Some fixed point theorems for single and multivalued mappings and applications

Mohamed Yassin Abdallah

ED SF n^{o} d'ordre :

UNIVERSITÉ CLERMONT AUVERGNE

ÉCOLE DOCTORALE DES SCIENCES FONDAMENTALES

Laboratoire de Mathématiques Blaise Pascal, UMR 6620

PROJET DE THÈSE

présentée pour obtenir le grade de

DOCTEUR D'UNIVERSITÉ

Spécialité : Mathématiques et applications

Par **Mohamed Yassin Abdallah**

Some fixed point theorems for single and multivalued mappings and applications

---

Soutenue publiquement le 16 Décembre 2022 devant le jury composé de :

Directeur :	Khalid LATRACH	Professeur, Université de Clermont Auvergne
Rapporteurs :	Jesús GARCÍA-FALSET	Professeur, Université de Valénsia
	Mohamed AZIZ TAOUDI	Professeur, Université Cadi Ayyad, École Nationale des Sciences Appliquées de Marrakesh
Examinateur :	Véronique BAGLAND	MCF, Université de Clermont Auvergne

Remerciement

Ce n'est pas une chose simple de résumer ici, en quelques lignes, combien ont pu compter pour moi les personnes qui ont participé de près ou de loin à ce doctorat, que ce soit en travaillant avec moi, ou bien en partageant des instants de vie.
Je tiens avant tout à exprimer ma reconnaissance et mes remerciements particuliers à mon directeur, le professeur Khaled Latrach, pour ses excellents conseils, son soutien constant, ses encouragements, ses qualités scientifiques et notamment sa patience dans la direction de ce travail. Je tiens à lui exprimer ma profonde gratitude pour m'avoir accepté en thèse.
Je remercie Mohamed AZIZ-TAOUDI et Jésus GARCIA-FALSET d'avoir eu l'obligeance de rapporter sur cette thèse malgré leurs nombreuses occupations. Vos remarques précieuses m'ont sans doute permis d'améliorer et préciser mon propos. Je voudrais également remercier Véronique BAGLAND d'avoir accepté de participer à mon jury.
Je ne saurais oublier de remercier mon ami et frère Abdul-majeed pour sa coopération permanente et son appui inconditionnel dans une ambiance de fraternité pour la réussite de ce projet.
J'adresse un merci particulier à tonton Ahemed Mohamed ainsi qu'à sa merveilleuse femme Saada Ahemed, à mes amis Bouh, Abdi-chakour pour les moments passés ensemble les dimanches, qui me remplissaient de joie et d'énergie pour le reste de la semaine.
Je remercie aussi à mes collègues de l'université de Djibouti et en particulier à mes collègues et amis de la faculté des sciences. Parmi eux Liban Ismael, Abdirazack Mohamed, Yahyeh lakios, Liban Ohtman, Kosad, Thomas, Yonis...
Mon coeur bat pour ma famille et belle famille, étendue et proche, Un immense merci à mes parents sans qui rien n'aurait été possible. En particulier à ma mère, qui a œuvré pour ma réussite, de par son amour, son soutien, tous les sacrifices consentis et ses précieux conseils, pour toute son assistance et sa présence dans ma vie, reçois à travers ce travail aussi modeste soit-il, l'expression de mes sentiments et de mon éternelle gratitude. J'exprime toute ma gratitude du font du cœur à ma belle-mère pour son amour infini pour moi et mes enfants, soutiens constant, son empressement à faire passer nos besoins avant les siens et son soin le plus attentif m'ont donné le luxe et l'assurance nécessaires pour m'aventurier dans le monde de la recherche en les sachant en sécurité et entourée de tendresse.
A mon brillant frère Mahamoud Yassin et adorable sœur Linda Yassin qui ont su m'encourager. Je tiens également à remercier mes belles soeurs, Deka, Mariamouna, Fozia et en particulier à Dr Djama et Farhan pour leurs soutiens et confiances à mon égard.
Je ne peux finir sans remercier et dédier cette thèse à ma merveilleuse femme Neima Elmi et la mère la plus tendre et la plus attentionnée, a été une compagnon de vie rêvé. Notre histoire n'est pas terminée, et j'attends avec impatience les nombreuses aventures à venir. L'amour un détail qui reste très sensible, celui qui change le parcours de notre vie, on ressent des choses étranges, jamais vue auparavant, des sentiments profondes nous envahissent, il construit un lien très fort entre-deux personnes pour les réunir sous les mêmes ambitions et fonder leur propre foyer. Ton sourire, tes encouragements, tes échanges ma permis te tenir jusqu'au bout. Merci pour tous, merci d'avoir supporté mon absence et d'élever avec amour nos deux adorables enfants Hikma et Yassin.

Contents

Chapter 1

Introduction générale

Depuis les années 1950, la théorie de points fixes est une branche très active des mathématiques et elle est au cœur de l'analyse puisqu'elle fournit un outil puissant pour établir l'existence de solutions à de nombreux problèmes non linéaires posés en théorie des jeux, économie, mécanique, biologie, etc... En fait, dans de nombreux problèmes stationnaires concrets, en général, on transforme le problème en un problème de point fixe et on cherche les solutions comme étant les points fixes du problème en utilisant des hypothèses sur les opérateurs intérvenants dans le problème ou sur la structure de l'espace de Banach. Les théorèmes du points fixes trouvent leurs origines au XIX siécle dans les travaux des grands mathématiciens. Citons à titre d'exemple : Picard, Banach, Brouwer, Schauder, etc...

Dans ce projet de thèse, on présente quelques résultats portant sur les théorèmes de points fixes pour une classe d'applications univoques et multivoques, ensuite on donne des applications pour des problèmes intervenant en neutronique et d'autres portant sur des équations intégrales de type Volterra. Ce travail est composé d'une introduction, cinq chapitres et d'une liste de références bibliographies.

• **Dans le chapitre** 2, nous rappelons les principaux concepts, définitions et résultats qui sont requis tout au long des chapitres suivants. Tout d'abord, on rappelle la définition des mesures de non compacités et, en particulier celle de Kuratowskii et celle de Hausdorff. On introduit aussi la mesure de non faible compacité de De Blasi, que nous utiliserons dans les chapitres 4-6. On donne les définitions des opérateurs contractants ou k-contractants, non expansifs, pseudo-contractifs, contractions non linéaires et les opérateurs ww-compacts et ws-compacts. Nous définissons également des opérateurs de superposition. Enfin, ce chapitre est clôturé en introduisant la définition d'opérateur multivoque, et quelques résultats de points fixes qui sont nécessaires dans la suite.

• **Dans le chapitre** 3, On établit des théorèmes de points fixes pour des opérateurs condensants dénombrables par rapport à une mesure de non compacité. La théorie des l'opérateurs condensants a commencé en 1967 avec le travail de Sadovskii [65]. Soient K un sous-ensemble borné, convexe fermé de l'epace de Banach X et $f : K \to K$ un opérateur univoque. Si f est continu et condensant par rapport à la mesure de non compacité μ c'est à dire pour tout sous-ensemble

borné C de K, $\mu(f(C)) \leq \mu(C)$, alors f admet un point fixe. Le résultat de Sadovskii a été amélioré en utilisant différentes idées: Daher [25] a montré qu'il est valide pour les opérateurs condensants dénombrables par rapport à une mesure de non compacité. Motivé par l'article de H. Monch [60], In-Soon Kim [42] obtient un théorème de type Larey-Schauder pour des opérateurs dénombrablement condensants univoques. Dans [41], Himmelberg, Porter et Van Vleck ont également étendu la définition des opérateurs condensants aux applications multivoques et ont obtenu une nouvelle version du théorème de Sadovskii. Un résultat analogue au théorème du point fixe de Daher a été obtenu par Agarwal et O'Regan [2] (pour les multivoques semi-continues supérieures) dans un espace de Banach séparable. Pour plus d'information sur les opérateurs dénombrablement condensants multivoques, on renvoie le lecteur aux références [20, 28]. Nos résultats pour les opérateurs condensant dénombrable présentés dans ce chapitre concernent précisement des théorèmes de points fixes de type Altman, de type Leray-Schauder, de type Krasnoselskii et de type Krasnoselskii-Shaefer (le cas du domaine non borné).

• **Le Chapitre** 4, est consacré au problèmes de points fixes pour les opérateurs ws-compact et ww-compact (voir Définition 2.1.13). En 2006, K. Latrach, M.A. Taoudi et A. Zeghal [51], ont introduit les opérateurs ws-compact et ww-compact et ont obtenu des nouveaux théorèmes de points fixes. En fait, l'intérêt des opérateurs ws-compacts réside dans le fait qu'ils permettent d'obtenir des théorèmes de points fixes de type Schauder, Darbo, Sadovskii, Krasnosel'skii ou encore de type Schaefer (dans des domaines non bornés) pour les opérateurs transformant un sous-ensemble faiblement compact dans lui-même sans que ceux-ci soient faiblement continus. Observant que, sur un espace de Banach, tout opérateur de Dunford-Pettis (voir Définition 2.1.14) est ws-compact et que tout opérateur faiblement compact sur un espace de Banach vérifient la propriété de Dunford-Pettis (voir Définition 2.1.14) est un opérateur ws-compact. L'objet de ce chapitre est de développer des théorèmes de points fixes de type Schauder et Krasnosel'skii pour des problèmes faisant intervenir deux ou trois opérateurs tel que un est ww-compacts, l'autre ws-compacts et le dernier est contractif.

• **Le chapitre** 5 est dédié aux opérateurs multivoques. Soient X, Y deux espaces normés. Un opérateur multivoque T de X vers Y est une correspondance qui associe à tout élément $x \in X$ un sous ensemble Tx de Y (on désigne généralement un operateur multivoque par $T : X \to \mathsf{P}(Y)$ où $\mathsf{P}(Y)$ est l'ensemble des parties de Y). Un opérateur multivoque T peut être traité comme un opérateur univoque de X dans $\mathsf{P}(Y)$ [46]. Nadler [66] a commencé le développement de la théorie du point fixe des opérateurs multivoques contractants pour la distance de Pompieu-Hausdorff et il a prouvé l'existence des points fixes dans les espaces métriques complets, en généralisant ainsi le principe de contraction de Banach. Ensuite, il y a eu de nombreux travaux concernant l'existence de points fixes pour les opérateurs multivoques. Des résultats de type Schauder [40] et de type Krasnosel'skii [6] (voir aussi [12, 15, 21, 29, 30, 36, 63, 64, 69]). Ce travail est motivé par un résultat de Himmelberg [40] cf. Théorème 2.3.1 qui affirme que, si M est un sous-ensemble convexe fermé non vide d'un espace de Banach X et $F : M \to \mathsf{P}_{cl,cv}(M)$ est un opérateur multivoque semi-continu supérieurement et $F(M)$ est relativement compact, alors il existe $x \in M$ tel que $x \in F(x)$. En utilisant ce résultat et en prenant des selections ws-compact et des perturbations ww-compact (voir Définition 2.1.13), on montre de nouveaux théorèmes de points fixes de type Krasnosel'skii pour la somme $F + B$ où F est un opérateur multivoque et B est une perturbation (selon le cas, elle

pourra être supposée univoque ou multivoque).

- **Dans le chapitre** 6, on va présenter quelques applications de nos résultats.

La première section de ce chapitre est consacrée à deux équations intégrales de type Volterra de la forme:

$$\varphi(t) = \mathcal{H}(t) + \int_a^t f(s, \varphi(s))ds, \quad \forall t \in [a, b] \tag{1.0.1}$$

où f est une application de $[a, b] \times X$ dans X et $\mathcal{H}$ une application $[a, b]$ dans X. Quant à la seconde, elle est de la forme

$$\varphi(t) = \mathcal{H}(\varphi(t)) + \int_0^t g(s, \varphi(s))ds \quad t \in [0, 1]; \tag{1.0.2}$$

où $g : [0, 1] \times X \to X$, $\mathcal{H} : X \to X$ sont deux applications et X est un espace de Banach. Notre objectif est d'étudier l'existence des solutions de l'équation (1.0.1) (resp. (1.0.2)) dans l'espace $C([a, b], X)$ (resp. $C([0, 1], X)$). L'idée consiste à transformer l'équation (1.0.1) (resp, (1.0.2)) sous la forme d'un problème de point fixe de type Krasnosel'skii et ensuite vérifier si toutes les hypothèses de la Proposition 3.3.2 sont satisfaites. Lorsque c'est le cas, on obtiendrait des solutions de l'équation (1.0.1) (resp,(1.0.2)).

La deuxième section, est dédiée à l'existence de solution de l'équation de transport des neutrons rétardés de la forme:

$$\begin{cases} v.\nabla_x f_0(x, v) + \sigma(x, v, f_0(x, v)) + \lambda f_0(x, v) = \\ \qquad\qquad + \int_{\mathbb{R}^N} \kappa_0(x, v, v')\Theta_0(x, v', f_0(x, v'))d\mu(v') + \sum_{i=1}^{d} \lambda_i \beta_i(x, v) f_i(x, v) \\ \lambda_i f_i(x, v) = \int_{\mathbb{R}^N} \kappa_i(x, v, v')\Theta_i(x, v', f_0(x, v'))d\mu(v'), \quad 1 \le i \le d \end{cases} \tag{1.0.3}$$

où $(x, v) \in \mathsf{D} \times \mathbb{R}^N$. Les conditions aux limites sont modélisées par l'équation

$$f_0^- = H(f_0^+). \tag{1.0.4}$$

Après quelques transformations, le problème (1.0.3)-(1.0.4) se réduit au problème de point fixe suivant

$$f_0 = \mathsf{F}(\lambda)f_0 + \mathsf{B}(\lambda)f_0, \quad f_{0|\Gamma_-} = H(f_{0|\Gamma_+}). \tag{1.0.5}$$

Pour montrer que l'équation (1.0.3)-(1.0.4) admet des solutions, il suffit de montrer que le problème de point fixe (1.0.5) admet des solutions. Pour cela on vérifie que (1.0.5) satisfait aux hypothèses du Corollaire 4.2.1.

Dans la dernière section, on étudie l'existence de solution de l'équation de transport avec des opérateurs de collisions élastiques et inélastiques de la forme :

$$\begin{aligned} v.\frac{\partial\varphi}{\partial x}(x,v) + \sigma(x,v,\varphi(x,v)) + \lambda\varphi(x,v) = & \int_V k_c(x,v,v')f(x,v',\varphi(x,v'))d\mu(v') \\ & + \sum_{j=1}^{l}\int_{\mathbb{S}^{N-1}} k_d^j(x,\rho_j,\omega,\omega')\varphi(x,\rho_j\omega')d\omega' \\ & + \int_{\mathbb{S}^{N-1}} k_e(x,\rho,\omega,\omega')g(x,\rho,\omega',\varphi(x,\rho,\omega'))d\omega' \end{aligned} \tag{1.0.6}$$

où $(x,v) \in \Omega \times V$, ($V$ est appelé l'espace des vitesses admissibles et Ω est un ensemble ouvert et borné de $\mathbb{R}^N$ ($N \geq 3$) muni de la mesure de Lebesgue dx). Les conditions aux limites sont modélisées par

$$\varphi_{|\Gamma_-} = 0, \tag{1.0.7}$$

Notre but dans cette section est de résoudre le problème stationnaire (1.0.6)-(1.0.7) dans L^p-espace ($1 \leq p < \infty$), la stratégie consiste à transformer le problème (1.0.6)-(1.0.7) au problème de point fixe suivant

$$\psi = A(\psi) + C(\psi)$$

où A, et C sont des opérateurs non linéaires. Pour résoudre le dernier problème, on utilise les arguments de compacité établis dans [16, 52], et les théorèmes du points fixes de Krasnosels'kii (voir, [1, Corollary 3.8]).

General introduction

The theory of fixed points, since 1950 is a very active branch of mathematics and is at the heart of nonlinear analysis since it provides powerful tool to establish existence of solutions for many nonlinear problems arising in mechanic, economics, biology etc.. In fact, in many concrete problems, we seek solutions as fixed point of the problem using hypotheses on the mappings involved in the problem or on the structure of the Banach space.

In this thesis project we present some fixed point results for single-valued as well as multivalued mappings. It is divided into four chapters.

• **In Chapter** 2, we recall the main concepts, definitions and results which are required throughout Chapters 3-6. First, we recall the definition of measures of non-compactness and measures of weak noncompactness. As examples we recall the measure of noncompactness of Kuratowskii and that of Hausdorff. The measure of weak noncompactness of De Blasi, which we will use in Chapters 3-6, is also recalled. We give the definitions of the contraction or k-contractive, nonexpansive, pseudocontractive, accretive and ψ-contraction operators, ws-compact and ww-compact operators. The definition of superposition operators and some of their properties are given. Further, the concept of multivalued mapping is introduced. Throughout this chapter, some fixed point results required in the sequel are stated.

• **In Chapter** 3, we discuss fixed point results for countably condensing mappings with respect to the measure of noncompactness. The theory of condensing operator began in 1967 with the paper of Sadovskii [65]. Let K be a nonempty closed, bounded and convex subset of a Banach space X and let $f : K \to K$ be a single valued maps. If f is continuous and condensing with respect to a measure of noncompactness μ, that is for all bounded subset C of K, $\mu(f(C)) < \mu(C)$, then f has a fixed point. Sadovskii's result was improved in the literature using different ideas: Daher [25] showed that it is still true for countably condensing maps with respect to a measure of noncompactness; motivated by H. Mönch paper [60], In-Soon Kim [42] derives a Leray-Schauder's type fixed point theorem for countably condensing mappings and applies it to discuss an eigenvalue problem. In [41], Himmelberg, Porter and Van Vleck extended the definition of condensing maps to multimaps and derived a multivalued version of Sadovskii's theorem; a multivalued analogue of Daher's fixed point theorem was obtained by Agarwal and O'Regan [2] (for upper semi-continuous multimaps) in a separable Banach space. For further results about countably condensing multimaps, we refer the reader to papers [20, 28] and the references therein. Here, beside a result of Altma's type, our analysis is focussed on results of Leray-Shauder's type, Krasnoselskii's type and Krasnoselskii-Shaefer's type (the case of unbounded domain).

• **Chapter** 4, is devoted to fixed point problem involving ws-compact and ww-compact (see Definition 2.1.13). In 2006, K. Latrach, M.A. Taoudi and A. Zeghal [51], introduced ws-compact and ww-compact operators and obtained new fixed point theorems. The interest of the class of ws-compact operators lies in the fact that they make possible to obtain fixed point theorems of types Schauder, Darbo, Sadovskii, Krasnosel'skii or even Schaefer type (in unbounded domains)

for operators transforming each weakly compact subsets into itself. Here we derive new fixed points results of Schauder's type and others of Krasnosel'skii 's for fixed point problems involving ws-compact and ww-compact mappings. Observing that, on a Banach space, every Dunford-Pettis operator (see Definition 2.1.14) is ws-compact and that every weakly compact operator on a Banach space with the Dunford-Pettis property (see Definition 2.1.14) is ws-compact operaors, we extend our results to these two classes of operators.

• **Chapter** 5 is dedicated to multivalued mappings, that is, mappings from X into $\mathcal{P}_\lambda(Y)$ (the set of all subsets of Y) where X and Y are two normed spaces. In [66], Nadler began the development of fixed point theory for contracting multivalued maps using the Pompieu-Hausdorff distance which is defined on $\mathsf{P}_{cl,bd}(X)$ (the set of all closed bounded subsets of Y). He proved existence fixed points for this class of mappings. Later, several works dedicated to fixed point theory for multivalued mappings were published. Recently, results involving the weak topology and the measure of weak noncompactness were derived (see, for example, [6, 12, 15, 21, 29, 30, 36, 63, 64, 69] and the reference therein). This work is motivated by a Himmelberg's result [40] cf. Theorem 2.3.1 which asserts that, if M is a nonempty closed, convex subset of a Banach space X and $F : M \to \mathsf{P}_{cl,cv}(M)$ is a upper semicontinuous multivalued map and $F(M)$ is relatively compact, then there exists $x \in M$ such that $x \in F(x)$. Using this result together with conditions ws-compact and ww-compact (see Definition 2.1.13), we prove several new Krasnosel'skii-type fixed set theorems for the sum $F + B$ where F is a multivalued mapping and B is a perturbation. Various kinds of perturbations were considered. The main tools used in this chapter are the measure of weak noncompactness, the concept of condensing multivalued (or single valued) mappings and the concept of ws-compact and ww-compact and Theorem 2.3.1.

• **In Chapter** 6 we present applications of our results to the study of the existence of solutions to some integro-differential equations.

The first section of this chapter deals with two Volterra integral equations:

$$\varphi(t) = \mathcal{H}(t) + \int_a^t f(s, \varphi(s))ds, \quad \forall t \in [a, b] \tag{1.0.8}$$

where f is a map from $[a, b] \times X$ into X and $\mathcal{H}$ maps $[a, b]$ into X.

$$\varphi(t) = \mathcal{H}(\varphi(t)) + \int_0^t g(s, \varphi(s))ds \quad t \in [0, 1]; \tag{1.0.9}$$

where $g : [0, 1] \times X \to X$, $\mathcal{H} : X \to X$ are two maps and X is a Banach space. Our objective is discuss existence of the solutions to equation (1.0.8) (resp. (1.0.9)) in the space $C([a, b], X)$ (resp. $C([0, 1], X)$). The idea of proof consists in transforming these two equations to two fixed point problems and we check that the involved operators satisfy the assumptions of Proposition 3.3.2.

In the second section, we will discuss the existence of solution to the following boundary value problem which models a transport equation with delayed neutrons

$$\begin{cases} v.\nabla_x f_0(x,v) + \sigma(x,v,f_0(x,v)) + \lambda f_0(x,v) = \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^N} \kappa_0(x,v,v')\Theta_0(x,v',f_0(x,v'))d\mu(v') + \sum_{i=1}^{d} \lambda_i\beta_i(x,v)f_i(x,v) \\ \lambda_i f_i(x,v) = \displaystyle\int_{\mathbb{R}^N} \kappa_i(x,v,v')\Theta_i(x,v',f_0(x,v'))d\mu(v'), \quad 1 \le i \le d \end{cases} \tag{1.0.10}$$

where $(x,v) \in \mathsf{D} \times \mathbb{R}^N$. The boundary conditions are modeled by

$$f_0^- = H(f_0^+). \tag{1.0.11}$$

After some transformations, Problem (1.0.10)-(1.0.11) reduces to the following fixed point problem

$$f_0 = \mathsf{F}(\lambda)f_0 + \mathsf{B}(\lambda)f_0, \quad f_{0|\Gamma_-} = H(f_{0|\Gamma_+}), \tag{1.0.12}$$

So, to solve Problem (1.0.10)-(1.0.11), it suffices to seek solutions to the fixed point problem (1.0.12). This can be made with the help of Corollary 4.2.1

In the last section, we will study the existence of solution of the transport equation with elastic and inelastic collision operators of the form :

$$\begin{aligned} v.\frac{\partial\varphi}{\partial x}(x,v) + \sigma(x,v,\varphi(x,v)) + \lambda\varphi(x,v) = & \int_V k_c(x,v,v')f(x,v',\varphi(x,v'))d\mu(v') \\ & + \sum_{j=1}^{l} \int_{\mathbb{S}^{N-1}} k_d^j(x,\rho_j,\omega,\omega')\varphi(x,\rho_j\omega')d\omega' \\ & + \int_{\mathbb{S}^{N-1}} k_e(x,\rho,\omega,\omega')g(x,\rho,\omega',\varphi(x,\rho,\omega'))d\omega' \end{aligned} \tag{1.0.13}$$

where $(x,v) \in \mathsf{D} \times V$, ($V$ is called the space of admissible velocities and Ω is an open and bounded set of $\mathbb{R}^N$ ($N \ge 3$) endowed with the Lebesgue measure dx). The boundary conditions are modeled by

$$\varphi_{|\Gamma_-} = 0. \tag{1.0.14}$$

The aim of this section is to solve the stationary Problem (1.0.13)-(1.0.14) in L^p-espace ($1 \le p < \infty$). The strategy consists in transforming problem (1.0.13)-(1.0.14) into the following fixed point problem

$$\psi = A(\psi) + C(\psi)$$

where A, and C are nonlinear operators. To solve the last problem we use the compactness arguments established in [16, 52], and the Krasnoselskii fixed point theorems (see, [1, Corollary 3.8]).

Chapter 2

Preliminary

In this chapter, we recall some definitions and results required in the next chapters.

2.1 Some definitions and properties

Let X be a Banach space endowed with the $\|\cdot\|$. We introduce the following notations:

$$\mathsf{P}(X) = \{D \subset X : D \text{ is a non-empty subsets of } X\},$$

$$\mathsf{B}(X) = \{D \in \mathsf{P}(X) \text{ such that } D \text{ is bounded}\},$$

$$\mathsf{C}_r(X) = \{D \in \mathsf{P}(X) \text{ such that } D \text{ is relatively compact}\}.$$

Let K be a subset of X. We denote by $\mathsf{B}_c(K)$ the set

$$\mathsf{B}_c(K) = \{D \subset K \text{ such that } D \text{ is a countable and bounded}\}$$

Definition 2.1.1 *We recall that a function $\beta : \mathsf{B}(X) \to \mathbb{R}^+$ is called measure of noncompactness on X if it satisfies the following properties :*

(a) Regularity: $\beta(B) = 0$ if and only if $B \in \mathsf{C}_r(X)$.

(b) Invariant under closure: $\beta(B) = \beta(\overline{B})$, $\forall B \in \mathsf{B}(X)$.

(c) Iinvariant under passage to the convex hull: $\beta(co(B)) = \beta(B)$, $\forall B \in \mathsf{B}(X)$.

(d) Monotonicity: for any B_1, $B_2 \in \mathsf{B}(X)$ we have $B_1 \subset B_2 \Longrightarrow \beta(B_1) \leq \beta(B_2)$.

(e) Homogeneity: $\beta(\lambda B) = |\lambda|\beta(B)$ for any real λ and $B \in \mathsf{B}(X)$.

(f) Algebraic semi-additivity: $\beta(B_1 + B_2) \leq \beta(B_1) + \beta(B_2)$, $\forall\, B_1,\, B_2 \in \mathsf{B}(X)$,

(g) if $(M_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of nonempty and closed sets in* $\mathsf{B}(X)$ *such that* $\lim_{n\to\infty}\beta(M_n) = 0$*, then the intersection set* $M_\infty = \cap_{n=1}^{\infty} M_n$ *is nonempty.*

The concept of measure of noncompactness was first introduced by Kuratowski in [45] (Kuratowskii measure of noncompactness). It was defined in the following way: $\alpha : \mathsf{B}(X) \to \mathbb{R}^+$,

$$B \mapsto \alpha(B) = \inf\left\{r > 0 : B \subset \cup_{i=1}^{n} D_i,\ \operatorname{diam}(D_i) \leq r\right\}.$$

Another classical example is the Hausdorff measure of noncompactness χ. It was introduced by Goldenstein et al. [37]. It is defined by $\chi : \mathsf{B}(X) \to \mathbb{R}^+$, $B \mapsto \chi(B) = \inf\left\{r > 0 : B \subset \cup_{i=1}^{n}\mathbb{B}_r(x_i),\ x_i \in X\right\}$.

Notation: Throughout this chapter, $\beta(\cdot)$ will denote a measure of noncompactness in the sense of Definition 2.1.1.

Definition 2.1.2 *Let* K *be a subset of a Banach space* X *and let* β *be a measure of noncompactness on* X*. Let* $f : K \to X$ *be a map.*

(a) We say that f *satisfies the Darbo condition for bounded countable sets if there exists* $k > 0$ *such that*

$$\beta(f(B)) \leq k\beta(B), \quad \forall B \in \mathsf{B}_c(K); \tag{2.1.1}$$

(b) f *is said to be countably* β*-*k*-contractive if it satisfies the condition* (2.1.1) *for some* $k \in [0, 1)$*;*

(c) f *is said to be countably* β*-condensing if* $\beta(f(B)) < \beta(B)$ *for all* $B \in \mathsf{B}_c(K)$ *with* $\beta(B) > 0$*.*

(d) f *is said to be countably* 1*-set-contractive if* $\beta(f(B)) \leq \beta(B)$ *for all* $B \in \mathsf{B}_c(K)$ *with* $\beta(B) > 0$*.*

It is clear that, every β-k-contractive (resp. β-condensing) map is countably β-k-contractive (resp. countably β-condensing) but the converse is not in general true.

Recall that a map $f : K \subset X \to X$ is said to be k-Lipschitzian if

$$\|f(x) - f(y)\| \leq k\,\|x - y\|$$

with $k \geq 0$. **If** $k = 1$**,** f **is called nonexpansive** . Further, if $k \in [0, 1)$, then f is called **a contraction (or** k**-contractive)**.

The following lemma is required in the sequel.

