbolic Navier-Stokes model, we begin by recalling some fundamental results on the existence, regularity and uniqueness known for this type of equations.

4.1.1 Reminder of the known results of existence and regularity of solutions

Brenier, Natalini and Puel in [16] considered for the first time the hyperbolic version of Navier-Stokes (3.4.5) on $\mathbb{R}^+ \times \mathbb{T}^2$, where \mathbb{T}^2 is the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. We recalled that the equation is given by

$$(4.1.1) \quad \begin{cases} \tau \partial_t^2 u^\tau + \partial_t u^\tau + u^\tau \cdot \nabla u^\tau - \nu \Delta u^\tau = -\nabla p^\tau, \\ \operatorname{div} u^\tau = 0, \\ u^\tau(0, x) = u_0^\tau(x), \quad \partial_t u^\tau(0, x) = u_1^\tau(x). \end{cases}$$

This equation can be rewritten in the form of:

$$(4.1.2) \quad \begin{cases} \partial_t u^\tau + \operatorname{div}(V^\tau) = -\nabla p^\tau, \\ \sqrt{\tau} \partial_t V^\tau + \frac{\nu}{\sqrt{\tau}} \nabla u^\tau = -\frac{1}{\sqrt{\tau}} (V^\tau - u^\tau \otimes u^\tau), \\ \operatorname{div} u^\tau = 0, \\ u^\tau(0, x) = u_0^\tau(x), \quad V^\tau(0, x) = V_0^\tau(x). \end{cases}$$

They demonstrated that if the initial data (u_0^τ, V_0^τ) are small in $H^2(\mathbb{T}^2)^2 \times H^1(\mathbb{T}^2)^4$, then they manage to obtain a global result of existence of the solutions, as well as the convergence of these solutions to the Navier-Stokes solutions with a regular initial data. In their result, the authors show that this convergence is not only formal. The result obtained in [16] is as follows

Theorem 4.1.1. *Let $T \geq 0$ and u_0 be a smooth divergence free vector field on \mathbb{T}^2 . Let also $\tau > 0$ and (u_0^τ, V_0^τ) be a sequence of smooth initial data on \mathbb{T}^2 for problem (4.1.2). Assume, moreover, that there exists a constant C independent of τ such that*

$$\|u_0^\tau\|_{H^1(\mathbb{T}^2)} \leq C, \quad \|V_0^\tau\|_{H^2(\mathbb{T}^2)} \leq \frac{C}{\sqrt{\tau}},$$

and

$$\|u_0^\tau\|_{H^2(\mathbb{T}^2)} \leq \frac{C_0}{K_s \sqrt{\tau}}, \quad \|u_0^\tau - u_0\|_{L^2(\mathbb{T}^2)}^2 \leq C\sqrt{\tau},$$

where $0 < C_0 < 1$ and K_s is the constant in Sobolev's inequality on $H^2(\mathbb{T})$. if u is the Navier–Stokes solution with u_0 initial data, then there exists a constant C_T such that

$$\sup_{t \in [0, T]} \|u^\tau - u\|_{L^2(\mathbb{T}^2)}^2 \leq C_T \sqrt{\tau}.$$

We recall that the Sobolev space has for definition in \mathbb{R}^n

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \widehat{f} \in L^1_{loc}(\mathbb{R}^n), \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < +\infty \right\},$$

where \mathcal{S}' is the space of tempered distributions in \mathbb{R}^n . The standard norm in H^s is defined by:

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

To prove their result, they used the hyperbolic energy method and Dafermos modulated energy for convergence. The energy obtained by multiplying the equation (4.1.1) by $u^\tau + 2\tau \partial_t u^\tau$ and then integrating into space is

$$E_\tau(t) = \int_{\mathbb{T}^2} \left(\frac{1}{2} |u^\tau(t) + \tau \partial_t u^\tau(t)|^2 + \frac{\tau^2}{2} |\partial_t u^\tau(t)|^2 + \tau |\nabla u^\tau(t)|^2 \right) dx.$$

In order, for them to have information about convergence, they modulated this energy by a solution of the Navier-Stokes equation u , obtaining the modulated Dafermos energy,

$$E_{\tau, u}(t) = \int_{\mathbb{T}^2} \left(\frac{1}{2} |u^\tau(t) - u(t) + \tau \partial_t u^\tau(t)|^2 + \frac{\tau^2}{2} |\partial_t u^\tau(t)|^2 + \tau |\nabla u^\tau(t)|^2 \right) dx.$$

This energy can control the norm $\|u^\tau(t) - u(t)\|_{L^2}$.

The method that the authors use to prove their result remains restricted to the two-dimensional framework and requires a lot of regularity on the initial data.

An improvement of this result was made by Paicu and Raugel in [108, 109], who approximated the Navier–Stokes equations with external force f which can depend on time by a hyperbolic perturbation

$$\partial_t u + u \cdot \nabla u - \Delta u = -\nabla p + f.$$

Using the same energy method introduced in [16] and a Strichartz inequality on high frequencies, the authors obtain a global existence and uniqueness results for this type of equation with initial data in $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$. For the convergence, the authors obtain a results that improve those of Brenier, Natalini and Puel.

Specifically, they achieve the following results. To simplify, the external force f is considered to be zero.

Theorem 4.1.2. *Let $\tau_0 > 0$, there exist a strictly positive constant K such that, if $0 < \tau < \tau_0$ and if the free-divergence vectors $u_0^\tau \in H^1(\mathbb{R}^2)^2$, $u_1^\tau \in L^2(\mathbb{R}^2)^2$ satisfy*

$$\|u_0^\tau\|_{L^2} + \|u_1^\tau\|_{L^2} + \|\nabla u_0^\tau\|_{L^2} \leq K,$$

then the equation (4.1.1) has a unique global integral solution

$$(u^\tau, \partial_t u^\tau) \in C^0(\mathbb{R}^+, H^1(\mathbb{R}^2)^2) \times C^0(\mathbb{R}^+, L^2(\mathbb{R}^2)^2).$$

When τ converges to 0, the formal limit of the (4.1.1) system is given by the Navier–Stokes system. In the following theorem, Paicu and Raugel show that this limit is not only formal.

Theorem 4.1.3. *For all strictly positive numbers R and T , there exist a strictly positive number $\tau_1 = \tau_1(R, T)$ such that, for $0 < \tau < \tau_1$, if the couple of free-divergence vectors $(u^\tau, \partial_t u^\tau) \in H^1(\mathbb{R}^2)^2$, $u_0^\tau \in L^2(\mathbb{R}^2)^2$ verify*

$$\|u_0^\tau\|_{L^2} + \sqrt{\tau}\|u_1^\tau\|_{L^2} \leq R,$$

then the system (4.1.1) has a unique integral solution

$$(u^\tau, \partial_t u^\tau) \in C^0(\mathbb{R}^+, H^1(\mathbb{R}^2)^2) \times C^0(\mathbb{R}^+, L^2(\mathbb{R}^2)^2).$$

For all $0 \leq t \leq T$,

$$\sqrt{\tau}\|\partial_t(t(u^\tau(t) - u(t)))\|_{L^2} + \|t(u^\tau(t) - u(t))\|_{H^1} \leq \tau e^K,$$

where K is a constant that depends only on R and T (see theorem 3 [108]).

In the context of two and three dimensions, we can also mention the work of Hachicha in [68], which obtained a result of global existence and uniqueness of solutions for the (4.1.1) system with well-chosen initial data. The result in dimension 2 is as follows

Theorem 4.1.4. *Let $0 < s, \delta < 1$ and $u_0 \in H^s(\mathbb{R}^2)^2$ be a divergence-free vector field. Let $(u_0^\tau, u_1^\tau)_{/\tau>0} \in H^{1+\delta}(\mathbb{R}^2)^2 \times H^\delta(\mathbb{R}^2)^2$ be a sequence of initial data for problem (4.1.1). Assume that*

$$\left\{ \begin{array}{l} \|u_0^\tau - u_0\|_{L^2} + \tau\|u_1^\tau\|_{L^2} + \tau^{\frac{1}{2}}\|u_0^\tau\|_{\dot{H}^1} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{\frac{1+\delta}{2}}\|u_0^\tau\|_{\dot{H}^{1+\delta}} + \tau^{\frac{\delta}{2}}\|u_0^\tau\|_{\dot{H}^\delta} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{1+\frac{\delta}{2}}\|u_1^\tau\|_{\dot{H}^\delta} = \mathcal{O}(1). \end{array} \right.$$

Then, for τ small enough, there exists a global solution u^τ to the system (4.1.1) that converges when τ goes to 0 in $L_{loc}^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2)^2)$ norm, towards the unique solution u to the incompress-

ible Navier-Stokes equations, with u_0 as initial data. Moreover, there exists a constant C_T , depending only on T and u , such that

$$\sup_{t \in [0; T]} \int_{\mathbb{R}^2} |u^\tau - u|^2 dx \leq C_T \tau^{(\frac{s}{2})^-}.$$

In dimension 3, the author imposed an additional condition of smallness on the initial data u_0^τ . It obtains a result similar to that of dimension 2.

Theorem 4.1.5. *Let $s > 0$, $\delta < 1$ and $u_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ be a divergence-free vector field. Let $(u_0^\tau, u_1^\tau)|_{\tau>0} \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$ be a sequence of initial data for problem (4.1.1) such that $\|u_0^\tau\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{16}$. Assume that*

$$\begin{cases} \|u_0^\tau - u_0\|_{\dot{H}^{\frac{1}{2}}} + \tau \|u_1^\tau\|_{\dot{H}^{\frac{1}{2}}} + \tau^{\frac{1}{2}} \|u_0^\tau\|_{\dot{H}^{\frac{3}{2}}} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{\frac{1+\delta}{2}} \|u_0^\tau\|_{\dot{H}^{\frac{3}{2}+\delta}} + \tau^{\frac{\delta}{2}} \|u_0^\tau\|_{\dot{H}^{\frac{1}{2}+\delta}} = \mathcal{O}(\tau^{\frac{s}{2}}), \\ \tau^{1+\frac{\delta}{2}} \|u_1^\tau\|_{\dot{H}^{\frac{1}{2}+\delta}} = \mathcal{O}(1). \end{cases}$$

Then, for τ small enough, there exists a global solution u^τ to the system (4.1.1) that converges, when τ goes to 0, in the norm $L_{loc}^\infty(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3)$, towards the unique solution u to the incompressible Navier-Stokes equations, with u_0 as initial data. Moreover, there exists a constant C_T avec depending only on T and u , such that

$$\sup_{t \in [0; T]} \int_{\mathbb{R}^3} |\Lambda^{\frac{1}{2}}(u^\tau - u)|^2 dx \leq C_T \tau^{(\frac{s}{2})^-}.$$

The main idea of the proof is to use the hyperbolic energy method inspired by the paper of Brenier, Natalini and Puel in [16]. Recall that the homogeneous Sobolev space is defined for all $s \in \mathbb{R}$ by

$$\dot{H}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \widehat{f} \in L_{loc}^1(\mathbb{R}^n), \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi < +\infty \right\},$$

where \mathcal{S}' is the space of tempered distribution in \mathbb{R}^n . The semi-norm is defined in H^s by

$$\|f\|_{\dot{H}^s} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

4.1.2 Contribution de la thèse

In chapter 3 of this thesis, for $\epsilon > 0$ and $\tau > 0$ we consider the following system:

$$(HNS) \quad \begin{cases} \tau \partial_t^2 U^\tau + \partial_t U^\tau + U^\tau \cdot \nabla U^\tau - \epsilon^2 \Delta U^\tau + \nabla P^\tau = 0, \\ \operatorname{div} U^\tau = 0, \\ U_{/t=0}^\tau = U_0^\tau, \quad \partial_t U_{/t=0}^\tau = U_1^\tau, \end{cases}$$

All the results mentioned above are obtained when their domain Ω is \mathbb{R}^n ($n = 2, 3$) or \mathbb{T}^n . In our case, we want to study the system (HNS) when the fluid evolves in a thin domain with

Dirichlet conditions at the boundary. Note $\mathcal{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\}$, such that ϵ is the width of the domain. The advantage of working in a thin field supplemented by the Dirichlet conditions at the edge is to make appear in our study the Prandtl type equations in their hyperbolic version, i.e. we will have a term of type ∂_t^2 in the classical Prandtl equation. The main idea of the result is to obtain the existence and uniqueness of a global solutions for the (HNS) system in a thin domain. To simplify our system, we eliminate the dependency τ , for this we perform the rescaling

$$(4.1.3) \quad U^\tau(t, X) = \tau^{-\frac{1}{2}}U(\tau^{-1}t, \tau^{-\frac{1}{2}}X), \quad P^\tau(t, X) = \tau^{-1}P(\tau^{-1}t, \tau^{-\frac{1}{2}}X).$$

We replace into the system (HNS), we can obtain

$$(4.1.4) \quad \begin{cases} \partial_t^2 U + \partial_t U + U \cdot \nabla U - \epsilon^2 \Delta U + \nabla P = 0, \\ \operatorname{div} U = 0, \\ U|_{t=0} = \sqrt{\tau}U_0^\tau(\sqrt{\tau}X) = U_0, \\ \partial_t U|_{t=0} = \tau^{\frac{3}{2}}U_1^\tau(\sqrt{\tau}X) = U_1 \end{cases}$$

this system is completed by the following Dirichlet boundary condition:

$$U|_{y=0} = 0 \quad \text{et} \quad U|_{y=\epsilon} = 0$$

and by the following initial data:

$$U|_{t=0} = U_0^\epsilon = \left(u_0 \left(x, \frac{y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{y}{\epsilon} \right) \right).$$

We look for solutions to our system of form

$$(4.1.5) \quad \begin{cases} U(t, x, y) = \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ P(t, x, y) = p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

Let $\mathcal{S} := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$. After a natural scaling change, the system (4.1.4) becomes the scaled anisotropic hyperbolic Navier-Stokes system:

$$(4.1.6) \quad \begin{cases} \partial_t^2 u^\epsilon + \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon + \partial_x p^\epsilon = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ \epsilon^2 (\partial_t^2 v^\epsilon + \partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) + \partial_y p^\epsilon = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ (u^\epsilon, v^\epsilon)|_{t=0} = (u_0, v_0) \quad \text{and} \quad \partial_t (u^\epsilon, v^\epsilon)|_{t=0} = (u_1, v_1), & \text{in } \mathcal{S} \\ (u^\epsilon, v^\epsilon)|_{y=0} = (u^\epsilon, v^\epsilon)|_{y=1} = 0. & \end{cases}$$

Taking $\epsilon \rightarrow 0$ in the system (4.1.6), we obtain the hyperbolic Prandtl equation:

$$(4.1.7) \quad \begin{cases} \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_y p = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x u + \partial_y v = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ u|_{t=0} = u_0, & \text{in } \mathcal{S} \\ \partial_t u|_{t=0} = u_1, & \text{in } \mathcal{S}, \end{cases}$$

where the velocity $U = (u, v)$ satisfies the Dirichlet non-slip boundary condition

$$(u, v)|_{y=0} = (u, v)|_{y=1} = 0.$$

The purpose is to justify the transition to the boundary of the system (4.1.6) to the system (4.1.7), when considering analytical initial data. Working with analytical data overcomes the difficulty of the nonlinear term $v \partial_y u$ in (4.1.7), in the general case where structural assumptions are not imposed on the initial data.

Our first result is to prove that the Prandtl system (4.1.7) with analytical and small initial data is globally well posed.

Theorem 4.1.6. *Let $a > 0$. There exists a constant $c_0 > 0$ sufficiently small, such that, for any data (u_0, u_1) verifying the compatibility condition $\int_0^1 u_0 dy = 0$, the smallness condition*

$$(4.1.8) \quad \left\| e^{a|D_x|}(u_0 + u_1) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} \partial_y u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \| e^{a|D_x|} u_1 \|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a,$$

then the system (4.1.7) has a unique global solution u satisfying the estimate

$$(4.1.9) \quad \begin{aligned} & \frac{1}{2} \| e^{\mathcal{R}t} (u + \partial_t u)_\phi \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \| e^{\mathcal{R}t} \partial_y u_\phi \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \| e^{\mathcal{R}t} (\partial_t u)_\phi \|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & + \frac{1}{2} \| e^{\mathcal{R}t} (\partial_t u)_\phi \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \leq C \| e^{a|D_x|} \partial_y u_0 \|_{\mathcal{B}^s} + C \| e^{a|D_x|} (u_0 + u_1) \|_{\mathcal{B}^s} + C \| e^{a|D_x|} u_1 \|_{\mathcal{B}^s}, \end{aligned}$$

where $C = C(s)$ and u_ϕ is defined by

$$(4.1.10) \quad \begin{cases} u_\phi(t, x, y) = e^{\phi(t, D_x)} u(t, x, y) \triangleq \mathcal{F}_h^{-1}(e^{\phi(t, \xi)} \hat{u}(t, \xi, y)), \\ \phi(t, \xi) = (a - \lambda \theta(t)) |\xi|, \end{cases}$$

The quantity $\theta(t)$ describes the evolution of the analyticity band over time, this quantity is positive for all $t \in \mathbb{R}_+^*$ satisfies Equation $\dot{\theta}(t) = \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$ and when $t = 0$ we have $\theta(0) = 0$. \mathcal{R} is a constant determined by the Poincaré inequality for the band \mathcal{S} . The functional space used to achieve this result is the Besov space of the Chemin-Lerner type.

$$\|u\|_{\tilde{L}_T^p(\mathcal{B}^s(\mathcal{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \left\| \Delta_q^h u(t) \right\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

Remark 4.1.1. *The main idea to prove the above two theorems is to control the new unknown u_ϕ defined by (4.1.10), where u is the horizontal velocity and u_ϕ is a weighted function of u in*

the dual Fourier variable with an exponential function of $(a - \lambda\theta(t))|\xi|$.

The second result is the global well-posedness of the perturbed Navier-Stokes system (4.1.6) with small analytical data in the tangential variable.

Theorem 4.1.7. *Let $a > 0$. We assume that our initial satisfies the following smallness condition*

$$(4.1.11) \quad \begin{aligned} & \|e^{a|D_x|}\partial_y(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|e^{a|D_x|}\partial_x(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} \\ & + \|e^{a|D_x|}(u_0 + u_1, \epsilon(v_0 + v_1))\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(u_1, \epsilon v_1)\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a, \end{aligned}$$

for some c_1 sufficiently small. Then System (4.1.6) has a unique global solution (u, v) , so that

$$(4.1.12) \quad \begin{aligned} & \frac{1}{2} \|e^{\mathcal{R}t}(u + \partial_t u, \epsilon(v + \partial_t v))_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \epsilon \|e^{\mathcal{R}t}\partial_x(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\ & + \frac{1}{2} \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & \leq C \left(\|e^{a|D_x|}\partial_y(u_0, \epsilon v_0)\|_{\mathcal{B}^s} + \epsilon \|e^{a|D_x|}\partial_x(u_0, \epsilon v_0)\|_{\mathcal{B}^s} \right. \\ & \quad \left. + \|e^{a|D_x|}(u_1, \epsilon v_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}(u_0 + u_1, \epsilon(v_0 + v_1))\|_{\mathcal{B}^s} \right), \end{aligned}$$

where $C = C(s)$ and (u_Θ, v_Θ) is given by (4.1.10) with $\phi = \Theta$.

The third result concern the study of the convergence from the scaled anisotropic perturbed Navier-Stokes system (4.1.6) to the limit system (4.1.7), so in this theorem, we proved that the convergence across globally in time.

Theorem 4.1.8. *Let $a > 0$, and $(u_0^\epsilon, v_0^\epsilon)$ satisfying (4.1.11). Let (u_0, u_1) satisfy $e^{a|D_x|}(u_0, u_1) \in (\mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{7}{2}})^2$, $e^{a|D_x|}\partial_y(u_0, u_1) \in (\mathcal{B}^{\frac{3}{2}})^2$, and the compatibility condition $\int_0^1 u_0 dy = 0$ and*

$$(4.1.13) \quad \begin{aligned} & \left\| e^{a|D_x|}(u_0 + u_1) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|}\partial_y u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{1}{2}}} \\ & \leq \frac{c_2 a}{2 + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{3}{2}}}}, \end{aligned}$$

for some c_2 sufficiently small, then we have

$$(4.1.14) \quad \begin{aligned} & \frac{1}{2} \|(R^1 + \partial_t R^1, \epsilon(R^2 + \partial_t R^2))_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \\ & + \epsilon \|\partial_x(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \frac{1}{2} \|(\partial_t R^1)_\varphi, \epsilon(\partial_t R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \\ & + \|(\partial_t R^1, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \leq \|e^{a|D_x|}((u_1^\epsilon - u_1), \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}} \\ & + C \left(\|e^{a|D_x|}\partial_y(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|e^{a|D_x|}\partial_x(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} \right. \\ & \quad \left. + \|e^{a|D_x|}((u_0^\epsilon - u_0) + (u_1^\epsilon - u_1), \epsilon(v_0^\epsilon - v_0) + \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}} + M\epsilon \right). \end{aligned}$$

where $C = C(s)$ and

$$(4.1.15) \quad \begin{cases} R^1 = u^\epsilon - u, \\ R^2 = v^\epsilon - v, \end{cases}$$

and v_0 is determined from u_0 via $\partial_x u + \partial_y v = 0$ and $v_0|_{y=0} = v_0|_{y=1} = 0$, $(R_\varphi^1, \epsilon R_\varphi^2)$ is given by (4.1.10).

Remark 4.1.2. *The main idea of proving the above theorem is to use analytical energy estimates, which are motivated by [112] and which originates from [27].*

4.2 Hydrostatic approximation of the magneto-hydrodynamics equations

The purpose of this section is to present the results obtained during this thesis on a hydrostatic approximation of the magneto-hydrodynamics equations in a thin domain completed with Dirichlet boundary conditions. Before presenting the results, we begin by recalling the fundamental results on the existence as well as the regularity and uniqueness of the solutions of these equations.

4.2.1 Known results

We start with the origin of these approximations, which were obtained after a systematic derivation of the boundary layer models in magneto-hydrodynamics by D. Gérard-Varet and M. Prestipino in [63]. Indeed, they considered the magneto-hydrodynamic system in a Ω subspace of \mathbb{R}^3 given by

$$(4.2.1) \quad \begin{cases} \partial_t U + U \nabla U - \frac{1}{Re} \Delta U = -\nabla P + SB \cdot \nabla B, \\ \partial_t B - \text{curl}(U \times B) - \frac{1}{Rm} \Delta B = 0, \\ \text{div } U = \text{div } B = 0. \end{cases}$$

The parameters Re and Rm represent the hydrodynamic and magnetic Reynolds numbers, respectively. It is assumed that these parameters are very large ($Re \gg 1$, and $Rm \sim Re$). This choice of parameter regime is necessary for the creation of a boundary layer. The parameter S is a coupling parameter given by

$$S = \frac{B_0^2}{\mu \rho V^2} = \frac{Ha}{Re Rm}, \quad \text{where } Ha = B_0 L \left(\frac{\sigma}{\eta} \right)^{\frac{1}{2}},$$

where the parameter Ha is the Hartmann number, B_0 and V are the typical amplitudes for magnetic and velocity fields, L is a typical length scale of the flow, ρ is the density of the fluid, μ is its magnetic permeability, and η is the viscosity coefficient.

In our study, we will focus on fluids that evolve in thin two-dimensional domains for example $\Omega_1 = \mathbb{R} \times (0,1)$. Consider the solutions of the system (4.2.1)

$$(4.2.2) \quad \begin{cases} U(t, x, y) \sim \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ B(t, x, y) \sim \left(b^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon c^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ P(t, x, y) \sim p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

The parameter ϵ denotes the size of the boundary layer, it is a very small parameter ($\epsilon \ll 1$).

Remark 4.2.1. *Note that the free-divergence condition on U and B , always remains valid by the scaling of the vertical components by the factor ϵ . Indeed*

$$\begin{aligned} \operatorname{div} U &= \partial_x u^\epsilon + \epsilon \frac{1}{\epsilon} \partial_y v^\epsilon = \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \\ \operatorname{div} B &= \partial_x b^\epsilon + \epsilon \frac{1}{\epsilon} \partial_y c^\epsilon = \partial_x b^\epsilon + \partial_y c^\epsilon = 0. \end{aligned}$$

If we replace U , B and P by the solutions proposed in (4.2.2) in the (4.2.1) system, we obtain

$$\begin{aligned} & \partial_t U + U \cdot \nabla U - \frac{1}{Re} \Delta U + \nabla P - SB \cdot \nabla B \\ &= \left(\partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y u^\epsilon - \frac{1}{Re} \partial_x^2 u^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 u^\epsilon + \partial_x p^\epsilon - Sb^\epsilon \partial_x b^\epsilon - S\epsilon c^\epsilon \frac{1}{\epsilon} \partial_y b^\epsilon, \right. \\ & \left. \partial_t \epsilon v^\epsilon + u^\epsilon \partial_x \epsilon v^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y \epsilon v^\epsilon - \frac{1}{Re} \partial_x^2 \epsilon v^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 \epsilon v^\epsilon + \frac{1}{\epsilon} \partial_y p^\epsilon - Sb^\epsilon \partial_x \epsilon c^\epsilon - S\epsilon c^\epsilon \frac{1}{\epsilon} \epsilon \partial_y c^\epsilon \right) \\ &= (0, 0), \end{aligned}$$

and

$$\begin{aligned} & \partial_t B + U \cdot \nabla B - B \cdot \nabla U - \frac{1}{Rm} \Delta B \\ &= \left(\partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y b^\epsilon - \frac{1}{Re} \partial_x^2 b^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 b^\epsilon - b^\epsilon \partial_x u^\epsilon - \epsilon c^\epsilon \frac{1}{\epsilon} \partial_y u^\epsilon, \right. \\ & \left. \partial_t \epsilon c^\epsilon + u^\epsilon \partial_x \epsilon c^\epsilon + \epsilon v^\epsilon \frac{1}{\epsilon} \partial_y \epsilon c^\epsilon - \frac{1}{Re} \partial_x^2 \epsilon c^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 \epsilon c^\epsilon - b^\epsilon \partial_x \epsilon v^\epsilon - \epsilon c^\epsilon \frac{1}{\epsilon} \epsilon \partial_y v^\epsilon \right) \\ &= (0, 0). \end{aligned}$$

So, the new system is of the form

$$(4.2.3) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \frac{1}{Re} \partial_x^2 u^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 u^\epsilon = -\partial_x p^\epsilon + S b^\epsilon \partial_x b^\epsilon + S c^\epsilon \partial_y b^\epsilon, \\ \partial_t v^\epsilon + \epsilon u^\epsilon \partial_x v^\epsilon + \epsilon v^\epsilon \partial_y v^\epsilon - \frac{\epsilon}{Re} \partial_x^2 v^\epsilon - \frac{1}{Re} \frac{1}{\epsilon} \partial_y^2 v^\epsilon = -\frac{1}{\epsilon} \partial_y p^\epsilon + \epsilon S b^\epsilon \partial_x c^\epsilon + S \epsilon c^\epsilon \partial_y c^\epsilon, \\ \partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + v^\epsilon \partial_y b^\epsilon - \frac{1}{Re} \partial_x^2 b^\epsilon - \frac{1}{Re} \frac{1}{\epsilon^2} \partial_y^2 b^\epsilon = b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y u^\epsilon, \\ \epsilon \partial_t c^\epsilon + \epsilon u^\epsilon \partial_x c^\epsilon + \epsilon v^\epsilon \partial_y c^\epsilon - \frac{1}{Re} \epsilon \partial_x^2 c^\epsilon - \frac{1}{Re} \frac{1}{\epsilon} \partial_y^2 c^\epsilon = \epsilon b^\epsilon \partial_x v^\epsilon + \epsilon c^\epsilon \partial_y v^\epsilon, \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \text{ et } \partial_x b^\epsilon + \partial_y c^\epsilon = 0. \end{cases}$$

This system is supplemented by the Dirichlet conditions at the boundary:

$$(u^\epsilon, v^\epsilon)|_{y=0,1} = (b^\epsilon, c^\epsilon)|_{y=0,1} = 0.$$

In the following, it is assumed that the constant $S = 1$. We define the size of our layer by $\epsilon = \frac{1}{\sqrt{Re}} = \frac{1}{\sqrt{Rm}}$, this means that our hydrodynamic and magnetic Reynolds numbers are identical. If we replace in the system (4.2.3), we obtain

$$(4.2.4) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon = -\partial_x p^\epsilon + b^\epsilon \partial_x b^\epsilon + c^\epsilon \partial_y b^\epsilon, \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) = -\partial_y p^\epsilon + \epsilon^2 (b^\epsilon \partial_x c^\epsilon + c^\epsilon \partial_y c^\epsilon), \\ \partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + v^\epsilon \partial_y b^\epsilon - \epsilon \partial_x^2 b^\epsilon - \partial_y^2 b^\epsilon = b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y u^\epsilon, \\ \epsilon (\partial_t c^\epsilon + u^\epsilon \partial_x c^\epsilon + v^\epsilon \partial_y c^\epsilon - \epsilon^2 \partial_x^2 c^\epsilon - \partial_y^2 c^\epsilon) = \epsilon (b^\epsilon \partial_x v^\epsilon + c^\epsilon \partial_y v^\epsilon), \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \text{ et } \partial_x b^\epsilon + \partial_y c^\epsilon = 0. \end{cases}$$

Taking the formal limit from ϵ to 0 in the (4.2.4) system, we obtain the following Prandtl system:

$$(4.2.5) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u = -\partial_x p + b \partial_x b + c \partial_y b, \\ \partial_y p = 0, \\ \partial_t b + u \partial_x b + v \partial_y b - \partial_y^2 b = b \partial_x u + c \partial_y u, \\ \partial_x u + \partial_y v = 0, \text{ and } \partial_x b + \partial_y c = 0. \end{cases}$$

The (4.2.5) system has been the subject of several recent studies, the existence and regularity of global and local solutions are difficult questions that researchers have tried to prove for this system. However, Liu, Xie and Yang in [95] managed to prove that this system is indeed posed locally in Sobolev space without asking for any monotonicity condition on tangential velocity. The only essential hypothesis in their work is that the horizontal component of the magnetic field admits a positive lower bound. This result gives a good understanding of the general physics according to which the magnetic field stabilizes the boundary layer.

On the other hand, when the initial data are analytical, N. Liu and P. Zhang in [99] proved a result of existence and global uniqueness of solutions for the magnetohydrodynamic system in two dimension in the upper space $\mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}^+\}$. Before stating their result

we recall that their system is of the form:

$$(4.2.6) \quad \begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - \partial_y^2 u_1 = -\partial_x p + b_1 \partial_x b_1 + b_2 \partial_y b_1, \\ \partial_t b_1 + u_1 \partial_x b_1 + u_2 \partial_y b_1 - \kappa \partial_y^2 b_1 = b_1 \partial_x u_1 + b_2 \partial_y u_1, \\ \partial_x u_1 + \partial_y u_2 = 0, \text{ and } \partial_x b_1 + \partial_y b_2 = 0, \\ (u_1, u_2)_{/y=0} = 0 \text{ and } (\partial_y b_1, b_2)_{/y=0} = 0, \\ \lim_{y \rightarrow +\infty} u_1 = U_1 \text{ and } \lim_{y \rightarrow +\infty} b_1 = B_1, \\ u_1/t=0 = u_{1,0}, \quad b_1/t=0 = b_{1,0}, \end{cases}$$

where (u_1, u_2) and (b_1, b_2) represent the fluid velocity and the magnetic field, respectively, the parameter κ is a positive constant representing the ratio between the hydrodynamic Reynolds number and the magnetic Reynolds number, $(U_1, B_1, p)(t, x)$ are the traces of the tangential fields and the pressure of the flow on the boundary and satisfy Bernoulli's law:

$$(4.2.7) \quad \begin{cases} \partial_t U_1 + U_1 \partial_x U_1 + \partial_x p = B_1 \partial_x B_1, \\ \partial_t B_1 + U_1 \partial_x B_1 = B_1 \partial_x U_1. \end{cases}$$

The aim of their work is to obtain a result of the global existence of solutions for this problem. To do this, they defined for any constant \bar{B}_κ , a plateau function $\chi \in C^\infty[0, \infty)$ defined by

$$\chi(y) = \begin{cases} y & \text{si } y \geq 2, \\ 0 & \text{si } y \leq 1, \end{cases} \text{ and made the following variable change :}$$

$$(4.2.8) \quad \begin{aligned} u &= u_1 - \chi'(y)U \quad \text{and} \quad v = u_2 + \chi(y)\partial_x U, \\ b &= b_1 - \chi'(y)B - \bar{B}_\kappa \quad \text{and} \quad c = b_2 + \chi(y)\partial_x B, \end{aligned}$$

where $U = U_1$ et $B = B_1 - \bar{B}_\kappa$. Since (u_1, u_2, b_1, b_2) solves the (4.2.6) system, then (u, v, b, c) solves

$$(4.2.9) \quad \begin{cases} \partial_t u - \partial_y^2 u - \bar{B}_\kappa \partial_x b + u \partial_x u - b \partial_x b + v \partial_y u - c \partial_y b + \chi'(U \partial_x u - B \partial_x b) \\ \quad + \chi'(\partial_x U u - \partial_x B b) + \chi(-\partial_x U \partial_y u + \partial_x B \partial_y b) + \chi''(U v - B c) = M_U, \\ \partial_t b - \kappa \partial_y^2 b - \bar{B}_\kappa \partial_x u + u \partial_x b - b \partial_x u + v \partial_y b - c \partial_y u + \chi'(U \partial_x b - B \partial_x u) \\ \quad + \chi'(\partial_x B u - \partial_x U b) + \chi(-\partial_x U \partial_y b + \partial_x B \partial_y u) + \chi''(B v - U c) = M_B, \\ \partial_x u + \partial_y v = 0, \text{ and } \partial_x b + \partial_y c = 0 \\ (u, v)_{/y=0} = 0 \text{ and } (\partial_y b, c)_{/y=0} = 0, \\ \lim_{y \rightarrow +\infty} (u, v) = 0 \text{ and } \lim_{y \rightarrow +\infty} (b, c) = 0, \\ u/t=0 = u_0 = u_{1,0} - \chi'U_0, \quad b/t=0 = b_0 = b_{1,0} - \chi'B_0 - \bar{B}_\kappa, \end{cases}$$

The terms M_U and M_B are supported for any $y \in [0, 2]$ and $t > 0$. These terms are defined by

$$(4.2.10) \quad \begin{aligned} M_U &= (1 - \chi')(\partial_t U - \bar{B}_\kappa \partial_x B) + \chi'''U + (1 - (\chi')^2 + \chi\chi'')(U \partial_x U - B \partial_x B), \\ M_B &= (1 - \chi')(\partial_t B - \bar{B}_\kappa \partial_x U) + \chi'''B + (1 - (\chi')^2 - \chi\chi'')(U \partial_x B - B \partial_x U). \end{aligned}$$

On the other hand, since $\partial_x u + \partial_y v = 0$ and $\partial_x b + \partial_y c = 0$, then there are two potential

functions (φ, ψ) such that

$$(u, b) = \partial_y(\varphi, \psi) \quad \text{and} \quad (v, c) = -\partial_x(\varphi, \psi).$$

These primitive functions can have the following boundary conditions

$$(\varphi, \psi)|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} (\varphi, \psi) = 0.$$

By integrating the equations (4.2.9) satisfied by (u, b) with respect to the vertical variable y on the interval $[y, \infty[$, we obtain

$$(4.2.11) \quad \left\{ \begin{array}{l} \partial_t \varphi - \partial_y^2 \varphi - \bar{B}_\kappa \partial_x \psi + u \partial_x \varphi - b \partial_x \psi + 2 \int_y^\infty (\partial_x \varphi \partial_y u - \partial_x \psi \partial_y b) dy' \\ \quad + \chi'(U \partial_x \varphi - B \partial_x \psi) + 2 \int_y^\infty \chi''(U \partial_x \varphi - B \partial_x \psi) dy' + \chi(-\partial_x U u + \partial_x B b) \\ \quad + 2 \chi'(\partial_x U \varphi - \partial_x B \psi) + 2 \int_y^\infty \chi''(\partial_x U \varphi - \partial_x B \psi) = m_U, \\ \partial_t \psi - \kappa \partial_y^2 \psi - \bar{B}_\kappa \partial_x \varphi + u \partial_x \psi - b \partial_x \varphi + \chi'(U \partial_x \psi - B \partial_x \varphi) \\ \quad + \chi(-\partial_x U b + \partial_x B u) = m_B, \\ \varphi|_{y=0} = \psi|_{y=0} = 0, \quad \text{and} \quad \lim_{y \rightarrow +\infty} \varphi = \lim_{y \rightarrow +\infty} \psi = 0, \\ \varphi|_{t=0} = \varphi_0 = - \int_y^\infty u_0 dy', \quad \psi|_{t=0} = \psi_0 = - \int_y^\infty b_0 dy', \end{array} \right.$$

where $(m_U, m_B) = - \int_y^\infty (M_U, M_B) dy'$ also supported in $y \in [0, 2]$ for any $t > 0$. The result achieved in their work is as follows:

Theorem 4.2.1 (N. Liu et P. Zhang). *Let $\kappa \in]0, 2[$, $\bar{B}_\kappa = \begin{cases} 1 & \text{if } \kappa = 1, \\ 0 & \text{else,} \end{cases}$ and $\epsilon, \delta > 0$.*

We assume that the far field states (U, B) satisfy

$$(4.2.12) \quad \|\langle t \rangle^{\frac{9}{4}} e^{\delta|D_x|}(U, B)\|_{\tilde{L}^\infty(\mathbb{R}^+; B_h^{\frac{3}{2}})} + \|\langle t \rangle^{\frac{7}{4}} e^{\delta|D_x|}(\partial_t U, \partial_t B, U, B)\|_{\tilde{L}^\infty(\mathbb{R}^+; B_h^{\frac{1}{2}})} \leq \epsilon,$$

$$(4.2.13) \quad \int_0^\infty \langle t \rangle^{\frac{5}{4}} \|e^{\delta|D_x|}(U, B)\|_{B_h^{\frac{1}{2}}} dt \leq \epsilon.$$

Let the initial data $(u_0, b_0, \varphi_0, \psi_0)$ satisfy the compatibility condition $\int_0^\infty u_0 dy = \int_0^\infty b_0 dy = 0$ and

$$(4.2.14) \quad \|e^{\frac{y^2}{8}} e^{\delta|D_x|}(u_0, b_0, \varphi_0, \psi_0)\|_{B^{\frac{1}{2}, 0}} < \infty \quad \text{and} \quad \|e^{\frac{y^2}{8}} e^{\delta|D_x|}(G_0, H_0)\|_{B^{\frac{1}{2}, 0}} \leq \sqrt{\epsilon},$$

where $G_0 = u_0 + \frac{y}{2\langle t \rangle} \varphi_0$ and $H_0 = b_0 + \frac{y}{2\kappa\langle t \rangle} \psi_0$. Then there exist positive constants λ, a and $\epsilon_0(\lambda, a, \kappa, \delta)$ so that for $\epsilon \leq \epsilon_0$ and $l_\kappa = \frac{\kappa(2-\kappa)}{4} \in]0, \frac{1}{4}]$, the system (4.2.9) has a unique global solution (u, b) which satisfies

$$\sup_{t \in [0, \infty[} \theta(t) \leq \frac{\delta}{2\lambda}.$$

The quantity $\theta(t)$ is a key quantity that allows us to describe the evolution of the analyticity band of (u, b) . In this case, this quantity is defined by

$$(4.2.15) \quad \begin{cases} \dot{\theta}(t) = \langle t \rangle^{\frac{1}{4}} \|e^{\Psi} \partial_y(G, H)_\phi(t)\|_{B^{\frac{1}{2},0}} + \frac{1}{\sqrt{\epsilon}} \langle t \rangle^{\frac{5}{4}} \|(U, B)_\phi(t)\|_{B_h^{\frac{1}{2}}} \\ \theta|_{t=0} = 0. \end{cases}$$

Here the phase function ϕ is defined by

$$\phi(t, \xi) = (\delta - \lambda\theta(t))|\xi|,$$

and the weighted function Ψ is determined by

$$\Psi(t, y) = \frac{y^2}{8\langle t \rangle} \quad \text{with} \quad \langle t \rangle = 1 + t.$$

Remark 4.2.2. *The main idea of proving the above theorem is to use analytic energy estimates, which are motivated by [112] and which originates from [27].*

This result is obtained in the Besov functional space of the Chemin-Lerner type.

$$(4.2.16) \quad \|u\|_{\tilde{L}^p(T_0, T; B^{s,0})} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_{T_0}^T \|\Delta_q^h u(t)\|_{L^2_+}^p dt \right)^{\frac{1}{p}},$$

where $L^2_+ = L^2(\mathbb{R}^2_+)$. In particular, when $T_0 = 0$, we simplify the notation of $\|u\|_{\tilde{L}^p(0, T; B^{s,0})}$ with $\|u\|_{\tilde{L}^p_T(B^{s,0})}$.

Remark 4.2.3. *The space $B_h^{\frac{3}{2}}$ denotes the Besov space $B_{2,1}^{\frac{3}{2}}(\mathbb{R}_h)$ and the space $B_h^{\frac{1}{2}}$ denotes the Besov space $B_{2,1}^{\frac{1}{2}}(\mathbb{R}_h)$.*

A second result in the direction of analytical data for the boundary layer magneto-hydrodynamic system is the result of S. Li and F. Xie in [88]. Their result concerns the study of the global well-posed character of the solutions of the magneto-hydrodynamic system (4.2.6) in the space of analytic functions. In this work, the authors are interested in the case where $B_1 = 1$, $\kappa = 1$ and $U_1(t, x) = f(t)$ in the (4.2.6) system. By Bernoulli's law, their pressure term satisfies $\partial_x p = -f'(t)$. Let $\chi \in C^\infty[0, \infty)$ a cut-off function defined by $\chi(y) = \begin{cases} 1 & \text{si } y \geq 2, \\ 0 & \text{si } y \leq 1, \end{cases}$, we note $U = u_1 - f(t)\chi(y)$. Then (U, u_2, b_1, b_2) solve the system:

$$(4.2.17) \quad \begin{cases} \partial_t U + (U + f(t)\chi(y))\partial_x U + u_2 \partial_y (U + f(t)\chi(y)) \\ \quad - \partial_y^2 U - b_1 \partial_x b_1 - b_2 \partial_y b_1 = m(t, y), \\ \partial_t b_1 + \partial_y (u_2 b_1 - (U + f(t)\chi(y))b_2) = \partial_y^2 b_1, \\ \partial_x U + \partial_y u_2 = 0, \text{ and } \partial_x b_1 + \partial_y b_2 = 0, \end{cases}$$

where $m(t, y) = (1 - \chi(y))f'(t) + f(t)\chi''(y)$.

If we place ourselves in the context of a shear flow we have that $(u^s(t, s), 0, 1, 0)$ is a trivial solution of the system (4.2.17) with $u^s(t, y)$ solving

$$(4.2.18) \quad \begin{cases} \partial_t u^s - \partial_y^2 u^s = m(t, y), \\ u^s(t, y)/_{y=0} = 0 \quad \text{et} \quad \lim_{y \rightarrow +\infty} u^s(t, y) = 0, \\ u^s(t, y)/_{t=0} = 0. \end{cases}$$

This allows us to see the solution of the system (4.2.17) as a perturbation (u, v, b, c) of $(u^s(t, y), 0, 1, 0)$. This solution is given by

$$\begin{aligned} U &= u^s + u \quad \text{et} \quad u_2 = v, \\ b_1 &= 1 + b \quad \text{et} \quad b_2 = c. \end{aligned}$$

As a result, the system (4.2.17) becomes

$$(4.2.19) \quad \begin{cases} \partial_t u + (u + u^s + f(t)\chi(y))\partial_x u + v\partial_y(u + u^s + f(t)\chi(y)) \\ \quad - \partial_y^2 u - (1 + b)\partial_x b - c\partial_y b = 0, \\ \partial_t b + (u + u^s + f(t)\chi(y))\partial_x b - c\partial_y(u + u^s + f(t)\chi(y)) \\ \quad - \partial_y^2 b - (1 + b)\partial_x u + v\partial_y b = 0, \\ \partial_x u + \partial_y v = 0, \quad \text{and} \quad \partial_x b + \partial_y c = 0, \\ (u, v, \partial_y b, c)/_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} (u, b) = 0, \\ u(0, x, y) = u_0 = u_{1,0} - f(0)\chi(y) \quad \text{et} \quad b(0, x, y) = b_0 = b_{1,0} - 1. \end{cases}$$

Similarly, as $\partial_x u + \partial_y v = 0$ and $\partial_x b + \partial_y c = 0$, there exist two potential functions (φ, ψ) such as $(u, b) = \partial_y(\varphi, \psi)$ and $(v, c) = -\partial_x(\varphi, \psi)$. Integrating the equations (4.2.19) satisfied by (u, b) with respect to the vertical variable y on the interval $[y, \infty[$, we obtain

$$(4.2.20) \quad \begin{cases} \partial_t \varphi + (u + u^s + f(t)\chi(y))\partial_x \varphi + 2 \int_y^\infty \partial_y(u + u^s + f(t)\chi(y)) \cdot \partial_x \varphi dy \\ \quad - (1 + b)\partial_x \psi - 2 \int_y^\infty \partial_y b \cdot \partial_x \psi dy - \partial_y^2 \varphi = 0, \\ \partial_t \psi + (u + u^s + f(t)\chi(y))\partial_x \psi - (1 + b)\partial_x \varphi - \partial_y^2 \psi = 0, \\ \varphi/_{y=0} = \psi/_{y=0} = 0, \quad \text{and} \quad \lim_{y \rightarrow +\infty} \varphi = \lim_{y \rightarrow +\infty} \psi = 0, \\ \varphi/_{t=0} = \varphi_0 = - \int_y^\infty u_0 dy', \quad \psi/_{t=0} = \psi_0 = - \int_y^\infty b_0 dy'. \end{cases}$$

The result obtained in their paper is as follows

Theorem 4.2.2 (S.Li et F.Xie). *Let $\delta > 0$, and $f \in H^1(\mathbb{R}_+)$ which satisfies:*

$$(4.2.21) \quad C_f = \int_0^\infty \langle t \rangle (|f(t)| + |f'(t)|) dt + \left(\int_0^\infty \langle t \rangle^3 (f^2(t) + (f'(t))^2) dt \right)^{\frac{1}{2}} < \infty,$$

with $\langle t \rangle = 1 + t$ and $u_0 = \partial_y \varphi_0$, $b_0 = \partial_y \psi_0$ satisfy the compatibility conditions $\int_0^\infty u_0 dy =$

$\int_0^\infty b_0 dy = 0$. Assume further that $G_0 = u_0 + \frac{y}{2(t)}\varphi_0$ and $H_0 = b_0 + \frac{y}{2(t)}\psi_0$ satisfy

$$(4.2.22) \quad \|e^{\frac{y^2}{8}} e^{2\delta|D_x|}(u_0, b_0)\|_{B^{\frac{1}{2},0}} < \epsilon \quad \text{et} \quad \|e^{\frac{y^2}{8}} e^{\delta|D_x|}(G_0, H_0)\|_{B^{\frac{1}{2},0}} \leq \infty,$$

for some sufficiently small ϵ . Then the system (4.2.19) admits a unique global-in-time solution (u, b) , which satisfies:

$$(4.2.23) \quad \|e^{\frac{y^2}{8(t)}} e^{\delta/2|D_x|}(u, b)\|_{L^\infty(\mathbb{R}_+; B^{\frac{1}{2},0})} + \|e^{\frac{y^2}{8(t)}} e^{\delta/2|D_x|}\partial_y(u, b)\|_{L^2(\mathbb{R}_+; B^{\frac{1}{2},0})} \leq C \|e^{\frac{y^2}{8}} e^{2\delta|D_x|}(u_0, b_0)\|_{B^{\frac{1}{2},0}},$$

where C is an independent constant of C_f .

Remark 4.2.4. The space used to obtain this result is the Besov space defined by

$$(4.2.24) \quad \|u\|_{\tilde{L}^p(T_0, T; B^{s,0})} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_{T_0}^T \|\Delta_q^h u(t)\|_{L^2_+}^p dt \right)^{\frac{1}{p}}.$$

Remark 4.2.5. The main idea of the proof of the above theorem is to use the analytical energy estimates, which are motivated by [112] and define the phase function

$$\phi(t, \xi) = (\delta - \lambda\theta(t))|\xi|.$$

The quantity $\theta(t)$ is a key quantity that allows us to describe the evolution of the analyticity band of (u, b) .

4.2.2 Outcome Statement of the Magnetohydrodynamic System

In chapter 4 of this thesis, for $\epsilon > 0$, we consider the following system

$$(MHD) \quad \begin{cases} \partial_t U + U \cdot \nabla U - \epsilon^2 \Delta U + \nabla P = B \cdot \nabla B, \\ \partial_t B + U \cdot \nabla B - \epsilon^2 \Delta B = B \cdot \nabla U, \\ \operatorname{div} U = \operatorname{div} B = 0, \\ U|_{t=0} = U_0, \quad B|_{t=0} = B_0, \end{cases}$$

All the results mentioned are obtained when the domain Ω is

$$\mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}^+\}.$$

In our case, we studied the magneto-hydrodynamic system for a fluid that evolves in thin domain with Dirichlet conditions at the boundary, our field of study is the following

$$\mathcal{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\},$$

where ϵ represents the width of the domain. The interest of being in a thin domain supplemented by the Dirichlet conditions at the boundary is the appearance of Prandtl-type equations for velocity and magnetic fields. The main idea of the result is to obtain the existence and unique-

ness of solutions for the (MHD) system in a thin domain.

The sustem (MHD) is completed by the Dirichlet boundary conditions

$$(U, B)|_{y=0} = 0 \quad \text{and} \quad (U, B)|_{y=\epsilon} = 0$$

Our initial data is of the following form

$$U|_{t=0} = U_0^\epsilon = \left(u_0 \left(x, \frac{y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{y}{\epsilon} \right) \right) \quad \text{in } \mathcal{S}^\epsilon,$$

and

$$B|_{t=0} = B_0^\epsilon = \left(b_0 \left(x, \frac{y}{\epsilon} \right), \epsilon c_0 \left(x, \frac{y}{\epsilon} \right) \right) \quad \text{in } \mathcal{S}^\epsilon.$$

We look for solutions for our system of the form

$$(4.2.25) \quad \begin{cases} U(t, x, y) = \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ B(t, x, y) = \left(b^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon c^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ P(t, x, y) = p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

Let $\mathcal{S} := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$. After a natural change of scale, the (MHD) system becomes an anisotropic system

$$(4.2.26) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon + \partial_x p^\epsilon = b^\epsilon \partial_x b^\epsilon + c^\epsilon \partial_y b^\epsilon, & \text{in }]0, \infty[\times \mathcal{S} \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) + \partial_y p^\epsilon = \epsilon^2 (b^\epsilon \partial_x c^\epsilon + c^\epsilon \partial_y c^\epsilon), & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + v^\epsilon \partial_y b^\epsilon - \epsilon^2 \partial_x^2 b^\epsilon - \partial_y^2 b^\epsilon = b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y u^\epsilon, & \text{in }]0, \infty[\times \mathcal{S} \\ \epsilon (\partial_t c^\epsilon + u^\epsilon \partial_x c^\epsilon + v^\epsilon \partial_y c^\epsilon - \epsilon^2 \partial_x^2 c^\epsilon - \partial_y^2 c^\epsilon) = \epsilon (b^\epsilon \partial_x v^\epsilon + c^\epsilon \partial_y v^\epsilon), & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x b^\epsilon + \partial_y c^\epsilon = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{t=0} = (u_0, v_0, b_0, c_0), & \text{in } \mathcal{S} \\ (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{y=0} = (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{y=1} = 0. & \end{cases}$$

Formally taking $\epsilon \rightarrow 0$ in the scaled system (4.2.26), we obtain the Prandtl system on u and also on b which is of the following form:

$$(4.2.27) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = b \partial_x b + c \partial_y b, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_y p = 0, & \text{in } \mathcal{S} \times]0, \infty[\\ \partial_t b + u \partial_x b + v \partial_y b - \partial_y^2 b = b \partial_x u + c \partial_y u, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x u + \partial_y v = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ \partial_x b + \partial_y c = 0, & \text{in }]0, \infty[\times \mathcal{S} \\ u|_{t=0} = u_0, & \text{in } \mathcal{S} \\ b|_{t=0} = b_0, & \text{in } \mathcal{S}, \end{cases}$$

where the velocity field $U = (u, v)$ and the magnetic field $B = (b, c)$ satisfy the Dirichlet

non-slip boundary condition

$$(u, v, b, c) |_{y=0} = (u, v, b, c) |_{y=1} = 0.$$

Our goal is to obtain the global existence of solutions for the (4.2.26) and (4.2.27) systems when our initial data are analytical, and then we want to show the convergence of the scaled anisotropic MHD system (4.2.26) to the limit system (4.2.27) when ϵ tends to zero. The first result is the global well-posedness of the limit MHD system (4.2.27), with small analytic data in the horizontal variable.

Theorem 4.2.3. *Let $a > 0$, we assume that for some constant c_0 sufficiently small independent of ϵ , and for any initial data (u_0, b_0) satisfying*

$$(4.2.28) \quad \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0a,$$

and the compatibility conditions $\int_0^1 (u_0, b_0) dy = 0$ are satisfied. Then the system limit (4.2.27) has a unique global solution (u, b) satisfying

$$(4.2.29) \quad \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \leq C\|e^{a|D_x|}(u_0, b_0)\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $C = C(s)$ and (u_ϕ, b_ϕ) are given by

$$(4.2.30) \quad \begin{cases} (u_\phi, b_\phi)(t, x, y) = e^{\phi(t, D_x)}(u(t, x, y), b(t, x, y)) \triangleq \mathcal{F}_h^{-1}(e^{\phi(t, \xi)}(\hat{u}(t, \xi, y), \hat{b}(t, \xi, y))) \\ \phi(t, \xi) = (a - \lambda\theta(t))|\xi|. \end{cases}$$

The quantity $\theta(t)$ describes the evolution of the decrease of the analyticity band over time, this quantity is positive for all $t \in \mathbb{R}_+^*$ satisfies the equation $\dot{\theta}(t) = \|\partial_y(u_\phi, b_\phi)\|_{\mathcal{B}^{\frac{1}{2}}}$ and when $t = 0$, we have $\theta(0) = 0$.

\mathcal{R} is a constant determined by the Poincaré inequality for the band \mathcal{S} , the functional space used to obtain this result is the Besov space of the Chemin-Lerner type.

$$\|u\|_{\tilde{L}_T^p(\mathcal{B}^s(\mathcal{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \|\Delta_q^h u(t)\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

The second result is the global well-posedness of the scaled system (4.2.26) with small analytic initial data in the horizontal variable x .

Theorem 4.2.4. *Let $a > 0$, there exist a constant c_1 sufficiently small independent of ϵ , such that for any initial data $(u_0^\epsilon, v_0^\epsilon, b_0^\epsilon, c_0^\epsilon) = (U_0^\epsilon, B_0^\epsilon)$ satisfying*

$$(4.2.31) \quad \|e^{a|D_x|}(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0, \epsilon c_0)\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1a,$$

then the system (4.2.26) has a unique global solution (U^ϵ, B^ϵ) so that

$$(4.2.32) \quad \|e^{\mathcal{R}t}(u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}(b_\varphi, \epsilon c_\varphi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y(u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \\ + \|e^{\mathcal{R}t}\partial_y(b_\varphi, \epsilon c_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \epsilon \|e^{\mathcal{R}t}\partial_x(u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})}$$

$$+ \epsilon \|e^{\mathcal{R}t} \partial_x (b_\varphi, \epsilon c_\varphi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \leq C \left(\|e^{a|D_x|}(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0, \epsilon c_0)\|_{\mathcal{B}^{\frac{1}{2}}} \right),$$

where $C = C(s)$ and $(u_\varphi^\epsilon, v_\varphi^\epsilon)$ and $(b_\varphi^\epsilon, c_\varphi^\epsilon)$ will given also by (4.2.30). The function φ is given by

$$\varphi(t, D_x) = (a - \lambda\tau(t))|D_x|,$$

such that $\tau(t)$ is the evolution of the analyticity band.

The third result concern the study of the convergence from the scaled anisotropic MHD system (4.2.26) to the limit system (4.2.27), so in this theorem we proved that the convergence is globally in time.

Theorem 4.2.5. *Let $a > 0$, and $(u_0^\epsilon, v_0^\epsilon, b_0^\epsilon, c_0^\epsilon)$, satisfying (6.2.7). Let u_0 and b_0 satisfying $e^{a|D_x|}u_0 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}}$, $e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$, (the same thing for b_0) and the compatibility condition $\int_0^1 (u_0, b_0) dy = 0$ are satisfied and*

$$\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_2 a}{1 + \|e^{a|D_x|}(u_0, b_0)\|_{\mathcal{B}^{\frac{3}{2}}}},$$

for some constant $c_2 > 0$ sufficiently small independent of ϵ , then we have

$$(4.2.33) \quad \begin{aligned} & \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(t)\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_y(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ & + \epsilon \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ & \leq C \left(\|e^{a|D_x|}(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0^\epsilon - b_0, \epsilon(c_0^\epsilon - c_0))\|_{\mathcal{B}^{\frac{1}{2}}} + M\epsilon \right), \end{aligned}$$

where $C = C(s)$ and

$$(4.2.34) \quad \begin{cases} (\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, q^\epsilon) = (u^\epsilon - u, v^\epsilon - v, p^\epsilon - p), \\ (\Phi^{1,\epsilon}, \Phi^{2,\epsilon}) = (b^\epsilon - b, c^\epsilon - c), \end{cases}$$

and v_0 is determined from u_0 via $\partial_x u + \partial_y v = 0$ and $v_0|_{y=0} = v_0|_{y=1} = 0$, and $(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)$, $(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)$ will be given by (4.2.30). The function φ is given by

$$\Theta(t, D_x) = (a - \mu\eta(t))|D_x|,$$

such that $\eta(t)$ is the evolution of the analyticity band, and $\mu \geq \lambda$ is a constant.

Remark 4.2.6. *The main idea of proving the above theorem is to use analytic energy estimates, which are motivated by [112] and which originates from [27].*

4.2.3 Statement of the results on the hyperbolic version of the magneto-hydrodynamic system

In the chapter 5 of this thesis, we are interested in the following hyperbolic version of the magneto-hydrodynamic system in two dimensions:

$$(4.2.35) \quad \begin{cases} \mathbb{J} \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = \mathbb{H}^2 (b_1 b_2 v - u b_2^2 - b_2 e) & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_y p = \mathbb{H}^2 (b_1 b_2 u - b_1^2 v + b_1 e) & (0, T) \times \mathbb{R} \times (0, 1), \\ \frac{\kappa}{\text{Pr}_m} \partial_t^2 b_1 + \partial_t b_1 + u \partial_x b_1 + v \partial_y b_1 - \frac{1}{\text{Pr}_m} \partial_y^2 b_1 = b_1 \partial_x u + b_2 \partial_y u & (0, T) \times \mathbb{R} \times (0, 1), \\ \frac{\kappa}{\text{Pr}_m} \partial_t^2 b_2 + \partial_t b_2 + u \partial_x b_2 + v \partial_y b_2 - \frac{1}{\text{Pr}_m} \partial_y^2 b_2 = b_1 \partial_x v + b_2 \partial_y v & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_t b_1 + \partial_y e = 0 \quad \text{and} \quad \partial_t b_2 - \partial_x e = 0 & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_x u + \partial_y v = 0 \quad \text{and} \quad \partial_x b_1 + \partial_y b_2 = 0 & (0, T) \times \mathbb{R} \times (0, 1), \end{cases}$$

coupled with the following initial conditions and limits:

$$(4.2.36) \quad (\text{IC}) \quad \begin{cases} (u, b_1, b_2)|_{t=0} = (\bar{u}, \bar{b}_1, \bar{b}_2) & \mathbb{R} \times (0, 1), \\ (\partial_t u, \partial_t b_1, \partial_t b_2)|_{t=0} = (\tilde{u}, \tilde{b}_1, \tilde{b}_2) & \mathbb{R} \times (0, 1), \end{cases}$$

$$(4.2.37) \quad (\text{BC}) \quad \begin{cases} (u, b_1, b_2, e)|_{y=1} = (0, 0, 0, 0) & (0, T) \times \mathbb{R}, \\ (u, b_1, b_2, e)|_{y=1} = (0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{e}) & (0, T) \times \mathbb{R}. \end{cases}$$

All state variables (u, v, b_1, b_2, e) in (4.2.35) depend on time $t \in (0, T)$ and space $(x, y) \in \mathbb{R} \times (0, 1)$. The vector fields $(u, v)^T \in \mathbb{R}^2$ and $(b_1, b_2)^T \in \mathbb{R}^2$ are divergence free and stand for the velocity and magnetic fields of plasma, respectively. The electric field assumes size $e \in \mathbb{R}$ and is perpendicular to the plane containing the plasma. All constants \mathbb{H} , κ , Pr_m and \mathbb{J} are positive and depend on standard dimensionless parameters of magnetohydrodynamics. If we assume that our initial data are analytical and sufficiently small in the horizontal variable, then the result obtained in Chapter 5 represents the existence and uniqueness of the overall solutions of the system (4.2.35).

Theorem 4.2.6. *Assume homogeneous boundary conditions in (4.2.36) and (4.2.37): $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{e}) = (0, 0, 0)$. For any $s > 2$, there exists a sufficiently small positive constant $\varepsilon_s \in [0, 1)$ (which depends uniquely upon s), such that the following result holds true. Let \bar{u} , \bar{b}_1 and \tilde{b}_1 be initial data of (4.2.35) that are analytic in the variable $x \in \mathbb{R}$ with radius of analyticity $\tau_0 > 0$:*

$$(4.2.38) \quad \begin{aligned} e^{\tau_0(1+|D_x|)} \bar{u} \quad \text{and} \quad e^{\tau_0(1+|D_x|)} \bar{b}_1 & \text{ belong to } H^{s+1,1}(\mathbb{R} \times (0, 1)), \\ e^{\tau_0(1+|D_x|)} \tilde{u} \quad \text{and} \quad e^{\tau_0(1+|D_x|)} \tilde{b}_1 & \text{ belong to } H^{s,0}(\mathbb{R} \times (0, 1)). \end{aligned}$$

If the following smallness condition on the initial data holds true

$$(4.2.39) \quad \begin{aligned} & \|e^{\tau_0(1+|D_x|)} \bar{u}\|_{H^{s+1,0}} + \|e^{\tau_0(1+|D_x|)} \partial_y \bar{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)} \tilde{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)} \bar{b}_1\|_{H^{s+1,0}} \\ & + \|e^{\tau_0(1+|D_x|)} \partial_y \bar{b}_1\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)} \tilde{b}_1\|_{H^{s,0}} \\ & \leq \left(\frac{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}} \min\{\tau_0, \tau_0^{-1}\}^{\frac{3}{2}}}{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{5}{2}} \max\{1, \mathbb{H}^2\} \max\{\text{Pr}_m^{-1}, \text{Pr}_m\}^{\frac{1}{2}}} \right) \varepsilon_s, \end{aligned}$$

then there exists a global-in-time analytic solution (u, b_1) of (4.2.35), which has a decaying

radius of analyticity $\tau : \mathbb{R}_+ \rightarrow (0, \tau_0]$ given by

$$(4.2.40) \quad \tau(t) := \tau_0 \exp \left\{ - \frac{t}{16 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\} > 0.$$

Furthermore, the analytic norms of the solution decay exponentially in time $t \in \mathbb{R}_+$ as follows:

$$(4.2.41) \quad \begin{aligned} & \|e^{\tau(t)(1+|D_x|)} u(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)} \partial_t u(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)} \partial_y u(t)\|_{H^{s,0}}^2 \\ & + \|e^{\tau(t)(1+|D_x|)} b_1(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)} \partial_t b_1(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)} \partial_y b_1(t)\|_{H^{s,0}}^2 \\ & \leq C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) \left\{ \|e^{\tau_0(1+|D_x|)} \bar{u}\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)} \tilde{u}\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)} \partial_y \bar{u}\|_{H^{s,0}}^2 \right. \\ & \left. + \|e^{\tau_0(1+|D_x|)} \bar{b}\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)} \tilde{b}\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)} \partial_y \bar{b}\|_{H^{s,0}}^2 \right\} \exp \left\{ - \frac{t}{8 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\}. \end{aligned}$$

where the constant $C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0)$ is defined by

$$C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) = 4^3 \frac{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3} \max\{\text{Pr}_m, \text{Pr}_m^{-1}\} \max\{\tau_0, \tau_0^{-1}\}^2.$$

4.3 Hydrostatic approximation of Primitive Equations

In this section, we seek to give the results obtained during the work of this thesis on the modeling of primitive equations in two-dimensional boundary layers. The model studied represents geophysical fluids on a large scale. In our case, we will take into account the strong stratification due to the Earth's gravity. Our model is of the following form:

$$(4.3.1) \quad \begin{cases} \partial_t u + U \cdot \nabla u - \nu_1 \partial_x^2 u - \nu_2 \partial_y^2 u + \partial_x p = 0, \\ \partial_t v + U \cdot \nabla v - \nu_1 \partial_x^2 v - \nu_2 \partial_y^2 v + \partial_y p = \frac{T}{F_r}, \\ \partial_t T + U \cdot \nabla T - \kappa_1 \partial_x^2 T - \kappa_2 \partial_y^2 T = 0 \\ \text{div } U = 0, \\ (U, T)|_{t=0} = (U_0, T_0), \end{cases}$$

4.3.1 Reminder of existing results

The results of this thesis concern the study of primitive equations in a thin domain supplemented by Dirichlet conditions at the edge. The structure of the thin domain as well as the choice of initial data cause horizontal viscosity's to disappear at the limit, when the thickness of the domain converges to zero. We will therefore have to study the hydrostatic approximation of the Navier-Stokes system, which is a Prandtl equation, coupled with a transport-diffusion equation on temperature. Before discussing the results obtained for this type of system, we briefly recall some of the results obtained previously.

We begin by recalling some results obtained when considering all non-zero viscosity and diffusivity. The first to mathematically process this model were Lions-Temam-Wang in [91–93]

in 1990. Their work concerns the study of the global existence of weak solutions for primitive equations taking into account viscosity and diffusivity. Regarding the uniqueness of the solution, Guillén-Gonzalez, Masmoudi and Rodriguez-Bellido in [67] proved the uniqueness of the weak solution for these equations in a 2-dimensional frame. They also obtained a local existence result in [101]. The global existence in two-dimensional for these equations was obtained by Bresch, Kazhikhov and Lemoine in [17].

When considering only a vertical viscosity that appears in the system and all the diffusivity, C. Cao, Q. Lin and E.S.Titi obtained a result of local and global existence with small initial data of the strong solution of the reduced 3D primitive geostrophic adjustment model with low dissipation in [21]. More specifically, they looked at the following model:

$$(4.3.2) \quad \begin{cases} \partial_t u + u\partial_x u + w\partial_z u + \epsilon_1 u - f_0 v - \nu\partial_z^2 u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + w\partial_z v + \epsilon_1 v + f_0 u - \nu\partial_z^2 v = 0, \\ \epsilon_2 w + \partial_z p + T = 0, \\ \partial_t T + u\partial_x T + w\partial_z T - \kappa\Delta T = 0 \\ \partial_x u + \partial_z w = 0 \\ (u, v, T)|_{t=0} = (u_0, v_0, T_0), \end{cases}$$

This model is studied in the domain $\{(x, z) \in \mathbb{R}^2 : 0 \leq z \leq H\}$, completed by the following boundary condition:

$$(\partial_z u, \partial_z v, w, T)|_{z=0, H=0}, \text{ and } u, v, w, T \text{ are periodic in } x \text{ with period } 1,$$

by $\partial_x u + \partial_z w = 0$, we deduce that

$$w(x, z) = - \int_0^z \partial_x u(x, s) ds.$$

In the (4.3.2) model, ϵ_1 and ϵ_2 are positive constants, ν is a positive constant that represents the vertical viscosity of the horizontal moment equations and f_0 is the Coriolis parameter. This work led by C. Cao, Q. Lin and E.S. Titi in [21] to the following results:

Theorem 4.3.1. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_x v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfy the symmetry conditions:*

$$\begin{aligned} &u, v \text{ et } T \text{ are periodic in } x \text{ and } z \text{ with period } 1; \\ &u, v \text{ are even in } z, \text{ and } T \text{ is odd in } z, \end{aligned}$$

with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Then there exists some time \mathcal{T} positive such that there exists a unique strong solution (u, v, T) of system (4.3.2), on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.

Remark 4.3.1. *The proof of this theorem is based on the standard Galerkin approximation. Indeed, the authors first establish the formal a priori estimates for the solutions of system (4.3.2). These estimates can be justified rigorously by deriving them first to the Galerkin approximation*

system and then passing to the limit using the Aubin-Lions compactness theorem.

The second result concerns the existence and global uniqueness of the strong solution of system (4.3.2), provided that the initial data is sufficiently small.

Theorem 4.3.2. *Suppose that $u_0, v_0, T_0, \partial_x u_0, \partial_y v_0, \partial_x T_0 \in H^1(\mathbb{T}^2)$ satisfies the symmetry conditions:*

$$\begin{aligned} u, v \text{ and } T \text{ are periodic in } x \text{ and } z \text{ with period } 1; \\ u, v \text{ are even in } z, \text{ and } T \text{ is odd in } z, \end{aligned}$$

with the compatibility condition $\int_0^1 \partial_x u_0 dz = 0$. Moreover, suppose that

$$\|u_0\|_{H^1} + \|v_0\|_{H^1} + C_0 \|T_0\|_{H^1} + \|\partial_x u_0\|_{H^1} + \|\partial_x v_0\|_{H^1} + C_0 \|\partial_x T_0\|_{H^1} \ll 1$$

for some $C_0 > 0$. Then for any time $\mathcal{T} > 0$, there exists a unique strong solution (u, v, T) of system (4.3.2) on the interval $[0, \mathcal{T}]$. Moreover, the unique strong solution (u, v, T) depends continuously on the initial data.

4.3.2 Contribution of the thesis

In The Chapter 8 of this thesis, we consider for $\epsilon > 0$ the system

$$(PE) \quad \begin{cases} \partial_t U + U \cdot \nabla U - \epsilon^2 \Delta U + \nabla P = \begin{pmatrix} 0 \\ \frac{T}{Fr} \end{pmatrix}, \\ \partial_t T + U \cdot \nabla T - \kappa_1 \partial_x^2 T - \kappa_2 \partial_y^2 T = 0, \\ \operatorname{div} U = 0, \\ U|_{t=0} = U_0, \quad T|_{t=0} = T_0, \end{cases}$$

In our study, we looked at primitive equations on a thin domain with Dirichlet conditions at the boundary. Our domain of study is

$$\mathcal{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\},$$

where ϵ represent the width of the domain. The general idea for dealing with this type of equation is to impose analyticity according to the horizontal variable. This allows us to control the nonlinear terms that give difficulty due to derivative losses in the horizontal variable. This method led us to find a result of the existence and global uniqueness of solutions for the (PE) system in a thin domain.

In the (PE) system, the constants κ_1 and κ_2 represents the horizontal and vertical thermal diffusivity. Throughout our results, these constants are given by $\kappa_1 = 1$ and $\kappa_2 = \epsilon^2$. Fr is the Froude number measuring the importance of stratification, which is assumed to be $Fr = \epsilon$.

The (PE) system supplemented by the Dirichlet conditions at the boundary

$$(U, T)|_{y=0} = 0 \quad \text{and} \quad (U, T)|_{y=\epsilon} = 0.$$

Our initial data has the form

$$U|_{t=0} = U_0^\epsilon = \left(u_0 \left(x, \frac{y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{y}{\epsilon} \right) \right) \quad \text{in} \quad \mathcal{S}^\epsilon,$$

et

$$T|_{t=0} = T_0 \left(x, \frac{y}{\epsilon} \right) \quad \text{in} \quad \mathcal{S}^\epsilon.$$

We are looking for solutions for our system of form

$$(4.3.3) \quad \begin{cases} U(t, x, y) = \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right), \\ T(t, x, y) = T^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \\ P(t, x, y) = p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

Performing scaling change $z = \frac{y}{\epsilon}$, i.e. $\mathcal{S} := \{(x, z) \in \mathbb{R}^2 : 0 < z < 1\}$, we obtain the primitive anisotropic system

$$(4.3.4) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_z u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_z^2 u^\epsilon + \partial_x p^\epsilon = 0, \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_z v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_z^2 v^\epsilon) + \partial_z p^\epsilon = T^\epsilon, \\ \partial_t T^\epsilon + u^\epsilon \partial_x T^\epsilon + v^\epsilon \partial_z T^\epsilon - \Delta T^\epsilon = 0, \\ \partial_x u^\epsilon + \partial_z v^\epsilon = 0, \\ (u^\epsilon, v^\epsilon, T^\epsilon)|_{t=0} = (u_0, v_0, T_0), \\ (u^\epsilon, v^\epsilon, T^\epsilon)|_{\bar{y}=0} = (u^\epsilon, v^\epsilon, T^\epsilon)|_{\bar{y}=1} = 0. \end{cases}$$

Formally taking $\epsilon \rightarrow 0$ in the system (4.3.4), we obtain the following hydrostatic limit for primitive equations, which are combination of a Prandtl-like system with a transport-diffusion equation of the temperature

$$(4.3.5) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_z u - \partial_z^2 u + \partial_x p = 0, \\ \partial_y p = T, \\ \partial_t T + u \partial_x T + v \partial_z T - \Delta T = 0, \\ \partial_x u + \partial_z v = 0, \\ (u, T)|_{t=0} = (u_0, T_0), \\ (u, v, T)|_{y=0,1} = 0, \end{cases}$$

Our goal is to obtain the global existence of solutions for the (4.3.4) and (4.3.5) systems when our initial data are analytical. Then we want to show the convergence of the primitive anisotropic equations scaled (4.3.4) to the limit system (4.3.5) when ϵ tends to zero.

The first result obtained concerns the existence of global solutions of the hydrostatic system

(4.3.5), with small analytical data according to the horizontal variable.

Theorem 4.3.3. *Let $a > 0$, $s > 0$ and assume that $e^{a|D_x|}(u_0, T_0) \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}} \cap \mathcal{B}^s$. There exist positive constants c_0, C and a decreasing function $\tilde{\phi} : \mathbb{R}_+ \rightarrow [\frac{2a}{3}, a]$ such that, if we suppose that the initial data (u_0, T_0) satisfy the compatibility condition $\int_0^1 u_0 dz = 0$ and the smallness assumption*

$$(4.3.6) \quad \left\| e^{a|D_x|} u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} T_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a \quad \text{and} \quad \left\| e^{a|D_x|} u_0 \right\|_{\mathcal{B}^{\frac{3}{2}}} + \left\| e^{a|D_x|} T_0 \right\|_{\mathcal{B}^{\frac{3}{2}}} \leq c_0,$$

then the system (4.3.5) has a unique global solution

$$(u, T) \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}^s) \cap C(\mathbb{R}_+; \mathcal{B}^s) \quad \text{with} \quad \partial_z u \in \tilde{L}^2(\mathbb{R}_+; \mathcal{B}^s),$$

satisfying

$$(4.3.7) \quad \left\| e^{\mathcal{R}t}(u_\phi, T_\phi) \right\|_{\tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}^s)} + \frac{1}{4} \left\| e^{\mathcal{R}t} \partial_z u_\phi \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \frac{1}{2} \left\| e^{\mathcal{R}t} \nabla T_\phi \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq 2C \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^s},$$

for any $0 \leq \mathcal{R} \leq \frac{1}{2\mathcal{K}}$, where $\phi(t, \xi) = \tilde{\phi}(t) |\xi|$ and where for any $f \in L^2(\mathcal{S})$,

$$f_\phi(t, x, z) = e^{\phi(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\phi(t, \xi)} \hat{f}(t, \xi, z)).$$

where $\mathcal{K} > 0$ is the Poincaré constant on the strip \mathcal{S} , in the sense that, for any $f \in L^2(\mathcal{S})$, $f|_{\partial\mathcal{S}} = 0$ and $\partial_z f \in L^2(\mathcal{S})$, we have

$$\|f\|_{L^2(\mathcal{S})} \leq \mathcal{K} \|\partial_z f\|_{L^2(\mathcal{S})}.$$

Furthermore, we have,

$$(4.3.8) \quad \left\| e^{\mathcal{R}t}(\partial_t u)_\phi \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \partial_z u_\phi \right\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \leq C \left(\left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^s} + \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^{s+1}} \right).$$

where $C = C(s)$, the functional space used to obtain this result is the Besov Chemin-Lerner space given by the norm

$$\|u\|_{\tilde{L}_T^p(\mathcal{B}^s(\mathcal{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \left\| \Delta_q^h u(t) \right\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

Remark 4.3.2. *We remark that the normal component v is uniquely determined from the incompressibility and the boundary condition*

$$(4.3.9) \quad v(t, x, z) = \int_0^z \partial_z v(t, x, z') dz' = - \int_0^z \partial_x u(t, x, z') dz'.$$

The second result shown is that the primitive anisotropic equations (4.3.4) with small analytical initial data in the tangential variable are generally well-posed.

Theorem 4.3.4. *Let $a > 0$, $s > 0$, $\epsilon > 0$ and assume that $e^{a|D_x|}(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon) \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}} \cap \mathcal{B}^s$. There exist positive constants c_1, C (independent of ϵ) and a decreasing function $\tilde{\Theta} : \mathbb{R}_+ \rightarrow$*

$[\frac{2a}{3}, a]$ such that, if we suppose that the initial data $(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon)$ satisfy

$$\left\| e^{a|D_x|}(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon) \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a \quad \text{and} \quad \left\| e^{a|D_x|}(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon) \right\|_{\mathcal{B}^{\frac{3}{2}}} \leq c_1,$$

then, for any $0 < \epsilon < \frac{1}{2C}$ the system (4.3.4) has a unique global solution $(u^\epsilon, v^\epsilon, T^\epsilon)$ satisfying,

$$\|e^{\mathcal{R}t}(u_\Theta^\epsilon, \epsilon v_\Theta^\epsilon, T_\Theta^\epsilon)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_z(u_\Theta^\epsilon, \epsilon v_\Theta^\epsilon, T_\Theta^\epsilon)\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_x T_\Theta^\epsilon\|_{\tilde{L}_t(\mathcal{B}^s)} \leq C \|e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon, T_0^\epsilon)\|_{\mathcal{B}^s},$$

where $C = C(s)$, for any $0 \leq \mathcal{R} \leq \frac{1}{2\mathcal{K}}$. Here, $\Theta(t, \xi) = \tilde{\Theta}(t) |\xi|$ and for any $f \in L^2(\mathbf{S})$,

$$f_\Theta(t, x, z) = e^{\Theta(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\Theta(t, \xi)} \hat{f}(t, \xi, z)).$$

The third result concerns the convergence of the solutions of the scaled anisotropic equation of primitive equations (4.3.4) to the limit system (4.3.5).

Theorem 4.3.5. *Let $a > 0$ and $0 < \epsilon \leq 1$. We suppose that the initial data (u_0, v_0, T_0) and $(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon)$ satisfy the assumptions of Theorems 4.3.3 and 4.3.4. Let (u, v, T) and $(u^\epsilon, v^\epsilon, T^\epsilon)$ be the respective solutions of the systems (4.3.5) and (4.3.4). Then, there exist a constant $M > 0$ independent of ϵ and a decreasing function $\tilde{\varphi} : \mathbb{R}_+ \rightarrow [\frac{a}{3}, a]$ such that*

$$\begin{aligned} & \| (u_\varphi^\epsilon - u_\varphi, \epsilon v_\varphi^\epsilon - \epsilon v_\varphi) \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \| \partial_z (u_\varphi^\epsilon - u_\varphi, \epsilon v_\varphi^\epsilon - \epsilon v_\varphi) \|_{\tilde{L}_t^2(\mathcal{B}^s)} + \epsilon \| (u_\varphi^\epsilon - u_\varphi, \epsilon v_\varphi^\epsilon - \epsilon v_\varphi) \|_{\tilde{L}_t^2(\mathcal{B}^{s+1})} \\ & \leq C \left(\| e^{a|D_x|}(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0)) \|_{\mathcal{B}^s} + C \| e^{a|D_x|}(T_0^\epsilon - T_0) \|_{\mathcal{B}^s} + M\epsilon \right). \end{aligned}$$

where $C = C(s)$ and $\varphi(t, \xi) = \tilde{\varphi}(t) |\xi|$ and where, for any $f \in L^2(\mathbf{S})$,

$$f_\varphi(t, x, z) = e^{\varphi(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\varphi(t, \xi)} \hat{f}(t, \xi, z)).$$

Remark 4.3.3. *The main idea of proving the above theorem is to use analytic energy estimates, which are motivated by [112] and which originates from [27].*

4.4 Study of the equations of homogeneous and inhomogeneous magnetohydrodynamics

In the second part of this thesis, we were interested in the study of the existence and regularity as well as the uniqueness of the solutions of the equation of magneto-hydrodynamics, when we have a condition of smallness of the unidirectional derivative of the initial velocity in certain scale-invariant spaces.

4.4.1 Reminder of the results

The results of this thesis are motivated by the result obtained by Y. Liu, M. Paicu and P. Zhang in [94]. The authors proved that as long as the unidirectional derivative of the initial velocity is small enough in some scale-invariant spaces, then the Navier–Stokes system admits

a global unique solution. Indeed, the authors studied the following system:

$$(4.4.1) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta_h u + \nabla p = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0}(t, x) = u_0(x), \end{cases}$$

where $\Delta_h = \partial_x^2 + \partial_y^2$. This system (4.4.1) is generally well posed if the initial data u_0 satisfy a smallness condition on its vertical derivative $\partial_3 u_0$ in certain critical spaces. The main result of their article is as follows:

Theorem 4.4.1. *Soit Λ_h^{-1} be a Fourier multiplier with symbol $|\xi_h|^{-1}$, let $u_0 \in \mathcal{B}^{0, \frac{1}{2}}$ be a solenoids vector field with $\Lambda_h^{-1} \partial_3 u_0 \in \mathcal{B}^{0, \frac{1}{2}}$. Then there exist some sufficiently small positive constant ϵ_0 and some universal positive constants L, M, N , so that if*

$$(4.4.2) \quad \|\Lambda_h^{-1} \partial_3 u_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(L(1 + \|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}}^4) \exp(M\mathcal{A}_N^4(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}))\right) \leq \epsilon_0,$$

with

$$\begin{aligned} \mathcal{A}_N(\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}) &= N^{\frac{1}{2}} \|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp(C\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2) \\ &\quad + \|u_{0,N}^h\|_{\mathcal{B}^{0, \frac{1}{2}}} \exp\left(N^2 \exp(C\|u_0^h\|_{\mathcal{B}^{0, \frac{1}{2}}}^2)\right) \end{aligned}$$

and N is taken large enough that $\|u_{0,N}^h\|_{\mathcal{B}^{0, \frac{1}{2}}}$ be small enough, then the system (4.4.1) has a unique global solution $u = v + e^{t\Delta_h} \begin{pmatrix} 0 \\ u_{0,hh}^3 \end{pmatrix}$ with

$$v \in C([0, \infty[; \mathcal{B}^{0, \frac{1}{2}}), \quad \nabla_h v \in L^2([0, \infty[; \mathcal{B}^{0, \frac{1}{2}})$$

and $u_{0,hh}^3 = \sum_{k \geq l-1} \Delta_k^h \Delta_l^v u^3$.

Remark 4.4.1. *The proof of this theorem consists to decompose the solution of the (4.4.1) system into the sum of a solution of the two-dimensional Navier–Stokes system with a parameter and a solution of the perturbed three-dimensional anisotropic Navier–Stokes system.*

Remark 4.4.2. *In the theorem 4.4.1, the authors obtain their result in the following homogeneous Besov space:*

$$\mathcal{B}^{0, \frac{1}{2}} \triangleq \left\{ f \in \mathcal{S}'_h(\mathbb{R}^3) : \|u\|_{\mathcal{B}^{0, \frac{1}{2}}} \triangleq \sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \|\Delta_q^v u\|_{L^2(\mathbb{R}^3)} < +\infty \right\}.$$

4.4.2 Contribution of the thesis

The work of the last chapter of this thesis focuses on the study of the inhomogeneous magneto-hydrodynamic system. The goal is to assume that the vertical derivative of the initial data is sufficiently small and then to prove the existence and global uniqueness of the solutions of the

3D inhomogeneous magneto-hydrodynamic system. We therefore consider the system

$$(4.4.3) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \rho(\partial_t u + u \cdot \nabla u) - \mu_1 \Delta u + \nabla p = b \cdot \nabla b, \\ \partial_t b - \mu_2 \Delta b - \text{curl}(u \times b) = 0, \\ \text{div } u = 0, \text{ div } b = 0, \\ \rho|_{t=0} = \rho_0(x), u|_{t=0} = u_0(x), b|_{t=0} = u_0(x), \end{cases}$$

where ρ is a scalar representing the density of the fluid, $u = (u_1, u_2, u_3)$ and $b = (b_1, b_2, b_3)$ represent respectively the velocity field and the magnetic field of the fluid, p is a scalar function indicating pressure. μ_1 is a parameter representing the kinematic viscosity coefficient and μ_2 is a parameter representing the magnetic scattering coefficient. Since μ_1 and μ_2 play no role in the existence and uniqueness of the global solution, so it is assumed that these parameters are worth 1 in the rest of the calculations.

When the density is constant in the (4.4.3) system, it becomes the next homogeneous standard MHD system

$$(4.4.4) \quad \begin{cases} \partial_t u^H + u^H \cdot \nabla u^H - \Delta u^H = -\nabla p^H + b^H \cdot \nabla b^H, \\ \partial_t b^H - \Delta b^H + u^H \cdot \nabla b^H - b^H \cdot \nabla u^H = 0, \\ \text{div } u^H = \text{div } b^H = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u^H|_{t=0} = u_0^H(x), b^H|_{t=0} = b_0^H(x). \end{cases}$$

Using Littlewood-Paley theory and adapting the method developed in the work of Y. Liu, M. Paicu and P. Zhang in [94] for homogeneous magneto-hydrodynamic equations, we obtain a result of the existence and global uniqueness of solutions in Besov spaces. The result is as follows:

Theorem 4.4.2. *Let $u_0^H = (u_0^{H,h}, u_0^{H,3}) \in \mathcal{B}_p^0(\mathbb{R}^3)$, $b_0^H = (b_0^{H,h}, b_0^{H,3}) \in \mathcal{B}_p^0(\mathbb{R}^3)$ are two divergence-free vector fields with $(u_0^{H,h}, b_0^{H,h}) \in H^1(\mathbb{R}^3)$, $\Lambda_h^{-1} \partial_3(u_0^H, b_0^H) \in \mathcal{B}_p^0(\mathbb{R}^3)$ ($2 < p < 4$), where Λ_h^{-1} is a Fourier multiplier with symbol $|\xi_h|^{-1}$. Then there exist some sufficiently small positive constant ε_0 and an universal positive constant C such that if (u_0^H, b_0^H) satisfies the smallness condition*

$$(4.4.5) \quad \begin{aligned} & \|\Lambda_h^{-1} \partial_3(u_0^H, b_0^H)\|_{\mathcal{B}_p^0} B(u_0^{H,h}, b_0^{H,h}) \times \exp \left(C \left(\|(u_0^{H,3}, b_0^{H,3})\|_{\mathcal{B}_p^0} \right. \right. \\ & \left. \left. + \|(u_0^{H,h}, b_0^{H,h})\|_{\mathcal{B}_p^0}^2 + \|(u_0^{H,h}, b_0^{H,h})\|_{H^1}^4 \right) B(u_0^{H,h}, u_0^{H,h}) \right) \leq \varepsilon_0. \end{aligned}$$

where

$$B(u_0^{H,h}, b_0^{H,h}) = \exp \left(C \left(\|(u_0^{H,h}, b_0^{H,h})\|_{\mathcal{B}_p^0} + \|(u_0^{H,h}, b_0^{H,h})\|_{H^1}^2 \exp(C \|(u_0^{H,h}, b_0^{H,h})\|_{L^2}) \right) \right).$$

Then system (4.4.4) has a unique global solution (u^{MHD}, b^{MHD}) which satisfies

$$u^H \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2), \quad b^H \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2).$$

Now let's move on to the inhomogeneous case of the system (4.4.3). In the following, it is assumed that the initial density verifies $\inf \rho_0 > 0$. By the principle of maximum for the transport equation, we have $\inf \rho(t, x) > 0$. We can then use the transformation $a \triangleq \frac{1}{\rho} - 1$ which allows us to obtain the system

$$(4.4.6) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1+a)(-\Delta u + \nabla p) = (1+a)(b \cdot \nabla b), \\ \partial_t b + u \cdot \nabla b - \Delta b = b \cdot \nabla u, \\ \operatorname{div} u = 0, \operatorname{div} b = 0, \\ a|_{t=0} = a_0(x), u|_{t=0} = u_0(x), b|_{t=0} = b_0(x). \end{cases}$$

The next theorem shows the global well-posedness result associated with the system (4.4.6).

Theorem 4.4.3. *Let $u_0 = (u_0^h, u_0^3) \in \mathcal{B}_p^0(\mathbb{R}^3)$, $b_0 = (b_0^h, b_0^3) \in \mathcal{B}_p^0(\mathbb{R}^3)$ are two divergence-free vector fields with $(u_0^h, b_0^h) \in H^1(\mathbb{R}^3)$, $\Lambda_h^{-1} \partial_3(u_0, b_0) \in \mathcal{B}_p^0$ ($2 < p < 4$), $a_0 \in B_p^{\frac{3}{p}}$, where Λ_h^{-1} is a Fourier multiplier with symbol $|\xi_h|^{-1}$. Then there exist some sufficiently small positive constant ε_0 and an universal positive constant C such that if (a_0, u_0, b_0) satisfies the smallness condition*

$$(4.4.7) \quad \left(\|a_0\|_{B_p^{\frac{3}{p}}} + \|\Lambda_h^{-1} \partial_3(u_0, b_0)\|_{\mathcal{B}_p^0} \right) L(u_0, b_0) \leq \varepsilon_1,$$

with

$$L(u_0, b_0) = B(u_0^h, b_0^h) \exp \left(C(1 + \|(u_0^3, b_0^3)\|_{\mathcal{B}_p^0} + \|(u_0^h, b_0^h)\|_{\mathcal{B}_p^0}^2 + \|(u_0^h, b_0^h)\|_{H^1}^4) B(u_0^h, b_0^h) \right)$$

and

$$B(u_0^h, b_0^h) = \exp(C(\|(u_0^h, b_0^h)\|_{\mathcal{B}_p^0} + \|(u_0^h, b_0^h)\|_{H^1}^2) \exp(C\|(u_0^h, b_0^h)\|_{L^2})).$$

Then system (4.4.6) has a unique global solution

$$a \in \tilde{L}^\infty(\mathbb{R}_+; B_p^{\frac{3}{p}}), \quad u \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2) \quad \text{and} \quad b \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2).$$

Remark 4.4.3. *The proof of this theorem consists to decompose the inhomogeneous solution into a homogeneous solution plus a remainder term, which is denoted in the last chapter by R .*

Remark 4.4.4. *In the theorems 4.4.2-4.4.2, we use the following homogenous Besov space:*

$$B_{p,r}^{s_1, s_2} \triangleq \{u \in \mathcal{S}'_h(\mathbb{R}^3) : \|u\|_{\mathcal{B}_p^{s_1, s_2}} \triangleq (2^{s_1 j} 2^{s_2 k} \|\Delta_j^h \Delta_k^v u\|_{L^p})_{\ell^1(\mathbb{Z})} < \infty\},$$

for simplicity we use the following notation

$$\mathcal{B}_p^0 \triangleq \mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}, \quad \mathcal{B}_p^1 \triangleq \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}} \quad \text{and} \quad \mathcal{B}_p^2 \triangleq \mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}, 2+\frac{1}{p}},$$

we mention that the space \mathcal{B}_p^0 is critical. We recall that the space $\mathcal{B}_p^{\frac{3}{p}}$ is defined by

$$B_p^{\frac{3}{p}} \triangleq \{u \in \mathcal{S}'_h(\mathbb{R}^3) : \|u\|_{B_p^{\frac{3}{p}}} \triangleq (2^{\frac{3j}{p}} \|\Delta_j u\|_{L^p(\mathbb{R}^3)})_{\ell^1(\mathbb{Z})} < \infty\}.$$

Chapter 5

Hyperbolic version of Navier-Stokes

In this chapter we present the results of the following paper:

N. Aarach, Global well-posedness of 2D Hyperbolic perturbation of the Navier-Stokes system in a thin strip, arXiv:2111.13052

5.1 Introduction

The Navier-Stokes equations are a type of nonlinear partial differential equation that describes the motion of Newtonian fluids. These equations have been an enormous research topic since their introduction in the 1930s. They are defined by the following Newtonian incompressible viscous fluids:

$$(5.1.1) \quad \begin{cases} \partial_t U + U \cdot \nabla U - \nu \Delta U + \nabla P = 0, \\ \operatorname{div} U = 0. \end{cases}$$

here U denotes the velocity field, P the scalar pressure function, and ν is the fluid's viscosity coefficient.

Nonetheless, System (5.1.1) creates a physical conundrum due to the fact that the velocity equation has an infinite propagation speed. To avoid this nonphysical aspect, Cattaneo in [24, 25] (also see Vernotte [128]) advocated modifying the heat equation to a hyperbolic form known as Cattaneo's heat transfer law. More specifically, they advocated replacing the Fourier law they proposed to replace Fourier's law which describes the following stress tensor

$$\tau(t) = -PId + \nu(\nabla U + (\nabla U)^\top)(t),$$

with the hyperbolic model shown below

$$\frac{1}{c^2} \partial_t^2 \theta + \frac{1}{\beta} \partial_t \theta - \Delta \theta = 0.$$

This equation is called **The Telegraph equation**. It has a finite propagation speed and is compatible with both the principle of relativity and the second law of thermodynamics, so it is a satisfactory physical model. As a result, it makes sense to think of a hyperbolic Navier-Stokes system by including the term $\tau \partial_t^2$ to the classical Navier-Stokes system (5.1.1), where τ is a small parameter we can assume that is equal to $1/c^2$ (c is the speed of light).

In our work, we consider a hyperbolic perturbation of the incompressible Navier-Stokes equations in $\mathbb{R} \times (0, \epsilon)$ such that the viscosity $\nu = \epsilon^2$ (this viscosity called vanishing viscosity). We study this system in a thin region with no-slip boundary conditions. We denote $\mathbf{S}^\epsilon = \{(x, Y) \in \mathbb{R}^2 : 0 < Y < \epsilon\}$ where ϵ is the width of the domain. Our system is of the following form:

$$(5.1.2) \quad \begin{cases} \tau \partial_t^2 U^{(\tau, \epsilon)} + \partial_t U^{(\tau, \epsilon)} + U^{(\tau, \epsilon)} \cdot \nabla U^{(\tau, \epsilon)} \\ \quad - \epsilon^2 \Delta U^{(\tau, \epsilon)} + \nabla P^{(\tau, \epsilon)} = 0, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ \operatorname{div} U^{(\tau, \epsilon)} = 0, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ U|_{t=0}^{(\tau, \epsilon)} = U_0^{(\tau, \epsilon)}, \quad \partial_t U|_{t=0}^{(\tau, \epsilon)} = U_1^{(\tau, \epsilon)}, & \text{in } \mathbf{S}^\epsilon \end{cases}$$

where $U^{(\tau, \epsilon)}(t, x, Y) = (U_1^{(\tau, \epsilon)}(t, x, Y), U_2^{(\tau, \epsilon)}(t, x, Y))$ denotes the velocity of the fluid and $P^{(\tau, \epsilon)}(t, x, Y)$ the scalar pressure function, which guarantees the divergence-free property of the velocity field $U^{(\tau, \epsilon)}$. The system (5.1.2) is complemented by the no-slip boundary condition

$$U|_{Y=0}^{(\tau, \epsilon)} = 0 \quad \text{and} \quad U|_{Y=\epsilon}^{(\tau, \epsilon)} = 0 \quad \text{in }]0, \infty[\times \mathbf{S}^\epsilon.$$

Here, in the equation of the velocity, the Laplacian is $\Delta = \partial_x^2 + \partial_Y^2$.

The dissipative hyperbolic Navier-Stokes equation (5.1.2) is obtained after relaxing the Euler equations and a change of scale variables. This perturbation, considered as a relaxation of Euler's equations, was considered by Brenier, Natalini, and Puel in [16]. They introduced a hyperbolic system of equations, based on a relaxation approximation of the incompressible Navier-Stokes equations, following the scheme described by Jin and Xin in [78].

$$(5.1.3) \quad \begin{cases} \partial_t U^\epsilon + \operatorname{div} V^\epsilon = \nabla Q^\epsilon \\ \partial_t V^\epsilon + \frac{\epsilon^2}{\tau} \nabla U^\epsilon = -\frac{1}{\tau} (V^\epsilon - U^\epsilon \otimes U^\epsilon) \\ \operatorname{div} U^\epsilon = 0 \\ (U^\epsilon, V^\epsilon)|_{t=0} = (U_0^\epsilon, V_0^\epsilon) \end{cases}$$

In their work they proved global existence and uniqueness for the perturbed Navier-Stokes equation with initial data in $\mathcal{H}^2(\mathbb{T}^2)^2 \times \mathcal{H}^1(\mathbb{T}^2)^2$, where \mathbb{T}^2 is the periodic square $\mathbb{R}^2/\mathbb{Z}^2$. Moreover, they proved the convergence of the solution of the perturbed Navier-Stokes to a smooth solution for Navier-Stokes.

Later this equation was considered by Paicu and Raugel in [108, 109]. In their work, they also proved a global existence and uniqueness result with significantly improved regularity for the initial data, when τ is small enough. In fact, they only require the regularity in $\mathcal{H}^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$. Also Hachicha in [68], obtained a global result of existence and uniqueness of the perturbed Navier-Stokes in two and three space dimensions and under suitable smallness assumptions on the initial data in the space $[(\mathcal{H}^{\frac{n}{2}+\delta} \cap \mathcal{H}^{\frac{n}{2}-1+\delta})(\mathbb{R}^n)]^n$ with $n = 2, 3$. Moreover, for all positive times T , she proved the convergence to perturbed Navier-Stokes towards solutions of the Navier-Stokes system (NS) with initial data in $\mathcal{H}^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$, $s > 0$.

We finally mention a recent result obtained by O. Coulaud, I. Hachicha and G. Raugel in [38]. They considered a hyperbolic quasi-linear version of the Navier-Stokes equations in \mathbb{R}^2 and proved the existence and uniqueness of solutions to these equations, and exhibit smallness assumptions on the data, under which the solutions are global in time in the 2D case.

The way these authors introduce their system of equations is by using methods that are usually devoted to the study of numerical patterns, and that can be applied to all preservation laws. In this article, we take the problem from another point of view. In order to describe hydrodynamical flows

on the earth, in geophysics, it is usually assumed that vertical motion is much smaller than horizontal motion and that the fluid layer depth is small compared to the radius of the sphere, thus, they are a good approximation of global atmospheric and oceanic flows. The thin domain in the system (5.1.2) is considered to take into account this anisotropy between horizontal and vertical directions. Under this assumption, it is believed that the dynamics of fluids on large scale tend towards a geostrophic balance (see [64], [71] or [118]).

The purpose of this paper is to show the existence and uniqueness of solutions to (5.1.2) in the thin domain $\mathbb{R} \times (0, \epsilon)$, For some analytically small initial data in the tangential variable. To simplify our system we eliminate the τ -dependency. For that purpose, we perform the re-scaling

$$(5.1.4) \quad U^{(\tau, \epsilon)}(t, X) = \tau^\alpha U^\epsilon(\tau^\beta t, \frac{X}{\sqrt{\tau}}), \quad P^{(\tau, \epsilon)}(t, X) = \tau^{\alpha'} P^\epsilon(\tau^\beta t, \frac{X}{\sqrt{\tau}}).$$

We replace in System (5.1.2), we find that $\alpha = -\frac{1}{2}$, $\beta = -1$ and $\alpha' = -1$, then our re-scaling 5.1.4 have the following form

$$(5.1.5) \quad U^{(\tau, \epsilon)}(t, X) = \frac{1}{\sqrt{\tau}} U^\epsilon(\frac{t}{\tau}, \frac{X}{\sqrt{\tau}}), \quad P^{(\tau, \epsilon)}(t, X) = \frac{1}{\tau} P^\epsilon(\frac{t}{\tau}, \frac{X}{\sqrt{\tau}}).$$

where $X = (x, Y)$. This scaling transforms the τ -dependent equations (5.1.2) into the following system of equations with initial data which depend on τ :

$$(5.1.6) \quad \begin{cases} \partial_t^2 U^\epsilon + \partial_t U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \epsilon^2 \Delta U^\epsilon + \nabla P^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ \operatorname{div} U^{(\tau, \epsilon)} = 0, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ U^\epsilon|_{t=0} = \sqrt{\tau} U_0^{\tau, \epsilon}(\sqrt{\tau} X) = U_0^\epsilon, & \text{in } \mathbf{S}^\epsilon \\ \partial_t U^\epsilon|_{t=0} = \tau^{\frac{3}{2}} U_1^{\tau, \epsilon}(\sqrt{\tau} X) = U_1^\epsilon, & \text{in } \mathbf{S}^\epsilon \\ U^\epsilon|_{y=0} = U^\epsilon|_{y=1} = 0, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \end{cases}$$

In a formal way, as in [114] and [4], taking into account this anisotropy, we also consider the initial data of the following form,

$$U^\epsilon|_{t=0} = U_0^\epsilon = \left(u_0 \left(x, \frac{Y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{Y}{\epsilon} \right) \right),$$

and

$$U^\epsilon|_{t=1} = U_1^\epsilon = \left(u_1 \left(x, \frac{Y}{\epsilon} \right), \epsilon v_1 \left(x, \frac{Y}{\epsilon} \right) \right).$$

In this paper, we look for solutions in the form

$$(5.1.7) \quad \begin{cases} U^\epsilon(t, x, Y) = \left(u^\epsilon \left(t, x, \frac{Y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{Y}{\epsilon} \right) \right) \\ P^\epsilon(t, x, Y) = p^\epsilon \left(t, x, \frac{Y}{\epsilon} \right). \end{cases}$$

Let $\mathbf{S} := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$, we can rewrite the system (5.1.6) as follows

$$(5.1.8) \quad \begin{cases} \partial_t^2 u^\epsilon + \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon + \partial_x p^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \epsilon^2 (\partial_t^2 v^\epsilon + \partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) + \partial_y p^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ (u^\epsilon, v^\epsilon)|_{t=0} = (u_0, v_0) \quad \text{and} \quad \partial_t (u^\epsilon, v^\epsilon)|_{t=0} = (u_1, v_1), & \text{in } \mathbf{S} \\ (u^\epsilon, v^\epsilon)|_{y=0} = (u^\epsilon, v^\epsilon)|_{y=1} = 0. & \end{cases}$$

Formally taking $\epsilon \rightarrow 0$ in the system (5.1.8), we obtain the following perturbation hydrostatic Navier-Stokes equations,

$$(5.1.9) \quad \begin{cases} \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_y p = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x u + \partial_y v = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ u|_{t=0} = u_0, & \text{in } \mathbf{S} \\ \partial_t u|_{t=0} = u_1, & \text{in } \mathbf{S}, \end{cases}$$

where the velocity $U = (u, v)$ satisfy the Dirichlet no-slip boundary condition

$$(5.1.10) \quad (u, v)|_{y=0} = (u, v)|_{y=1} = 0.$$

Now let us state our main results.

The first result obtained in this paper is the global well-posedness of System (5.1.9) with small analytic data in the tangential variable. The global well-posedness and the global analyticity of the solutions to the classical 2-D perturbed hydrostatic Navier-Stokes system are well-known (see [108] for instance).

Theorem 5.1.1. *Let $a > 0$ and λ large enough. There exists a constant $c_0 > 0$ sufficiently small, such that, for any data (u_0, u_1) verifying the compatibility condition $\int_0^1 u_0 dy = 0$, and the smallness condition*

$$(5.1.11) \quad \left\| e^{a|D_x|} (u_0 + u_1) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} \partial_y u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} u_1 \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a < \frac{a}{\lambda},$$

then the system (5.1.9) has a unique global solution u satisfying the estimate $\forall t \geq 0$

$$(5.1.12) \quad \left\| e^{\mathcal{R}t} (u + \partial_t u)_\phi \right\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \left\| e^{\mathcal{R}t} \partial_y u_\phi \right\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \left\| e^{\mathcal{R}t} (\partial_t u)_\phi \right\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ + \left\| e^{\mathcal{R}t} (\partial_t u)_\phi \right\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \leq C \left(\left\| e^{a|D_x|} \partial_y u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} (u_0 + u_1) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} u_1 \right\|_{\mathcal{B}^{\frac{1}{2}}} \right),$$

where u_ϕ is given by (5.2.3), \mathcal{R} is a constant smaller than the constant that comes out of the Poincaré inequality for the domain \mathbf{S} (see (5.3.11)), and the functional spaces will be presented in Section 5.2.

The second result is the global well-posedness of the perturbed Navier-Stokes system (5.1.8) with small analytic data in the tangential variable.

Theorem 5.1.2. *Let $a > 0$, $\epsilon > 0$ and λ large enough. We assume that our initial data satisfy the following smallness condition*

$$(5.1.13) \quad \left\| e^{a|D_x|} \partial_y (u_0^\epsilon, \epsilon v_0^\epsilon) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \left\| e^{a|D_x|} \partial_x (u_0^\epsilon, \epsilon v_0^\epsilon) \right\|_{\mathcal{B}^{\frac{1}{2}}} \\ + \left\| e^{a|D_x|} (u_0^\epsilon + u_1^\epsilon, \epsilon (v_0^\epsilon + v_1^\epsilon)) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} (u_1^\epsilon, \epsilon v_1^\epsilon) \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a < \frac{a}{\lambda},$$

for some c_1 sufficiently small. Then System (5.1.8) has a unique global solution (u^ϵ, v^ϵ) , so that

$$\begin{aligned}
(5.1.14) \quad & \|e^{\mathcal{R}t}(u + \partial_t u, \epsilon(v^\epsilon + \partial_t v^\epsilon))_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y(u^\epsilon, \epsilon v^\epsilon)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \epsilon\|e^{\mathcal{R}t}\partial_x(u^\epsilon, \epsilon v^\epsilon)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \\
& + \|e^{\mathcal{R}t}(\partial_t u^\epsilon, \epsilon\partial_t v^\epsilon)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}(\partial_t u^\epsilon, \epsilon\partial_t v^\epsilon)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y(u^\epsilon, \epsilon v^\epsilon)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
& + \epsilon\|e^{\mathcal{R}t}\partial_x(u^\epsilon, \epsilon v^\epsilon)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \leq C \left(\|e^{a|D_x|}\partial_y(u_0^\epsilon, \epsilon v_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon\|e^{a|D_x|}\partial_x(u_0^\epsilon, \epsilon v_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} \right. \\
& \left. + \|e^{a|D_x|}(u_1^\epsilon, \epsilon v_1^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(u_0^\epsilon + u_1^\epsilon, \epsilon(v_0^\epsilon + v_1^\epsilon))\|_{\mathcal{B}^{\frac{1}{2}}} \right),
\end{aligned}$$

where $(u_\Theta^\epsilon, v_\Theta^\epsilon)$ is given by (5.5.1).

The main idea to prove the above two theorems is to control the new unknown u_ϕ defined by (5.2.3), where u is the horizontal velocity and u_ϕ is a weighted function of u in the dual Fourier variable with an exponential function of $(a - \lambda\theta(t))|\xi|$. By the classical Cauchy-Kovalevskaya theorem, one expects the radius of analyticity of the solutions to decay in time and so the exponent, which corresponds to the width of the analytical band, is allowed to vary with time. Using energy estimates on the equation satisfied by u_ϕ and the control of the quantity which describes 'the loss of the analytical radius', we shall show that the analytical band persists globally in time. Consequently, our result is a global Cauchy-Kovalevskaya type theorem.

The third result concerns the study of the convergence from the scaled anisotropic perturbed Navier-Stokes system (5.1.8) to the limit system (5.1.9), so in this theorem, we proved that the convergence occurs globally in time.

Theorem 5.1.3. *Let $a > 0$ and $\epsilon > 0$, $\exists M > 0$, and $(u_0^\epsilon, v_0^\epsilon)$ satisfying (5.1.13). Let (u_0, u_1) satisfy $e^{a|D_x|}(u_0, u_1) \in (\mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{7}{2}})^2$, $e^{a|D_x|}\partial_y(u_0, u_1) \in (\mathcal{B}^{\frac{3}{2}})^2$, and the compatibility condition $\int_0^1 u_0 dy = 0$ and*

$$\begin{aligned}
(5.1.15) \quad & \left\| e^{a|D_x|}(u_0 + u_1) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|}\partial_y u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{1}{2}}} \\
& \leq \frac{c_2 a}{2 + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{3}{2}}}},
\end{aligned}$$

for some c_2 sufficiently small, then we have

$$\begin{aligned}
(5.1.16) \quad & \left(\|(R^1 + \partial_t R^1, \epsilon(R^2 + \partial_t R^2))_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \epsilon\|\partial_x(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \right. \\
& \left. + \|(\partial_t R^1)_\varphi, \epsilon(\partial_t R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \right) + \|(\partial_t R^1, \epsilon\partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
& + \epsilon\|\partial_x(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \leq C \left(\|e^{a|D_x|}((u_1^\epsilon - u_1), \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}} \right. \\
& + \|e^{a|D_x|}\partial_y(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon\|e^{a|D_x|}\partial_x(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} \\
& \left. + \|e^{a|D_x|}((u_0^\epsilon - u_0) + (u_1^\epsilon - u_1), \epsilon(v_0^\epsilon - v_0) + \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}} + M\epsilon \right).
\end{aligned}$$

where

$$(5.1.17) \quad \begin{cases} R^1 = u^\epsilon - u, \\ R^2 = v^\epsilon - v, \end{cases}$$

and v_0 is determined from u_0 via $\partial_x u + \partial_y v = 0$ and $v_0|_{y=0} = v_0|_{y=1} = 0$, $(R_\varphi^1, \epsilon R_\varphi^2)$ is given by (5.6.6).

We remark that without the smallness conditions (5.1.13) and (5.1.15), we can not prove the convergence from the solutions to System (5.1.8) to the solutions System (5.1.9) on a fixed time interval $[0, t]$

for $t < T^*$, where T^* is the lifetime of the solution of the hydrostatic perturbed Navier-Stokes equation with the large initial data u_0 .

Remarque 5.1.1. *The main idea to prove the above theorem is to use analytic energy estimates, which are motivated by [112] and which originates from [27].*

Organisation of the paper: Our paper will be divided into several sections as follows. In section 5.2, we present some basic notions of the Littlewood-Paley Theory and some technical lemmas. In Section 5.3, we prove the global wellposedness of System (5.1.9) for small data in the analytic framework. Section 5.5 is devoted to the study of System (5.1.8) and the proof of Theorem 5.1.2. In section 5.4 we present some proposition stating the propagation for any \mathcal{B}^s regularity. In Section 5.6, we prove the convergence of System (5.1.8) towards System (5.1.9) when $\epsilon \rightarrow 0$. Finally, in the last section, we give the proofs of some technical estimates.

We end this introduction by the notations that will be used in all that follows. By $f \lesssim g$, we mean that there is a uniform constant C , which may be different from line to line, such that $f \leq Cg$. We denote by $\langle f, g \rangle_{L^2}$ the inner product of f and g in $L^2(\mathbf{S})$. Finally, we denote by $(d_q)_{q \in \mathbb{Z}}$ (resp. $(d_q(t))_{q \in \mathbb{Z}}$) to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{q \in \mathbb{Z}} d_q = 1$ (resp. $\sum_{q \in \mathbb{Z}} d_q(t) = 1$).

5.2 Littlewood-Paley Theory and Some technical lemmas

5.2.1 Littlewood-Paley Theory

To introduce the result of this paper, we will recall some elements of the Littlewood-Paley theory and also introduce the function space and technique used in the proof of our result. We define the dyadic operator in the horizontal variable, (of x variable) and for all $q \in \mathbb{Z}$, we recall from [15] that

$$\begin{aligned}\Delta_q^h f(x, y) &= \mathcal{F}_h^{-1} \left(\varphi(2^{-q} |\xi|) \widehat{f}(\xi, y) \right), \\ S_q^h f(x, y) &= \mathcal{F}_h^{-1} \left(\psi(2^{-q} |\xi|) \widehat{f}(\xi, y) \right).\end{aligned}$$

where ψ and φ are a smooth function such that

$$\begin{aligned}\text{supp } \varphi &\subset \{z \in \mathbb{R} / \frac{3}{4} \leq |z| \leq \frac{8}{3}\} \quad \text{and} \quad \forall z \neq 0, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q} z) = 1, \\ \text{supp } \psi &\subset \{z \in \mathbb{R} / |z| \leq \frac{4}{3}\} \quad \text{and} \quad \psi(z) + \sum_{q \geq 0} \varphi(2^{-q} z) = 1,\end{aligned}$$

and

$$\forall q, q' \in \mathbb{Z}, |q - q'| \geq 2, \quad \text{supp } \varphi(2^{-q} \cdot) \cap \text{supp } \varphi(2^{-q'} \cdot) = \emptyset.$$

And in all that follows, $\mathcal{F}_h f$ and \widehat{f} always denote the partial Fourier transform of the distribution f with respect to the horizontal variable (of x variable), that is, $\widehat{f}(\xi, y) = \mathcal{F}_{x \rightarrow \xi}(f)(\xi, y) = \mathcal{F}_h(f)(\xi, y)$. We refer to [15] and [18] for a more detailed construction of the dyadic decomposition. Combining the definition of the dyadic operator to the fact that

$$(5.2.1) \quad \forall z \in \mathbb{R}, \quad \psi(z) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} z) = 1,$$

implies that all tempered distributions can be decomposed with respect to the horizontal frequencies as

$$f = \sum_{q \in \mathbb{Z}} \Delta_q^h f.$$

We now introduce the function spaces used throughout the paper. As in [114], we define the anisotropic Besov-type spaces \mathcal{B}^s , $s \in \mathbb{R}$ as follows.

Definition 5.2.1. *Let $s \in \mathbb{R}$ and $\mathbf{S} = \mathbb{R} \times]0, 1[$. For all tempered distributions $u \in \mathcal{S}'_h(\mathbf{S})$, i.e., f belongs to $\mathcal{S}'(\mathbf{S})$ and $\lim_{q \rightarrow -\infty} \|S_q f\|_{L^\infty} = 0$, we set*

$$\|f\|_{\mathcal{B}^{s,0}} \triangleq \|(2^{qs} \|\Delta_q^h f\|_{L^2})_{q \in \mathbb{Z}}\|_{\ell^1(\mathbb{Z})}.$$

(i) For $s \leq \frac{1}{2}$, we define

$$\mathcal{B}^{s,0}(\mathbf{S}) \triangleq \{f \in \mathcal{S}'_h(\mathbf{S}) : \|f\|_{\mathcal{B}^{s,0}} < +\infty\}.$$

(ii) For $s \in]k - \frac{1}{2}, k + \frac{1}{2}]$, with $k \in \mathbb{N}^*$, we define $\mathcal{B}^{s,0}(\mathbf{S})$ as the subset of distributions f in $\mathcal{S}'_h(\mathbf{S})$ such that $\partial_x^k f \in \mathcal{B}^{s-k,0}(\mathbf{S})$.

For a better use of the smoothing effect given by the diffusion terms, we will work in the following Chemin-Lerner type spaces and also the time-weighted Chemin-Lerner type spaces.

Definition 5.2.2. *Let $p \in [1, +\infty]$ and $T \in]0, +\infty]$. Then, the space $\tilde{L}_T^p(\mathcal{B}^{s,0}(\mathbf{S}))$ is the closure of $C([0, T]; \mathcal{S}(\mathbf{S}))$ under the norm*

$$\|f\|_{\tilde{L}_T^p(\mathcal{B}^{s,0}(\mathbf{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \|\Delta_q^h f(t)\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

with the usual change if $p = +\infty$.

Definition 5.2.3. *Let $p \in [1, +\infty]$ and let $\delta \in L_{loc}^1(\mathbb{R}_+)$ be a non negative function. Then, the space $\tilde{L}_{t,\delta(t)}^p(\mathcal{B}^{s,0}(\mathbf{S}))$ is the closure of $C([0, T]; \mathcal{S}(\mathbf{S}))$ under the norm*

$$\|f\|_{\tilde{L}_{t,\delta(t)}^p(\mathcal{B}^{s,0}(\mathbf{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^t \delta(t') \|\Delta_q^h f(t')\|_{L^2}^p dt' \right)^{\frac{1}{p}}.$$

Remark 5.2.1. *To simplify the notation even further, in all that follows, we shall denote*

$$B^{s,0} \triangleq B^s.$$

The following Bernstein lemma gives important properties of a distribution u when its Fourier transform is well localized. We refer the reader to [26] for the proof of this lemma.

