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## UNIVERSITÉ D'ORLÉANS

### ÉCOLE DOCTORALE MATHÉMATIQUES, INFORMATIQUE, PHYSIQUE THÉORIQUE ET INGÉNIERIE DES SYSTÈMES

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**Alexis KAGAN**

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Discipline/ Spécialité : **Mathématiques**

<h3>Traces de marches aléatoires en milieux aléatoires sur des arbres</h3>
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**THÈSE** dirigée par :

**M. Pierre ANDREOLETTI**

Maître de conférences, Université d'Orléans

**RAPPORTEURS :**

**M. Élie AÏDÉKON**

Professeur, Fudan University

**M. Yueyun HU**

Professeur, Université Sorbonne Paris Nord

---

**JURY :**

**M. Romain ABRAHAM**

Professeur, Université d'Orléans, Président du jury

**M. Élie AÏDÉKON**

Professeur, Fudan University

**M. Pierre ANDREOLETTI**

Maître de conférences, Université d'Orléans

**M. Yueyun HU**

Professeur, Université Sorbonne Paris Nord

**M. Shen LIN**

Maître de conférences, Sorbonne Université

**M. Pascal MAILLARD**

Professeur, Université Toulouse III - Paul Sabatier

# Traces de marches aléatoires en milieux aléatoires sur des arbres

Alexis Kagan

Institut Denis Poisson, Université d'Orléans

Sous la direction de Pierre Androletti

## Résumé

Cette thèse est consacrée à l'étude d'une marche aléatoire au plus proche voisin  $\mathbb{X}$  récurrente nulle sur une marche aléatoire branchante  $(\mathbb{T}, (V(x); x \in \mathbb{T}))$  où  $\mathbb{T}$  est un arbre de Bienaymé-Galton-Watson sur-critique. Une attention particulière est accordée à sa trace, le sous-arbre  $\mathcal{R}_n = \{X_0, \dots, X_n\}$  de  $\mathbb{T}$  des sommets visités par  $\mathbb{X}$  jusqu'à l'instant  $n$ .

Dans une première partie, nous nous intéressons au volume de certaines traces contraintes à la fois le long des trajectoires de  $\mathbb{X}$  et le long de celles du potentiel branchant  $V$  dans le régime lent [HS16b]. Il s'agit typiquement du nombre de sommets visités au moins  $n^b$  (avec  $b \in [0, 1)$ ) fois par la marche  $\mathbb{X}$  jusqu'à l'instant  $n$  et dont le potentiel  $V$  est contraint d'évoluer dans un ensemble donné. Sous quelques hypothèses, certaines naturelles, d'autres plus techniques, nous donnons le comportement quand  $n$  tend vers l'infini du volume de ces traces générales, mettant en lumière les interactions entre la marche aléatoire  $\mathbb{X}$  et son potentiel branchant  $V$ . Nous exhibons également quelques exemples de traces contraintes nous paraissant pertinents. Un exemple important est celui des potentiels hauts : il s'agit de forcer la marche  $\mathbb{X}$  à visiter des sommets de potentiel typiquement supérieur à  $(\log n)^\alpha$  avec  $\alpha \in [1, 2)$ . Ces sommets sont nombreux à l'instant  $n$  (lorsque  $n$  tend vers l'infini), de l'ordre de  $n^{1-b}e^{-c(\log n)^{\alpha-1}}$  avec  $c > 0$ .

Dans une deuxième partie, nous nous focalisons sur le volume de certains sous-ensembles de  $\mathcal{R}_n^{\times k} = \mathcal{R}_n \times \dots \times \mathcal{R}_n$ ,  $k \geq 2$ , dans le cas de la marche diffusive [HS07b]. Nous supposons que le diamètre  $\mathbf{L}_n - 1$  (pour la topologie naturelle sur  $\mathbb{T}^{\times k}$ ) d'un tel sous-ensemble est déterministe et nous montrons que sous une hypothèse d'hérédité, le volume de ce dernier est proportionnel à  $(n^{1/2}\mathbf{L}_n)^k$  lorsque  $n$  tend vers l'infini. Nous étudions également cette « constante » de proportionnalité, qui est une variable aléatoire non-déterministe, mettant ainsi en lumière les liens entre les différents sommets de  $\mathcal{R}_n$ . Enfin, nous appliquons ces résultats au fameux problème de généalogie suivant : tirons  $k \geq 2$  sommets uniformément et sans remise dans un sous-ensemble de l'arbre  $\mathcal{R}_n$ . A quoi ressemble leur arbre généalogique ? Sous certaines hypothèses portant sur ce sous-ensemble, nous répondons à cette question assez précisément.

## Summary

This thesis is devoted to the study of a nearest-neighbour random walk  $\mathbb{X}$ , null recurrent, on a branching random walk  $(\mathbb{T}, (V(x); x \in \mathbb{T}))$ , where  $\mathbb{T}$  is a super-critical Bienaymé-Galton-Watson tree. A particular attention is paid to its range, that is to say the sub-tree  $\mathcal{R}_n = \{X_0, \dots, X_n\}$  of vertices of  $\mathbb{T}$  visited by the random walk  $\mathbb{X}$  up to the time  $n$ .

In a first part, we are interested in the volume of some ranges with both constraints on the trajectories of the random walk  $\mathbb{X}$  and on those of the underlying branching potential  $V$  in the slow regime [HS16b]. This is typically the number of vertices of  $\mathbb{T}$  visited more than  $n^b$  times (with  $b \in (0, 1)$ ) by the walk  $\mathbb{X}$  up to the time  $n$  with potential forced to stay in a given set. Under a few hypothesis, some are natural, others more technical, we give the asymptotic behavior of this generalized range and by studying it, we do wish to highlight interactions between the random walk  $\mathbb{X}$  and its random branching potential  $V$ . We also present some examples of such ranges we do believe to be relevant. An important example is to enforce the vertices visited by the random walk  $\mathbb{X}$  to have a potential typically larger than  $(\log n)^\alpha$  with  $\alpha \in [1, 2)$ . These vertices are numerous at time  $n$  (when  $n$  goes to infinity), of order  $n^{1-b} e^{-c(\log n)^{\alpha-1}}$  with  $c > 0$ . In a second part, we focus on the volume of some subsets of  $\mathcal{R}_n^{\times k} = \mathcal{R}_n \times \dots \times \mathcal{R}_n$ ,  $k \geq 2$ , in the diffusive regime [HS07b]. The diameter  $\mathbf{L}_n - 1$  (for the usual topology on  $\mathbb{T}^{\times k}$ ) of such subset is assumed to be deterministic and we show, under an hereditary assumption, that the volume of the latter is proportional to  $(n^{1/2} \mathbf{L}_n)^k$  when  $n$  goes to infinity. We also study this «proportionality constant», which is actually a non-deterministic random variable, allowing to highlight the links between the vertices visited by the random walk  $\mathbb{X}$ . Finally, we apply these results to the following well known genealogy problem : pick  $k \geq 2$  vertices uniformly and without replacement in a subset of  $\mathcal{R}_n$ . What does their genealogical tree look like ? Under some hypothesis on this subset, we do answer this question quite precisely.



# Table des matières

<b>Introduction</b>	<b>7</b>
1 Arbres marqués . . . . .	8
1.1 Définitions et notations . . . . .	8
1.2 Arbres de Bienaymé-Galton-Watson marqués . . . . .	9
2 Marches aléatoires en milieux aléatoires . . . . .	10
2.1 La marche lente . . . . .	11
2.2 La marche (sous-)diffusive . . . . .	13
2.3 La marche aléatoire et son potentiel branchant . . . . .	14
3 Trace de marches aléatoires en milieux aléatoires . . . . .	17
3.1 Définitions et exemples . . . . .	17
3.2 La trace usuelle . . . . .	19
3.3 La trace lourde . . . . .	22
3.4 Traces sur potentiels hauts et généralisations . . . . .	23
4 Généalogie de la marche aléatoire diffusive . . . . .	27
4.1 Généalogie des arbres de Galton-Watson . . . . .	27
4.2 Généalogie de la marche diffusive : les petites générations . . . . .	28
4.3 Généalogie de la marche diffusive : les générations critiques . . . . .	32
5 Annexe . . . . .	33
5.1 Preuve de (9) . . . . .	34
5.2 Preuve de (10) . . . . .	35
<b>I Generalized range in the slow regime</b>	<b>39</b>
1 Introduction . . . . .	40
1.1 First results : examples . . . . .	42
1.2 A general result (informal statement) . . . . .	45
1.3 A general result (full statement) . . . . .	50
2 Proof of the theorems . . . . .	54
2.1 Preliminary material . . . . .	54
2.2 Proofs of Theorems 1.1 to 1.4 . . . . .	55
2.3 Proof of Theorem 1.5 . . . . .	74

3	Proof of Proposition 1 . . . . .	77
3.1	Preliminary . . . . .	77
3.2	Lower bound for $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$ . . . . .	79
3.3	Upper bound for $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$ . . . . .	89
4	Technical estimates for one-dimensional random walk . . . . .	94
4.1	Two Laplace transforms . . . . .	94
4.2	Additional technical estimates . . . . .	100
<b>II</b>	<b>Genealogy in the diffusive case</b>	<b>109</b>
1	Introduction . . . . .	110
1.1	Randomly biased random walk on trees . . . . .	110
1.2	Genealogy of uniformly chosen vertices in the range . . . . .	112
1.3	The tiny generations . . . . .	113
1.4	The small generations : examples . . . . .	114
1.5	The small generations : a general result . . . . .	121
1.6	Further discussion : the critical generations . . . . .	123
2	Proofs of the theorems . . . . .	124
2.1	Proofs of Theorems 1.2 to 1.5 . . . . .	125
2.2	Proof of Theorem 1.6 and Theorem 1.7 . . . . .	131
3	Proofs of Propositions 2 and 3 . . . . .	135
3.1	The range on $\mathfrak{E}^{k,\cdot}$ . . . . .	138
3.2	The range on $\Delta^k \setminus \mathfrak{E}^{k,\cdot}$ . . . . .	154
	<b>Bibliographie</b>	<b>163</b>



# Introduction

## Contents

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<b>1</b>	<b>Arbres marqués . . . . .</b>	<b>8</b>
1.1	Définitions et notations . . . . .	8
1.2	Arbres de Bienaymé-Galton-Watson marqués . . . . .	9
<b>2</b>	<b>Marches aléatoires en milieux aléatoires . . . . .</b>	<b>10</b>
2.1	La marche lente . . . . .	11
2.2	La marche (sous-)diffusive . . . . .	13
2.3	La marche aléatoire et son potentiel branchant . . . . .	14
<b>3</b>	<b>Trace de marches aléatoires en milieux aléatoires . . . . .</b>	<b>17</b>
3.1	Définitions et exemples . . . . .	17
3.2	La trace usuelle . . . . .	19
3.3	La trace lourde . . . . .	22
3.4	Traces sur potentiels hauts et généralisations . . . . .	23
<b>4</b>	<b>Généalogie de la marche aléatoire diffusive . . . . .</b>	<b>27</b>
4.1	Généalogie des arbres de Galton-Watson . . . . .	27
4.2	Généalogie de la marche diffusive : les petites générations . . . . .	28
4.3	Généalogie de la marche diffusive : les générations critiques . . . . .	32
<b>5</b>	<b>Annexe . . . . .</b>	<b>33</b>
5.1	Preuve de (9) . . . . .	34
5.2	Preuve de (10) . . . . .	35

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## 1 Arbres marqués

Dans cette section, nous présentons en premier lieu la notion d'arbre planaire enraciné et marqué selon J. Neveu [Nev86] puis nous donnons une construction possible d'un arbre de Bienaymé-Galton-Watson marqué.

### 1.1 Définitions et notations

Commençons par quelques définitions. Considérons l'ensemble des mots  $\mathcal{U} = \bigcup_{\ell \in \mathbb{N}} (\mathbb{N}^*)^\ell$ , c'est-à-dire l'ensemble des suites de longueur finie d'entiers strictement positifs et on note  $|u|$  la longueur d'un mot  $u \in \mathcal{U}$ . On désigne par  $e$  l'unique mot de longueur 0, appelé racine. Si  $u = (i_1, \dots, i_\ell) \in \mathcal{U}$  est un mot de longueur  $\ell$  alors l'ensemble  $\{e\} \cup \{(i_1, \dots, i_m); 1 \leq m \leq \ell\}$  des mots de longueur inférieure à celle de  $u$  et ayant le même suffixe que ce dernier constitue la collection des ancêtres de  $u$ . Le mot  $(i_1, \dots, i_{\ell-1})$  est appelé parent de  $u$  et on le note  $u^*$ . La notion d'ancêtre induit une relation d'ordre partielle  $\leq$  sur  $\mathcal{U}$  : pour tout mot  $u$  et  $v$ ,  $u \leq v$  si et seulement si  $u$  est un ancêtre de  $v$ . Dans ce cas, on dira également que  $v$  est un descendant de  $u$ . Pour tout  $j \in \mathbb{N}^*$ , le descendant  $(i_1, \dots, i_\ell, j)$  du mot  $u = (i_1, \dots, i_\ell)$  est le  $j$ -ème enfant de  $u$ , noté  $u^j$ . On notera  $u < v$  si et seulement si  $u \leq v$  et  $u \neq v$ . On peut également définir sur  $\mathcal{U}$  l'ordre lexicographique  $\preceq$  : pour tout mot  $u = (i_1, \dots, i_\ell)$  et  $v = (j_1, \dots, j_{\ell'})$ , on a  $u \preceq v$  si et seulement si  $u = v$  ou si  $u \neq v$  et il existe  $m \in \{1, \dots, \min(\ell, \ell')\}$  tel que pour tout  $p \in \{1, \dots, m-1\}$ ,  $i_p = j_p$  et  $i_m < j_m$ . On écrira  $u \prec v$  si et seulement si  $u \preceq v$  et  $u \neq v$ . Notons que  $u \leq v$  implique  $u \preceq v$  mais qu'en général, le réciproque est fausse.

Définissons maintenant la notion d'arbre planaire selon J. Neveu. Un arbre  $\mathbf{T}$  est sous-ensemble de  $\mathcal{U}$  vérifiant les propriétés suivantes : *i*)  $e \in \mathbf{T}$ ; *ii*) si  $(i_1, \dots, i_\ell) \in \mathbf{T}$  alors  $(i_1, \dots, i_m) \in \mathbf{T}$  pour tout  $m \in \{1, \dots, \ell\}$ ; *iii*) si  $u = (i_1, \dots, i_\ell) \in \mathbf{T}$  alors il existe  $N_u \in \mathbb{N}$  tel que  $(i_1, \dots, i_\ell, j) \in \mathbf{T}$  si et seulement si  $j \in \{1, \dots, N_u\}$ .

Un mot  $u$  appartenant à l'arbre  $\mathbf{T}$  est appelé sommet. *i*) nous dit que la racine  $e$  est un sommet de  $\mathbf{T}$  et par convention, c'est un ancêtre de tout sommet de l'arbre  $\mathbf{T}$ . *ii*) affirme que tout ancêtre d'un sommet  $u \in \mathbf{T}$  est nécessairement un sommet de  $\mathbf{T}$ . *iii*) nous dit que qu'un certain nombre  $N_u \in \mathbb{N}$  des enfants de  $u \in \mathbf{T}$  sont des sommets de  $\mathbf{T}$  et lorsque  $N_u \in \mathbb{N}^*$ , le mot  $u^j$  est un sommet de  $\mathbf{T}$  pour tout  $j \in \{1, \dots, N_u\}$ .

L'ensemble  $\{u \in \mathbf{T}; |u| = \ell\}$  constitue la génération  $\ell$  de l'arbre  $\mathbf{T}$  et pour tout sommet  $u \in \mathbf{T}$  tel que  $|u| = \ell$ ,  $u_i$  désigne l'unique ancêtre de  $u$  dans la génération  $i \in \{1, \dots, \ell\}$ . L'ensemble  $\llbracket e, u \rrbracket := \{v \in \mathbf{T}; v \leq u\} = \{e, u_1, \dots, u_\ell\}$  est alors appelé ligne généalogique du sommet  $u$ . Pour tout sommet  $u$  et  $v$  dans l'arbre  $\mathbf{T}$ , l'ancêtre commun le plus récent de  $u$  et  $v$ , noté  $u \wedge v$ , est l'unique sommet  $z \in \llbracket e, u \rrbracket \cap \llbracket e, v \rrbracket$  tel que  $\max_{w \in \llbracket e, u \rrbracket \cap \llbracket e, v \rrbracket} |w| = |z|$ .

Nous pouvons attribuer une marque à chaque sommet d'un arbre. Soit  $\mathbf{T}$  un arbre et  $(\mathbf{a}_u; u \in \mathbf{T})$  une collection d'éléments, qu'on appelle des marques, d'un espace mesurable  $\mathbf{M}$ . Pour tout sommet  $u \in \mathbf{T}$ , l'élément  $\mathbf{a}_u$  est une marque attribuée au sommet  $u$  et le couple  $(\mathbf{T}, (\mathbf{a}_u; u \in \mathbf{T}))$  définit ainsi un arbre marqué.

## 1.2 Arbres de Bienaymé-Galton-Watson marqués

Nous présentons maintenant une classe d'arbres aléatoires très connue, la classe des arbres de Bienaymé-Galton-Watson marqués. Soit  $(N, M)$  une variable aléatoire à valeurs dans  $\mathbb{N} \times \mathbf{M}$ , sous une probabilité  $\mathbf{P}$ . On construit l'arbre aléatoire marqué  $(\mathfrak{T}, (\mathbf{a}_u; u \in \mathfrak{T}))$ , génération par génération, de la façon suivante :

- la génération 0 est constituée d'un unique sommet, la racine  $e$  et on pose  $\mathbf{a}_e := 0$ ;
- Soit  $n \in \mathbb{N}^*$  et supposons que la génération  $n - 1$  est construite. Si cette dernière est vide alors la génération  $n$  l'est aussi. Sinon, chaque sommet  $u$  dans la génération  $n - 1$  donne naissance, indépendamment des autres sommets de sa génération, à  $N_u$  enfants marqués  $(u^1, \mathbf{a}_{u^1}), \dots, (u^{N_u}, \mathbf{a}_{u^{N_u}})$ , suivant la loi de  $(N, M)$  formant ainsi la génération  $n$ .

Cette procédure s'arrête uniquement en cas d'extinction, c'est-à-dire lorsqu'une génération est vide. Autrement, elle continue indéfiniment.

La loi de  $N$  est appelée loi de reproduction de l'arbre aléatoire  $\mathfrak{T}$  et est naturellement liée à probabilité d'extinction de ce dernier. On suppose que  $\mathbf{P}(N = 1) < 1$ . Soit  $Z_n := \sum_{x \in \mathbb{T}; |x|=n} 1$  le nombre de sommets dans la génération  $n$  avec la convention  $\sum_{x \in \emptyset} = 0$ . On définit la probabilité d'extinction  $p_{\text{ext}}$  de l'arbre  $\mathfrak{T}$  par  $p_{\text{ext}} = \mathbf{P}(\exists n \in \mathbb{N} : Z_n = 0)$ .  $p_{\text{ext}}$  est en réalité le plus petit point fixe de l'application  $s \in [0, 1] \mapsto \mathbf{E}[s^N]$  et  $p_{\text{ext}} = 1$  si et seulement si  $\mathbf{E}[N] \leq 1$ . L'arbre  $\mathfrak{T}$  est dit sous-critique si  $\mathbf{E}[N] < 1$ , critique si  $\mathbf{E}[N] = 1$  et sur-critique si  $\mathbf{E}[N] > 1$ .

A partir de maintenant,  $(\mathbb{T}, (A_x; x \in \mathbb{T}))$  désigne un arbre de Bienaymé-Galton-Watson marqué sur-critique tel que  $A_x \in \mathbb{R}$  (c'est-à-dire en prenant  $\mathbf{M} = \mathbb{R}$  dans la construction précédente) et on note  $\mathbf{P}^*(\cdot) := \mathbf{P}(\cdot | \text{non-extinction de } \mathbb{T})$ .