Lemma 2.1.1 *Let* K *be a nonempty subset of a Banach space* X *and let* $f : K \to X$ *be a* k*-Lipschitizian map. Then, for each* $C \in \mathsf{B}_c(K)$*, we have* $\beta(f(C)) \leq k\beta(C)$*.*

Proof. It is well known that if f is k-Lipschitizian, then, for each bounded subset C of K, we have $\beta(f(C)) \leq k\beta(C)$. The result follows from the fact that countably bounded subsets of K are also bounded in K. □

We recall the definition of ϕ-contraction mappings.

Definition 2.1.3 *Let X be a Banach space. A mapping $B : X \to X$ is said to be a Φ-contraction if there exists a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(r) < r$ for any $r > 0$ and, for all $x, y \in X$, we have*

$$\|B(x) - B(y)\| \leq \phi(\|x - y\|).$$

The function ϕ is called the Φ-function of B.

Moreover, if the function ϕ is nondecreasing, then B is called a nonlinear contraction . □

Remark 2.1.1 *We note that if $\phi(r) = \alpha r$, $\alpha \in (0, 1)$, then B is α-contractive. Hence, any α-contractive map, for some $\alpha \in (0, 1)$, is a nonlinear contraction, however, the converse is, in general, not true.* □

Recall that it is well known (cf. [17]) that each nonlinear contraction on a Banach space $B : X \to X$ possesses a unique fixed point and, moreover, $(I - B)$ is invertible and, for each subset K of X, $(I - B)$ is a homeomorphism from K onto $(I - B)(K)$.

It is clear that, every β-k-contractive (resp. β-condensing) map is countably β-k-contractive (resp. countably β-condensing) but the converse is, in general, false.

The following lemma is required in the sequel.

Lemma 2.1.2 *Let K be a nonempty subset of the Banach space X . If the map $g : K \subset X \to X$ is a nonlinear ϕ-contraction, then $(I - g)$ is a homeomorphism from K onto $(I - g)(K)$.*

Proof. Since g is a nonlinear ϕ-contraction, there exists a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(r) < r$ for any $r > 0$ and, for all $x, y \in X$, we have

$$\|g(x) - g(y)\| \leq \phi(\|x - y\|).$$

Let $x, y \in K$, $x \neq y$ we have

$$\begin{aligned} \|(I - g)x - (I - g)y\| &= \|(x - y) - (g(x) - g(y))\| \\ &\geq \|x - y\| - \|g(x) - g(y)\| \\ &\geq \|x - y\| - \phi(\|x - y\|) > 0. \end{aligned}$$

Hence, $(I-g)$ is an injective map on K and therefore $(I-g)^{-1}$ exists from $(I-g)(K)$ to K. To show the continuity of $(I-g)^{-1}$ suppose that there exists a point x and a sequence $(x_n)_{n\in\mathbb{N}}$ of points of K such that $(I-g)(x_n) \to (I-g)(x)$ and $\lim_{n\to+\infty} \sup_{k\geq n} \|x_k - x\| = a$. Due to inequality

$$\|(I-g)x_n - (I-g)x\| \geq \|x_n - x\| - \phi(\|x_n - x\|),$$

we see that $0 \geq a - \phi(a)$. Since $\phi(a) < a$, we deduce that $a = 0$ and so $(I-g)^{-1}$ is continuous. □

Definition 2.1.4 *Let X be a Banach space and let $f : K \subset X \to X$ be a mapping on X .*

(a) We say that f is pseudocontractive if, for all $x, y \in K$ and, $r > 0$, one has $\|x-y\| \leq \|r(f(y)-f(x)) + (1+r)(x-y)\|$.

(b) We say that f is accretive if the inequality

$$\|x - y + \lambda(f(x) - f(y))\| \geq \|x - y\| \quad \textit{holds for all, } \lambda > 0, \textit{ and}$$

$x, y \in K$.

It is not difficult to check that every nonexpansive map is pseudocontractive. For a proof, let $x, y \in K$ and $r > 0$. We have

$$\begin{aligned}\|r(f(y)-f(x)) + (1+r)(x-y)\| &\geq (1+r)\|x-y\| - r\|f(x)-f(y)\| \\ &\geq (1+r)\|x-y\| - r\|x-y\| \\ &\geq \|x-y\|\end{aligned}$$

□

Remark 2.1.2 Note that there exists connection between pseudocontractive operators and accretive operators which we shall use later. It may be asserted as follows: f is pseudocontractive if and only if $I - f$ is accretive where I is the identity operator. □

We recall also the following elementary lemmas where the proofs can be found, for example, in [48].

Lemma 2.1.3 *Let K be a subset of the Banach space X and let $g : K \to X$ be a k-contractive for some $k \in (0,1)$. Then the operator $(I-g)$ is a homeomorphism from K onto $(I-g)(K)$ and*

$$\left\|(I-g)^{-1}(x) - (I-g)^{-1}(y)\right\| \leq \frac{1}{1-k}\|x-y\|, \quad \forall\, x, y \in (I-g)(K)$$

Proof. Let $x, y \in X$, we have

$$\begin{aligned}\|(I-g)(x)-(I-g)(y)\| &\geq \|x-y\| - \|g(x)-g(y)\| \\ &\geq (1-k)\|x-y\|.\end{aligned}$$

This shows that the map $(I-g)$ is injective and therefore invertible with inverse $(I-g)^{-1} : (I-g)(K) \to K$. The continuity of $(I-g)^{-1}$ results from the previous inequality. Let $u, v \in (I-g)(K)$. By substituting in the previous inequality $(I-g)^{-1}(u)$ by x and $(I-g)^{-1}(v)$ by y, we obtain

$$\left\|(I-g)^{-1}(u)-(I-g)^{-1}(v)\right\| \leq \frac{1}{1-k}\|u-v\|.$$

□

Definition 2.1.5 *Let K be a subset of a Banach space X. The mapping $f : K \to X$ is said to be expansive, if there exists a constant $h > 1$ such that, for all $x, y \in K$ we have $\|f(x)-f(y)\| \geq h\|x-y\|$.*

Lemma 2.1.4 *Let K be a subset of the Banach space X and let $g : K \to X$ be a continuous expansive mapping with constant $h > 1$. Then the operator $(I-g)$ is a homeomorphism from K onto $(I-g)(K)$ and*

$$\left\|(I-g)^{-1}(x)-(I-g)^{-1}(y)\right\| \leq \frac{1}{h-1}\|x-y\|, \quad \forall\, x, y \in (I-g)(K).$$

Proof. We note that $(I-g)$ is a continuous mapping. Let $x, y \in X$, we have

$$\|(I-g)(x)-(I-g)(y)\| = \|(g(x)-g(y))-(x-y)\| \geq (h-1)\|x-y\|. \tag{2.1.2}$$

Hence, $(I-g)$ is an injective map on K and therefore $(I-g)^{-1}$ exists from $(I-g)(K)$ to K.Let $u, v \in (I-g)(K)$. By substituting in the equation (2.1.2) $(I-g)^{-1}(u)$ by x and $(I-g)^{-1}(v)$ by y, we obtain

$$\left\|(I-g)^{-1}(u)-(I-g)^{-1}(v)\right\| \leq \frac{1}{h-1}\|u-v\|.$$

□

In this note, we give the following definition:

Definition 2.1.6 *An operator $f : D(f) \subset X \to X$ is said ψ-expansive if there is a function $\psi : [0,\infty) \to [0,\infty)$ such that for every $x, y \in D(f)$, the inequality $\|f(x)-f(y)\| \geq \psi(\|x-y\|)$ holds. The function ψ satisfies*

- $\psi(0) = 0$;
- $\psi(r) > 0$ *for any* $r > 0$;
- *either it is continuous or it is nondecreasing.*

Clearly, if we take $\phi(t) = \alpha t$ with $\alpha > 1$, then f is an expansive mapping in the sense of Definition 2.1.5.

We need also the following result. We think that it is probably known but, since we have any reference, we shall give a proof.

Lemma 2.1.5 *Let K be a nonempty subset of a Banach space X, β a measure of noncompactness on X and $g : K \to X$ a map. If g is a nonlinear ϕ-contraction, then, for each bounded subset S of K with $\beta(S) > 0$, we have $\beta(g(S)) < \beta(S)$.*

Proof. Let $\varepsilon > 0$ be an arbitrary real. Suppose that $S \subseteq \cup_{i=1}^{n} S_i$ with $\operatorname{diam}(S_i) \leq \beta(S) + \varepsilon$. Then we have $g(S) \subseteq \cup_{i=1}^{n} g(S_i) = \cup_{i=1}^{n} F_i$ where $F_i := g(S_i)$. Let $y_1, y_2 \in F_i$ where $i \in \{1, 2, \cdots, n\}$, there exists $x_1, x_2 \in S_i$ such that $y_1 = g(x_1)$ and $y_2 = g(x_2)$. Since g is a nonlinear ϕ-contraction, one can write $\|g(x_2) - g(x_1)\| \leq \phi(\|x_2 - x_1\|) \leq \phi(\beta(S) + \varepsilon)$. Hence, we conclude that $\operatorname{diam}(F_i) \leq \phi(\beta(S) + \varepsilon)$ and therefore $\beta(g(S)) \leq \phi(\beta(S) + \varepsilon) < \beta(S) + \varepsilon$ because g is a nonlinear ϕ-contraction. Since ε is an arbitrary real number, we infer that $\beta(g(S)) < \beta(S)$. □

Definition 2.1.7 *Let X be a normed topological vector space and let K be a subset of X.*

- *We say that K is absorbing if, for every $x \in X$, there exists $t \geq 0$ such that $x \in tK$.*
- *We say that K is balanced if $\lambda K \subset K$ for every λ with $|\lambda| \leq 1$.*

Now we can define the Minkowski's Functionals.

Definition 2.1.8 (Minkowski's functional) *Let K be a convex absorbing subset of a normed space X. We define the Minkowski functional of K, $\mu_K(\cdot)$, by $\mu_K(x) = \inf\{t > 0 : x \in tK\}$.*

Note that for each $x \in X$, the set of real $\eta > 0$ such that $x \in \eta K$ is not empty (because K is absorbing), so the definition of $\mu_K(\cdot)$ has a meaning.

Lemma 2.1.6 *[70] Let K be an open convex absorbing subset of a normed space X, $\theta \in K$. Then the Minkowski functional $\mu_K(\cdot)$ of K is nonnegative and continuous on X. Moreover,*

(a) $\mu_K(\lambda x) = \lambda \mu_K(x)$, $\forall \lambda \geq 0$;

(b) $x \in K$ *if and only if* $0 \leq \mu_K(x) < 1$;

(c) $x \in \partial K$ *if and only if* $\mu_K(x) = 1$;

(d) $x \notin \overline{K}$ *if and only if* $\mu_K(x) > 1$.

Definition 2.1.9 *Let K be a subset of X and $f : K \to X$. We say that f has a fixed point in K if there exists $x \in K$ such that $f(x) = x$. The set of all fixed points of f is denoted by* $\mathsf{Fix}(f)$.

We now recall the following result due to Daher [25]. It is an extension of Sadovskii's fixed point theorem to countably β-condensing maps .

Theorem 2.1.1 (Daher) *Let K be a nonempty, bounded, closed and convex subset of a Banach space X and let β be a measure of noncompactness on X. If $f : K \to K$ is a continuous countably β-condensing map, then* $\mathsf{Fix}(f) \neq \emptyset$.

We recall also the following result (cf. [71]).

Theorem 2.1.2 *Let K be a closed subset of X and let $f : K \to X$ be an expansive mapping. If $f(K) \supset K$, then* $\mathsf{Fix}(f) \neq \emptyset$.

We will introducing the definition of semi-closed mappings.

Definition 2.1.10 *Let K be a nonempty subset of a Banach space X. A map $f : K \to X$ is said to be semi-closed if for every closed subset F of K, the set $(I - f)(F)$ is closed.*

Lemma 2.1.7 *Let K be nonempty closed, bounded and convex subset of a Banach space X, and let $f : K \to K$ be a continuous, countably 1-set-contractive map. If f is semi-closed and $f(K)$ is bounded, then* $\mathsf{Fix}(f) \neq \emptyset$.

Proof. Let $z \in K$ and define the maps f_n by $f_n := t_n f + (1 - t_n)z$ where $n \in \mathbb{N}$ and $(t_n)_{n\in\mathbb{N}}$ a sequence of $(0, 1)$ such that $t_n \to 1$ as $n \to +\infty$. The convexity of K implies that f_n maps K into itself. Let $C \in \mathbf{B}_c(K)$. Then we have

$$\beta(f_n(C)) \leq \beta\big(t_n f(C) + (1 - t_n)z\big) \leq t_n \beta(f(C)) \leq t_n \beta(C).$$

If $\beta(C) \neq 0$, then $\beta(f_n(C)) < \beta(C)$ and therefore f_n is countably β-condensing. Hence, according to Theorem 2.1.1, f_n has a fixed point $x_n \in K$, that is $x_n = t_n f(x_n) + (1 - t_n)z$. Note that, $\forall x \in K$, we have

$$\|f(x) - f_n(x)\| = \|f(x) - t_n f(x) + (1 - t_n)z\| = |1 - t_n| \, \|f(x) - z\| .$$

Moreover, the boundedness of $f(K)$ implies $\|f(x) - f_n(x)\| \to 0$ as $n \to \infty$. Now using the equality above we get $\|(I - f)(x_n)\| = \|(f_n(x_n) - f)(x_n)\| = |(1 - t_n)| \, \|z - f(x_n)\| \to 0$ as $n \to \infty$. Next, using the fact that f is semi-closed and K is closed, we infer that $0 \in (I - f)(K)$. Hence, $\mathsf{Fix}(f) \neq \emptyset$. □

Let X be a Banach space. Let us first introduce the following two subsets

$$\mathsf{W}_r(X) = \{D \in \mathsf{B}(X) \text{ such that } D \text{ is relatively weakly compact}\},$$

and

$$\mathsf{W}(X) = \{D \in \mathsf{B}(X) \text{ such that } D \text{ is weakly compact}\}.$$

In the sequel, we denote by $\mathbb{B}_r$ the closed ball in X centred at 0 with radius r.

Definition 2.1.11 *A map $\mu : \mathsf{B}(X) \to [0, +\infty[$ is said to be a measure of weak noncompactness on X if it satisfies the following conditions:*

(a) The family $\ker\mu := \{M \in \mathsf{B}(X) : \mu(M) = 0\}$ is nonempty and $\ker\mu$ is contained in $\mathsf{W}_r(X)$.

(b) Monotonicity: $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$, for all $M_1, M_2 \in \mathsf{B}(X)$

(c) Invariance under passage to the closed convex hull: $\forall M \in \mathsf{B}(X)$, $\mu(\overline{co}(M)) = \mu(M)$ where $\overline{co}$ denotes the closed convex hull of M.

(d) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$. for all $M_1, M_2 \in \mathsf{B}(X)$

(e) Homogeneity $\mu(\alpha M) = |\alpha|\,\mu(M)$ for any real number α and $M \in \mathsf{B}(X)$

(f) Generalized Cantor intersection theorem: If $(M_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ such that $\lim_{n\to\infty} \mu(M_n) = 0$, then $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$ and $\mu(\bigcap_{n=1}^{\infty} M_n) = 0$. □

The family $\ker\mu$ given in first assertion is called the kernel of the measure of weak noncompactness $\mu(\cdot)$. It should be noticed that the containment $M \subseteq \overline{M^w} \subseteq \overline{co}(M)$ together with the item (3) of Definition 2.1.11 implies that $\mu(\overline{M^w}) = \mu(M)$ where $\overline{M^w}$ stands for the weak closure of M.

For the axiomatic definition of the measure of weak noncompactness we refer, for example, to [10].

We recall that the first example of measure of weak noncompactness was introduced by De Blasi in 1977 (cf. [27]). The De Blasi measure of weak noncompactness $\omega(\cdot)$ is defined by

$$\omega \; : \; \mathsf{B}(X) \to [0, +\infty[, \quad A \longmapsto \omega(A)$$

with

$$\omega(A) = \inf\left\{t > 0 \; : \; \text{there exists} \;\; K \in \mathsf{W}(X) \;\; \text{such that } A \subseteq K + t\mathbb{B}_1\right\}$$

where $\mathbb{B}_1$ denotes the closed unit ball of X. It satisfies all axioms listed in Definition 2.1.11 and more. For the proofs of the properties of $\omega(\cdot)$ we refer to [27, 32] or [48, p. 225].

Notation : In the remainder of this report $\omega(\cdot)$ will always denote the measure of weak noncompactness of De Blasi.

Definition 2.1.12 *Let X be a Banach space and $B : X \to X$ a single-valued mapping. We say that B is weakly sequentially continuous if it is sequentially continuous for the weak topology $\sigma(X, X^*)$ of X and B is said to be weakly continuous if it is continuous for the weak topology $\sigma(X, X^*)$ of X.*

For the definition and properties of the weak topology $\sigma(X, X^*)$ on a normed space X we refer the reader to [19] or [55, p.124]

We recall the following definition introduced in [51, p. 260]:

Definition 2.1.13 *Let X be a Banach space and let $f : D(f) \subset X \to X$ be a continuous mapping.*

(i) We say that f is ws*-compact if for each weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ of $D(f)$ the sequence $(f(x_n))_{n\in\mathbb{N}}$ has a strongly convergent subsequence.*

(ii) We say that f is ww*-compact if for each weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ of $D(f)$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence.*

Remark 2.1.3 *For the properties* ws*-compact or* ww*-compact we refer, for example, to [35, 51, 52] or [48, pp. 221-222]. We note that with the help of Eberlein-Šmulian (theorem [31, p. 430]),* ww*-compact operators map weakly compact sets into weakly compact sets. Next since bounded linear operators are weakly continuous, so each bounded linear operator is* ww*-compact.* □

We now recall the following two theorems concerning these two classes of operators established established in [51].

Theorem 2.1.3 *Let K be a nonempty closed convex subset of a Banach space X. Assume that $f : K \to K$ is a* ws*-compact mapping. If $f(K)$ is relatively weakly compact, then $\mathsf{Fix}(f) \neq \emptyset$.*

Theorem 2.1.4 *Let K be a nonempty closed bounded convex subset of a Banach space X. Suppose that $f : K \to K$ and $g : X \to K$ such that:*

(a) f is ws*-compact and $f(K)$ is relatively weakly compact,*

(b) g is ww*-compact and k-contractive for some $k \in (0, 1)$,*

(c) $f(K) + g(K) \subset K$.

Then there exists $x \in K$ such that $f(x) + g(x) = x$.

Let X, Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y.

Definition 2.1.14 *Let X and Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$.*

(a) We recall that T is called weakly compact if it carries norm bounded subset of X to relatively weakly compact subset of Y.

*(b) We say that T is a **Dunford-Pettis** operator if T maps weakly compact subsets of X into norm compact sets of Y.* □

Theorem 2.1.5 (Krein-Smulian) *Let X be a Banach space and K a weakly compact subset of X (that is, K is compact when X is endowed with the weak topology). Then the closed convex hull of K in X is weakly compact.*

For a proof of Krein-Smulian's theorem, we refer, for example, to [31, p. 434].

We recall the following lemma established in [35, pp. 3438-3439].

Lemma 2.1.8 *Let X be a Banach space and let $B : X \to X$ be a* ww*-compact mapping. If B is a Φ-contraction with Φ-function ϕ, then*

$$\omega(B(S)) \leq \phi(\omega(S)), \quad \text{for all } S \in \mathsf{B}(X).$$

Note that if B is a k-contractive and ww-compact maps, then according to Remark 2.1.1, B is a Φ-contraction with Φ-function ϕ where, for each $x \in X$, $\phi(x) = kx$. Hence, the estimate in Lemma 2.1.8 may be written simply as

$$\omega(B(S)) \leq k\,\omega(S), \quad \text{for all } S \in \mathsf{B}(X). \tag{2.1.3}$$

It should be noticed that Equation (2.1.3) is in fact valid for all k-Lipschitzian and ww-compact maps (cf. [3]).

Remark 2.1.4 *It should be noticed that each bounded linear operator from a normed space X into itself is* ww*-compact (use Remark 2.1.3). Morever, it is $\|T\|$-Lipschitzian where $\|T\|$ is norm of T (in the operator topology), so according to* (2.1.3)*, we have*

$$\omega(T(M)) \leq \|T\|\,\omega(M), \quad \text{for all } M \in \mathsf{B}(X). \tag{2.1.4}$$

□

In Lemma 2.1.8, if we replace the hypothesis f is ww-compact by f is weakly sequentially continuous, the result remains valid. This motivated by the following Lemma.

Lemma 2.1.9 *Let X be a Banach space and let $B : X \to X$ be a weakly sequentially continuous mapping. If B is a Φ-contraction with Φ-function ϕ, then*

$$\omega(B(S)) \leq \phi(\omega(S)), \quad \text{for all } S \in \mathsf{B}(X).$$

Proof. Let $S \in \mathsf{B}(X)$ and $r > \omega(S)$. There exists $r_0 \in [0, r)$ and an element $K \in \mathsf{W}(X)$ such that $S \subset K + \mathbb{B}_{r_0}$. Since f is a Φ-contraction with Φ-function ϕ, we have

$$f(S) \subset f(K) + \mathbb{B}_{\phi(r_0)} \subset \overline{f(K)}^{w} + \mathbb{B}_{\phi(r_0)}$$

where $\overline{f(K)}^{w}$ denote the closure of $f(K)$ in the weak topology. Since f is weakly sequentially continuous, using Eberlin-Smulian's theorem (see, for example [31, Theorem 1, p.430]), we infer that $\overline{f}^{w}$ is weakly compact. So

$$\omega(f(S)) \leq \phi(r_0) < \phi(r).$$

Now, using the continuity of ϕ and letting r goes to $\omega(S)$, we get

$$\omega(f(S)) \leq \phi(\omega(S)).$$

□

2.2 Superposition operators

Now we shall recall some facts concerning superposition operators required below. Let Ω be a subset of $\mathbb{R}^N$. A function $g : \Omega \times \mathbb{C} \longrightarrow \mathbb{C}$ is said to be a Carathéodory function on $\Omega \times \mathbb{C}$ if

$$\begin{cases} t \longrightarrow g(t,u) \text{ is measurable on } \Omega \text{ for all } u \in \mathbb{C}, \\ u \longrightarrow g(t,u) \text{ is continuous on } \mathbb{C} \text{ for almost all } t \in \Omega. \end{cases}$$

If g is a Carathéodory function, we can define the operator $\mathcal{N}_g$ on the set of functions $\psi : \Omega \longrightarrow \mathbb{C}$ by

$$(\mathcal{N}_g\psi)(z) := g(z, \psi(z))$$

for every $z \in \Omega$. The operator $\mathcal{N}_g$ is called the Nemytskii operator generated by g. In L^p-spaces the Nemytskii operator has been extensively investigated (see [8, 24, 44] and the references therein). However, we recall the following result due to Krasnosel'skii which states a basic fact for the theory of these operators on L^p.

Lemma 2.2.1 *Assume that g satisfies the Carathéodory conditions. If the operator $\mathcal{N}_g$ acts from L^p into L^p, then $\mathcal{N}_g$ is continuous and maps bounded sets into bounded sets. Moreover, there is a constant $\eta > 0$ and a function $h(.) \in L_p^+$ (the positive cone of L_p) such that*

$$|g(x,y)| \leq h(x) + \eta\,\|y\| \quad \text{a.e. in } x, \text{ for all } y \in \mathbb{R}.$$

Proof. For the proof we refer, for example, to [24, p. 35] or [44]. □

2.3 Multivalued maps

Now we introduce notations and definitions which are required in the paper. Let X be Banach space and define the sets

$$\mathsf{P}_{cv}(X) = \{M \subset X : M \text{ is nonempty and convex}\},$$

$$\mathsf{P}_{cl,cv}(X) = \{M \subset X : M \text{ is nonempty, convex and closed}\}.$$

Before going further we introduce the following definitions which are required below.

Definition 2.3.1 *Let (X, d) and (Y, d) be two metric spaces and let $F : X \to \mathsf{P}_{cl,cv}(Y)$ be a multivalued map. We say that F is upper semicontinuous if, for every open set U of Y, the set $F^{-1}(U)$ is open in X, where $F^{-1}(U) = \{x \in X : F(x) \subset U\}$.*

Definition 2.3.2 *Let X and Y be two metric spaces and $F : X \to \mathsf{P}_{cl,cv}(Y)$ be a multivalued map. A single valued map $f : X \to Y$ is called a selection of F if for every $x \in X$, $f(x) \in F(x)$.*

Definition 2.3.3 *Let M be a nonempty closed, convex subset of a Hausdorff locally convex linear topological space X, $\mu(\cdot)$ a measure of weak noncompactness on X and $F : M \to \mathsf{P}(X)$ a multivalued mapping. We say that F is condensing with respect to $\mu(\cdot)$ (or μ-condensing) if*

(a) $F(M)$ is bounded,

(b) $\mu(F(D)) < \mu(D)$, for all bounded subset D of M with $\mu(D) > 0$.

We recall the following lemma established in [3].

Lemma 2.3.1 *Let X be a Banach spaces and let $B : X \to X$ be a* ww*-compact map. If B is k-lipschitzian with $k \in \mathbb{R}$, then*

$$\omega(B(S)) \leq k\,\omega(S), \quad \text{for all } S \in \mathsf{B}(X).$$

We need also the following lemma.

Lemma 2.3.2 *Let X be a Banach space and let $B : X \to X$ be a k-contractive map for some $k \in [0, 1)$. If B is a* ww*-compact, then $(I - B)^{-1}$ is a* ww*-compact.*

Proof. We know that $I - B$ is invertible and $(I - B)^{-1} : X \to X$ is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points belonging to X which converges weakly to some $x \in X$. Thus the set $N = \left\{x_n : n \in \mathbb{N}\right\}$ is relatively weakly compact and therefore $\omega(N) = 0$. Moreover, it follows from the equality $(I - B)^{-1} = I + B(I - B)^{-1}$ that $\omega((I - B)^{-1}N) \leq \omega(N) + \omega(B(I - B)^{-1}N)$. Next, the use of Lemma 2.1.8 implies that $\omega((I - B)^{-1}N) \leq k\omega((I - B)^{-1}N)$ which is a contradiction. Thus $\omega((I - B)^{-1}N) = 0$ which prove that $(I - B)^{-1}N$ is weakly compact. Hence, the sequence $((I - B)^{-1}x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence which ends the proof. □

It is not difficult to check that the result of Lemma 2.3.2 remains valid if we replace the hypothesis "B is a ww-compact map" by "B is weakly sequentially continuous map". That is

Lemma 2.3.3 *Let X be a Banach space and let $B : X \to X$ be a k-contractive map for some $k \in [0, 1)$. If B is a weakly sequentially continuous map, then $(I - B)^{-1}$ is a weakly sequentially continuous map.*

The proof of this lemma is similar to that of Lemma 2.3.2 it suffices to replace in the proof Lemma 2.1.8 by Lemma 2.1.9.

Definition 2.3.4 *Let M be a subset of a Banach space X and let $B : M \to \mathsf{P}(M)$ be a multivalued mapping. A point $x \in M$ is called a fixed point of B if $x \in B(M)$.*

We close this section by recalling the following result owing to Himmelberg [40] which will play a crucial role in our further analysis.

Theorem 2.3.1 *Let X be a Hausdorff locally convex linear topological space and let $M \in \mathsf{P}_{cl,cv}(X)$. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a upper semicontinuous multivalued mapping such that $F(M)$ is relatively compact. Then there exists $x \in M$ such that $x \in F(x)$.*

Chapter 3

Fixed point results for countably condensing mappings

This section is devoted to fixed points results for countably condensing mappings. It is organised as follows. In Section 3.1, we give a fixed point of Altman's type. In Section 3.2, we present some fixed point results of Leray-Schauder's type. Section 3.3 deals with results of Krasnosel'skii's type where various kinds of perturbations were considered. In Section 3.4, results of Krasnosel'skii-Schaefer's type were discussed.

3.1 A fixed point result of Altman's type

Theorem 3.1.1 *Let K be a bounded open absorbing convex subset of a Banach space X, and let $f : \overline{K} \to X$ be a countably β-condensing map. If $\forall x \in \partial K$, $\|f(x) - x\|^2 \geq \|f(x)\|^2 - \|x\|^2$, then* $\mathsf{Fix}(f) \neq \emptyset$.

Proof. Define the function $r(\cdot)$ on X by $r(x) = (\max\{1, \mu_K(x)\})^{-1}x$ where $,\mu_K(x)$ is the Minkowski functional of K. Let $\lambda = (\max\{1, ,\mu_K(x)\})^{-1}$. It follows from Lemma 2.1.6 that $\lambda \in (0, 1]$. One can check easily that $r : X \to \overline{K}$ is a retraction (i.e. r is continuous, $r(x) = x$, $\forall x \in \overline{K}$). Moreover, we have $(r \circ f)(\overline{K}) \subset \overline{K}$.