Lemma 5.2.1. *Let $k \in \mathbb{N}$, $d \in \mathbb{N}^*$ and $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. There exists a constant $C > 0$ such that, for any $a, b \in \mathbb{R}$, $1 \leq a \leq b \leq +\infty$, for any $\lambda > 0$ and for any $f \in L^a(\mathbb{R}^d)$, we have*

$$\text{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq r_1 \lambda\} \implies \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a},$$

and

$$\text{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^d \mid r_1 \lambda \leq |\xi| \leq r_2 \lambda\} \implies C^{-k} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^k \lambda^k \|f\|_{L^a}.$$

Finally to deal with the estimate concerning the product of two distributions, we shall frequently use on Bony's decomposition (see [18]) in the horizontal variable (x variable) that for f, g two tempered distributions :

$$(5.2.2) \quad fg = T_f^h g + T_g^h f + R^h(f, g),$$

where

$$T_f^h g = \sum_q S_{q-1}^h f \Delta_q^h g, \quad T_g^h f = \sum_q S_{q-1}^h g \Delta_q^h f$$

and the remainder term satisfies

$$R^h(f, g) = \sum_q \tilde{\Delta}_q^h f \Delta_q^h g \quad \text{with} \quad \tilde{\Delta}_q^h f = \sum_{|q-q'| \leq 1} \Delta_{q'}^h f.$$

5.2.2 Technical lemmas

Our main difficulty relies on finding a way to estimate the nonlinear terms, which allows exploiting the smoothing effect given by the above function spaces. Using the method introduced by Chemin in [27] (see also [30], [110] or [114]), for any $f \in L^2(\mathbf{S})$, we define the following auxiliary function, which allows to control the analyticity of u in the horizontal variable x ,

$$(5.2.3) \quad \begin{cases} f_\phi(t, x, y) = e^{\phi(t, D_x)} f(t, x, y) \triangleq \mathcal{F}_h^{-1}(e^{\phi(t, \xi)} \mathcal{F}_h(f)(t, \xi, y)) \\ \phi(t, \xi) = (a - \lambda\theta(t))|\xi|. \end{cases}$$

where the quantity $\theta(t)$ is the analyticity radius, which describes the evolution of the analytic band of f , satisfies

$$(5.2.4) \quad \forall t > 0, \quad \dot{\theta}(t) \geq 0 \quad \text{and} \quad \theta(0) = 0.$$

Remark 5.2.2. *In the proofs we need to choose that $\theta(t)$ is satisfying an ordinary differential equation*

$$\dot{\theta}(t) = \|e^{(a-\lambda\theta(t))|D_x|} \partial_y f\|_{\mathcal{B}^{\frac{1}{2}}}$$

The local existence of $\theta(t)$ of t between $[0, T[$ is given by the Cauchy-Lipschitz theorem. In all our computation we will be in time interval $[0, \tilde{T}^[$ where \tilde{T}^* is maximal lifespan of the solution.*

In what follows, we shall always assume that $t < T^*$, with T^* is determined by

$$(5.2.5) \quad T^* \triangleq \sup \left\{ T > 0 : \forall t \in [0, T) : \|f_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{4C^2} \quad \text{and} \quad \theta(t) \leq \frac{a}{\lambda} \right\}.$$

By virtue of (5.2.3) for any $t < T^*$, there holds the following convex inequality

$$(5.2.6) \quad \phi(t, \xi) \leq \phi(t, \xi - \eta) + \phi(t, \eta) \quad \forall \xi, \eta \in \mathbb{R}.$$

Remark 5.2.3. *To obtain a solution of our system, we use the Friedrich scheme which allows us to build a sequence of solutions, and then we go to the limit on the sequence of solutions. Indeed, for that the quantity $\theta^n(t)$ satisfying the following ordinary differential equation :*

$$\dot{\theta}^n(t) = \|e^{(a-\lambda\theta^n(t))|D_x|} \partial_y J_n u\|_{\mathcal{B}^{\frac{1}{2}}},$$

with

$$J_n u = \sum_{|k| < n} \mathbf{1}_{\{|\xi| < n\}} \sin(k\pi y) \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{u}^k(t, \xi) d\xi,$$

define the frequency cut-off operators in the **all variables**, where $u(t, x, y) = \sum_n \sin(n\pi y) u^n(t, x)$ and

$u^n(t, x) = \frac{1}{2\pi} \int_0^1 u(t, x, y) \sin(n\pi y) dy$. Let consider

$$X^n = \begin{pmatrix} \theta^n(t) \\ u^n \\ \partial_t u^n \end{pmatrix},$$

then X^n verify a differential equation in time on Banach space $\mathbb{R} \times B^{1/2,0} \times B^{1/2,0}$, we obtain the following equation:

$$(5.2.7) \quad \begin{cases} \partial_t X^n + J_n A(D) X^n = - \begin{pmatrix} \|e^{(a-\lambda\theta^n(t))|D_x|} \partial_y J_n u^n\|_{\mathcal{B}^{\frac{1}{2}}} \\ 0 \\ J_n (J_n u^n \partial_x J_n u^n)_\phi + J_n (J_n v^n \partial_y J_n u^n)_\phi + \partial_x J_n p_\phi^n \end{pmatrix} \\ \partial_x u_\phi^n + \partial_y v_\phi^n = 0 \end{cases}$$

where

$$A(D) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -\partial_y^2 & 1 \end{pmatrix}$$

which is an ordinary differential equation. The solution is obtained on the interval $[0, T_n^*[$ by Cauchy-Lipschitz. Since we have that $J_n^2 = J_n$, we can deduce that $J_n X^n$ is also a solution (5.2.7). The uniqueness of the solutions implies that $X^n = J_n X^n$. So we can write

$$(5.2.8) \quad \begin{cases} \partial_t X^n + J_n A(D) X_\phi^n = - \begin{pmatrix} \|e^{(a-\lambda\theta^n(t))|D_x|} \partial_y J_n u\|_{\mathcal{B}^{\frac{1}{2}}} \\ 0 \\ J_n (u^n \partial_x u^n)_\phi + J_n (v^n \partial_y u^n)_\phi + \partial_x p_\phi^n \end{pmatrix} \\ \partial_x u_\phi^n + \partial_y v_\phi^n = 0 \end{cases}$$

We note that all analytical estimates obtained in all sections 5.3-5.6 are valid also for this equation. So, (u^n) will have a convergent sub-sequence that we still denote by (u^n) , then we have that (u^n) converge to u in $L_{loc}^\infty(\mathbb{R}^+, H^{-s})$ for all $s \in]0, 1[$, which u is the solution of our final system.

Before stating the obtained result, we need the following lemma to characterize the product $(fg)_\phi$, indeed this product will be useful in all the rest of the paper.

Corollary 5.2.1. Let $f \in L_x^2$, $g \in L_x^2$, we define $f^+ = \mathcal{F}_\xi^{-1}(|\mathcal{F}_x(f)|)$ then, we have

$$|\widehat{(fg)_\phi}(\xi)| \leq \widehat{f_\phi^+ g_\phi^+}(\xi) \quad \text{and} \quad \|f^+\|_{L_x^2} = \|f\|_{L_x^2}$$

Proof. Let us consider f , and g two functions in L_x^2 , we have

$$\begin{aligned} |\widehat{(fg)_\phi}(\xi)| &= e^{\phi(\xi)} |\widehat{f}(\cdot) * \widehat{g}(\cdot)(\xi)| \\ &\leq e^{\phi(\xi)} \int |\widehat{f}(\xi - \eta)| |\widehat{g}(\eta)| d\eta, \end{aligned}$$

By virtue of the definition of Function ϕ we have $e^{\phi(\xi)} > 0$ and $e^{\phi(\xi)} \leq e^{\phi(\xi-\eta)} e^{\phi(\eta)}$, thus

$$\begin{aligned} |\widehat{(fg)_\phi}(\xi)| &\leq \int e^{\phi(\xi-\eta)} |\widehat{f}(\xi - \eta)| e^{\phi(\eta)} |\widehat{g}(\eta)| d\eta \\ &\leq \int |\widehat{f}_\phi(\xi - \eta)| |\widehat{g}_\phi(\eta)| d\eta \\ &\leq |\widehat{f}_\phi * \widehat{g}_\phi|(\xi) = \mathcal{F}_\xi(f_\phi^+ g_\phi^+) = \widehat{f_\phi^+ g_\phi^+}(\xi) \end{aligned}$$

The second point of the lemma is trivial. □

Lemma 5.2.2. *For smooth functions we have*

$$\|(fg)_\phi\|_{L^2} \leq \|(f_\phi^+)\|_{L^\infty} \|(g_\phi^+)\|_{L^2}.$$

Proof. Let us consider f , and g two functions in L_x^2 , we have by using Plancherel theorem that

$$\begin{aligned} \|(fg)_\phi\|_{L^2}^2 &= \int |\mathcal{F}_\xi(fg)_\phi(\xi, y)|^2 d\xi \\ &\leq \int \left| \int e^{\phi(\xi)} \mathcal{F}_\xi(f)(\xi - \eta) \mathcal{F}_\xi(g)(\eta) d\eta \right|^2 d\xi \\ &\leq \int \int e^{2\phi(\xi)} |\mathcal{F}_\xi(f)(\xi - \eta)|^2 |\mathcal{F}_\xi(g)(\eta)|^2 d\eta d\xi. \end{aligned}$$

We recall that the function ϕ verify the following inequality $e^{\phi(\xi)} \leq e^{\phi(\xi-\eta)} e^{\phi(\eta)}$, thus

$$\begin{aligned} \|(fg)_\phi\|_{L^2}^2 &\leq \|f_\phi^+ g_\phi^+\|_{L^2}^2 \\ &\leq \|(f_\phi^+)\|_{L^\infty}^2 \|(g_\phi^+)\|_{L^2}^2, \end{aligned}$$

□

Corollary 5.2.2. *For any f and g in L_x^2 , we have*

$$|\widehat{(Tfg)_\phi}(\xi)| \leq \widehat{(T_{f^+} g_\phi^+)}(\xi) \quad \text{and} \quad |\widehat{R(f, g)_\phi}(\xi)| \leq \widehat{R(f_\phi^+, g_\phi^+)}(\xi).$$

We next present the weighted energy estimate for the linear heat equations

Lemma 5.2.3. *Let f and g two smooth enough functions on $\mathbb{R} \times (0, 1)$, satisfying the Dirichlet boundary condition 5.1.10, then we have*

1.

$$(5.2.9) \quad \begin{aligned} \frac{d}{dt} \langle \Delta_q^h f_\phi, \Delta_q^h g_\phi \rangle_{L^2} &= \langle \Delta_q^h (\partial_t f)_\phi, \Delta_q^h g_\phi \rangle + \langle \Delta_q^h f_\phi, \Delta_q^h (\partial_t g)_\phi \rangle \\ &\quad - 2\lambda \dot{\theta}(t) \langle \Delta_q^h |D_x|^{\frac{1}{2}} f_\phi, \Delta_q^h |D_x|^{\frac{1}{2}} g_\phi \rangle, \end{aligned}$$

In particular if $f = g$, we obtain that

$$(5.2.10) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q^h f_\phi\|_{L^2}^2 = \langle \Delta_q^h (\partial_t f)_\phi, \Delta_q^h f_\phi \rangle_{L^2} - \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} f_\phi\|_{L^2}^2,$$

2.

$$(5.2.11) \quad \begin{aligned} \langle \Delta_q^h (\partial_t^2 f)_\phi, \Delta_q^h (\partial_t f)_\phi \rangle_{L^2} &= \frac{1}{2} \frac{d}{dt} \|\Delta_q^h (\partial_t f)_\phi\|_{L^2}^2 \\ &\quad + \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t f)_\phi\|_{L^2}^2, \end{aligned}$$

3.

$$(5.2.12) \quad \begin{aligned} - \langle \Delta_q^h \partial_y^2 f_\phi, \Delta_q^h (\partial_t f)_\phi \rangle_{L^2} &= \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y f_\phi\|_{L^2}^2 \\ &\quad + \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y f_\phi\|_{L^2}^2, \end{aligned}$$

Proof. To prove the first assertion, we apply the rules of the derivation of a product, we obtain

$$\begin{aligned}
\frac{d}{dt} \langle \Delta_q^h f_\phi, \Delta_q^h g_\phi \rangle_{L^2} &= \langle \Delta_q^h \partial_t f_\phi, \Delta_q^h g_\phi \rangle_{L^2} + \langle \Delta_q^h f_\phi, \Delta_q^h \partial_t g_\phi \rangle_{L^2} \\
&= \langle \Delta_q^h (\partial_t f)_\phi, \Delta_q^h g_\phi \rangle_{L^2} - \lambda \dot{\theta}(t) \langle \Delta_q^h |D_x| f_\phi, \Delta_q^h g_\phi \rangle_{L^2} \\
&+ \langle \Delta_q^h f_\phi, \Delta_q^h (\partial_t g)_\phi \rangle_{L^2} - \lambda \dot{\theta}(t) \langle \Delta_q^h f_\phi, \Delta_q^h |D_x| g_\phi \rangle_{L^2} \\
&= \langle \Delta_q^h (\partial_t f)_\phi, \Delta_q^h g_\phi \rangle_{L^2} + \langle \Delta_q^h f_\phi, \Delta_q^h (\partial_t g)_\phi \rangle_{L^2} - 2\lambda \dot{\theta}(t) \langle \Delta_q^h |D_x|^{\frac{1}{2}} f_\phi, \Delta_q^h |D_x|^{\frac{1}{2}} g_\phi \rangle_{L^2} \quad \square
\end{aligned}$$

By using the rules of the derivation of a product and Parseval equality, we find

$$\begin{aligned}
&\langle \Delta_q^h (\partial_t^2 f)_\phi, \Delta_q^h (\partial_t f)_\phi \rangle_{L^2} \\
&= \int \Delta_q^h (e^{\phi(t, |D_x|)} \partial_t^2 f) \overline{\Delta_q^h (e^{\phi(t, |D_x|)} \partial_t f)} dx \\
&= \int \Delta_q^h (e^{\widehat{\phi(t, |D_x|)}} \partial_t^2 f) \overline{\Delta_q^h (e^{\widehat{\phi(t, |D_x|)}} \partial_t f)} d\xi_h \\
&= \frac{1}{2} \frac{d}{dt} \int |\Delta_q^h (e^{\widehat{\phi(t, |D_x|)}} \partial_t f)|^2 d\xi_h \\
&\quad + \frac{1}{2} \int 2\lambda \dot{\theta}(t) \varphi(2^{-q}\xi) |\xi_h| e^{\phi(t, |\xi_h|)} \partial_t \widehat{f}(\xi) \overline{\varphi(2^{-q}\xi) e^{\phi(t, |\xi_h|)} \partial_t \widehat{f}(\xi)} d\xi_h \\
&= \frac{1}{2} \frac{d}{dt} \|\Delta_q^h (\partial_t f)_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \int |\Delta_q^h |D_x|^{\frac{1}{2}} (e^{\widehat{\phi(t, |D_x|)}} \partial_t f)|^2 d\xi_h \\
&= \frac{1}{2} \frac{d}{dt} \|\Delta_q^h (\partial_t f)_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t f)_\phi\|_{L^2}^2.
\end{aligned}$$

By using integration by parts and replacing in (5.2.10) by $\partial_y f$, we can find the estimate

$$\begin{aligned}
\langle \Delta_q^h (-\partial_y^2 f)_\phi, \Delta_q^h (\partial_t f)_\phi \rangle_{L^2} &= \langle \Delta_q^h (\partial_y f)_\phi, \Delta_q^h (\partial_t \partial_y f)_\phi \rangle_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y f_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y f_\phi\|_{L^2}^2.
\end{aligned}$$

Note that here we use the fact that

$$\Delta_q^h (\partial_x^\alpha f_\phi) = \partial_x^\alpha (\Delta_q^h f)_\phi.$$

Before starting the proof of the mains lemmas we recall from Plancherel's formula and Fubini's theorem the following inequality.

Corollary 5.2.3. *Let f, g and h be smooth functions on $\mathbb{R} \times (0, 1)$, we have*

$$\|\Delta_q^h f_\phi\|_{L^2} \leq C \|f_\phi\|_{L^2}$$

and

$$\begin{aligned}
&|\langle \Delta_q^h (fg)_\phi, \Delta_q^h h_\phi \rangle_{L^2}| \leq \sum_{|q-q'|\leq 4} |\langle \Delta_q^h (S_{q'-1} f \Delta_{q'} g)_\phi, \Delta_q^h h_\phi \rangle_{L^2}| \\
&+ \sum_{|q-q'|\leq 4} |\langle \Delta_q^h (S_{q'-1} g \Delta_{q'} f)_\phi, \Delta_q^h h_\phi \rangle_{L^2}| + \sum_{q'\geq q-3} |\langle \Delta_q^h (\tilde{\Delta}_{q'} f \Delta_{q'} g)_\phi, \Delta_q^h h_\phi \rangle_{L^2}|,
\end{aligned}$$

as well as

$$|\langle \Delta_q^h (S_{q'-1} f \Delta_{q'} g)_\phi, \Delta_q^h h_\phi \rangle_{L^2}| \leq \begin{cases} C \|S_{q'-1} f_\phi^+\|_{L^\infty} \|\Delta_{q'} g_\phi^+\|_{L^2} \|\Delta_q^h h_\phi\|_{L^2} \\ C \|S_{q'-1} f_\phi^+\|_{L_h^\infty(L_v^2)} \|\Delta_{q'} g_\phi^+\|_{L_h^2(L_v^\infty)} \|\Delta_q^h h_\phi\|_{L^2}, \end{cases}$$

and

$$\left| \left\langle \Delta_q^h (\tilde{\Delta}_{q'} f \Delta_{q'} g)_\phi, \Delta_q^h h_\phi \right\rangle_{L^2} \right| \leq \begin{cases} C \|\tilde{\Delta}_{q'} f_\phi^+\|_{L^\infty} \|\Delta_{q'} g_\phi^+\|_{L^2} \|\Delta_q^h h_\phi\|_{L^2} \\ C \|\tilde{\Delta}_{q'} f_\phi^+\|_{L_h^\infty(L_v^2)} \|\Delta_{q'} g_\phi^+\|_{L_h^2(L_v^\infty)} \|\Delta_q^h h_\phi\|_{L^2} \\ C \|\tilde{\Delta}_{q'} f_\phi^+\|_{L_h^2(L_v^\infty)} \|\Delta_{q'} g_\phi^+\|_{L^2} \|\Delta_q^h h_\phi\|_{L_h^\infty(L_v^2)}. \end{cases}$$

Remark 5.2.4. Using this corollary, can assume from now without losing generality, that $\hat{f} \geq 0$.

Lemma 5.2.4. Let f be smooth function on $\mathbb{R} \times (0, 1)$. If $f = 0$ when $y = 0$ then we have

$$(5.2.13) \quad \|f\|_{L_v^\infty} \leq C \|f\|_{L_v^2}^{\frac{1}{2}} \|\partial_y f\|_{L_v^2}^{\frac{1}{2}}.$$

Proof. Let f be a smooth function on $\mathbb{R} \times (0, 1)$, while $f = 0$ on $y = 0$ then we can write

$$(f(x, y))^2 = \int_0^y \partial_y (f(x, y')^2) dy' = 2 \int_0^y f(x, y') \partial_y f(x, y') dy',$$

by using Cauchy-Schwarz inequality, we obtain

$$(f(x, y))^2 \leq 2 \|f\|_{L^2} \|\partial_y f\|_{L^2}.$$

Then,

$$\|f(x, \cdot)\|_{L_v^\infty} \leq \sqrt{2} \|f\|_{L_v^2}^{\frac{1}{2}} \|\partial_y f\|_{L_v^2}^{\frac{1}{2}},$$

□

We also apply the following Poincaré inequality for a smooth function f vanishing on the boundary (for $y = 0, 1$)

$$\|f\|_{L_v^\infty} \leq \|\partial_y f\|_{L_v^2} \quad \text{and} \quad \|f\|_{L_v^2} \leq \|\partial_y f\|_{L_v^2}.$$

Now, let us state our lemmas, which we use to prove our results.

Lemma 5.2.5. Let $s \in (0, 1)$, and A, B and C be smooth enough functions on $[0, T^*) \times \mathbb{R} \times (0, 1)$, with A vanishing on the boundary, let ϕ be defined as in (5.2.3) with $\theta(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$. For any $t \in [0, T^*[$ and for any $B, C \in \tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$, we have

$$(5.2.14) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (A \partial_x B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

with $d_q \in \ell^1(\mathbb{Z})$, $\sum_{q \in \mathbb{Z}} d_q \leq 1$ and where \mathcal{R} is the Poincaré constant.

Remark 5.2.5. We can note that $\theta(t)$ verifying an ordinary differential equation

$$\dot{\theta}(t) = \|e^{(a-\lambda\theta(t))|D_x|} \partial_y A\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|e^{(a-\lambda\theta(t))|D_x|} \partial_y B\|_{\mathcal{B}^{\frac{1}{2}}}$$

The local existence of $\theta(t)$ of t between $[0, \tilde{T}^*[$ is given by Cauchy Lipschitz.

Proof. As in [15], using Bony's homogeneous decomposition of $A \partial_x B$ into para-products in the horizontal variable, we can write

$$A \partial_x B = T_A^h \partial_x B + T_{\partial_x B}^h A + R^h(A, \partial_x B)$$

where,

$$T_A^h \partial_x B = \sum_{q \in \mathbb{Z}} S_{q-1}^h A \Delta_q^h \partial_x B \quad \text{and} \quad R^h(A, \partial_x B) = \sum_{|q'-q| \leq 1} \Delta_q^h A \Delta_{q'}^h \partial_x B = \sum_q \tilde{\Delta}_q^h A \cdot \Delta_q^h \partial_x B.$$

We have the following bound

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (A \partial_x B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \leq M_{1,q} + M_{2,q} + M_{3,q},$$

where

$$\begin{aligned} M_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_A^h \partial_x B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ M_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x B}^h A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ M_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(A, \partial_x B))_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

We start by getting the estimate of the first term $M_{1,q}$, for that we need to use the definition of $T_A^h \partial_x B$ and the support properties given in [18, Proposition 2.10] and Corollary 5.2.2. We infer

$$(5.2.15) \quad M_{1,q} \leq \sum_{|q-q'| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h A_\phi^+(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_x B_\phi^+(t')\|_{L^2} \|\Delta_q^h C_\phi(t')\|_{L^2} dt'.$$

Using Bernstein Lemma 5.2.1 we have

$$\|\Delta_{q'}^h A_\phi^+\|_{L_h^\infty} \lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h A_\phi^+\|_{L_h^2}.$$

Taking the L_v^∞ norm and using that $\|f\|_{L_v^p L_h^q} \leq \|f\|_{L_h^q L_v^p}$, for $q \geq p \geq 1$, we obtain

$$\|\Delta_{q'}^h A_\phi^+\|_{L^\infty} \lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h A_\phi^+\|_{L_v^\infty L_h^2} \lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h A_\phi\|_{L_h^2 L_v^\infty}.$$

While using the inclusion $H_y^1 \hookrightarrow L_y^\infty$,

$$\|\Delta_{q'}^h A_\phi\|_{L_v^\infty} \lesssim \|\Delta_{q'}^h A_\phi\|_{L_v^2}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y A_\phi\|_{L_v^2}^{\frac{1}{2}},$$

and the Poincaré inequality on the interval $\{0 < y < 1\}$ on A (as we have that $A = 0$ when $y = 0, 1$)

$$\|\Delta_{q'}^h A_\phi\|_{L^2} \lesssim \|\Delta_{q'}^h \partial_y A_\phi\|_{L^2},$$

we obtain

$$\begin{aligned} \|\Delta_{q'}^h A_\phi(t')\|_{L^\infty} &\lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h A_\phi(t')\|_{L_h^2(L_v^\infty)} \\ &\lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h A_\phi(t')\|_{L^2}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y A_\phi(t')\|_{L^2}^{\frac{1}{2}} \\ (5.2.16) \quad &\lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h \partial_y A_\phi(t')\|_{L^2} \lesssim d_q(A_\phi) \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \\ (5.2.17) \quad &\lesssim d_q(A_\phi) \dot{\theta}(t). \end{aligned}$$

Here and in all that follows, we always denote $(d_q(A_\phi))_{q \in \mathbb{Z}}$ to be a generic element of $\ell^1(\mathbb{Z})$ so that

$\sum_{q \in \mathbb{Z}} d_q(A_\phi) \leq 1$. Then,

$$\|S_{q'-1}^h A_\phi(t')\|_{L^\infty} \lesssim \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

and replacing this result in our estimate (5.2.15), and combining with Hölder's inequality, imply that

$$\begin{aligned} M_{1,q} &\lesssim \sum_{|q-q'|\leq 4} \int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_{q'}^h \partial_x B_\phi(t')\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} e^{\mathcal{R}t'} \|\Delta_{q'}^h B_\phi(t')\|_{L^2} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

We recall that $\dot{\theta}(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$, and using Definition 5.2.3, we have

$$\begin{aligned} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &= \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q(s+\frac{1}{2})} d_q(C_\phi) \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Then,

$$(5.2.18) \quad M_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(B_\phi) 2^{(q-q')(s-\frac{1}{2})} \right).$$

Similarly, by Lemma 5.2.1 and considering the support properties to the Fourier transform given in [18, Proposition 2.10] of the terms in $T_{\partial_x B}^h A$, we obtain

$$\begin{aligned} M_{2,q}(t) &\lesssim \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x B}^h A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ &\lesssim \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x B_\phi^+\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h A_\phi^+\|_{L_h^2(L_v^\infty)} \|\Delta_q^h C_\phi\|_{L^2} dt'. \end{aligned}$$

As in (5.2.16), we can write

$$\|\Delta_{q'}^h A_\phi^+\|_{L_h^2(L_v^\infty)} \lesssim \|\Delta_{q'}^h \partial_y A_\phi\|_{L^2} \lesssim 2^{-\frac{q'}{2}} d_{q'}(A_\phi) \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

and using Bernstein's inequality we have

$$\|S_{q'-1}^h \partial_x B_\phi^+(t')\|_{L_h^\infty(L_v^2)} \lesssim \sum_{l \leq q'-2} 2^l 2^{\frac{l}{2}} \|\Delta_l^h B_\phi\|_{L^2}.$$

By using Hölder's inequality, we have

$$\begin{aligned}
M_{2,q} &\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{-\frac{q'}{2}} d_{q'}(A_\phi) e^{\mathcal{R}t'} \|S_{q'-1}^h \partial_x B_\phi(t')\|_{L_h^\infty(L_v^2)} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\
&\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{-\frac{q'}{2}} e^{\mathcal{R}t'} \sum_{l\leq q'-2} 2^l 2^{\frac{l}{2}} \|\Delta_l^h B_\phi\|_{L^2} \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2} dt' \\
&\lesssim \sum_{|q-q'|\leq 4} 2^{-\frac{q'}{2}} \sum_{l\leq q'-2} \left(\int_0^t e^{2\mathcal{R}t'} 2^{3l} \|\Delta_l^h B_\phi\|_{L^2}^2 \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^t e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}}.
\end{aligned}$$

Yet we observe from Definition 5.2.3, and $s < 1$ we have

$$\begin{aligned}
&\sum_{l\lesssim q'-2} \left(\int_0^t 2^{3l} e^{2\mathcal{R}t'} \|\Delta_l^h B_\phi\|_{L^2}^2 \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{l\lesssim q'-2} 2^{\frac{3l}{2}} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_l^h B_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{l\lesssim q'-2} 2^{\frac{3l}{2}} 2^{-l(s+\frac{1}{2})} 2^{l(s+\frac{1}{2})} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_l^h B_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim 2^{q'(1-s)} \sum_{l\lesssim q'-2} \tilde{d}_l(B_\phi) 2^{(q'-l)(s-1)} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
(5.2.19) \quad &\lesssim 2^{q'(1-s)} d_{q'}(B_\phi) \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

where

$$d_{q'}(B_\phi) = \sum_{l\leq q'-2} 2^{-(q'-l)(1-s)} \tilde{d}_l(B_\phi) \in \ell^1(\mathbb{Z}).$$

We remark that $d_{q'}(B_\phi) \in \ell^1(\mathbb{Z})$ because $s < 1$ and then $d_{q'}(B_\phi)$ is a convolution between summable sequences. So that it comes out

$$(5.2.20) \quad M_{2,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} \tilde{d}_{q'}(C_\phi) 2^{(q-q')(s-\frac{1}{2})} \right),$$

with d_q a summable sequence.

To end this proof, it remains to estimate $M_{3,q}$ (is the rest term). Using the definition of $R^h(A, \partial_x B)$, the support properties given in [18, Proposition 2.10], and Bernstein's lemma 5.2.1, we can write

$$\begin{aligned}
M_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(A, \partial_x B))_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\
&\lesssim \sum_{q'\geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\tilde{\Delta}_{q'}^h A_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h \partial_x B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L_h^\infty(L_v^2)} dt'.
\end{aligned}$$

By using Bernstein's Lemma 5.2.1, we can find that

$$(5.2.21) \quad \|\Delta_{q'}^h \partial_x B_\phi\|_{L^2} \lesssim 2^{q'} \|\Delta_{q'}^h B_\phi\|_{L^2} \quad \text{and} \quad \|\Delta_q^h C_\phi\|_{L_h^\infty(L_v^2)} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h C_\phi\|_{L_h^2(L_v^2)},$$

and by using Poincaré inequality on A (having that $A = 0$ when $y = 0, 1$), we obtain

$$(5.2.22) \quad \begin{aligned} \|\tilde{\Delta}_{q'}^h A_\phi\|_{L_h^2(L_v^\infty)} &\lesssim \|\tilde{\Delta}_{q'}^h A_\phi\|_{L_h^2(L_v^2)}^{\frac{1}{2}} \|\tilde{\Delta}_{q'}^h \partial_y A_\phi\|_{L_h^2(L_v^2)}^{\frac{1}{2}} \\ &\lesssim \|\tilde{\Delta}_{q'}^h \partial_y A_\phi\|_{L^2}^{\frac{1}{2}} \|\tilde{\Delta}_{q'}^h \partial_y A_\phi\|_{L^2}^{\frac{1}{2}} = \|\tilde{\Delta}_{q'}^h \partial_y A_\phi\|_{L^2} \\ &\lesssim d_{q'}(A_\phi) 2^{-\frac{q'}{2}} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}. \end{aligned}$$

Then, we replace on (5.2.20) we achieve

$$\begin{aligned} M_{3,q} &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} 2^{q'(1-\frac{1}{2})} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} 2^{\frac{q'}{2}} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt'. \end{aligned}$$

Since $s < 1$, we have by using Hölder's inequality

$$\begin{aligned} M_{3,q} &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t'} B_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t'} C_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_{q'}^h B_\phi\|_{L^2}^2 \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h C_\phi\|_{L^2}^2 \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q(C_\phi) 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\sum_{q' \geq q-3} d_{q'}(B_\phi) 2^{(q-q')s} \right) dt'. \end{aligned}$$

Then,

$$(5.2.23) \quad M_{3,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{q' \geq k-3} d_{q'}(B_\phi) 2^{(q-q')s} \right)$$

with d_q a summable sequence of positive numbers. Lemma 5.2.5 is then proved by summing Estimates (5.2.18), (5.2.20) and (5.2.23). \square

Lemma 5.2.6. *Let A, B and C be smooth functions on $[0, T^*) \times \mathbb{R} \times (0, 1)$, with A vanishing on the boundary, $s \in]0, 1[$, and ϕ be defined as in (5.2.3), with $\theta(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$. For any $t \in [0, T^*)$ and for any $B, C \in \tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$, we have*

$$(5.2.24) \quad \begin{aligned} &\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h \left(\int_0^y \partial_x B dy' \cdot \partial_y A \right)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ &\lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Proof. As in [15], using homogeneous Bony's decomposition of $(\int_0^y \partial_x B dy') \cdot \partial_y A$ into para-products in the horizontal variable and remainders, we can write

$$\partial_y A \cdot \int_0^y \partial_x B dy' = T_{\partial_y A}^h \int_0^y \partial_x B dy' + T_{\int_0^y \partial_x B dy'}^h \partial_y A + R^h(\partial_y A, \int_0^y \partial_x B dy')$$

where

$$\begin{aligned} T_{\partial_y A}^h \int_0^y \partial_x B dy' &= \sum_{q \in \mathbb{Z}} S_{q-1}^h \partial_y A \cdot \Delta_q^h \int_0^y \partial_x B dy' \\ R^h(\partial_y A, \int_0^y \partial_x B dy') &= \sum_{|q'-q| \leq 1} \Delta_q^h \partial_y A \cdot \Delta_{q'}^h \int_0^y \partial_x B dy' \\ &= \sum_q \tilde{\Delta}_q^h \partial_y A \cdot \Delta_q^h \int_0^y \partial_x B dy'. \end{aligned}$$

We replace this decomposition and we obtain the following bound of

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (\partial_y A \cdot \int_0^y \partial_x B dy')_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \leq N_{1,q} + N_{2,q} + N_{3,q},$$

where

$$\begin{aligned} N_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y A}^h \int_0^y \partial_x B dy')_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ N_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\int_0^y \partial_x B dy'}^h \partial_y A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ N_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(\partial_y A, \int_0^y \partial_x B dy'))_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

We start by getting the estimate of the first term $N_{1,q}$, for that we need to use the definition of $T_{\partial_y A}^h \int_0^y \partial_x B dy'$, the support properties given in [18, Proposition 2.10] and again thanks to the Corollary 5.2.2 we infer

$$(5.2.25) \quad N_{1,q} \leq \sum_{|q-q'| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_y A_\phi(t')\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h \int_0^y \partial_x B_\phi(t') dy'\|_{L_h^2(L_v^\infty)} \|\Delta_q^h C_\phi(t')\|_{L^2} dt'.$$

We have by applying Bernstein's Lemma 5.2.1 that

$$(5.2.26) \quad \begin{aligned} \|\Delta_q^h \partial_y A_\phi(t')\|_{L_h^\infty(L_v^2)} &\lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_y A_\phi(t')\|_{L^2} \\ &\lesssim d_q(A_\phi) \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}, \end{aligned}$$

where $(d_q(t))_{q \in \mathbb{Z}}$ is a generic element of $\ell^1(\mathbb{Z})$ such that $\sum d_q(t) \leq 1$. Then

$$\|S_{q'-1}^h \partial_y A_\phi(t')\|_{L_h^\infty(L_v^2)} \lesssim \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

and

$$\|\Delta_{q'}^h \int_0^y \partial_x B_\phi(t') dy'\|_{L_h^2(L_v^\infty)} \lesssim 2^{q'} \|\Delta_{q'}^h \int_0^y B_\phi dy'\|_{L_h^2(L_v^\infty)} \lesssim 2^{q'} \|\Delta_{q'}^h B_\phi\|_{L^2},$$

where we have used that $\|\int_0^y f dy'\|_{L_y^\infty} \leq \int_0^1 |f| dy' \leq \|f\|_{L_y^2}$. As a result, it comes out

$$\begin{aligned} N_{1,q} &\leq \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t e^{2\mathcal{R}t'} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \end{aligned}$$

where in the last step we used Hölder's inequality, and we note that $\dot{\theta}(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$. Using definition 5.2.3, we achieve

$$\left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim d_{q'}(B_\phi) 2^{-q'(s+\frac{1}{2})} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

and

$$\left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim d_{q'}(C_\phi) 2^{-q'(s+\frac{1}{2})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then,

$$(5.2.27) \quad N_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(B_\phi) 2^{(q-q')(s-\frac{1}{2})} \right)$$

Along the same way for $s < 1$, we obtain

$$\begin{aligned} N_{2,q}(t) &\leq \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\int_0^t \partial_x B dy'}^h \partial_y A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ &\lesssim \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \int_0^y \partial_x B_\phi dy'\|_{L^\infty} \|\Delta_{q'}^h \partial_y A_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{-\frac{q'}{2}} \sum_{l \lesssim q'-2} \left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_l^h \int_0^y \partial_x B_\phi\|_{L^\infty}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

By using the fact that $\|\int_0^y f dy'\|_{L_y^\infty} \leq \int_0^1 |f| dy' \leq \|f\|_{L_y^2}$, we have

$$\|\Delta_l^h \int_0^y \partial_x B_\phi(t') dy'\|_{L_h^\infty(L_v^\infty)} \lesssim 2^l \|\Delta_l^h \int_0^y B_\phi dy'\|_{L_h^\infty(L_v^\infty)} \lesssim 2^l \|\Delta_l^h B_\phi\|_{L_h^\infty(L_v^2)}.$$

Combining with Bernstein's lemma 5.2.1, we can find

$$\|\Delta_l^h \int_0^y \partial_x B_\phi(t') dy'\|_{L_h^\infty(L_v^\infty)} \lesssim 2^l \|\Delta_l^h B_\phi\|_{L_h^\infty(L_v^2)} \lesssim 2^{\frac{3l}{2}} \|\Delta_l^h B_\phi\|_{L^2}.$$

We observe from Definition 5.2.3, and we use the same thing as in (5.2.19), then for $s < 1$ we have

$$\begin{aligned}
(5.2.28) \quad & \sum_{l \lesssim q'-2} \left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_l^h \int_0^y \partial_x B_\phi\|_{L^\infty}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim \sum_{l \lesssim q'-2} \left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') 2^{3l} \|\Delta_l^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim \sum_{l \lesssim q'-2} 2^{\frac{3l}{2}} 2^{-l(s+\frac{1}{2})} 2^{l(s+\frac{1}{2})} \left(\int_0^t e^{2\mathcal{R}t'} \dot{\theta}(t') \|\Delta_l^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim 2^{q'(1-s)} \sum_{l \lesssim q'-2} \tilde{d}_l(B_\phi) 2^{(q'-l)(s-1)} \|B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& \lesssim 2^{q'(1-s)} d_{q'}(B_\phi) \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

where

$$d_{q'}(B_\phi) = \sum_{l \leq q'-2} 2^{-(q'-l)(1-s)} \tilde{d}_l(B_\phi) \in \ell^1(\mathbb{Z}).$$

So that it comes out

$$(5.2.29) \quad N_{2,q} \leq d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'| \leq 4} d_{q'}(B_\phi) 2^{(q-q')(s-\frac{1}{2})} \right)$$

is a suitable sequence of positive constants.

To end this proof, it remains to estimate $N_{3,q}$ (is the rest term). Using the support properties given in [18, Proposition 2.10], the definition of $R^h(\partial_y A, \int_0^y \partial_x B dy')$ and Bernstein's Lemma 5.2.1, we can write

$$\begin{aligned}
N_{3,q} & \leq \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\tilde{\Delta}_{q'}^h \partial_y A_\phi\|_{L^2} \|\Delta_{q'}^h \int_0^y \partial_x B_\phi dy'\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h C_\phi\|_{L_h^\infty(L_v^2)} dt' \\
& \leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} 2^{q'(1-\frac{1}{2})} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_{q'}^h C_\phi\|_{L^2} dt' \\
& \leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} 2^{\frac{q'}{2}} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_{q'}^h C_\phi\|_{L^2} dt'.
\end{aligned}$$

Since $s < 1$, we have by applying Hölder's inequality

$$\begin{aligned}
N_{3,q} & \leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t'} B_\phi\|_{L^2} \|\Delta_{q'}^h e^{\mathcal{R}t'} C_\phi\|_{L^2} dt' \\
& \leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
& \quad \times \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\end{aligned}$$

Definition 5.2.3 tells us

$$\left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim d_q(C_\phi) 2^{-q(s+\frac{1}{2})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then we obtain

$$(5.2.30) \quad N_{3,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{q' \geq q-3} d_{q'}(B_\phi) 2^{(q-q')s} \right)$$

is a suitable sequence of positive constants.

Lemma 5.2.6 is then proved by summing Estimates (5.2.27), (5.2.29) and (5.2.30). \square

Lemma 5.2.7. *For any $s \in]0, 1[$ and $t \leq T^*$, and ϕ be defined as in (5.2.3), with*

$$\dot{\theta}(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_y B_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Then, for any $t \in [0, T^)$ and for any $A, B \in \tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$ that satisfies $\partial_x A = -\partial_y B$, A and B vanishing on the boundary (so that $B(t, x, y) = -\int_0^y \partial_x A(t, x, s) ds$), we have*

$$\begin{aligned} \epsilon^2 \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (B \partial_y B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ \lesssim d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} \epsilon B_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) \|e^{\mathcal{R}t} \epsilon C_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Proof. As in [15], using Bony's homogeneous decomposition of the term $B \partial_y B$ into para-products in the horizontal variable and remainders, we can write

$$B \partial_y B = T_{\partial_y B}^h B + T_B^h \partial_y B + R^h(\partial_y B, B)$$

We replace, we obtain the following bound of $\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (B \partial_y B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'$

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (B \partial_y B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \leq L_{1,q} + L_{2,q} + L_{3,q},$$

where

$$\begin{aligned} L_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_B^h \partial_y B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ L_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y B}^h B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ L_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(B, \partial_y B))_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

We start by getting the estimate of the first term $L_{1,q}$. Due to $\partial_y B = -\partial_x A$, one has

$$(5.2.31) \quad \epsilon^2 L_{1,q} \lesssim \epsilon^2 \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h B_\phi(t)\|_{L^\infty} \|\Delta_{q'}^h \partial_x A_\phi(t)\|_{L^2} \|\Delta_q^h C_\phi(t)\|_{L^2}.$$

By using Bernstein's Lemma 5.2.1, we have

$$\|S_{q'-1}^h B_\phi(t')\|_{L_h^\infty} \lesssim \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h B_\phi(t')\|_{L_h^2}.$$

Taking the L_v^∞ norm and using that $\|f\|_{L_v^p L_h^q} \leq \|f\|_{L_h^q L_v^p}$, for $p \geq q \geq 1$ we obtain

$$\|S_{q'-1}^h B_\phi(t')\|_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h B_\phi(t')\|_{L_v^\infty(L_h^2)} \lesssim \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h B_\phi(t')\|_{L_h^2(L_v^\infty)}.$$

While using the inclusion $H_y^1 \hookrightarrow L_v^\infty$,

$$\|\Delta_l^h B_\phi(t')\|_{L_v^\infty} \lesssim \|\Delta_l^h B_\phi(t')\|_{L_v^2}^{\frac{1}{2}} \|\Delta_l^h \partial_y B_\phi(t')\|_{L_v^2}^{\frac{1}{2}},$$

and Poincaré inequality on the interval $\{0 < y < 1\}$ on B (we use the fact that $B = 0$ when $y = 0, 1$) gives

$$(5.2.32) \quad \|S_{q'-1}^h B_\phi(t')\|_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h \partial_y B_\phi(t')\|_{L^2} \lesssim \|\partial_y B_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}$$

Then, we replace in (5.2.31), we get by using Hölder's inequality

$$\begin{aligned} \epsilon^2 L_{1,q} &\lesssim \epsilon^2 \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|\partial_y B_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h e^{\mathcal{R}t'} A_\phi(t)\|_{L^2} \|\Delta_q^h C_\phi(t)\|_{L^2} \\ &\lesssim \epsilon \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \epsilon \|\partial_y B_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h A_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \epsilon \|\partial_y B_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t'} C_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

Hence we deduce from the Definition 5.2.1 that

$$\left(\int_0^t \epsilon \|\partial_y B_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t'} A_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim d_{q'}(A_\phi) 2^{-q'(s+\frac{1}{2})} \|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then,

$$(5.2.33) \quad L_{1,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

where

$$d_q^2 = d_q(C_\phi) \sum_{|q'-q| \leq 4} d_{q'}(A_\phi) 2^{(q-q')(s-\frac{1}{2})}$$

Now we move to get the estimate of the second term $L_{2,q}$. Due to $\partial_y B_\phi = -\partial_x A_\phi$, we can achieve

$$L_{2,q}(t) \leq \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y B}^h B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'$$

$$\lesssim \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x A_\phi\|_{L^\infty} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt'.$$

Along the same way as we did to estimate $L_{1,q}$, we can obtain by using (5.2.32) that

$$\|S_{q'-1}^h \partial_x A_\phi(t')\|_{L^\infty} \lesssim 2^{q'} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Then,

$$\begin{aligned} L_{2,q}(t) &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t e^{2\mathcal{R}t'} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}, \end{aligned}$$

in the last step we use Hölder's inequality. Then thanks to Definition 5.2.1, we arrive at

$$\left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim d_{q'}(B_\phi) 2^{-q'(s+\frac{1}{2})} \|e^{\mathcal{R}t'} B_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

We conclude our estimate by

$$(5.2.34) \quad \epsilon^2 L_{2,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} \epsilon B_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \epsilon C_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(B_\phi) 2^{(q-q')(s-\frac{1}{2})} \right)$$

To end this proof, it remains to estimate $L_{3,q}$ (is the rest term). Due to $\partial_y B = -\partial_x A$, we get, by applying lemma 5.2.1 that

$$\begin{aligned} L_{3,q} &\lesssim \sum_{q'\geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\tilde{\Delta}_{q'}^h \partial_x A_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L_h^\infty(L_v^2)} dt' \\ &\lesssim \sum_{q'\geq q-3} 2^{\frac{q'}{2}} \int_0^t e^{2\mathcal{R}t'} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} 2^{\frac{q'}{2}} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q'\geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

which together with Definition 5.2.3 and $s < 1$ ensures that

$$(5.2.35) \quad \epsilon^2 L_{3,q} \lesssim d_q 2^{-2qs} \|e^{\mathcal{R}t} \epsilon B_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \epsilon C_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{q' \geq q-3} d_{q'}(B_\phi) 2^{(q-q')s} \right)$$

Lemma 5.2.7 is then proved by summing Estimates (5.2.33), (5.2.34) and (5.2.35). \square

5.3 Global existence of the perturbed hydrostatic system (5.1.9)

The goal of this section is to prove the global well-posedness of the limit system of the Perturbed Navier-Stokes equation, we remark that the local smooth solution of the limit system follows a standard parabolic regularization method similar to the Perturbation NS system, First, we remark that the Dirichlet boundary condition

$$(u, v)_{/y=0} = (u, v)_{/y=1} = 0,$$

and the incompressible condition $\partial_x u + \partial_y v = 0$ imply that :

$$(5.3.1) \quad v(t, x, y) = \int_0^y \partial_y v(t, x, s) ds = - \int_0^y \partial_x u(t, x, s) ds$$

Due to the compatibility condition $\partial_x \int_0^1 u_0 dy = 0$, we deduce from $\partial_x u + \partial_y v = 0$ that

$$(5.3.2) \quad \partial_x \int_0^1 u(t, x, y) dy = - \int_0^1 \partial_y v(t, x, y) dy = v(t, x, 0) - v(t, x, 1) = 0,$$

which together with the fact: $u(t, x, y) \rightarrow 0$ as $|x| \rightarrow \infty$, ensure that

$$(5.3.3) \quad \int_0^1 u(t, x, y) dy = 0.$$

Then by integrating the equations $\partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = 0$ on $y \in [0, 1]$ and using the fact that $\partial_y p = 0$, we obtain

$$(5.3.4) \quad \partial_x p = \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \partial_x \int_0^1 (u)^2(t, x, y) dy$$

In view of System (5.1.9), we can transform it into an equation of order one in time, so if we define $V = (u, \partial_t u)$, Then V satisfies the following system

$$(5.3.5) \quad \begin{cases} \partial_t V + A(D)V = - \begin{pmatrix} 0 \\ u \partial_x u + v \partial_y u + \partial_x p \end{pmatrix} \\ \partial_y p = 0 \\ \partial_x u + \partial_y v = 0 \\ (u, v)_{/y=0} = (u, v)_{/y=1} = 0 \end{cases}$$

where

$$V = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \quad \text{and} \quad A(D) = \begin{pmatrix} 0 & -1 \\ -\partial_y^2 & 1 \end{pmatrix}$$

Then in view of (5.2.3) we observe that V_ϕ verifies

$$(5.3.6) \quad \begin{cases} \partial_t V_\phi + \lambda \dot{\theta}(t) |D_x| V_\phi + A(D) V_\phi = - \begin{pmatrix} 0 \\ (u \partial_x u)_\phi + (v \partial_y u)_\phi + \partial_x p_\phi \end{pmatrix} \\ \partial_y p_\phi = 0 \\ \partial_x u_\phi + \partial_y v_\phi = 0 \\ (u_\phi, v_\phi)_{/y=0} = (u_\phi, v_\phi)_{/y=1} = 0 \end{cases}$$

Where $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$.

The main idea of this technique consists in the fact that if we differentiate, with respect to the time variable a function of the type $e^{\phi(t, D_x)} u(t, x, y)$, we obtain an additional 'good term' which plays a smoothing role. More precisely, we have

$$\frac{\partial}{\partial t} \left(e^{\phi(t, D_x)} V(t, x, y) \right) = -\lambda \dot{\theta}(t) |D_x| e^{\phi(t, D_x)} V(t, x, y) + e^{\phi(t, D_x)} \partial_t V(t, x, y),$$

where $-\dot{\theta}(t) |D_x| e^{\phi(t, D_x)} u(t, x, y)$ gives a smoothing effect if $\dot{\theta}(t) \geq 0$. This smoothing effect allows to obtain our global existence and stability results in the analytic framework.

Proof of global well-posedness of system (5.1.9) By applying the dyadic operator in the horizontal variable Δ_q^h to (5.3.6) and taking the L^2 inner product of the resulting equation with $\Delta_q^h(V_\phi)$ we obtain

$$(5.3.7) \quad \langle \Delta_q^h \partial_t V_\phi, \Delta_q^h V_\phi \rangle_{L^2} + \lambda \dot{\theta}(t) \langle \Delta_q^h |D_x| V_\phi, \Delta_q^h V_\phi \rangle_{L^2} + \langle \Delta_q^h A(D) V_\phi, \Delta_q^h V_\phi \rangle_{L^2} \\ = - \langle \Delta_q^h (u \partial_x u + v \partial_y u)_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2}.$$

In what follows, we shall use the technical lemmas in Section 5.2, to handle every term in the display (5.3.7).

By applying the result of Lemma 5.2.3, we find that

$$(5.3.8) \quad \langle \Delta_q^h \partial_t V_\phi, \Delta_q^h V_\phi \rangle_{L^2} + \lambda \dot{\theta}(t) \langle \Delta_q^h |D_x| V_\phi, \Delta_q^h V_\phi \rangle_{L^2} = \langle \Delta_q^h \partial_t u_\phi, \Delta_q^h u_\phi \rangle_{L^2} \\ + \langle \Delta_q^h \partial_t (\partial_t u)_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} + \lambda \dot{\theta}(t) \left(\langle \Delta_q^h |D_x| u_\phi, \Delta_q^h u_\phi \rangle_{L^2} + \langle \Delta_q^h |D_x| (\partial_t u)_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \right) \\ = \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q^h u_\phi\|_{L^2}^2 + \|\Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right) + \lambda \dot{\theta}(t) \left(\|\Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2 + \|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi\|_{L^2}^2 \right),$$

and by using the fact that $(\partial_t u)_\phi = \partial_t u_\phi + \lambda \dot{\theta}(t) |D_x| u_\phi$, we have

$$(5.3.9) \quad \langle \Delta_q^h A(D) V_\phi, \Delta_q^h V_\phi \rangle_{L^2} = - \langle \Delta_q^h (\partial_t u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \\ + \langle \Delta_q^h (\partial_t u)_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} = -\frac{1}{2} \frac{d}{dt} \|\Delta_q^h u_\phi\|_{L^2}^2 - \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2 \\ + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 + \|\Delta_q^h (\partial_t u)_\phi\|_{L^2}^2.$$

Summing the two equalities (5.3.8) and (5.3.9), we obtain

$$(5.3.10) \quad \langle \Delta_q^h \partial_t V_\phi, \Delta_q^h V_\phi \rangle_{L^2} + \lambda \dot{\theta}(t) \langle \Delta_q^h |D_x| V_\phi, \Delta_q^h V_\phi \rangle_{L^2} + \langle \Delta_q^h A(D) V_\phi, \Delta_q^h V_\phi \rangle_{L^2} = \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right. \\ \left. + \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 \right) + \lambda \dot{\theta}(t) \left(\|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi\|_{L^2}^2 + \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 \right) + \|\Delta_q^h (\partial_t u)_\phi\|_{L^2}^2.$$

Now, for the pressure term, using the Dirichlet boundary condition $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$ and $\partial_t(u, v)|_{y=0} = \partial_t(u, v)|_{y=1} = 0$, the incompressibility condition $\partial_x u + \partial_y v = 0$, and performing an integration by parts, we get

$$\begin{aligned} D &= \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} = - \langle \Delta_q^h p_\phi, \Delta_q^h (\partial_t \partial_x u)_\phi \rangle_{L^2} \\ &= \langle \Delta_q^h p_\phi, \Delta_q^h (\partial_t \partial_y v)_\phi \rangle_{L^2} \\ &= - \langle \Delta_q^h \partial_y p_\phi, \Delta_q^h (\partial_t v)_\phi \rangle_{L^2} = 0, \end{aligned}$$

In the last step we use the fact that $\partial_y p_\phi = 0$. Thus,

$$D = \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} = 0.$$

While due to $u|_{y=0} = u|_{y=1} = 0$, by applying Poincaré's inequality, we have

$$(5.3.11) \quad k \|\Delta_q^h u_\phi\|_{L^2}^2 \leq \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2,$$

where k is the Poincaré's constant. Then, multiplying (5.3.10) by $e^{2\mathcal{R}t}$ (\mathcal{R} is a constant smaller than Poincaré's constant denoted by k), we achieve

$$\begin{aligned} (5.3.12) \quad & \frac{1}{2} \frac{d}{dt} \left(\|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 \right) + \|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \\ & + \lambda \dot{\theta}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 \\ & \leq \mathcal{R} \left(\|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 \right) + \left| \langle \Delta_q^h (u \partial_x u)_\phi, e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \right| \\ & \quad + \left| \langle \Delta_q^h (v \partial_y u)_\phi, e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \right|. \end{aligned}$$

In what follows, we shall always assume that $t < T^*$, where T^* is given by

$$(5.3.13) \quad T^* \triangleq \sup \left\{ t > 0, \quad \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{4C^2}, \quad \theta(t) \leq \frac{a}{\lambda} \right\}.$$

Next, we define

$$\begin{cases} J_1^q(t) = \left| \langle \Delta_q^h (u \partial_x u)_\phi, e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \right| \\ J_2^q(t) = \left| \langle \Delta_q^h (v \partial_y u)_\phi, e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \right|. \end{cases}$$

To estimate those two non linear terms we need to use Lemmas 5.2.5 and 5.2.6. In view of Lemma 5.2.5, we replace C_ϕ by $(\partial_t u)_\phi$ and $A = B = u$, then we conclude the following estimate of J_1^q for any $t < T^*$

$$(5.3.14) \quad \begin{aligned} \int_0^t J_1^q(t') dt' &= \int_0^t \left| \langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \right| dt' \\ &\leq C 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \end{aligned}$$

where $\dot{\theta}(t) = \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$.

In view of Lemma 5.2.6, we replace C_ϕ by $(\partial_t u)_\phi$, $A = u$ and $B = u$, then we conclude the following estimate of J_2^q for any $t < T^*$

$$\int_0^t J_2^q(t') dt' = \left| \langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi \rangle_{L^2} \right|$$

$$(5.3.15) \quad \leq C2^{-2qs}d_q^2\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where $\dot{\theta}(t) = \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$.

Then we deduce from (5.3.12) by integrating with respect to the time interval, that

$$(5.3.16) \quad \int_0^t \frac{1}{2} \frac{d}{dt} \left(\|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi(t')\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi(t')\|_{L^2}^2 \right) dt' + \int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi(t')\|_{L^2}^2 dt' \\ + \lambda \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi(t')\|_{L^2}^2 dt' + \lambda \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi(t')\|_{L^2}^2 dt' \\ \leq \mathcal{R} (\|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L^2}^2) + C2^{-2qs}d_q^2\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

We multiply the estimate (5.3.16) by 2, we obtain

$$(5.3.17) \quad \int_0^t \frac{d}{dt} \left(\|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi(t')\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi(t')\|_{L^2}^2 \right) dt' + 2 \int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi(t')\|_{L^2}^2 dt' \\ + 2\lambda \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi(t')\|_{L^2}^2 dt' + 2\lambda \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi(t')\|_{L^2}^2 dt' \\ \leq 2\mathcal{R} (\|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L^2}^2) + C2^{-2qs}d_q^2\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

Now we still have to get some information of the norm $\|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$, for that we need to apply the dyadic operator Δ_q^h to the equation

$$(5.3.18) \quad e^{\phi(t, D_x)} (\partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p) = 0,$$

and then, we take the L^2 inner product of the resulting equation (5.3.18) with $\Delta_q^h u_\phi$, we obtain

$$(5.3.19) \quad \langle \Delta_q^h (\partial_t^2 u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} + \langle \Delta_q^h (\partial_t u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h u_\phi \rangle_{L^2} \\ = - \langle \Delta_q^h (u \partial_x u + v \partial_y u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2}.$$

In what follows, we shall use again the technical lemmas in Section 5.2, to handle term by term in the estimate (5.3.19). We start by the term

$$I_1 = \langle \Delta_q^h (\partial_t^2 u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} \quad \text{and} \quad I_2 = \langle \Delta_q^h (\partial_t u)_\phi, \Delta_q^h u_\phi \rangle_{L^2}$$

so by using integration by parts, we find

$$I_1 = \frac{d}{dt} \int \Delta_q^h (\partial_t u)_\phi \Delta_q^h u_\phi dX - \int \Delta_q^h (\partial_t u)_\phi \Delta_q^h (\partial_t u)_\phi dX \\ + 2\lambda \dot{\theta}(t) \int \Delta_q^h |D_x| (\partial_t u)_\phi \Delta_q^h u_\phi dX,$$

and

$$I_2 = \frac{1}{2} \frac{d}{dt} \|\Delta_q^h u_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2.$$

Whereas due to the boundary condition, and by integrating by part, we achieve

$$\langle \Delta_q^h (-\partial_y^2 u_\phi), \Delta_q^h u_\phi \rangle_{L^2} = \langle \Delta_q^h \partial_y u_\phi, \Delta_q^h \partial_y u_\phi \rangle_{L^2} = \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2.$$

Now, by using the Dirichlet boundary condition $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$, and the incompressibility condition $\partial_x u + \partial_y v = 0$ and the relation $\partial_y p_\phi = 0$, we can find by integrating by parts the estimate of the pressure

$$\begin{aligned} \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2} \right| &= \left| \langle \Delta_q^h p_\phi, \Delta_q^h \partial_x u_\phi \rangle_{L^2} \right| \\ &= \left| \langle \Delta_q^h p_\phi, \Delta_q^h \partial_y v_\phi \rangle_{L^2} \right| \\ &= \left| \langle \Delta_q^h \partial_y p_\phi, \Delta_q^h v_\phi \rangle_{L^2} \right| = 0. \end{aligned}$$

Then using Lemma 5.2.1 and multiplying (5.3.19) by $e^{2\mathcal{R}t}$, we achieve

$$\begin{aligned} & \frac{d}{dt} \int e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi \Delta_q^h u_\phi dX - \int e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi \Delta_q^h (\partial_t u)_\phi dX + 2\lambda \dot{\theta}(t) \int e^{2\mathcal{R}t} \Delta_q^h |D_x| (\partial_t u)_\phi \Delta_q^h u_\phi dX \\ & + \frac{1}{2} \frac{d}{dt} \|e^{\mathcal{R}t} \Delta_q^h u_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 \\ & = 2\mathcal{R} \int e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi \Delta_q^h u_\phi dx + \mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h u_\phi\|_{L^2}^2 \\ (5.3.20) \quad & - \langle \Delta_q^h (u \partial_x u)_\phi, e^{2\mathcal{R}t} \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h (v \partial_y u)_\phi, e^{2\mathcal{R}t} \Delta_q^h u_\phi \rangle_{L^2}. \end{aligned}$$

Next, we note that

$$\begin{cases} L_1^q(t) = \left| \langle \Delta_q^h (u \partial_x u)_\phi, e^{2\mathcal{R}t} \Delta_q^h u_\phi \rangle_{L^2} \right| \\ L_2^q(t) = \left| \langle \Delta_q^h (v \partial_y u)_\phi, e^{2\mathcal{R}t} \Delta_q^h u_\phi \rangle_{L^2} \right| \\ L_3^q(t) = \left| 2\lambda \dot{\theta}(t) \int e^{2\mathcal{R}t} \Delta_q^h |D_x| (\partial_t u)_\phi \Delta_q^h u_\phi dX \right|. \end{cases}$$

In view, of Lemmas 5.2.5-5.2.6, we can deduce for $t < T^*$ that

$$(5.3.21) \quad \begin{aligned} \int_0^t L_1^q(t') dt' &= \int_0^t \left| \langle \Delta_q^h (u \partial_x u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h u_\phi \rangle_{L^2} \right| dt' \\ &\leq C 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \end{aligned}$$

and

$$(5.3.22) \quad \begin{aligned} \int_0^t L_2^q(t') dt' &= \int_0^t \left| \langle \Delta_q^h (v \partial_y u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h u_\phi \rangle_{L^2} \right| dt' \\ &\leq C 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2. \end{aligned}$$

Then we still have to estimate L_3^q , therefore by cutting the derivative $|D_x|$ into two half derivative, we achieve the

$$(5.3.23) \quad \begin{aligned} L_3^q(t) &= \left| 2\lambda \dot{\theta}(t) \int e^{2\mathcal{R}t'} \Delta_q^h |D_x| (\partial_t u)_\phi \Delta_q^h u_\phi dX \right| \\ &\leq 2\lambda \dot{\theta}(t) \int \left| e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi \right| \left| e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} u_\phi \right| dX \\ &\leq \lambda \dot{\theta}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2, \end{aligned}$$

Next, by taking the integration over time of (5.3.20), and we sum it with (5.3.17), we gather that

(5.3.24)

$$\begin{aligned}
& \int_0^t \frac{d}{dt} \left(\frac{1}{2} \|e^{\mathcal{R}t'} \Delta_q^h (u + \partial_t u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \frac{1}{2} \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right) dt' + \int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 dt' \\
& + \int_0^t \lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2 dt' + \int_0^t \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 dt' + \int_0^t 2\lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi\|_{L^2}^2 dt' \\
& + \int_0^t 2\lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 dt' \leq \mathcal{R} \int_0^t \left[\|e^{\mathcal{R}t'} \Delta_q^h (u + \partial_t u)_\phi\|_{L^2}^2 + 2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 \right. \\
& + \left. \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right] dt' + \lambda \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi\|_{L^2}^2 dt' + \lambda \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2 dt' \\
& + 2C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}^2_{t, \dot{\theta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}^2_{t, \dot{\theta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).
\end{aligned}$$

We begin with by observing that the term in the square brackets of the right-hand side in (5.3.24) can be absorbed by the dissipation $\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 dt' + \int_0^t \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 dt'$. Indeed, since the value of \mathcal{R} is smaller than $\min\{\frac{1}{8}, \frac{k}{8}\}$, we have that

$$\begin{aligned}
& \mathcal{R} \left[\|e^{\mathcal{R}t'} \Delta_q^h (u + \partial_t u)_\phi\|_{L^2}^2 + 2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right] \\
& \leq \mathcal{R} \left[2 \|e^{\mathcal{R}t'} \Delta_q^h u_\phi\|_{L^2}^2 + 2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 + 3 \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right] \\
& \leq \frac{k}{8} \|e^{\mathcal{R}t'} \Delta_q^h u_\phi\|_{L^2}^2 + \frac{1}{4} \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \frac{1}{4} \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2.
\end{aligned}$$

To absorb this last term, we shall then invoke the Poincaré's inequality in $y \in (0, 1)$: $k \| \Delta_q^h u_\phi \|_{L^2} \leq \| \Delta_q^h \partial_y u_\phi \|_{L^2}$. Thus

$$\begin{aligned}
& \mathcal{R} \left[\|e^{\mathcal{R}t'} \Delta_q^h (u + \partial_t u)_\phi\|_{L^2}^2 + 2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right] \\
(5.3.25) \quad & \leq \frac{1}{2} \left(\|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right).
\end{aligned}$$

We replace the result obtained (5.3.25) in (5.3.24), we deduce that

$$\begin{aligned}
(5.3.26) \quad & \int_0^t \frac{d}{dt} \left(\frac{1}{2} \|e^{\mathcal{R}t'} \Delta_q^h (u + \partial_t u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \frac{1}{2} \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 \right) dt' \\
& + \frac{1}{2} \int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi\|_{L^2}^2 dt' + \frac{1}{2} \int_0^t \|e^{\mathcal{R}t'} \Delta_q^h \partial_y u_\phi\|_{L^2}^2 dt' \\
& + \int_0^t \lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\phi\|_{L^2}^2 dt' + \int_0^t 2\lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 dt' \\
& \leq 2C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}^2_{t, \dot{\theta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}^2_{t, \dot{\theta}(t)}(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).
\end{aligned}$$

In the last estimate (5.3.26), we remark that the term $\int_0^t \lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2 dt'$ has disappeared because of the second term that comes out of the estimate (5.3.23). We can make it appear by using Poincaré's inequality on the term $\int_0^t 2\lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 dt'$, we have

$$\begin{aligned}
& \int_0^t 2\lambda \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 dt' \\
& \geq \lambda \left(k \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} u_\phi\|_{L^2}^2 dt' + \int_0^t \dot{\theta}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\phi\|_{L^2}^2 dt' \right).
\end{aligned}$$

Now we multiply the obtained result in (5.3.24) by 2^{2qs} for $s \in]0, 1[$. Summing with respect to

$q \in \mathbb{Z}$, we find that for $t < T^*$

$$\begin{aligned}
(5.3.27) \quad & \left(\|e^{\mathcal{R}t}(u + \partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \right) \\
& + \left(\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right) + \sqrt{k\lambda} \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{\lambda} \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& + \sqrt{\lambda} \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \leq C \left(\|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^s} + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^s} \right) \\
& + \sqrt{2}C \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{2}C \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

Taking $\lambda \geq 4C^2$ and $k\lambda \geq 8C^2$ in the above inequality leads to

$$\begin{aligned}
(5.3.28) \quad \mathcal{E}_{s,\frac{\lambda}{4}}(u)(t) & \leq C \|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^s} + C \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^s} \\
& + C \|e^{a|D_x|}u_1\|_{\mathcal{B}^s}, \quad \text{for } t < T^*.
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}_{s,\frac{\lambda}{4}}(u)(t) & = \left(\|e^{\mathcal{R}t}(u + \partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \right) \\
& + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
& + \frac{\sqrt{k\lambda}}{2} \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{\sqrt{\lambda}}{2} \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{\sqrt{\lambda}}{2} \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}
\end{aligned}$$

We recall that we already defined $\dot{\theta}(t) = \|\partial_y u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ with $\theta(0) = 0$. Then, for any $0 < t < T^*$, Inequality (5.3.28) yields

$$\begin{aligned}
\theta(t) & = \int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\
& \leq \int_0^t e^{-\mathcal{R}t'} \|e^{\mathcal{R}t'} \partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\
& \leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|e^{\mathcal{R}t'} \partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\
& \leq C \frac{1}{\mathcal{R}^{\frac{1}{2}}} \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
& \leq C \left(\|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{1}{2}}}^2 \right) \\
& < \frac{a}{2\lambda}
\end{aligned}$$

We deduce from the continuity argument that $T^* = +\infty$ and we have (5.3.28) is valid for any $t \in \mathbb{R}_+$.

5.4 Propagation of the regularity and of the vorticity of the hyperbolic Prandtl equation (5.1.9)

In this section, we present first a proposition stating the propagation for any \mathcal{B}^s regularity on the solution of the perturbed hyperbolic Navier-Stokes equations (5.1.9). The second proposition allows us to control two derivatives in the normal direction ∂_y^2 in any \mathcal{B}^s , despite the difficulties raised by the

boundary conditions. Those propositions will be useful in the last section when we prove the global convergence of Theorem 5.1.3. In what follows, we shall always assume that $t < T_a^*$, where T_a^* given by

$$(5.4.1) \quad T_a^* \triangleq \sup \left\{ t > 0, \quad \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{4C^2} \quad \theta(t) \leq \frac{a}{\lambda} \right\}.$$

Proposition 5.4.1. *We assume that the condition (5.1.11) is satisfied, then for any $s > 0$ and $u_0, u_1 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}} \cap \mathcal{B}^s$, there exist a small constant C , such that for*

$$\lambda = C(1 + \|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} (u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} u_1\|_{\mathcal{B}^{\frac{3}{2}}}),$$

we have

$$(5.4.2) \quad \|e^{\mathcal{R}t} (u + \partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ \leq C \left(\|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^s} + \|e^{a|D_x|} (u_0 + u_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|} u_1\|_{\mathcal{B}^s} \right), \quad \text{for } t < T_a^*.$$

where $C = C(s)$.

Proof of Proposition 5.4.1: We first deduce from Lemma 5.2.5 that for any $s > 0$ we have by replacing $A = B = u$ and $C = \partial_t u$

$$(5.4.3) \quad \int_0^t \left| \left\langle \Delta_q^h e^{\mathcal{R}t'} (T_u^h \partial_x u + R^h(u, \partial_x u))_\phi, e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi \right\rangle_{L^2} \right| dt' \\ \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then we only have to prove that

$$\int_0^t \left| \left\langle \Delta_q^h e^{\mathcal{R}t'} (T_{\partial_x u}^h u)_\phi, e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi \right\rangle_{L^2} \right| dt' \\ \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \quad \text{for } s > 0.$$

Indeed, in view of (5.2.16), we infer

$$\int_0^t \left| \left\langle \Delta_q^h e^{\mathcal{R}t'} (T_{\partial_x u}^h u)_\phi, e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\phi \right\rangle_{L^2} \right| dt' \\ \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x u_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h u_\phi(t')\|_{L^2} \|\Delta_q^h (\partial_t u)_\phi(t')\|_{L^2} dt' \\ \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \int_0^t e^{2\mathcal{R}t'} \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi(t')\|_{L^2} \|\Delta_q^h (\partial_t u)_\phi(t')\|_{L^2} dt' \\ \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ \times \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h (\partial_t u)_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}},$$

which leads to

(5.4.4)

$$\int_0^t \left| \left\langle \Delta_q^h(u\partial_x u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h(\partial_t u)_\phi \right\rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

While it follows from the proof of Lemma 5.2.6 that

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h(T_{\partial_y u}^h v + R^h(v, \partial_y u))_\phi, e^{2\mathcal{R}t'} \Delta_q^h(\partial_t u)_\phi \right\rangle_{L^2} \right| dt' \\ & \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \end{aligned}$$

so we have yet to determine the estimate of $\int_0^t \left| \left\langle \Delta_q^h(T_v^h \partial_y u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h(\partial_t u)_\phi \right\rangle_{L^2} \right| dt'$, we have

$$\|\Delta_q^h v_\phi(t)\|_{L^\infty} \lesssim d_q(u_\phi) 2^{\frac{q}{2}} \|u_\phi(t)\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

so that

$$\|S_{q'-1}^h v_\phi(t')\|_{L^\infty} \lesssim 2^{\frac{q'}{2}} \|u_\phi(t)\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

which implies that

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h(T_v^h \partial_y u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h(\partial_t u)_\phi \right\rangle_{L^2} \right| dt' \\ & \lesssim \sum_{|q'-q|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h v_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_y u_\phi(t')\|_{L^2} \|\Delta_q^h(\partial_t u)_\phi(t')\|_{L^2} dt' \\ & \lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \Delta_{q'}^h \partial_y u_\phi\|_{L_t^2(L^2)} \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t'}(\partial_t u)_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ & \lesssim d_q^2 2^{-2qs} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

By summing all the terms we obtain

$$(5.4.5) \quad \begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h(v\partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi \right\rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ & \times \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right). \end{aligned}$$

Along the same way we can obtain

$$(5.4.6) \quad \int_0^t \left| \left\langle \Delta_q^h(u\partial_x u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

and

$$(5.4.7) \quad \begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h(v\partial_y u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ & \times \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right). \end{aligned}$$

By virtue of (5.4.4), (5.4.5) (5.4.6) and (5.4.7), we deduce from (5.3.12) and (5.3.20) that for $t < T_a^*$

$$\begin{aligned}
(5.4.8) \quad & \|e^{\mathcal{R}t}(u + \partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
& + \sqrt{k\lambda}\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \sqrt{\lambda}\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{\lambda}\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& \leq C\left(\|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^s} + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^s}\right) \\
& \quad + \sqrt{2}C\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{2}C\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}}\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}} \\
& + \sqrt{2}C\|u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}}\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{4}}\|\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^{\frac{1}{2}} + \sqrt{2}C\|(\partial_t u)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}}\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{4}}\|\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^{\frac{1}{2}}.
\end{aligned}$$

Applying Young's inequality yields

$$\begin{aligned}
& C\|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{4}}\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^{\frac{1}{2}}\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}} \\
& \leq C\|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}}\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{10}\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}.
\end{aligned}$$

Then we achieve

$$\begin{aligned}
(5.4.9) \quad & \|e^{\mathcal{R}t}(u + \partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
& + \sqrt{k\lambda}\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + c\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \sqrt{\lambda}\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{\lambda}\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& \leq C\|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^s} + C\|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^s} + C\|e^{a|D_x|}u_1\|_{\mathcal{B}^s} \\
& \quad + C(2 + \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}})\|e^{\mathcal{R}t}(u, \partial_t u)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

Therefore if we take

$$\begin{aligned}
& \lambda \geq 4C^2(2 + \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}) \\
(5.4.10) \quad & k\lambda \geq 8C^2(2 + \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}),
\end{aligned}$$

we obtain

$$\mathcal{E}_{s,\frac{\lambda}{4}}(u)(t) \leq C\left(\|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^s} + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^s}\right),$$

where

$$\begin{aligned}
\mathcal{E}_{s,\frac{\lambda}{4}}(u)(t) & = \|e^{\mathcal{R}t}(u + \partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\
& \quad + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + c\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
& \quad + \frac{\sqrt{k\lambda}}{2}\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{\sqrt{\lambda}}{2}\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{\sqrt{\lambda}}{2}\|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}
\end{aligned}$$

which in particular implies that under Condition (5.1.11), there holds

$$\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq \sqrt{C}\left(\|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{3}{2}}}\right).$$

Then by taking $\lambda = C(2 + \|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{3}{2}}})$, therefore the condition of the proposition is satisfied and then the proposition is proved for any $t > 0$.

Proposition 5.4.2. *We assume that the condition (5.1.11) is satisfied, then for any $s > 0$, there exist*

a small constant C , such that for

$$\lambda = C(1 + \|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{3}{2}}}),$$

we have

$$(5.4.11) \quad \begin{aligned} & \|e^{\mathcal{R}t}\partial_y(u + \partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y^2 u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\ & + \|e^{\mathcal{R}t}\partial_y(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y^2 u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq C \left(\|e^{a|D_x|}\partial_y^2 u_0\|_{\mathcal{B}^s} \right. \\ & \quad + \|e^{a|D_x|}\partial_y(u_0 + u_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}\partial_y u_1\|_{\mathcal{B}^s} \\ & \quad \left. + \|e^{a|D_x|}u_0\|_{\mathcal{B}^{s+2}} + \|e^{a|D_x|}u_0\|_{\mathcal{B}^{s+1}} + \|e^{a|D_x|}u_1\|_{\mathcal{B}^{s+1}} \right), \quad \text{for } t < T^*. \end{aligned}$$

Proof. of Proposition 5.4.2: We start by applying the partial derivative ∂_y to (5.1.9), we obtain

$$\partial_t^2 \partial_y u + \partial_t \partial_y u + \partial_y(u \partial_x u) + \partial_y(v \partial_y u) - \partial_y^3 u + \partial_x \partial_y p = 0.$$

Due to the divergence-free condition of U (it means that $\partial_x u + \partial_y v = 0$), we have

$$\begin{aligned} \partial_y(u \partial_x u) + \partial_y(v \partial_y u) &= \partial_y u \partial_x u + u \partial_x \partial_y u + \partial_y v \partial_y u + v \partial_y^2 u \\ &= \partial_y u \partial_x u + u \partial_x \partial_y u - \partial_x u \partial_y u + v \partial_y^2 u \\ &= u \partial_x \partial_y u + v \partial_y^2 u, \end{aligned}$$

so our equation becomes

$$(5.4.12) \quad \partial_t^2 w + \partial_t w + u \partial_x w + v \partial_y w - \partial_y^2 w + \partial_x q = 0$$

where $w = \partial_y u$ and $q = \partial_y p$. from which, using that $-\partial_y w + \partial_x p$ is vanishing on the boundary, we get, by using a similar derivation of (5.3.12) and (5.3.20). It mean that we take compute the scalar product of our equation with $\Delta_q^h(w + 2\partial_t w)_\phi$, so that

$$(5.4.13) \quad \begin{aligned} & \frac{d}{dt} \left(\|e^{\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t}\Delta_q^h \partial_y w_\phi\|_{L^2}^2 \right) + 2 \|e^{\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi\|_{L^2}^2 \\ & \quad + 2\lambda\dot{\theta}(t') \|e^{\mathcal{R}t}\Delta_q^h |D_x|^{\frac{1}{2}}(\partial_t w)_\phi\|_{L^2}^2 + 2\lambda\dot{\theta}(t') \|e^{\mathcal{R}t}\Delta_q^h |D_x|^{\frac{1}{2}}\partial_y w_\phi\|_{L^2}^2 \\ & \leq 2 \left(|\langle \Delta_q^h(u \partial_x w)_\phi, e^{2\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi \rangle|_{L^2} + |\langle \Delta_q^h(v \partial_y w)_\phi, e^{2\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi \rangle|_{L^2} \right) \\ & + 2\mathcal{R} \left(\|e^{\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t}\Delta_q^h \partial_y w_\phi\|_{L^2}^2 \right) + 2 \left| \int_{\mathbb{R}} \Delta_q^h \partial_x p_\phi \cdot e^{2\mathcal{R}t}\Delta_q^h(\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \right|, \end{aligned}$$

and

$$(5.4.14) \quad \begin{aligned} & \frac{d}{dt} \mathcal{A}(w_\phi)(t) + \frac{1}{2} \frac{d}{dt} \|e^{\mathcal{R}t}\Delta_q^h w_\phi\|_{L^2}^2 + \lambda\dot{\theta}(t) \|e^{\mathcal{R}t}\Delta_q^h |D_x|^{\frac{1}{2}} w_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t}\Delta_q^h \partial_y w_\phi\|_{L^2}^2 \\ & = - \langle \Delta_q^h(u \partial_x w)_\phi, e^{2\mathcal{R}t}\Delta_q^h w_\phi \rangle_{L^2} - \langle \Delta_q^h(v \partial_y w)_\phi, e^{2\mathcal{R}t}\Delta_q^h w_\phi \rangle_{L^2} + \mathcal{R} \left(\|e^{\mathcal{R}t}\Delta_q^h w_\phi\|_{L^2}^2 \right. \\ & \quad \left. + 2 \int e^{2\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi \Delta_q^h w_\phi dx \right) - \int_{\mathbb{R}} \Delta_q^h \partial_x p_\phi \cdot e^{2\mathcal{R}t}\Delta_q^h(w(t, x, 1) - w(t, x, 0))_\phi dx \end{aligned}$$

where

$$\mathcal{A}(w_\phi)(t) = \int e^{2\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi \Delta_q^h w_\phi dx - \int e^{2\mathcal{R}t}\Delta_q^h(\partial_t w)_\phi \Delta_q^h(\partial_t w)_\phi dx$$

$$+ 2\lambda\dot{\theta}(t) \int e^{2\mathcal{R}t} \Delta_q^h |D_x| (\partial_t w)_\phi \Delta_q^h w_\phi dx$$

Now, We start to estimate the pressure term, for date we denote

$$K_q = \int_{\mathbb{R}} \Delta_q^h \partial_x p_\phi \cdot \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx.$$

In view of (5.3.4) and Lemma 5.2.3, we write

$$\begin{aligned} K_q &= \int_{\mathbb{R}} \Delta_q^h \partial_x p_\phi \cdot \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \\ &= \int_{\mathbb{R}} \Delta_q^h \left((w_\phi(t, x, 1) - w_\phi(t, x, 0)) \cdot \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0)) \right)_\phi dx \\ &\quad - \int_{\mathbb{R}} \Delta_q^h \partial_x \left(\int_0^1 (u)_\phi^2 dy \right) \cdot \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \Delta_q^h \left((w(t, x, 1) - w(t, x, 0))_\phi \right)^2 dx + \lambda\dot{\theta}(t) \int_{\mathbb{R}} \Delta_q^h |D_x| \left((w(t, x, 1) - w(t, x, 0))_\phi \right)^2 dx \\ &\quad - \int_{\mathbb{R}} \Delta_q^h \partial_x \left(\int_0^1 (u)_\phi^2 dy \right) \cdot \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \\ &\quad - \mathcal{R} \int_{\mathbb{R}} \Delta_q^h \left((w(t, x, 1) - w(t, x, 0))_\phi \right)^2 dx \end{aligned}$$

If we use Lemma 5.2.3 to the following quantity $\int_{\mathbb{R}} \Delta_q^h \partial_x \left(\int_0^1 (u)^2 dy \right) \cdot e^{2\phi} \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx$, we can obtain that

$$\begin{aligned} &\int_{\mathbb{R}} \Delta_q^h e^\phi \partial_x \left(\int_0^1 (u)^2 dy \right) \cdot \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \\ &= \frac{d}{dt} \int_{\mathbb{R}} \Delta_q^h e^\phi \partial_x \left(\int_0^1 (u)^2 dy \right) \cdot e^\phi \Delta_q^h (w(t, x, 1) - w(t, x, 0)) dx \\ &\quad - 2 \int_{\mathbb{R}} \Delta_q^h e^\phi \partial_x \left(\int_0^1 (u \partial_t u) dy \right) \cdot e^\phi \Delta_q^h (w(t, x, 1) - w(t, x, 0)) dx \\ &\quad + 2\lambda\dot{\theta}(t) \int_{\mathbb{R}} \Delta_q^h e^\phi |D_x| \partial_x \left(\int_0^1 (u)^2 dy \right) \cdot e^\phi \Delta_q^h (w(t, x, 1) - w(t, x, 0)) dx. \end{aligned}$$

So we multiply our equation by $e^{2\mathcal{R}t}$, we achieve that

$$\begin{aligned} |e^{2\mathcal{R}t} K_q(t)| &\leq \frac{1}{2} \frac{d}{dt} \|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L_v^\infty(L_h^2)}^2 + \lambda\dot{\theta}(t) \|\Delta_q^h e^{\mathcal{R}t} |D_x|^{\frac{1}{2}} w_\phi\|_{L_v^\infty(L_h^2)}^2 \\ &\quad + \frac{d}{dt} \left| \int_{\mathbb{R}} e^{2\mathcal{R}t} \Delta_q^h e^\phi \partial_x \left(\int_0^1 (u)^2 dy \right) \cdot e^\phi \Delta_q^h (w(t, x, 1) - w(t, x, 0)) dx \right| \\ &\quad + 2 \int_{\mathbb{R}} |e^{2\mathcal{R}t} \Delta_q^h e^\phi \partial_x \left(\int_0^1 (u \partial_t u) dy \right) \cdot e^\phi \Delta_q^h (w(t, x, 1) - w(t, x, 0))| dx \\ &\quad + 2\lambda\dot{\theta}(t) \int_{\mathbb{R}} |e^{2\mathcal{R}t} \Delta_q^h e^\phi |D_x| \partial_x \left(\int_0^1 (u)^2 dy \right) \cdot e^\phi \Delta_q^h (w(t, x, 1) - w(t, x, 0))| dx \\ &\quad + 2\mathcal{R} \left| \int_{\mathbb{R}} e^{2\mathcal{R}t} \Delta_q^h e^\phi \partial_x \left(\int_0^1 (u)^2 dy \right) \cdot e^\phi \Delta_q^h (w(t, x, 1) - w(t, x, 0)) dx \right| \\ &\quad + \mathcal{R} \left| \int_{\mathbb{R}} \Delta_q^h \left((w(t, x, 1) - w(t, x, 0))_\phi \right)^2 dx \right|. \end{aligned}$$

from which, we infer

$$\begin{aligned}
|e^{2\mathcal{R}t}K_q(t)| &\leq \frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} \|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L_v^\infty(L_h^2)}^2 + \|\Delta_q^h e^{\mathcal{R}t} \partial_x(u^2)_\phi\|_{L_v^1(L_h^2)}^2 \right) \\
&+ \left(\lambda \dot{\theta}(t) \|\Delta_q^h e^{\mathcal{R}t} |D_x|^{\frac{1}{2}} w_\phi\|_{L_v^\infty(L_h^2)}^2 + \lambda \dot{\theta}(t) \|\Delta_q^h e^{\mathcal{R}t} |D_x|^{\frac{1}{2}} \partial_x(u^2)_\phi\|_{L_v^1(L_h^2)}^2 \right) \\
&+ C \left(\|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L_v^\infty(L_h^2)}^2 + \|\Delta_q^h e^{\mathcal{R}t} (u \partial_t u)_\phi\|_{L_v^1(L_h^2)}^2 + \lambda \dot{\theta}(t) \|\Delta_q^h e^{\mathcal{R}t} |D_x|^{\frac{1}{2}} w_\phi\|_{L_v^\infty(L_h^2)}^2 \right) \\
&+ \mathcal{R} \left(\frac{3}{2} \|\Delta_q^h e^{\mathcal{R}t} w_\phi\|_{L_v^\infty(L_h^2)}^2 + \|\Delta_q^h e^{\mathcal{R}t} \partial_x(u^2)_\phi\|_{L_v^1(L_h^2)}^2 \right).
\end{aligned}$$

By applying Bony's decomposition, we have for any $s > 0$

$$\begin{aligned}
\|e^{\mathcal{R}t} \Delta_q^h \partial_x(u^2)_\phi\|_{L_t^2(L_v^1(L_h^2))} &\lesssim \|e^{\mathcal{R}t} \Delta_q^h \partial_x(u^2)_\phi\|_{L_t^2(L^2)} \\
&\lesssim d_q 2^{-qs} \|u_\phi\|_{L_t^2(L^\infty)} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} \\
&\lesssim d_q 2^{-qs} \|\partial_y u_\phi\|_{L_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})}
\end{aligned}$$

and

$$\begin{aligned}
\|e^{\mathcal{R}t} \Delta_q^h \partial_x(u \partial_t u)_\phi\|_{L_t^2(L_v^1(L_h^2))} &\lesssim \|e^{\mathcal{R}t} \Delta_q^h \partial_x(u \partial_t u)_\phi\|_{L_t^2(L^2)} \\
&\lesssim d_q 2^{-qs} \left(\|u_\phi\|_{L_t^2(L^\infty)} \|e^{\mathcal{R}t} \partial_t u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} + \|\partial_t u_\phi\|_{L_t^2(L^\infty)} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} \right) \\
&\lesssim d_q 2^{-qs} \left(\|\partial_y u_\phi\|_{L_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_t u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} + \|\partial_t \partial_y u_\phi\|_{L_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} \right).
\end{aligned}$$

While notice that

$$\int_0^1 \Delta_q^h \partial_y u_\phi(t, x, y) dy = 0,$$

then for any fixed $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, there exist $Y_0^q(t, x)$ so that $\Delta_q^h \partial_y u_\phi(t, x, Y_0^q(t, x)) = 0$. So we have

$$(\Delta_q^h \partial_y u_\phi(t, x, y))^2 \leq 2 \|\Delta_q^h \partial_y u_\phi\|_{L_v^2} \|\Delta_q^h \partial_y^2 u_\phi\|_{L_v^2},$$

which implies that

$$\|\Delta_q^h \partial_y u_\phi(t, x, y)\|_{L_v^\infty(L_h^2)} \leq 2 \|\Delta_q^h \partial_y u_\phi\|_{L^2} \|\Delta_q^h \partial_y^2 u_\phi\|_{L^2}.$$

As a result, it comes out

(5.4.15)

$$\begin{aligned}
\int_0^t |e^{2\mathcal{R}t} K_q(t')| dt' &\leq d_q^2 2^{-2qs} \left(C\mathbb{E}(0) + C \|e^{\mathcal{R}t} w_\phi\|_{L_t^\infty(\mathcal{B}^s)}^2 + \frac{1}{4} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{L_t^\infty(\mathcal{B}^s)}^2 \right) \\
&+ \frac{1}{2} \|\partial_y u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})}^2 \|e^{\mathcal{R}t} u_\phi\|_{L_t^\infty(\mathcal{B}^{s+1})}^2 + \lambda \|\partial_y u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^2 \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C \|e^{\mathcal{R}t} w_\phi\|_{L_t^2(\mathcal{B}^s)}^2 \\
&+ \frac{1}{4} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{L_t^2(\mathcal{B}^s)}^2 + \|\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^2 \|e^{\mathcal{R}t} \partial_t u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})}^2 + \|\partial_t \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})}^2 \\
&+ C \lambda \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \frac{\lambda}{4} \|e^{\mathcal{R}t} \partial_y^2 u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \\
&\frac{3\mathcal{R}}{8} \|e^{\mathcal{R}t} w_\phi\|_{L_t^2(\mathcal{B}^s)}^2 + \frac{3\mathcal{R}}{8} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{L_t^2(\mathcal{B}^s)}^2 + \mathcal{R} \|\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})}^2
\end{aligned}$$

where

$$\mathbb{E}(0) = \|e^{a|D_x|} w_0\|_{\mathcal{B}^s}^2 + \|e^{a|D_x|} \partial_y w_0\|_{\mathcal{B}^s}^2 + \|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{1}{2}}}^2 \|e^{a|D_x|} u_0\|_{\mathcal{B}^{s+1}}^2$$

Along the same way we obtain

$$(5.4.16) \quad \int_0^t |e^{2\mathcal{R}t'} \int_{\mathbb{R}} \Delta_q^h \partial_x p_\phi \cdot \Delta_q^h (w(t, x, 1) - w(t, x, 0))_\phi dx| dt' \\ \leq d_q^2 2^{-2qs} \left(C \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + C \|u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^2 \|e^{\mathcal{R}t} u_\phi\|_{L_t^\infty(\mathcal{B}^{s+1})}^2 + \frac{1}{4} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right).$$

It follows from the proof of Lemma 5.2.5, for any $s > 0$

$$\int_0^t \left| \left\langle \Delta_q^h (T_u^h \partial_x w + R^h(u, \partial_x w))_\phi, e^{2\mathcal{R}t'} \Delta_q^h (\partial_t w)_\phi \right\rangle_{L^2} \right| dt' \\ \leq C \tilde{d}_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

While we deduce from Lemma 5.2.1 and Definition 5.2.3

$$\int_0^t \left| \left\langle \Delta_q^h (T_{\partial_x w}^h u)_\phi, e^{2\mathcal{R}t'} \Delta_q^h (\partial_t w)_\phi \right\rangle_{L^2} \right| dt' \\ \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x w_\phi(t')\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h u_\phi(t')\|_{L_h^2(L_v^\infty)} \|\Delta_q^h (\partial_t w)_\phi(t')\|_{L^2} dt' \\ \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \int_0^t e^{2\mathcal{R}t'} \|w_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_y u_\phi(t')\|_{L^2} \|\Delta_q^h (\partial_t w)_\phi(t')\|_{L^2} dt' \\ \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \|w_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h w_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ \times \left(\int_0^t \|w_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h (\partial_t w)_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Then we conclude for any $s > 0$

$$(5.4.17) \quad \int_0^t \left| \left\langle \Delta_q^h (u \partial_x w)_\phi, e^{2\mathcal{R}t'} \Delta_q^h (\partial_t w)_\phi \right\rangle_{L^2} \right| dt' \\ \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

In the same way we have

$$(5.4.18) \quad \int_0^t \left| \left\langle \Delta_q^h (u \partial_x w)_\phi, e^{2\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} \right| dt' \\ \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

On the other hand, we deduce from Lemma 5.2.1 that for any $s > 0$

$$\int_0^t \left| \left\langle \Delta_q^h (T_v^h \partial_y w)_\phi, e^{2\mathcal{R}t'} \Delta_q^h (\partial_t w)_\phi \right\rangle_{L^2} \right| dt' \\ \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h v_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_y w_\phi(t')\|_{L^2} \|\Delta_q^h (\partial_t w)_\phi(t')\|_{L^2} dt' \\ \lesssim \sum_{|q'-q| \leq 4} 2^{\frac{q'}{2}} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} e^{\mathcal{R}t'} \|\Delta_{q'}^h \partial_y w_\phi\|_{L_t^2(L^2)} \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h (\partial_t w)_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}}$$

$$\leq Cd_q^2 2^{-2qs} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

In the same way we obtain that

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h R^h(v, \partial_y w)_\phi, e^{2\mathcal{R}t'} \Delta_q^h (\partial_t w)_\phi \right\rangle_{L^2} dt' \right. \\ & \leq Cd_q^2 2^{-2qs} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Finally, we use Lemma 5.2.5, we find

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h (T_{\partial_y w}^h v)_\phi, e^{2\mathcal{R}t'} \Delta_q^h (\partial_t w)_\phi \right\rangle_{L^2} dt' \right. \\ & \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_y w_\phi(t')\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h v_\phi(t')\|_{L_h^2(L_v^\infty)} \|\Delta_q^h (\partial_t w)_\phi(t')\|_{L^2} dt' \\ & \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|\partial_y w_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \|\Delta_q^h (\partial_t w)_\phi\|_{L^2} dt' \\ & \leq Cd_q^2 2^{-2qs} \|\partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{L_t^\infty(\mathcal{B}^{s+1})} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{2t}^2(\mathcal{B}^s)}. \end{aligned}$$

Then by summing we deduce that

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h (v \partial_y w)_\phi, e^{2\mathcal{R}t'} \Delta_q^h (\partial_t w)_\phi \right\rangle_{L^2} dt' \right. \\ & \leq Cd_q^2 2^{-2qs} \|\partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{L_t^\infty(\mathcal{B}^{s+1})} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & (5.4.19) \\ & + Cd_q^2 2^{-2qs} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \end{aligned}$$

Along the same way we find that

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h (T_v^h \partial_y w + R^h(v, \partial_y w))_\phi, e^{2\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} dt' \right. \\ & \lesssim d_q^2 2^{-2qs} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Then we still have to estimate $T_{\partial_y w}^h v$, so by integration by part we can obtain

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h (T_{\partial_y w}^h v)_\phi, e^{2\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} dt' \right. \\ & \leq \int_0^t \left| \left\langle \Delta_q^h (T_w^h \partial_y v)_\phi, e^{2\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} dt' \right. \\ & \quad \left. + \int_0^t \left| \left\langle \Delta_q^h (T_w^h v)_\phi, e^{2\mathcal{R}t'} \Delta_q^h \partial_y w_\phi \right\rangle_{L^2} dt' \right. \end{aligned}$$

Due to the free divergence $\partial_x u + \partial_y v = 0$ we deduce

$$\begin{aligned} & \int_0^t \left| \left\langle \Delta_q^h (T_w^h \partial_y v)_\phi, e^{2\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} dt' \right. \\ & \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h w_\phi(t')\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h \partial_x u_\phi(t')\|_{L_h^2(L_v^\infty)} \|\Delta_q^h w_\phi(t')\|_{L^2} dt' \\ & \lesssim \sum_{|q'-q| \leq 4} \int_0^t 2^{q'} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h w_\phi\|_{L^2} \|e^{\mathcal{R}t'} \Delta_q^h w_\phi(t')\|_{L^2} dt' \\ & \leq Cd_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2. \end{aligned}$$

While we observe that

$$\begin{aligned}
& \int_0^t \left| \left\langle \Delta_q^h (T_w^h v)_\phi, e^{2\mathcal{R}t'} \Delta_q^h \partial_y w_\phi \right\rangle_{L^2} \right| dt' \\
& \lesssim \sum_{|q'-q|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h w_\phi(t')\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h v_\phi(t')\|_{L_h^2(L_v^\infty)} \|\Delta_q^h \partial_y w_\phi(t')\|_{L^2} dt' \\
& \lesssim \sum_{|q'-q|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|w_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \|\Delta_q^h \partial_y w_\phi\|_{L^2} dt' \\
& \leq C d_q^2 2^{-2qs} \|w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}.
\end{aligned}$$

By summarizing the above estimates, we obtain

$$\begin{aligned}
(5.4.20) \quad & \int_0^t \left| \left\langle \Delta_q^h (v \partial_y w)_\phi, e^{2\mathcal{R}t'} \Delta_q^h w_\phi \right\rangle_{L^2} \right| dt' \leq C d_q^2 2^{-2qs} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& + C d_q^2 2^{-2qs} \left(\|w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right)
\end{aligned}$$

By inserting the resulting estimates (5.4.15)-(5.4.20) in (5.4.13) + (5.4.14) and multiplying by 2^{2qs} for $s > 0$, and then integrating over time, and summing with respect to $q \in \mathbb{Z}$, we find that for $t < T^*$

$$\begin{aligned}
(5.4.21) \quad & \left(\|e^{\mathcal{R}t} (w + \partial_t w)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \right) + \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
& + \sqrt{k\lambda} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \sqrt{\lambda} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{\lambda} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& \leq C \left(\mathcal{L}(0) + \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}} \right. \\
& \quad \left. + \|w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{4}} \|\partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^{\frac{1}{2}} + \|(\partial_t w)_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{4}} \|\partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^{\frac{1}{2}} \right) \\
& + \|\partial_t u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} + \|\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_t u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} + \|\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+2})} \\
& \quad \sqrt{\lambda} \left(C \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) + C \sqrt{\lambda} \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& \quad + C \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \frac{1}{8} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + C \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \frac{1}{8} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}
\end{aligned}$$

where

$$\mathcal{L}(0) = \mathbf{E}(0) + \|e^{a|D_x|} \partial_y w_0\|_{\mathcal{B}^s} + \|e^{a|D_x|} (w_0 + w_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|} w_1\|_{\mathcal{B}^s}.$$

Applying Young's inequality yields

$$\begin{aligned}
& C \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{4}} \|e^{\mathcal{R}t'} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^{\frac{1}{2}} \|e^{\mathcal{R}t'} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^{\frac{1}{2}} \\
& \leq C \|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t'} w_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{8} \|e^{\mathcal{R}t'} \partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}.
\end{aligned}$$

Then we achieve

$$(5.4.22) \quad \left(\|e^{\mathcal{R}t} (w + \partial_t w)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \frac{3}{2} \|e^{\mathcal{R}t} \partial_y w_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \right) + \|e^{\mathcal{R}t} (\partial_t w)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}$$

$$\begin{aligned}
& +\sqrt{k\lambda}\|e^{\mathcal{R}t}w_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}+\frac{1}{2}\|e^{\mathcal{R}t}\partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}+\sqrt{\lambda}\|e^{\mathcal{R}t}(\partial_t w)_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}+\sqrt{\lambda}\|e^{\mathcal{R}t}\partial_y w_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& \leq \sqrt{C}\left(\mathcal{L}(0)+\left(\sqrt{2}+\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}\right)\|e^{\mathcal{R}t}w_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}+\sqrt{\lambda}\left(1+\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}\right)\|e^{\mathcal{R}t}w_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}\right. \\
& \quad \left.+\left(\sqrt{2}+\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}\right)\|e^{\mathcal{R}t}(\partial_y w,\partial_t w)_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}\right. \\
& \quad \left.+\|\partial_t u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})}+\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})}+\|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+2})}\right).
\end{aligned}$$

Therefore if we take

$$(5.4.23) \quad \begin{cases} \lambda \geq 8C\left(2+\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}\right) \\ k\lambda \geq 8C\left(2+\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}\right), \end{cases}$$

we obtain by using the

$$\begin{aligned}
\mathcal{E}_{s,\frac{\lambda}{4}}(w)(t) & \leq C\left(\mathcal{L}(0)+\|e^{a|D_x|}u_0\|_{\mathcal{B}^{s+2}}+\|e^{a|D_x|}u_0\|_{\mathcal{B}^{s+1}}+\|e^{a|D_x|}u_1\|_{\mathcal{B}^{s+1}}\right. \\
& \quad \left.+\sqrt{\lambda}\left(1+\|\partial_y u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}\right)\|e^{\mathcal{R}t}w_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}_{s,\frac{\lambda}{4}}(w)(t) & = \left(\|e^{\mathcal{R}t}(w+\partial_t w)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}+\|e^{\mathcal{R}t}(\partial_t w)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}+\frac{3}{2}\|e^{\mathcal{R}t}\partial_y w_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)}\right) \\
& \quad +\|e^{\mathcal{R}t}(\partial_t w)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}+\frac{\sqrt{k\lambda}}{4}\|e^{\mathcal{R}t}w_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}+\frac{1}{2}\|e^{\mathcal{R}t}\partial_y w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
& \quad +\frac{\sqrt{\lambda}}{4}\|e^{\mathcal{R}t}(\partial_t w)_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}+\frac{\sqrt{\lambda}}{4}\|e^{\mathcal{R}t}\partial_y w_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}
\end{aligned}$$

Then, taking $\lambda = 8C(2+\|e^{a|D_x|}\partial_y u_0\|_{\mathcal{B}^{\frac{3}{2}}}+\|e^{a|D_x|}(u_0+u_1)\|_{\mathcal{B}^{\frac{3}{2}}}+\|e^{a|D_x|}u_1\|_{\mathcal{B}^{\frac{3}{2}}})$, therefore the condition of the proposition is satisfied and then the proposition is proved. \square

As a matter of fact, it remains to present the estimate of $\|\Delta_q^h(\partial_t^2 u)_\phi\|_{L^2}$, this estimate will serve us in the proof of the last theorem 5.1.3. Indeed by applying Δ_q^h to (5.3.18) and take the L^2 inner product of resulting equation with $\Delta_q^h(\partial_t^2 u)_\phi$. That yields

$$\begin{aligned}
\|\Delta_q^h(\partial_t^2 u)_\phi\|_{L^2}^2 & = \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} - \langle \Delta_q^h(\partial_t u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} - \langle \Delta_q^h(u\partial_x u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \\
& \quad - \langle \Delta_q^h(v\partial_y u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2}.
\end{aligned}$$

The fact that $(\partial_t u)_\phi = \partial_t u_\phi + \lambda\theta(t)|D_x|u_\phi$ implies

$$\langle \Delta_q^h(\partial_t u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} = \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \lambda\dot{\theta}(t)2^q \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2,$$

from which, we deduce that

$$\|\Delta_q^h(\partial_t^2 u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 & = \left| \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right| \\
I_2 & = \left| \langle \Delta_q^h(u\partial_x u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right|
\end{aligned}$$

$$I_3 = \left| \langle \Delta_q^h(v\partial_y u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right|$$

$$I_4 = \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right|.$$

Since $\partial_x u + \partial_y v = 0$, using (5.3.1) and integrations by parts, we find

$$I_4 = \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right| = 0.$$

For I_1 , I_2 and I_3 we have

$$I_2 = \left| \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right| \leq C \|\Delta_q^h(\partial_y^2 u)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t^2 u)_\phi\|_{L^2}^2$$

$$I_2 = \left| \langle \Delta_q^h(u\partial_x u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right| \leq C \|\Delta_q^h(u\partial_x u)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t^2 u)_\phi\|_{L^2}^2$$

$$I_3 = \left| \langle \Delta_q^h(v\partial_y u)_\phi, \Delta_q^h(\partial_t^2 u)_\phi \rangle_{L^2} \right| \leq C \|\Delta_q^h(v\partial_y u)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t^2 u)_\phi\|_{L^2}^2.$$

Then, we deduce that

$$\|\Delta_q^h(\partial_t^2 u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 \leq C (\|\Delta_q^h(u\partial_x u)_\phi\|_{L^2}^2 + \|\Delta_q^h(v\partial_y u)_\phi\|_{L^2}^2 + \|\Delta_q^h \partial_y^2 u_\phi\|_{L^2}^2).$$

Multiplying the result by $e^{2\mathcal{R}t}$ and integrating over $[0, t]$, we get

$$\|e^{\mathcal{R}t} \Delta_q^h(\partial_t^2 u)_\phi\|_{L_t^2(L^2)}^2 + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h(\partial_t u)_\phi\|_{L_t^\infty(L^2)}^2$$

$$\leq C \left(\|\Delta_q^h e^{a|D_x|} u_1\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h(u\partial_x u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(v\partial_y u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y^2 u_\phi\|_{L_t^2(L^2)}^2 \right).$$

Multiplying the above inequality by 2^{3q} , then taking the square root of the resulting estimate, and finally summing up the obtained equations with respect to $q \in \mathbb{Z}$, we obtain

$$(5.4.24) \quad \|e^{\mathcal{R}t}(\partial_t^2 u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \frac{1}{4} \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq C \left(\|e^{a|D_x|} u_1\|_{\mathcal{B}^{\frac{3}{2}}} \right.$$

$$\left. + \|e^{\mathcal{R}t}(u\partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(v\partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t} \partial_y^2 u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right).$$

Next, to deal with the nonlinear terms in (5.4.24), we need the following lemma that gives the estimates of the terms $\|e^{\mathcal{R}t}(u\partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}$ and $\|e^{\mathcal{R}t}(v\partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}$, the proof will be given in the appendix 5.8.

Lemma 5.4.1.

$$(5.4.25) \quad \|e^{\mathcal{R}t}(u\partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \leq C \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})};$$

and

$$(5.4.26) \quad \|e^{\mathcal{R}t}(v\partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \leq C \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} + \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{5}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}.$$

Inserting the above estimates into (5.4.24) and then using the smallness condition $\|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{4C^2}$ and propositions 5.4.1 and 5.4.2, we finally obtain

$$\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq C \left(\|e^{a|D_x|} u_1\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_y u_0\|_{\mathcal{B}^{\frac{5}{2}}} \right.$$

$$\left. + \|e^{a|D_x|}(u_0 + u_1)\|_{\mathcal{B}^{\frac{5}{2}}} + \|e^{a|D_x|} \partial_y^2 u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|} \partial_y(u_0 + u_1)\|_{\mathcal{B}^{\frac{3}{2}}} \right).$$

5.5 Global well-posedness of System (5.1.8)

The goal of this section is to prove the Theorem 5.1.2 and to establish the global well-posedness of the system (5.1.8) with small analytic data. As in Section 2, for any locally bounded function Θ on $\mathbb{R}_+ \times \mathbb{R}$ and any $u \in L^2(\mathcal{S})$, we define the analyticity in the horizontal variable x by means of the following auxiliary function

$$(5.5.1) \quad u_\Theta^\epsilon(t, x, y) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\epsilon^{\Theta(t, \xi)} \widehat{u}^\epsilon(t, \xi, y)).$$

The width of the analyticity band Θ is defined by

$$\Theta(t, \xi) = (a - \lambda\tau(t))|\xi|,$$

where $\lambda > 0$ with be precised later and $\tau(t)$ will be chosen in such a way that $\Theta(t, \xi) > 0$, for any $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}$ and $\dot{\Theta}(t) = \Theta'(t) = -\lambda\dot{\tau}(t) \leq 0$. In our paper, we will choose

$$(5.5.2) \quad \dot{\tau}(t) = \|\partial_y u_\Theta^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_y v_\Theta^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} \quad \text{with} \quad \tau(0) = 0.$$

In what follows, for the sake of the simplicity, we will neglect the script ϵ and write (u_Θ, v_Θ) instead of $(u_\Theta^\epsilon, v_\Theta^\epsilon)$. In view of System (5.1.8), we can transform it like a equation of order one in time, so if we define $U = (u, \partial_t u)$ and $V = (v, \partial_t v)$, Then U and V satisfy the following equation

$$(5.5.3) \quad \begin{cases} \partial_t U + A_\epsilon(D)U = - \begin{pmatrix} 0 \\ u\partial_x u + v\partial_y u + \partial_x p \end{pmatrix} \\ \epsilon^2 (\partial_t V + A_\epsilon(D)V) = - \begin{pmatrix} 0 \\ \epsilon^2(u\partial_x v + v\partial_y v) + \partial_y p \end{pmatrix} \\ \partial_x u + \partial_y v = 0 \\ (u, v)/_{y=0} = (u, v)/_{y=1} = 0 \end{cases}$$

where

$$U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \quad \text{and} \quad A_\epsilon(D) = \begin{pmatrix} 0 & -1 \\ -\epsilon^2 \partial_x^2 - \partial_y^2 & 1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} v \\ \partial_t v \end{pmatrix}$$

Then in view of (5.2.3) we observe that $(U, V)_\Theta$ verifies

$$(5.5.4) \quad \begin{cases} \partial_t U_\Theta + \lambda\dot{\tau}(t)|D_x|U_\Theta + A_\epsilon(D)U_\Theta = - \begin{pmatrix} 0 \\ (u\partial_x u)_\Theta + (v\partial_y u)_\Theta + \partial_x p_\Theta \end{pmatrix} \\ \epsilon^2 (\partial_t V_\Theta + \lambda\dot{\tau}(t)|D_x|V_\Theta + A_\epsilon(D)V_\Theta) = - \begin{pmatrix} 0 \\ \epsilon^2(u\partial_x v)_\Theta + \partial_y p_\Theta \end{pmatrix} \\ \partial_x u_\Theta + \partial_y v_\Theta = 0 \\ (u_\Theta, v_\Theta)/_{y=0} = (u_\Theta, v_\Theta)/_{y=1} = 0 \end{cases}$$

Where $|D_x|$ denote the Fourier multiplier of the symbol $|\xi|$. In what follows, we recall that we use C to denote a generic positive constant which can change from line to line.