Revenons quelques instants à la suite  $(Z_n)$ . Sous  $\mathbf{P}$ , la suite  $(Z_n / \mathbf{E}[N]^n)$  est une martingale positive, elle converge donc  $\mathbf{P}$ -presque sûrement vers une variable aléatoire  $M_\infty \in [0, \infty)$ . D'après le théorème de Kesten-Stigum [KS66],  $\mathbf{P}(M_\infty = 0) = p_{\text{ext}} \iff \mathbf{E}[M_\infty] = 1 \iff \mathbf{E}[N \log^+ N] < \infty$  où  $\log^+$  est la partie positive de  $\log$ . Si l'une des conditions précédentes est vérifiée alors  $\mathbf{P}^*(M_\infty > 0) = 1$  et donc  $\mathbf{P}^*$ -presque sûrement,  $(Z_n)$  a une croissance exponentielle. Lorsque ce n'est pas le cas, E. Seneta [Sen68] et C. C. Heyde [Hey70] ont prouvé qu'il existe toujours une suite  $(c_n)$  de réels strictement positifs et une variables aléatoire  $M'_\infty$  telles que  $c_{n+1}/c_n \rightarrow \mathbf{E}[N]$  et la suite  $(Z_n/c_n)$  converge  $\mathbf{P}$ -presque sûrement vers  $M'_\infty$ . De plus,  $\mathbf{P}(M'_\infty = 0) = p_{\text{ext}}$ .

L'arbre marqué aléatoire sur-critique avec marques réelles  $\mathcal{E} = (\mathbb{T}, (A_x; x \in \mathbb{T}))$  défini ci-dessus est appelé environnement aléatoire et c'est sur une réalisation de  $\mathbb{T}$  que nous définissons une marche aléatoire au plus proche voisin dont les probabilités de transition dépendent des marques de ce dernier.

## 2 Marches aléatoires en milieux aléatoires

Dans cette section, nous définissons une marche aléatoire biaisée  $\mathbb{X}$  sur une réalisation de l'environnement  $\mathcal{E}$  puis nous présentons quelques résultats importants pour cette marche, notamment un critère de récurrence/transience ainsi que des propriétés de localisation.

Pour des raisons pratiques, nous ajoutons un sommet  $e^*$  à  $\mathbb{T}$ . Il s'agit du parent de la racine  $e$ . Nous considérons ainsi, sous la famille de probabilités  $\{\mathbb{P}_x^\mathcal{E}; x \in \mathbb{T} \cup \{e^*\}\}$ , la marche aléatoire au plus proche voisin  $\mathbb{X} := (X_j)_{j \in \mathbb{N}}$ , à valeurs dans  $\mathbb{T} \cup \{e^*\}$  et réfléchi en  $e^*$ , dont les probabilités de transition sont, pour tout  $x \in \mathbb{T} \setminus \{e^*\}$

$$p^\mathcal{E}(x, x^*) = \frac{1}{1 + \sum_{i=1}^{N_x} e^{-A_{x^i}}} \quad \text{et pour tout } 1 \leq i \leq N_x, \quad p^\mathcal{E}(x, x^i) = \frac{e^{-A_{x^i}}}{1 + \sum_{i=1}^{N_x} e^{-A_{x^i}}}.$$

Autrement,  $p^\mathcal{E}(x, u) = 0$  et par définition,  $p^\mathcal{E}(e^*, e) = 1$ . Pour tout  $x \in \mathbb{T} \cup \{e^*\}$ , la mesure de probabilité  $\mathbb{P}_x^\mathcal{E}$  est appelée probabilité quenched issu de  $x$ , c'est-à-dire  $\mathbb{P}_x^\mathcal{E}(X_0 = x) = 1$  et  $\mathbb{P}^\mathcal{E} := \mathbb{P}_e^\mathcal{E}$  désigne la probabilité quenched issue de la racine  $e$ . Enfin, nous définissons les probabilités annealed  $\mathbb{P}$  et  $\mathbb{P}^*$  par

$$\mathbb{P}(\cdot) := \mathbf{E}[\mathbb{P}^\mathcal{E}(\cdot)] \quad \text{et} \quad \mathbb{P}^*(\cdot) := \mathbf{E}^*[\mathbb{P}^\mathcal{E}(\cdot)].$$

La marche aléatoire  $\mathbb{X}$  est appelée marche aléatoirement biaisée sur l'arbre de Bienaymé-Galton-Watson marqué sur-critique  $\mathcal{E}$ . La famille de marques aléatoires  $\{A_x; x \in \mathbb{T}\}$  représente le biais aléatoire vers la racine  $e$  (ou son parent  $e^*$ ) et comme nous le verrons par la suite, ce biais joue un rôle central dans l'étude de la transience et de la récurrence de la marche aléatoire  $\mathbb{X}$ . Les premières références à ces marches remontent à R. Lyons et R. Pemantle [LP92] et M.V. Menshikov et D. Petritis [MP02]. Un cas particulier important de cette marche aléatoire, introduit pour la première fois par R. Lyons (voir [Lyo90] et [Lyo92]) est celui de la marche  $\lambda$ -biaisée sur  $\mathbb{T}$  : il s'agit du cas où les marques  $A_z$  sont déterministes et constantes égales à  $\log \lambda$  avec  $\lambda > 0$ . Les probabilités de transition deviennent alors :  $p^\mathcal{E}(x, x^*) = \lambda/(\lambda + N_x)$  et  $p^\mathcal{E}(x, x^i) = 1/(\lambda + N_x)$  pour tout  $1 \leq i \leq N_x$ . Cette marche est transiente sauf quand le biais vers la racine est assez fort : si  $\lambda \geq \mathbf{E}[N]$  alors  $\mathbf{P}^*$ -presque sûrement, la marche  $\lambda$ -biaisée est récurrente (récurrente positive si  $\lambda > \mathbf{E}[N]$ ). Quand  $\lambda = \mathbf{E}[N]$ , Y. Peres et O. Zeitouni [PZ06] puis E. Aïdékon et L. de Raphélis [AdR17] ont montré le théorème central limite suivant :  $\mathbf{P}^*$ -presque sûrement, le processus  $(|X_{[nt]}|/\sqrt{\sigma^2 n})_{t \geq 0}$  converge en loi vers le mouvement brownien réfléchi standard, avec  $\sigma^2 = \lambda(\lambda - 1)/\mathbf{E}[N(N - 1)] \in (0, \infty)$ . Le cas transient a notamment été étudié par R. Lyons, R. Pemantle et Y. Peres (voir [LPP96b] et [LPP96a]) puis par E. Aïdékon [Aid14]. En particulier, il est montré dans [LPP96a] que la vitesse de fuite de la marche existe et est déterministe :  $\mathbf{P}^*$ -presque sûrement,  $\lim_{n \rightarrow \infty} |X_n|/n := v_\lambda \geq 0$ . De plus, la vitesse  $v_\lambda$  est strictement positive si et seulement si  $\lambda \in (\lambda_c, \mathbf{E}[N])$ , avec  $\lambda_c > 0$ . E. Aïdékon [Aid14] donna plus tard une expression explicite de cette vitesse  $v_\lambda$ .

Revenons au cas des marches aléatoires non déterministes, posons

$$V(e) = 0 \quad \text{et pour tout } x \in \mathbb{T} \setminus \{e\}, \quad V(x) := \sum_{e < z \leq x} A_z.$$

Sous  $\mathbf{P}$ ,  $(\mathbb{T}, (V(x); x \in \mathbb{T}))$  est une marche aléatoire branchante,  $V(x)$  est appelé potentiel du sommet  $x$  et correspond à la position de ce dernier par rapport à la racine  $e$ . Définissons la log-transformée de Laplace  $\psi$  associée à ce potentiel aléatoire

$$\psi(t) := \log \mathbf{E} \left[ \sum_{|x|=1} e^{-tV(x)} \right], \quad (1)$$

en supposant que  $\psi$  est bien définie dans un voisinage de  $[0, 1]$  et que  $\psi'(1)$  existe.

Voici un critère de transience/réurrence pour la marche aléatoire  $\mathbb{X}$  en fonction des fluctuations de  $\psi$  dû à G. faraud [Far11] (lui même précédé par R. Lyons et R. Pemantle [LP92]) : si  $\inf_{t \in [0,1]} \psi(t) > 0$ , alors  $\mathbf{P}^*$ -presque sûrement,  $\mathbb{X}$  est transiente, sinon, elle est récurrente. E. Aïdékon [Aïd08] a porté une attention particulière au cas transient, montrant en particulier que sous certaines hypothèses, la marche  $\mathbb{X}$  a une vitesse de fuite strictement positive. Dans le cas récurrent, le critère de G. faraud peut-être détaillé. Supposons que  $\inf_{t \in [0,1]} \psi(t) \leq 0$ .

- Si  $\inf_{t \in [0,1]} \psi(t) < 0$  ou  $\inf_{t \in [0,1]} \psi(t) = 0$  et  $\psi'(1) > 0$  alors  $\mathbf{P}^*$ -presque sûrement,  $\mathbb{X}$  est récurrente positive ;
- si  $\inf_{t \in [0,1]} \psi(t) = 0$  et  $\psi'(1) \leq 0$  alors  $\mathbf{P}^*$ -presque sûrement,  $\mathbb{X}$  est récurrente nulle.

Dans la suite de cette thèse, nous nous consacrerons entièrement au cas récurrent nulle (deuxième point ci-dessus). Ce cas est particulièrement riche, la marche aléatoire  $\mathbb{X}$  pouvant exhiber deux comportements radicalement différents.

## 2.1 La marche lente

Commençons par présenter le régime lent pour la marche aléatoire  $\mathbb{X}$ . Plaçons nous dans le cas appelé traditionnellement frontière pour  $(V(x))_{x \in \mathbb{T}}$  :

$$\inf_{t \in [0,1]} \psi(t) = \psi(1) = \psi'(1) = 0. \quad (\mathbf{BC})$$

Sous  $(\mathbf{BC})$ ,  $\mathbb{X}$  est appelée marche lente sur  $\mathcal{E}$ . L'appellation « marche lente » apparaît pour la première fois dans l'article de Y. Hu et Z. Shi [HS07a] pour décrire une marche qui peine à atteindre les hautes générations de l'arbre  $\mathbb{T}$ . Plus tard, G. faraud, Y. Hu et Z. Shi ([FHS11], Théorème 1.1) montrèrent que sous  $(\mathbf{BC})$ ,  $\mathbf{P}^*$ -presque sûrement

$$\frac{\max_{1 \leq j \leq n} |X_j|}{(\log n)^3} \xrightarrow{n \rightarrow \infty} \frac{8}{3\pi^2 \sigma^2}, \quad (2)$$

où  $\sigma^2 := \mathbf{E}[\sum_{|x|=1} V(x)^2 e^{-V(x)}] \in (0, \infty)$ . La plus grande génération atteinte par  $(X_j)_{j \leq n}$  est donc de l'ordre de  $(\log n)^3$ . Il est par ailleurs montré par Y. Hu et Z. Shi ([HS16b], Corollaire 2.3) que  $\max_{1 \leq j \leq n} |X_j|$  est  $\log n$  fois plus grand  $|X_n|$  : sous  $\mathbb{P}^*$ ,  $(|X_n|/(\log n)^2)_{n \geq 2}$  converge en loi vers une variable aléatoire strictement positive (la loi est explicite).

Sous **(BC)**, la marche est en fait très fortement biaisée vers la racine et ce biais est connu : notons  $\mathcal{L}_x^n = \sum_{j=1}^n \mathbb{1}_{\{X_j=x\}}$  le temps local du sommet  $x \in \mathbb{T}$  à l'instant  $n$ . Y. Hu et Z. Shi ont montré ([HS16b], Proposition 2.4) que le nombre de fois que la marche  $\mathbb{X}$  visite la racine  $e$  avant l'instant  $n$  est de l'ordre de  $n/\log n$  : en  $\mathbb{P}^*$ -probabilité

$$\frac{n}{\log n} \mathcal{L}_e^n \xrightarrow[n \rightarrow \infty]{} \frac{\sigma^2}{4} \frac{1 + W_1}{D_\infty}, \quad (3)$$

où  $W_1 := \sum_{|x|=1} e^{-V(x)}$  et  $D_\infty := \lim_{n \rightarrow \infty} D_n$  avec  $D_n := \sum_{|x|=n} V(x) e^{-V(x)}$ .

Accordons un instant à la suite  $(D_n)$ . Il est aisé de voir que sous **(BC)**,  $(D_n)$  est une  $(\mathcal{F}_n)$ -martingale où  $\mathcal{F}_n := \sigma((\mathbb{T}, V(x); x \in \mathbb{T}, |x| \leq n))$ .  $(D_n)$  est appelée martingale dérivée. Cette dernière converge  $\mathbf{P}^*$ -presque sûrement vers une variable aléatoire  $D_\infty \in (0, \infty)$  (J.D. Biggins et A.E. Kyprianou [BK04] montrent que  $D_\infty \geq 0$ , X. Chen [Che15] prouve que  $D_\infty > 0$  sous certaines hypothèses).  $D_\infty$  peut également être obtenue via la martingale additive  $W_n := \sum_{|x|=n} e^{-V(x)}$  : E. Aïdékon et Z. Shi [AS14a] ont montré qu'en  $\mathbf{P}^*$ -probabilité,  $(\pi \sigma^2 n/2)^{1/2} W_n \rightarrow D_\infty$  quand  $n \rightarrow \infty$ . Une preuve alternative récente de ce résultat est proposée par P. Boutaud et P. Maillard [BM19]. En particulier,  $W_n$  est de l'ordre de  $1/n$  et R. Lyons [Lyo97] montrait déjà quelques années auparavant que  $\mathbf{P}^*$ -presque sûrement,  $\lim_{n \rightarrow 0} W_n = 0$ .

Revenons à la marche lente sur  $\mathcal{E}$  et pour terminer, voici un résultat intéressant dû à P. Andreatti et P. Debs [AD14a] : toutes les générations inférieures à  $\tilde{\gamma}^{-1} \log n$  où  $\tilde{\gamma} := \sup\{a \in \mathbb{R}; \inf_{t \geq 0} (\psi(-t) - at) > 0\}$  sont entièrement visitées par la marche  $(X_j)_{j \leq n}$  : avec grande probabilité, pour tout  $G \in (0, \tilde{\gamma}^{-1})$ ,  $\{x \in \mathbb{T}; |x| \leq G \log n\} = \{x \in \mathcal{R}_n; |x| \leq G \log n\}$ . De plus, la valeur  $\tilde{\gamma}^{-1}$  est optimale : si on note  $G_n := \sup\{k \geq 0; \forall |x| = k, \mathcal{L}_x^n \geq 1\}$  la plus grande génération de l'arbre  $\mathbb{T}$  entièrement visitée par  $\mathbb{X}$  jusqu'à l'instant  $n$  alors ([AD14a], Théorème 1.1),  $\mathbf{P}^*$ -presque sûrement

$$\frac{G_n}{\log n} \xrightarrow[n \rightarrow \infty]{} \frac{1}{\tilde{\gamma}}. \quad (4)$$

Notons que sous **(BC)**,  $G_n$  est seulement  $\log n$  fois plus petite que les générations typiques de la marche  $\mathbb{X}$  à l'instant  $n$  et  $(\log n)^2$  fois plus petite que les plus grandes générations atteintes avant l'instant  $n$ .

Le régime lent n'est pas le seul régime pour la marche aléatoire récurrente nulle  $\mathbb{X}$  sur l'arbre  $\mathbb{T}$ , cette dernière pouvant être (sous-)diffusive.

## 2.2 La marche (sous-)diffusive

Plaçons nous à présent dans le cas

$$\inf_{t \in [0,1]} \psi(t) = \psi(1) = 0 \quad \text{et} \quad \psi'(1) < 0, \quad (\mathbf{D})$$

et posons

$$\kappa := \inf\{t > 1; \psi(t) = 0\}, \quad (5)$$

en supposant que  $\kappa \in (1, \infty)$ . D'après Y. Hu et Z. Shi ([HS07b], Théorème 1.2),  $\mathbb{P}^*$ -presque sûrement

$$\frac{\log \max_{1 \leq j \leq n} |X_j|}{\log n} \xrightarrow{n \rightarrow \infty} 1 - \frac{1}{\kappa \wedge 2}, \quad (6)$$

avec  $\kappa \wedge 2 = \min(\kappa, 2)$ . Autrement dit, sous  $(\mathbf{D})$ , la plus grande génération visitée par la marche  $\mathbb{X}$  jusqu'à l'instant  $n$  est de l'ordre de  $n^{1 - \frac{1}{\kappa \wedge 2}}$ . En particulier, la marche aléatoire  $\mathbb{X}$  est d'autant plus rapide que  $\kappa$  est grand. Il est également montré par E. Aïdékon et L. de Raphélis [AdR17] (pour  $\kappa > 2$ ) et L. de Raphélis [dR22] (pour  $\kappa \in (1, 2]$ ) que  $|X_n|$  et  $\max_{1 \leq j \leq n} |X_j|$  sont du même ordre : les suites  $(|X_n|/n^{1 - \frac{1}{\kappa \wedge 2}})_{n \in \mathbb{N}^*}$  (si  $\kappa \neq 2$ ) et  $((\log n)^{1/2} |X_n|/n^{1/2})_{n \in \mathbb{N}^*}$  (si  $\kappa = 2$ ) converge en loi vers une variable aléatoire strictement positive et explicite. Ainsi, sous  $(\mathbf{D})$ , la marche aléatoire  $\mathbb{X}$  est sous-diffusive si  $\kappa \in (1, 2]$  et diffusive si  $\kappa > 2$ . Naturellement, cette dernière est bien moins biaisée vers la racine que la marche lente : d'après P. Andreoletti et P. Debs [AD14a] et Y. Hu [Hu17] plus récemment,  $\mathcal{L}_e^n$  est de l'ordre de  $n^{\frac{1}{\kappa \wedge 2}}$ . La racine  $e$  est bien entendu un des sommets les plus visités par la marche  $\mathbb{X}$ , peut-on caractériser précisément les sommets les plus visités ? Cette question nous mène naturellement au problème des sites favoris, problème introduit par P. Erdős et P. Révész [ER84] pour la marche simple sur  $\mathbb{Z}$ . Pour tout  $n \in \mathbb{N}$ , notons  $\mathbb{F}(n) := \{x \in \mathbb{T}; \mathcal{L}_x^n = \max_{y \in \mathbb{T}} \mathcal{L}_y^n\}$  l'ensemble des sites favoris de la marche  $\mathbb{X}$  avant l'instant  $n$ , c'est-à-dire l'ensemble des sommets de  $\mathbb{T}$  les plus visités par  $(X_j)_{j \leq n}$ . Soit  $T^1 = \inf\{i \geq 1; X_i = e^*\}$  le premier instant de passage en  $e^*$  et  $\mathcal{M} := \{x \in \mathbb{T}; \mathbb{E}^e[\mathcal{L}_x^{T^1}] = \max_{y \in \mathbb{T}} \mathbb{E}^e[\mathcal{L}_y^{T^1}]\}$  une version moyennée (sur une excursion et sous la probabilité quenched) de l'ensemble des sites favoris. Un résultat montré par D. Chen, L. de Raphélis et Y. Hu [CdRH16] dans le cas de la marche diffusive ( $\kappa > 2$ ) lie l'ensemble  $\mathcal{M}$  à  $\mathbb{F}(n)$  :  $\mathbb{P}^*$ -presque sûrement, pour  $n$  assez grand,  $\mathbb{F}(n) \subset \mathcal{M}$ . Ils prouvèrent également que  $\mathbb{P}^*$ -presque sûrement,  $|\mathbb{F}(n)| = \min(3, |\mathcal{M}|)$   $n$ -infinitement souvent où  $|\mathcal{D}|$  désigne le cardinal de l'ensemble  $\mathcal{D}$ . Nous reviendrons brièvement sur l'ensemble  $\mathcal{M}$  dans la suite. Mentionnons qu'il existe des approximations de la queue de distribution de  $\max_{x \in \mathbb{T}} N_x^{\sigma_1}$  où  $N_x^T := \sum_{j=1}^T \mathbb{1}_{\{X_j = x^*, X_j = x\}}$  est le nombre de passage sur l'arête orientée  $(x^*, x)$  avant l'instant  $T$  et  $\sigma_1 := \inf\{j \geq 1; X_j = e^*, X_j = e\}$  le premier instant de passage sur l'arête  $(e^*, e)$ . X. Chen et L. de Raphélis [CdR20] ont montré que pour tout  $\kappa > 1$ , il existe une constante  $c_\kappa > 0$  telle que  $t^{(\kappa \vee 2)/2} \mathbb{P}(\max_{x \in \mathbb{T}} N_x^{\sigma_1} \geq t) \rightarrow c_\kappa$  si  $\kappa \neq 2$ ,  $t(\log t)^{1/2} \mathbb{P}(\max_{x \in \mathbb{T}} N_x^{\sigma_1} \geq t) \rightarrow c_\kappa$  si  $\kappa = 2$  avec

$\kappa \vee 2 = \max(\kappa, 2)$ .