We claim that the map $r(\cdot)$ is a countably 1-set-contraction. Indeed, since $\lambda \in (0, 1]$, for $C \in \mathbf{B}_c(K)$, we have $r(C) = \lambda C \subset co\{C, \theta\}$ and then $\beta(r(C)) \leq \beta(co\{C, \theta\}) = \beta(C)$ which proves our claim. Moreover, for any $C \in \mathbf{B}_c(K)$ with $\beta(C) > 0$, we have $\beta(r \circ f(C)) \leq \beta(f(C)) < \beta(C)$. Hence $r \circ f$ is countably β-condensing. Applying Theorem 2.1.1 we deduce that there exists $x_1 \in \overline{K}$ such that $x_1 = (r \circ f)(x_1) \in \overline{K}$.

• If $f(x_1) \in \overline{K}$, then $r(f(x_1)) = f(x_1)$ and then $f(x_1) = x_1$.

• If $f(x_1) \notin K$, then, by Lemma 2.1.6, $\mu_K(f(x_1)) \geq 1$ and therefore $x_1 = r(f(x_1)) = (\mu_K(f(x_1)))^{-1} f(x_1)$. Hence

$$\mu_K(x_1) = \mu_K(r(f(x_1))) = \mu_K\Big(\frac{f(x_1)}{\mu_K(f(x_1))}\Big) = \frac{\mu_K(f(x_1))}{\mu_K(f(x_1))} = 1$$

which proves that $x_1 \in \partial K$ (use Lemma 2.1.6).

Let us now suppose that $x_1 \in \partial K$. If $f(x_1) \in \overline{K}$, then $x_1 = r \circ f(x_1) = f(x_1)$. If $f(x_1) \notin \overline{K}$, then, by Lemma 2.1.6, we have $\mu_K(f(x_1)) > 1$ and thus $x_1 = r \circ f(x_1) = (\mu_K(f(x_1)))^{-1} f(x_1)$. Letting $k = \mu_K(f(x_1))$, one obtains $f(x_1) = kx_1$, and consequently,

$$\|f(x_1) - x_1\|^2 = (k-1)^2 \|x_1\|^2 \geq \|f(x_1)\|^2 - \|x_1\|^2 = (k^2 - 1)\|x_1\|^2 .$$

This implies that $(k-1)^2 \geq (k^2 - 1)$, and therefore $k \leq 1$. This contradicts the fact that $k = \mu_K(f(x_1)) > 1$. So, we have necessarily that $\mu_K(f(x_1)) \leq 1$ which prove that x_1 is a fixed point of f. □

An immediate consequence of Theorem 3.1.1 we have:

Corollary 3.1.1 *Let K be a bounded open absorbing convex subset of a Banach space X, and let $f : \overline{K} \to X$ be a countably β-condensing map. Assume that one of following condition is satisfied:*

(a) $\|f(x)\| \leq \|x\|$, $\forall x \in \partial K$ (Roth's condition type);

(b) $\|x - f(x)\| \geq \|f(x)\|$, $\forall x \in \partial K$ (Petryshyn's condition type);

(c) for Hilbert space X, $\langle f(x), x\rangle \leq \|x\|^2$, $\forall x \in \partial K$ (Krasnosel'skii's condition type).

Then $\mathsf{Fix}(f) \neq \emptyset$ *in* $\overline{K}$.

3.2 Fixed point results of Leray-Schauder's type

We start this section by establishing a result of Schaefer type for countably β-condensing operators .

Proposition 3.2.1 *Let X be a Banach space and let $f : X \to X$ be a map. If f is continuous, countably β-condensing, then*

(a) either $\mathsf{Fix}(f) \neq \emptyset$ *or,*

(b) the set $\{x \in X : x = \lambda f(x),$ for some $\lambda \in (0,1)\}$ is unbounded.

Proof. Let $\tau > 0$ and let ζ_τ be the radial retraction mapping from X into $\mathbb{B}_\tau$ (the closed ball with center θ and radius τ). One can define the mapping $J_\tau : \mathbb{B}_\tau \to \mathbb{B}_\tau$ by $J_\tau(x) = \zeta_\tau(f(x))$. It is clear that J_τ is continuous. Hence for each $x \in \mathbb{B}_\tau$, we have two possibilities:

• If $\|f(x)\| \leq \tau$, then $\zeta_\tau(f(x)) = f(x) \in f(\mathbb{B}_\tau) \subset \text{co}(f(\mathbb{B}_\tau) \cup \{0\})$.

• If $\|f(x)\| > \tau$, alors $\zeta_\tau(f(x)) = \dfrac{\tau}{\|f(x)\|} f(x) + (1 - \dfrac{\tau}{\|f(x)\|})0$ is an element of $\text{co}(f(\mathbb{B}_\tau) \cup \{0\})$.

Hence, in all cases we have $J_\tau(\mathbb{B}_\tau) \subset \text{co}(f(\mathbb{B}_\tau) \cup \{0\})$, and so, for any countably subset C of $\mathbb{B}_\tau$, we have $J_\tau(C) \subset \text{co}(f(C) \cup \{0\})$. Next, using the properties if $\beta(\cdot)$ and the fact that f is countably β-condensing we get

$$\beta(J_\tau(C)) \leq \beta(f(C) \cup \{0\}) < \beta(C).$$

This shows that J_τ is countably β-condensing. Now the use of Theorem 2.1.1 guarantees that there exists $z \in \mathbb{B}_\tau$ such that

$$J_\tau(z) = z = \zeta_\tau(f(z)).$$

Note that, if $\|f(z)\| \leq \tau$, then we have $\zeta_\tau(f(z)) = f(z) = z$, and so z is a fixed point of f.

Now if $\|f(z)\| > \tau$, then $\zeta_\tau(f(z)) = \dfrac{\tau}{\|f(z)\|} f(z) = z$, and therefore z is a solution to the equation $x = \lambda f(x)$ with $\lambda = \frac{\tau}{\|f(z)\|} \in (0, 1)$.

Hence, either, for some real $\tau > 0$, we obtain a solution of $f(x) = x$, or for each $\tau > 0$, we obtain an eigenvector of norm τ for some eigenvalue in $(0, 1)$. In the second case the set of such eigenvectors is unbounded. □

Theorem 3.2.1 *Let K be a nonempty convex closed and bounded subset of a Banach space X, and let $U \subset K$ be an open subset of K with $p \in U$. Let $f : \overline{U} \to K$ be a continuous countably β-condensing map and assume that $f(\overline{U})$ is bounded. Then*

(a) either $\mathsf{Fix}(f) \neq \emptyset$*, or*

(b) there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = \lambda f(u) + (1 - \lambda)p$,

here ∂U denotes the boundary of U in X.

We note that, since K is closed in X, the boundaries of U in X and in K are the same.

Proof. Suppose (b) is false and f has no fixed point on ∂U. Then $z \notin \lambda f(z) + (1 - \lambda)p$ for all $z \in \partial U$ and $\lambda \in (0, 1)$. Let

$$\Lambda := \{x \in \overline{U} : x = \lambda f(x) + (1 - \lambda)p \text{ for some } \lambda \in [0, 1]\}.$$

Note that $\Lambda \neq \emptyset$ is nonempty since $p \in \Lambda$ and, by the continuity of f, it is closed. Notice $\Lambda \cap \partial U = \emptyset$. By Uryshon's theorem (see, for example, Theorem 1.1.1 in [48]), there exists a

continuous mapping $\zeta : \overline{U} \to [0,1]$ separating Λ and ∂U, i.e., $\zeta(\Lambda) = 1$ and $\zeta(\partial U) = 0$. Define the function $\eta : K \to K$ by

$$\eta(x) := \begin{cases} \zeta(x)f(x) + (1-\zeta(x))p & \text{if } x \in \overline{U} \\ p & \text{if } x \in K\backslash\overline{U}. \end{cases}$$

Note that η is continuous. Further, for any subset $C \in \mathbf{B}_c(K)$ with $\beta(C) > 0$, the use of the inclusion $\eta(C) \subset \mathrm{co}(f(C \cap U) \cup \{ p\})$ implies that

$$\beta(\eta(C)) \leq \beta(f(C \cap U)).$$

Hence, if $\beta(C \cap U) = 0$, then $C \cap U$ is relatively compact, so $f(C \cap U)$ is relatively compact too, and therefore $\beta(\eta(C)) \leq \beta(f(C \cap U)) = 0 < \beta(C)$. Assume now that $\beta(C \cap U) \neq 0$, then $\beta(f(C \cap U)) < \beta(C \cap U) \leq \beta(C)$. Thus in both cases we have $\beta(\eta(C)) < \beta(C)$ which prove that η is countably β-condensing. It follows from Theorem 2.1.1 that there exists $x \in K$ such that $\eta(x) = x$. Since $p \in U$, the point $x \in U$, hence, $x = \zeta(x)f(x) + (1-\zeta(x))p$. Because $\zeta(x) \in [0,1]$, one sees that $x \in \Lambda$ and therefore $\zeta(x) = 1$ that is $x = f(x)$. □

If we take $p = 0$, then we obtain the following corollary.

Corollary 3.2.1 *Let K be a nonempty convex closed and bounded subset of a Banach space X, $U \subset K$ an open subset of K with $\theta \in U$. Let $f : \overline{U} \to K$ be a continuous countably β-condensing. If $f(\overline{U})$ is bounded, then*

(a) either $\mathsf{Fix}(f) \neq \emptyset$*, or*

(b) there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda f(u)$.

Proposition 3.2.2 *Let K be a nonempty convex closed and bounded subset of a Banach space X, U an open subset of K such that $\theta \in U$ and $f : \overline{U} \to K$ a continuous, countably 1-set-contractive map. If $f(\overline{U})$ is bounded and f is semi-closed, then*

(a) either $\mathsf{Fix}(f) \neq \emptyset$*, or*

(b) there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda f(u)$.

Proof. Suppose that (b) is false and f has no fixed point on ∂U. Define the maps f_n, $n \in \mathbb{N}$, by $f_n = t_n f$ where $(t_n)_{n\in\mathbb{N}}$ is a sequence of $(0,1)$ such that $t_n \to 1$ as $n \to +\infty$. The convexity of K together with the fact that $\theta \in K$ implies that f_n maps $\overline{U}$ into K. Because $t_n \in (0,1)$, the set $f_n(\overline{U})$ is bounded. Assume that $\lambda_n f_n(y_n) = y_n$ for some $y_n \in \partial U$ and some $\lambda_n \in (0,1)$. Therefore we have $y_n = \lambda_n t_n f(y_n)$, which contradicts our assumption because $\lambda_n t_n \in (0,1)$. Since f is countably 1-set-contractive, f_n is countably β-condensing. It follows from Theorem 2.1.1 that f_n has a fixed point say $x_n \in \overline{U}$. This yields $\|x_n - f(x_n)\| = \|t_n f(x_n) - f(x_n)\| = |t_n - 1|\,\|f(x_n)\|$. Using the fact that $f(\overline{U})$ is bounded, we deduce that $\|x_n - f(x_n)\| \to 0$ as $n \to +\infty$. Since f is semi-closed and K is closed, we conclude that $0 \in (I-f)(K)$, which ends the proof. □

3.3 Fixed point results for sums of mappings

We start our discussion by etablishing the following elementary results of Krasnosel'skii type.

Proposition 3.3.1 *Let K be a nonempty convex closed and bounded subset of a Banach space X, and let $f : K \to K$ and $g : K \to K$ be two continuous maps. Assume that f and g satisfy*

(a) $f(K)$ is relatively compact;

(b) g is countably β-condensing;

(c) $f(K) + g(K) \subseteq K$.

Then $\mathsf{Fix}(f + g) \neq \emptyset$.

Proof. Let $C \in \mathbf{B}_c(K)$. Using that fact that $f(C)$ is relatively compact, we get $\beta((f + g)(C)) \leq \beta(f(C)) + \beta(g(C)) \leq \beta(g(C)) < \beta(C)$. Hence the operator $f + g$ is countably β-condensing. The use of Theorem 2.1.1 concludes the proof. □

Proposition 3.3.2 *Let K be a nonempty convex closed and bounded subset of a Banach space X, and let $f : K \to K$ and $g : X \to X$ be two continuous maps. Assume that f and g satisfy*

(a) f is countably β-η-contractive for some $\eta \in (0, 1/2)$;

(b) g is $(1 - \eta)$-contractive;

(c) $(x = g(x) + f(y), y \in K) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(f + g) \neq \emptyset$.

Proof. It is well known that by conditions (b) and (c) the map $(I - g)^{-1} f$ is well defined and maps K into itself. Using Lemmas 2.1.1 and 2.1.3, we get

$$\beta((I - g)^{-1} f)(C)) \leq \frac{\eta}{1 - \eta} \beta(C) \text{ for all } C \in \mathbf{B}_c(K).$$

Since $\eta \in (0, 1/2)$, we have $\dfrac{\eta}{1 - \eta} < 1$. Hence, $(I - g)^{-1} f$ is countably β-condensing. Now, applying Theorem 2.1.1, we conclude that there exists $z \in K$ such that $z = (I - g)^{-1} f(z)$ which ends the proof. □

Theorem 3.3.1 *Let K be a nonempty convex closed and bounded subset of a Banach space X, and let $f : K \to X$ and $g : X \to X$ be two continuous maps. Assume that f and g satisfy*

(a) g *is pseudocontractive and countably* β*-*k*-contractive for some* $k \in [0, 1)$*;*

(b) $(I - g)$ *is* ψ*-expansive,*

(c) f *is countably* β*-*s*-contractive for some* $s \in (0, 1 - k)$*,*

(d) $\Big(x = g(x) + f(y), y \in K\Big) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(f + g) \neq \emptyset$.

Proof. (1) We show that the operator $I - g : X \to (I - g)(X)$ is bijective. Consider $x, y \in X$ with $x \neq y$. Since $(I - g)$ is ψ-expansive, we have

$$\|(I - g)(x) - (I - g)(y)\| \geq \psi(\|x - y\|).$$

Thus $(I - g)$ is one-to-one, so $I - g : X \to (I - g)(X)$ is bijective. Arguing as in the point (b) of the proof of Theorem 3.3 in [34] one sees that $(I - g)^{-1} f(K) \subset K$ and so $f(K) \subset (I - g)(K)$.

(2) We claim that $(I - g)^{-1} f$ is continuous. Indeed, it is clear that $(I - g)^{-1} f : K \to K \iff (I - g)^{-1} : f(K) \to K \iff (I - g)^{-1} : (I - g)(K) \to K$.

Let show that $(I - g)^{-1} : (I - g)(K) \to K$ is continuous. We first assume that ψ is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of $(I - g)(K)$ such that $(x_n)_{n\in\mathbb{N}}$ converges to a point $x_0 \in (I - g)(K)$ and set $y_n = (I - g)^{-1} x_n$ and $y_0 = (I - g)^{-1} x_0$. So we have $(I - g)y_n = x_n$ and $(I - g)y_0 = x_0$. Because $(I - g)$ is ψ-expansive, we get

$$\psi(\|y_n - y_0\|) \leq \|(I - g)(y_n) - (I - g)(y_0)\| = \|x_n - x_0\|.$$

Since ψ is continuous with $\psi(0) = 0$ and $\psi(r) > 0$, for all $r > 0$, we get $(I-g)^{-1}(x_n) \to (I-g)^{-1}(x_0)$, which proves the continuity of $(I - g)^{-1}$.

We suppose now that ψ is increasing. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of $(I - g)(K)$ such that $(x_n)_{n\in\mathbb{N}}$ converges to some point $x_0 \in (I - g)(K)$ and set $y_n = (I - g)^{-1} x_n$ and $y_0 = (I - g)^{-1} x_0$. It is assumed that the sequence $(y_n)_{n\in\mathbb{N}}$ does not converge to y_0. Hence, there exists $\epsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ from $(x_n)_{n\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$, we have

$$\left\|(I - g)^{-1}(x_{n_k}) - (I - g)^{-1}(x_0)\right\| > \epsilon.$$

We put $\delta := \psi(\epsilon) > 0$. Because $(x_n)_{n\in\mathbb{N}}$ converges to x_0, we have $\left\|x_{n_k} - x\right\| < \delta$ for k large enough. So

$$\delta = \psi(\epsilon) \leq \psi(\left\|(I - g)^{-1}(x_{n_k}) - (I - g)^{-1}(x_0)\right\|) \leq \left\|x_{n_k} - x\right\| < \delta,$$

which is absurd. Hence the operator $(I - g)^{-1}$ is continuous and so is $(I - g)^{-1} f$.

(3) We shall show now that $(I - g)^{-1} f$ is countably β-condensing. Using the equation

$$(I - g)^{-1} f = f + g(I - g)^{-1} f \tag{3.3.1}$$

we get, for all $C \in \mathbf{B}_c(K)$

$$\beta((I-g)^{-1}f(C)) \leq \beta(f(C)) + \beta(g(I-g)^{-1}f(C)) \leq s\beta(C) + k\beta((I-g)^{-1}f(C))$$

and therefore

$$\beta((I-g)^{-1}f(C)) \leq \frac{s}{1-k}\beta(C).$$

This proves that $(I-g)^{-1}f : K \to K$ is countably β-condensing.

To conclude, it suffices to apply Theorem 2.1.1 to the map $(I-g)^{-1}f$. □

Corollary 3.3.1 *Let K be a nonempty convex closed and bounded subset of a Banach space X, and let $f : K \to X$ and $g : X \to X$ be two continuous maps. Assume that f and g satisfy*

(a) g is pseudocontractive and, for all $B \in \mathbf{B}(X)$, $g(B)$ is a relatively compact;

(b) $(I-g)$ is ψ-expansive;

(c) f is countably β-condensing;

(d) $(x = g(x) + f(y), y \in K) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(f+g) \neq \emptyset$.

Proof. Arguing as in the proof of Theorem 3.3.1 one sees that the operator $\zeta : K \to K$ defined by $\zeta(x) = (I-g)^{-1}f(x)$ is well defined and continuous. Using (3.3.1) one sees that $\zeta = f + g\zeta$, so, for any subset $C \in \mathbf{B}_c(K)$, we have $\beta(\zeta(C)) \leq \beta(f(C)) + \beta(g(\zeta(C))) < \beta(C)$ (use the hypothesis (b) and the fact that g maps bounded sets into relatively compact ones). This shows that ζ is countably β-condensing. The result is now follows from Theorem 2.1.1. □

Theorem 3.3.2 *Let K be a nonempty convex closed and bounded subset of a Banach space X and let $f : K \to X$ and $g : X \to X$ be continuous maps. Assume that f and g satisfy*

(a) $f(K)$ is relatively compact;

(b) g is pseudocontractive and countably 1-set-contractive;

(c) $g_{|K}$ is nonexpansive;

(d) if $(x_n)_{n\in\mathbb{N}}$ is a sequence of element of K such that $(I-g)(x_n) \to y$, then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence;

(e) if $\lambda \in (0,1)$ and $x = \lambda g(x) + f(y)$, for some $y \in K$, then $x \in K$.

Then $\mathsf{Fix}(f+g) \neq \emptyset$.

Before establishing this result, we first recall the following facts in connection with accretive maps. To this end, we denote by $[\cdot,\cdot]_s$ the directional derivative of the function $x \to \|x\|$, that is,

$$[x,y]_s = \lim_{\lambda \searrow 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}, \quad \forall x, y \in X.$$

A characterization of $[\cdot,\cdot]_s$ is given by (see, for example, [11, 14])

$$[x,y]_s = \max\{\langle y, x^*\rangle \,:\, x^* \in J(x)\}, \quad \forall x, y \in X,$$

where $J(\cdot)$ denotes the duality mapping from X into 2^{X^*} defined by

$$J(x) = \{x^* \in X^* \text{ such that } \langle x^*, x\rangle = \|x\| \text{ and } \left\|x^*\right\| \leq 1\}.$$

It is well known that an operator f is accretive if and only if, for any $x, y \in D(A)$, there exists $x^* \in J(x-y)$ such that

$$\langle f(x) - f(y), x^*\rangle \geq 0.$$

For more information, we refer the reader, for example, to the books [11, 14] and the references there in.

Proof of Theorem 3.2. It is clear that λg is continuous and β-λ-contractive for any $\lambda \in (0,1)$.

We claim that, for any $\lambda \in (0,1)$, the operator λg is pseudocontractive. Indeed, let $\lambda \in (0,1)$. Since g is pseudocontractive, by Remark 2.1.2, the operator $I-g$ is accretive; so, as it was remembered before the proof, for every $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-g)x - (I-g)y, j(x-y)\rangle \geq 0.$$

Hence, one can write

$$\begin{aligned}
\big[(I-\lambda g)(x) - (I-\lambda g)(y), x-y\big]_s &\geq \langle (I-\lambda g)(x) - (I-\lambda g)(y), j(x-y)\rangle \\
&= \langle \lambda x - \lambda g(x) - \lambda y + \lambda g(y), j(x-y)\rangle \\
&\qquad + \langle (1-\lambda)(x-y), j(x-y)\rangle \\
&= \lambda \langle (I-g)(x) - (I-g)(y), j(x-y)\rangle \\
&\qquad + (1-\lambda)\langle \lambda x - y, j(x-y)\rangle \\
&\geq (1-\lambda)\|x-y\|^2 \geq 0.
\end{aligned}$$

Hence, the operator λg is pseudocontractive. On the other hand, the same inequality yields that

$$\begin{aligned}\|(x-\lambda g(x))-(y-\lambda g(y))\|\,\|x-y\| &\geq \langle x-\lambda g(x)-(y-\lambda g(y)),x-y\rangle_s\\ &\geq (1-\lambda)\,\|x-y\|^2 \geq 0.\end{aligned}$$

Consequently,

$$\|(x-\lambda g(x))-(y-\lambda g(y))\| \geq \psi(\|x-y\|),$$

where $\psi(t)=(1-\lambda)t$. This proves that the map $I-\lambda g$ is ψ-expansive. The above arguments show that f and λg satisfy the conditions of Theorem 3.3.1. Hence, for each $\lambda \in (0,1)$, there exists $x_\lambda \in K$ such that

$$x_\lambda = \lambda g(x_\lambda) + f(x_\lambda).$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of $(0,1)$ such that $\lambda_n \to 1$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K such that

$$x_n = \lambda_n g(x_n) + f(x_n), \quad \forall n \geq 0. \tag{3.3.2}$$

Using the fact that $f(K)$ is relatively compact and passing eventually to a subsequence, we may assume that $(f(x_n))_{n\in\mathbb{N}}$ converges to some $y \in K$. Accordingly

$$f(x_n) = x_n - \lambda_n g(x_n) = (I-\lambda_n g)(x_n) \to y.$$

Since K is bounded and $g_{|K}$ is nonexpansive, we infer that the sequence $\big(g(x_n)\big)_{n\in\mathbb{N}}$ is bounded. Consequently,

$$\|(x_n-g(x_n))-(x_n-\lambda_n g(x_n))\| = (1-\lambda_n)\,\|g(x_n)\| \to 0 \quad \text{as } n \to +\infty,$$

and therefore

$$x_n - g(x_n) \to y.$$

In view of (d) there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which converges to some $z \in K$. The use of (3.3.2) and the fact that f and g are continuous, imply that $f(z)+g(z)=z$. □

Theorem 3.3.3 *Let K be a nonempty convex closed and bounded subset of a Banach space X. Suppose that $f : K \to X$ and $g : X \to X$ are two maps such that*

(a) f is continuous, countably β-k-contractive for some $k \in [0,1)$, and $f(K)$ is relatively compact;

(b) g is a nonlinear ϕ-contraction with $\phi(r) < (1-k)r$ for $r > 0$;

(c) $\Big(x = g(x) + f(y), y \in K\Big) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(f+g) \neq \emptyset$.

Proof. By Lemma 2.1.2, $I-g$ has a continuous inverse on X, and then $J := (I-g)^{-1}f$ is well defined on K. By rotinuous calculations, one sees that J is continuous and $J(K) \subset K$. Let $C \in \mathbf{B}_c(K)$ be such that $\beta(C) > 0$. Using equation (3.3.1) together with the hypotheses (a), (b) and Lemma 2.1.5, we get

$$\beta(J(C)) \leq \beta(f(C)) + \beta(gJ(C)) \leq k\beta(C) + \phi(\beta(J(C))). \tag{3.3.3}$$

If $k = 0$, inequality (3.3.3) becomes $\beta(J(C)) \leq \phi(\beta(J(C)))$, and then $\beta(J(C)) = 0$ (use the fact that g is a nonlinear ϕ-contraction). Otherwise, the use of the fact, for $r > 0$, $\phi(r) < (1-k)r$, inequality (3.3.3) becomes

$$\beta(J(C)) < k\beta(C) + (1-k)\beta(J(C)), \text{and therefore } \beta(J(C)) < \beta(C).$$

In both cases, J is countably β-condensing. Now the use of Theorem 2.1.1 achieves the proof. □

Theorem 3.3.4 *Let K be a bounded open absorbing convex set of a Hilbert space H, and let $f : \overline{K} \to H$ and $g : H \to H$ be two mappings such that*

(a) f is countably β-k-contractive, for some $k \in (0,1)$;

(b) g is expansive with constant $h > k+1$;

(c) $f(\overline{K}) \subset (I-g)(X)$;

(d) $\theta \in K$ and $\langle x, G(f(x)\rangle \leq \|x\|^2$ for each $x \in \partial K$, where $G = (I-g)^{-1}$.

Then $\mathsf{Fix}(f+g) \neq \emptyset$ *in* $\overline{K}$.

Proof. For each $x \in \overline{K}$, by (c), we see that there exists $y \in H$ such that $y - g(y) = f(x)$. By Lemma 2.1.4, we have $y = (G \circ f(x)) \in H$ with $G(x) := (I-g)^{-1}(x)$. It follows from Lemma 2.1.4, (b) and (a) that $G \circ f : \overline{K} \to H$ is $\dfrac{k}{k-1}$-contractive because $\dfrac{k}{h-1} < 1$ and therefore from Lemma 2.1.1 it is countably β-condensing. In what follows, we shall check that the condition (c) of Corollary 3.1.1 is satisfied. Indeed for $x \in \partial K$, the use of the equality $\|\rho(f(x)) - x\|^2 = \|\rho(f(x))\|^2 + \|x\|^2 - 2\langle\rho(f(x)), x\rangle$ together with the hypothesis (d) we derive the estimate

$$\|\rho(f(x))\|^2 - \|x\|^2 - \|\rho(f(x)) - x\|^2 = 2[(x, \rho(f(x)) - \|x\|^2] \leq 0.$$

This shows that the condition (c) of Corollary 3.1.1 is satisfied. This ends the proof. □

3.4 Fixed point results of Krasnoselskii-Schaefer's type

Let X be a Banach space. We say that a map f from X into itself is completely continuous if it maps each bounded set into a relatively compact one .

Theorem 3.4.1 *Let X be a Banach space and let $f, g : X \to X$ be two continuous mappings. If f and g satisfy the conditions*

(a) g is pseudocontractive and β-k-contractive, for some $k \in (0,1)$;

(b) $I - g$ is ψ-expansive where ψ is either strictly increasing or $\lim_{r\to\infty} \psi(r) = \infty$;

(c) f is countably β-s-contractive for some $s \in (0, 1-k)$.

Then

(i) either $\mathsf{Fix}(f+g) \neq \emptyset$, or

(ii) the set $\{x \in X \text{ such that } x = \lambda g(\frac{x}{\lambda}) + \lambda f(x) \text{ for some } \lambda \in (0,1)\}$ is unbounded.

Proof. It is easy to check that $x \in X$ is a solution of the equation $x = f(x) + g(x)$ if and only if x is a fixed point of the map $(I-g)^{-1}f$, whenever it is well defined. According to the proof of Theorem 3.2 in [35] and Theorem 3.3 in [34], we can prove that $(I-g)^{-1}f : X \to X$ is well defined and continuous. By Proposition 3.2.1 we only need to show that $(I-g)^{-1}f$ is countably β-condensing. Let $C \in \mathbf{B}_c(X)$ such that $\beta(C) > 0$. We first show that $(I-g)^{-1}f(C)$ is bounded. Let $x, y \in (I-g)^{-1}f(C)$. Hence, there exist $z_1, z_2 \in C$ such that $x = (I-g)^{-1}f(z_1)$, $y = (I-g)^{-1}f(z_2)$. Then, $x - g(x) = f(z_1)$, $y - g(y) = f(z_2)$. Since $I - g$ is ψ-expansive, we can write

$$\begin{aligned}\psi(\|x-y\|) &\leq \|x - g(x) - (y - g(y))\| \\ &= \|f(z_1) - f(z_2)\| \\ &\leq \text{ diam } (f(C)) < +\infty.\end{aligned}$$

If $(I-g)^{-1}f(C)$ is not a bounded set, then there exist $x_n, y_n \in (I-g)^{-1}f(C)$ such that $\|x_n - y_n\| \to +\infty$. Hence,

$$\psi(\|x_n - y_n\|) \leq \text{ diam } (f(C)).$$

If ψ is such that $\lim_{r\to\infty} \psi(r) = \infty$, then necessarily diam $(f(C)) = +\infty$, which is a contradiction. Else, if ψ is strictly increasing, then ψ has an inverse on $[0, +\infty)$, which is strictly increasing as well. Then

$$\|x_n - y_n\| \leq \psi^{-1}(\text{ diam } (f(C))) < +\infty,$$

which gives another contradiction, Hence, in any case, $(I-g)^{-1}f(C)$ is bounded.