By applying the dyadic operator in the horizontal variable Δ_q^h to (5.5.4) and taking the L^2 inner product of the resulting equation with $\Delta_q^h U_\Theta$ and $\Delta_q^h V_\Theta$ we obtain

$$(5.5.5) \quad \langle \Delta_q^h \partial_t U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} + \lambda \dot{\tau}(t) \langle \Delta_q^h |D_x| U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} + \langle \Delta_q^h A_\epsilon(D) U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} \\ = - \langle \Delta_q^h (u \partial_x u + v \partial_y u)_\Theta, \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\Theta, \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2}$$

and

$$(5.5.6) \quad \epsilon^2 \langle \Delta_q^h \partial_t V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} + \epsilon^2 \lambda \dot{\tau}(t) \langle \Delta_q^h |D_x| V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} + \epsilon^2 \langle \Delta_q^h B_\epsilon(D) V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} \\ = - \epsilon^2 \langle \Delta_q^h (u \partial_x v + v \partial_y v)_\Theta, \Delta_q^h (\partial_t v)_\Theta \rangle_{L^2} - \langle \Delta_q^h \partial_y p_\Theta, \Delta_q^h (\partial_t v)_\Theta \rangle_{L^2}.$$

As in (5.3.10) and by using Lemma 5.2.3, we gather

$$(5.5.7) \quad \langle \Delta_q^h \partial_t U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} + \lambda \dot{\tau}(t) \langle \Delta_q^h |D_x| U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} = \langle \Delta_q^h \partial_t u_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} \\ + \langle \Delta_q^h \partial_t (\partial_t u)_\Theta, \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2} + \lambda \dot{\tau}(t) \left(\langle \Delta_q^h |D_x| u_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} + \langle \Delta_q^h |D_x| (\partial_t u)_\Theta, \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2} \right) \\ = \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q^h u_\Theta\|_{L^2}^2 + \|\Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2 \right) + \lambda \dot{\tau}(t) \left(\|\Delta_q^h |D_x|^{\frac{1}{2}} u_\Theta\|_{L^2}^2 + \|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\Theta\|_{L^2}^2 \right),$$

and

$$(5.5.8) \quad \epsilon^2 \langle \Delta_q^h \partial_t V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} + \epsilon^2 \lambda \dot{\tau}(t) \langle \Delta_q^h |D_x| V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} = \epsilon^2 \langle \Delta_q^h \partial_t v_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} \\ + \epsilon^2 \langle \Delta_q^h \partial_t (\partial_t v)_\Theta, \Delta_q^h (\partial_t v)_\Theta \rangle_{L^2} + \epsilon^2 \lambda \dot{\tau}(t) \left(\langle \Delta_q^h |D_x| v_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} + \langle \Delta_q^h |D_x| (\partial_t v)_\Theta, \Delta_q^h (\partial_t v)_\Theta \rangle_{L^2} \right) \\ = \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q^h \epsilon v_\Theta\|_{L^2}^2 + \|\Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2 \right) + \lambda \dot{\tau}(t) \left(\|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon v_\Theta\|_{L^2}^2 + \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon (\partial_t v)_\Theta\|_{L^2}^2 \right).$$

Next, we use that $(\partial_t u)_\Theta = \partial_t u_\Theta + \lambda \dot{\tau}(t) |D_x| u_\Theta$ and $(\partial_t v)_\Theta = \partial_t v_\Theta + \lambda \dot{\tau}(t) |D_x| v_\Theta$

$$\langle \Delta_q^h A_\epsilon(D) U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} = - \langle \Delta_q^h (\partial_t u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h \partial_x^2 u_\Theta, \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2} \\ - \langle \Delta_q^h \partial_y^2 u_\Theta, \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2} + \langle \Delta_q^h (\partial_t u)_\Theta, \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2}$$

and

$$\epsilon^2 \langle \Delta_q^h B_\epsilon(D) V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} = - \epsilon^2 \langle \Delta_q^h (\partial_t v)_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} - \epsilon^4 \langle \Delta_q^h \partial_x^2 v_\Theta, \Delta_q^h (\partial_t v)_\Theta \rangle_{L^2} \\ - \epsilon^2 \langle \Delta_q^h \partial_y^2 v_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} + \epsilon^2 \langle \Delta_q^h (\partial_t v)_\Theta, \Delta_q^h v_\Theta \rangle_{L^2}.$$

We use again the result of Lemma 5.2.3, we obtain

$$(5.5.9) \quad \langle \Delta_q^h A_\epsilon(D) U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} = - \frac{1}{2} \frac{d}{dt} \|\Delta_q^h u_\Theta\|_{L^2}^2 - \lambda \dot{\tau}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} u_\Theta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \epsilon \partial_x u_\Theta\|_{L^2}^2 \\ + \lambda \dot{\tau}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon \partial_x u_\Theta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y u_\Theta\|_{L^2}^2 + \lambda \dot{\tau}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\Theta\|_{L^2}^2 + \|\Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2,$$

and

$$(5.5.10) \quad \epsilon^2 \langle \Delta_q^h B_\epsilon(D) V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} = - \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \epsilon v_\Theta\|_{L^2}^2 - \lambda \dot{\tau}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon v_\Theta\|_{L^2}^2 + \frac{\epsilon^2}{2} \frac{d}{dt} \|\Delta_q^h \epsilon \partial_x v_\Theta\|_{L^2}^2 \\ + \epsilon^2 \lambda \dot{\tau}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon \partial_x v_\Theta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \epsilon \partial_y v_\Theta\|_{L^2}^2 + \lambda \dot{\tau}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon \partial_y v_\Theta\|_{L^2}^2 + \|\Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2.$$

We sum (5.5.7) with (5.5.9), we achieve

$$\begin{aligned}
& \langle \Delta_q^h \partial_t U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} + \lambda \dot{\tau}(t) \langle \Delta_q^h |D_x| U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} + \langle \Delta_q^h A_\epsilon(D) U_\Theta, \Delta_q^h U_\Theta \rangle_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2 + \|\Delta_q^h \epsilon \partial_x u_\Theta\|_{L^2}^2 + \|\Delta_q^h \partial_y u_\Theta\|_{L^2}^2 \right) + \|\Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2 \\
(5.5.11) \quad &+ \lambda \dot{\tau}(t) \left(\|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\Theta\|_{L^2}^2 + \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon \partial_x u_\Theta\|_{L^2}^2 + \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\Theta\|_{L^2}^2 \right).
\end{aligned}$$

Next we sum (5.5.8) with (5.5.10), we obtain

$$\begin{aligned}
& \epsilon^2 \langle \Delta_q^h \partial_t V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} + \epsilon^2 \lambda \dot{\tau}(t) \langle \Delta_q^h |D_x| V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} + \epsilon^2 \langle \Delta_q^h B_\epsilon(D) V_\Theta, \Delta_q^h V_\Theta \rangle_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2 \epsilon^2 + \|\Delta_q^h \epsilon \partial_x v_\Theta\|_{L^2}^2 + \|\Delta_q^h \epsilon \partial_y v_\Theta\|_{L^2}^2 \right) + \|\Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2 \\
(5.5.12) \quad &+ \lambda \dot{\tau}(t) \left(\|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon (\partial_t v)_\Theta\|_{L^2}^2 + \epsilon^2 \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon \partial_x v_\Theta\|_{L^2}^2 + \|\Delta_q^h |D_x|^{\frac{1}{2}} \epsilon \partial_y v_\Theta\|_{L^2}^2 \right).
\end{aligned}$$

By using the Dirichlet boundary condition $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$, and the incompressibility condition $\partial_x u + \partial_y v = 0$ and the relation, we can perform integration by parts, we get

$$\left| \langle \Delta_q^h \nabla p_\Theta, \Delta_q^h (\partial_t u, \partial_t v)_\Theta \rangle_{L^2} \right| = 0.$$

We insert the resulting equality (5.5.11) in (5.5.5) and (5.5.12) in (5.5.6), and then multiplying by $e^{2\mathcal{R}t}$, we achieve

$$\begin{aligned}
(5.5.13) \quad & \frac{1}{2} \frac{d}{dt} \left(\|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_x u_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \epsilon \partial_y v_\Theta\|_{L^2}^2 \right. \\
& \left. + \epsilon^4 \|e^{\mathcal{R}t} \Delta_q^h \partial_x v_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t} \Delta_q^h \partial_x u_\Theta\|_{L^2}^2 \right) + \|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2 \\
& + \lambda \dot{\tau}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u)_\Theta\|_{L^2}^2 + \epsilon^2 \lambda \dot{\tau}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t v)_\Theta\|_{L^2}^2 + \epsilon^2 \lambda \dot{\tau}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_x u_\Theta\|_{L^2}^2 \\
& + \lambda \dot{\tau}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y u_\Theta\|_{L^2}^2 + \lambda \dot{\tau}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_y v_\Theta\|_{L^2}^2 + \epsilon^4 \lambda \dot{\tau}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}} \partial_x v_\Theta\|_{L^2}^2 \\
& \leq \mathcal{R} \left(\|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \epsilon \partial_y v_\Theta\|_{L^2}^2 \right. \\
& \left. + \epsilon^4 \|e^{\mathcal{R}t} \Delta_q^h \partial_x v_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t} \Delta_q^h \partial_x u_\Theta\|_{L^2}^2 \right) + \left| \langle \Delta_q^h (u \partial_x u)_\Theta, e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2} \right| \\
& + \left| \langle \Delta_q^h (v \partial_y u)_\Theta, e^{2\mathcal{R}t} \Delta_q^h (\partial_t u)_\Theta \rangle_{L^2} \right| + \epsilon^2 \left| \langle \Delta_q^h (u \partial_x v)_\Theta, e^{2\mathcal{R}t} \Delta_q^h (\partial_t v)_\Theta \rangle_{L^2} \right| \\
& \quad + \epsilon^2 \left| \langle \Delta_q^h (v \partial_y v)_\Theta, e^{2\mathcal{R}t} \Delta_q^h (\partial_t v)_\Theta \rangle_{L^2} \right|.
\end{aligned}$$

In what follows, we shall always assume that $t < T_1^*$, where T_1^* given by

$$(5.5.14) \quad T_1^* \triangleq \sup \left\{ t > 0, \quad \tau(t) \leq \frac{a}{\lambda} \right\}.$$

Then we deduce from Lemmas 5.2.5-5.2.7 for any $t < T_1^*$, and integrating over time (5.5.13), that

$$\begin{aligned}
(5.5.15) \quad & \int_0^t \frac{1}{2} \frac{d}{dt'} \left(\|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h (\partial_y u, \epsilon \partial_y v)_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t'} \Delta_q^h (\partial_x u, \epsilon \partial_x v)_\Theta\|_{L^2}^2 \right) dt' \\
& + \int_0^t \left(\|e^{\mathcal{R}t'} \Delta_q^h (\partial_t u)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h \epsilon (\partial_t v)_\Theta\|_{L^2}^2 \right) dt' + \lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 dt' \\
& + \lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_y u, \epsilon \partial_y v)_\Theta\|_{L^2}^2 dt' + \epsilon^2 \lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_x u, \epsilon \partial_x v)_\Theta\|_{L^2}^2 dt'
\end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{R} \left(\|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h(\partial_y u, \epsilon \partial_y v)_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t'} \Delta_q^h(\partial_x u, \epsilon \partial_x v)_\Theta\|_{L^2}^2 \right) \\
&\quad + C2^{-2qs} d_q^2 \left(\|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t}(\partial_t u)_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right. \\
&\quad \left. + \|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t}(\epsilon \partial_t v)_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right).
\end{aligned}$$

We multiply the estimate (5.5.15) by 2, we have

$$\begin{aligned}
(5.5.16) \quad &\int_0^t \frac{d}{dt} \left(\|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h(\partial_y u, \epsilon \partial_y v)_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t'} \Delta_q^h(\partial_x u, \epsilon \partial_x v)_\Theta\|_{L^2}^2 \right) dt' \\
&+ 2 \int_0^t \left(\|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h(\partial_t v)_\Theta\|_{L^2}^2 \right) dt' + 2\lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 dt' \\
&+ 2\lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}}(\partial_y u, \epsilon \partial_y v)_\Theta\|_{L^2}^2 dt' + 2\epsilon^2 \lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}}(\partial_x u, \epsilon \partial_x v)_\Theta\|_{L^2}^2 dt' \\
&\leq 2\mathcal{R} \left(\|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h(\partial_y u, \epsilon \partial_y v)_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t'} \Delta_q^h(\partial_x u, \epsilon \partial_x v)_\Theta\|_{L^2}^2 \right) \\
&\quad + 2C2^{-2qs} d_q^2 \left(\|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t}(\partial_t u)_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right. \\
&\quad \left. + \|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t}(\epsilon \partial_t v)_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right).
\end{aligned}$$

Now we still have to get some information of the norm $\|\partial_y u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}}$ and $\|\partial_x u_\Theta\|_{\mathcal{B}^{\frac{1}{2}}}$, for that we need to apply the dyadic operator Δ_q^h to the equation

$$\begin{aligned}
(5.5.17) \quad &e^{\Theta(t, |D_x|)} \left(\partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u - \epsilon^2 \partial_x^2 u + \partial_x p \right) = 0 \\
&e^{\Theta(t, |D_x|)} \left(\epsilon^2 (\partial_t^2 v + \partial_t v + u \partial_x v + v \partial_y v - \partial_y^2 v - \epsilon^2 \partial_x v) + \partial_y p \right) = 0,
\end{aligned}$$

and then, we take the L^2 inner product of the resulting equation (5.5.17) with $\Delta_q^h u_\Theta$ and $\Delta_q^h v_\Theta$, we obtain

$$\begin{aligned}
(5.5.18) \quad &\langle \Delta_q^h(\partial_t^2 u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} + \langle \Delta_q^h(\partial_t u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} \\
&\quad - \langle \Delta_q^h \partial_y^2 u_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h \partial_x^2 u_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} \\
&\quad = - \langle \Delta_q^h(u \partial_x u + v \partial_y u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\Theta, \Delta_q^h u_\Theta \rangle_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
(5.5.19) \quad &\langle \Delta_q^h(\epsilon \partial_t^2 v)_\Theta, \Delta_q^h \epsilon v_\Theta \rangle_{L^2} + \langle \Delta_q^h(\epsilon \partial_t v)_\Theta, \Delta_q^h \epsilon v_\Theta \rangle_{L^2} \\
&\quad - \langle \Delta_q^h \epsilon \partial_y^2 v_\Theta, \Delta_q^h \epsilon v_\Theta \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h \epsilon \partial_x^2 v_\Theta, \Delta_q^h \epsilon v_\Theta \rangle_{L^2} \\
&\quad = -\epsilon^2 \langle \Delta_q^h(u \partial_x v + v \partial_y v)_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} - \langle \Delta_q^h \partial_y p_\Theta, \Delta_q^h v_\Theta \rangle_{L^2},
\end{aligned}$$

In what follows, we shall use again the technical lemmas in Section 5.2, to handle term by term in the estimate (5.5.18) and (5.5.19). We start by the complicate term $I_1 = \langle \Delta_q^h(\partial_t^2 u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2}$ and $I_2 = \langle \Delta_q^h(\epsilon \partial_t^2 v)_\Theta, \Delta_q^h \epsilon v_\Theta \rangle_{L^2}$, so by using integration by parts, we find

$$\begin{aligned}
I_1 &= \frac{d}{dt} \int \Delta_q^h(\partial_t u)_\Theta \Delta_q^h u_\Theta dX - \int \Delta_q^h(\partial_t u)_\Theta \Delta_q^h(\partial_t u)_\Theta dX \\
&\quad + 2\lambda \dot{\tau}(t) \int \Delta_q^h |D_x|(\partial_t u)_\Theta \Delta_q^h u_\Theta dX
\end{aligned}$$

$$I_2 = \frac{d}{dt} \int \Delta_q^h(\epsilon \partial_t v)_\Theta \Delta_q^h \epsilon v_\Theta dX - \int \Delta_q^h(\epsilon \partial_t v)_\Theta \Delta_q^h(\epsilon \partial_t v)_\Theta dX \\ + 2\lambda \dot{\tau}(t) \int \Delta_q^h |D_x|(\epsilon \partial_t v)_\Theta \Delta_q^h \epsilon v_\Theta dX$$

Whereas due to the boundary condition, and by integrating by part, we achieve

$$-\langle \Delta_q^h(\partial_y^2 u_\Theta + \epsilon^2 \partial_x^2 u_\Theta), \Delta_q^h u_\Theta \rangle_{L^2} = \|\Delta_q^h \partial_y u_\Theta\|_{L^2}^2 + \epsilon^2 \|\Delta_q^h \partial_x u_\Theta\|_{L^2}^2 \\ \langle \Delta_q^h(-\epsilon \partial_y^2 v_\Theta - \epsilon^3 \partial_x^2 v_\Theta), \epsilon \Delta_q^h v_\Theta \rangle_{L^2} = \|\Delta_q^h \epsilon \partial_y v_\Theta\|_{L^2}^2 + \epsilon^2 \|\Delta_q^h \epsilon \partial_x v_\Theta\|_{L^2}^2.$$

Now, by using the Dirichlet boundary condition $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$, and the incompressibility condition $\partial_x u + \partial_y v = 0$, we can find by integrating by parts the estimate of the pressure

$$\left| \langle \Delta_q^h \nabla p_\Theta, \Delta_q^h(u, v)_\Theta \rangle_{L^2} \right| = \left| \langle \Delta_q^h p_\Theta, \Delta_q^h \operatorname{div}(u, v)_\Theta \rangle \right| = 0.$$

Then by multiplying (5.5.18) and (5.5.19) by $e^{2\mathcal{R}t}$, and integrating the resulting inequality over time, we achieve

$$(5.5.20) \\ \int \left(\frac{d}{dt} \left(e^{2\mathcal{R}t} \Delta_q^h(\partial_t u)_\Theta \Delta_q^h u_\Theta \right) - e^{2\mathcal{R}t} \Delta_q^h(\partial_t u)_\Theta \Delta_q^h(\partial_t u)_\Theta + 2\lambda \dot{\tau}(t) e^{2\mathcal{R}t} \Delta_q^h |D_x|(\partial_t u)_\Theta \Delta_q^h u_\Theta \right) dX \\ + \int \left(\frac{d}{dt} \left(e^{2\mathcal{R}t} \Delta_q^h(\epsilon \partial_t v)_\Theta \Delta_q^h(\epsilon v)_\Theta \right) - e^{2\mathcal{R}t} \Delta_q^h(\epsilon \partial_t v)_\Theta \Delta_q^h(\epsilon \partial_t v)_\Theta + 2\lambda \dot{\tau}(t) e^{2\mathcal{R}t} \Delta_q^h |D_x|(\epsilon \partial_t v)_\Theta \Delta_q^h(\epsilon v)_\Theta \right) dX \\ + \frac{1}{2} \frac{d}{dt} \|e^{\mathcal{R}t} \Delta_q^h u_\Theta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|e^{\mathcal{R}t} \Delta_q^h \epsilon v_\Theta\|_{L^2}^2 + \lambda \dot{\tau}(t) \|e^{\mathcal{R}t} \Delta_q^h |D_x|^{\frac{1}{2}}(u_\Theta, \epsilon v_\Theta)\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\Theta\|_{L^2}^2 \\ + \|e^{\mathcal{R}t} \Delta_q^h \partial_y \epsilon v_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t} \Delta_q^h \partial_x u_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t} \Delta_q^h \partial_x \epsilon v_\Theta\|_{L^2}^2 \\ = \mathcal{R} \left(2 \int e^{2\mathcal{R}t} \Delta_q^h(\partial_t u)_\Theta \Delta_q^h u_\Theta dX + 2 \int e^{2\mathcal{R}t} \Delta_q^h(\epsilon \partial_t v)_\Theta \Delta_q^h(\epsilon v)_\Theta dX + \|e^{\mathcal{R}t} \Delta_q^h(u_\Theta, \epsilon v_\Theta)\|_{L^2}^2 \right) \\ - \langle \Delta_q^h(u \partial_x u + v \partial_y u)_\Theta, e^{2\mathcal{R}t} \Delta_q^h u_\Theta \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h(u \partial_x v + v \partial_y v)_\Theta, e^{2\mathcal{R}t} \Delta_q^h v_\Theta \rangle_{L^2}.$$

In view, of Lemma 5.2.5-5.2.6 for $t < T_1^*$, and by summing (5.5.16) with (5.5.20), we obtain

$$(5.5.21) \quad \mathbf{B}(u, v)(t) \leq \mathcal{R} \int_0^t \left[\|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u + u, \epsilon(\partial_t v + v))_\Theta\|_{L^2}^2 + 2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 \right. \\ \left. + 2\epsilon^2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 \right] dt' + \lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}}(u, \epsilon v)_\Theta\|_{L^2}^2 dt' \\ + \lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 dt' + 2C2^{-2qs} d_q^2 \left(\|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) \\ \times \left(\|e^{\mathcal{R}t}(\partial_t u)_\Theta\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t}(\epsilon \partial_t v)_\Theta\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) + \|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \Big).$$

where

$$\mathbf{B}(u, v)(t) = \int_0^t \frac{d}{dt} \left(\frac{1}{2} \|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u + u, \epsilon(\partial_t v + v))_\Theta\|_{L^2}^2 + \frac{1}{2} \|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u, \epsilon(\partial_t v))_\Theta\|_{L^2}^2 \right. \\ \left. + \|e^{\mathcal{R}t'} \Delta_q^h \partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 \right) dt' + \int_0^t \left(\|e^{\mathcal{R}t'} \Delta_q^h(\partial_t u, \epsilon \partial_t v)_\Theta\|_{L^2}^2 \right. \\ \left. + \|e^{\mathcal{R}t'} \Delta_q^h \partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 + \epsilon^2 \|e^{\mathcal{R}t'} \Delta_q^h \partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 + \lambda \dot{\tau}(t') \|e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}}(u, \epsilon v)_\Theta\|_{L^2}^2 \right) dt'$$

$$\begin{aligned}
& + 2\lambda(\dot{\tau}(t'))\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 + \dot{\tau}(t')\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(\partial_y u, \epsilon\partial_y v)_\Theta\|_{L^2}^2 \\
& + 2\epsilon^2\lambda\dot{\tau}(t')\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(\partial_x u, \epsilon\partial_x v)_\Theta\|_{L^2}^2)dt'.
\end{aligned}$$

We begin with by observing that the term in the square brackets in (5.5.21) can be absorbed by the dissipation

$$\int_0^t \|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 dt' + \int_0^t \|e^{\mathcal{R}t'}\Delta_q^h(\partial_y u, \epsilon\partial_y v)_\Theta\|_{L^2}^2 dt' + \epsilon^2 \int_0^t \|e^{\mathcal{R}t'}\Delta_q^h(\partial_x u, \epsilon\partial_x v)_\Theta\|_{L^2}^2 dt'.$$

Indeed, since the value of \mathcal{R} is smaller than $\min\{\frac{1}{8}, \frac{k}{8}\}$, we have that

$$\begin{aligned}
& \mathcal{R} \left[\|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u + u, \epsilon(\partial_t v + v))_\Theta\|_{L^2}^2 + 2\|e^{\mathcal{R}t'}\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 + 2\epsilon^2\|e^{\mathcal{R}t'}\Delta_q^h\partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 \right. \\
& \left. + \|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 \right] \leq \mathcal{R} \left[\|e^{\mathcal{R}t'}\Delta_q^h(u, \epsilon v)_\Theta\|_{L^2}^2 + 2\|e^{\mathcal{R}t'}\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 \right. \\
& \left. + 2\epsilon^2\|e^{\mathcal{R}t'}\Delta_q^h\partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 + 2\|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 \right] \leq \frac{k}{8}\|e^{\mathcal{R}t'}\Delta_q^h(u, \epsilon v)_\Theta\|_{L^2}^2 \\
& + \frac{1}{4}\|e^{\mathcal{R}t'}\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 + \frac{\epsilon^2}{4}\|e^{\mathcal{R}t'}\Delta_q^h\partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 + \frac{1}{4}\|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2.
\end{aligned}$$

To absorb those terms, we shall then invoke the Poincaré inequality in $y \in (0, 1)$: $k\|\Delta_q^h(u, \epsilon v)_\Theta\|_{L^2} \leq \|\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}$. Thus

$$\begin{aligned}
(5.5.22) \quad & \mathcal{R} \left[\|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u + u, \epsilon(\partial_t v + v))_\Theta\|_{L^2}^2 + 2\|e^{\mathcal{R}t'}\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 \right. \\
& \left. + 2\epsilon^2\|e^{\mathcal{R}t'}\Delta_q^h\partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 \right] \\
& \leq \frac{1}{2} \left(\|e^{\mathcal{R}t'}\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 + \|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 + \epsilon^2\|e^{\mathcal{R}t'}\Delta_q^h(\partial_x u, \epsilon\partial_x v)_\Theta\|_{L^2}^2 \right).
\end{aligned}$$

We replace the obtained result (5.5.22) in (5.5.21), we deduce that

$$\begin{aligned}
(5.5.23) \quad \mathbf{F}(u, v)(t) & \leq 2C2^{-2qs}d_q^2 \left(\|e^{\mathcal{R}t}u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \times \left(\|e^{\mathcal{R}t}(\partial_t u)_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right. \right. \\
& \left. \left. + \|e^{\mathcal{R}t}(\epsilon\partial_t v)_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) + \|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right),
\end{aligned}$$

where

$$\begin{aligned}
(5.5.24) \quad \mathbf{F}(u, v)(t) & = \int_0^t \frac{d}{dt} \left(\frac{1}{2}\|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u + u, \epsilon(\partial_t v + v))_\Theta\|_{L^2}^2 + \frac{1}{2}\|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon(\partial_t v))_\Theta\|_{L^2}^2 \right. \\
& \left. + \|e^{\mathcal{R}t'}\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 + \epsilon^2\|e^{\mathcal{R}t'}\Delta_q^h\partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 \right) dt' + \int_0^t \left(\frac{1}{2}\|e^{\mathcal{R}t'}\Delta_q^h(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 \right. \\
& \left. + \frac{1}{2}\|e^{\mathcal{R}t'}\Delta_q^h\partial_y(u, \epsilon v)_\Theta\|_{L^2}^2 + \frac{\epsilon^2}{2}\|e^{\mathcal{R}t'}\Delta_q^h\partial_x(u, \epsilon v)_\Theta\|_{L^2}^2 + k\lambda\dot{\tau}(t')\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(u, \epsilon v)_\Theta\|_{L^2}^2 \right. \\
& \left. + \lambda\dot{\tau}(t')\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(\partial_t u, \epsilon\partial_t v)_\Theta\|_{L^2}^2 + \lambda\dot{\tau}(t')\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(\partial_y u, \epsilon\partial_y v)_\Theta\|_{L^2}^2 \right. \\
& \left. + \epsilon^2\lambda\dot{\tau}(t')\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(\partial_x u, \epsilon\partial_x v)_\Theta\|_{L^2}^2 \right) dt'.
\end{aligned}$$

If we look at the third line of the equality (5.5.24), we remark that we have a new term

$$k\lambda \int_0^t \dot{\tau}(t')\|e^{\mathcal{R}t'}\Delta_q^h|D_x|^{\frac{1}{2}}(u, \epsilon v)_\Theta\|_{L^2}^2 dt',$$

this term handle from the Poincaré inequality applied to the following term

$$\lambda \int_0^t \dot{\tau}(t') \|e^{\mathcal{R}t'} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_y u, \epsilon \partial_y v)_\Theta\|_{L^2}^2 dt'.$$

Multiplying (5.5.21) by 2^{2qs} for $s \in]0, 1[$ and then integrating over time, and summing with respect to $q \in \mathbb{Z}$, we find that for $t < T_1^*$

$$(5.5.25) \quad \mathbf{E}_{s,\lambda,k}(u, v)(t) \leq C \left(\|e^{a|D_x|} \partial_y(u_0, \epsilon v_0)\|_{\mathcal{B}^s} + \epsilon \|e^{a|D_x|} \partial_x(u_0, \epsilon v_0)\|_{\mathcal{B}^s} \right. \\ \left. + \|e^{a|D_x|}(u_1, \epsilon v_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}(u_0 + u_1, \epsilon(v_0 + v_1))\|_{\mathcal{B}^s} \right. \\ \left. + \sqrt{2}C \|e^{\mathcal{R}t}((\partial_t u)_\Theta, \epsilon(\partial_t v)_\Theta)\|_{\tilde{L}_{t,\tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{2}C \|e^{\mathcal{R}t}(u, \epsilon v)_\Theta\|_{\tilde{L}_{t,\tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \right.$$

where

$$\mathbf{E}_{s,\lambda,k}(u, v)(t) = \|e^{\mathcal{R}t}(u + \partial_t u, \epsilon(v + \partial_t v))_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\ + \epsilon \|e^{\mathcal{R}t} \partial_x(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\ + \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \sqrt{k\lambda} \|e^{\mathcal{R}t}(u, \epsilon v)_\Theta\|_{\tilde{L}_{t,\tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ + \|e^{\mathcal{R}t} \partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \epsilon \|e^{\mathcal{R}t} \partial_x(u, \epsilon v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \sqrt{\lambda} \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_{t,\tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ + \sqrt{\lambda} \|e^{\mathcal{R}t} \partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_{t,\tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \sqrt{\lambda} \|e^{\mathcal{R}t} \partial_x(u, \epsilon v)_\Theta\|_{\tilde{L}_{t,\tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}$$

Taking $\lambda \geq 2C^2$ in the above inequality leads to

$$(5.5.26) \quad \|e^{\mathcal{R}t}(u + \partial_t u, \epsilon(v + \partial_t v))_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\ + \epsilon \|e^{\mathcal{R}t} \partial_x(u, \epsilon v)_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}(\partial_t u, \epsilon \partial_t v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ + \epsilon \|e^{\mathcal{R}t} \partial_x(u, \epsilon v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq C \left(\|e^{a|D_x|} \partial_y(u_0, \epsilon v_0)\|_{\mathcal{B}^s} + \epsilon \|e^{a|D_x|} \partial_x(u_0, \epsilon v_0)\|_{\mathcal{B}^s} \right. \\ \left. + \|e^{a|D_x|}(u_1, \epsilon v_1)\|_{\mathcal{B}^s} + \|e^{a|D_x|}(u_0 + u_1, \epsilon(v_0 + v_1))\|_{\mathcal{B}^s} \right), \quad \text{for } t < T_1^*.$$

We recall that we already defined $\tau(t)$ in (5.5.2). Then, for any $0 < t < T_1^*$, Inequality (5.3.28) yields for $s = \frac{1}{2}$

$$\tau(t) = \int_0^t \|\partial_y u_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_y v_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\ \leq \int_0^t e^{-\mathcal{R}t'} \|e^{\mathcal{R}t'} \partial_y(u, \epsilon v)_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\ \leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \left(\int_0^t \|e^{\mathcal{R}t'} \partial_y(u, \epsilon v)_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\ \leq C \|e^{\mathcal{R}t} \partial_y(u, \epsilon v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ \leq C \left(\|e^{a|D_x|} \partial_y(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|e^{a|D_x|} \partial_x(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(u_0 + u_1, \epsilon(v_0 + v_1))\|_{\mathcal{B}^{\frac{1}{2}}} \right. \\ \left. + \|e^{a|D_x|}(u_1, \epsilon v_1)\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \frac{a}{2\lambda}$$

A continuity argument implies that $T_1^* = +\infty$ and we have (5.3.28) is valid for any $t \in \mathbb{R}_+$.

5.6 The convergence to the perturbed hydrostatic Navier-Stokes equations

In this section, we justify the limit from the scaled perturbed anisotropic Navier-Stokes system to the perturbed hydrostatic Navier-Stokes system in a 2D thin domain. As in the sections 3 and 4, the main idea will be to obtain a control of the difference between the two solutions in analytic spaces, by using energy estimates with exponential weights in the Fourier variable. As previously, the exponent of the exponential weight is depending on time but shall take into account now the "loss of the analyticity" for both solutions, of the re-scaled perturbed Navier-Stokes system and respectively of the perturbed hydrostatic Navier-Stokes equations. To this end, we introduce

$$(5.6.1) \quad \begin{cases} R^{1,\epsilon} = u^\epsilon - u, \\ R^{2,\epsilon} = v^\epsilon - v, \\ q^\epsilon = p^\epsilon - p. \end{cases}$$

Then, Systems (5.1.8) and (5.1.9) imply that $(R^{1,\epsilon}, R^{2,\epsilon}, q^\epsilon)$ verifies

$$(5.6.2) \quad \begin{cases} \partial_t^2 R^{1,\epsilon} + \partial_t R^{1,\epsilon} - \epsilon^2 \partial_x^2 R^{1,\epsilon} - \partial_y^2 R^{1,\epsilon} + \partial_x q^\epsilon = F^{1,\epsilon} & \text{in } \mathcal{S} \times]0, \infty[, \\ \epsilon^2 (\partial_t^2 R^{2,\epsilon} + \partial_t R^{2,\epsilon} - \epsilon^2 \partial_x^2 R^{2,\epsilon} - \partial_y^2 R^{2,\epsilon}) + \partial_y q^\epsilon = F^{2,\epsilon}, \\ \partial_x R^{1,\epsilon} + \partial_y R^{2,\epsilon} = 0 \\ (R^{1,\epsilon}, R^{2,\epsilon})|_{t=0} = (u_0^\epsilon - u_0, v_0^\epsilon - v_0), \\ \partial_t (R^{1,\epsilon}, R^{2,\epsilon})|_{t=0} = (u_1^\epsilon - u_1, v_1^\epsilon - v_1), \\ (R^{1,\epsilon}, R^{2,\epsilon})|_{y=0} = (R^{1,\epsilon}, R^{2,\epsilon})|_{y=1} = 0, \end{cases}$$

where the remaining terms $F^{i,\epsilon}$, with $i = 1, 2$, are determined by

$$(5.6.3) \quad \begin{cases} F^{1,\epsilon} = \epsilon^2 \partial_x^2 u - (u^\epsilon \partial_x u^\epsilon - u \partial_x u) - (v^\epsilon \partial_y u^\epsilon - v \partial_y u), \\ F^{2,\epsilon} = -\epsilon^2 (\partial_t^2 v + \partial_t v - \epsilon^2 \partial_x^2 v - \partial_y^2 v + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon). \end{cases}$$

As $(R^{1,\epsilon}, R^{2,\epsilon})$ satisfies the boundary condition and also the free divergence, therefore these two conditions allows us to write

$$(5.6.4) \quad R^{2,\epsilon}(t, x, y) = \int_0^y \partial_y R^{2,\epsilon}(t, x, s) ds = - \int_0^y \partial_x R^{1,\epsilon}(t, x, s) ds$$

If we replace y by 1 in (5.6.4), we deduce from the incompressibility condition $\partial_x R^{1,\epsilon} + \partial_y R^{2,\epsilon} = 0$ that

$$\partial_x \int_0^1 R^{1,\epsilon}(t, x, y) dy = - \int_0^1 \partial_y R^{2,\epsilon}(t, x, y) dy = R^{2,\epsilon}(t, x, 0) - R^{2,\epsilon}(t, x, 1) = 0.$$

In what follows, for simplicity, we shall neglect the subscript ϵ in $(R^{1,\epsilon}, R^{2,\epsilon}, q^\epsilon)$. In view of System (5.6.2), we can transform it like an equation of order one in time, so if we define $G = (R^1, \partial_t R^1)$ and

$H = (R^2, \partial_t R^2)$, Then G and H satisfy the following equation

$$(5.6.5) \quad \begin{cases} \partial_t G + A_\epsilon(D)G = \begin{pmatrix} 0 \\ F^1 - \partial_x q \end{pmatrix} \\ \epsilon^2 (\partial_t H + A_\epsilon(D)H) = \begin{pmatrix} 0 \\ F^2 - \partial_y q \end{pmatrix} \\ \partial_x R^1 + \partial_y R^2 = 0 \\ (R^1, R^2)/_{y=0} = (R^1, R^2)/_{y=1} = 0 \end{cases}$$

where

$$G = \begin{pmatrix} R^1 \\ \partial_t R^1 \end{pmatrix} \quad \text{and} \quad A_\epsilon(D) = \begin{pmatrix} 0 & -1 \\ -\epsilon^2 \partial_x^2 - \partial_y^2 & 1 \end{pmatrix}$$

and

$$H = \begin{pmatrix} R^2 \\ \partial_t R^2 \end{pmatrix}.$$

In view of (5.2.3), we define for any suitable function f

$$(5.6.6) \quad f_\varphi(t, x, y) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{\varphi(t, \xi)} \widehat{f}(t, \xi, y) \right) \quad \text{where} \quad \varphi(t, \xi) = (a - \mu \eta(t)) |\xi|,$$

where $\mu \geq \lambda$ will be determined later, and $\eta(t)$ is given by

$$\eta(t) = \int_0^t \left(\|(\partial_y u_\Theta^\epsilon, \epsilon \partial_x u_\Theta^\epsilon)(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \right) dt'.$$

We can observe that, if we take c_0 and c_1 small enough in Theorems 5.1.1 and 5.1.2 then $\varphi(t) \geq 0$ and

$$0 \leq \varphi(t, \xi) \leq \min(\phi(t, \xi), \Theta(t, \xi)).$$

Then in view of (5.6.6), we observe that $(G, H)_\varphi$ verifies

$$(5.6.7) \quad \begin{cases} \partial_t G_\varphi + \mu \dot{\eta}(t) |D_x| G_\varphi + A_\epsilon(D) G_\varphi = \begin{pmatrix} 0 \\ F_\varphi^1 + \partial_x q_\varphi \end{pmatrix} \\ \epsilon^2 (\partial_t H_\varphi + \mu \dot{\eta}(t) |D_x| H_\varphi + A_\epsilon(D) H_\varphi) = - \begin{pmatrix} 0 \\ F_\varphi^2 + \partial_y q_\varphi \end{pmatrix} \\ \partial_x R_\varphi^1 + \partial_y R_\varphi^2 = 0 \\ (R_\varphi^1, R_\varphi^2)/_{y=0} = (R_\varphi^1, R_\varphi^2)/_{y=1} = 0 \end{cases}$$

Where $|D_x|$ denote the Fourier multiplier of the symbol $|\xi|$. In what follows, we recall that we use "C" to denote a generic positive constant which can change from line to line. Thanks to Theorems 5.1.1 and 5.1.2, Propositions 5.4.1 and 5.4.2, we deduce for $0 < s < 1$ there exist some $M > 0$ such that

$$(5.6.8) \quad \|u_\varphi^\epsilon\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} + \|(u + \partial_t u)_\varphi\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{7}{2}})} + \|\partial_y^2 u_\varphi\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{7}{2}})} \\ + \|\partial_y u_\varphi\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{7}{2}})} + \|(\partial_t^2 u)_\varphi\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{3}{2}})} \leq M,$$

where u_φ^ϵ and u_φ are respectively determined by (5.5.1) and (5.2.3) and $M \geq 1$ is a constant independent to ϵ .

Proof of Theorem 5.1.3 We apply the dyadic operator in the horizontal variable Δ_q^h to (5.6.7) and taking the L^2 inner product of the resulting equation with $\Delta_q^h G_\varphi$ and $\Delta_q^h H_\varphi$ we obtain

$$(5.6.9) \quad \langle \Delta_q^h \partial_t G_\varphi, \Delta_q^h G_\varphi \rangle_{L^2} + \mu \dot{\eta}(t) \langle \Delta_q^h |D_x| G_\varphi, \Delta_q^h G_\varphi \rangle_{L^2} + \langle \Delta_q^h A_\epsilon(D) G_\varphi, \Delta_q^h G_\varphi \rangle_{L^2} \\ = \langle \Delta_q^h (F^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} - \langle \Delta_q^h \partial_x q_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2}$$

and

$$(5.6.10) \quad \langle \Delta_q^h \partial_t H_\varphi, \Delta_q^h H_\varphi \rangle_{L^2} + \mu \dot{\eta}(t) \langle \Delta_q^h |D_x| H_\varphi, \Delta_q^h H_\varphi \rangle_{L^2} + \langle \Delta_q^h B_\epsilon(D) H_\varphi, \Delta_q^h H_\varphi \rangle_{L^2} \\ = - \langle \Delta_q^h (F^2)_\varphi, \Delta_q^h (\partial_t R^2)_\varphi \rangle_{L^2} - \langle \Delta_q^h \partial_y q_\varphi, \Delta_q^h (\partial_t R^2)_\varphi \rangle_{L^2}.$$

Due to the free divergence condition, we have

$$\left| \langle \Delta_q^h \nabla q_\varphi, \Delta_q^h (\partial_t R^1, \partial_t R^2)_\varphi \rangle_{L^2} \right| = 0.$$

Then by using Lemma 5.2.1, we achieve

$$(5.6.11) \quad \frac{1}{2} \frac{d}{dt} \left(\|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2}^2 + \|\Delta_q^h \partial_y R_\varphi^1\|_{L^2}^2 + \|\Delta_q^h \epsilon (\partial_t R^2)_\varphi\|_{L^2}^2 + \|\Delta_q^h \epsilon \partial_y R_\varphi^2\|_{L^2}^2 \right. \\ \left. + \epsilon^4 \|\Delta_q^h \partial_x R_\varphi^2\|_{L^2}^2 + \epsilon^2 \|\Delta_q^h \partial_x R_\varphi^1\|_{L^2}^2 \right) + \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2}^2 + \|\Delta_q^h \epsilon (\partial_t R^2)_\varphi\|_{L^2}^2 \\ + \mu \dot{\eta}(t') \|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t R^1)_\varphi\|_{L^2}^2 + \epsilon^2 \mu \dot{\eta}(t') \|\Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t R^2)_\varphi\|_{L^2}^2 + \epsilon^2 \mu \dot{\eta}(t') \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_x R_\varphi^1\|_{L^2}^2 \\ + \mu \dot{\eta}(t') \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y R_\varphi^1\|_{L^2}^2 + \mu \dot{\eta}(t') \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_y R_\varphi^2\|_{L^2}^2 + \epsilon^4 \mu \dot{\eta}(t') \|\Delta_q^h |D_x|^{\frac{1}{2}} \partial_x R_\varphi^2\|_{L^2}^2 \\ \leq \left| \langle \Delta_q^h (F^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle \right|_{L^2} + \left| \langle \Delta_q^h (F^2)_\varphi, \Delta_q^h (\partial_t R^2)_\varphi \rangle \right|_{L^2}.$$

Now we still have to take the inner product in L^2 with $\Delta_q^h R_\varphi^1$ and $\Delta_q^h R_\varphi^2$ to the equation

$$(5.6.12) \quad e^{\varphi(t, D_x)} (\partial_t^2 R^1 + \partial_t R^1 - \partial_y^2 R^1 - \epsilon^2 \partial_x^2 R^1 + \partial_x q - F^1) = 0 \\ e^{\varphi(t, D_x)} (\epsilon^2 (\partial_t^2 R^2 + \partial_t R^2 - \partial_y^2 R^2 - \epsilon^2 \partial_x^2 R^2) + \partial_y q - F^2) = 0,$$

we obtain

$$(5.6.13) \quad \langle \Delta_q^h (\partial_t^2 R^1)_\varphi, \Delta_q^h R_\varphi^1 \rangle_{L^2} + \langle \Delta_q^h (\partial_t R^1)_\varphi, \Delta_q^h R_\varphi^1 \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 R_\varphi^1, \Delta_q^h R_\varphi^1 \rangle_{L^2} \\ - \epsilon^2 \langle \Delta_q^h \partial_x^2 R_\varphi^1, \Delta_q^h R_\varphi^1 \rangle_{L^2} = \langle \Delta_q^h (F^1)_\varphi, \Delta_q^h R_\varphi^1 \rangle_{L^2} - \langle \Delta_q^h \partial_x q_\varphi, \Delta_q^h R_\varphi^1 \rangle_{L^2},$$

and

$$(5.6.14) \quad \langle \Delta_q^h (\epsilon \partial_t^2 R^2)_\varphi, \Delta_q^h \epsilon R_\varphi^2 \rangle_{L^2} + \langle \Delta_q^h (\epsilon \partial_t R^2)_\varphi, \Delta_q^h \epsilon R_\varphi^2 \rangle_{L^2} - \langle \Delta_q^h \epsilon \partial_y^2 R_\varphi^2, \Delta_q^h \epsilon R_\varphi^2 \rangle_{L^2} \\ - \epsilon^2 \langle \Delta_q^h \epsilon \partial_x^2 R_\varphi^2, \Delta_q^h \epsilon R_\varphi^2 \rangle_{L^2} = \langle F^2 \rangle_\varphi, \Delta_q^h \epsilon R_\varphi^2 \rangle_{L^2} - \langle \Delta_q^h \partial_y q_\varphi, \Delta_q^h \epsilon R_\varphi^2 \rangle_{L^2},$$

In what follows, we shall use again the technical lemmas from Section 5.2, to handle term by term in the estimate (5.6.13) and (5.6.14). We start by the complicate term $I_1 = \langle \Delta_q^h (\partial_t^2 R^1)_\varphi, \Delta_q^h R_\varphi^1 \rangle_{L^2}$ and $I_2 = \langle \Delta_q^h (\epsilon \partial_t^2 R^2)_\varphi, \Delta_q^h \epsilon R_\varphi^2 \rangle_{L^2}$, so by using integration by parts, we find

$$I_1 = \frac{d}{dt} \int \Delta_q^h (\partial_t R^1)_\varphi \Delta_q^h R_\varphi^1 dx - \int \Delta_q^h (\partial_t R^1)_\varphi \Delta_q^h (\partial_t R^1)_\varphi dx \\ + 2\mu \dot{\eta}(t) \int \Delta_q^h |D_x| (\partial_t R^1)_\varphi \Delta_q^h R_\varphi^1 dx \\ I_2 = \frac{d}{dt} \int \Delta_q^h (\epsilon \partial_t R^2)_\varphi \Delta_q^h \epsilon R_\varphi^2 dx - \int \Delta_q^h (\epsilon \partial_t R^2)_\varphi \Delta_q^h (\epsilon \partial_t R^2)_\varphi dx$$

$$+ 2\mu\dot{\eta}(t) \int \Delta_q^h |D_x| (\epsilon \partial_t R^2)_\varphi \Delta_q^h \epsilon R_\varphi^2 dx$$

Then by using Lemma 5.2.1, we achieve

$$\begin{aligned}
& \frac{d}{dt} \int \Delta_q^h (\partial_t R^1)_\varphi \Delta_q^h R_\varphi^1 dx - \int \Delta_q^h (\partial_t R^1)_\varphi \Delta_q^h (\partial_t R^1)_\varphi dx \\
& + 2\mu\dot{\eta}(t) \int \Delta_q^h |D_x| (\partial_t R^1)_\varphi \Delta_q^h R_\varphi^1 dx + \frac{d}{dt} \int \Delta_q^h (\partial_t R^2)_\varphi \Delta_q^h R_\varphi^2 dx \\
& - \int \Delta_q^h (\partial_t R^2)_\varphi \Delta_q^h (\partial_t R^2)_\varphi dx + 2\mu\dot{\eta}(t) \int \Delta_q^h |D_x| (\partial_t R^2)_\varphi \Delta_q^h R_\varphi^2 dx \\
& + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h R_\varphi^1\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \epsilon R_\varphi^2\|_{L^2}^2 + \mu\dot{\tau}(t) \|\Delta_q^h |D_x|^{\frac{1}{2}} (R_\varphi^1, R_\varphi^2)\|_{L^2}^2 \\
(5.6.15) \quad & + \|\Delta_q^h \partial_y R_\varphi^1\|_{L^2}^2 + \|\Delta_q^h \partial_y \epsilon R_\varphi^2\|_{L^2}^2 + \epsilon^2 \|\Delta_q^h \partial_x R_\varphi^1\|_{L^2}^2 + \epsilon^2 \|\Delta_q^h \partial_x \epsilon R_\varphi^2\|_{L^2}^2 \\
& = \langle \Delta_q^h (F^1)_\varphi, \Delta_q^h R_\varphi^1 \rangle_{L^2} + \langle \Delta_q^h (F^2)_\varphi, \Delta_q^h R_\varphi^2 \rangle_{L^2}.
\end{aligned}$$

Now we claim that

$$\begin{aligned}
(5.6.16) \quad & \int_0^t \left| 2\mu\dot{\eta}(t) \int \left(\Delta_q^h |D_x| (\partial_t R^1)_\varphi \Delta_q^h R_\varphi^1 + \Delta_q^h |D_x| (\partial_t R^2)_\varphi \Delta_q^h R_\varphi^2 \right) dx \right| \\
& \leq 2^{-q} d_q^2 \mu \left(\|(\partial_t R^1, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} \|\partial_y (R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} \right),
\end{aligned}$$

$$\begin{aligned}
(5.6.17) \quad & \int_0^t \left| \langle \Delta_q^h F_\varphi^1, \Delta_q^h R_\varphi^1 \rangle_{L^2} \right| dt' + \int_0^t \left| \langle \Delta_q^h F_\varphi^1, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\
& \lesssim 2^{-q} d_q^2 \left(\epsilon \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \left(\|\epsilon R_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\epsilon (\partial_t R^1)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right) \right. \\
& \quad \left. + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_y R_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \left(\|R_\varphi^1\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} \right. \right. \\
& \quad \left. \left. + \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} \right) + \|(R^1, \partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^2 \right),
\end{aligned}$$

and

$$\begin{aligned}
(5.6.18) \quad & \int_0^t \left| \langle \Delta_q^h F_\varphi^2, \Delta_q^h R_\varphi^2 \rangle_{L^2} \right| dt' + \int_0^t \left| \langle \Delta_q^h F_\varphi^2, \Delta_q^h (\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\
& \lesssim 2^{-q} d_q^2 \left\{ \|R_\varphi^1\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^2 + \epsilon^2 \|(R_\varphi^2, (\partial_t R^2)_\varphi)\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^2 + \epsilon^2 \|(R_\varphi^2, \epsilon (\partial_t R^2)_\varphi)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right. \\
& \quad \times \left(\|(\partial_t^2 u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|(\partial_t u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \epsilon \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{7}{2}})} + \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) \\
& \quad \left. + \epsilon^2 \|(R_\varphi^2, (\partial_t R^2)_\varphi)\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} \left[\|R_\varphi^2\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} + \|u_\varphi^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right. \right. \\
& \quad \left. \left. + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \left(\|\partial_y R_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) \right] \right\},
\end{aligned}$$

the proof of those estimates will be presented later in the last section of the chapter.

By virtue of (5.6.8), (5.6.17) and (5.6.18), we infer

$$\begin{aligned}
(5.6.19) \quad & \sum_{i=1}^2 \int_0^t \left| \langle \Delta_q^h F_\varphi^i, \Delta_q^h R_\varphi^i \rangle_{L^2} \right| dt' + \int_0^t \left| \langle \Delta_q^h F_\varphi^i, \Delta_q^h (\partial_t R^i)_\varphi \rangle_{L^2} \right| dt' \\
& \lesssim d_q^2 2^{-q} \left\{ M \epsilon \|(\epsilon R^1, \epsilon \partial_t R^1)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + M^{\frac{1}{2}} \|\partial_y R_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|(R^1, \partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \|(R^1, \partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^2 + \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^2 + M\epsilon \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
& + M^{\frac{3}{2}}\epsilon \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} + M^{\frac{1}{2}}\|\epsilon \partial_y R_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} \Big\}.
\end{aligned}$$

Then by summing $2 \times (5.6.11)$ with (5.6.15) and multiplying the resulting by 2^q and then integrating over time, and summing with respect to $q \in \mathbb{Z}$, we find that for $t < T^*$

$$\begin{aligned}
(5.6.20) \quad \mathcal{K}(t) & \leq \sqrt{C}\mathcal{E}(0) + \sqrt{C} \left(\sqrt{M}\epsilon \|(\epsilon R^1, \epsilon \partial_t R^1)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} + M^{\frac{1}{4}} \|\partial_y R_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \| (R^1, \partial_t R^1)_\varphi \|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^{\frac{1}{2}} \right. \\
& + \|(R^1, \partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} + \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} + \sqrt{M}\epsilon \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \\
& \left. + M^{\frac{3}{4}}\epsilon^{\frac{1}{2}} \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^{\frac{1}{2}} + M^{\frac{1}{4}} \|\epsilon \partial_y R_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}^{\frac{1}{2}} \right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}(t) & = \left(\|(R^1 + \partial_t R^1, \epsilon(R^2 + \partial_t R^2))_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|(\partial_t R^1)_\varphi, \epsilon(\partial_t R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \right. \\
& + \|\partial_y(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|\partial_x(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \Big) + \|(\partial_t R^1, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
& + \sqrt{\mu} \|(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \|\partial_y(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|\partial_x(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
& + \sqrt{\mu} \|(\partial_t R^1, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \sqrt{\mu} \|\partial_y(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \sqrt{\mu} \|\partial_x(R^1, \epsilon R^2)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}(0) & = \|e^{a|D_x|} \partial_y(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \sqrt{C}\epsilon \|e^{a|D_x|} \partial_x(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} \\
& + \sqrt{C} \|e^{a|D_x|} ((u_0^\epsilon - u_0) + (u_1^\epsilon - u_1), \epsilon(v_0^\epsilon - v_0) + \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}} \\
& + \sqrt{C} \|e^{a|D_x|} ((u_1^\epsilon - u_1), \epsilon(v_1^\epsilon - v_1))\|_{\mathcal{B}^{\frac{1}{2}}}
\end{aligned}$$

Applying Young's inequality gives rise to

$$(5.6.21) \quad \mathcal{K}(t) \leq \sqrt{C} \left(\mathcal{E}(0) + M(\epsilon + \|(R^1, \partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)} + \|(\epsilon R^2, \epsilon \partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta}^2(\mathcal{B}^1)}) \right)$$

Taking $\mu \geq CM^2$ leads to (5.3.1), this completes the proof of the theorem 5.1.3.

5.7 Proof of the estimates (5.6.17) and (5.6.18)

5.7.1 Proof of the estimate (5.6.17)

We first observe that

$$F_\varphi^1 = (\epsilon^2 \partial_x^2 u - (u^\epsilon \partial_x R^1 + R^1 \partial_x u) - (v^\epsilon \partial_y R^1 + R^2 \partial_y u))_\varphi,$$

so, we define

$$\begin{aligned}
G_1^q & = \int_0^t \left| \langle \Delta_q^h F_\varphi^1, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\
& = \int_0^t \left| \langle \Delta_q^h (\epsilon^2 \partial_x^2 u - (u^\epsilon \partial_x R^1 + R^1 \partial_x u) - (v^\epsilon \partial_y R^1 + R^2 \partial_y u))_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\
& \leq I_1^q + I_2^q + I_3^q,
\end{aligned}$$

where

$$\begin{aligned} I_1^q &= \int_0^t \left| \langle \Delta_q^h(\epsilon^2 \partial_x^2 u)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ I_2^q &= \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x R^1 + R^1 \partial_x u)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ I_3^q &= \int_0^t \left| \langle \Delta_q^h(v^\epsilon \partial_y R^1 + R^2 \partial_y u)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt'. \end{aligned}$$

We first observe that

$$(5.7.1) \quad I_1^q \leq C d_q^2 2^{-q} \epsilon^2 \|u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}.$$

For I_2^q , we write

$$I_2^q = \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x R^1 + R^1 \partial_x u)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \leq I_{21}^q + I_{22}^q,$$

where

$$\begin{aligned} I_{21}^q &= \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x R^1)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ I_{22}^q &= \int_0^t \left| \langle \Delta_q^h(R^1 \partial_x u)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt'. \end{aligned}$$

Lemma 5.2.5 implies

$$(5.7.2) \quad I_{21}^q \leq C d_q^2 2^{-q} \|R_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)},$$

For I_{22} , using Bony's decomposition for the horizontal variable, we write

$$R^1 \partial_x u = T_{\partial_x u}^h R^1 + T_{R^1}^h \partial_x u + R^h(R^1, \partial_x u),$$

and then, we have the following bound

$$I_{22}^q = \int_0^t \left| \langle \Delta_q^h(T_{\partial_x u}^h R^1 + T_{R^1}^h \partial_x u + R^h(R^1, \partial_x u))_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \leq I_{22,1}^q + I_{22,2}^q + I_{22,3}^q$$

with

$$\begin{aligned} I_{22,1}^q &= \int_0^t \left| \langle \Delta_q^h(T_{\partial_x u}^h R^1)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ I_{22,2}^q &= \int_0^t \left| \langle \Delta_q^h(T_{R^1}^h \partial_x u)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ I_{22,3}^q &= \int_0^t \left| \langle \Delta_q^h(R^h(R^1, \partial_x u))_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt'. \end{aligned}$$

Using the support properties given in [[18], Proposition 2.10] and the definition of $T_{R^1}^h \partial_x u$, we have

$$I_{22,2}^q \leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h R_\varphi^1\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L^2} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt'.$$

As we have $R^1 = u^\epsilon - u$, so

$$\|S_{q'-1}^h R_\varphi^1\|_{L^\infty} = \|S_{q'-1}^h(u^\epsilon - u)_\varphi\|_{L^\infty} \leq \|S_{q'-1}^h u_\varphi^\epsilon\|_{L^\infty} + \|S_{q'-1}^h u_\varphi\|_{L^\infty},$$

then,

$$\begin{aligned} I_{22,2}^q &= \int_0^t |\langle \Delta_q^h (T_{\partial_x u}^h R^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle| \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h u_\varphi^\epsilon\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L^2} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\quad + \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h u_\varphi\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L^2} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} dt'. \end{aligned}$$

We note that we get by applying Bernstein lemma 5.2.1

$$\begin{aligned} \|\Delta_{q'}^h u_\varphi^\epsilon\|_{L^\infty} &\leq 2^{\frac{q'}{2}} \|\Delta_{q'}^h u_\varphi^\epsilon\|_{L_h^2(L_v^\infty)} \\ &\leq C d_{q'}(u_\varphi^\epsilon) \|u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \|\partial_y u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}, \end{aligned}$$

Here and in all that follows, we always denote $(d_{q'}(u_\varphi^\epsilon))_{q' \in \mathbb{Z}}$ to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{q' \in \mathbb{Z}} d_{q'}(u_\varphi^\epsilon) \leq 1$, then

$$\|S_{q'-1}^h u_\varphi^\epsilon\|_{L^\infty} \lesssim \|u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \|\partial_y u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \quad \text{and} \quad \|S_{q'-1}^h u_\varphi\|_{L^\infty} \lesssim \|u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}.$$

So

$$\begin{aligned} I_{22,2}^q &\lesssim \sum_{|q-q'|\leq 4} \int_0^t \|u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \|\partial_y u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L^2} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\quad + \sum_{|q-q'|\leq 4} \int_0^t \|u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L^2} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} \|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L_t^2(L^2)} \left(\int_0^t \|\partial_y u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\quad + \sum_{|q-q'|\leq 4} \|u_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L_t^2(L^2)} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\leq C 2^{-q} d_q^2 \left(\|u_\varphi^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \right) \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

Now, we recall that

$$\|S_{q-1}^h \partial_x u_\varphi\|_{L^\infty} \leq \sum_{l \leq q-2} 2^{\frac{3l}{2}} \|\Delta_l^h \partial_y u_\varphi\|_{L^2} \lesssim 2^q \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}},$$

so we can deduce

$$\begin{aligned} I_{22,1}^q &= \int_0^t |\langle \Delta_q^h (T_{\partial_x u}^h R^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle| \leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h \partial_x u_\varphi\|_{L^\infty} \|\Delta_{q'}^h R_\varphi^1\|_{L^2} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} \\ &\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} \end{aligned}$$

$$\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}$$

Using the definition of $\dot{\eta}(t)$ and Definition 5.2.3 we have

$$\left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h R_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(R_\varphi^1) \|R_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Then,

$$I_{22,1}^q \lesssim 2^{-q} d_q^2 \|R_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}$$

where

$$d_q^2 = d_q \left(\sum_{|q-q'|\leq 4} d_{q'} \right)$$

In a similar way, we have

$$\begin{aligned} I_{22,3}^q &= \int_0^t |\langle \Delta_q^h (R^h(R^1, \partial_x u))_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle| dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h R_\varphi^1\|_{L^2} \|\tilde{\Delta}_{q'}^h \partial_x u_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2}^2 dt' \right) \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h (\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right) \\ &\lesssim d_q^2 2^{-q} \|R_\varphi^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \end{aligned}$$

Then we conclude that

$$(5.7.3) \quad I_{22}^q \lesssim d_q^2 2^{-q} \left(\| (R^1, \partial_t R^1)_\varphi \|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}^2 + (\|u_\varphi^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \right.$$

$$(5.7.4) \quad \left. + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \right) \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}$$

For the term I_3^q , we write

$$(5.7.5) \quad I_3^q = \int_0^t \left| \langle v^\epsilon \partial_y R^1 + R^2 \partial_y u \rangle_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \leq I_{31}^q + I_{32}^q,$$

where

$$I_{31}^q = \int_0^t \left| \langle \Delta_q^h (v^\epsilon \partial_y R^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt'$$

$$I_{32}^q = \int_0^t \left| \langle \Delta_q^h (R^2 \partial_y u)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt'.$$

Since

$$v^\epsilon \partial_y R^1 = (R^2 + v) \partial_y R^1 = R^2 \partial_y R^1 + v \partial_y R^1,$$

we get

$$I_{31}^q \leq I_{31,1}^q + I_{31,2}^q,$$

with

$$\begin{aligned} I_{31,1}^q &= \int_0^t \left| \langle \Delta_q^h (R^2 \partial_y R^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ I_{31,2}^q &= \int_0^t \left| \langle \Delta_q^h (v \partial_y R^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt'. \end{aligned}$$

Lemma 5.2.6 implies

$$(5.7.6) \quad I_{31,1}^q \lesssim d_q^2 2^{-q} \|R_\varphi^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)}.$$

For the term $I_{31,2}^q$, we apply Bony's decomposition with respect to the horizontal variable

$$v \partial_y R^1 = T_v^h \partial_y R^1 + T_{\partial_y R^1}^h v + R^h(v, \partial_y R^1).$$

Using (5.3.1), we have

$$\begin{aligned} \|S_{q'-1}^h v_\varphi\|_{L^\infty} &= \|S_{q'-1}^h \int_0^y \partial_x u_\varphi(t, x, s) ds\|_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h u_\varphi\|_{L^2}^{\frac{1}{2}} \|\Delta_l^h \partial_y u_\varphi\|_{L^2}^{\frac{1}{2}} \\ &\lesssim 2^{\frac{q'}{2}} \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}, \end{aligned}$$

from which, we infer

$$\begin{aligned} (5.7.7) \quad &\int_0^t \left| \langle \Delta_q^h (T_v^h \partial_y R^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h v_\varphi\|_{L^\infty} \|\Delta_{q'}^h \partial_y R_\varphi^1\|_{L^2} \|\Delta_{q'}^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\leq \sum_{|q'-q| \leq 4} \int_0^t 2^{\frac{q'}{2}} \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_y R_\varphi^1\|_{L^2} \|\Delta_{q'}^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim d_q^2 2^{-q} \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y R_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)}, \end{aligned}$$

where $\{d_q\}$ forms a suitable sequence.

As we have $R^1 = u^\epsilon - u$, so

$$\|S_{q'-1}^h \partial_y R_\varphi^1\|_{L_h^\infty(L_v^2)} = \|S_{q'-1}^h \partial_y (u^\epsilon - u)_\varphi\|_{L_h^\infty(L_v^2)} \leq \|S_{q'-1}^h \partial_y u_\varphi^\epsilon\|_{L_h^\infty(L_v^2)} + \|S_{q'-1}^h \partial_y u_\varphi\|_{L_h^\infty(L_v^2)},$$

then,

$$\begin{aligned} &\int_0^t \left| \langle \Delta_q^h (T_{\partial_y R^1}^h v)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h \partial_y R_\varphi^1\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h \partial_y u_\varphi^\epsilon\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h (\partial_t R^1)_\varphi\|_{L^2} dt' \\ &+ \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h \partial_y u_\varphi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h (\partial_t R^1)_\varphi\|_{L^2} dt', \end{aligned}$$

We note that we get by applying Bernstein lemma 5.2.1

$$\|S_{q'-1}^h \partial_y u_\varphi^\epsilon\|_{L_h^\infty(L_v^2)} \lesssim \|\partial_y u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \quad \text{and} \quad \|S_{q'-1}^h u_\varphi\|_{L_h^\infty(L_v^2)} \lesssim \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}.$$

So,

$$\begin{aligned} I_{22,2}^q &\lesssim \sum_{|q-q'|\leq 4} \int_0^t \|\partial_y u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &+ \sum_{|q-q'|\leq 4} \int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} \|\partial_y u_\varphi^\epsilon\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\Delta_{q'}^h v_\varphi\|_{\tilde{L}_t^\infty(L_h^2(L_v^\infty))}^{\frac{1}{2}} \|\Delta_{q'}^h v_\varphi\|_{\tilde{L}_t^2(L_h^2(L_v^\infty))}^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &+ \sum_{|q-q'|\leq 4} \|\partial_y u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_x \partial_y u_\varphi\|_{L_t^2(L^2)} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\leq C 2^{-q} d_q^2 \left(\|\partial_y u_\varphi^\epsilon\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} + \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \right) \|\partial_y u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^1)}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^1)}^{\frac{1}{2}} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

where $\{d_q\}$ forms a suitable sequence.

Finally, by using that

$$\|\Delta_{q'}^h v_\varphi(t, x, y)\|_{L_h^2(L_v^\infty)} \lesssim d_{q'}(u_\phi) \|u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}},$$

we have

$$\begin{aligned} &\int_0^t \left| \langle \Delta_q^h(R^h(v, \partial_y R^1))_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h v_\varphi\|_{L_h^2(L_v^\infty)} \|\tilde{\Delta}_{q'}^h \partial_y R_\varphi^1\|_{L^2} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|u_\varphi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\tilde{\Delta}_{q'}^h \partial_y R_\varphi^1\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim d_q^2 2^{-q} \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y R_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}, \end{aligned}$$

Then we obtain the following estimates,

$$(5.7.8) \quad I_{31,2}^q \lesssim 2^{-q} d_q^2 \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y R_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

Now we estimate the term I_{32}^q in (5.7.5). Bony's decomposition with respect to the horizontal variable implies

$$I_{32}^q = \int_0^t \left| \langle \Delta_q^h(T_{R^2}^h \partial_y u + T_{\partial_y u}^h R^2 + R^h(R^2, \partial_y u))_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \leq I_{32,1}^q + I_{32,2}^q + I_{32,3}^q,$$

where

$$\begin{aligned} I_{32,1}^q &= \int_0^t \left| \langle \Delta_q^h(T_{R^2}^h \partial_y u)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \\ I_{32,2}^q &= \int_0^t \left| \langle \Delta_q^h(T_{\partial_y u}^h R^2)_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt' \end{aligned}$$

$$I_{32,3}^q = \int_0^t \left| \langle \Delta_q^h(R^h(R^2, \partial_y u))_\varphi, \Delta_q^h(\partial_t R^1)_\varphi \rangle_{L^2} \right| dt'.$$

We first observe that

$$I_{32,1}^q \lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h R_\varphi^2\|_{L^\infty} \|\Delta_{q'}^h \partial_y u_\varphi\|_{L^2} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt'.$$

Due to the fact that $R^2(t, x, y) = -\int_0^y \partial_x R^1(t, x, s) ds$, we deduce

$$\begin{aligned} I_{32,1}^q &\lesssim \sum_{|q'-q| \leq 4} \int_0^t 2^{\frac{-q'}{2}} \|S_{q'-1}^h R_\varphi^2\|_{L^\infty} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \int_0^y \|S_{q'-1}^h \partial_x R_\varphi^1(t, x, s)\|_{L^\infty}^2 ds dt' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}, \end{aligned}$$

Taking into account the definition of $\dot{\eta}(t)$ and Definition 5.2.3 we obtain

$$\left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Then,

$$(5.7.9) \quad I_{32,1}^q \lesssim 2^{-q} d_q^2 \|R_\varphi^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}$$

Now, for $I_{32,2}^q$, we have

$$\begin{aligned} I_{32,2}^q &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h \partial_y u_\varphi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h R_\varphi^1\|_{L_h^2(L_v^\infty)} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t 2^{\frac{q'}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} 2^{\frac{q'}{2}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q^2 2^{-q} \|R_\varphi^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

We end by estimating $I_{32,3}^q$, in the same way, we have

$$\begin{aligned} I_{32,3}^q &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h R_\varphi^2\|_{L_h^2(L_v^\infty)} \|\tilde{\Delta}_{q'}^h \partial_y u_\varphi\|_{L^2} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h R_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h(\partial_t R^1)_\varphi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q^2 2^{-q} \|R_\varphi^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}, \end{aligned}$$

Summing all the resulting estimate, we can achieve

$$(5.7.10) \quad I_{32}^q \lesssim d_q^2 2^{-q} \|R_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \|(\partial_t R^1)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

By summing up (5.7.1)-(5.7.10), we conclude the proof of (5.6.17).

Remark 5.7.1. *For the proof when we have R^1 and not $\partial_t R^1$, we just easily adapt the proof by replacing R^1 instead of $\partial_t R^1$.*

5.7.2 Proof of the estimate (5.6.18)

We first deduce from $\partial_x u + \partial_y v = 0$, that

$$(5.7.11) \quad \begin{aligned} & \epsilon^2 \int_0^t \left| \langle \Delta_q^h(\partial_t v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-q} \epsilon^2 \|(\partial_t u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ & \epsilon^2 \int_0^t \left| \langle \Delta_q^h(\partial_t^2 v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \lesssim d_q^2 2^q \epsilon^2 \|(\partial_t^2 u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}, \\ & \epsilon^2 \int_0^t \left| \langle \Delta_q^h(\partial_y^2 v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-q} \epsilon^2 \|\partial_y^2 u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}, \\ & \epsilon^4 \int_0^t \left| \langle \Delta_q^h(\partial_x^2 v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-q} \epsilon^4 \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{7}{2}})} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}. \end{aligned}$$

Then now we still have to control

$$J_4^q = \epsilon^2 \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x v^\epsilon)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt'$$

and

$$J_5^q = \epsilon^2 \int_0^t \left| \langle \Delta_q^h(v^\epsilon \partial_y v^\epsilon)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt'.$$

We start first by J_4^q , we have

$$J_4^q \leq \epsilon^2 (J_{41}^q + J_{42}^q),$$

where

$$\begin{aligned} J_{41}^q &= \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x R^2)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ J_{42}^q &= \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt'. \end{aligned}$$

It follows from Lemma 5.2.5 that

$$(5.7.12) \quad \begin{aligned} J_{41}^q &= \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x R^2)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ &\lesssim d_q^2 2^{-q} \|R_\varphi^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

For the second term, Bony's decomposition for the horizontal variable gives

$$J_{42}^q = \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \leq J_{421}^q + J_{422}^q + J_{423}^q,$$

with

$$\begin{aligned} J_{421}^q &= \int_0^t \left| \langle \Delta_q^h(T_{u^\epsilon}^h \partial_x v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ J_{422}^q &= \int_0^t \left| \langle \Delta_q^h(T_{\partial_x v}^h u^\epsilon)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ J_{423}^q &= \int_0^t \left| \langle \Delta_q^h(R^h(u^\epsilon, \partial_x v))_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt'. \end{aligned}$$

Due to

$$\|S_{q'-1}^h u^\epsilon\|_{L^\infty} \lesssim \|u^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_y u^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

and the relation (5.3.1), we have

$$\begin{aligned} J_{421}^q &= \int_0^t \left| \langle \Delta_q^h(T_{u^\epsilon}^h \partial_x v)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h u^\epsilon\|_{L^\infty} \|\Delta_{q'}^h \partial_x v_\varphi\|_{L^2} \|\Delta_q^h(\partial_t R^2)_\varphi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|u^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_y u^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} 2^{2q'} \|\Delta_{q'}^h u_\varphi\|_{L^2} \|\Delta_q^h(\partial_t R^2)_\varphi\|_{L^2} dt' \\ &\lesssim d_q^2 2^{-q} \|u^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

While again thanks to (5.3.1), we find

$$\|S_{q'-1}^h \partial_x v_\varphi\|_{L^\infty} \lesssim \int_0^y \|S_{q'-1}^h \partial_x(\partial_x u_\varphi(t, x, s))\|_{L^\infty} ds \lesssim 2^{\frac{q'}{2}} \|\partial_y u_\varphi\|_{\mathcal{B}^2},$$

which leads to

$$\begin{aligned} J_{422}^q &= \int_0^t \left| \langle \Delta_q^h(T_{\partial_x v}^h u^\epsilon)_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ &\lesssim d_q^2 2^{-q} \|u^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

Along the same way, we obtain

$$\begin{aligned} J_{423}^q &= \int_0^t \left| \langle \Delta_q^h(R^h(\partial_x v, u^\epsilon))_\varphi, \Delta_q^h(\partial_t R^2)_\varphi \rangle_{L^2} \right| \\ &\lesssim d_q^2 2^{-q} \|u^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

This gives rise to

$$(5.7.13) \quad J_{42}^q \lesssim d_q^2 2^{-q} \|u^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|(\partial_t R^2)_\varphi\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Now for J_5^q , We first note that

$$v^\epsilon \partial_y v^\epsilon = v \partial_y R^2 + R^2 \partial_y R^2 + v \partial_y v + R^2 \partial_y v.$$

Lemma 5.2.5 yields

$$\epsilon^2 \int_0^t \left| \langle \Delta_q^h (R^2 \partial_y R^2)_\varphi, \Delta_q^h (\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-q} \| (R_\varphi^1, \epsilon (\partial_t R^2)_\varphi) \|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}^2.$$

From (5.7.8), we have

$$\int_0^t \left| \langle \Delta_q^h (v \partial_y R^2)_\varphi, \Delta_q^h (\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-q} \| u_\varphi \|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \| \partial_y R_\varphi^2 \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \| (\partial_t R^2)_\varphi \|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

As for (5.7.3), we obtain

$$\begin{aligned} & \int_0^t \left| \langle \Delta_q^h (R^2 \partial_x u)_\varphi, \Delta_q^h (\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ & \lesssim d_q^2 2^{-q} \| (\partial_t R^2)_\varphi \|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \| u_\varphi \|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \| \partial_y R_\varphi^2 \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}. \end{aligned}$$

Then, we deduce from the proof of (5.7.8) that

$$\begin{aligned} & \int_0^t \left| \langle \Delta_q^h (v \partial_y v)_\varphi, \Delta_q^h (\partial_t R^2)_\varphi \rangle_{L^2} \right| dt' \\ & \lesssim d_q^2 2^{-q} \| u_\varphi \|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \| \partial_y v_\varphi \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \| (\partial_t R^2)_\varphi \|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \\ & \lesssim d_q^2 2^{-q} \| u_\varphi \|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \| \partial_y u_\varphi \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \| (\partial_t R^2)_\varphi \|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

As a result, it comes out

$$\begin{aligned} J_5^q & \lesssim d_q^2 2^{-q} \left(\| (R_\varphi^1, \epsilon (\partial_t R^2)_\varphi) \|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}^2 \right. \\ (5.7.14) \quad & \left. + \epsilon^2 \| u_\varphi \|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} (\| \partial_y R_\varphi^2 \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \| \partial_y u_\varphi \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}) \| (\partial_t R^2)_\varphi \|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \right). \end{aligned}$$

By summing up (5.7.11)-(5.7.14), we conclude the proof of (5.6.18)

Remark 5.7.2. For the proof when we have R^2 and not $\partial_t R^2$, we just adapt the proof by replacing R^2 instead of $\partial_t R^2$.

5.8 Proof of lemma 5.4.1

In this appendix we present the proof of the lemma 5.4.1, we start by giving the proof of the estimate (5.4.25). By using the Bony's decomposition (5.2.2), we can write

$$u \partial_x u = T_u^h \partial_x u + T_{\partial_x u}^h u + R^h(u, \partial_x u).$$

Then

$$\begin{aligned} & \left(\int_0^t \| e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u)_\phi \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim \left(\int_0^t \| e^{\mathcal{R}t'} \Delta_q^h (T_u^h \partial_x u)_\phi \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ & + \left(\int_0^t \| e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x u}^h u)_\phi \|_{L^2}^2 dt' \right)^{\frac{1}{2}} + \left(\int_0^t \| e^{\mathcal{R}t'} \Delta_q^h (R^h(u, \partial_x u))_\phi \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \end{aligned}$$

We use the definition of T^h , we have

$$\left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim \left(\int_0^t \sum_{|q'-q| \leq 4} \|e^{\mathcal{R}t'} (S_{q'-1}^h u \Delta_{q'}^h \partial_x u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.$$

The lemma 5.2.2 allows us to obtain

(5.8.1)

$$\left(\int_0^t \sum_{|q'-q| \leq 4} \|e^{\mathcal{R}t'} (S_{q'-1}^h u \Delta_{q'}^h \partial_x u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim \sum_{|q'-q| \leq 4} \left(\int_0^t \|S_{q'-1}^h u_\phi^+\|_{L_h^\infty(L_v^2)}^2 \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_x u_\phi\|_{L_h^2(L_v^\infty)}^2 dt' \right)^{\frac{1}{2}}$$

(5.8.2)

$$\lesssim \sum_{|q'-q| \leq 4} \left(\int_0^t \|S_{q'-1}^h u_\phi^+\|_{L_h^\infty(L_v^2)}^2 2^{2q'} \|e^{\mathcal{R}t'} \Delta_{q'}^h u_\phi\|_{L_h^2(L_v^\infty)}^2 dt' \right)^{\frac{1}{2}}$$

Using Bernstein Lemma 5.2.1 we have

$$\|\Delta_{q'}^h u_\phi\|_{L_h^\infty} \lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h u_\phi\|_{L_h^2}.$$

So we can deduce that

$$\begin{aligned} \|\Delta_{q'}^h u_\phi\|_{L_h^\infty(L_v^2)} &\lesssim 2^{\frac{q'}{2}} \|\Delta_{q'}^h u_\phi\|_{L^2} \\ &\lesssim d_{q'}(u_\phi) \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \end{aligned}$$

Here and in all that follows, we always denote $(d_q(u_\phi))_{q \in \mathbb{Z}}$ to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{q \in \mathbb{Z}} d_q(u_\phi) \leq 1$.

Then,

$$\|S_{q'-1}^h u_\phi(t')\|_{L_h^\infty(L_v^2)} \lesssim \|u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}.$$

While using the inclusion $H_y^1 \hookrightarrow L_y^\infty$,

$$\|\Delta_{q'}^h u_\phi\|_{L_v^\infty} \lesssim \|\Delta_{q'}^h u_\phi\|_{L_v^2}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y u_\phi\|_{L_v^2}^{\frac{1}{2}},$$

and the Poincaré inequality on the interval $\{0 < y < 1\}$ on u (as we have that $u = 0$ when $y = 0, 1$)

$$\|\Delta_{q'}^h u_\phi\|_{L^2} \lesssim \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2},$$

we obtain

$$(5.8.3) \quad \|\Delta_{q'}^h u_\phi(t')\|_{L_h^2(L_v^\infty)} \lesssim \|\Delta_{q'}^h \partial_y u_\phi(t')\|_{L^2}.$$

We replace in (5.8.1) we obtain

$$\begin{aligned} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_u^h \partial_x u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \sum_{|q'-q| \leq 4} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} 2^{q'} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_y u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q'-q| \leq 4} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} 2^{q'} \left(2^{-\frac{5q'}{2}} d_{q'}(\partial_y u_\phi) \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \right) \\ (5.8.4) \quad &\leq C d_q 2^{-\frac{3q}{2}} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}, \end{aligned}$$

where

$$d_q = \sum_{|q'-q|\leq 4} d_{q'} (\partial_y u_\phi) 2^{\frac{3}{2}(q-q')}.$$

Along the same way we can obtain that

$$\left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x u}^h u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \leq C d_q 2^{-\frac{3q}{2}} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}.$$

We still have to find the estimate of the reminder term, we have

$$\begin{aligned} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \sum_{q' \geq q-3} \left(\int_0^t \|\tilde{\Delta}_{q'}^h u_\phi\|_{L_h^\infty(L_v^2)}^2 \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_x u_\phi\|_{L_h^2(L_v^\infty)}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{q' \geq q-3} \left(\int_0^t \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 2^{2q'} \|e^{\mathcal{R}t'} \Delta_{q'}^h u_\phi\|_{L_h^2(L_v^\infty)}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

By using the Poincaré inequality we achieve

$$\|e^{\mathcal{R}t'} \Delta_{q'}^h u_\phi\|_{L_h^2(L_v^\infty)} \lesssim \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_y u_\phi\|_{L^2}.$$

Then

$$\begin{aligned} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (R^h(u, \partial_x u))_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \sum_{q' \geq q-3} \left(\int_0^t \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 2^{2q'} \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_y u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\leq C d_q 2^{-\frac{3q}{2}} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}. \end{aligned}$$

By summing all the resulting estimates, and then multiplying by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} we obtain the proof of the term $\|e^{\mathcal{R}t} (u \partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}$.

Now we give the proof of the second estimate in (5.4.25). By using the Bony's decomposition (5.2.2), we can write

$$v \partial_y u = T_v^h \partial_y u + T_{\partial_y u}^h v + R^h(v, \partial_y u).$$

Then

$$\begin{aligned} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (v \partial_y u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_y u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &+ \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y u}^h v)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} + \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_y u))_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \end{aligned}$$

We use the definition of T^h , we have

$$\left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_y u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim \left(\int_0^t \sum_{|q'-q|\leq 4} \|e^{\mathcal{R}t'} (S_{q'-1}^h v \Delta_{q'}^h \partial_y u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.$$

Lemma 5.2.2 allows us to obtain

(5.8.5)

$$\left(\int_0^t \sum_{|q'-q|\leq 4} \|e^{\mathcal{R}t'} (S_{q'-1}^h v \Delta_{q'}^h \partial_y u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim \sum_{|q'-q|\leq 4} \left(\int_0^t \|S_{q'-1}^h v_\phi^+\|_{L^\infty}^2 \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_y u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.$$

We have by applying Bernstein Lemma 5.2.1 that

$$\|\Delta_{q'}^h v_\phi(t')\|_{L^\infty} \lesssim 2^{\frac{3q}{2}} \|\Delta_{q'}^h \int_0^y u_\phi(t', \cdot, s) ds\|_{L_h^2(L_v^\infty)},$$

we use the fact that $\|\int_0^y f dy'\|_{L_y^\infty} \leq \int_0^1 |f| dy' \leq \|f\|_{L_y^2}$. As a result, it comes out

$$\begin{aligned} \|\Delta_{q'}^h v_\phi(t')\|_{L^\infty} &\lesssim 2^{\frac{3q}{2}} \|\Delta_{q'}^h u_\phi(t')\|_{L^2} \\ &\leq C d_{q'}(u_\phi) 2^{q'} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}, \end{aligned}$$

where $(d_{q'}(u_\phi))_{q' \in \mathbb{Z}}$ is a generic element of $\ell^1(\mathbb{Z})$ such that $\sum d_{q'}(u_\phi) \leq 1$. Then

$$\|S_{q'-1}^h v_\phi(t')\|_{L^\infty} \lesssim 2^{q'} \|u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

We replace in (5.8.5) we obtain

$$\begin{aligned} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_y u)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \sum_{|q'-q| \leq 4} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} 2^{q'} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_y u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q'-q| \leq 4} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} 2^{q'} \left(2^{-\frac{5q'}{2}} d_{q'}(\partial_y u_\phi) \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \right) \\ (5.8.6) \quad &\leq C d_q 2^{-\frac{3q}{2}} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}, \end{aligned}$$

where

$$d_q = \sum_{|q'-q| \leq 4} d_{q'}(\partial_y u_\phi) 2^{\frac{3}{2}(q-q')}.$$

Along the same way we can obtain that

$$\left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y u}^h v)_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \leq C d_q 2^{-\frac{3q}{2}} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}.$$

We finish our proof by giving the estimate of the last term $\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_y u))_\phi\|_{L^2}^2 dt'$. We use the definition of R^h , we have

$$\begin{aligned} \left(\int_0^t \|e^{\mathcal{R}t'} \Delta_q^h (R^h(v, \partial_y u))_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \sum_{q' \geq q-3} \left(\int_0^t \|\tilde{\Delta}_{q'}^h v_\phi\|_{L^\infty}^2 \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_y u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{q' \geq q-3} \left(\int_0^t \|u_\phi\|_{\mathcal{B}^{\frac{5}{2}}}^2 2^{-2q'} \|e^{\mathcal{R}t'} \Delta_{q'}^h \partial_y u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\leq C d_q 2^{-\frac{3q}{2}} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{5}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}. \end{aligned}$$

By summing all the resulting estimates, and then multiplying by $2^{\frac{3q}{2}}$ and taking the sum over \mathbb{Z} we obtain the proof of the term $\|e^{\mathcal{R}t} (v \partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}$.

Chapter 6

Magnetohydrodynamics system

In this chapter we present the results of the following paper:

N. Aarach, Hydrostatic approximation of the 2D MHD system in a thin strip with a small analytic data, Journal of Mathematical Analysis and Applications Volume 509, Issue 2, 2022

6.1 Introduction

The dynamics of an electrically conducting liquid near a wall has been a topic of constant interest, at least since the pioneering work of Hartmann [69]. It is relevant to many domains of active research, such as dynamo theory [48] or nuclear fusion [132]. In this paper, we consider the global well-posedness of the following two-dimensional Navier-Stokes system coupled with an approximation of the Maxwell equation, this system is called the MHD system. We study this system in a thin domain provided by Dirichlet boundary conditions. Then let $\mathbf{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\}$ where ϵ is the width of the domain. So, our system is of the following form:

$$(6.1.1) \quad \begin{cases} \partial_t U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \epsilon^2 \Delta U^\epsilon + \nabla P^\epsilon = B^\epsilon \cdot \nabla B^\epsilon, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ \partial_t B^\epsilon + U^\epsilon \cdot \nabla B^\epsilon - \epsilon^2 \Delta B^\epsilon = B^\epsilon \cdot \nabla U^\epsilon, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ \operatorname{div} U^\epsilon = \operatorname{div} B^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ U^\epsilon_{/t=0} = U_0^\epsilon, \quad B^\epsilon_{/t=0} = B_0^\epsilon, & \text{in } \mathbf{S}^\epsilon \end{cases}$$

where $U^\epsilon(t, x, Y) = (u^\epsilon(t, x, Y), v^\epsilon(t, x, Y))$ is the fluid's velocity vector, and $P^\epsilon(t, x, Y)$ is the pressure function that guarantees the velocity field's free divergence, $B^\epsilon(t, x, Y) = (b^\epsilon(t, x, Y), c^\epsilon(t, x, Y))$ represents the magnetic field of the fluid. System (6.1.1) is completed by the following boundary condition

$$U^\epsilon_{|Y=0} = U^\epsilon_{|Y=\epsilon} = 0 \quad \text{and} \quad B^\epsilon_{|Y=0} = B^\epsilon_{|Y=\epsilon} = 0.$$

In our System (6.1.1) the Laplace is given by $\Delta = \partial_x^2 + \partial_Y^2$.

The purpose of this work is to rely on this recent progress to gain insight and perspective in the study of the Magnetohydrodynamic system (MHD), which couples the Navier-Stokes equation with small viscosity and an approximation of the Maxwell equation for the electromagnetic field. The purpose of this paper is to gain some insight into the analysis of MHD boundary layer models. It is primarily intended for mathematicians, either applied or interested in the theory of fluid PDE's. The electrohydrodynamics, or mechanical fluids ionized in the presence of electric fields (electrostatic) without a magnetic field, falls between the mechanics of classic fluids and the magnetohydrodynamics. This system describes the

evolution of a conducting fluid under the effect of a magnetic field in a strip (such as, for instance, the liquid iron in the Earth's core under the influence of the Earth's magnetic field).

In geophysics, it is common to assume that vertical motion is much smaller than horizontal motion and that the fluid layer depth is small in comparison to the radius of the sphere. As a result, these assumptions used to describe hydrodynamical flows on the earth and are a good approximation of oceanic and global magnetospheric flow. The thin domain in System (6.1.1) considered to take into account this anisotropy between horizontal and vertical directions. Under this assumption, it is believed that the dynamics of fluids on a large scale tend towards a geostrophic balance (see [64], [71] or [118]). Formally, as in [114], taking into account this anisotropy, we also consider the initial data in the following form,

$$U_{|t=0}^\epsilon = U_0^\epsilon = \left(u_0 \left(x, \frac{Y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{Y}{\epsilon} \right) \right) \quad \text{in } \mathbf{S}^\epsilon,$$

and

$$B_{|t=0}^\epsilon = B_0^\epsilon = \left(b_0 \left(x, \frac{Y}{\epsilon} \right), \epsilon c_0 \left(x, \frac{Y}{\epsilon} \right) \right) \quad \text{in } \mathbf{S}^\epsilon.$$

Then in our paper, we look for solutions of System (6.1.1) in the following form :

$$(6.1.2) \quad \begin{cases} U^\epsilon(t, x, Y) = \left(u^\epsilon \left(t, x, \frac{Y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{Y}{\epsilon} \right) \right), \\ B^\epsilon(t, x, Y) = \left(b^\epsilon \left(t, x, \frac{Y}{\epsilon} \right), \epsilon c^\epsilon \left(t, x, \frac{Y}{\epsilon} \right) \right), \\ P^\epsilon(t, x, Y) = p^\epsilon \left(t, x, \frac{Y}{\epsilon} \right). \end{cases}$$

Let $\mathbf{S} = \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$, we start by giving the two equations obtained by replacing U^ϵ by u^ϵ and v^ϵ and B^ϵ by b^ϵ and c^ϵ

$$(6.1.3) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon + \partial_x p^\epsilon = b^\epsilon \partial_x b^\epsilon + c^\epsilon \partial_y b^\epsilon, & \text{in }]0, \infty[\times \mathbf{S} \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_y^2 v^\epsilon) + \partial_y p^\epsilon = \epsilon^2 (b^\epsilon \partial_x c^\epsilon + c^\epsilon \partial_y c^\epsilon), & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_t b^\epsilon + u^\epsilon \partial_x b^\epsilon + v^\epsilon \partial_y b^\epsilon - \epsilon^2 \partial_x^2 b^\epsilon - \partial_y^2 b^\epsilon = b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y u^\epsilon, & \text{in }]0, \infty[\times \mathbf{S} \\ \epsilon (\partial_t c^\epsilon + u^\epsilon \partial_x c^\epsilon + v^\epsilon \partial_y c^\epsilon - \epsilon^2 \partial_x^2 c^\epsilon - \partial_y^2 c^\epsilon) = \epsilon (b^\epsilon \partial_x v^\epsilon + c^\epsilon \partial_y v^\epsilon), & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x b^\epsilon + \partial_y c^\epsilon = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{t=0} = (u_0, v_0, b_0, c_0), & \text{in } \mathbf{S} \\ (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{y=0} = (u^\epsilon, v^\epsilon, b^\epsilon, c^\epsilon)|_{y=1} = 0. & \end{cases}$$

Formally taking $\epsilon \rightarrow 0$ in the scaled system (6.1.3), we obtain the Prandtl system on u and also for b which is of the following form :

$$(6.1.4) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = b \partial_x b + c \partial_y b, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_y p = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_t b + u \partial_x b + v \partial_y b - \partial_y^2 b = b \partial_x u + c \partial_y u, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x u + \partial_y v = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x b + \partial_y c = 0, & \text{in }]0, \infty[\times \mathbf{S} \\ u|_{t=0} = u_0, & \text{in } \mathbf{S} \\ b|_{t=0} = b_0, & \text{in } \mathbf{S}, \end{cases}$$

where (u, v) and (b, c) satisfying the Dirichlet boundary condition

$$(6.1.5) \quad (u, v, b, c)|_{y=0} = (u, v, b, c)|_{y=1} = 0.$$

Our goal is to achieve the global existence of the solution for Systems (6.1.3) et (6.1.4), then we want to show the convergence of the scaled MHD system (6.1.3) to the limit system when ϵ tends towards zero.

Remark 6.1.1. *We can also get the equation satisfied by c when $\epsilon \rightarrow 0$. We remark that System (6.1.4) implies the following equation for c :*

$$\partial_t c + u\partial_x c + v\partial_y c - \partial_y^2 c = b\partial_x v + c\partial_y v \quad \text{in }]0, \infty[\times \mathbf{S}.$$

Indeed, by taking the derivative of the equation satisfied by b with respect to x variable, and using the free divergence of U and B , we obtain

$$\partial_x (\partial_t b + u\partial_x b + v\partial_y b - \partial_y^2 b - b\partial_x u - c\partial_y u) = -\partial_y (\partial_t c + u\partial_x c + v\partial_y c - \partial_y^2 c - b\partial_x v - c\partial_y v) = 0.$$

From which we obtain

$$\partial_t c + u\partial_x c + v\partial_y c - \partial_y^2 c - b\partial_x v - c\partial_y v = m(t, x),$$

but as c is zero at the boundary then $m(t, x) = 0$ this ensure the equation satisfied by c .

We remark that in System (6.1.4), we have to deal with the same difficulties as for the Prandtl equations due to its degenerate form and the nonlinear term $v\partial_y u$ which will lead to the loss of one derivative in the tangential direction in the process of energy estimates. For a more complete survey on this very challenging problem we refer the reader to the works [13, 52, 53, 59, 101] and references therein. To overcome this difficulty, one has to impose a monotony hypothesis on the normal derivative of the velocity or analytic regularity on the velocity. Lately, this result was proved via the energy method in [13, 101] independently by taking care of the cancellation property in the convection terms of (PE). Sammartino and Caffisch [124] solved the problem for analytic solutions on a half-space and later, the analyticity in the normal variable y was removed by Lombardo, Cannone, and Sammartino in [100]. The main argument used in [100, 124] is to apply the abstract Cauchy-Kowalewskaya (CK) theorem. We also mention a well-posedness result of the Prandtl system for a class of data with Gevrey regularity [60]. Lately, for a class of convex data, Gérard-Varet, Masmoudi, and Vicol [62] proved the well-posedness of the Prandtl system in the Gevrey class. We also want to remark that unlike the case of Prandtl equations, in the system (6.1.4), the pressure term is determined by $\partial_y p = 0$ and by the compatibility condition $\int_0^1 u dy = 0$. Indeed, the pressure act as a Lagangean multiplier associated to the compatibility condition, and can be computed by

$$\partial_x p = \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \partial_x \int_0^1 (u)^2(t, x, y) dy + \partial_x \int_0^1 (b)^2(t, x, y) dy.$$

One of the difficulties of the chapter is to find a way to deal with the term pressure, for this we have used analytical methods to remove this difficulty. We also want to recall some results concerning the system (6.1.3). Concerning the strong solutions for the 2D case, the locale existence result was established by Guillén-González, Masmoudi and Rodriguez-Bellido [67], while the global existence for the 2D case was achieved by Bresch, Kazhikhov, and Lemoine in [17] and by Temam and Ziane in [127]. In our paper, we also want to establish the global well-posedness of the system (6.1.3) in a 2D case in a thin domain.

6.2 Littlewood-Paley Theory and Functional Framework

To introduce the result of this paper, we will recall some elements of the Littlewood-Paley theory and introduce the functions space and technique using for the proof of our result. So we define the dyadic operator in the horizontal variable, (of x variable) and for all $q \in \mathbb{Z}$, we recall from [15] that

$$\begin{aligned}\Delta_q^h a(x, y) &= \mathcal{F}_h^{-1} (\varphi(2^{-q} |\xi|) \widehat{a}(\xi, y)), \\ S_q^h a(x, y) &= \mathcal{F}_h^{-1} (\psi(2^{-q} |\xi|) \widehat{a}(\xi, y)).\end{aligned}$$

where ψ and φ are a smooth function such that

$$\begin{aligned}\text{supp } \varphi &\subset \{z \in \mathbb{R} / \frac{3}{4} \leq |z| \leq \frac{8}{3}\} \quad \text{and} \quad \forall z > 0, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q} z) = 1, \\ \text{supp } \psi &\subset \{z \in \mathbb{R} / |z| \leq \frac{4}{3}\} \quad \text{and} \quad \psi(z) + \sum_{q \geq 0} \varphi(2^{-q} z) = 1.\end{aligned}$$

and

$$\forall q, q' \in \mathbb{Z}, |q - q'| \geq 2, \quad \text{supp } \varphi(2^{-q} \cdot) \cap \text{supp } \varphi(2^{-q'} \cdot) = \emptyset.$$

And in all that follows, $\mathcal{F}a$ and \widehat{a} always denote the partial Fourier transform of the distribution a with respect to the horizontal variable (of x variable), that is, $\widehat{a}(\xi, y) = \mathcal{F}_{x \rightarrow \xi}(a)(\xi, y)$. We refer to [15] and [18] for a more detailed construction of the dyadic decomposition. Combined the definition of the dyadic operator to

$$(6.2.1) \quad \forall z \in \mathbb{R}, \quad \psi(z) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} z) = 1,$$

implies that all tempered distributions can be decomposed with respect to the horizontal frequencies as

$$a = \sum_{q \in \mathbb{Z}} \Delta_q^h a.$$

We now introduce the function spaces used throughout the paper. As in [114], we define the Besov-type spaces $\mathcal{B}^{s,0}$, $s \in \mathbb{R}$ as follows.

Definition 6.2.1. *Let $s \in \mathbb{R}$ and $\mathbf{S} = \mathbb{R} \times]0, 1[$. For any $a \in \mathcal{S}'_h(\mathbf{S})$, i.e., a belongs to $\mathcal{S}'(\mathbf{S})$ and $\lim_{q \rightarrow -\infty} \|S_q^h a\|_{L^\infty} = 0$, we set*

$$\|a\|_{\mathcal{B}^{s,0}} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q^h a\|_{L^2}.$$

(i) For $s \leq \frac{1}{2}$, we define

$$\mathcal{B}^{s,0}(\mathbf{S}) \triangleq \{a \in \mathcal{S}'_h(\mathbf{S}) : \|a\|_{\mathcal{B}^{s,0}} < +\infty\}.$$

(ii) For $s \in]k - \frac{1}{2}, k + \frac{1}{2}]$, with $k \in \mathbb{N}^*$, we define $\mathcal{B}^{s,0}(\mathbf{S})$ as the subset of distributions u in $\mathcal{S}'_h(\mathbf{S})$ such that $\partial_x^k a \in \mathcal{B}^{s-k,0}(\mathbf{S})$.

For better use of the smoothing effect given by the diffusion terms, we work in the following Chemin-Lerner type spaces and the time-weighted Chemin-Lerner type spaces.

Definition 6.2.2. *Let $p \in [1, +\infty]$ and $T \in]0, +\infty[$. Then, the space $\tilde{L}_T^p(\mathcal{B}^{s,0}(\mathbf{S}))$ is the closure of $C([0, T]; \mathcal{S}(\mathbf{S}))$ under the norm*

$$\|a\|_{\tilde{L}_T^p(\mathcal{B}^{s,0}(\mathbf{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \|\Delta_q^h a(t)\|_{L^2}^p dt \right)^{\frac{1}{p}},$$

with the usual change if $p = +\infty$.

Definition 6.2.3. Let $p \in [1, +\infty]$ and let $f \in L^1_{loc}(\mathbb{R}_+)$ be a nonnegative function. Then, the space $\tilde{L}^p_{t,f(t)}(\mathcal{B}^{s,0}(\mathbf{S}))$ is the closure of $C([0, T]; \mathcal{S}(\mathbf{S}))$ under the norm

$$\|a\|_{\tilde{L}^p_{t,f(t)}(\mathcal{B}^{s,0}(\mathbf{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^t f(t') \|\Delta_q^h a(t')\|_{L^2}^p dt' \right)^{\frac{1}{p}}.$$

Remark 6.2.1. To simplify the notation even further, in all that follows, we shall denote

$$B^{s,0} \triangleq B^s.$$

The following Bernstein lemma gives important properties of a distribution u when its Fourier transform is well localized. We refer the reader to [26] for the proof of this lemma.

Lemma 6.2.1. Let $k \in \mathbb{N}$, $d \in \mathbb{N}^*$ and $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. There exists a constant $C > 0$ such that, for any $p, q \in \mathbb{R}$, $1 \leq p \leq q \leq +\infty$, for any $\lambda > 0$ and for any $a \in L^p(\mathbb{R}^d)$, we have

$$\text{supp } (\hat{a}) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq r_1 \lambda\} \implies \sup_{|\alpha|=k} \|\partial^\alpha a\|_{L^q} \leq C^k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|a\|_{L^p},$$

and

$$\text{supp } (\hat{a}) \subset \{\xi \in \mathbb{R}^d \mid r_1 \lambda \leq |\xi| \leq r_2 \lambda\} \implies C^{-k} \lambda^k \|a\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha a\|_{L^p} \leq C^k \lambda^k \|a\|_{L^p}.$$

Finally, to deal with the estimate concerning the product of two distribution, we shall frequently use Bony's decomposition (see [18]) in the horizontal variable (of the x variable) that for f, g two tempered distributions :

$$(6.2.2) \quad fg = T_f^h g + T_g^h f + R^h(f, g),$$

where

$$T_f^h g = \sum_q S_{q-1}^h f \Delta_q^h g, \quad T_g^h f = \sum_q S_{q-1}^h g \Delta_q^h f,$$

and the rest term satisfied

$$R^h(f, g) = \sum_q \tilde{\Delta}_q^h f \Delta_q^h g \quad \text{with} \quad \tilde{\Delta}_q^h f = \sum_{|q-q'| \leq 1} \Delta_{q'}^h f.$$

6.2.1 Main results

Our main difficulty relies on finding a way to estimate the nonlinear terms, which allows exploiting the smoothing effect given by the above function spaces. Using the method introduced by Chemin in [27] (see also [30], [110] or [114]), for any $f \in L^2(\mathbf{S})$, we define the following auxiliary function, which allows to control the analyticity of f in the horizontal variable x ,

$$(6.2.3) \quad f_\phi(t, x, y) = e^{\phi(t, D_x)} f(t, x, y) \triangleq \mathcal{F}_h^{-1}(e^{\phi(t, \xi)} \hat{f}(t, \xi, y)) \quad \text{with} \quad \phi(t, \xi) = (a - \lambda\theta(t))|\xi|,$$

where the quantity $\theta(t)$, which describes the evolution of the analytic band of f , satisfies

$$(6.2.4) \quad \forall t > 0, \dot{\theta}(t) \geq 0 \quad \text{and} \quad \theta(0) = 0.$$

The main idea of this technique consists in the fact that if we differentiate, with respect to the time variable, a function of the type $e^{\phi(t, D_x)} f(t, x, y)$, we obtain an additional “good term” which plays the smoothing role. More precisely, we have

$$\frac{d}{dt} \left(e^{\phi(t, D_x)} f(t, x, y) \right) = -\lambda \dot{\theta}(t) |D_x| e^{\phi(t, D_x)} f(t, x, y) + e^{\phi(t, D_x)} \partial_t f(t, x, y),$$

where $-\lambda \dot{\theta}(t) |D_x| e^{\phi(t, D_x)} f(t, x, y)$ gives a smoothing effect if $\dot{\theta}(t) \geq 0$. This smoothing effect allows to obtain our global existence and stability results in the analytic framework. Besides, MHD system is known to be very unstable.

Before starting the obtained result, we need the following lemma to characterize the product $(fg)_\phi$, indeed this product be useful in all the rest of the paper.

Lemma 6.2.2. *Let $f \in L_x^2$, $g \in L_x^2$, we define $f^+ = \mathcal{F}_\xi^{-1}(|\mathcal{F}_x(f)|)$ then, we have*

$$|\widehat{(fg)_\phi}(\xi)| \leq \widehat{f^+ g^+}(\xi) \quad \text{and} \quad \|f^+\|_{L_x^2} = \|f\|_{L_x^2}.$$

Proof. Let as consider f , and g two functions in L_x^2 , we have

$$\begin{aligned} |\widehat{(fg)_\phi}(\xi)| &= e^{\phi(\xi)} |\widehat{f}(\cdot) * \widehat{g}(\cdot)|(\xi) \\ &\leq e^{\phi(\xi)} \int |\widehat{f}(\xi - \eta)| |\widehat{g}(\eta)| d\eta, \end{aligned}$$

By virtue of the definition of the function ϕ we have $e^{\phi(\xi)} > 0$ and $e^{\phi(\xi)} \leq e^{\phi(\xi - \eta)} e^{\phi(\eta)}$, thus

$$\begin{aligned} |\widehat{(fg)_\phi}(\xi)| &\leq \int e^{\phi(\xi - \eta)} |\widehat{f}(\xi - \eta)| e^{\phi(\eta)} |\widehat{g}(\eta)| d\eta \\ &\leq \int |\widehat{f}_\phi(\xi - \eta)| |\widehat{g}_\phi(\eta)| d\eta \\ &\leq \widehat{f}_\phi * \widehat{g}_\phi(\xi) = \widehat{f^+ g^+}(\xi). \end{aligned}$$

The second point of the lemma is trivial. □

Corollary 6.2.1. *For any f and g in L_x^2 , we have*

$$|\widehat{(Tfg)_\phi}| \leq \widehat{(Tf^+ g^+)_\phi} \quad \text{and} \quad |\widehat{R(f, g)_\phi}| \leq \widehat{R(f^+, g^+)_\phi}.$$

Our main results are the following theorems.

The first result is the global well-posedness of the limit MHD system (6.1.4), with small analytic data in the horizontal variable.

Theorem 6.2.1 (Global well-posedness of the MHD limit system). *. . Let $a > 0$ and λ large enough, we assume that for some constant c_0 sufficiently small independent of ϵ , and for any initial data (u_0, b_0) satisfying*

$$(6.2.5) \quad \|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} b_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a < \frac{a}{\lambda},$$

and the compatibility conditions $\int_0^1 (u_0, b_0) dy = 0$ are satisfied. Then the limit system (6.1.4) has a unique global solution (u, b) satisfying

$$(6.2.6) \quad \|e^{\mathcal{R}t} (u_\phi, b_\phi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_y (u_\phi, b_\phi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \leq C \|e^{a|D_x|} (u_0, b_0)\|_{\mathcal{B}^{\frac{1}{2}}},$$

where (u_ϕ, v_ϕ) will be given by (6.2.3) and \mathcal{R} is a constant smaller than Poincaré constant on the domain $\mathbf{S} = \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$.

The second result is the global well-posedness of the scaled system (6.1.3) with small analytic initial data in the horizontal variable x . The main interesting point here is that the smallness of data is independent of ϵ and there holds the global uniform estimate (6.2.8) with respect to the parameter ϵ .

Theorem 6.2.2 (Global well-posedness of the scaled system). *Let $a > 0$ and $\epsilon > 0$, there exist a constant c_1 sufficiently small independent of ϵ , such that for any initial data $(u_0^\epsilon, v_0^\epsilon, b_0^\epsilon, c_0^\epsilon) = (U_0^\epsilon, B_0^\epsilon)$ satisfying*

$$(6.2.7) \quad \|e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0^\epsilon, \epsilon c_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a,$$

then System (6.1.3) has a unique global solution (U^ϵ, B^ϵ) so that

$$(6.2.8) \quad \begin{aligned} & \|e^{\mathcal{R}t}(u_\varphi^\epsilon, \epsilon v_\varphi^\epsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}(b_\varphi^\epsilon, \epsilon c_\varphi^\epsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y(u_\varphi^\epsilon, \epsilon v_\varphi^\epsilon)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \\ & + \|e^{\mathcal{R}t}\partial_y(b_\varphi^\epsilon, \epsilon c_\varphi^\epsilon)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \epsilon \|e^{\mathcal{R}t}\partial_x(u_\varphi^\epsilon, \epsilon v_\varphi^\epsilon)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} + \epsilon \|e^{\mathcal{R}t}\partial_x(b_\varphi^\epsilon, \epsilon c_\varphi^\epsilon)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^{\frac{1}{2}})} \\ & \leq C \left(\|e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0^\epsilon, \epsilon c_0^\epsilon)\|_{\mathcal{B}^{\frac{1}{2}}} \right), \end{aligned}$$

where $(u_\varphi^\epsilon, v_\varphi^\epsilon)$ and $(b_\varphi^\epsilon, c_\varphi^\epsilon)$ will given also by (6.2.3).

The main idea to prove the above two theorems is to control the new unknown (u_ϕ, b_ϕ) defined by (6.2.3), where u is the horizontal velocity and b is the horizontal component of the fluid's magnetic field. u_ϕ and b_ϕ is a weighted function of u and b in the dual Fourier variable with an exponential function of $(a - \lambda\theta(t))|\xi|$, which is equivalent to the analyticity of the solution in the horizontal variable. By the classical Cauchy-Kowalewskaya theorem, one expects the radius of the analyticity of the solutions to decay in time and so the exponent, which corresponds to the width of the analyticity band, is allowed to vary with time. Using energy estimates on the equation satisfied by (u_ϕ, b_ϕ) and the control of the quantity which describes 'the loss of the analyticity radius', we shall show that 'the analyticity persists globally in time. Consequently, our result is a global Cauchy-Kowalewskaya type theorem.

Remark 6.2.2. *To obtain a solution of our systems, we use the Friedrich's schemes which allows us to build a sequence of solutions, and then we go to the limit on the sequence of solutions. Indeed, for that we define our the quantity $\theta^n(t)$ satisfying the following ordinary differential equation :*

$$\dot{\theta}^n(t) = \|e^{(a - \lambda\theta^n(t))|D_x|}\partial_y(J_n u, J_n b)\|_{\mathcal{B}^{\frac{1}{2}}},$$

with

$$(6.2.9) \quad \begin{cases} J_n u = \sum_{|k| < n} \mathbf{1}_{\{|\xi| < n\}} \sin(k\pi y) \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{u}^k(t, \xi) d\xi, \\ J_n b = \sum_{|k| < n} \mathbf{1}_{\{|\xi| < n\}} \sin(k\pi y) \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{b}^k(t, \xi) d\xi, \end{cases}$$

define the frequency cut-off operators in the **all variables**, where

$$(6.2.10) \quad \begin{cases} u(t, x, y) = \sum_n \sin(n\pi y) u^n(t, x), \\ b(t, x, y) = \sum_n \sin(n\pi y) b^n(t, x), \end{cases}$$

and

$$(6.2.11) \quad \begin{cases} u^n(t, x) = \frac{1}{2\pi} \int_0^1 u(t, x, y) \sin(n.\pi.y) dy, \\ b^n(t, x) = \frac{1}{2\pi} \int_0^1 b(t, x, y) \sin(n.\pi.y) dy. \end{cases}$$

Then we achieve the following equation

$$(6.2.12) \quad \begin{cases} \partial_t u^n - J_n \partial_y^2 u^n + J_n (J_n u^n \nabla J_n u^n) + \nabla J_n p_n = J_n (J_n b^n \nabla J_n b^n), \\ \partial_t b^n - J_n \partial_y^2 b^n + J_n (J_n u^n \nabla J_n b^n) = J_n (J_n b^n \nabla J_n u^n), \end{cases}$$

which is an ordinary differential equation, the solution is obtained on the interval $[0, T_n^*$ by Cauchy-Lipschitz. Since we have that $J_n^2 = J_n$, we can deduce that $J_n u^n$ and $J_n b^n$ are also a solution (6.2.12). The uniqueness of the solution implies that $u^n = J_n u^n$ and $b^n = J_n b^n$. So we can write

$$(6.2.13) \quad \begin{cases} \partial_t u^n - J_n \partial_y^2 u^n + J_n (u^n \nabla u^n) + \nabla p_n = J_n (b^n \nabla b^n), \\ \partial_t b^n - J_n \partial_y^2 b^n + J_n (u^n \nabla b^n) = J_n (b^n \nabla u^n), \end{cases}$$

We note that all the analyticity estimates got it in Sections 5.3-5.6 are valid also for this equation. So, (u^n) and (b^n) will have a convergent sub-sequence that we still denote by (u^n) and (b^n) , then we have that (u^n) and (b^n) converge to u and b in $L_{loc}^\infty(\mathbb{R}^+, H^{-s})$ for all $s \in]0, 1[$, which u and b are the solution of our final system.

The third result concerns the study of the convergence from the scaled anisotropic MHD system (6.1.3) to the limit system (6.1.4), so in this theorem we proved that the convergence is globally in time.

Theorem 6.2.3 (Convergence to the MHD limit system). *Let $a > 0$ and $\epsilon > 0$, $\exists M > 0$ and $(u_0^\epsilon, v_0^\epsilon, b_0^\epsilon, c_0^\epsilon)$, satisfying (6.2.7). Let u_0 and b_0 satisfying $e^{a|D_x|} u_0 \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}}$, $e^{a|D_x|} \partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$, (the same thing for b_0) and the compatibility condition $\int_0^1 (u_0, b_0) dy = 0$ are satisfied and*

$$\|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} b_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_2 a}{1 + \|e^{a|D_x|} (u_0, b_0)\|_{\mathcal{B}^{\frac{3}{2}}}},$$

for some c_2 sufficiently small independent of ϵ , then we have

$$(6.2.14) \quad \begin{aligned} & \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(t)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_y (\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ & + \epsilon \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y (\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ & \leq C \left(\|e^{a|D_x|} (u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} (b_0^\epsilon - b_0, \epsilon(c_0^\epsilon - c_0))\|_{\mathcal{B}^{\frac{1}{2}}} + M\epsilon \right). \end{aligned}$$

where

$$(6.2.15) \quad \begin{cases} (\Psi^{1,\epsilon}, \Psi^{2,\epsilon}) = (u^\epsilon - u, v^\epsilon - v), \\ (\Phi^{1,\epsilon}, \Phi^{2,\epsilon}) = (b^\epsilon - b, c^\epsilon - c), \end{cases}$$

and v_0 is determined from u_0 via $\partial_x u + \partial_y v = 0$ and $v_0|_{y=0} = v_0|_{y=1} = 0$, and $(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)$, $(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)$ will be given by (6.5.6).

Remarque 6.2.1. *The main idea to prove the above theorems is to use analytic energy estimates, which are motivated by [112] and which originates from [27].*

6.3 Global well posedness of the limit system

The goal of this section is to prove the global well posedness of the limit system of the MHD equation, we remark that the local smooth solution of the limit system follow a standard parabolic regularization method similar to the MHD system. First we remark that the Dirichlet boundary condition

$$(u, v)_{/y=0} = (u, v)_{/y=1} = (b, c)_{/y=0} = (b, c)_{/y=1} = 0,$$

and the incompressible conditions $\partial_x u + \partial_y v = 0$ and $\partial_x b + \partial_y c = 0$ imply that :

$$(6.3.1) \quad v(t, x, y) = \int_0^y \partial_y v(t, x, s) ds = - \int_0^y \partial_x u(t, x, s) ds,$$

$$(6.3.2) \quad c(t, x, y) = \int_0^y \partial_y c(t, x, s) ds = - \int_0^y \partial_x b(t, x, s) ds.$$

We want now to find the equation for the pressure. The Dirichlet boundary conditions $(u, v, b, c)_{/y=0} = (u, v, b, c)_{/y=1} = 0$ we deduce from the incompressibility condition $\partial_x u + \partial_y v = 0$ and $\partial_x b + \partial_y c = 0$ that

$$\begin{aligned} \partial_x \int_0^1 u(t, x, y) dy &= - \int_0^1 \partial_y v(t, x, y) dy = v(t, x, 0) - v(t, x, 1) = 0, \\ \partial_x \int_0^1 b(t, x, y) dy &= - \int_0^1 \partial_y c(t, x, y) dy = c(t, x, 0) - c(t, x, 1) = 0. \end{aligned}$$

The compatibility condition $\partial_x \int_0^1 (u_0, b_0) dy = 0$ and the fact that $u(t, x, y) \rightarrow 0$ and $b(t, x, y) \rightarrow 0$ as $|x| \rightarrow \infty$, ensure that

$$\int_0^1 u(t, x, y) dy = 0 \quad \text{and} \quad \int_0^1 b(t, x, y) dy = 0.$$

Then by integrating the equations $\partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = b \partial_x b + c \partial_y b$ and $\partial_t b + u \partial_x b + v \partial_y b - \partial_y^2 b = b \partial_x u + c \partial_y u$, for $y \in [0, 1]$ and using the fact that $\partial_y p = 0$, we obtain

$$\partial_x p = \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \partial_x \int_0^1 (u)^2(t, x, y) dy + \partial_x \int_0^1 (b)^2(t, x, y) dy.$$

Remark 6.3.1. We consider the equation for the magnetic field $B = (b, c)$,

$$\partial_t B + U \nabla B - B \nabla U - \partial_y^2 B = 0.$$

Taking the trace of this equation we find that $\partial_y^2 B = 0$ on the boundary for a smooth solutions of (6.1.4). Using that B satisfies the divergence free condition $\partial_x b + \partial_y c = 0$, we obtain

$$\partial_x (\partial_y b) + \partial_y^2 c = 0.$$

Since $\partial_y^2 c = 0$ on the boundaries of the domain, then $\partial_x (\partial_y b) = 0$ also on the boundary. This implies that that $(\partial_y b)(t, x, 0) = m(t)$ and using the fact that $\partial_y b(t, x, y) \rightarrow 0$ as $|x| \rightarrow \infty$, then we deduce that $(\partial_y b)(t, x, 0) = 0$. Using a similar argument we obtain $(\partial_y b)(t, x, 1) = 0$.

So, we have obtained

$$\partial_y b(t, x, 1) = \partial_y b(t, x, 0) = 0,$$

for smooth solutions of (6.1.4).

Remark 6.3.2. Using the previous boundary condition, we obtain that the average $\int_0^1 b(t, x, y) dy = 0$

for all $t \geq 0$. Indeed, we have

$$\begin{aligned} & \partial_t \int_0^1 b(t, x, y) dy + \int_0^1 u \partial_x b dy + \int_0^1 v \partial_y b dy - \int_0^1 \partial_y^2 b(t, x, y) dy - \int_0^1 b \partial_x u dy - \int_0^1 c \partial_y u dy \\ &= \partial_t \int_0^1 b(t, x, y) dy - \partial_y b(t, x, 1) + \partial_y b(t, x, 0) = 0. \end{aligned}$$

Let $(u_\phi, v_\phi, b_\phi, c_\phi)$ be defined as in (6.2.3) and (6.2.4). By direct calculations from the limit system (6.1.4), we show that $(u_\phi, v_\phi, b_\phi, c_\phi)$ verify the following system

$$(6.3.3) \quad \begin{cases} \partial_t u_\phi + \lambda \dot{\theta}(t) |D_x| u_\phi + (u \partial_x u)_\phi + (v \partial_y u)_\phi - \partial_y^2 u_\phi + \partial_x p_\phi = (b \partial_x b)_\phi + (c \partial_y b)_\phi, \\ \partial_y p_\phi = 0, \\ \partial_t b_\phi + \lambda \dot{\theta}(t) |D_x| b_\phi + (u \partial_x b)_\phi + (v \partial_y b)_\phi - \partial_y^2 b_\phi = (b \partial_x u)_\phi + (c \partial_y u)_\phi, \\ \partial_x u_\phi + \partial_y v_\phi = 0, \\ \partial_x b_\phi + \partial_y c_\phi = 0, \\ u_\phi|_{t=0} = e^{\alpha |D_x|} u_0, \\ b_\phi|_{t=0} = e^{\alpha |D_x|} b_0, \end{cases}$$

where $|D_x|$ denotes the Fourier multiplier of symbol $|\xi|$. In what follows, we recall that we use C to denote a generic positive constant that can change from line to line.

Applying the dyadic operator Δ_q^h to the system (6.3.3), then taking the $L^2(\mathbf{S})$ scalar product of the first and the third equations of the obtained system with $\Delta_q^h u_\phi$ and $\Delta_q^h b_\phi$ respectively, we get

$$\begin{aligned} & \langle \Delta_q^h \partial_t u_\phi, \Delta_q^h u_\phi \rangle_{L^2} + \lambda \dot{\theta}(t) \langle |D_x| \Delta_q^h u_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h u_\phi \rangle_{L^2} + \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2} \\ &= - \langle \Delta_q^h (u \partial_x u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h (v \partial_y u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} + \langle \Delta_q^h (b \partial_x b + c \partial_y b)_\phi, \Delta_q^h u_\phi \rangle, \end{aligned}$$

and

$$\begin{aligned} & \langle \Delta_q^h \partial_t b_\phi, \Delta_q^h b_\phi \rangle_{L^2} + \lambda \dot{\theta}(t) \langle |D_x| \Delta_q^h b_\phi, \Delta_q^h b_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 b_\phi, \Delta_q^h b_\phi \rangle_{L^2} \\ &= - \langle \Delta_q^h (u \partial_x b)_\phi, \Delta_q^h b_\phi \rangle_{L^2} - \langle \Delta_q^h (v \partial_y b)_\phi, \Delta_q^h b_\phi \rangle_{L^2} + \langle \Delta_q^h (b \partial_x u + c \partial_y u)_\phi, \Delta_q^h b_\phi \rangle. \end{aligned}$$

Thanks to the Dirichlet boundary condition ($u_\phi|_{y=0} = u_\phi|_{y=1} = 0$) and due to the free divergence of $U = (u, v)$ (it mean that $\operatorname{div} U_\phi = \partial_x u_\phi + \partial_y v_\phi = 0$), we get by using the integration by part that

$$\begin{aligned} \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2} &= - \langle \Delta_q^h p_\phi, \Delta_q^h \partial_x u_\phi \rangle_{L^2} \\ &= \langle \Delta_q^h p_\phi, \Delta_q^h \partial_y v_\phi \rangle_{L^2} \\ &= - \langle \Delta_q^h \partial_y p_\phi, \Delta_q^h v_\phi \rangle_{L^2} = 0. \quad (\text{because } \partial_y p_\phi = 0) \end{aligned}$$

We recall that we have by integrating by part that

$$\begin{aligned} \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h u_\phi \rangle_{L^2} &= - \langle \Delta_q^h \partial_y u_\phi, \Delta_q^h \partial_y u_\phi \rangle_{L^2} = - \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2, \\ \langle \Delta_q^h \partial_y^2 b_\phi, \Delta_q^h b_\phi \rangle_{L^2} &= - \langle \Delta_q^h \partial_y b_\phi, \Delta_q^h \partial_y b_\phi \rangle_{L^2} = - \|\Delta_q^h \partial_y b_\phi\|_{L^2}^2. \end{aligned}$$

Replacing this in the obtained estimate, we get

$$(6.3.4) \quad \frac{1}{2} \frac{\partial}{\partial t} \|\Delta_q^h u_\phi(t)\|_{L^2}^2 + \lambda \dot{\theta}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi(t) \right\|_{L^2}^2 + \|\Delta_q^h \partial_y u_\phi(t)\|_{L^2}^2 = - \langle \Delta_q^h (u \partial_x u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} \\ - \langle \Delta_q^h (v \partial_y u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} + \langle \Delta_q^h (b \partial_x b)_\phi, \Delta_q^h u_\phi \rangle + \langle \Delta_q^h (c \partial_y b)_\phi, \Delta_q^h u_\phi \rangle,$$

and

$$(6.3.5) \quad \frac{1}{2} \frac{\partial}{\partial t} \|\Delta_q^h b_\phi(t)\|_{L^2}^2 + \lambda \dot{\theta}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h b_\phi(t) \right\|_{L^2}^2 + \|\Delta_q^h \partial_y b_\phi(t)\|_{L^2}^2 = - \langle \Delta_q^h (u \partial_x b)_\phi, \Delta_q^h b_\phi \rangle_{L^2} \\ - \langle \Delta_q^h (v \partial_y b)_\phi, \Delta_q^h b_\phi \rangle_{L^2} + \langle \Delta_q^h (b \partial_x u)_\phi, \Delta_q^h b_\phi \rangle + \langle \Delta_q^h (c \partial_y u)_\phi, \Delta_q^h b_\phi \rangle.$$

Multiplying (6.3.4) and (6.3.5) with $e^{2\mathcal{R}t}$, and integrating with respect to the time variable, we have

$$(6.3.6) \quad \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h u_\phi\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L_t^2(L^2)}^2 \\ = \mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h u_\phi\|_{L_t^2(L^2)}^2 + \frac{1}{2} \left\| \Delta_q^h e^{a|D_x|} u_0 \right\|_{L^2}^2 + I_1 + I_2 + I_3 + I_4,$$

and

$$(6.3.7) \quad \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h b_\phi\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h b_\phi(t') \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \partial_y b_\phi\|_{L_t^2(L^2)}^2 \\ = \mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h b_\phi\|_{L_t^2(L^2)}^2 + \frac{1}{2} \left\| \Delta_q^h e^{a|D_x|} b_0 \right\|_{L^2}^2 + D_1 + D_2 + D_3 + D_4.$$

We begin by observing that the terms $\mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h u_\phi\|_{L_t^2(L^2)}^2$ and $\mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h b_\phi\|_{L_t^2(L^2)}^2$ can be absorbed by the dissipation $\|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L_t^2(L^2)}^2$ and $\|e^{\mathcal{R}t} \Delta_q^h \partial_y b_\phi\|_{L_t^2(L^2)}^2$. Choosing \mathcal{R} smaller than $\frac{k}{8}$, where k is the Poincaré constant that comes out from the inequality $k \|u_\phi\|_{L^2} \leq \|\partial_y u_\phi\|_{L^2}$, we can achieve that

$$(6.3.8) \quad \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h u_\phi\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi(t') \right\|_{L^2}^2 dt' + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h \partial_y u_\phi\|_{L_t^2(L^2)}^2 \\ \leq C \left\| \Delta_q^h e^{a|D_x|} u_0 \right\|_{L^2}^2 + I_1 + I_2 + I_3 + I_4,$$

and

$$(6.3.9) \quad \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h b_\phi\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h b_\phi(t') \right\|_{L^2}^2 dt' + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h \partial_y b_\phi\|_{L_t^2(L^2)}^2 \\ \leq C \left\| \Delta_q^h e^{a|D_x|} b_0 \right\|_{L^2}^2 + D_1 + D_2 + D_3 + D_4,$$

Next, by using the technical Lemmas in the subsection 6.3.1, yield that

$$|I_1| = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \\ |I_2| = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \\ |I_3| = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (b \partial_x b)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

$$|I_4| = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (c \partial_y b)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

and

$$\begin{aligned} |D_1| &= \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x b)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \\ |D_2| &= \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y b)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \\ |D_3| &= \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (b \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \\ |D_4| &= \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (c \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2. \end{aligned}$$

Multiplying (6.3.8) and (6.3.9) by 2^{2qs} (for $s \in]0, 1[$), and taking square root of the resulting inequality, and finally by summing up the resulting ones over \mathbb{Z} , we obtain

$$(6.3.10) \quad \begin{aligned} &\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\leq C \left(\|e^{a|D_x|} u_0\|_{\mathcal{B}^s} + \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} (u, b)_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right), \end{aligned}$$

and

$$(6.3.11) \quad \begin{aligned} &\|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\leq C \left(\|e^{a|D_x|} b_0\|_{\mathcal{B}^s} + \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} (u_\phi, b_\phi)\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right). \end{aligned}$$

Thus, choosing

$$(6.3.12) \quad C \geq \max \left\{ 4, \frac{1}{2\mathcal{R}} \right\},$$

taking the sum of (6.3.10) and (6.3.11), we have

$$(6.3.13) \quad \begin{aligned} &\|e^{\mathcal{R}t} (u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t} (u_\phi, b_\phi)\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\leq C \|e^{a|D_x|} (u_0, b_0)\|_{\mathcal{B}^s} + 2C \|e^{\mathcal{R}t} (u_\phi, b_\phi)\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

We set

$$(6.3.14) \quad T^* \triangleq \sup \left\{ t > 0 : \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} + \|b_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{2C^2} \text{ and } \theta(t) \leq \frac{a}{\lambda} \right\}.$$

We choose initial data such that

$$C \left(\|e^{a|D_x|} u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|} b_0\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \min \left\{ \frac{1}{2C^2}, \frac{a}{2\lambda} \right\},$$

combining with the fact that $\theta(0) = 0$, we deduce that $T^* > 0$. We choose now $\sqrt{2\lambda} \geq 2C$. For any

$0 < t < T^*$, we deduce from (6.3.13) that

$$(6.3.15) \quad \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq C \left\| e^{a|D_x|}(u_0, b_0) \right\|_{\mathcal{B}^s}.$$

We deduce from (6.3.15), using (6.3.12), that, for any $0 < t < T^*$,

$$\|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} \leq C \left\| e^{a|D_x|}(u_0, b_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq C \left(\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \frac{1}{2C^2}.$$

Now, we recall that we already defined $\dot{\theta}(t) = \|\partial_y(u_\phi, b_\phi)(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ with $\theta(0) = 0$. Then, for any $0 < t < T^*$, Inequality (6.3.15) yields

$$\begin{aligned} \theta(t) &= \int_0^t \|\partial_y(u_\phi, b_\phi)(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\ &\leq \int_0^t e^{-\mathcal{R}t'} \|e^{\mathcal{R}t'} \partial_y(u_\phi, b_\phi)(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\ &\leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \left(\int_0^t \|e^{\mathcal{R}t'} \partial_y(u_\phi, b_\phi)(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\ &\leq C \left(\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{1}{2}}} \right) \\ &< \frac{a}{2\lambda}. \end{aligned}$$

A continuity argument implies that $T^* = +\infty$ and we have (6.3.15) is valid for any $t \in \mathbb{R}_+$.