De même que dans le cas frontière, P. Debs et P. Andreatti ont montré que toutes les générations inférieures à  $(\tilde{\gamma}(\kappa \wedge 2))^{-1} \log n$  sont entièrement visitées par la marche  $(X_j)_{j \leq n}$  : avec grande probabilité, pour tout  $G \in (0, (\tilde{\gamma}(\kappa \wedge 2))^{-1})$ ,  $\{x \in \mathbb{T}; |x| \leq G \log n\} = \{x \in \mathcal{R}_n; |x| \leq G \log n\}$ . De plus, la valeur  $(\tilde{\gamma}(\kappa \wedge 2))^{-1}$  est optimale : si on note  $G_n := \sup\{k \geq 0; \forall |x| = k, \mathcal{L}_x^n \geq 1\}$  la plus grande génération de l'arbre  $\mathbb{T}$  entièrement visitée par  $\mathbb{X}$  jusqu'à l'instant  $n$  alors ([AD14a], Théorème 1.1),  $\mathbb{P}^*$ -presque sûrement

$$\frac{G_n}{\log n} \xrightarrow{n \rightarrow \infty} \frac{1}{\tilde{\gamma}(\kappa \wedge 2)}. \quad (7)$$

Notons que contrairement au cas frontière,  $G_n$  est ici beaucoup plus petite que les générations typiques de la marche aléatoire  $\mathbb{X}$  à l'instant  $n$ .

Nous venons de voir que le potentiel aléatoire  $V$  joue un rôle central dans le comportement de la marche aléatoire  $\mathbb{X}$ , notamment via les fluctuations de  $\psi$  (voir (1)). La prochaine sous-section est ainsi dédiée aux interactions entre la marche aléatoire  $\mathbb{X}$  et son potentiel  $V$ .

### 2.3 La marche aléatoire et son potentiel branchant

Dans cette sous-section, nous faisons état des liens entre la marche  $\mathbb{X}$  et son potentiel  $V$ . Une attention particulière est accordée aux interactions entre les processus  $(X_j)_{j \leq n}$  et  $(V(x))_{x \in \mathbb{T}}$ .

#### Le temps local

Le potentiel  $V$  apparaît naturellement lorsqu'on regarde le temps local  $\mathcal{L}_x^{T^1}$  d'un sommet  $x \in \mathbb{T}$  avant le premier passage en  $e^*$  :  $\mathbb{E}^\mathcal{E}[\mathcal{L}_x^{T^1}] = e^{-U(x)}$  où  $U(x) := V(x) + \log p^\mathcal{E}(x, x^*)$  est le potentiel décalé. L'ensemble  $\mathcal{M} = \{x \in \mathbb{T}; \mathbb{E}^\mathcal{E}[\mathcal{L}_x^{T^1}] = \max_{y \in \mathbb{T}} \mathbb{E}^\mathcal{E}[\mathcal{L}_y^{T^1}]\}$  introduit précédemment dans le cadre du problème des sites favorisés de la marche diffusive n'est en réalité rien d'autre que l'ensemble  $\{x \in \mathbb{T}; U(x) = \min_{y \in \mathbb{T}} U(y)\}$  des minimiseurs du potentiel décalé  $U$ . La position minimale de  $U$  a été abondamment étudiée. J.M. Hammersley [Ham74], J.F.C. Kingman [Kin75] ou encore J.D. Biggins [Big76] ont montré que  $\mathbf{P}^*$ -presque sûrement,  $\min_{|x|=n} U(x)/n \rightarrow \mathbf{c}$  quand  $n \rightarrow \infty$  où  $\mathbf{c} = 0$  lorsque  $\psi'(1) = 0$  et  $\mathbf{c} > 0$  quand  $\psi'(1) < 0$ . De plus, Y.Hu et Z.Shi [HS09] donnent un développement à l'ordre deux de  $\min_{|x|=n} U(x)$  : en  $\mathbf{P}^*$ -probabilité,  $(\min_{|x|=n} U(x) - \mathbf{c}n)/\log n \rightarrow 3/2$  quand  $n \rightarrow \infty$ . Notons que E. Aïdékon [Aïd13] a prouvé que  $(\min_{|x|=n} U(x) - \mathbf{c}n - (3/2) \log n)_{n \geq 2}$  converge en loi vers une variable aléatoire explicite dépendant de  $D_\infty$ .

#### Les descentes de potentiel

Le comportement de la marche aléatoire  $\mathbb{X}$  est directement lié aux fluctuations du potentiel  $V$ . Il est facile de se convaincre que pour tout  $x \in \mathbb{T}$ , moins l'application  $V_x : i \in \{0, \dots, |x|\} \mapsto V(x_i)$  est régulière plus le temps local de la marche  $\mathbb{X}$  le long de ligne généalogique de  $x$  est élevé, ralentissant la marche. Pour tout sommet  $x \in \mathbb{T}$ , posons  $H_x := \sum_{e < z \leq x} e^{V(z) - V(x)}$ .  $H_x$  est



un indicateur important de la régularité de  $V_x$ . En effet,  $e^{\bar{V}(x)-V(x)} \leq H_x \leq |x|e^{\bar{V}(x)-V(x)}$  avec  $\bar{V}(x) := \max_{e < z \leq x} V(z)$  donc plus  $H_x$  est petit, plus  $V_x$  est régulière et réciproquement. Dans le cas frontière **(BC)**, les descentes (exponentielles) de potentiel  $H$  sont typiquement assez grandes : nous savons que  $H_x \geq e^{V(x_1)-V(x)}$  donc pour tout  $\delta > 0$  et tout entier  $n \geq 2$ ,  $\mathbf{E}[\sum_{|x|=n} e^{-V(x)} (H_x)^\delta] \geq \mathbf{E}[\sum_{|z|=1} e^{-V(z)} \sum_{|x|=n; x>z} e^{(1+\delta)(V(z)-V(x))}] = e^{(n-1)\psi(1+\delta)}$  et par définition,  $\psi(1+\delta) \in (0, \infty]$  donc  $\sup_{n \in \mathbb{N}^*} \mathbf{E}[\sum_{|x|=n} e^{-V(x)} (H_x)^\delta] = \infty$ . Dans le cas **(D)**, les descentes (exponentielles) de potentiel  $H$  sont beaucoup plus petites, elles sont d'autant plus petites que  $\kappa$  est grand : d'après P. Andreoletti et R. Diel ([AD20], Lemme 2.2), pour tout  $\varepsilon > 0$ ,  $\sup_{n \in \mathbb{N}^*} \mathbf{E}[\sum_{|x|=n} e^{-V(x)} (H_x)^{\kappa-1-\varepsilon}] < \infty$ .

Faisons un lien entre les temps locaux de la marche  $\mathbb{X}$  et les descentes (exponentielles) de potentiel  $H$ . On rappelle que  $N_x^{T^1} = \sum_{j=1}^{T^1} \mathbf{1}_{\{X_j=x, X_j=x^*\}}$  est le nombre de fois que la marche aléatoire  $\mathbb{X}$  passe par l'arête orientée  $(x^*, x)$  avant le premier passage en  $e^*$ . On note  $R_{T^1} := \sum_{x \in \mathbb{T}} \mathbf{1}_{\{N_x^{T^1} \geq 1\}}$  le nombre de sommets de  $\mathbb{T}$  visités par la marche  $\mathbb{X}$  avant le premier passage en  $e^*$ . Dans l'étude de la trace de  $\mathbb{X}$ , on cherche à contrôler le ratio  $\mathbb{E}[(R_{T^1})^2]/\mathbb{E}[R_{T^1}]^2$ .  $(R_{T^1})^2$  n'est rien d'autre que le nombre de couples  $(x, y)$  de sommets de  $\mathbb{T}$  visités pendant la première excursion au dessus de  $e^*$ . Visiter les sommets  $x$  et  $y$  dans la même excursion demande inévitablement à la marche de se rendre au sommet  $x \wedge y$  (l'ancêtre commun le plus récent de  $x$  et  $y$ , voir 1.1) plusieurs fois et le temps local de ce dernier est nécessairement grand. Notons que  $\mathbb{E}^\delta[(R_{T^1})^2] = \sum_{x \neq y} \mathbb{P}^\delta(N_x^{T^1} \wedge N_y^{T^1} \geq 1) + \mathbb{E}^\delta[R_{T^1}]$  et d'après la propriété de Markov forte

$$H_{x \wedge y} e^{V(x \wedge y)} \leq \frac{\mathbb{P}^\delta(N_x^{T^1} \wedge N_y^{T^1} \geq 1)}{\mathbb{P}^\delta(N_x^{T^1} \geq 1) \mathbb{P}^\delta(N_y^{T^1} \geq 1)} \leq 2H_{x \wedge y} e^{V(x \wedge y)},$$

ce grand temps local se manifestant par l'apparition du facteur  $H_{x \wedge y}$ . On doit donc contrôler le temps local le long de la ligne généalogique de chaque sommet visité via les descentes de potentiel :  $\mathbb{E}[(R_{T^1})^2]/\mathbb{E}[R_{T^1}]^2 \leq \mathbf{E}[\sum_{z \in \mathbb{T}} e^{-V(z)} H_z] + 1$ . Dans le cas frontière **(BC)**, Y. Hu et Z. Shi ([HS16b], Théorème 2.8) ont montré qu'une barrière pour les descentes (exponentielles) de potentiel ne peut-être franchie par la marche aléatoire  $\mathbb{X}$  : pour tout  $\delta \in (0, 1)$

$$\mathbb{P}^*(\forall j \leq n, X_j \in \mathcal{O}_{n/(\log n)^{1+\delta}}) \xrightarrow{n \rightarrow \infty} 1 \quad (8)$$

où nous rappelons que pour tout  $\lambda > 0$ ,  $\mathcal{O}_\lambda = \{x \in \mathbb{T}; \forall e < z \leq x, H_z \leq \lambda\}$ . Dans le cas **(D)**, pour obtenir une estimation logarithmique de la trace lourde (cette trace est définie dans la sous-section 3.3), P. Andreoletti et R. Diel [AD20] remarquent qu'il est possible de se restreindre à l'ensemble  $\{z \in \mathbb{T}; \sum_{e < u \leq z} H_u \leq n^{1-\delta}\}$  avec  $\delta \in (0, 1)$ .

Terminons cette sous-section par quelques résultats sur le processus  $(V(X_j))_{j \in \mathbb{N}}$ .

**Le processus  $(V(X_j))_{j \in \mathbb{N}}$**

Ce processus suscite naturellement beaucoup d'intérêt, principalement dans le régime lent. Ce dernier est moins intéressant dans le cas de la marche (sous-)diffusive, les potentiels très grands

étant facilement atteints par la marche. En effet, pour tout  $\kappa > 1$  il existe  $0 < c_1 < c_2$  tel que pour tout  $\varepsilon > 0$

$$\mathbb{P}^*(c_1 n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon} \leq V(X_n) \leq c_2 n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}) \xrightarrow[n \rightarrow \infty]{} 1, \quad (9)$$

$$\mathbb{P}^*(c_1 n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon} \leq \max_{1 \leq j \leq n} V(X_j) \leq c_2 n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}) \xrightarrow[n \rightarrow \infty]{} 1.$$

Autrement dit,  $V(X_n)$  se comporte comme  $\max_{1 \leq j \leq n} V(X_j)$  et est également du même ordre que  $|X_n|$ . Une preuve de (9) se trouve dans l'annexe 5.

Le processus  $(V(X_j))_{j \in \mathbb{N}}$  est nettement plus riche dans le cas de la marche lente. Dans le cas frontière Y.Hu et Z.Shi conjecturaient ([HS16a], Conjecture 1.2) une convergence en loi de  $(V(X_n)/\log n)_{n \geq 2}$ . Effectivement, cette suite converge en loi, sous  $\mathbb{P}^*$ , vers une variable aléatoire dont la loi  $\mu_V$  de support  $(0, \infty)$  est donnée par : pour tout borélien  $\mathcal{D} \subset (0, \infty)$

$$\mu_V(\mathcal{D}) = \int_0^\infty \frac{du}{(2\pi u)^{1/2}} \mathbf{P}(\mathfrak{s} < u^{-1/2}, u^{1/2} \mathbf{m}_1 \in \mathcal{D}), \quad (10)$$

où  $(\mathbf{m}_t)_{t \in [0,1]}$  est un méandre brownien et  $\mathfrak{s} := \sup_{t \in [0,1]} (\sup_{s \in [0,t]} \mathbf{m}_s - \mathbf{m}_t)$ . Notons qu'un résultat de Y. Hu, Z. Shi et M. Yor ([HSY15], Théorème 1.1) nous assure que  $\mu_V$  est bien une mesure de probabilité. Une preuve de (10) se trouve dans l'annexe 5.

Sous (BC) en revanche,  $\max_{1 \leq j \leq n} V(X_j)$  est  $\log n$  fois plus grand que  $V(X_n)$  :  $\mathbb{P}^*$ -presque sûrement ([HS16a], Théorème 1.1)

$$\frac{1}{(\log n)^2} \max_{1 \leq j \leq n} V(X_j) \xrightarrow[n \rightarrow \infty]{} \frac{1}{2}. \quad (11)$$

Remarquons que  $\max_{1 \leq j \leq n} V(X_j)$  est du même ordre que  $|X_n|$ , comme dans le cas de marche (sous-)diffusive.

Toujours dans le cas frontière pour le potentiel  $V$ , un autre aspect intéressant est le nombre de fois que le processus  $(V(X_j))_{j \leq n}$  dépasse une valeur donnée. Pour tout  $\alpha \in (1, 2)$ , en  $\mathbb{P}^*$ -probabilité

$$\sum_{j=0}^n \mathbf{1}_{\{V(X_j) \geq (\log n)^\alpha\}} = n e^{-(1+o(1))(\log n)^{\alpha-1}}. \quad (12)$$

Cette somme n'est en réalité rien d'autre que le nombre total de fois que la marche aléatoire  $\mathbb{X}$  visite les sommets dont le potentiel est supérieur à  $(\log n)^\alpha$  avant l'instant  $n$ . Le chapitre I est entre autres dédié à ce type de traces.

Nous consacrons la prochaine section à la trace d'une marche aléatoire et plus particulièrement au comportement asymptotique de son volume.

### 3 Trace de marches aléatoires en milieux aléatoires

Nous commençons par définir la notion de trace d'une marche aléatoire et donnons quelques résultats qui nous semblent intéressants pour des marches à valeurs dans  $\mathbb{Z}^d$  et dans des arbres réguliers. Les trois dernières sous-sections 3.2, 3.3 et 3.4 sont entièrement dédiées à la marche aléatoire en milieux aléatoires récurrentes nulle  $\mathbb{X}$ . La sous-section 3.2 aborde la trace usuelle, la trace lourde est traitée dans la sous-section 3.3 et enfin une trace généralisée est présentée dans la sous-section 3.4.

#### 3.1 Définitions et exemples

Soit  $\mathfrak{D}$  un ensemble dénombrable et  $\mathfrak{X} := (\mathfrak{X}_j)_{j \in \mathbb{N}}$  une marche aléatoire à valeurs dans  $\mathfrak{D}$ . On appelle trace (usuelle) de  $\mathfrak{X}$  jusqu'à l'instant  $n$  l'ensemble des points distincts visités par la marche aléatoire  $\mathfrak{X}$  jusqu'à l'instant  $n$  et on note  $\mathfrak{R}_n$  son cardinal (que nous appellerons trace également).

Les deux premiers exemples concernent des marches aléatoires sur  $\mathbb{Z}^d$ , le dernier se concentre sur une marche à valeurs dans un arbre régulier.

##### La marche simple sur $\mathbb{Z}^d$

Nous commençons avec la marche simple sur  $\mathbb{Z}^d$  avec  $d \in \mathbb{N}^*$ , c'est-à-dire  $\mathfrak{D} = \mathbb{Z}^d$  et  $\mathfrak{X}$  est la marche aléatoire au plus proche voisin sur  $\mathfrak{D}$  sautant uniformément sur l'un des ses  $2d$  voisins. Le cas  $d = 1$  est très simple puisque  $\mathfrak{R}_n = 1 + \max_{1 \leq j \leq n} \mathfrak{X}_j - \min_{1 \leq j \leq n} \mathfrak{X}_j$  et d'après le principe d'invariance de Donsker, en loi,  $\mathfrak{R}_n/n^{1/2} \rightarrow G_1 + G_2$  où pour tout  $i \in \{1, 2\}$ ,  $G_i$  suit la loi de la valeur absolue d'une variable gaussienne standard.

Passons directement au cas  $d \geq 3$ . Dans ce cas, la marche aléatoire  $\mathfrak{X}$  est transiente, le temps local de chaque sommet du réseau  $\mathbb{Z}^d$  est donc fini presque sûrement et ainsi  $\mathfrak{R}_n = \sum_{i \in \mathbb{Z}^d} \mathbb{1}_{\{\exists j \leq n: \mathfrak{x}_j = i\}}$  est de l'ordre de  $n$  : A. Dvoretzky et P. Erdős [ED51] ont en effet montré qu'il existe une constante  $\gamma_d > 0$  telle que  $\mathfrak{R}_n/n \rightarrow \gamma_d$  presque sûrement.

Terminons par le cas critique  $d = 2$ . Dans ce cas, toujours d'après A. Dvoretzky et P. Erdős,  $\mathfrak{R}_n$  est de l'ordre de  $n/\log n$  : presque sûrement,  $\mathfrak{R}_n/(n/\log n) \rightarrow \pi$  quand  $n \rightarrow \infty$ . Pourquoi la trace  $\mathfrak{R}_n$  se comporte-t-elle différemment quand  $d = 2$ ? et pourquoi  $\pi n/\log n$ ? D'abord, contrairement à la marche simple sur  $\mathbb{Z}^d$  avec  $d \geq 3$ ,  $\mathfrak{X}$  est récurrente (nulle) et donc le temps local de chaque sommet du réseau  $\mathbb{Z}^d$  est infini presque sûrement. On peut ainsi s'attendre à ce que la marche aléatoire  $\mathfrak{X}$  visite moins de sommets qu'en dimension supérieure. Par ailleurs, P. Erdős et S. Taylor [ET60] ont montré que  $(\pi \mathcal{L}_{(0,0)}^n / \log n)_{n \geq 2}$  converge en loi vers une variable exponentielle de paramètre 1 où  $\mathcal{L}_i^n$  est le temps local de  $i \in \mathbb{Z}^2$  à l'instant  $n$ . De plus, il est prouvé par P. Auer [Aue90] que pour tout  $i \in \mathbb{Z}^2$ ,  $\mathcal{L}_i^n / \mathcal{L}_{(0,0)}^n \rightarrow 1$  presque sûrement. Autrement dit, la marche aléatoire  $\mathfrak{X}$  passe un temps de l'ordre de  $\pi^{-1} \log n$  sur chaque point du réseau  $\mathbb{Z}^2$  visité avant l'instant  $n$ , ce qui explique le comportement en  $\pi n/\log n$  de  $\mathfrak{R}_n$ . Mentionnons que parmi ces points visités par la marche  $\mathfrak{X}$  avant l'instant  $n$ , nombreux sont ceux visités plus de  $c\pi^{-1}(\log n)^2$  fois avec  $c \in (0, 1)$ . En effet, si on note  $R_n^{c\pi^{-1}(\log n)^2}$  le cardinal de l'ensemble de

ces derniers, alors  $R_n^{c\pi^{-1}(\log n)^2}$  est de l'ordre de  $n^{1-c}$ . En effet, d'après ([Ros05], Théorème 1.2), presque sûrement,  $\log R_n^{c\pi^{-1}(\log n)^2} / \log n \rightarrow 1 - c$  quand  $n \rightarrow \infty$ .