Next using equation (3.3.1), one sees that

$$\begin{aligned}\beta((I-g)^{-1}f(C)) &\leq \beta(f(C)) + \beta(g(I-g)^{-1}f(C)) \\ &\leq s\beta(C) + k\beta((I-g)^{-1}f(C)) \\ &\leq \frac{s}{1-k}\beta(C) \\ &< \beta(C).\end{aligned}$$

which shows that $(I-g)^{-1}f$ is countably β-condensing. Now the use of Proposition 3.2.1 completes the proof. □

Theorem 3.4.2 *Let X be a Banach space and let $f,\ g:\ X \to X$ be two continuous mappings. If f and g satisfy the conditions*

(a) f is completely continuous;

(b) g is pseudocontractive and countably β-condensing;

(c) $I-g$ is ψ-expansive where ψ is either strictly increasing or $\lim_{r\to\infty}\psi(r) = \infty$.

Then

(i) either $\mathsf{Fix}(f+g) \neq \emptyset$, or

(ii) the set $\{x \in X \text{ such that } x = \lambda g(\frac{x}{\lambda}) + \lambda f(x) \text{ for some } \lambda \in (0,1)\}$ is unbounded.

Proof. In view of the proof of a Theorem 3.4.1, the map $(I-g)^{-1}f : X \to X$ is well defined, and continuous. Let $C \in \mathbf{B}_c(X)$ be such that $\beta(C) > 0$. Using equality (3.3.1), we get

$$\begin{aligned}\beta((I-g)^{-1}f(C)) &\leq \beta(f(C)) + \beta(g(I-g)^{-1}f(C)) \\ &= \beta(g(I-g)^{-1}f(C)) \\ &< \beta((I-g)^{-1}f(C)),\end{aligned}$$

which is a contradiction. So $(I-g)^{-1}f$ maps bounded sets into relatively compact ones, so $(I-g)^{-1}f$ is completely continuous on X. Now the use of Theorem 3.5.2 in [48] completes the proof. □

Theorem 3.4.3 *Let X be a Banach space. Suppose that $f, g : X \to X$ are two continuous mappings satisfying the conditions*

(a) f is countably β-k-contractive, for some $k \in [0,1)$;

(b) g is a nonlinear ϕ-contraction;

(c) ϕ *satisfies, for each* $r > 0$, $\phi(r) < (1-k)r$ *and* $\lim_{r\to\infty}[r - \phi(r)] = +\infty$.

Then

(i) either $\mathsf{Fix}(f+g) \neq \emptyset$, *or*

(ii) the set $\{x \in X \text{ such that } x = \lambda g(\frac{x}{\lambda}) + \lambda f(x) \text{ for some } \lambda \in (0,1)\}$ *is unbounded.*

Arguing as in the proofs of the previous theorems, one sees that $J := (I-G)^{-1}f$ is well defined and continuous. Let us prove that $J := (I-g)^{-1}f$ maps bounded sets into a bounded sets. Let A be a bounded set and let $x, y \in (I-g)^{-1}f(A)$. Hence, there exist $z_1, z_2 \in A$ such that $x = (I-g)^{-1}f(z_1)$, $y = (I-g)^{-1}f(z_2)$ or again $x - g(x) = f(z_1)$, $y - g(y) = f(z_2)$. By assumption (b) and the boundedness of $f(A)$, we have

$$\begin{aligned} \|(x-y) - (g(x)-g(y))\| &= \|f(z_1) - f(z_2)\| \\ \|x-y\| - \|g(x)-g(y)\| &\leq \|f(z_1) - f(z_2)\| \\ \|x-y\| - \phi(\|x-y\|) &\leq \|f(z_1) - f(z_2)\| \\ &\leq \text{ diam } (f(A)) < +\infty. \end{aligned}$$

If $J(A) = (I-g)^{-1}f(A)$ is unbounded, then there exist $x_n, y_n \in (I-g)^{-1}f(A)$ such that $\|x_n - y_n\| \to +\infty$.
Using assumption (c) we infer that $\|x_n - y_n\| - \phi(\|x_n - y_n\|) \to +\infty$. This is a contradiction with diam $(f(A)) < +\infty$. Hence $J(A)$ is bounded.

Let $C \in \mathbf{B}_c(X)$ with $\beta(C) > 0$. By equation (3.3.1), we have $J(C) \subset f(C) + g(J(C))$. Since f is countably β-k-contractive and g satisfies $\beta(g(J(C))) \leq \phi(\beta(J(C)))$ we obtain

$$\beta(J(C)) \leq \beta(f(C)) + \beta(gJ(C)) \leq k\beta(C) + \phi(\beta(J(C))). \tag{3.4.1}$$

If $k = 0$, inequality (3.4.1) becomes $\beta(J(C)) \leq \phi(\beta(J(C)))$ and therefore $\beta(J(C)) = 0$. Otherwise, by recalling the assumption that $\phi(r) < (1-k)r$ for $r > 0$, inequality (3.4.1) becomes $\beta(J(C)) < \beta(C)$. In both cases, J is countably β-condensing. Now, applying Proposition 3.2.1 to J we obtain the desired result. □

Chapter 4

Some fixed point theorems for ws-compact and ww-compact operators

This chapter is dedicated to fixed point results for Schauder's type and Krasnoselkii's type for ws-compact and ww-compact operators. As a consequence we derive similar results for Dunford-Pettis operators and weakly compact operators on spaces with the Dunford-Pettis Property because these two classes of operators are ws-compact operators.

4.1 Fixed point theorems of Schauder's type

In this section we shall present some fixed point theorems involving two or three mappings.

Theorem 4.1.1 *Let X be a Banach space and let K be a nonempty closed convex subset of X. Let $f : X \to K$ and $g : K \to X$ be two continuous maps and set $F = f \circ g$. Assume that*

(a) $F(K)$ is relatively weakly compact,

(b) f satisfies ws-compact,

(c) g satisfies ww-compact.

Then $\mathsf{Fix}(F) \neq \emptyset$.

Proof. Let $C = \overline{\mathrm{co}}(F(K))$. Since K is a closed convex subset of X satisfying $F(K) \subseteq K$, then $C \subseteq K$ and therefore $F(C) \subseteq F(K) \subseteq \overline{co}(F(K)) = C$. This shows that F maps C into itself. Since $F(K)$ is relatively weakly compact, by Krein-Šmulian theorem [31, p. 434], C is weakly compact too.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C. Using the fact that C is weakly compact and g satisfies ww-compact, we infer that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(g(x_{n_k}))_{k\in\mathbb{N}}$ converges weakly in C. Next, using the fact that f is satisfies ws-compact, we conclude that there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ such that the sequence $(F(x_{n_{k_j}}))_{j\in\mathbb{N}}$ converges strongly in C. This shows that F satisfies condition ws-compact. Because $F(C)$ is relatively weakly compact, the use of Theorem 2.1.3 concludes the proof. □

Remark 4.1.1 *Note that, the identity operator I of the space X belongs to $\mathcal{L}(X)$, so it satisfies* ww*-compact (cf. Remark 2.1.3). Hence, if we take $g = I$ in Theorem 4.1.1, then we recapture Theorem 2.1.3.* □

Remark 4.1.2 *By definition, linear Dunford-Pettis operators map weakly compact sets into norm compact sets, so they satisfy condition* ws*-compact.* □

Corollary 4.1.1 *Let X be a Banach space and let K be a nonempty closed convex subset of X. Let $f : X \to K$ and $g : K \to X$ be two continuous maps and set $F = f \circ g$. Assume that*

(a) $F(K)$ is relatively weakly compact,

(b) f is a linear Dunford-Pettis operator,

(c) g satisfies ww*-compact.*

Then $\mathsf{Fix}(F) \neq \emptyset$.

Proof. We note that according to Remark 4.1.2, the operator f satisfies condition ws-compact, so the result follows from Theorem 4.1.1. □

Remark 4.1.3 *A Banach space X has the Dunford-Pettis property if every linear weakly compact operator defined on X takes weakly compact sets into norm compact sets. It is clear that if X is a Banach space with the Dunford-Pettis property, then every linear weakly compact operator on X is a Dunford-Pettis operator.* □

Making use of this observation we get the following result.

Corollary 4.1.2 *Let X be a Banach space with the Dunford-Pettis property and let K be a nonempty closed convex subset of X. Let $f : X \to K$ and $g : K \to X$ be two continuous maps and set $F = f \circ g$. Assume that*

(a) f is a linear weakly compact operator,

(b) g satisfies ww*-compact and $g(K)$ is bounded.*

Then $\mathsf{Fix}(F) \neq \emptyset$.

Proof. Since X is a Banach space with Dunford-Pettis property and f is a weakly compact linear operator, according to Remark 4.1.3, f is a Dunford-Pettis operator. To complete the proof, it suffices to show that $F(K)$ is relatively weakly compact. This follows from the fact that $g(K)$ is bounded and hypothesis (a). Now the use of Corollary 4.1.1 ends the proof. □

4.2 Fixed point theorems of Krasnosel'skii's type

Theorem 4.2.1 *Let X be a Banach space, K a nonempty closed bounded convex subset of X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $f : X \to K$, $g : K \to X$ and $B : K \to X$ be continuous maps and set $F = f \circ g$. Suppose that*

(a) f satisfies ws*-compact,*

(b) there exists $\gamma \in (0, 1)$ such that $\mu(F(S) + B(S)) \leq \gamma\mu(S)$ for all $S \subset K$ with $\mu(S) > 0$,

(c) g satisfies ww*-compact,*

(d) B is a nonlinear contraction with Φ-function ϕ,

(e) $\Big(x = B(x) + F(y),\ y \in K\Big) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(F + B) \neq \emptyset$.

Proof. Since B is a nonlinear contraction with Φ-function ϕ, it is known that $(I-B)$ is a homeomorphism from K onto $(I - B)(K)$. Let y be a point in K and define the map $K \ni x \mapsto B(x) + F(y)$. It is clear that it is a nonlinear contraction with Φ-function ϕ. Hence, according to Boyd-Wong's theorem (see, for example, [17] or [48, Theorem 2.7.2, p. 52]), the equation $z = B(z)+F(y)$ has a unique solution $x \in X$. Using hypothesis (e) we infer that $x \in K$ and therefore $x = (I - B)^{-1}F(y) \in K$. Hence,

$$(I - B)^{-1}F(K) \subset K. \tag{4.2.1}$$

We define a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of K by

$$K_0 = K \ \text{ and } \ K_{n+1} = \overline{co}((I - B)^{-1}F(K_n)).$$

It is clear that $(K_n)_{n\in\mathbb{N}}$ is a sequence of nonempty closed convex subsets of K. Moreover, the inclusion (4.2.1) shows that $(K_n)_{n\in\mathbb{N}}$ is decreasing (in the sense of the inclusion). Furthermore, elementary calculations show that

$$(I - B)^{-1}F = F + B(I - B)^{-1}F. \tag{4.2.2}$$

Hence, using equation (4.2.2) we get

$$(I-B)^{-1}F(K_n) \subset F(K_n) + B(\overline{\text{co}}((I-B)^{-1}F(K_n))) \subset F(K_n) + B(K_{n+1}).$$

Since $(K_n)_{n\in\mathbb{N}}$ is decreasing, we get $(I-B)^{-1}F(K_n) \subset F(K_n) + B(K_n)$. Thus, assumption (b) yields

$$\mu(K_{n+1}) \leq \mu(F(K_n) + B(K_n)) \leq \gamma\mu(K_n).$$

By induction we get $\mu(K_{n+1}) \leq \gamma^n\mu(K)$ and therefore $\lim_{n\to+\infty} \mu(K_n) = 0$ because $\gamma \in (0,1)$. Hence, it follows from the generalized Cantor intersection theorem that $K_\infty = \bigcap_{n\geq 0} K_n$ is a nonempty convex weakly compact subset of K. Moreover, we have $(I-B)^{-1}F(K_\infty) \subset K_\infty$ and consequently the set $(I-B)^{-1}F(K_\infty)$ is relatively weakly compact.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K_∞, so it has a weakly convergent subsequence which we denote again $(x_n)_{n\in\mathbb{N}}$. Using hypothesis (c), one sees that there exists a subsequence denoted by $(x_{n_k})_{k\in\mathbb{N}}$ such that $(g(x_{n_k}))_{k\in\mathbb{N}}$ is weakly convergent. Next, using the fact that f satisfies ws-compact, we conclude that there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ such that $(f(g(x_{n_{k_j}})))_{j\in\mathbb{N}}$ converges strongly in K_∞. Because $(I-B)^{-1}$ is continuous, the sequence $(I-B)^{-1}(f(g(x_{n_{k_j}})))_{j\in\mathbb{N}}$ converges strongly in K_∞. Hence, the map $(I-B)^{-1}F$ satisfies condition ws-compact. So, invoking the fact that $(I-B)^{-1}F(K_\infty)$ is relatively weakly compact and Theorem 2.1.3, we conclude that $\mathsf{Fix}((I-B)^{-1}F) \neq \emptyset$, which concludes the proof. □

As an immediate consequence of Theorem 4.2.1 and Remark 4.1.2 we have:

Corollary 4.2.1 *Let X be a Banach space, K a nonempty closed bounded convex subset of X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $f : X \to K$, $g : K \to X$ and $B : K \to X$ be continuous maps and set $F = f \circ g$. Suppose that*

(a) f is a Dunford-Pettis operator,

(b) there exists $\gamma \in (0,1)$ such that $\mu(F(S) + B(S)) \leq \gamma\mu(S)$ for all $S \subset K$ with $\mu(S) > 0$,

(c) g satisfies ww*-compact,*

(d) B is a nonlinear contraction with Φ-function ϕ,

(e) $\Big(x = B(x) + F(y),\ y \in K\Big) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(F+B) \neq \emptyset$.

Note that if X is Banach space with the Dunford-Pettis property, then, according to Remark 4.1.3, every weakly compact linear operator on X is a Dunford-Pettis operator. This yields the following result.

Corollary 4.2.2 *Let X be a Banach space with the Dunford-Pettis property, K a nonempty closed bounded convex subset of X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $f : X \to K$, $g : K \to X$ and $B : K \to X$ be continuous maps and set $F = f \circ g$. Suppose that*

(a) f is a weakly compact linear operator,

(b) there exists $\gamma \in (0,1)$ such that $\mu(F(S) + B(S)) \leq \gamma\mu(S)$ for all $S \subset K$ with $\mu(S) > 0$,

(c) g satisfies ww*-compact and $g(K)$ is bounded,*

(d) B is a nonlinear contraction with Φ-function ϕ,

(e) $\Big(x = B(x) + F(y),\ y \in K\Big) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(F + B) \neq \emptyset$.

Remark 4.2.1 *Note that according to Remark 2.1.2, if B is a contractive mapping, then Theorem 4.2.1 and Corollaries 4.2.1 and 4.2.2 hold true.* □

Theorem 4.2.2 *Let X be a Banach space, K a nonempty closed bounded convex subset of X. Let $f : K \to K$, $g : K \to K$ and $B : K \to X$ be continuous maps and set $F = f \circ g$. Suppose that*

(a) f satisfies ws*-compact,*

(b) $F(K)$ is relatively weakly compact,

(c) g satisfies ww*-compact,*

(d) B is k-contractive for some $k \in (0,1)$ and satisfies ww*-compact,*

(e) $\Big(x = B(x) + F(y),\ y \in K\Big) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(F + B) \neq \emptyset$.

Proof. Because B is k-contractive, we know that $(I - B)$ is a homeomorphism from K onto $(I - B)(K)$. Let y be a point in K and define the map
$K \ni x \mapsto B(x) + F(y)$. It is clear that it is k-contractive, hence by the Banach contraction principle, the equation $z = B(z) + F(y)$ has a unique solution $x \in X$. Using hypothesis (e), we infer that $x \in K$ and therefore $x = (I - B)^{-1}F(y) \in K$. Hence $(I - B)^{-1}F(K) \subset K$. We define a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of K by

$$K_0 = K \ \text{ and } \ K_{n+1} = \overline{co}((I - B)^{-1}F(K_n)).$$

Thus, $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty closed convex subsets of K. Further, using equation (4.2.2), we see that

$$(I-B)^{-1}F(K_n) \subset F(K_n) + B\Big(\overline{co}\big((I-B)^{-1}F(K_n)\big)\Big) \subset F(K_n) + B(K_n).$$

The properties of the measure of weak noncompactness $\omega(\cdot)$, assumptions (b) and (d) and Eq. (2.1.3) yield that

$$\omega(K_{n+1}) \leq \omega\big(F(K_n)\big) + \omega\big(B(K_n)\big) \leq k\omega(K_n).$$

Hence, for all $n \in \mathbb{N}$, we have $\omega(K_n) \leq k^n\omega(K)$ and therefore $\lim\limits_{n\to+\infty} \omega(K_n) = 0$. By the generalized Cantor intersection theorem we conclude that $K_\infty = \bigcap_{n\geq 0} K_n$ is a nonempty convex weakly compact subset of K and
$(I-B)^{-1}F(K_\infty) \subset K_\infty$. This implies that $(I-B)^{-1}F(K_\infty)$ is a relatively weakly compact subset of X.

To complete the proof, it suffices to argue as in the last paragraph of the proof of Theorem 4.2.1. □

We recall now the following result established in [33].

Lemma 4.2.1 *Let X be a Banach space and let K be a nonempty closed bounded and convex subset of X. If $B : K \to K$ is a nonexpansive mapping such that $I - B$ is ψ-expansive, then B has a unique fixed point in K.*

Theorem 4.2.3 *Let X be a Banach space and let K be a nonempty closed bounded convex subset of X. Let $f : K \to K$, $g : K \to K$ and $B : K \to X$ be continuous maps and set $F = f \circ g$. Suppose that*

(a) f satisfies ws*-compact and $f(K)$ is relatively weakly compact,*

(b) g satisfies ww*-compact,*

(c) B is nonexpansive and ω-condensing,

(d) $I - B$ is ψ-expansive,

(e) $\Big(x = B(x) + F(y),\ y \in K\Big) \Longrightarrow x \in K$.

Then $\mathsf{Fix}(F + B) \neq \emptyset$.

Proof. Note that according to the proof of Theorem 3.2 in [35], under assumptions (c) and (d) the operator $(I-B)$ is invertible, $(I-B)^{-1}$ is continuous and its domain contains the range of F. Since f and g are continuous, the operator $(I-B)^{-1}F : K \to K$ is continuous and maps K into itself.

We claim that $(I-B)^{-1}F$ satisfies condition ws-compact. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence of points of K, then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(g(x_{n_k}))_{k\in\mathbb{N}}$ converges in the weak topology because g satisfies ww-compact. Next, using the fact that f satisfies ws-compact we infer that there exists a subsequence $(g(x_{n_{k_j}}))_{j\in\mathbb{N}}$ of $(g(x_{n_k}))_{k\in\mathbb{N}}$ such that $\left(f(g((x_{n_{k_j}})))\right)_{j\in\mathbb{N}}$ converges in the strong topology. Next, using the continuity of the operator $(I-B)^{-1}$ we conclude that the sequence $\Big((I-B)^{-1}f(g((x_{n_{k_j}})))\Big)_{j\in\mathbb{N}}$ converges in the strong topology. Consequently, the operator $(I-B)^{-1}F$ satisfies condition ws-compact.

We claim that the subset $(I-B)^{-1}F(K)$ is relatively weakly compact. To see this, we first note that $(I-B)^{-1}F(K)$ is bounded and $(I-B)^{-1}F(K) \subseteq K$. Moreover, since $g(K) \subset K$, we conclude that $F(K)$ is relatively weakly compact (use assumption (a)). If $(I-B)^{-1}F(K)$ is not relatively weakly compact, then using (4.2.2), the fact that B is ω-condensing and the properties of ω we get

$$\begin{aligned}\omega((I-B)^{-1}F(K)) &= \omega\Big((F + B(I-B)^{-1}F)(K)\Big)\\ &\leq \omega(F(K)) + \omega(B(I-B)^{-1}F(K))\\ &= \omega(B(I-B)^{-1}F(K))\\ &< \omega((I-B)^{-1}F(K))\end{aligned}$$

which is a contradiction. This yields that $\omega((I-B)^{-1}F(K)) = 0$ and therefore $(I-B)^{-1}F(K)$ is relatively weakly compact. To complete the proof it suffices to apply Theorem 2.1.3 to the map $((I-B)^{-1}\circ f)\circ g = (I-B)^{-1}F$. □

Theorem 4.2.4 *Let X be a Banach space and let K be a nonempty closed bounded convex subset of X. Let $f : K \to K$, $g : K \to K$ and $B : K \to X$ be continuous maps and set $F = f\circ g$. Suppose that*

(a) f satisfies ws*-compact,*

(b) g satisfies ww*-compact,*

(c) B is pseudocontractive and $I-B$ is ψ-expansive,

(d) $\omega(F(S)+B(S)) < \omega(S)$ for all $S \subset K$ such that $\omega(S) > 0$,

(e) $\Big(x = B(x)+F(y),\ y\in K\Big) \Longrightarrow x \in K$.

Then Fix$(F+B) \neq \emptyset$.

Proof. As in the proofs of the above theorems, it can be seen easily that $x \in K$ is a solution for the equation $x = F(x) + B(x)$ if, and only if, x is a fixed point for the operator $(I - B)^{-1}F$ whenever it is well-defined. According to the proof of Theorem 3.3 in [34], we can prove that $(I - B)^{-1}F$ is well defined.

The map $(I - B)^{-1}F$ satisfies ws-compact. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in K. Since g satisfies condition ww-compact, then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(g(x_{n_k}))_{k\in\mathbb{N}}$ is weakly convergent. Because f satisfies ws-compact, there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ such that $\big(f(g(x_{n_{k_j}}))\big)_{k\in\mathbb{N}}$ converges in the strong topology. This shows that F satisfies ws-compact. Next, using the continuity of $(I-B)^{-1}$ we infer that $(I-B)^{-1}F$ satisfies the condition ws-compact.

Next, we prove that there exists a subset C of M such that C is weakly compact and the map $(I - B)^{-1}F : C \to C$ is continuous.

Given an element $\zeta \in K$, we define the set

$$\Lambda := \Big\{A \subset K : A \text{ is closed convex, } \zeta \in A \text{ and } (I - B)^{-1}F(A) \subset A\Big\},$$

and put $M := \bigcap_{A\in\Lambda} A$. Let $C := \overline{\text{co}}((I - B)^{-1}F(M \cup \{\zeta\})$. Because $\zeta \in M$ and $(I - B)^{-1}F(M) \subset M$, it follows that $C \subset M$. This implies that

$$(I - B)^{-1}F(C) \subset (I - B)^{-1}F(M) \subset C. \tag{4.2.3}$$

Moreover, since $\zeta \in C$, we obtain that $C \in \Lambda$. This yields that $C = M$. If C is not relatively weakly compact, then the use of (4.2.2), (4.2.3) and the properties of ω gives

$$\begin{aligned}
\omega(C) &= \omega\Big(\overline{\text{co}}\Big((I - B)^{-1}F(M \cup \{\zeta\})\Big)\Big) \\
&= \omega((I - B)^{-1}F(M)) = \omega((I - B)^{-1}F(C)) \\
&= \omega\Big((F + B\,(I - B)^{-1}\,F)(C)\Big) \\
&\leq \omega(F(C) + B(C)) \\
&< \omega(C).
\end{aligned}$$

This implies $\omega(C) = 0$. Hence C is relatively weakly compact. Because $(I - B)^{-1}F(C) \subset C$, we infer that $(I-B)^{-1}F(C)$ is relatively weakly compact too. To complete the proof it suffices to apply Theorem 2.1.3 to the map $(I - B)^{-1}\,F : C \to C$. □

Chapter 5

Some fixed points results for perturbed multivalued mappings

This chapter is devoted to some fixed point results for perturbed multivalued mapping. It is organised as follows: In Section 5.1 deals with contractive perturbations while Section 5.2 is concerned with condensing perturbations. In Section 5.3 we give two fixed set results using the concept of Mönch-sets.

5.1 Contractive single valued perturbations

In this section we establish a fixed point theorem for the sum of some multivalued mappings and single valued mappings.

Theorem 5.1.1 *Let M be a nonempty closed, convex subset of a Banach space X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a upper semicontinuous multivalued mapping and $B : X \to X$ a bounded linear single valued mapping. Suppose that*

(a) all selections of F are ws*-compact,*

(b) $F(M)$ is relatively weakly compact,

(c) $\|B\| < 1$,

(d) $F(M) + B(M) \subset M$.

Then there exists $y \in M$ such that $y \in F(y) + B(y)$.

Proof. It is clear that according to assumption (c), the operator $I - B$ is invertible and the operator $(I - B)^{-1} \in \mathcal{L}(X)$. Hence, according to Theorem 3.10 in [19], $(I - B)^{-1}$ is weakly continuous.

Define the operator $N : M \to \mathsf{P}_{cl,cv}(M)$ by

$$x \mapsto N(x) = (I - B)^{-1}F(x).$$

Since, for each $x \in M$, $F(x) \in \mathsf{P}_{cl,cv}(M)$ and $(I - B)^{-1} \in \mathcal{L}(X)$, the subset $N(x)$ is convex. Futher, using the continuity of $I - B$ and the fact that $F(x)$ is closed, we conclude that $N(x)$ is also closed. Hence, for each $x \in M$, $N(x) \in \mathsf{P}_{cl,cv}(X)$.

Let $x \in M$ and $g \in (I - B)^{-1}(F(x))$. It is clear that there exists $y \in F(x)$ such that $g = (I - B)^{-1}(y)$ and therefore $g = B(g) + y \subseteq B(g) + F(x)$. Using the hypothesis (d) one sees that $g \in M$. Hence, $N(M) \subset M$. On the other hand, because $F(M)$ is relatively weakly compact and $(I-B)^{-1}$ is weakly continuous, the set $N(M)$ is relatively weakly compact.

Next, set $K = \overline{co}(N(M))$. Using the Krein-Ŝmulian theorem 2.1.5 (ref [31, p. 434]) we infer that K is a weakly compact convex subset of X. Moreover, because M is closed and convex, we have $K \subset M$ and consequently we have $N(K) \subseteq N(M) \subset \overline{co}(N(M)) = K$. This proves that $N(K) \subseteq K$ and therefore $N(K)$ is relatively weakly compact. Because $(I - B)^{-1}$ is continuous, the operator N is upper semicontinuous.

We claim that $N(K)$ is relatively compact. To see this, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in K. Since K is weakly compact, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x$ ($x \in K$ because K is weakly closed), as $n \to +\infty$. Let $(y_{n_k})_{k\in\mathbb{N}}$ be a sequence in $N(K)$ such that, for each $k \in \mathbb{N}$, $y_{n_k} \in N(x_{n_k})$, that is $y_{n_k} \in (I - B)^{-1}F(x_{n_k})$. Hence, there exists a selection f of F such that $y_{n_k} = (I - B)^{-1}f(x_{n_k})$, for each $k \in \mathbb{N}$. Since f satisfies condition ws-compact, we infer that the sequence $\left(f(x_{n_k})\right)_{k\geq 0}$ has a strongly convergent subsequence in K, say $\left(f(x_{n_{k_j}})\right))_{j\geq 0}$. Moreover, since the operator $(I-B)^{-1}$ is continuous, the sequence $\left((I-B)^{-1}f(x_{n_{k_j}})\right)_{j\geq 0}$ converges strongly in K. This proves that the sequence $\left(N(x_{n_k})\right)_{k\in\mathbb{N}}$ has a strongly convergente sequence. Hence $N(K)$ is relatively compact.