In what follows, we present a proposition states the propagation for any \mathcal{B}^s (where $0 < s \leq \frac{5}{2}$) regularity on the solution of the hydrostatic MHD equations (6.1.4), and also the estimate of the term $\|\Delta_q^h(\partial_t u)_\phi\|_{L^2}$. This proposition will be useful in the last section when we prove the global convergence of the Theorem 6.2.3

Proposition 6.3.1. *Let $a > 0$. If $e^{a|D_x|}u_0 \in \mathcal{B}^{\frac{5}{2}}$, $e^{a|D_x|}\partial_y u_0 \in \mathcal{B}^{\frac{3}{2}}$, $e^{a|D_x|}b_0 \in \mathcal{B}^{\frac{3}{2}}$ and*

$$(6.3.16) \quad \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{c_1 a}{1 + \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{3}{2}}}}$$

for some c_1 sufficiently small, then there exists a positive constant C so that for $\lambda \geq C^2(1 + \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{3}{2}}})$, and $1 \leq s \leq \frac{5}{2}$, one has

$$(6.3.17) \quad \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}^\infty(\mathbb{R}^+, \mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}^2(\mathbb{R}^+, \mathcal{B}^s)} \leq C \left\| e^{a|D_x|}(u_0, b_0) \right\|_{\mathcal{B}^s},$$

and

$$(6.3.18) \quad \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(\partial_t b)_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \\ \leq C \left(\|e^{a|D_x|}\partial_y(u_0, b_0)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}(u_0, b_0)\|_{\mathcal{B}^{\frac{5}{2}}} \right).$$

Proof. We start by giving proof of (6.3.17). In the same way, by applying the dyadic operator to the system (6.3.3) and taking the L^2 inner product, then integrating with respect to the time variable, we have

$$(6.3.19) \quad \frac{1}{2} \|e^{\mathcal{R}t}\Delta_q^h u_\phi(t)\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t}\Delta_q^h \partial_y u_\phi(t)\|_{L_t^2(L^2)}^2 \\ = \mathcal{R} \|e^{\mathcal{R}t}\Delta_q^h u_\phi(t)\|_{L_t^2(L^2)}^2 + \left\| \Delta_q^h e^{a|D_x|}u_0 \right\|_{L^2}^2 + I_1 + I_2,$$

and

$$(6.3.20) \quad \begin{aligned} & \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h b_\phi(t)\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h b_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \partial_y^2 b_\phi(t)\|_{L_t^2(L^2)}^2 \\ & = \mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h b_\phi(t)\|_{L_t^2(L^2)}^2 + \left\| \Delta_q^h e^{a|D_x|} b_0 \right\|_{L^2}^2 + I_3 + I_4. \end{aligned}$$

To absorb the terms $\mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h u_\phi(t)\|_{L_t^2(L^2)}^2$ and $\mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h b_\phi(t)\|_{L_t^2(L^2)}^2$ in (6.3.19)-(6.3.20), we proceed in the same way as we did in (6.3.8)-(6.3.9), we obtain

$$(6.3.21) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h u_\phi(t)\|_{L_t^\infty(L^2)}^2 + 2\lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \partial_y^2 u_\phi(t)\|_{L_t^2(L^2)}^2 \\ & \leq C \left\| \Delta_q^h e^{a|D_x|} u_0 \right\|_{L^2}^2 + I_1 + I_2, \end{aligned}$$

and

$$(6.3.22) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h b_\phi(t)\|_{L_t^\infty(L^2)}^2 + 2\lambda \int_0^t \dot{\theta}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h b_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \partial_y^2 b_\phi(t)\|_{L_t^2(L^2)}^2 \\ & \leq C \left\| \Delta_q^h e^{a|D_x|} b_0 \right\|_{L^2}^2 + I_3 + I_4. \end{aligned}$$

where

$$\begin{aligned} |I_1| & = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u + v \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right|, \\ |I_2| & = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (b \partial_x b + c \partial_y b)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right|, \\ |I_3| & = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x b + v \partial_y b)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle dt' \right|, \\ |I_4| & = \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h (b \partial_x u + c \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle dt' \right|. \end{aligned}$$

All the estimates obtained in the proof of the theorem 6.2.1, are true when $s \in]0, 1[$. We need now give the proof of all this estimate when our $s > 0$. We first remark from the proof of the lemma 6.3.1, that for any $s > 0$, we have

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_u^h \partial_x u + R^h(u, \partial_x u))_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

we only need to prove that

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x u}^h u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \quad \text{for } s > 0.$$

Indeed, we use Bernstein lemma 6.2.1 we get

$$\|\Delta_q^h u_\phi(t')\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h u_\phi(t')\|_{L_h^2(L_h^\infty)},$$

the inclusion $H_y^1 \hookrightarrow L^\infty$ give

$$\|\Delta_q^h u_\phi(t')\|_{L_h^2(L_h^\infty)} \lesssim \|\Delta_q^h u_\phi(t')\|_{L^2}^{\frac{1}{2}} \|\Delta_q^h \partial_y u_\phi(t')\|_{L^2}^{\frac{1}{2}}$$

and the Poincaré inequality on the interval $\{0 < y < 1\}$ on u (we have that $u = 0$ when $y = 0$) allows us to have

$$(6.3.23) \quad \|\Delta_q^h \partial_x u_\phi(t')\|_{L^\infty} \lesssim d_q(u_\phi) 2^{q'} \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

we infer,

$$\begin{aligned} \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x u}^h u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' &\lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x u_\phi\|_{L^\infty} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \int_0^t e^{2\mathcal{R}t'} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2. \end{aligned}$$

While it follows from the proof of Lemma 6.3.2 that

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y u}^h v + R^h(v, \partial_y u))_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

Based at the proof of Lemma 6.3.2 (Estimate (6.3.39)), we have

$$\|\Delta_q^h v_\phi(t)\|_{L^\infty} \leq d_q 2^{\frac{q}{2}} \|u_\phi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}},$$

we replace in the inequality of the term $T_v^h \partial_y u$, we obtain

$$\begin{aligned} &\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_v^h \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h v_\phi\|_{L^\infty} \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{\frac{q'}{2}} \int_0^t e^{2\mathcal{R}t'} \|u_\phi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\ &\lesssim 2^{-2qs} d_q^2 \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

As a result, it comes out

$$\begin{aligned} &\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u + v \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ &\quad \times \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right). \end{aligned}$$

Now, we will get the estimate of the second term I_2 . In the same way, we remark from the proof of Lemma 6.3.1 that we only need to prove

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x b}^h b)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \leq C 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \quad \text{for } s > 0.$$

So, in views of (6.3.23), we infer

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x b}^h b)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x b_\phi\|_{L^\infty} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt'$$

$$\begin{aligned}
&\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \int_0^t e^{2\mathcal{R}t'} \|\partial_y b_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\
&\lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

Based on the proof of Lemme 6.3.2 (Estimate (6.3.39)), we have

$$\|\Delta_q^h c_\phi(t)\|_{L^\infty} \leq d_q 2^{\frac{q}{2}} \|b_\phi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

so that there holds

$$\begin{aligned}
&\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_c^h \partial_y b)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \\
&\lesssim \sum_{|q'-q|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h c_\phi\|_{L^\infty} \|\Delta_{q'}^h \partial_y b_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\
&\lesssim \sum_{|q'-q|\leq 4} 2^{\frac{q'}{2}} \int_0^t e^{2\mathcal{R}t'} \|b_\phi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y b_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} dt' \\
&\lesssim 2^{-2qs} d_q^2 \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

As a result, it comes out

$$\begin{aligned}
&\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (b\partial_x b + c\partial_y b)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle \right| dt' \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
&\quad \times \left(\|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right).
\end{aligned}$$

We do the same thing to estimate I_3 and I_4 , then we have

$$\begin{aligned}
I_3 + I_4 &\lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
&\quad \times \left(\|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right).
\end{aligned}$$

Applying Young's inequality yields

$$\begin{aligned}
&\|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y (b_\phi, u_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
&\leq C \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \frac{1}{10} \|e^{\mathcal{R}t} \partial_y (b_\phi, u_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_y (u_\phi, b_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
&\leq C \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \frac{1}{10} \|e^{\mathcal{R}t} \partial_y (b_\phi, u_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.
\end{aligned}$$

So we replace all the results obtained in (6.3.21) and (6.3.22) and taking the square-root and multi-

plying by 2^{qs} , then summing with respect to $q \in \mathbb{Z}$, we obtain

(6.3.24)

$$\begin{aligned}
& \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
(6.3.25) \quad & \leq C \left\| e^{a|D_x|}u_0 \right\|_{\mathcal{B}^s} + C \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& + C \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} + \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \right) + \frac{1}{10} \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)},
\end{aligned}$$

and

(6.3.26)

$$\begin{aligned}
& \|e^{\mathcal{R}t}b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t}b_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\
(6.3.27) \quad & \leq C \left\| e^{a|D_x|}b_0 \right\|_{\mathcal{B}^s} + C \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& + C \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} + \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \right) + \frac{1}{10} \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)}.
\end{aligned}$$

We take the sum of (6.3.24) and (6.3.26), we have

$$\begin{aligned}
& \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t}\partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq C \left\| e^{a|D_x|}(u_0, b_0) \right\|_{\mathcal{B}^s} \\
& + C \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + C \|e^{\mathcal{R}t}b_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& + C \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} + \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \right) + \frac{1}{5} \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)}.
\end{aligned}$$

Therefore, if we take

$$(6.3.28) \quad \sqrt{2\lambda} \geq 2C \left(1 + \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} + \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \right),$$

then we obtain

$$(6.3.29) \quad \|e^{\mathcal{R}t}(u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq C \left\| e^{a|D_x|}(u_0, b_0) \right\|_{\mathcal{B}^s}.$$

In particular Estimate (6.3.29) implies that under the condition (6.3.28), there holds

$$\|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} + \|b_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq C \left\| e^{a|D_x|}(u_0, b_0) \right\|_{\mathcal{B}^{\frac{3}{2}}},$$

then by taking $\lambda \geq C(1 + \|e^{a|D_x|}(u_0, b_0)\|_{\mathcal{B}^{\frac{3}{2}}})$, (6.3.28) verified. So under the condition (6.3.16), we have that (6.3.28) holds, and then (6.3.29) is valid for any $t > 0$. This completes the proof of the proposition.

Now we still have to prove the second estimate (6.3.18). For that, we apply the dyadic operator Δ_q^h to (6.3.3) and taking the L^2 inner product of the resulting equation with $\Delta_q^h(\partial_t u)_\phi$ and $\Delta_q^h(\partial_t b)_\phi$. That yields

$$\begin{aligned}
& \langle \Delta_q^h(\partial_t u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} + \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \\
& = - \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} - \langle \Delta_q^h(v \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} + \langle \Delta_q^h(b \partial_x b + c \partial_y b)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle,
\end{aligned}$$

and

$$\begin{aligned} & \langle \Delta_q^h(\partial_t b)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 b_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} \\ &= - \langle \Delta_q^h(u \partial_x b)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} - \langle \Delta_q^h(v \partial_y b)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} + \langle \Delta_q^h(b \partial_x u + c \partial_y u)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle. \end{aligned}$$

The fact that $(\partial_t u)_\phi = \partial_t u_\phi + \lambda \dot{\theta}(t) |D_x| u_\phi$ implies

$$\begin{aligned} \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} &= - \left(\frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) 2^q \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 \right), \\ \langle \Delta_q^h \partial_y^2 b_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} &= - \left(\frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y b_\phi\|_{L^2}^2 + \lambda \dot{\theta}(t) 2^q \|\Delta_q^h \partial_y b_\phi\|_{L^2}^2 \right), \end{aligned}$$

from which, we deduce that

$$\|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2 \leq I_1 + I_2 + I_3,$$

and

$$\|\Delta_q^h(\partial_t b)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_y b_\phi\|_{L^2}^2 \leq D_1 + D_2,$$

where

$$\begin{aligned} I_1 &= \left| \langle \Delta_q^h(u \partial_x u + b \partial_x b)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right|, \\ I_2 &= \left| \langle \Delta_q^h(v \partial_y u + c \partial_y b)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right|, \\ I_3 &= \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right|, \\ D_1 &= \left| \langle \Delta_q^h(u \partial_x b + b \partial_x u)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} \right|, \\ D_2 &= \left| \langle \Delta_q^h(v \partial_y b + c \partial_y u)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} \right|. \end{aligned}$$

Since $\partial_x u + \partial_y v = 0$, using (6.3.1) and integrating by parts, we find

$$\begin{aligned} I_3 &= \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| = \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u_\phi + \lambda \dot{\theta}(t) |D_x| u_\phi) \rangle_{L^2} \right| \\ &= \left| \langle \Delta_q^h p_\phi, \Delta_q^h(\partial_t \partial_x u_\phi + \lambda \dot{\theta}(t) |D_x| \partial_x u_\phi) \rangle_{L^2} \right| \\ &= \left| \langle \Delta_q^h p_\phi, \Delta_q^h(\partial_t \partial_y v_\phi + \lambda \dot{\theta}(t) |D_x| \partial_y v_\phi) \rangle_{L^2} \right| \\ &= \left| \langle \Delta_q^h \partial_y p_\phi, \Delta_q^h(\partial_t v)_\phi \rangle_{L^2} \right| = 0 \text{ (because } \partial_y p = 0). \end{aligned}$$

For I_1, I_2, D_1 and D_2 we have

$$\begin{aligned} I_1 &= \left| \langle \Delta_q^h(u \partial_x u + b \partial_x b)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq \left| \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| + \left| \langle \Delta_q^h(b \partial_x b)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \\ &\leq \|\Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \|\Delta_q^h(b \partial_x b)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2, \end{aligned}$$

and

$$I_2 = \left| \langle \Delta_q^h(v \partial_y u + c \partial_y b)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq \|\Delta_q^h(v \partial_y u)_\phi\|_{L^2}^2 + \|\Delta_q^h(c \partial_y b)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2,$$

and

$$D_1 = \left| \langle \Delta_q^h(u\partial_x b + b\partial_x u)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} \right| \leq \|\Delta_q^h(u\partial_x b)_\phi\|_{L^2}^2 + \|\Delta_q^h(b\partial_x u)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t b)_\phi\|_{L^2}^2,$$

and

$$D_2 = \left| \langle \Delta_q^h(v\partial_y b + c\partial_y u)_\phi, \Delta_q^h(\partial_t b)_\phi \rangle_{L^2} \right| \leq \|\Delta_q^h(v\partial_y b)_\phi\|_{L^2}^2 + \|\Delta_q^h(c\partial_y u)_\phi\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t b)_\phi\|_{L^2}^2.$$

Then, we deduce that

$$\begin{aligned} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \frac{d}{dt} \|\Delta_q^h \partial_y(u_\phi, b_\phi)\|_{L^2}^2 + \|\Delta_q^h(\partial_t b)_\phi\|_{L^2}^2 &\leq C \left(\|\Delta_q^h(u\partial_x u)_\phi\|_{L^2}^2 \right. \\ &+ \|\Delta_q^h(v\partial_y u)_\phi\|_{L^2}^2 + \|\Delta_q^h(b\partial_x b)_\phi\|_{L^2}^2 + \|\Delta_q^h(c\partial_y b)_\phi\|_{L^2}^2 + \|\Delta_q^h(u\partial_x b)_\phi\|_{L^2}^2 \\ &+ \left. \|\Delta_q^h(b\partial_x u)_\phi\|_{L^2}^2 + \|\Delta_q^h(c\partial_y u)_\phi\|_{L^2}^2 + \|\Delta_q^h(b\partial_x u)_\phi\|_{L^2}^2 \right). \end{aligned}$$

Multiplying the result by $e^{2\mathcal{R}t}$ and integrating over $[0, t]$, we get

$$\begin{aligned} &\|e^{\mathcal{R}t} \Delta_q^h(\partial_t u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(\partial_t b)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_y(u_\phi, b_\phi)\|_{L_t^\infty(L^2)}^2 \leq C \left(\|\Delta_q^h \partial_y e^{a|D_x|}(u_0, b_0)\|_{L^2}^2 \right. \\ &+ \|e^{\mathcal{R}t} \Delta_q^h(u\partial_x u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(v\partial_y u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(b\partial_x b)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(c\partial_y b)_\phi\|_{L_t^2(L^2)}^2 \\ &+ \left. \|e^{\mathcal{R}t} \Delta_q^h(u\partial_x b)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(v\partial_y b)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(b\partial_x u)_\phi\|_{L_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h(c\partial_y u)_\phi\|_{L_t^2(L^2)}^2 \right). \end{aligned}$$

Multiplying the above inequality by 2^{3q} , then taking the square root of the resulting estimate, and finally summing up the obtained equations with respect to $q \in \mathbb{Z}$, we obtain

$$\begin{aligned} (6.3.30) \quad &\|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(\partial_t b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t} \partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq C \left(\|\partial_y e^{a|D_x|}(u_0, b_0)\|_{\mathcal{B}^{\frac{3}{2}}} \right. \\ &+ \|e^{\mathcal{R}t}(u\partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(v\partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(b\partial_x b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(c\partial_y b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ &+ \left. \|e^{\mathcal{R}t}(u\partial_x b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(v\partial_y b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(b\partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(c\partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right). \end{aligned}$$

Next, it follows from the law of product in anisotropic Besov spaces and Poincaré inequality that

$$\begin{aligned} &\|e^{\mathcal{R}t}(u\partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \leq \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}, \\ &\|e^{\mathcal{R}t}(b\partial_x b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \leq \|b_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}, \\ &\|e^{\mathcal{R}t}(v\partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \leq \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} + \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{5}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}, \\ &\|e^{\mathcal{R}t}(c\partial_y b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \leq \|b_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} + \|b_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{5}{2}})} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}. \end{aligned}$$

and

$$\begin{aligned} &\|e^{\mathcal{R}t}(u\partial_x b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(b\partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ &\leq \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} + \|b_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})}, \end{aligned}$$

and

$$\begin{aligned} &\|e^{\mathcal{R}t}(v\partial_y b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(c\partial_y u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \leq \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \\ &+ \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{5}{2}})} \|e^{\mathcal{R}t} \partial_y b_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|b_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} + \|b_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{5}{2}})} \|e^{\mathcal{R}t} \partial_y u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}. \end{aligned}$$

Inserting the above estimates into (6.3.30) and then using the smallness condition $\|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} + \|b_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \leq$

$\frac{1}{2C^2}$, we finally obtain

$$\begin{aligned} & \|e^{\mathcal{R}t}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}(\partial_t b)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|e^{\mathcal{R}t}\partial_y(u_\phi, b_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \\ & \leq C \left(\|e^{a|D_x|}\partial_y(u_0, b_0)\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{5}{2}}} + \|e^{a|D_x|}b_0\|_{\mathcal{B}^{\frac{5}{2}}} \right). \end{aligned}$$

this complete the proof of Proposition 6.3.1. \square

6.3.1 Technicals lemmas

Lemma 6.3.1. *Let A, B and C be a smooth function on $[0, T] \times \mathbb{R} \times (0, 1)$ vanishing on the boundary, and $s \in]0, 1[$, $T > 0$ and ϕ be defined as in (6.2.3), with $\hat{\theta}(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$. There exists $C \geq 1$ such that, for any $t > 0$, $\phi(t, \xi) > 0$ and for any $B, C \in \tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$, we have*

$$(6.3.31) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (A \partial_x B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t, \hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Proof. We define I the integral given by

$$I(t) = \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (A \partial_x B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'.$$

As in [15], using Bony's homogeneous decomposition of $A \partial_x B$ into para-products in the horizontal variable and remainders as in Definition of a tempered distribution, we can write

$$A \partial_x B = T_A^h \partial_x B + T_{\partial_x B}^h A + R^h(A, \partial_x B),$$

where,

$$T_A \partial_x B = \sum_{q \in \mathbb{Z}} S_{q-1}^h A \Delta_q^h \partial_x B \quad \text{and} \quad R^h(A, \partial_x B) = \sum_{|q'-q| \leq 1} \tilde{\Delta}_q^h A \Delta_{q'}^h \partial_x B.$$

We have the following bound of I

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (A \partial_x B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \leq I_{1,q} + I_{2,q} + I_{3,q},$$

where

$$\begin{aligned} I_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_A^h \partial_x B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt', \\ I_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x B}^h A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt', \\ I_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(A, \partial_x B))_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

We start by getting the estimate of the first term $I_{1,q}$, for that we need to use the support properties given in [[18], Proposition 2.10] and the definition of $T_A^h \partial_x B$, and the lemma 6.2.2, we infer

$$(6.3.32) \quad I_{1,q} \leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h A_\phi^+(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_x B_\phi(t')\|_{L^2} \|\Delta_q^h C_\phi(t')\|_{L^2} dt'.$$

By the Poincaré inequality on the interval $\{0 < y < 1\}$, we have the inclusion $\dot{H}_y^1 \hookrightarrow L_y^\infty$ and

$$(6.3.33) \quad \|\Delta_q^h A_\phi^+(t')\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h A_\phi^+(t')\|_{L_h^2(L_v^\infty)} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_y A_\phi(t')\|_{L^2} \lesssim d_q(A_\phi) \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $\{d_q(A_\phi)\}$ is a square-summable sequence with $\sum d_q(A_\phi)^2 = 1$. Then,

$$\|S_{q'-1}^h A_\phi^+(t')\|_{L^\infty} \lesssim \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

then, we replace this result in our estimate (6.3.32), and combining with Hölder inequality, imply that

$$\begin{aligned} I_{1,q} &\lesssim \sum_{|q-q'|\leq 4} \int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_{q'}^h \partial_x B_\phi(t')\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} e^{\mathcal{R}t'} \|\Delta_{q'}^h B_\phi(t')\|_{L^2} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

we note that $\dot{\theta}(t) = \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}$, using Definition 6.2.3, we have

$$\begin{aligned} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q(s+\frac{1}{2})} d_q(C_\phi) \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Then,

$$(6.3.34) \quad I_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\theta}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(B_\phi) 2^{(q-q')(s-\frac{1}{2})} \right).$$

Now we move to get the estimate of the second term, by using the support properties given in [[18], Proposition 2.10] and the definition of $T_{\partial_x B}^h A$, and Lemma 6.2.2, we can estimate $I_{2,q}$ in a similar way as we did for $I_{1,q}$.

$$\begin{aligned} I_{2,q}(t) &\leq \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x B}^h A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x B_\phi^+\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h A_\phi^+\|_{L_h^2(L_v^\infty)} \|\Delta_q^h C_\phi\|_{L^2} dt'. \end{aligned}$$

As in (6.3.33), we can write

$$\|\Delta_{q'}^h A_\phi^+\|_{L_h^2(L_v^\infty)} \lesssim \|\Delta_{q'}^h \partial_y A_\phi^+\|_{L^2} \lesssim d_{q'}(A_\phi) 2^{-\frac{q'}{2}} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Since $s < 1$ and we use the fact that $d_{q'}(A_\phi) \leq 1$, we have

$$\begin{aligned} I_{2,q} &\leq \sum_{|q-q'|\leq 4} 2^{-\frac{q'}{2}} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x B_\phi\|_{L_h^\infty(L_v^2)} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\leq \sum_{|q-q'|\leq 4} \left(\int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x B_\phi\|_{L_h^\infty(L_v^2)}^2 \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \end{aligned}$$

Yet we observe from Definition 5.2.1, and $s < 1$ we have

$$\begin{aligned} &\sum_{l \lesssim q'-2} \left(\int_0^t 2^{3l} e^{2\mathcal{R}t'} \|\Delta_l^h B_\phi\|_{L^2}^2 \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{l \lesssim q'-2} 2^{\frac{3l}{2}} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_l^h B_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{l \lesssim q'-2} 2^{\frac{3l}{2}} 2^{-l(s+\frac{1}{2})} 2^{l(s+\frac{1}{2})} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_l^h B_\phi(t')\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{q'(1-s)} \sum_{l \lesssim q'-2} \tilde{d}_l(B_\phi) \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\hat{\theta}}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ (6.3.35) \quad &\lesssim 2^{q'(1-s)} d_{q'}(B_\phi) \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Then, we find the following estimate for $I_{2,q}$

$$I_{2,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(B_\phi) 2^{(q-q')(s-\frac{1}{2})} \right),$$

is a summable sequence of positive constants. Thus,

$$(6.3.36) \quad I_{2,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where we recall that $\hat{\theta}(t) = \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$.

To end this proof, it remains to estimate $I_{3,q}$ (is the remainder term). Using the support properties given in [[18], Proposition 2.10], the definition of $R^h(A, \partial_x B)$ and Bernstein lemma 6.2.1, we can write

$$I_{3,q} \leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} 2^{q'(1-\frac{1}{2})} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt'$$

$$\leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} 2^{\frac{q'}{2}} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h B_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt'.$$

Since $0 < s < 1$, we have

$$\begin{aligned} I_{3,q} &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t'} B_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t'} C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q(C_\phi) 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\sum_{q' \geq q-3} d_{q'}(B_\phi) 2^{(q-q')s} \right) \\ &\lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{q' \geq q-3} d_{q'}(B_\phi) 2^{(q-q')s} \right),$$

is a summable sequence of positive constants. So,

$$(6.3.37) \quad I_{3,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} B_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Lemma 6.3.1 is proved by summing Estimates (6.3.34), (6.3.36) and (6.3.37). \square

Lemma 6.3.2. *Let A, B and C be a smooth function on $[0, T] \times \mathbb{R} \times (0, 1)$ vanishing on the boundary, and $s \in]0, 1[$, $T > 0$ and ϕ be defined as in (6.2.3), with $\dot{\theta}(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ and $B(t, x, y) = -\int_0^y \partial_x A(t, x, y') dy'$. There exist $C \geq 1$ such that, for any $t > 0$ such that $\phi(t, \xi) > 0 \forall \xi$ and for any $B, C \in \tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$, we have*

$$(6.3.38) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (B \partial_y A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Proof. We define the function $K(t)$ by the following formula

$$K(t) = \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (B \partial_y A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'.$$

As in [15], using Bony's homogeneous decomposition of $B \partial_y A$ into para-products in the horizontal variable and remainders as in definition of a tempered distribution, we can write

$$B \partial_y A = T_B^h \partial_y A + T_{\partial_y A}^h B + R^h(B, \partial_y A),$$

where,

$$T_B \partial_y A = \sum_{q \in \mathbb{Z}} S_{q-1}^h B \Delta_q^h \partial_y A \quad \text{and} \quad R^h(B, \partial_y A) = \sum_{|q'-q| \leq 1} \tilde{\Delta}_q^h B \Delta_{q'}^h \partial_y A.$$

We have the following bound of K

$$K(t) = \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (B \partial_y A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \leq K_{1,q} + K_{2,q} + K_{3,q},$$

where

$$\begin{aligned} K_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_B^h \partial_y A)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt', \\ K_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y A}^h B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt', \\ K_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(B, \partial_y A))_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

We start by getting the estimate of the first term $K_{1,q}$, for that we need to use the support properties given in [[18], Proposition 2.10] and the definition of $T_B^h \partial_y A$, and Lemma 6.2.2, we infer

$$K_{1,q} \leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h B_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_y A_\phi(t')\|_{L^2} \|\Delta_q^h C_\phi(t')\|_{L^2} dt'.$$

By the Poincaré inequality on the interval $\{0 < y < 1\}$, we have the inclusion $\dot{H}_y^1 \hookrightarrow L_y^\infty$ and

$$\begin{aligned} \|\Delta_q^h B_\phi(t')\|_{L^\infty} &\lesssim \int_0^1 \|\Delta_q^h \partial_x A_\phi(t', x, y')\|_{L_h^\infty} dy' \lesssim 2^{\frac{q}{2}} \int_0^1 \|\Delta_q^h \partial_x A_\phi(t', x, y')\|_{L_h^2} dy' \\ (6.3.39) \quad &\lesssim 2^{\frac{q}{2}} 2^q \int_0^1 \|\Delta_q^h A_\phi(t', x, y')\|_{L_h^2} dy' \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h A_\phi(t')\|_{L^2}, \end{aligned}$$

and

$$\|\Delta_{q'}^h \partial_y A_\phi(t')\|_{L^2} \leq d_{q'}(A_\phi) 2^{-\frac{q'}{2}} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Then, we replace this result in the estimate of $K_{1,q}$, and combining with Hölder inequality, imply that

$$\begin{aligned} K_{1,q} &\lesssim \sum_{|q-q'|\leq 4} 2^{-\frac{q'}{2}} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h B_\phi(t')\|_{L^\infty} \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h C_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{-\frac{q'}{2}} \sum_{l \leq q'-2} \left(\int_0^t 2^{\frac{3l}{2}} e^{2\mathcal{R}t'} \|\Delta_l^h A_\phi(t')\|_{L^2}^2 \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi(t')\|_{L^2}^2 \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}}. \end{aligned}$$

we note that $\dot{\theta}(t) = \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}$, we use the same thing as we do in (6.3.35), we obtain by using Definition 6.2.3 that

$$\begin{aligned} &\sum_{l \leq q'-2} 2^{3l} \left(\int_0^t e^{2\mathcal{R}t'} \|\Delta_l^h A_\phi(t')\|_{L^2}^2 \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{q'(1-s)} d_q(A_\phi) \|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \end{aligned}$$

and

$$\begin{aligned} \left(\int_0^t e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q(s+\frac{1}{2})} d_q(C_\phi) \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

So,

$$(6.3.40) \quad K_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(A_\phi) 2^{(q-q')(s-\frac{1}{2})} \right).$$

Now we move to get the estimate of the second term, by using the support properties given in [[18], Proposition 2.10] and the definition of $T_{\partial_y A}^h B$, and Lemma 6.2.2, we can estimate $K_{2,q}$ in a similar way as we did for $K_{1,q}$.

$$\begin{aligned} K_{2,q}(t) &\leq \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y A}^h B)_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_y A_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h B_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h C_\phi\|_{L^2} dt'. \end{aligned}$$

As in (6.3.39), we can write

$$\|\Delta_{q'}^h B_\phi\|_{L_h^2(L_v^\infty)} \lesssim 2^{q'} \int_0^y \|\Delta_{q'}^h A_\phi(t, x, y')\|_{L_h^2} dy' \lesssim 2^{q'} \|\Delta_{q'}^h A_\phi(t)\|_{L^2}.$$

Since $0 < s < 1$, we have

$$\begin{aligned} K_{2,q} &\leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_y A_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h B_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} e^{\mathcal{R}t'} \|S_{q'-1}^h \partial_y A_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h A_\phi(t)\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} e^{\mathcal{R}t'} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h A_\phi(t)\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2} dt' \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h A_\phi(t)\|_{L^2} \|e^{\mathcal{R}t'} \Delta_q^h C_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h A_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

we note that $\dot{\theta}(t) = \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}$, using Definition 6.2.3, we have

$$\begin{aligned} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q(s+\frac{1}{2})} d_q(C_\phi) \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Then,

$$(6.3.41) \quad K_{2,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'} 2^{(q-q')(s-\frac{1}{2})} \right),$$

is a summable sequence of positive constants. We recall that $\dot{\theta}(t) = \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}}$.

To end this proof, it remains to estimate $K_{3,q}$ (is the remainder term). Using the support properties given in [[18], Proposition 2.10], the definition of $R^h(B, \partial_y A)$ and Bernstein lemma 6.2.1, we can write

$$\begin{aligned} K_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(B, \partial_y A))_\phi, e^{\mathcal{R}t'} \Delta_q^h C_\phi \right\rangle_{L^2} \right| dt' \\ &\leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\tilde{\Delta}_{q'}^h B_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h \partial_y A_\phi\|_{L^2} \|\Delta_q^h C_\phi\|_{L^2} dt'. \end{aligned}$$

Similar calculations as in (6.3.39) imply

$$\|\Delta_q^h B_\phi(t)\|_{L_v^\infty(L_h^2)} \leq \int_0^1 \|\Delta_q^h \partial_x A_\phi(t, \cdot, y')\|_{L_h^2} dy' \lesssim 2^q \int_0^1 \|\Delta_q^h A_\phi(t, \cdot, y')\|_{L_h^2} dy' \lesssim 2^q \|\Delta_q^h A_\phi(t)\|_{L^2}.$$

Since $0 < s < 1$, we have

$$\begin{aligned} K_{3,q} &\leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^q \|e^{\mathcal{R}t'} \tilde{\Delta}_{q'}^h A_\phi(t')\|_{L^2} \|\Delta_{q'}^h \partial_y A_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t'} C_\phi\|_{L^2} dt' \\ &\leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|e^{\mathcal{R}t'} \tilde{\Delta}_{q'}^h A_\phi(t')\|_{L^2} \|\partial_y A_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t'} C_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h A_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

we note that $\dot{\theta}(t) = \|\partial_y A_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$, using Definition 6.2.3, we have

$$\begin{aligned} \left(\int_0^t \|\partial_y A_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_q^h C_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q(s+\frac{1}{2})} d_q(C_\phi) \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

So,

$$(6.3.42) \quad K_{3,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} A_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} C_\phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(C_\phi) \left(\sum_{q' \geq q-3} d_{q'}(A_\phi) 2^{(q-q')s} \right),$$

is a summable sequence of positive constants. Lemma 6.3.1 is proved by summing Estimates (6.3.40), (6.3.41) and (6.3.42). \square

Lemma 6.3.3. For any $s \in]0, 1[$ and $t \leq T^*$, there exist $C \geq 1$ such that,

$$(6.3.43) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (b \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt' \leq C \tilde{d}_q 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

and

$$(6.3.44) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (c \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt' \leq C \tilde{d}_q 2^{-2qs} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t, \theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Proof. At first, we start proving the first estimate (6.3.43) of lemma 6.3.3. We define the time-dependent function $L(t)$

$$L(t) = \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (b \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt'.$$

Bony's decomposition for the horizontal variable into the para-products of the term $b \partial_x u$ implies

$$(6.3.45) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (b \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt' \leq L_{1,q} + L_{2,q} + L_{3,q},$$

with

$$\begin{aligned} L_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_b^h \partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt', \\ L_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_x u}^h b)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt', \\ L_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(b, \partial_x u))_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

We start by getting the estimate of the first term $L_{1,q}$, Using the support properties given in [[18], Proposition 2.10] and the definition of $T_b^h \partial_x u$, and Lemma 6.2.2, we infer

$$(6.3.46) \quad L_{1,q} \leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h b_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\phi(t')\|_{L^2} \|\Delta_q^h b_\phi(t')\|_{L^2} dt'.$$

By the Poincaré inequality on the interval $\{0 < y < 1\}$, we have the inclusion $\dot{H}_y^1 \hookrightarrow L_y^\infty$ and

$$(6.3.47) \quad \|\Delta_q^h b_\phi(t')\|_{L^\infty} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h b_\phi(t')\|_{L_h^2(L_v^\infty)} \lesssim 2^{\frac{q}{2}} \|\Delta_q^h \partial_y b_\phi(t')\|_{L^2} \lesssim d_q(b_\phi) \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $\{d_q(b_\phi)\}$ is a square-summable sequence with $\sum d_q(u_\phi)^2 = 1$. Then,

$$\|S_{q'-1}^h b_\phi(t')\|_{L^\infty} \lesssim \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

then, we replace this result in our estimate (6.3.46), and combining with Hölder inequality, imply that

$$\begin{aligned} L_{1,q} &\lesssim \sum_{|q-q'|\leq 4} \int_0^t \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_{q'}^h \partial_x u_\phi(t')\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h b_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} e^{\mathcal{R}t'} \|\Delta_{q'}^h u_\phi(t')\|_{L^2} \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} e^{\mathcal{R}t'} \|\Delta_q^h b_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

We note that $\dot{\theta}(t) = \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}$, using Definition 6.2.3, we have

$$\begin{aligned} \left(\int_0^t \|\partial_y b_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q'(s+\frac{1}{2})} d_{q'}(u_\phi) \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\dot{\theta}}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Then,

$$(6.3.48) \quad L_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(b_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'}(u_\phi) 2^{(q-q')(s-\frac{1}{2})} \right).$$

Using the support properties given in [[18], Proposition 2.10] and the definition of $T_b^h \partial_x u$, and Lemma 6.2.2, we can estimate $L_{2,q}$ in a similar way as we did for $L_{1,q}$. As in (6.3.49), we can write

$$(6.3.49) \quad \|\Delta_q^h \partial_x u_\phi(t')\|_{L^\infty} \lesssim 2^q 2^{\frac{q}{2}} \|\Delta_q^h \partial_y u_\phi(t')\|_{L^2} \lesssim d_q(u_\phi) 2^q \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $\{d_q(b_\phi)\}$ is a square-summable sequence with $\sum d_q(u_\phi)^2 = 1$. Then,

$$\begin{aligned} L_{2,q} &= \int_0^t e^{2\mathcal{R}t'} \left| \langle \Delta_q^h (T_{\partial_x u}^h b)_\phi, \Delta_q^h b_\phi \rangle_{L^2} \right| dt' \\ &\lesssim \int_0^t e^{2\mathcal{R}t'} \sum_{|q-q'|\leq 4} \|S_{q'}^h \partial_x u_\phi\|_{L^\infty} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h b_\phi\|_{L^2} dt' \\ &\leq \int_0^t e^{2\mathcal{R}t'} \sum_{|q-q'|\leq 4} 2^q \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h b_\phi\|_{L^2} \|\Delta_q^h b_\phi\|_{L^2} dt'. \end{aligned}$$

Since $0 < s < 1$, we have

$$\begin{aligned} L_{2,q} &\lesssim \sum_{|q-q'|\leq 4} 2^q \left(\int_0^t \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\dot{\theta}}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\dot{\theta}}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

So,

$$(6.3.50) \quad L_{2,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

where

$$d_q^2 = d_q(b_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'} 2^{(q-q')(s-\frac{1}{2})} \right),$$

is a summable sequence of positive constants.

To end this proof, it remains to estimate $L_{3,q}$. Using the support properties given in [[18], Proposition 2.10], the definition of $R^h(b, \partial_x u)$ and Bernstein lemma 6.2.1, we can write

$$\begin{aligned}
L_{3,q} &= \int_0^t e^{2\mathcal{R}t'} \left| \langle \Delta_q^h (R^h(b, \partial_x u))_\phi, \Delta_q^h b_\phi \rangle_{L^2} \right| dt' \\
&\lesssim \int_0^t e^{2\mathcal{R}t'} 2^{\frac{q}{2}} \sum_{q' \geq q-3} \|\Delta_{q'}^h b_\phi\|_{L_n^\infty(L^\infty)} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \|\Delta_q^h b_\phi\|_{L^2} dt' \\
&\lesssim \int_0^t e^{2\mathcal{R}t'} 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{q'(1-\frac{1}{2})} \|\partial_y b_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h b_\phi\|_{L^2} dt' \\
&\lesssim \int_0^t e^{2\mathcal{R}t'} 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \|\partial_y b_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_q^h b_\phi\|_{L^2} dt'.
\end{aligned}$$

Since $0 < s < 1$, we have

$$\begin{aligned}
L_{3,q} &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq k-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y b_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t'} u_\phi\|_{L^2}^2 dt' \right) \times \left(\int_0^t \|\partial_y b_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\sum_{q' \geq q-3} d_{q'}(u_\phi) 2^{(q-q')s} \right).
\end{aligned}$$

So,

$$(6.3.51) \quad L_{3,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(b_\phi) \left(\sum_{q' \geq q-3} d_{q'}(u_\phi) 2^{(q-q')s} \right),$$

is a summable sequence of positive constants.

By summing the estimates (6.3.48), (6.3.50) and (6.3.51), we achieved the proof of (6.3.43) .

We now prove the estimate (6.3.44). Using Bony's decomposition for the horizontal variable, we have

$$(6.3.52) \quad \int_0^t \left| \langle e^{\mathcal{R}t'} \Delta_q^h (c \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \rangle_{L^2} \right| dt' \leq C_{1,q} + C_{2,q} + C_{3,q},$$

where

$$\begin{aligned}
C_{1,q} &= \int_0^t \left| \langle e^{\mathcal{R}t'} \Delta_q^h (T_c^h \partial_y u)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \rangle_{L^2} \right| dt', \\
C_{2,q} &= \int_0^t \left| \langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y u}^h c)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \rangle_{L^2} \right| dt', \\
C_{3,q} &= \int_0^t \left| \langle e^{\mathcal{R}t'} \Delta_q^h (R^h(c, \partial_y u))_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \rangle_{L^2} \right| dt'.
\end{aligned}$$

Identity (6.3.1), and Bernstein lemma 6.2.1, imply

$$(6.3.53) \quad \|\Delta_q^h c_\phi(t)\|_{L^\infty} \leq \int_0^1 \|\Delta_q^h \partial_x b_\phi(t, \cdot, y')\|_{L_n^\infty} dy' \lesssim 2^{\frac{3q}{2}} \int_0^1 \|\Delta_q^h b_\phi(t, \cdot, y')\|_{L_n^2} dy' \lesssim 2^{\frac{3q}{2}} \|\Delta_q^h b_\phi(t)\|_{L^2},$$

Along the same way as we did in (6.3.35), we get for $s < 1$ that

$$\begin{aligned}
C_{1,q} &\lesssim \int_0^t \sum_{|q'-q|\leq 4} 2^{-\frac{q'}{2}} e^{2\mathcal{R}t} \|S_{q'-1}^h c_\phi\|_{L^\infty} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h b_\phi\|_{L^2} dt' \\
&\lesssim \int_0^t \sum_{|q'-q|\leq 4} 2^{-\frac{q'}{2}} \sum_{l\leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h e^{\mathcal{R}t'} b_\phi\|_{L^2} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t'} b_\phi\|_{L^2} dt' \\
&\lesssim \sum_{|q'-q|\leq 4} 2^{-\frac{q'}{2}} \sum_{l\leq q'-2} \left(\int_0^t 2^{3l} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_l^h e^{\mathcal{R}t'} b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^t \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t'} b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

So,

$$(6.3.54) \quad C_{1,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(b_\phi) \left(\sum_{|q'-q|\leq 4} d_{q'}(b_\phi) 2^{(q-q')(s-1)} \right),$$

is a summable sequence of positive constants.

Now we move to get the estimate of the second term, by using the support properties given in [18], Proposition 2.10] and the definition of $T_{\partial_y u}^h c$, we can estimate $C_{2,q}$ in a similar way as we did for $C_{1,q}$.

$$\begin{aligned}
C_{2,q}(t) &\leq \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (T_{\partial_y u}^h c)_\phi, e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt' \\
&\leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_y u_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h c_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h b_\phi\|_{L^2} dt'.
\end{aligned}$$

As in (6.3.53), we can write

$$\|\Delta_{q'}^h c_\phi\|_{L_h^2(L_v^\infty)} \lesssim 2^{q'} \int_0^y \|\Delta_{q'}^h b_\phi(t, x, y')\|_{L_h^2} dy' \lesssim 2^{q'} \|\Delta_{q'}^h b_\phi(t)\|_{L^2}.$$

Since $0 < s < 1$, we have

$$\begin{aligned}
C_{2,q} &\leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_y u_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h c_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_q^h b_\phi\|_{L^2} dt' \\
&\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} e^{\mathcal{R}t'} \|S_{q'-1}^h \partial_y u_\phi\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h b_\phi(t)\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h b_\phi\|_{L^2} dt' \\
&\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} e^{\mathcal{R}t'} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h b_\phi(t)\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h b_\phi\|_{L^2} dt' \\
&\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h b_\phi(t)\|_{L^2} \|e^{\mathcal{R}t'} \Delta_q^h b_\phi\|_{L^2} dt'
\end{aligned}$$

$$\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.$$

We note that $\dot{\theta}(t) = \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}$, using the definition (6.2.3), we have

$$\begin{aligned} \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q(s+\frac{1}{2})} d_q(b_\phi) \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Thus,

$$(6.3.55) \quad C_{2,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

where

$$d_q^2 = d_q(b_\phi) \left(\sum_{|q-q'|\leq 4} d_{q'} 2^{(q-q')(s-\frac{1}{2})} \right),$$

is a summable sequence of positive constants. We recall that $\dot{\theta}(t) = \|\partial_y(u_\phi, b_\phi)\|_{\mathcal{B}^{\frac{1}{2}}}$.

To end this proof, it remains to estimate $C_{3,q}$ (is the rest term). Using the support properties given in [[18], Proposition 2.10], the definition of $R^h(B, \partial_y A)$ and Bernstein lemma 6.2.1, we can write

$$\begin{aligned} C_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (R^h(c, \partial_y u)), e^{\mathcal{R}t'} \Delta_q^h b_\phi \right\rangle_{L^2} \right| dt' \\ &\leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\Delta_{q'}^h c_\phi\|_{L_h^2(L_v^\infty)} \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2} \|\Delta_q^h b_\phi\|_{L^2} dt'. \end{aligned}$$

Similar calculations as in (6.3.39) imply

$$\|\Delta_q^h c_\phi(t)\|_{L_v^\infty(L_h^2)} \leq \int_0^1 \|\Delta_q^h \partial_x b_\phi(t, \cdot, y')\|_{L_h^2} dy' \lesssim 2^q \int_0^1 \|\Delta_q^h b_\phi(t, \cdot, y')\|_{L_h^2} dy' \lesssim 2^q \|\Delta_q^h b_\phi(t)\|_{L^2}.$$

Since $0 < s < 1$, we have

$$\begin{aligned} C_{3,q} &\leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^q \|e^{\mathcal{R}t'} \Delta_{q'}^h b_\phi(t')\|_{L^2} \|\Delta_{q'}^h \partial_y u_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t'} b_\phi\|_{L^2} dt' \\ &\leq 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|e^{\mathcal{R}t'} \Delta_{q'}^h b_\phi(t')\|_{L^2} \|\partial_y u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t'} b_\phi\|_{L^2} dt' \\ &\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_{q'}^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|e^{\mathcal{R}t'} \Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

we note that $\dot{\theta}(t) = \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}$, using Definition 6.2.3, we have

$$\begin{aligned} \left(\int_0^t \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} &\lesssim \left(\int_0^t \dot{\theta}(t') e^{2\mathcal{R}t'} \|\Delta_q^h b_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-q(s+\frac{1}{2})} d_q(b_\phi) \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

Thus,

$$(6.3.56) \quad C_{3,q} \lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} b_\phi\|_{\tilde{L}_{t,\hat{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

where

$$d_q^2 = d_q(b_\phi) \left(\sum_{q' \geq q-3} d_{q'}(b_\phi) 2^{(q-q')s} \right),$$

is a summable sequence of positive constants.

The proof of Lemma 6.3.3 is then completed by summing Estimates (6.3.54), (6.3.55) and (6.3.56). \square

6.4 Global well posedness of the 2D MHD system in a thin domain

The goal of this section is to prove Theorem 6.2.2 and to establish the global well-posedness of the system (6.1.3) with small analytic data. As in section 3 for any locally bounded function φ on $\mathbb{R}_+ \times \mathbb{R}$ and any $f \in L^2(\mathbf{S})$, we define the analytically in the horizontal variable x through the following auxiliary function

$$(6.4.1) \quad f_\varphi^\epsilon(t, x, y) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\epsilon^{\varphi(t,\xi)} \widehat{f}^\epsilon(t, \xi, y)).$$

The width of the analytically band φ is defined by

$$\varphi(t, \xi) = (a - \lambda\tau(t))|\xi|,$$

where $\lambda > 0$ with be precised later and $\tau(t)$ will be chosen in such a way that $\varphi(t, \xi) > 0$, for any $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}$ and $\dot{\varphi}(t) = \varphi'(t) = -\lambda\dot{\tau}(t) \geq 0$. In our paper, we will choose :

$$(6.4.2) \quad \dot{\tau}(t) = \|\partial_y u_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_y v_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} \quad \text{with} \quad \tau(0) = 0.$$

In what follows, for the sake of simplicity, we will neglect the script ϵ and write $(u_\Theta, v_\Theta, T_\Theta)$ instead of $(u_\varphi^\epsilon, v_\varphi^\epsilon, b_\varphi^\epsilon, c_\varphi^\epsilon)$. Direct calculations from (6.1.3) and (6.4.1) show that $(u_\varphi, v_\varphi, b_\varphi, c_\varphi)$ satisfies the system :

$$(6.4.3) \quad \begin{cases} \partial_t u_\varphi + \lambda\dot{\tau}(t)|D_x|u_\varphi + (u\partial_x u)_\varphi + (v\partial_y u)_\varphi - \epsilon^2 \partial_x^2 u_\varphi - \partial_y^2 u_\varphi + \partial_x p_\varphi = (b\partial_x b)_\varphi + (c\partial_y b)_\varphi, \\ \epsilon^2 (\partial_t v_\varphi + \lambda\dot{\tau}(t)|D_x|v_\varphi + (u\partial_x v)_\varphi + (v\partial_y v)_\varphi - \epsilon^2 \partial_x^2 v_\varphi - \partial_y^2 v_\varphi) + \partial_y p_\varphi = \epsilon^2 ((b\partial_x c)_\varphi + (c\partial_y c)_\varphi), \\ \partial_t b_\varphi + \lambda\dot{\tau}(t)|D_x|b_\varphi + (u\partial_x b)_\varphi + (v\partial_y b)_\varphi - \epsilon^2 \partial_x^2 b_\varphi - \partial_y^2 b_\varphi = (b\partial_x u)_\varphi + (c\partial_y u)_\varphi, \\ \epsilon^2 (\partial_t c_\varphi + \lambda\dot{\tau}(t)|D_x|c_\varphi + (u\partial_x c)_\varphi + (v\partial_y c)_\varphi - \epsilon^2 \partial_x^2 c_\varphi - \partial_y^2 c_\varphi) = \epsilon^2 ((b\partial_x v)_\varphi + (c\partial_y v)_\varphi), \\ \partial_x u_\varphi + \partial_y v_\varphi = 0 \quad \text{and} \quad \partial_x b_\varphi + \partial_y c_\varphi = 0, \\ (u_\varphi, v_\varphi, b_\varphi, c_\varphi)|_{y=0} = (u_\varphi, v_\varphi, b_\varphi, c_\varphi)|_{y=1} = 0, \\ (u_\varphi, v_\varphi, b_\varphi, c_\varphi)|_{t=0} = (u_0, v_0, b_0, c_0). \end{cases}$$

Where $|D_x|$ denote the Fourier multiplier of the symbol $|\xi|$. In what follows, we recall that we use C to denote a generic positive constant that can change from line to line.

Applying the dyadic operator Δ_q^h to the system (6.4.3), then taking the $L^2(\mathbf{S})$ ($\mathbf{S} = \mathbb{R} \times]0, 1[$) scalar product of the first, second, third and the fourth equations of the obtained system with $\Delta_q^h u_\varphi$, $\Delta_q^h v_\varphi$

$\Delta_q^h b_\phi$ and $\Delta_q^h c_\phi$ respectively, we get :

$$\begin{aligned} & \langle \Delta_q^h \partial_t(u_\varphi, \epsilon v_\varphi), \Delta_q^h(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} + \lambda \dot{\tau}(t) \langle |D_x| \Delta_q^h(u_\varphi, \epsilon v_\varphi), \Delta_q^h(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} \\ & - \langle \Delta_q^h \partial_y^2(u_\varphi, \epsilon v_\varphi), \Delta_q^h(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h \partial_x^2(u_\varphi, \epsilon v_\varphi), \Delta_q^h(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} + \langle \Delta_q^h \nabla p_\varphi, \Delta_q^h(u_\varphi, v_\varphi) \rangle_{L^2} \\ & = - \langle \Delta_q^h(u \partial_x u + v \partial_y u)_\varphi, \Delta_q^h u_\varphi \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h(u \partial_x v + v \partial_y v)_\varphi, \Delta_q^h v_\varphi \rangle_{L^2} \\ & \quad + \langle \Delta_q^h(b \partial_x b + c \partial_y b)_\varphi, \Delta_q^h u_\varphi \rangle_{L^2} + \epsilon^2 \langle \Delta_q^h(b \partial_x c + c \partial_y c)_\varphi, \Delta_q^h v_\varphi \rangle_{L^2}, \end{aligned}$$

and

$$\begin{aligned} & \langle \Delta_q^h \partial_t(b_\varphi, \epsilon c_\varphi), \Delta_q^h(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} + \lambda \dot{\tau}(t) \langle |D_x| \Delta_q^h(b_\varphi, \epsilon c_\varphi), \Delta_q^h(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} \\ & - \langle \Delta_q^h \partial_y^2(b_\varphi, \epsilon c_\varphi), \Delta_q^h(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h \partial_x^2(b_\varphi, \epsilon c_\varphi), \Delta_q^h(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} \\ & = - \langle \Delta_q^h(u \partial_x b + v \partial_y b)_\varphi, \Delta_q^h b_\varphi \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h(u \partial_x c + v \partial_y c)_\varphi, \Delta_q^h c_\varphi \rangle_{L^2} \\ & \quad + \langle \Delta_q^h(b \partial_x u + c \partial_y u)_\varphi, \Delta_q^h b_\varphi \rangle_{L^2} + \epsilon^2 \langle \Delta_q^h(b \partial_x v + c \partial_y v)_\varphi, \Delta_q^h c_\varphi \rangle_{L^2}. \end{aligned}$$

Thanks to the Dirichlet boundary condition and due to the fact that U is divergence-free (means that $\operatorname{div} U = \partial_x u + \partial_y v = 0$), we get by using the integration by part that :

$$\begin{aligned} \langle \Delta_q^h \nabla p_\varphi, \Delta_q^h(u_\varphi, v_\varphi) \rangle_{L^2} &= - \langle \Delta_q^h p_\varphi, \Delta_q^h \operatorname{div}(u_\varphi, v_\varphi) \rangle_{L^2} \\ &= \langle \Delta_q^h p_\varphi, \Delta_q^h(\partial_y v_\varphi + \partial_x u_\varphi) \rangle_{L^2} \\ &= 0. \quad (\text{because } \partial_y v_\varphi + \partial_x u_\varphi = 0). \end{aligned}$$

We recall that we have by integrating by part that :

$$\begin{aligned} \langle \Delta_q^h \partial_y^2(u_\varphi, \epsilon v_\varphi), \Delta_q^h(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} &= - \langle \Delta_q^h \partial_y(u_\varphi, \epsilon v_\varphi), \Delta_q^h \partial_y(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} = - \|\Delta_q^h \partial_y(u_\varphi, \epsilon v_\varphi)\|_{L^2}^2, \\ \epsilon^2 \langle \Delta_q^h \partial_x^2(u_\varphi, \epsilon v_\varphi), \Delta_q^h(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} &= -\epsilon^2 \langle \Delta_q^h \partial_x(u_\varphi, \epsilon v_\varphi), \Delta_q^h \partial_x(u_\varphi, \epsilon v_\varphi) \rangle_{L^2} = -\epsilon^2 \|\Delta_q^h \partial_x(u_\varphi, \epsilon v_\varphi)\|_{L^2}^2, \\ \langle \Delta_q^h \partial_y^2(b_\varphi, \epsilon c_\varphi), \Delta_q^h(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} &= - \langle \Delta_q^h \partial_y(b_\varphi, \epsilon c_\varphi), \Delta_q^h \partial_y(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} = - \|\Delta_q^h \partial_y(b_\varphi, \epsilon c_\varphi)\|_{L^2}^2, \\ \epsilon^2 \langle \Delta_q^h \partial_x^2(b_\varphi, \epsilon c_\varphi), \Delta_q^h(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} &= -\epsilon^2 \langle \Delta_q^h \partial_x(b_\varphi, \epsilon c_\varphi), \Delta_q^h \partial_x(b_\varphi, \epsilon c_\varphi) \rangle_{L^2} = -\epsilon^2 \|\Delta_q^h \partial_x(b_\varphi, \epsilon c_\varphi)\|_{L^2}^2, \end{aligned}$$

we replace in the obtained estimate, we get :

$$\begin{aligned} (6.4.4) \quad & \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(u_\varphi, \epsilon v_\varphi)(t)\|_{L^2}^2 + \lambda \dot{\tau}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h(u_\varphi, \epsilon v_\varphi) \right\|_{L^2}^2 + \|\Delta_q^h \partial_y(u_\varphi, \epsilon v_\varphi)(t)\|_{L^2}^2 \\ & \epsilon^2 \|\Delta_q^h \partial_x(u_\varphi, \epsilon v_\varphi)(t)\|_{L^2}^2 = - \langle \Delta_q^h(u \partial_x u + v \partial_y u)_\varphi, \Delta_q^h u_\varphi \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h(u \partial_x v + v \partial_y v)_\varphi, \Delta_q^h v_\varphi \rangle_{L^2} \\ & \quad + \langle \Delta_q^h(b \partial_x b + c \partial_y b)_\varphi, \Delta_q^h u_\varphi \rangle_{L^2} + \epsilon^2 \langle \Delta_q^h(b \partial_x c + c \partial_y c)_\varphi, \Delta_q^h v_\varphi \rangle_{L^2}, \end{aligned}$$

and

$$\begin{aligned} (6.4.5) \quad & \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(b_\varphi, \epsilon c_\varphi)(t)\|_{L^2}^2 + \lambda \dot{\tau}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h(b_\varphi, \epsilon c_\varphi) \right\|_{L^2}^2 + \|\Delta_q^h \partial_y(b_\varphi, \epsilon c_\varphi)(t)\|_{L^2}^2 + \\ & \epsilon^2 \|\Delta_q^h \partial_x(b_\varphi, \epsilon c_\varphi)(t)\|_{L^2}^2 = - \langle \Delta_q^h(u \partial_x b + v \partial_y b)_\varphi, \Delta_q^h b_\varphi \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h(u \partial_x c + v \partial_y c)_\varphi, \Delta_q^h c_\varphi \rangle_{L^2} \\ & \quad + \langle \Delta_q^h(b \partial_x u + c \partial_y u)_\varphi, \Delta_q^h b_\varphi \rangle_{L^2} + \epsilon^2 \langle \Delta_q^h(b \partial_x v + c \partial_y v)_\varphi, \Delta_q^h c_\varphi \rangle_{L^2}. \end{aligned}$$

Multiplying (6.3.4) and (6.3.5) with $e^{2\mathcal{R}t}$ and integrating with respect to the time variable, we have the resulting estimates

(6.4.6)

$$\begin{aligned} & \frac{1}{2} \left\| e^{\mathcal{R}t} \Delta_q^h(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h(u_\varphi, \epsilon v_\varphi) \right\|_{L^2}^2 dt' + \left\| e^{\mathcal{R}t} \Delta_q^h \partial_y(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \\ & + \epsilon^2 \left\| e^{\mathcal{R}t} \Delta_q^h \partial_x(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \leq \mathcal{R} \left\| e^{\mathcal{R}t} \Delta_q^h(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^2(L^2)}^2 + C \left\| \Delta_q^h(u_\varphi, \epsilon v_\varphi)(0) \right\|_{L^2}^2 + F_1 + F_2 + F_3 + F_4, \end{aligned}$$

and

(6.4.7)

$$\begin{aligned} & \frac{1}{2} \left\| e^{\mathcal{R}t} \Delta_q^h(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h(b_\varphi, \epsilon c_\varphi) \right\|_{L^2}^2 dt' + \left\| e^{\mathcal{R}t} \Delta_q^h \partial_y(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \\ & + \epsilon^2 \left\| e^{\mathcal{R}t} \Delta_q^h \partial_x(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \leq \mathcal{R} \left\| e^{\mathcal{R}t} \Delta_q^h(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^2(L^2)}^2 + C \left\| \Delta_q^h(b_\varphi, \epsilon c_\varphi)(0) \right\|_{L^2}^2 + F_5 + F_6. \end{aligned}$$

where

$$\begin{cases} F_1 = \left| \int_0^t \langle e^{\mathcal{R}t} \Delta_q^h(u \partial_x u)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \rangle_{L^2} + \langle e^{\mathcal{R}t} \Delta_q^h(v \partial_y u)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \rangle_{L^2} dt' \right| \\ F_2 = \epsilon^2 \left| \int_0^t \langle e^{\mathcal{R}t} \Delta_q^h(u \partial_x v)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \rangle_{L^2} + \langle e^{\mathcal{R}t} \Delta_q^h(v \partial_y v)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \rangle_{L^2} dt' \right| \\ F_3 = \left| \int_0^t \langle e^{\mathcal{R}t} \Delta_q^h(b \partial_x b)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \rangle_{L^2} + \langle e^{\mathcal{R}t} \Delta_q^h(c \partial_y b)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \rangle_{L^2} dt' \right| \\ F_4 = \epsilon^2 \left| \int_0^t \langle e^{\mathcal{R}t} \Delta_q^h(b \partial_x c)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \rangle_{L^2} + \langle e^{\mathcal{R}t} \Delta_q^h(c \partial_y c)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \rangle_{L^2} dt' \right|, \end{cases}$$

and

$$\begin{cases} F_5 = \left| \int_0^t \langle \Delta_q^h(u \partial_x b + v \partial_y b)_\varphi, \Delta_q^h b_\varphi \rangle_{L^2} + \langle \Delta_q^h(b \partial_x u + c \partial_y u)_\varphi, \Delta_q^h b_\varphi \rangle_{L^2} dt' \right| \\ F_6 = \epsilon^2 \left| \int_0^t \langle \Delta_q^h(u \partial_x c + v \partial_y c)_\varphi, \Delta_q^h c_\varphi \rangle_{L^2} + \langle \Delta_q^h(b \partial_x v + c \partial_y v)_\varphi, \Delta_q^h c_\varphi \rangle_{L^2} dt' \right|. \end{cases}$$

We start by observing that the terms $\mathcal{R} \left\| e^{\mathcal{R}t} \Delta_q^h(u, \epsilon v)_\phi \right\|_{L_t^2(L^2)}^2$ and $\mathcal{R} \left\| e^{\mathcal{R}t} \Delta_q^h(b, \epsilon c)_\phi \right\|_{L_t^2(L^2)}^2$ can be absorbed by the dissipation $\left\| e^{\mathcal{R}t} \Delta_q^h \partial_y(u, \epsilon v)_\phi \right\|_{L_t^2(L^2)}^2$ and $\left\| e^{\mathcal{R}t} \Delta_q^h \partial_y(b, \epsilon c)_\phi \right\|_{L_t^2(L^2)}^2$. So, choosing \mathcal{R} smaller than $\frac{k}{8}$, (here k is the Poincaré constant that comes out from the inequality $k \|(u, \epsilon v)_\phi\|_{L^2} \leq \|\partial_y(u, \epsilon v)_\phi\|_{L^2}$), we can achieve that

(6.4.8)

$$\begin{aligned} & \frac{1}{2} \left\| e^{\mathcal{R}t} \Delta_q^h(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h(u_\varphi, \epsilon v_\varphi) \right\|_{L^2}^2 dt' + \frac{1}{2} \left\| e^{\mathcal{R}t} \Delta_q^h \partial_y(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \\ & + \epsilon^2 \left\| e^{\mathcal{R}t} \Delta_q^h \partial_x(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \leq C \left\| \Delta_q^h(u_\varphi, \epsilon v_\varphi)(0) \right\|_{L^2}^2 + F_1 + F_2 + F_3 + F_4, \end{aligned}$$

and

(6.4.9)

$$\begin{aligned} & \frac{1}{2} \left\| e^{\mathcal{R}t} \Delta_q^h(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^\infty(L^2)}^2 + \lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h(b_\varphi, \epsilon c_\varphi) \right\|_{L^2}^2 dt' + \frac{1}{2} \left\| e^{\mathcal{R}t} \Delta_q^h \partial_y(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \\ & + \epsilon^2 \left\| e^{\mathcal{R}t} \Delta_q^h \partial_x(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^2(L^2)}^2 \leq C \left\| \Delta_q^h(b_\varphi, \epsilon c_\varphi)(0) \right\|_{L^2}^2 + F_5 + F_6. \end{aligned}$$

After giving the estimate of the nonlinear terms, we begin by giving the following lemma that handles to estimate F_2 , F_4 and F_6 .

Lemma 6.4.1. For any $s \in]0, 1[$ and $t \leq T^*$, and φ be defined as in (6.4.1), with

$$\dot{\tau}(t) = \|\partial_y u_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_y v_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Then, there exists $C \geq 1$ such that, for any $t > 0$, $\varphi(t, \xi) > 0$ and for any $u \in \tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$ that satisfies $v(t, x, y) = -\int_0^t \partial_x u(t, x, s) ds$, we have

$$\epsilon^2 \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_y v)_\varphi, e^{\mathcal{R}t'} \Delta_q^h v_\varphi \right\rangle_{L^2} \right| dt' \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Next, by using the lemmas 6.3.2, 6.3.1, 6.4.1 and 6.3.3

$$\begin{aligned} F_1 &= \left| \int_0^t \left\langle e^{\mathcal{R}t} \Delta_q^h (u \partial_x u)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \right\rangle_{L^2} + \left\langle e^{\mathcal{R}t} \Delta_q^h (v \partial_y u)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \right\rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\varphi\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \end{aligned}$$

$$\begin{aligned} F_2 &= \epsilon^2 \left| \int_0^t \left\langle e^{\mathcal{R}t} \Delta_q^h (u \partial_x v)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \right\rangle_{L^2} + \left\langle e^{\mathcal{R}t} \Delta_q^h (v \partial_y v)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \right\rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\varphi, \epsilon v_\varphi)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \end{aligned}$$

$$\begin{aligned} F_3 &= \left| \int_0^t \left\langle e^{\mathcal{R}t} \Delta_q^h (b \partial_x b)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \right\rangle_{L^2} + \left\langle e^{\mathcal{R}t} \Delta_q^h (c \partial_y b)_\varphi, e^{\mathcal{R}t} \Delta_q^h u_\varphi \right\rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\varphi, b_\varphi)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \end{aligned}$$

$$\begin{aligned} F_4 &= \epsilon^2 \left| \int_0^t \left\langle e^{\mathcal{R}t} \Delta_q^h (b \partial_x c)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \right\rangle_{L^2} + \left\langle e^{\mathcal{R}t} \Delta_q^h (c \partial_y c)_\varphi, e^{\mathcal{R}t} \Delta_q^h v_\varphi \right\rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (b_\varphi, \epsilon v_\varphi)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \end{aligned}$$

and

$$\begin{aligned} F_5 &= \left| \int_0^t \left\langle \Delta_q^h (u \partial_x b + v \partial_y b)_\varphi, \Delta_q^h b_\varphi \right\rangle_{L^2} + \left\langle \Delta_q^h (b \partial_x u + c \partial_y u)_\varphi, \Delta_q^h b_\varphi \right\rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\varphi, b_\varphi)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2, \end{aligned}$$

$$\begin{aligned} F_6 &= \epsilon^2 \left| \int_0^t \left\langle \Delta_q^h (u \partial_x c + v \partial_y c)_\varphi, \Delta_q^h c_\varphi \right\rangle_{L^2} + \left\langle \Delta_q^h (b \partial_x v + c \partial_y v)_\varphi, \Delta_q^h c_\varphi \right\rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\varphi, \epsilon c_\varphi)\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2. \end{aligned}$$

Multiplying (6.4.8) and (6.4.9) by 2^{2qs} (for $s \in]0, 1[$), taking the square root of the resulting inequality, and finally summing up the resulting ones over \mathbb{Z} , we obtain

$$(6.4.10) \quad \frac{1}{2} \|e^{\mathcal{R}t} (u_\varphi, \epsilon v_\varphi)(t)\|_{L_t^\infty(\mathcal{B}^s)} + \sqrt{\lambda} \left\| e^{\mathcal{R}t} (u_\varphi, \epsilon v_\varphi) \right\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_y (u_\varphi, \epsilon v_\varphi)(t)\|_{L_t^2(\mathcal{B}^s)}$$

$$+ \epsilon \left\| e^{\mathcal{R}t} \partial_x(u_\varphi, \epsilon v_\varphi)(t) \right\|_{L_t^2(\mathcal{B}^s)} \leq C \left\| (u_\varphi, \epsilon v_\varphi)(0) \right\|_{\mathcal{B}^s} + C \left\| e^{\mathcal{R}t}(u_\varphi, b_\varphi, \epsilon v_\varphi) \right\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

and

$$(6.4.11) \quad \frac{1}{2} \left\| e^{\mathcal{R}t}(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^\infty(\mathcal{B}^s)} + \sqrt{\lambda} \left\| e^{\mathcal{R}t'}(b_\varphi, \epsilon c_\varphi) \right\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{2} \left\| e^{\mathcal{R}t} \partial_y(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^2(\mathcal{B}^s)} \\ + \epsilon \left\| e^{\mathcal{R}t} \partial_x(b_\varphi, \epsilon c_\varphi)(t) \right\|_{L_t^2(\mathcal{B}^s)} \leq C \left\| (b_\varphi, \epsilon c_\varphi)(0) \right\|_{\mathcal{B}^s} + C \left\| e^{\mathcal{R}t}(u_\varphi, b_\varphi, \epsilon c_\varphi) \right\|_{\tilde{L}_{t, \dot{\tau}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Thus, choosing our constant C such that

$$(6.4.12) \quad C \geq \max \left\{ 4, \frac{1}{2\mathcal{R}} \right\},$$

and taking the sum of the two estimates (6.4.10) and (6.4.11), we obtain

$$\frac{1}{2} \left\| e^{\mathcal{R}t}(u_\varphi, b_\varphi, \epsilon(v_\varphi, c_\varphi)) \right\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{\lambda} \left\| e^{\mathcal{R}t}(u_\varphi, b_\varphi, \epsilon(v_\varphi, c_\varphi)) \right\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ + \frac{1}{2} \left\| e^{\mathcal{R}t} \partial_y(u_\varphi, b_\varphi, \epsilon(v_\varphi, c_\varphi)) \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \epsilon \left\| e^{\mathcal{R}t} \partial_x(u_\varphi, b_\varphi, \epsilon(v_\varphi, c_\varphi)) \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ \leq 2C \left\| e^{a|D_x|}(u_0, b_0, \epsilon v_0, \epsilon c_0) \right\|_{\mathcal{B}^s} + 2C \left\| e^{\mathcal{R}t}(u_\varphi, b_\varphi, \epsilon(v_\varphi, c_\varphi)) \right\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

We set

$$(6.4.13) \quad T^* \triangleq \sup \left\{ t > 0 : \|u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{1}{2C} \text{ and } \tau(t) \leq \frac{a}{\lambda} \right\},$$

and we choose the initial data such that

$$C \left(\|e^{a|D_x|}(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0, \epsilon c_0)\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \min \left\{ \frac{1}{2C}, \frac{a}{2\lambda} \right\}.$$

The fact that $\tau(0) = 0$ implies already that $T^* > 0$. If $\sqrt{\lambda} = 2C$, for any $0 < t < T^*$, we have

$$(6.4.14) \quad \left\| e^{\mathcal{R}t}(u_\varphi, \epsilon v_\varphi) \right\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t}(b_\varphi, \epsilon c_\varphi) \right\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \partial_y(b_\varphi, \epsilon c_\varphi) \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \partial_y(u_\varphi, \epsilon v_\varphi) \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ + \epsilon \left\| e^{\mathcal{R}t} \partial_x(u_\varphi, \epsilon v_\varphi) \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \epsilon \left\| e^{\mathcal{R}t} \partial_x(b_\varphi, \epsilon c_\varphi) \right\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq 2C \left(\left\| e^{a|D_x|}(u_0, \epsilon v_0) \right\|_{\mathcal{B}^s} + \left\| e^{a|D_x|}(u_0, \epsilon v_0) \right\|_{\mathcal{B}^s} \right).$$

From (6.4.14) and (6.4.12), we get that, for any $0 < t < T^*$,

$$\|u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \leq \left\| e^{\mathcal{R}t}(u_\varphi, \epsilon v_\varphi) \right\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \leq C \left(\left\| e^{a|D_x|}(u_0, \epsilon v_0) \right\|_{\mathcal{B}^s} + \left\| e^{a|D_x|}(b_0, \epsilon c_0) \right\|_{\mathcal{B}^s} \right) \\ \leq C \left(\left\| e^{a|D_x|}(u_0, \epsilon v_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|}(b_0, \epsilon c_0) \right\|_{\mathcal{B}^s} \right) < \frac{1}{2C}.$$

Now, we recall that we already defined $\dot{\tau}(t) = \|\partial_y u_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_y v_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ with $\tau(0) = 0$. Then, for any $0 < t < T^*$, Inequality (6.4.14) yields

$$\tau(t) = \int_0^t \left(\|\partial_y u_\varphi^\epsilon(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_y v_\varphi^\epsilon(t')\|_{\mathcal{B}^{\frac{1}{2}}} \right) dt' \\ \leq \int_0^t e^{-\mathcal{R}t'} \left(\left\| e^{\mathcal{R}t'} \partial_y u_\varphi^\epsilon(t') \right\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \left\| e^{\mathcal{R}t'} \partial_y v_\varphi^\epsilon(t') \right\|_{\mathcal{B}^{\frac{1}{2}}} \right) dt'$$

$$\begin{aligned}
&\leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \left(\int_0^t (\|e^{\mathcal{R}t'} \partial_y u_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|e^{\mathcal{R}t'} \partial_y v_\varphi^\epsilon(t)\|_{\mathcal{B}^{\frac{1}{2}}})^2 dt' \right)^{\frac{1}{2}} \\
&\leq C \|e^{\mathcal{R}t} (\epsilon \partial_y v_\varphi^\epsilon, \partial_y u_\varphi^\epsilon)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
&\leq C \left(\|e^{a|D_x|}(u_0, \epsilon v_0)\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}(b_0, \epsilon c_0)\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \frac{a}{2\lambda}.
\end{aligned}$$

A continuity argument implies that $T^* = +\infty$ and we have (6.4.14) is valid for any $t \in \mathbb{R}_+$.

6.5 The convergence to the limit system MHD

In this section, we justify the limit from the scaled anisotropic MHD system to the hydrostatic MHD system in a 2D thin domain. As in the first section, our main idea is to get control of the difference between the two solutions (U^ϵ, B^ϵ) and (U, B) of the systems (6.1.3) and (6.1.4) (respectively), in analytic space with some small initial data. To this end, we introduce :

$$(6.5.1) \quad \begin{cases} (\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, q^\epsilon) = (u^\epsilon - u, v^\epsilon - v, p^\epsilon - p), \\ (\Phi^{1,\epsilon}, \Phi^{2,\epsilon}) = (b^\epsilon - b, c^\epsilon - c). \end{cases}$$

Then, Systems (6.1.3) and (6.1.4) imply that $(\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, q^\epsilon, \Phi^{1,\epsilon}, \Phi^{2,\epsilon})$ verifies

$$(6.5.2) \quad \begin{cases} \partial_t \Psi^{1,\epsilon} - \epsilon^2 \partial_x^2 \Psi^{1,\epsilon} - \partial_y^2 \Psi^{1,\epsilon} + \partial_x q^\epsilon = R^{1,\epsilon}, \\ \epsilon^2 (\partial_t \Psi^{2,\epsilon} - \epsilon^2 \partial_x^2 \Psi^{2,\epsilon} - \partial_y^2 \Psi^{2,\epsilon}) + \partial_y q^\epsilon = R^{2,\epsilon}, \\ \partial_t \Phi^{1,\epsilon} - \epsilon^2 \partial_x^2 \Phi^{1,\epsilon} - \partial_y^2 \Phi^{1,\epsilon} = R^{3,\epsilon}, \\ \partial_t \Phi^{2,\epsilon} - \epsilon^2 \partial_x^2 \Phi^{2,\epsilon} - \partial_y^2 \Phi^{2,\epsilon} = R^{4,\epsilon}, \\ \partial_x \Psi^{1,\epsilon} + \partial_y \Psi^{2,\epsilon} = 0 \quad \partial_x \Phi^{1,\epsilon} + \partial_y \Phi^{2,\epsilon} = 0 \\ (\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, \Phi^{1,\epsilon}, \Phi^{2,\epsilon})|_{t=0} = (u_0^\epsilon - u_0, v_0^\epsilon - v_0, b_0^\epsilon - b_0, c_0^\epsilon - c), \\ (\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, \Phi^{1,\epsilon}, \Phi^{2,\epsilon})|_{y=0} = (\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, \Phi^{1,\epsilon}, \Phi^{2,\epsilon})|_{y=1} = 0, \end{cases}$$

where v_0 is a function of u_0 and c_0 is a function of b_0 , using (6.3.1) and the remaining terms $R^{i,\epsilon}$, with $i = 1, 2, 3, 4$, are determined by the rest

$$(6.5.3) \quad \begin{cases} R^{1,\epsilon} = \epsilon^2 \partial_x^2 u - (u^\epsilon \partial_x u^\epsilon - u \partial_x u) - (v^\epsilon \partial_y u^\epsilon - v \partial_y u) + (b^\epsilon \partial_x b^\epsilon - b \partial_x b) + (c^\epsilon \partial_y b^\epsilon - c \partial_y b), \\ R^{2,\epsilon} = -\epsilon^2 (\partial_t v - \epsilon^2 \partial_x^2 v - \partial_y^2 v + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon + b^\epsilon \partial_x c^\epsilon + c^\epsilon \partial_y c^\epsilon), \\ R^{3,\epsilon} = \epsilon^2 \partial_x^2 b - (u^\epsilon \partial_x b^\epsilon - u \partial_x b) - (v^\epsilon \partial_y b^\epsilon - v \partial_y b) + (b^\epsilon \partial_x u^\epsilon - b \partial_x u) + (c^\epsilon \partial_y u^\epsilon - c \partial_y u), \\ R^{4,\epsilon} = -\epsilon^2 (\partial_t c - \epsilon^2 \partial_x^2 c - \partial_y^2 c + u^\epsilon \partial_x c^\epsilon + v^\epsilon \partial_y c^\epsilon + b^\epsilon \partial_x u^\epsilon + c^\epsilon \partial_y v^\epsilon). \end{cases}$$

As $(\Psi^{1,\epsilon}, \Psi^{2,\epsilon}, q^\epsilon, \Phi^{1,\epsilon}, \Phi^{2,\epsilon})$ satisfies the boundary condition and also the divergence-free condition, therefore these two conditions allow us to write

$$(6.5.4) \quad \Psi^{2,\epsilon}(t, x, y) = \int_0^y \partial_y \Psi^{2,\epsilon}(t, x, s) ds = - \int_0^y \partial_x \Psi^{1,\epsilon}(t, x, s) ds,$$

$$(6.5.5) \quad \Phi^{2,\epsilon}(t, x, y) = \int_0^y \partial_y \Phi^{2,\epsilon}(t, x, s) ds = - \int_0^y \partial_x \Phi^{1,\epsilon}(t, x, s) ds.$$

If we replace y by 1 in (6.5.4) and (6.5.5), we deduce from the incompressibility condition $\partial_x \Psi^{1,\epsilon} + \partial_y \Psi^{2,\epsilon} = 0$ $\partial_x \Phi^{1,\epsilon} + \partial_y \Phi^{2,\epsilon} = 0$ that

$$\begin{aligned}\partial_x \int_0^1 \Psi^{1,\epsilon}(t, x, y) dy &= - \int_0^1 \partial_y \Psi^{2,\epsilon}(t, x, y) dy = \Psi^{2,\epsilon}(t, x, 1) - \Psi^{2,\epsilon}(t, x, 0) = 0 \\ \partial_x \int_0^1 \Phi^{1,\epsilon}(t, x, y) dy &= - \int_0^1 \partial_y \Phi^{2,\epsilon}(t, x, y) dy = \Phi^{2,\epsilon}(t, x, 1) - \Phi^{2,\epsilon}(t, x, 0) = 0.\end{aligned}$$

Now for suitable function f , we define

$$(6.5.6) \quad f_\Theta(t, x, y) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(e^{\Theta(t, \xi)} \widehat{f}(t, \xi, y) \right) \quad \text{where} \quad \Theta(t, \xi) = (a - \mu\eta(t)) |\xi|,$$

where $\mu \geq \lambda$ will be determined later, and $\eta(t)$ is given by

$$\eta(t) = \int_0^t \left(\|(\partial_y u_\varphi^\epsilon, \epsilon \partial_x u_\varphi^\epsilon)(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \|\partial_y u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} \right) dt'.$$

We can observe that, if we take c_0 and c_1 small enough in Theorems 6.2.1 and 6.2.2 then $\Theta(t) \geq 0$ and

$$0 \leq \Theta(t, \xi) \leq \min(\phi(t, \xi), \varphi(t, \xi)).$$

In what follows, for simplicity, we drop the script ϵ and we will write $(\Psi_\Theta^1, \Psi_\Theta^2, q_\Theta, \Phi_\Theta^1, \Phi_\Theta^2)$ instead of $(\Psi_\Theta^{1,\epsilon}, \Psi_\Theta^{2,\epsilon}, q_\Theta^\epsilon, \Phi_\Theta^{1,\epsilon}, \Phi_\Theta^{2,\epsilon})$. Direct calculations show that $(\Psi_\Theta^1, \Psi_\Theta^2, q_\Theta, \Phi_\Theta^1, \Phi_\Theta^2)$ satisfies

$$(6.5.7) \quad \begin{cases} \partial_t \Psi_\Theta^1 + \mu |D_x| \dot{\eta}(t) \Psi_\Theta^1 - \epsilon^2 \partial_x^2 \Psi_\Theta^1 - \partial_y^2 \Psi_\Theta^1 + \partial_x q_\Theta = R_\Theta^1, \\ \epsilon^2 (\partial_t \Psi_\Theta^2 + \mu |D_x| \dot{\eta}(t) \Psi_\Theta^2 - \epsilon^2 \partial_x^2 \Psi_\Theta^2 - \partial_y^2 \Psi_\Theta^2) + \partial_y q_\Theta = R_\Theta^2, \\ \partial_t \Phi_\Theta^1 + \mu |D_x| \dot{\eta}(t) \Phi_\Theta^1 - \epsilon^2 \partial_x^2 \Phi_\Theta^1 - \partial_y^2 \Phi_\Theta^1 = R_\Theta^3, \\ \epsilon^2 (\partial_t \Phi_\Theta^2 + \mu |D_x| \dot{\eta}(t) \Phi_\Theta^2 - \epsilon^2 \partial_x^2 \Phi_\Theta^2 - \partial_y^2 \Phi_\Theta^2) = R_\Theta^4, \\ \partial_x \Psi_\Theta^1 + \partial_y \Psi_\Theta^2 = 0 \quad \partial_x \Phi_\Theta^1 + \partial_y \Phi_\Theta^2 = 0, \\ (\Psi_\Theta^1, \Psi_\Theta^2, \Phi_\Theta^1, \Phi_\Theta^2)|_{t=0} = (u_0^\epsilon - u_0, v_0^\epsilon - v_0, b_0^\epsilon - b_0, c_0^\epsilon - c), \\ (\Psi_\Theta^1, \Psi_\Theta^2, \Phi_\Theta^1, \Phi_\Theta^2)|_{y=0} = (\Psi_\Theta^1, \Psi_\Theta^2, \Phi_\Theta^1, \Phi_\Theta^2)|_{y=1} = 0. \end{cases}$$

As in the previous sections, we will use C to denote a generic positive constant that can change from line to line. So, thanks to Theorems 6.2.1 and 6.2.2, and the proposition 6.3.1 we deduce that

$$(6.5.8) \quad \|(u_\varphi^\epsilon, b_\varphi^\epsilon)\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})} + \|(u_\phi, b_\phi)\|_{\tilde{L}^\infty(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}})} + \|\partial_y(u_\phi, b_\phi)\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{5}{2}})} + \|(\partial_t(u, b))_\phi\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{\frac{3}{2}})} \leq M,$$

where u_Θ^ϵ and u_ϕ are respectively determined by (6.4.3) and (6.3.3) and $M \geq 1$ is a constant independent of ϵ . Now we return to get the proof of the last theorem, for that we start by applying the dyadic operator in the system (6.5.7) and then taking the $L^2(\mathbf{S})$ (such that $\mathbf{S} = \mathbb{R} \times]0, 1[$) scalar product of all the equations then we obtain that

$$\begin{aligned}& \langle \Delta_q^h \partial_t(\Psi_\Theta^1, \epsilon \Psi_\Theta^2), \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} + \mu \dot{\eta}(t) \langle |D_x| \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2), \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} \\ & + \langle \Delta_q^h \nabla q_\Theta, \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} - \langle \Delta_q^h \partial_y^2(\Psi_\Theta^1, \epsilon \Psi_\Theta^2), \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} \\ & - \epsilon^2 \langle \Delta_q^h \partial_x^2(\Psi_\Theta^1, \epsilon \Psi_\Theta^2), \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} = \langle \Delta_q^h R_\Theta^1, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2} + \langle \Delta_q^h R_\Theta^2, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2},\end{aligned}$$

and

$$\begin{aligned}& \langle \Delta_q^h \partial_t(\Phi_\Theta^1, \epsilon \Phi_\Theta^2), \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2) \rangle_{L^2} + \mu \dot{\eta}(t) \langle |D_x| \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2), \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2) \rangle_{L^2} \\ & - \langle \Delta_q^h \partial_y^2(\Phi_\Theta^1, \epsilon \Phi_\Theta^2), \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2) \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h \partial_x^2(\Phi_\Theta^1, \epsilon \Phi_\Theta^2), \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2) \rangle_{L^2} \\ & = \langle \Delta_q^h R_\Theta^3, \Delta_q^h \Phi_\Theta^1 \rangle_{L^2} + \langle \Delta_q^h R_\Theta^4, \Delta_q^h \Phi_\Theta^2 \rangle_{L^2}.\end{aligned}$$

Thanks to the Dirichlet boundary condition and due to the free divergence of Ψ ($\operatorname{div} \Psi = \partial_x \Psi^1 + \partial_y \Psi^2 = 0$), we get by using the integration by part that

$$\begin{aligned} \langle \Delta_q^h \nabla q_\Theta, \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} &= - \langle \Delta_q^h q_\Theta, \Delta_q^h \operatorname{div}(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} \\ &= \langle \Delta_q^h q_\Theta, \Delta_q^h(\partial_y \Psi_\Theta^2 + \partial_x \Psi_\Theta^1) \rangle_{L^2} \\ &= 0. \quad (\text{because } \partial_y \Psi_\Theta^2 + \partial_x \Psi_\Theta^1 = 0). \end{aligned}$$

We recall that we have by integrating by part that

$$\begin{aligned} \langle \Delta_q^h \partial_y^2(\Psi_\Theta^1, \epsilon \Psi_\Theta^2), \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} &= - \|\Delta_q^h \partial_y(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{L^2}^2, \\ \epsilon^2 \langle \Delta_q^h \partial_x^2(\Psi_\Theta^1, \epsilon \Psi_\Theta^2), \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2) \rangle_{L^2} &= -\epsilon^2 \|\Delta_q^h \partial_x(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{L^2}^2, \\ \langle \Delta_q^h \partial_y^2(\Phi_\Theta^1, \epsilon \Phi_\Theta^2), \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2) \rangle_{L^2} &= - \|\Delta_q^h \partial_y(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{L^2}^2, \\ \epsilon^2 \langle \Delta_q^h \partial_x^2(\Phi_\Theta^1, \epsilon \Phi_\Theta^2), \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2) \rangle_{L^2} &= -\epsilon^2 \|\Delta_q^h \partial_x(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{L^2}^2, \end{aligned}$$

we replace in the obtained estimate and then integrating with respect to the time variable, we have the result estimates

$$(6.5.9) \quad \begin{aligned} \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)(t)\|_{L_t^\infty(L^2)}^2 + \mu \int_0^t \dot{\eta}(t') \|e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{L^2}^2 dt' \\ + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h \partial_y(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)(t)\|_{L_t^2(L^2)}^2 + \epsilon^2 \|e^{\mathcal{R}t} \Delta_q^h \partial_x(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)(t)\|_{L_t^2(L^2)}^2 \\ \leq C \|\Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)(0)\|_{L^2}^2 + G_1^q + G_2^q, \end{aligned}$$

and

$$(6.5.10) \quad \begin{aligned} \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(t)\|_{L_t^\infty(L^2)}^2 + \mu \int_0^t \dot{\eta}(t') \|e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{L^2}^2 dt' \\ + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h \partial_y(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(t)\|_{L_t^2(L^2)}^2 + \epsilon^2 \|e^{\mathcal{R}t} \Delta_q^h \partial_x(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(t)\|_{L_t^2(L^2)}^2 \\ \leq C \|\Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(0)\|_{L^2}^2 + G_3^q + G_4^q, \end{aligned}$$

where

$$\begin{aligned} \|\Delta_q^h(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)(0)\|_{L^2}^2 &= \|\Delta_q^h e^{a|D_x|}(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{L^2}^2, \\ \|\Delta_q^h(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(0)\|_{L^2}^2 &= \|\Delta_q^h e^{a|D_x|}(b_0^\epsilon - b_0, \epsilon(c_0^\epsilon - c_0))\|_{L^2}^2. \end{aligned}$$

Next, we claim that G_i , $i = 1, \dots, 4$ satisfy :

$$(6.5.11) \quad \begin{aligned} G_1^q &= \int_0^t \left| \langle \Delta_q^h R_\Theta^1, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2} \right| dt' \lesssim 2^{-q} \tilde{d}_q \epsilon \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\epsilon \Psi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\ &+ 2^{-q} \tilde{d}_q \left(\|u_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Psi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|b_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right) \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \\ &+ 2^{-q} \tilde{d}_q \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}^2 + 2^{-q} \tilde{d}_q \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Phi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

(6.5.12)

$$\begin{aligned}
G_2^q &= \int_0^t \left| \langle \Delta_q^h R_\Theta^2, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt' \\
&\lesssim 2^{-q} \tilde{d}_q \left\{ \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} + \epsilon^2 \|\Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \left(\|\Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} + \|\Phi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \right) \right. \\
&\quad + \|\Phi_\Theta^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} + \epsilon^2 \|(\partial_y \Psi_\Theta^2, \epsilon \partial_x \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \left(\|(\partial_t u)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \right) \\
&\quad + \|u_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} + \|u_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \left(\|\partial_y \Psi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) \\
&\quad \left. + \|b_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \left(\|\partial_y \Phi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) + \|b_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} \right\}.
\end{aligned}$$

(6.5.13)

$$\begin{aligned}
G_3^q &= \int_0^t \left| \langle \Delta_q^h R_\Theta^3, \Delta_q^h \Phi_\Theta^1 \rangle_{L^2} \right| dt' \lesssim 2^{-q} \tilde{d}_q \epsilon \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\epsilon \partial_x \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
&\quad + 2^{-q} \tilde{d}_q \left(\|u_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|b_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Psi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right) \|\Phi_\Theta^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \\
&\quad + 2^{-q} \tilde{d}_q \|\Phi_\Theta^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}^2 + 2^{-q} \tilde{d}_q \|\Psi_\Theta^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \|\Phi_\Theta^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}.
\end{aligned}$$

(6.5.14)

$$\begin{aligned}
G_4^q &= \int_0^t \left| \langle \Delta_q^h R_\Theta^4, \Delta_q^h \Phi_\Theta^2 \rangle_{L^2} \right| dt' \\
&\lesssim 2^{-q} \tilde{d}_q \left\{ \|(\Psi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}^2 + \epsilon^2 \|\Phi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \left(\|\Phi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} + \|\Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \right) \right. \\
&\quad + \epsilon^2 \|(\partial_y \Phi_\Theta^2, \epsilon \partial_x \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \left(\|(\partial_t b)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \right) \\
&\quad + \epsilon^2 \|\Phi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \left(\|b_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} + \|u_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \left(\|\partial_y \Phi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) \right) \\
&\quad \left. + \|b_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \left(\|\partial_y \Psi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) + \|u_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} \right\}.
\end{aligned}$$

By virtue of (6.5.8), (6.5.11), (6.5.12), (6.5.13) and (6.5.14), we infer

$$\begin{aligned}
(6.5.15) \quad \sum_{i=1}^2 G_i^q &= \sum_{i=1}^4 \int_0^t \left| \langle \Delta_q^h R_\Theta^i, \Delta_q^h \Psi_\Theta^i \rangle_{L^2} \right| dt' \\
&\lesssim \tilde{d}_q 2^{-q} \left(M \epsilon \|(\epsilon \partial_x (\Psi_\Theta^1, \epsilon \Psi_\Theta^2), \epsilon \partial_y \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right. \\
&\quad + M^{\frac{1}{2}} \|\partial_y (\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \\
&\quad + M^{\frac{3}{2}} \epsilon \|\epsilon \Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} + \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}^2 \\
&\quad + \|(\epsilon \Psi_\Theta^2, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}^2 + \|(\Psi_\Theta^1, \Phi_\Theta^1)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}^2 \\
&\quad \left. + M^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} + M^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \|\partial_y \Phi_\Theta^2\|_{\tilde{L}_t(\mathcal{B}^{\frac{1}{2}})} \|\epsilon \Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \right).
\end{aligned}$$

Multiplying the above inequality (6.5.9) and (6.5.10) by 2^q , and summing the obtained inequalities with respect to $q \in \mathbb{Z}$, we come to

$$\begin{aligned}
&\frac{1}{2} \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)(t)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \mu^{\frac{1}{2}} \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} + \mu^{\frac{1}{2}} \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \\
&\quad + \frac{1}{2} \|\partial_y (\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|(\Psi_\Theta^1, \epsilon \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \frac{1}{2} \|\partial_y (\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon \|(\Phi_\Theta^1, \epsilon \Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|e^{a|D_x|}(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + C\|e^{a|D_x|}(b_0^\epsilon - b_0, \epsilon(c_0^\epsilon - c_0))\|_{\mathcal{B}^{\frac{1}{2}}} \\
(6.5.16) \quad &+ C\left(M\epsilon\|(\epsilon\partial_x(\Psi_\Theta^1, \epsilon\Psi_\Theta^2), \epsilon\partial_y\Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \right. \\
&+ M^{\frac{1}{2}}\|\partial_y(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}\|(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \\
&+ M^{\frac{3}{2}}\epsilon\|\epsilon\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \|(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2 \\
&+ \|(\epsilon\Psi_\Theta^2, \epsilon\Phi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2 + \|(\Psi_\Theta^1, \Phi_\Theta^1)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2 \\
&\left. + M^{\frac{1}{2}}\|\partial_y\Phi_\Theta^1\|_{\tilde{L}_t(\mathcal{B}^{\frac{1}{2}})}\|(\Psi_\Theta^1)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + M^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\|\partial_y\Phi_\Theta^2\|_{\tilde{L}_t(\mathcal{B}^{\frac{1}{2}})}\|(\epsilon\Psi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \right).
\end{aligned}$$

Young's inequality leads to

$$\begin{aligned}
&\frac{1}{2}\|(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|(\Phi_\Theta^1, \epsilon\Phi_\Theta^2)(t)\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \mu^{\frac{1}{2}}\|(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \mu^{\frac{1}{2}}\|(\Phi_\Theta^1, \epsilon\Phi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \\
(6.5.17) \quad &+ \frac{1}{2}\|\partial_y(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon\|(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \frac{1}{2}\|\partial_y(\Phi_\Theta^1, \epsilon\Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \epsilon\|(\Phi_\Theta^1, \epsilon\Phi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \\
&\leq C\|e^{a|D_x|}(u_0^\epsilon - u_0, \epsilon(v_0^\epsilon - v_0))\|_{\mathcal{B}^{\frac{1}{2}}} + C\|e^{a|D_x|}(b_0^\epsilon - b_0, \epsilon(c_0^\epsilon - c_0))\|_{\mathcal{B}^{\frac{1}{2}}} \\
&+ CM\left(\epsilon + \|(\Psi_\Theta^1, \epsilon\Psi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \|(\Phi_\Theta^1, \epsilon\Phi_\Theta^2)\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \right).
\end{aligned}$$

Then by taking $\mu = CM$, we can complete the proof Theorem 6.2.3. \square

Lemma 6.5.1. *We use the same assertion used in Theorem 6.2.3, and the result from Remark 6.1.1 we obtain the convergence of $\Phi^{2,\epsilon} = c^\epsilon - c$ to 0 when $\epsilon \rightarrow 0$.*

Proof. of Estimate (6.5.11). According to (6.5.3), we write

$$\begin{aligned}
R_\Theta^1 &= (\epsilon^2\partial_x^2u - (u^\epsilon\partial_xu^\epsilon - u\partial_xu) - (v^\epsilon\partial_yu^\epsilon - v\partial_yu) + (b^\epsilon\partial_xb^\epsilon - b\partial_xb) + (c^\epsilon\partial_yb^\epsilon - c\partial_yb))_\Theta \\
&= \epsilon^2\partial_x^2u_\Theta - ((u^\epsilon\partial_x(u^\epsilon - u) + (u^\epsilon - u)\partial_xu) - (v^\epsilon\partial_y(u^\epsilon - u) + (v^\epsilon - v)\partial_yu))_\Theta \\
&\quad + ((b^\epsilon\partial_x(b^\epsilon - b) + (b^\epsilon - b)\partial_xb) - (c^\epsilon\partial_y(b^\epsilon - b) + (c^\epsilon - c)\partial_yb))_\Theta \\
R_\Theta^1 &= Q_\Theta^1 + ((b^\epsilon\partial_x\Phi^1 + \Phi^1\partial_xb) - (c^\epsilon\partial_y\Phi^1 + \Phi^2\partial_yb))_\Theta.
\end{aligned}$$

We have already bounded $\int_0^t |\langle \Delta_q^h Q_\Theta^1, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2}| dt'$ (see the proof of the estimate (4.76) in [4]), so

$$\begin{aligned}
(6.5.18) \quad &\int_0^t |\langle \Delta_q^h Q_\Theta^1, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2}| dt' \lesssim d_q^2 2^{-q} \left(\epsilon \|\partial_y u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \| \epsilon w_\varphi^1 \|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right. \\
&\quad \left. + \|u_\varphi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2 \right).
\end{aligned}$$

So, we still have to get the estimate of

$$\int_0^t |\langle \Delta_q^h (b^\epsilon\partial_x\Phi^1 + \Phi^1\partial_xb)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' \quad \text{and} \quad \int_0^t |\langle \Delta_q^h (c^\epsilon\partial_y\Phi^1 + \Phi^2\partial_yb)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt'.$$

Then, we start by giving the estimate of the term $\int_0^t |\langle \Delta_q^h (b^\epsilon\partial_x\Phi^1 + \Phi^1\partial_xb)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt'$.

It follow from the lemma 6.3.1 that

$$(6.5.19) \quad \int_0^t |\langle \Delta_q^h (b^\epsilon\partial_x\Phi^1)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' \leq C d_q^2 2^{-q} \|\Phi_\Theta\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \|\Psi_\Theta\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

We note

$$I_1^q = \int_0^t |\langle \Phi^1 \partial_x b \rangle_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt',$$

by applying Bony's decomposition (6.2.2) for the horizontal variable to $\Phi^1 \partial_x b$ we obtain

$$\Phi^1 \partial_x b = T_{\Phi^1}^h \partial_x b + T_{\partial_x b}^h \Phi^1 + R^h(\partial_x b, \Phi^1).$$

and then, we have the following bound

$$I_1^q = \int_0^t \left| \langle \Delta_q^h (T_{\Phi^1}^h \partial_x b + T_{\partial_x b}^h \Phi^1 + R^h(\partial_x b, \Phi^1)) \rangle_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2} \right| dt' \leq I_{1,1}^q + I_{1,2}^q + I_{1,3}^q,$$

with

$$\begin{aligned} I_{1,2}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\partial_x b}^h \Phi^1) \rangle_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2} \right| dt', \\ I_{1,1}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\Phi^1}^h \partial_x b) \rangle_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2} \right| dt', \\ I_{1,3}^q &= \int_0^t \left| \langle \Delta_q^h (R^h(\partial_x b, \Phi^1)) \rangle_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle_{L^2} \right| dt'. \end{aligned}$$

Using the support properties given in [[18], Proposition 2.10] and the definition of $T_{\Phi^1}^h \partial_x b$, we have

$$\begin{aligned} I_{1,1}^q &\leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h \Phi_\Theta^1\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h \partial_x b_\Theta\|_{L_h^2(L_v^\infty)} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} \\ &\leq \sum_{|q-q'|\leq 4} \int_0^t 2^{\frac{q'}{2}} \|S_{q'-1}^h \Phi_\Theta^1\|_{L^2} \|\Delta_{q'}^h \partial_x b_\Theta\|_{L_h^2(L_v^\infty)} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}. \end{aligned}$$

Since

$$\|\Delta_{q'}^h \partial_x b_\Theta\|_{L_h^2(L_v^\infty)} \lesssim 2^{q'} \|\Delta_{q'}^h b_\Theta\|_{L^2}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y b_\Theta\|_{L^2}^{\frac{1}{2}},$$

then,

$$\begin{aligned} I_{1,1}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\Phi^1}^h \partial_x b) \rangle_\Theta, \Delta_q^h \Psi_\Theta^1 \right| dt' \\ &\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{\frac{q'}{2}} \|S_{q'-1}^h \Phi_\Theta^1\|_{L^2} \|b_\Theta\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{\frac{q'}{2}} \left(\int_0^t \|S_{q'-1}^h \Phi_\Theta^1\|_{L^2}^2 \|b_\Theta\|_{\mathcal{B}^{\frac{3}{2}}}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}}^2 \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q^2 2^{-q} \|b_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)}, \end{aligned}$$

where $\{d_q^2\}$ is a summable sequence, which implies

$$(6.5.20) \quad I_{1,1}^q = \int_0^t \left| \langle \Delta_q^h (T_{\Phi^1}^h \partial_x b) \rangle_\Theta, \Delta_q^h \Psi_\Theta^1 \right| dt' \lesssim d_q^2 2^{-q} \|b_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^1)}.$$

While observing that

$$\|S_{q'-1}^h \partial_x b_\Theta\|_{L^\infty} \leq \sum_{l \leq q-2} 2^{\frac{3l}{2}} \|\Delta_l^h b_\Theta\|_{L^2}^{\frac{1}{2}} \|\Delta_l^h \partial_y b_\Theta\|_{L^2}^{\frac{1}{2}} \lesssim 2^q \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}},$$

so we can deduce

$$\begin{aligned}
I_{1,2}^q &= \int_0^t |\langle \Delta_q^h (T_{\partial_x b}^h \Phi^1)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| \leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h \partial_x b_\Theta\|_{L^\infty} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} \\
&\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{q'} \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} \\
&\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\end{aligned}$$

Using the definition of $\dot{\eta}(t)$ and Definition 6.2.3 we have

$$\left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(\Psi_\Theta^1) \|\Psi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Then,

$$(6.5.21) \quad I_{1,2}^q \lesssim 2^{-q} d_q^2 \|\Psi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Phi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)},$$

where

$$d_q^2 = d_q(\Psi_\Theta^1) \left(\sum_{|q-q'|\leq 4} d_{q'}(\Phi_\Theta^1) \right).$$

In a similar way, we have

$$\begin{aligned}
I_{1,3}^q &= \int_0^t |\langle \Delta_q^h (R^h(\Phi^1, \partial_x b))_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' \lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2} \|\tilde{\Delta}_{q'}^h \partial_x b_\Theta\|_{L_h^2(L_v^\infty)} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t 2^{\frac{q'}{2}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2} \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\end{aligned}$$

Using the definition of $\dot{\eta}(t)$ and Definition 6.2.3 we have

$$\left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(\Psi_\Theta^1) \|\Psi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Then,

$$(6.5.22) \quad I_{1,3}^q \lesssim 2^{-q} d_q^2 \|\Psi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Phi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)},$$

where

$$d_q^2 = d_q(\Psi_\Theta^1) \left(\sum_{|q-q'|\leq 4} d_{q'}(\Phi_\Theta^1) \right).$$

Summing the estimates (6.5.20), (6.5.21) and (6.5.22) we obtain

$$(6.5.23) \quad I_1^q \lesssim d_q^2 2^{-q} \left(\|b_\Theta\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\Phi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)} \right) \|\Psi_\Theta^1\|_{\tilde{L}_{t,\dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Now we move to get the estimate of the term $\int_0^t |\langle \Delta_q^h (c^\epsilon \partial_y \Phi^1 + \Phi^2 \partial_y b)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' = J_1^q + J_2^q$, we start by J_1^q , we recall that $c^\epsilon = c + \Phi^2$, then

$$\begin{aligned} J_2^q &= \int_0^t |\langle \Delta_q^h (c^\epsilon \partial_y \Phi^1)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' \\ &= \int_0^t |\langle \Delta_q^h (\Phi^2 \partial_y \Phi^1)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' + \int_0^t |\langle \Delta_q^h (c \partial_y \Phi^1)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' \\ &= J_{1,1}^q + J_{1,2}^q. \end{aligned}$$

It follow from the lemma 6.3.2 that

$$(6.5.24) \quad J_{1,1}^q = \int_0^t |\langle \Delta_q^h (\Phi^2 \partial_y \Phi^1)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt' \leq C d_q^2 2^{-q} \|\Phi_\Theta\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \|\Psi_\Theta\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

For the second term $J_{1,2}^q$, we apply the Bony's decomposition (6.2.2) for the horizontal variable to $c \partial_y \Phi^1$, we obtain

$$c \partial_y \Phi^1 = T_c^h \partial_y \Phi^1 + T_{\partial_y \Phi^1}^h c + R^h(c, \partial_y \Phi^1),$$

so, we have the following bound

$$J_{1,2}^q = \int_0^t \left| \langle \Delta_q^h (T_c^h \partial_y \Phi^1 + T_{\partial_y \Phi^1}^h c + R^h(c, \partial_y \Phi^1))_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle \right| dt' \leq J_{1,21}^q + J_{1,22}^q + J_{1,23}^q,$$

where

$$\begin{aligned} J_{1,21}^q &= \int_0^t |\langle \Delta_q^h (T_c^h \partial_y \Phi^1)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt', \\ J_{1,22}^q &= \int_0^t |\langle \Delta_q^h (T_{\partial_y \Phi^1}^h c)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt', \\ J_{1,23}^q &= \int_0^t |\langle \Delta_q^h (R^h(c, \partial_y \Phi^1))_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt'. \end{aligned}$$

Using the support properties given in [[18], Proposition 2.10] and the definition of $T_c^h \partial_y \Phi^1$, we have

$$J_{1,21}^q \leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h c_\Theta\|_{L^\infty} \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt'.$$

Due to $\partial_x b + \partial_y c = 0$ and Poincaré inequality, we can write $c(t, x, y) = -\int_0^y \partial_x b(t, x, s) ds$, then we deduce from the lemma 6.2.1 that Since

$$\begin{aligned} \|\Delta_{q'}^h c_\Theta\|_{L^\infty} &\leq \int_0^1 \|\Delta_{q'}^h \partial_x b_\Theta(t, x, s)\|_{L_h^\infty} ds \\ (6.5.25) \quad &\leq 2^{\frac{3q}{2}} \int_0^1 \|\Delta_{q'}^h b_\Theta(t, x, s)\|_{L_h^2} ds \leq 2^{\frac{3q}{2}} \|\Delta_{q'}^h b_\Theta(t, x, s)\|_{L^2}, \end{aligned}$$

then,

$$\|S_{q'-1}^h c_\Theta\|_{L^\infty} \leq 2^{\frac{q'}{2}} \|b_\Theta(t')\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

from which we infer

$$J_{1,21}^q \lesssim \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h c_\Theta\|_{L^\infty} \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt'$$

$$\begin{aligned}
&\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{\frac{q'}{2}} \|b_\Theta(t')\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim \sum_{|q-q'|\leq 4} \int_0^t 2^{\frac{q'}{2}} \|b_\Theta(t')\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L^2} \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim \sum_{|q-q'|\leq 4} 2^{\frac{q'}{2}} \|b_\Theta(t')\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L_t^2(L^2)} \left(\int_0^t \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim d_q^2 2^{-q} \|b_\Theta(t')\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)},
\end{aligned}$$

where $\{\tilde{d}_q\}$ is a summable sequence, which implies

$$(6.5.26) \quad J_{1,21}^q \lesssim d_q^2 2^{-q} \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

While observing that

$$\|\Delta_q^h c_\Theta\|_{L_h^2(L_v^\infty)} \leq d_q \|b_\Theta(t')\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

so we can deduce

$$\begin{aligned}
J_{1,22}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\partial_y \Phi_\Theta^1}^h c)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle \right| dt' \\
&\leq \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h \partial_y \Phi_\Theta^1\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h c_\Theta\|_{L_h^2(L_v^\infty)} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim \sum_{|q-q'|\leq 4} \int_0^t \|S_{q'-1}^h \partial_y \Phi_\Theta^1\|_{L_h^\infty(L_v^2)} d_{q'}(b_\Theta) \|b_\Theta(t')\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim \sum_{|q-q'|\leq 4} \|b_\Theta(t')\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|S_{q'-1}^h \partial_y \Phi_\Theta^1\|_{L_t^2(L_h^\infty(L_v^2))} \left(\int_0^t \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim d_q^2 2^{-q} \|b_\Theta(t')\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.
\end{aligned}$$

Using the definition of $\eta(t)$ and Definition 6.2.3 we have

$$\left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(\Psi_\Theta^1) \|\Psi_\Theta^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

Then,

$$(6.5.27) \quad J_{1,22}^q \lesssim 2^{-q} d_q^2 \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)},$$

where

$$d_q^2 = d_q(\Psi_\Theta^1) \left(\sum_{|q-q'|\leq 4} d_{q'} \right).$$

In a similar way, we have

$$J_{1,23}^q = \int_0^t \left| \langle \Delta_q^h (R^h(c, \partial_y \Phi^1))_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle \right| dt'$$

$$\begin{aligned}
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L^2} \|\tilde{\Delta}_{q'}^h c_\Theta\|_{L_t^2(L_v^\infty)} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|b_\Theta(t')\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta(t')\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \|b_\Theta(t')\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\Delta_{q'}^h \partial_y \Phi_\Theta^1\|_{L_t^2(L^2)} \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\end{aligned}$$

Using the definition of $\dot{\eta}(t)$ and Definition 6.2.3 we have

$$\left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(\Psi_\Theta^1) \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

As a consequence, we arrive at

$$(6.5.28) \quad J_{1,2,3}^q \lesssim d_q^2 2^{-q} \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Summing the estimates (6.5.26), (6.5.27) and (6.5.28) we obtain

$$(6.5.29) \quad J_{1,2}^q \lesssim d_q^2 2^{-q} \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^1\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Now we will get the estimate of the last term $J_2^q = \int_0^t |\langle \Delta_q^h(\Phi^2 \partial_y b)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt'$. For this end, we apply the Bony's decomposition (6.2.2) for the horizontal variable to $\Phi^2 \partial_y b$, we obtain

$$\Phi^2 \partial_y b = T_{\Phi^2}^h \partial_y b + T_{\partial_y b}^h \Phi^2 + R^h(\partial_y b, \Phi^2),$$

so, we have the following bound

$$J_2^q = \int_0^t \left| \langle \Delta_q^h(T_{\Phi^2}^h \partial_y b + T_{\partial_y b}^h \Phi^2 + R^h(\partial_y b, \Phi^2))_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle \right| dt' \leq J_{2,1}^q + J_{2,2}^q + J_{2,3}^q,$$

where

$$\begin{aligned}
J_{2,1}^q &= \int_0^t |\langle \Delta_q^h(T_{\Phi^2}^h \partial_y b)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt', \\
J_{2,2}^q &= \int_0^t |\langle \Delta_q^h(T_{\partial_y b}^h \Phi^2)_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt', \\
J_{2,3}^q &= \int_0^t |\langle \Delta_q^h(R^h(\partial_y b, \Phi^2))_\Theta, \Delta_q^h \Psi_\Theta^1 \rangle| dt'.
\end{aligned}$$

Using the support properties given in [[18], Proposition 2.10] and the definition of $T_{\Phi^2}^h \partial_y b$, we have

$$\begin{aligned}
J_{2,1}^q &\leq \sum_{|q-q'| \leq 4} \int_0^t \|S_{q'-1}^h \Phi_\Theta^2\|_{L^\infty} \|\Delta_{q'}^h \partial_y b_\Theta\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim \sum_{|q-q'| \leq 4} \int_0^t d_{q'} 2^{-\frac{q'}{2}} \|S_{q'-1}^h \Phi_\Theta^2\|_{L^\infty} \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt'.
\end{aligned}$$

Due to $\partial_x \Phi^1 + \partial_y \Phi^2 = 0$ and Poincaré inequality, we can write $\Phi^2(t, x, y) = -\int_0^y \partial_x \Phi^1(t, x, s) ds$, then

we deduce from the lemma 6.2.1 that Since

$$\begin{aligned} \|\Delta_{q'}^h \Phi_\Theta^2\|_{L^\infty} &\leq \int_0^1 \|\Delta_{q'}^h \partial_x \Phi_\Theta^1(t, x, s)\|_{L_h^\infty} ds \\ &\leq 2^{\frac{3q}{2}} \int_0^1 \|\Delta_{q'}^h \Phi_\Theta^1(t, x, s)\|_{L_h^2} ds \leq 2^{\frac{3q}{2}} \|\Delta_{q'}^h \Phi_\Theta^1(t, x, s)\|_{L^2}, \end{aligned}$$

from which, we infer

$$\begin{aligned} &\left(\int_0^t \|S_{q'-1}^h \Phi_\Theta^2\|_{L^\infty}^2 \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\ &\sum_{l \leq q'-2} 2^{\frac{3l}{2}} \left(\int_0^t \|\Delta_l^h \Phi_\Theta^1\|_{L^2}^2 \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{l \leq q'-2} d_l 2^{\frac{l}{2}} \|\Phi^1(t)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \\ &\lesssim 2^{\frac{q'}{2}} d_{q'}(\Phi^1) \|\Phi^1(t)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

We replace in $J_{2,1}^q$, we obtain

$$(6.5.30) \quad J_{2,1}^q \lesssim d_q^2 2^{-q} \|\Phi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Now let get the estimate of the second term $J_{2,2}$

$$\begin{aligned} J_{2,2}^q &= \int_0^t \left| \left\langle \Delta_q^h (T_{\partial_y b}^h \Phi^2)_\Theta, \Delta_q^h \Psi_\Theta^1 \right\rangle \right| dt' \\ &\leq \sum_{|q-q'| \leq 4} \int_0^t \|S_{q'-1}^h \partial_y b_\Theta\|_{L_h^\infty(L_v^2)} \|\Delta_{q'}^h \Phi_\Theta^2\|_{L_h^2(L_v^\infty)} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'| \leq 4} 2^{q'} \int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'| \leq 4} 2^{q'} \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

Using the definition of $\dot{\eta}(t)$ and Definition 6.2.3 we have

$$\left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(\Psi_\Theta^1) \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Then,

$$(6.5.31) \quad J_{2,2}^q \lesssim 2^{-q} d_q^2 \|\Phi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)},$$

where

$$d_q^2 = d_q(\Psi_\Theta^1) \left(\sum_{|q-q'| \leq 4} d_{q'}(\Phi_\Theta^1) \right).$$

In a similar way, we have

$$J_{2,3}^q = \int_0^t \left| \left\langle \Delta_q^h (R^h(\Phi^2, \partial_y b))_\Theta, \Delta_q^h \Psi_\Theta^1 \right\rangle \right| dt'$$

$$\begin{aligned}
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} \int_0^t \|\tilde{\Delta}_{q'}^h \partial_y b_\Theta\|_{L^2} \|\Delta_{q'}^h \Phi_\Theta^2\|_{L_h^2(L_v^\infty)} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2} \|\Delta_q^h \Psi_\Theta^1\|_{L^2} dt' \\
&\lesssim 2^{\frac{q}{2}} \sum_{q' \geq q-3} 2^{\frac{q'}{2}} \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Phi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\end{aligned}$$

Using the definition of $\dot{\eta}(t)$ and Definition 6.2.3 we have

$$\left(\int_0^t \|\partial_y b_\Theta\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h \Psi_\Theta^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \lesssim 2^{-q} d_q(\Psi_\Theta^1) \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

As a consequence, we arrive at

$$(6.5.32) \quad J_{2,3}^q \lesssim 2^{-q} d_q^2 \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Phi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

Summing the estimates (6.5.30), (6.5.31) and (6.5.32) we obtain

$$(6.5.33) \quad J_2^q \lesssim d_q^2 2^{-q} \|\Psi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Phi_\Theta^1\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)}.$$

By summing all the resulting estimates we obtain the proof of the estimate (6.5.11). \square

We will now study the second term G_2^q .