### Une marche aléatoire branchante sur $\mathbb{Z}^d$

On s'intéresse maintenant à une marche aléatoire étudiée par J.-F. Le Gall et S. Lin ([GL13], [GL15]). Il s'agit d'une marche aléatoire sur  $\mathbb{Z}^d$  indexée par un arbre de Galton-Watson critique conditionné par sa taille. Soit  $(\mathfrak{T}, (\mathbf{a}_u; u \in \mathfrak{T}))$  un arbre de Galton-Watson marqué défini dans la section 1.2 avec  $\mathbf{M} = \mathbb{Z}^d$ ,  $\mathbf{a}_u$  est donc une variable aléatoire à valeurs dans le réseau  $\mathbb{Z}^d$ . On note  $\mathfrak{V}(u) = \sum_{e < z \leq u} \mathbf{a}_z$  le potentiel branchant du sommet  $u \in \mathfrak{T} \setminus \{e\}$ ,  $\mathfrak{V}(e) = 0$ . On suppose que la loi de reproduction  $N$  est géométrique (sur  $\mathbb{N}$ ) de paramètre  $1/2$  (on a donc  $\mathbf{E}[N] = 1$  et  $\mathfrak{T}$  est critique). On note  $\mathfrak{T}(n)$  l'arbre aléatoire  $\mathfrak{T}$  conditionné à avoir  $n$  sommets,  $\mathfrak{T}(n)$  est donc uniformément distribué sur l'ensemble des arbres planaires à  $n$  sommets. Enfin, pour tout  $0 \leq j \leq n-1$ ,  $u_n(j)$  désigne le  $j$ -ème sommet de l'arbre  $\mathfrak{T}(n)$  pour l'ordre lexicographique, avec la convention  $u_n(0) = e$  (voir la sous-section 1.1).

Pour tout  $n \in \mathbb{N}^*$ , on considère la marche aléatoire  $\mathfrak{X}^n = (\mathfrak{X}_j^n)_{0 \leq j \leq n-1}$  sur  $\mathcal{D} = \mathbb{Z}^d$  définie par  $\mathfrak{X}_j^n := \mathfrak{V}(u_n(j))$ . Autrement dit,  $\mathfrak{X}_j^n$  correspond à la position sur le réseau  $\mathbb{Z}^d$  du sommet  $u_n(j) \in \mathfrak{T}(n)$  par rapport à la racine  $e$ . La trace s'écrit donc ici  $\mathfrak{R}_n = \sum_{i \in \mathbb{Z}^d} \mathbb{1}_{\{\exists 0 \leq j \leq n-1: \mathfrak{X}_j^n = i\}}$ . Sous quelques hypothèses de régularité sur la variable aléatoire  $(N, M)$ , J.-F. Le Gall et S. Lin [GL15] prouvent que pour tout  $d \leq 3$ ,  $(\mathfrak{R}_n/n^{d/4})$  converge en loi. Remarquons que lorsque  $d = 2$ , on retrouve la même normalisation que pour la marche simple sur  $\mathbb{Z}$ . Dans l'article [GL13], ces derniers montrent qu'il existe une constante  $\mathfrak{r} > 0$  telle que pour tout  $d \geq 5$ ,  $\mathfrak{R}_n/n \rightarrow \mathfrak{r}$  en probabilité. De plus, dans la dimension critique  $d = 4$ ,  $(\mathfrak{R}_n/(n/\log n))$  converge dans  $L^2$  et on retrouve la même normalisation que pour la marche simple sur  $\mathbb{Z}^2$ . La trace  $\mathfrak{R}_n$  est donc plus petite que celle de la marche simple sur  $\mathbb{Z}^d$ , la marche indexée par un arbre de Galton-Watson interagit vraisemblablement plus souvent avec elle-même.

### Une marche aléatoire branchante sur l'arbre $\mathfrak{b}$ -ary

Soit  $\mathfrak{b} \geq 2$  un entier. On note  $\mathbb{W}_{\mathfrak{b}}$  l'arbre  $\mathfrak{b}$ -ary infini enraciné en  $e$ , c'est-à-dire l'arbre infini dont chaque sommet a  $\mathfrak{b}$  enfants. On présente ici une marche aléatoire étudiée par T. Duquesne, R. Khanfir, S. Lin et N. Torri [DKLT22]. Il s'agit d'une marche aléatoire sur l'arbre  $\mathbb{W}_{\mathfrak{b}}$  et indexée par un arbre de Galton-Watson critique conditionné par sa taille. Soit  $\mathfrak{T}$  un arbre de Galton-Watson défini dans la section 1.2 avec  $\mathbf{E}[N] = 1$ . Comme vu précédemment,  $\mathfrak{T}$  est critique et s'éteint  $\mathbf{P}$ -presque sûrement. On note  $\mathfrak{T}(n)$  l'arbre aléatoire  $\mathfrak{T}$  conditionné à avoir  $n$  sommets et pour tout  $0 \leq j \leq n-1$ ,  $u_n(j)$  est le  $j$ -ème sommet de  $\mathfrak{T}(n)$  pour l'ordre lexicographique. Pour tout  $n \in \mathbb{N}^*$ , on considère la marche aléatoire  $\mathfrak{X}^n = (\mathfrak{X}_{u_n(j)}^n)_{0 \leq j \leq n-1}$  sur  $\mathcal{D} = \mathbb{W}_{\mathfrak{b}}$  dont les sauts sont ceux d'une marche aléatoire récurrente nulle sur  $\mathbb{W}_{\mathfrak{b}}$  partant de la racine  $e$ :  $\mathfrak{X}_{u_n(0)}^n = \mathfrak{X}_e^n = e$ . Si  $\mathfrak{X}_{u_n(j)^*}^n = e$  alors  $\mathfrak{X}_{u_n(j)}^n = e^i$  avec probabilité  $1/\mathfrak{b}$  pour tout  $1 \leq i \leq \mathfrak{b}$ , si  $\mathfrak{X}_{u_n(j)^*}^n = z \neq e$  alors  $\mathfrak{X}_{u_n(j)}^n = z^*$  avec probabilité  $1/2$  et  $\mathfrak{X}_{u_n(j)}^n = z^i$  avec probabilité  $1/(2\mathfrak{b})$  pour tout  $1 \leq i \leq \mathfrak{b}$  (on rappelle que pour tout  $z \in \mathbb{W}_{\mathfrak{b}}$ ,  $z^*$  désigne le parent de  $z$  et  $z^i$  le  $i$ -ème enfant de  $z$ , voir la sous-section 1.1). La trace s'écrit ici  $\mathfrak{R}_n = \sum_{z \in \mathbb{W}_{\mathfrak{b}}} \mathbb{1}_{\{\exists 0 \leq j \leq n-1: \mathfrak{X}_{u_n(j)}^n = z\}}$ . Il est alors prouvé dans

[DKLT22], sous certaines hypothèses sur la loi de reproduction, qu'il existe une constante  $\mathfrak{r}_b > 0$  telle qu'en probabilité,  $\mathfrak{R}_n/n \rightarrow \mathfrak{r}_b$ .

Revenons au cas de la marche aléatoire récurrente nulle  $\mathbb{X}$  sur  $(\mathbb{T}, (V(x); x \in \mathbb{T}))$  (voir la sous-section 1.2 pour la définition de la marche aléatoire branchante et la section 2 pour la marche aléatoire  $\mathbb{X}$ ). Dans les trois prochaines sous-sections, nous présentons des résultats asymptotiques pour différentes traces de la marche aléatoire  $\mathbb{X}$  jusqu'à l'instant  $n$ .

### 3.2 La trace usuelle

Rappelons que pour tout sommet  $x \in \mathbb{T}$ ,  $\mathcal{L}_x^n = \sum_{j=1}^n \mathbf{1}_{\{X_j=x\}}$  désigne le temps local de  $x$  à l'instant  $n$ . On note  $R_n := \sum_{x \in \mathbb{T}} \mathbf{1}_{\{\mathcal{L}_x^n \geq 1\}}$  la trace usuelle de la marche aléatoire  $\mathbb{X}$  à l'instant  $n$ , c'est-à-dire le nombre de sommets distincts visités par  $\mathbb{X}$  jusqu'à l'instant  $n$ .

Nous commençons par la marche (sous-)diffusive et nous remarquons que  $R_n$  est de l'ordre de  $n$ , comme la marche aléatoire simple (et transiente) sur  $\mathbb{Z}^d$  avec  $d \geq 3$ .

#### La marche (sous-)diffusive

On rappelle que sous **(D)**, la marche aléatoire  $\mathbb{X}$  est sous-diffusive si  $\kappa \in (1, 2]$  et diffusive si  $\kappa > 2$  (la définition de  $\kappa$  se trouve dans l'équation (5)). D'après l'article de E. Aïdékon et L. de Raphélis [AdR17] dans un premier temps puis celui de L. de Raphélis [dR22], pour tout  $\kappa \in (1, \infty)$ , en  $\mathbb{P}^*$ -probabilité

$$\frac{R_n}{n} \xrightarrow[n \rightarrow \infty]{} \mathbf{E} \left[ \frac{1}{2 + 2 \sum_{j \geq 1} e^{-S_j}} \right], \quad (13)$$

où  $(S_j - S_{j-1})_{j \geq 1}$  est la suite de variables aléatoires réelles et i.i.d sous  $\mathbf{P}$  vérifiant, pour toute fonction  $h : \mathbb{R} \rightarrow \mathbb{R}$  mesurable bornée,  $\mathbf{E}[h(S_1)] = \mathbf{E}[\sum_{|x|=1} e^{-V(x)} h(V(x))]$ .

Le point de vue développé dans ces deux articles est celui des arbres de Galton-Watson multitypes. Soient  $i_0 \in \mathbb{N}$  et  $((N_i)_{i \in \mathbb{N}}, D)$  une suite de variables aléatoires à valeurs dans  $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ . Un arbre de Galton-Watson multitypes  $(\mathfrak{T}_{\text{MT}}, (\beta(u); u \in \mathfrak{T}_{\text{MT}}), i_0)$  de type initial  $i_0$  est la donnée d'un arbre aléatoire  $\mathfrak{T}_{\text{MT}}$  et d'une collection de types  $(\beta(u); u \in \mathfrak{T}_{\text{MT}})$  avec  $\beta_u \in \mathbb{N}$  dont la construction est la même que pour l'arbre de Galton-Watson  $\mathfrak{T}$  de la section 1.2 sauf qu'on assigne à chaque sommet  $u$  un type  $\beta(u)$  (avec la convention  $\beta(e) = i_0$ ). Chaque sommet  $u$  donne naissance, indépendamment des sommets de sa génération à  $N_u$  sommets typés  $(u^1, \beta(u^1)), \dots, (u^{N_u}, \beta(u^{N_u}))$  suivant la loi de  $(N_{\beta(u)}, D)$ . Ainsi, le nombre d'enfants d'un sommet  $u \in \mathfrak{T}_{\text{MT}}$  dépend de son type  $\beta(u)$ .

En réalité, la motivation principale de E. Aïdékon et L. de Raphélis est l'étude de la trace  $\mathcal{R}_n = \{x \in \mathbb{T}; \mathcal{L}_x^n \geq 1\}$ . Ces derniers ont cependant besoin d'une estimation précise de son cardinal  $R_n$  et utilisent le fait que pour tout  $N \in \mathbb{N}^*$ ,  $(\mathbb{T}_{\text{MT}}, (\beta(x); x \in \mathbb{T}_{\text{MT}}), N) := (\mathcal{R}_{T^N} \setminus \{e^*\}, (N_x^{T^N}; x \in \mathcal{R}_{T^N} \setminus \{e^*\}))$  est, sous la probabilité annealed  $\mathbb{P}$ , un arbre de Galton-Watson multitypes de type initial  $N$  où  $T^0 = 0$  et pour tout  $j \in \mathbb{N}^*$ ,  $T^j = \inf\{i > T^{j-1}; X_j = e^*\}$ .

Présentons brièvement les idées permettant d'aboutir à (13). Commençons par noter que sous  $\mathbb{P}^*$ , avec grande probabilité,  $\mathcal{R}_n \subset \mathcal{R}_{T^{j_n}}$  où  $(j_n)$  est une suite d'entiers positifs bien choisie et  $\mathcal{R}_{T^{j_n}}$  est, comme nous venons de le mentionner, un arbre de Galton-Watson multitypes de type initial  $j_n$ . Leur idée, centrée autour des sommets  $u$  de type  $\beta(u) = 1$ , est d'associer à  $\mathcal{R}_n$  deux forêts de Galton-Watson 2-types  $\mathbb{F}_n^1$  et  $\mathbb{F}_n^2$  (c'est-à-dire deux familles d'arbres de Galton-Watson 2-types définies dans [AdR17] et [dR17]) construites à partir de copies indépendantes de l'arbre  $(\mathbb{T}_{\text{MT}}, (\beta(u); x \in \mathbb{T}_{\text{MT}}), 1)$ . Dans ces deux forêts, un sommet  $x$  tel que  $\beta(x) = 1$  est de type 1, autrement, il est de type 0 (et est nécessairement une feuille, c'est-à-dire qu'il n'a pas d'enfant). Si  $R_n^1$  désigne le nombre de sommets  $x \in \mathbb{T}$  de type  $\beta(x) = 1$  visités par la marche aléatoire  $\mathbb{X}$  avant l'instant  $n$  et  $\tau_{\mathcal{R}_n}^1(j)$  est le  $j$ -ème instant de visite d'un tel sommet alors  $R_n^1 = \sup\{j \geq 1; \tau_{\mathcal{R}_n}^1(j) \leq n\}$ . Cette relation reste vraie pour la forêt  $\mathbb{F}_n^1$  : si  $\tau_{\mathbb{F}_n^1}^1(j)$  désigne l'indice du  $j$ -ème sommet de type 1 dans  $\mathbb{F}_n^1$  pour l'ordre lexicographique (voir (1.1)) alors  $R_n^1 = \sup\{j \geq 1; \tau_{\mathbb{F}_n^1}^1(j) \leq n\}$ . Quant à la forêt  $\mathbb{F}_n^2$ , elle est construite de façon à ce que son  $j$ -ème sommet pour l'ordre lexicographique est le  $j$ -ème sommet distinct de  $\mathbb{T}$  visité par la marche  $\mathbb{X}$  jusqu'à l'instant  $n$ . En particulier,  $R_n^1 = \sup\{j \geq 1; \tau_{\mathbb{F}_n^2}^1(j) \leq R_n\}$ . On en déduit que  $\tau_{\mathbb{F}_n^2}^1(R_n^1)/\tau_{\mathbb{F}_n^1}^1(R_n^1 + 1) \leq R_n/n \leq \tau_{\mathbb{F}_n^2}^1(R_n^1 + 1)/\tau_{\mathbb{F}_n^1}^1(R_n^1)$  et (13) découle du Lemme 2.1 de [AdR17].

Nous passons maintenant au régime lent (BC). La trace est plus petite que sous (D),  $R_n$  étant de l'ordre de  $n/\log n$ , donc du même ordre que la marche aléatoire simple sur  $\mathbb{Z}^2$ . Dans la sous-partie 3.1, nous expliquons le changement de normalisation pour la marche simple sur  $\mathbb{Z}^2$  entre autres par le fait que pour tout  $d \geq 2$ , cette dernière est la seule à être récurrente (nulle). Cette explication n'est plus valable ici, les marches lente et (sous-)diffusive étant toutes les deux récurrentes nulles.

Pour éviter toute ambiguïté, nous définissons ici deux notations utilisées dans les prochaines sous-sections : on écrira  $a_n \sim b_n$  si et seulement si  $(a_n/b_n)$  converge vers un réel non nul. La deuxième notation n'a pas de sens mathématique précis et relève de l'heuristique : on écrira «  $\approx$  » pour signifier « de l'ordre de ».

### La marche lente

Le résultat suivant est dû à P. Andreatti et X. Chen ([AC18], Théorème 1.5) : dans le cas frontière pour le potentiel branchant (BC), en  $\mathbb{P}^*$ -probabilité

$$\frac{\log n}{n} R_n \xrightarrow{n \rightarrow \infty} \frac{\sigma^2}{4} \Lambda, \quad (14)$$

où nous rappelons que  $\sigma^2 = \mathbf{E}[\sum_{|x|=1} V(x)^2 e^{-V(x)}]$  et  $\Lambda$  est une constante explicite faisant intervenir deux méandres browniens indépendants.

Le contexte de la marche aléatoire en milieux aléatoires est très différent de celui de la marche simple sur  $\mathbb{Z}^d$  mais la marche lente sur  $\mathbb{T}$  et la marche simple sur  $\mathbb{Z}^2$  ont la même normalisation. Pourquoi  $n/\log n$ ? Dans le cas de la marche sur  $\mathbb{Z}^2$  (sous partie 3.1) nous tentions de justifier cette normalisation en disant que le temps local jusqu'à l'instant  $n$  de chaque point du réseau

$\mathbb{Z}^2$  est de l'ordre de  $\log n$ . Dans le cas de la marche lente, c'est le biais vers la racine  $e$  (voir (3)) qui nous fournit cette normalisation.

Nous donnons ici une brève description des sommets typiques de l'environnement permettant d'obtenir le comportement de  $R_n$ . Soit  $(\ell_n)$  une suite d'entiers positifs. Nous nous intéressons d'abord à la trace  $R_{T^n}(\ell_n) := \sum_{|x|=\ell_n} \mathbb{1}_{\{\mathcal{L}_x^{T^n} \geq 1\}}$  où nous rappelons que  $T^0 = 0$  et pour tout  $n \in \mathbb{N}^*$ ,  $T^n := \inf\{j > T^{n-1}; X_j = e^*\}$  le  $n$ -ième instant de passage en  $e^*$ .  $R_{T^n}(\ell_n)$  n'est rien d'autre que le nombre de sommets dans la générations  $\ell_n$  de l'arbre  $\mathbb{T}$  et visités par la marche  $\mathbb{X}$  pendant les  $n$  premières excursions au dessus de  $e^*$ . Dans un premier temps, il faut remarquer que seules les générations  $\ell_n \sim (\log n)^2$  contribuent significativement à la trace ([AD14b], Théorème 1.1). On choisit ensuite soigneusement les sommets pertinents : les sommets  $x \in \mathbb{T}$  dont les descentes (exponentielles) de potentiel le long de la ligne généalogique ne sont pas trop grandes en demandant typiquement  $H_z = \sum_{e < w \leq z} e^{V(w)-V(z)} \leq n/(\log n)^\delta$  pour tout  $e < z \leq x$ , avec  $\delta > 1$  (ce qui est raisonnable au regard du résultat (8)) ainsi que ceux dont le potentiel  $V(x)$  est plus grand que  $\log n + \log \log n$ . Notons  $A_1$  l'ensemble de ces sommets pertinents et  $R_{T^n}^{A_1}(\ell_n)$  la trace associée. Intéressons nous maintenant à la moyenne quenched  $\mathbb{E}^\mathcal{E}[R_{T^n}^{A_1}(\ell_n)] = \sum_{|x|=\ell_n} (1 - (1 - a_x)^n) \mathbb{1}_{\{x \in A_1\}}$  où  $a_x = e^{-V(x)}/H_x$ . La condition  $V(x) \geq \log n + \log \log n$  créer de la quasi-indépendance le long des trajectoires de la marche  $\mathbb{X}$ , nous permettant d'obtenir  $\mathbb{E}^\mathcal{E}[R_{T^n}^{A_1}(\ell_n)] \approx n \sum_{|x|=\ell_n} a_x \mathbb{1}_{\{x \in A_1\}}$  et faisant ainsi apparaître un facteur  $n$ . Nous savons que  $\sum_{|x|=\ell_n} e^{-V(x)} \approx 1/\ell_n^{1/2}$  (il s'agit du résultat de E. Aïdékon et Z. Shi [AS14b] énoncé précédemment) et les sommets de  $A_1$  sont pertinents justement car  $\sum_{|x|=\ell_n} e^{-V(x)} \mathbb{1}_{\{x \in A_1\}} \approx 1/\ell_n^{1/2}$ . Il nous faut donc déterminer le coût de la division par  $H_x$ . Notons que  $e^{\bar{V}(x)-V(x)} \leq H_x \leq \ell_n e^{\bar{V}(x)-V(x)}$  pour tout  $|x| = \ell_n$ , où  $\bar{V}(x) = \max_{e < z \leq x} V(z)$ . Cherchant à minimiser  $H_x$ , nous conservons seulement les sommets  $x$  satisfaisant  $\bar{V}(x) \approx V(x)$  et le coût de ce choix est  $1/\ell_n^{1/2}$ . Autrement dit,  $\sum_{|x|=\ell_n} a_x \mathbb{1}_{\{x \in A_1\}} \approx \sum_{|x|=\ell_n} a_x \mathbb{1}_{\{x \in A_1, \bar{V}(x) \approx V(x)\}} \approx (1/\ell_n^{1/2})^2 \sim 1/(\log n)^2$  d'où  $\mathbb{E}^\mathcal{E}[R_{T^n}^{A_1}(\ell_n)] \approx n/(\log n)^2$ . L'étape suivante est de montrer que  $R_{T^n}(\ell_n) \approx \mathbb{E}^\mathcal{E}[R_{T^n}^{A_1}(\ell_n)]$  en contrôlant la variance de  $R_{T^n}(\ell_n)$ . Contrôler la variance, c'est contrôler le rapport  $\mathbb{E}[R_{T^1}(\ell_n)^2]/\mathbb{E}[R_{T^1}(\ell_n)]^2$  c'est-à-dire minimiser le temps local le long de la ligne généalogique de chaque sommet  $x$  visités par  $\mathbb{X}$ , ce qui, comme nous l'avons vu dans la sous-section 2.3, est rendu possible par la condition  $\max_{e < z \leq x} H_z \leq n/(\log n)^\delta$  (l'avantage par rapport à la barrière (8) est qu'ici, nous prenons  $\delta$  aussi grand que souhaité, le choix de sa valeur ayant une importance mineure, voir le lemme 2.2 de [AC18]) nous permettant ainsi d'obtenir  $R_{T^n}(\ell_n) \approx n/(\log n)^2$ . Seules les générations de l'ordre de  $(\log n)^2$  contribuant significativement à la trace usuelle,  $R_{T^n} \approx \sum_{\ell_n \sim (\log n)^2} R_{T^n}(\ell_n) \approx (\log n)^2 \times (n/(\log n)^2) = n$ . La dernière étape utilise le biais aléatoire vers la racine : en  $\mathbb{P}^*$  probabilité,  $T^{n/\log n} \sim n$  (voir (3)) faisant de  $n/\log n$  la bonne normalisation pour la trace  $R_n$ .