Now applying Theorem 2.3.1, one sees that there exists $\zeta \in K \subseteq M$ such that $\zeta \in (I - B)^{-1}F(\zeta)$ or equivalently $\zeta \in F(\zeta) + B(\zeta)$. This ends the proof. □

Remark 5.1.1 *(a) In Theorem 5.1.1 above we consider a linear perturbation B satisfying $\|B\| < 1$. In fact, this result holds true if we suppose that there exists a nonnegative integer p such that $\|B^p\| < 1$. In this case the operator $I - B$ is invertible, $(I - B)^{-1} = (I - B^p)^{-1} \sum_{k=0}^{k=p-1} B^k$ and $I - B$ is an homeomorphism from M onto $(I - B)^{-1}M$. So, the arguments used in the proof above apply.*

(b) Note also that Theorem 5.1.1 remains valid if we suppose that, for some nonnegative integer p, B^p is a nonlinear contraction. Here again the operator $I - B$ is invertible, $(I - B)^{-1} = (I -$

$B^p)^{-1} \sum_{k=0}^{k=p-1} B^k$ *and* $I-B$ *is an homeomorphism from* M *onto* $(I-B)^{-1}M$. *(cf. [61] or [48, p. 155]).*
□

We note that one of the main tools in the proof of Theorem 5.1.1 is the fact $(I-B)^{-1}$ is weakly continuous. If we suppose that B is a weakly continuous k-contractive mapping B for some $k \in [0,1)$, as it is shown in the next lemma, the map $(I-B)^{-1}$ is also weakly continuous.

Theorem 5.1.2 *Let M be a nonempty closed, convex subset of a Banach space X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a upper semicontinuous multivalued mapping and $B : X \to X$ a weakly sequentially continuous single valued mapping satisfying* ww*-compact. Suppose that*

(a) all selections of F are ws*-compact,*

(b) $F(M)$ is relatively weakly compact,

(c) B is a k-contractive map, for some $k \in [0,1)$,

(d) for each $x \in M$, $(I-B)^{-1}F(x) \in \mathsf{P}_{cv}(M)$,

(e) $F(M) + B(M) \subset M$.

Then there exists $y \in M$ such that $y \in F(y) + B(y)$.

Remark 5.1.2 *Note that in Theorem 5.1.1, since $(I-B)^{-1}$ is a linear homeomorphism operator, it maps each closed convex set $F(x)$, $x \in M$, onto a closed convex subset of M. In Theorem 5.1.2, even if, for all $x \in M$, $(I-B)^{-1}F(x)$ is closed, it is not necessary convex. So the hypothesis (d) was added to guarantee the convexity of $(I-B)^{-1}F(x)$ for all $x \in M$.* □

Proof. It is clear that according to assumption (c) and Lemma 2.3.3, the operator $I-B$ is invertible and its inverse $(I-B)^{-1}$ is a weakly sequentially continuous mapping.

Note that, for all $x \in M$, $F(x) \in \mathsf{P}_{cl,cv}(M)$, and therefore by assumption (d) we conclude that $(I-B)^{-1}F(x) \in \mathsf{P}_{cl,cv}(M)$. Hence, we can define the mapping $N : M \to \mathsf{P}_{cl,cv}(M)$ by

$$x \mapsto N(x) = (I-B)^{-1}F(x).$$

Let $x \in M$ and $z \in (I-B)^{-1}F(x)$. It is clear that there exists $y \in F(x)$ such that $z = (I-B)^{-1}(y)$ and therefore $z = B(z) + y \subseteq B(z) + F(x)$. Using the hypothesis (e) one sees that $z \in M$. Hence, $N(M) \subset M$. On the other hand, because $F(M)$ is relatively weakly compact and $(I-B)^{-1}$ is weakly sequentially continuous on $F(M)$, hence the use of Theorem 1.1 in [62] shows that $(I-B)^{-1}$ is weakly continuous on $F(M)$. This yields that $N(M)$ is relatively weakly compact.

Next, set $K = \overline{co}(N(M))$. It follows from Krein-Ŝmulian's theorem 2.1.5 that the set K is a convex weakly compact subset of X. Moreover, because $M \in \mathsf{P}_{cl,cv}(X)$, we have $K \subset M$ and therefore $N(K) \subseteq N(M) \subset \overline{co}(N(M)) = K$. This proves that $N(K) \subseteq K$ and then $N(K)$ is relatively weakly compact. Note also that N is is upper semicontinuous because $(I - B)^{-1}$ is continuous and F is upper semicontinuous.

To complete the proof we have just to show that $N(K)$ is relatively compact and to apply Theorem 2.3.1. The proof of the compactness of $N(K)$ uses condition ws-compact and is similar to the last part of the proof of Theorem 5.1.1. □

5.2 Condensing perturbations

In this section we establish some fixed point results in the spirit of Theorems 5.1.1 and 5.1.2 for condensing perturbations with respect to a measure of weak noncompactness.

Theorem 5.2.1 *Let M be a nonempty closed,bounded and convex subset of a Banach space X and $\mu(\cdot)$ a measure of weak non-compactness on X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ and $B : X \to \mathsf{P}_{cl,cv}(M)$ be two upper semicontinuous multivalued mappings. Suppose that*

(a) all selections of F and B are ws*-compact,*

(b) $F(M)$ is relatively weakly compact,

(c) B is μ-condensing,

(d) $F(M) + B(M) \subset M$.

Then there exists $y \in M$ such that $y \in F(y) + B(y)$.

Proof. Let $\zeta \in M$ and set $G := F + B$. We define the family Λ of subsets of M by

$$\Lambda := \Big\{D \subseteq M : D \text{ is closed convex, } \zeta \in D \,\text{and}\, G : D \to \mathsf{P}_{cl,cv}(D)\Big\}.$$

Set

$$K = \bigcap_{D\in\Lambda} D \qquad \text{and} \qquad K^* = \overline{co}(G(K) \cup \{\zeta\}).$$

It is clear that K is a closed, convex subset of M, $\zeta \in K$ and, for each $D \in \Lambda$, we have $G(K) \subseteq G(D) \subseteq D$, hence $G(K) \subseteq \bigcap_{D\in\Lambda} D = K$ which proves that $K \in \Lambda$. Since $G(K) \cup \{\zeta\} \subseteq K$, the use of the relation $K^* = \overline{co}(G(K) \cup \{\zeta\}) \subseteq \overline{co}(K) = K$ yields that $K^* \subseteq K$. Moreover, because $G(K^*) \subseteq G(K) \subseteq K^*$, we infer that $K^* \in \Lambda$ and then $K \subseteq K^*$. This proves that $K = K^*$.

We claim that K is relatively weakly compact. If it is not the case, then $\mu(K) > 0$. Using the properties of $\mu(\cdot)$ and the fact that B is μ-condensing we can write

$$\begin{aligned}\mu(K) &= \mu(\overline{co}(G(K) \cup \{\zeta\})) \leq \mu(G(K)) \\ &= \mu(F(K) + B(K)) \leq \mu(B(K)) < \mu(K)\end{aligned}$$

which is a contraction. Hence $\mu(K) = 0$ and so K is relatively weakly compact. Next, because $G(K) \subset K$, the set $G(K)$ is relatively weakly compact too.

We claim that $G(K)$ is relatively compact. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in K. Since K is weakly compact, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x$ as $n \to +\infty$. Let $(w_{n_k})_{k\in\mathbb{N}}$ be a sequence in $F(K)$ such that, for each $k \in \mathbb{N}$, $w_{n_k} \in F(x_{n_k})$. By hypothesis, there exists a selection f of F such that $w_{n_k} = f(x_{n_k})$ satisfying ws-compact. Also, let $(z_{n_k})_{k\in\mathbb{N}}$ be a sequence in $B(K)$ such that, for each $k \in \mathbb{N}$, $z_{n_k} \in B(x_{n_k})$. By hypothesis, there exists a selection g of B such that $z_{n_k} = g(x_{n_k})$ satisfying ws-compact. Since f and g satisfy condition ws-compact, we infer that the sequence $(z_{n_k} + w_{n_k})_{k\in\mathbb{N}}$ with $z_{n_k} + w_{n_k} = f(x_{n_k}) + g(x_{n_k})$, has a strongly convergent subsequence in K. Hence $G(K)$ is relatively compact.

Now, since the map $G = F + B$ is upper semicontinuous, applying Theorem 2.3.1, we conclude that there exists $z \in K \subseteq M$ such that $z \in G(z)$ which ends the proof. □

In the following we shall give some single-valued perturbations for which the mapping $F + B$ is ω-condensing. To this end we introduce the following lemma which appears in [48, p. 248] in french. For completeness we give a proof.

Lemma 5.2.1 *Let X be a Banach space and let $B : X \to X$ be a weakly sequentially continuous map on X. If B is a nonlinear contraction, then B is ω-condensing.*

Proof. Let $S \in \mathbf{B}(X)$ such that $\omega(S) = \tau$ and $\varepsilon > 0$. By definition of ω, there exists $W \in \mathsf{W}$ such that $S \subseteq W + (\tau + \varepsilon)\mathbb{B}_1$. So, for $x \in S$, there exist $y \in W$ and $z \in (\tau + \varepsilon)\mathbb{B}_1$ such that $x = y + z$ and then

$$\|B(x) - B(y)\| \leq \phi(\|x - y\|) \leq \phi(\tau + \varepsilon)$$

where ϕ is the Φ-function of B. Hence, $B(S) \subseteq B(W) + \mathbb{B}_{\phi(\tau+\varepsilon)}$. On the other hand, since B is weakly sequentially continuous and W is weakly compact, according to Theorem 2.1 in [62], the map B is weakly continuous on W. Hence $B(W)$ is weakly compact, and therefore $\omega(B(S)) \leq \phi(\tau + \varepsilon)$. Since ε is arbitrary, we get $\omega(B(S)) \leq \phi(\tau) < \tau = \omega(S)$. This ends the proof. □

Theorem 5.2.2 *Let M be a nonempty closed, bounded and convex subset of a Banach space X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a upper semicontinuous multivalued mapping and let $B : M \to X$ be a weakly sequentially continuous single valued mapping. Suppose that*

(a) all selections of F are ws*-compact,*

(b) $F(M)$ is relatively weakly compact,

(c) B is a nonlinear contraction with Φ-function ϕ,

(d) for each $x \in M$, $(I-B)^{-1}F(x) \subset \mathsf{P}_{cv}(M)$.

(e) $F(M) + B(M) \subset M$.

Then there exists $x \in M$ such that $x \in F(x) + B(x)$.

Proof. Let K be a bounded subset of M such that $\omega(K) \neq 0$. Since $F(M)$ is relatively weakly compact, we have

$$\omega((F+B)(K)) \leq \omega(F(K)) + \omega(B(K)) = \omega(B(K)). \tag{5.2.1}$$

According to Lemma 5.2.1, the operator B is ω-condensing. Consequently, Equation (5.2.1) may be written in the form

$$\omega((F+B)(K)) < \omega(K).$$

This shows that the mapping $F + B$ is ω-condensing.

Let $k_0 \in M$ and consider the set Π defined by

$$\Pi := \{A \subset M \text{ such that } (F+B)(A) \subseteq A,\ k_0 \in A \text{ and } A \text{ is closed, convex}\}.$$

As in the proof of Theorem 5.2.1, we show that the set

$$C := \bigcap_{A \in \Pi} A = \overline{co}((F+B)(C) \cup \{k_0\}) \tag{5.2.2}$$

belongs to Π and consequently

$$(F+B)(C) \subset C. \tag{5.2.3}$$

The set C is weakly compact. Indeed, using the properties of ω and the fact that $F + B$ is ω-condensing, we can write

$$\omega(C) = \omega(\overline{co}\{(B+F)(C) \cup \{k_0\}\}) = \omega((B+F)(C)) < \omega(C)$$

which is a contradiction. Hence $\omega(C) = 0$ and so C is weakly compact.

Since B is a nonlinear contraction, it is well known that $(I-B)$ is continuous, invertible and $(I-B)^{-1}$ is continuous. Moreover, the use of the hypothesis (d) shows that $(I-B)^{-1}F(x) \in \mathsf{P}_{cl,cv}(C)$ (here

we use the fact the preimage of each closed subset of C by $(I-B)^{-1}$ is closed). So, we can define the map $N : C \to \mathsf{P}_{cl,cv}(C)$ by

$$x \mapsto N(x) = (I-B)^{-1}F(x).$$

Since F is upper semicontinuous and $(I-B)^{-1}$ is continuous, we conclude that N is upper semicontinuous. On the other hand, because $F(C)$ is relatively weakly compact and the map $(I-B)^{-1}$ is weakly continuous, we conclude that the set $N(C)$ is relatively weakly compact.

We claim that $N(C)$ is compact. To see this, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in C. Since C is weakly compact, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x$ as $n \to +\infty$ ($x \in C$ because C is weakly closed). Let $(y_{n_k})_{k\in\mathbb{N}}$ be a sequence in $N(K)$ such that, for each $k \in \mathbb{N}$, $y_{n_k} \in N(x_{n_k})$, that is $y_{n_k} \in (I-B)^{-1}F(x_{n_k})$. Hence, there exists a selection f of F such that $y_{n_k} = (I-B)^{-1}f(x_{n_k})$, for each $k \in \mathbb{N}$. Since f satisfies condition ws-compact, we infer that the sequence $\left(f(x_{n_k})\right)_{k\geq 0}$ has a strongly convergent subsequence in K, say $\left(f(x_{n_{k_j}})\right))_{j\geq 0}$. Moreover, since the operator $(I-B)^{-1}$ is continuous, the sequence $\left((I-B)^{-1}f(x_{n_{k_j}})\right)_{j\geq 0}$ converges strongly in C. This proves that the sequence $\left(N(x_{n_k})\right)_{k\in\mathbb{N}}$ has a strongly convergente sequence. Hence $N(C)$ is relatively compact.

Now applying Theorem 2.3.1, one sees that there exists $\zeta \in K \subseteq M$ such that $\zeta \in (I-B)^{-1}F(\zeta)$ or equivalently $\zeta \in F(\zeta) + B(\zeta)$. □

Let us now introduce the following class of mappings introduced in [53].

Definition 5.2.1 *Let X be a normed space. A mapping $B : D(B) \subseteq X \to X$ is said to be a separate contraction if there exist two functions $\phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

(a) $\psi(0) = 0$, ψ is strictly increasing,

(b) $\|B(x) - B(y)\| \leq \phi(\|x-y\|)$ for all $x, y \in D(B)$,

(c) $\psi(r) \leq r - \phi(r)$ for all $r > 0$.

Corollary 5.2.1 *Let M be a nonempty closed, bounded and convex subset of a Banach space X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a upper semicontinuous multivalued mapping and let $B : M \to X$ be a weakly sequentially continuous single valued mapping. Suppose that*

(a) all selections of F are ws-compact,

(b) $F(M)$ is relatively weakly compact,

(c) B is a separate contraction,

(d) for each $x \in M$, $(I-B)^{-1}F(x) \subset \mathsf{P}_{cv}(M)$.

(e) $F(M) + B(M) \subset M$.

Then there exists $x \in M$ *such that* $x \in F(x) + B(x)$.

Proof. This is an immediate consequence of Theorem 5.2.2, it suffices to observe that any separate contraction mapping is a nonlinear contraction. □

We now introduce the following class of mappings due to Burton [18].

Definition 5.2.2 *Let* X *be a normed space. A mapping* $B : D(B) \subseteq X \to X$ *is said to be a large contraction if,* $\forall\, x, y \in D(B)$ *with* $x \neq y$, *we have* $\|B(x) - B(y)\| < \|x - y\|$ *and if, for any* $\varepsilon > 0$, *there exists a real* $\delta = \delta(\varepsilon) < 1$ *such that*

$$(x, y \in X,\ \varepsilon \leq \|x - y\|) \Longrightarrow (\|B(x) - By)\| \leq \delta \|x - y\|).$$

Corollary 5.2.2 *Let* M *be a nonempty closed, bounded and convex subset of a Banach space* X. *Let* $F : M \to \mathsf{P}_{cl,cv}(M)$ *be a upper semicontinuous multivalued mapping and let* $B : M \to X$ *be a weakly sequentially continuous single valued mapping. Suppose that*

(a) all selections of F *are* ws*-compact,*

(b) $F(M)$ *is relatively weakly compact,*

(c) B *is a large contraction,*

(d) for each $x \in M$, $(I - B)^{-1}F(x) \subset \mathsf{P}_{cv}(M)$.

(e) $F(M) + B(M) \subset M$.

Then there exists $x \in M$ *such that* $x \in F(x) + B(x)$.

Proof. Note that according to Lemma 1.1 in [53], B is a separate contraction mapping. Hence, the result follows from Corollary 5.2.1. □

5.3 Countably condensing perturbations

In this section, we present fixed set results for the sum of two upper semicontinuous multivalued mappings using the concept of Mönch-sets.

Theorem 5.3.1 *Let M be a nonempty closed, convex subset of a Banach space X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ and $B : M \to \mathsf{P}_{cl,cv}(M)$ be two upper semicontinuous multivalued mappings. Suppose that*

(a) all selections of F satisfy ws*-compact,*

(b) $F(M)$ is relatively weakly compact,

(c) For each $x \in M$, $F(x) \cap B(x) \neq \emptyset$,

(d) B maps compact sets into relatively compact sets,

(e) there exists $x_0 \in M$ such that

$$\begin{cases} D \subset M,\ D = co(\{x_0\} \cup B(D)) \\ and\ \overline{D} = \overline{C}\ with\ C \subset D\ countable \end{cases} \Longrightarrow \overline{D}\ is\ compact,$$

(f) $F(M) + B(M) \subset M$.

Then there exists $y \in M$ such that $y \in F(y) + B(y)$.

Proof. Let $x_0 \in M$ and define the iterative sequence of sets $(D_n)_{n\in\mathbb{N}}$ by

$$D_0 = \{x_0\},\ D_n = co(\{x_0\} \cup B(D_{n-1})) \text{ for } n = 1, 2, \cdots . \text{ and } D = \bigcup_{n=0}^{\infty} D_n.$$

Using the hypothesis (b) and by a standard argument (cf. the proof of Theorem 3.1 in [20]), we infer that $D = \text{co}(\{x_0\} \cup B(D))$ and $\overline{D}$ is compact. Since $B(D) \subset \overline{D}$ we infer that $B(D)$ is relatively compact.

Next, set $O = F(M) \cap B(D)$. It is clear that O is nonempty (use hypothesis (c)). Since $B(D)$ is relatively compact, we deduce that O is also relatively compact. Thus $\overline{O}$ is compact and therefore, by hypothesis (d), $B(\overline{O})$ is relatively compact.

On the other hand, because $\overline{O} \subset M$, we have $F(\overline{O}) \subset F(M)$ and the use of hypothesis (b) shows that $F(\overline{O})$ is relatively weakly compact.

We claim that $F(\overline{O})$ is relatively compact. To see this, it suffices to use the assumption (a) and to argue as at the end of the proof of Theorem 5.1.1 or Theorem 5.2.1.

According to the steps above, the set $(F + B)(\overline{O})$ is relatively compact. Now applying Theorem 2.3.1, we conclude that there exists $z \in \overline{O}$ (and then $z \in M$) such that $z \in (F + B)(z)$ which ends the proof. □

Definition 5.3.1 *Let M be a subset of a Banach space X and let $\mu(\cdot)$ be a measure of noncompactness on X. Let $F : M \to \mathsf{P}_{cl,cv}(M)$ be a multivalued mapping. The map F is said to be countably μ-condensing if $\mu(F(M)) < \infty$ and $\mu(F(C)) < \mu(C)$ for any countable bounded subset C of M with $\mu(C) > 0$.*

Before going further, we first recall the following definition introduced in [21].

Definition 5.3.2 *Let M be a convex subset of X and $F : M \to \mathsf{B}(M)$ be a given map. We say that $A \subset M$ is a Mönch-set for F if there exists $x_0 \in M$ such that $A = co(\{x_0\} \cup F(A))$ and there exists a countable set $C \subset A$ with $\overline{A} = \overline{C}$.*

Corollary 5.3.1 *Let M be a nonempty closed, convex bounded subset of a Banach space X and $\mu(\cdot)$ a measure of non-compactness on X. Assume that $F : M \to \mathsf{P}_{cl,cv}(M)$ and $B : M \to \mathsf{P}_{cl,cv}(M)$ be two upper semicontinuous multivalued mappings. Suppose that the hypotheses (a), (b), (c), (d) and (f) of Theorem 5.3.1 are satisfied. If B is countably μ-condensing, then there exists $y \in M$ such that $y \in F(y) + B(y)$.*

Proof. We prove that B satisfies hypothesis (e) of Theorem 5.3.1. Fix x_0 in M and let D be a Mönch-set for B contained in M, that is, $D = co(\{x_0\} \cup B(D))$ and

$$\overline{D} = \overline{C} \tag{5.3.1}$$

with C is a countable subset of D. Since $C \subset co(\{x_0\} \cup B(D))$, each point of C can be written as a finite combination of points belonging to the set $\{x_0\} \cup B(D)$. Therefore, there exists a countable set $N \subset D$ such that

$$C \subset co(\{x_0\} \cup B(N)). \tag{5.3.2}$$

We claim that $\mu(C) = 0$. Indeed, the use of (5.3.2), the fact that B is countably μ-condensing and the properties of μ, gives

$$\mu(C) \leq \mu(co(\{x_0\} \cup B(N))) = \mu(\{x_0\} \cup B(N)) = \mu(B(N)) < \mu(N). \tag{5.3.3}$$

If $\mu(N) = 0$, then the proof is finished. Otherwise, combining (5.3.1) and (5.3.3) we obtain

$$\mu(C) < \mu(N) \leq \mu(D) = \mu(\overline{D}) = \mu(\overline{C}) = \mu(C) \tag{5.3.4}$$

which is a contradiction. Hence, $\mu(C) = 0$ which proves our claim. Accordingly, the use of (5.3.1) implies that $\overline{D}$ is compact. Keeping in mind that $D = co(\{x_0\} \cup B(D))$, we infer that $B(D) \subset \overline{D}$ and therefore $B(D)$ is relatively compact.

Set $O = F(M) \cap B(D)$. Note that, by the hypothesis (c), O is nonempty and $(F + B)(\overline{O})$ is relatively compact. Now arguing as at the end of the proof of Theorem 5.3.1, we conclude that there exists $x \in \overline{O}$ (and then $x \in M$) such that $x \in (F + B)(x)$. □

Chapter 6

Applications

In this Section 6.1 we shall apply the results of Chapter 2 to solve two nonlinear Volterra integral equations. In Section 6.2 the results of Chapter 4 were applied to solve a nonlinear transport equations with delayed neutrons while Section 6.3, with the help of Theorem 2.1.4 we discuss existence result for a nonlinear transport equation with partly elastic collisions using.

6.1 Volterra integral equation

In this section we shall present two examples. Let us first recall the following result of Ambrosetti's type established in [57].

Lemma 6.1.1 *Let X be a Banach space and let A be a subset of $C([0,T],X)$ with $T > 0$. If A is bounded and equicontinuous, then*

$$\beta(A) = \sup_{t\in[0,T]} \beta(A(t)) = \beta(A([0,T]))$$

where β is a measure of noncompactness on X (see Definition 2.1.1), $A(t) = \{h(t) : h \in A\}$ and $A[0,T] = \cup_{t\in[0,T]}\{h(t) : h \in A\}$.

Example 1

We consider the following Voltera problem

$$\varphi(t) = \mathcal{H}(t) + \int_a^t f(s,\varphi(s))ds, \quad \forall t \in [a,b] \tag{6.1.1}$$

where f is a map from $[a,b]\times X$ into X and $\mathcal{H}$ maps $[a,b]$ into X. Here X is a Banach space.

Our objective is to show the the problem (6.1.1) has a solution belonging to the space $C([a,b],X)$.

To this end, we introduce the following hypotheses :

(a) f is a L^p-Carathéodory function, $p\in(1,\infty)$, that is

  (i) the map $t\to f(t,z)$ is measurable (Bochner) for all $z\in X$,

  (ii) the map $z\to f(s,z)$ is continuous for almost all $s\in[a,b]$,

  (iii) for each $r>0$, there exists $\mu_r\in L^p([a,b],\mathbb{R})$ such that $\|z\|\le r$ implies $\|f(s,z)\|\le \mu_r(s)$ for almost all $s\in[a,b]$;

(b) $\mathcal{H}\in C([a,b],X)$;

(c) for any countably bounded subset D of $C([a,b],X)$ and for each $t\in[a,b]$, we have

$$\beta(f([a,b]\times D))\le\zeta\beta(D)$$

where $\zeta>0$ and $f([a,b]\times D):=\{f(s,\varphi(s))\ : a\le s\le b,\ \varphi\in D\}$.

(d) let $M=\|\mathcal{H}\|_0+(b-a)^{1/q}\,\|\mu_r\|_p$.

Theorem 6.1.1 *Let $f:[a,b]\times X\to X$ be a L^p-Carathéodory function and assume that the conditions $(a)-(d)$ hold true. Denote by U the closed ball of $C([a,b],X)$ centered at 0 with radius M. If $\zeta(b-a)<1$, then Problem* (6.1.1) *has a solution $\varphi\in U$ where*

$$U=\{\varphi\in C([a,b],X):\|\varphi\|_0\le M\}.$$

Proof. Let T be the map defined by :

$$C([a,b],X)\ni\varphi\mapsto T\varphi(t)=\mathcal{H}(t)+\int_a^t f(s,\varphi(s))ds.$$

Note that for any $\varphi\in C([a,b],X)$, there exists $r>0$ such that $\|\varphi\|_0\le r$ and since f is L^p-Carathéodory, there exists $\mu_r\in L^p[a,b]$ with $\|f(s,\varphi(s))\|_X\le\mu_r(s)$, for almost all $s\in[a,b]$. Hence, for $t_1,t_2\in[a,b]$, we have

$$\begin{aligned}\|T\varphi(t_1)-T\varphi(t_2)\|_X&\le\|\mathcal{H}(t_1)-\mathcal{H}(t_2)\|_X+|t_1-t_2|^{1/q}\,\|\mu_r\|_p\\&\to 0\text{ as }t_1\to t_2.\end{aligned}\tag{6.1.2}$$

Consequently T maps $C([a,b],X)$ into itself.

Now Eq. (6.1.1) is equivalent to the fixed point problem

$$\varphi=T\varphi.$$

We shall apply Theorem Daher. First we prove that $T : U \to C([a,b],X)$ is continuous. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of U such that $\varphi_n \to \varphi$ in $C([a,b],X)$. We have to show that $T\varphi_n \to T\varphi$ in $C([a,b],X)$. We know that $\|\varphi_n\|_0 \leq M$, $\|\varphi\|_0 \leq M$, and there exists $\mu_M \in L^p[a,b]$ such that $\|f(s,\varphi_n(s))\|_X \leq \mu_M(s)$ and $\|f(s,\varphi(s))\|_X \leq \mu_M(s)$, for almost all $s \in [a,b]$. The Lebesgue dominated convergence theorem implies $T\varphi_n(s) \to T\varphi(s)$ point-wise on $[a,b]$. This means

$$\forall t, \forall \epsilon > 0 \ \exists N \text{ such that } \|T\varphi_n(t) - T\varphi(t)\| \leq \epsilon/3.$$

Thus, using the continuity of $T\varphi_n(\cdot)$ and $T\varphi(\cdot)$ (see (6.1.2)), for $\epsilon > 0$ (taken in the last step), there exists $\delta > 0$ such that, for all $t_1, t_2 \in [a,b]$ with $|t_2 - t_1| < \delta$, we have

$$\|T\varphi_n(t_1) - T\varphi_n(t_2)\|_X \leq \epsilon/3, \quad \text{for all } n \in \mathbb{N} \tag{6.1.3}$$

and

$$\|T\varphi(t_1) - T\varphi(t_2)\|_X \leq \epsilon/3. \tag{6.1.4}$$

Now using (6.1.3), (6.1.4) together with the fact that $[a,b]$ is compact we conclude that for all $t \in [a,b]$ and for all $s \in [a,b]$ such that $|t-s| < \delta$, for n large enough, we have

$$\begin{aligned}\|T\varphi_n(t) - T\varphi(t)\|_X &\leq \|T\varphi_n(t) - T\varphi_n(s)\|_X + \|T\varphi_n(s) - T\varphi(s)\|_X \\ &\quad + \|T\varphi(s) - T\varphi(t)\|_X \\ &\leq \epsilon\end{aligned}$$

and therefore $\|T\varphi - T\varphi_n\|_0 \leq \varepsilon$. This proves the continuity of T.

Now, we show that T is countably β-condensing. Indeed, let D be a countably bounded subset of U, we have

$$\begin{aligned}\beta(T(D)(t)) &= \beta\Big(\Big\{\mathcal{H}(t) + \int_a^t f(s,\varphi(s))ds : \varphi \in D\Big\}\Big) \\ &\leq \beta\Big(\Big\{\int_a^t f(s,\varphi(s))ds : \varphi \in D\Big\}\Big) \\ &\leq \beta\Big((t-a)\overline{conv}(\{f(s,\varphi(s)) : \varphi \in D\})\Big) \\ &\leq (t-a)\beta(\overline{conv}(\{f([a,t]\times D)\})) \\ &= (t-a)\beta(\{f([a,t]\times D)\}).\end{aligned}$$

Because U is bounded and equicontinuous, D is also bounded and equicontinuous. Using condition (c) together with Lemma 6.1.1, we get

$$\beta(T(D)(t)) \leq \zeta(b-a)\beta(D) < \beta(D) \text{ with } \beta(D) > 0.$$

Hence T is countably β-condensing.