Proof. of the estimate (6.5.12). Using the definition of R_Θ^2 , we write

$$\begin{aligned}
R_\Theta^2 &= -\epsilon^2 (\partial_t v - \epsilon^2 \partial_x^2 v - \partial_y^2 v + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_y v^\epsilon + b^\epsilon \partial_x c^\epsilon + c^\epsilon \partial_y c^\epsilon)_\Theta \\
&= Q_\Theta^2 + (b^\epsilon \partial_x c^\epsilon)_\Theta + (c^\epsilon \partial_y c^\epsilon)_\Theta.
\end{aligned}$$

We have already do the proof of the estimate $\int_0^t \left| \langle \Delta_q^h Q_\Theta^2, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt'$ (see the proof of the estimate (4.79) in [4]), then

$$\begin{aligned}
(6.5.34) \quad &\int_0^t \left| \langle \Delta_q^h Q_\Theta^2, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt' \\
&\lesssim d_q^2 2^{-q} \left(\|\Psi_\Theta^1, \epsilon \Psi_\Theta^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} + \epsilon^2 \|\Psi_\Theta^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \|\Psi_\Theta^2\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^1)} \right. \\
&\quad + \epsilon^2 \|(\partial_y \Psi_\Theta^2, \epsilon \partial_x \Psi_\Theta^2)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \left(\|(\partial_t u)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} + \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{5}{2}})} \right) \\
&\quad \left. + \|u_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} + \|u_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} (\|\partial_y \Psi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})}) \right).
\end{aligned}$$

So, we still have to get the estimate of

$$\begin{aligned}
J_3^q &= \int_0^t \left| \langle \Delta_q^h (b^\epsilon \partial_x c^\epsilon)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle \right| dt', \\
J_4^q &= \int_0^t \left| \langle \Delta_q^h (c^\epsilon \partial_y c^\epsilon)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle \right| dt'.
\end{aligned}$$

Then, we start by giving the estimate of the term $J_3^q = \int_0^t \left| \langle \Delta_q^h (b^\epsilon \partial_x c^\epsilon)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle \right| dt'$. We write

$$J_3^q \leq \epsilon^2 (J_{31}^q + J_{32}^q),$$

where

$$\begin{aligned} J_{31}^q &= \int_0^t \left| \langle \Delta_q^h (b^\epsilon \partial_x \Phi^2)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt' \\ J_{32}^q &= \int_0^t \left| \langle \Delta_q^h (b^\epsilon \partial_x c)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt'. \end{aligned}$$

It follows from Lemma 6.3.1 that

$$J_{31}^q \lesssim d_q^2 2^{-q} \|\epsilon \Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)} \|\epsilon \Phi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}.$$

For the second term, Bony's decomposition for the horizontal variable gives

$$J_{32}^q = \int_0^t \left| \langle \Delta_q^h (b^\epsilon \partial_x c)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt' \leq J_{321}^q + J_{322}^q + J_{323}^q,$$

with

$$\begin{aligned} J_{321}^q &= \int_0^t \left| \langle \Delta_q^h (T_{b^\epsilon}^h \partial_x c)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt', \\ J_{322}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\partial_x c}^h b^\epsilon)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt', \\ J_{323}^q &= \int_0^t \left| \langle \Delta_q^h (R^h(b^\epsilon, \partial_x c))_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt'. \end{aligned}$$

Using the estimate

$$\|S_{q'-1}^h b_\Theta^\epsilon\|_{L^\infty} \lesssim \|b_\Theta^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}},$$

and the relation (6.3.1), we have

$$\begin{aligned} J_{321}^q &= \int_0^t \left| \langle \Delta_q^h (T_{b^\epsilon}^h \partial_x c)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h b_\Theta^\epsilon\|_{L^\infty} \|\Delta_{q'}^h \partial_x c_\Theta\|_{L^2} \|\Delta_q^h \Psi_\Theta^2\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|b_\Theta^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_y b_\Theta^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} 2^{2q'} \|\Delta_{q'}^h b_\Theta\|_{L^2} \|\Delta_q^h \Psi_\Theta^2\|_{L^2} dt' \\ &\lesssim d_q^2 2^{-q} \|b_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|\Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

So, we obtain

$$(6.5.35) \quad J_{321}^q \lesssim d_q^2 2^{-q} \|b_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|\Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}.$$

In a similar way, the fact that

$$\|S_{q'-1}^h \partial_x c_\Theta\|_{L^\infty} \lesssim \int_0^y \|S_{q'-1}^h \partial_x (\partial_x b_\Theta(t, x, s))\|_{L^\infty} ds \lesssim 2^{\frac{q'}{2}} \|\partial_y b_\Theta\|_{\mathcal{B}^2},$$

leads to

$$(6.5.36) \quad J_{322}^q = \int_0^t \left| \langle \Delta_q^h (T_{\partial_x c}^h b^\epsilon)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-q} \|b_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|\Psi_\Theta^2\|_{\tilde{L}_{t, \eta(t)}^2(\mathcal{B}^1)}.$$

For the last term J_{323}^q , we have

$$(6.5.37) \quad J_{323}^q = \int_0^t \left| \langle \Delta_q^h (R^h(\partial_x c, b^\epsilon))_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle_{L^2} \right| \lesssim d_q^2 2^{-q} \|b_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

Summing inequalities (6.5.35), (6.5.36) and (6.5.37) finally yields

$$J_{32}^q \lesssim d_q^2 2^{-q} \|b_\Theta^\epsilon\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^2)} \|\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

To end, we need to estimate the last term $J_4^q = \int_0^t \langle \Delta_q^h (c^\epsilon \partial_y c^\epsilon)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle dt'$. We first note that

$$c^\epsilon \partial_y c^\epsilon = c \partial_y c + \Phi^2 \partial_y \Phi^2 + c \partial_y \Phi^2 + \Phi^2 \partial_y c.$$

We first deduce from the lemma (6.4.1) that

$$\epsilon^2 \int_0^t |\langle \Delta_q^h (\Phi^2 \partial_y \Phi^2)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle| dt' \lesssim d_q^2 2^{-q} \|\Phi_\Theta^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \|\epsilon \Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

We deduce also from the proof of (6.5.29) that

$$\begin{aligned} \int_0^t |\langle \Delta_q^h (c \partial_y c)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle| dt' &\lesssim d_q^2 2^{-q} \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y c_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \\ &\lesssim d_q^2 2^{-q} \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \|\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}, \end{aligned}$$

and

$$\int_0^t |\langle \Delta_q^h (c \partial_y \Phi^2)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle| dt' \lesssim 2^{-q} d_q^2 \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \|\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}.$$

And (6.5.23) ensure that

$$\begin{aligned} \int_0^t |\langle \Delta_q^h (\Phi^2 \partial_x b)_\Theta, \Delta_q^h \Psi_\Theta^2 \rangle| dt' \\ \lesssim d_q^2 2^{-q} \left(\|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_y \Phi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\Phi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} \right) \|\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}. \end{aligned}$$

As a result, we find

$$(6.5.38) \quad \begin{aligned} \epsilon^2 J_4^q &\lesssim d_q^2 2^{-q} \left(\|\Phi_\Theta^1, \epsilon \Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2 \right. \\ &\quad \left. + \epsilon^2 \|b_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \left(\|\partial_y b_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} + \|\partial_y \Phi_\Theta^2\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{3}{2}})} \right) \|\Psi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)} + \|\epsilon \Phi_\Theta^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^1)}^2 \right). \end{aligned}$$

By summing all the resulting estimates we obtain the proof of the estimate (6.5.12). \square

For the proof of the estimates G_3^q and G_4^q it is the same as for G_1^q and G_2^q .

Chapter 7

MHD Hyperbolic

In this chapter we present the results of the following paper:

N. Aarach, F. De Anna, M. Paicu and N. Zhu, On the role of the displacement current and the Cattaneo's law on boundary layers of plasma, arXiv:2210.12994

7.1 Introduction

The mathematical study of electrically conducting fluids and hot plasma has received for many years numerous investigations. The understanding of the underlying equations (MHD equations or Navier-Stokes-Maxwell equations) has provided a fascinating number of implications, both on technological processes and physical experiments. Among the remarkable variety of applications, MHD flows are ubiquitous in contexts like astronomy (hydrodynamics of plasma in neutron stars and white dwarfs), nuclear fusion reactors (self-cooled liquid metal blankets) and metallurgic (liquid metal stirring).

In this paper we are interested in deriving and analysing a family of partial differential equations that mathematically account for boundary layers of plasma and electrically conducting fluids, when the corresponding characteristic speed is of relativistic order. This specific hydrodynamics near a wall surface has been a topic of constant interest in astrophysics. For instance, these boundary layers are expected to be dominant sources of X-ray production in neutron stars [123], gravitational radiation [104] and magnetic reconnection [129].

The mathematical treatment of boundary layers in electrically conducting fluids has a long history, which dates back to the pioneering work of Hartmann [69]. Hartmann studied a duct flow of a viscous electrically-conducting fluid under the influence of a transverse magnetic field. Oriented at the right angle, the magnetic field produced additional viscosity, separating the channel into two main regions, boundary-layer region (Hartmann layers) and central core region.

Afterwards, many theoretical investigations and experiments have been developed around this theory, most of them under the assumption that the electromagnetic variations of plasma are non-relativistic (i.e. the characteristic speed of plasma has magnitude consistently lower than the speed of light). This hypothesis relaxes several terms of the Maxwell's equations, in particular, it neglects the so-called displacement current in the Ampere's law (a source of the magnetic field related to the ratio between the characteristic speed of plasma and the speed of light).

The lack of the displacement current may or may not be satisfactory, depending on the modelling context. In neutron stars, for instance, strong time-dependent electric field could develop, when the plasma density falls below a critical value [84]. Thus, the associated displacement current makes up for the deficit of the plasma density and plays a major role in the evolution of the magnetic field.

To the best of our knowledge, it still remains an open problem to mathematically understand boundary layers of plasma, whose magnetic field in the Navier-Stokes-Maxwell's equations is affected by the displacement current. This chapter is therefore a first mathematical attempt to address this issue. In details, we derive and analyse the following system of PDEs (written in dimensionless form):

$$(7.1.1) \quad \begin{cases} \mathbb{J}\partial_t^2 u + \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = \mathbb{H}^2(b_1 b_2 v - u b_2^2 - b_2 e) & (t, x, y) \in (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_y p = \mathbb{H}^2(b_1 b_2 u - b_1^2 v + b_1 e) & (0, T) \times \mathbb{R} \times (0, 1), \\ \frac{\kappa}{\text{Pr}_m} \partial_t^2 b_1 + \partial_t b_1 + u\partial_x b_1 + v\partial_y b_1 - \frac{1}{\text{Pr}_m} \partial_y^2 b_1 = b_1 \partial_x u + b_2 \partial_y u & (0, T) \times \mathbb{R} \times (0, 1), \\ \frac{\kappa}{\text{Pr}_m} \partial_t^2 b_2 + \partial_t b_2 + u\partial_x b_2 + v\partial_y b_2 - \frac{1}{\text{Pr}_m} \partial_y^2 b_2 = b_1 \partial_x v + b_2 \partial_y v & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_t b_1 + \partial_y e = 0 \quad \text{and} \quad \partial_t b_2 - \partial_x e = 0 & (0, T) \times \mathbb{R} \times (0, 1), \\ \partial_x u + \partial_y v = 0 \quad \text{and} \quad \partial_x b_1 + \partial_y b_2 = 0 & (0, T) \times \mathbb{R} \times (0, 1), \end{cases}$$

coupled with initial and boundary conditions

$$(7.1.2) \quad (\text{IC}) \begin{cases} (u, b_1, b_2)|_{t=0} = (\bar{u}, \bar{b}_1, \bar{b}_2) & \mathbb{R} \times (0, 1), \\ (\partial_t u, \partial_t b_1, \partial_t b_2)|_{t=0} = (\tilde{u}, \tilde{b}_1, \tilde{b}_2) & \mathbb{R} \times (0, 1), \end{cases} \quad (\text{BC}) \begin{cases} (u, b_1, b_2, e)|_{y=0} = (0, 0, 0, 0) & (0, T) \times \mathbb{R}, \\ (u, b_1, b_2, e)|_{y=1} = (0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{e}) & (0, T) \times \mathbb{R}. \end{cases}$$

7.1.1 Overview of System (7.1.1)

For the sake of a clear presentation, we consider here a rather simple geometry, as well as simple boundary conditions. We assume indeed that the conducting fluid is restricted to the whole half space, in other words System (7.1.1) represents the behaviour of the fluid on a thin layer near a (flat) boundary line ($y = 0$). The velocity field $(u, v)^T$ satisfies no-slip boundary conditions, while the magnetic field $(b_1, b_2)^T$ and the electric-field intensity e are assumed constant on the boundary (a scenario which is typical when the surrounding medium is an insulator).

All state variables (u, v, b_1, b_2, e) in (7.1.1) depend on time $t \in (0, T)$ and space $(x, y) \in \mathbb{R} \times (0, 1)$. The vector fields $(u, v)^T \in \mathbb{R}^2$ and $(b_1, b_2)^T \in \mathbb{R}^2$ are divergence free and stand for the velocity and magnetic fields of plasma, respectively. The electric field assumes size $e \in \mathbb{R}$ and is perpendicular to the plane containing the plasma.

All constants \mathbb{H} , κ , Pr_m and \mathbb{J} are positive and depend on standard dimensionless parameters of magnetohydrodynamics. More precisely, \mathbb{H} stands for the asymptotic of the ratio between the Hartmann number $\text{Ha} > 0$ and the Reynolds number Re , as $\text{Re} \rightarrow \infty$. The magnetic Prandtl number $\text{Pr}_m > 0$ is assumed in this work constant and represents the ratio between the Reynolds number Re and the magnetic Reynolds number Re_m . Hence viscous and magnetic diffusions are proportional, a regime typical of heavier white dwarfs, in which Reynolds and magnetic Reynolds numbers range between 10^{14} to 10^{15} (cf. Section 2 in [76]). Furthermore, the proportionality between Ha and Re_m reflects a threshold for the initiation of magnetic advection and subsequent reconnection (cf. for instance Section 3.3 in [103], for the binary star AE Aquarii).

7.1.2 Novelities of the model

Although System (7.1.1) differs intrinsically from previous models (such as the Prandtl-MHD equations, cf. (7.1.3)), the major novelties reside in particular within the terms $\kappa/\text{Pr}_m \partial_{tt}^2 b_1$ and $\kappa/\text{Pr}_m \partial_{tt}^2 b_2$ for the equations of b_1 and b_2 (due to the displacement current [50]), as well as within the term $\mathbb{J}\partial_t^2 u$

in the equation for u . The role of the underlying constant $\kappa > 0$ is exploited in details in Section 7.2 (cf. Theorem 7.2.1 and Theorem 7.2.2), it relates however to the ratio $(U_0/c)^2$ between the characteristic speed of plasma $U_0 > 0$ and the speed of light “ c ”, for high value of $\text{Re} \gg 1$.

The constant $\mathbb{J} \geq 0$ together with the second time derivative $\partial_t^2 u$ are derived from a well-known hyperbolic extension of the Navier-Stokes equations, a model which is known as Navier-Stokes with Cattaneo’s law (cf. [6, 16, 38, 108, 111, 122]). This extension was first proposed in fluid-dynamics by Carrassi and Morro [22] (inspired by the original work of Cattaneo in the study of heat diffusion [24, 25]). As most compelling reason to introduce this term, a positive value of $\mathbb{J} > 0$ avoids indeed an infinite speed of propagation of u , which would be quite unnatural when considering the evolution of fluids at large scale.

The general concern of this paper is twofold. First, we aim to derive model (7.1.1) from suitable asymptotics of the full Navier-Stokes-Maxwell’s equations under Cattaneo’s law (cf. Section 7.1.4).

Secondly we aim at studying the underlying well-posedness theory and prove the existence of global-in-time smooth solutions for system (7.1.1), by considering initial data that are small and highly regular (more precisely analytic in the horizontal coordinate $x \in \mathbb{R}$, cf. Section 7.1.5).

7.1.3 A brief overview of the analysis of MHD boundary layers

The analysis of boundary layers in magnetohydrodynamics have received from the mathematical community numerous investigations during the past decades. To the best of our knowledge, however, the derivation of system (7.1.1) is new, hence there has not been related analytical results, up to now. The majority of the results concerns classical MHD-equations and the underlying boundary-layer theory, in which the displacement current is indeed neglected. In this paragraph we shall hence focus on the various contributions that dealt with certain equations that shear at least similarities with system (7.1.1).

One of the systems that has received most attention was provided by Gérard-Varet and Prestipino in [63]. Omitting the notation of the several dimensionless parameters, the equations read as follows:

$$(7.1.3) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u + \partial_x p = b_1 \partial_x b_1 + b_2 \partial_y b_1, \\ \partial_y p = 0, \\ \partial_t b_1 + u \partial_x b_1 + v \partial_y b_1 - \partial_y^2 b_1 = b_1 \partial_x u + b_2 \partial_y u, \\ \partial_t b_2 + u \partial_x b_2 + v \partial_y b_2 - \partial_y^2 b_2 = b_1 \partial_x v + b_2 \partial_y v, \\ \partial_x u + \partial_y v = 0, \quad \partial_x b_1 + \partial_y b_2 = 0. \end{cases}$$

The authors derived this system as boundary asymptotic of the classical MHD equations, under a stringent regime of the coupling parameters (which we also assume in this paper): the Hartmann number Ha , the Reynolds number Re and the magnetic Reynolds number Re_m were all proportional and assumed high values.

The major differences between Systems (7.1.1) and (7.1.3) can be recognized with the forcing term of the momentum equation in u , a non-constant pressure in the vertical variable due to $\partial_y p \neq 0$ and the second time derivatives $\partial_t^2 u$, $\partial_t^2 b_1$ and $\partial_t^2 b_2$ on the equations for the magnetic field (which are due to the displacement current, cf. also Remark 7.2.1).

System (7.1.3) retains most terms of the original MHD equations and it reduces to the widespread Prandtl equations for purely hydrodynamic flows, when the magnetic field (b_1, b_2) is null. Among the mathematical community, there has been hence an increasing interest to transfer well-known analytical results of the Prandtl theory to the corresponding Prandtl-MHD equations [63, 95, 99, 116]: existence and

uniqueness of solutions, regularity analysis (Sobolev, analytic, Gevrey), stability of certain equilibria (such as shear flows).

At a first glance, one may think that System (7.1.3) shall satisfy reduced (or at least similar) properties than classical Prandtl. However, in [63], Gérard-Varet and Prestipino overturned this statement, when dealing with the stability of (7.1.3) around certain equilibria. The authors showed indeed how System (7.1.3) is linearly stable around to a suitable family of shear flows in which both plasma's velocity and magnetic field are parallel to the bounding flat surface:

$$(u(t, x, y), v(t, x, y), b_1(t, x, y), b_2(t, x, y)) = (U_s(y), 0, 1, 0),$$

This stability holds already at the level of Sobolev regularity, a fact that is in sharp contrast with the Sobolev instability of the classical Prandtl equations (cf. [59, 61]). Hence, in terms of Sobolev stability, Prandtl-MHD has better properties than Prandtl. To better understand this unusual characteristic, one shall first recall that the major difficulties of the Prandtl equations reside in the convective term $v\partial_y u$ (this vertical component v has a lower regularity in the tangential variable than u and is determined by the divergence free condition $\partial_x u + \partial_y v = 0$). However, when dealing with the Prandtl-MHD equations (7.1.3), one can get rid of this “bad term”, by introducing a new modified velocity field $\tilde{u} = u + U'_s \phi$, where ϕ stands for the potential generating the magnetic field (i.e. $b_1 = \partial_y \phi$ and $b_2 = -\partial_x \phi$). This mathematical artifact has clarified certain observations in physics, in particular the fact that a magnetic field has a stabilizing effect on the flow of plasma and provides therefore a mechanism for containment.

Away from shear flows, Liu, Xie and Tong proposed in [96] a generalisation of this Sobolev stability, when dealing with the full nonlinear version of equations (7.1.3). The authors showed indeed that a modified velocity similar to \tilde{u} could still be defined, as long as the tangential component b_1 of the magnetic field never vanishes (a condition known as “nondegeneracy of b_1 ”). This result was local in time and was extended globally by Liu and Zhang in [99], under a smallness condition on the initial data.

After [96], a remaining open problem was to understand if the nondegeneracy of b_1 was somehow necessary in order to recover the mentioned Sobolev stability. The same authors Liu, Xie and Tong in [97] provided a surprisingly positive answer to this dilemma: when linearising equations (7.1.3) around a family of shear flows of the form

$$(u(t, x, y), v(t, x, y), b_1(t, x, y), b_2(t, x, y)) = (U_s(t, y), 0, B_s(t, y), 0),$$

in which B_s vanishes (together with some of its derivatives), it was shown that the corresponding system is indeed ill-posed in Sobolev spaces.

This result in [97] opened a further variety of questions, which regarded in particular the following aspects:

- If the asymptotic limit ($\text{Re}, \text{Re}_m \rightarrow \infty$) of the Prandtl-MHD equations (7.1.3) is well posed in Sobolev spaces, does a formal mathematical expansion reveal the corresponding boundary layers within the solutions of the original MHD equations (for high values of $\text{Re} \gg 1$ and $\text{Re}_m \gg 1$)?
- If the nondegeneracy condition $b_1 \neq 0$ is necessary for the Sobolev stability, can one consider higher regularities (such as Gevrey), in order to relax this constraint?

Liu, Xie and Yang in [96] provided a positive answer to the first question, as long as the tangential magnetic component $b_1 \geq \delta > 0$ remains strictly positive. With some additional technical condition on the initial data, the authors showed that the differences between the smooth solutions of the original MHD equations and the boundary layers (which depend on $\text{Re} > 0$ and $\text{Re}_m > 0$) converge to the smooth solution of the limit case $\text{Re}, \text{Re}_m \rightarrow \infty$. The convergence is indeed uniform in L^∞ (both in time and space) because of the Sobolev stability.

For what concerns higher regularities, Li and Yang addressed in [87] solutions with Gevrey regularities in “ x ” of order $3/2$ (a function space between analytic and Sobolev functions). Gevrey functions have significantly influenced and impacted the analysis of the Prandtl equations, since they still allow to cope with smooth test functions (in contrast with analytic regularity). Without any assumption on the tangent component b_1 of the magnetic field, Li and Yang showed in [87] that initial data in Gevrey $3/2$ generate local-in-time smooth solutions of (7.1.3). It still remains an interesting open problem to establish if this result is optimal or if one can further enlarge the regularity (for instance towards Gevrey 2, the optimal value of Prandtl, cf. [49]).

7.1.4 Statement of our formal results

The well-posedness results of classical Navier-Stokes-Maxwell equations without Cattaneo’s law can be found in [14, 58, 82, 102]. The approach of our modelling is first to introduce a suitable form of the Navier-Stokes-Maxwell equations with Cattaneo’s law (cf. System (7.2.1) and its dimensionless form (7.2.5)) and secondly to examine the asymptotics when some related dimensionless parameters converge to ∞ (that are indeed popular in MHD). More precisely, we derive System (7.2.1) for high values of the Reynolds number Re , the magnetic Reynolds numbers Re_m and the Hartmann number Ha (for details, cf. Sections 7.2.2-7.2.4, as well as Theorem 7.1.1 and Theorem 7.1.2, below).

The nature of the derived equations depends on certain hypotheses on the characteristic speed $U_0 > 0$ of Plasma (we refer to (7.2.5) for the explicit relation between the starting equations and U_0). We establish indeed boundary layers with thickness of two types: Prandtl or Hartmann.

To clarify our result, we shall first recall that in magnetohydrodynamics several types of boundary layers can occur, depending on the angle of orientation of the magnetic field boundary. Among the most relevant, Prandtl and Hartmann layers stand out, since they also differ on their thickness. Conventional Prandtl layers are indeed purely hydrodynamic and are characterised by a length, which is proportional to $1/\sqrt{\text{Re}}$. On the other hand, Hartmann layers are mainly attributable to the magnetic field, having a thickness inversely proportional to the Hartmann number Ha . Depending on the magnitude of an imposed magnetic field, Hartmann layers may be as thin as one desires, thus the velocity field of plasma usually increases much more rapidly over a short distance from the boundary.

Our first result in Theorem 7.1.1 shows that, when the characteristic speed of Plasma U_0 is proportional to the Reynolds number Re , then System (7.1.1) stands for a boundary layer within a region of thickness $1/\sqrt{\text{Re}}$ (Prandtl).

Theorem 7.1.1. *Consider the Navier-Stokes-Maxwell equations with Cattaneo’s law in (7.2.5). Assume that the following relations between the dimensionless parameters in system (7.2.5) are satisfied:*

$$\lim_{\text{Re} \rightarrow +\infty} \frac{\text{Ha}}{\text{Re}} = \mathbb{H} \in \mathbb{R}, \quad \text{Pr}_m := \frac{\text{Re}_m}{\text{Re}} \text{ is fixed}, \quad \lim_{\text{Re} \rightarrow +\infty} \left(\frac{U_0}{c} \right)^2 \frac{1}{\text{Re}} = \kappa \in \mathbb{R}, \quad \mathbb{J} := \kappa \mathcal{J} \text{ is fixed.}$$

Furthermore, assume that the initial data $\mathbf{B}' = (\mathbf{B}'_1, \mathbf{B}'_2)^T \in \mathbb{R}^2$ and $\mathbf{E}' \in \mathbb{R}$ are such that

$$\mathbf{B}'_1 = \mathbf{b}_1 \text{ is fixed, while } \mathbf{B}'_2 = \frac{1}{\sqrt{\text{Re}}} \mathbf{b}_2, \quad \mathbf{E}' = \frac{1}{\sqrt{\text{Re}}} \mathbf{e},$$

for some $\mathbf{b}_2 \in \mathbb{R}$ and $\mathbf{e} \in \mathbb{R}$. Then System (7.1.1) appears as boundary layer of equations (7.2.5) in the region

$$(t', x', y') \in (0, T) \times \mathbb{R} \times \left(0, \frac{1}{\sqrt{\text{Re}}} \right),$$

when $\text{Re} \rightarrow +\infty$ (and thus also when $\text{Re}, \text{Ha} \rightarrow +\infty$).

Since (t, x, y) are the variables of the boundary layer in (7.1.1), we clarify that (t', x', y') are now the variables of the starting Navier-Stokes-Maxwell equations (7.2.5).

The relation $U_0^2/(c^2\text{Re}) \rightarrow \kappa > 0$ is of course questionable, since it would imply that the characteristic speed U_0 takes values that are much higher than the speed of light ' c '. In case U_0/c is fixed (we treat this case in Theorem 7.1.2), the contribution of the displacement current would indeed vanish, thus an ansatz typical of the Prandtl theory would probably lead to the standard Prandtl-MHD equations (7.1.1) (as derived in [63]) rather than system (7.1.1).

Theorem 7.1.1 would seem therefore to suggest that when the characteristic speed of Plasma is constant, System (7.1.1) is ineffective at the limit $\text{Re} \rightarrow \infty$. With the next Theorem 7.1.2, we counteract this statement, by showing that System (7.1.1) remains an accurate boundary layer, when considering a different scaling (and thus a different region of the layer).

Theorem 7.1.2. *Consider the Navier-Stokes-Maxwell equations with Cattaneo's law in (7.2.5). Assume that the following relations between the dimensionless parameters in system (7.2.5) are satisfied:*

$$\lim_{\text{Re} \rightarrow +\infty} \frac{\text{Ha}}{\text{Re}} = \mathbb{H} \in \mathbb{R} \quad \text{and} \quad \text{Pr}_m := \frac{\text{Re}_m}{\text{Re}}, \quad \kappa := \left(\frac{U_0}{c}\right)^2, \quad \mathbb{J} := \kappa \mathcal{J} \quad \text{are all fixed.}$$

Furthermore, assume that the initial data $\mathbf{B}' = (\mathbf{B}'_1, \mathbf{B}'_2)^T$ are such that

$$\mathbf{B}'_2 = \mathbf{b}_2 \in \mathbb{R} \text{ is fixed, while } \mathbf{B}'_1 = \sqrt{\frac{\text{Ha}}{\mathbb{H}}} \mathbf{b}_1, \quad \mathbf{E}' = \sqrt{\frac{\text{Ha}}{\mathbb{H}}} \mathbf{e},$$

for some $\mathbf{b}_1 \in \mathbb{R}$ and $\mathbf{e} \in \mathbb{R}$. Then System (7.1.1) appears as boundary layer in the region

$$(t', x', y') \in \left(0, \frac{\mathbb{H}}{\text{Ha}} T'\right) \times \mathbb{R} \times \left(0, \frac{\mathbb{H}}{\text{Ha}}\right),$$

when $\text{Re} \rightarrow +\infty$ (and thus also when $\text{Re}, \text{Ha} \rightarrow +\infty$).

Remark 7.1.1. *The domain of the boundary layer in Theorem 7.1.2 is not only close to $y' = 0$ but also to the time origin $t' = 0$. This is not surprising from a mathematical point of view. Indeed, at the asymptotic limit $\text{Re} \rightarrow \infty$, the Navier-Stokes-Maxwell's equations with Cattaneo's law (7.2.5) switch from hyperbolic to parabolic in the variables (t, y) . This leads to a loss of initial data on the time derivative of the velocity field and the magnetic field, thus the appearance of boundary layers near the origin in time.*

From a physical point of view, one would wonder if this domain is rather an artifact of the equations, since such a short time would not be observed in real applications. We counter this statement through the following remark: although $t' \in (0, \delta T)$ reflects a short range of time, the solution (u, v, b_1, b_2, e) of (7.1.1) contributes to dynamics of Plasma $(U'_1, U'_2, B'_1, B'_2, E')$ with a rescaled magnitude of order $1/\delta$ (for more details cf. transformation (7.2.14)). This property would suggest that the solution (u, v, b_1, b_2, e) might still impact the evolution of Plasma near bounding surface for larger time $t' > \delta T$. The formal proof of this statement is however beyond the interest of this paper.

We conclude this introduction with a short overview of the sections concerning our modelling. In Section 7.2.1 and Section 7.2.2 we introduce the Navier-Stokes-Maxwell equations with Cattaneo's law in a suitable dimensionless form. Next, in Section 7.2.3, we prove Theorem 7.1.1 and derive System (7.1.1) as a boundary layer with thickness of Prandtl type. Finally, we address the Hartmann origin of System (7.1.1), by proving Theorem 7.1.2 in Section 7.2.4.

7.1.5 Statement of our mathematical results

In this subsection we investigate the well-posedness problem of the derived system. Since the considered model is an extension of the standard Prandtl equations, it presents similar analytical challenges, in particular the lack of diffusion (and thus of regularising effects) on the variable $x \in \mathbb{R}$. The major nonlinearities can indeed generate strong instabilities in the horizontal direction, specifically under the occurrence of high oscillations of the solutions (the contribution of the high frequencies in $x \in \mathbb{R}$). It is rather common in the scientific community to address the analysis of boundary layers by considering therefore highly regular initial data, such as analytic in $x \in \mathbb{R}$ [2, 112, 114]. We postpone the precise definition of this functional framework to Section 7.3, we shall however mention that these are functions whose frequencies $\xi \in \mathbb{R}$ under Fourier transform decay like $\exp(-\tau|\xi|)$, for some $\tau > 0$ known as radius of analyticity. Our analytical result asserts that if the initial data are indeed analytic and are sufficiently small, then there exists a global-in-time analytic solution of equations (7.1.1), whose radius of analyticity decays exponentially in time.

Theorem 7.1.3. *Assume homogeneous boundary conditions in (7.1.2): $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{e}) = (0, 0, 0)$. For any $s > 2$, there exists a sufficiently small positive constant $\varepsilon_s \in [0, 1)$ (which depends uniquely upon s), such that the following result holds true. Let \bar{u} , \bar{b}_1 and \tilde{b}_1 be initial data of (7.1.1) that are analytic in the variable $x \in \mathbb{R}$ with radius of analyticity $\tau_0 > 0$:*

$$(7.1.4) \quad \begin{aligned} e^{\tau_0(1+|D_x|)}\bar{u} \quad \text{and} \quad e^{\tau_0(1+|D_x|)}\bar{b}_1 \quad \text{belong to} \quad H^{s+1,1}(\mathbb{R} \times (0, 1)), \\ e^{\tau_0(1+|D_x|)}\tilde{u} \quad \text{and} \quad e^{\tau_0(1+|D_x|)}\tilde{b}_1 \quad \text{belong to} \quad H^{s,0}(\mathbb{R} \times (0, 1)). \end{aligned}$$

If the following smallness condition on the initial data holds true

$$(7.1.5) \quad \begin{aligned} & \|e^{\tau_0(1+|D_x|)}\bar{u}\|_{H^{s+1,0}} + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\tilde{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\bar{b}_1\|_{H^{s+1,0}} + \\ & + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{b}_1\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\tilde{b}_1\|_{H^{s,0}} \leq \left(\frac{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}}}{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{5}{2}}} \frac{\min\{\tau_0, \tau_0^{-1}\}^{\frac{3}{2}}}{\max\{1, \mathbb{H}^2\} \max\{\text{Pr}_m^{-1}, \text{Pr}_m\}^{\frac{1}{2}}} \right) \varepsilon_s, \end{aligned}$$

then there exists a unique global-in-time analytic solution (u, b_1) of (7.1.1), which has a decaying radius of analyticity $\tau : \mathbb{R}_+ \rightarrow (0, \tau_0]$ given by

$$(7.1.6) \quad \tau(t) := \tau_0 \exp \left\{ - \frac{t}{16 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\} > 0.$$

Furthermore, the analytic norms of the solution decay exponentially in time $t \in \mathbb{R}_+$ as follows:

$$(7.1.7) \quad \begin{aligned} & \|e^{\tau(t)(1+|D_x|)}u(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_t u(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_y u(t)\|_{H^{s,0}}^2 + \\ & + \|e^{\tau(t)(1+|D_x|)}b_1(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_t b_1(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_y b_1(t)\|_{H^{s,0}}^2 \\ & \leq C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) \left\{ \|e^{\tau_0(1+|D_x|)}\bar{u}\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)}\tilde{u}\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{u}\|_{H^{s,0}}^2 + \right. \\ & \left. + \|e^{\tau_0(1+|D_x|)}\bar{b}_1\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)}\tilde{b}_1\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)}\partial_y\bar{b}_1\|_{H^{s,0}}^2 \right\} \exp \left\{ - \frac{t}{8 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\}. \end{aligned}$$

where the constant $C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0)$ is defined by

$$C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) = 4^3 \frac{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3} \max\{\text{Pr}_m, \text{Pr}_m^{-1}\} \max\{\tau_0, \tau_0^{-1}\}^2.$$

Some remarks are here in order. The statement considers uniquely the state variables (u, b_1) . Indeed all others variables (v, b_2, e) are determined by the divergence-free conditions, Faraday's law and the

homogeneous boundary conditions :

$$v(t, x, y) = - \int_0^y \partial_x u(t, x, z) dz, \quad b_2(t, x, y) = - \int_0^y \partial_x b_1(t, x, z) dz \quad e(t, x, y) = - \int_0^y \partial_t b_1(t, x, z) dz.$$

We refer to Section 7.3.2 for more details, and to (7.3.3) for an explicit form of the equations in terms of (u, b_1) .

The function space described by (7.1.4) is analytic in the variable $x \in \mathbb{R}$, since $e^{\tau_0(1+|D_x|)}$ is a Fourier multiplier that enforces an exponential decay on the frequencies of the initial data (more details in Section 7.3.1). The range $s > 2$ of the related norms is however a pure artifact of our analysis rather than a real restriction. This condition simplifies indeed certain estimates (cf. for instance Lemma 7.4.1 together with (7.4.14) where $\sigma_2 - 1/2 = s - 2 > 0$). Nevertheless we could also consider $s \in [0, 2)$, since $e^{\tau_0(1+|D_x|)}\bar{u}$ (and similarly all the other initial data) can always be recasted as $e^{-\epsilon(1+|D_x|)}e^{(\tau_0+\epsilon)(1+|D_x|)}\bar{u}$, where the Fourier multiplier $e^{-\epsilon(1+|D_x|)}$ is a regularising operator (of course we would need a slightly higher radius of analyticity). We do not pursue this approach just for the sake of a short presentation.

The small parameter ε_s in (7.1.5) depends uniquely on $s > 2$ and can be explicitly defined as (cf. (7.3.20))

$$\varepsilon_s := \frac{s-2}{2^{2s+14}} \frac{1}{1 + \frac{s-2}{\sqrt{s-1}}}.$$

Moreover the right hand-side of (7.1.5) and the size of the initial data decrease proportionally to $\tau_0^{3/2}$ (when the radius of analyticity $\tau_0 < 1$). This aspect is revealed and supported by the function framework in (7.1.4), which converges to standard Sobolev spaces when τ_0 vanishes (our model may still be ill-posed in Sobolev, as Prandtl).

Let us now comment on the relation between the smallness condition (7.1.5) and the different physical parameters from System (7.1.1). When $\mathbb{H} \gg 1$ the nonlinearities in the right-hand side of the u -equation become predominant, hence a more restrictive condition on the initial data is natural in order to achieve analytic stability. Moreover, although the constants \mathbb{J} and κ/Pr_m play a major role in (7.1.5), since they inherently decrease the size of the initial data, when they converge towards 0. At a first glance, this property would seem questionable, since we would expect that when these constants vanish (i.e. both the inertial term of the velocity field and the displacement current are neglected) we should recover similar equations to Prandtl-MHD in (7.1.3). Consequently, the equations would switch from hyperbolic in the y -direction (with damping mechanisms) to parabolic, a setting which is usually more stable. This observation is however inaccurate, since our model (7.1.1) owns a more involved structure than Prandtl-MHD (7.1.3), which can be highlighted in the following aspects:

- The right-hand side of the momentum equation (first equation in (7.1.1)) has terms that are trilinear in the solution (contrary to the bilinear ones in (7.1.3)). These terms further increase the instabilities of our model, when compared with the ones of boundary layers in MHD.
- The contribution of the pressure is not trivial as in Prandtl-MHD, since $\partial_y p$ is not identically zero and it encompasses further trilinear terms. These are indeed the most challenging terms to estimate.

Our analysis and our smallness condition (7.1.5) therefore suggest that the contributions \mathbb{J} , κ/Pr_m of the displacement current and the Cattaneo's law have not only a role as derivation of our model, but they may rather have a stabilizing effect on the underlying solutions (at least at the level of analytic regularities).

The part of this paper concerning the analysis of System (7.1.1) is organized as follows. In Section 7.3.1 we define the function spaces of the analytic solutions, while in Section 7.3.2 we provide a compact formulation of System (7.1.1). Section 7.3.4 is devoted to the sketch of the proof of Theorem 7.1.3,

postponing the more technical parts. Our approach is indeed based on a suitable estimate of the norms of the solutions (cf. Proposition 7.3.1, whose proof is formally developed in Section 7.4).

7.2 Derivation of the model

This section is devoted to prove Theorem 7.1.1 and Theorem 7.1.2. To this end, in Section 7.2.1 we introduce a suitable form of the Navier-Stokes-Maxwell equations with Cattaneo's law, that we recast in their dimensionless form in Section 7.2.2. Section 7.2.3 is hence devoted to prove Theorem 7.1.1, while in Section 7.2.4 we deal with Theorem 7.1.2.

7.2.1 The Navier-Stokes-Maxwell equations with Cattaneo's law

We begin by recalling the widespread form of the two-dimensional Navier-Stokes-Maxwell system with Cattaneo's law:

$$(7.2.1) \quad \begin{cases} \frac{\rho\nu\mathcal{J}}{c^2}\partial_\tau^2\vec{U} + \rho(\partial_\tau\vec{U} + \vec{U} \cdot \nabla\vec{U}) - \rho\nu\Delta\vec{U} + \nabla P = \vec{J} \times \vec{B} & \text{balance of linear momentum,} \\ \operatorname{div}\vec{U} = 0 & \text{conservation of mass,} \\ \partial_\tau\vec{B} + \operatorname{curl}\vec{E} = 0 & \text{Faraday's law,} \\ \vec{J} = \sigma(\vec{E} + \vec{U} \times \vec{B}) & \text{Ohm's law,} \\ \frac{1}{c^2}\partial_\tau\vec{E} + \mu_0\vec{J} = \operatorname{curl}\vec{B} & \text{Ampere's law,} \\ \operatorname{div}\vec{B} = 0 & \text{Gauss's law for magnetism,} \\ \operatorname{div}\vec{E} = 0 & \text{Gauss's law for electric field.} \end{cases}$$

The system and the corresponding state variables depends upon $(\tau, X, Y) \in (0, T) \times \mathbb{R} \times \mathbb{R}_+$ (instead of (t, x, y) , which are the variables of the boundary layers), for a positive time $T > 0$. The following boundary conditions are also prescribed:

$$\vec{U}(\tau, X, 0) = 0 \in \mathbb{R}^2, \quad \vec{B}(\tau, X, 0) = \vec{\mathbf{B}} \in \mathbb{R}^2 \quad \text{and} \quad E(\tau, X, 0) = \mathbf{E} \in \mathbb{R},$$

i.e. the velocity field satisfies a no-slip boundary condition, whereas the surrounding medium in $(\tau, X, Y) \in (0, T) \times \mathbb{R} \times \{y < 0\}$ is an insulator with a prescribed fixed magnetic field $\vec{\mathbf{B}} \in \mathbb{R}^2$.

The constants $\rho > 0$ and $\nu > 0$ are the density of the fluid and the kinematic viscosity, respectively, while $c > 0$ stands for the speed of light. The first term $(\rho\nu\mathcal{J}/c^2)\partial_\tau^2\vec{U}$ in the balance of linear momentum is due to the Cattaneo's law [6, 16, 38, 108, 111, 122] and depends on a general inertial constant $\mathcal{J} > 0$. This law develops around a first-order Taylor expansion of a delayed relation on the Cauchy stress tensor

$$\mathbb{S}(\tau + \tau_{\text{rel}}, \cdot) = \nu \frac{\nabla u + \nabla u^T}{2}(\tau, \cdot)$$

where for us the relaxation time is given by $\tau_{\text{rel}} = \rho\nu\mathcal{J}/c^2$. This particular form (in terms of $\mathcal{J} > 0$ and not directly in $\tau_{\text{rel}} > 0$), will be important indeed when rescaling our system for the boundary layers.

We have denoted by $\sigma > 0$ the electrical conductivity, by $\mu_0 > 0$ the magnetic permeability. We further write $\vec{U}(\tau, X, Y) = (U_1(\tau, X, Y), U_2(\tau, X, Y))^T \in \mathbb{R}^2$ and $\vec{B}(\tau, X, Y) = (B_1(\tau, X, Y), B_2(\tau, X, Y))^T \in \mathbb{R}^2$ the velocity field and magnetic field of the media, respectively. The scalar pressure $P(\tau, X, Y) \in \mathbb{R}$ is the Lagrangian multiplier that ensures the incompressibility of the velocity field. The current density $\vec{J} = (0, 0, J(\tau, X, Y))^T$ and the electric field $\vec{E} = (0, 0, E(\tau, X, Y))^T$ are considered as three dimensional

vector fields, being perpendicular to the plane in which the fluid motion occurs. Since we are dealing with the two dimensional version of the equations, we shall clarify the employed notation:

$$\vec{J} \times \vec{B} = J \vec{B}^\perp = J \begin{pmatrix} -B_2 \\ B_1 \end{pmatrix}, \quad \text{curl} \vec{E} = \begin{pmatrix} \partial_Y E \\ -\partial_X E \end{pmatrix}, \quad \vec{U} \times \vec{B} = \begin{pmatrix} 0 \\ 0 \\ U_1 B_2 - U_2 B_1 \end{pmatrix}, \quad \text{curl} \vec{B} = \begin{pmatrix} 0 \\ 0 \\ \partial_X B_2 - \partial_Y B_1 \end{pmatrix}.$$

The positive parameters ν , μ_0 and ε_0 correspond to the kinematic viscosity, the magnetic permeability and permittivity of free space, respectively. Furthermore, the parameter σ represents the electrical conductivity of the medium.

Some of the terms in (7.2.1) are redundant, indeed we can recast the overall system as five equations depending on \vec{U} , \vec{B} and \vec{E} . First, we formulate Faraday's law in (7.2.1) only in terms of the magnetic field \vec{B} , making use of Ohm's law (7.2.1):

$$(7.2.2) \quad \partial_\tau \vec{B} = -\text{curl} \vec{E} = \text{curl}(\vec{U} \times \vec{B}) - \frac{1}{\sigma} \text{curl} \vec{J}.$$

Furthermore, to get rid of the current density in $\text{curl} \vec{J}$, we apply the curl operator to Ampere's law:

$$\frac{1}{c^2} \partial_\tau (\text{curl} \vec{E}) + \mu_0 \text{curl} \vec{J} = \text{curl} \text{curl} \vec{B},$$

which leads to

$$\text{curl} \vec{J} = \frac{1}{\mu_0 c^2} \partial_\tau^2 \vec{B} + \frac{1}{\mu_0} (\nabla \text{div} \vec{B} - \Delta \vec{B}) = \frac{1}{\mu_0 c^2} \partial_\tau^2 \vec{B} + \frac{1}{\mu_0} (\nabla \text{div} \vec{B} - \Delta \vec{B})$$

Thus, we can plug this last relation in equation (7.2.2), to finally obtain an hyperbolic form of Ampere's law in terms of the magnetic field \vec{B} :

$$(7.2.3) \quad \frac{1}{\sigma \mu_0 c^2} \partial_\tau^2 \vec{B} + \partial_\tau \vec{B} - \frac{1}{\sigma \mu_0} \Delta \vec{B} = \text{curl}(\vec{U} \times \vec{B}) = \vec{B} \cdot \nabla \vec{U} - \vec{U} \cdot \nabla \vec{B}.$$

Similarly, we can get rid of \vec{J} also in the balance of linear momentum in (7.2.1) through

$$\frac{\rho \nu \mathcal{J}}{c^2} \partial_\tau^2 \vec{U} + \rho (\partial_t \vec{U} + \vec{U} \cdot \nabla \vec{U}) - \nu \Delta \vec{U} + \nabla P = \sigma (\vec{E} + \vec{U} \times \vec{B}) \times \vec{B} = \sigma \vec{E} \times \vec{B} + \sigma (\vec{B} (\vec{U} \cdot \vec{B}) - \vec{U} |\vec{B}|^2).$$

We are now able to reduce the number of equations in (7.2.1). By considering the electric field $\vec{E}(\tau, X, Y) = (0, 0, E(\tau, X, Y))^T$ (whose divergence is always null) and recalling the definition of the vector field $\vec{B}^T = (-B_2, B_1)^T$, we finally gather for $(\tau, X, Y) \in (0, T) \times \mathbb{R} \times \mathbb{R}_+$:

$$(7.2.4) \quad \begin{cases} \frac{\rho \nu \mathcal{J}}{c^2} \partial_\tau^2 \vec{U} + \rho (\partial_t \vec{U} + \vec{U} \cdot \nabla \vec{U}) - \nu \Delta \vec{U} + \nabla P = \sigma (\vec{B} (\vec{U} \cdot \vec{B}) - \vec{U} |\vec{B}|^2) + \sigma E \vec{B}^T, \\ \text{div} \vec{U} = 0, \\ \frac{1}{\sigma \mu_0 c^2} \partial_\tau^2 \vec{B} + \partial_\tau \vec{B} - \frac{1}{\sigma \mu_0} \Delta \vec{B} = \vec{B} \cdot \nabla \vec{U} - \vec{U} \cdot \nabla \vec{B}, \\ \partial_\tau \vec{B} + \text{curl} \vec{E} = 0, \\ \text{div} \vec{B} = 0, \end{cases}$$

with boundary conditions

$$\vec{U}(\tau, X, 0) = 0 \in \mathbb{R}^2, \quad \vec{B}(\tau, X, 0) = \mathbf{B} \in \mathbb{R}^2 \quad \text{and} \quad E(\tau, X, 0) = \mathbf{E} \in \mathbb{R}.$$

Before performing an asymptotic analysis of equations (7.2.4) to derive the boundary-layer model (7.1.1),

it is reasonable to first recast equations (7.2.4) in their dimensionless form.

7.2.2 The equations in dimensionless form

We shall first briefly recall some dimensionless parameters which are well-known in the magnetohydrodynamic theory. We refer to [45,121] for additional details and an exhaustive overview of the underlying physics.

Throughout this manuscript we denote by $U_0 \in \mathbb{R}_+$ and by $B_0 \in \mathbb{R}_+$ the sizes of the characteristic speed and magnetic field of the fluid, respectively. We further denote by $l_0 \in \mathbb{R}_+$ and $t_0 \in \mathbb{R}_+$ the underlying length and time scales (thus $U_0 = l_0/t_0$). The ratio between the sizes of the inertial and viscous terms is given by the Reynolds number $\text{Re} = l_0 U_0 / \nu$, while $\text{Re}_m = l_0 U_0 \sigma \mu_0$ stands for the magnetic Reynolds number and measures the coupling between the flow and the magnetic field. The Hartmann number $\text{Ha} = B_0 l_0 \sqrt{\sigma / \rho \nu}$ represents the ratio between the magnetic and viscous forces.

Hence, we can introduce the change of variables $t' = \tau/t_0 \in (0, T')$ (with $T' = T/t_0$), $x' = X/l_0 \in \mathbb{R}$, $y' = Y/l_0 \in \mathbb{R}_+$ as well as the new state variables $\vec{U}' = \vec{U}/U_0$, $\vec{B}' = \vec{B}/B_0$ and $E' = EB_0 U_0$. In this new framework the equations in (7.2.4) become

$$(7.2.5) \quad \begin{cases} \left(\frac{U_0}{c}\right)^2 \frac{\mathcal{J}}{\text{Re}} \partial_t^2 \vec{U}' + \partial_{t'} \vec{U}' + \vec{U}' \cdot \nabla \vec{U}' - \frac{1}{\text{Re}} \Delta \vec{U}' + \nabla P = \\ \quad = \frac{\text{Ha}^2}{\text{Re}} (\vec{B}' (\vec{U}' \cdot \vec{B}') - \vec{U}' |\vec{B}'|^2 + E' \vec{B}'^T), & (t', x', y') \in (0, T') \times \mathbb{R} \times \mathbb{R}_+, \\ \text{div } \vec{U}' = 0, & (0, T') \times \mathbb{R} \times \mathbb{R}_+, \\ \left(\frac{U_0}{c}\right)^2 \frac{1}{\text{Re}_m} \partial_t^2 \vec{B}' + \partial_{t'} \vec{B}' - \frac{1}{\text{Re}_m} \Delta \vec{B}' = \vec{B}' \cdot \nabla \vec{U}' - \vec{U}' \cdot \nabla \vec{B}', & (0, T') \times \mathbb{R} \times \mathbb{R}_+, \\ \partial_{t'} \vec{B}' + \text{curl } \vec{E}' = 0, & (0, T') \times \mathbb{R} \times \mathbb{R}_+, \\ \text{div } \vec{B}' = 0, & (0, T') \times \mathbb{R} \times \mathbb{R}_+, \end{cases}$$

with boundary conditions

$$(7.2.6) \quad \vec{U}'(t', x', 0) = 0 \in \mathbb{R}^2, \quad \vec{B}'(t', x', 0) = \vec{\mathbf{B}}' := \frac{\vec{\mathbf{B}}}{B_0} \in \mathbb{R}^2 \quad \text{and} \quad E'(t, x, 0) = \mathbf{E}' := \mathbf{E} B_0 U_0 \in \mathbb{R},$$

where all spatial derivatives ∇ , Δ and div are now in terms of (x', y') . The behaviour of solutions of the system is therefore quantified by the Reynolds number Re and the Hartmann number in the momentum equation, as well as by the characteristic speed U_0 and the magnetic Reynolds number Re_m in the third equation.

Remark 7.2.1. *Let us comment on a major difference between the parameters of the Navier-Stokes-Maxwell equations given by (7.2.5) and classic MHD system (i.e. when $c \sim +\infty$). The forcing term on the right-hand side of the balance of linear momentum in (7.2.5) is driven by the dimensionless constant Ha^2/Re . At a first glance, this constant differs from $\text{Ha}^2/(\text{Re} \text{Re}_m)$ in the linear momentum of MHD, i.e.*

$$\partial_{t'} \vec{U}' + \vec{U}' \cdot \nabla \vec{U}' - \frac{1}{\text{Re}} \Delta \vec{U}' + \nabla P = \frac{\text{Ha}^2}{\text{Re} \text{Re}_m} \left(\vec{B}' \cdot \nabla \vec{B}' - \nabla \frac{|\vec{B}'|^2}{2} \right) = \frac{\text{Ha}^2}{\text{Re} \text{Re}_m} \text{curl} \vec{B}' \times \vec{B}',$$

This observation is however imprecise. Indeed, by neglecting $1/c^2 \partial_t E$ in (7.2.1) (thus also $U_0^2/(c^2 \text{Re}_m) \partial_t^2 \vec{B}'$

in the third equation of (7.2.5)), Ohm's law together with Ampere's law imply

$$\operatorname{curl} B' \times B' = \mu_0 l_0 U_0 \sigma (U' \times B' + E') \wedge B' = \operatorname{Re}_m (\vec{B}' (\vec{U}' \cdot \vec{B}') - \vec{U}' |\vec{B}'|^2 + E' \vec{B}'^T),$$

which provides the additional constant $1/\operatorname{Re}_m$ to yield $\operatorname{Ha}^2/(\operatorname{Re} \operatorname{Re}_m)$. System (7.2.5) is therefore an extension of the MHD-equations in dimensionless form.

In the forthcoming sections, we aim at obtaining system (7.1.1), by sending Re , Re_m and Ha towards ∞ in (7.2.5). To this end, we consider suitable conditions on U_0 and Ha , as well as some valid rescalings of the variables (t', x', y') near the boundary. In particular, in section 7.2.3 we derive System (7.1.1) as boundary layer with thickness of Prandtl type, while in section 7.2.3 the equations are revealed as a layer with thickness of Hartmann type.

7.2.3 appearance of Boundary layers with thickness of Prandtl type

The goal of this section is to prove Theorem 7.1.1, analyzing the asymptotic limit of Equations (7.2.5) on a thin layer near the boundary, under suitable assumptions on the dimensionless parameters. Boundary layers commonly appear in fluid dynamics at high values of the Reynolds number $\operatorname{Re} \gg 1$, however we shall also here clarify the asymptotic of the magnetic Reynolds number Re_m and the Hartmann number Ha . We assume here that the ratio between the magnetic Reynolds number and the Reynolds number (known as magnetic Prandtl number) is fixed

$$\frac{\operatorname{Re}_m}{\operatorname{Re}} = \operatorname{Pr}_m > 0.$$

thus also the magnetic Reynolds number assumes high values $\operatorname{Re}_m \gg 1$. Similarly, we address the case in which the Hartmann number Ha diverges to ∞ proportionally to Re .

We can then summarise the statement of Theorem 7.1.1 as follows.

Theorem 7.2.1. *Assume that the following relations between the dimensionless parameters in system (7.2.5) are satisfied:*

(7.2.7)

$$\lim_{\operatorname{Re} \rightarrow +\infty} \frac{\operatorname{Ha}}{\operatorname{Re}} = \mathbb{H} \in \mathbb{R}, \quad \frac{\operatorname{Re}_m}{\operatorname{Re}} = \operatorname{Pr}_m \text{ is fixed}, \quad \lim_{\operatorname{Re} \rightarrow +\infty} \left(\frac{U_0}{c} \right)^2 \frac{1}{\operatorname{Re}} = \kappa \in \mathbb{R}, \quad \mathbb{J} := \kappa \mathcal{J} \text{ is fixed.}$$

Furthermore, assume that the initial data $\mathbf{B}' = (\mathbf{B}'_1, \mathbf{B}'_2)^T \in \mathbb{R}^2$ and $\mathbf{E}' \in \mathbb{R}$ in (7.2.6) are such that

$$\mathbf{B}'_1 = \mathbf{b}_1 \text{ is fixed, while } \mathbf{B}'_2 = \frac{1}{\sqrt{\operatorname{Re}}} \mathbf{b}_2, \quad \mathbf{E}' = \frac{1}{\sqrt{\operatorname{Re}}} \mathbf{e},$$

for some $\mathbf{b}_2 \in \mathbb{R}$ and $\mathbf{e} \in \mathbb{R}$. Then System (7.1.1) appears as boundary layer in the region

$$(7.2.8) \quad (t', x', y') \in \mathbb{R}_+ \times \mathbb{R} \times \left[0, \frac{1}{\sqrt{\operatorname{Re}}} \right],$$

when $\operatorname{Re} \rightarrow +\infty$ (and thus also when $\operatorname{Re}, \operatorname{Ha} \rightarrow +\infty$).

Remark 7.2.2. *Before addressing the proof of this Theorem, we shall first clarify certain aspects and terminologies of its statement.*

- The third condition in (7.2.7) is rather unphysical, since it implies that the characteristic speed U_0 converges towards ∞ for high values of $\operatorname{Re} \gg 1$. This relation seems however necessary in

order to avoid losing contribution from the equations of (b_1, b_2) . A more physical scenario is addressed in Section 7.2.4.

- Although the boundary layer appears in the region given by (7.2.8), System (7.1.1) is written in terms of rescaled variables (t, x, y) in (7.2.9) and rescaled functions (u, b_1, b_2, p, e) in (7.2.10). Therefore, the domain of System (7.1.1) does not depend on the Reynolds number: $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 1]$.
- By saying that “System (7.1.1) appears as boundary layer”, we mean that as long as the triple (\vec{U}', \vec{B}', E') (which depends on Re) converges towards a profile $((u, 0)^T, (b_1, b_2)^T, e)$ when $\text{Re} \gg 1$ (under a suitable rescaling), then (u, b_1, b_2, e) must satisfy system (7.1.1). The convergence is well known in the purely hydrodynamic regime ($\text{Ha} = 0$), a fact that have been highly studied through the stability theory of the Prandtl equations. We infer that this convergence holds true also when $\text{Ha} \neq 0$, but that is beyond the scope of this paper.

Proof. We denote by $\varepsilon^2 = 1/\text{Re}$ the inverse of the Reynolds number, which converges towards 0 when Re converges to ∞ . The parameter $\varepsilon \ll 1$ represents the size of the region in which the boundary layer occurs. We derive System (7.1.1) as a rescaled version of the asymptotic limit of (7.2.5) within the domain $(t, x', y') \in [0, T] \times \mathbb{R} \times [0, \varepsilon]$, by (informally) sending ε towards 0. To this end, we shall first introduce the change of variables

$$(7.2.9) \quad t = t' \in [0, T], \quad x = x' \in \mathbb{R}, \quad \text{and} \quad y = y'/\varepsilon \in [0, 1],$$

as well as the following new state variables

$$(7.2.10) \quad \begin{aligned} u(t, x, y) &:= U_1(t, x, \varepsilon y), & v(t, x, y) &:= U_2(t, x, \varepsilon y)/\varepsilon, & b_1(t, x, y) &:= B_1(t, x, \varepsilon y), \\ b_2(t, x, y) &:= B_2(t, x, \varepsilon y)/\varepsilon, & p(t, x, y) &:= P(t, x, \varepsilon y), & e(t, x, y) &:= E(t, x, \varepsilon y)/\varepsilon. \end{aligned}$$

Hence, we can develop System (7.2.5) in terms of u, v, e and p , as well as the variables (t, x, y) :

$$\left\{ \begin{aligned} &\left(\frac{U_0}{c}\right)^2 \frac{\varepsilon^2}{\kappa} \mathbb{J} \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \varepsilon^2 \partial_x^2 u - \partial_y^2 u + \partial_x p = \text{Ha}^2 \varepsilon^4 (b_1 b_2 v - u b_2^2 - b_2 e), \\ &\varepsilon \left(\frac{U_0}{c}\right)^2 \frac{\varepsilon^2}{\kappa} \mathbb{J} \partial_t^2 v + \partial_t v + u \partial_x v + v \partial_y v - \varepsilon^3 \partial_x^2 v - \varepsilon \partial_y^2 v + \frac{1}{\varepsilon} \partial_y p = \text{Ha}^2 \varepsilon^3 (b_2 b_1 u - b_1^2 v + b_1 e), \\ &\partial_x u + \partial_y v = 0, \\ &\left(\frac{U_0}{c}\right)^2 \frac{\varepsilon^2}{\text{Pr}_m} \partial_t^2 b_1 + \partial_t b_1 - \frac{\varepsilon^2}{\text{Pr}_m} \partial_x^2 b_1 - \frac{1}{\text{Pr}_m} \partial_y^2 b_1 = b_1 \partial_x u + b_2 \partial_y u - u \partial_x b_1 - v \partial_y b_1, \\ &\left(\frac{U_0}{c}\right)^2 \frac{\varepsilon^3}{\text{Pr}_m} \partial_t^2 b_2 + \varepsilon \partial_t b_2 - \frac{\varepsilon^3}{\text{Pr}_m} \partial_x^2 b_2 - \frac{\varepsilon}{\text{Pr}_m} \partial_y^2 b_2 = \varepsilon b_1 \partial_x v + \varepsilon b_2 \partial_y v - \varepsilon u \partial_x b_2 - \varepsilon v \partial_y b_2, \\ &\partial_t b_1 + \partial_y e = 0, \\ &\varepsilon (\partial_t b_2 - \partial_x e) = 0, \\ &\partial_x b_1 + \partial_y b_2 = 0. \end{aligned} \right.$$

with boundary conditions

$$u(t, x, 0) = 0, \quad v(t, x, 0) = 0, \quad b_1(t, x, 0) = \mathbf{b}_1, \quad b_2(t, x, 0) = \mathbf{b}_2 \quad \text{and} \quad e(t, x, 0) = \mathbf{e} \in \mathbb{R}.$$

We now remark that the conditions in (7.2.7) imply that $\text{Ha}^2 \varepsilon^4 \rightarrow \mathbb{H}^2$ and $(U_0/c)^2 \varepsilon^2 = (U_0/c)^2 / \text{Re} \rightarrow \kappa$ as $\varepsilon \rightarrow 0$. System (7.1.1) appears therefore by multiplying the second equation in (7.2.5) by ε , dividing the fifth equation by ε and finally sending ε towards 0. \square

7.2.4 Boundary layers with thickness of Hartmann type

In the modeling of Section 7.2.3, we have imposed that the characteristic velocity U_0 blows up at high value of the Reynolds number (cf. the third relation in (7.2.7)). This is a major drawback on the physics of the system. We can adjust this nonphysical scenario, by considering a different type of rescaling than the one introduced in (7.2.9) and (7.2.10). The corresponding thickness of the boundary layer will now be of Hartmann type, since it is inversely proportional to Ha (cf. (7.2.13) and (7.2.14)). Since Ha and Re are also in this section proportional, the thickness behaves like $1/\text{Re}$. This is much smaller than $1/\sqrt{\text{Re}}$ as in (7.2.8), so the nature of this thickness reminds the one of Hartmann layers. (Of course we have slightly abused the notation about Hartmann, since Hartmann layers usually occur when the magnetic field is oriented at some specific angles, whereas our modelling treats more general scenarios).

Although we introduce a different scaling, the derived equations remain the same as in System (7.1.1). We exploit this aspect in the following statement.

Theorem 7.2.2. *Assume that the following relations between the dimensionless parameters in system (7.2.5) are satisfied:*

$$(7.2.11) \quad \lim_{\text{Re} \rightarrow +\infty} \frac{\text{Ha}}{\text{Re}} = \mathbb{H} \in \mathbb{R} \quad \text{and} \quad \text{Pr}_m := \frac{\text{Re}_m}{\text{Re}}, \quad \kappa := \left(\frac{U_0}{c} \right)^2, \quad \mathbb{J} := \kappa \mathcal{J} \quad \text{are all fixed.}$$

Furthermore, assume that the initial data $\mathbf{B}' = (\mathbf{B}'_1, \mathbf{B}'_2)^T$ in (7.2.6) are such that

$$\mathbf{B}'_2 = \mathbf{b}_2 \in \mathbb{R} \text{ is fixed, while } \mathbf{B}'_1 = \sqrt{\frac{\text{Ha}}{\mathbb{H}}} \mathbf{b}_1, \quad \mathbf{E}' = \sqrt{\frac{\text{Ha}}{\mathbb{H}}} \mathbf{e},$$

for some $\mathbf{b}_1 \in \mathbb{R}$ and $\mathbf{e} \in \mathbb{R}$. Then System (7.1.1) appears as boundary layer in the region

$$(7.2.12) \quad (t', x', y') \in \left(0, \frac{\mathbb{H}}{\text{Ha}} T' \right) \times \mathbb{R} \times \left(0, \frac{\mathbb{H}}{\text{Ha}} \right),$$

when $\text{Re} \rightarrow +\infty$ (and thus also when $\text{Re}, \text{Ha} \rightarrow +\infty$).

Remark 7.2.3. *Some remarks are here in order:*

- We do not impose any condition on U_0 in terms of the Reynolds number, therefore this characteristic speed can range within physical values below the speed of light.
- The interval $[0, \mathbb{H}/\text{Ha}]$ in (7.2.12) can be replaced by $[0, 1/\text{Re}]$, since our assumptions ensure that $\mathbb{H}/\text{Ha} \approx 1/\text{Re}$ when $\text{Re} \gg 1$.
- The domain in (7.2.12) represents a region close to the boundary of the domain $(t', x', y') \in \mathbb{R} \times \mathbb{R}_+^2$. From the scaling of the new variables (cf. (7.2.13)) the domain of model (7.1.1) shall be better understood as an asymptotic expansion of a different domain, namely $(t', x', y') \in [0, T/\text{Ha}] \times [-\sqrt{X_0/\text{Ha}}, \sqrt{X_0/\text{Ha}}] \times [0, \mathbb{H}/\text{Ha}]$. In particular, the singular behaviour of the solutions given by (7.1.1) appears close to the boundary of the domain $\mathbb{R} \times \mathbb{R}_+^2$ and close to the origin both in time $t' = 0$ and in space $x' = 0$. This particular region of the domain is motivated by the fact that the term $(U_0/c)^2 1/\text{Re}_m \partial_{t'}^2 \vec{B}'$ of the displacement current in (7.2.5) is now vanishing. As a result, when $\text{Re} \gg 1$, the limit system of (7.2.5) requires different boundary conditions both in space (i.e. in $y' = 0$) as well as in time (i.e. in $t' = 0$).

Proof of Theorem 7.2.2. We denote by $\delta = \mathbb{H}/\text{Ha} \ll 1$ the size of the boundary layer. We thus introduce the change of variables

$$(7.2.13) \quad t = \frac{t'}{\delta} \in \mathbb{R}_+, \quad x = \frac{x'}{\sqrt{\delta}} \in \mathbb{R}, \quad y = \frac{y'}{\delta} \in [0, 1]$$

and the functions

(7.2.14)

$$\begin{aligned} u(t, x, y) &:= \sqrt{\delta} U'_1(\delta t, \sqrt{\delta}x, \delta y), \quad v(t, x, y) := U'_2(\delta T, \sqrt{\delta}x, \delta y), \quad b_1(t, x, y) := \sqrt{\delta} B'_1(\delta t, \sqrt{\delta}x, \delta y), \\ b_2(\tau, X, Y) &:= B'_2(\delta t, \sqrt{\delta}x, \delta y), \quad p(t, x, y) := \delta P'(\delta t, \sqrt{\delta}x, \delta y), \quad e(t, x, y) := \sqrt{\delta} E'(\delta t, \sqrt{\delta}x, \delta y). \end{aligned}$$

Hence, by recasting system (7.2.4) in terms of the new functions and variables, we deduce that (u, v) , (b_1, b_2) and e are solutions of

(7.2.15)

$$\left\{ \begin{aligned} &\delta^{-\frac{3}{2}} \left(\mathbb{J} \frac{\text{Ha}}{\text{Re}} \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u \right) - \frac{\text{Ha}}{\text{Re}\mathbb{H}} \delta^{-\frac{1}{2}} \partial_x^2 u - \frac{\text{Ha}}{\text{Re}\mathbb{H}} \delta^{-\frac{3}{2}} \partial_y^2 u + \delta^{-\frac{3}{2}} \partial_x p = \\ &\qquad\qquad\qquad = \frac{\text{Ha}^2}{\text{Re}} \delta^{-\frac{1}{2}} (b_1 b_2 v - u b_2^2 - b_2 e), \\ &\delta^{-1} \left(\mathbb{J} \frac{\text{Ha}}{\text{Re}} \partial_t^2 v + \partial_t v + u \partial_x v + v \partial_y v \right) - \frac{\text{Ha}}{\text{Re}\mathbb{H}} \partial_x^2 v - \frac{\text{Ha}}{\text{Re}\mathbb{H}} \delta^{-1} \partial_y^2 v + \delta^{-2} \partial_y p = \frac{\text{Ha}^2}{\text{Re}} \delta^{-1} (b_2 b_1 u - b_1^2 v + b_1 e), \\ &\delta^{-1} (\partial_x u + \partial_y v) = 0, \\ &\left(\frac{U_0}{c} \right)^2 \frac{\text{Ha}}{\text{Re}\mathbb{H}} \frac{\delta^{-\frac{3}{2}}}{\text{Pr}_m} \partial_t^2 b_1 + \delta^{-\frac{3}{2}} \partial_t b_1 - \frac{\delta^{-\frac{1}{2}}}{\text{Pr}_m} \frac{\text{Ha}}{\text{Re}\mathbb{H}} \partial_x^2 b_1 - \frac{\delta^{-\frac{3}{2}}}{\text{Pr}_m} \frac{\text{Ha}}{\text{Re}\mathbb{H}} \partial_y^2 b_1 = \delta^{-\frac{3}{2}} (b_1 \partial_x u + b_2 \partial_y u - u \partial_x b_1 - v \partial_y b_1), \\ &\left(\frac{U_0}{c} \right)^2 \frac{\text{Ha}}{\text{Re}\mathbb{H}} \frac{\delta^{-1}}{\text{Pr}_m} \partial_t^2 b_2 + \delta^{-1} \partial_t b_2 - \frac{1}{\text{Pr}_m} \frac{\text{Ha}}{\text{Re}\mathbb{H}} \partial_x^2 b_2 - \frac{\delta^{-1}}{\text{Pr}_m} \frac{\text{Ha}}{\text{Re}\mathbb{H}} \partial_y^2 b_2 = \delta^{-1} (b_1 \partial_x v + b_2 \partial_y v - u \partial_x b_2 - v \partial_y b_2), \\ &\delta^{-\frac{3}{2}} (\partial_t b_1 + \partial_y e) = 0, \\ &\delta^{-1} (\partial_t b_2 - \partial_x e) = 0, \\ &\delta^{-1} (\partial_x b_1 + \partial_y b_2) = 0, \end{aligned} \right.$$

with boundary conditions

$$u(t, x, 0) = 0, \quad v(t, x, 0) = 0, \quad b_1(t, x, 0) = \mathbf{b}_1, \quad b_2(t, x, 0) = \mathbf{b}_2 \quad \text{and} \quad e(t, x, 0) = \mathbf{e}.$$

We thus multiply the first, fourth and sixth equations in (7.2.15) by $\delta^{3/2}$, as well as the second, third, fifth, seventh and eighth equations by δ . Finally, remarking that

$$\lim_{\text{Re} \rightarrow \infty} \frac{\text{Ha}}{\text{Re}\mathbb{H}} = 1, \quad \lim_{\text{Re} \rightarrow \infty} \frac{\text{Ha}^2}{\text{Re}} = \lim_{\text{Re} \rightarrow \infty} \frac{\text{Ha}^2}{\text{Re}} \delta = \lim_{\text{Re} \rightarrow \infty} \frac{\text{Ha}}{\text{Re}} \mathbb{H} = \mathbb{H}^2,$$

we finally derive the main system (7.1.1), by also denoting $\kappa = (U_0/c)^2$. □

7.3 Existence of Analytic Solutions of System (7.1.1)

The main goal of the present paragraph is to prove the existence of certain smooth solutions for the derived System (7.1.1) (cf. Theorem 7.3.1). The analysis of these equations has similar challenges as the ones of the classical Prandtl system, in particular the fact that the system lacks of regularizing effects on the horizontal variable $x \in \mathbb{R}$ (the dissipative mechanisms of the system is indeed only on the variable $y \in (0, 1)$). In order to cope with this difficulty, it is rather common to impose high regularities on the initial data, along such horizontal variable. Our work addresses in particular the case of analytic functions. Before stating our main result, we shall first clarify the definition of analytic solutions, that we will use throughout the next sections. Furthermore, we provide a suitable reduction of the overall system, that will simplify our forthcoming analysis.

7.3.1 Analytic functions in the horizontal direction

A function $f = f(x)$ is analytic in $x \in \mathbb{R}$, if its Fourier transform $\mathcal{F}_x(f)(\xi) = \hat{f}(\xi)$ decays exponentially to zero as $e^{-\tau|\xi|}$, for some $\tau > 0$, when the frequency $\xi \in \mathbb{R}$ diverges to $\pm\infty$. For a fixed $\tau > 0$ (which stands for the radius of analyticity of f), this function space is indeed a Banach space. Between the

several equivalent norms, we will make use of the one given by $\|e^{\tau|D_x|}f\|_{H^s(\mathbb{R})}$, where $H^s(\mathbb{R})$ is a Sobolev space with regularity $s > 2$ and $e^{\tau|D_x|}$ stands for the Fourier multiplier $\mathcal{F}_x(e^{\tau|D_x|}f)(\xi) = e^{\tau|\xi|}\hat{f}(\xi)$.

Thus, we are interested in solutions (u, v, b_1, b_2) which depend upon $(t, x, y) \in (0, T) \times \mathbb{R} \times (0, 1)$ and are analytic in the variable $x \in \mathbb{R}$, as described by the following function space:

$$(7.3.1) \quad \begin{aligned} & e^{\tau(t)|D_x|}u, e^{\tau(t)|D_x|}b_1, e^{\tau(t)|D_x|}b_2, \in C([0, T], H^{s+1,0}(\mathbb{R} \times (0, 1))), \\ & e^{\tau(t)|D_x|}\partial_t u, e^{\tau(t)|D_x|}\partial_t b_1, e^{\tau(t)|D_x|}\partial_t b_2, e^{\tau(t)|D_x|}v, e^{\tau(t)|D_x|}e \in C([0, T], H^{s,0}(\mathbb{R} \times (0, 1))). \end{aligned}$$

The space $C([0, T], H^{s,0}(\mathbb{R} \times (0, 1)))$ (which we will abbreviate from now on by $C([0, T], H^{s,0})$) is anisotropic in space, namely it has H^s regularity in $x \in \mathbb{R}$ and only L^2 -regularity in $y \in (0, 1)$. The corresponding norm on a general function $g = g(t, x, y)$ is given by

$$\sup_{t \in (0, T)} \int_0^1 \int_{\mathbb{R}} (1 + |\xi|)^{2s} |\hat{g}(t, \xi, y)|^2 d\xi dy < \infty, \quad \text{where} \quad \hat{g}(t, \xi, y) = \mathcal{F}_x(g)(t, \xi, y) := \int_{\mathbb{R}} e^{-ix\xi} f(t, x, y) dx$$

stands for the Fourier transform in the horizontal variable $x \in \mathbb{R}$.

The radius of analyticity $\tau(t) > 0$ of the solutions in (7.3.1) is explicitly defined in Theorem 7.3.1 (cf. (7.3.5)). It depends on the radius of analyticity of the initial data and coincides with it at initial time $t = 0$. One shall furthermore remark that $\tau(t)$ decreases in $t \in (0, T)$, a fact that expresses (roughly speaking) the degrading mechanisms of the regularity in the horizontal variable.

7.3.2 Reduced system

We remark that system (7.1.1) can be shortened, since both v and b_2 in (7.1.1) are determined by the divergence-free relations $\partial_x u + \partial_y v = 0$, $\partial_x b_1 + \partial_y b_2 = 0$ and the boundary conditions $v|_{y=0} = 0$, $b_2|_{y=0} = 0$:

$$(7.3.2) \quad v(t, x, y) = - \int_0^y \partial_x u(t, x, z) dz, \quad \text{and} \quad b_2(t, x, y) = - \int_0^y \partial_x b_1(t, x, z) dz.$$

The above identities are well-defined, since we cope with solutions that are smooth in $x \in \mathbb{R}$ and the corresponding derivatives $\partial_x u(t, x, \cdot)$, $\partial_x b_1(t, x, \cdot)$ are L^2 -integrable in $y \in (0, 1)$ (for any $(t, x) \in (0, T) \times \mathbb{R}$). Moreover, for the sake of a compact presentation, we will shorten from now on the identities in (7.3.2) as $v = - \int_0^y \partial_x u$ and $b_2 = - \int_0^y \partial_x b_1$.

Similarly, the magnitude e of the electric field can be recast just in terms of b_1 , making use of the relation:

$$\begin{cases} \partial_t b_1 + \partial_y e = 0 & \text{in } (0, T) \times \mathbb{R} \times (0, 1), \\ e = 0 & \text{on } (0, T) \times \mathbb{R} \times \{0\}, \end{cases} \iff e(t, x, y) = - \int_0^y \partial_t b_1(t, x, z) dz$$

for any $(t, x, y) \in (0, T) \times \mathbb{R} \times (0, 1)$.

Because of these aspects, the equations of b_2 and e are redundant in system (7.1.1) and we can reduce the considered model uniquely in terms of (u, b_1) :

$$(7.3.3) \quad \begin{cases} \mathbb{J}\partial_{tt}^2 u + \partial_t u + u\partial_x u + v\partial_y u - \partial_{yy}^2 u + \partial_x p = \mathbb{H}^2 \left\{ b_1 b_2 v - u b_2^2 + b_2 \left(\int_0^y \partial_t b_1 \right) \right\}, \\ \partial_y p = \mathbb{H}^2 \left\{ b_1 b_2 u - b_1^2 v - b_1 \left(\int_0^y \partial_t b_1 \right) \right\}, \\ \frac{\kappa}{\text{Pr}_m} \partial_{tt}^2 b_1 + \partial_t b_1 + u\partial_x b_1 + v\partial_y b_1 - \frac{1}{\text{Pr}_m} \partial_{yy}^2 b_1 = b_1 \partial_x u + b_2 \partial_y u, \end{cases}$$

in $(t, x, y) \in (0, T) \times \mathbb{R} \times (0, 1)$, coupled with (7.3.2) and the following initial and boundary conditions:

$$\begin{aligned} u|_{t=0} &= \bar{u}, \quad \partial_t u|_{t=0} = \tilde{u}, \quad b_1|_{t=0} = \bar{b}_1, \quad \partial_t b_1|_{t=0} = \tilde{b}_1 && \text{in } \mathbb{R} \times (0, 1), \\ u|_{y=0} &= u|_{y=1} = b_1|_{y=0} = b_1|_{y=1} = 0 && \text{on } (0, T) \times \mathbb{R}. \end{aligned}$$

7.3.3 Statement of the result

The function space being introduced, we can state our result, which asserts the existence of global-in-time analytic solutions with small initial data.

Theorem 7.3.1. *For any $s > 2$, there exists a sufficiently small positive constant $\varepsilon_s \in [0, 1)$ (which depends uniquely upon s), such that the following result holds true. Let \bar{u} , \bar{b}_1 , \tilde{u} and \tilde{b}_1 be initial data that are analytic in the variable $x \in \mathbb{R}$ with radius of analyticity $\tau_0 > 0$:*

$$\begin{aligned} e^{\tau_0(1+|D_x|)}\bar{u} \quad \text{and} \quad e^{\tau_0(1+|D_x|)}\bar{b}_1 && \text{belong to } H^{s+1,1}(\mathbb{R} \times (0, 1)), \\ e^{\tau_0(1+|D_x|)}\tilde{u} \quad \text{and} \quad e^{\tau_0(1+|D_x|)}\tilde{b}_1 && \text{belong to } H^{s,0}(\mathbb{R} \times (0, 1)). \end{aligned}$$

If the following smallness condition on the initial data holds true

$$(7.3.4) \quad \begin{aligned} &\|e^{\tau_0(1+|D_x|)}\bar{u}\|_{H^{s+1,0}} + \|e^{\tau_0(1+|D_x|)}\partial_y \bar{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\tilde{u}\|_{H^{s,0}} + \|e^{\tau_0(1+|D_x|)}\bar{b}_1\|_{H^{s+1,0}} + \|e^{\tau_0(1+|D_x|)}\partial_y \bar{b}_1\|_{H^{s,0}} \\ &+ \|e^{\tau_0(1+|D_x|)}\tilde{b}_1\|_{H^{s,0}} \leq \delta := \left(\frac{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}}}{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}}} \frac{\min\{\tau_0, \tau_0^{-1}\}^{\frac{3}{2}}}{\max\{1, \mathbb{H}^2\} \max\{\text{Pr}_m^{-1}, \text{Pr}_m\}^{\frac{1}{2}}} \right) \varepsilon_s, \end{aligned}$$

then there exists a global-in-time analytic solution (u, b_1) of (7.3.3), which has a decaying radius of analyticity $\tau : \mathbb{R}_+ \rightarrow (0, \tau_0]$ given by

$$(7.3.5) \quad \tau(t) := \tau_0 \exp \left\{ - \frac{t}{16 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\} > 0.$$

Furthermore, the analytic norms of the solution decay exponentially in time $t \in \mathbb{R}_+$ as follows:

$$(7.3.6) \quad \begin{aligned} &\|e^{\tau(t)(1+|D_x|)}u(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_t u(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_y u(t)\|_{H^{s,0}}^2 \\ &+ \|e^{\tau(t)(1+|D_x|)}b_1(t)\|_{H^{s+1,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_t b_1(t)\|_{H^{s,0}}^2 + \|e^{\tau(t)(1+|D_x|)}\partial_y b_1(t)\|_{H^{s,0}}^2 \\ &\leq C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) \left\{ \|e^{\tau_0(1+|D_x|)}\bar{u}\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)}\tilde{u}\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)}\partial_y \bar{u}\|_{H^{s,0}}^2 \right. \\ &\left. + \|e^{\tau_0(1+|D_x|)}\bar{b}_1\|_{H^{s+1,0}}^2 + \|e^{\tau_0(1+|D_x|)}\tilde{b}_1\|_{H^{s,0}}^2 + \|e^{\tau_0(1+|D_x|)}\partial_y \bar{b}_1\|_{H^{s,0}}^2 \right\} \exp \left\{ - \frac{t}{8 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \right\}. \end{aligned}$$

where the constant $C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0)$ is defined by

$$C(\mathbb{J}, \kappa, \text{Pr}_m, \tau_0) = 4^3 \frac{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3} \max\{\text{Pr}_m, \text{Pr}_m^{-1}\} \max\{\tau_0, \tau_0^{-1}\}^2$$

In what follows, we shall first describe the main idea for the proof and postpone the detailed estimates (that are rather involved) to the subsequent sections.

7.3.4 Proof of Theorem 7.3.1

Our approach is entirely performed in terms of a-priori estimates. Indeed, without loss of generality, we can assume that the regular initial data generates a local-in-time analytic solution, whose largest lifespan is denoted by $T > 0$. If $T < +\infty$, the norms on the left-hand side of (7.3.6) would blow up, thus our aim is to propagate the smallness condition of the initial data to any time $t \in (0, T)$. This implies in particular that $T = +\infty$ and thus the local solution is in reality global in time.

The core of the proof relies on a suitable transformation of the state variables (u, b_1) in system (7.3.3), which aims to highlight the time behaviour of the underlying radius of analyticity. More precisely, for a general non decreasing function $\eta : \mathbb{R}_+ := [0, +\infty) \rightarrow [0, \tau_0]$ in $C^2(\mathbb{R}_+)$, with $\eta(0) = 0$, we introduce the transformation $f \rightarrow f_\eta$ to a general function $f = f(t, x, y)$, by applying the Fourier multiplier $e^{(\tau_0 - \eta(t))(1 + |D_x|)}$:

$$(7.3.7) \quad f_\eta(t, x, y) := e^{(\tau_0 - \eta(t))(1 + |D_x|)} f(t, x, y), \quad \text{i.e.} \quad \mathcal{F}_x(f_\eta)(t, \xi, y) = e^{(\tau_0 - \eta(t))(1 + |\xi|)} \mathcal{F}_x(f)(t, \xi, y).$$

Here \mathcal{F}_x stands for the Fourier transform uniquely on the variable $x \in \mathbb{R}$, while τ_0 is the radius of analyticity of the initial data.

The Fourier multiplier and the related transformation (7.3.7) are well defined, as long as f is analytic in $x \in \mathbb{R}$ for fixed $(t, y) \in (0, T) \times (0, 1)$, with radius of analyticity given by $\tau_0 - \eta(t)$ (or larger). In particular this positive radius degrades (or stays constant) as time $t \in \mathbb{R}_+$ increases, since we restrict the function η within the interval $[0, \tau_0]$.

Our approach is to apply the mentioned transformation to both u and b_1 and eventually to determine an optimal function η in (7.3.7), such that the new functions u_η and $b_{1,\eta}$ fulfill a specific dissipative energy estimate (cf. Proposition 7.3.1 and (7.3.17) for the final form of η). This energy controls higher-order Sobolev norms of the transformed state variables u_η and $b_{1,\eta}$ and thus of the analytic norms of the solutions u and b_1 themselves.

Roughly speaking, for a general function η , the transformation produces some additional damping mechanisms and dissipations to the system, but at the same time introduces further nonlinearities that could complicate the overall analysis. Our goal is therefore to select a suitable function η , so that the damping mechanism are indeed predominant. In this regime, a suitable "high-order" energy occurs that allows to control the $H^{s,0}$ -norms of u_η and $b_{1,\eta}$ (or equivalently of $e^{(\tau_0 - \eta(t))|D_x|}u$ and $e^{(\tau_0 - \eta(t))|D_x|}b_1$, for the analyticity).

We proceed now to formalize the described strategy and we begin by stating the following proposition, that provides the mentioned energy inequality (with higher-order Sobolev norms) for a general function $\eta \in C^2(\mathbb{R}_+)$.

Proposition 7.3.1. *Denote by $\mathfrak{m} = \mathfrak{m}(\mathbb{J}, \kappa, \text{Pr}_m)$, $\mathfrak{M} = \mathfrak{M}(\mathbb{J}, \kappa, \text{Pr}_m)$ and by $\mathcal{R} = \mathcal{R}(\mathbb{J}, \kappa, \text{Pr}_m)$ the following three parameters:*

$$\mathfrak{m} := \min \left\{ 1, \mathbb{J}, \frac{\kappa}{\text{Pr}_m} \right\}, \quad \mathfrak{M} := \max \left\{ 1, \mathbb{J}, \frac{\kappa}{\text{Pr}_m} \right\}, \quad \mathcal{R} := \frac{1}{4\mathfrak{M}} = \frac{1}{4 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}}, \quad .$$

There exists a constant $D_s \geq 1$, which depends uniquely on the regularity $s > 2$, such that the following

inequality holds true for any function $\eta \in C^2(\mathbb{R})$, with $\eta(0) = 0$:

$$\begin{aligned}
(7.3.8) \quad & \underbrace{e^{\mathcal{R}t} \left(\mathcal{E}_s + \mathfrak{m} \eta' \mathcal{E}_{s+\frac{1}{2}} + \mathfrak{m}^2 (\eta')^2 \mathcal{E}_{s+1} \right)}_{\text{Energy}}(t) + \underbrace{\int_0^t e^{\mathcal{R}\tilde{t}} \left(\mathcal{D}_s + \eta' \mathcal{D}_{s+\frac{1}{2}} + \mathfrak{m} (\eta')^2 \mathcal{D}_{s+1} + \mathfrak{m}^2 (\eta')^3 \mathcal{D}_{s+\frac{3}{2}} \right)}_{\text{Dissipation}}(\tilde{t}) d\tilde{t} \\
& - \underbrace{\int_0^t e^{\mathcal{R}\tilde{t}} \left\{ \left(\mathbb{J} + \frac{\kappa}{\text{Pr}_m} \right) \eta'' \mathcal{E}_{s+\frac{1}{2}} + 2 \left(\mathbb{J}^2 + \frac{\kappa^2}{\text{Pr}_m^2} \right) \eta' \eta'' \mathcal{E}_{s+1} \right\}}_{\text{Rest due to the hyperbolicity of the system}}(\tilde{t}) d\tilde{t} \leq \underbrace{\mathcal{E}_s(0) + \mathfrak{M} \eta'(0) \mathcal{E}_{s+\frac{1}{2}}(0) + \mathfrak{M}^2 \eta'(0)^2 \mathcal{E}_{s+1}(0)}_{\text{Initial energy}} \\
& + D_s \max\{1, \mathbb{H}^2\} \underbrace{\left\{ \max \left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \int_0^t e^{\mathcal{R}\tilde{t}} \sqrt{\mathcal{E}_s(\tilde{t})} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} + \int_0^t e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} \right.}_{\text{Estimate of the bilinear terms}} \\
& \quad \left. + \max \left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \int_0^t e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \sqrt{\mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) \mathcal{D}_{s+\frac{3}{2}}(\tilde{t})} d\tilde{t} \right\}}_{\text{Estimate of the trilinear terms}}.
\end{aligned}$$

The functionals \mathcal{E}_s , $\mathcal{E}_{s+1/2}$ and \mathcal{E}_{s+1} are defined in terms of the following Sobolev norms for the transformed solution $(u_\eta, b_{1,\eta})$:

$$\begin{aligned}
(7.3.9) \quad \mathcal{E}_s &:= \frac{\mathbb{J}^2}{2} \left(\|\partial_t u_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 + \|\partial_y u_\eta\|_{H^{s,0}}^2 \right) \\
& \quad + \frac{\kappa^2}{2\text{Pr}_m^2} \|\partial_t b_{1,\eta}\|_{H^{s,0}}^2 + \frac{1}{2} \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\|_{H^{s,0}}^2 + \frac{\kappa}{\text{Pr}_m^2} \|\partial_y b_{1,\eta}\|_{H^{s,0}}^2, \\
\mathcal{E}_{s+\frac{1}{2}} &:= \frac{1}{2} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \frac{1}{2} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2, \quad \mathcal{E}_{s+1} := \|u_\eta\|_{H^{s+1,0}}^2 + \|b_{1,\eta}\|_{H^{s+1,0}}^2.
\end{aligned}$$

Furthermore the dissipative functionals \mathcal{D}_s , $\mathcal{D}_{s+1/2}$, \mathcal{D}_{s+1} and $\mathcal{D}_{s+\frac{3}{2}}$ are defined by

$$\begin{aligned}
(7.3.10) \quad \mathcal{D}_s &:= \frac{1}{2} \|\partial_y u_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\partial_t u_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\partial_y b_{1,\eta}\|_{H^{s,0}}^2 + \frac{1}{2} \|(\partial_t b)_{1,\eta}\|_{H^{s,0}}^2, \\
\mathcal{D}_{s+\frac{1}{2}} &:= \frac{1}{2} \|\partial_t u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \frac{1}{2} \|(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + 2 \|\partial_y u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \|\partial_t(u_\eta)\|_{H^{s+\frac{1}{2},0}}^2 + \frac{3}{8} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \\
& \quad + \frac{1}{2} \|(\partial_t b_1)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \frac{1}{2} \|(\partial_t b_1)_\eta + b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2 + 2 \|\partial_y b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2 + \|\partial_t(b_{1,\eta})\|_{H^{s+\frac{1}{2},0}}^2 + \frac{3}{8} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2, \\
\mathcal{D}_{s+1} &:= \frac{3}{4} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \frac{3}{4} \|\partial_t b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2, \quad \mathcal{D}_{s+\frac{3}{2}}(t) := \|u_\eta\|_{H^{s+\frac{3}{2},0}}^2 + \|b_{1,\eta}\|_{H^{s+\frac{3}{2},0}}^2.
\end{aligned}$$

Since the proof of this Proposition is rather technical, we postpone it to the forthcoming sections and we focus this paragraph to the remaining steps to prove Theorem 7.3.1. We shall however first provide some remarks on the main inequality (7.3.8), and highlight in particular the dissipative mechanisms due to η , as well as the more challenging terms, that we are indeed left to estimate.

An explicit relation on the constant $D_s \geq 1$ is formally determined later on (cf. (7.4.21)):

$$(7.3.11) \quad D_s = \frac{2^{2s+6}}{s-2} \left(1 + \frac{s-2}{\sqrt{s-1}} \right) \geq 1.$$

By assuming that Proposition 7.3.1 holds true, the proof of Theorem 7.3.1 follows with some straightforward steps. Indeed, our main goal is to determine a suitable function $\eta \in \mathcal{C}^2(\mathbb{R}_+)$ in (7.3.8) and a small parameter $\varepsilon_s > 0$ for the initial condition (7.3.4), that ensure the following relations:

- (a) the terms on the left-hand side of (7.3.8) are all non-negative and thus "support" the $H^{s,0}$ -energy inequality,

(b) the right-hand side of (7.3.8) can eventually be absorbed by some of the positive terms of the left-hand side, under a suitable smallness condition on the initial data.

For what concerns part (a), the only term that (for a general η) could reach negative values is the first integral at the second line of (7.3.8), namely

$$(7.3.12) \quad - \int_0^t e^{\mathcal{R}\tilde{t}} \left\{ \left(\mathbb{J} + \frac{\kappa}{\text{Pr}_m} \right) \eta'' \mathcal{E}_{s+\frac{1}{2}} + 2 \left(\mathbb{J}^2 + \frac{\kappa^2}{\text{Pr}_m^2} \right) \eta' \eta'' \mathcal{E}_{s+1} \right\} (\tilde{t}) d\tilde{t}.$$

The sign of this integral is entangled with the sign of the weights $-\eta''(t)$ and $-2\eta'(t)\eta''(t)$, $t \in \mathbb{R}$. It is natural therefore to calibrate the function $\eta \in \mathcal{C}^2(\mathbb{R})$, in such a way that this integral provides a positive dissipation or at least vanishes. In other words, we shall seek for a function $\eta \in \mathcal{C}^2(\mathbb{R}_+)$ such that

$$(7.3.13) \quad \tau_0 - \eta(t) > 0, \quad \eta'(t) \geq 0, \quad -\eta''(t) \geq 0, \quad \text{for any } t \in \mathbb{R}_+.$$

Among the several functions satisfying (7.3.13), we consider a specific family of the form $\eta(t) := \tau_0(1 - e^{-\lambda t})$, where λ is (momentarily) an arbitrary positive constant (the exact value of λ for our analysis will be shortly be determined in (7.3.17)). Indeed, we remark that

$$\tau_0 - \eta(t) = \tau_0 e^{-\lambda t} > 0, \quad \eta'(t) = \lambda \tau_0 e^{-\lambda t} > 0, \quad -\eta''(t) = \lambda^2 \tau_0 e^{-\lambda t} > 0, \quad \text{for any } t \in \mathbb{R}_+.$$

In doing so, we can recast the main inequality (7.3.8) into

$$(7.3.14) \quad \begin{aligned} & e^{\mathcal{R}t} \mathcal{E}_s(t) + \mathfrak{m} \lambda \tau_0 e^{(\mathcal{R}-\lambda)t} \mathcal{E}_{s+\frac{1}{2}} + \mathfrak{m}^2 \lambda^2 \tau_0^2 e^{(\mathcal{R}-2\lambda)t} \mathcal{E}_{s+1}(t) + \int_0^t e^{\mathcal{R}\tilde{t}} \mathcal{D}_s(\tilde{t}) d\tilde{t} + \tau_0 \lambda \int_0^t e^{(\mathcal{R}-\lambda)\tilde{t}} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} + \\ & + \mathfrak{m} \tau_0^2 \lambda^2 \int_0^t e^{(\mathcal{R}-2\lambda)\tilde{t}} \mathcal{D}_{s+1}(\tilde{t}) d\tilde{t} + \mathfrak{m}^2 \tau_0^3 \lambda^3 \int_0^t e^{(\mathcal{R}-3\lambda)\tilde{t}} \mathcal{D}_{s+\frac{3}{2}}(\tilde{t}) d\tilde{t} \leq \mathcal{E}_s(0) + \mathfrak{M} \tau_0^2 \lambda \mathcal{E}_{s+\frac{1}{2}}(0) + \\ & + \mathfrak{M}^2 \tau_0 \lambda^2 \mathcal{E}_{s+1}(0) + D_s \max\{1, \mathbb{H}^2\} \left\{ \max\left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \int_0^t e^{\mathcal{R}\tilde{t}} \sqrt{\mathcal{E}_s(\tilde{t})} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} + \right. \\ & \left. + \int_0^t e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} + \max\left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \int_0^t e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \sqrt{\mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) \mathcal{D}_{s+\frac{3}{2}}(\tilde{t})} d\tilde{t} \right\}, \end{aligned}$$

where we have dropped the integral (7.3.12) on the left-hand side of the inequality, since it is positive and furthermore it does not support the next steps of our analysis. Our goal is therefore to determine a suitable $\lambda > 0$ in (7.3.14) and a suitable parameter $\varepsilon_s > 0$ in the smallness condition (7.3.4) for the initial data, such that all integrals in the third line of (7.3.14) can be absorbed by the dissipative terms of the inequality. Consequently, we shall first reformulate these integrals in accordance with the dissipative terms. First

$$(7.3.15) \quad \begin{aligned} & \int_0^t e^{\mathcal{R}\tilde{t}} \sqrt{\mathcal{E}_s(\tilde{t})} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} \leq \sup_{\tilde{t} \in (0,t)} \left(e^{2\lambda\tilde{t}} \mathcal{E}_s(\tilde{t}) \right)^{\frac{1}{2}} \int_0^t e^{(\mathcal{R}-\lambda)\tilde{t}} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t}, \\ & \int_0^t e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} \leq \sup_{\tilde{t} \in (0,t)} \left(e^{\lambda\tilde{t}} \mathcal{E}_s(\tilde{t}) \right) \int_0^t e^{(\mathcal{R}-\lambda)\tilde{t}} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t}. \end{aligned}$$

We hence remark that we can bound $e^{2\lambda t} \mathcal{E}_s(\tilde{t})$ and $e^{\lambda t} \mathcal{E}_s(\tilde{t})$ with the energy $e^{\mathcal{R}t} \mathcal{E}_s(t)$ in (7.3.14), as long as $\lambda > 0$ is considered within the range $\lambda \leq \mathcal{R}/2 = 1/(8 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\})$. Major difficulties arise however from the last integrals at the fourth line of (7.3.14), since this term involves the dissipation

$\mathcal{D}_{s+3/2}$, which has indeed the highest Sobolev regularity. We deal with this integral, by observing that (7.3.16)

$$\begin{aligned} & \max \left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \int_0^t e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \sqrt{\mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) \mathcal{D}_{s+\frac{3}{2}}(\tilde{t})} d\tilde{t} = \\ & = \max \left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \int_0^t e^{2\lambda\tilde{t}} \mathcal{E}_s(\tilde{t}) \sqrt{e^{(\mathcal{R}-\lambda)\tilde{t}} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t})} \sqrt{e^{(\mathcal{R}-3\lambda)\tilde{t}} \mathcal{D}_{s+\frac{3}{2}}(\tilde{t})} d\tilde{t}, \\ & \leq \sup_{\tilde{t} \in (0,t)} \left(e^{2\lambda\tilde{t}} \mathcal{E}_s(\tilde{t}) \right) \left\{ \int_0^t e^{(\mathcal{R}-\lambda)\tilde{t}} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} + \frac{1}{4} \max \left\{ \frac{1}{\mathbb{J}}, \frac{\text{Pr}_m^2}{\kappa} \right\} \int_0^t e^{(\mathcal{R}-3\lambda)\tilde{t}} \mathcal{D}_{s+\frac{3}{2}}(\tilde{t}) d\tilde{t} \right\}, \end{aligned}$$

where we can still bound $e^{2\lambda\tilde{t}} \mathcal{E}_s(\tilde{t})$ with the energy $e^{\mathcal{R}t} \mathcal{E}_s(t)$, as long as $\lambda \leq \mathcal{R}/2$.

We are now in the condition to set a specific value of λ , namely

$$(7.3.17) \quad \lambda := \frac{\mathcal{R}}{4} = \frac{1}{16 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}}, \quad \text{namely} \quad \eta(t) = \tau_0 \left(1 - e^{-\frac{t}{16 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}}} \right)$$

(we do not consider the threshold $\lambda = \mathcal{R}/2$, since the exponential function in front of \mathcal{E}_{s+1} in (7.3.14) would in that case vanish, not allowing us to derive an exponential decay of the related norms). In particular, coupling (7.3.15) and (7.3.16) together with (7.3.14), we obtain

$$(7.3.18) \quad \begin{aligned} & e^{\mathcal{R}t} \mathcal{E}_s(t) + \frac{\mathfrak{m}\mathcal{R}\tau_0}{4} e^{\frac{3}{4}\mathcal{R}t} \mathcal{E}_{s+\frac{1}{2}}(t) + \frac{\mathfrak{m}^2\mathcal{R}^2\tau_0}{4^2} e^{\frac{\mathcal{R}}{2}t} \mathcal{E}_{s+1}(t) + \\ & + \left(\frac{\tau_0\mathcal{R}}{4} - D_s \max\{1, \mathbb{H}^2\} \sup_{\tilde{t} \in (0,t)} \left\{ \max \left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \left(e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right)^{\frac{1}{2}} + 2e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right\} \right) \int_0^t e^{\frac{3\mathcal{R}}{4}\tilde{t}} \mathcal{D}_{s+\frac{1}{2}}(\tilde{t}) d\tilde{t} + \\ & + \left(\frac{\mathfrak{m}^2\tau_0^3\mathcal{R}^3}{4^3} - \frac{D_s}{4} \max\{1, \mathbb{H}^2\} \max \left\{ \frac{1}{\mathbb{J}}, \frac{\text{Pr}_m^2}{\kappa} \right\} \sup_{\tilde{t} \in (0,t)} e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right) \int_0^t e^{\frac{\mathcal{R}}{4}\tilde{t}} \mathcal{D}_{s+\frac{3}{2}}(\tilde{t}) d\tilde{t} \\ & \leq \mathcal{E}_s(0) + \frac{\mathfrak{M}\mathcal{R}\tau_0}{4} \mathcal{E}_{s+\frac{1}{2}}(0) + \frac{\mathfrak{M}^2\mathcal{R}^2\tau_0^2}{4^2} \mathcal{E}_{s+1}(0), \end{aligned}$$

where we have omitted to write the positive dissipative integrals in \mathcal{D}_s and \mathcal{D}_{s+1} on the left-hand side. Now, we remark that $\mathcal{E}_s(0) + \frac{\mathfrak{M}\mathcal{R}\tau_0}{4} \mathcal{E}_{s+\frac{1}{2}}(0) + \frac{\mathfrak{M}^2\mathcal{R}^2\tau_0^2}{4^2} \mathcal{E}_{s+1}(0)$ can be estimated in terms of the initial data and the smallness condition (7.3.4). Indeed, recalling the definition of \mathcal{E}_s , $\mathcal{E}_{s+1/2}$ and \mathcal{E}_{s+1} in (7.3.9), we observe that

$$\begin{aligned} \mathcal{E}_s(0) & \leq \max \left\{ 1, \mathbb{J}^2, \left(\frac{\kappa}{\text{Pr}_m} \right)^2 \right\} \max \left\{ 1, \frac{1}{\text{Pr}_m} \right\} \left(\frac{3}{2} \|\tilde{u}\|_{H^{s,0}}^2 + \frac{1}{2} \|\bar{u}\|_{H^{s,0}}^2 + \right. \\ & \quad \left. + \|\partial_y \bar{u}\|_{H^{s,0}}^2 + \frac{3}{2} \|\tilde{b}_1\|_{H^{s,0}}^2 + \frac{1}{2} \|\bar{b}_1\|_{H^{s,0}}^2 + \|\partial_y \bar{b}_1\|_{H^{s,0}}^2 \right) \\ & \leq 2 \max \left\{ 1, \mathbb{J}, \frac{\kappa}{\text{Pr}_m} \right\}^2 \max \left\{ 1, \frac{1}{\text{Pr}_m} \right\} \left(\|\tilde{u}\|_{H^{s,0}} + \|\bar{u}\|_{H^{s+1,0}} + \right. \\ & \quad \left. + \|\partial_y \bar{u}\|_{H^{s,0}} + \|\tilde{b}_1\|_{H^{s,0}} + \|\bar{b}_1\|_{H^{s+1,0}} + \|\partial_y \bar{b}_1\|_{H^{s,0}} \right)^2 \\ & \leq 2 \max \left\{ 1, \mathbb{J}, \frac{\kappa}{\text{Pr}_m} \right\}^2 \max \left\{ 1, \frac{1}{\text{Pr}_m} \right\} \delta^2, \end{aligned}$$

where we recall that $\delta > 0$ is the small parameter bounding the norms of the initial data and it is defined in (7.3.4) by

$$\delta := \left(\frac{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}}}{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{5}{2}}} \frac{\min\{\tau_0, \tau_0^{-1}\}^{\frac{3}{2}}}{\max\{\text{Pr}_m^{-1}, \text{Pr}_m\}^{\frac{1}{2}}} \right) \varepsilon_s,$$

for a small parameter ε_s that depends only on $s > 2$ (and that we have not determined, yet). Similarly, we have that $\mathcal{E}_{s+1/2}(0) \leq (1/2)\delta^2$ and $\mathcal{E}_{s+1}(0) \leq \delta^2$. Hence, we obtain the following estimate of the

right-hand side in (7.3.18):

$$\begin{aligned}
(7.3.19) \quad \mathcal{E}_s(0) + \frac{\mathfrak{M}\mathcal{R}\tau_0}{4}\mathcal{E}_{s+\frac{1}{2}}(0) + \frac{\mathfrak{M}^2\mathcal{R}^2\tau_0^2}{4^2}\mathcal{E}_{s+1}(0) &\leq \mathcal{E}_s(0) + \frac{\tau_0}{4^2}\mathcal{E}_{s+\frac{1}{2}}(0) + \frac{\tau_0^2}{4^4}\mathcal{E}_{s+1}(0) \\
&\leq \max\{1, \tau_0\}^2 \left(2 \max\left\{1, \mathbb{J}, \frac{\kappa}{\text{Pr}_m}\right\}^2 \max\left\{1, \frac{1}{\text{Pr}_m}\right\} \delta^2 + \frac{1}{4^2}\delta^2 + \frac{1}{4^4}\delta^2 \right) \\
&\leq 4 \max\{1, \tau_0\}^2 \max\left\{1, \mathbb{J}, \frac{\kappa}{\text{Pr}_m}\right\}^2 \max\left\{1, \frac{1}{\text{Pr}_m}\right\} \delta^2 \\
&\leq \frac{4 \max\{1, \tau_0\}^2 \min\{\tau_0, \tau_0^{-1}\}^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{\max\{1, \mathbb{H}^2\}^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3 \max\{\text{Pr}_m^{-1}, \text{Pr}_m\} \min\{1, \text{Pr}_m\}} \varepsilon_s^2 \\
&\leq \frac{4 \min\{1, \tau_0\}^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{\max\{1, \mathbb{H}^2\}^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3 \max\{1, \text{Pr}_m\}} \varepsilon_s^2.
\end{aligned}$$

where, in the last line, we have used the identity $\max\{\text{Pr}_m^{-1}, \text{Pr}_m\} \min\{1, \text{Pr}_m\} = \max\{1, \text{Pr}_m\}$, as well as the inequality $\max\{1, \tau_0\}^2 \min\{\tau_0, \tau_0^{-1}\}^3 \leq \min\{1, \tau_0\}^2$. Inequality (7.3.18) together with (7.3.19) allow us to conclude the proof by means of a bootstrap method. Indeed, by introducing the small parameter $\varepsilon_s > 0$ in (7.3.4) and a maximal time $T^* \in (0, T)$, such that

$$\begin{aligned}
(7.3.20) \quad \varepsilon_s^2 &:= \frac{1}{2^{16} D_s^2} \quad \left(\text{thus } \varepsilon_s := \frac{s-2}{2^{2s+14}} \frac{1}{1 + \frac{s-2}{\sqrt{s-1}}}, \text{ because of (7.3.11)} \right), \\
T^* &:= \sup \left\{ t \in (0, T) : e^{\mathcal{R}t} \mathcal{E}_s(t) \leq \frac{1}{2^{12} D_s^2} \frac{\min\{1, \tau_0\}^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{\max\{1, \mathbb{H}^2\}^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3 \max\{1, \text{Pr}_m\}} < 1 \right\},
\end{aligned}$$

we have that, for any time $t \in (0, T^*)$, the constants in front of the dissipative terms of (7.3.18) are indeed positive:

$$\begin{aligned}
&\frac{\tau_0 \mathcal{R}}{4} - D_s \max\{1, \mathbb{H}^2\} \sup_{\tilde{t} \in (0, t)} \left\{ \max\left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} \left(e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right)^{\frac{1}{2}} + 2e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right\} = \\
&= \frac{\tau_0}{4^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} - D_s \max\{1, \mathbb{H}^2\} \sup_{\tilde{t} \in (0, t)} \left\{ \left(e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right)^{\frac{1}{2}} \left(\max\left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} + 2 \left(e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right)^{\frac{1}{2}} \right) \right\} \\
&\geq \frac{\tau_0}{4^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} - D_s \max\{1, \mathbb{H}^2\} \sup_{\tilde{t} \in (0, t)} \left\{ \left(e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right)^{\frac{1}{2}} \left(\max\left\{ \frac{1}{\sqrt{\mathbb{J}}}, \frac{\text{Pr}_m}{\sqrt{\kappa}} \right\} + 2 \right) \right\} \\
&\geq \frac{\tau_0}{4^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} - D_s \max\{1, \mathbb{H}^2\} \sup_{\tilde{t} \in (0, t)} \left\{ \left(e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \right)^{\frac{1}{2}} \frac{4 \max\{1, \text{Pr}_m\}^{\frac{1}{2}}}{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{1}{2}}} \right\} \\
&\geq \frac{\tau_0}{4^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} - \frac{D_s \max\{1, \mathbb{H}^2\} \min\{1, \tau_0\}^{\frac{3}{2}} \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}}}{2^6 D_s \max\{1, \mathbb{H}^2\} \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{3}{2}} \max\{1, \text{Pr}_m\}^{\frac{1}{2}} \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{1}{2}}} \frac{4 \max\{1, \text{Pr}_m\}^{\frac{1}{2}}}{\min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{1}{2}}} \\
&\geq \frac{\min\{1, \tau_0\}}{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \left(\frac{1}{4^2} - \frac{1}{2^4} \frac{\min\{1, \tau_0\}^{\frac{1}{2}} \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}}{\max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^{\frac{1}{2}}} \right) \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\mathfrak{m}^2 \tau_0^3 \mathcal{R}^3}{4^3} - \frac{D_s}{4} \max\{1, \mathbb{H}^2\} \max\left\{ \frac{1}{\mathbb{J}}, \frac{\text{Pr}_m^2}{\kappa} \right\} \sup_{\tilde{t} \in (0, t)} e^{\mathcal{R}\tilde{t}} \mathcal{E}_s(\tilde{t}) \\
&\geq \frac{\tau_0^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^2}{4^6 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3} - \frac{D_s \max\{1, \mathbb{H}^2\} \max\{1, \text{Pr}_m\}}{4 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}} \frac{\min\{1, \tau_0\}^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{2^{12} D_s^2 \max\{1, \mathbb{H}^2\}^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3 \max\{1, \text{Pr}_m\}} \\
&\geq \frac{\tau_0^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^2}{4^6 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3} - \frac{\min\{1, \tau_0\}^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^2}{2^{14} \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3} \geq 0.
\end{aligned}$$

Hence the energy inequality (7.3.18) and the estimate (7.3.19) at initial time $t = 0$ imply that the

functional $e^{t/4}\mathcal{E}_s(t)$ stays small for any $t \in (0, T^*)$:

$$e^{\mathcal{R}t}\mathcal{E}_s(t) \leq \frac{1}{2^{14}} \frac{\min\{1, \tau_0\}^3 \min\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3}{D_s^2 \max\{1, \mathbb{H}^2\}^2 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\}^3 \max\{1, \text{Pr}_m\}}.$$

From the definition of T^* in (7.3.20), we finally deduce that $T^* = T$, which must be infinite since the analytic norm does not blow up at this time. Accordingly, the solution (u, b_1) is indeed global in time.

Finally, inequality (7.3.6) about the exponential decay of the norms of the solution follows directly from the estimate (7.3.18), which implies

$$\begin{aligned} e^{\frac{\mathcal{R}}{2}t} \min\left\{1, \frac{\mathfrak{m}\mathcal{R}}{4}\right\}^2 \min\{1, \tau_0\} \left(\mathcal{E}_s(t) + \mathcal{E}_{s+1/2}(t) + \mathcal{E}_{s+1}(t)\right) &\leq \\ &\leq \max\{1, \tau_0\}^2 \left(\mathcal{E}_s(0) + \mathcal{E}_{s+1/2}(0) + \mathcal{E}_{s+1}(0)\right). \end{aligned}$$

We hence achieve (7.3.6), by manipulating $\mathfrak{m} = \min\{\mathbb{J}, \kappa/\text{Pr}_m\}$ and $\mathbb{R} = 1/(4 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\})$, as well as by recalling the definition of the functions \mathcal{E}_s , $\mathcal{E}_{s+1/2}$ and \mathcal{E}_{s+1} in (7.3.9). This concludes the proof of Theorem 7.3.1.

7.4 Proof of Proposition 7.3.1

The core of our approach being showed, it remains to prove the ‘‘high-order’’ energy estimate described by Proposition 7.3.1.