Passons maintenant à la trace lourde. Il s'agit de l'ensemble des sommets visités très souvent par la marche aléatoire  $\mathbb{X}$ .

### 3.3 La trace lourde

Soit  $T \in \mathbb{N}^*$  et  $x \in \mathbb{T}$ . On rappelle que  $\mathcal{L}_x^T = \sum_{j=1}^T \mathbf{1}_{\{X_j=x\}}$  est le temps local du sommet  $x$  à l'instant  $T$ . Pour tout réel  $b \geq 0$ , la trace lourde  $\mathcal{R}_{T^n}^{(b)}$  de la marche  $\mathbb{X}$  jusqu'à l'instant  $T^n$  est l'ensemble  $\{x \in \mathbb{T}; \mathcal{L}_x^{T^n} \geq n^b\}$  des sommets de l'arbre  $\mathbb{T}$  visités au moins  $n^b$  fois. On note  $R_{T^n}^{(b)}$  son cardinal. Remarquons que  $\mathcal{R}_{T^n}^{(0)} = \mathcal{R}_{T^n}$  mais qu'en général,  $\mathcal{R}_{T^n}^{(b)}$  n'est pas un sous-arbre de  $\mathbb{T}$  pour  $b > 0$ . La trace  $R_{T^n}^{(b)}$  a été étudiée à la fois dans le cas de la marche (sous-)diffusive et dans le cas de la marche lente par P. Andreatti et R. Diel [AD20] puis un résultat plus précis a été démontré par X. Chen [Che22] dans le cas de la marche lente. Quelque soit le régime de la marche  $\mathbb{X}$ , le comportement de  $R_{T^n}^{(b)}$  dépend naturellement de  $b$  et il est aisé de voir que  $\mathbb{P}^*(R_{T^n}^{(b)} = 0) \rightarrow 1$  quand  $n \rightarrow \infty$  pour tout  $b > 1$ . Le résultat suivant est dû à P. Andreatti et R. Diel ([AD20], Théorème 1.1) : pour tout  $b \in [0, 1]$ , en  $\mathbb{P}^*$ -probabilité

$$\frac{\log^+ R_{T^n}^{(b)}}{\log n} \xrightarrow{n \rightarrow \infty} \xi_b \quad (15)$$

où  $\xi_b = 1 - b$  sous (BC),  $\xi_b = \kappa(1 - b)$  sous (D) et  $\kappa \in (1, 2]$ ,  $\xi = \max(2 - \kappa b, 1 - b)$  sous (D) et  $\kappa > 2$ .  $\log^+(x) = \log(\max(x, 1))$  est la partie positive de  $\log(x)$ .

Le résultat ci-dessus (15) nous dit que  $R_{T^n}^{(b)} \approx n/n^{1-\xi_b}$  signifiant que la contrainte « le sommet  $x \in \mathbb{T}$  doit être visité au moins  $n^b$  fois par la marche aléatoire  $\mathbb{X}$  » produit un facteur  $n^{-(1-\xi_b)}$ . Tentons d'expliquer brièvement l'apparition de ce facteur, par exemple dans le cas (D) avec  $\kappa \in (1, 2]$ . L'idée est de montrer que  $R_{T^n}^{(b)}$  se comporte comme sa moyenne annealed  $\mathbb{E}[R_{T^n}^{(b)}]$  en contrôlant sa variance comme précédemment. A nouveau, le choix des sommets pertinents est crucial : d'après (6) et [Hu17], on peut se restreindre aux générations plus petites que  $n^{\kappa-1+\varepsilon}$  pour tout  $\varepsilon > 0$ . Par ailleurs, on ne garde que les sommets  $x \in \mathbb{T}$  dont les descentes (exponentielles) de potentiel sont raisonnables :  $\sum_{z \leq x} H_z \leq n^{1-\delta}$  avec  $\delta > 0$ , et dont le potentiel  $V(x)$  est supérieur à  $4 \log n$ . Notons  $A_2$  cette ensemble. Nous avons alors  $\mathbb{E}[R_{T^n}^{(b)}] \approx n \mathbf{E}[\sum_{|x| \leq n^{\kappa-1}} e^{-V(x)} h_n(H_x) \mathbf{1}_{\{x \in A_2\}}]$ , le facteur  $n$  provenant de la condition  $V(x) \geq 4 \log n$  qui a permis d'obtenir de l'indépendance le long des trajectoires de  $\mathbb{X}$  et  $h_n(t) = (1 - 1/t)^{n^b-1}/t$ . On cherche ici un équilibre subtil : nous devons minimiser le temps local le long de la ligne généalogique de chaque sommet  $x$  visité par  $\mathbb{X}$  via la condition  $\sum_{z \leq x} H_z \leq n^{1-\delta}$  (pour contrôler la variance de la trace) mais également maximiser celui du sommet  $x$ . Maximiser le temps local de  $x$ , c'est maximiser la fonction  $h_n$ , son maximum étant atteint pour  $t = n^b$ . Donc seuls les sommets  $x \in A_2$  tels que  $H_x \approx n^b$  comptent pour la trace lourde, ce qui donne  $\mathbb{E}[R_{T^n}^{(b)}] \approx n^{1-b} \sum_{k \leq n^{\kappa-1+\varepsilon}} \mathbf{E}[\sum_{|x|=k} e^{-V(x)} \mathbf{1}_{\{x \in A_2, H_x \approx n^b\}}]$ . La condition  $H_x \approx n^b$  a déjà coûté un facteur  $n^b$  mais coûte également un facteur  $n^{b(\kappa-1)}$  via l'espérance précédente :  $\mathbf{E}[\sum_{|x|=k} e^{-V(x)} \mathbf{1}_{\{x \in A_2, H_x \approx n^b\}}] \approx 1/n^{b(\kappa-1)}$ . On obtient finalement  $\mathbb{E}[R_{T^n}^{(b)}] \approx n^{1-b} \times n^{\kappa-1+\varepsilon} \times n^{-b(\kappa-1)} = n^{\kappa(1-b)+\varepsilon}$ .

X.Chen [Che22] a affiné le résultat (15) dans le cas de la marche lente : Soit  $N_x^T$  le edge-temps local du sommet  $x$  à l'instant  $T$ , c'est-à-dire  $N_x^T = \sum_{j=1}^T \mathbf{1}_{\{X_{j-1}=x^*, X_j=x\}}$ ,  $\sigma_0 = 0$  et pour tout  $n \in \mathbb{N}^*$ ,  $\sigma_n := \inf\{i > \sigma_{n-1}; X_{i-1} = e^*, X_i = e\}$ . On note  $\mathbf{R}_{\sigma_n}^{(b)}$  le nombre de sommets  $x \in \mathbb{T}$  tels



que  $N_x^{\sigma_n} \geq n^b$  avec  $b \in (0, 1)$ . Pour tout  $b \in (0, 1)$  il existe une constante  $\Lambda(b) > 0$  explicite telle qu'en  $\mathbb{P}^*$ -probabilité

$$\frac{\mathbf{R}_{\sigma_n}^{(b)}}{n^{1-b}} \xrightarrow[n \rightarrow \infty]{} \Lambda(b)D_\infty.$$

Finalement, dans un article en préparation, X. Chen et L. de Raphélis prouvent, sous **(D)**, la convergence en loi de  $(\mathbf{R}_{\sigma_n}^{(b)}/n^{\xi_b})_{n \in \mathbb{N}^*}$  à la fois sous la probabilité annealed et sous la probabilité quenched.

Clôtureons cette section en présentant des résultats asymptotiques pour une trace généralisée dans le régime lent.

### 3.4 Traces sur potentiels hauts et généralisations

Les résultats présentés dans cette sous-section sont détaillés dans le chapitre I et sont tirés de l'article [AK21] écrit avec Pierre Andreatti et dans lequel nous nous focalisons sur une trace généralisée pour la marche aléatoire  $\mathbb{X}$  dans le régime lent **(BC)**. Il s'agit d'une trace contrainte à la fois le long des trajectoires de la marche aléatoire  $\mathbb{X}$  et le long de celles du potentiel branchant  $V$ . Une de nos principales motivations est de mieux comprendre les interactions entre  $\mathbb{X}$  et son potentiel  $V$ .

Nous commençons cette sous-section par un exemple important de contrainte sur  $V$ , celle des potentiels hauts puis nous détaillons une trace contrainte générale.

Nous avons vu précédemment (voir la sous-section (3.2)) que les sommets  $x$  contribuant significativement à la trace usuelle sont ceux dont le potentiel  $V(x)$  est plus grand que  $\log n + \log \log n$ . Nous pouvons montrer de la même façon qu'ajouter la contrainte  $V(x) \geq A \log n$  avec  $A > 0$  n'a pas d'impact majeur sur le comportement de la trace : si  $R_{n,K} := \sum_{x \in \mathbb{T}} \mathbb{1}_{\{T_x \leq n\}} \mathbb{1}_{\{V(x) \geq K\}}$  (avec  $K > 0$ ) désigne le nombre de sommets visités par la marche  $\mathbb{X}$  jusqu'à l'instant  $n$  et dont le potentiel est supérieur à  $K$  alors pour tout  $A > 0$ , il existe une constante  $C(A) > 0$  telle qu'en  $\mathbb{P}^*$ -probabilité

$$\frac{\log n}{n} R_{n, A \log n} \xrightarrow[n \rightarrow \infty]{} C(A). \quad (16)$$

La preuve de la convergence (16) est identique à celle de X. Chen et P. Andreatti permettant d'obtenir (14). En revanche, la constante  $C(A)$ , bien que similaire à celle de la trace usuelle, fait intervenir une fonction décroissante en  $A$ .

Par ailleurs, nous savons d'après (11) qu'il n'y a aucun sommet dont le potentiel est plus grand que  $c(\log n)^2$  avec  $c > 1/2$  :  $\mathbb{P}^*$ -presque sûrement, il existe  $n_0 \in \mathbb{N}^*$  tel que pour tout  $n \geq n_0$ ,  $R_{n, c(\log n)^2} = 0$ .

Il est donc naturel de se demander comment choisir  $K$  pour que la condition  $V(x) \geq K$  ait un impact significatif mais non excessif sur la trace? Prenons par exemple  $K = (\log n)^\alpha$  avec

$\alpha \in (1, 2)$ . Nous avons ([AK21], Théorème 1.1), en  $\mathbb{P}^*$ -probabilité

$$\frac{\log^+ R_{n,(\log n)^\alpha} - \log n}{(\log n)^{\alpha-1}} \xrightarrow{n \rightarrow \infty} -1. \quad (17)$$

Autrement dit,  $R_{n,(\log n)^\alpha} = ne^{-(1+o(1))(\log n)^{\alpha-1}}$ , ce qui est naturellement plus petit que le  $n/\log n$  de la trace usuelle tout en prouvant que les sommets visités par la marche  $\mathbb{X}$  dont le potentiel est capable d'atteindre de telles valeurs sont nombreux : en  $\mathbb{P}^*$  probabilité,  $\log R_{n,(\log n)^\alpha} \sim \log R_n$ . Cette différence dans la normalisation peut s'expliquer par le fait que  $(X_j)_{j \leq n}$  passe la majorité de son temps sur des sommets de génération de l'ordre de  $(\log n)^2$ . Or, les sommets dont le potentiel est supérieur à  $(\log n)^\alpha$  avec  $\alpha \in (1, 2)$  y sont très peu présents. Ces derniers sont nombreux dans les générations de l'ordre de  $(\log n)^{\alpha+1}$ , générations peu visitées par la marche  $(X_j)_{j \leq n}$  et d'autant plus difficiles à visiter que  $\alpha$  est grand.

Dans le même esprit, nous pouvons compter le nombre  $R_{n,K}^\nu$  de sommets  $x \in \mathbb{T}$  visités par la marche avant l'instant  $n$  satisfaisant  $V(x_{\nu(|x|)}) \geq K$  pour une certaine fonction  $\nu : \mathbb{N} \rightarrow \mathbb{N}$  croissante vérifiant  $\nu(i) \leq i$ . On rappelle que  $|x|$  correspond à la génération de  $x$  et que pour tout  $\ell \leq |x|$ ,  $x_\ell$  désigne l'unique ancêtre de  $x$  à la génération  $\ell$ . Prenons  $\nu(i) = \nu_\beta(i) = \lfloor i/\beta \rfloor$  avec  $\beta \geq 1$ . Nous montrons ([AK21], Théorème 1.2), en  $\mathbb{P}^*$ -probabilité

$$\frac{\log^+ R_{n,(\log n)^\alpha}^{\nu_\beta} - \log n}{(\log n)^{\alpha-1}} \xrightarrow{n \rightarrow \infty} -1 - \tilde{\rho}((\beta - 1)\frac{\pi^2}{4}). \quad (18)$$

où  $\tilde{\rho} : t \in [0, \infty) \mapsto \sqrt{t} - \rho(t) > 0$  est croissante et  $\rho$  est une fonction continue telle que  $\rho(0) = 0$  (voir (I.6) pour une expression de  $\rho(t)$  pour tout  $t > 0$ ). Notons que  $R_{n,K}^{\nu_1} = R_{n,K}$  et puisque  $\tilde{\rho}(0) = 0$ , (18) est bien vraie pour  $\beta = 1$ . Ce résultat implique donc que pour tout  $\beta \geq 1$ ,  $R_{n,(\log n)^\alpha}^{\nu_\beta} \approx R_{n,(\log n)^\alpha} \times e^{-\tilde{\rho}(\pi^2(\beta-1)/4)(\log n)^{\alpha-1}}$ .

Cette contrainte sur les trajectoires du potentiel  $V$  fait apparaître le terme du second ordre  $\tilde{\rho}(\pi^2(\beta - 1)/4)$  dans (18), terme qui était donc absent dans (17). Ce terme du second ordre est assez grand :  $\lim_{t \rightarrow 0} \tilde{\rho}(t)/t = 3/4$ . Autrement dit, une faible translation le long de la ligne généalogique dans la condition de potentiel haut a un impact majeur sur la trace.

D'une façon globale, nous nous intéressons aussi à des contraintes générales sur les trajectoires de  $V$ , considérons par exemple

$$R(n, \mathbf{A}_n) := \sum_{x \in \mathbb{T}} \mathbb{1}_{\{T_x \leq n\}} \mathbb{1}_{\{(V(x_1), \dots, V(x)) \in \mathbf{A}_n^{|x|}\}}, \quad (19)$$

où  $\mathbf{A}_n := \{\mathbf{A}_n^k \subset \mathbb{R}^k; k \in \mathbb{N}^*\}$  est une collection donnée d'ensembles. Pour tout  $\lambda > 0$  et  $t \in \mathbb{R}$ , on définit la transformée de Laplace tronquée  $\psi_\lambda(t, \mathbf{A}_n)$  associée à la collection  $\mathbf{A}_n$  par :

$$\psi_\lambda(t, \mathbf{A}_n) := \mathbf{E} \left[ \sum_{x \in \mathbb{T}} e^{-(V(x)+t)} \mathbb{1}_{\{x \in \mathcal{O}_\lambda, (V(x_1)+t, \dots, V(x)+t) \in \mathbf{A}_n^{|x|}\}} \right], \quad (20)$$

où nous rappelons que  $\mathcal{O}_\lambda = \{x \in \mathbb{T}; \max_{e < z \leq x} H_z \leq \lambda\}$ . Sous certaines hypothèses, nous sommes capables de donner le comportement asymptotique de  $R(n, \mathbf{A}_n)$  : si  $h_n := |\log \psi_n(0, \mathbf{A}_n)|$  est assez grand (au sens de (I.13)) alors ([AK21], Théorème 1.5) en  $\mathbb{P}^*$ -probabilité

$$\frac{\log^+ R(n, \mathbf{A}_n) - \log n}{h_n} \xrightarrow[n \rightarrow \infty]{} -1. \quad (21)$$

Autrement dit,  $R(n, \mathbf{A}_n) \approx ne^{-h_n}$  et le coût de la contrainte imposée aux trajectoires du potentiel  $V$  se retrouve dans le paramètre exponentiel  $e^{-h_n}$ . Notons enfin que la normalisation  $h_n$  de la trace dépend très simplement de la contrainte  $\mathbf{A}_n$ .

Revenons brièvement sur les hypothèses permettant d'obtenir (21). Pour simplifier, on se place dans le cas où  $\mathbf{A}_n^k$  s'écrit comme un produit :  $\mathbf{A}_n^k = A_{n,1} \times \cdots \times A_{n,k}$  où pour tout  $j \in \{1, \dots, k\}$ ,  $A_{n,j} \subset \mathbb{R}$  et on suppose que  $A_{n,k} \subset [3 \log n, \infty)$  (il s'agit d'une hypothèse raisonnable d'après (16)).

Pour prouver (21), nous faisons quatre hypothèses. Les deux dernières, (A3) et (A4), sont très techniques et nous n'en parlerons pas ici. La première hypothèse, (A1), est une borne inférieure sur  $\psi_n(0, \mathbf{A}_n)$  : pour tout  $\varepsilon > 0$  et  $n$  assez grand

$$\psi_n(0, \mathbf{A}_n) \geq \frac{1}{n^\varepsilon}.$$

L'idée derrière (A1) est la suivante : nous nous appuyons sur le fait que  $\mathbb{E}[R_n(\mathbf{A}_n)] \approx n\psi_n(0, \mathbf{A}_n)$  pour demander que la condition sur les trajectoires de  $V$  ne soit pas trop difficile à satisfaire pour la marche  $\mathbb{X}$ , dans le sens suivant :  $\liminf_{n \rightarrow \infty} \log \mathbb{E}[R_n(\mathbf{A}_n)/n] / \log n > 0$ .

Passons maintenant à l'hypothèse (A2). Définissons  $\mathbf{A}_n^{\ell+} := \{\mathbf{A}_n^{\ell+k} \subset \mathbb{R}^{\ell+k}; k \in \mathbb{N}^*\}$  la collection  $\mathbf{A}_n$  décalée d'un entier  $\ell \leq (\log n)^3$  (on rappelle que les plus grandes générations atteintes par  $(X_j)_{j \leq n}$  sont de l'ordre de  $(\log n)^3$ , voir (2)), soit  $z \in \mathbb{T}$  dans la génération  $\ell$  et enfin posons  $\lambda_n := ne^{-5h_n}$  (on rappelle que  $h_n = |\log \psi_n(0, \mathbf{A}_n)|$ ). L'hypothèse (A2) est une borne supérieure pour  $\psi_{\lambda_n}(V(z), \mathbf{A}_n^{\ell+})$  : pour tout  $\varepsilon \in (0, 1)$ ,  $\delta, B > 0$ , pour  $n \in \mathbb{N}^*$  assez grand, tout  $\ell \leq (\log n)^3$ , et tout sommet  $|z| = \ell$  tel  $V(z) \geq -B$ ,  $H_z \leq n$  et pour tout  $j \in \{1, \dots, \ell\}$ ,  $V(z_j) \in A_{n,j}$

$$\psi_{\lambda_n}(V(z), \mathbf{A}_n^{\ell+}) \leq e^{-\delta V(z)} \psi_n(0, \mathbf{A}_n)^{1-\varepsilon}.$$

Autrement dit, la contrainte  $\mathbf{A}_n$  doit être suffisamment souple pour accepter à la fois une translation du potentiel mais également une translation de génération. En effet, par définition, la contrainte  $A_{n,i}$  agit sur le potentiel  $V(x_i)$  de l'ancêtre  $x_i$  de  $x$  dans la génération  $i \in \{1, \dots, |x|\}$ . On veut donc qu'un décalage de génération  $\ell \leq (\log n)^3$  ait un impact raisonnable sur la trace.