Note also that $T(U)$ is bounded because

$$\|T\varphi\|_0 \leq \|\mathcal{H}\|_0 + (b-a)^{1/q}\,\|\mu_r\|_p \quad \text{for all } \varphi \in U$$

where q stands for the conjugate exponent of p.

Finally, applying Theorem Daher, we conclude that T has a fixed point in U, or equivalently, (6.1.1) has a solution in U. □

Example 2

Now we shall discuss the existence of solutions to the following Volterra integral equation

$$\varphi(t) = \mathcal{H}(\varphi(t)) + \int_0^t f(s, \varphi(s))ds \quad t \in [0, 1]; \tag{6.1.5}$$

here $f : [0, 1] \times X \to X$, $\mathcal{H} : X \to X$ are two maps and X is a Banach space.

Our objective is to show the the Problem (6.1.5) has a solution belonging to the space $C([0, 1], X)$.

Let us introduce the following hypotheses :

(a) f is a L^p-Carathéodory function, $p \in (1, \infty)$, that is

  (i) the map $t \to f(t, z)$ is measurable (Bochner) for all $z \in X$,

  (ii) the map $z \to f(s, z)$ is continuous for almost all $s \in [0, t]$,

  (iii) for each $r > 0$, there exists $\mu_r \in L^p([0, t], \mathbb{R})$ such that $\|z\| \leq r$ implies $\|f(s, z)\| \leq \mu_r(s)$ for almost all $s \in [0, t]$;

(b) there exists $\zeta \in (0, 1)$ such that for any countably bounded subset D of $C([0, 1], X)$ and for each $t \in [0, 1]$, we have

$$\beta(f([0, t] \times D)) \leq \zeta\beta(D)$$

  where $f([0, t] \times D) := \{f(s, \varphi(s)) \ : 0 \leq s \leq t, \ \varphi \in D\}$;

(c) let $\tau > 0$ be such that $\frac{1}{\zeta}\Big(\|\mu_r\|_q + \|\mathcal{H}(0)\| \Big) \leq \tau$;

(d) $\|\mathcal{H}(u) - \mathcal{H}(v))\| \leq (1 - \zeta) \|u - v\|$ for all $u, v \in X$.

The real ζ appearing in conditions (b), (c) and (d) is the same and belongs to $(0, 1)$.

Theorem 6.1.2 *Assume that the conditions* $(a) - (d)$ *hold true and denote by* B_τ *the closed ball of* $C([0, 1], X)$ *centered at* 0 *with radius* τ *. If* $\zeta \in (0, 1/2)$*, then* (6.1.5) *has a solution in* B_τ*.*

Proof. Let B_τ be the closed ball of $C([0, 1], X)$ centered at the origin with radius $\tau > 0$.Clearly B_τ is a closed, bounded and equicontinuous subset of $C([0, 1], X)$. Define the mappings Π and S on B_τ by

$$\Pi\varphi(t) = \int_0^t f(s, \varphi(s))ds \quad \text{and} \quad S\varphi(t) = \mathcal{H}(\varphi(t))$$

It is clear that Π maps B_τ into $C([0,1],X)$ (argue as in the previous example). The continuity of Π is contained in the proof of the continuity of the map T in the previous example.

Now, we show that Π is countably β-ζ-contractive map. Let D be a countably bounded subset of B_τ, we have

$$\begin{aligned}\beta(\Pi(D)(t)) &= \beta\Big(\Big\{\int_0^t f(s,\varphi(s))ds : \varphi \in D\Big\}\Big)\\ &\leq \beta\Big(t\overline{conv}(\{f(s,\varphi(s)) : \varphi \in D\})\Big)\\ &\leq t\beta(\overline{conv}(\{f([0,t]\times D)\}))\\ &= t\beta(\{f([0,t]\times D)\})\end{aligned}$$

Since D is bounded and equicontinuous (because B_τ is equicontinuous and bounded), using condition (b) together with Lemma 6.1.1, we get

$$\beta(\Pi(D)(t)) \leq \zeta t\beta(D).$$

Hence Π is countably β-ζ-contractive.

We show that if $\varphi = \Pi\phi + S\varphi$, $\phi \in B_\tau$, then $\varphi \in B_\tau$.

Let $\varphi \in C([0,1],X)$ and $\phi \in B_\tau$ such that $\varphi = S\varphi + \Pi\phi$. For all $t \in [0,1]$ we have

$$\varphi(t) = \mathcal{H}(\varphi(t)) + \int_0^t f(s,\phi(s))ds$$

and therefore

$$\begin{aligned}\|\varphi(t)\| &\leq \|\mathcal{H}(\psi(t)) - \mathcal{H}(0)\| + \|\mathcal{H}(0)\| + \int_1^t \|f(s,\phi(s))\|\,ds\\ &\leq (1-\zeta)\,\|\varphi(t)\| + \|\mathcal{H}(0)\| + \|\mu_\tau\|_p\,.\end{aligned}$$

Hence

$$\|\varphi\|_0 \quad \leq (1-\zeta)\,\|\varphi\|_0 + \|\mathcal{H}(0)\| + \|\mu_\tau\|_p\,.$$

This writes simply as

$$\zeta\,\|\varphi\|_0 \quad \leq \|\mathcal{H}(0)\| + \|\mu_\tau\|_p\,.$$

By condition (b), we get $\|\varphi\|_0 \leq \dfrac{1}{\zeta}\tau\,\zeta = \tau$ and so φ belongs to B_τ. Moreover, we know from the hypothesis (d) that S is $(1-\zeta)$-contractive. Finally, applying Proposition 3.3.2 one sees that there exists $\phi \in B_\tau$ such that $\Pi\phi + S\phi = \phi$. □

6.2 A transport equation with delayed neutrons

This section deals with existence of solutions to the following nonlinear boundary value problem which involves a multidimensional transport equation with delayed neutrons in a bounded spatial domain. More precisely, we shall discuss existence of solutions to the nonlinear boundary value problem:

$$\begin{cases} v.\nabla_x f_0(x,v) + \sigma(x,v,f_0(x,v)) + \lambda f_0(x,v) = \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^N} \kappa_0(x,v,v')\Theta_0(x,v',f_0(x,v'))d\mu(v') + \sum_{i=1}^{d} \lambda_i \beta_i(x,v) f_i(x,v) \\ \lambda_i f_i(x,v) = \displaystyle\int_{\mathbb{R}^N} \kappa_i(x,v,v')\Theta_i(x,v',f_0(x,v'))d\mu(v'), \quad 1 \le i \le d \end{cases} \tag{6.2.1}$$

where $(x,v) \in \mathsf{D} \times \mathbb{R}^N$. Here D is a smooth open bounded convex subset of $\mathbb{R}^N$, $\mu(\cdot)$ is a positive Radon measure on $\mathbb{R}^N$ such that $\mu(\{0\}) = 0$. The function f_0 represents the neutrons density, $f_i(x,v)$ represents the density of the i-th group delayed neutron emitters and $\{\lambda_i, 1 \le i \le d\}$ are radioactive decay constants [56]. The functions $\sigma(\cdot,\cdot,\cdot)$ and $\kappa_i(\cdot,\cdot,\cdot)$ are called, respectively, the collision frequency and the scattering kernel which are, in general, nonlinear functions of f_0 while $\beta_i(x,v)$, $i = 1, 2, \cdots, d$, are essentially bounded functions which denote physical parameters related to the i-th group delayed neutrons. The maps Θ_i, $i = 0, 1, \cdots, d$ are nonlinear functions of the density of neutrons f_0.
The boundary conditions are modeled by

$$f_0^- = H(f_0^+) \tag{6.2.2}$$

where f_0^- (resp. f_0^+) is the restriction of f_0 to Γ_- (resp. Γ_+), with Γ_- (resp. Γ_+) is the incoming (resp. the outgoing) part of the phase space boundary and H is a linear bounded operator from a suitable space on Γ_- to a similar one on Γ_+. The well known classical boundary conditions (vacuum boundary conditions, specular reflections, diffuse reflections, periodic and mixed type boundary conditions) are special examples of our framework.

we apply Corollary 4.2.1 to solve the boundary value problem (6.2.1)-(6.2.2). The functional setting of the problem is L^p-spaces with $p \in [1, +\infty)$. We note that to solve Problem (6.2.1)-(6.2.2) in L^p-spaces with $p \in (1, +\infty)$ is not difficult, it uses the compactness results established in [49] and requires the use of the classical Krasnosel'skii's fixed point theorem. However, in L^1-space, Problem (6.2.1)-(6.2.2) involves a Dunford-Pettis operator, and its proof requires Corollary 4.2.1.

6.2.1 Basic facts related to the problem

Before going further, we first gather some definitions and results related to the usual neutron transport equation with delayed neutrons required in the sequel.

Let D be a smooth open bounded convex subset of $\mathbb{R}^n$ and let μ be a positive Radon measure on $\mathbb{R}^N$ such that $\mu(\{0\}) = 0$.

The boundary of the phase space writes as $\partial\mathsf{D} \times \mathbb{R}^N := \Gamma_- \cup \Gamma_+$ with

$$\Gamma_\pm = \{(x, v) \in \partial\mathsf{D} \times \mathbb{R}^N, \pm v.\nu_x \geq 0\},$$

where ν_x stands for the outer unit normal vector at $x \in \partial\mathsf{D}$.

Definition 6.2.1 *Let $(x, v) \in \overline{\mathsf{D}} \times \mathbb{R}^N$. We set*

$$\begin{aligned} t^\pm(x, v) &= \sup\{t > 0,\ x \pm sv \in \mathsf{D},\ 0 < s < t\} \\ &= \inf\{t > 0,\ x \pm tv \notin \mathsf{D}\} \end{aligned}$$

and

$$\tau(x, v) := t^+(x, v) + t^-(x, v) \textit{ for all } (x, v) \in \overline{\mathsf{D}} \times \mathbb{R}^N.$$

□

We now introduce the following functional space

$$\mathsf{W}_p = \left\{\psi \in \mathsf{X}_p \text{ such that } v.\nabla_x f \in \mathsf{X}_p\right\},$$

where

$$\mathsf{X}_p := L^p(\mathsf{D} \times \mathbb{R}^N;\, dx \otimes d\mu(v)) \quad (1 \leq p < \infty).$$

It is well known (cf. [22, 23, 13] or [39]) that any function f in W_p possesses traces $f^\pm$ on $\Gamma_\pm$ belonging to $L^\pm_{p,\mathrm{loc}}(\Gamma_\pm; |v.\nu_x| d\gamma_x d\mu(v))$, $d\gamma_x$ being the Lebesgue measure on $\partial\mathsf{D}$. In applications, suitable L^p-spaces for the traces are

$$L^\pm_p := L^p(\Gamma_\pm; |v.\nu_x| d\gamma_x d\mu(v)).$$

So, we define the set

$$\widetilde{\mathsf{W}}_p = \{f \in \mathsf{W}_p;\ f^- \in L^-_p\}.$$

It is well known that if $f \in \mathsf{W}_p$ and $f^- \in L^-_p$, then $f^+ \in L^+_p$ and vice versa [22, 23, 39]. More precisely we have the identity

$$\widetilde{\mathsf{W}}_p = \{f \in \mathsf{W}_p;\ f^- \in L^-_p\} = \{f \in \mathsf{W}_p;\ f^+ \in L^+_p\}.$$

The boundary conditions may be written abstractly as an operator H relating the incoming and the outgoing fluxes, that is

$$H(f^+) = f^-, \quad D(H) = L_p^+ \quad \text{and} \quad H(L_p^+) \subseteq L_p^-.$$

Let $H \in \mathcal{L}(L_p^+, L_p^-)$ be a positive boundary operator (the positivity is taken in the lattice sense, that is H transforms the positive cone of L_p^+ into the positive cone of L_p^-). Define the free streaming operator T_H

$$\begin{cases} T_H \ : \ D(T_H) \subseteq \mathsf{X}_p \longrightarrow \mathsf{X}_p, \\ T_H f(x,v) = -v.\nabla_x f(x,v) \end{cases}$$

with domain

$$D(T_H) = \left\{ f \in \widetilde{\mathsf{W}}_p \ \text{ such that } \ f^- = H(f^+) \right\}.$$

Let $\lambda \in \mathbb{C}$ and let g be a given function of X_p. We consider the following boundary value problem

$$\begin{cases} v.\nabla_x f(x,v) + \lambda f(x,v) = g(x,v) \\ f^- = H(f^+) \end{cases} \tag{6.2.3}$$

where the unknown f is to be found in $D(T_H)$. For $\mathrm{Re}\lambda > 0$, the solution of equation (6.2.3) can be given formally by

$$f(x,v) = f(x - t^-(x,v)v, v)e^{-\lambda t^-(x,v)} + \int_0^{t^-(x,v)} e^{-\lambda s} g(x - sv, v)ds. \tag{6.2.4}$$

Moreover, for $(x,v) \in \Gamma_+$, the equation (6.2.4) becomes

$$f^+ = f^- e^{-\lambda \tau(x,v)} + \int_0^{\tau(x,v)} e^{-\lambda s} g(x - sv, v)ds. \tag{6.2.5}$$

To allow the abstract formulation of (6.2.4) and (6.2.5), let us define the following operators depending on the parameter λ:

$$\begin{aligned} M_\lambda : L_p^- \to L_p^+, \quad u \to M_\lambda u := ue^{-\lambda\tau(x,v)}; \\ B_\lambda : L_p^- \to \mathsf{X}_p, \quad u \to M_\lambda u := ue^{-\lambda t^-(x,v)}; \end{aligned}$$

$$\begin{cases} G_\lambda : \mathsf{X}_p \to L_p^+, \\ g \to G_\lambda g := \displaystyle\int_0^{\tau(x,v)} e^{-\lambda s} g(x - sv, v)ds; \end{cases}$$

and

$$\begin{cases} C_\lambda : \mathsf{X}_p \to \mathsf{X}_p, \\ g \to C_\lambda g := \displaystyle\int_0^{t^-(x,v)} e^{-\lambda s} g(x - sv, v)ds; \end{cases}$$

Remark 6.2.1 *We note that for $\lambda > 0$, the operator M_λ, B_λ, G_λ and C_λ are bounded and positive (in the lattice sense).* □

Using these operators and the fact that f must satisfy the boundary condition, Eq. (6.2.5) becomes

$$f^+ = M_\lambda H f^+ + G_\lambda g.$$

The solution of this equation reduces to the invertibility of the operator $I - M_\lambda H$ So, if $(I - M_\lambda H)^{-1}$ exists, then

$$f^+ = (I - M_\lambda H)^{-1} G_\lambda g. \tag{6.2.6}$$

On the other hand, Eq (6.2.4) can be written as follows:

$$f = B_\lambda H f^+ + C_\lambda g.$$

Substituting (6.2.6) in this equation we obtain

$$f = B_\lambda H (I - M_\lambda H)^{-1} G_\lambda g + C_\lambda g.$$

Thus

$$(\lambda - T_H)^{-1} = B_\lambda H (I - M_\lambda H)^{-1} G_\lambda + C_\lambda. \tag{6.2.7}$$

In the remainder of this section we will use the following notation: For any real number τ, we set

$$\mathbb{C}_\tau = \Big\{ \lambda \in \mathbb{C} \text{ such that } \mathrm{Re}\lambda > \tau \Big\}. \tag{6.2.8}$$

Remark 6.2.2 (see [49]) *We note that, for any $\lambda \in \mathbb{C}_0$, the operator C_λ is nothing else but the resolvent of the free streaming operator with absorbing boundary conditions ($H = 0$) (i.e., $C_\lambda = (\lambda - T_0)^{-1}$).* □

Let X be a Banach space and let X^* be its topological dual space. We call the normalized duality map of X the map $J : X \to 2^{X^*}$ defined by

$$J(x) = \Big\{ x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 \text{ with } \|x^*\| = \|x\| \Big\}.$$

Let $x \in X$, by the Hahn-Banach theorem $J(x) \neq \emptyset$. A linear operator A with domain and range both in X is called dissipative if for every $x \in D(A)$ there is a $x^* \in J(x)$ such that $\mathrm{Re}\langle Ax, x^* \rangle \leq 0$.

Let $(\Omega, \sum, \mu)$ be a measure space and $L^p(\Omega, d\mu)$ with $1 < p < \infty$, $J(0) = \{0\}$ and for $0 \neq u \in L^p(\Omega, d\mu)$, $J(u)$ is a singleton, that is ,

$$J(u) = \Big\{ \|u\|^{2-p} \, |u|^{p-2} \, u \Big\}.$$

For $p = 1$, to show the dissipativity of the operator A, it is sufficient to show that $\mathrm{Re}\langle Au, \mathrm{sign}(u)\rangle \leq 0$ for all $u \in D(A)$ where

$$\mathrm{sign}(\psi)(x) = \begin{cases} 1 & \text{when} \quad \psi(x) > 0, \\ 0 & \text{when} \quad \psi(x) = 0, \\ -1 & \text{when} \quad \psi(x) < 0. \end{cases}$$

The following observation is required below: For any $\psi \in D(A)$ we have

$$\begin{aligned} \langle \|\psi\|_1 \, \mathrm{sign}(\psi), \psi\rangle &= \int_\Omega \|\psi\|_1 \, \mathrm{sign}(\psi)(x)\, \psi(x) dx \\ &= \|\psi\|_1 \int_\Omega \mathrm{sign}(\psi)(x)\, \psi(x) dx \\ &= \|\psi\|_1 \int_\Omega |\psi(x)|\, dx = \|\psi\|_1^2 . \end{aligned}$$

Hence, for any $\psi \in D(A)$ we have

$$\|\psi\|_1^2 = \langle \|\psi\|_1 \, \mathrm{sign}(\psi), \psi\rangle.$$

Remark 6.2.3 It is well known that. For all boundary operator H such that $\|H\| \leq 1$, the free streaming operator T_H is dissipative, that is for all $\psi \in D(T_H)$, we have $\langle T_H\psi, |\psi|^{p-2}\,\psi\rangle \leq 0$.

Lemma 6.2.1 *Let $1 \leq p < +\infty$ and assume that $\|H\| \leq 1$. Then, for any $\lambda \in \mathbb{C}_0$, we have*

$$\left\|(\lambda - T_H)^{-1}\right\| \leq \frac{1}{\mathrm{Re}\lambda}.$$

Proof. For $p \in (1 + \infty)$, we refer to Lemma 2.2 in [49]. Consider now the case $p = 1$. We first check that T_H is dissipative. We have

$$\begin{aligned} \langle T_H\psi, \mathrm{sign}(\psi)\rangle &= -\int_{\mathsf{D}\times\mathbb{R}^N} v\nabla_x\psi(x,v)\mathrm{sign}(\psi)(x,v)dxd\mu(v) \\ &= -\int_{\mathsf{D}\times\mathbb{R}^N} v\nabla_x(|\psi(x,v)|)dxd\mu(v) \\ &= -\int_{\partial\mathsf{D}\times\mathbb{R}^N} |\psi|\, v\nu_x d\gamma(x)\, d\mu(v) \\ &= \|\psi_-\|_{L^{1,-}} - \|\psi_+\|_{L^{1,+}} \\ &= \left\|H\psi^+\right\|_{L^{1,-}} - \|\psi_+\|_{L^{1,+}} . \end{aligned}$$

Hence, for every $\psi \in D(T_H)$,

$$\langle T_H\psi,\, \|\psi\|_{\mathsf{X}_1}\, \mathrm{sign}(\psi)\rangle = \|\psi\|_{\mathsf{X}_1}\, \langle T_H\psi,\, \mathrm{sign}(\psi)\rangle \leq 0$$

because $\|H\| \leq 1$. This proves that T_H is dissipative.

Let $\psi \in D(T_H)$ and set $\varphi = \lambda\psi - T_H\psi$. Using the equality

$$\mathrm{Re}\lambda \, \|\psi\|_{\mathsf{X}_1}^2 = \mathrm{Re}[\lambda\langle\psi, \psi^*\rangle]$$

with $\psi^* = \|\psi\|_{\mathsf{X}_1} \, \mathrm{sign}(\psi)$, one sees that

$$\begin{aligned} \mathrm{Re}\lambda \, \|\psi\|_{\mathsf{X}_1}^2 &\leq \mathrm{Re}[\langle\lambda\psi, \psi^*\rangle] \\ &\leq \mathrm{Re}[\langle\lambda\psi, \psi^*\rangle] - \langle T_H\psi, \, \psi^*\rangle \\ &= \mathrm{Re}[\langle\lambda\psi - T_H\psi, \psi^*\rangle] \leq \|\varphi\|_{\mathsf{X}_1} \, \|\psi\|_{\mathsf{X}_1} . \end{aligned}$$

Accordingly, if $\lambda \in \mathbb{C}_0$, then $\|\psi\|_{\mathsf{X}_1} = \left\|(\lambda - T_H)^{-1}\varphi\right\|_{\mathsf{X}_1} \leq \dfrac{\|\varphi\|_{\mathsf{X}_1}}{\mathrm{Re}\lambda}$ which ends the proof. □

Let us now introduce the following assumptions:

(A1) For $p \in (1, +\infty)$, the operator H is compact and satisfies

$$\|H\|_{\mathcal{L}(L^{p,+}, L^{p,-})} < 1.$$

(A2) For $p = 1$, the operator H is weakly compact and satisfies

$$\|H\|_{\mathcal{L}(L^{1,+}, L^{1,-})} < 1.$$

(A3) For each $e \in S^{N-1}$, $\mu\{v \in \mathbb{R}^N, \, v.e = 0\} = 0$

where S^{N-1} denotes the unit sphere of $\mathbb{R}^N$.

This hypothesis means that the hyperplanes of $\mathbb{R}^N$ (through the origin) have zero μ-measure.

The physical collision operators K, used in nuclear reactor theory for all types of moderators (gas, liquid or solid), are bounded and most of them are of the form

$$\mathsf{X}_p \ni \phi \mapsto \int_{\mathbb{R}^N} \kappa(x, v, v')\phi(x, v')d\mu(v') \in \mathsf{X}_p, \tag{6.2.9}$$

where $k(\cdot, \cdot, \cdot)$ is a non-negative measurable function (see, for example, [26, 43, 39] or [58] and the references therein).

Note that K is local with respect to the space variable $x \in \mathsf{D}$. So, it may be regarded as an operator valued mapping from D into $\mathcal{L}(L^p(\mathbb{R}^N; d\mu(v)))$, that is,

$$x \in \mathsf{D} \longrightarrow K(x) \in \mathcal{L}(L^p(\mathbb{R}^N; d\mu(v))),$$

where

$$K(x) \, : \, \varphi \in L^p(\mathbb{R}^N; d\mu(v)) \longrightarrow K(x)\varphi = \int_{\mathbb{R}^N} \kappa(x, v, v')\varphi(v')d\mu(v').$$

6.2.2 Compactness properties

Let us first recall the definition of collectively compact (resp. collectively weakly compact) operators on Banach spaces.

Case $p \in (1, \infty)$

Definition 6.2.2 *Let X and Y be two normed spaces. A set C of $\mathcal{L}(X, Y)$ is said to be collectively compact if, and only if,*

the set $C(\mathbb{B}_1) = \{U(x),\ U \in C,\ x \in \mathbb{B}_1\}$ is relatively compact in Y [7].

Now we are ready to state the definition of regular collisions operators [59] (see also [58]).

Definition 6.2.3 *Let $p \in (1, +\infty)$. A collision operator K is said to be regular on X_p if:*

(a) $\{K(x) : x \in \mathsf{D}\}$ is a set of collectively compact operators on $L^p(\mathbb{R}^N; d\mu(v))$, i.e.

$$\left\{K(x)\varphi;\ x \in \mathsf{D},\ \|\varphi\|_p \leq 1\right\} \text{ is relatively compact in } L^p(\mathbb{R}^N; d\mu(v)).$$

(b) For every $\varphi' \in L^q(\mathbb{R}^N; d\mu(v))$,

$$\left\{K'(x)\varphi';\ x \in \mathsf{D},\ \|\varphi\|_q \leq 1\right\} \text{ is relatively compact in } L^q(\mathbb{R}^N; d\mu(v)).$$

Here $K'(x)$ denotes the dual operator of $K(x)$ and $q = \dfrac{p}{p-1}$. □

We now recall the following result established in [49, 16, 58] which will play a crucial role.

Proposition 6.2.1 *Let K be a regular collision operator in X_p with nonnegative kernel and assume that the conditions* (A1) *and* (A3) *are satisfied. Then, for any $\lambda \in \mathbb{C}_0$, the operator $(\lambda - T_H)^{-1}K$ is compact on X_p with $p \in (1, +\infty)$,*

Proof. Item (a) is Theorem 3.1 in [49]. □

Case $p = 1$

Definition 6.2.4 *Let X and Y be two normed spaces. A set W of $\mathcal{L}(X, Y)$ is said to be collectively weakly compact if, and only if, the set*
$W(\mathbb{B}_1) = \{U(x),\ U \in W,\ x \in \mathbb{B}_1\}$ *is relatively weakly compact in Y.* □

Now, recall regular collision operators in X_1 [54, 70].

Definition 6.2.5 *We say that K is a regular collision operator on X_1 if, for almost all $x \in \mathsf{D}$, the operator*

$$\phi \in L^1(\mathbb{R}^N; d\mu(v)) \longrightarrow \int_{\mathbb{R}^N} \kappa(x, v, v')\phi(v')d\mu(v') \in L^1(\mathbb{R}^N; d\mu(v))$$

is weakly compact on $L^1(\mathbb{R}^N; d\mu(v))$ and the family of such operators on $L^1(\mathbb{R}^N; d\mu(v))$ indexed by $x \in \mathsf{D}$ is collectively weakly compact. □

We now recall the following result established in [49, 58] which will play a crucial role.

Proposition 6.2.2 *Let K be a regular collision operator in X_p with nonnegative kernel and assume that the conditions* (A2) *and* (A3) *are satisfied. Then, for any $\lambda \in \mathbb{C}_0$, $(\lambda - T_H)^{-1}K$ is a Dunford-Pettis operator on X_1.*

Proof. see Lemma 3.3 in [16].

Let $\mathsf{B}(\mathsf{X}_1)$ denote the collection of all nonempty bounded subsets of X_1 and let $\mathsf{W}(\mathsf{X}_1)$ be the subset of $\mathsf{B}(\mathsf{X}_1)$ consisting of all relatively weakly compact subsets of X_1. Inspired by measures of weak noncompactness introduced in the works [16] and [50], we define the map $\zeta : \mathsf{B}(\mathsf{X}_1) \to [0, +\infty)$ by

$$\zeta(M) = \zeta_1(M) + \zeta_2(M),$$

where

$$\zeta_1(M) = \limsup_{\varepsilon \to 0} \left\{ \sup_{\psi \in M} \left[\iint_E |\psi(x, v)| dx d\mu(v),\ |E| \leq \varepsilon \right] \right\}, \tag{6.2.10}$$

with $|E|$ denotes the measure of $E \subset D \times \mathbb{R}^n$ with respect to $dxd\mu(v)$ and

$$\zeta_2(M) = \lim_{m \to +\infty} \left\{ \sup_{\psi \in M} \left[\int_D \int_{|v| \geq m} |\psi(x, v)| dx d\mu(v) \right] \right\}. \tag{6.2.11}$$

Arguing as in the proof of Proposition 2.8 in [16] or Proposition 2 in [50] we establish that $\zeta(\cdot)$ is a regular measure of weak noncompactness on X_1. In fact, it satisfies axioms of Definition 2.1.11 and more. To avoid repetitions we will not recall them.

6.2.3 Existence results

For $i \in \{1, 2, \cdots, d\}$, let Λ_i denote the bounded multiplication operators from X_p into itself defined by $\Lambda_i f_i = \lambda_i \beta_i(x, v) f_i$. Since the functions $\beta_i(\cdot, \cdot)$, $i = 1, \cdots, N$, belong to $L^\infty(\mathsf{D} \times \mathbb{R}^N;\, dx \otimes d\mu(v))$, the operators Λ_i, $i = 1, \cdots, N$ are bounded. We also denote by K_i, $0 \le i \le d$, the operator defined from X_p to X_p by

$$f \mapsto K_i f(x, v) := \int_{\mathbb{R}^N} \kappa_i(x, v, v') f(x, v') d\mu(v'),\ 0 \le i \le d, \tag{6.2.12}$$

where $\kappa_i : \mathsf{D} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$. Moreover, $\mathcal{N}_g$ stands for the Nemytskii operator generated by the function g (for $g = -\sigma, \Theta_0, \cdots, \Theta_d$).