7.4.1 Estimates related to the equation of u

In this section we deal with the momentum equation of u in the main system (7.3.3), which satisfies

$$(7.4.1) \quad \mathbb{J}\partial_t^2 u + \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = \mathbb{H}^2 \left\{ b_1 b_2 v - u b_2^2 + b_2 \left(\int_0^y \partial_t b_1 \right) \right\}.$$

Recalling the definition of $u_\eta(t, \cdot) = e^{(\tau_0 - \eta(t))(1 + |D_x|)} u(t, \cdot)$ and the value $\mathcal{R} = 1/(4 \max\{1, \mathbb{J}, \kappa/\text{Pr}_m\})$, we remark that the function $e^{\mathcal{R}t/2} u_\eta(t, x, y)$ is solution of

$$(7.4.2) \quad \begin{aligned} \mathbb{J}\partial_t(e^{\frac{\mathcal{R}}{2}t}(\partial_t u)_\eta) + \mathbb{J}e^{\frac{\mathcal{R}}{2}t}\eta'(t)(1 + |D_x|)(\partial_t u)_\eta + e^{\frac{\mathcal{R}}{2}t}(\partial_t u)_\eta + e^{\frac{\mathcal{R}}{2}t}(u\partial_x u)_\eta \\ + e^{\frac{\mathcal{R}}{2}t}(v\partial_y u)_\eta + e^{\frac{\mathcal{R}}{2}t}\partial_x p_\eta - \partial_y^2(e^{\frac{\mathcal{R}}{2}t}u_\eta) = \frac{\mathbb{J}\mathcal{R}}{2}e^{\frac{\mathcal{R}}{2}t}(\partial_t u)_\eta + e^{\frac{\mathcal{R}}{2}t}F_\eta, \end{aligned}$$

where the forcing term F_η in (7.3.3) is generated by applying the Fourier multiplier $e^{(\tau_0 - \eta(t))(1 + |D_x|)}$ to the right-hand side of (7.4.1):

$$(7.4.3) \quad F_\eta = \mathbb{H}^2(b_1 b_2 v)_\eta - \mathbb{H}^2(u b_2^2)_\eta + \mathbb{H}^2\left(b_2 \left(\int_0^y \partial_t b_1 \right)\right)_\eta.$$

We can further derive an equivalent form of this equation, by developing the time derivative $e^{\mathcal{R}t/2}(\partial_t u)_\eta$ in the third term of (7.4.2) by means of $e^{\mathcal{R}t/2}(\partial_t(u_\eta) + \eta'(t)(1 + |D_x|)u_\eta)$. Thus equation (7.4.2) can also be recasted as

$$(7.4.4) \quad \begin{aligned} \partial_t(e^{\frac{\mathcal{R}}{2}t}(\mathbb{J}(\partial_t u)_\eta + u_\eta)) + e^{\frac{\mathcal{R}}{2}t}\eta'(t)(1 + |D_x|)(\mathbb{J}(\partial_t u)_\eta + u_\eta) + e^{\frac{\mathcal{R}}{2}t}(u\partial_x u)_\eta \\ + e^{\frac{\mathcal{R}}{2}t}(v\partial_y u)_\eta + e^{\frac{\mathcal{R}}{2}t}\partial_x p_\eta - \partial_y^2(e^{\frac{\mathcal{R}}{2}t}u_\eta) = \frac{\mathcal{R}}{2}e^{\frac{\mathcal{R}}{2}t}(\mathbb{J}(\partial_t u)_\eta + u_\eta) + e^{\frac{\mathcal{R}}{2}t}F_\eta. \end{aligned}$$

Next, we take the $H^{s,0}$ -inner product between (7.4.2) and $e^{\mathcal{R}t/2}\mathbb{J}(\partial_t u)_\eta$ and adding the result with the $H^{s,0}$ -inner product between (7.4.4) and $e^{\mathcal{R}t/2}(\mathbb{J}(\partial_t u)_\eta + u_\eta)$, we gather that

$$(7.4.5) \quad \begin{aligned} & \frac{d}{dt} \left[e^{\mathcal{R}t} \left(\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|\partial_y u_\eta\|_{H^{s,0}}^2 \right) \right] + e^{\mathcal{R}t} \left(\|\partial_y u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 \right) \\ & + \eta'(t) e^{\mathcal{R}t} \left\{ \mathbb{J}^2 \|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + 2\mathbb{J} \|\partial_y u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \right\} = \mathcal{R} e^{\mathcal{R}t} \left[\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 \right. \\ & \quad \left. + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|\partial_y u_\eta\|_{H^{s,0}}^2 \right] + e^{\mathcal{R}t} \left\{ - \langle (u \partial_x u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right. \\ & \quad \left. - \langle (v \partial_y u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} + \langle F_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} - \langle \partial_x p_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right\}. \end{aligned}$$

We begin by observing that the term in the square brackets of the right-hand side in (7.4.5) can be absorbed by the dissipation $e^{\mathcal{R}t} (\|\partial_y u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|(\partial_t u)_\eta\|_{H^{s,0}}^2)$ of the left-hand side. Indeed, since the value of \mathbb{R} is smaller than $\min\{1/4, 1/(4\mathbb{J})\}$, we have that

$$\begin{aligned} & \mathcal{R} e^{\mathcal{R}t} \left[\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|\partial_y u_\eta\|_{H^{s,0}}^2 \right] \\ & \leq \mathcal{R} e^{\mathcal{R}t} \left[\frac{3\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \|u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|\partial_y u_\eta\|_{H^{s,0}}^2 \right] \\ & \leq \frac{3\mathbb{J}}{8} e^{\mathcal{R}t} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{4} e^{\mathcal{R}t} \|u_\eta\|_{H^{s,0}}^2 + \frac{1}{4} e^{\mathcal{R}t} \|\partial_y u_\eta\|_{H^{s,0}}^2. \end{aligned}$$

To absorb this last term, we shall then invoke the Poincaré inequality in $y \in (0, 1)$: $\|u_\eta\|_{H^{s,0}} \leq \|\partial_y u_\eta\|_{H^{s,0}}$ (here the homogeneous boundary condition $u = 0$ in $y = 0$ comes into play). Thus

$$\mathcal{R} e^{\mathcal{R}t} \left[\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|\partial_y u_\eta\|_{H^{s,0}}^2 \right] \leq e^{\mathcal{R}t} \left(\frac{\mathbb{J}}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\partial_y u_\eta\|_{H^{s,0}}^2 \right),$$

which corresponds to half dissipation on the left-hand side of (7.4.5).

We can summarize what obtained with the following estimate:

$$(7.4.6) \quad \begin{aligned} & \frac{d}{dt} \left[e^{\mathcal{R}t} \left(\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|\partial_y u_\eta\|_{H^{s,0}}^2 \right) \right] + \frac{1}{2} e^{\mathcal{R}t} \left(\|\partial_y u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 \right) \\ & \quad + \eta'(t) e^{\mathcal{R}t} \left\{ \mathbb{J}^2 \|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + 2\mathbb{J} \|\partial_y u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \right\} \\ & \leq e^{\mathcal{R}t} \left\{ - \langle (u \partial_x u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} + \langle (v \partial_y u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right. \\ & \quad \left. + \langle F_\eta, 2(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} - \langle \partial_x p_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right\}. \end{aligned}$$

The left-hand side of (7.4.6) already provides information on several norms of the solution. However these norms are of Sobolev regularities lower than $s+1/2$, while our final estimate (7.3.8) in Proposition 7.3.1 incorporates higher regularities, such as $H^{s+3/2,0}$ in $\mathcal{D}_{s+3/2}$. We shall therefore perform a further development of the above inequality. To this end, we first isolate $\frac{\eta'(t)}{2} e^{\mathcal{R}t} (\mathbb{J}^2 \|(\partial_t u)_\eta\|_{H^{s+1/2,0}}^2 + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+1/2,0}}^2)$ in the second line of (7.4.6) and, recalling the formula $(\partial_t u)_\eta = \partial_t(u_\eta) + \eta'(t)(1 + |D_x|)u_\eta$, we remark that

$$\begin{aligned} & \frac{\eta'(t)}{2} e^{\mathcal{R}t} \left\{ \mathbb{J}^2 \|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \right\} = \\ & = \frac{\eta'(t)}{2} e^{\mathcal{R}t} \left\{ \mathbb{J}^2 \|\partial_t(u_\eta) + \eta'(t)(1 + |D_x|)u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \|\mathbb{J}(\partial_t(u_\eta) + \mathbb{J}\eta'(t)(1 + |D_x|)u_\eta + u_\eta)\|_{H^{s+\frac{1}{2},0}}^2 \right\} \\ & = \frac{\eta'(t)}{2} e^{\mathcal{R}t} \left\{ \mathbb{J}^2 \|\partial_t(u_\eta)\|_{H^{s+\frac{1}{2},0}}^2 + \mathbb{J}^2 \eta'(t)^2 \|u_\eta\|_{H^{s+\frac{3}{2},0}}^2 + \mathbb{J}^2 2\eta'(t) \langle \partial_t(u_\eta), u_\eta \rangle_{H^{s+1,0}} + \mathbb{J}^2 \|\partial_t(u_\eta)\|_{H^{s+\frac{1}{2},0}}^2 \right. \\ & \quad \left. + \mathbb{J}^2 \eta'(t)^2 \|u_\eta\|_{H^{s+\frac{3}{2},0}}^2 + \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \mathbb{J}^2 2\eta'(t) \langle \partial_t(u_\eta), u_\eta \rangle_{H^{s+1,0}} \right\} \end{aligned}$$

$$+ \mathbb{J}2\langle \partial_t(u_\eta), u_\eta \rangle_{H^{s+\frac{1}{2},0}} + \mathbb{J}2\eta'(t)\|u_\eta\|_{H^{s+1,0}}^2 \Big\}.$$

In this last identity, we have used the relation $\|(1 + |D_x|)u_\eta\|_{H^{s+\frac{1}{2},0}} = \|u_\eta\|_{H^{s+\frac{3}{2},0}}$, as well as the inner products $\langle \partial_t(u_\eta), (1 + |D_x|)u_\eta \rangle_{H^{s+\frac{1}{2},0}} = \langle \partial_t(u_\eta), u_\eta \rangle_{H^{s+1,0}}$ and $\langle (1 + |D_x|)u_\eta, u_\eta \rangle_{H^{s+\frac{1}{2},0}} = \|u_\eta\|_{H^{s+1,0}}^2$. Thus, the isolated term satisfies

$$\begin{aligned} \frac{\eta'(t)}{2} e^{\mathcal{R}t} \Big\{ \mathbb{J}^2 \|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \Big\} &= \mathbb{J}^2 e^{\mathcal{R}t} \eta'(t)^2 \frac{d}{dt} \|u_\eta\|_{H^{s+1,0}}^2 \\ &+ \frac{\mathbb{J}}{2} e^{\mathcal{R}t} \eta'(t) \frac{d}{dt} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + e^{\mathcal{R}t} \Big\{ \mathbb{J}^2 \eta'(t) \|\partial_t(u_\eta)\|_{H^{s+\frac{1}{2},0}}^2 + \mathbb{J}^2 \eta'(t)^3 \|u_\eta\|_{H^{s+\frac{3}{2},0}}^2 \\ &+ \frac{\eta'(t)}{2} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \mathbb{J} \eta'(t)^2 \|u_\eta\|_{H^{s+1,0}}^2 \Big\}. \end{aligned}$$

and by bringing the time derivative in front of $\mathbb{J}e^{\mathcal{R}t}\eta'(t)^2$ and $\mathbb{J}\eta'(t)/2$ also

$$(7.4.7) \quad \begin{aligned} \frac{\eta'(t)}{2} e^{\mathcal{R}t} \Big\{ \mathbb{J}^2 \|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \Big\} &= \frac{d}{dt} \left[\mathbb{J}^2 \eta'(t)^2 e^{\mathcal{R}t} \|u_\eta\|_{H^{s+1,0}}^2 + \frac{\mathbb{J}}{2} \eta'(t) e^{\mathcal{R}t} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \right] \\ &- 2\mathbb{J}^2 e^{\mathcal{R}t} \eta'(t) \eta''(t) \|u_\eta\|_{H^{s+1,0}}^2 - \frac{\mathbb{J}}{2} \eta''(t) e^{\mathcal{R}t} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + e^{\mathcal{R}t} \Big\{ \mathbb{J}^2 \eta'(t) \|\partial_t(u_\eta)\|_{H^{s+\frac{1}{2},0}}^2 \\ &+ \mathbb{J}^2 \eta'(t)^3 \|u_\eta\|_{H^{s+\frac{3}{2},0}}^2 + \left(\frac{1}{2} - \frac{\mathbb{J}\mathcal{R}}{2} \right) \eta'(t) \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + (\mathbb{J} - \mathcal{R}\mathbb{J}^2) \eta'(t)^2 \|u_\eta\|_{H^{s+1,0}}^2 \Big\}. \end{aligned}$$

The Sobolev norm of $H^{s+3/2,0}$ has now appeared (first term of the third line). Furthermore, recalling that $\mathcal{R} \leq (1/4) \min\{1, \mathbb{J}^{-1}\}$ both $(1/2 - \mathbb{J}\mathcal{R}/2) \geq 3/8$ and $\mathbb{J} - \mathcal{R}\mathbb{J}^2 \geq 3\mathbb{J}/4$ are positive. Needless to say, this high regularity comes with a price, namely the appearance of certain terms which depend on the second time derivative $\eta''(t)$ (and are also in our main estimate (7.3.8)). By coupling (7.4.6) together with (7.4.7), we eventually gather the estimate

$$(7.4.8) \quad \begin{aligned} &\frac{d}{dt} \left[e^{\mathcal{R}t} \underbrace{\left(\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|\partial_y u_\eta\|_{H^{s,0}}^2 + \eta'(t) \frac{\mathbb{J}}{2} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \right)}_{u\text{-terms in } \mathcal{E}_s(t)} \underbrace{\frac{\mathbb{J}}{2} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2}_{u\text{-term in } \mathcal{E}_{s+\frac{1}{2}}(t)} \right. \\ &\left. + \eta'(t)^2 \underbrace{\mathbb{J}^2 \|u_\eta\|_{H^{s+1,0}}^2}_{u\text{-term in } \mathcal{E}_{s+1}(t)} \right] + \eta'(t) e^{\mathcal{R}t} \Big\{ \underbrace{\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 + 2\mathbb{J} \|\partial_y u_\eta\|_{H^{s+\frac{1}{2},0}}^2}_{u\text{-terms in } \mathcal{D}_{s+\frac{1}{2}}(t)} \\ &\quad + \underbrace{\mathbb{J}^2 \|\partial_t(u_\eta)\|_{H^{s+\frac{1}{2},0}}^2 + \frac{3}{8} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2}_{u\text{-terms in } \mathcal{D}_{s+\frac{1}{2}}(t)} \Big\} + \mathbb{J} \eta'(t)^2 e^{\mathcal{R}t} \underbrace{\frac{3}{4} \|u_\eta\|_{H^{s+1,0}}^2}_{u\text{-term in } \mathcal{D}_{s+1}(t)} + \mathbb{J}^2 \eta'(t)^3 e^{\mathcal{R}t} \underbrace{\|u_\eta\|_{H^{s+\frac{3}{2},0}}^2}_{u\text{-terms in } \mathcal{D}_{s+\frac{3}{2}}(t)} \\ &- 2\mathbb{J}^2 e^{\mathcal{R}t} \eta'(t) \eta''(t) \underbrace{\|u_\eta\|_{H^{s+1,0}}^2}_{u\text{-term in } \mathcal{E}_{s+1}(t)} - \underbrace{\mathbb{J} e^{\mathcal{R}t} \eta''(t) \frac{1}{2} \|u_\eta\|_{H^{s+\frac{1}{2},0}}^2}_{u\text{-term in } \mathcal{E}_{s+\frac{1}{2}}(t)} + \frac{e^{\mathcal{R}t}}{2} \underbrace{\left(\|\partial_y u_\eta\|_{H^{s,0}}^2 + \mathbb{J} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 \right)}_{u\text{-term in } \mathcal{D}_s(t)} \\ &\leq e^{\mathcal{R}t} \Big\{ - \langle (u \partial_x u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} - \langle (v \partial_y u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \\ &\quad + \langle F_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} - \langle \partial_x p_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \Big\}. \end{aligned}$$

We next proceed to estimate each term on the right-hand side of (7.4.8). For each estimated term, we will determine a suitable lower bound of the constant $D_s \geq 1$ in the main inequality (7.3.8) of Proposition 7.3.1. This lower bound will increase at any step. The last term will therefore provide the exact form of D_s .

Throughout the forthcoming analysis, we will repeatedly use the following estimates, which recast the

H^s -Sobolev norm of u_η and $b_{1,\eta}$ in terms of \mathcal{E}_s .

$$\begin{aligned}
(7.4.9) \quad & \|u_\eta\|_{H^{s,0}} \leq \|u_\eta + \mathbb{J}(\partial_t u)_\eta\|_{H^{s,0}} + \|\mathbb{J}(\partial_t u)_\eta\|_{H^{s,0}} \\
& \leq 2 \left(\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s,0}}^2 \right)^{\frac{1}{2}} \leq 2\sqrt{\mathcal{E}_s}, \\
& \|b_{1,\eta}\|_{H^{s,0}} \leq \left\| b_{1,\eta} + \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta \right\|_{H^{s,0}} + \frac{\kappa}{\text{Pr}_m} \|(\partial_t b_1)_\eta\|_{H^{s,0}} \\
& \leq 2 \left(\frac{\kappa^2}{2\text{Pr}_m^2} \|(\partial_t b_1)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\|_{H^{s,0}}^2 \right)^{\frac{1}{2}} \leq 2\sqrt{\mathcal{E}_s}.
\end{aligned}$$

Similarly, we can connect the $H^{s+1/2}$ -norms of u_η and $b_{1,\eta}$ in terms of $\mathcal{D}_{s+\frac{1}{2}}$:

$$(7.4.10) \quad \|u_\eta\|_{H^{s+\frac{1}{2},0}} \leq 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}}, \quad \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \leq 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}}.$$

The first term on the right-hand side of (7.4.8), that we deal with, is the convection

$$\begin{aligned}
(7.4.11) \quad & |e^{\mathcal{R}t} \langle (u\partial_x u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}}| \\
& \leq e^{\mathcal{R}t} \| (u\partial_x u)_\eta \|_{H^{s-\frac{1}{2},0}} \left(\|\mathbb{J}(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}} + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}} \right) \\
& \leq e^{\mathcal{R}t} \| (u\partial_x u)_\eta \|_{H^{s-\frac{1}{2},0}} 2 \left(\frac{\mathbb{J}^2}{2} \|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \frac{1}{2} \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}}^2 \right)^{\frac{1}{2}} \\
& \leq 2e^{\mathcal{R}t} \| (u\partial_x u)_\eta \|_{H^{s-\frac{1}{2},0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.
\end{aligned}$$

In order to cope with $(u\partial_x u)_\eta$ in $H^{s-1/2,0}$, we shall transfer the η -transformation i.e. the Lagrangian multiplier $e^{(\tau_0-\eta(t))(1+|D_x|)}$ to each component u and $\partial_x u$. Of course, $(u\partial_x u)_\eta \neq u_\eta \partial_x u_\eta$ in general. However, we are here controlling a Sobolev norm and not the functions themselves, pointwise. The following product law therefore allows us to transfer the mentioned Lagrangian multiplier in terms of pure Sobolev estimates:

Lemma 7.4.1. *Let $f, g : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$ be two functions such that f_η, g_η and $\partial_y f_\eta$ belong to $H^{\sigma_1,0}(\mathbb{R} \times (0, 1))$ with $\sigma_1 > 1/2$ (and thus also to $H^{\sigma_2,0}(\mathbb{R} \times (0, 1))$, for any $\sigma_2 \leq \sigma_1$). Furthermore, assume that $f \equiv 0$ in $y = 0$ in the sense of trace. Then*

$$\|(fg)_\eta\|_{H^{\sigma_1,0}} \leq \frac{2^{\sigma_1-\frac{1}{2}}}{\sqrt{\sigma_2-\frac{1}{2}}} \left(\|\partial_y f_\eta\|_{H^{\sigma_1,0}} \|g_\eta\|_{H^{\sigma_2,0}} + \|\partial_y f_\eta\|_{H^{\sigma_2,0}} \|g_\eta\|_{H^{\sigma_1,0}} \right)$$

for any regularities $\sigma_2 \in (1/2, \sigma_1]$.

We postpone the technical proof of this lemma 7.4.1 to the appendix (cf. Lemma 7.5.1). Addressing our original estimate (7.4.11), we are in the position to apply Lemma 7.4.1 with the regularities $\sigma_1 = \sigma_2 = s - 1/2 > 1/2$ and the functions $f(\cdot) = u(t, \cdot)$, $g(\cdot) = \partial_x u(t, \cdot)$. We deduce therefore that

$$\begin{aligned}
\| (u\partial_x u)_\eta \|_{H^{s-\frac{1}{2},0}} & \leq \frac{2^{s-1}}{\sqrt{s-1}} 2 \|\partial_y u_\eta\|_{H^{s-\frac{1}{2},0}} \|\partial_x u_\eta\|_{H^{s-\frac{1}{2},0}} \leq \frac{2^s}{\sqrt{s-1}} \|\partial_y u_\eta\|_{H^{s,0}} \|u_\eta\|_{H^{s+\frac{1}{2},0}} \\
& \leq \frac{2^{s+1}}{\sqrt{s-1}} \|\partial_y u_\eta\|_{H^{s,0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)},
\end{aligned}$$

where in the last inequality we have indeed applied (7.4.10). Plugging this inequality to the original estimate (7.4.11), we eventually gather that the convective term is bounded by

$$|e^{\mathcal{R}t} \langle (u\partial_x u)_\eta, u_\eta \rangle_{H^{s,0}}| \leq \frac{2^{s+2}}{\sqrt{s-1}} e^{\mathcal{R}t} \frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+\frac{1}{2}}(t).$$

We shall now remark that the above right-hand side is indeed in the first integrand of the third line of our main inequality (7.3.8) of Proposition 7.3.1. A necessary condition for the validity of this Proposition is that the related constant D_s must satisfy $D_s \geq 2^{s+2}/\sqrt{s-1}$.

Next, we deal with the second term on the right-hand side of (7.4.6), more precisely

$$\begin{aligned} e^{\mathcal{R}t} \left| \langle (v\partial_y u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| &\leq e^{\mathcal{R}t} \|(v\partial_y u)_\eta\|_{H^{s-\frac{1}{2},0}} \left(\mathbb{J}\|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}} + \|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}} \right) \\ &\leq e^{\mathcal{R}t} \|(v\partial_y u)_\eta\|_{H^{s-\frac{1}{2},0}} 2\sqrt{D_{s+\frac{1}{2}}(t)}. \end{aligned}$$

We apply Lemma 7.4.1 once more with regularities $\sigma_1 = \sigma_2 = s - 1/2 > 1/2$, but with functions $f = v$ and $g = \partial_y u$. Hence

$$\begin{aligned} e^{\mathcal{R}t} \left| \langle (v\partial_y u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| &\leq \frac{2^{s-1}}{\sqrt{s-1}} e^{\mathcal{R}t} 2 \|\partial_y v_\eta\|_{H^{s-\frac{1}{2},0}} \|\partial_y u_\eta\|_{H^{s-\frac{1}{2},0}} 2\sqrt{D_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{s+1}}{\sqrt{s-1}} e^{\mathcal{R}t} \|\partial_x u_\eta\|_{H^{s-\frac{1}{2},0}} \|\partial_y u_\eta\|_{H^{s-\frac{1}{2},0}} \sqrt{D_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{s+1}}{\sqrt{s-1}} e^{\mathcal{R}t} \|\partial_y u_\eta\|_{H^{s,0}} \|u_\eta\|_{H^{s+\frac{1}{2},0}} \sqrt{D_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{s+1}}{\sqrt{s-1}} e^{\mathcal{R}t} \|\partial_y u_\eta\|_{H^{s,0}} \mathcal{D}_{s+\frac{1}{2}}(t) \leq \frac{2^{s+2}}{\sqrt{s-1}} e^{\mathcal{R}t} \frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+\frac{1}{2}}(t), \end{aligned}$$

which is in the first integrand of the third line of our energy inequality (7.3.8). Hence D_s must satisfy at least $D_s \geq 2^{s+3}/\sqrt{s-1}$.

Next, we aim at estimating each component of the function F_η in (7.4.6) (see also (7.4.3)). We begin with

$$\begin{aligned} e^{\mathcal{R}t} \left| \langle (b_1 b_2 v)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| &\leq e^{\mathcal{R}t} \|(b_1 b_2 v)_\eta\|_{H^{s-\frac{1}{2},0}} \left(\mathbb{J}\|(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}} + \mathbb{J}\|(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}} \right) \\ &\leq e^{\mathcal{R}t} \|(b_1 b_2 v)_\eta\|_{H^{s-\frac{1}{2},0}} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \end{aligned}$$

Thanks to Lemma 7.4.1, with $\sigma_1 = s - 1/2$, $\sigma_2 = s - 1$, $f = v$ and $g = b_1 b_2$, we deduce that

$$\begin{aligned} e^{\mathcal{R}t} \left| \langle (b_1 b_2 v)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| &\leq e^{\mathcal{R}t} \frac{2^{s-1}}{\sqrt{s-3/2}} \left(\|\partial_y v_\eta\|_{H^{s-\frac{1}{2},0}} \|(b_1 b_2)_\eta\|_{H^{s-1,0}} + \|\partial_y v_\eta\|_{H^{s-1,0}} \|(b_1 b_2)_\eta\|_{H^{s-\frac{1}{2},0}} \right) 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\ &\leq e^{\mathcal{R}t} \frac{2^s}{\sqrt{s-3/2}} \left(\|u_\eta\|_{H^{s+\frac{1}{2},0}} \|(b_1 b_2)_\eta\|_{H^{s-1,0}} + \|u_\eta\|_{H^{s,0}} \|(b_1 b_2)_\eta\|_{H^{s-\frac{1}{2},0}} \right) \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \end{aligned}$$

Next, we apply Lemma 7.4.1 twice, first with regularities $\sigma_1 = \sigma_2 = s - 1$ and functions $f = b_2$, $g = b_1$ (to deal with the term $\|(b_1 b_2)_\eta\|_{H^{s-1}}$), secondly with regularities $\sigma_1 = \sigma_2 = s - 1/2$ and same functions $f = b_2$, $g = b_1$ (to deal with the term $\|(b_1 b_2)_\eta\|_{H^{s-1/2}}$). By recalling the divergence-free condition $\partial_y b_{2,\eta} = -\partial_x b_{1,\eta}$, we gather

$$\begin{aligned} e^{\mathcal{R}t} \left| \langle (b_1 b_2 v)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| &\leq e^{\mathcal{R}t} \frac{2^s}{\sqrt{s-3/2}} \left\{ \|u_\eta\|_{H^{s+\frac{1}{2},0}} \frac{2^{s-\frac{3}{2}}}{\sqrt{s-3/2}} 2 \|b_{1,\eta}\|_{H^{s-1,0}} \|\partial_x b_{1,\eta}\|_{H^{s-1,0}} \right. \\ &\quad \left. + \frac{2^{s-1}}{\sqrt{s-1}} 2 \|u_\eta\|_{H^{s,0}} \|b_{1,\eta}\|_{H^{s-\frac{1}{2},0}} \|\partial_x b_{1,\eta}\|_{H^{s-\frac{1}{2},0}} \right\} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{2s}}{s-3/2} e^{\mathcal{R}t} \left\{ \|b_{1,\eta}\|_{H^{s,0}}^2 \|u_\eta\|_{H^{s+\frac{1}{2},0}} + \|u_\eta\|_{H^{s,0}} \|b_{1,\eta}\|_{H^{s,0}} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \right\} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}. \end{aligned}$$

Thus, by recalling (7.4.9) and (7.4.10), we deduce that

$$\begin{aligned}
& e^{\mathcal{R}t} \left| \langle (b_1 b_2 v)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| \leq \\
& \leq \frac{2^{2s}}{s-3/2} e^{\mathcal{R}t} \left\{ 2^2 \mathcal{E}_s(t) 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} + 2^2 \mathcal{E}_s(t) 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \right\} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& \leq \frac{2^{2s+4}}{s-3/2} e^{\mathcal{R}t} \mathcal{E}_s(t) \mathcal{D}_{s+\frac{1}{2}}(t),
\end{aligned}$$

which is one of integral in the third line of (7.4.1). Hence D_s must satisfy $D_s \geq 2^{s+3}/\sqrt{s-1} + 2^{2s+4}/(s-3/2)$.

The remaining components of F_η are dealt with an analogous procedure. Thanks to Lemma 7.4.1, we have indeed

$$\begin{aligned}
e^{\mathcal{R}t} \left| \langle (b_2^2 u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| & \leq e^{\mathcal{R}t} \left\| (b_2^2 u)_\eta \right\|_{H^{s-\frac{1}{2},0}} \left(\|\mathbb{J}(\partial_t u)_\eta + u_\eta\|_{H^{s+\frac{1}{2},0}} + \|\mathbb{J}(\partial_t u)_\eta\|_{H^{s+\frac{1}{2},0}} \right) \\
& \leq e^{\mathcal{R}t} \left\| (b_2^2 u)_\eta \right\|_{H^{s-\frac{1}{2},0}} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.
\end{aligned}$$

We invoke once more Lemma 7.4.1, with $\sigma_1 = s-1/2$, $\sigma_2 = s-1$, $f = b_2$ and $g = ub_2$. Thanks to the divergence-free condition $\partial_y b_{2,\eta} = -\partial_x b_{1,\eta}$, we deduce that

$$\begin{aligned}
& e^{\mathcal{R}t} \left| \langle (b_2^2 u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| \\
& \leq e^{\mathcal{R}t} \frac{2^{s-1}}{\sqrt{s-3/2}} \left(\|\partial_y b_{2,\eta}\|_{H^{s-\frac{1}{2},0}} \|(ub_2)_\eta\|_{H^{s-1,0}} + \|\partial_y b_{2,\eta}\|_{H^{s-1,0}} \|(ub_2)_\eta\|_{H^{s-\frac{1}{2},0}} \right) 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& \leq \frac{2^s}{\sqrt{s-3/2}} e^{\mathcal{R}t} \left(\|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \|(ub_2)_\eta\|_{H^{s-1,0}} + \|b_{1,\eta}\|_{H^{s,0}} \|(ub_2)_\eta\|_{H^{s-\frac{1}{2},0}} \right) \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.
\end{aligned}$$

We are now in the position to apply Lemma 7.4.1 to cope with $\|(ub_2)_\eta\|_{H^{s-1,0}}$ and $\|(ub_2)_\eta\|_{H^{s-1/2,0}}$. We first consider regularities $\sigma_1 = \sigma_2 = s-1$ and functions $f = b_2$, $g = u$ and secondly regularities $\sigma_1 = \sigma_2 = s-1/2$, with functions $f = b_2$, $g = u$:

$$\begin{aligned}
e^{\mathcal{R}t} \left| \langle (b_2^2 u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| & \leq \frac{2^s}{\sqrt{s-3/2}} e^{\mathcal{R}t} \left\{ \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \frac{2^{s-\frac{3}{2}}}{\sqrt{s-3/2}} 2\|\partial_x b_{1,\eta}\|_{H^{s-1,0}} \|u_\eta\|_{H^{s-1,0}} \right. \\
& \quad \left. + \frac{2^{s-1}}{\sqrt{s-1}} 2\|b_{1,\eta}\|_{H^{s,0}} \|\partial_x b_{1,\eta}\|_{H^{s-\frac{1}{2},0}} \|u_\eta\|_{H^{s-\frac{1}{2},0}} \right\} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& \leq \frac{2^{2s}}{s-3/2} e^{\mathcal{R}t} \left\{ \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \|b_{1,\eta}\|_{H^{s,0}} \|u_\eta\|_{H^{s,0}} + \|b_{1,\eta}\|_{H^{s,0}} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \|u_\eta\|_{H^{s,0}} \right\} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& \leq \frac{2^{2s+1}}{s-3/2} e^{\mathcal{R}t} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \|b_{1,\eta}\|_{H^{s,0}} \|u_\eta\|_{H^{s,0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.
\end{aligned}$$

Thus, recalling (7.4.9) and (7.4.10), we obtain

$$\begin{aligned}
& e^{\mathcal{R}t} \left| \langle (b_2^2 u)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} \right| \\
& \leq \frac{2^{2s+1}}{s-3/2} e^{\mathcal{R}t} \left\{ 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} 2^2 \mathcal{E}_s(t) \right\} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& = \frac{2^{2s+4}}{s-3/2} e^{\mathcal{R}t} \mathcal{E}_s(t) \mathcal{D}_{s+\frac{1}{2}}(t).
\end{aligned}$$

We need therefore to impose $D_s \geq 2^{s+3}/\sqrt{s-1} + 2^{2s+5}/(s-3/2)$.

The last term of F_η is finally estimated as follows:

$$\begin{aligned}
|e^{\mathcal{R}t} \langle (b_2 \int_0^y \partial_t b_1)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}}| &\leq e^{\mathcal{R}t} \| (b_2 \int_0^y \partial_t b_1)_\eta \|_{H^{s-\frac{1}{2},0}} \| 2\mathbb{J}(\partial_t u)_\eta + u_\eta \|_{H^{s+\frac{1}{2},0}} \\
&\leq \frac{2^s}{\sqrt{s-1}} e^{\mathcal{R}t} \| \partial_x b_{1,\eta} \|_{H^{s-\frac{1}{2},0}} \| \int_0^y (\partial_t b_1)_\eta \|_{H^{s-\frac{1}{2},0}} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
&\leq \frac{2^{s+1}}{\sqrt{s-1}} e^{\mathcal{R}t} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \| \int_0^y (\partial_t b_1)_\eta \|_{H^{s-\frac{1}{2},0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.
\end{aligned}$$

Since $y \in (0, 1)$, one has that $\| \int_0^y (\partial_t b_1)_\eta \|_{H^{s-\frac{1}{2},0}} \leq \| (\partial_t b_1)_\eta \|_{H^{s-\frac{1}{2},0}} \leq \| (\partial_t b_1)_\eta \|_{H^{s,0}} \leq (2/\sqrt{\mathbb{J}}) \sqrt{\mathcal{E}_s(t)}$, hence

$$|e^{\mathcal{R}t} \langle (b_2 \int_0^y \partial_t b_1)_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}}| \leq \frac{2^{s+2}}{\sqrt{s-1}} e^{\mathcal{R}t} \frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+\frac{1}{2}}(t),$$

which is indeed in the second integral of (7.4.1). We shall impose

$$D_s \geq 3 \cdot 2^{s+2} / \sqrt{s-1} + 2^{2s+5} / (s-3/2).$$

7.4.2 Estimates of the pressure

To complete the estimates of the momentum equation, we shall finally address the term in (7.4.8) related to the pressure. First, we remark that

$$\begin{aligned}
-\langle \partial_x p_\eta, 2\mathbb{J}(\partial_t u)_\eta + u_\eta \rangle_{H^{s,0}} &= \langle \partial_x p_\eta, 2\mathbb{J}(\partial_t \partial_x u)_\eta + \partial_x u_\eta \rangle_{H^{s,0}} = -\langle p_\eta, 2\mathbb{J}(\partial_t \partial_y v)_\eta + \partial_y v_\eta \rangle_{H^{s,0}} \\
&= \langle \partial_y p_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}}.
\end{aligned}$$

Furthermore, making use of the second equation in (7.3.3), we can decompose this term as

$$\begin{aligned}
e^{\mathcal{R}t} \langle \partial_y p_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}} &= e^{\mathcal{R}t} \left\{ \langle (b_1 b_2 u)_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}} + \langle (b_1^2 v)_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}} \right. \\
(7.4.12) \quad &\quad \left. - \langle (b_1 \int_0^y \partial_t b_1)_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}} \right\}.
\end{aligned}$$

We begin with by estimating the inner product $\langle (b_1 b_2 u)_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}}$, where we can first localise a dissipation of the form $\mathcal{D}_{s+1/2}(t)^{1/2}$:

$$\begin{aligned}
(7.4.13) \quad |e^{\mathcal{R}t} \langle (b_1 b_2 u)_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}}| &\leq e^{\mathcal{R}t} \| (b_1 b_2 u)_\eta \|_{H^{s+\frac{1}{2},0}} \| 2\mathbb{J}(\partial_t v)_\eta + v_\eta \|_{H^{s-\frac{1}{2},0}} \\
&\leq e^{\mathcal{R}t} \| (b_1 b_2 u)_\eta \|_{H^{s+\frac{1}{2},0}} \left(\mathbb{J} \int_0^y \partial_x (\partial_t u)_\eta \|_{H^{s-\frac{1}{2},0}} + \| \mathbb{J} \int_0^y \partial_x (\partial_t u)_\eta + \int_0^y \partial_x u_\eta \|_{H^{s-\frac{1}{2},0}} \right) \\
&\leq e^{\mathcal{R}t} \| (b_1 b_2 u)_\eta \|_{H^{s+\frac{1}{2},0}} \left(\mathbb{J} \| \partial_x (\partial_t u)_\eta \|_{H^{s-\frac{1}{2},0}} + \| \mathbb{J} \partial_x (\partial_t u)_\eta + \partial_x u_\eta \|_{H^{s-\frac{1}{2},0}} \right) \\
&\leq e^{\mathcal{R}t} \| (b_1 b_2 u)_\eta \|_{H^{s+\frac{1}{2},0}} \left(\mathbb{J} \| (\partial_t u)_\eta \|_{H^{s+\frac{1}{2},0}} + \| \mathbb{J}(\partial_t u)_\eta + u_\eta \|_{H^{s+\frac{1}{2},0}} \right) \\
&\leq e^{\mathcal{R}t} \| (b_1 b_2 u)_\eta \|_{H^{s+\frac{1}{2},0}} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}
\end{aligned}$$

We shall now address the trilinear term $\|(b_1 b_2 u)_\eta\|_{H^{s+1/2,0}}$ and we apply Lemma 7.4.1 with $\sigma_1 = s+1/2$, $\sigma_2 = s-3/2$, $f = b_2$ and $g = b_1 u$:

(7.4.14)

$$\begin{aligned} \|(b_1 b_2 u)_\eta\|_{H^{s+1/2,0}} &\leq \frac{2^s}{\sqrt{s-2}} \left\{ \|\partial_y b_{2,\eta}\|_{H^{s+1/2,0}} \|(ub_1)_\eta\|_{H^{s-3/2,0}} + \|\partial_y b_{2,\eta}\|_{H^{s-3/2,0}} \|(ub_1)_\eta\|_{H^{s+1/2,0}} \right\} \\ &\leq \frac{2^s}{\sqrt{s-2}} \left\{ \|\partial_x b_{1,\eta}\|_{H^{s+1/2,0}} \|(ub_1)_\eta\|_{H^{s-3/2,0}} + \|\partial_x b_{1,\eta}\|_{H^{s-3/2,0}} \|(ub_1)_\eta\|_{H^{s+1/2,0}} \right\} \\ &\leq \frac{2^s}{\sqrt{s-2}} \left\{ \|b_{1,\eta}\|_{H^{s+3/2,0}} \|(ub_1)_\eta\|_{H^{s-3/2,0}} + \|b_{1,\eta}\|_{H^{s-1/2,0}} \|(ub_1)_\eta\|_{H^{s+1/2,0}} \right\} \\ &\leq \frac{2^s}{\sqrt{s-2}} \left\{ 2\sqrt{\mathcal{D}_{s+3/2}(t)} \|(ub_1)_\eta\|_{H^{s-3/2,0}} + 2\sqrt{\mathcal{E}_s(t)} \|(ub_1)_\eta\|_{H^{s+1/2,0}} \right\}. \end{aligned}$$

Next, we apply twice Lemma 7.4.1, in order to deal with $\|(ub_1)_\eta\|_{H^{s-3/2,0}}$ and $\|(ub_1)_\eta\|_{H^{s+1/2,0}}$. For both cases we consider functions $f = u$ and $g = b_1$, however the regularities are considered $\sigma_1 = s-3/2 = \sigma_2$ and $\sigma_1 = s+1/2$, $\sigma_2 = s$, respectively. We gather

$$\begin{aligned} \|(b_1 b_2 u)_\eta\|_{H^{s+1/2,0}} &\leq \frac{2^s}{\sqrt{s-2}} \left\{ 2\sqrt{\mathcal{D}_{s+3/2}(t)} \frac{2^{s-2}}{\sqrt{s-2}} \|\partial_y u_\eta\|_{H^{s-3/2,0}} \|b_{1,\eta}\|_{H^{s-3/2,0}} + \right. \\ &\quad \left. + 2\sqrt{\mathcal{E}_s(t)} \frac{2^s}{\sqrt{s-1/2}} \left(\|\partial_y u_\eta\|_{H^{s+1/2,0}} \|b_{1,\eta}\|_{H^{s,0}} + \|\partial_y u_\eta\|_{H^{s,0}} \|b_{1,\eta}\|_{H^{s+1/2,0}} \right) \right\} \\ &\leq \frac{2^s}{\sqrt{s-2}} \left\{ \frac{2^{s+1}}{\sqrt{s-2}} \sqrt{\mathcal{D}_{s+3/2}(t)} \frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{E}_s(t)} 2\sqrt{\mathcal{E}_s(t)} + \right. \\ &\quad \left. + \frac{2^{s+1}}{\sqrt{s-1/2}} \sqrt{\mathcal{E}_s(t)} \left(\frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{D}_{s+1/2}(t)} 2\sqrt{\mathcal{E}_s(t)} + \frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{E}_s(t)} 2\sqrt{\mathcal{D}_{s+1/2}(t)} \right) \right\} \\ &\leq \frac{2^{2s+3}}{s-2} \frac{1}{\sqrt{\mathbb{J}}} \left\{ \sqrt{\mathcal{D}_{s+3/2}(t)} + \sqrt{\mathcal{D}_{s+1/2}(t)} \right\} \mathcal{E}_s(t). \end{aligned}$$

Plugging the last inequality into (7.4.13), we eventually obtain

(7.4.15)

$$\left| e^{\mathcal{R}t} \langle (b_1 b_2 u)_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}} \right| \leq \frac{2^{2s+4}}{s-2} \frac{e^{\mathcal{R}t}}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{D}_{s+3/2}(t) \mathcal{D}_{s+1/2}(t)} \mathcal{E}_s(t) + \frac{2^{2s+4}}{s-2} \frac{e^{\mathcal{R}t}}{\sqrt{\mathbb{J}}} \mathcal{E}_s(t) \mathcal{D}_{s+1/2}(t).$$

We shall thus impose $D_s \geq 3 \cdot 2^{s+2}/\sqrt{s-1} + 2^{2s+5}/(s-3/2) + 2^{2s+4}/(s-2)$.

Coming back to (7.4.12), we infer that a similar approach as the one used to show (7.4.15) implies that

$$\left| e^{\mathcal{R}t} \langle (b_1^2 v)_\eta, 2(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}} \right| \leq \frac{2^{2s+4}}{s-2} \frac{e^{\mathcal{R}t}}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{D}_{s+3/2}(t) \mathcal{D}_{s+1/2}(t)} \mathcal{E}_s(t) + \frac{2^{2s+4}}{s-2} \frac{e^{\mathcal{R}t}}{\sqrt{\mathbb{J}}} \mathcal{E}_s(t) \mathcal{D}_{s+1/2}(t),$$

thus we must impose $D_s \geq 3 \cdot 2^{s+2}/\sqrt{s-1} + 2^{2s+5}/(s-3/2) + 2^{2s+5}/(s-2)$.

To conclude the estimates of the momentum equation, we consider $\langle (b_1(\int_0^y \partial_t b_1))_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}}$, the last term of (7.4.12).

$$\begin{aligned} e^{\mathcal{R}t} \langle (b_1(\int_0^y \partial_t b_1))_\eta, 2\mathbb{J}(\partial_t v)_\eta + v_\eta \rangle_{H^{s,0}} &\leq e^{\mathcal{R}t} \left\| (b_1(\int_0^y \partial_t b_1))_\eta \right\|_{H^{s+1/2,0}} 2\sqrt{\mathcal{D}_{s+1/2}(t)} \\ &\leq e^{\mathcal{R}t} \frac{2^s}{\sqrt{s-1/2}} \left(\|b_{1,\eta}\|_{H^{s+1/2,0}} \|\partial_t b_{1,\eta}\|_{H^{s,0}} + \|b_{1,\eta}\|_{H^{s,0}} \|\partial_t b_{1,\eta}\|_{H^{s+1/2,0}} \right) 2\sqrt{\mathcal{D}_{s+1/2}(t)} \\ &\leq e^{\mathcal{R}t} \frac{2^{s+1}}{\sqrt{s-1/2}} \left(2\sqrt{\mathcal{D}_{s+1/2}(t)} \frac{2\text{Pr}_m}{\kappa} \sqrt{\mathcal{E}_s(t)} + 2\sqrt{\mathcal{E}_s(t)} \frac{2\text{Pr}_m}{\kappa} \sqrt{\mathcal{D}_{s+1/2}(t)} \right) \sqrt{\mathcal{D}_{s+1/2}(t)} \\ &\leq e^{\mathcal{R}t} \frac{2^{s+4}}{\sqrt{s-1/2}} \frac{\text{Pr}_m}{\kappa} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+1/2}(t). \end{aligned}$$

We shall impose

$$(7.4.16) \quad D_s \geq 3 \cdot \frac{2^{s+2}}{\sqrt{s-1}} + \frac{2^{2s+5}}{(s-3/2)} + \frac{2^{2s+5}}{(s-2)} + \frac{2^{2s+4}}{\sqrt{s-1/2}}.$$

This concludes the estimates related to the momentum equation.

7.4.3 Estimates related to the equation of b_1

In this section we cope with the equation of b_1 in the main system (7.3.3), more precisely we deal with

$$(7.4.17) \quad \frac{\kappa}{\text{Pr}_m} \partial_t b_1^2 + \partial_t b_1 + u \partial_x b_1 + v \partial_y b_1 - \frac{1}{\text{Pr}_m} \partial_y^2 b_1 = b_1 \partial_x u + b_2 \partial_y u.$$

By coupling the definition of $b_{1,\eta} = e^{(\tau_0 - \eta)(1 + |D_x|)} b_1$ and equation (7.4.17), we remark that the function $(t, x, y) \in (0, T) \times \mathbb{R} \rightarrow e^{\mathcal{R}t/2} b_{1,\eta}(t, x, y)$ is solution of the following equation

$$(7.4.18) \quad \begin{aligned} & \frac{\kappa}{\text{Pr}_m} \partial_t (e^{\frac{\mathcal{R}}{2}t} (\partial_t b_1)_\eta) + e^{\frac{\mathcal{R}}{2}t} (\partial_t b_1)_\eta + \eta'(t) (1 + |D_x|) (e^{\frac{\mathcal{R}}{2}t} (\partial_t b_1)_\eta) + e^{\frac{\mathcal{R}}{2}t} (u \partial_x b_1)_\eta + \\ & + e^{\frac{\mathcal{R}}{2}t} (v \partial_y b_1)_\eta - e^{\frac{\mathcal{R}}{2}t} \partial_y^2 (e^{\frac{\mathcal{R}}{2}t} b_{1,\eta}) = \frac{\kappa}{\text{Pr}_m} \mathcal{R} e^{\frac{\mathcal{R}}{2}t} (\partial_t b_1)_\eta + e^{\frac{\mathcal{R}}{2}t} (b_1 \partial_x u)_\eta + e^{\frac{\mathcal{R}}{2}t} (b_2 \partial_y u)_\eta. \end{aligned}$$

With a similar approach as the one used to prove inequality (7.4.8), we infer that the following estimate holds true:

$$(7.4.19) \quad \begin{aligned} & \frac{d}{dt} \left[e^{\mathcal{R}t} \left(\underbrace{\frac{\kappa^2}{2\text{Pr}_m^2} \|(\partial_t b_1)_\eta\|_{H^{s,0}}^2 + \frac{1}{2} \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\|_{H^{s,0}}^2 + \frac{\kappa}{\text{Pr}_m^2} \|\partial_y b_{1,\eta}\|_{H^{s,0}}^2}_{b_1\text{-terms in } \mathcal{E}_s(t)} + \frac{\kappa}{\text{Pr}_m} \eta'(t) \frac{1}{2} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2}_{b_1\text{-term in } \mathcal{E}_{s+\frac{1}{2}}(t)} \right. \right. \\ & \left. \left. + \eta'(t)^2 \frac{\kappa^2}{\text{Pr}_m^2} \|b_{1,\eta}\|_{H^{s+1,0}}^2 \right) \right] + \eta'(t) e^{\mathcal{R}t} \left\{ \underbrace{\frac{\kappa^2}{2\text{Pr}_m^2} \|(\partial_t b_1)_\eta\|_{H^{s+\frac{1}{2},0}}^2 + \frac{1}{2} \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + u_\eta \right\|_{H^{s+\frac{1}{2},0}}^2}_{b_1\text{-terms in } \mathcal{D}_{s+\frac{1}{2}}(t)} \right. \\ & \left. + \underbrace{\frac{2\kappa}{2\text{Pr}_m^2} \|\partial_y b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2 + \frac{\kappa^2}{\text{Pr}_m^2} \|\partial_t (b_{1,\eta})\|_{H^{s+\frac{1}{2},0}}^2 + \frac{3}{8} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2}_{b_1\text{-terms in } \mathcal{D}_{s+\frac{1}{2}}(t)} \right\} + \frac{\kappa}{\text{Pr}_m} \eta'(t)^2 e^{\mathcal{R}t} \frac{3}{4} \|u_\eta\|_{H^{s+1,0}}^2 \\ & \left. + \frac{\kappa^2}{\text{Pr}_m^2} \eta'(t)^3 e^{\mathcal{R}t} \underbrace{\|b_{1,\eta}\|_{H^{s+\frac{3}{2},0}}^2}_{b_1\text{-terms in } \mathcal{D}_{s+\frac{3}{2}}(t)} - 2 \frac{\kappa^2}{\text{Pr}_m^2} e^{\mathcal{R}t} \eta'(t) \eta''(t) \underbrace{\|b_{1,\eta}\|_{H^{s+1,0}}^2}_{b_1\text{-term in } \mathcal{E}_{s+1}(t)} - \frac{\kappa}{\text{Pr}_m} e^{\mathcal{R}t} \eta''(t) \frac{1}{2} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}}^2}_{b_1\text{-term in } \mathcal{E}_{s+\frac{1}{2}}(t)} \right. \\ & \left. + \frac{e^{\mathcal{R}t}}{2} \left(\underbrace{\frac{1}{\text{Pr}_m} \|\partial_y b_{1,\eta}\|_{H^{s,0}}^2 + \frac{\kappa}{\text{Pr}_m} \|(\partial_t b_1)_\eta\|_{H^{s,0}}^2}_{b_1\text{-term in } \mathcal{D}_s(t)} \right) \leq e^{\mathcal{R}t} \left\{ \left| \langle (u \partial_x b_1)_\eta, 2(\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| \right. \\ & \left. + \left| \left\langle (v \partial_y b_1)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\rangle_{H^{s,0}} \right| + \left| \left\langle (b_1 \partial_x u)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\rangle_{H^{s,0}} \right| \\ & \left. + \left| \left\langle (b_2 \partial_x v)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\rangle_{H^{s,0}} \right| \right\}. \end{aligned}$$

We next proceed to estimate each term on the right-hand side of (7.4.19). For each estimated term, we will determine a suitable (increasing) lower bound of the constant D_s . The last term will therefore provide the exact form of D_s . We recall that from the estimate of the momentum equation, $D_s \geq 3 \cdot 2^{s+2}/\sqrt{s-1} + 2^{2s+5}/(s-3/2) + 2^{2s+5}/(s-2) + 2^{2s+4}/\sqrt{s-1/2}$, momentarily. The first term in

(7.4.19) we deal with is the convection

$$\begin{aligned}
(7.4.20) \quad & e^{\mathcal{R}t} \left| \langle (u \partial_x b_1)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| \\
& \leq e^{\mathcal{R}t} \| (u \partial_x b_1)_\eta \|_{H^{s-\frac{1}{2},0}} \left(\frac{\kappa}{\text{Pr}_m} \| (\partial_t b_1)_\eta \|_{H^{s+\frac{1}{2},0}} + \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\|_{H^{s+\frac{1}{2},0}} \right) \\
& \leq e^{\mathcal{R}t} \| (u \partial_x b_1)_\eta \|_{H^{s-\frac{1}{2},0}} 2 \left(\frac{\kappa^2}{2\text{Pr}_m^2} \| (\partial_t b_1)_\eta \|_{H^{s+\frac{1}{2},0}}^2 + \frac{1}{2} \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\|_{H^{s+\frac{1}{2},0}}^2 \right) \\
& \leq 2e^{\mathcal{R}t} \| (u \partial_x b_1)_\eta \|_{H^{s-\frac{1}{2},0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}
\end{aligned}$$

We are in the position to apply Lemma 7.4.1 with the regularities $\sigma_1 = \sigma_2 = s - 1/2 > 1/2$ and the functions $f(\cdot) = u(t, \cdot)$, $g(\cdot) = \partial_x u(t, \cdot)$. We deduce therefore that

$$\begin{aligned}
\| (u \partial_x b_1)_\eta \|_{H^{s-\frac{1}{2},0}} & \leq \frac{2^{s-1}}{\sqrt{s-1}} 2 \| \partial_y u_\eta \|_{H^{s-\frac{1}{2},0}} \| \partial_x b_{1,\eta} \|_{H^{s-\frac{1}{2},0}} \leq \frac{2^s}{\sqrt{s-1}} \| \partial_y u_\eta \|_{H^{s,0}} \| b_{1,\eta} \|_{H^{s+\frac{1}{2},0}} \\
& \leq \frac{2^s}{\sqrt{s-1}} \frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{E}_s(t)} 2 \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.
\end{aligned}$$

Plugging the above estimate to the original convective term (7.4.20), we eventually gather that

$$e^{\mathcal{R}t} \left| \langle (u \partial_x b_1)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| \leq \frac{2^{s+2}}{\sqrt{s-1}} e^{\mathcal{R}t} \frac{1}{\sqrt{\mathbb{J}}} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+\frac{1}{2}}(t).$$

Therefore we require $D_s \geq 2^{s+4}/\sqrt{s-1} + 2^{2s+5}/(s-3/2) + 2^{2s+5}/(s-2) + 2^{2s+4}/\sqrt{s-1/2}$. Next, we treat the second term on the right-hand side of (7.4.6):

$$e^{\mathcal{R}t} \left| \langle (v \partial_y b_1)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| \leq e^{\mathcal{R}t} \| (v \partial_y b_1)_\eta \|_{H^{s-\frac{1}{2},0}} 2 \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.$$

We apply Lemma 7.4.1 once more with regularities $\sigma_1 = \sigma_2 = s - 1/2 > 1/2$, with functions $f = v$ and $g = \partial_y b_1$. Hence

$$\begin{aligned}
e^{\mathcal{R}t} \left| \langle (v \partial_y b_1)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| & \leq \frac{2^{s-1}}{\sqrt{s-1}} e^{\mathcal{R}t} 2 \| \partial_y v_\eta \|_{H^{s-\frac{1}{2},0}} \| \partial_y b_{1,\eta} \|_{H^{s-\frac{1}{2},0}} 2 \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& \leq \frac{2^{s+1}}{\sqrt{s-1}} e^{\mathcal{R}t} \| \partial_x u_\eta \|_{H^{s-\frac{1}{2},0}} \| \partial_y b_{1,\eta} \|_{H^{s-\frac{1}{2},0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& \leq \frac{2^{s+1}}{\sqrt{s-1}} e^{\mathcal{R}t} \| u_\eta \|_{H^{s+\frac{1}{2},0}} \| \partial_y b_{1,\eta} \|_{H^{s-\frac{1}{2},0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\
& \leq \frac{2^{s+2}}{\sqrt{s-1}} e^{\mathcal{R}t} \| \partial_y b_{1,\eta} \|_{H^{s,0}} \mathcal{D}_{s+\frac{1}{2}}(t) \leq \frac{2^{s+2}}{\sqrt{s-1}} e^{\mathcal{R}t} \frac{\text{Pr}_m}{\sqrt{\kappa}} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+\frac{1}{2}}(t),
\end{aligned}$$

which is still on the second integral of (7.4.1). Hence D_s must satisfy at least $D_s \geq 5 \cdot 2^{s+2}/\sqrt{s-1} + 2^{2s+5}/(s-3/2) + 2^{2s+5}/(s-2) + 2^{2s+4}/\sqrt{s-1/2}$. Next, we deal with

$$\begin{aligned}
& e^{\mathcal{R}t} \left| \langle (b_1 \partial_x u)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| \\
& \leq e^{\mathcal{R}t} \| (b_1 \partial_x u)_\eta \|_{H^{s-\frac{1}{2},0}} \left(\left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\|_{H^{s+\frac{1}{2},0}} + \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta \right\|_{H^{s+\frac{1}{2},0}} \right) \\
& \leq e^{\mathcal{R}t} \| (b_1 \partial_x u)_\eta \|_{H^{s-\frac{1}{2},0}} 2 \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}.
\end{aligned}$$

Applying Lemma 7.4.1 with regularities $\sigma_1 = \sigma_2 = s - 1/2 > 1/2$, with functions $f = b_1$ and $g = \partial_x u$, we gather

$$\begin{aligned} e^{\mathcal{R}t} \left| \langle (b_1 \partial_x u)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| &\leq \frac{2^s e^{\mathcal{R}t}}{\sqrt{s-1}} \|\partial_y b_1\|_{H^{s-\frac{1}{2},0}} \|\partial_x u_\eta\|_{H^{s-\frac{1}{2},0}} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{s+2} e^{\mathcal{R}t}}{\sqrt{s-1}} \|\partial_y b_1\|_{H^{s-\frac{1}{2},0}} \mathcal{D}_{s+\frac{1}{2}}(t) \leq \frac{2^{s+2} e^{\mathcal{R}t}}{\sqrt{s-1}} \frac{\text{Pr}_m}{\sqrt{\kappa}} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+\frac{1}{2}}(t). \end{aligned}$$

Hence D_s must satisfy at least $D_s \geq 6 \cdot 2^{s+2} / \sqrt{s-1} + 2^{2s+5} / (s-3/2) + 2^{2s+5} / (s-2) + 2^{2s+4} / \sqrt{s-1/2}$. Finally, we deal with

$$\begin{aligned} &e^{\mathcal{R}t} \left| \langle (b_2 \partial_y u)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| \\ &\leq e^{\mathcal{R}t} \|(b_2 \partial_y u)_\eta\|_{H^{s-\frac{1}{2},0}} \left(\left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \right\|_{H^{s+\frac{1}{2},0}} + \left\| \frac{\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta \right\|_{H^{s+\frac{1}{2},0}} \right) \\ &\leq e^{\mathcal{R}t} \|(b_2 \partial_y u)_\eta\|_{H^{s-\frac{1}{2},0}} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)}. \end{aligned}$$

Applying Lemma 7.4.1 with regularities $\sigma_1 = \sigma_2 = s - 1/2 > 1/2$, with functions $f = b_1$ and $g = \partial_x u$, we gather

$$\begin{aligned} e^{\mathcal{R}t} \left| \langle (b_2 \partial_y u)_\eta, \frac{2\kappa}{\text{Pr}_m} (\partial_t b_1)_\eta + b_{1,\eta} \rangle_{H^{s,0}} \right| &\leq \frac{2^s e^{\mathcal{R}t}}{\sqrt{s-1}} \|\partial_y b_{2,\eta}\|_{H^{s-\frac{1}{2},0}} \|\partial_y u_\eta\|_{H^{s-\frac{1}{2},0}} 2\sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{s+1} e^{\mathcal{R}t}}{\sqrt{s-1}} \|\partial_x b_{1,\eta}\|_{H^{s-\frac{1}{2},0}} \|\partial_y u_\eta\|_{H^{s-\frac{1}{2},0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{s+1} e^{\mathcal{R}t}}{\sqrt{s-1}} \|b_{1,\eta}\|_{H^{s+\frac{1}{2},0}} \|\partial_y u_\eta\|_{H^{s-\frac{1}{2},0}} \sqrt{\mathcal{D}_{s+\frac{1}{2}}(t)} \\ &\leq \frac{2^{s+2} e^{\mathcal{R}t}}{\sqrt{s-1}} \|\partial_y u_\eta\|_{H^{s-\frac{1}{2},0}} \mathcal{D}_{s+\frac{1}{2}}(t) \leq \frac{2^{s+2} e^{\mathcal{R}t}}{\sqrt{s-1}} \frac{1}{\sqrt{\mathcal{J}}} \sqrt{\mathcal{E}_s(t)} \mathcal{D}_{s+\frac{1}{2}}(t). \end{aligned}$$

Hence D_s must satisfy at least $D_s \geq 7 \cdot 2^{s+2} / \sqrt{s-1} + 2^{2s+5} / (s-3/2) + 2^{2s+5} / (s-2) + 2^{2s+4} / \sqrt{s-1/2}$. Since, we have concluded our estimates, we are now in the position to set a value of D_s . For instance, as compact form. we can consider D_s as

$$(7.4.21) \quad D_s = \frac{2^{2s+6}}{s-2} \left(1 + \frac{s-2}{\sqrt{s-1}} \right)$$

This concludes the proof of Proposition 7.3.1.

7.5 A suitable product law

This appendix is devoted to the proof of Lemma 7.4.1. We recall here the statement.

Lemma 7.5.1. *Let $f, g : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$ be two functions such that f_η, g_η and $\partial_y f_\eta$ belong to $H^{\sigma_1,0}(\mathbb{R} \times (0, 1))$ with $\sigma_1 > 1/2$ (and thus also to $H^{\sigma_2,0}(\mathbb{R} \times (0, 1))$, for any $\sigma_2 \leq \sigma_1$). Furthermore, assume that $f \equiv 0$ in $y = 0$ in the sense of trace. Then*

$$\|(fg)_\eta\|_{H^{\sigma_1,0}} \leq C \frac{2^{\sigma_1 - \frac{1}{2}}}{\sqrt{\sigma_2 - \frac{1}{2}}} \left(\|\partial_y f_\eta\|_{H^{\sigma_1,0}} \|g_\eta\|_{H^{\sigma_2,0}} + \|\partial_y f_\eta\|_{H^{\sigma_2,0}} \|g_\eta\|_{H^{\sigma_1,0}} \right)$$

for any regularities $\sigma_2 \in (1/2, \sigma_1]$.

Proof. According to the definition of $(fg)_\eta$ and the anisotropic Sobolev norm $H^{\sigma_1,0}$, we have

$$\begin{aligned}
\|(fg)_\eta\|_{H^{\sigma_1,0}}^2 &= \int_0^1 \int_{\mathbb{R}} (1+|\xi|)^{2\sigma_1} e^{2(\tau_0-\eta(t))(1+|\xi|)} |\widehat{fg}(\xi, y)|^2 d\xi dy \\
&= \int_0^1 \int_{\mathbb{R}} |(1+|\xi|)^{\sigma_1} e^{(\tau_0-\eta(t))(1+|\xi|)} \widehat{f} * \widehat{g}(\xi, y)|^2 d\xi dy \\
(7.5.1) \quad &= \int_0^1 \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (1+|\xi|)^{\sigma_1} e^{(\tau_0-\eta(t))(1+|\xi|)} \widehat{f}(\xi-\zeta, y) \widehat{g}(\zeta, y) d\zeta \right|^2 d\xi dy.
\end{aligned}$$

We claim that

$$(7.5.2) \quad (1+|\xi|)^{2\sigma_1} \leq 2^{2\sigma_1-1} [(1+|\xi-\zeta|)^{2\sigma_1} + (1+|\zeta|)^{2\sigma_1}],$$

for $\sigma_1 > \frac{1}{2}$. Indeed, we have

$$\begin{aligned}
(1+|\xi|)^{2\sigma_1} &\leq (1+|\xi-\zeta|+|\zeta|)^{2\sigma_1} \\
&\leq [(1+|\xi-\zeta|)^{2\sigma_1} + |\zeta|^{2\sigma_1}]^{2\sigma_1 \times \frac{1}{2\sigma_1}} (1+1)^{(1-\frac{1}{2\sigma_1}) \times 2\sigma_1},
\end{aligned}$$

where in the last step we have used the Hölder inequality $(a+b) \leq (a^\theta + b^\theta)^{1/\theta} 2^{1-1/\theta}$, with $\theta = 2\sigma_1$. Continuing our estimate, we have therefore

$$\begin{aligned}
(1+|\xi|)^{2\sigma_1} &\leq 2^{2\sigma_1-1} ((1+|\xi-\zeta|)^{2\sigma_1} + |\zeta|^{2\sigma_1}) \\
&\leq 2^{2\sigma_1-1} ((1+|\xi-\zeta|)^{2\sigma_1} + (1+|\zeta|)^{2\sigma_1}),
\end{aligned}$$

which is indeed (7.5.2). Next, we bring inequality (7.5.2) to (7.5.1) and we use the fact that $e^{(\tau_0-\eta(t))(1+|\xi|)} \leq e^{(\tau_0-\eta(t))(1+|\xi-\zeta|)} e^{(\tau_0-\eta(t))(1+|\zeta|)}$, thus

$$\begin{aligned}
\|(fg)_\eta\|_{H^{\sigma_1,0}}^2 &= 2^{2\sigma_1-1} \int_0^1 \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (1+|\xi-\zeta|)^{\sigma_1} e^{(\tau_0-\eta(t))(1+|\xi|)} \widehat{f}(\xi-\zeta, y) \widehat{g}(\zeta, y) d\zeta \right|^2 d\xi dy \\
&\quad + 2^{2\sigma_1-1} \int_0^1 \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (1+|\zeta|)^{\sigma_1} e^{(\tau_0-\eta(t))(1+|\xi|)} \widehat{f}(\xi-\zeta, y) \widehat{g}(\zeta, y) d\zeta \right|^2 d\xi dy \\
&\leq 2^{2\sigma_1-1} \int_0^1 \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (1+|\xi-\zeta|)^{\sigma_1} e^{(\tau_0-\eta(t))(1+|\xi-\zeta|)} \widehat{f}(\xi-\zeta, y) e^{(\tau_0-\eta(t))(1+|\zeta|)} \widehat{g}(\zeta, y) d\zeta \right|^2 d\xi dy \\
&\quad + 2^{2\sigma_1-1} \int_0^1 \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{(\tau_0-\eta(t))(1+|\xi-\zeta|)} \widehat{f}(\xi-\zeta, y) (1+|\zeta|)^{\sigma_1} e^{(\tau_0-\eta(t))(1+|\zeta|)} \widehat{g}(\zeta, y) d\zeta \right|^2 d\xi dy.
\end{aligned}$$

We are now in the condition to apply the Young's inequality for the convolution of functions, more precisely:

$$\|(fg)_\eta\|_{H^{\sigma_1,0}}^2 \leq C 2^{2\sigma_1-1} \|(1+|D_x|)^{\sigma_1} f_\eta\|_{L_x^2 L_y^\infty}^2 \|g_\eta\|_{L_x^\infty L_y^2}^2 + C 2^{2\sigma_1-1} \|(1+|D_x|)^{\sigma_1} g_\eta\|_{L^2}^2 \|f_\eta\|_{L_x^\infty L_y^\infty}^2.$$

Next, since $\sigma_2 > 1/2$, we have that

$$\begin{aligned}
|f_\eta(x, y)| &\leq C \left| \int_{\mathbb{R}} e^{ix\xi} \widehat{f}_\eta(\xi, y) d\xi \right| \\
&\leq C \left(\int_{\mathbb{R}} \frac{1}{(1+|\xi|)^{2\sigma_2}} d\xi \right)^{\frac{1}{2}} \times \left(\int_{\mathbb{R}} |(1+|D_x|)^{\sigma_2} f_\eta(x, y)|^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{C}{(\sigma_2 - \frac{1}{2})^{\frac{1}{2}}} \|(1+|D_x|)^{\sigma_2} f_\eta\|_{L_x^2},
\end{aligned}$$

then we use that $f(t, x, y) = \int_0^y \partial_y f(t, x, z) dz$, we obtain

$$\begin{aligned} \|f_\eta\|_{L_x^\infty L_y^\infty}^2 &\leq \frac{C}{(\sigma_2 - \frac{1}{2})} \|(1 + |D_x|)^{\sigma_2} f_\eta\|_{L_x^2 L_y^\infty}^2 \\ &\leq \frac{C}{\sigma_2 - \frac{1}{2}} \|\partial_y f_\eta\|_{H^{\sigma_2, 0}}^2. \end{aligned}$$

Moreover

$$\begin{aligned} \|(1 + |D_x|)^{\sigma_1} f_\eta\|_{L_x^2 L_y^\infty}^2 &= \int_{\mathbb{R}} \sup_{y \in (0, 1)} \left| \int_0^y (1 + |D_x|)^{\sigma_1} \partial_y f_\eta(x, z) dz \right|^2 dx \\ &\leq \int_{\mathbb{R}} \sup_{y \in (0, 1)} y \int_0^y (1 + |D_x|)^{2\sigma_1} |\partial_y f_\eta(x, z)|^2 dz dx \\ &\leq C \int_{\mathbb{R}} \int_0^1 (1 + |D_x|)^{2\sigma_1} |\partial_y f_\eta(x, z)|^2 dz dx = C \|\partial_y f_\eta\|_{H^{\sigma_1, 0}}^2. \end{aligned}$$

Summarising, we finally deduce that

$$\|(fg)_\eta\|_{H^{\sigma_1, 0}}^2 \leq C \frac{2^{2\sigma_1 - 1}}{\sigma_2 - \frac{1}{2}} \|\partial_y f_\eta\|_{H^{\sigma_1, 0}}^2 \|g_\eta\|_{H^{\sigma_2, 0}}^2 + C \frac{2^{2\sigma_1 - 1}}{\sigma_2 - \frac{1}{2}} \|\partial_y f_\eta\|_{H^{\sigma_2, 0}}^2 \|g_\eta\|_{H^{\sigma_1, 0}}^2.$$

By applying the square root to the above inequality, we conclude the proof of the Lemma. □

Chapter 8

Primitive equations

In this chapter we present the results of the following paper:

N. Aarach and V. S. Ngo, Hydrostatic approximation of the 2D primitive equations in a thin strip, arXiv:2006.16025.

8.0.1 Primitive equations in a thin strip

The primitive equations describe the motion of large scale fluids on the earth (hundreds to thousands of kilometers), typically an ocean or the atmosphere. The study of geophysical fluids of that scale involve two important phenomena: the effect of the vertical stratification due to the gravity and the effect of the rotation of the earth at large scales.

The first effect naturally appears when we consider a fluid of variable density (hot and cold air for instance). The fluid has then a vertical distribution where heavier layers stay under lighter ones and the gravity constantly maintains this structure against the internal movements of the fluid.

The second effect becomes important at large scale when the fluid can “sense” the rotation of the earth. More precisely, two additional force terms will appear in the equations: the centrifugal force, which is included in the gravity gradient term, and the Coriolis force, which becomes large when the rotation is rapid (or when the scale is large) and which induces a vertical rigidity in the fluid movements. The latter is well known as the phenomenon of Taylor-Proudman columns. The estimate of the importance of this rigidity leads to the comparison between the typical time scale of the system and the Brunt-Väisälä frequency by means of the introduction of the Froude number Fr . For more details about physical considerations, we refer to [28], [39], [117], and [19].

In [54], Embid and Majda considered specific scale simplifications by choosing the same scale for the rotation and the stratification. More precisely, the Froude number is supposed to be $Fr = \epsilon F$, where $F > 0$ is a constant (which will also be called “Froude number”) and ϵ is the Rossby number. Using the moment, energy, mass conservation laws, they obtained the following primitive equations in \mathbb{R}^3

$$\begin{cases} \partial_t U_\epsilon + v_\epsilon \cdot \nabla U_\epsilon - LU_\epsilon + \frac{1}{\epsilon} \mathcal{A}U_\epsilon = \frac{1}{\epsilon} (-\nabla \Phi_\epsilon, 0), \\ \operatorname{div} v_\epsilon = 0, \\ U_\epsilon|_{t=0} = U_0, \end{cases}$$

where the unknowns are $U_\epsilon = (v_\epsilon, \rho_\epsilon)$, which represents the velocity and the temperature of the fluid, and Φ_ϵ the pressure. The operator L is defined by $LU_\epsilon = (\nu \Delta v_\epsilon, \nu' \Delta \rho_\epsilon)$ and the skew-symmetric matrix

\mathcal{A} is given by

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

Another important remark concerning geophysical fluids is the difference between the horizontal scale (generally of order of hundreds to thousands of kilometers) and the vertical scale (generally a few kilometers for oceans and 10-20 kilometers for the atmosphere). In order to take into account this anisotropy, one can consider anisotropic viscosities as in the works of Chemin, Desjardins, Gallagher and Grenier (see [28] and the references therein for instance) or in the work of Charve and Ngo (see [23] for instance). Another direction consists in studying the fluids in thin domains where a dimension is supposed to be very small and goes to zero, which is the main consideration of our work.

In this paper, we will neglect the effect of the rotation and only focus on the effect of the vertical stratification. The combined effect of the rotation and the stratification in the full primitive equations will be studied in a forthcoming paper. In the two-dimensional case, these considerations lead to the following non-rotating primitive equations in the thin strip $\mathbf{S}^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\}$, where ϵ is supposed to be very small.

$$(8.0.1) \quad \begin{cases} \partial_t U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \epsilon^2 \Delta U^\epsilon + \nabla P^\epsilon = \frac{1}{\epsilon} (0, T^\epsilon) & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ \partial_t T^\epsilon + U^\epsilon \cdot \nabla T^\epsilon - \Delta_\epsilon T^\epsilon = 0 & \text{in }]0, \infty[\times \mathbf{S}^\epsilon \\ \operatorname{div} U^\epsilon = 0 & \text{in }]0, \infty[\times \mathbf{S}^\epsilon, \end{cases}$$

Here, $U^\epsilon(t, x, y) = (U_1^\epsilon(t, x, y), U_2^\epsilon(t, x, y))$ denotes the velocity of the fluid and $P^\epsilon(t, x, y)$ the pressure which guarantees the divergence-free property of the velocity field U^ϵ ; $T^\epsilon(t, x, y)$ is the temperature of the fluid and the Froude number is supposed to be $F = 1$. System (8.0.1) is complemented by the no-slip boundary condition

$$U_{|y=0}^\epsilon = U_{|y=\epsilon}^\epsilon = 0 \quad \text{and} \quad T_{|y=0}^\epsilon = T_{|y=\epsilon}^\epsilon = 0.$$

In the equation of the velocity, the Laplacian is $\Delta = \partial_x^2 + \partial_y^2$ and in the equation of the temperature, the anisotropic Laplacian $\Delta_\epsilon = \partial_x^2 + \epsilon^2 \partial_y^2$ reflects the difference between the horizontal and the vertical scales. We will explain our choice of the diffusion term Δ_ϵ in the next section.

8.0.2 Hydrostatic limit of non-rotating primitive equations

In this framework, it is believed that the fluid behavior tends towards a geostrophic balance (see [64], [71] or [118]). In a formal way, as in [114], taking into account this anisotropy, we consider the initial data in the following form,

$$U_{|t=0}^\epsilon = U_0^\epsilon = \left(u_0 \left(x, \frac{y}{\epsilon} \right), \epsilon v_0 \left(x, \frac{y}{\epsilon} \right) \right) \quad \text{in } \mathbf{S}^\epsilon$$

and

$$T_{|t=0}^\epsilon = T_0^\epsilon \left(x, \frac{y}{\epsilon} \right)$$

and we look for solutions in the form

$$(8.0.2) \quad \begin{cases} U^\epsilon(t, x, y) = \left(u^\epsilon \left(t, x, \frac{y}{\epsilon} \right), \epsilon v^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \right) \\ T^\epsilon(t, x, y) = T^\epsilon \left(t, x, \frac{y}{\epsilon} \right) \\ P^\epsilon(t, x, y) = p^\epsilon \left(t, x, \frac{y}{\epsilon} \right). \end{cases}$$

In order to study the limit when $\epsilon \rightarrow 0$, we perform the rescaling $z = \frac{y}{\epsilon}$ and bring our problem to the fixed domain $\mathbf{S} = \{(x, z) \in \mathbb{R}^2 : 0 < z < 1\}$. We rewrite System (8.0.1) as follows

$$(8.0.3) \quad \begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_z u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_z^2 u^\epsilon + \partial_x p^\epsilon = 0 & \text{in }]0, \infty[\times \mathbf{S} \\ \epsilon^2 (\partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_z v^\epsilon - \epsilon^2 \partial_x^2 v^\epsilon - \partial_z^2 v^\epsilon) + \partial_z p^\epsilon = T^\epsilon & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_t T^\epsilon + u^\epsilon \partial_x T^\epsilon + v^\epsilon \partial_z T^\epsilon - \partial_x^2 T^\epsilon - \partial_z^2 T^\epsilon = 0 & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x u^\epsilon + \partial_z v^\epsilon = 0 & \text{in }]0, \infty[\times \mathbf{S} \\ (u^\epsilon, v^\epsilon, T^\epsilon)|_{t=0} = (u_0^\epsilon, v_0^\epsilon, T_0^\epsilon) & \text{in } \mathbf{S} \\ (u^\epsilon, v^\epsilon, T^\epsilon)|_{z=0} = (u^\epsilon, v^\epsilon, T^\epsilon)|_{z=1} = 0. & \end{cases}$$

Formally taking $\epsilon \rightarrow 0$ in System (8.0.3), we obtain the following hydrostatic limit for primitive equations, which are a combination of a Prandtl-like system with a transport-diffusion equation of the temperature

$$(8.0.4) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_z u - \partial_z^2 u + \partial_x p = 0 & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_z p = T & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_t T + u \partial_x T + v \partial_z T - \partial_x^2 T - \partial_z^2 T = 0 & \text{in }]0, \infty[\times \mathbf{S} \\ \partial_x u + \partial_z v = 0 & \text{in }]0, \infty[\times \mathbf{S} \\ (u, T)|_{t=0} = (u_0, T_0) & \text{in } \mathbf{S} \\ (u, v, T)|_{z=0} = (u, v, T)|_{z=1} = 0. & \end{cases}$$

We want to recall some results on the well-posedness of System (8.0.1). This system was studied by Lions, Temam and Wang in [91–93], where the authors considered full viscosity and diffusivity, and established the global existence of weak solutions. In the two-dimensional case, the local existence of strong solutions was proved by Guillén-González, Masmoudi and Rodriguez-Bellido [67], while the global existence was achieved by Bresch, Kazhikhov and Lemoine in [17] and by Temam and Ziane in [127]. In our work, we will study the global well posedness of System (8.0.1) in the 2D thin strip \mathbf{S}^ϵ when ϵ is close to zero. The equivalent result is also available for the rescaled System (8.0.3).

Concerning the hydrostatic limit system (8.0.4), we remark the same difficulties as for the Prandtl equations due to its degenerate form and the non-local nonlinear term $v \partial_z u$ which leads to the loss of one derivative in the tangential direction when one wants to perform energy estimates. For a more complete survey on this challenging problem, we refer the reader to the works [13, 52, 53, 59, 101] and the references therein. The main ideas to overcome this difficulty consist in imposing a monotonicity hypothesis on the normal derivative of the velocity or an analytic regularity on the velocity. In the pioneering work [106], Oleinik and Samokhin used the Croco transformation under the monotonicity assumption to transform Prandtl equations into a new quasilinear system and established the local existence of solutions in the Sobolev functional framework. However, the nonlinearity of the Croco variables induces certain difficulties for understanding the nature of Prandtl equations.

Later, in [124], Sammartino and Caffisch solved the problem for analytic solutions (analytic in both

tangential and normal directions) on a half space without using the monotonicity assumption and the Croco transformation. The analyticity in the normal variable was then removed by Lombardo, Cannone and Sammartino in [100]. The main argument used in [100, 124] is to apply the abstract Cauchy-Kowalewskaya (CK) theorem. We also mention the well-posedness results of the Prandtl equations in Gevrey classes [60, 62]. Under the monotonicity assumption, recently, Alexandre, Wang, Xu and Yang [13] and Masmoudi and Wong [101] obtained the existence of local smooth solutions for Prandtl equations by performing direct energy estimates in weighted Sobolev spaces and by exploiting the cancellation properties of the “bad terms”, without using the Croco transformation.

We remark that unlike the case of Prandtl equations, in System (8.0.4), the pressure term is not defined by the outer flow using the Bernoulli’s law but by temperature via the relation $\partial_z p = T$. One of the novelties of the paper is to find a way to treat the pressure term using the temperature equation to obtain the global well posedness of our system. In the case where the temperature is constant, the well-posedness of the hydrotatic Navier-Stokes equations was studied in Gevrey classes by Gérard-Varet, Masmoudi and Vicol in [60] and recently in [114], Paicu, Zhang and Zhang proved the global well-posedness of the hydrostatic Navier-Stokes system for small analytic data. We remark that the well-posedness of hydrostatic limit systems is still open in Sobolev settings. In this work, we will apply the method from [114] in the case of the hydrostatic primitive equations, where the temperature is not constant.

Remark 8.0.1. 1. *We first remark that the tangential pressure term is of order 1 in the first equation of (8.0.4). Let us suppose for now that we have the necessary regularity to perform the following calculations. Taking the L^2 -scalar product of $\partial_x p$ with u and performing integrations by parts, we obtain*

$$\begin{aligned} \langle \partial_x p, u \rangle_{L^2} &= - \langle p, \partial_x u \rangle_{L^2} = \langle p, \partial_z v \rangle_{L^2} \\ &= - \langle \partial_z p, v \rangle_{L^2} = - \left\langle T, \int_0^z -\partial_x u \, dz' \right\rangle_{L^2} \\ &= - \left\langle \partial_x T, \int_0^z u \, dz' \right\rangle_{L^2}. \end{aligned}$$

So, in order to control $\partial_x p$, we need a control of $\partial_x T$ (or more precisely, we need at least a control of $|D_x|^{\frac{1}{2}} T$ as explained in the following calculation). That is the reason why we consider an anisotropic Laplacian term Δ_ϵ in the temperature equation.

2. *We remark a particular case where we can still get a control of $\partial_x p$ without the consideration of an anisotropic diffusion on the temperature. Indeed, if we suppose that the pressure satisfies a hydrostatic law of the type $\partial_z p = f(t)T$, where the function $f(t)$ is integrable, then we can perform the following estimate*

$$\begin{aligned} \langle \partial_x p, u \rangle_{L^2} &= - \langle p, \partial_x u \rangle_{L^2} = \langle p, \partial_z v \rangle_{L^2} \\ &= - \langle \partial_z p, v \rangle_{L^2} = \left\langle f(t)T, \int_0^z \partial_x u \, dz' \right\rangle_{L^2} \\ &\cong -f(t) \left\langle |D_x|^{\frac{1}{2}} T, \int_0^z |D_x|^{\frac{1}{2}} u \, dz' \right\rangle_{L^2}. \end{aligned}$$

The method of [114] (that we will resume in the next section) and the functional settings used in our paper allow to gain an addition “diffusion-type” term, which implies the necessary controls on $|D_x|^{\frac{1}{2}} T$ and $|D_x|^{\frac{1}{2}} u$.

8.0.3 Functional framework

In order to introduce our results, we will briefly recall some elements of the Littlewood-Paley theory and introduce the function spaces and techniques that we are going to use throughout this paper. Let ψ be an even smooth function in $C_0^\infty(\mathbb{R})$ such that the support is contained in the ball $B_{\mathbb{R}}(0, \frac{4}{3})$ and ψ is equal to 1 on a neighborhood of the ball $B_{\mathbb{R}}(0, \frac{3}{4})$. Let $\chi(z) = \psi(\frac{z}{2}) - \psi(z)$. Thus, the support of χ is contained in the ring $\{z \in \mathbb{R} : \frac{3}{4} \leq |z| \leq \frac{8}{3}\}$, and χ is identically equal to 1 on the ring $\{z \in \mathbb{R} : \frac{4}{3} \leq |z| \leq \frac{3}{2}\}$. The functions ψ and χ enjoy the following important properties

$$(8.0.5) \quad \forall z \in \mathbb{R}, \quad \psi(z) + \sum_{j \in \mathbb{N}} \chi(2^{-j}z) = 1,$$

$$(8.0.6) \quad \forall z \neq 0, \quad \sum_{j \in \mathbb{Z}} \chi(2^{-j}z) = 1.$$

Here, (8.0.5) is used to define non-homogeneous Sobolev (and Besov) spaces and (8.0.6) is important for the definition of homogeneous Sobolev (and Besov) spaces. We also remark that χ satisfies the following quasi-orthogonal relation

$$\forall j, j' \in \mathbb{Z}, |j - j'| \geq 2, \quad \text{supp } \chi(2^{-j}\cdot) \cap \text{supp } \chi(2^{-j'}\cdot) = \emptyset.$$

Let \mathcal{F}_h and \mathcal{F}_h^{-1} be the Fourier transform and the inverse Fourier transform respectively in the horizontal direction. We will also use the notation $\widehat{u} = \mathcal{F}_h u$. We introduce the following definitions of the homogeneous dyadic cut-off operators.

Definition 8.0.1. *For all tempered distribution u in the horizontal direction (of x variable) and for all $q \in \mathbb{Z}$, we set*

$$\begin{aligned} \Delta_q^h u(x, z) &= \mathcal{F}_h^{-1} (\chi(2^{-q}|\xi|)\widehat{u}(\xi, z)), \\ S_q^h u(x, z) &= \sum_{l \leq q-1} \Delta_l^h u(x, z). \end{aligned}$$

We refer to [15] and [18] for a more detailed construction of the dyadic decomposition. This definition, combined with the equality (8.0.6), implies that all tempered distributions can be decomposed with respect to the horizontal frequencies as

$$u = \sum_{q \in \mathbb{Z}} \Delta_q^h u.$$

The following Bernstein lemma gives important properties of a distribution u when its Fourier transform is well localized. We refer the reader to [26, Lemma 2.1.1] for the proof and for other comments.

Lemma 8.0.1. *Let $k \in \mathbb{Z}$, $d \in \mathbb{N}^*$ and $r_1, r_2 \in \mathbb{R}$ satisfy $0 < r_1 < r_2$. There exists a constant $C > 0$ such that, for any $a, b \in \mathbb{R}$, $1 \leq a \leq b \leq +\infty$, for any $\lambda > 0$ and for any $u \in L^a(\mathbb{R}^d)$, we have*

$$\text{supp } (\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq r_1 \lambda\} \implies \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a},$$

and

$$\text{supp } (\widehat{u}) \subset \{\xi \in \mathbb{R}^d \mid r_1 \lambda \leq |\xi| \leq r_2 \lambda\} \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.$$

We now introduce the function spaces used throughout the paper. As in [114], we define the Besov-type spaces $\mathcal{B}^{s,0}$, $s \in \mathbb{R}$ as follows.

Definition 8.0.2. Let $s \in \mathbb{R}$ and $\mathbf{S} = \mathbb{R} \times]0, 1[$. For any $u \in \mathcal{S}'_h(\mathbf{S})$, i.e., u belongs to $\mathcal{S}'(\mathbf{S})$ and $\lim_{q \rightarrow -\infty} \|S_q^h u\|_{L^\infty} = 0$, we set

$$\|u\|_{\mathcal{B}^{s,0}} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \|\Delta_q^h u\|_{L^2}.$$

(i) For $s \leq \frac{1}{2}$, we define

$$\mathcal{B}^{s,0}(\mathbf{S}) \triangleq \{u \in \mathcal{S}'_h(\mathbf{S}) : \|u\|_{\mathcal{B}^{s,0}} < +\infty\}.$$

(ii) For $s \in]k - \frac{1}{2}, k + \frac{1}{2}]$, with $k \in \mathbb{N}^*$, we define $\mathcal{B}^{s,0}(\mathbf{S})$ as the subset of distributions u in $\mathcal{S}'_h(\mathbf{S})$ such that $\partial_x^k u \in \mathcal{B}^{s-k,0}(\mathbf{S})$.

For a better use of the smoothing effect given by the diffusion terms, we will need the following time-weighted Chemin-Lerner-type spaces.

Definition 8.0.3. Let $p \in [1, \infty]$ and let $f \in L^1_{loc}(\mathbb{R}_+)$ be a non-negative function. Then, the space $\tilde{L}^p_{t,f(t)}(\mathcal{B}^{s,0}(\mathbf{S}))$ is the closure of $C([0, t]; C_0^\infty(\mathbf{S}))$ under the norm

$$\|u\|_{\tilde{L}^p_{t,f(t)}(\mathcal{B}^{s,0}(\mathbf{S}))} \triangleq \sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^t f(t') \|\Delta_q^h u(t')\|_{L^2}^p dt' \right)^{\frac{1}{p}},$$

with the usual change if $p = \infty$. In the case where $f \equiv 1$, we will simply use the notation $\tilde{L}^p_t(\mathcal{B}^{s,0}(\mathbf{S}))$.

The following estimates are a direct consequence of the above definition.

Proposition 8.0.4. Let $p \in [1, \infty]$ and let $f \in L^1_{loc}(\mathbb{R}_+)$ be a non-negative function. If $u \in \tilde{L}^p_{t,f(t)}(\mathcal{B}^{s,0}(\mathbf{S}))$ then, there exists a sequence $(d_q(u))_{q \in \mathbb{Z}}$ such that $\sum d_q(u) = 1$ and

$$\left(\int_0^t f(t') \|\Delta_q^h u(t')\|_{L^2}^p dt' \right)^{\frac{1}{p}} \leq d_q(u) 2^{-qs} \|u\|_{\tilde{L}^p_{t,f(t)}(\mathcal{B}^{s,0}(\mathbf{S}))}, \quad \forall q \in \mathbb{Z}.$$

Remark 8.0.2. To simplify the notation even further, in all that follows, we shall denote

$$\mathcal{B}^{s,0} \triangleq \mathcal{B}^s.$$

8.0.4 Main results

The difficulty when one wants to estimate the nonlinear terms relies in finding a way to compensate the loss of one derivative. The main idea here is to exploit the smoothing effect given by the above function spaces, which allows to gain “one half derivative”. Using the method introduced by Chemin in [27] (see also [30], [110] or [114]), for any $f \in L^2(\mathbf{S})$, we define the following auxiliary function, which allows to control the analyticity of f in the horizontal variable x ,

$$(8.0.7) \quad f_\phi(t, x, z) = e^{\phi(t, D_x)} f(t, x, z) \triangleq \mathcal{F}_h^{-1}(e^{\phi(t, \xi)} \widehat{f}(t, \xi, z)) \quad \text{with} \quad \phi(t, \xi) = (a - \lambda \rho(t)) |\xi|,$$

where the quantity $\rho(t)$, which describes the evolution of the analytic band of f , satisfies

$$(8.0.8) \quad \forall t > 0, \dot{\rho}(t) \geq 0 \quad \text{and} \quad \rho(0) = 0.$$

Here, the constants $a > 0$ and $\lambda > 0$ are given and in particular, a represents the width of the initial analytic band. We will chose λ according to the necessity of the “bootstrap” process in the next sections.

We remark that if we differentiate, with respect to the time variable, a function of the type $e^{\phi(t, D_x)} f(t, x, z)$, we obtain an additional “good term” which allows to gain “one half derivative”. More precisely, we have

$$(8.0.9) \quad \frac{d}{dt} \left(e^{\phi(t, D_x)} f(t, x, z) \right) = -\lambda \dot{\rho}(t) |D_x| e^{\phi(t, D_x)} f(t, x, z) + e^{\phi(t, D_x)} \partial_t f(t, x, z),$$

where $-\lambda\dot{\rho}(t)|D_x|e^{\phi(t,D_x)}f(t,x,z)$ plays a smoothing role if $\dot{\rho}(t) \geq 0$. This smoothing effect allows to obtain our global existence and stability results in the analytic framework.

From now on, let $\mathcal{K} > 0$ be the Poincaré constant on the strip \mathcal{S} , in the sens that, for any $f \in L^2(\mathcal{S})$, $f|_{\partial\mathcal{S}=0}$ and $\partial_z f \in L^2(\mathcal{S})$, we have

$$\|f\|_{L^2(\mathcal{S})} \leq \mathcal{K} \|\partial_z f\|_{L^2(\mathcal{S})}.$$

Our main results are the following theorems.

Theorem 8.0.5 (Global well-posedness of the hydrostatic limit system). *Let $a > 0$, $s > 0$ and assume that $e^{a|D_x|}(u_0, T_0) \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}} \cap \mathcal{B}^s$. There exist positive constants c_0, C and a decreasing function $\tilde{\phi} : \mathbb{R}_+ \rightarrow [\frac{2a}{3}, a]$ such that, if we suppose that the initial data (u_0, T_0) satisfy the compatibility condition $\int_0^1 u_0 dz = 0$ and the smallness assumption*

$$\left\| e^{a|D_x|} u_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} + \left\| e^{a|D_x|} T_0 \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_0 a \quad \text{and} \quad \left\| e^{a|D_x|} u_0 \right\|_{\mathcal{B}^{\frac{3}{2}}} + \left\| e^{a|D_x|} T_0 \right\|_{\mathcal{B}^{\frac{3}{2}}} \leq c_0,$$

then the system (8.0.4) has a unique global solution

$$(u, T) \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}^s) \cap C(\mathbb{R}_+; \mathcal{B}^s) \quad \text{with} \quad \partial_z u \in \tilde{L}^2(\mathbb{R}_+; \mathcal{B}^s),$$

satisfying

$$(8.0.10) \quad \left\| e^{\mathcal{R}t}(u_\phi, T_\phi) \right\|_{\tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \partial_z u_\phi \right\|_{\tilde{L}^2_t(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \nabla T_\phi \right\|_{\tilde{L}^2_t(\mathcal{B}^s)} \leq C \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^s},$$

for any $0 \leq \mathcal{R} \leq \frac{1}{2\mathcal{K}}$, where $\phi(t, \xi) = \tilde{\phi}(t)|\xi|$ and where for any $f \in L^2(\mathbf{S})$,

$$f_\phi(t, x, z) = e^{\phi(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\phi(t, \xi)} \hat{f}(t, \xi, z)).$$

Furthermore, we have,

$$(8.0.11) \quad \left\| e^{\mathcal{R}t}(\partial_t u)_\phi \right\|_{\tilde{L}^2_t(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \partial_z u_\phi \right\|_{\tilde{L}^\infty_t(\mathcal{B}^s)} \\ \leq C \left(\left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^s} + \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^{s+1}} \right).$$

Remark 8.0.3. *We remark that the normal component v is uniquely determined from the incompressibility and the boundary condition*

$$(8.0.12) \quad v(t, x, z) = \int_0^z \partial_z v(t, x, z') dz' = - \int_0^z \partial_x u(t, x, z') dz'.$$

Theorem 8.0.6 (Global well-posedness of the primitive system). *Let $a > 0$, $s > 0$, $\epsilon > 0$ and assume that $e^{a|D_x|}(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon) \in \mathcal{B}^{\frac{1}{2}} \cap \mathcal{B}^{\frac{3}{2}} \cap \mathcal{B}^s$. There exist positive constants c_1, C (independent of ϵ) and a decreasing function $\tilde{\Theta} : \mathbb{R}_+ \rightarrow [\frac{2a}{3}, a]$ such that, if we suppose that the initial data $(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon)$ satisfy*

$$\left\| e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon, T_0^\epsilon) \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq c_1 a \quad \text{and} \quad \left\| e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon, T_0^\epsilon) \right\|_{\mathcal{B}^{\frac{3}{2}}} \leq c_1,$$

then, for any $0 < \epsilon < \frac{1}{2C}$ System (8.0.3) has a unique global solution $(u^\epsilon, v^\epsilon, T^\epsilon)$ satisfying,

$$\left\| e^{\mathcal{R}t}(u_\Theta^\epsilon, \epsilon v_\Theta^\epsilon, T_\Theta^\epsilon) \right\|_{\tilde{L}^\infty_t(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \partial_z(u_\Theta^\epsilon, \epsilon v_\Theta^\epsilon, T_\Theta^\epsilon) \right\|_{\tilde{L}^2_t(\mathcal{B}^s)} + \left\| e^{\mathcal{R}t} \partial_x T_\Theta^\epsilon \right\|_{\tilde{L}^2_t(\mathcal{B}^s)} \leq C \left\| e^{a|D_x|}(u_0^\epsilon, \epsilon v_0^\epsilon, T_0^\epsilon) \right\|_{\mathcal{B}^s},$$

for any $0 \leq \mathcal{R} \leq \frac{1}{2\mathcal{K}}$. Here, $\Theta(t, \xi) = \tilde{\Theta}(t)|\xi|$ and for any $f \in L^2(\mathbf{S})$,

$$f_\Theta(t, x, z) = e^{\Theta(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\Theta(t, \xi)} \hat{f}(t, \xi, z)).$$

Theorem 8.0.7 (Convergence to the hydrostatic limit system). *Let $a > 0$ and $0 < \epsilon \leq 1$. We suppose that the initial data (u_0, v_0, T_0) and $(u_0^\epsilon, v_0^\epsilon, T_0^\epsilon)$ satisfy the assumptions of Theorems 8.0.5 and 8.0.6. Let (u, v, T) and $(u^\epsilon, v^\epsilon, T^\epsilon)$ be the respective solutions of Systems (8.0.4) and (8.0.3). Then, there exist a constant $M > 0$ independent of ϵ and a decreasing function $\tilde{\varphi} : \mathbb{R}_+ \rightarrow [\frac{a}{3}, a]$ such that*

$$\begin{aligned} & \| (u_\varphi^\epsilon - u_\varphi, \epsilon v_\varphi^\epsilon - \epsilon v_\varphi) \|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \| \partial_z (u_\varphi^\epsilon - u_\varphi, \epsilon v_\varphi^\epsilon - \epsilon v_\varphi) \|_{\tilde{L}_t^2(\mathcal{B}^s)} + \epsilon \| (u_\varphi^\epsilon - u_\varphi, \epsilon v_\varphi^\epsilon - \epsilon v_\varphi) \|_{\tilde{L}_t^2(\mathcal{B}^{s+1})} \\ & \leq C \left(\| e^{a|D_x|} (u_0^\epsilon - u_0, \epsilon (v_0^\epsilon - v_0)) \|_{\mathcal{B}^s} + C \| e^{a|D_x|} (T_0^\epsilon - T_0) \|_{\mathcal{B}^s} + M\epsilon \right). \end{aligned}$$

where $\varphi(t, \xi) = \tilde{\varphi}(t) |\xi|$ and where, for any $f \in L^2(\mathbf{S})$,

$$f_\varphi(t, x, z) = e^{\varphi(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\varphi(t, \xi)} \hat{f}(t, \xi, z)).$$

8.0.5 Organisation of the paper

Our chapter will be divided into several sections as follows. In the next section, we establish the needed nonlinear estimates which will be used throughout this chapter. we start by in the giving the proofs of Lemmas 8.1.1 and 8.1.3 in Section 8.1. In Section 8.2, we prove the global wellposedness of System (8.0.4) for small data in analytic framework. Section 8.3 is devoted to the study of System (8.0.3) and the proof of Theorem 8.0.6. Finally, in Section 8.4, we prove the convergence of System (8.0.3) towards System (8.0.4) when ϵ goes to 0.

We end this introduction by the notations that will be used in all that follows. For $f \lesssim g$, we mean that there is a positive constant C , which may be different from line to line, such that $f \leq Cg$. We denote by $\langle f, g \rangle_{L^2}$ the inner product of f and g in $L^2(\mathcal{S})$. Finally, we denote by $(d_q)_{q \in \mathbb{Z}}$ (resp. $(d_q(t))_{q \in \mathbb{Z}}$) to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{q \in \mathbb{Z}} d_q = 1$ (resp. $\sum_{q \in \mathbb{Z}} d_q(t) = 1$).

8.1 Nonlinear estimates

The proofs of our main theorems rely on the following lemmas 8.1.1 and 8.1.3 which give controls of the nonlinear terms in our analytic norms.

Lemma 8.1.1. *Let $s > 0$ and $\phi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$. There exists a constant $C \geq 1$ such that, for any functions u, w and \bar{w} which are defined on $\mathbb{R}_+ \times \mathbf{S}$, $u|_{\partial \mathbf{S}} = w|_{\partial \mathbf{S}} = \bar{w}|_{\partial \mathbf{S}} = 0$ and satisfy*

$$\partial_z u_\phi(t), \partial_z w_\phi(t) \in \mathcal{B}^{\frac{1}{2}}, \quad \forall t \geq 0, \quad \text{and} \quad w, \bar{w} \in \tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}}),$$

we have, for any $0 \leq \mathcal{R} \leq \frac{1}{2\bar{\kappa}}$ and for any $q \in \mathbb{Z}$,

$$\begin{aligned} & \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \\ & \leq C d_q^2 2^{-2qs} \left(\| e^{\mathcal{R}t} u_\phi \|_{\tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \| e^{\mathcal{R}t} w_\phi \|_{\tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) \| e^{\mathcal{R}t} \bar{w}_\phi \|_{\tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \end{aligned}$$

where u_ϕ is determined by (8.0.7), where

$$f \in L_{loc}^1(\mathbb{R}_+) \quad \text{and} \quad f(t) \geq \max \left\{ \| \partial_z u_\phi(t) \|_{\mathcal{B}^{\frac{1}{2}}}, \| \partial_z w_\phi(t) \|_{\mathcal{B}^{\frac{1}{2}}} \right\}, \quad \forall t \geq 0,$$

and $(d_q)_{q \in \mathbb{Z}}$ is a positive sequence with $\sum_{q \in \mathbb{Z}} d_q = 1$.

Remark 8.1.1. In the case where $0 < s < 1$, we can relax the condition $f(t) \geq \|\partial_z w_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$ and obtain, for any $w, \bar{w} \in \tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})$, the following estimate (as in [114])

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Before starting the calculations, we remark that one should be careful when dealing with the product of the type $(fg)_\phi$, since in general, we have $(fg)_\phi \neq f_\phi g_\phi$. However, we still can “compare” this two products as in the following lemma.

Lemma 8.1.2. For any $f, g \in L_x^2$, we set

$$f^+ = \mathcal{F}_\xi^{-1}(|\mathcal{F}_x(f)|) \quad \text{and} \quad g^+ = \mathcal{F}_\xi^{-1}(|\mathcal{F}_x(g)|).$$

Then, we have

$$|(\widehat{fg})_\phi(\xi)| \leq \widehat{f_\phi^+ g_\phi^+}(\xi).$$

Proof. We have

$$|(\widehat{fg})_\phi(\xi)| = e^{\phi(\xi)} |\widehat{f}(\cdot) * \widehat{g}(\cdot)|(\xi) \leq e^{\phi(\xi)} \int |\widehat{f}(\xi - \eta)| |\widehat{g}(\eta)| d\eta.$$

From the definition of the function ϕ , we have $e^{\phi(\xi)} \leq e^{\phi(\xi-\eta)} e^{\phi(\eta)}$, then

$$\begin{aligned} |(\widehat{fg})_\phi(\xi)| &\leq \int e^{\phi(\xi-\eta)} |\widehat{f}(\xi - \eta)| e^{\phi(\eta)} |\widehat{g}(\eta)| d\eta \\ &\leq \int |\widehat{f}_\phi(\xi - \eta)| |\widehat{g}_\phi(\eta)| d\eta \\ &\leq |\widehat{f}_\phi| * |\widehat{g}_\phi|(\xi) = \widehat{f_\phi^+} * \widehat{g_\phi^+} = \widehat{f_\phi^+ g_\phi^+}(\xi). \end{aligned}$$

□

Remark 8.1.2. For any $f \in L_x^2(\mathbb{R})$, we have $\|f^+\|_{L_x^2} = \|f\|_{L_x^2}$. Since the norms of the function spaces used in our work are based on the L^2 -norm (with respect to the space variables), without loss of generality, from now on, we can assume that $\widehat{f} \geq 0$.

Proof of Lemma 8.1.1.

We first recall the Bony homogeneous decomposition into paraproducts and remainders (see for instance [15, Chapter 2]) in the tangential direction. For two tempered distributions X and Y , we have

$$XY = \mathbb{T}_X^h Y + \mathbb{T}_Y^h X + \mathcal{R}^h(X, Y),$$

where

$$\mathbb{T}_X^h Y = \sum_{q \in \mathbb{Z}} S_{q-1}^h X \Delta_q^h Y \quad \text{and} \quad \mathcal{R}^h(X, Y) = \sum_{|q'-q| \leq 1} \Delta_q^h X \Delta_{q'}^h Y.$$

Using this decomposition, we write

$$\int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \leq A_{1,q} + A_{2,q} + A_{3,q},$$

where

$$\begin{aligned} A_{1,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (\mathbb{T}_u^h \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \\ A_{2,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (\mathbb{T}_{\partial_x w}^h u)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \\ A_{3,q} &= \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (\mathcal{R}^h(u, \partial_x w))_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt'. \end{aligned}$$

Using the support properties of the Fourier transform of the dyadic operators Δ_q^h (see for example [18, Proposition 2.10]), Lemma 8.1.2 and Remark 8.1.2, we get

$$A_{1,q} \leq \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h u_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h \partial_x w_\phi(t')\|_{L^2} \|\Delta_{q'}^h \bar{w}_\phi(t')\|_{L^2} dt'.$$

For any $l \in \mathbb{Z}$, Sobolev inclusion $\dot{H}_z^1([0, 1]) \hookrightarrow L_z^\infty([0, 1])$ and the definition of $\mathcal{B}^{\frac{1}{2}}$ yield

$$(8.1.1) \quad \|\Delta_l^h u_\phi(t')\|_{L^\infty} \lesssim 2^{\frac{l}{2}} \|\Delta_l^h u_\phi(t')\|_{L_x^2(L_z^\infty)} \lesssim 2^{\frac{l}{2}} \|\Delta_l^h \partial_x u_\phi(t')\|_{L^2} \lesssim d_q(u_\phi) \|\partial_x u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}},$$

where $\{d_q(u_\phi)\}$ is a positive sequence with $\sum d_q(u_\phi) = 1$ and then,

$$\|S_{q'-1}^h u_\phi(t')\|_{L^\infty} = \left\| \sum_{l \leq q'-2} \Delta_l^h u_\phi(t') \right\|_{L^\infty} \lesssim \|\partial_x u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Using Cauchy-Schwarz inequality and Proposition 8.0.4, we obtain

$$\begin{aligned} (8.1.2) \quad A_{1,q} &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_0^t \|\partial_x u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_{q'}^h w_\phi(t')\|_{L^2} e^{\mathcal{R}t'} \|\Delta_{q'}^h \bar{w}_\phi(t')\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_x u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_x u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 e^{2\mathcal{R}t'} \|\Delta_{q'}^h \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$d_q^2 = d_q(\bar{w}_\phi) \sum_{|q-q'|\leq 4} d_{q'}(w_\phi) 2^{(q-q')(s-\frac{1}{2})}.$$

Using the symmetry of $\mathbb{T}_X^h Y$ and $\mathbb{T}_Y^h X$, we can estimate $A_{2,q}$ in the similar way as $A_{1,q}$. First, Bernstein lemma 8.0.1 and similar calculations as in (8.1.1) give

$$\|S_{q'-1}^h \partial_x w_\phi(t')\|_{L^\infty} \lesssim 2^{q'} \|S_{q'-1}^h w_\phi(t')\|_{L^\infty} \lesssim 2^{q'} \|\partial_x w_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Then, we obtain

$$(8.1.3) \quad A_{2,q} \lesssim \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x w_\phi(t')\|_{L^\infty} \|\Delta_{q'}^h u_\phi(t')\|_{L^2} \|\Delta_{q'}^h \bar{w}_\phi(t')\|_{L^2} dt'$$

$$\begin{aligned}
&\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t \|\partial_z w_\phi\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_z w_\phi\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_q^h \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$d_q^2 = d_q(\bar{w}_\phi) \sum_{|q-q'|\leq 4} d_{q'}(u_\phi) 2^{(q-q')(s-\frac{1}{2})}.$$

To end the proof, it remains to estimate $A_{3,q}$. Using the support localization properties given in [18, Proposition 2.10], the definition of $\mathcal{R}^h(u, \partial_x w)$, Sobolev inclusion $\dot{H}_z^1([0, 1]) \hookrightarrow L_z^\infty([0, 1])$, Young inequality and Bernstein lemma 8.0.1, we can write

$$\begin{aligned}
\left| \langle \Delta_q^h(\mathcal{R}^h(u, \partial_x w))_\phi, \Delta_q^h \bar{w}_\phi \rangle_{L^2} \right| &\lesssim \sum_{q' \geq q-3} \|\Delta_{q'}^h u_\phi\|_{L_x^2 L_z^\infty} \|\Delta_{q'}^h \partial_x w_\phi\|_{L^2} \|\Delta_q^h \bar{w}_\phi\|_{L_x^\infty L_z^2} \\
&\lesssim \sum_{q' \geq q-3} 2^{\frac{q+q'}{2}} \left(2^{\frac{q'}{2}} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \right) \|\Delta_{q'}^h w_\phi\|_{L^2} \|\Delta_q^h \bar{w}_\phi\|_{L^2}.
\end{aligned}$$

The definition of the $\mathcal{B}^{\frac{1}{2}}$ -norm implies that

$$2^{\frac{q'}{2}} \|\Delta_{q'}^h \partial_x u_\phi\|_{L^2} \leq \|\partial_x u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}, \quad \forall q' \in \mathbb{Z}.$$

Thus, as for the term $A_{1,q}$, we can perform the following estimates

$$\begin{aligned}
(8.1.4) \quad A_{3,q} &\lesssim \sum_{q' \geq q-3} 2^{\frac{q+q'}{2}} \int_0^t \|\partial_x u_\phi(t')\|_{\mathcal{B}^{\frac{1}{2}}} e^{\mathcal{R}t'} \|\Delta_{q'}^h w_\phi(t')\|_{L^2} e^{\mathcal{R}t'} \|\Delta_q^h \bar{w}_\phi(t')\|_{L^2} dt' \\
&\lesssim \sum_{q' \geq q-3} 2^{\frac{q+q'}{2}} \left(\int_0^t \|\partial_x u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_{q'}^h w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_x u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} e^{2\mathcal{R}t'} \|\Delta_q^h \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{q' \geq q-3} 2^{\frac{q+q'}{2}} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_q^h \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim 2^{-2qs} d_q^2 \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$d_q^2 = d_q(\bar{w}_\phi) \sum_{q' \geq q-3} d_{q'}(w_\phi) 2^{(q-q')s}.$$

Here, we remark that we can write the above sum as a convolution product and using Young inequality, if we set

$$\bar{d}_q = \sum_{q' \geq q-3} d_{q'}(w_\phi) 2^{(q-q')s},$$

then $(\bar{d}_q)_{q \in \mathbb{Z}}$ is an element of ℓ^1 .

Lemma 8.1.1 is then a consequence of Estimates (8.1.2), (8.1.3) and (8.1.4). \square

Lemma 8.1.3. *Let $s > 0$ and $\phi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$. There exists a constant $C \geq 1$ such that, for any*

(u, v, w, \bar{w}) , which are defined on $\mathbb{R}_+ \times \mathbf{S}$, $(u, w, \bar{w})|_{\partial \mathbf{S}} = 0$ and satisfy, for any $t \geq 0$,

$$u_\phi(t) \in \mathcal{B}^{\frac{3}{2}}, \quad \partial_z u_\phi(t), \partial_z w_\phi(t) \in \mathcal{B}^{\frac{1}{2}}, \quad \int_0^1 \partial_x u dz = 0, \quad \text{and} \quad u, w, \bar{w} \in \tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}}),$$

with

$$f \in L_{loc}^1(\mathbb{R}_+), \quad f(t) \geq \max \left\{ \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}, \|\partial_z w_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \right\} \quad \text{and} \quad v(t, x, z) = - \int_0^z \partial_x u(t, x, z') dz',$$

we have, for any $\mathcal{R} \geq 0$ and for any $q \in \mathbb{Z}$,

$$(8.1.5) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_z w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \\ \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} \partial_z w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right),$$

where $(d_q)_{q \in \mathbb{Z}}$ is a positive sequence with $\sum_{q \in \mathbb{Z}} d_q = 1$.

Before proving this lemma, we remark that the incompressibility condition $\partial_x u + \partial_z v = 0$ implies a ‘‘transfer of regularity’’ from u to v and so we can control v by mean of u . Indeed, we have the following result

Lemma 8.1.4. *Let u, v be defined on \mathbf{S} with $u|_{\partial \mathbf{S}} = v|_{\partial \mathbf{S}} = 0$ and $\partial_x u + \partial_z v = 0$ such that the following terms are well defined. Then, for any $q \in \mathbb{Z}$,*

$$\|\Delta_q^h v_\phi\|_{L_x^2 L_z^\infty} \leq 2^q \|\Delta_q^h u_\phi\|_{L^2} \\ \|\Delta_q^h v_\phi\|_{L^\infty} \leq 2^{\frac{3q}{2}} \|\Delta_q^h u_\phi\|_{L^2}.$$

Proof. To prove the first estimate, we write

$$v = - \int_0^z \partial_x u dz'.$$

Then we have,

$$\|\Delta_q^h v_\phi\|_{L_x^2 L_z^\infty} \leq \int_0^1 \|\partial_x \Delta_q^h u_\phi\|_{L_x^2} dz' \lesssim 2^q \|\Delta_q^h u_\phi\|_{L^2}.$$

Finally, we remark that we can obtain the second estimate from the first one, using Bernstein lemma 8.0.1. \square

Proof of Lemma 8.1.3.