Pour prouver (21), nous montrons que  $R(n, \mathbf{A}_n)$  se concentre autour de sa moyenne annealed  $\mathbb{E}[R(n, \mathbf{A}_n)]$ . C'est la borne inférieure qui demande le plus d'attention, voici quelques idées permettant de l'obtenir. Commençons par regarder la trace  $R(T^n, \mathbf{A}_n)$  après  $n$  passages en  $e^*$

et définissons une nouvelle trace  $\bar{R}(T^n, \mathbf{A}_n, \lambda_n)$  qui conserve seulement les sommets  $x$  tels que  $\bar{V}(x) = V(x)$  et dont les descentes (exponentielles) de potentiel  $(H_z; e < z \leq x)$  le long de la ligne généalogique de  $x$  sont inférieures à  $\lambda_n = ne^{-5h_n}$  (nous demandons donc davantage de contrôle que ce que nous propose la barrière (8)). Par définition,  $R(T^n, \mathbf{A}_n) \geq \bar{R}(T^n, \mathbf{A}_n, \lambda_n)$  et c'est sur cette dernière trace que nous nous attarderons.

Nous avons supposé ici que pour tout  $k \in \mathbb{N}^*$ ,  $A_{n,k} \subset [3 \log n, \infty)$ . Autrement dit,  $\bar{R}(T^n, \mathbf{A}_n, \lambda_n)$  compte des sommets  $x \in \mathbb{T}$  vérifiant  $V(x) \geq 3 \log n$ . Or, sous  $\mathbb{P}^*$ , avec grande probabilité, un sommet  $x \in \mathbb{T}$  tel que  $V(x) \geq 3 \log n$  n'est pas visité ou est visité durant une unique excursion au dessus de  $e^*$ . De ce fait, on déduit l'indépendance le long de la trajectoire de la marche aléatoire  $\mathbb{X}$  nécessaire pour prouver que  $\bar{R}(T^n, \mathbf{A}_n, \lambda_n)$  est du même ordre que  $n\mathbb{E}^\mathcal{G}[\bar{R}(T^1, \mathbf{A}_n, \lambda_n)]$ . La condition  $\mathcal{O}_{\lambda_n}$  a permis ici de contrôler la variance de la trace. Nous montrons ensuite que cette espérance quenched se concentre autour de  $\mathbb{E}[\bar{R}(T^1, \mathbf{A}_n, \lambda_n)]$  et c'est précisément à cette étape (que nous ne détaillons pas ici) que les hypothèses (A1) et (A2) sont indispensables. Pour ce faire, nous créons cette fois de l'indépendance le long des trajectoires du potentiel  $V$  en sacrifiant les générations inférieures à  $h_n$  nous permettant ainsi d'appliquer une inégalité de concentration pour  $\mathbb{E}^\mathcal{G}[\bar{R}(T^1, \mathbf{A}_n, \lambda_n)]$ . Ce sacrifice s'avère avoir un coût raisonnable, c'est-à-dire le facteur  $e^{-\varepsilon h_n}$  pour  $\varepsilon \in (0, 1)$ . Notons que grâce à la condition  $\bar{V}(x) = V(x)$  nous avons  $H_x = \sum_{e < z \leq x} e^{V(z) - V(x)} \leq |x|$  d'où

$$\mathbb{E}[\bar{R}_{T^1}(\mathbf{A}_n, \mathcal{O}_{\lambda_n})] \geq \mathbf{E}\left[\sum_{x \in \mathbb{T}} e^{-V(x)} / |x| \mathbb{1}_{\{(V(x_1), \dots, V(x)) \in \mathbf{A}_n^{|x|}, \bar{V}(x) = V(x), x \in \mathcal{O}_{\lambda_n}\}}\right],$$

et l'hypothèse (A3) (dont nous n'avons pas parlé) permet d'obtenir que l'espérance ci-dessus est de l'ordre de  $\psi_n(0, \mathbf{A}_n)$ . Finalement, le passage à  $R(n, \mathbf{A}_n)$  est rendu possible grâce au biais vers  $e^*$  (voir (3)).

Bien entendu, les quatre hypothèses (A1), (A2), (A3) et (A4) sont satisfaites par nos deux exemples précédents (et par d'autres exemples détaillés dans [AK21]). Le résultat principal (Théorème 1.5), plus général que (19), concerne des traces mélangeant contraintes sur les trajectoires de la marche  $\mathbb{X}$  et contraintes sur celles du potentiel branchant  $V$ , c'est-à-dire de la forme

$$\mathcal{R}_n(g_n, \mathbf{f}^n) = \sum_{x \in \mathbb{T}} g_n(\mathcal{L}_x^n) f^{n, |x|}(V(x_1), \dots, V(x)). \quad (22)$$

L'étude de ces traces généralisées nous permet entre autres de mieux comprendre les interactions entre la marche aléatoire  $\mathbb{X}$  son potentiel branchant  $V$  dans le régime lent, interactions que nous avons évoquées dans la sous-section 2.3.

Il est légitime de se demander s'il est possible d'obtenir une convergence presque sûre plutôt qu'en probabilité pour la trace généralisée définie dans (22) voir même gagner en précision en s'affranchissant du logarithme (voir (21) et le Théorème 1.5). Ces questions pourront faire partie

d'un futur projet de recherche.

Nous consacrons la dernière section de cette introduction à l'étude de la généalogie de sommets tirés uniformément dans l'arbre  $\mathcal{R}_n = \{X_0, \dots, X_n\}$  lorsque la marche aléatoire  $\mathbb{X}$  est diffusive (voir (2.2)).

## 4 Généalogie de la marche aléatoire diffusive

Les résultats présentés dans cette partie sont détaillés dans le chapitre II et sont tirés de l'article [Kag22].

Soit  $\mathfrak{T}$  un arbre aléatoire,  $T$  un nombre entier strictement positif et notons  $Z_T$  le nombre de sommets dans la générations  $\ell_n$  de l'arbre  $\mathfrak{T}$ . Soit  $k \geq 2$  un entier. Sachant l'événement  $\{Z_T \geq k\}$ , tirons  $k$  sommets distincts  $\mathcal{X}_T^{(1)}, \dots, \mathcal{X}_T^{(k)}$  uniformément et sans remise dans la générations  $T$ . A quoi ressemble l'arbre généalogique de ces  $k$  sommets ? Cette question, posée pour la première fois par W. Bühler [Bü71] a reçu beaucoup d'attention et continue d'en recevoir aujourd'hui. Dans cette section, nous y répondons dans un premier temps lorsque que  $\mathfrak{T}$  est un arbre de Galton-Watson en nous appuyant sur une littérature riche. Nous focalisons ensuite notre attention sur le cas  $\mathfrak{T} = \{X_0, \dots, X_n\}$  lorsque la marche aléatoire  $\mathbb{X}$  est diffusive.

### 4.1 Généalogie des arbres de Galton-Watson

Soit  $\mathfrak{T}$  un arbre de Galton-Watson (voir la sous-section (1.2) pour une définition). Le cas  $k = 2$  est très bien documenté. Lorsque que  $\mathfrak{T}$  est sur-critique (la moyenne de la loi de reproduction est strictement plus grande que 1), il est montré par K.B. Athreya [Ath12b] que les sommets  $\mathcal{X}_T^{(1)}$  et  $\mathcal{X}_T^{(2)}$  tirés dans la générations  $T$  partagent un ancêtre commun pour la dernière fois dans l'une des premières générations de l'arbre : si  $\mathcal{M}_T := |\mathcal{X}_T^{(1)} \wedge \mathcal{X}_T^{(2)}|$  désigne la génération du plus récent ancêtre commun de  $\mathcal{X}_T^{(1)}$  et  $\mathcal{X}_T^{(2)}$  alors  $(\mathcal{M}_T)$  converge en loi vers une variable aléatoire réelle positive et explicite.

Lorsque que  $\mathfrak{T}$  est critique (la moyenne de la loi de reproduction est égale à 1), la littérature est plus riche. Il est montré respectivement par A. M. Zubkov [Zub76], R. Durrett [Dur78] et K.B. Athreya [Ath12a] que  $\mathcal{X}_T^{(1)}$  et  $\mathcal{X}_T^{(2)}$  partagent un ancêtre commun pour la dernière fois dans une générations proche de  $T$  :  $(\mathcal{M}_T/T)$  converge en loi vers une variable aléatoire explicite à valeurs dans  $[0, 1]$ .

Mentionnons enfin le cas sous-critique (la moyenne de la loi de reproduction est strictement inférieure à 1), ce dernier étant assez proche du cas critique. En effet, d'après K.B. Athreya [Ath12a],  $(T - \mathcal{M}_T)$  converge en loi vers une variable aléatoire réelle positive et explicite.

Lorsque  $k \geq 2$ , S. Harris, S. Johnston et M. Roberts [HJR17] puis S. Johnston [Joh19] ont décrit de façon exhaustive la généalogie des sommets  $\mathcal{X}_T^{(1)}, \dots, \mathcal{X}_T^{(k)}$  quand  $\mathfrak{T}$  est un arbre de Galton-Watson en temps continu, à  $T$  fixé et quand  $T$  tend vers l'infini dans les régimes sur-critique, critique et sous-critique.

Intéressons nous maintenant au cas de la marche aléatoire en milieu aléatoire diffusive  $\mathbb{X}$ , c'est-à-dire au cas  $\mathfrak{T} = \mathcal{R}_n$ , où nous rappelons que  $\mathcal{R}_T = \{x \in \mathbb{T}; \mathcal{L}_x^T \geq 1\}$  désigne le sous-arbre de  $\mathbb{T}$  des sommets visités par la marche aléatoire  $\mathbb{X}$  jusqu'à l'instant  $T$ .

Soit  $k \in \mathbb{N}^*$ . Pour tout ensemble non vide  $\mathcal{D}$ , on note  $\mathcal{D}^{\times k}$  le  $k$ -produit cartésien  $\mathcal{D} \times \cdots \times \mathcal{D}$ . Pour tout  $k \geq 2$ ,  $\Delta^k := \{\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \mathbb{T}^{\times k}; \forall i \neq j, x^{(i)} \neq x^{(j)}\}$  est l'ensemble des  $k$ -uplets de sommets distincts de  $\mathbb{T}$ . Soit  $\mathcal{D}$  un sous-ensemble de  $\mathbb{T}$  de cardinal  $|\mathcal{D}|$  supérieur ou égal à  $k$ . Posons  $\Delta^k(\mathcal{D}) := \Delta^k \cap \mathcal{D}^{\times k}$  l'ensemble, supposé non vide, des  $k$ -uplets de sommets distincts de  $\mathcal{D}$ . Lorsque  $|\mathcal{D}| < \infty$ , le cardinal  $|\Delta^k(\mathcal{D})|$  de l'ensemble  $\Delta^k(\mathcal{D})$  est égale à  $|\mathcal{D}|(|\mathcal{D}| - 1) \times \cdots \times (|\mathcal{D}| - k + 1)$ .

Soit  $\mathcal{D}_n$  un sous-ensemble de  $\mathcal{R}_n$  de cardinal  $D_n$  supérieur ou égale à  $k$ . On se donne une variable aléatoire  $\mathcal{X}^{(n)} = (\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)})$  à valeurs dans l'ensemble  $\Delta^k$  dont la loi est donnée par : pour tout  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ , si  $\mathbb{P}^*(D_n \geq k) > 0$  alors

$$\mathbb{P}^*(\mathcal{X}^{(n)} = \mathbf{x}) = \frac{1}{\mathbb{P}^*(D_n \geq k)} \mathbb{E}^* \left[ \frac{\mathbb{1}_{\{\mathbf{x} \in \Delta^k(\mathcal{D}_n)\}}}{|\Delta^k(\mathcal{D}_n)|} \mathbb{1}_{\{D_n \geq k\}} \right]. \quad (23)$$

Sinon,  $\mathbb{P}^*(\mathcal{X}^{(n)} = \mathbf{x}) = 0$ . Autrement dit, les sommets  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  de l'arbre  $\mathcal{R}_n$  sont tirés uniformément et sans remise dans  $\mathcal{D}_n$ .

En pratique, nous nous intéressons aux ensembles  $\mathcal{D}_n$  de la forme  $\{x \in \mathcal{R}_n; \ell_n \leq |x| \leq \mathfrak{L}_n\}$  où  $(\mathfrak{L}_n)$  est une suite d'entiers strictement positifs telle que  $\mathfrak{L}_n \rightarrow \infty$  quand  $n \rightarrow \infty$  mais  $\mathfrak{L}_n \leq n^{1/2}$  et  $(\ell_n)$  est une suite d'entiers positifs telle que  $\ell_n \leq \mathfrak{L}_n$  pour tout  $n \in \mathbb{N}^*$  et  $\lim_{n \rightarrow \infty} \mathbb{P}^*(D_n \geq k) = 1$ . Rappelons que les plus grandes générations visitées par la marche aléatoire diffusive  $\mathbb{X}$  sont de l'ordre de  $n^{1/2}$  (voir la sous-section 2.2), d'où le choix  $\mathfrak{L}_n \leq n^{1/2}$ .

L'arbre généalogique de ces  $k$  sommets est très différent suivant si  $\mathfrak{L}_n = o(n^{1/2})$  (il s'agit du cas des petites générations) ou non (c'est-à-dire le cas des générations critiques).

## 4.2 Généalogie de la marche diffusive : les petites générations

Commençons par le cas  $\mathfrak{L}_n = o(n^{1/2})$ , c'est-à-dire le cas des petites générations. Ce dernier se sépare en deux sous-cas : celui des toutes petites générations, c'est-à-dire telles que  $\mathfrak{L}_n \leq (2\tilde{\gamma})^{-1} \log n$  (on rappelle que  $\tilde{\gamma}$  dépend de la log-transformée de Laplace (1) et est défini dans la section 2) et celui des petites générations modérées, c'est-à-dire telles que  $\mathfrak{L}_n > (2\tilde{\gamma})^{-1} \log n$ .

### Les toutes petites générations

Nous évoquons dans la section consacrée à la marche (sous-)diffusive 2.2 le résultat de P. Andreatti et P. Debs [AD14a] affirmant qu'avec grande probabilité, pour tout  $G \in (0, (2\tilde{\gamma})^{-1})$ ,  $\{x \in \mathbb{T}; |x| \leq G \log n\} = \{x \in \mathcal{R}_n; |x| \leq G \log n\}$ . De plus, la valeur  $(2\tilde{\gamma})^{-1}$  est optimale. On en déduit que pour tout  $G \in (0, (2\tilde{\gamma})^{-1})$  et  $\ell \leq G \log n$ ,  $R_n(\ell) = \sum_{|x|=\ell} \mathbb{1}_{\{\mathcal{L}_x^n \geq 1\}} = \sum_{|x|=\ell} 1$ . De plus,  $(e^{-\ell\psi(0)} \sum_{|x|=\ell} 1)_\ell$  est une martingale positive et puisque que l'arbre de Galton-Watson  $\mathbb{T}$

est sur-critique, pour toute suite  $(l_n)$  d'entiers telle que  $l_n \rightarrow \infty$  et  $l_n \leq G \log n$ , en  $\mathbb{P}^*$ -probabilité

$$\frac{R_n(l_n)}{e^{l_n \psi(0)}} \xrightarrow[n \rightarrow \infty]{} M_\infty,$$

où  $M_\infty \in (0, \infty)$   $\mathbf{P}^*$ -presque sûrement dès que  $\mathbf{E}[M_\infty] = 1$  comme évoqué dans la section 1.2.

Lorsque  $k = 2$ , comme l'avons vu dans la sous-section 4.1, K.B. Athreya ([Ath12b], Théorème 2) montre que les sommets  $\mathcal{X}^{(1,n)}$  et  $\mathcal{X}^{(2,n)}$  partagent un ancêtre commun pour la dernière fois dans l'une des premières générations de l'arbre  $\mathbb{T}$ . Plus généralement, il est possible, en adaptant la démarche de S. Johnston ([Joh19], Théorème 3.5) au cas discret, de décrire la généalogie des sommets  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  pour  $k \geq 2$ .

Plaçons nous maintenant dans le cas des petites générations modérées. Il existe un nombre réel  $0 < \delta_0 < 2\tilde{\gamma}$  ( $\delta_0$  est défini explicitement dans le Chapitre II et dépend uniquement de la log-transformée de Laplace  $\psi$  définie en (1)) tel que pour tout  $G \in ((2\tilde{\gamma})^{-1}, \delta_0^{-1})$ , si  $l_n = G \log n$ , alors il nous est impossible de donner avec précision le comportement asymptotique de  $R_n(l_n)$  notamment à cause de la forte dépendance le long des trajectoires de la marche aléatoires  $\mathbb{X}$  dans ce type de petites générations. L'étude de la trace dans ces générations pourra faire partie d'un futur projet de recherche.

### Les petites générations modérées

Supposons à présent que  $\mathcal{L}_n \geq \delta_0^{-1} \log n$  et  $\mathcal{L}_n = o(n^{1/2})$ . En réalité, nous supposons que la suite  $(n^{-1/2} \mathbf{\Lambda}(\mathcal{L}_n))_{n \in \mathbb{N}^*}$  tend vers 0 quand  $n$  tend vers l'infini où pour tout  $t > 0$ ,  $\mathbf{\Lambda}(t) = t \Lambda_{l_0}(t)$  et  $\Lambda_{l_0}$  est une fonction qui croît très lentement (voir l'hypothèse de petites générations 2).

Soit  $k \geq 2$  un entier. Pour tout  $k$ -uplet  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ , nous définissons la plus petite génération  $\mathcal{S}^k(\mathbf{x})$  dans laquelle aucun sommet de  $\mathbf{x}$  ne partage d'ancêtre commun :

$$\mathcal{S}^k(\mathbf{x}) := \min \{m \in \mathbb{N}^*; \forall i \neq j : |x^{(i)} \wedge x^{(j)}| < m\},$$

où on rappelle que  $x \wedge y$  est le plus récent ancêtre commun de  $x$  et  $y$ . Posons  $\mathcal{C}_m^k := \{\mathbf{x} \in \Delta^k; \mathcal{S}(\mathbf{x}) \leq m\}$  avec  $m \in \mathbb{N}^*$ . Considérons la trace tronquée suivante : si  $D_n \geq k$

$$\mathcal{A}^k(\mathcal{D}_n, \mathbf{1}_{\mathcal{C}_m^k}) := \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} \mathbf{1}_{\mathcal{C}_m^k}(\mathbf{x}),$$

sinon  $\mathcal{A}^k(\mathcal{D}_n, \mathbf{1}_{\mathcal{C}_m^k}) = 0$ . Cette trace n'est rien d'autre que le nombre de  $k$ -uplets de sommets de l'ensemble  $\mathcal{D}_n$  ne partageant aucun ancêtre commun à la génération  $m$ . Nous montrons dans un premier temps que ces sommets sont très nombreux : pour tout entier  $k \geq 2$ , si  $\kappa > 4k$  et sous quelques hypothèses d'intégrabilité que nous n'évoquerons pas ici, en  $\mathbb{P}^*$ -probabilité

$$\frac{\mathcal{A}^k(\mathcal{D}_n, \mathbf{1}_{\mathcal{C}_m^k})}{|\Delta^k(\mathcal{D}_n)|} \mathbf{1}_{\{D_n \geq k\}} \xrightarrow[n \rightarrow \infty]{} \frac{\mathcal{A}_\infty^k(\mathbf{1}_{\mathcal{C}_m^k})}{(W_\infty)^k}, \quad (24)$$

où  $\mathcal{A}_\infty^k(\mathbf{1}_{\mathcal{C}_m^k}) := \lim_{l \rightarrow \infty} \sum_{\mathbf{x} \in \Delta_l^k} \mathbf{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq m\}} \prod_{i=1}^k e^{-V(x^{(i)})}$  et  $W_\infty$  est la limite de la martingale additive  $W_l = \sum_{|x|=l} e^{-V(x)}$ . Contrairement au cas frontière **(BC)**,  $\mathbf{P}^*(W_\infty > 0) = 1$  ici. Il est de plus montré par Q. Liu [Liu00] que  $\lim_{t \rightarrow \infty} t^\kappa \mathbf{P}(W_\infty > t)$  existe. Notons que  $\lim_{m \rightarrow \infty} \mathcal{A}_\infty^k(\mathbf{1}_{\mathcal{C}_m^k}) = (W_\infty)^k$  et ainsi, la suite  $(\mathcal{S}^k(\mathcal{X}^{(n)}))$  converge en loi sous  $\mathbb{P}^*$  :

$$\mathbb{P}^*(\mathcal{S}^k(\mathcal{X}^{(n)}) \leq m) = \mathbb{E}^* \left[ \frac{\mathcal{A}^k(\mathcal{D}_n, \mathbf{1}_{\mathcal{C}_m^k})}{|\Delta^k(\mathcal{D}_n)|} \mathbf{1}_{\{D_n \geq k\}} \right] \xrightarrow{n \rightarrow \infty} \mathbf{E}^* \left[ \frac{\mathcal{A}_\infty^k(\mathbf{1}_{\mathcal{C}_m^k})}{(W_\infty)^k} \right]. \quad (25)$$

On déduit de ce résultat que deux sommets ou plus parmi  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  ne peuvent partager un ancêtre commun que dans l'une des premières générations de l'arbre et on obtient ainsi un résultat analogue à celui évoqué dans la sous-section 4.1 lorsque  $\mathfrak{T}$  est un arbre de Galton-Watson sur-critique.