Now Problem (6.2.1)-(6.2.2) may be written abstractly as

$$\begin{cases} (\lambda - T_H) f_0 = \mathcal{N}_{-\sigma} + K_0 \mathcal{N}_{\Theta_0} f_0 + \displaystyle\sum_{i=1}^{d} \lambda_i \Lambda_i f_i \\ \lambda_i f_i = K_i \mathcal{N}_{\Theta_i} f_0 \quad (1 \le i \le d). \end{cases} \tag{6.2.13}$$

Since, for $i = 1, \cdot, \cdot, \cdot, d$, the operator Λ_i is bounded, we can replace $\lambda_i f_i(x, \xi)$ of the first equation of (6.2.13) by $K_i \mathcal{N}_{\Theta_i} f_0$. Moreover, according to Remark 6.2.2, each $\lambda \in \mathbb{C}$ such that $\lambda \in \mathbb{C}_0$ belongs to $\rho(T_H)$. Hence, for such λ, Problem (6.2.1)-(6.2.2) reduces to the following fixed point problem

$$f_0 = \mathsf{F}(\lambda) f_0 + \mathsf{B}(\lambda) f_0, \quad f_{0|\Gamma_-} = H(f_{0|\Gamma_+}), \tag{6.2.14}$$

where

$$\begin{cases} \mathsf{F}(\lambda) = (\lambda - T_H)^{-1} K_0 \mathcal{N}_{\Theta_0} \quad \text{and} \\ \mathsf{B}(\lambda) = (\lambda - T_H)^{-1} \mathcal{N}_{-\sigma} + \displaystyle\sum_{i=1}^{d} (\lambda - T_H)^{-1} \Lambda_i K_i \mathcal{N}_{\Theta_i}. \end{cases} \tag{6.2.15}$$

Because, for each $i \in \{1, \cdots, d\}$, $f_i = \dfrac{K_i \mathcal{N}_{\Theta_i} f_0}{\lambda_i}$; so to solve Problem (6.2.1)-(6.2.2), it suffices to search the solutions of the fixed point problem (6.2.14).

Let $r > 0$, $p \in [1, +\infty)$ and denote by $\mathbb{B}_r^p$ the ball defined by

$$\mathbb{B}_r^p := \Big\{ x \in \mathsf{X}_p \ \text{ such that } \ \|x\|_{\mathsf{X}_p} \le r \Big\}.$$

We now introduce the following assumptions

(A4) For each $i \in \{0, 1, \cdot, \cdot, \cdot, d\}$, the map $\Theta_i(\cdot, \cdot, \cdot)$ is a Carathéodory function and $\mathcal{N}_{\Theta_i}$ acts from X_p and into itself.

(A5) For each $r > 0$, the function $\sigma(\cdot, \cdot, \cdot)$ satisfies

$$|\sigma(x, v, \psi_1(x, v)) - \sigma(x, v, \psi_2(x, v))| \leq \sigma_0(x, v)\,|\psi_1 - \psi_2|$$

$\forall\, \psi_1, \psi_2 \in \mathbb{B}_r^p$, $\sigma(\cdot, \cdot, \cdot)$ is a Carathéodory function and $\mathcal{N}_{-\sigma}$ acts from X_p into itself, where $\sigma_0(., .) \in L^\infty(\mathsf{D} \times \mathbb{R}^N; dx \otimes d\mu(v))$.

(A6) For each $r > 0$ and each $i \in \{1, 2, \cdot, \cdot, \cdot, d\}$, the function $\Theta_i(\cdot, \cdot, \cdot)$ satisfies

$$|\Theta_i(x, v, \psi_1(x, v)) - \Theta_i(x, v, \psi_2(x, v))| \leq \rho_i(x, v)\,|\psi_1 - \psi_2|$$

$\forall\, \psi_1, \psi_2 \in \mathbb{B}_r^p$, where $\rho_i(., .) \in L^\infty(\mathsf{D} \times \mathbb{R}^N; dx \otimes d\mu(v))$.

Theorem 6.2.1 *Let $1 \leq p < \infty$, K_0 be a regular collision operator on X_p and assume that* (A1)-(A6) *hold true. Then, for each $r > 0$, there exists a real $\theta(r) > 0$ such that, for all $\lambda \in \mathbb{C}_{\theta(r)}$, the boundary value problem* (6.2.1)-(6.2.2) *has at least one solution with $f_0 \in \mathbb{B}_r^p$.*

Proof. Let $\lambda \in \mathbb{C}_0$, $p \in [1, +\infty)$ and let $r > 0$. It is clear that the operators $\mathsf{B}(\lambda)$ and $\mathsf{F}(\lambda)$ are well defined and are continuous.

Now let $\varphi, \psi \in \mathbb{B}_r^p$. Since the Nemytskii operator generated by $-\sigma$, $\Theta_0, \cdots, \Theta_d$ map $\mathbb{B}_r^p$ into bounded sets, easy calculations using Lemma 6.2.1 yield

$$\|\mathsf{F}(\lambda)\varphi + \mathsf{B}(\lambda)\psi\|_{\mathsf{X}_p} \leq \frac{1}{\mathrm{Re}\lambda}\left(\|K_0\|\mathcal{M}_{r,0} + \mathcal{M}_{r,\sigma} + \sum_{i=1}^{d} |\lambda_i|\|\beta_i\|_\infty \|K_i\|\mathcal{M}_{r,i}\right),$$

where $\mathcal{M}_{r,i}$, $0 \leq i \leq \mathsf{d}$, (resp. $\mathcal{M}_{r,\sigma}$) denotes the upper bound of $\mathcal{N}_{\Theta_i}$ (resp. $\mathcal{N}_{-\sigma}$) on $\mathbb{B}_r^p$.

Let $\mathcal{M}_r = \max(\mathcal{M}_{r,0}, ..., \mathcal{M}_{r,d}, \mathcal{M}_{r,\sigma})$ and let τ_1 be a nonnegative real number such that

$$\frac{1}{\tau_1}\left(1 + \|K_0\| + \sum_{i=1}^{d} |\lambda_i|\|\beta_i\|_\infty \|K_i\|\right) \leq \frac{r}{\mathcal{M}_r}.$$

Hence, for all $\lambda \in \mathbb{C}_{\tau_1}$ we have

$$\|\mathsf{F}(\lambda)\varphi + \mathsf{B}(\lambda)\psi\|_{\mathsf{X}_p} \leq r,$$

and therefore

$$\mathsf{F}(\lambda)(\mathbb{B}_r^p) + \mathsf{B}(\lambda)(\mathbb{B}_r^p) \subset \mathbb{B}_r^p.$$

Next, we show that, for appropriate value of $\lambda \in \mathbb{C}_0$, the operator $\mathbf{B}(\lambda)$ is a contraction on X_p. Indeed, let $\psi_1, \psi_2 \in \mathsf{X}_p$ and let $\lambda \in \mathbb{C}_0$. Using Lemma 6.2.1 and (A5)-(A6), we obtain

$$\|\mathsf{B}(\lambda)\psi_1 - \mathsf{B}(\lambda)\psi_2\|_{\mathsf{X}_p} \leq \frac{1}{\mathrm{Re}\lambda}\left(\|\sigma_0\|_\infty + \sum_{i=1}^{d} |\lambda_i|\|\beta_i\|_\infty\|K_i\|\|\rho_i\|_\infty\right)\|\psi_1 - \psi_2\|_{\mathsf{X}_p}.$$

Let τ_2 be a nonnegative real number such that

$$k := \frac{1}{\tau_2}(\|\sigma_0\|_\infty + \sum_{i=1}^{d} |\lambda_i|\|\beta_i\|_\infty\|K_i\|\|\rho_i\|_\infty) < 1.$$

Hence, for all $\lambda \in \mathbb{C}_{\tau_2}$, $\mathsf{B}(\lambda)$ is k-contractive.

From now, we must treat separately the cases $p \in (1, +\infty)$ and $p = 1$.

(a) Let $p \in (1, +\infty)$.
We show that the operator $\mathsf{F}(\lambda)$ is compact. Since K_0 is a regular collision operator, then according to Proposition 6.2.1 (a) $(\lambda - T_H)^{-1}K_0$ is compact. Therefore, by Lemma 2.2.1 and assumption (A4), we infer that $\mathcal{N}_{\Theta_0}$ is continuous and consequently $\mathsf{F}(\lambda)$ is compact on X_p.

Set $\theta(r) = \max(\tau_1, \tau_2)$. Then, for any $\lambda \in \mathbb{C}_{\theta(r)}$, the operators $\mathsf{B}(\lambda)$ and $\mathsf{F}(\lambda)$ satisfy the conditions of the classical Krasnosel'skii fixed point theorem. So, the fixed point Problem (6.2.14) has a solution $f_0 \in \mathbb{B}_r^p$ and therefore the boundary value problem (6.2.1)-(6.2.2) has at least one solution $(f_0, ..., f_d)$ with $f_0 \in \mathbb{B}_r^p$.

(b) Consider now the case $p = 1$.
We shall show that the operators $\mathsf{F}(\lambda)$ and $\mathsf{B}(\lambda)$ appearing in Eq. (6.2.15) satisfy the hypotheses of Corollary 4.2.1. To do so, we first observe that, according to Proposition 6.2.2, $(\lambda - T_H)^{-1}K_0$ is a Dunford-Pettis operator. Hence the condition (a) of Corollary 4.2.1 is satisfied.

Let $S \subset \mathbb{B}_r^1$. Since the maps Θ_i, $i = 0, \cdot, \cdot, \cdot, d$, satisfy the condition (A4), by Lemma 2.2.1, there exist $\eta_i > 0$ and $h_i(.) \in \mathsf{X}_1^+$ (the positive cone of X_1) such that

$$|\Theta_i(x, v, f(x, v))| \leq h_i(x, v) + \eta_i|f(x, v)|.$$

for all $f \in S$ and for almost all $(x, v) \in \mathsf{D} \times \mathbb{R}^N$. Accordingly,

$$\begin{aligned}\int_E |(\mathcal{N}_{\Theta_i} f)(x, v)|dxd\mu(v) \leq \quad & \int_E h_i(x, v)dxd\mu(v) \\ & + \eta_i \int_E |f(x, v)|dxd\mu(v),\end{aligned} \tag{6.2.16}$$

and

$$\int_{\mathsf{D}}\int_{|v|\geq m}|(\mathcal{N}_{\Theta_i}f)(x,v)|dxd\mu(v) \quad \leq \int_{\mathsf{D}}\int_{|v|\geq m}h_i(x,v)dxd\mu(v)$$
$$+\eta_i\int_{\mathsf{D}}\int_{|v|\geq m}|f(x,v)|dxd\mu(v), \tag{6.2.17}$$

for all measurable subset E of $\mathsf{D}\times\mathbb{R}^N$, $m>0$ and $0\leq i\leq d$. This together with the estimate (2.1.4) implies that

$$\begin{aligned}\zeta_1(\mathsf{F}(\lambda)(S)+\mathsf{B}(\lambda)(S)) &\leq \zeta_1(\mathsf{F}(\lambda)(S))+\zeta_1(\mathsf{B}(\lambda)(S))\\ &\leq \left\|(\lambda-T_H)^{-1}\right\|\\ &\left(\|K_0\|\zeta_1(\mathcal{N}_{\mathbf{\Theta}_0}S)+\zeta_1(\mathcal{N}_{-\sigma}S)+\sum_{i=1}^{d}\|\Lambda_iK_i\|\zeta_1(\mathcal{N}_{\mathbf{\Theta}_i}S)\right)\\ &\leq \frac{1}{\mathrm{Re}\lambda}\left(\eta_0\|K_0\|+\eta_\sigma+\sum_{i=1}^{d}\eta_i\|\beta_i\|_\infty\|K_i\|\right)\zeta_1(S).\end{aligned}$$

Now, let τ_3 be a nonnegative real number such that

$$\gamma:=\frac{1}{\tau_3}(\eta_0\|K_0\|+\eta_\sigma+\sum_{i=1}^{d}\eta_i|\lambda_i|\|\beta_i\|_\infty\|K_i\|)<1.$$

Hence, for all $\lambda\in\mathbb{C}_{\tau_3}$ and for all subset S of $\mathbb{B}_r^1$, we have

$$\zeta_1\Big(\mathsf{F}(\lambda)(S)+\mathsf{B}(\lambda)(S)\Big)\leq\gamma\zeta_1(S).$$

Similarly, by using (6.2.17), we obtain

$$\zeta_2\Big(\mathsf{F}(\lambda)(S)+\mathsf{B}(\lambda)(S)\Big)\leq\gamma\zeta_2(S).$$

Hence the last two estimates yield

$$\zeta\Big(\mathsf{F}(\lambda)(S)+\mathsf{B}(\lambda)(S)\Big)\leq\gamma\zeta(S),$$

for all subset $S\subset\mathbb{B}_r^1$.

Next, since Θ_0 satisfies (A4), it follows from Lemma 3.2 in [16] that $\mathcal{N}_{\Theta_0}$ satisfies condition ww-compact.

Set $\theta(r)=\max(\tau_1,\tau_2,\tau_3)$. The steps above show that, for all $\lambda\in\mathbb{C}_{\theta(r)}$, the assumptions of Corollary 4.2.1 are satisfied and so the fixed point problem (6.2.14) has a solution $f_0\in\mathbb{B}_r^1$. Therefore, the boundary value problem (6.2.1)-(6.2.2) has at least one solution $(f_0,...,f_d)$ with $f_0\in\mathbb{B}_r^1$. □

6.3 A transport equation with partly elastic collision operators

The purpose of this section is to discuss existence of solutions to the following boundary value problem

$$v.\frac{\partial\varphi}{\partial x}(x,v)+\sigma(x,v,\varphi(x,v))+\lambda\varphi(x,v)=\int_V k_c(x,v,v')f(x,v',\varphi(x,v'))d\mu(v')$$
$$+\sum_{j=1}^{l}\int_{\mathbb{S}^{N-1}} k_d^j(x,\rho_j,\omega,\omega')\varphi(x,\rho_j\omega')d\omega'$$
$$+\int_{\mathbb{S}^{N-1}} k_e(x,\rho,\omega,\omega')g(x,\rho,\omega',\varphi(x,\rho,\omega'))d\omega' \tag{6.3.1}$$

where $(x,v)\in D\times V$, (D is an open smooth bounded subset of $\mathbb{R}^N$ ($N\geq 3$) endowed with the Lebesgue measure dx while V stands for the admissible velocities space). The subset V is given by

$$V=\{v=\rho\omega,\omega\in\mathbb{S}^{N-1},0\leq\rho_{\min}\leq\rho\leq\rho_{\max}<\infty\}=:I\times\mathbb{S}^{N-1}.$$

It is endowed with the Lebesgue measure $dv=\rho^{N-1}d\rho d\omega$ where $d\omega$ is the Lebesgue measure on the unit sphere $\mathbb{S}^{N-1}$. The functions $\sigma(\cdot,\cdot,\cdot)$ and $f(\cdot,\cdot,\cdot)$ are measurable nonlinear of φ and the function $\varphi(x,v)$ represents the number density of particles having the position x and the velocity v. The function $\sigma(\cdot,\cdot,\cdot)$ is called the collision frequency and the functions $k_c(\cdot,\cdot,\cdot,\cdot)$, $k_e(\cdot,\cdot,\cdot,\cdot,\cdot)$ and $k_d^j(\cdot,\cdot,\cdot,\cdot,\cdot)$, $j=1,\cdots,l$, denote the scattering kernels of the operators K_c, K_e and $K_d=\sum_{j=1}^{l}K_d^j$ (called classical, elastic and inelastic collision operators respectively).

In our framework, the boundary conditions are modled by

$$\varphi_{|\Gamma_-}=0, \tag{6.3.2}$$

where, Γ_- denotes the incoming part of the boundary of the phase space $\Omega\times V$ and define by

$$\Gamma_-=\{(x,v)\in\partial\Omega\times V;-v.\nu_x>0\}$$

where ν_x stands for the outer unit normal vector at $x\in\partial\Omega$.

In this model, the collision operator is given by

$$K=K_c+K_e+K_d.$$

where K_c denotes the collision operator involved in the classical neutron transport theory (see, for example, [59]). It is given by

$$K_c\varphi(x,v)=\int_V k_c(x,v,v')\varphi(x,v')d\mu(v'),$$

The operator K_d is called the inelastic collision operator, it is given by

$$K_d\varphi(x,v) = \sum_{j=1}^{l} K_d^j\varphi(x,v) = \sum_{j=1}^{l}\int_{\mathbb{S}^{N-1}} k_d^j(x,\rho_j,\omega,\omega')\varphi(x,\rho_j\omega')d\omega',$$

where each operator K_d^j ($j = 1,...,l$) describes an event in which a discrete energy E_j is lost by a neutron at x with initial speed ρ_j and final speed ρ. The speed ρ_j is defined by

$$E_j = \frac{1}{2}M(\rho_j)^2 - \frac{1}{2}M(\rho)^2,$$

while M stands for the mass of neutron. The operator K_e is called the elastic collision operator. It is given by

$$K_e\varphi(x,v) = \int_{\mathbb{S}^{N-1}} k_e(x,\rho,\omega,\omega')\varphi(x,\rho\omega')d\omega'.$$

It describes the collisions which do not vary the kinetic energy of neutrons. It acts only through the angular part of the velocity variable .

The goal of this section is complete the analysis started in the papers [47, 68, 4, 5] where several analysis of the spectral properties and the properties of solution to Cauchy problem governed by these operators for the problem. Our aim in this section to solve the stationary Problem (6.3.1)-(6.3.2) in L^p-spaces ($1 \leq p < \infty$). We introduce the functionnal setting of the problem (6.3.1)-(6.3.2) and we establish some existence results of the compactness to be able to solve the problem (6.3.1)-(6.3.2).

6.3.1 Notations

The aim of this section is to introduce the different notations and some preliminary results which we will need in the sequel.

We define the streaming operator T with absorbing boundary conditions (that is $H = 0$) by

$$\begin{cases} T \ : \ D(T) \subset L^p(\mathsf{D}\times V) \longrightarrow L^p(\mathsf{D}\times V) \\ \qquad \varphi \longmapsto T\varphi(x,v) = -v.\dfrac{\partial\varphi}{\partial x}(x,v) \\ D(T) = \left\{\varphi \in L^p(\mathsf{D}\times V); v.\frac{\partial\varphi}{\partial x} \in L^p(\mathsf{D}\times V) \text{ such that } \varphi_{|\Gamma_-} = 0\right\}, \end{cases}$$

where

$$L^p(\mathsf{D}\times V) := L^p(\mathsf{D}\times V, dxdv), \quad 1 \le p < \infty.$$

Let $\psi \in L^p(D\times V)$ and consider the resolvent equation for the operator T,

$$(\lambda - T)\psi = \varphi \tag{6.3.3}$$

where λ is a complex number and the unknown ψ must be sought in $D(T)$. Proceeding as in [49], one sees that, for $\mathrm{Re}\lambda > 0$, the solution of Eq. (6.3.3) is formally given by

$$\psi(x,v) \quad = \quad (\lambda - T)^{-1}\varphi(x,v) = \int_0^{s(x,v)} e^{-\lambda s}\varphi(x - sv, v)ds$$

where $s(x,v) = \inf\{s > 0;\ x - sv \notin \mathsf{D}\}$. In fact, $s(x,v)$ is the time required by a particle having the position x and the velocity v to go out of D.

Remark 6.3.1 *As already observed, the operator T is nothing else but the operator T_0 (the boundary operator is $H = 0$), so the proof of Lemma 6.2.1 shows that T is dissipative and, for any $\lambda \in \mathbb{C}_0$ (see* (6.2.8)*), the following estimate holds true*

$$\left\|(\lambda - T)^{-1}\right\| \le \frac{1}{\mathrm{Re}\lambda}. \tag{6.3.4}$$

□

We note that, following (6.3.4), the resolvent set of the operator T contains the complex half plane $\mathbb{C}_0$.

We recall that In [9], Appell and De Pascale proved that the measure of weak non-compactness of De Blasi $\omega(\cdot)$ can be expressed in L^1-spaces as

$$\omega(M) = \limsup_{\epsilon\to 0}\Big\{\sup_{\psi\in M}\Big(\int_D |\psi(t)|\,dt :\ |D| < \epsilon\Big)\Big\},$$

for every subset $M \in \mathfrak{B}(L^1(\mathsf{D}\times V))$. Here $|D|$ denotes the Lebesgue measure of D.

6.3.2 Compactness properties

In the paper [52], the spectral properties of the operator $T + K_e + K_d + K_c$ was discussed in the space L^1. In [68], this analysis was extended to L^p-spaces with $p \in [1, +\infty)$. Our objective in this section is to complete this analysis by investigating compactness properties of the operators $(\lambda - T)^{-1}K_e$ and $K_e(\lambda - T)^{-1}$ in the space L^p for $p \ge 1$ and $\lambda \in \mathbb{C}_0$.

Here we will split our discussion into two cases : the case $1 < p < \infty$ and the case $p = 1$.

Case $p \in (1, \infty)$

In this subsection, we shall be interested in the compactness properties of the operators $(\lambda - T)^{-1}K_e$ and $K_e(\lambda - T)^{-1}$ on the spaces

$$L^p(\mathsf{D} \times I \times \mathbb{S}^{N-1}) := L^p(\mathsf{D} \times I \times \mathbb{S}^{N-1}; \rho^{N-1} dx d\rho d\omega)$$

with $1 < p < +\infty$.

Let us first note that the collision operator K_c is a bounded linear operator on $L^p(\mathsf{D} \times V)$ defined by

$$\begin{cases} K_c : L^p(\mathsf{D} \times V) \to L^p(\mathsf{D} \times V) \\ \qquad \psi \to \displaystyle\int_V \kappa_c(x, v, v')\psi(x, v')dv', \end{cases}$$

where the kernel $\kappa_c(\cdot, \cdot, \cdot)$ is a nonnegative measurable function on $\mathsf{D} \times V \times V$.

It is clear that the collision operator K_c is local in x. So, we may regard K_c as an operator on $L^p(V; dv)$ depending on a parameter $x \in \mathsf{D}$. One can define the mapping

$$K_c \ : \ \mathsf{D} \to \mathcal{L}(L^p(V, dv)), \quad x \mapsto K_c(x),$$

where

$$K_c(x) : \varphi \in L^p(V, dv) \mapsto K_c(x)\varphi = \int_V k_c(x, v, v')\varphi(v')dv'.$$

Remark 6.3.2 *It should be noticed that the operator K_c is nothing other than the collision operator K considered in Section 6.2. So, if K_c is regular, and the hypothesis* (A3) *holds true, $(\lambda - T)^{-1}K_c$ is compact on X_p (see Proposition 6.2.1 or [58, Theorem 4.1, p. 57]).*

The elastic collision operator K_e is defined by

$$\begin{cases} K_e : L^p(\mathsf{D} \times I \times \mathbb{S}^{N-1}) \longrightarrow L^p(\mathsf{D} \times I \times \mathbb{S}^{N-1}) \\ \psi \mapsto \displaystyle\int_{\mathbb{S}^{N-1}} k_e(x, \rho, \omega, \omega')\psi(x, \rho, \omega')d\omega'. \end{cases} \tag{6.3.5}$$

where the kernel $k_e(\cdot, \cdot, \cdot, \cdot)$ is a nonnegative measurable function on $\mathsf{D} \times I \times \mathbb{S}^{N-1} \times \mathbb{S}^{N-1}$.

We introduce the following hypothesis:

(A7) for all $\psi \in L^p(\mathbb{S}^{N-1}, d\omega)$, the set

$$\Big\{ \int_{\mathbb{S}^{N-1}} k_e(x, \rho, \omega, \omega')\psi(\omega')d\omega' : (x, \rho) \in \mathsf{D} \times I, \|\psi\|_{L^p(\mathbb{S}^{N-1}, d\omega)} \leq 1 \Big\}$$

is relatively compact set in $L^p(\mathbb{S}^{N-1}, d\omega)$ and, for every, $\psi' \in L^q(\mathbb{S}^{N-1}, d\omega)$,

$$\Big\{ \int_{\mathbb{S}^{N-1}} k_e(x, \rho, \omega, \omega')\psi'(\omega)d\omega : (x, \rho) \in \mathsf{D} \times I \Big\}$$

is relatively compact set in $L^q(\mathbb{S}^{N-1}, d\omega)$.

The hypothesis (A7) was introduced by M. Sbihi in [67] (see also [68]).

Under condition (A7), we have

$$\|K_e\|_{\mathcal{L}(L^p(\mathsf{D}\times I\times\mathbb{S}^{N-1}, dxdv))} = ess \sup_{(x,\rho)\in\mathsf{D}\times I} \|K_e(x, \rho)\|_{\mathcal{L}(L^p(\mathbb{S}^{N-1}, d\omega))} \tag{6.3.6}$$

where $K_e(x, \rho)$ is the operator defined by

$$L^p(\mathbb{S}^{N-1}) \ni \psi \longmapsto \int_{\mathbb{S}^{N-1}} k_e(x, \rho, \omega, \omega')\psi(\omega')d\omega'.$$

We recall also the following lemma owing to M. Sbihi [67].

Lemma 6.3.1 *An elastic collision operator K_e satisfying* (A7) *can be approximated in the norm operator topology by collision operators with kernels of the form*

$$k_e^i(x, \rho, \omega, \omega') = \sum_{i\in J} \alpha_i(x, \rho) f_i(\omega) g_i(\omega')$$

where $\alpha_i \in L^\infty(\mathsf{D} \times I)$, $f_i \in L^p(\mathbb{S}^{N-1})$ and $g_i \in L^q(\mathbb{S}^{N-1})$, $i \in J$ with q denotes the exponent conjugate of p while J is a finite set.

Proposition 6.3.1 *Let $\lambda \in \mathbb{C}_0$. If K_e is a regular collision operator in $L^p(\mathsf{D} \times I \times \mathbb{S}^{N-1})$ satisfying* (A7)*, then the operators $(\lambda - T)^{-1}K_e$ is compact.*

Proof. Since K_e is regular operator, according to Lemma 6.3.2, it is enough to establish the theorem for a collision operator with a kernel of the form

$$k_e^i(x, \rho, \omega, \omega') = \sum_{i\in J} \alpha_i(x, \rho) f_i(\omega) g_i(\omega')$$

where $\alpha_i \in L^\infty(\mathsf{D} \times I)$, $f_i \in L^p(\mathbb{S}^{N-1})$ and $g_i \in L^q(\mathbb{S}^{N-1})$.

By approximating, we assume that the functions f and g are constants ($f(\cdot) \in L^\infty(\mathbb{S}^{N-1})$ and $g(\cdot) \in L^\infty(\mathbb{S}^{N-1})$. In such case, it suffices to consider the case $K_e(\lambda - T)^{-1}$ and $(\lambda - T)^{-1}K_e$ map $L^s(\mathsf{D} \times I \times \mathbb{S}^{N-1})$ into itself for all $s \in [1, \infty]$ so that (by interpolation) it suffices to give a proof for $p = 2$. We consider first $K_e(\lambda - T)^{-1}$. Let M_g be the averaging operator

$$M_g : \varphi \in L^2(\mathsf{D} \times I \times \mathbb{S}^{N-1}) \to \int_{\mathbb{S}^{N-1}} \varphi(x, \rho, \omega')d\omega' \in L^2(\mathsf{D}).$$

It suffices to show that

$$M_g(\lambda - T)^{-1} : L^2(\mathsf{D} \times I \times \mathbb{S}^{N-1}) \to L^2(\mathsf{D}) \text{ is compact.}$$

This amounts to

$$M_g : D(T) = \left\{\varphi \in L^2(\mathsf{D} \times V);\ v.\frac{\partial \varphi}{\partial x} \in L^2(\mathsf{D} \times V),\ \varphi_{|\Gamma_-} = 0\right\} \to L^2(\mathsf{D}).$$

We denote

$$W^2(\mathbb{R}^n_x \times \mathbb{R}^+ \times \mathbb{S}^{N-1}) = \left\{\varphi \in L^2(\mathbb{R}^n_x \times \mathbb{R}^+ \times \mathbb{S}^{N-1}) \text{ such that } v.\frac{\partial \varphi}{\partial x} \in L^2(\mathbb{R}^n_x \times \mathbb{R}^+ \times \mathbb{S}^{N-1})\right\}.$$

Furthermore, since D is convex, it follows from the lemma of extention that there exists a continuous extension operator (see[38]) such that $\mathcal{E} : W^2(\mathsf{D} \times \mathbb{R}^+ \times \mathbb{S}^{N-1}) \to W^2(\mathbb{R}^n_x \times \mathbb{R}^+ \times \mathbb{S}^{N-1})$. Let $\mathcal{O} \subset W^2(\mathsf{D} \times \mathbb{R}^+ \times \mathbb{S}^{N-1})$ be bounded. Then $\mathcal{E}(\mathcal{O}) \subset W^2(\mathbb{R}^n_x \times \mathbb{R}^+ \times \mathbb{S}^{N-1})$. Applying Theorem 3.1 in [58], we conclude that the set

$$\{\mathcal{E}\varphi_{|\mathsf{D}}, \varphi \in \mathcal{O}\}$$

is relatively compact in $L^2(\mathsf{D})$. And consequently

$$K_e^*((\lambda - T)^{-1})^* \text{ is compact.}$$

Now using the Schauder theorem [31, Theorem 2, p. 485], we conclude that $(\lambda - T)^{-1}K_e$ is compact by duality. □

Case $p = 1$

In this subsection we are concerned with the transport operator $T + K_e$ on the spaces

$$L^1(\mathsf{D} \times I \times \mathbb{S}^{N-1}) := L^1(\mathsf{D} \times I \times \mathbb{S}^{N-1}; \rho^{N-1}dxd\rho d\omega)$$

The elastic collision operator K_e on $L^1(\mathsf{D} \times I \times \mathbb{S}^{N-1})$ defined by

$$\begin{cases} K_e : L^1(\mathsf{D}\times I\times \mathbb{S}^{N-1}) \longrightarrow L^1(\mathsf{D}\times I\times \mathbb{S}^{N-1}) \\ \Phi \mapsto \displaystyle\int_{\mathbb{S}^{N-1}} k_e(x,\rho,\omega,\omega')\Phi(x,\rho,\omega')d\omega'. \end{cases} \tag{6.3.7}$$

where the kernel $k_e(\cdot,\cdot,\cdot,\cdot)$ is a nonnegative measurable function on $\mathsf{D}\times I\times \mathbb{S}^{N-1}\times \mathbb{S}^{N-1}$.