As in the proof of Lemma 8.1.1, we decompose the term on the left-hand side of (8.1.5) as follows

$$(8.1.6) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_z w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \leq B_{1,q} + B_{2,q} + B_{3,q},$$

with

$$B_{1,q} = \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (\mathbb{T}_v^h \partial_z w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \\ B_{2,q} = \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (\mathbb{T}_{\partial_z w}^h v)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt'$$

$$B_{3,q} = \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (\mathcal{R}^h(v, \partial_z w))_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt'.$$

As for the term $A_{1,q}$ in the proof of Lemma 8.1.1, we have the following estimate

$$\left| \left\langle e^{\mathcal{R}t} \Delta_q^h (\mathbb{T}_v^h \partial_z w)_\phi, e^{\mathcal{R}t} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| \lesssim \sum_{|q'-q| \leq 4} e^{\mathcal{R}t} \|S_{q'-1}^h v_\phi\|_{L_x^2 L_z^\infty} \|\Delta_{q'}^h \partial_z w_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} \bar{w}_\phi\|_{L_x^\infty L_z^2}.$$

Lemma 8.1.4, Poincaré and Cauchy-Schwarz inequalities imply

$$\begin{aligned} \|S_{q'-1}^h v_\phi\|_{L_x^2 L_z^\infty} &\leq \sum_{l \leq q'-2} 2^l \|\Delta_l^h u_\phi\|_{L^2} \lesssim \sum_{l \leq q'-2} 2^{\frac{3l}{4}} \|\Delta_l^h u_\phi\|_{L^2}^{\frac{1}{2}} 2^{\frac{l}{4}} \|\Delta_l^h \partial_z u_\phi\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \left(\sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h u_\phi\|_{L^2} \right)^{\frac{1}{2}} \left(\sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h \partial_z u_\phi\|_{L^2} \right)^{\frac{1}{2}} \lesssim \|u_\phi\|_{\mathcal{B}^{\frac{3}{2}}} \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}. \end{aligned}$$

Thus,

$$\begin{aligned} (8.1.7) \quad B_{1,q} &\lesssim \sum_{|q'-q| \leq 4} \int_0^t 2^{\frac{q}{2}} \|u_\phi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_{q'}^h e^{\mathcal{R}t} \partial_z w_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} \bar{w}_\phi\|_{L^2} \\ &\lesssim \sum_{|q'-q| \leq 4} \left(\int_0^t \|u_\phi\|_{\mathcal{B}^{\frac{3}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t} \partial_z w_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} 2^{\frac{q}{2}} \left(\int_0^t \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h e^{\mathcal{R}t} \bar{w}_\phi\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim d_q^2 2^{-2qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \end{aligned}$$

where

$$d_q^2 = d_q(\bar{w}_\phi) \sum_{|q'-q| \leq 4} d_{q'}(\partial_z w_\phi) 2^{(q-q')s}.$$

For the second term on the right-hand side of (8.1.6), using similar calculations as in (8.1.1), we get

$$\|S_{q'-1}^h \partial_z w_\phi\|_{L_x^\infty L_z^2} \leq \|\partial_z w_\phi\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Thus, Lemma 8.1.4, Young and Cauchy-Schwarz inequalities imply

$$\begin{aligned} (8.1.8) \quad B_{2,q} &\lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_z w_\phi\|_{L_x^\infty L_z^2} \|\Delta_{q'}^h v_\phi\|_{L_x^2 L_z^\infty} \|\Delta_q^h \bar{w}_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|\partial_z w_\phi\|_{\mathcal{B}^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h e^{\mathcal{R}t'} u_\phi\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t'} \bar{w}_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q-q'| \leq 4} 2^{q'} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_q^h \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}, \end{aligned}$$

where

$$d_q^2 = d_q(\bar{w}_\phi) \sum_{|q'-q| \leq 4} d_{q'}(u_\phi) 2^{(q-q')(s-\frac{1}{2})}.$$

Now, for the third term on the right-hand side of (8.1.6), using Lemma 8.1.4, we obtain

$$\begin{aligned}
(8.1.9) \quad B_{3,q} &\lesssim \sum_{q' \geq q-3} \int_0^t e^{2\mathcal{R}t'} \|\Delta_{q'}^h v_\phi\|_{L_x^2 L_z^\infty} \|\Delta_{q'}^h \partial_z w_\phi\|_{L^2} \|\Delta_{q'}^h \bar{w}_\phi\|_{L_x^\infty L_z^2} dt' \\
&\lesssim \sum_{q' \geq q-3} \int_0^t \left(2^{q'} \|\Delta_{q'}^h e^{\mathcal{R}t'} u_\phi\|_{L^2}\right) \left(2^{-\frac{q'}{2}} \|\partial_z w_\phi\|_{\mathcal{B}^{\frac{1}{2}}}\right) \left(2^{\frac{q}{2}} \|\Delta_{q'}^h e^{\mathcal{R}t'} \bar{w}_\phi\|_{L^2}\right) dt' \\
&\lesssim \sum_{q' \geq q-3} 2^{\frac{q+q'}{2}} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h u_\phi\|_{L^2}^2 dt'\right)^{\frac{1}{2}} \left(\int_0^t f(t') e^{2\mathcal{R}t'} \|\Delta_{q'}^h \bar{w}_\phi\|_{L^2}^2 dt'\right)^{\frac{1}{2}} \\
&\lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},
\end{aligned}$$

where

$$d_q^2 = d_q(\bar{w}_\phi) \sum_{q' \geq q-3} d_{q'}(u_\phi) 2^{(q-q')s}.$$

The proof of Estimate (8.1.5) is completed. \square

8.2 Global wellposedness of the hydrostatic limit system

The goal of this section is to prove Theorem 8.0.5. We remark that the construction of a local smooth solution of System (8.0.4) follows a standard parabolic regularization method, similar to the case of Prandtl system, which features an additional horizontal smoothing term of the type $\gamma \partial_x^2$ and then taking $\gamma \rightarrow 0$. The difficulty here relies on the presence of the unknown pressure term $\partial_x p$ in the first equation of (8.0.4). However, as in [21], we can reformulate the problem by writing v and $\partial_x p$ as functions of u and T . From the Dirichlet boundary condition $(u, v)|_{z=0} = (u, v)|_{z=1} = 0$ and the incompressibility condition $\partial_x u + \partial_z v = 0$, we get

$$v(t, x, z) = \int_0^z \partial_z v(t, x, z') dz' = - \int_0^z \partial_x u(t, x, z') dz'.$$

For the pressure term, due to the Dirichlet boundary condition $(u, v, T)|_{z=0} = 0$, we deduce from the incompressibility condition $\partial_x u + \partial_z v = 0$ that

$$(8.2.1) \quad \partial_x \int_0^1 u(t, x, z) dz' = - \int_0^1 \partial_z v(t, x, z') dz' = v(t, x, 0) - v(t, x, 1) = 0.$$

Integrating the equation $\partial_z p = T$ with respect z , we obtain

$$(8.2.2) \quad p(t, x, z) = p(t, x, 0) + \int_0^z T(t, x, z') dz'.$$

Next, differentiating (8.2.2) with respect to x and using the first equation of the system (8.0.4), we get

$$\begin{aligned}
\partial_x p(t, x, 0) &= - \int_0^z \partial_x T(t, x, z') dz' + \partial_x p(t, x, z) \\
&= - \int_0^z \partial_x T(t, x, z') dz' - (\partial_t u + u \partial_x u + v \partial_z u - \partial_z^2 u)(t, x, z)
\end{aligned}$$

Now, we set $c(t) = \int_0^1 u(t, x, z) dz$. Integrating the above equation with respect to $z \in [0, 1]$ and

performing integration by parts lead to

$$\partial_x p(t, x, 0) = - \int_0^1 \int_0^z \partial_x T(t, x, z') dz' dz + \partial_z u(t, x, 1) - \partial_z u(t, x, 0) - \dot{c}(t) - \partial_x \int_0^1 u^2(t, x, z) dz$$

that yields

$$\begin{aligned} \partial_x p(t, x, z) &= \int_0^z \partial_x T(t, x, z') dz' - \int_0^1 \int_0^z \partial_x T(t, x, z') dz' dz \\ &\quad + \partial_z u(t, x, 1) - \partial_z u(t, x, 0) - \dot{c}(t) - \partial_x \int_0^1 u^2(t, x, z) dz. \end{aligned}$$

Let $\phi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\phi(t, \xi) = (a - \lambda\rho(t)) |\xi|.$$

Here $\lambda > 0$ and ρ is a function satisfying (8.0.8), both will be determined later. Applying the operator defined in (8.0.7) to System (8.0.4) and taking into account (8.0.9), we obtain

$$(8.2.3) \quad \begin{cases} \partial_t u_\phi + \lambda\dot{\rho}(t) |D_x| u_\phi + (u\partial_x u)_\phi + (v\partial_z u)_\phi - \partial_z^2 u_\phi + \partial_x p_\phi = 0 & \text{in } \mathbf{S} \times]0, \infty[, \\ \partial_z p_\phi = T_\phi \\ \partial_t T_\phi + \lambda\dot{\rho}(t) |D_x| T_\phi + (u\partial_x T)_\phi + (v\partial_z T)_\phi - \Delta T_\phi = 0 \\ \partial_x u_\phi + \partial_z v_\phi = 0, \\ (u_\phi, T_\phi)|_{t=0} = (e^{a|D_x|} u_0, e^{a|D_x|} T_0) \\ (u_\phi, v_\phi, T_\phi)|_{z=0} = (u_\phi, v_\phi, T_\phi)|_{z=1} = 0. \end{cases}$$

where $|D_x|$ denotes the Fourier multiplier of symbol $|\xi|$. In what follows, we recall that for the sake of the simplicity, we use “ C ” to denote a generic positive constant which can change from line to line and $(d_q)_{q \in \mathbb{Z}}$ (resp. $(d_q(t))_{q \in \mathbb{Z}}$) a generic element of $\ell^1(\mathbb{Z})$ such that $\sum_{q \in \mathbb{Z}} d_q = 1$ (resp. $\sum_{q \in \mathbb{Z}} d_q(t) = 1$).

We will now perform *a priori* estimates needed to prove Theorem 8.0.5. We remark that the continuity with respect to time variable of the solutions in \mathcal{B}^s can be proved by a similar argument as in [107]. We will only focus on estimates in analytic norms. Applying the dyadic operator Δ_q^h to the system (8.2.3), then taking the $L^2(\mathbf{S})$ -scalar product of the first and the third equations of the obtained system with $\Delta_q^h u_\phi$ and $\Delta_q^h T_\phi$ respectively, we get

$$(8.2.4) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^h u_\phi(t)\|_{L^2}^2 + \lambda\dot{\rho}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 + \|\Delta_q^h \partial_z u_\phi(t)\|_{L^2}^2 \\ = - \langle \Delta_q^h (u\partial_x u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h (v\partial_z u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2}, \end{aligned}$$

and

$$(8.2.5) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^h T_\phi(t)\|_{L^2}^2 + \lambda\dot{\rho}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h T_\phi \right\|_{L^2}^2 + \|\Delta_q^h \nabla T_\phi(t)\|_{L^2}^2 \\ = - \langle \Delta_q^h (u\partial_x T)_\phi, \Delta_q^h T_\phi \rangle_{L^2} - \langle \Delta_q^h (v\partial_z T)_\phi, \Delta_q^h T_\phi \rangle_{L^2}. \end{aligned}$$

Let $\mathcal{R} > 0$. Multiplying (8.2.4) and (8.2.5) with $e^{2\mathcal{R}t}$ and taking remarking that

$$e^{2\mathcal{R}t} \frac{d}{dt} f(t) = \frac{d}{dt} (f(t)e^{2\mathcal{R}t}) - 2\mathcal{R}f(t)e^{2\mathcal{R}t},$$

we have

$$\frac{1}{2} \frac{d}{dt} \|e^{\mathcal{R}t} \Delta_q^h u_\phi(t)\|_{L^2}^2 - \mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h u_\phi(t)\|_{L^2}^2 + \lambda\dot{\rho}(t) \left\| e^{\mathcal{R}t} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi(t)\|_{L^2}^2$$

$$= -e^{2\mathcal{R}t} \langle \Delta_q^h(u\partial_x u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - e^{2\mathcal{R}t} \langle \Delta_q^h(v\partial_z u)_\phi, \Delta_q^h u_\phi \rangle_{L^2} - e^{2\mathcal{R}t} \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2},$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|e^{\mathcal{R}t} \Delta_q^h T_\phi(t)\|_{L^2}^2 - \mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h T_\phi(t)\|_{L^2}^2 + \lambda \dot{\rho}(t) \left\| e^{\mathcal{R}t} |D_x|^{\frac{1}{2}} \Delta_q^h T_\phi \right\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \nabla T_\phi(t)\|_{L^2}^2 \\ & = -e^{2\mathcal{R}t} \langle \Delta_q^h(u\partial_x T)_\phi, \Delta_q^h T_\phi \rangle_{L^2} - e^{2\mathcal{R}t} \langle \Delta_q^h(v\partial_z T)_\phi, \Delta_q^h T_\phi \rangle_{L^2}. \end{aligned}$$

For $0 \leq \mathcal{R} \leq \frac{1}{2K}$, we apply Poincaré inequality, then we integrate the above identities with respect to the time variable and get

$$(8.2.6) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h u_\phi(t)\|_{L_t^\infty(L^2)}^2 + 2\lambda \int_0^t \dot{\rho}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi(t)\|_{L_t^2(L^2)}^2 \\ & \leq \|\Delta_q^h u_\phi(0)\|_{L^2}^2 + D_1 + D_2 + D_3, \end{aligned}$$

and

$$(8.2.7) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h T_\phi(t)\|_{L_t^\infty(L^2)}^2 + 2\lambda \int_0^t \dot{\rho}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h T_\phi \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \nabla T_\phi(t)\|_{L_t^2(L^2)}^2 \\ & \leq \|\Delta_q^h T_\phi(0)\|_{L^2}^2 + D_4 + D_5. \end{aligned}$$

From now on, we set

$$(8.2.8) \quad \dot{\rho}(t) = \|\partial_z u_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \|\partial_z T_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Lemmas 8.1.1 and 8.1.3 yield

$$|D_1| = 2 \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h(u\partial_x u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

$$\begin{aligned} |D_2| &= 2 \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h(v\partial_z u)_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \\ &\leq C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + d_q^2 2^{-2qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2, \end{aligned}$$

$$\begin{aligned} |D_4| &= 2 \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h(u\partial_x T)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle dt' \right| \\ &\leq C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ &\leq C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right) \end{aligned}$$

and

$$\begin{aligned} |D_5| &= 2 \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h(v\partial_z T)_\phi, e^{\mathcal{R}t'} \Delta_q^h T_\phi \right\rangle dt' \right| \\ &\leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right) \end{aligned}$$

$$\leq Cd_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right) + d_q^2 2^{-2qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2.$$

Concerning the pressure term, using the Dirichlet boundary condition $(u, v, T)|_{z=0} = 0$, the incompressibility condition $\partial_x u + \partial_z v = 0$, the relation $\partial_z p = T$ and Poincaré inequality, we can perform integrations by parts and get

$$(8.2.9) \quad \begin{aligned} |\langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle| &= |\langle \Delta_q^h p_\phi, \Delta_q^h \partial_x u_\phi \rangle| = |\langle \Delta_q^h p_\phi, \Delta_q^h \partial_z v_\phi \rangle| = |\langle \Delta_q^h \partial_z p_\phi, \Delta_q^h v_\phi \rangle| \\ &= |\langle \Delta_q^h T_\phi, \Delta_q^h v_\phi \rangle| = \left| \left\langle \Delta_q^h T_\phi, \Delta_q^h \int_0^z \partial_x u_\phi dz' \right\rangle \right| = \left| \left\langle \Delta_q^h \partial_x T_\phi, \Delta_q^h \int_0^z u_\phi dz' \right\rangle \right|, \end{aligned}$$

and so we can have the following bound

$$|\langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle| \leq \|\Delta_q^h \partial_x T_\phi\|_{L^2} \|\Delta_q^h u_\phi\|_{L^2} \leq C \|\Delta_q^h \partial_x T_\phi\|_{L^2}^2 + \frac{1}{4} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2.$$

Thus,

$$|D_3| = 2 \left| \int_0^t \left\langle e^{\mathcal{R}t'} \Delta_q^h \partial_x p_\phi, e^{\mathcal{R}t'} \Delta_q^h u_\phi \right\rangle dt' \right| \leq Cd_q^2 2^{-2qs} \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \frac{1}{4} \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi(t)\|_{L_t^2 L^2}^2.$$

Now, we recall that for a positive sequence (a_1, \dots, a_n) , $n \in \mathbb{N}^*$, we have

$$(8.2.10) \quad \frac{1}{\sqrt{n}} \sum_{j=1}^n a_n \leq \sqrt{\sum_{j=1}^n a_n^2} \leq \sum_{j=1}^n a_n.$$

Inequality (8.2.10) allows us to take the square root of each terms on the two sides of (8.2.6) and (8.2.7) (with a cost of a constant multiplier on the right-hand side, which will be included in the generic constant C). Summing the two obtained inequalities, we get

$$\begin{aligned} &\|e^{\mathcal{R}t} \Delta_q^h u_\phi(t)\|_{L_t^\infty(L^2)} + \sqrt{2\lambda} \left(\int_0^t \dot{\rho}(t') \|e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} + \frac{1}{2} \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi(t)\|_{L_t^2(L^2)} \\ &\leq \|e^{a|D_x|} \Delta_q^h u_0\|_{L^2} + Cd_q 2^{-qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + Cd_q 2^{-qs} \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\quad + d_q 2^{-qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}, \end{aligned}$$

and

$$\begin{aligned} &\|e^{\mathcal{R}t} \Delta_q^h T_\phi(t)\|_{L_t^\infty(L^2)} + \sqrt{2\lambda} \left(\int_0^t \dot{\rho}(t') \|e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h T_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} + \|e^{\mathcal{R}t} \Delta_q^h \nabla T_\phi(t)\|_{L_t^2(L^2)} \\ &\leq \|e^{a|D_x|} \Delta_q^h T_0\|_{L^2} + Cd_q 2^{-qs} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + Cd_q 2^{-qs} \|e^{\mathcal{R}t} T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ &\quad + d_q 2^{-qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}. \end{aligned}$$

Multiplying the above inequalities by 2^{qs} and then summing with respect to $q \in \mathbb{Z}$, we obtain

$$(8.2.11) \quad \begin{aligned} &\|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\leq \|e^{a|D_x|} u_0\|_{\mathcal{B}^s} + C \|e^{\mathcal{R}t} u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + C \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}, \end{aligned}$$

and

$$(8.2.12) \quad \begin{aligned} & \|e^{\mathcal{R}t}T_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t}T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t}\nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & \leq \left\| e^{a|D_x|}T_0 \right\|_{\mathcal{B}^s} + C \|e^{\mathcal{R}t}u_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + C \|e^{\mathcal{R}t}T_\phi\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t}\partial_z T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)}. \end{aligned}$$

Without loss of generality, we can suppose that $C \geq 2$. Multiplying Inequality (8.2.12) by $2C$ and then adding the obtained result to Inequality (8.2.11), we get

$$\begin{aligned} & \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t}\partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & \leq 2C \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^s} + 3C^2 \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + 2C \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t}(\partial_z u_\phi, \partial_z T_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)}. \end{aligned}$$

From now on, we will fix λ and \mathcal{R} such that

$$0 < \mathcal{R} < \frac{1}{2\mathcal{K}} \quad \text{and} \quad \sqrt{\lambda} \geq 9C^2.$$

We obtain from the above inequality that

$$(8.2.13) \quad \begin{aligned} & \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{\lambda} \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_{t,\rho(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{2} \|e^{\mathcal{R}t}\partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t}\nabla T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ & \leq 2C \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^s} + 2C \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t}(\partial_z u_\phi, \partial_z T_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^s)}. \end{aligned}$$

Let

$$t^* = \sup \left\{ t > 0 : \|u_\phi(t)\|_{\mathcal{B}^{\frac{3}{2}}} \leq \frac{1}{16C^4}, \text{ and } \rho(t) \leq \frac{a}{3\lambda} \right\}.$$

For small initial data such that

$$\begin{cases} \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}} < \frac{a\sqrt{2\mathcal{R}}}{16C\lambda} \\ \|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{3}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{3}{2}}} \leq \frac{1}{64C^5}, \end{cases}$$

the continuity with respect to the time variable in $\mathcal{B}^{\frac{3}{2}}$ and the fact that $\rho(0) = 0$ imply that $t^* > 0$. For $s = \frac{3}{2}$ and for any $0 < t < t^*$, we have

$$\|u_\phi(t)\|_{\mathcal{B}^{\frac{3}{2}}} \leq \|e^{\mathcal{R}t}(u_\phi, T_\phi)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} < 2C \left\| e^{a|D_x|}(u_0, T_0) \right\|_{\mathcal{B}^{\frac{3}{2}}} \leq 2C \cdot \frac{1}{64C^5} = \frac{1}{32C^4}.$$

For $s = \frac{1}{2}$ and for any $0 < t < t^*$, we deduce from (8.2.13) that

$$\begin{aligned} \rho(t) &= \int_0^t \|\partial_z(u_\phi, T_\phi)(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \leq \int_0^t e^{-\mathcal{R}t'} \|e^{\mathcal{R}t'} \partial_z(u_\phi, T_\phi)(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\ &\leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \left(\int_0^t \|e^{\mathcal{R}t'} \partial_z(u_\phi, T_\phi)(t')\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2\mathcal{R}}} \|e^{\mathcal{R}t} \partial_z(u_\phi, T_\phi)\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\ &\leq \frac{4C}{\sqrt{2\mathcal{R}}} \left(\|e^{a|D_x|}u_0\|_{\mathcal{B}^{\frac{1}{2}}} + \|e^{a|D_x|}T_0\|_{\mathcal{B}^{\frac{1}{2}}} \right) < \frac{a}{4\lambda}. \end{aligned}$$

We deduce that $t^* = +\infty$ and that (8.0.10) is verified for any $t \in \mathbb{R}_+$.

In order to end the proof of Theorem 8.0.5, we need to prove Inequality (8.0.11). Applying the operator ϕ and then Δ_q^h to the first equation of (8.0.4) and taking the L^2 inner product of resulting equation with $\Delta_q^h(\partial_t u)_\phi$ yield

$$\begin{aligned} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 &= \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} - \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \\ &\quad - \langle \Delta_q^h(v \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2}. \end{aligned}$$

An integration by parts gives

$$\begin{aligned} \langle \Delta_q^h \partial_z^2 u_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} &= \langle \Delta_q^h \partial_z^2 u_\phi, \partial_t \Delta_q^h u_\phi + \lambda \dot{\rho}(t) |D_x| \Delta_q^h u_\phi \rangle_{L^2} \\ &= - \left(\frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \lambda \dot{\rho}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi \right\|_{L^2}^2 \right), \end{aligned}$$

and thus,

$$\|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \lambda \dot{\rho}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi \right\|_{L^2}^2 \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \left| \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \\ I_2 &= \left| \langle \Delta_q^h(v \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \\ I_3 &= \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right|. \end{aligned}$$

For I_1 and I_2 , we simply have

$$\begin{aligned} I_1 &= \left| \langle \Delta_q^h(u \partial_x u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq C \|\Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \frac{1}{6} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 \\ I_2 &= \left| \langle \Delta_q^h(v \partial_z u)_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq C \|\Delta_q^h(v \partial_z u)_\phi\|_{L^2}^2 + \frac{1}{6} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2. \end{aligned}$$

Now, using similar calculations as in (8.2.9), we find

$$I_3 = \left| \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h(\partial_t u)_\phi \rangle_{L^2} \right| \leq \|\Delta_q^h \partial_x T_\phi\|_{L^2} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2} \leq C \|\Delta_q^h \partial_x T_\phi\|_{L^2}^2 + \frac{1}{6} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2.$$

Then, we deduce that

$$\begin{aligned} \|\Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \frac{d}{dt} \|\Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \lambda \dot{\rho}(t) \left\| |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi \right\|_{L^2}^2 \\ \leq C \left(\|\Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \|\Delta_q^h(v \partial_z u)_\phi\|_{L^2}^2 + \|\Delta_q^h \partial_x T_\phi\|_{L^2}^2 \right). \end{aligned}$$

Multiplying the above inequality by $e^{2\mathcal{R}t}$, we get

$$\begin{aligned} \|e^{\mathcal{R}t} \Delta_q^h(\partial_t u)_\phi\|_{L^2}^2 + \frac{d}{dt} \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 - 2\mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi\|_{L^2}^2 + \lambda \dot{\rho}(t) \left\| e^{\mathcal{R}t} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi \right\|_{L^2}^2 \\ \leq C \left(\|e^{\mathcal{R}t} \Delta_q^h(u \partial_x u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h(v \partial_z u)_\phi\|_{L^2}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_x T_\phi\|_{L^2}^2 \right). \end{aligned}$$

Integrating over $[0, t]$, we obtain

(8.2.14)

$$\begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h (\partial_t u)_\phi\|_{\tilde{L}_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi\|_{\tilde{L}_t^\infty(L^2)}^2 - 2\mathcal{R} \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\phi\|_{\tilde{L}_t^2(L^2)}^2 + \lambda \int_0^t \dot{\rho} \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h \partial_z u_\phi \right\|_{L^2}^2 dt' \\ & \leq \|\Delta_q^h \partial_z e^{a|D_x|} u_0\|_{L^2}^2 + C \left(\|e^{\mathcal{R}t} \Delta_q^h (u \partial_x u)_\phi\|_{\tilde{L}_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h (v \partial_z u)_\phi\|_{\tilde{L}_t^2(L^2)}^2 + \|e^{\mathcal{R}t} \Delta_q^h \partial_x T_\phi\|_{\tilde{L}_t^2(L^2)}^2 \right). \end{aligned}$$

We recall that Inequality (8.2.10) allows to take the square root of each terms in the above inequality, with the price of a constant multiplier that will be included in the generic constant C . So, taking the square root of each terms of (8.2.14), multiplying the obtained inequality by 2^{qs} and then summing with respect to $q \in \mathbb{Z}$, we obtain

$$(8.2.15) \quad \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \leq \|e^{a|D_x|} \partial_z u_0\|_{\mathcal{B}^s} + 2\mathcal{R} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + C \left(\|e^{\mathcal{R}t} (u \partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t} (v \partial_z u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right).$$

We will accept for now the following estimates. The proof of these estimates will be given later.

Lemma 8.2.1. *Under the hypotheses of Theorem 8.0.5, we have the following inequalities*

$$\begin{aligned} \|e^{\mathcal{R}t} (u \partial_x u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} & \leq C \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{s+1})}; \\ \|e^{\mathcal{R}t} (v \partial_z u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} & \leq C \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{s+1})} + \|u_\phi\|_{\tilde{L}^\infty(\mathcal{B}^{s+1})} \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})}. \end{aligned}$$

Now, using Estimate (8.0.10), we have

$$\begin{aligned} \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{1}{2}})} & \leq 2C \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} \\ \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} & \leq 2C \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} \\ \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} & \leq 2C \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^s} \\ \|e^{\mathcal{R}t} \partial_x T_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} & \leq 2C \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^s} \\ \|u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+1})} & \leq 2C \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^{s+1}} \\ \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{s+1})} & \leq 2C \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^{s+1}}. \end{aligned}$$

Inserting all the above estimates into (8.2.15), we finally obtain the existence of a constant $\overline{C} > 0$ such that

$$\begin{aligned} & \|e^{\mathcal{R}t} (\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \|e^{\mathcal{R}t} \partial_z u_\phi\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} \\ & \leq \overline{C} \left(\left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^s} + \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} \left\| e^{a|D_x|} (u_0, T_0) \right\|_{\mathcal{B}^{s+1}} \right). \end{aligned}$$

Theorem 8.0.5 is proved. \square

Proof of Lemma 8.2.1.

The proof of this lemma is very similar to the proof of Lemmas 8.1.1 and 8.1.3. We will give the main calculations without going into details. For the first inequality, we decompose

$$(u \partial_x u)_\phi = (\mathbb{T}_u^h \partial_x u)_\phi + (\mathbb{T}_{\partial_x u}^h u)_\phi + (\mathcal{R}^h(u, \partial_x u))_\phi.$$

We have

$$\begin{aligned}
\|e^{\mathcal{R}t}\Delta_q^h(\mathbb{T}_u^h\partial_x u)_\phi\|_{L_t^2(L^2)} &\leq \sum_{|q-q'|\leq 4} \left(\int_0^t \|S_{q'-1}^h u_\phi\|_{L_x^\infty L_z^2}^2 \|e^{\mathcal{R}t'}\Delta_{q'}^h\partial_x u_\phi\|_{L_x^2 L_z^\infty}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \sum_{|q-q'|\leq 4} 2^{q'} \|e^{\mathcal{R}t}\Delta_{q'}^h\partial_z u_\phi\|_{L_t^2(L^2)} \\
&\lesssim d_q 2^{-qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t}\partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{s+1})}
\end{aligned}$$

where

$$d_q = \sum_{|q-q'|\leq 4} d_{q'}(\partial_z u_\phi) 2^{q'-q},$$

$$\begin{aligned}
\|e^{\mathcal{R}t}\Delta_q^h(\mathbb{T}_{\partial_x u}^h u)_\phi\|_{L_t^2(L^2)} &\leq \sum_{|q-q'|\leq 4} \left(\int_0^t \|S_{q'-1}^h \partial_x u_\phi\|_{L_x^\infty L_z^2}^2 \|e^{\mathcal{R}t'}\Delta_{q'}^h u_\phi\|_{L_x^2 L_z^\infty}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{|q-q'|\leq 4} \left(\int_0^t 2^{2q'} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 \|e^{\mathcal{R}t'}\Delta_{q'}^h\partial_z u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \sum_{|q-q'|\leq 4} 2^{q'} \|e^{\mathcal{R}t}\Delta_{q'}^h\partial_z u_\phi\|_{L_t^2(L^2)} \\
&\lesssim d_q 2^{-qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t}\partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{s+1})}
\end{aligned}$$

where

$$d_q = \sum_{|q-q'|\leq 4} d_{q'}(\partial_z u_\phi) 2^{q'-q}.$$

Finally, we have

$$\begin{aligned}
\|e^{\mathcal{R}t}\Delta_q^h(\mathcal{R}^h(u, \partial_x u))_\phi\|_{L_t^2(L^2)} &\leq \sum_{q'\geq q-3} \left(\int_0^t \|\Delta_{q'}^h \partial_x u_\phi\|_{L_x^\infty L_z^2}^2 \|e^{\mathcal{R}t'}\Delta_{q'}^h u_\phi\|_{L_x^2 L_z^\infty}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \sum_{q'\geq q-3} \left(\int_0^t 2^{2q'} \|u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 \|e^{\mathcal{R}t'}\Delta_{q'}^h\partial_z u_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\
&\lesssim \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \sum_{q'\geq q-3} 2^{q'} \|e^{\mathcal{R}t}\Delta_{q'}^h\partial_z u_\phi\|_{L_t^2(L^2)} \\
&\lesssim d_q 2^{-qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} \|e^{\mathcal{R}t}\partial_z u_\phi\|_{\tilde{L}_t^2(\mathcal{B}^{s+1})},
\end{aligned}$$

where

$$d_q = \sum_{q'\geq q-3} d_{q'}(\partial_z u_\phi) 2^{-(q-q')}.$$

Here, for the third estimate, we remark that the sequence $(d_q)_{q\in\mathbb{Z}}$ can be considered as the convolution of two summable sequences and is then also summable.

For the second inequality, we can perform the same calculations, while taking into account the ‘‘transfer of regularity’’ given in Lemma 8.1.4. \square

8.3 Global well-posedness of the 2D non-rotating primitive equations in a thin strip

In this section, we will prove Theorem 8.0.6 and establish the global well-posedness of System (8.0.3) for small analytic data. We will use the same technique as in the previous section but will consider a different auxiliary function to control the evolution of the analyticity of the solutions. This auxiliary function is chosen to be adapted to the primitive system (8.0.3). Let $\Theta : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\Theta(0, \xi) = 0, \quad \Theta(t, \xi) = (a - \lambda\tau(t))|\xi|, \quad \forall t > 0, \forall \xi \in \mathbb{R},$$

where $\lambda > 0$ and $\tau(t)$ will be determined later. For any function $f \in L^2(\mathbf{S})$, we define

$$\Theta : f \mapsto f_\Theta; \quad f_\Theta(t, x, z) = e^{\Theta(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\Theta(t, \xi)} \widehat{f}(t, \xi, z)).$$

In what follows, for the sake of the simplicity, we will neglect the index ϵ and write $(u_\Theta, v_\Theta, T_\Theta)$ instead of $(u_\Theta^\epsilon, v_\Theta^\epsilon, T_\Theta^\epsilon)$. Applying the operator Θ to the system (8.0.3), we obtain

$$(8.3.1) \quad \begin{cases} \partial_t u_\Theta + \lambda \dot{\tau}(t) |D_x| u_\Theta + (u \partial_x u)_\Theta + (v \partial_z u)_\Theta - \epsilon^2 \partial_x^2 u_\Theta - \partial_z^2 u_\Theta + \partial_x p_\Theta = 0, \\ \epsilon^2 (\partial_t v_\Theta + (u \partial_x v)_\Theta + (v \partial_z v)_\Theta - \epsilon^2 \partial_x^2 v_\Theta - \partial_z^2 v_\Theta) + \partial_z p_\Theta = T_\Theta, \\ \partial_t T_\Theta + (u \partial_x T)_\Theta + (v \partial_z T)_\Theta - \Delta T_\Theta = 0, \\ \partial_x u_\Theta + \partial_z v_\Theta = 0, \\ (u_\Theta, v_\Theta, T_\Theta)|_{z=0} = (u_\Theta, v_\Theta, T_\Theta)|_{z=1} = 0, \\ (u_\Theta, v_\Theta, T_\Theta)|_{t=0} = (e^{a|D_x|} u_0, e^{a|D_x|} v_0, e^{a|D_x|} T_0). \end{cases}$$

We remark that the pressure term is not really an unknown and can be determined as function of (u_Θ, T_Θ) as we did for the hydrostatic limit system (see also [21] for more details). We recall also that we always use “ C ” to denote a generic positive constant which can change from line to line.

Applying the operator Δ_q^h to the system (8.3.1), then taking the $L^2(\mathcal{S})$ scalar product of the first three equations of the obtained system with $\Delta_q^h u_\Theta$, $\Delta_q^h v_\Theta$ and $\Delta_q^h T_\Theta$ respectively and summing up the first and second equations, we get

$$(8.3.2) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(u_\Theta, \epsilon v_\Theta)(t)\|_{L^2}^2 + \lambda \dot{\tau}(t) \langle |D_x| \Delta_q^h(u_\Theta, \epsilon v_\Theta), \Delta_q^h(u_\Theta, \epsilon v_\Theta) \rangle_{L^2} \\ & + \|\partial_z \Delta_q^h(u_\Theta, \epsilon v_\Theta)\|_{L^2}^2 + \epsilon^2 \|\partial_x \Delta_q^h(u_\Theta, \epsilon v_\Theta)\|_{L^2}^2 \\ & = - \langle \Delta_q^h(u \partial_x u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} - \langle \Delta_q^h(v \partial_z u)_\Theta, \Delta_q^h u_\Theta \rangle_{L^2} - \langle \nabla \Delta_q^h p_\Theta, \Delta_q^h(u_\Theta, v_\Theta) \rangle_{L^2} \\ & - \epsilon^2 \langle \Delta_q^h(u \partial_x v)_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} - \epsilon^2 \langle \Delta_q^h(v \partial_z v)_\Theta, \Delta_q^h v_\Theta \rangle_{L^2} + \langle \Delta_q^h T_\Theta, \Delta_q^h v_\Theta \rangle_{L^2}, \end{aligned}$$

and

$$(8.3.3) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q^h T_\Theta(t)\|_{L^2}^2 + \lambda \dot{\tau}(t) \langle |D_x| \Delta_q^h T_\Theta, \Delta_q^h T_\Theta \rangle_{L^2} + \|\nabla \Delta_q^h T_\Theta\|_{L^2}^2 \\ & = - \langle \Delta_q^h(u \partial_x T)_\Theta, \Delta_q^h T_\Theta \rangle_{L^2} - \langle \Delta_q^h(v \partial_z T)_\Theta, \Delta_q^h T_\Theta \rangle_{L^2}. \end{aligned}$$

As in the previous section, we will multiply (8.3.2) and (8.3.3) by $e^{2\mathcal{R}t}$, then we integrate the obtained equations with respect to the time variable and get, for any $0 < \mathcal{R} < \frac{1}{2K}$,

$$(8.3.4) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h(u_\Theta, \epsilon v_\Theta)(t)\|_{L_t^\infty(L^2)}^2 + 2\lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h(u_\Theta, \epsilon v_\Theta)(t') \right\|_{L^2}^2 dt' \\ & + \|e^{\mathcal{R}t'} \partial_z \Delta_q^h(u_\Theta, \epsilon v_\Theta)\|_{L_t^2(L^2)}^2 + \epsilon^2 \|e^{\mathcal{R}t'} \partial_x \Delta_q^h(u_\Theta, \epsilon v_\Theta)\|_{L_t^2(L^2)}^2 \\ & \leq \|\Delta_q^h(u_\Theta, \epsilon v_\Theta)(0)\|_{L^2}^2 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6, \end{aligned}$$

and

$$(8.3.5) \quad \begin{aligned} & \|e^{\mathcal{R}t} \Delta_q^h T_\Theta(t)\|_{L_t^\infty(L^2)}^2 + 2\lambda \int_0^t \dot{\tau}(t') \left\| e^{\mathcal{R}t'} |D_x|^{\frac{1}{2}} \Delta_q^h T_\Theta \right\|_{L^2}^2 dt' + \|e^{\mathcal{R}t} \Delta_q^h \nabla T_\Theta(t)\|_{L_t^2 L^2}^2 \\ & \leq \| \Delta_q^h T_\Theta(0) \|_{L^2}^2 + F_7 + F_8. \end{aligned}$$

From now on, we will fix

$$(8.3.6) \quad \dot{\tau}(t) = \|\partial_z u_\Theta(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \|\partial_z T_\Theta(t)\|_{\mathcal{B}^{\frac{1}{2}}} + \epsilon \|\partial_z v_\Theta(t)\|_{\mathcal{B}^{\frac{1}{2}}}.$$

Using Lemma 8.1.1, we get the following controls

$$|F_1| = 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x u)_\Theta, e^{\mathcal{R}t'} \Delta_q^h u_\Theta \rangle_{L^2} dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

$$|F_4| = 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x (\epsilon v))_\Theta, e^{\mathcal{R}t'} \Delta_q^h (\epsilon v)_\Theta \rangle_{L^2} dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\Theta, \epsilon v_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2$$

and

$$|F_7| = 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x T)_\Theta, e^{\mathcal{R}t'} \Delta_q^h T_\Theta \rangle dt' \right| \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} (u_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Next, Lemma 8.1.3 implies the following controls

$$\begin{aligned} |F_2| &= 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_z u)_\Theta, e^{\mathcal{R}t'} \Delta_q^h u_\Theta \rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} u_\Theta\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|u_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} \partial_z u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right), \end{aligned}$$

$$\begin{aligned} |F_5| &= 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_z (\epsilon v))_\Theta, e^{\mathcal{R}t'} \Delta_q^h (\epsilon v)_\Theta \rangle_{L^2} dt' \right| \\ &\leq C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} (u_\Theta, \epsilon v_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|u_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} \partial_z (\epsilon v)_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right) \end{aligned}$$

and

$$\begin{aligned} |F_8| &= 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_z T)_\Theta, e^{\mathcal{R}t'} \Delta_q^h T_\Theta \rangle dt' \right| \\ &\leq C d_q^2 2^{-2qs} \left(\|e^{\mathcal{R}t} (u_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|u_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|e^{\mathcal{R}t} \partial_z T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right). \end{aligned}$$

For the term F_3 , using the divergence-free property $\partial_x u_\Theta + \partial_z v_\Theta = 0$ and an integration by parts, we deduce that

$$|F_3| = 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \nabla \Delta_q^h p_\Theta, e^{\mathcal{R}t'} \Delta_q^h (u_\Theta, v_\Theta) \rangle_{L^2} dt' \right| = 0.$$

For the last term F_4 , the boundary condition $(u_\Theta, v_\Theta)|_{y \in \{0,1\}} = 0$ and the relation

$$v_\Theta(t, x, y) = - \int_0^y \partial_x u_\Theta(t, x, s) ds$$

imply

$$\begin{aligned} \left| \langle e^{\mathcal{R}t} \Delta_q^h T_\Theta, e^{\mathcal{R}t} \Delta_q^h v_\Theta \rangle_{L^2} \right| &= \left| \left\langle e^{\mathcal{R}t} \Delta_q^h T_\Theta, e^{\mathcal{R}t} \Delta_q^h \int_0^y -\partial_x u_\Theta ds \right\rangle_{L^2} \right| \\ &\leq \|\Delta_q^h e^{\mathcal{R}t} \partial_x T_\Theta\|_{L^2} \|\Delta_q^h e^{\mathcal{R}t} u_\Theta\|_{L^2} \\ &\leq C \|\Delta_q^h e^{\mathcal{R}t} \partial_x T_\Theta\|_{L^2}^2 + \frac{1}{16} \|\Delta_q^h e^{\mathcal{R}t} u_\Theta\|_{L^2}^2. \end{aligned}$$

Then, Poincaré and Cauchy-Schwartz inequalities yield

$$|F_6| = 2 \left| \int_0^t \langle e^{\mathcal{R}t'} \Delta_q^h T_\Theta, e^{\mathcal{R}t'} \Delta_q^h v_\Theta \rangle_{L^2} dt' \right| \leq d_q^2 2^{-2qs} \left(C \|e^{\mathcal{R}t} \partial_x T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \frac{1}{8} \|e^{\mathcal{R}t} \Delta_q^h \partial_z u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right).$$

We now multiply (8.3.4) and (8.3.5) by 2^{2qs} and we recall that we can take square root of each terms with the cost of a multiplier which will be included in the generic constant C . Summing the resulting inequalities with respect to $q \in \mathbb{Z}$, we have

$$\begin{aligned} &\|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{2\lambda} \|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_x T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\quad + \|e^{\mathcal{R}t} \partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \epsilon \|e^{\mathcal{R}t} \partial_x(u_\Theta, \epsilon v_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq \left\| e^{a|D_x|}(u_0, \epsilon v_0, T_0) \right\|_{\mathcal{B}^s} \\ &\quad + C \|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \frac{1}{8} \|e^{\mathcal{R}t} \partial_z u_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\quad + C \|u_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}. \end{aligned}$$

Without loss of generalit, we can suppose that $C \geq 2$ and we choose $\sqrt{\lambda} > 3C$. For $\epsilon \leq \frac{1}{2C}$, we can simply the above inequality as follows

$$(8.3.7) \quad \begin{aligned} &\|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^s)} + \sqrt{\lambda} \|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_{t, \tau(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + \|e^{\mathcal{R}t} \partial_x T_\Theta\|_{\tilde{L}_t^2(\mathcal{B}^s)} \\ &\quad + \frac{1}{2} \|e^{\mathcal{R}t} \partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)} \leq \left\| e^{a|D_x|}(u_0, \epsilon v_0, T_0) \right\|_{\mathcal{B}^s} \\ &\quad + C \|u_\Theta\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^2(\mathcal{B}^s)}. \end{aligned}$$

Let

$$t^* \triangleq \sup \left\{ t > 0 : \|u_\Theta\|_{\mathcal{B}^{\frac{3}{2}}} \leq \frac{1}{4C} \text{ and } \tau(t) \leq \frac{a}{3\lambda} \right\},$$

For initial data such that

$$\begin{cases} \left\| e^{a|D_x|}(u_0, \epsilon v_0, T_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} \leq \frac{a\sqrt{2\mathcal{R}}}{8\lambda} \\ \left\| e^{a|D_x|}(u_0, \epsilon v_0, T_0) \right\|_{\mathcal{B}^{\frac{3}{2}}} \leq \frac{1}{8C} \end{cases}$$

the fact that $\tau(0) = 0$ and the continuity in time in \mathcal{B}^s imply that $t^* > 0$.

For $s = \frac{3}{2}$ and for any $0 < t < t^*$, from (8.3.7) we have

$$\|u_\Theta\|_{\mathcal{B}^{\frac{3}{2}}} \leq \|e^{\mathcal{R}t}(u_\Theta, \epsilon v_\Theta, T_\Theta)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq \left\| e^{a|D_x|}(u_0, \epsilon v_0, T_0) \right\|_{\mathcal{B}^{\frac{3}{2}}} \leq \frac{1}{8C}.$$

For $s = \frac{1}{2}$ and for any $0 < t < t^*$, Inequality (8.3.7) also yields

$$\tau(t) = \int_0^t \|\partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta)(t')\|_{\mathcal{B}^{\frac{1}{2}}} dt'$$

$$\begin{aligned}
&\leq \int_0^t e^{-\mathcal{R}t'} \left\| e^{\mathcal{R}t'} \partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta)(t') \right\|_{\mathcal{B}^{\frac{1}{2}}} dt' \\
&\leq \left(\int_0^t e^{-2\mathcal{R}t'} dt' \right)^{\frac{1}{2}} \left(\int_0^t \left\| e^{\mathcal{R}t'} \partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta)(t') \right\|_{\mathcal{B}^{\frac{1}{2}}}^2 dt' \right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2\mathcal{R}}} \left\| e^{\mathcal{R}t} \partial_z(u_\Theta, \epsilon v_\Theta, T_\Theta) \right\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{1}{2}})} \\
&\leq \frac{2}{\sqrt{2\mathcal{R}}} \left\| e^{\alpha|D_x|}(u_0, \epsilon v_0, T_0) \right\|_{\mathcal{B}^{\frac{1}{2}}} < \frac{a}{4\lambda}.
\end{aligned}$$

A continuity argument implies that $t^* = +\infty$ and allows to close the proof of Theorem 8.0.6.

8.4 Convergence to the hydrostatic limit system

In this section, we will prove Theorem 8.0.7 and justify the approximation of the scaled non-rotating primitive equations and the hydrostatic limit system in a two-dimensional thin strip. To this end, we introduce the following difference quantities

$$w^{\epsilon,1} = u^\epsilon - u, \quad w^{\epsilon,2} = v^\epsilon - v, \quad \theta^\epsilon = T^\epsilon - T, \quad q^\epsilon = p^\epsilon - p$$

where $(u^\epsilon, v^\epsilon, T^\epsilon, p^\epsilon)$ and (u, v, T, p) are respectively solutions of the systems (8.0.3) and (8.0.4). We deduce that $(w^{\epsilon,1}, w^{\epsilon,2}, \theta^\epsilon, q^\epsilon)$ satisfies the following system

$$(8.4.1) \quad \begin{cases} \partial_t w^{\epsilon,1} - \epsilon^2 \partial_x^2 w^{\epsilon,1} - \partial_z^2 w^{\epsilon,1} + \partial_x q^\epsilon = R^{1,\epsilon} \\ \epsilon^2 (\partial_t w^{\epsilon,2} - \epsilon^2 \partial_x^2 w^{\epsilon,2} - \partial_z^2 w^{\epsilon,2}) + \partial_z q^\epsilon = \theta^\epsilon + R^{2,\epsilon} \\ \partial_t \theta^\epsilon - \partial_x^2 \theta^\epsilon - \partial_z^2 \theta^\epsilon = R^{3,\epsilon} \\ \partial_x w^{\epsilon,1} + \partial_z w^{\epsilon,2} = 0 \\ (w^{\epsilon,1}, w^{\epsilon,2}, \theta^\epsilon)|_{t=0} = (u_0^\epsilon - u_0, v_0^\epsilon - v_0, T_0^\epsilon - T_0) \\ (w^{\epsilon,1}, w^{\epsilon,2}, \theta^\epsilon)|_{z=0} = (w^{\epsilon,1}, w^{\epsilon,2}, \theta^\epsilon)|_{z=1} = 0, \end{cases}$$

where the remaining terms $R^{i,\epsilon}$, with $i = 1, 2, 3$, are given by

$$(8.4.2) \quad \begin{cases} R^{1,\epsilon} = \epsilon^2 \partial_x^2 u - (u^\epsilon \partial_x u^\epsilon - u \partial_x u) - (v^\epsilon \partial_z u^\epsilon - v \partial_z u), \\ R^{2,\epsilon} = -\epsilon^2 (\partial_t v - \epsilon^2 \partial_x^2 v - \partial_z^2 v + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_z v^\epsilon), \\ R^{3,\epsilon} = -(u^\epsilon \partial_x T^\epsilon + v^\epsilon \partial_z T^\epsilon) + (u \partial_x T + v \partial_z T). \end{cases}$$

Let $\varphi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$\varphi(0, \xi) = 0, \quad \varphi(t, \xi) = (a - \mu\eta(t))|\xi|, \quad \forall t > 0, \forall \xi \in \mathbb{R},$$

where $\mu \geq \lambda > 0$ and $\eta(t)$ will be determined later. For any function $f \in L^2(\mathbf{S})$, we define

$$\varphi : f \mapsto f_\varphi; \quad f_\varphi(t, x, z) = e^{\varphi(t, D_x)} f(t, x, z) = \mathcal{F}_h^{-1}(e^{\varphi(t, \xi)} \widehat{f}(t, \xi, z)).$$

In what follows, for the sake of the simplicity, we will drop the index ϵ and write $(w_\varphi^1, w_\varphi^2, \theta_\varphi, q_\varphi, R_\varphi^i)$

instead of $(w_\varphi^{\epsilon,1}, w_\varphi^{\epsilon,2}, \theta_\varphi^\epsilon, q^\epsilon, R_\varphi^{i,\epsilon})$. Direct calculations show that $(w_\varphi^1, w_\varphi^2, \theta_\varphi, q_\varphi)$ satisfies

$$(8.4.3) \quad \begin{cases} \partial_t w_\varphi^1 + \mu |D_x| \dot{\eta}(t) w_\varphi^1 - \epsilon^2 \partial_x^2 w_\varphi^1 - \partial_z^2 w_\varphi^1 + \partial_x q_\varphi = R_\varphi^1 \\ \epsilon^2 (\partial_t w_\varphi^2 + \mu |D_x| \dot{\eta}(t) w_\varphi^2 - \epsilon^2 \partial_x^2 w_\varphi^2 - \partial_z^2 w_\varphi^2) + \partial_z q_\varphi = \theta_\varphi + R_\varphi^2 \\ \partial_t \theta_\varphi + \mu |D_x| \dot{\eta}(t) \theta_\varphi - \partial_x^2 \theta_\varphi - \partial_z^2 \theta_\varphi = R_\varphi^3 \\ \partial_x w_\varphi^1 + \partial_z w_\varphi^2 = 0 \\ (w_\varphi^1, w_\varphi^2, \theta_\varphi)|_{t=0} = e^{a|D_x|} (u_0^\epsilon - u_0, v_0^\epsilon - v_0, T_0^\epsilon - T_0) \\ (w_\varphi^1, w_\varphi^2, \theta_\varphi)|_{z=0} = (w_\varphi^1, w_\varphi^2, \theta_\varphi)|_{z=1} = 0. \end{cases}$$

As in the previous sections, we will use “ C ” to denote a generic positive constant which can change from line to line.

Applying the dyadic operator Δ_q^h to System (8.4.3), then taking the $L^2(\mathcal{S})$ scalar product of the first, second and the third equations of the obtained system with $\Delta_q^h w_\varphi^1$, $\Delta_q^h w_\varphi^2$ and $\Delta_q^h \theta_\varphi$ respectively, we obtain

$$(8.4.4) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)(t)\|_{L^2}^2 + \mu \dot{\eta}(t) \langle |D_x| \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2), \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2) \rangle_{L^2} \\ + \|\partial_z \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)\|_{L^2}^2 + \epsilon^2 \|\partial_x \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)\|_{L^2}^2 = \langle \Delta_q^h R_\varphi^1, \Delta_q^h w_\varphi^1 \rangle_{L^2} \\ + \langle \Delta_q^h R_\varphi^2, \Delta_q^h w_\varphi^2 \rangle_{L^2} - \langle \Delta_q^h \nabla q_\varphi, \Delta_q^h(w_\varphi^1, w_\varphi^2) \rangle_{L^2} + \langle \Delta_q^h \theta_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \end{aligned}$$

and

$$(8.4.5) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q^h \theta_\varphi(t)\|_{L^2}^2 + \mu \dot{\eta}(t) \langle |D_x| \Delta_q^h \theta_\varphi, \Delta_q^h \theta_\varphi \rangle_{L^2} + \|\nabla \Delta_q^h \theta_\varphi\|_{L^2}^2 = \langle \Delta_q^h R_\varphi^3, \Delta_q^h \theta_\varphi \rangle_{L^2}.$$

Integrating (8.4.4) and (8.4.5) with respect to the time variable, we have

$$(8.4.6) \quad \begin{aligned} \|\Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)\|_{L_t^\infty(L^2)}^2 + \mu \int_0^t \dot{\eta}(t') \langle |D_x| \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2), \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2) \rangle_{L^2} dt' \\ + \|\partial_z \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)\|_{L_t^2(L^2)}^2 + \epsilon^2 2^q \|\Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)\|_{L_t^2(L^2)}^2 \\ \leq \|\Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)(0)\|_{L^2}^2 + G_1^q + G_2^q + G_3^q + G_4^q, \end{aligned}$$

and

$$(8.4.7) \quad \begin{aligned} \|\Delta_q^h \theta_\varphi\|_{L_t^\infty(L^2)}^2 + \mu \int_0^t \dot{\eta}(t') \langle |D_x| \Delta_q^h \theta_\varphi, \Delta_q^h \theta_\varphi \rangle_{L^2} dt' + \|\partial_z \Delta_q^h \theta_\varphi\|_{L_t^2(L^2)}^2 + 2^q \|\Delta_q^h \theta_\varphi\|_{L_t^2(L^2)}^2 \\ \leq \|\Delta_q^h \theta_\varphi(0)\|_{L^2}^2 + G_5^q, \end{aligned}$$

where the terms G_i , $i = 1, \dots, 5$, will be precised and controlled in what follows.

We will now estimate the linear terms on the right-hand side of the above inequalities. First of all, for pressure term, using the incompressibility property $\partial_x w_\varphi^1 + \partial_z w_\varphi^2 = 0$, we perform an integration by parts and get

$$(8.4.8) \quad G_3^q = 2 \left| \int_0^t \langle \nabla \Delta_q^h q_\varphi, \Delta_q^h(w_\varphi^1, w_\varphi^2) \rangle_{L^2} dt' \right| = 2 \left| \int_0^t \langle \Delta_q^h q_\varphi, \operatorname{div}(\Delta_q^h(w_\varphi^1, w_\varphi^2)) \rangle_{L^2} dt' \right| = 0.$$

For the temperature term, the boundary condition $(u_\Theta, v_\Theta)|_{z=0} = (u_\Theta, v_\Theta)|_{z=1} = 0$ and the incompressibility property $\partial_x w_\varphi^1 + \partial_z w_\varphi^2 = 0$ allow to write w_φ^2 as function of w_φ^1

$$w_\varphi^2 = - \int_0^z \partial_x w_\varphi^1(x, z') dz'.$$

Then, using integration by parts, Poincaré inequality and Bernstein lemma 8.0.1, we have

$$\begin{aligned}
(8.4.9) \quad G_4^q &= 2 \left| \int_0^t \langle \Delta_q^h \theta_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} dt' \right| = 2 \left| \int_0^t \left\langle \Delta_q^h \theta_\varphi, -\Delta_q^h \int_0^z \partial_x w_\varphi^1(x, s) ds \right\rangle_{L^2} dt' \right| \\
&\leq 2 \int_0^t \left| \left\langle \Delta_q^h \partial_x \theta_\varphi, \Delta_q^h \int_0^z w_\varphi^1(x, s) ds \right\rangle_{L^2} \right| dt' \leq 2 \int_0^t \|\Delta_q^h \partial_x \theta_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\
&\leq C 2^q \|\Delta_q^h \theta_\varphi\|_{L_t^2(L^2)}^2 + \frac{1}{16} \|\partial_z \Delta_q^h w_\varphi^1\|_{L_t^2(L^2)}^2
\end{aligned}$$

The goal of the main part of this section is to estimate the nonlinear terms on the right-hand side of (8.4.6) and (8.4.7), which are

$$\begin{aligned}
G_1^q &= 2 \left| \int_0^t \langle \Delta_q^h R_\varphi^1, \Delta_q^h w_\varphi^1 \rangle_{L^2} dt' \right| \\
G_2^q &= 2 \left| \int_0^t \langle \Delta_q^h R_\varphi^2, \Delta_q^h w_\varphi^2 \rangle_{L^2} dt' \right| \\
G_5^q &= 2 \left| \int_0^t \langle \Delta_q^h R_\varphi^3, \Delta_q^h \theta_\varphi \rangle_{L^2} dt' \right|,
\end{aligned}$$

with R_φ^i being defined in (8.4.2). From now on, we will set

$$\eta(t) = \int_0^t \left(\|(\partial_z u_\Theta^\epsilon, \epsilon \partial_x u_\Theta^\epsilon, \partial_z T_\Theta^\epsilon)(t')\|_{\mathcal{B}^{\frac{1}{2}}} + \|\partial_z(u_\phi, T_\phi)(t')\|_{\mathcal{B}^{\frac{1}{2}}} \right) dt' = \rho(t) + \tau(t),$$

where ρ and τ are defined in the previous sections. We remark that, under the hypotheses of Theorems 8.0.5 and 8.0.6, we have

$$\frac{a}{3} \leq \varphi(t, \xi) \leq \min \{ \phi(t, \xi), \Theta(t, \xi) \}.$$

Before giving the estimates of G_1^q , G_2^q and G_5^q , we remark that we need a slightly different version of Lemmas 8.1.1 and 8.1.3 where we can relax the condition $f(t) \geq \|\partial_z w_\phi(t)\|_{\mathcal{B}^{\frac{1}{2}}}$. More precisely, we will prove the following lemma.

Lemma 8.4.1. *Let $0 < s < \frac{1}{2}$ and $\phi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$. There exists a constant $C \geq 1$ such that, for any (u, v, w, \bar{w}) , which are defined on $\mathbb{R}_+ \times \mathcal{S}$, $(u, w, \bar{w})|_{\partial \mathcal{S}} = 0$ and satisfy, for any $t \geq 0$,*

$$u_\phi(t) \in \mathcal{B}^{\frac{3}{2}}, \quad \partial_z u_\phi(t), \partial_z w_\phi(t) \in \mathcal{B}^{\frac{1}{2}}, \quad \int_0^1 \partial_x u dz = 0, \quad \text{and} \quad u, w, \bar{w} \in \tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}}),$$

with

$$f \in L_{loc}^1(\mathbb{R}_+), \quad f(t) \geq \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \quad \text{and} \quad v(t, x, z) = - \int_0^z \partial_x u(t, x, z') dz',$$

we have, for any $\mathcal{R} \geq 0$ and for any $q \in \mathbb{Z}$,

$$(8.4.10) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (u \partial_x w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt' \leq C d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t, f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

$$(8.4.11) \quad \int_0^t \left| \left\langle e^{\mathcal{R}t'} \Delta_q^h (v \partial_z w)_\phi, e^{\mathcal{R}t'} \Delta_q^h \bar{w}_\phi \right\rangle_{L^2} \right| dt'$$

$$\leq C d_q^2 2^{-2qs} \|u_\phi\|_{L_x^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where $(d_q)_{q \in \mathbb{Z}}$ is a positive sequence with $\sum_{q \in \mathbb{Z}} d_q = 1$.

Proof of Lemma 8.4.1.

1. In order to prove Estimate (8.4.10), we can use Lemma 8.1.1 and we only need to modify the calculations that we did for the $A_{2,q}$ on the page 258. We use Sobolev inclusion $\dot{H}_z^1([0, 1]) \hookrightarrow L_z^\infty([0, 1])$ and write

$$A_{2,q} \lesssim \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_x w_\phi\|_{L_x^\infty L_z^2} \|\Delta_{q'}^h \partial_z u_\phi\|_{L^2} \|\Delta_{q'}^h \bar{w}_\phi\|_{L^2} dt'.$$

Now, using the definition of $S_{q'-1}^h$ and of the $\mathcal{B}^{\frac{1}{2}}$ -norm and Bernstein lemma 8.0.1, we get

$$A_{2,q} \lesssim \sum_{|q-q'|\leq 4} \int_0^t e^{2\mathcal{R}t'} \left(\sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h w_\phi\|_{L^2} \right) d_{q'}(\partial_z u_\phi) 2^{-\frac{q'}{2}} \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h \bar{w}_\phi\|_{L^2} dt'.$$

Since $d_{q'}(\partial_z u_\phi) \leq 1$, using Proposition 8.0.4, we can write

$$\begin{aligned} A_{2,q} &\lesssim \sum_{|q-q'|\leq 4} 2^{-\frac{q'}{2}} \left(\int_0^t \sum_{l \leq q'-2} 2^{3l} \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_l^h e^{\mathcal{R}t'} w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h e^{\mathcal{R}t'} \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \sum_{|q-q'|\leq 4} 2^{-\frac{q'}{2}} \left(\sum_{l \leq q'-2} d_l(w) 2^{l(1-s)} \right) \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} d_q(\bar{w}) 2^{-q(s+\frac{1}{2})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}. \end{aligned}$$

We remark that the sequence $\{\bar{d}_{q'}\}_{q' \in \mathbb{Z}}$ with

$$\bar{d}_{q'} = \sum_{l \leq q'-2} d_l(w) 2^{(q'-l)(s-1)}$$

can be written as a convolution product of two summable sequences if $0 < s < 1$. Thus, we finally obtain

$$A_{2,q} \lesssim d_q^2 2^{-2qs} \|e^{\mathcal{R}t} w_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})},$$

where

$$d_q^2 = d_q(\bar{w}) \sum_{|q-q'|\leq 4} \bar{d}_{q'} 2^{(q-q')(s-\frac{1}{2})}.$$

2. To prove Estimate (8.4.11), the same modifications can be done to the terms $B_{2,q}$ and $B_{3,q}$ in the proof of Lemma 8.1.3 on page 261. We will show these modifications for $B_{2,q}$. We recall that, using Lemma 8.1.4, we have

$$\begin{aligned} B_{2,q} &\lesssim \sum_{|q'-q|\leq 4} \int_0^t e^{2\mathcal{R}t'} \|S_{q'-1}^h \partial_z w_\phi\|_{L_x^\infty L_z^2} \|\Delta_{q'}^h v_\phi\|_{L_x^2 L_z^\infty} \|\Delta_{q'}^h \bar{w}_\phi\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q|\leq 4} \int_0^t \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h e^{\mathcal{R}t'} \partial_z w_\phi\|_{L^2} 2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} \|\Delta_{q'}^h e^{\mathcal{R}t'} \bar{w}_\phi\|_{L^2} dt'. \end{aligned}$$

Now, using Poincaré inequality, we can write

$$2^{q'} \|\Delta_{q'}^h u_\phi\|_{L^2} = 2^{\frac{3q'}{4}} \|\Delta_{q'}^h u_\phi\|_{\tilde{L}^2} 2^{\frac{q'}{4}} \|\Delta_{q'}^h \partial_z u_\phi\|_{\tilde{L}^2}^{\frac{1}{2}} \leq \|u_\phi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Then, Cauchy-Schwartz inequality implies

$$\begin{aligned} B_{2,q} &\lesssim \sum_{|q'-q|\leq 4} \int_0^t \|u_\phi\|_{\mathcal{B}^{\frac{3}{2}}}^{\frac{1}{2}} \sum_{l\leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h e^{\mathcal{R}t'} \partial_z w_\phi\|_{L^2} \left(\|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\Delta_q^h e^{\mathcal{R}t'} \bar{w}_\phi\|_{L^2} \right) dt' \\ &\lesssim \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \sum_{l\leq q'-2} 2^{\frac{l}{2}} \left(\int_0^t \|\Delta_l^h e^{\mathcal{R}t'} \partial_z w_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_z u_\phi\|_{\mathcal{B}^{\frac{1}{2}}}^2 \|\Delta_q^h e^{\mathcal{R}t'} \bar{w}_\phi\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \left(\sum_{l\leq q'-2} d_l(w) 2^{l(\frac{1}{2}-s)} \right) \|e^{\mathcal{R}t} \partial_z w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \left(d_q(\bar{w}) 2^{-q(s+\frac{1}{2})} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \right) \end{aligned}$$

We remark that the sequence $\{\bar{d}_{q'}\}_{q'\in\mathbb{Z}}$ with

$$\bar{d}_{q'} = \sum_{l\leq q'-2} d_l(w) 2^{(q'-l)(s-\frac{1}{2})}$$

can be written as a convolution product of two summable sequences if $0 < s < \frac{1}{2}$. Thus, we obtain

$$B_{2,q} \lesssim d_q^2 2^{-2qs} \|u_\phi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|e^{\mathcal{R}t} \partial_z w_\phi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|e^{\mathcal{R}t} \bar{w}_\phi\|_{\tilde{L}_{t,f(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.$$

Finally, we remark that the term $B_{3,q}$ can be controlled in the similar way and we obtain Estimate (8.4.11). \square

8.4.1 Control of G_1^q

We start by observing that we can write

$$R_\varphi^1 = (\epsilon^2 \partial_x^2 u)_\varphi - (u^\epsilon \partial_x w^1)_\varphi - (w^1 \partial_x u)_\varphi - (v^\epsilon \partial_z w^1)_\varphi - (w^2 \partial_z u)_\varphi,$$

so,

$$G_1^q \leq I_1^q + I_2^q + I_3^q + I_4^q + I_5^q,$$

where I_i^q will be precised and controlled in what follows.

Using Bernstein lemma 8.0.1 and Poincaré inequality, we can write

$$\|\Delta_q^h (\partial_x u)_\varphi\|_{L^2} \lesssim 2^{-q} \|\Delta_q^h u_\varphi\|_{L^2} \lesssim 2^{-q} \|\Delta_q^h \partial_z u_\varphi\|_{L^2}$$

Then, using Poincaré inequality and Bernstein lemma 8.0.1, we have

(8.4.12)

$$\begin{aligned} I_1^q &= 2\epsilon^2 \int_0^t \left| \langle \Delta_q^h (\partial_x^2 u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' = 2\epsilon^2 \int_0^t \|\Delta_q^h (\partial_x^2 u)_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim \epsilon^2 2^{2q} \|\Delta_q^h u_\varphi\|_{L_t^2(L^2)} \|\Delta_q^h w_\varphi^1\|_{L_t^2(L^2)} \lesssim \epsilon^2 2^{2q} \|\Delta_q^h \partial_z u_\varphi\|_{L_t^2(L^2)} \|\Delta_q^h w_\varphi^1\|_{L_t^2(L^2)} \\ &\lesssim d_q^2 2^{-2qs} \epsilon \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})} \|\epsilon w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{1}{2}})} \lesssim d_q^2 2^{-2qs} \left(C\epsilon^2 \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \frac{1}{4} \|\epsilon w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right) \end{aligned}$$

For I_2^q , Estimate (8.4.10) of Lemma 8.4.1 implies

$$(8.4.13) \quad I_2^q = 2 \int_0^t \left| \langle \Delta_q^h (u^\epsilon \partial_x w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \|w_\varphi^1\|_{\tilde{L}_{t, \tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2,$$

The term I_3^q can be controlled in the exact same way as we did to prove Estimate (8.4.11). Indeed, using the Bony decomposition, we can write

$$\int_0^t \left| \langle \Delta_q^h (w^1 \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \leq \bar{B}_{1,q} + \bar{B}_{2,q} + \bar{B}_{3,q},$$

with

$$\begin{aligned} \bar{B}_{1,q} &= \int_0^t \left| \langle \Delta_q^h (\mathbb{T}_{w^1}^h \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ \bar{B}_{2,q} &= \int_0^t \left| \langle \Delta_q^h (\mathbb{T}_{\partial_x u}^h w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ \bar{B}_{3,q} &= \int_0^t \left| \langle \Delta_q^h (\mathcal{R}^h(w^1, \partial_x u))_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt'. \end{aligned}$$

Using Poincaré inequality and Bernstein lemma 8.0.1, we can write

$$\begin{aligned} \bar{B}_{1,q} &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h w_\varphi^1\|_{L^\infty} \|\Delta_{q'}^h \partial_x u_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h \partial_x w_\varphi^1\|_{L^2} 2^{q'} \|\Delta_{q'}^h u_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt'. \end{aligned}$$

Thus, we can control $\bar{B}_{1,q}$ in the exact same way as what we did for $B_{2,q}$ above. Similarly, $\bar{B}_{2,q}$ can be controlled in the same way as $B_{1,q}$ and $\bar{B}_{3,q}$ as $B_{3,q}$. We obtain

$$(8.4.14) \quad \begin{aligned} I_3^q &= 2 \int_0^t \left| \langle \Delta_q^h (w^1 \partial_x u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \|u_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|w_\varphi^1\|_{\tilde{L}_{t, \tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ &\lesssim d_q^2 2^{-2qs} \left(\|u_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|w_\varphi^1\|_{\tilde{L}_{t, \tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right). \end{aligned}$$

For I_4^q , using Estimate (8.4.11) of Lemma 8.4.1, we also have

$$(8.4.15) \quad \begin{aligned} I_4^q &= 2 \int_0^t \left| \langle \Delta_q^h (v^\epsilon \partial_z w^1)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ &\lesssim d_q^2 2^{-2qs} \|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)} \|w_\varphi^1\|_{\tilde{L}_{t, \tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\ &\lesssim d_q^2 2^{-2qs} \left(\|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|w_\varphi^1\|_{\tilde{L}_{t, \tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right). \end{aligned}$$

For the term I_5^q , we will not use Lemma 8.4.1 and we will use a slightly different control. We first recall the following horizontal decomposition into paraproducts and remainders

$$\begin{aligned} I_5^q &= 2 \int_0^t \left| \langle \Delta_q^h (w^2 \partial_z u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ &= 2 \int_0^t \left| \langle \Delta_q^h (T_{w^2}^h \partial_z u + T_{\partial_z u}^h w^2 + R^h(w^2, \partial_z u))_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \end{aligned}$$

$$\leq 2(I_{5,1}^q + I_{5,2}^q + I_{5,3}^q),$$

where

$$\begin{aligned} I_{5,1}^q &= \int_0^t \left| \langle \Delta_q^h (T_{w^2}^h \partial_z u)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ I_{5,2}^q &= \int_0^t \left| \langle \Delta_q^h (T_{\partial_z u}^h w^2)_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt' \\ I_{5,3}^q &= \int_0^t \left| \langle \Delta_q^h (R^h(w^2, \partial_z u))_\varphi, \Delta_q^h w_\varphi^1 \rangle_{L^2} \right| dt'. \end{aligned}$$

For $I_{5,1}^q$, Lemma 8.1.4 yields

$$\|S_{q'-1}^h w_\varphi^2\|_{L^\infty} \lesssim \sum_{l \leq q'-2} \|\Delta_l^h w_\varphi^1\|_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h w_\varphi^1\|_{L^2}.$$

Then, we can write

$$\begin{aligned} I_{5,1}^q &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h w_\varphi^2\|_{L^2} \|\Delta_{q'}^h \partial_z u_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \left(\sum_{l \leq q'-2} 2^{\frac{3l}{2}} \|\Delta_l^h w_\varphi^1\|_{L^2} \right) \|\Delta_{q'}^h \partial_z u_\varphi\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt'. \end{aligned}$$

Thus, $I_{5,1}^q$ can be controlled in the same way as $A_{2,q}$ in the proof of Estimate (8.4.10) of Lemma 8.4.1 and if $0 < s < 1$, we have

$$(8.4.16) \quad I_{5,1}^q \lesssim d_q^2 2^{-2qs} \|w_\varphi^1\|_{\dot{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

For $I_{5,2}^q$, using Lemma 8.1.4, we write

$$\|\Delta_{q'}^h w_\varphi^2\|_{L_x^2 L_z^\infty} \lesssim 2^{q'} \|\Delta_{q'}^h w_\varphi^1\|_{L^2}.$$

Since

$$\|S_{q'-1}^h \partial_z u_\varphi\|_{L_x^\infty L_z^2} \lesssim \|\partial_z u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}},$$

as in the estimate of the term $A_{1,q}$ in the proof of Lemma 8.1.1, we have,

$$\begin{aligned} (8.4.17) \quad I_{5,2}^q &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h \partial_z u_\varphi\|_{L_x^\infty L_z^2} \|\Delta_{q'}^h w_\varphi^2\|_{L_x^2 L_z^\infty} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} \int_0^t \|\partial_z u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} 2^{q'} \|\Delta_{q'}^h w_\varphi^1\|_{L^2} \|\Delta_q^h w_\varphi^1\|_{L^2} dt' \\ &\lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left(\int_0^t \|\partial_z u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_{q'}^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_z u_\varphi\|_{\mathcal{B}^{\frac{1}{2}}} \|\Delta_q^h w_\varphi^1\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_q^2 2^{-2qs} \|w_\varphi^1\|_{\dot{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2. \end{aligned}$$

The last term $I_{5,3}^q$ can also be treated in the similar way as the term $A_{3,q}$ in the proof of Lemma 8.1.1

and we will get

$$(8.4.18) \quad I_{5,3}^q \lesssim d_q^2 2^{-2qs} \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Summing Estimate (8.4.16), (8.4.17) and (8.4.18) will imply that

$$(8.4.19) \quad I_5^q \lesssim d_q^2 2^{-2qs} \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2.$$

Summing Estimates (8.4.12), (8.4.13), (8.4.14), (8.4.15), and (8.4.19), we obtain

$$(8.4.20) \quad G_1^q \lesssim d_q^2 2^{-q} \left(C\epsilon^2 \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \frac{1}{4} \|\epsilon w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C \|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|w_\varphi^1\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).$$

8.4.2 Control of G_2^q

We recall that

$$R_\varphi^2 = -\epsilon^2 (\partial_t v - \partial_z^2 v - \epsilon^2 \partial_x^2 v + u^\epsilon \partial_x v^\epsilon + v^\epsilon \partial_z v^\epsilon)_\varphi.$$

Then, we can bound G_2^q as follows

$$G_2^q \leq J_1^q + J_2^q + J_3^q + J_4^q + J_5^q,$$

where J_i^q will be precised and controlled in what follows.

Using Young inequality, Sobolev inclusion $\dot{H}_z^1([0, 1]) \hookrightarrow L_z^\infty([0, 1])$ and Bernstein lemma 8.0.1, we have

$$\begin{aligned} J_1^q &= 2\epsilon^2 \int_0^t \left| \langle \Delta_q^h(\partial_t v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim \epsilon^2 \|\Delta_q^h(\partial_t v)_\varphi\|_{L_t^2(L_x^\infty L_z^2)} \|w_\varphi^2\|_{L_t^2(L_x^2 L_z^\infty)} \\ &\lesssim \epsilon^2 2^{\frac{3}{2}q} \|\Delta_q^h(\partial_t v)_\varphi\|_{L_t^2(L^2)} \|\Delta_q^h \partial_z w_\varphi^2\|_{L_t^2(L^2)} \lesssim \epsilon^2 2^{\frac{3}{2}q} \|\Delta_q^h(\partial_t \partial_z v)_\varphi\|_{L_t^2(L^2)} \|\Delta_q^h \partial_z w_\varphi^2\|_{L_t^2(L^2)} \\ &\lesssim \epsilon^2 2^{\frac{3}{2}q} \|\Delta_q^h(\partial_t \partial_x u)_\varphi\|_{L_t^2(L^2)} \|\Delta_q^h \partial_z w_\varphi^2\|_{L_t^2(L^2)} \lesssim \epsilon^2 2^{\frac{3}{2}q} \|\Delta_q^h(\partial_t u)_\varphi\|_{L_t^2(L^2)} \|\Delta_q^h \partial_z w_\varphi^2\|_{L_t^2(L^2)} \\ &\lesssim d_q^2 2^{-2qs} \epsilon^2 \left(C \|(\partial_t u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \frac{1}{100} \|\partial_z w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right). \end{aligned}$$

Similarly, we have

$$J_2^q = 2\epsilon^2 \int_0^t \left| \langle \Delta_q^h(\partial_z^2 v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \epsilon^2 \left(C \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \frac{1}{100} \|\partial_z w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right),$$

and

$$J_3^q = 2\epsilon^4 \int_0^t \left| \langle \Delta_q^h(\partial_x^2 v)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \lesssim d_q^2 2^{-2qs} \epsilon^4 \left(C \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \frac{1}{100} \|\partial_z w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right).$$

For J_4^q , we can use similar estimates as in the proof of Estimate (8.4.11) of Lemma 8.4.1. We first apply Bony's decomposition for the horizontal variable and write

$$J_4^q = 2\epsilon^2 \int_0^t \left| \langle \Delta_q^h(u^\epsilon \partial_x v^\epsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \leq 2(J_{41}^q + J_{42}^q + J_{43}^q),$$

with

$$\begin{aligned} J_{41}^q &= \epsilon^2 \int_0^t \left| \langle \Delta_q^h (T_{u^\epsilon}^h \partial_x v^\epsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \\ J_{42}^q &= \epsilon^2 \int_0^t \left| \langle \Delta_q^h (T_{\partial_x v^\epsilon}^h u^\epsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \\ J_{43}^q &= \epsilon^2 \int_0^t \left| \langle \Delta_q^h (R^h(u^\epsilon, \partial_x v^\epsilon))_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt'. \end{aligned}$$

We remark that Lemma 8.1.4 and Bernstein lemma 8.0.1 imply

$$\|\Delta_{q'}^h \partial_x v_\varphi^\epsilon\|_{L_x^2 L_z^\infty} \leq 2^{2q'} \|\Delta_q^h u_\varphi^\epsilon\|_{L^2}.$$

We also remark that, using Poincaré inequality, Bernstein lemma 8.0.1 and then Cauchy-Schwartz inequality, we have

$$\|S_{q'-1}^h u_\varphi^\epsilon\|_{L_x^\infty L_z^2} \lesssim \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h u_\varphi^\epsilon\|_{L^2} \lesssim \sum_{l \leq q'-2} 2^{\frac{l}{2}} \|\Delta_l^h u_\varphi^\epsilon\|_{L^2}^{\frac{1}{2}} \|\Delta_l^h \partial_z u_\varphi^\epsilon\|_{L^2}^{\frac{1}{2}} \lesssim \|u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}}.$$

Then, using Young inequality, we get

$$\begin{aligned} J_{41}^q &\lesssim \epsilon^2 \sum_{|q'-q| \leq 4} \int_0^t \|S_{q'-1}^h u_\varphi^\epsilon\|_{L_x^\infty L_z^2} \|\Delta_{q'}^h \partial_x v_\varphi^\epsilon\|_{L_x^2 L_z^\infty} \|\Delta_q^h w_\varphi^2\|_{L^2} dt' \\ &\lesssim \epsilon^2 \sum_{|q'-q| \leq 4} \int_0^t \|u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} \|\partial_z u_\varphi^\epsilon\|_{\mathcal{B}^{\frac{1}{2}}}^{\frac{1}{2}} 2^{2q'} \|\Delta_{q'}^h u_\varphi^\epsilon\|_{L^2} \|\Delta_q^h w_\varphi^2\|_{L^2} dt'. \end{aligned}$$

Cauchy-Schwartz and Poincaré inequalities finally imply

$$J_{41}^q \lesssim d_q^2 2^{-2qs} \epsilon^2 \left(\|u_\varphi^\epsilon\|_{L^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_z u_\varphi^\epsilon\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \|w_\varphi^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right)$$

Similar estimates also lead to

$$\begin{aligned} J_{42}^q &\lesssim d_q^2 2^{-2qs} \epsilon^2 \left(\|u_\varphi^\epsilon\|_{L^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_z u_\varphi^\epsilon\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \|w_\varphi^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right) \\ J_{43}^q &\lesssim d_q^2 2^{-2qs} \epsilon^2 \left(\|u_\varphi^\epsilon\|_{L^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_z u_\varphi^\epsilon\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \|w_\varphi^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right). \end{aligned}$$

Summing the above inequalities, we obtain

$$J_4^q \lesssim d_q^2 2^{-2qs} \epsilon^2 \left(\|u_\varphi^\epsilon\|_{L^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_z u_\varphi^\epsilon\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \|w_\varphi^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).$$

Now, for J_5^q , using the divergence-free property, we can write $v^\epsilon \partial_z v^\epsilon = v^\epsilon \partial_x u^\epsilon$. Thus, J_5^q can be controlled in the exact same way as J_4^q and we get

$$\begin{aligned} J_5^q &= 2\epsilon^2 \int_0^t \left| \langle \Delta_q^h (v^\epsilon \partial_z v^\epsilon)_\varphi, \Delta_q^h w_\varphi^2 \rangle_{L^2} \right| dt' \\ &\lesssim d_q^2 2^{-2qs} \epsilon^2 \left(\|u_\varphi^\epsilon\|_{L^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_z u_\varphi^\epsilon\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \|w_\varphi^2\|_{\tilde{L}_{t,\eta(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right). \end{aligned}$$

We deduce that

$$(8.4.21) \quad G_2^q \lesssim d_q^2 2^{-2qs} \epsilon^2 \left(C \|\partial_t u\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + C \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + C \epsilon^2 \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{5}{2}})}^2 \right. \\ \left. + \frac{1}{20} \|\partial_z w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|u_\varphi^\epsilon\|_{L^\infty(\mathcal{B}^{\frac{1}{2}})} \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})}^2 + \|w_\varphi^2\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).$$

8.4.3 Control of G_5^q

To estimate the last term G_5^q , we will write

$$R_\varphi^3 = -(u^\epsilon \partial_x \theta + w^1 \partial_x T)_\varphi - (v^\epsilon \partial_z \theta + w^2 \partial_z T)_\varphi,$$

and so,

$$G_5^q = 2 \int_0^t \left| \langle \Delta_q^h R_\varphi^3, \Delta_q^h \theta_\varphi \rangle_{L^2} \right| dt' \leq 2 (L_1^q + L_2^q + L_3^q + L_4^q),$$

where

$$L_1^q = \int_0^t \left| \langle \Delta_q^h (u^\epsilon \partial_x \theta)_\varphi, \Delta_q^h \theta_\varphi \rangle_{L^2} \right| dt' \\ L_2^q = \int_0^t \left| \langle \Delta_q^h (w^1 \partial_x T)_\varphi, \Delta_q^h \theta_\varphi \rangle_{L^2} \right| dt' \\ L_3^q = \int_0^t \left| \langle \Delta_q^h (v^\epsilon \partial_z \theta)_\varphi, \Delta_q^h \theta_\varphi \rangle_{L^2} \right| dt' \\ L_4^q = \int_0^t \left| \langle \Delta_q^h (w^2 \partial_z T)_\varphi, \Delta_q^h \theta_\varphi \rangle_{L^2} \right| dt'.$$

Using Lemma 8.4.1, we immediately obtain

$$L_1^q \lesssim d_q^2 2^{-2qs} \|\theta_\varphi\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2$$

and

$$L_3^q \lesssim d_q^2 2^{-2qs} \left(\|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_z \theta_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|\theta_\varphi\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).$$

The term L_2^q can be controlled in the same way as we did for the term I_3^q (page 278). We have

$$L_2^q \lesssim d_q^2 2^{-2qs} \left(\|T_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + \|\theta_\varphi\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).$$

Finally, the term L_4^q can be controlled in the same way as we did for the term $I_{5,1}^q$ (page 279) and we obtain

$$L_4^q \lesssim d_q^2 2^{-2qs} \left(\|w_\varphi^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + \|\theta_\varphi\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right).$$

Thus, we deduce that

$$(8.4.22) \quad G_5^q \lesssim d_q^2 2^{-2qs} \left(C \|w_\varphi^1\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 + C \|\theta_\varphi\|_{\tilde{L}_{t,\tilde{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}^2 \right. \\ \left. + C \|T_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 + C \|u_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \|\partial_z \theta_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^s)}^2 \right).$$

End of the proof of Theorem 8.0.7. We put Estimates (8.4.20), (8.4.21), (8.4.8), (8.4.9) and (8.4.22) into Inequalities (8.4.6) and (8.4.7), we multiply (8.4.7) by $2C$ and then we sum the obtained inequalities. Now, we recall that we can take the square-root of each term of the resulting inequality, with the cost of a constant multiplier which will be included in the generic constant C . Without loss of generality, we can always consider that $C \geq 2$. We obtain

(8.4.23)

$$\begin{aligned}
& \|\Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)\|_{L_t^\infty(L^2)} + \sqrt{\mu} \left(\int_0^t \dot{\eta}(t') \left\| |D_x|^{\frac{1}{2}} \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi) \right\|_{L^2} dt' \right)^{\frac{1}{2}} \\
& + \frac{3}{4} \|\partial_z \Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)\|_{L_t^2(L^2)} + \epsilon 2^{\frac{q}{2}} \|\Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2)\|_{L_t^2(L^2)} + 2^{\frac{q}{2}} \|\Delta_q^h \theta_\varphi\|_{L_t^2(L^2)} \\
& \leq 2C \|\Delta_q^h(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)(0)\|_{L^2} + d_q 2^{-qs} \left(C\epsilon \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})} + \frac{1}{2} \|\epsilon w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{1}{2}})} \right. \\
& + C \|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)} + C\epsilon \|(\partial_t u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})} + C\epsilon \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})} \\
& + C\epsilon^2 \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{5}{2}})} + \frac{1}{4} \epsilon \|\partial_z w_\varphi^2\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \epsilon \|u_\varphi^\epsilon\|_{L^\infty(\mathcal{B}^{\frac{1}{2}})}^{\frac{1}{2}} \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})} \\
& \left. + C \|(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} + C \|T_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{1}{2}} \|\partial_z w_\varphi^1\|_{\tilde{L}_t^2(\mathcal{B}^s)} + C \|u_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})}^{\frac{3}{2}} \|\partial_z \theta_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^s)} \right).
\end{aligned}$$

Now, from the results of the previous sections, we deduce the existence of a constant $M \geq 1$ such that

$$\|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{1}{2}})} + \|u_\varphi^\epsilon\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} + \|(\partial_t u)_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})} + \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{3}{2}})} + \|\partial_z u_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{5}{2}})} \leq M$$

and we deduce also that, for $\|(u_0, T_0)\|_{\mathcal{B}^{\frac{3}{2}}}$ small enough,

$$\|T_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} + \|u_\varphi\|_{L_t^\infty(\mathcal{B}^{\frac{3}{2}})} \leq \frac{1}{4C}.$$

Multiplying (8.4.23) by 2^{qs} and then summing with respect to q , we obtain

$$\begin{aligned}
& \|(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)\|_{L_t^\infty(\mathcal{B}^s)} + \sqrt{\mu} \|(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& + \frac{1}{4} \|\partial_z(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)\|_{\tilde{L}_t^2(\mathcal{B}^s)} + \frac{\epsilon}{2} \|(w_\varphi^1, \epsilon w_\varphi^2)\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{1}{2}})} + \|\theta_\varphi\|_{\tilde{L}_t^2(\mathcal{B}^{s+\frac{1}{2}})} \\
& \leq 2C \|(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)(0)\|_{\mathcal{B}^s} + CM\epsilon + C \|(w_\varphi^1, \epsilon w_\varphi^2, \theta_\varphi)\|_{\tilde{L}_{t, \dot{\eta}(t)}^2(\mathcal{B}^{s+\frac{1}{2}})}.
\end{aligned}$$

Then by taking $\mu \geq C^2$, we can complete the proof Theorem 8.0.7. \square

Chapter 9

Homogeneous and inhomogeneous MHD system

In this chapter we present the results of the following paper:

N. Aarach and N. Zhu, Global well-posedness of 3D homogeneous and inhomogeneous MHD system with small unidirectional derivative. Accepted in the Calculus of Variations and Partial Differential Equations Journal.

9.1 Introduction

We study in this chapter the global well-posedness of the following 3D inhomogeneous MHD system,

$$(IMHD) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \rho(\partial_t u + u \cdot \nabla u) - \mu_1 \Delta u + \nabla p = b \cdot \nabla b, \\ \partial_t b - \mu_2 \Delta b - \text{curl}(u \times b) = 0, \\ \text{div } u = 0, \text{ div } b = 0, \\ \rho|_{t=0} = \rho_0(x), u|_{t=0} = u_0(x), b|_{t=0} = u_0(x), \end{cases}$$

where ρ is a scalar standing for the density, $u = (u^1, u^2, u^3)$ and $b = (b^1, b^2, b^3)$ represent the velocity and magnetic field of the fluid respectively. p is a scalar function denoting the pressure. The parameter μ_1 represent the kinematic viscosity coefficient and μ_2 is a parameter representing the magnetic diffusive coefficient. Since the specific value of the positive coefficients μ_1, μ_2 plays no role in the argument of this paper. For simplicity, we suppose $\mu_1 = \mu_2 = 1$ in the rest of this paper. This system is derived by coupling the inhomogeneous Navier-Stokes system and the Maxwell equation. Such a system describes in particular the motion of several conducting incompressible immiscible fluids (without surface tension) in presence of a magnetic field.

It is worth mentioning that when there is no electromagnetic field, that is, $b \equiv 0$, then System (IMHD) reduces to the following incompressible inhomogeneous Navier-Stokes equations,

$$(INS) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla p = 0, \\ \text{div } u = 0, \\ \rho|_{t=0} = \rho_0(x), u|_{t=0} = u_0(x), \end{cases}$$

which have been studied by many researchers. For the well-posedness of (INS), Kazhikhov in [83] solved this problem with an additional condition $\inf \rho_0 > 0$ based on the energy method. Then Simon in [126] removed the lower-bound assumption on ρ_0 and constructed a global weak solution of (INS) with finite energy. See also [90] for the general result with variable viscosity. For the case of bounded domain Ω with homogeneous Dirichlet boundary condition for the velocity field. Ladyženskaja and Solonnikov in [85] first addressed the question of unique resolvability. Under the assumptions that $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ with $p > d$, and u_0 is divergence-free and vanishes on $\partial\Omega$, $\rho_0 \in C^1(\Omega)$ which is bounded and away from zero, then they proved:

- Global well-posedness in dimension $d = 2$;
- Local well-posedness in dimension $d = 3$. If in addition u_0 is small in $W^{2-\frac{2}{p},p}(\Omega)$, then global well-posedness holds.

Then Abidi in [7] and Danchin in [40, 41] studied this well-posedness problem with lower regularity initial data which satisfies the smallness condition in some critical Besov space. Later, Paicu and Zhang in [113] relaxed the smallness condition in the case that only a_0 and the horizontal component u^h of the velocity field are suitable small in some Besov space. Chemin et. al. in [33] further improved the result of [113] to the critical anisotropic Besov spaces. For more results on the system (INS), one can see [8, 10, 11, 37, 42–44, 66, 72, 113, 115, 133] and references therein.

When the density is constant in System (IMHD), it degenerates to the following classical homogeneous MHD system:

$$(MHD) \quad \begin{cases} \partial_t u^H + u^H \cdot \nabla u^H - \Delta u^H = -\nabla p^H + b^H \cdot \nabla b^H, \\ \partial_t b^H - \Delta b^H + u^H \cdot \nabla b^H - b^H \cdot \nabla u^H = 0, \\ \operatorname{div} u^H = \operatorname{div} b^H = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u^H|_{t=0} = u_0^H(x), \quad b^H|_{t=0} = b_0^H(x). \end{cases}$$

For this system, Duvaut and Lions in [51] established the local existence and uniqueness of the solution in the classical Sobolev spaces $H^s(\mathbb{R}^d)$, $s \geq d$, and they also proved the global existence of the solution for small initial data. Sermange and Temam [125] proved the global unique solution in dimension two. For the MHD system with partial Laplace dissipation, Jiu and Niu [79] proved the local well-posedness in Sobolev space H^s ($s \geq 3$) with only Laplace dissipation in the velocity equation. Later, this result was generalized to the lower regularity initial data in [32, 55, 56, 86, 130]. For the global solution with large initial data, due to the well-known problem of the global well-posedness of the 3D incompressible Navier-Stokes equations is still open, so the case of the MHD system is more difficult and also open. Many recent papers study the stability problem of the 3D MHD system near some equilibrium state, which can be found in [12, 20, 46, 70, 89, 131] and references therein.

The present chapter is devoted to studying the full inhomogeneous MHD system (IMHD). Compared with the Navier-Stokes equations, the dynamic motion of the fluid and the magnetic field interact with each other. The hydrodynamic and electrodynamics effects in the motion are strongly coupled. Hence the problems of inhomogeneous MHD system are considerably more complicated. Motivated by Lions' work [90] where the global existence result for the density-dependent Navier-Stokes equations was obtained, Gerbeau and Le Bris [57] (see also Desjardins and Le Bris [47]) proved a global-in-time existence of a weak solution for the density-dependent MHD equations in a bounded domain. The global existence of strong solutions with small initial data in the critical Besov spaces was established by Abidi and Paicu [9]. Chen, Tan and Wang [36] extended the local existence in presence of vacuum by the Galerkin method, energy method and the domain expansion technique. Huang and Wang [73] proved the existence and uniqueness of a global strong solution for the initial-boundary problem with initial vacuum in dimension two by using a critical Sobolev inequality of logarithmic type. Recently, Gui in [65] proved that the 2D MHD equations with a constant viscosity are globally well-posed for a generic family of variations of the initial data and an inhomogeneous electrical conductivity.

Notice that in [98], the authors proved that as long as the one-directional derivative of the initial velocity is sufficiently small in some scaling invariant spaces, then the classical Navier-Stokes system has a global unique solution. The goal of this paper is to extend this type of result to the 3-D homogeneous and inhomogeneous MHD system in L^p functional framework. For the MHD system, the situation is more complicated compared with the Navier-Stokes system due to the strong coupling effect between the velocity and the magnetic fields.

Before we present our main result, we first recall the functional space framework which we are going to use in what follows. The definitions of the space we are going to work with require anisotropic dyadic decomposition of the Fourier variables (see [29, 34, 107]). Let us recall from [15, 33] that

$$\begin{aligned}\Delta_j^h f &= \mathcal{F}^{-1}(\varphi(2^{-j}|\xi_h|)\widehat{f}), & \Delta_k^v f &= \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_3|)\widehat{f}), \\ S_j^h f &= \mathcal{F}^{-1}(\chi(2^{-j}|\xi_h|)\widehat{f}), & S_k^v f &= \mathcal{F}^{-1}(\chi(2^{-k}|\xi_3|)\widehat{f}), \\ \Delta_j f &= \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{f}), & S_j f &= \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{f}),\end{aligned}$$

where $\xi_h = (\xi_1, \xi_2)$, $\mathcal{F}(f)$ and \widehat{f} denote the Fourier transform of the distribution f . The functions $\chi(\xi)$ and $\varphi(\xi)$ are smooth such that

$$\begin{aligned}\text{supp}\varphi &\subset \left\{ \zeta \in \mathbb{R} : \frac{3}{4} \leq |\zeta| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \zeta > 0, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\zeta) = 1, \\ \text{supp}\chi &\subset \left\{ \zeta \in \mathbb{R} : |\zeta| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\zeta) + \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\zeta) = 1.\end{aligned}$$

Then we give the definition of the classical Besov space.

Definition 9.1.1. Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $f \in \mathcal{S}'_h(\mathbb{R}^3)$, which means that $f \in \mathcal{S}'(\mathbb{R}^3)$ and $\lim_{j \rightarrow \infty} \|S_j f\|_\infty = 0$, we set

$$B_{p,r}^s \triangleq \{f \in \mathcal{S}'_h(\mathbb{R}^3) : \|f\|_{B_{p,r}^s} \triangleq (2^{js} \|\Delta_j f\|_{L^p(\mathbb{R}^3)})_{\ell^r(\mathbb{Z})} < \infty\}.$$

Remark 9.1.1. What we defined in Definition 9.1.1 is the homogeneous Besov space, we remark that in many books and other papers, the notation is $\dot{B}_{p,r}^s$, for the sake of simplicity, we write it as $B_{p,r}^s$.

Remark 9.1.2. To simplify the notation even further, in all that follows, we shall denote

$$B_p^s \triangleq B_{p,1}^s.$$

We will also use the space-time space introduced by Chemin and Lerner (see, e.g., [15]).

Definition 9.1.2. For $t > 0$, $s \in \mathbb{R}$ and $1 \leq p, q, r \leq \infty$, the space-time space $\widetilde{L}^q([0, T]; B_{p,r}^s)$ is defined through the norm

$$(9.1.1) \quad \|f\|_{\widetilde{L}^q([0, T]; B_{p,r}^s)} \triangleq \left(\sum_{j \in \mathbb{Z}} 2^{srj} \|\Delta_j f\|_{L_t^q L_x^p}^r \right)^{\frac{1}{r}}.$$

Then we give the definition of the anisotropic Besov space, which will be used frequently throughout this paper.

Definition 9.1.3. Let p be in $[1, +\infty]$, $s_1, s_2 \in \mathbb{R}$ and u in $\mathcal{S}'_h(\mathbb{R}^3)$, we set

$$\|u\|_{B_p^{s_1, s_2}} \triangleq (2^{s_1 j} 2^{s_2 k} \|\Delta_j^h \Delta_k^v u\|_{L^p})_{\ell^1(\mathbb{Z})}.$$

Notations. In all that follows, we shall denote

$$\mathcal{B}_p^0 \triangleq \mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}, \quad \mathcal{B}_p^1 \triangleq \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}} \quad \text{and} \quad \mathcal{B}_p^2 \triangleq \mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}} \cap \mathcal{B}_p^{-1+\frac{2}{p}, 2+\frac{1}{p}},$$

we mention that the space \mathcal{B}_p^0 is critical. Similar as (9.1.1), we can define the anisotropic Chemin-Lerner space as:

$$(9.1.2) \quad \|f\|_{\tilde{L}^q([0, T]; \mathcal{B}_p^{s_1, s_2})} \triangleq \sum_{j, k \in \mathbb{Z}} 2^{s_1 j} 2^{s_2 k} \|\Delta_j^h \Delta_k^v f\|_{L_t^q L_x^p}.$$

The space $\tilde{L}^q([0, T]; \mathcal{B}_p^{s_1, s_2})$ may be linked with the more classical space $L^q([0, T]; \mathcal{B}_p^{s_1, s_2})$ via the Minkowski inequality that:

$$(9.1.3) \quad \begin{aligned} \|u\|_{L^q([0, T]; \mathcal{B}_p^{s_1, s_2})} &\leq \|u\|_{\tilde{L}^q([0, T]; \mathcal{B}_p^{s_1, s_2})} && \text{if } q \geq 1, \\ \|u\|_{\tilde{L}^q([0, T]; \mathcal{B}_p^{s_1, s_2})} &\leq \|u\|_{L^q([0, T]; \mathcal{B}_p^{s_1, s_2})} && \text{if } q = 1. \end{aligned}$$

Then we present our main results of this paper. The first theorem following obtains the global well-posedness for homogeneous MHD system (MHD).

Theorem 9.1.1. *Let $u_0^H = (u_0^{H, h}, u_0^{H, 3}) \in \mathcal{B}_p^0(\mathbb{R}^3)$, $b_0^H = (b_0^{H, h}, b_0^{H, 3}) \in \mathcal{B}_p^0(\mathbb{R}^3)$ are two divergence-free vector fields with $(u_0^{H, h}, b_0^{H, h}) \in H^1(\mathbb{R}^3)$, $\Lambda_h^{-1} \partial_3(u_0^H, b_0^H) \in \mathcal{B}_p^0(\mathbb{R}^3)$ ($2 < p < 4$), where Λ_h^{-1} is the Fourier multiplier with symbol $|\xi_h|^{-1}$. Then there exist some sufficiently small positive constant ε_0 and universal positive constant C such that if (u_0^H, b_0^H) satisfies the smallness condition*

$$(9.1.4) \quad \|\Lambda_h^{-1} \partial_3(u_0^H, b_0^H)\|_{\mathcal{B}_p^0} M(u_0^H, b_0^H) \leq \varepsilon_0.$$

where

$$\begin{aligned} M(u_0^H, b_0^H) &= B(u_0^{H, h}, b_0^{H, h}) \times \exp \left(C(1 + \|(u_0^{H, 3}, b_0^{H, 3})\|_{\mathcal{B}_p^0} \right. \\ &\quad \left. + \|(u_0^{H, h}, b_0^{H, h})\|_{\mathcal{B}_p^0}^2 + \|(u_0^{H, h}, b_0^{H, h})\|_{H^1}^4) B(u_0^{H, h}, u_0^{H, h}) \right), \end{aligned}$$

and

$$B(u_0^{H, h}, b_0^{H, h}) = \exp \left(C(\|(u_0^{H, h}, b_0^{H, h})\|_{\mathcal{B}_p^0} + \|(u_0^{H, h}, b_0^{H, h})\|_{H^1}^2 \exp(C\|(u_0^{H, h}, b_0^{H, h})\|_{L^2})) \right).$$

Then System (MHD) has a unique global solution (u^H, b^H) which satisfies

$$u^H \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2), \quad b^H \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2).$$

Remarque 9.1.1. *We mention (9.1.4) is a relative smallness condition that is triple-exponential in $\|(u_0^{H, h}, b_0^{H, h})\|_{L^2}$.*

Then we consider the inhomogeneous (MHD) system (IMHD). In the following, we suppose that the initial density verifies $\inf \rho_0(x) > 0$ on \mathbb{R}^3 , and thus, by the maximum principle for the transport equation, we have $\inf \rho(t, x) > 0$ on \mathbb{R}^3 . Then we can use the transformation $a \triangleq \frac{1}{\rho} - 1$ which allows us the following system:

$$(IMHD') \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1+a)(-\Delta u + \nabla p) = (1+a)(b \cdot \nabla b), \\ \partial_t b + u \cdot \nabla b - \Delta b = b \cdot \nabla u, \\ \operatorname{div} u = 0, \operatorname{div} b = 0, \\ a|_{t=0} = a_0(x), u|_{t=0} = u_0(x), b|_{t=0} = b_0(x). \end{cases}$$

The next theorem shows the global well-posedness result associated with the system (IMHD').

Theorem 9.1.2. *Let $u_0 = (u_0^h, u_0^3) \in \mathcal{B}_p^0(\mathbb{R}^3)$, $b_0 = (b_0^h, b_0^3) \in \mathcal{B}_p^0(\mathbb{R}^3)$ are two divergence-free vector fields with $(u_0^h, b_0^h) \in H^1(\mathbb{R}^3)$, $\Lambda_h^{-1} \partial_3(u_0, b_0) \in \mathcal{B}_p^0$ ($2 < p < 4$), $a_0 \in B_p^{\frac{3}{p}}$, where Λ_h^{-1} is the Fourier multiplier with symbol $|\xi_h|^{-1}$. Then there exist some sufficiently small positive constant ε_0 and universal positive constant C such that if (a_0, u_0, b_0) satisfies the smallness condition*

$$(9.1.5) \quad \left(\|a_0\|_{B_p^{\frac{3}{p}}} + \|\Lambda_h^{-1} \partial_3(u_0, b_0)\|_{\mathcal{B}_p^0} \right) L(u_0, b_0) \leq \varepsilon_0,$$

with

$$L(u_0, b_0) = B(u_0^h, b_0^h) \exp \left(C(1 + \|(u_0^3, b_0^3)\|_{\mathcal{B}_p^0} + \|(u_0^h, b_0^h)\|_{\mathcal{B}_p^0}^2 + \|(u_0^h, b_0^h)\|_{H^1}^4) B(u_0^h, b_0^h) \right)$$

and

$$B(u_0^h, b_0^h) = \exp(C(\|(u_0^h, b_0^h)\|_{\mathcal{B}_p^0} + \|(u_0^h, b_0^h)\|_{H^1}^2) \exp(C\|(u_0^h, b_0^h)\|_{L^2})).$$

Then system (IMHD') has a unique global solution

$$a \in \tilde{L}^\infty(\mathbb{R}_+; B_p^{\frac{3}{p}}), \quad u \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2) \quad \text{and} \quad b \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2).$$

Let us end this section with the notations that we shall use in this context.

Throughout this paper, C stands for some real positive constant which may vary from line to line. By $a \lesssim b$, we mean that there is a uniform constant C , such that $a \leq Cb$. We denote by ℓ^1 the space of summable sequence with the norm $\|\{b_q\}\|_{\ell^1} = \sum_q |b_q|$. Finally, $\|f(x)\|_{L^p(\mathbb{R}^3)}$ denotes the norm in the Lebesgue space $L^p(\mathbb{R}^3)$ while $\|f(x_h, x_3)\|_{L_v^q L_h^p(\mathbb{R}^3)} := \left\| \|f(x_h, x_3)\|_{L_h^p(\mathbb{R}_h^2)} \right\|_{L_v^q(\mathbb{R})}$ denotes the norm in the anisotropic Lebesgue space $L_v^q L_h^p(\mathbb{R}^3)$, and $\nabla_h \triangleq (\partial_{x_1}, \partial_{x_2})$, $\nabla \triangleq (\nabla_{x_h}, \partial_{x_3})$ with $x_h \triangleq (x_1, x_2)$.

9.2 Preliminaries

In this section, we shall introduce some useful lemmas which will be used frequently in the proof of our main results. We first recall the following anisotropic Bernstein inequalities from [35, 107].

Lemma 9.2.1 (Anisotropic Bernstein inequality). *Let \mathbf{B}_h (resp. \mathbf{B}_v) be a ball of \mathbb{R}_h^2 (resp. \mathbb{R}_v), and \mathbf{C}_h (resp. \mathbf{C}_v) be a ring of \mathbb{R}_h^2 (resp. \mathbb{R}_v), $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there holds: If the support of \hat{f} is included in $2^j \mathbf{B}_h \times \mathbb{R}$, then*

$$(9.2.1) \quad \|\partial_{x_h}^\alpha f\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{j(|\alpha| + 2(\frac{1}{p_2} - \frac{1}{p_1}))} \|f\|_{L_h^{p_2}(L_v^{q_1})}.$$

If the support of \hat{f} is included in $\mathbb{R}^2 \times 2^k \mathbf{B}_v$, then

$$(9.2.2) \quad \|\partial_3^\beta f\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(\beta + (\frac{1}{q_2} - \frac{1}{q_1}))} \|f\|_{L_h^{p_1}(L_v^{q_2})}.$$

If the support of \widehat{f} is included in $2^j \mathbf{C}_h \times \mathbb{R}$, then

$$(9.2.3) \quad \|f\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-jN} \sup_{|\alpha|=N} \|\partial_{x_h}^\alpha f\|_{L_h^{p_1}(L_v^{q_1})}.$$

If the support of \widehat{f} is included in $\mathbb{R}^2 \times 2^k \mathbf{C}_v$, then

$$(9.2.4) \quad \|f\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \|\partial_3^N f\|_{L_h^{p_1}(L_v^{q_1})}.$$

To make connections between anisotropic Besov spaces and the classical homogeneous Besov spaces, we need the following lemma:

Lemma 9.2.2. *Let s and t be positive real numbers. Then for any $p \in [1, \infty]$, one has*

$$\|f\|_{\mathcal{B}_p^{s,t}} \lesssim \|f\|_{B_p^{s+t}}.$$

Next, we provide a product law in the anisotropic Besov space, which is useful in the following estimates. The proof can be found in Lemma 2.3 of [33].

Lemma 9.2.3. *Let $p \geq q \geq 1$ with $\frac{1}{p} + \frac{1}{q} \leq 1$, and $s_1 \leq \frac{2}{q}, s_2 \leq \frac{2}{p}$ with $s_1 + s_2 > 0$. Let $\sigma_1 \leq \frac{1}{q}, \sigma_2 \leq \frac{1}{p}$. Then for f in $\mathcal{B}_q^{s_1, \sigma_1}(\mathbb{R}^3)$ and g in $\mathcal{B}_p^{s_2, \sigma_2}(\mathbb{R}^3)$, the product fg belongs to $\mathcal{B}_p^{s_1+s_2-\frac{2}{q}, \sigma_1+\sigma_2-\frac{1}{q}}(\mathbb{R}^3)$, and*

$$\|fg\|_{\mathcal{B}_p^{s_1+s_2-\frac{2}{q}, \sigma_1+\sigma_2-\frac{1}{q}}} \lesssim \|f\|_{\mathcal{B}_q^{s_1, \sigma_1}} \|g\|_{\mathcal{B}_p^{s_2, \sigma_2}}.$$

Remark 9.2.1. *From Lemma 9.2.3, we can obtain the following special cases:*

$$(9.2.5) \quad \|fg\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} \lesssim \|f\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} \|g\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \quad \text{with } 1 \leq p < 4.$$

Then we provide a lemma which gives some interpolation inequalities in the anisotropic Besov space.

Lemma 9.2.4. *Assume f and g belong to $\mathcal{B}_p^0 \cap \mathcal{B}_p^2$, then we have*

$$(9.2.6) \quad \|f\|_{\mathcal{B}_p^1} \lesssim \|f\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|f\|_{\mathcal{B}_p^2}^{\frac{1}{2}},$$

$$(9.2.7) \quad \|f \cdot \nabla g\|_{\mathcal{B}_p^0} \lesssim \|f\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|f\|_{\mathcal{B}_p^2}^{\frac{1}{2}} \|g\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|g\|_{\mathcal{B}_p^2}^{\frac{1}{2}}.$$

Proof. First we verify (9.2.6), according to the definition of \mathcal{B}_p^1 ,

$$(9.2.8) \quad \|f\|_{\mathcal{B}_p^1} \triangleq \|f\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} + \|f\|_{\mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}}}.$$

According to the definition of the anisotropic Besov space,

$$(9.2.9) \quad \begin{aligned} \|f\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} &= \sum_{j,k \in \mathbb{Z}} 2^{\frac{2}{p}j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \\ &= \sum_{j,k \in \mathbb{Z}} \left(2^{(-1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \times \left(2^{(1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{j,k \in \mathbb{Z}} 2^{(-1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \times \left(\sum_{j,k \in \mathbb{Z}} 2^{(1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \\ &\leq C \|f\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \|f\|_{\mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}}. \end{aligned}$$

Similarly,

(9.2.10)

$$\begin{aligned}
\|f\|_{\mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}}} &= \sum_{j,k \in \mathbb{Z}} 2^{(-1+\frac{2}{p})j} 2^{(1+\frac{1}{p})k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \\
&= \sum_{j,k \in \mathbb{Z}} \left(2^{(-1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \times \left(2^{(-1+\frac{2}{p})j} 2^{(2+\frac{1}{p})k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{j,k \in \mathbb{Z}} 2^{(-1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \times \left(\sum_{j,k \in \mathbb{Z}} 2^{(-1+\frac{2}{p})j} 2^{(2+\frac{1}{p})k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \\
&\leq C \|f\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \|f\|_{\mathcal{B}_p^{-1+\frac{2}{p}, 2+\frac{1}{p}}}^{\frac{1}{2}}.
\end{aligned}$$

Inserting the estimates (9.2.9) and (9.2.10) into (9.2.8) leads to (9.2.6).

Next we prove (9.2.7). According to product law (9.2.5) and Lemma 9.2.1,

$$\begin{aligned}
\|f \cdot \nabla g\|_{\mathcal{B}_p^0} &\leq \|f^h \cdot \nabla_h g\|_{\mathcal{B}_p^0} + \|f^3 \partial_3 g\|_{\mathcal{B}_p^0} \\
&\lesssim \|f^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|\nabla_h g\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|f^3\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|\partial_3 g\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} \\
&\lesssim \|f\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|g\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} + \|f\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|g\|_{\mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}}}.
\end{aligned}$$

Then by (9.2.6), we obtain

$$\|f \cdot \nabla g\|_{\mathcal{B}_p^0} \lesssim \|f\|_{\mathcal{B}_p^{\frac{1}{2}}} \|f\|_{\mathcal{B}_p^{\frac{1}{2}}} \|g\|_{\mathcal{B}_p^{\frac{1}{2}}} \|g\|_{\mathcal{B}_p^{\frac{1}{2}}},$$

which completes the proof of this lemma. \square

In order to estimate the pressure term in System (IMHD³), we need the following lemma which in the case of the density a is small. The proof can be found in Lemma 2.4 of [33].

Lemma 9.2.5. *Let $p \in (1, 4)$, we consider a function ρ such that $\|\rho\|_{\mathcal{B}_p^{\frac{3}{p}}}$ is small enough. If Π satisfies*

$$(9.2.11) \quad \operatorname{div}((1 + \rho)\nabla\Pi - f) = 0$$

with f in \mathcal{B}_p^0 , then $\nabla\Pi$ satisfies

$$\|\nabla\Pi\|_{\mathcal{B}_p^0} \lesssim \|f\|_{\mathcal{B}_p^0} \quad \text{and thus} \quad \|(1 + \rho)\nabla\Pi\|_{\mathcal{B}_p^0} \lesssim \|f\|_{\mathcal{B}_p^0}.$$

Then, we provide a lemma which is a classical estimate for transport equation (See Section 3.2 of [15] for instance).

Lemma 9.2.6. *Let p be in $[1, \infty]$ and $s \in (0, 1)$. Then given an initial data a_0 in $B_p^s(\mathbb{R}^3)$ and a vector field u in $L^1([0, T]; Lip(\mathbb{R}^3))$ with $\operatorname{div} u = 0$, then the unique solution a of*

$$\partial_t a + u \cdot \nabla a = 0, \quad a|_{t=0} = a_0,$$

satisfies

$$(9.2.12) \quad \|a\|_{\tilde{L}^\infty([0, T]; B_p^s)} \leq \|a_0\|_{B_p^s} \exp\left(C \int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau\right).$$

Finally, we prove an embedding lemma which shows the Lipschitz norm can be bounded by the Besov space \mathcal{B}_p^2 with $2 < p < \infty$.

Lemma 9.2.7. *Assume f to be a vector field belonging to \mathcal{B}_p^2 for $2 < p < \infty$, then we have*

$$(9.2.13) \quad \|\nabla f\|_{L^\infty} \lesssim \|f\|_{\mathcal{B}_p^2}.$$

Proof. Using anisotropic Littlewood-Paley decomposition,

$$\|\nabla f\|_{L^\infty} = \sum_{j,k \in \mathbb{Z}} \|\Delta_j^h \Delta_k^v \nabla f\|_{L^\infty} \leq \sum_{j,k \in \mathbb{Z}} \|\Delta_j^h \Delta_k^v \nabla_h f\|_{L^\infty} + \sum_{j,k \in \mathbb{Z}} \|\Delta_j^h \Delta_k^v \partial_3 f\|_{L^\infty}.$$

Applying Bernstein's inequality from Lemma 9.2.1,

$$\sum_{j,k \in \mathbb{Z}} \|\Delta_j^h \Delta_k^v \nabla_h f\|_{L^\infty} \lesssim \sum_{j,k \in \mathbb{Z}} 2^j 2^{\frac{2}{p}j} 2^{\frac{k}{p}} \|\Delta_j^h \Delta_k^v f\|_{L^p} \leq \|f\|_{\mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}}}.$$

Similarly,

$$\sum_{j,k \in \mathbb{Z}} \|\Delta_j^h \Delta_k^v \partial_3 f\|_{L^\infty} \lesssim \sum_{j,k \in \mathbb{Z}} 2^k 2^{\frac{2}{p}j} 2^{\frac{k}{p}} \|\Delta_j^h \Delta_k^v f\|_{L^p} \leq \|f\|_{\mathcal{B}_p^{\frac{2}{p}, 1+\frac{1}{p}}}.$$

Then according to the definition of $\mathcal{B}_p^{\frac{2}{p}, 1+\frac{1}{p}}$, we have

$$\begin{aligned} \|f\|_{\mathcal{B}_p^{\frac{2}{p}, 1+\frac{1}{p}}} &= \sum_{j,k \in \mathbb{Z}} 2^{\frac{2}{p}j} 2^{(1+\frac{1}{p})k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \\ &= \sum_{j,k \in \mathbb{Z}} \left(2^{(-1+\frac{2}{p})j} 2^{(2+\frac{1}{p})k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \times \left(2^{(1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{j,k \in \mathbb{Z}} 2^{(-1+\frac{2}{p})j} 2^{(2+\frac{1}{p})k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \times \left(\sum_{j,k \in \mathbb{Z}} 2^{(1+\frac{2}{p})j} 2^{\frac{1}{p}k} \|\Delta_j^h \Delta_k^v f\|_{L^p} \right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{\mathcal{B}_p^{-1+\frac{2}{p}, 2+\frac{1}{p}}}^{\frac{1}{2}} \|f\|_{\mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}}. \end{aligned}$$

Thus we complete the proof of this lemma. \square

9.3 Sketch of the proof

In this section, we show the strategy to prove our main results and give the proof of Theorem 9.1.1 and Theorem 9.1.2. Motivated by the study of the global large solution to the classical 3D Navier-Stokes system, here we are going to decompose the solution of (MHD) as a sum of a solution to the Quasi-2D MHD system and a solution to the 3D perturbed system. For the inhomogeneous case, we decompose the solution of the 3D inhomogeneous MHD system (IMHD') into a solution of the 3D homogeneous MHD system and putting a solution to the 3D perturbed system.

Before we proceed to derive the energy estimate, we first recall the classical two-dimensional Biot-Savart's law which will be used frequently in this paper. For $u^h = (u^1, u^2)$ and $b^h = (b^1, b^2)$, we decompose it into:

$$u^h = u_{\text{curl}}^h + u_{\text{div}}^h \quad \text{with} \quad u_{\text{curl}}^h \triangleq \nabla_h^\perp \Delta_h^{-1} \text{curl}_h u^h \quad \text{and} \quad u_{\text{div}}^h \triangleq \nabla_h \Delta_h^{-1} \text{div}_h u^h,$$

and

$$b^h = b_{\text{curl}}^h + b_{\text{div}}^h \quad \text{with} \quad b_{\text{curl}}^h \triangleq \nabla_h^\perp \Delta_h^{-1} \text{curl}_h b^h \quad \text{and} \quad b_{\text{div}}^h \triangleq \nabla_h \Delta_h^{-1} \text{div}_h b^h,$$

where $\text{curl}_h u^h \triangleq \partial_1 u^2 - \partial_2 u^1$, $\text{div}_h u^h \triangleq \partial_1 u^1 + \partial_2 u^2$ and $\text{curl}_h b^h \triangleq \partial_1 b^2 - \partial_2 b^1$, $\text{div}_h b^h \triangleq \partial_1 b^1 + \partial_2 b^2$.

Consider the following Quasi-2D MHD system satisfied by $\bar{u}^h = \bar{u}^h(t, x_h; x_3) \in \mathbb{R}^2$ and $\bar{b}^h = \bar{b}^h(t, x_h; x_3) \in \mathbb{R}^2$ which x_3 can be regarded as a parameter:

$$(9.3.1) \quad \begin{cases} \partial_t \bar{u}^h + \bar{u}^h \cdot \nabla_h \bar{u}^h - \Delta \bar{u}^h + \nabla_h \bar{p} = \bar{b}^h \cdot \nabla_h \bar{b}^h, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \bar{b}^h - \Delta \bar{b}^h + \bar{u}^h \cdot \nabla_h \bar{b}^h = \bar{b}^h \cdot \nabla_h \bar{u}^h, \\ \text{div}_h \bar{u}^h = \text{div}_h \bar{b}^h = 0, \\ \bar{u}^h|_{t=0} = \bar{u}_0^h(x) = u_{0,\text{curl}}^h, \quad \bar{b}^h|_{t=0} = \bar{b}_0^h(x) = b_{0,\text{curl}}^h. \end{cases}$$

Concerning the system (Quasi-2D MHD), we have the following a priori estimates:

Proposition 9.3.1. *Let $(\bar{u}_0^h, \bar{b}_0^h) \in H^1 \cap \mathcal{B}_p^0$ with $\Lambda_h^{-1} \partial_3(\bar{u}_0^h, \bar{b}_0^h) \in \mathcal{B}_p^0$. Then System (Quasi-2D MHD) has a unique global solution (\bar{u}^h, \bar{b}^h) that satisfies:*

$$(9.3.1) \quad \|(\bar{u}^h, \bar{b}^h)\|_{L^\infty(0,T;L^2)}^2 + \|\nabla(\bar{u}^h, \bar{b}^h)\|_{L^2(0,T;L^2)}^2 \leq \|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2,$$

$$(9.3.2) \quad \begin{aligned} & \|\nabla(\bar{u}^h, \bar{b}^h)\|_{L^\infty(0,T;L^2)}^2 + \|\nabla^2(\bar{u}^h, \bar{b}^h)\|_{L^2(0,T;L^2)}^2 \\ & \leq C \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp\left(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2)\right), \end{aligned}$$

$$(9.3.3) \quad \begin{aligned} & \|(\bar{u}^h, \bar{b}^h)\|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + c_0 \|(\bar{u}^h, \bar{b}^h)\|_{L^1(0,T;\mathcal{B}_p^2)} \\ & \leq C \|(\bar{u}_0^h, \bar{b}_0^h)\|_{\mathcal{B}_p^0} + \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp(C \|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2) \triangleq M_A \end{aligned}$$

and

$$(9.3.4) \quad \begin{aligned} & \|\Lambda_h^{-1} \partial_3(\bar{u}^h, \bar{b}^h)\|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + c_0 \|\Lambda_h^{-1} \partial_3(\bar{u}^h, \bar{b}^h)\|_{L^1(0,T;\mathcal{B}_p^2)} \leq C \|\Lambda_h^{-1} \partial_3(\bar{u}_0^h, \bar{b}_0^h)\|_{\mathcal{B}_p^0} \\ & \quad \times \exp(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{\mathcal{B}_p^0} + \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp(C \|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2)) \triangleq M_B, \end{aligned}$$

for some harmless positive constant c_0 and C_0 independent of time T .

The proof of Proposition 9.3.1 will be given in Section 4.