Nous étendons ensuite le résultat (24) à d'autres fonctions en considérant la trace généralisée suivante : pour tout  $n \in \mathbb{N}^*$

$$\mathcal{A}^k(\mathcal{D}_n, f) := \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} f(\mathbf{x}), \quad (26)$$

où  $f : \mathbf{x} \in \Delta^k \rightarrow \mathbb{R}^+$  est une fonction bornée.  $\mathcal{A}^k(\mathcal{D}_n, f)$  n'est rien d'autre que le nombre  $k$ -uplets de sommets distincts du sous ensemble  $\mathcal{D}_n$  de l'arbre  $\mathcal{R}_n$  satisfaisant la contrainte jointe  $f$ . En pratique, nous considérons des contraintes  $f$  vérifiant la propriété suivante : il existe un entier  $\mathfrak{g} \in \mathbb{N}^*$  tel que pour tout entier  $p \geq \mathfrak{g}$  et tout  $\mathbf{x} \in \Delta^k$ , si  $\min_{1 \leq i \leq k} |x^{(i)}| \geq p$  et  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \mathcal{C}_p^k$  alors

$$f(x^{(1)}, \dots, x^{(k)}) = f((x^{(1)})_p, \dots, (x^{(k)})_p). \quad (27)$$

Autrement dit, nous demandons à ce que la contrainte imposée soit héréditaire pour chaque sommet à partir d'une génération donnée.

Nous montrons dans [Kag22] que  $\mathcal{A}^k(\mathcal{D}_n, f)$  est de l'ordre de  $(n^{1/2} \mathbf{L}_n)^k$  où  $\mathbf{L}_n - 1 = \mathfrak{L}_n - \ell_n$  est la hauteur de l'ensemble  $\mathcal{D}_n$  dans l'arbre  $\mathbb{T}$ , c'est-à-dire le diamètre de  $\mathcal{D}_n$  pour la topologie naturelle sur  $\mathbb{T}$ . Pour tout entier  $k \geq 2$ , si  $\kappa > 4k$  et  $f$  satisfait (27) alors en loi, sous  $\mathbb{P}^*$

$$\frac{\mathcal{A}^k(\mathcal{D}_n, f)}{(n^{1/2} \mathbf{L}_n)^k} \xrightarrow{n \rightarrow \infty} \mathcal{A}_\infty^k(f) \left( \frac{c_\infty c_0^{1/2}}{p^\mathfrak{e}(e, e^*) W_\infty} \right)^k |\mathcal{N}|^k, \quad (28)$$

où  $c_\infty := \lim_{l \rightarrow \infty} \mathbf{E}[(\sum_{i=1}^l e^{-S_j})^{-1}]$ ,  $c_0 := \mathbf{E}[\sum_{x \neq y; |x|=|y|=1} e^{-V(x)-V(y)}] / (1 - e^{\psi(2)})$ , et  $\mathcal{N}$  suit, sous  $\mathbb{P}^\mathfrak{e}$ , une loi normale centrée réduite. De plus,

$$\mathcal{A}_\infty^k(f) := \lim_{l \rightarrow \infty} \sum_{\mathbf{x} \in \Delta_l^k} f(\mathbf{x}) \prod_{i=1}^k e^{-V(x^{(i)})}. \quad (29)$$



La convergence dans (29) a lieu  $\mathbf{P}^*$ -presque sûrement et dans  $L^2(\mathbf{P}^*)$ .

Nous nous apercevons également que la contrainte n'a pas d'influence sur la normalisation de la trace : si  $g \neq 0$  et satisfait (27) alors en  $\mathbb{P}^*$ -probabilité

$$\frac{\mathcal{A}^k(\mathcal{D}_n, f)}{\mathcal{A}^k(\mathcal{D}_n, g)} \mathbb{1}_{\{D_n \geq k\}} \xrightarrow{n \rightarrow \infty} \frac{\mathcal{A}_\infty^k(f)}{\mathcal{A}_\infty^k(g)}.$$

Autrement dit, quelque soit la contrainte  $f$  imposée au sommet de  $\mathcal{D}_n$ , la trace  $\mathcal{A}^k(\mathcal{D}_n, f)$  se comporte comme  $(\mathbf{L}_n \max_{x \in \mathcal{R}_n} |x|)^k$  et c'est la valeur limite  $\mathcal{A}_\infty^k(f)$  qui contient toute l'information sur les interactions entre les sommets de l'arbre.

L'hypothèse d'hérédité (27) sur la fonction  $f$  est en réalité assez souple et permet d'étudier le comportement asymptotique d'un vaste nombre de traces différentes. On caractérise l'arbre généalogique des  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  de la façon suivante : pour tout  $m \leq \mathfrak{L}_n$  et tout  $i_1, i_2 \in \{1, \dots, k\}$ ,  $i_1 \sim_m i_2$  si et seulement si  $\mathcal{X}^{(i_1,n)}$  et  $\mathcal{X}^{(i_2,n)}$  partagent un ancêtre commun dans la génération  $m$ .  $\sim_m$  est une relation d'équivalence sur  $\{1, \dots, k\}$  et on définit ainsi le processus  $\pi^{k,n} = (\pi_m^{k,n})_{m \leq \mathfrak{L}_n}$  à valeurs dans l'ensemble des partitions de  $\{1, \dots, k\}$  tel que les blocs de  $\pi_m^{k,n}$  sont les classes d'équivalence de  $\sim_m$ .  $\pi^{k,n}$  est appelé arbre généalogique de  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$ . Soit  $\mathcal{G}^{(i,n)} = |\mathcal{X}^{(i,n)}|$  la génération du sommet  $\mathcal{X}^{(i,n)}$ . Par définition,

$$\pi_0^{k,n} = \{\{1, \dots, k\}\} \text{ and } \pi_m^{k,n} = \{\{1\}, \dots, \{k\}\} \text{ pour tout } m \in \{\max_{1 \leq i \leq k} \mathcal{G}^{(i,n)}, \dots, \mathfrak{L}_n\}.$$

En choisissant convenablement notre fonction  $f$ , nous sommes en mesure de donner, sous la probabilité annealed  $\mathbb{P}^*$ , la loi asymptotique de l'arbre généalogique des sommets  $\mathcal{X}^{(n,1)}, \dots, \mathcal{X}^{(n,k)}$ . En particulier, on observe la présence d'un caractère héréditaire dans cet arbre généalogique dû à l'environnement aléatoire, caractère absent dans le cas des toutes petites générations. Nous développons cette idée dans le chapitre II.

Nous donnons maintenant quelques idées de preuve permettant d'aboutir à (28). Commençons par le découpage suivant :

$$\mathcal{A}^k(\mathcal{D}_n, f) = \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} f(\mathbf{x}) \mathbb{1}_{\{\mathcal{S}^k(\mathbf{x}) > a_n\}} + \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} f(\mathbf{x}) \mathbb{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq a_n\}},$$

avec  $a_n := (2\delta_0)^{-1} \log n$ . On s'aperçoit que les  $k$ -uplets de  $\mathcal{D}_n$  dont au moins deux sommets partagent un ancêtre commun dans une génération supérieure à  $a_n$  ne contribuent pas à la trace :

$$\mathbb{E}^* \left[ \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} \mathbb{1}_{\{\mathcal{S}^k(\mathbf{x}) > a_n\}} \right] = o((n^{1/2} \mathbf{L}_n)^k).$$

Nous étudions ensuite  $\mathcal{A}^k(\mathcal{D}_n, f \mathbb{1}_{\mathcal{E}_{a_n}^k}) = \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} f(\mathbf{x}) \mathbb{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq a_n\}}$ . Pour cela, nous considérons en premier lieu cette trace sur l'arbre  $\mathcal{R}_T^s$  pour tout entier  $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$ , avec

$\varepsilon_1 \in (0, 1)$  :

$$\mathcal{A}_n^k(\mathcal{R}_{T^s}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) := \sum_{m_1, \dots, m_k = \ell_n}^{\mathfrak{L}_n} \sum_{\mathbf{x} \in \Delta^k(\mathcal{R}_{T^s})} f(\mathbf{x}) \mathbf{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq a_n, |x^{(1)}|=m_1, \dots, |x^{(k)}|=m_k\}},$$

Nous prouvons qu'en  $\mathbb{P}^*$ -probabilité

$$\frac{1}{s^k} \sum_{\mathbf{x} \in \Delta^k(\mathcal{R}_{T^s})} f(\mathbf{x}) \mathbf{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq a_n, |x^{(1)}|=m_1, \dots, |x^{(k)}|=m_k\}} \xrightarrow{n \rightarrow \infty} (c_\infty)^k \mathcal{A}_\infty^k(f), \quad (30)$$

uniformément en  $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$  et en  $m_1, \dots, m_k \in \{\ell_n, \dots, \mathfrak{L}_n\}$ . Il s'agit de montrer que la somme ci-dessus se concentre autour de sa moyenne quenched. Notons que pour tout sommet  $x \in \mathbb{T}$  tel que  $|x| \geq \delta_0^{-1} \log n$ , on a  $V(x) \geq 3 \log n$  avec grande probabilité. De ce fait, on déduit que tous les sommets du  $k$ -uplet  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in (\mathcal{R}_{T^s})^{\times k}$  sont en réalité visités durant une unique excursion au dessus de la racine et que seuls les  $k$ -uplets dont les sommets sont visités durant des excursions distinctes contribuent à la trace. On obtient ainsi l'indépendance le long des trajectoires de la marche aléatoire  $\mathbb{X}$  indispensable à la convergence de la trace vers sa moyenne quenched. Quant à la convergence de  $\mathbb{E}^\varepsilon[\frac{1}{s^k} \sum_{\mathbf{x} \in \Delta^k(\mathcal{R}_{T^s})} f(\mathbf{x}) \mathbf{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq a_n, |x^{(1)}|=m_1, \dots, |x^{(k)}|=m_k\}}]$  vers  $(c_\infty)^k \mathcal{A}_\infty^k(f)$ , elle est assurée par l'événement  $\{\mathcal{S}^k(\mathbf{x}) \leq a_n\}$ , qui ne fait rien d'autre que d'empêcher les lignes généalogiques des sommets  $x^{(1)}, \dots, x^{(k)}$  de se croiser entre les générations  $\min_{1 \leq i \leq k} m_i$  et  $\max_{1 \leq i \leq k} m_i$ , inhibition qui est transmise de génération en génération à travers  $f$  grâce à l'hypothèse d'hérédité (27).

Pour faire le lien avec la trace sur  $\mathcal{R}_n$ , il suffit d'appliquer (30) avec  $s = \mathcal{L}_e^n$  puis nous concluons en utilisant un résultat de Y. Hu ([Hu17], Corollaire 1.2) et le fait que  $\mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k})$  et  $\mathcal{A}_n^k(\mathcal{R}_{T^{\mathcal{L}_e^n}}, f \mathbf{1}_{\mathcal{C}_{a_n}^k})$  sont en réalité très proches dans le sens suivant : en  $\mathbb{P}^*$ -probabilité

$$\frac{1}{(\mathcal{L}_e^n \mathbf{L}_n)^k} \left( \mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) - \mathcal{A}_n^k(\mathcal{R}_{T^{\mathcal{L}_e^n}}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) \right) \xrightarrow{n \rightarrow \infty} 0.$$

Nous achevons cette section en évoquant brièvement le cas des générations critiques.

### 4.3 Généalogie de la marche diffusive : les générations critiques

Supposons sans perte de généralité que  $\ell_n = \mathfrak{L}_n = n^{1/2}$  et  $k = 2$ . La généalogie des sommets  $\mathcal{X}^{(1,n)}$  et  $\mathcal{X}^{(2,n)}$  tirés dans la génération  $\ell_n$  selon la loi définie dans (23) est différente dans le cas des générations critiques. Commençons par énoncer un résultat dû à E. Aïdékon et L. de Raphélis ([AdR17], Théorème 6.1) : en loi (sous  $\mathbb{P}^*$  et  $\mathbf{P}^*$ -presque sûrement sous  $\mathbb{P}^\varepsilon$ ), pour la topologie de Gromov-Hausdorff

$$\frac{1}{(\tilde{\sigma}^2 n)^{1/2}} \mathcal{R}_n \xrightarrow{n \rightarrow \infty} \mathcal{T}_{|B|}, \quad (31)$$

avec  $\tilde{\sigma}^2 := (1 - \psi(2))/(c_0(1 - e^{\psi(2)}))$ ,  $\mathcal{R}_n/(\tilde{\sigma}^2 n^{1/2})$  n'étant rien d'autre que l'arbre  $\mathcal{R}_n$  dont les branches sont de longueur  $1/(\tilde{\sigma}^2 n^{1/2})$  et  $\mathcal{T}_{|B|}$  est l'arbre réel codé par le mouvement brownien réfléchi standard  $|B| := (|B_t|)_{t \in [0,1]}$  sur l'intervalle  $[0, 1]$ . On pourra consulter l'article de J.-F. Le Gall [Gal06] pour plus d'information sur les arbres réels.

La définition même de  $|B|$  suggère que  $\mathcal{X}^{(1,n)}$  et  $\mathcal{X}^{(2,n)}$  (deux sommets tirés dans l'ensemble  $\{x \in \mathcal{R}_{T_n^{1/2}}; |x| = \ell_n\}$  selon la loi définie dans (23)) peuvent partager un ancêtre commun dans une génération proche de  $\ell_n$  mais aussi dans une génération proche de la racine  $e$ . En effet, nous montrons, en notant  $\tilde{\mathcal{X}}^{(n)} = (\mathcal{X}^{(1,n)}, \mathcal{X}^{(2,n)})$  que

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}^*(\varepsilon \ell_n < \mathcal{S}^k(\tilde{\mathcal{X}}^{(n)}) \leq \ell_n) > 0,$$

mais aussi

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}^*(\mathcal{S}^k(\tilde{\mathcal{X}}^{(n)}) \leq 1/\varepsilon) > 0.$$

De plus, on observe que ces deux scénarios sont les seuls possibles :

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^*(1/\varepsilon < \mathcal{S}^k(\tilde{\mathcal{X}}^{(n)}) \leq \varepsilon \ell_n) = 0.$$

En revanche, les techniques classiques notamment utilisées dans le cas des petites générations ne permettent ni de calculer  $\lim_{n \rightarrow \infty} \mathbb{P}^*(a \ell_n \leq \mathcal{S}^k(\tilde{\mathcal{X}}^{(n)}) \leq b \ell_n)$  avec  $0 < a < b \leq 1$  ni  $\lim_{n \rightarrow \infty} \mathbb{P}^*(\mathcal{S}^k(\tilde{\mathcal{X}}^{(n)}) \leq m)$  pour  $m \in \mathbb{N}^*$ .

Une possibilité est d'exprimer le quotient  $\mathcal{A}^2(\mathcal{D}_n, \mathbb{1}_{\{a \ell_n < \mathcal{S}^k(\cdot) \leq b \ell_n\}})/\mathcal{A}^2(\mathcal{D}_n, 1)$  et le quotient  $\mathcal{A}^2(\mathcal{D}_n, \mathbb{1}_{\{\mathcal{S}^k(\cdot) \leq m\}})/\mathcal{A}^2(\mathcal{D}_n, 1)$  comme une fonctionnelle d'un processus codant l'arbre  $\mathcal{R}_{T_n}$  dont on connaîtrait, en loi, le comportement asymptotique. Il s'agit d'un travail en cours avec Pierre Andreatti et Loïc de Raphélis.

Qu'en est-il de la généalogie dans le cas de la marche sous-diffusive, c'est-à-dire lorsque  $\kappa \in (1, 2]$ ? L. de Raphélis montre ([dR22], Théorème 1) une convergence du type (31) : il existe une constante  $c_\kappa^* > 0$  telle que si  $\kappa = 2$ ,  $(\mathcal{R}_n(\log n)^{1/2}/(c_\kappa^* n^{1/2}))$  converge en loi vers  $\mathcal{T}_{|B|}$  et si  $\kappa \in (1, 2)$ ,  $(\mathcal{R}_n/(c_\kappa^* n^{1-1/\kappa}))$  converge en loi vers  $\mathcal{T}_{H_t \in [0,1]}$  où  $(H_t)_{t \in \mathbb{R}^+}$  est le processus des hauteurs d'un processus de Lévy stable d'indice  $\kappa$  (voir [DLG05] pour la notion de processus des hauteurs). En particulier,  $\mathcal{T}_{H_t \in [0,1]}$  code (presque) la généalogie d'un processus de branchement à espace d'états continu de mécanisme de branchement  $\lambda \mapsto \lambda^\kappa$  et on peut donc espérer une réponse intéressante à notre question. Nous tenterons d'y répondre dans un futur projet de recherche.

## 5 Annexe

Dans cette section, nous proposons une preuve des résultats (9) et (10). On rappelle que le premier résultat, qui concerne la marche (sous-)diffusive, implique qu'avec grande probabilité,

$\max_{1 \leq j \leq n} V(X_j)$  et  $V(X_n)$  se comportent comme les générations typiques de la marche  $\mathbb{X}$ , c'est-à-dire en  $n^{1-\frac{1}{\kappa\wedge 2}}$ . Le deuxième résultat, qui concerne cette fois la marche lente, implique que  $V(X_n)$  est  $\log n$  fois plus petit que les générations typiques de la marche  $\mathbb{X}$ , c'est-à-dire de l'ordre de  $\log n$ .