We introduce the following hypotheses:

(A8) $$\int_{\mathbb{S}^{N-1}} |k_e(\cdot,\cdot,\omega,\cdot)|\, d\omega \in L^1(\mathsf{D}\times I\times \mathbb{S}^{N-1})$$

(A9) $$\left\{ |k_e(x,\rho,\cdot,\omega')| : (x,\rho,\omega') \in \mathsf{D}\times I\times \mathbb{S}^{N-1} \right\} \text{ is weakly compact in } L^1(\mathbb{S}^{N-1}, d\omega)$$

Proceeding as in the proof of Lemma 3.2 in [52] we prove the following proposition. The proof is omitted.

Proposition 6.3.2 *Let K_e be a regular collision operator in $L^1(D\times V)$ and assume that K_e satisfies assumptions* (A8) *and* (A9)*. Then, for any $\lambda \in \mathbb{C}_0$, $(\lambda - T)K_e$ is a Dunford-Pettis operator.*

Next, let K_c be the bounded linear operator defined by

$$\begin{cases} K_c : L^1(\mathsf{D}\times V) \longrightarrow L^1(\mathsf{D}\times V) \\ \psi \mapsto \displaystyle\int_V k_c(x,v,v')\psi(x,v')d\mu(v'). \end{cases} \tag{6.3.8}$$

We recall the following result established in [52, Lemma 3.2] .

Lemma 6.3.2 *If K_c is a regular collision operator in $L^1(\mathsf{D}\times V)$. Then, for any $\lambda \in \mathbb{C}_0$, $(\lambda - T)^{-1}K_c$ is a Dunford-Pettis operator.* □

6.3.3 Existence results

In this section we discuss existence results for the boundary value problem (6.3.1)-(6.3.2) on the space $L^p(\mathsf{D}\times V)$ with $p \in [1,\infty)$.

Now we introduce the assumption .

(A10) The functions f and g satisfy the Carathéodory conditions and the Nemytskii operators N_f and N_g act from $L^p(\mathsf{D}\times V)$ into itself.

Lemma 6.3.3 *If* (A10) *holds true, then the operators* N_f *and* N_g *are* ww*-compact.*

Proof. For the proof we refer, for example, to [52]. □

Let $\mathbf{r} > 0$, we denote by $\mathbb{B}^p_{\mathbf{r}}$ the set

$$\mathbb{B}^p_{\mathbf{r}} = \Big\{\psi \in \mathsf{X}_p \ \text{ such that } \ \|\psi\|_{L^p(\mathsf{D}\times V)} \leq \mathbf{r}\Big\}.$$

We are now ready to state our existence result.

Theorem 6.3.1 *Let* K_c *and* K_e *be regular collision operators on* $L^p(D\times V)$ *and assume that* (A3)-(A10) *are satisfied. Then, for all* $\lambda \in \mathbb{C}_r$ *such that the boundary value problem* (6.3.1)-(6.3.2) *has at least one solution on* $\mathbb{B}^p_{\mathbf{r}}$.

Proof. Let $p \in [1, +\infty)$ and $\mathbf{r} > 0$. Since $\mathbb{C}_0 \subset \rho(T)$, for each $\lambda \in \mathbb{C}_0$, Problem (6.3.1)-(6.3.2) may be transformed into the following fixed point problem

$$\psi = (\mathsf{F}_\lambda)(\psi) + \mathsf{G}_\lambda(\psi), \qquad \varphi_{|\Gamma_-} = 0,$$

where

$$\mathsf{F}_\lambda = (\lambda - T)^{-1}K_c\mathcal{N}_f + (\lambda - T)^{-1}K_e\mathcal{N}_g \quad \text{and} \quad \mathsf{G}_\lambda = (\lambda - T)^{-1}(\mathcal{N}_{-\sigma} + K_d).$$

It is clear that F_λ and G_λ are a continous operators.

Let $\varphi, \psi \in \mathbb{B}^p_{\mathbf{r}}$ and for any $\lambda \in \mathbb{C}_0$ and taking into account of the hypotheses (A3) – (A10) together with the estimate (6.3.4) and the fact that K_c, K_e and K_d are a bounded collision operators, we get

$$\|\mathsf{F}_\lambda(\varphi) + \mathsf{G}_\lambda(\psi)\|_{L^p(\mathsf{D}\times V)} \leq \frac{1}{\mathrm{Re}\lambda}\Big((\|K_c\|\, N(\mathbf{r}) + \|K_e\|\, N'(\mathbf{r})) + (\|K_d\| + N''(\mathbf{r}))\Big).$$

where $N(\mathbf{r}), N'(\mathbf{r})$ and $N''(\mathbf{r})$ denote the upper bounds of $\mathcal{N}_f, \mathcal{N}_g$ and $\mathcal{N}_{-\sigma}$, respectively, on $\mathbb{B}^p_{\mathbf{r}}$.

Let θ_1 be the real number defined by

$$\theta_1 := \frac{(\|K_c\|\, N(\mathbf{r}) + \|K_e\|\, N'(\mathbf{r})) + (\|K_d\| + N''(\mathbf{r}))}{\mathbf{r}}.$$

So, for any $\lambda \in \mathbb{C}_{\theta_1}$, we have

$$\|\mathsf{F}_\lambda(\varphi) + \mathsf{G}_\lambda(\psi)\|_{L^p(\mathsf{D}\times V)} \leq \mathbf{r}.$$

and therefore

$$\mathsf{F}_\lambda(\mathbb{B}_{\mathbf{r}}^p) + \mathsf{G}_\lambda(\mathbb{B}_{\mathbf{r}}^p) \subset \mathbb{B}_{\mathbf{r}}^p.$$

Next, we shall check that, for suitable λ in $\mathbb{C}_0$, the operator G_λ is a contraction on $L^p(\mathsf{D}\times V)$. To see this, let $\psi_1, \psi_2 \in L^p(\mathsf{D}\times V)$ and let $\lambda \in \mathbb{C}_0$. Using (A5) and the estimate of Eq. (6.3.4), one sees that

$$\begin{aligned}\|\mathsf{G}_\lambda(\psi_1) - \mathsf{G}_\lambda(\psi_2)\|_{L^p(\mathsf{D}\times V)} &\leq \frac{\|K_d\|}{\mathrm{Re}\lambda}\,\|\psi_1 - \psi_2\|_{L^p(\mathsf{D}\times V)} \\ &\quad + \frac{1}{\mathrm{Re}\lambda}\,\|\mathcal{N}_{-\sigma}(\psi_1) - \mathcal{N}_{-\sigma}(\psi_2)\|_{L^p(\mathsf{D}\times V)} \\ &\leq \frac{\|K_d\| + \|\rho\|_\infty}{\mathrm{Re}\lambda}\,\|\psi_1 - \psi_2\|_{L^p(\mathsf{D}\times V)}\,.\end{aligned}$$

Let θ_2 be a nonnegative real number such that

$$\alpha := \frac{\|K_d\| + \|\rho\|_\infty}{\theta_2} < 1.$$

Hence, for any λ such that $\mathrm{Re}\lambda > \theta_2$, the operator G_λ is α-contractive.

The rest of the proof is divided into two steps following $1 < p < \infty$ or $p = 1$.

a) Let $1 < p < \infty$.
We first observe that the operator F_λ is compact. Indeed, since K_c and K_e are a regular operators, according to Lemma 6.3.1 and Proposition 6.3.1, we infer that the operators $(\lambda - T)^{-1}K_c$ and $(\lambda - T)^{-1}K_e$ are compact. Further, using (A10) together with Lemma 2.2.1 one sees that N_f and N_g are continuous and consequently F_λ is compact. Next, set $\theta_{\mathbf{r}} = \max(\theta_1, \theta_2)$, the operators F_λ and G_λ satisfy the conditions of the classical Krasnosel'skii fixed point theorem and therefore Problem (6.3.1)-(6.3.2) has at least a solution ψ in $\mathbb{B}_{\mathbf{r}}^p$ for all λ such that $\mathrm{Re}\lambda > \theta_{\mathbf{r}}$.

b) For $p = 1$.
The operator $\mathsf{G}_\lambda = (\lambda - T)^{-1}\mathcal{N}_{-\sigma} + (\lambda - T)^{-1}K_d$ is ww-compact. Indeed, since $(\lambda - T)^{-1}$ and $(\lambda - T)^{-1}K_d$ are linear oerator, according to Remark 2.1.4, they are ww-compact. Moreover, by Lemma 6.3.3, we know that $\mathcal{N}_{-\sigma}$ is ww-compact. It is not dificult to check that the composition and the sum of ww-compact operator are also ww-compact. So, we conclude that G_λ is ww-compact.

We claim that the operator F_λ is ws-compact. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in $L^1(\mathsf{D}\times V)$ and set $U = \{x_0, x_2, \cdots\}$. It is clear that U is relativelly compact in $L^1(\mathsf{D}\times V)$. Using Lemma 6.3.3 and the fact that the $(\lambda - T)^{-1}K_c$ and $(\lambda - T)^{-1}K_e$ are Dunford-Pettis operators we conclude that the sets $(\lambda - T)^{-1}K_c\mathcal{N}_f(U)$ and $(\lambda - T)^{-1}K_e\mathcal{N}_g(U)$ are relatively compacts and therefore the set $(\lambda - T)^{-1}K_c\mathcal{N}_f(U) + (\lambda - T)^{-1}K_e\mathcal{N}_g(U)$ is relatively compact. This proves that F_λ is ws-compact.

Next, let $\lambda \in \mathbb{C}_{\lambda_r}$ and consider the sequence $(M_n)_{\in\mathbb{N}}$ be the sequence of nonempty closed bounded convex subsets of $\mathbb{B}^1_{\mathbf{r}}$ defined by

$$M_0 = \mathbb{B}^1_{\mathbf{r}} \qquad \text{and} \qquad M_{n+1} = \overline{co}(\mathsf{F}_\lambda(M_n) + \mathsf{G}_\lambda(M_n)).$$

We note that for $\lambda \in \mathbb{C}_{\lambda_r}$, so using the convexity of $\mathbb{B}^1_{\mathbf{r}}$ and the inclusion $\mathsf{F}_\lambda(\mathbb{B}^1_{\mathbf{r}}) + \mathsf{G}_\lambda(\mathbb{B}^1_{\mathbf{r}}) \subset \mathbb{B}^1_{\mathbf{r}}$ we check easily by induction that $(M_n)_{n\in\mathbb{N}}$ is a decreasing sequence of $\mathcal{B}(L^1(\mathsf{D} \times V))$ (in the sens of the inclusion).

Next, using the properties of $\omega(\cdot)$, we get

$$\begin{aligned}
\omega(M_{n+1}) &= \omega\Big(\overline{co}(\mathsf{F}_\lambda(M_n) + \mathsf{G}_\lambda(M_n))\Big) = \omega(\mathsf{F}_\lambda(M_n) + \mathsf{G}_\lambda(M_n)) \\
&\leq \omega(\mathsf{F}_\lambda(M_n) + \omega(\mathsf{G}_\lambda(M_n)) \\
&\leq \frac{1}{\mathrm{Re}\lambda}(\eta \|K_c\| + \eta \|K_e\| + \|K_d\| + \zeta)\omega(M_n).
\end{aligned}$$

Let Θ_3 be a nonnegative real number such that

$$\gamma := \frac{1}{\Theta_3}(\eta \|K_c\| + \eta \|K_e\| + \|K_d\| + \zeta) < 1.$$

Hence, for all λ satisfying $\mathrm{Re}\lambda > \Theta_3$ and all subset M_n of $\mathbb{B}^1_{\mathbf{r}}$, we have

$$\omega(\mathsf{F}_\lambda(M_n) + \mathsf{G}_\lambda(M_n)) \leq \gamma\omega(M_n). \tag{6.3.9}$$

It follows from the estimate above that, for any λ satisfying $\mathrm{Re}(\lambda) \geq \Theta_3$, $\omega(M_{n+1}) \leq \gamma\omega(M_n)$ and therefore $\omega(M_n) \leq \gamma^n\omega(M_0)$. The fact that $\gamma \in (0, 1)$ implies $\lim_{n\to\infty} \omega(M_n) = 0$. Now invoking Definition 2.1.11 (e), we infer that $\mathcal{M} = \bigcap_{n=1}^{\infty} M_n$ is a nonempty weakly compact subset of $\mathbb{B}^1_{\mathbf{r}}$. Moreover, the inclusion $\mathsf{F}_\lambda(\mathcal{M}) + \mathsf{G}_\lambda(\mathcal{M}) \subseteq \mathcal{M}$ is obvious.

To conclude the proof, it suffices to consider the fixed point problem $\psi = (\mathsf{F}_\lambda)(\psi) + \mathsf{G}_\lambda(\psi)$ on the set $\mathcal{M}$.We first note that $\mathsf{F}_\lambda(\mathcal{M}) + \mathsf{G}_\lambda(\mathcal{M}) \subseteq \mathcal{M}$ (see above). Moreover, we established above that F_λ is ws-compact on $\mathbb{B}^1_{\mathbf{r}}$, so it is ws-compact on $\mathcal{M}$. Since $\mathsf{F}_\lambda(\mathcal{M}) + \mathsf{G}_\lambda(\mathcal{M}) \subseteq \mathcal{M}$, we have $\mathsf{F}_\lambda(\mathcal{M}) \subseteq \mathcal{M}$ and therefore $\mathsf{F}_\lambda(\mathcal{M})$ is relatively weakly compact. Further, we proved above that G_λ is ww-compact and α-contractive. Hence all the conditions of Theorem 2.1.4 are satisfied which ends the proof. □

Bibliography

[1] Abdallah MY, Latrach K, Zeghal A. *Some fixed point theorems in Banach space and aplication to a transport equation with delayed neutrons.* Mediterr. J. Math. 2021; 18(4): 23 pp.

[2] R. P. Agarwal, D. O'Regan, *A generalization of the Krasnoselskii-Petryshyn compression and expansion theorem: an essential map approach.* J. Korean Math. Soc. 38 (2001), 669-681.

[3] Agarwal, R.P., O'Regan, D., Taoudi, M.A.: Browder-Krasnoleskii-type fixed point theorems in Banach spaces. Fixed Point Theory Appl. 243716, 20 pp (2010)

[4] Al-Izeri, AM, Latrach, K.*On the asymptotic spectrum of a transport operator with elastic and inelastic collision operators.* Acta. Math. Sci. 2020; 40(B): 805-823.

[5] Al-Izeri, AM, Latrach, K.*On the solution of a class of linear transport equation with partly elastic collision operators*. Submitted Acta Mathematica Scientia.

[6] M. A. Al-Thagafi, N. Shahzad, *Krasnosel'skii-type fixed-set results*. Fixed Point Theory (2010) Article ID 394139, 9 pages.

[7] Anselone, P.M., Palmer, T.W.: Collectively compact sets of linear operators. Pacific J. Math. **25**, 417-422 (1968)

[8] Appell, J.: The superposition operator in function spaces – A survey. Exposition. Math. **6**, 209-270 (1988)

[9] Appel J, De Pascale E. *Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili.* Boll. Unione Mat. Ital. 1984; B (6): 497-515.

[10] Banaś, J., Rivero, J.: On measures of weak noncompactness. Ann. Mat. Pura Appl. **151**, 213-224 (1988)

[11] Barroso, C.S.: Krasnoslskii's fixed point theorem for weakly continuous maps. Nonlinear Anal. **55**, 25-31 (2003)

[12] I. Basoc, T. Cardinali, *A hybrid nonlinear alternative theorem and some hybrid fixed point theorems for multimaps*. J. Fixed Point Theory Appl. **17** (2015), 413-424.

[13] Beals, R., Protopopescu, V.: Abstract time-dependent transport equations. J. Math. Anal. Appl. **121**, 370-405 (1987)

[14] Ph. Bénilan, M. G. Crandall, A. Pazy, *Evolution equations governed by accretive operators.* Forthcoming.

[15] M. Biondini, T. Cardinali, *Existence of solutions for a nonlinear integral equation via a hybrid fixed point theorem*. Results Math. **71** (2017), no. 3-4, 1259-1276.

[16] Boumhamdi, M., Latrach, K., Zeghal, A.: Existence results for a nonlinear transport equation with unbounded admissible velocities space. Mediterr. J. Math. **13**, 3155-3171 (2016)

[17] Boyd, D.W. Wong, J.S.W.: On nonlinear contractions. Proc. Amer. Math. Soc. **20**, 458-464 (1969)

[18] T. A. Burton, *Integral equations, implicit functions, and fixed points.* Proc. Amer. Math. Soc. **124** (1996), no. 8, 2383-2390.

[19] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, New York, 2011.

[20] T. Cardinali, P. Rubbioni *Countably condensing multimaps and fixed points.* Electron. J. Qual. Theory Differ. Equ. 83 (2012), 1-9.

[21] T. Cardinali, D. O'Regan and P. Rubbioni, Monch sets and fixed point theorems for multimaps in locally convex topological vector spaces, Fixed Point Theory, 18 (2017), No. 1, 147-154

[22] Cessenat, M.: Théorèmes de traces L_p pour des espaces de fonctions de la neutronique. C. R. Acad. Sci. Paris sér. I Math. **299**, 831-834 (1984)

[23] Cessenat, M.: Théorèmes de traces pour des espaces de fonctions de la neutronique. C. R. Acad. Sci. Paris sér. I Math. **300**, 89-92 (1985)

[24] Chow, S.N., Hale, J.K.: Methods of bifurcation theory. Springer Verlag, Berlin, (1982)

[25] S. J. Daher, *On a fixed point principle of Sadovskii.* Nonlinear Anal. T. M. A. 2 (1978), 643-645.

[26] Dautray, R., Lions, J.L., Analyse mathématique et calcul numérique. Tome 9, Paris, Masson, (1988)

[27] De Blasi, F.S.: On a property of the unit sphere in a Banach space. Bull. Math. Soc. Sci. Math. R.S. Roumanie **21**, 259-262 (1977)

[28] B. C. Dhage, *A fixed point theorem for multi-valued mappings in ordered Banach spaces with applications I.* Nonlinear Analysis Forum 10(1), (2005), 105-126.

[29] B. C. Dhage, *Multivalued operators and fixed-point theorems in Banach algebras. II.* Comput. Math. Appl. **48** (2004), no. 10-11, 1461?1476.

[30] B. C. Dhage, *Mutli-valued mappings and fixed points II*. Tamkanc J. Math. **37**, (2006), 27-46.

[31] Dunford, N., Schwartz, J.T.: Linear operators, Part 1: general theory. Interscience, New York, (1958)

[32] Emmanuele, G.: Measure of weak noncompactness and fixed point theorems. Bull. Math. Soc. Sci. Math. R.S. Roumanie **25**, 353-358 (1981)

[33] Garcia-Falset, J.: Existence of fixed points for the sum of two operators. Math. Nachr. **283**, 1736-1757 (2010)

[34] J. Garcia-Falset, K. Latrach, *Krasnosel'skii-type fixed point theorems for weakly sequentially continuous mappings.* Bull. London Math. Soc. 44 (2012), 25-38.

[35] J. Garcia-Falset, K. Latrach, E. Moreno-Galvez, and M-A, Taoudi. *Schaefer-Krasnoselskii fixed point theorems using a usual measure of weak noncompactness.* J. Differ. Equ. 252 (2012), 3436-3452.

[36] J. R. Graef, J. Henderson and A. Ouahab, *Multivalued versions of a Krasnosel'skii-type fixed point theorem*. J. Fixed Point Theory Appl. DOI 10.1007/s11784-016-0328-6

[37] L. S. Goldenstein, I. T. Gohberg, A. S. Markus, *Investigations of somme properties of bounded linear operators with their q-norms.* Ucen. Zap. Kishinovsk. 29 (1957), 29-36.

[38] Golse F, Lions PL, Perthame B, Sentis R. *Regularity of the moment of the solution of a neutron transport equation.* J. Funct. Anal. 1988; 76: 110-125.

[39] Greenberg, W., Van der Mee, C., Protopopescu, V.: Boundary value problems in abstract kinetic theory. Birkhauser, Basel, (1987)

[40] C. J. Himmelberg, *Fixed points of compact multifunctions*. J. Math. Anal. Appl. **38** (1972), 205-207.

[41] C. J. Himmelberg, J. R. Porter, J. R. Van Vleck, *Fixed point theorems for condensing multifunction.* Proc. Amer. Math. Soc. 23 (1969), 635-641.

[42] In-Sook Kim, *Fixed points of countably condensing mappings and its application to nonlinear eigenvalue problems*. J. Korean Math. Soc. 43 (2003), 1-9.

[43] Kaper, H.G., Lekkerkerker, C.G., Hejtmanek, J.: Spectral methods in linear transport theory. Birkhauser Verlag, Basel, (1982)

[44] Krasnoselskii M.A., et al.: Integral operators in spaces of summable functions. Noordhoff, Leyden, (1976)

[45] K. Kuratowski, *Sur les espaces complete.* Fund. Math. 15 (1930), 301-209.

[46] K. Kuratowski, *Topology.*, 1-2, Acad. Press (1966-1968).

[47] Larsen EW, Zweifel PF. *On the spectrum of the linear transport operatior.* J. Mathematical Phys. 1974; 15: 1987-1997.

[48] K. Latrach, *Introduction à la théorie des points fixes métrique et topologique: avec applications et exercices corrigés,* Ellipses Édition Marketing S. A., 2017.

[49] Latrach, K.: Compactness results for transport equations and applications. Math. Models Methods Appl. Sci. **11**, 1181-1202 (2001)

[50] Latrach, K., Oummi, H., Zeghal, A.: Existence results for a class of nonlinear singular transport equations in bounded spatial domains. Math. Methods Appl. Sci. **43**, 1685-1700 (2020)

[51] Latrach, K., Taoudi, M.A., Zeghal, A.: Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations. J. Differential Equations **221**, 256-271 (2006)

[52] Latrach, K., Zeghal, A.: Existence results for a nonlinear transport equation in bounded geometry on L^1-spaces. Appl. Math. Comput. **219**, 1163-1172. (2012)

[53] Y. Liu, Z. Li, *Schaefer type theorem and periodic solutions of evolution equations*. J. Math. Anal. Appl. **316** (2006), 237-255.

[54] Lods, B.: On linear kinetic equations involving unbounded cross-sections: Math. Methods Appl. Sci. **27**, 1049-1075 (2004)

[55] E.R. Megginson, An introduction to Banach Space Theory, Graduate Texts in Mathematics, **183**. Springer-Verlag, New York, 1998.

[56] Marti, J.T.: Mathematical foundations of kinetics in neutron transport theory. Nucleonik. **8**, 159-163 (1966)

[57] A. R. Michell, C. K. L. Smith, *An existence theorem for weak solutions of differential equation in Banach spaces.* In: Lakshmikantham, V (ed.) Nonlinear Equation in Abstract Spaces, pp. 387–404. Academic Press, San Diego (1978)

[58] Mokhtar-Kharroubi, M.: Mathematical topics in neutron transport theory. New aspects. Series on advances in Mathematics for Applied Sciences, Vol. 46, World (1997)

[59] Mokhtar-Kharroubi, M.: Optimal spectral theory of the linear Boltzmann equation. J. Funct. Anal. **226**, 21-247 (2005)

[60] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces.* Nonlinear Anal. 4 (1980), 985-999.

[61] M. Z. Nashed, J. S. W. Wong, *Some variants of a fixed point theorem of Krasnosel'skii and applications to nonlinear integral equations*. J. Math. Mech. **18** (1969), 767-777.

[62] D. O'Regan, *Fixed point theory of Mönch type for weakly sequentially upper semicontinuous maps.* Bull. Austral. Math. Soc. **61** (3) (2000), 439-449.

[63] E. A. Ok, *Fixed set theorems of Krasnoselskii type.* Proc. Amer. Math. Soc. **137** (2009), 511-518.

[64] S. Reich, *Fixed points of condensing functions.* J. Math. Anal. Appl. 41 (1973), 460-467.

[65] B.N. Sadovski. *A fixed point principle*, Functional Anal. Appl. 1 (1967), 151-153.

[66] Sam B. Nadler, Jr. *Multi-valued contraction mappings.* Pacific J. Math. Vol. **30**, N0. 2, (1969),

[67] Sbihi M. Spectral theory of neutron transport semigroups with partly elastic collision operators. J. Math. Phys. 2006; 47: 123502 (12 pages).

[68] Sbihi M. Analyse Spectrale De Modèles Neutroniques". Thèse de Doctorat de l'université de Franche-Comté: Besançon"; 2005.

[69] P. Somyot, T. Tipphawan, *Fixed point theorems of Krasnosel'skii type for the sum of two multivalued mappings in Banach spaces*. J. Nonlinear Convex Anal. **14** (2013), 183-191.

[70] Takác, P.: A spectral mapping theorem for the exponential function in linear transport theory. Transport Theory Statist. Phys. **14**, 653-667 (1985)

[71] T. Xiang, R. Yuan, *A class of expansive-type Krasnosel'skii fixed point theorems.* Nonlinear Analysis 71 (2009), 3229-3239.

Quelques théorèmes de points fixes pour des applications univoques et multivoques et applications

Résumé On présente dans cette thèse quelques théorèmes de points fixes pour des applications univoques et multivoques dans les espaces de Banach. Dans la première, des théorèmes de points fixes de type Altman, Sadovskii, Leray-Schauder et Krasnosel'skii ont été établis pour les applications dénombrablement condensantes. Ensuite des résultats de points fixes faisant intervenir deux ou trois applications ont été présentés. Ces résultats reposent sur les concepts d'applications ws-compactes et ww-compactes. Pour les applications multivoques on a établi de nombreux résultats de points fixes de type Krasnosel'skii en considérant divers types de perturbations (applications univoques et mutivoques). Dans la dernière partie, on a présenté des applications de ces résultats aux équations non linéaires intégrales de Volterra et à deux modèles d'équations de transport neutronique stationnaires.

Mots-clés: Théorèmes de point fixe, mesure de non-compacité et de non faible compacité, applications dénombrablement condensantes, applications multivoques semi-continues supérieurement, opérateurs de Dunford-Pettis, équations intégrales non linéaires de Volterra, équations stationnaires de transport neutronique.

Some fixed point theorems for single and multivalued mapping and applications

Abstract In this thesis, we present some fixed point theorems for single and multivalued mappings in Banach spaces. In the first part, fixed point theorems of the Altman, Sadovskii, Leray-Schauder and Krasnosel'skii type have been established for countably condensing maps. Further, some fixed point results involving two or three maps were presented. These results are based on the concepts of ws-compact and ww-compact maps. For multivalued mapping, many fixed point results of Krasnosel'skii's type have been established for various kind of perturbations (singlevalued and multivalued maps). In the last part of this work, we present applications of our results to solve nonlinear integral Volterra equations and two models of stationary neutron transport equations.

Keywords: Fixed point theorem, measure of noncompactness and weak noncompactness, condensing mappings, countably condensing mappings, upper semicontinuous multivalued mappings, Dunford-Pettis operators, Nonlinear integral equations of Volterra, stationary neutron transport equations.

Index