In order to study the well-posedness of system (MHD), we decompose the solution u^H and b^H of (MHD) as $u^H = \begin{pmatrix} \bar{u}^h \\ 0 \end{pmatrix} + v$, $b^H = \begin{pmatrix} \bar{b}^h \\ 0 \end{pmatrix} + c$, where the couple (\bar{u}^h, \bar{b}^h) is determined by the system (Quasi-2D MHD). According to the result given by Proposition 9.3.1, we only need to estimate the remainder part (v, c) . It is easy to verify that the remainder (v, c) satisfies

$$(9.3.5) \quad \begin{cases} \partial_t v^h + v \cdot \nabla v^h + \bar{u}^h \cdot \nabla_h v^h + v \cdot \nabla \bar{u}^h - \Delta v^h = -\nabla_h q + c \cdot \nabla c^h + \bar{b}^h \cdot \nabla_h c^h + c \cdot \nabla \bar{b}^h, \\ \partial_t v^3 + v^h \cdot \nabla_h v^3 - v^3 \text{div}_h v^h + \bar{u}^h \cdot \nabla_h v^3 - \Delta v^3 \\ \quad = -\partial_3(q - \bar{p}) + c^h \cdot \nabla_h c^3 - c^3 \text{div}_h c^h + \bar{b}^h \cdot \nabla_h c^3, \\ \partial_t c^h + v \cdot \nabla c^h + \bar{u}^h \cdot \nabla_h c^h + v \cdot \nabla \bar{b}^h - \Delta c^h = c \cdot \nabla v^h + c \cdot \nabla \bar{u}^h + \bar{b}^h \cdot \nabla_h v^h, \\ \partial_t c^3 + v^h \cdot \nabla_h c^3 - v^3 \text{div}_h c^h + \bar{u}^h \cdot \nabla_h c^3 - \Delta c^3 = c^h \cdot \nabla_h v^3 - c^3 \text{div}_h v^h + \bar{b}^h \cdot \nabla_h v^3, \\ \text{div } v = 0, \quad \text{div } c = 0, \\ v|_{t=0} = v_0(x) = (-\nabla_h \Delta_h^{-1}(\partial_3 u_0^3), u_0^3), \quad c|_{t=0} = c_0(x) = (-\nabla_h \Delta_h^{-1}(\partial_3 b_0^3), b_0^3). \end{cases}$$

From (Quasi-2D MHD), we know

$$(9.3.6) \quad \partial_3 \bar{p} = -\partial_3 (-\Delta)^{-1} \operatorname{div}_h (\bar{u}^h \cdot \nabla_h \bar{u}^h - \bar{b}^h \cdot \nabla_h \bar{b}^h).$$

Similarly from (9.3.5), using divergence-free condition of v ,

$$(9.3.7) \quad \begin{aligned} q = & (-\Delta)^{-1} \operatorname{div}_h (v^h \cdot \nabla_h v^h + v^3 \partial_3 v^h + \bar{u}^h \cdot \nabla_h v^h + v^h \cdot \nabla_h \bar{u}^h + v^3 \partial_3 \bar{u}^h) \\ & - (-\Delta)^{-1} \operatorname{div}_h (c^h \cdot \nabla_h c^h + c^3 \partial_3 c^h + \bar{b}^h \cdot \nabla_h c^h + c^h \cdot \nabla_h \bar{b}^h + c^3 \partial_3 \bar{b}^h) \\ & + (-\Delta)^{-1} \partial_3 (v^h \cdot \nabla_h v^3 - v^3 \operatorname{div}_h v^h + \bar{u}^h \cdot \nabla_h v^3 - \partial_3 \bar{p}) \\ & - c^h \cdot \nabla_h c^3 + c^3 \operatorname{div}_h c^h - \bar{b}^h \cdot \nabla_h c^3). \end{aligned}$$

We shall give the estimate of the solution (v, c) of System (9.3.5). We start first by giving a proposition that shows the global estimate in time for the couple (v^h, c^h) of System (9.3.5). For that, we define two functions corresponding to (v^h, c^h)

$$(9.3.8) \quad v_\mu^h(t) = v^h(t) e^{-\mu \int_0^t f(\tau) d\tau} \quad \text{and} \quad c_\mu^h(t) = c^h(t) e^{-\mu \int_0^t f(\tau) d\tau}$$

with $f(t) = \|(v^3, c^3)(t)\|_{\mathcal{B}_p^2} + \|(\bar{u}^h, \bar{b}^h)(t)\|_{\mathcal{B}_p^2}$ and μ be a positive constant will be defined later and taken large enough. Then we have the following proposition.

Proposition 9.3.2. *Assume that (u^{MHD}, b^{MHD}, p^H) be a smooth solution of the MHD system (MHD) associated with a smooth initial data (u_0^{MHD}, b_0^{MHD}) which satisfies*

$$\operatorname{div} (u_0^{MHD}, b_0^{MHD}) = 0.$$

Then we have for any $t > 0$,

$$(9.3.9) \quad \begin{aligned} & \|v_\mu^h(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \mu \int_0^t f(\tau) \|v_\mu^h(t)\|_{\mathcal{B}_p^0} d\tau + c_0 \|v_\mu^h\|_{L^1(0,t;\mathcal{B}_p^2)} \\ & \leq C \|v_0^h\|_{\mathcal{B}_p^0} + V_\mu^h + V_{vis}^h + V_{other}^h, \end{aligned}$$

where

$$\begin{aligned} V_\mu^h &= C \int_0^t \|v^3\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c^3\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau \\ &+ C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|c_\mu^h\|_{\mathcal{B}_p^0} d\tau, \end{aligned}$$

$$\begin{aligned} V_{vis}^h &= \epsilon \int_0^t \|v^3\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^3\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau \\ &+ \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau, \end{aligned}$$

$$\begin{aligned} V_{other}^h &= C \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|v_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau \\ &+ C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|\partial_3 \bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau + C \int_0^t \|c_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau \end{aligned}$$

and

$$(9.3.10) \quad \begin{aligned} & \|c_\mu^h(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \mu \int_0^t f(\tau) \|c_\mu^h(t)\|_{\mathcal{B}_p^0} d\tau + c_0 \|c_\mu^h\|_{L^1(0,t;\mathcal{B}_p^0)} \\ & \leq C \|c_0^h\|_{\mathcal{B}_p^0} + C_\mu^h + C_{vis}^h + C_{other}^h, \end{aligned}$$

where

$$\begin{aligned} C_\mu^h &= C \int_0^t \|v^3\|_{\mathcal{B}_p^2} \|c_\mu^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c^3\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau \\ &\quad + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|c_\mu^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau, \\ C_{vis}^h &= \epsilon \int_0^t \|v^3\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^3\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau \\ &\quad + \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau, \\ C_{other}^h &= C \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau \\ &\quad + C \int_0^t \|v_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau + C \int_0^t \|c_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau. \end{aligned}$$

The constant μ is large number and ϵ suitable small which will be verified exactly later in the proof of Theorem 9.1.1 in this section.

Remark 9.3.1. V_μ^h and C_μ^h in the above proposition will be bound by the term $\mu \int_0^t f(\tau) \|v_\mu^h(t)\|_{\mathcal{B}_p^0} d\tau$ on the left-hand-side of (9.3.9) and $\mu \int_0^t f(\tau) \|c_\mu^h(t)\|_{\mathcal{B}_p^0} d\tau$ on the left-hand-side of (9.3.10) with sufficiently large μ .

Remark 9.3.2. V_{vis}^h and C_{vis}^h in the above proposition will be bounded by the viscous dissipation terms in the left hand side of (9.3.9) and (9.3.10).

Next we will give a proposition which shows the estimate for the couple (v^3, c^3) , to do that we define the auxiliary function v_λ^3 and c_λ^3 as

$$(9.3.11) \quad v_\lambda^3(t) = v^3(t) e^{-\lambda \int_0^t h(\tau) d\tau} \quad \text{and} \quad c_\lambda^3(t) = c^3(t) e^{-\lambda \int_0^t h(\tau) d\tau}$$

with $h(t) = \|(\bar{u}^h, \bar{b}^h)(t)\|_{\mathcal{B}_p^2}$ and λ be a positive constant will be defined later and taken large enough. Then we have the following proposition.

Proposition 9.3.3. Assume that (u^{MHD}, b^{MHD}, p^H) is a smooth solution of the MHD system (MHD) associated with the smooth initial data (u_0^{MHD}, b_0^{MHD}) which satisfies the divergence-free condition. Then we have for any $t > 0$,

$$(9.3.12) \quad \begin{aligned} & \|v_\lambda^3(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \lambda \int_0^t h(\tau) \|v_\lambda^3(t)\|_{\mathcal{B}_p^0} d\tau + c_0 \|v_\lambda^3\|_{L^1(0,t;\mathcal{B}_p^0)} \\ & \leq \|v_0^3\|_{\mathcal{B}_p^0} + V_\lambda^3 + V_{vis}^3 + V_{other}^3. \end{aligned}$$

with

$$V_\lambda^3 = C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau$$

$$\begin{aligned}
V_{vis}^3 &= \epsilon \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau \\
&\quad + \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau,
\end{aligned}$$

$$\begin{aligned}
V_{other}^3 &= C \int_0^t \|v^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{u}_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{u}^h\|_{\mathcal{B}_p^2} d\tau \\
&\quad + C \int_0^t \|\bar{b}_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{b}^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\lambda^h\|_{\mathcal{B}_p^2} d\tau \\
&\quad + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|v_\lambda^h\|_{\mathcal{B}_p^0} d\tau \\
&\quad + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|c_\lambda^h\|_{\mathcal{B}_p^0} d\tau
\end{aligned}$$

and

$$\begin{aligned}
(9.3.13) \quad &\|c_\lambda^3(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \lambda \int_0^t h(\tau) \|c_\lambda^3(t)\|_{\mathcal{B}_p^0} d\tau + c_0 \|c_\lambda^3\|_{L^1(0,t;\mathcal{B}_p^2)} \\
&\leq \|c_0^3\|_{\mathcal{B}_p^0} + C_\lambda^3 + C_{vis}^3 + C_{other}^3,
\end{aligned}$$

with

$$\begin{aligned}
C_\lambda^3 &= C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau, \\
C_{vis}^3 &= \epsilon \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau \\
&\quad + \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau,
\end{aligned}$$

$$C_{other}^3 = C \int_0^t \|v^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau,$$

for a large number λ and suitable small ϵ which will be verified in the proof of Theorem 9.1.1 in this section.

Remark 9.3.3. V_λ^3 and C_λ^3 in the above proposition will be bounded by the term $\lambda \int_0^t h(\tau) \|v_\lambda^3(t)\|_{\mathcal{B}_p^0} d\tau$ on the left-hand-side of (9.3.12) and $\lambda \int_0^t h(\tau) \|c_\lambda^3(t)\|_{\mathcal{B}_p^0} d\tau$ on the left-hand-side of (9.3.13) with sufficiently large λ .

Remark 9.3.4. V_{vis}^3 and C_{vis}^3 in the above proposition will be bound by the viscous dissipation terms in the left-hand-side of (9.3.12) and (9.3.13).

Now we will show the proof of Theorem 9.1.1 with the help of the result of in Proposition 9.3.1, Proposition 9.3.2 and Proposition 9.3.3. The proof of these propositions will be presented in Section 4-6.

Proof of Theorem 9.1.1. Our goal is to prove the existence of global solution to the system (MHD) under the smallness condition in the vertical derivative (9.1.4). Let T^* denote the maximal time of existence

of the solution (u^{MHD}, b^{MHD}, p^H) to (MHD) associated with the smooth initial data (u_0^{MHD}, b_0^{MHD}) . Let us consider T^{MHD} defined by

$$(9.3.14) \quad T^{MHD} \triangleq \sup \left\{ t < T^* : \|(v^h, c^h)\|_{\tilde{L}^\infty(0, t; \mathcal{B}_p^0)} + \|(v^h, c^h)\|_{L^1(0, t; \mathcal{B}_p^2)} \leq \eta \right\},$$

where $\eta > 0$ will be chosen sufficiently small later. Let $\lambda > 16C$, $\epsilon < \min\{\frac{c_0}{16C}, \frac{c_0}{16\eta}\}$ in Proposition 9.3.3, absorbing the terms in V_λ^3 , V_{vis}^3 , C_λ^3 , C_λ^3 by the left-hand-side of (9.3.12) and (9.3.13), one can deduce for any $T > 0$,

$$\begin{aligned} & \|(v^3, c^3)\|_{\tilde{L}^\infty(0, T; \mathcal{B}_p^0)} + \frac{c_0}{2} \|(v^3, c^3)\|_{L^1(0, T; \mathcal{B}_p^2)} \leq \|(v_0^3, c_0^3)\|_{\mathcal{B}_p^0} \\ & + C \left(\|v^h\|_{L^1(0, T; \mathcal{B}_p^2)} \|v^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} + \|c^h\|_{L^1(0, T; \mathcal{B}_p^0)} \|c^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} \right. \\ & + \|\bar{u}^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\bar{u}^h\|_{L^1(0, T; \mathcal{B}_p^2)} + \|\bar{b}^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\bar{b}^h\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & + \|v^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|v^h\|_{L^1(0, T; \mathcal{B}_p^2)} + \|c^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|c^h\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & + \|\bar{u}^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|v^h\|_{L^1(0, T; \mathcal{B}_p^2)} + \|\bar{b}^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|c^h\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & + \|\bar{u}^h\|_{L^1(0, T; \mathcal{B}_p^2)} \|v^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} + \|\bar{b}^h\|_{L^1(0, T; \mathcal{B}_p^2)} \|c^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \\ & \left. + \|v^h\|_{L^1(0, T; \mathcal{B}_p^2)} \|c^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} + \|c^h\|_{L^1(0, T; \mathcal{B}_p^2)} \|v^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} \right) \\ & \times \exp(C \|(\bar{u}^h, \bar{b}^h)\|_{L^1(0, T; \mathcal{B}_p^2)}). \end{aligned}$$

Choosing $\eta < \frac{c_0}{32C_1}$, where C_1 is a constant such that $C \exp(C \|(\bar{u}^h, \bar{b}^h)\|_{L^1(0, T; \mathcal{B}_p^2)}) < C_1$. Combining with the global bound of \bar{u}^h and \bar{b}^h in Proposition 9.3.1, we have

$$(9.3.15) \quad \begin{aligned} & \|(v^3, c^3)\|_{\tilde{L}^\infty(0, T; \mathcal{B}_p^0)} + \frac{c_0}{2} \|(v^3, c^3)\|_{L^1(0, T; \mathcal{B}_p^2)} \leq C \left(1 + \|(v_0^3, c_0^3)\|_{\mathcal{B}_p^0} + \|(\bar{u}_0^h, \bar{b}_0^h)\|_{\mathcal{B}_p^0}^2 \right. \\ & \left. + \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^4 \right) \exp(C \|(\bar{u}_0^h, \bar{b}_0^h)\|_{\mathcal{B}_p^0} + \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2})) \triangleq M_3(v^3, c^3) \end{aligned}$$

where the constant C is independent of time T .

Next we move to estimate v_μ^h and c_μ^h . Let $\mu > 16C$ and $\epsilon < \frac{c_0}{16C}$ in Proposition 9.3.2, by using the definition of v_μ^h and c_μ^h and absorbing the terms in V_μ^h , V_{vis}^h , C_μ^h , C_λ^h by the left-hand-side of (9.3.9) and (9.3.10), we have

$$\begin{aligned} & \|(v^h, c^h)\|_{\tilde{L}^\infty(0, T; \mathcal{B}_p^0)} + \frac{c_0}{2} \|(v^h, c^h)\|_{L^1(0, T; \mathcal{B}_p^2)} \leq C \left(\|(v_0^h, c_0^h)\|_{\mathcal{B}_p^0} \right. \\ & + \|v^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|v^h\|_{L^1(0, T; \mathcal{B}_p^2)} + \|c^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|c^h\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & + \|v^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\partial_3 \bar{u}^h\|_{L^1(0, T; \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} + \|c^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\partial_3 \bar{b}^h\|_{L^1(0, T; \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} \\ & + \|\bar{u}^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\partial_3 \bar{u}^h\|_{L^1(0, T; \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} + \|\bar{b}^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\partial_3 \bar{b}^h\|_{L^1(0, T; \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} \\ & + \|v^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|c^h\|_{L^1(0, T; \mathcal{B}_p^2)} + \|c^h\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|v^h\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & \left. + \|v^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\partial_3 \bar{b}^h\|_{L^1(0, T; \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} + \|c^3\|_{L^\infty(0, T; \mathcal{B}_p^0)} \|\partial_3 \bar{u}^h\|_{L^1(0, T; \mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} \right) \\ & \times \exp(C \|(\bar{u}^h, \bar{b}^h)\|_{L^1(0, T; \mathcal{B}_p^2)} + C \|(v^3, c^3)\|_{L^1(0, T; \mathcal{B}_p^2)}). \end{aligned}$$

By choosing $\eta < \frac{c_0}{16C_2}$, where $C_2 = C_1 \exp\left(\frac{2M_3(v^3, c^3)}{c_0}\right)$, we have

$$\|(v^h, c^h)\|_{\tilde{L}^\infty(0, T; \mathcal{B}_p^0)} + \frac{c_0}{2} \|(v^h, c^h)\|_{L^1(0, T; \mathcal{B}_p^2)}$$

$$(9.3.16) \quad \leq C \left(\| (v_0^h, c_0^h) \|_{\mathcal{B}_p^0} + M_3(v^3, c^3) \times \left(\| \partial_3 \bar{u}^h \|_{L^1(0,T;\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} + \| \partial_3 \bar{b}^h \|_{L^1(0,T;\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}})} \right) \right) \\ \times \exp(C \| (\bar{u}^h, \bar{b}^h) \|_{L^1(0,T;\mathcal{B}_p^2)} + C \| (v^3, c^3) \|_{L^1(0,T;\mathcal{B}_p^2)}).$$

Recall that we have obtained the global bounded of \bar{u}^h and \bar{b}^h in Proposition 9.3.1 and also the estimate of the couple (v^3, c^3) in (9.3.15). According to the choosing of (v_0^h, c_0^h) , we have

$$(9.3.17) \quad \| (v^h, c^h) \|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + \frac{c_0}{2} \| (v^h, c^h) \|_{L^1(0,T;\mathcal{B}_p^2)} \leq C_h \| \Lambda_h^{-1} \partial_3 (u_0^H, b_0^H) \|_{\mathcal{B}_p^0} B(u_0^{H,h}, b_0^{H,h}) \\ \times \exp \left[C \left(1 + \| (u_0^{H,3}, b_0^{H,3}) \|_{\mathcal{B}_p^0} + \| (u_0^{H,h}, b_0^{H,h}) \|_{\mathcal{B}_p^0}^2 + \| (u_0^{H,h}, b_0^{H,h}) \|_{H^1}^4 \right) B(u_0^{H,h}, u_0^{H,h}) \right],$$

where

$$B(u_0^{H,h}, b_0^{H,h}) = \exp \left(C \left(\| (u_0^{H,h}, b_0^{H,h}) \|_{\mathcal{B}_p^0} + \| (u_0^{H,h}, b_0^{H,h}) \|_{H^1}^2 \exp(\| (u_0^{H,h}, b_0^{H,h}) \|_{L^2}) \right) \right).$$

We now claim that $T^* = +\infty$ if the initial data (u_0^H, b_0^H) satisfies the smallness condition (9.1.4). If not, from (9.3.17) and (9.1.4), we can infer

$$(9.3.18) \quad \| (v^h, c^h) \|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + \| (v^h, c^h) \|_{L^1(0,T;\mathcal{B}_p^2)} \leq C_0 \varepsilon_0.$$

In particular if we choose ε_0 in (9.1.4) small enough such that $\varepsilon_0 \leq \frac{\eta}{2C_0}$, one has

$$(9.3.19) \quad \| v^h \|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + \| v^h \|_{L^1(0,T;\mathcal{B}_p^2)} \leq \frac{\eta}{2} \quad \text{for } \forall T \leq T^{MHD},$$

which contradicts the induction hypothesis (9.3.14), and which in turn shows that $T^* = \infty$ under the assumption (9.1.4). This completes the proof of Theorem 9.1.1. \square

In the rest of this section we focus on the inhomogeneous MHD system. For this case, we only need to decompose the solution of System (IMHD') into a solution of the homogeneous MHD system plus a solution of the perturbed system. More precisely, defining $R_1 = u - u^{MHD}$, $R_2 = b - b^{MHD}$, $q = p - p^H$, $u_0^{MHD} = u_0$ and $b_0^{MHD} = b_0$ taking the difference between (IMHD') and (MHD), we can obtain the remaining term (R, q) satisfies the following system:

$$(9.3.20) \quad \begin{cases} \partial_t a + (R_1 + u^{MHD}) \cdot \nabla a = 0, \\ \partial_t R_1 - \Delta R_1 = -(a+1) \nabla q + a(\Delta R_1 + \Delta u^{MHD} - \nabla p^H) - R_1 \cdot \nabla R_1 \\ \quad - u^{MHD} \cdot \nabla R_1 - R_1 \cdot \nabla u^{MHD} + (1+a)(R_2 \cdot \nabla R_2 + b^{MHD} \cdot \nabla R_2 + R_2 \cdot \nabla b^{MHD}), \\ \partial_t R_2 - \Delta R_2 = -R_1 \cdot \nabla R_2 - R_1 \cdot \nabla b^{MHD} - u^{MHD} \cdot \nabla R_2 \\ \quad + R_2 \cdot \nabla R_1 + b^{MHD} \cdot \nabla R_1 + R_2 \cdot \nabla u^{MHD}, \\ \operatorname{div} R_1 = 0, \operatorname{div} R_2 = 0, \\ a|_{t=0} = a_0(x), R_1|_{t=0} = 0, R_2|_{t=0} = 0. \end{cases}$$

Based on the result in Theorem 9.1.1, the proof of Theorem 9.1.2 is transferred to estimate the remainder term (a, R_1, R_2) in System (9.3.20).

Proof of Theorem 9.1.2. Let us denote T^{**} the maximal time of existence of the solution (u, b, a) to (IMHD') associated with the smooth initial data (u_0, b_0, a_0) . Consider T^{IMHD} defined by

$$T^{IMHD} \triangleq \sup \left\{ t < T^{**} : \| a \|_{L^\infty(0,t;\mathcal{B}_p^{\frac{3}{p}})} + \| (R_1, R_2) \|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \| (R_1, R_2) \|_{L^1(0,t;\mathcal{B}_p^2)} \leq \alpha \right\},$$

where $\alpha > 0$ will be chosen sufficiently small later. Let us define the functions $R_{1,\kappa}$ and $R_{2,\kappa}$ associated with R_1 and R_2 respectively,

$$R_{1,\kappa}(t) = R_1(t)e^{-\kappa \int_0^t L(\tau) d\tau} \quad \text{and} \quad R_{2,\kappa}(t) = R_2(t)e^{-\kappa \int_0^t L(\tau) d\tau},$$

with $L(t) = \|u^{MHD}\|_{\mathcal{B}_p^2} + \|b^{MHD}\|_{\mathcal{B}_p^2}$. After calculation, we deduce that $R_{1,\kappa}$ and $R_{2,\kappa}$ verify the following equations,

$$(9.3.21) \quad \begin{cases} \partial_t R_{1,\kappa} - \Delta R_{1,\kappa} + \kappa L(t) R_{1,\kappa} = -(1+a)\nabla q_\kappa + g_{1,\kappa}, \\ \partial_t R_{2,\kappa} - \Delta R_{2,\kappa} + \kappa L(t) R_{2,\kappa} = g_{2,\kappa}, \end{cases}$$

where

$$\begin{cases} g_{1,\kappa} = a(\Delta R_1 + \Delta u^{MHD} - \nabla p^H)_\kappa - (R_1 \cdot \nabla R_1)_\kappa - (u^{MHD} \cdot \nabla R_1)_\kappa \\ \quad - (R_1 \cdot \nabla u^{MHD})_\kappa + (1+a)(R_2 \cdot \nabla R_2 + b^{MHD} \cdot \nabla R_2 + R_2 \cdot \nabla b^{MHD})_\kappa, \\ g_{2,\kappa} = (R_2 \cdot \nabla R_1 - R_1 \cdot \nabla R_2)_\kappa + (b^{MHD} \cdot \nabla R_1 - R_1 \cdot \nabla b^{MHD})_\kappa \\ \quad + (R_2 \cdot \nabla u^{MHD} - u^{MHD} \cdot \nabla R_2)_\kappa. \end{cases}$$

Applying the dyadic operator $\Delta_q^h \Delta_j^v$ to (9.3.21), and taking the L^p estimate, we deduce from Lemma 9.2.1 that for some positive constant c_0 ,

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\Delta_q^h \Delta_j^v (R_{1,\kappa}, R_{2,\kappa})(t)\|_{L^p}^p + c_0(2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v (R_{1,\kappa}, R_{2,\kappa})(t)\|_{L^p}^p \\ & + \kappa L(t) \|\Delta_q^h \Delta_j^v (R_{1,\kappa}, R_{2,\kappa})(t)\|_{L^p}^p \leq \|\Delta_q^h \Delta_j^v (-(1+a)\nabla q_\kappa + g_{1,\kappa})\|_{L^p} \\ & \quad \times \|\Delta_q^h \Delta_j^v (R_{1,\kappa}, R_{2,\kappa})\|_{L^p}^{p-1} + \|\Delta_q^h \Delta_j^v g_{2,\kappa}\|_{L^p} \|\Delta_q^h \Delta_j^v (R_{1,\kappa}, R_{2,\kappa})\|_{L^p}^{p-1}. \end{aligned}$$

Simplifying by $\|\Delta_q^h \Delta_j^v (R_{1,\kappa}, R_{2,\kappa})\|_{L^p}^{p-1}$ in the above inequality, and integrating from 0 to T in time, multiplying by $2^{(-1+\frac{2}{p})q} 2^{\frac{1}{p}j}$ and then taking summation in q, j , we obtain

$$(9.3.22) \quad \begin{aligned} & \| (R_{1,\kappa}, R_{2,\kappa})(t) \|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + \kappa \int_0^T L(\tau) \| (R_{1,\kappa}, R_{2,\kappa}) \|_{\mathcal{B}_p^0} d\tau + c_0 \| (R_{1,\kappa}, R_{2,\kappa}) \|_{L^1(0,T;\mathcal{B}_p^2)} \\ & \leq \int_0^T \| -(1+a)\nabla q_\kappa + g_{1,\kappa} \|_{\mathcal{B}_p^0} d\tau + \int_0^T \| g_{2,\kappa} \|_{\mathcal{B}_p^0} d\tau. \end{aligned}$$

By Lemma 9.2.5, we can deduce that

$$\| (1+a)\nabla q_\kappa - g_{1,\kappa} \|_{\mathcal{B}_p^0} \leq C \| g_{1,\kappa} \|_{\mathcal{B}_p^0}.$$

Thus now we only need to estimate $g_{1,\kappa}$ and $g_{2,\kappa}$. We start by showing the estimate of $g_{2,\kappa}$, using Lemma 9.2.4 and Young's inequality,

$$(9.3.23) \quad \begin{aligned} & \| (R_2 \cdot \nabla R_1 - R_1 \cdot \nabla R_2)_\kappa \|_{\mathcal{B}_p^0} \\ & \leq \| R_2 \|_{\mathcal{B}_p^0}^{\frac{1}{2}} \| R_2 \|_{\mathcal{B}_p^2}^{\frac{1}{2}} \| R_{1,\kappa} \|_{\mathcal{B}_p^0}^{\frac{1}{2}} \| R_{1,\kappa} \|_{\mathcal{B}_p^2}^{\frac{1}{2}} + \| R_1 \|_{\mathcal{B}_p^0}^{\frac{1}{2}} \| R_1 \|_{\mathcal{B}_p^2}^{\frac{1}{2}} \| R_{2,\kappa} \|_{\mathcal{B}_p^0}^{\frac{1}{2}} \| R_{2,\kappa} \|_{\mathcal{B}_p^2}^{\frac{1}{2}} \\ & \leq \epsilon (\| R_2 \|_{\mathcal{B}_p^0} \| R_{1,\kappa} \|_{\mathcal{B}_p^2} + \| R_1 \|_{\mathcal{B}_p^0} \| R_{2,\kappa} \|_{\mathcal{B}_p^2}) + C (\| R_2 \|_{\mathcal{B}_p^2} \| R_{1,\kappa} \|_{\mathcal{B}_p^0} + \| R_1 \|_{\mathcal{B}_p^2} \| R_{2,\kappa} \|_{\mathcal{B}_p^0}). \end{aligned}$$

Along the same way, we have

$$(9.3.24) \quad \| (b^{MHD} \cdot \nabla R_1 - R_1 \cdot \nabla b^{MHD})_\kappa \| \leq \epsilon \| b^{MHD} \|_{\mathcal{B}_p^0} \| R_{1,\kappa} \|_{\mathcal{B}_p^2} + C \| b^{MHD} \|_{\mathcal{B}_p^2} \| R_{1,\kappa} \|_{\mathcal{B}_p^0},$$

and

$$(9.3.25) \quad \|(u^{MHD} \cdot \nabla R_2 - R_2 \cdot \nabla u^{MHD})_\kappa\| \leq \epsilon \|u^{MHD}\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2} + C \|u^{MHD}\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0}.$$

Then summing up the estimates (9.3.23)-(9.3.25) we obtain the following estimate for $g_{2,\kappa}$,

$$(9.3.26) \quad \begin{aligned} \|g_{2,\kappa}\|_{\mathcal{B}_p^0} &\leq \epsilon (\|R_2\|_{\mathcal{B}_p^0} \|R_{1,\kappa}\|_{\mathcal{B}_p^2} + \|R_1\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2}) + C (\|R_2\|_{\mathcal{B}_p^2} \|R_{1,\kappa}\|_{\mathcal{B}_p^0} \\ &\quad + \|R_1\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0}) + \epsilon (\|b^{MHD}\|_{\mathcal{B}_p^0} \|R_{1,k}\|_{\mathcal{B}_p^2} + \|u^{MHD}\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2}) \\ &\quad + C (\|b^{MHD}\|_{\mathcal{B}_p^2} \|R_{1,k}\|_{\mathcal{B}_p^0} + \|u^{MHD}\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0}). \end{aligned}$$

Now we move to estimate $g_{1,\kappa}$, recall that

$$\begin{aligned} g_{1,\kappa} &= a(\Delta R_1 + \Delta u^{MHD} - \nabla p^H)_\kappa - (R_1 \cdot \nabla R_1)_\kappa - (u^{MHD} \cdot \nabla R_1)_\kappa \\ &\quad - (R_1 \cdot \nabla u^{MHD})_\kappa + (1+a)(R_2 \cdot \nabla R_2 + b^{MHD} \cdot \nabla R_2 + R_2 \cdot \nabla b^{MHD})_\kappa. \end{aligned}$$

In the same manner we can obtain the estimate of the terms $(u^{MHD} \cdot \nabla R_1)_\kappa$, $(R_1 \cdot \nabla u^{MHD})_\kappa$ and $(R_1 \cdot \nabla R_1)_\kappa$, then we still need to estimate the other terms. Denote that

$$f_k = (1+a)(R_2 \cdot \nabla R_2 + b^{MHD} \cdot \nabla R_2 + R_2 \cdot \nabla b^{MHD})_\kappa,$$

we can obtain

$$\begin{aligned} \|f_k\|_{\mathcal{B}_p^0} &\leq \|(R_2 \cdot \nabla R_2 + b^{MHD} \cdot \nabla R_2 + R_2 \cdot \nabla b^{MHD})_\kappa\|_{\mathcal{B}_p^0} \\ &\quad + \|a(R_2 \cdot \nabla R_2 + b^{MHD} \cdot \nabla R_2 + R_2 \cdot \nabla b^{MHD})_\kappa\|_{\mathcal{B}_p^0} \\ &\triangleq f_{1,\kappa} + f_{2,\kappa}. \end{aligned}$$

By using Lemma 9.2.4 and Young's inequality we can obtain

$$(9.3.27) \quad \begin{aligned} \|f_{1,\kappa}\|_{\mathcal{B}_p^0} &\leq \epsilon \|R_2\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2} + C \|R_2\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0} \\ &\quad + \epsilon \|b^{MHD}\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2} + C \|b^{MHD}\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0}. \end{aligned}$$

To estimate $f_{2,\kappa}$, using the product law (9.2.5), Lemmas 9.2.2-9.2.4 and Young's inequality,

$$(9.3.28) \quad \begin{aligned} \|f_{2,\kappa}\|_{\mathcal{B}_p^0} &\leq \|a\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|(R_2 \cdot \nabla R_2 + b^{MHD} \cdot \nabla R_2 + R_2 \cdot \nabla b^{MHD})_\kappa\|_{\mathcal{B}_p^0} \\ &\leq \|a\|_{\mathcal{B}_p^{\frac{3}{p}}} \left(\epsilon \|R_2\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2} + C \|R_2\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0} \right. \\ &\quad \left. + \epsilon \|b^{MHD}\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2} + C \|b^{MHD}\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0} \right). \end{aligned}$$

Similarly,

$$(9.3.29) \quad \|(a\Delta R_1)_\kappa\|_{\mathcal{B}_p^0} \leq \|a\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|\Delta R_{1,\kappa}\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} \lesssim \|a\|_{\mathcal{B}_p^{\frac{3}{p}}} \|R_{1,\kappa}\|_{\mathcal{B}_p^2},$$

and

$$(9.3.30) \quad \|(a\Delta u^{MHD})_\kappa\|_{\mathcal{B}_p^0} \lesssim \|a\|_{\mathcal{B}_p^{\frac{3}{p}}} \|u_\kappa^{MHD}\|_{\mathcal{B}_p^2}, \quad \|(a\nabla p^H)_\kappa\|_{\mathcal{B}_p^0} \lesssim \|a\|_{\mathcal{B}_p^{\frac{3}{p}}} \|\nabla p_\kappa^H\|_{\mathcal{B}_p^2}.$$

Then summing the estimates (9.3.27)-(9.3.30) up, we obtain the following estimate for $g_{1,\kappa}$,

$$\|g_{1,\kappa}\|_{\mathcal{B}_p^0} \leq \epsilon (\|R_1\|_{\mathcal{B}_p^0} \|R_{1,\kappa}\|_{\mathcal{B}_p^2} + \|R_2\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2}) + C (\|R_1\|_{\mathcal{B}_p^2} \|R_{1,\kappa}\|_{\mathcal{B}_p^0}$$

$$\begin{aligned}
(9.3.31) \quad & + \|R_2\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0} + \epsilon (\|b^{MHD}\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2} + \|u^{MHD}\|_{\mathcal{B}_p^0} \|R_{1,k}\|_{\mathcal{B}_p^2}) \\
& + C (\|b^{MHD}\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0} + \|u^{MHD}\|_{\mathcal{B}_p^2} \|R_{1,k}\|_{\mathcal{B}_p^0}) + \|a\|_{\mathcal{B}_p^{\frac{3}{2}}} (\|u_\kappa^{MHD}\|_{\mathcal{B}_p^2} \\
& + \|\nabla p_\kappa^H\|_{\mathcal{B}_p^2} + \|R_{1,\kappa}\|_{\mathcal{B}_p^2}) + \|a\|_{\mathcal{B}_p^{\frac{3}{2}}} (\epsilon \|R_2\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2} \\
& + C \|R_2\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0} + \epsilon \|b^{MHD}\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2} + C \|b^{MHD}\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0}).
\end{aligned}$$

Inserting the estimates (9.3.31) and (9.3.26) to (9.3.22), one can deduce

$$\begin{aligned}
& \| (R_{1,\kappa}, R_{2,\kappa})(t) \|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + \kappa \int_0^T L(\tau) \| (R_{1,\kappa}, R_{2,\kappa}) \|_{\mathcal{B}_p^0} d\tau + c_0 \| (R_{1,\kappa}, R_{2,\kappa}) \|_{L^1(0,T;\mathcal{B}_p^2)} \\
& \leq \int_0^T \|a\|_{\mathcal{B}_p^{\frac{3}{2}}} \left(\|u_\kappa^{MHD}\|_{\mathcal{B}_p^2} + \|\nabla p_\kappa^H\|_{\mathcal{B}_p^2} + \|R_{1,\kappa}\|_{\mathcal{B}_p^2} \right) d\tau + \int_0^T \|a\|_{\mathcal{B}_p^{\frac{3}{2}}} \left(\epsilon \|R_2\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2} \right. \\
& \quad \left. + C \|R_2\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0} + \epsilon \|b^{MHD}\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2} + C \|b^{MHD}\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0} \right) d\tau \\
& \quad + \epsilon \int_0^T \left(\|R_1\|_{\mathcal{B}_p^0} \|R_{1,\kappa}\|_{\mathcal{B}_p^2} + \|R_2\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2} \right) d\tau \\
& \quad + C \int_0^T \left(\|R_1\|_{\mathcal{B}_p^2} \|R_{1,\kappa}\|_{\mathcal{B}_p^0} + \|R_2\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0} \right) d\tau \\
& \quad + \epsilon \int_0^T \left(\|(u^{MHD}, b^{MHD})\|_{\mathcal{B}_p^0} \|R_{2,k}\|_{\mathcal{B}_p^2} + \|(u^{MHD}, b^{MHD})\|_{\mathcal{B}_p^0} \|R_{1,k}\|_{\mathcal{B}_p^2} \right) d\tau \\
& \quad + C \int_0^T \left(\|(u^{MHD}, b^{MHD})\|_{\mathcal{B}_p^2} \|R_{2,k}\|_{\mathcal{B}_p^0} + \|(u^{MHD}, b^{MHD})\|_{\mathcal{B}_p^2} \|R_{1,k}\|_{\mathcal{B}_p^0} \right) d\tau \\
& \quad + \epsilon \int_0^T \left(\|R_2\|_{\mathcal{B}_p^0} \|R_{1,\kappa}\|_{\mathcal{B}_p^2} + \|R_1\|_{\mathcal{B}_p^0} \|R_{2,\kappa}\|_{\mathcal{B}_p^2} \right) d\tau \\
& \quad + C \int_0^T \left(\|R_2\|_{\mathcal{B}_p^2} \|R_{1,\kappa}\|_{\mathcal{B}_p^0} + \|R_1\|_{\mathcal{B}_p^2} \|R_{2,\kappa}\|_{\mathcal{B}_p^0} \right) d\tau.
\end{aligned}$$

Noticing that we have shown in Theorem 9.1.1 that $u^{MHD} \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2)$ and $b^{MHD} \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2)$. To finish the proof, we need to do some estimate of the transport equation satisfied by a . Recall that

$$\partial_t a + (u^{MHD} + R_1) \cdot \nabla a = 0,$$

Using Lemma 9.2.6 we can obtain

$$\|a\|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^{\frac{3}{2}})} \leq \|a_0\|_{\mathcal{B}_p^{\frac{3}{2}}} \exp \left(C \int_0^T \|u^{MHD}(\tau)\|_{\mathcal{B}_p^2} + \|R_1(\tau)\|_{\mathcal{B}_p^2} d\tau \right).$$

According to Theorem 9.1.1 and the definition of T^{IMHD} in (9.3.14), we have

$$\exp \left(C \int_0^T \|u^{MHD}(\tau)\|_{\mathcal{B}_p^2} + \|R_1(\tau)\|_{\mathcal{B}_p^2} d\tau \right) \leq C'$$

where C' is a constant independent of time t . Choosing ϵ, α small such that

$$\epsilon \leq \min \left\{ \frac{c_0}{16 \|(u^{MHD}, b^{MHD})\|_{\mathcal{B}_p^0}}, \frac{c_0}{16 C' \alpha \|a_0\|_{\mathcal{B}_p^0}}, \frac{c_0}{16 C' \|a_0\|_{\mathcal{B}_p^0} \|(u^{MHD}, b^{MHD})\|_{\mathcal{B}_p^0}}, \frac{c_0}{16 \alpha} \right\}$$

$$\alpha \leq \min \left\{ \frac{1}{16 C C' \|a_0\|_{\mathcal{B}_p^0}}, \frac{1}{16 C C' \|a_0\|_{\mathcal{B}_p^0} \|(u^{MHD}, b^{MHD})\|_{\mathcal{B}_p^0}}, \frac{c_0}{16 \alpha} \right\},$$

and κ be large enough (say $\kappa > 16C$), then we have

$$\|(R_{1,\kappa}, R_{2,\kappa})\|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + c_0\|(R_{1,\kappa}, R_{2,\kappa})\|_{L^1(0,T;\mathcal{B}_p^2)} \leq C_a \|a_0\|_{B_p^{\frac{3}{p}}}.$$

Finally, by the definition of $R_{1,\kappa}$ and $R_{2,\kappa}$, we deduce the estimate of the couple (R_1, R_2)

$$\begin{aligned} & \|(R_1, R_2)\|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + \|(R_1, R_2)\|_{L^1(0,T;\mathcal{B}_p^2)} \\ & \leq C \|a_0\|_{B_p^{\frac{3}{p}}} \exp\left(\int_0^T \|u^{MHD}\|_{\mathcal{B}_p^2} + \|b^{MHD}\|_{\mathcal{B}_p^2} d\tau\right) \leq \tilde{C}_a \|a_0\|_{B_p^{\frac{3}{p}}}, \end{aligned}$$

where \tilde{C}_a is a positive constant depending on the initial datum but independent of time. From this estimate and combining with the smallness condition in Theorem 9.1.2, we can also prove that $T^{IMHD} = +\infty$ and the global solution to (a, u, b) of the (IMHD') satisfies

$$a \in \tilde{L}^\infty(\mathbb{R}_+; B_p^{\frac{3}{p}}), \quad u \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2) \quad \text{and} \quad b \in \tilde{L}^\infty(\mathbb{R}_+; \mathcal{B}_p^0) \cap L^1(\mathbb{R}_+; \mathcal{B}_p^2).$$

□

9.4 Proof of Proposition 9.3.1

In this section we present the proof of Proposition 9.3.1, we start by the standard L^2 energy estimates for the equations satisfied by u and b ,

$$(9.4.1) \quad \frac{1}{2} \frac{d}{dt} \|\bar{u}^h(t)\|_{L^2}^2 + \|\nabla \bar{u}^h\|_{L^2}^2 = -\langle \nabla_h \bar{p}, \bar{u}^h \rangle_{L^2} - \langle \bar{u}^h \cdot \nabla_h \bar{u}^h, \bar{u}^h \rangle_{L^2} + \langle \bar{b}^h \cdot \nabla_h \bar{b}^h, \bar{u}^h \rangle_{L^2}$$

and

$$(9.4.2) \quad \frac{1}{2} \frac{d}{dt} \|\bar{b}^h(t)\|_{L^2}^2 + \|\nabla \bar{b}^h\|_{L^2}^2 = -\langle \bar{u}^h \cdot \nabla_h \bar{b}^h, \bar{b}^h \rangle_{L^2} + \langle \bar{b}^h \cdot \nabla_h \bar{u}^h, \bar{b}^h \rangle_{L^2}.$$

Since we have the divergence-free condition of \bar{u}^h and \bar{b}^h , integrating by parts yield that

$$\langle \nabla_h \bar{p}, \bar{u}^h \rangle_{L^2} = -\int \bar{p} \operatorname{div}_h \bar{u}^h dx = 0,$$

$$\langle \bar{u}^h \cdot \nabla_h \bar{u}^h, \bar{u}^h \rangle_{L^2} = \int \bar{u}^h \cdot \nabla_h \bar{u}^h \cdot \bar{u}^h dx = -\frac{1}{2} \int \operatorname{div}_h \bar{u}^h |\bar{u}^h|^2 dx = 0,$$

and similarly $\langle \bar{u}^h \cdot \nabla_h \bar{b}^h, \bar{b}^h \rangle_{L^2} = 0$. We remark that if we integrate by part the term $\langle \bar{b}^h \cdot \nabla_h \bar{b}^h, \bar{u}^h \rangle_{L^2}$, we find that it equal to $-\langle \bar{b}^h \cdot \nabla_h \bar{u}^h, \bar{b}^h \rangle_{L^2}$, indeed

$$\begin{aligned} \langle \bar{b}^h \cdot \nabla_h \bar{b}^h, \bar{u}^h \rangle_{L^2} &= \int \bar{b}^h \cdot \nabla_h \bar{b}^h \cdot \bar{u}^h dx = \sum_{i,j=1}^2 \int (\bar{b}^h)^i \partial_i (\bar{b}^h)^j (\bar{u}^h)^j dx \\ &= -\sum_{i,j=1}^2 \int \partial_i (\bar{b}^h)^i (\bar{b}^h)^j (\bar{u}^h)^j dx - \sum_{i,j=1}^2 \int (\bar{b}^h)^i (\bar{b}^h)^j \partial_i (\bar{u}^h)^j dx \\ &= -\int \operatorname{div}_h \bar{b}^h \cdot \bar{u}^h \cdot \bar{b}^h dx - \int \bar{b}^h \cdot \nabla_h \bar{u}^h \cdot \bar{b}^h dx \end{aligned}$$

$$= - \left\langle \bar{b}^h \cdot \nabla_h \bar{u}^h, \bar{b}^h \right\rangle_{L^2}.$$

By summing up the two resulting estimates (9.4.1) and (9.4.2), and then integrating on time on $(0, T)$, we obtain

$$\|(\bar{u}^h, \bar{b}^h)\|_{L^\infty(0, T; L^2)} + \|\nabla(\bar{u}^h, \bar{b}^h)\|_{L^2(0, T; L^2)} \leq \|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2},$$

which verifies (9.3.1). Then we give the estimate of $\partial_3(\bar{u}^h, \bar{b}^h)$. Applying ∂_3 to (Quasi-2D MHD), we get

$$\begin{cases} \partial_t \partial_3 \bar{u}^h + \partial_3(\bar{u}^h \cdot \nabla_h \bar{u}^h) - \Delta \partial_3 \bar{u}^h + \nabla_h \partial_3 \bar{p} = \partial_3(\bar{b}^h \cdot \nabla_h \bar{b}^h), \\ \partial_t \partial_3 \bar{b}^h + \partial_3(\bar{u}^h \cdot \nabla_h \bar{b}^h) - \Delta \partial_3 \bar{b}^h = \partial_3(\bar{b}^h \cdot \nabla_h \bar{u}^h). \end{cases}$$

Defining $\bar{u}_\sigma^h(t) = \bar{u}^h(t) e^{-\sigma(\int_0^t \|\nabla \bar{u}^h\|_{L^2}^2 + \|\nabla \bar{b}^h\|_{L^2}^2 d\tau)}$ and $\bar{b}_\sigma^h(t) = \bar{b}^h(t) e^{-\sigma(\int_0^t \|\nabla \bar{u}^h\|_{L^2}^2 + \|\nabla \bar{b}^h\|_{L^2}^2 d\tau)}$, then we have

$$(9.4.3) \quad \begin{cases} \partial_t \partial_3 \bar{u}_\sigma^h + \sigma(\|\nabla \bar{u}^h\|_{L^2}^2 + \|\nabla \bar{b}^h\|_{L^2}^2) \partial_3 \bar{u}_\sigma^h + \partial_3(\bar{u}^h \cdot \nabla_h \bar{u}^h)_\sigma \\ \quad - \Delta \partial_3 \bar{u}_\sigma^h + \nabla_h \partial_3 \bar{p}_\sigma = \partial_3(\bar{b}^h \cdot \nabla_h \bar{b}^h)_\sigma, \\ \partial_t \partial_3 \bar{b}_\sigma^h + \sigma(\|\nabla \bar{u}^h\|_{L^2}^2 + \|\nabla \bar{b}^h\|_{L^2}^2) \partial_3 \bar{b}_\sigma^h + \partial_3(\bar{u}^h \cdot \nabla_h \bar{b}^h)_\sigma \\ \quad - \Delta \partial_3 \bar{b}_\sigma^h = \partial_3(\bar{b}^h \cdot \nabla_h \bar{u}^h)_\sigma. \end{cases}$$

Multiplying System (9.4.3) by $(\partial_3 \bar{u}_\sigma^h, \partial_3 \bar{b}_\sigma^h)$, and then integrating over \mathbb{R}^2 with respect to x_h , we assert that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_3 \bar{u}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 + \sigma(\|\nabla \bar{u}^h\|_{L_{x_h}^2}^2 + \|\nabla \bar{b}^h\|_{L_{x_h}^2}^2) \|\partial_3 \bar{u}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 \\ & \quad - \frac{1}{2} \partial_3^2 \|\partial_3 \bar{u}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 + \|\partial_3^2 \bar{u}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 + \|\nabla_h \partial_3 \bar{u}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 \\ & = - \int_{\mathbb{R}^2} \partial_3 \bar{u}_\sigma^h \cdot \nabla_h \bar{u}^h \cdot \partial_3 \bar{u}_\sigma^h dx_h + \int_{\mathbb{R}^2} \partial_3(\bar{b}^h \cdot \nabla_h \bar{b}^h)_\sigma \cdot \partial_3 \bar{u}_\sigma^h dx_h - \int_{\mathbb{R}^2} \nabla_h \partial_3 \bar{p}_\sigma \cdot \partial_3 \bar{u}_\sigma^h dx_h, \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_3 \bar{b}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 + \sigma(\|\nabla \bar{u}^h\|_{L_{x_h}^2}^2 + \|\nabla \bar{b}^h\|_{L_{x_h}^2}^2) \|\partial_3 \bar{b}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 \\ & \quad - \frac{1}{2} \partial_3^2 \|\partial_3 \bar{b}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 + \|\partial_3^2 \bar{b}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 + \|\nabla_h \partial_3 \bar{b}_\sigma^h(t, \cdot, x_3)\|_{L_{x_h}^2}^2 \\ & = - \int_{\mathbb{R}^2} \partial_3 \bar{u}_\sigma^h \cdot \nabla_h \bar{b}^h \cdot \partial_3 \bar{b}_\sigma^h dx_h + \int_{\mathbb{R}^2} \partial_3(\bar{b}^h \cdot \nabla_h \bar{u}^h)_\sigma \cdot \partial_3 \bar{b}_\sigma^h dx_h. \end{aligned}$$

Summing up these two resulting estimates and making use of the fact that

$$\int_{\mathbb{R}^2} \bar{b}^h \cdot \nabla_h \partial_3 \bar{b}_\sigma^h \cdot \partial_3 \bar{u}_\sigma^h dx_h = - \int_{\mathbb{R}^2} \bar{b}^h \cdot \nabla_h \partial_3 \bar{u}_\sigma^h \cdot \partial_3 \bar{b}_\sigma^h dx_h,$$

then we just need to estimate the following four terms:

$$\int_{\mathbb{R}^2} \partial_3 \bar{u}_\sigma^h \cdot \nabla_h \bar{u}^h \cdot \partial_3 \bar{u}_\sigma^h dx_h, \quad \int_{\mathbb{R}^2} \partial_3 \bar{b}_\sigma^h \cdot \nabla_h \bar{b}^h \cdot \partial_3 \bar{u}_\sigma^h dx_h$$

and

$$\int_{\mathbb{R}^2} \partial_3 \bar{u}_\sigma^h \cdot \nabla_h \bar{b}^h \cdot \partial_3 \bar{b}_\sigma^h dx_h, \quad \int_{\mathbb{R}^2} \partial_3 \bar{b}_\sigma^h \cdot \nabla_h \bar{u}^h \cdot \partial_3 \bar{b}_\sigma^h dx_h.$$

We start by the first term. Using Hölder's inequality, interpolation and Young's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_3 \bar{u}_\sigma^h \cdot \nabla_h \bar{u}^h \cdot \partial_3 \bar{u}_\sigma^h dx_h &\leq C \|\partial_3 \bar{u}_\sigma^h\|_{L^4_{x_h}} \|\nabla_h \bar{u}^h\|_{L^2_{x_h}} \|\partial_3 \bar{u}_\sigma^h\|_{L^4_{x_h}} \\
&\leq C \|\partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}} \|\nabla_h \bar{u}^h\|_{L^2_{x_h}} \|\nabla_h \partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}} \\
&\leq \frac{1}{10} \|\nabla_h \partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}}^2 + C \|\partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}}^2 \|\nabla_h \bar{u}^h\|_{L^2_{x_h}}^2.
\end{aligned}$$

In the same manner, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_3 \bar{b}_\sigma^h \cdot \nabla_h \bar{b}^h \cdot \partial_3 \bar{u}_\sigma^h dx_h &\leq \|\partial_3 \bar{b}_\sigma^h\|_{L^4_{x_h}} \|\nabla_h \bar{b}^h\|_{L^2_{x_h}} \|\partial_3 \bar{u}_\sigma^h\|_{L^4_{x_h}} \\
&\leq C \|\partial_3 \bar{b}_\sigma^h\|_{L^2_{x_h}}^{\frac{1}{2}} \|\nabla_h \partial_3 \bar{b}_\sigma^h\|_{L^2_{x_h}}^{\frac{1}{2}} \|\nabla_h \bar{b}^h\|_{L^2_{x_h}} \|\partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}}^{\frac{1}{2}} \|\nabla_h \partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}}^{\frac{1}{2}} \\
&\leq C \|\partial_3 \bar{b}_\sigma^h\|_{L^2_{x_h}} \|\nabla_h \partial_3 \bar{b}_\sigma^h\|_{L^2_{x_h}} \|\nabla_h \bar{b}^h\|_{L^2_{x_h}} + C \|\nabla_h \bar{b}^h\|_{L^2_{x_h}} \|\partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}} \|\nabla_h \partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}} \\
&\leq \frac{1}{10} \|\nabla_h \partial_3 (\bar{u}_\sigma^h, \bar{b}_\sigma^h)\|_{L^2_{x_h}}^2 + C (\|\partial_3 \bar{b}_\sigma^h\|_{L^2_{x_h}}^2 + \|\partial_3 \bar{u}_\sigma^h\|_{L^2_{x_h}}^2) \|\nabla_h \bar{b}^h\|_{L^2_{x_h}}^2.
\end{aligned}$$

Choosing σ large (say $\sigma > 2 + 2C$), we get

$$\begin{aligned}
\frac{d}{dt} \|\partial_3 (\bar{u}_\sigma^h, \bar{b}_\sigma^h)(t, \cdot, x_3)\|_{L^2_{x_h}}^2 - \partial_3^2 \|\partial_3 (\bar{u}_\sigma^h, \bar{b}_\sigma^h)(t, \cdot, x_3)\|_{L^2_{x_h}}^2 \\
+ \|\partial_3^2 (\bar{u}_\sigma^h, \bar{b}_\sigma^h)(t, \cdot, x_3)\|_{L^2_{x_h}}^2 + \|\nabla_h \partial_3 (\bar{u}_\sigma^h, \bar{b}_\sigma^h)(t, \cdot, x_3)\|_{L^2_{x_h}}^2 \leq 0.
\end{aligned}$$

Finally integrating in x_3 and time variable t , we obtain

$$\|\partial_3 (\bar{u}_\sigma^h, \bar{b}_\sigma^h)(t)\|_{L^2}^2 + \int_0^t \|\nabla \partial_3 (\bar{u}_\sigma^h, \bar{b}_\sigma^h)(\tau)\|_{L^2}^2 d\tau \leq \|\partial_3 (\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2$$

and then according to the definition of \bar{u}_σ^h , we deduce that for any $T > 0$,

$$\|\partial_3 (\bar{u}^h, \bar{b}^h)\|_{L^2}^2 + \int_0^T \|\nabla \partial_3 (\bar{u}^h, \bar{b}^h)\|_{L^2}^2 d\tau \leq \|\partial_3 (\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2 \exp(C \|\bar{u}_0^h, \bar{b}_0^h\|_{L^2}^2).$$

We still need to estimate the horizontal derivative of the solution (\bar{u}^h, \bar{b}^h) . For that we apply the curl_h to System (Quasi-2D MHD), one can obtain

$$\begin{cases} \partial_t \bar{\omega}^h + \bar{u}^h \cdot \nabla_h \bar{\omega}^h - \Delta \bar{\omega}^h = \bar{b}^h \cdot \nabla_h \bar{j}^h, \\ \partial_t \bar{j}^h + \bar{u}^h \cdot \nabla_h \bar{j}^h - \Delta \bar{j}^h = \bar{b}^h \cdot \nabla_h \bar{\omega}^h + Q. \end{cases}$$

with $\bar{\omega}^h = \text{curl}_h \bar{u}^h$, $\bar{j}^h = \text{curl}_h \bar{b}^h$ and

$$Q = 2\partial_1 (\bar{b}^h)^1 (\partial_2 (\bar{u}^h)^1 + \partial_1 (\bar{u}^h)^2) - 2\partial_1 (\bar{u}^h)^1 (\partial_2 (\bar{b}^h)^1 + \partial_1 (\bar{b}^h)^2).$$

Taking L^2 inner product with $\bar{\omega}^h$ and \bar{j}^h , using the fact that $\text{div}_h \bar{u}^h = \text{div}_h \bar{b}^h = 0$, we can assert that

$$\|(\bar{\omega}^h, \bar{j}^h)\|_{L^2}^2 + \int_0^T \|\nabla \bar{\omega}^h(\tau)\|_{L^2}^2 d\tau + \int_0^T \|\nabla \bar{j}^h(\tau)\|_{L^2}^2 d\tau \leq \|(\bar{\omega}_0^h, \bar{j}_0^h)\|_{L^2}^2 + \int_0^T \langle Q, \bar{j}^h \rangle_{L^2} d\tau.$$

By the definition of Q and Hölder inequality, we have

$$\begin{aligned}
\langle Q, \bar{j}^h \rangle_{L^2} &\leq C \int_{\mathbb{R}} \|\nabla_h \bar{u}^h\|_{L^4_{x_h}} \|\nabla_h \bar{b}^h\|_{L^4_{x_h}} \|\bar{j}^h\|_{L^2_{x_h}} dx_3 \\
&\leq C \int_{\mathbb{R}} \|\nabla_h \bar{u}^h\|_{L^2_{x_h}}^{\frac{1}{2}} \|\nabla_h \nabla_h \bar{u}^h\|_{L^2_{x_h}}^{\frac{1}{2}} \|\nabla_h \bar{b}^h\|_{L^2_{x_h}}^{\frac{1}{2}} \|\nabla_h \nabla_h \bar{b}^h\|_{L^2_{x_h}}^{\frac{1}{2}} \|\bar{j}^h\|_{L^2_{x_h}} dx_3 \\
&\leq C \|\nabla_h \bar{u}^h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla_h \bar{u}^h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \bar{b}^h\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla_h \bar{b}^h\|_{L^2}^{\frac{1}{2}} \|\bar{j}^h\|_{L^\infty_v L^2_{x_h}} \\
&\leq C \|\nabla_h(\bar{u}^h, \bar{b}^h)\|_{L^2} \|\nabla_h^2(\bar{u}^h, \bar{b}^h)\|_{L^2} \|\bar{j}^h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \bar{j}^h\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Using the 2D Biot-Savart law $\|\nabla_h \bar{u}^h\|_{L^2} \leq \|\bar{\omega}^h\|_{L^2}$ and Young's inequality, we find

$$\langle Q, \bar{j}^h \rangle_{L^2} \leq \epsilon \|\nabla_h(\bar{\omega}^h, \bar{j}^h)\|_{L^2}^2 + \|(\bar{\omega}^h, \bar{j}^h)\|_{L^2}^3 \|\partial_3 \bar{j}^h\|_{L^2}.$$

Noticing that we have $\|(\bar{\omega}^h, \bar{j}^h)\|_{L^2} \leq \|\nabla(\bar{u}^h, \bar{b}^h)\|_{L^2}$, then

$$\int_0^T \|(\bar{\omega}^h, \bar{j}^h)\|_{L^2} \|\partial_3 \bar{j}^h\|_{L^2} d\tau \leq C,$$

where the constant C only depend on the initial datum.

Then by applying the Grönwall Lemma, we obtain

$$\begin{aligned}
&\|(\bar{\omega}^h, \bar{j}^h)\|_{L^\infty(0,T;L^2)}^2 + \|\nabla(\bar{\omega}^h, \bar{j}^h)\|_{L^2(0,T;L^2)}^2 \\
&\leq C \|(\bar{\omega}_0^h, \bar{j}_0^h)\|_{L^2}^2 \exp\left(\int_0^T \|(\bar{\omega}^h, \bar{j}^h)\|_{L^2} \|\partial_3 \bar{j}^h\|_{L^2} d\tau\right) \\
&\leq C \|(\bar{\omega}_0^h, \bar{j}_0^h)\|_{L^2}^2 \exp(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2)).
\end{aligned}$$

According to the Biot-Savat law,

$$\begin{aligned}
&\|\nabla_h(\bar{u}^h, \bar{b}^h)\|_{L^\infty(0,T;L^2)}^2 + \|\nabla_h \nabla(\bar{u}^h, \bar{b}^h)\|_{L^2(0,T;L^2)}^2 \\
&\leq C \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp\left(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2)\right).
\end{aligned}$$

This completes the proof of the second part of this proposition.

Now we prove (9.3.3), for that we apply the dyadic bloc $\Delta_q^h \Delta_j^v$ to system (Quasi-2D MHD) and taking the L^2 inner product of the resulting equation with $|\Delta_q^h \Delta_j^v \bar{u}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{u}^h$ and $|\Delta_q^h \Delta_j^v \bar{b}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{b}^h$ respectively, we obtain

$$\begin{aligned}
&\frac{1}{p} \frac{d}{dt} \|\Delta_q^h \Delta_j^v \bar{u}^h(t)\|_{L^p}^p - \int_{\mathbb{R}^3} \Delta \Delta_q^h \Delta_j^v \bar{u}^h \cdot |\Delta_q^h \Delta_j^v \bar{u}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{u}^h dx \\
(9.4.4) \quad &= - \int_{\mathbb{R}^3} \Delta_q^h \Delta_j^v (\bar{u}^h \cdot \nabla_h \bar{u}^h) \cdot |\Delta_q^h \Delta_j^v \bar{u}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{u}^h dx \\
&+ \int_{\mathbb{R}^3} \Delta_q^h \Delta_j^v (\bar{b}^h \cdot \nabla_h \bar{b}^h) \cdot |\Delta_q^h \Delta_j^v \bar{u}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{u}^h dx \\
&+ \int_{\mathbb{R}^3} \nabla_h \Delta_q^h \Delta_j^v \bar{p} \cdot |\Delta_q^h \Delta_j^v \bar{u}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{u}^h dx
\end{aligned}$$

and

$$\frac{1}{p} \frac{d}{dt} \|\Delta_q^h \Delta_j^v \bar{b}^h(t)\|_{L^p}^p - \int_{\mathbb{R}^3} \Delta \Delta_q^h \Delta_j^v \bar{b}^h \cdot |\Delta_q^h \Delta_j^v \bar{b}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{b}^h dx$$

$$\begin{aligned}
&= - \int_{\mathbb{R}^3} \Delta_q^h \Delta_j^v (\bar{u}^h \cdot \nabla_h \bar{b}^h) \cdot |\Delta_q^h \Delta_j^v \bar{b}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{b}^h dx \\
&\quad + \int_{\mathbb{R}^3} \Delta_q^h \Delta_j^v (\bar{b}^h \cdot \nabla_h \bar{u}^h) \cdot |\Delta_q^h \Delta_j^v \bar{b}^h|^{p-2} \Delta_q^h \Delta_j^v \bar{b}^h dx.
\end{aligned}$$

Simplifying by $\|\Delta_q^h \Delta_j^v (\bar{u}^h, \bar{b}^h)(t)\|_{L^p}^{p-1}$, and using Lemma 9.2.1,

$$\begin{aligned}
&- \int_{\mathbb{R}^3} \Delta \Delta_q^h \Delta_j^v f \cdot |\Delta_q^h \Delta_j^v f|^{p-2} \Delta_q^h \Delta_j^v f dx \\
&= - \int_{\mathbb{R}^3} \Delta_h \Delta_q^h \Delta_j^v f \cdot |\Delta_q^h \Delta_j^v f|^{p-2} \Delta_q^h \Delta_j^v f dx - \int_{\mathbb{R}^3} \partial_3^2 \Delta_q^h \Delta_j^v f \cdot |\Delta_q^h \Delta_j^v f|^{p-2} \Delta_q^h \Delta_j^v f dx \\
&\geq c_0 (2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v f\|_{L^p}^p,
\end{aligned}$$

for some positive constant $c_0 > 0$, we obtain

$$\begin{aligned}
(9.4.5) \quad &\frac{d}{dt} \|\Delta_q^h \Delta_j^v \bar{u}^h(t)\|_{L^p} + c_0 (2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v \bar{u}^h\|_{L^p} \leq C \|\Delta_q^h \Delta_j^v (\bar{u}^h \cdot \nabla_h \bar{u}^h)\|_{L^p} \\
&\quad + C \|\Delta_q^h \Delta_j^v (\bar{b}^h \cdot \nabla_h \bar{b}^h)\|_{L^p} + C \|\nabla_h \Delta_q^h \Delta_j^v \bar{p}\|_{L^p},
\end{aligned}$$

and

$$\begin{aligned}
(9.4.6) \quad &\frac{d}{dt} \|\Delta_q^h \Delta_j^v \bar{b}^h(t)\|_{L^p} + c_0 (2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v \bar{b}^h\|_{L^p} \\
&\leq C \|\Delta_q^h \Delta_j^v (\bar{u}^h \cdot \nabla_h \bar{b}^h)\|_{L^p} + C \|\Delta_q^h \Delta_j^v (\bar{b}^h \cdot \nabla_h \bar{u}^h)\|_{L^p}.
\end{aligned}$$

Using the divergence-free condition $\operatorname{div}_h \bar{u}^h = 0$, we have

$$(9.4.7) \quad \nabla_h \bar{p} = \nabla_h (-\Delta_h)^{-1} \operatorname{div}_h (\bar{u}^h \cdot \nabla_h \bar{u}^h - \bar{b}^h \cdot \nabla_h \bar{b}^h),$$

where the operator $\nabla_h (-\Delta_h)^{-1} \operatorname{div}_h$ is bounded in L^p ($1 < p < \infty$), then

$$\|\nabla_h \Delta_q^h \Delta_j^v \bar{p}\|_{L^p} \lesssim \|\Delta_q^h \Delta_j^v (\bar{u}^h \cdot \nabla_h \bar{u}^h)\|_{L^p} + \|\Delta_q^h \Delta_j^v (\bar{b}^h \cdot \nabla_h \bar{b}^h)\|_{L^p}.$$

Integrating (9.4.5) and (9.4.6) from 0 to T in time and then multiplying the resulting equation by $2^{(-1+\frac{2}{p})q} 2^{\frac{1}{p}j}$ and taking the summation in q, j , we obtain

$$(9.4.8) \quad \|\bar{u}^h\|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + c_0 \|\bar{u}^h\|_{L^1(0,T;\mathcal{B}_p^2)} \leq \|\bar{u}_0^h\|_{\mathcal{B}_p^0} + \int_0^T \left(\|\bar{u}^h \cdot \nabla_h \bar{u}^h\|_{\mathcal{B}_p^0} + \|\bar{b}^h \cdot \nabla_h \bar{b}^h\|_{\mathcal{B}_p^0} \right) d\tau$$

and

$$(9.4.9) \quad \|\bar{b}^h\|_{\tilde{L}^\infty(0,T;\mathcal{B}_p^0)} + c_0 \|\bar{b}^h\|_{L^1(0,T;\mathcal{B}_p^2)} \leq \|\bar{b}_0^h\|_{\mathcal{B}_p^0} + \int_0^T \left(\|\bar{u}^h \cdot \nabla_h \bar{b}^h\|_{\mathcal{B}_p^0} + \|\bar{b}^h \cdot \nabla_h \bar{u}^h\|_{\mathcal{B}_p^0} \right) d\tau.$$

By using the product law, we find

$$\begin{aligned}
\|\bar{u}^h \cdot \nabla_h \bar{u}^h\|_{\mathcal{B}_p^0} &\leq C \|\bar{u}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \|\nabla_h \bar{u}^h\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \leq C \|\bar{u}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}}^2, \\
\|\bar{b}^h \cdot \nabla_h \bar{b}^h\|_{\mathcal{B}_p^0} &\leq C \|\bar{b}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \|\nabla_h \bar{b}^h\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \leq C \|\bar{b}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}}^2, \\
\|\bar{b}^h \cdot \nabla_h \bar{u}^h\|_{\mathcal{B}_p^0} &\leq C \|\bar{b}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \|\nabla_h \bar{u}^h\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} \leq C \left(\|\bar{b}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}}^2 + \|\bar{u}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}}^2 \right).
\end{aligned}$$

Then according to the global bound (9.3.1), (9.3.2) and the interpolation in Besov space, we have

$\bar{u}^h, \bar{b}^h \in L^2(0, T; B_p^{\frac{2}{p}, \frac{1}{p}})$. Combining with the estimate (9.4.8) and (9.4.9), one can deduce

$$\begin{aligned} & \|(\bar{u}^h, \bar{b}^h)\|_{\tilde{L}^\infty(0, T; \mathcal{B}_p^0)} + c_0 \|(\bar{u}^h, \bar{b}^h)\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & \leq C \left(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{\mathcal{B}_p^0} + \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp \left(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1}^2 \exp \left(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{L^2}^2 \right) \right) \right). \end{aligned}$$

Now we will verify the last equality (9.3.4). For that we define

$$\bar{u}_\gamma^h(t) = \bar{u}^h(t) e^{-\gamma \int_0^t m(\tau) d\tau} \quad \text{and} \quad \bar{b}_\gamma^h(t) = \bar{b}^h(t) e^{-\gamma \int_0^t m(\tau) d\tau}$$

with $m(t) = \|\bar{u}^h\|_{\mathcal{B}_p^2} + \|\bar{b}^h\|_{\mathcal{B}_p^2}$. From System (Quasi-2D MHD), we can derive the equations for \bar{u}_γ^h and \bar{b}_γ^h , and then applying the operator $\Lambda_h^{-1} \partial_3$, we obtain

$$(9.4.10) \quad \begin{cases} \partial_t \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h + \gamma m(t) \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h + \Lambda_h^{-1} \partial_3 (\bar{u}^h \cdot \nabla_h \bar{u}^h)_\gamma \\ \quad - \Delta \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h + \nabla_h \Lambda_h^{-1} \partial_3 \bar{p}_\gamma = \Lambda_h^{-1} \partial_3 (\bar{b}^h \cdot \nabla_h \bar{b}^h)_\gamma, \\ \partial_t \Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h + \gamma m(t) \Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h + \Lambda_h^{-1} \partial_3 (\bar{u}^h \cdot \nabla_h \bar{b}^h)_\gamma \\ \quad - \Delta \Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h = \Lambda_h^{-1} \partial_3 (\bar{b}^h \cdot \nabla_h \bar{u}^h)_\gamma. \end{cases}$$

Applying the dyadic operator $\Delta_q^h \Delta_j^v$ to the system (9.4.10), and taking the L^2 scalar product of the first and the second equations of the resulting system with

$$|\Delta_q^h \Delta_j^v \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h|^{p-2} \Delta_q^h \Delta_j^v \Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h \quad \text{and} \quad |\Delta_q^h \Delta_j^v \Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h|^{p-2} \Delta_q^h \Delta_j^v \Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h$$

respectively. Integrating from 0 to T in time, and multiplying by $2^{(-1+\frac{2}{p})q} 2^{\frac{1}{p}j}$ and taking the summation in q, j , we obtain

$$(9.4.11) \quad \begin{aligned} & \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\tilde{L}^\infty(0, T; \mathcal{B}_p^0)} + \gamma \int_0^T m(\tau) \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{\mathcal{B}_p^0} d\tau + c_0 \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & \leq \|\Lambda_h^{-1} \partial_3 \bar{u}_0^h\|_{\mathcal{B}_p^0} + \int_0^T \|\Lambda_h^{-1} \partial_3 (\bar{u}^h \cdot \nabla_h \bar{u}^h)_\gamma\|_{\mathcal{B}_p^0} + \|\Lambda_h^{-1} \partial_3 (\bar{b}^h \cdot \nabla_h \bar{b}^h)_\gamma\|_{\mathcal{B}_p^0} d\tau \\ & \quad + \int_0^T \|\Lambda_h^{-1} \partial_3 \nabla_h \bar{p}_\gamma\|_{\mathcal{B}_p^0} d\tau \end{aligned}$$

and

$$(9.4.12) \quad \begin{aligned} & \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{\tilde{L}^\infty(0, T; \mathcal{B}_p^0)} + \gamma \int_0^T m(\tau) \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{\mathcal{B}_p^0} d\tau + c_0 \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{L^1(0, T; \mathcal{B}_p^2)} \\ & \leq \|\Lambda_h^{-1} \partial_3 \bar{b}_0^h\|_{\mathcal{B}_p^0} + \int_0^T \|\Lambda_h^{-1} \partial_3 (\bar{u}^h \cdot \nabla_h \bar{b}^h)_\gamma\|_{\mathcal{B}_p^0} + \|\Lambda_h^{-1} \partial_3 (\bar{b}^h \cdot \nabla_h \bar{u}^h)_\gamma\|_{\mathcal{B}_p^0} d\tau. \end{aligned}$$

According to the formulation of the pressure \bar{p} in (9.4.7), and using the boundedness of Riesz transform in L^p , we find

$$\begin{aligned} \|\Lambda_h^{-1} \partial_3 \nabla_h \bar{p}_\gamma\|_{\mathcal{B}_p^0} & \leq \|\Lambda_h^{-1} \partial_3 \operatorname{div}_h (\bar{u}^h \otimes \bar{u}^h - \bar{b}^h \otimes \bar{b}^h)_\gamma\|_{\mathcal{B}_p^0} \\ & \lesssim \|\bar{u}^h \otimes \partial_3 \bar{u}_\gamma^h\|_{\mathcal{B}_p^0} + \|\bar{b}^h \otimes \partial_3 \bar{b}_\gamma^h\|_{\mathcal{B}_p^0} \\ & \lesssim \|\bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|\partial_3 \bar{u}_\gamma^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|\bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|\partial_3 \bar{b}_\gamma^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}}. \end{aligned}$$

By Lemma 9.2.4, we have

$$\|\bar{b}^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}} \lesssim \|\bar{b}^h\|_{B_p^0}^{\frac{1}{2}} \|\bar{b}^h\|_{B_p^2}^{\frac{1}{2}}$$

and

$$\begin{aligned} \|\partial_3 \bar{b}_\gamma^h\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}} &\lesssim \|\partial_3 \bar{b}_\gamma^h\|_{B_p^{-2+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \|\partial_3 \bar{b}_\gamma^h\|_{B_p^{\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \\ &\lesssim \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^{-1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^{1+\frac{2}{p}, \frac{1}{p}}}^{\frac{1}{2}} \\ &\lesssim \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^0}^{\frac{1}{2}} \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^2}^{\frac{1}{2}}. \end{aligned}$$

Using Young's inequality we conclude that

$$\begin{aligned} \|\Lambda_h^{-1} \partial_3 \nabla_h \bar{p}_\gamma\|_{B_p^0} &\lesssim \|\bar{u}^h\|_{B_p^0}^{\frac{1}{2}} \|\bar{u}^h\|_{B_p^2}^{\frac{1}{2}} \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{B_p^0}^{\frac{1}{2}} \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{B_p^2}^{\frac{1}{2}} \\ &\quad + \|\bar{b}^h\|_{B_p^0}^{\frac{1}{2}} \|\bar{b}^h\|_{B_p^2}^{\frac{1}{2}} \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^0}^{\frac{1}{2}} \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^2}^{\frac{1}{2}} \\ &\lesssim \frac{1}{4C} \|\bar{u}^h\|_{B_p^0} \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{B_p^2} + \|\bar{u}^h\|_{B_p^2} \|\Lambda_h^{-1} \partial_3 \bar{u}_\gamma^h\|_{B_p^0} \\ &\quad + \frac{1}{4C'} \|\bar{b}^h\|_{B_p^0} \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^2} + \|\bar{b}^h\|_{B_p^2} \|\Lambda_h^{-1} \partial_3 \bar{b}_\gamma^h\|_{B_p^0}. \end{aligned}$$

Combining this result with the estimates (9.4.11) and (9.4.12), choosing γ large enough (say $\gamma > 16C$), we can obtain

$$\|\Lambda_h^{-1} \partial_3(\bar{u}_\gamma^h, \bar{b}_\gamma^h)\|_{\tilde{L}^\infty(0, T; B_p^0)} + c_0 \|\Lambda_h^{-1} \partial_3(\bar{u}_\gamma^h, \bar{b}_\gamma^h)\|_{L^1(0, T; B_p^2)} \leq \|\Lambda_h^{-1} \partial_3(\bar{u}_0^h, \bar{b}_0^h)\|_{B_p^0}.$$

Then according to the definition of \bar{u}_γ^h and \bar{b}_γ^h , we can assert that

$$\begin{aligned} &\|\Lambda_h^{-1} \partial_3(\bar{u}^h, \bar{b}^h)\|_{\tilde{L}^\infty(0, T; B_p^0)} + c_0 \|\Lambda_h^{-1} \partial_3(\bar{u}^h, \bar{b}^h)\|_{L^1(0, T; B_p^2)} \\ &\leq \|\Lambda_h^{-1} \partial_3(\bar{u}_0^h, \bar{b}_0^h)\|_{B_p^0} \exp\left(C \int_0^T \|(\bar{u}^h, \bar{b}^h)\|_{B_p^2} d\tau\right) \\ &\leq C \|\Lambda_h^{-1} \partial_3(\bar{u}_0^h, \bar{b}_0^h)\|_{B_p^0} \exp\left(C \left(\|(\bar{u}_0^h, \bar{b}_0^h)\|_{B_p^0} + \|(\bar{u}_0^h, \bar{b}_0^h)\|_{H^1} \exp(C \|\nabla \bar{u}_0^h, \nabla \bar{b}_0^h\|_{L^2}^2)\right)\right), \end{aligned}$$

which completes the proof of the Proposition 9.3.1.

9.5 Proof of Proposition 9.3.2

In this section we will give the proof of Proposition 9.3.2 which shows the global estimate for the horizontal component (v^h, c^h) of the system (MHD).

We start by giving the proof of the second estimate satisfied by c_μ^h in (9.3.10), for that we need to use the function c_μ^h which verifies

$$(9.5.1) \quad \partial_t c_\mu^h + \mu c_\mu^h - \Delta c_\mu^h = G_\mu,$$

with

$$G_\mu = (c \cdot \nabla v^h + c \cdot \nabla \bar{u}^h + \bar{b}^h \cdot \nabla_h v^h - v \cdot \nabla c^h - \bar{u}^h \cdot \nabla_h c^h - v \cdot \nabla \bar{b}^h)_\mu.$$

Applying the dyadic operator $\Delta_q^h \Delta_j^v$ to (9.5.1), and taking the L^p scalar product, we deduce from the

Lemma 9.2.1 that there exists some positive constant c_0 ,

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|\Delta_q^h \Delta_j^v c_\mu^h(t)\|_{L^p}^p + \mu h(t) \|\Delta_q^h \Delta_j^v c_\mu^h\|_{L^p}^p + c_0(2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v c_\mu^h\|_{L^p}^p \\ & \leq \|\Delta_q^h \Delta_j^v G_\mu^h\|_{L^p} \|\Delta_q^h \Delta_j^v c_\mu^h\|_{L^p}^{p-1}. \end{aligned}$$

Integrating from 0 to t in time, multiplying by $2^{(-1+\frac{2}{p})q} 2^{\frac{1}{p}j}$ and then taking the summation in q, j , we obtain

$$(9.5.2) \quad \|c_\mu^h(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \mu \int_0^t h(\tau) \|c_\mu^h\|_{\mathcal{B}_p^0} d\tau + c_0 \|c_\mu^h\|_{L^1(0,t;\mathcal{B}_p^2)} \leq \|c^h(0)\|_{\mathcal{B}_p^0} + \int_0^t \|G_\mu(\tau)\|_{\mathcal{B}_p^0} d\tau.$$

According to the definition of the function G_μ , we have

$$(9.5.3) \quad \begin{aligned} \|G_\mu\|_{\mathcal{B}_p^0} & \leq \|(v \cdot \nabla c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c \cdot \nabla v^h)_\mu\|_{\mathcal{B}_p^0} + \|(\bar{u}^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} \\ & \quad + \|(\bar{b}^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} + \|(v \cdot \nabla \bar{b}^h)_\mu\|_{\mathcal{B}_p^0} + \|(c \cdot \nabla \bar{u}^h)_\mu\|_{\mathcal{B}_p^0} \\ & = G_1 + G_2 + G_3 + G_4, \end{aligned}$$

where

$$\begin{aligned} G_1 & \triangleq \|(v \cdot \nabla c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c \cdot \nabla v^h)_\mu\|_{\mathcal{B}_p^0}, \\ G_2 & \triangleq \|(\bar{u}^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0}, \\ G_3 & \triangleq \|(v \cdot \nabla \bar{b}^h)_\mu\|_{\mathcal{B}_p^0} \quad \text{and} \quad G_4 \triangleq \|(c \cdot \nabla \bar{u}^h)_\mu\|_{\mathcal{B}_p^0}. \end{aligned}$$

G_1 can be estimated by,

$$\begin{aligned} G_1 & = \|(v \cdot \nabla c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c \cdot \nabla v^h)_\mu\|_{\mathcal{B}_p^0} \\ & \leq \|(v^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} + \|(v^3 \partial_3 c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^3 \partial_3 v^h)_\mu\|_{\mathcal{B}_p^0} \\ & \leq G_{1,1} + G_{1,2}. \end{aligned}$$

where

$$G_{1,1} = \|(v^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} \quad \text{and} \quad G_{1,2} = \|(v^3 \partial_3 c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^3 \partial_3 v^h)_\mu\|_{\mathcal{B}_p^0}.$$

According to the product law (9.2.5) and Lemma 9.2.4, we have

$$\begin{aligned} G_{1,1} & = \|(v^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} \\ & \leq \|v^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} \|\nabla_h c_\mu^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} + \|c^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} \|\nabla_h v_\mu^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \\ & \leq \|v^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}}} + \|c^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^{1+\frac{2}{p}, \frac{1}{p}}} \\ & \leq \|v^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} + \|c^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2}, \end{aligned}$$

and

$$\begin{aligned} G_{1,2} & = \|(v^3 \partial_3 c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^3 \partial_3 v^h)_\mu\|_{\mathcal{B}_p^0} \\ & \leq \|v^3\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|\partial_3 c_\mu^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|c^3\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|\partial_3 v_\mu^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} \\ & \leq \|v^3\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|c_\mu^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}}} + \|c^3\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \|v_\mu^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, 1+\frac{1}{p}}} \\ & \leq \|v^3\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|v^3\|_{\mathcal{B}_p^2}^{\frac{1}{2}} \|c_\mu^h\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|c_\mu^h\|_{\mathcal{B}_p^2}^{\frac{1}{2}} + \|c^3\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|c^3\|_{\mathcal{B}_p^2}^{\frac{1}{2}} \|v_\mu^h\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|v_\mu^h\|_{\mathcal{B}_p^2}^{\frac{1}{2}} \end{aligned}$$

$$\leq \epsilon(\|v^3\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2} + \|c^3\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2}) + C(\|v^3\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0} + \|c^3\|_{\mathcal{B}_p^2}\|v_\mu^h\|_{\mathcal{B}_p^0}).$$

Then by summing the resulting estimates of $G_{1,1}$ and $G_{1,2}$ and integrating over time, we obtain

$$(9.5.4) \quad \int_0^t G_1(\tau)d\tau \leq C \int_0^t \|v^h\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|v^3\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0}d\tau \\ + C \int_0^t \|c^3\|_{\mathcal{B}_p^2}\|v_\mu^h\|_{\mathcal{B}_p^0}d\tau + \epsilon \int_0^t \|v^3\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2}d\tau + \epsilon \int_0^t \|c^3\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2}d\tau,$$

where we have used Young's inequality with a positive constant $\epsilon > 0$ suitable small.

The same conclusion can be drawn for G_2 ,

$$(9.5.5) \quad \int_0^t G_2(\tau)d\tau = \int_0^t \|(\bar{u}^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0}d\tau \\ \leq \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0}d\tau \\ + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2}\|v_\mu^h\|_{\mathcal{B}_p^0}d\tau.$$

For G_3 , similarly we have

$$G_3 = \|(v \cdot \nabla \bar{b}^h)_\mu\|_{\mathcal{B}_p^0} \\ \leq \|(v^h \cdot \nabla_h \bar{b}^h)_\mu\|_{\mathcal{B}_p^0} + \|(v^3 \cdot \partial_3 \bar{b}^h)_\mu\|_{\mathcal{B}_p^0} \\ \leq \|v_\mu^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}}\|\nabla_h \bar{b}^h\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}} + \|v_\mu^3\|_{\mathcal{B}_p^{-1+\frac{2}{p}, \frac{1}{p}}}\|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \\ \leq \|v_\mu^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}}\|\bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} + \|v_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \\ \leq \|v_\mu^h\|_{\mathcal{B}_p^0}^{\frac{1}{2}}\|v_\mu^h\|_{\mathcal{B}_p^2}^{\frac{1}{2}}\|\bar{b}^h\|_{\mathcal{B}_p^0}^{\frac{1}{2}}\|\bar{b}^h\|_{\mathcal{B}_p^2}^{\frac{1}{2}} + \|v_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} \\ \leq \epsilon\|v_\mu^h\|_{\mathcal{B}_p^2}\|\bar{b}^h\|_{\mathcal{B}_p^0} + C\|v_\mu^h\|_{\mathcal{B}_p^0}\|\bar{b}^h\|_{\mathcal{B}_p^2} + \|v_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}}.$$

Then integrating from 0 to t with respect to time variable,

$$(9.5.6) \quad \int_0^t G_3(\tau)d\tau \leq \epsilon \int_0^t \|v_\mu^h\|_{\mathcal{B}_p^2}\|\bar{b}^h\|_{\mathcal{B}_p^0}d\tau + C \int_0^t \|v_\mu^h\|_{\mathcal{B}_p^0}\|\bar{b}^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|v_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}}d\tau.$$

In the same way we have

$$(9.5.7) \quad \int_0^t G_4(\tau)d\tau \leq \epsilon \int_0^t \|c_\mu^h\|_{\mathcal{B}_p^2}\|\bar{u}^h\|_{\mathcal{B}_p^0}d\tau + C \int_0^t \|c_\mu^h\|_{\mathcal{B}_p^0}\|\bar{u}^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|c_\mu^3\|_{\mathcal{B}_p^0}\|\bar{u}^h\|_{\mathcal{B}_p^2}d\tau.$$

To conclude, we integrate (9.5.3) from 0 to t in time and then combining with the above estimates (9.5.4), (9.5.5), (9.5.6) and (9.5.7), we obtain

$$(9.5.8) \quad \int_0^t \|G_\mu(\tau)\|_{\mathcal{B}_p^0}d\tau \leq C \int_0^t \|v^h\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|v^3\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0}d\tau \\ + C \int_0^t \|c^3\|_{\mathcal{B}_p^2}\|v_\mu^h\|_{\mathcal{B}_p^0}d\tau + \epsilon \int_0^t \|v^3\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2}d\tau + \epsilon \int_0^t \|c^3\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2}d\tau \\ + \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2}d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0}d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2}d\tau$$

$$+ C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|v_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau + C \int_0^t \|c_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau.$$

Combining the estimate (9.5.8) with (9.5.2) we establish the inequality (9.3.10) of the Proposition 9.3.2. Now we show the first estimate (9.3.9) of Proposition 9.3.2, which we can deduce from Systems (9.3.5) and (9.3.8) that v_μ^h verifies the following equation,

$$(9.5.9) \quad \partial_t v_\mu^h + \mu v_\mu^h - \Delta v_\mu^h = L_\mu,$$

with

$$L_\mu = -(v \cdot \nabla v^h + v \cdot \nabla \bar{u}^h + \bar{u}^h \cdot \nabla_h v^h - c \cdot \nabla c^h - \bar{b}^h \cdot \nabla_h c^h - v \cdot \nabla \bar{b}^h)_\mu - \nabla_h q_\mu.$$

Applying the dyadic operator $\Delta_q^h \Delta_j^v$ to (9.5.9), and taking the L^p estimate, we deduce from Lemma 9.2.1 that, for some positive constant $c_0 > 0$,

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\Delta_q^h \Delta_j^v v_\mu^h(t)\|_{L^p}^p + \mu h(t) \|\Delta_q^h \Delta_j^v v_\mu^h(t)\|_{L^p}^p + c_0(2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v v_\mu^h(t)\|_{L^p}^p \\ \leq \|\Delta_q^h \Delta_j^v L_\mu^h(t)\|_{L^p} \|\Delta_q^h \Delta_j^v v_\mu^h(t)\|_{L^p}^{p-1}. \end{aligned}$$

Simplify by $\|\Delta_q^h \Delta_j^v v_\mu^h(t)\|_{L^p}^{p-1}$ in our equation, and integrating from 0 to t in time, multiplying by $2^{(-1+\frac{2}{p})q} 2^{\frac{1}{p}j}$ and then taking the summation in q, j , we obtain

$$(9.5.10) \quad \|v_\mu^h(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \mu \int_0^t h(\tau) \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau + c_0 \|v_\mu^h\|_{L^1(0,t;\mathcal{B}_p^0)} \leq \|v_0^h\|_{\mathcal{B}_p^0} + \int_0^t \|L_\mu(\tau)\|_{\mathcal{B}_p^0} d\tau.$$

According to the definition of the function L_μ , we have

$$(9.5.11) \quad \begin{aligned} \|L_\mu\|_{\mathcal{B}_p^0} &\leq \|(v \cdot \nabla v^h)_\mu\|_{\mathcal{B}_p^0} + \|(v \cdot \nabla \bar{u}^h)_\mu\|_{\mathcal{B}_p^0} + \|(\bar{u}^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} \\ &\quad + \|(c \cdot \nabla c^h)_\mu\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} + \|(c \cdot \nabla \bar{b}^h)_\mu\|_{\mathcal{B}_p^0} + \|\nabla_h q_\mu\|_{\mathcal{B}_p^0} \\ &= L_1 + L_2 + L_3 + L_4 + L_5, \end{aligned}$$

where

$$\begin{aligned} L_1 &\triangleq \|(v \cdot \nabla v^h)_\mu\|_{\mathcal{B}_p^0} + \|(c \cdot \nabla c^h)_\mu\|_{\mathcal{B}_p^0}, \\ L_2 &\triangleq \|(\bar{u}^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0}, \\ L_3 &\triangleq \|(v \cdot \nabla \bar{u}^h)_\mu\|_{\mathcal{B}_p^0}, \quad L_4 \triangleq \|(c \cdot \nabla \bar{b}^h)_\mu\|_{\mathcal{B}_p^0} \quad \text{and} \quad L_5 \triangleq \|\nabla_h q_\mu\|_{\mathcal{B}_p^0}. \end{aligned}$$

We start first to give the estimate of L_1 ,

$$\begin{aligned} L_1 &\triangleq \|(v \cdot \nabla v^h)_\mu\|_{\mathcal{B}_p^0} + \|(c \cdot \nabla c^h)_\mu\|_{\mathcal{B}_p^0} \\ &\leq \|(v^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} + \|(v^3 \partial_3 v^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^3 \partial_3 c^h)_\mu\|_{\mathcal{B}_p^0} \\ &\triangleq L_{1,1} + L_{1,2}, \end{aligned}$$

where

$$L_{1,1} = \|(v^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0}$$

and

$$L_{1,2} = \|(v^3 \partial_3 v^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^3 \partial_3 c^h)_\mu\|_{\mathcal{B}_p^0}.$$

Similar as the estimate of $G_{1,1}$ and $G_{1,2}$, we can deduce that

$$L_{1,1} \leq \|v^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} + \|c^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2},$$

and

$$L_{1,2} \leq \epsilon (\|v^3\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} + \|c^3\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2}) + C (\|v^3\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} + \|c^3\|_{\mathcal{B}_p^2} \|c_\mu^h\|_{\mathcal{B}_p^0}).$$

Then by summing the resulting estimates of $L_{1,1}$ and $L_{1,2}$ and integrating over time, we obtain the estimate of $\int_0^t L_1(\tau) d\tau$ that:

$$(9.5.12) \quad \int_0^t L_1(\tau) d\tau \leq C \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|v^3\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau \\ + C \int_0^t \|c^3\|_{\mathcal{B}_p^2} \|c_\mu^h\|_{\mathcal{B}_p^0} d\tau + \epsilon \int_0^t \|v^3\|_{\mathcal{B}_p^0} \|v_\mu^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^3\|_{\mathcal{B}_p^0} \|c_\mu^h\|_{\mathcal{B}_p^2} d\tau,$$

where in the last step we used Young's inequality with an arbitrary positive constant $\epsilon > 0$ which will be chosen suitable small.

Along the same way, we can bound L_2 as

$$(9.5.13) \quad \int_0^t L_2(\tau) d\tau = \int_0^t \|(\bar{u}^h \cdot \nabla_h v^h)_\mu\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h c^h)_\mu\|_{\mathcal{B}_p^0} d\tau \\ \leq \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|v_\mu^h\|_{\mathcal{B}_p^0} d\tau \\ + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|c_\mu^h\|_{\mathcal{B}_p^0} d\tau.$$

According to the result obtained in (9.5.6) and (9.5.7), we can deduce that

$$(9.5.14) \quad \int_0^t L_3(\tau) d\tau \leq \epsilon \int_0^t \|v_\mu^h\|_{\mathcal{B}_p^2} \|\bar{u}^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|v_\mu^h\|_{\mathcal{B}_p^0} \|\bar{u}^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|v_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau,$$

and

$$(9.5.15) \quad \int_0^t L_4(\tau) d\tau \leq \epsilon \int_0^t \|c_\mu^h\|_{\mathcal{B}_p^2} \|\bar{b}^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c_\mu^h\|_{\mathcal{B}_p^0} \|\bar{b}^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c_\mu^3\|_{\mathcal{B}_p^0} \|\partial_3 \bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau.$$

To conclude, it remains to estimate the last term L_5 . According to the definition of q in (9.3.7)

$$(9.5.16) \quad \nabla_h q_\mu = \nabla_h (-\Delta)^{-1} \operatorname{div}_h (v^h \cdot \nabla_h v^h + v^3 \partial_3 v^h + \bar{u}^h \cdot \nabla_h v^h + v^h \cdot \nabla_h \bar{u}^h + v^3 \partial_3 \bar{u}^h)_\mu \\ - \nabla_h (-\Delta)^{-1} \operatorname{div}_h (c^h \cdot \nabla_h c^h + c^3 \partial_3 c^h + \bar{b}^h \cdot \nabla_h c^h + c^h \cdot \nabla_h \bar{b}^h + c^3 \partial_3 \bar{b}^h)_\mu \\ + \nabla_h (-\Delta)^{-1} \partial_3 (v^h \cdot \nabla_h v^3 - v^3 \operatorname{div}_h v^h + \bar{u}^h \cdot \nabla_h v^3 + \partial_3 \bar{p}) \\ - c^h \cdot \nabla_h c^3 + c^3 \operatorname{div}_h c^h - \bar{b}^h \cdot \nabla_h c^3)_\mu.$$

Because of the boundedness of Riesz transforms, we only need to estimate the last two lines of the right hand side in (9.5.16) in \mathcal{B}_p^0 . By Lemma 9.2.3 and Lemma 9.2.4, we have

$$\|\nabla_h (-\Delta)^{-1} \partial_3 (v^h \cdot \nabla_h v^3 - c^h \cdot \nabla_h c^3)_\mu\|_{\mathcal{B}_p^0} \leq \|(v^h \cdot \nabla_h v^3)_\mu\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h c^3)_\mu\|_{\mathcal{B}_p^0} \\ \lesssim \|v_\mu^h\|_{\mathcal{B}_p^0} \|v^3\|_{\mathcal{B}_p^2} + \|c_\mu^h\|_{\mathcal{B}_p^0} \|c^3\|_{\mathcal{B}_p^2}$$

and

$$\begin{aligned} \|\nabla_h(-\Delta)^{-1}\partial_3(v^3\operatorname{div}_h v^h - c^3\operatorname{div}_h c^h)_\mu\|_{\mathcal{B}_p^0} &\lesssim \|(v^3\operatorname{div}_h v^h)_\mu\|_{\mathcal{B}_p^0} + \|(c^3\operatorname{div}_h c^h)_\mu\|_{\mathcal{B}_p^0} \\ &\lesssim \epsilon(\|v^3\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2} + \|c^3\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2}) + C(\|v^3\|_{\mathcal{B}_p^2}\|v_\mu^h\|_{\mathcal{B}_p^0} + \|c^3\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0}). \end{aligned}$$

Then we still have to show the boundedness of the terms $\nabla_h(-\Delta)^{-1}\partial_3(\bar{u}^h \cdot \nabla_h v^3)_\mu$ and $\nabla_h(-\Delta)^{-1}\partial_3(\bar{b}^h \cdot \nabla_h c^3)_\mu$ and $\partial_3\bar{p}$. Due to the divergence-free condition of \bar{u}^h , \bar{b}^h , v and c , we find

$$\begin{aligned} \nabla_h(-\Delta)^{-1}\partial_3(\bar{u}^h \cdot \nabla_h v^3)_\mu &= \nabla_h(-\Delta)^{-1}\operatorname{div}_h(\partial_3\bar{u}^h v^3 + \bar{u}^h \partial_3 v^3)_\mu \\ &= \nabla_h(-\Delta)^{-1}\operatorname{div}_h(\partial_3\bar{u}^h v^3 - \bar{u}^h \operatorname{div}_h v^3)_\mu, \end{aligned}$$

and

$$\begin{aligned} \nabla_h(-\Delta)^{-1}\partial_3(\bar{b}^h \cdot \nabla_h c^3)_\mu &= \nabla_h(-\Delta)^{-1}\operatorname{div}_h(\partial_3\bar{b}^h c^3 + \bar{b}^h \partial_3 c^3)_\mu \\ &= \nabla_h(-\Delta)^{-1}\operatorname{div}_h(\partial_3\bar{b}^h c^3 - \bar{b}^h \operatorname{div}_h c^3)_\mu. \end{aligned}$$

Along the same way, we have

$$\begin{aligned} \nabla_h(-\Delta)^{-1}\partial_3(\bar{u}^h \cdot \nabla_h v^3 - \bar{b}^h \cdot \nabla_h c^3)_\mu &\lesssim \epsilon(\|\bar{u}^h\|_{\mathcal{B}_p^0}\|v_\mu^3\|_{\mathcal{B}_p^2} + \|\bar{b}^h\|_{\mathcal{B}_p^0}\|c_\mu^3\|_{\mathcal{B}_p^2}) \\ &\lesssim C(\|\bar{u}^h\|_{\mathcal{B}_p^2}\|v_\mu^3\|_{\mathcal{B}_p^0} + \|\bar{b}^h\|_{\mathcal{B}_p^2}\|c_\mu^3\|_{\mathcal{B}_p^0} + \|v_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3\bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} + \|c_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3\bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}}). \end{aligned}$$

For $\partial_3\bar{p}$, recall that

$$\bar{p} = (-\Delta_h)^{-1}\operatorname{div}_h\operatorname{div}_h(\bar{u}^h \otimes \bar{u}^h - \bar{b}^h \otimes \bar{b}^h).$$

We can bound it by

$$\|\partial_3\bar{p}_\mu\|_{\mathcal{B}_p^0} \lesssim \|\bar{u}_\mu^h\|_{\mathcal{B}_p^0}\|\partial_3\bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} + \|\bar{b}_\mu^h\|_{\mathcal{B}_p^0}\|\partial_3\bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}}.$$

To conclude, we integrate (9.5.11) from 0 to t in time and then combining with the above estimates (9.5.12), (9.5.13), (9.5.14) and (9.5.15), we obtain

$$\begin{aligned} \int_0^t \|L_\mu\|_{\mathcal{B}_p^0} &\leq \epsilon \int_0^t \|v^3\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^3\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2} d\tau \\ &\quad + C \int_0^t \|v^h\|_{\mathcal{B}_p^0}\|v_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|v_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3\bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau \\ &\quad + C \int_0^t \|v^3\|_{\mathcal{B}_p^2}\|v_\mu^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c^3\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2}\|v_\mu^h\|_{\mathcal{B}_p^0} d\tau \\ &\quad + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2}\|c_\mu^h\|_{\mathcal{B}_p^0} d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0}\|c_\mu^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{u}_\mu^h\|_{\mathcal{B}_p^0}\|\partial_3\bar{u}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau \\ (9.5.17) \quad &\quad + C \int_0^t \|\bar{b}_\mu^h\|_{\mathcal{B}_p^0}\|\partial_3\bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau + C \int_0^t \|c_\mu^3\|_{\mathcal{B}_p^0}\|\partial_3\bar{b}^h\|_{\mathcal{B}_p^{\frac{2}{p}, \frac{1}{p}}} d\tau. \end{aligned}$$

Combining the estimate (9.5.17) with (9.5.10) we establishing the second inequality (9.3.9), which completes the proof of Proposition 9.3.2.

9.6 Proof of Proposition 9.3.3

In this section, we will give the proof of Proposition 9.3.2 which shows the global estimate for the vertical component (v^3, c^3) of system (MHD).

We first start by giving the proof of the second estimate (9.3.13) in Proposition 9.3.3. If we replace c^3 by c_λ^3 in (9.3.5), we deduce that c_λ^3 verifies the following equation,

$$(9.6.1) \quad \partial_t c_\lambda^3 + \mu c_\lambda^3 - \Delta c_\lambda^3 = M_\lambda,$$

with

$$M_\lambda = -(v^h \cdot \nabla_h c^3 - v^3 \operatorname{div}_h c^h + \bar{u}^h \cdot \nabla_h c^3 - c^h \cdot \nabla_h v^3 + c^3 \operatorname{div}_h v^h - \bar{b}^h \cdot \nabla_h v^3)_\lambda.$$

Applying the dyadic operator $\Delta_q^h \Delta_j^v$ to (9.6.1), and taking the L^p estimate, we deduce from Lemma 9.2.1 that for some positive constant c_0 ,

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\Delta_q^h \Delta_j^v c_\lambda^3(t)\|_{L^p}^p + \mu h(t) \|\Delta_q^h \Delta_j^v c_\lambda^3\|_{L^p}^p + c_0(2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v c_\lambda^3\|_{L^p}^p \\ \leq \|\Delta_q^h \Delta_j^v M_\lambda\|_{L^p} \|\Delta_q^h \Delta_j^v c_\lambda^3\|_{L^p}^{p-1}. \end{aligned}$$

Simplify by $\|\Delta_q^h \Delta_j^v c_\lambda^3(t)\|_{L^p}^{p-1}$, and integrating from 0 to t in time, multiplying by $2^{(-1+\frac{2}{p})q} 2^{\frac{1}{p}j}$ and then taking the summation in q, j , we obtain

$$(9.6.2) \quad \|c_\lambda^3(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \mu \int_0^t h(\tau) \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + c_0 \|c_\lambda^3\|_{L^1(0,t;\mathcal{B}_p^2)} \leq \|c_0^3\|_{\mathcal{B}_p^0} + \int_0^t \|M_\lambda(\tau)\|_{\mathcal{B}_p^0} d\tau.$$

According to the definition of the function M_λ , we have

$$(9.6.3) \quad \begin{aligned} \|M_\lambda\|_{\mathcal{B}_p^0} &= \|(v^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} + \|(v^3 \operatorname{div}_h c^h)_\lambda\|_{\mathcal{B}_p^0} + \|(\bar{u}^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} \\ &\quad + \|(c^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} + \|(c^3 \operatorname{div}_h v^h)_\lambda\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} \\ &= M_1 + M_2 + M_3 + M_4, \end{aligned}$$

where

$$\begin{aligned} M_1 &= \|(v^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0}, \\ M_2 &= \|(v^3 \operatorname{div}_h c^h)_\lambda\|_{\mathcal{B}_p^0} + \|(c^3 \operatorname{div}_h v^h)_\lambda\|_{\mathcal{B}_p^0}, \\ M_3 &= \|(\bar{u}^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} \quad \text{and} \quad M_4 = \|(\bar{b}^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0}. \end{aligned}$$

According to Lemma 9.2.3 and Lemma 9.2.4, one can deduce

$$\begin{aligned} M_1 &= \|(v^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} \\ &\leq \|v^h\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|v^h\|_{\mathcal{B}_p^2}^{\frac{1}{2}} \|c_\lambda^3\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|c_\lambda^3\|_{\mathcal{B}_p^2}^{\frac{1}{2}} + \|c^h\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|c^h\|_{\mathcal{B}_p^2}^{\frac{1}{2}} \|v_\lambda^3\|_{\mathcal{B}_p^0}^{\frac{1}{2}} \|v_\lambda^3\|_{\mathcal{B}_p^2}^{\frac{1}{2}}. \end{aligned}$$

Using Young's inequality and integrating from 0 to t in time we obtain the estimate of M_1 that

$$(9.6.4) \quad \begin{aligned} \int_0^t M_1(\tau) d\tau &\leq \epsilon \left(\int_0^t \|v^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau \right) \\ &\quad + C \left(\int_0^t \|v^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + \int_0^t \|c^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau \right), \end{aligned}$$

with ϵ is an arbitrary positive constant which will be chosen suitable small.

The same conclusion can be drawn for M_2 :

$$\int_0^t M_2(\tau) d\tau = \int_0^t \|(v^3 \operatorname{div}_h c^h)_\lambda\|_{\mathcal{B}_p^0} + \|(c^3 \operatorname{div}_h v^h)_\lambda\|_{\mathcal{B}_p^0} d\tau$$

$$(9.6.5) \quad \leq C \int_0^t \|v_\lambda^3\|_{\mathcal{B}_p^0} \|c^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c_\lambda^3\|_{\mathcal{B}_p^0} \|v^h\|_{\mathcal{B}_p^2} d\tau.$$

Also to get the bound of M_3 and M_4 in the same way,

$$(9.6.6) \quad \int_0^t M_3(\tau) d\tau \leq \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau,$$

and

$$(9.6.7) \quad \int_0^t M_4(\tau) d\tau \leq \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau.$$

To conclude, we integrate (9.6.3) from 0 to t in time and then combining with the above Estimates (9.6.4), (9.6.5), (9.6.6) and (9.6.7), we obtain

$$(9.6.8) \quad \begin{aligned} \int_0^t \|M_\lambda(\tau)\|_{\mathcal{B}_p^0} d\tau &\leq \epsilon \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau \\ &+ C \int_0^t \|v^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau + \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau \\ &+ C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau. \end{aligned}$$

Combining Estimates (9.6.8) with (9.6.2) we finish the proof of the second inequality (9.3.13) of Proposition 9.3.3.

Finally, we investigate the first estimate (9.3.12) of Proposition 9.3.3, we can deduce from Systems (9.3.5) and (9.3.11) that v_λ^3 verifies the following equation,

$$(9.6.9) \quad \partial_t v_\lambda^3 + \lambda v_\lambda^3 - \Delta v_\lambda^3 = K_\lambda,$$

with

$$L_\lambda = -(v^h \cdot \nabla_h v^3 - v^3 \operatorname{div}_h v^h + \bar{u}^h \cdot \nabla_h v^3 - c^h \cdot \nabla_h c^3 - c^3 \operatorname{div}_h c^h + \bar{b}^h \cdot \nabla_h c^3)_\lambda - \partial_3(q - \bar{p})_\lambda.$$

Applying the dyadic operator $\Delta_q^h \Delta_j^v$ to (9.6.9), and taking the L^p estimate, from Lemma 9.2.1 we have for some positive constant $c_0 > 0$,

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\Delta_q^h \Delta_j^v v_\lambda^3(t)\|_{L^p}^p + \mu h(t) \|\Delta_q^h \Delta_j^v v_\lambda^3\|_{L^p}^p + c_0(2^{2q} + 2^{2j}) \|\Delta_q^h \Delta_j^v v_\lambda^3\|_{L^p}^p \\ \leq \|\Delta_q^h \Delta_j^v K_\lambda(t)\|_{L^p} \|\Delta_q^h \Delta_j^v v_\lambda^3\|_{L^p}^{p-1}. \end{aligned}$$

Simplify by $\|\Delta_q^h \Delta_j^v v_\lambda^3(t)\|_{L^p}^{p-1}$, and integrating from 0 to t in time, multiplying by $2^{(-1+\frac{2}{p})q} 2^{\frac{1}{2}j}$ and taking the summation in q, j , we obtain

$$(9.6.10) \quad \|v_\lambda^3(t)\|_{\tilde{L}^\infty(0,t;\mathcal{B}_p^0)} + \mu \int_0^t h(\tau) \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau + c_0 \|v_\lambda^3\|_{L^1(0,t;\mathcal{B}_p^2)} \leq \|v_0^3\|_{\mathcal{B}_p^0} + \int_0^t \|K_\lambda(\tau)\|_{\mathcal{B}_p^0} d\tau.$$

According to the definition of the function K_λ , we have

$$(9.6.11) \quad \begin{aligned} \|K_\lambda\|_{\mathcal{B}_p^0} &= \|(v^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} + \|(v^3 \operatorname{div}_h v^h)_\lambda\|_{\mathcal{B}_p^0} + \|(\bar{u}^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} \\ &+ \|(c^3 \operatorname{div}_h c^h)_\lambda\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} + \|\partial_3 q_\lambda\|_{\mathcal{B}_p^0} + \|\partial_3 \bar{p}_\lambda\|_{\mathcal{B}_p^0} \\ &= K_1 + K_2 + K_3 + K_4 + K_5, \end{aligned}$$

where

$$K_1 = \|(v^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} + \|(c^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0}, \quad K_2 = \|(v^3 \operatorname{div}_h v^h)_\lambda\|_{\mathcal{B}_p^0} + \|(c^3 \operatorname{div}_h c^h)_\lambda\|_{\mathcal{B}_p^0},$$

$$K_3 = \|(\bar{u}^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0}, \quad K_4 = \|\partial_3 \bar{p}_\lambda\|_{\mathcal{B}_p^0} \quad \text{and} \quad K_5 = \|\partial_3 q_\mu\|_{\mathcal{B}_p^0}.$$

Similar to the estimate of M_1 , we have the bounded estimate of K_1 that

$$(9.6.12) \quad \int_0^t K_1(\tau) d\tau \leq \epsilon \left(\int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau \right) \\ + C \left(\int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau + \int_0^t \|c^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau \right),$$

with ϵ is an arbitrary positive constant suitable small.

Along the same way, we can bound the term K_2 by

$$(9.6.13) \quad \int_0^t K_2(\tau) d\tau = \int_0^t \|(v^3 \operatorname{div}_h v^h)_\lambda\|_{\mathcal{B}_p^0} + \|(c^3 \operatorname{div}_h c^h)_\lambda\|_{\mathcal{B}_p^0} d\tau \\ \leq C \int_0^t \|v_\lambda^3\|_{\mathcal{B}_p^0} \|v^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c_\lambda^3\|_{\mathcal{B}_p^0} \|c^h\|_{\mathcal{B}_p^2} d\tau.$$

According to the result obtained in (9.6.6) and (9.6.7), we can deduce that

$$(9.6.14) \quad \int_0^t K_3(\tau) d\tau = \int_0^t \|(\bar{u}^h \cdot \nabla_h v^3)_\lambda\|_{\mathcal{B}_p^0} + \|(\bar{b}^h \cdot \nabla_h c^3)_\lambda\|_{\mathcal{B}_p^0} d\tau \\ \leq \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau \\ + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau.$$

For K_4 , notice that

$$\bar{p} = (-\Delta_h)^{-1} \operatorname{div}_h \operatorname{div}_h (\bar{u}^h \otimes \bar{u}^h - \bar{b}^h \otimes \bar{b}^h),$$

by using the boundness of the Riesz transform and Lemma 9.2.3,

$$(9.6.15) \quad \int_0^t K_4(\tau) d\tau \leq C \int_0^t \|\bar{u}_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{u}_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{b}_\lambda^h\|_{\mathcal{B}_p^2} d\tau.$$

Finally, we estimate the last term K_5 , according to the definition of q in (9.3.7), we have

$$(9.6.16) \quad \partial_3 q_\lambda = \partial_3 (-\Delta)^{-1} \operatorname{div}_h (v^h \cdot \nabla_h v^h + v^3 \partial_3 v^h + \bar{u}^h \cdot \nabla_h v^h + v^h \cdot \nabla_h \bar{u}^h + v^3 \partial_3 \bar{u}^h)_\lambda \\ - \partial_3 (-\Delta)^{-1} \operatorname{div}_h (c^h \cdot \nabla_h c^h + c^3 \partial_3 c^h + \bar{b}^h \cdot \nabla_h c^h + c^h \cdot \nabla_h \bar{b}^h + c^3 \partial_3 \bar{b}^h)_\lambda \\ + \partial_3 (-\Delta)^{-1} \partial_3 (v^h \cdot \nabla_h v^3 - v^3 \operatorname{div}_h v^h + \bar{u}^h \cdot \nabla_h v^3 + \partial_3 \bar{p} \\ - c^h \cdot \nabla_h c^3 + c^3 \operatorname{div}_h c^h - \bar{b}^h \cdot \nabla_h c^3)_\lambda.$$

Because of the boundedness of the Riesz transform, the terms in the last two lines of (9.6.16) can be bounded similarly to K_1 - K_4 , we only need to estimate the first and second line of the right hand side in (9.6.16) in \mathcal{B}_p^0 . By Lemma 9.2.3 and Lemma 9.2.4, we have

$$\int_0^t \|\partial_3 (-\Delta)^{-1} \operatorname{div}_h (v^h \cdot \nabla_h v^h - c^h \cdot \nabla_h c^h)_\lambda\|_{\mathcal{B}_p^0} d\tau$$

$$\begin{aligned}
&\leq C \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\lambda^h\|_{\mathcal{B}_p^2} d\tau, \\
&\int_0^t \|\partial_3(-\Delta)^{-1} \operatorname{div}_h(\bar{u}^h \cdot \nabla_h v^h - \bar{b}^h \cdot \nabla_h c^h)_\lambda\|_{\mathcal{B}_p^0} d\tau \\
&\leq C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c_\lambda^h\|_{\mathcal{B}_p^2} d\tau,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \|\partial_3(-\Delta)^{-1} \operatorname{div}_h(v^h \cdot \nabla_h \bar{u}^h - c^h \cdot \nabla_h \bar{b}^h)_\lambda\|_{\mathcal{B}_p^0} d\tau \\
&\leq C \int_0^t \|v_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{u}^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{b}^h\|_{\mathcal{B}_p^2} d\tau.
\end{aligned}$$

By using Lemma 9.2.3 we obtain

$$\begin{aligned}
&\int_0^t \|\partial_3(-\Delta)^{-1} \operatorname{div}_h(v^3 \partial_3 v^h - c^3 \partial_3 c^h)_\lambda\|_{\mathcal{B}_p^0} d\tau \leq \epsilon \int_0^t \|v_\lambda^3\|_{\mathcal{B}_p^2} \|v^h\|_{\mathcal{B}_p^0} d\tau \\
&\quad + C \int_0^t \|v_\lambda^3\|_{\mathcal{B}_p^0} \|v^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c_\lambda^3\|_{\mathcal{B}_p^2} \|c^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c_\lambda^3\|_{\mathcal{B}_p^0} \|c^h\|_{\mathcal{B}_p^2} d\tau.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^t \|\partial_3(-\Delta)^{-1} \operatorname{div}_h(v^3 \partial_3 \bar{u}^h - c^3 \partial_3 \bar{b}^h)_\lambda\|_{\mathcal{B}_p^0} d\tau \leq \epsilon \int_0^t \|v_\lambda^3\|_{\mathcal{B}_p^2} \|\bar{u}^h\|_{\mathcal{B}_p^0} d\tau \\
&\quad + C \int_0^t \|v_\lambda^3\|_{\mathcal{B}_p^0} \|\bar{u}^h\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c_\lambda^3\|_{\mathcal{B}_p^2} \|\bar{b}^h\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|c_\lambda^3\|_{\mathcal{B}_p^0} \|\bar{b}^h\|_{\mathcal{B}_p^2} d\tau.
\end{aligned}$$

Integrating (9.6.11) from 0 to t in time, and then inserting all the resulting estimates obtained of K_i ($i = 1, \dots, 5$)

$$\begin{aligned}
&\int_0^t \|K_\lambda\|_{\mathcal{B}_p^0} \leq \epsilon \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|v^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau \\
&\quad + C \int_0^t \|c^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + \epsilon \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v_\lambda^3\|_{\mathcal{B}_p^2} d\tau + \epsilon \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c_\lambda^3\|_{\mathcal{B}_p^2} d\tau \\
&\quad + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|v_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|c_\lambda^3\|_{\mathcal{B}_p^0} d\tau + C \int_0^t \|\bar{u}_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{u}^h\|_{\mathcal{B}_p^2} d\tau \\
&\quad + C \int_0^t \|\bar{b}_\lambda^h\|_{\mathcal{B}_p^0} \|\bar{b}^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|v^h\|_{\mathcal{B}_p^0} \|v_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|c^h\|_{\mathcal{B}_p^0} \|c_\lambda^h\|_{\mathcal{B}_p^2} d\tau \\
&\quad + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^0} \|v_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^0} \|c_\lambda^h\|_{\mathcal{B}_p^2} d\tau + C \int_0^t \|\bar{u}^h\|_{\mathcal{B}_p^2} \|v_\lambda^h\|_{\mathcal{B}_p^0} d\tau \\
(9.6.17) \quad &\quad + C \int_0^t \|\bar{b}^h\|_{\mathcal{B}_p^2} \|c_\lambda^h\|_{\mathcal{B}_p^0} d\tau.
\end{aligned}$$

Combining Estimates (9.6.17) with (9.6.10) we obtain Inequality (9.3.12), which completes the proof of Proposition 9.3.3.

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