### 5.1 Preuve de (9)

Commençons par la borne inférieure de  $V(X_n)$ . Pour tout  $t > 0$  et  $\varepsilon \in (0, 1 - (\kappa \wedge 2)^{-1})$

$$\begin{aligned} \mathbb{P}^\mathcal{E}(V(X_n) < tn^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}) &\leq \mathbb{P}^\mathcal{E}(V(X_n) < tn^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}, n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon} \leq |X_n| \leq n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}) \\ &\quad + \mathbb{P}^\mathcal{E}(|X_n| < n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}) + \mathbb{P}^\mathcal{E}(|X_n| > n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}). \end{aligned}$$

De ([AdR17], Théorème 6.1) et ([dR22], Théorème 1), on déduit que la somme de probabilités  $\mathbb{P}^*(|X_n| < n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}) + \mathbb{P}^*(|X_n| > n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}) \rightarrow 0$  quand  $n \rightarrow \infty$ . Par ailleurs

$$\mathbb{P}^\mathcal{E}(V(X_n) < tn^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}, n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon} \leq |X_n| \leq n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}) \leq \sum_{|x|=n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}}^{n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}} \mathbb{E}^\mathcal{E}[\mathcal{L}_x^n \mathbf{1}_{\{V(x) < tn^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}\}}],$$

où nous avons utilisé le fait que  $\mathbf{1}_{\{X_n=x\}} \leq \sum_{j=1}^n \mathbf{1}_{\{X_j=x\}} = \mathcal{L}_x^n$  et puisque  $n \leq T^n$ ,  $\mathbb{E}^\mathcal{E}[\mathcal{L}_x^n] \leq \mathbb{E}^\mathcal{E}[\mathcal{L}_x^{T^n}] = ne^{-V(x)} + n \sum_{y^*; y=x} e^{-V(y)}$ . Nous rappelons que  $T^n$  est le  $n$ -ème instant de passage en  $e^*$ . On en déduit que la probabilité ci-dessus est plus petite que

$$\begin{aligned} 2\mathbf{E}^* \left[ \sum_{|x|=n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}}^{n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}} e^{-V(x)} \mathbf{1}_{\{V(x) < tn^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}\}} \right] &\leq 2\mathbf{E}^* \left[ \sum_{|x|=n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}}^{n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}} e^{-(1+r_0)V(x)+r_0tn^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}} \right] \\ &\leq 2n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon} e^{-n^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}(|\psi(1+r_0)|-tr_0)}, \end{aligned}$$

où on a choisi  $r_0 > 0$  tel que  $\psi(1+r_0) \in (-\infty, 0)$ , ce qui est possible car  $\psi$  est bien définie dans un voisinage de 1 et  $\psi'(1) < 0$ . On voit donc que pour tout  $t < |\psi(1+r_0)|/r_0$ ,  $\mathbb{P}^*(V(X_n) < tn^{1-\frac{1}{\kappa\wedge 2}-\varepsilon}) \rightarrow 0$  quand  $n \rightarrow \infty$  donnant ainsi la borne inférieure pour  $V(X_n)$  et  $\max_{1 \leq i \leq n} V(X_j)$ . Pour la borne supérieure, on procède de la même façon : pour tout  $\varepsilon, s > 0$ , la probabilité  $\mathbb{P}^\mathcal{E}(\max_{1 \leq i \leq n} V(X_j) > sn^{1-\frac{1}{\kappa\wedge 2}+\varepsilon})$  est plus petite que

$$\begin{aligned} \sum_{|x| \leq n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}} \mathbb{E}^\mathcal{E}[\mathcal{L}_x^n \mathbf{1}_{\{n \leq T^n \frac{1}{\kappa\wedge 2} + \varepsilon'\}} \mathbf{1}_{\{V(x) > sn^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}\}}] &+ \mathbb{P}^\mathcal{E} \left( \max_{j \leq T^n \frac{1}{\kappa\wedge 2} + \varepsilon} |X_j| > n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon} \right) \\ &+ \mathbb{P}^\mathcal{E}(n > T^n \frac{1}{\kappa\wedge 2} + \varepsilon'), \end{aligned}$$

avec  $\varepsilon' = \frac{\varepsilon}{2\kappa}$  et  $\tilde{\varepsilon} = \frac{\varepsilon}{2(\kappa\wedge 2)}$  et d'après ([AD20], Lemme 3.2), [AdR17], [dR22], la somme de probabilités  $\mathbb{P}^*(\max_{j \leq T^n \frac{1}{\kappa\wedge 2} + \varepsilon} |X_j| > n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}) + \mathbb{P}^*(n > T^n \frac{1}{\kappa\wedge 2} + \varepsilon')$  tend vers 0 quand  $n \rightarrow \infty$ .

Enfin, l'espérance annealed de la somme ci-dessus est inférieure à

$$\begin{aligned} & 2n^{\frac{1}{\kappa\wedge 2}+\varepsilon'} \mathbf{E} \left[ \sum_{|x| \leq n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}} e^{-V(x)} \mathbf{1}_{\{V(x) > sn^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}\}} \right] \\ & \leq 2n^{\frac{1}{\kappa\wedge 2}+\varepsilon'} \mathbf{E} \left[ \sum_{|x| \leq n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}} e^{-r_1 V(x) - s(1-r_1)n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}} \right] \leq 2n^{n+\varepsilon+\varepsilon'} e^{n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon} \psi(r_1) - s(1-r_1)n^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}}, \end{aligned}$$

où on a choisi  $r_1 \in (0, 1)$  tel que  $\psi(r_1) \in [0, \infty)$ , ce qui est possible car  $\psi$  est bien définie dans un voisinage de  $[0, 1]$ . On voit donc que pour tout  $s > \psi(r_1)/(1-r_1)$ ,  $\mathbf{P}^*(\max_{1 \leq j \leq n} V(X_j) > sn^{1-\frac{1}{\kappa\wedge 2}+\varepsilon}) \rightarrow 0$  quand  $n \rightarrow \infty$  donnant ainsi la borne supérieure pour  $\max_{1 \leq i \leq n} V(X_j)$  et  $V(X_n)$ .

## 5.2 Preuve de (10)

Nous faisons l'hypothèse raisonnable suivante : pour tout  $A, d > 0$ ,  $\mathbf{P}^*(\min_{|z| \leq A(\log n)^3} (V(z) - V(z^*)) > -d(\log n)) \rightarrow 1$  quand  $n \rightarrow \infty$ .

Soit  $a, A > 0$  deux réels. Regardons d'abord la probabilité  $\mathbb{P}^\mathcal{E}(V(X_n) > a \log n, X_n \in \mathcal{O}_n, |X_n| \leq A(\log n)^3)$  où nous rappelons que  $x \in \mathcal{O}_\lambda$  si et seulement si  $H_z = \sum_{e < w \leq z} e^{V(w)-V(z)} \leq \lambda$  pour tout  $e < z \leq x$ . Nous avons

$$\mathbb{P}^\mathcal{E}(V(X_n) > a \log n, X_n \in \mathcal{O}_n, |X_n| \leq A(\log n)^3) = \sum_{|x| \leq A(\log n)^3} \mathbb{P}^\mathcal{E}(X_n = x) \mathbf{1}_{\{V(x) > a \log n, x \in \mathcal{O}_n\}},$$

et d'après ([HS16b], Théorème 2.1), en  $\mathbf{P}^*$ -probabilité

$$\sum_{x \in \mathcal{O}_n} \left| \mathbb{P}^\mathcal{E}(X_n = x) - \frac{2}{\mathcal{Z}_n} (e^{-V(x)} + \sum_{y: y^* = x} e^{-V(y)}) \right| \xrightarrow{n \rightarrow \infty} 0$$

où  $\mathcal{Z}_n := \tilde{W}_n + 2 \sum_{k \geq 1} W_k$  avec  $\tilde{W}_n \rightarrow 1$  en  $\mathbf{P}^*$ -probabilité (voir [HS16b], Lemme 3.4). Soit  $B > 0$  un réel. On déduit de la convergence vers 0 ci-dessus et de l'égalité précédente qu'en  $\mathbf{P}^*$ -probabilité

$$\mathbf{1}_{\mathcal{D}_{a,n}} \left( \mathbb{P}^\mathcal{E}(V(X_n) > a \log n, X_n \in \mathcal{O}_n, |X_n| \leq A(\log n)^3) - \frac{2}{\mathcal{Z}_n} \sum_{k \leq A(\log n)^3} (M_{k,a}^n + \tilde{M}_{k,a}^n) \right) \xrightarrow{n \rightarrow \infty} 0$$

où  $\mathcal{D}_{a,n} := \{ \inf_{z \in \mathbb{T}} V(z) \geq -B, \min_{|z| \leq A(\log n)^3} (V(z) - V(z^*)) > -(a/2)(\log n) \}$ ,

$$M_{k,a}^n := \sum_{|u|=k} e^{-V(x)} \mathbf{1}_{\{V(u) > a \log n, x \in \mathcal{O}_n, \underline{V}(u) \geq -B\}},$$

et

$$\tilde{M}_{k,a}^n := \sum_{|u|=k} \mathbf{1}_{\{V(u) > a \log n, x \in \mathcal{O}_n, \underline{V}(u) \geq -B\}} \sum_{x: x^* = u} e^{-V(x)} \mathbf{1}_{\{V_u(x) \geq -(a/2) \log n\}}.$$

L'étape suivantes est de montrer que ces deux variables aléatoires sont très proches lorsque  $n \rightarrow \infty$  : il existe  $\gamma_a > 0$  tel que pour tout  $\varepsilon > 0$  et  $n$  assez grand

$$\mathbf{P}\left(\left|\sum_{k \leq A(\log n)^3} M_{k,a}^n - \sum_{k \leq A(\log n)^3} \tilde{M}_{k,a}^n\right| > \varepsilon\right) \leq n^{-\gamma_a}.$$

En effet, d'après l'inégalité de Markov et en remarquant que  $c_{a,n} M_{k,a}^n = \mathbf{E}[\tilde{M}_{k,a}^n | \mathcal{F}_k]$  avec  $c_{a,n} := \mathbf{E}[\sum_{|x|=1} e^{-V(x)} \mathbf{1}_{\{V(x) \geq -(a/2) \log n\}}] \in (0, 1]$  où nous rappelons que  $\mathcal{F}_k = \sigma((\mathbb{T}, V(x); |x| \leq k))$ , la probabilité ci-dessus est inférieure à

$$\frac{1}{\varepsilon} \sum_{k \leq A(\log n)^3} \mathbf{E}[(M_{k,a}^n - \tilde{M}_{k,a}^n)^2]^{1/2} = \frac{1}{\varepsilon} \sum_{k \leq A(\log n)^3} (\mathbf{E}[(\tilde{M}_{k,a}^n)^2] - (2c_{a,n} - 1)\mathbf{E}[(M_{k,a}^n)^2])^{1/2}.$$

De plus,  $(\tilde{M}_{k,a}^n)^2$  est inférieur à

$$\begin{aligned} & \sum_{\substack{u \neq v \\ |u|=|v|=k}} \mathbf{1}_{\{V(u) \wedge V(v) > a \log n, u, v \in \mathcal{O}_n, \underline{V}(u) \wedge \underline{V}(v) \geq -B\}} \sum_{x; x^*=u} e^{-V(x)} \mathbf{1}_{\{V_u(x) \geq -(a/2) \log n\}} \\ & \times \sum_{y; y^*=v} e^{-V(y)} \mathbf{1}_{\{V_v(y) \geq -(a/2) \log n\}} + \sum_{|u|=k} \mathbf{1}_{\{V(u) > a \log n, u \in \mathcal{O}_n, \underline{V}(u) \geq -B\}} \left( \sum_{\substack{x \neq y \\ x^*=y^*=u}} e^{-V(x)-V(y)} \right. \\ & \left. + \sum_{x; x^*=u} e^{-2V(x)} \mathbf{1}_{\{V_u(x) \geq -(a/2) \log n\}} \right), \end{aligned}$$

d'où

$$\begin{aligned} \mathbf{E}[(\tilde{M}_{k,a}^n)^2] & \leq \mathbf{E}[(c_{a,n} M_{k,a}^n)^2] + (c'_0 + n^{a/2}) \mathbf{E}\left[\sum_{|u|=k} e^{-2V(u)} \mathbf{1}_{\{V(u) > a \log n\}}\right] \\ & \leq \mathbf{E}[(c_{a,n} M_{k,a}^n)^2] + n^{-a}(c'_0 + n^{a/2}), \end{aligned}$$

où  $c'_0 := \mathbf{E}[\sum_{x \neq y; |x|=|y|=1} e^{-V(x)-V(y)}]$  et donc pour  $n$  assez grand

$$\sum_{k \leq A(\log n)^3} \mathbf{E}[(M_{k,a}^n - \tilde{M}_{k,a}^n)^2]^{1/2} \leq A(\log n)^3 ((1 - c_{a,n})^2 \sup_{k \leq A(\log n)^3} \mathbf{E}[(M_{k,a}^n)^2] + 2n^{-a/2})^{1/2}.$$

Notons d'une part que  $\sup_{k \leq A(\log n)^3} \mathbf{E}[(M_{k,a}^n)^2]/(\log n)^3 < \infty$  et d'autre part

$$1 - c_{a,n} = \mathbf{E}\left[\sum_{|x|=1} e^{-V(x)} \mathbf{1}_{\{V(x) < -(a/2) \log n\}}\right] \leq n^{-a(1-r_1)/2} e^{\psi(r_1)},$$

où  $r_1 \in (0, 1)$  est choisi tel que  $\psi(r_1) \in [0, \infty)$  ce qui prouve l'existence de  $\gamma_a$ . Ainsi en  $\mathbf{P}^*$ -probabilité

$$\mathbf{1}_{\mathcal{D}_{a,n}} \left( \mathbb{P}^{\mathcal{E}}(V(X_n) > a \log n, X_n \in \mathcal{O}_n, |X_n| \leq A(\log n)^3) - \frac{4}{\mathcal{Z}_n} \sum_{k \leq A(\log n)^3} M_{k,a}^n \right) \xrightarrow{n \rightarrow \infty} 0.$$

La suite de la preuve reprend le raisonnement de Y. Hu et Z. Shi ([HS16b], Corollaire 2.3) et un théorème de convergence fonctionnelle dû à T. Madaule ([Mad12], Théorème 1.2) : en  $\mathbf{P}^*$ -probabilité,  $4 \sum_{k \geq 1} M_{k,a}^n / \mathcal{Z}_n \rightarrow 1 - (2/\pi)^{1/2} \mathbf{E}[\min(\mathfrak{s}^{-1}, a\mathfrak{m}_1^{-1})]$  où  $(\mathfrak{m}_t)_{t \in [0,1]}$  est un méandre brownien et  $\mathfrak{s} := \sup_{t \in [0,1]} (\sup_{s \in [0,t]} \mathfrak{m}_s - \mathfrak{m}_t)$ . Notons que

$$\left(\frac{2}{\pi}\right)^{1/2} \mathbf{E}[\min(\mathfrak{s}^{-1}, a\mathfrak{m}_1^{-1})] = \int_0^\infty \frac{du}{(2\pi u)^{1/2}} \mathbf{P}(\mathfrak{s} < u^{-1/2}, u^{1/2}\mathfrak{m}_1 \leq a) = \mu_V((0, a]).$$

On conclut en utilisant le fait (voir (2) et (8)) que pour tout  $\varepsilon > 0$ , pour  $n$  assez grand,  $\mathbb{P}^*(X_n \notin \mathcal{O}_n) + \mathbb{P}^*(|X_n| > A_\varepsilon(\log n)^3) \leq \varepsilon/2$ . De plus, nous savons que  $\mathbf{P}^*$ -presque sûrement,  $\inf_{|z|=n} V(z) \rightarrow +\infty$  quand  $n \rightarrow \infty$  d'où l'existence d'un réel  $B_\varepsilon > 0$  tel que  $\mathbf{P}^*(\inf_{z \in \mathbb{T}} V(x) < -B_\varepsilon) \leq \varepsilon/4$  et donc, par hypothèse, pour  $n$  assez grand,  $1 - \mathbf{P}^*(\mathcal{D}_{a,n}) \leq \varepsilon/2$ , ce qui donne le résultat voulu.





# Chapitre I

## Generalized range for slow random walks on trees

**Abstract :** We are interested in the set of visited vertices of a tree  $\mathbb{T}$  by a randomly biased random walk  $\mathbb{X} := (X_n, n \in \mathbb{N})$ . The aim is to study a generalized range, that is to say the volume of the trace of  $\mathbb{X}$  with both constraints on the trajectories of  $\mathbb{X}$  and on the trajectories of the underlying branching random potential  $\mathbb{V} := (V(x), x \in \mathbb{T})$ . Focusing on slow regime's random walks (see [HS16b], [AC18]), we prove a general result and detail examples. These examples exhibit many different behaviors for a wide variety of ranges, showing the interactions between the trajectories of  $\mathbb{X}$  and the ones of  $\mathbb{V}$ . This chapter is based on the article [AK21] written with Pierre Andreatti.

### Contents

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<b>1</b>	<b>Introduction</b>	<b>40</b>
1.1	First results : examples	42
1.2	A general result (informal statement)	45
1.3	A general result (full statement)	50
<b>2</b>	<b>Proof of the theorems</b>	<b>54</b>
2.1	Preliminary material	54
2.2	Proofs of Theorems 1.1 to 1.4	55
2.3	Proof of Theorem 1.5	74
<b>3</b>	<b>Proof of Proposition 1</b>	<b>77</b>
3.1	Preliminary	77
3.2	Lower bound for $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$	79
3.3	Upper bound for $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$	89
<b>4</b>	<b>Technical estimates for one-dimensional random walk</b>	<b>94</b>
4.1	Two Laplace transforms	94
4.2	Additional technical estimates	100

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## 1 Introduction

The construction of the process we are interested in starts with a supercritical Galton-Watson tree  $\mathbb{T}$  with offspring distributed as a random variable  $\nu$  such that  $\mathbb{E}[\nu] > 1$ . We adopt the following usual notations for tree-related quantities : the root of  $\mathbb{T}$  is denoted by  $e$ , for any  $x \in \mathbb{T}$ ,  $\nu_x$  denotes the number of descendants of  $x$ , the parent of a vertex  $x$  is denoted by  $x^*$  and its children by  $\{x^i, 1 \leq i \leq \nu_x\}$ . For technical reasons, we add to the root  $e$ , a parent  $e^*$  which is not considered as a vertex of the tree. We denote by  $|x|$  the generation of  $x$ , that is the length of the path from  $e$  to  $x$  and we write  $x < y$  when  $y$  is a descendant of  $x$ , also  $x \leq y$  signifying that  $x$  can also be equal to  $y$ . Finally, we write  $\mathbb{T}_n$  for the tree truncated at generation  $n$ . We then introduce a real-valued branching random walk indexed by  $\mathbb{T} : (V(x), x \in \mathbb{T})$ . We suppose that  $V(e) = 0$  and for any generation  $n$ , conditionally to  $\mathcal{E}_n = \{\mathbb{T}_n, (V(x), x \in \mathbb{T}_n)\}$ , the vectors of increments  $((V(x^i) - V(x), i \leq \nu_x), |x| = n)$  are assumed to be i.i.d. Finally, we denote by  $\mathbf{P}$  the distribution of  $\mathcal{E} = \{\mathbb{T}, (V(x), x \in \mathbb{T})\}$  and  $\mathbf{P}^*$ , the probability conditioned on the survival set of the tree  $\mathbb{T}$ .

We can now introduce the main process of this work which is a random walk  $(X_n)_{n \in \mathbb{N}}$  on  $\mathbb{T} \cup \{e^*\}$  : for a given realization of the environment  $\mathcal{E}$ ,  $(X_n)_{n \in \mathbb{N}}$  is a Markov chain with transition probabilities given by

$$\begin{aligned} \mathbb{P}^{\mathcal{E}}(X_{n+1} = e | X_n = e^*) &= 1, \\ \forall x \in \mathbb{T} \setminus \{e^*\}, \quad \mathbb{P}^{\mathcal{E}}(X_{n+1} = x^* | X_n = x) &= \frac{e^{-V(x)}}{e^{-V(x)} + \sum_{i=1}^{\nu_x} e^{-V(x^i)}}, \\ \forall j \leq \nu_x, \quad \mathbb{P}^{\mathcal{E}}(X_{n+1} = x^j | X_n = x) &= \frac{e^{-V(x^j)}}{e^{-V(x)} + \sum_{i=1}^{\nu_x} e^{-V(x^i)}}. \end{aligned}$$

The measure  $\mathbb{P}^{\mathcal{E}}$  is usually referred to as the quenched distribution of the walk  $(X_n)_{n \in \mathbb{N}}$  in contrast to the annealed distribution  $\mathbb{P}$  which is the measure  $\mathbb{P}^{\mathcal{E}}$  integrated with respect to the law of  $\mathcal{E}$  :

$$\mathbb{P}(\cdot) = \int \mathbb{P}^{\mathcal{E}}(\cdot) \mathbf{P}(d\mathcal{E}).$$

Similarly,  $\mathbb{P}^*$  is the annealed probability conditioned on the survival set of the tree  $\mathbb{T}$  (defined by replacing  $\mathbf{P}$  by  $\mathbf{P}^*$  in the above probability). For  $x \in \mathbb{T} \cup \{e^*\}$ , we use the notation  $\mathbb{P}_x^{\mathcal{E}}$  for the conditional probability  $\mathbb{P}^{\mathcal{E}}(\cdot | X_0 = x)$ ; when there is no subscript, the walk is supposed to start at the root  $e$ . Recurrent criteria for these walks is determined from the fluctuations of log-Laplace transform

$$\psi(s) := \log \mathbf{E} \left[ \sum_{|z|=1} e^{-sV(z)} \right], s > 0. \quad (\text{I.1})$$

If  $\inf_{0 \leq s \leq 1} \psi(s) > 0$  then  $(X_n, n)$  is  $\mathbf{P}$  almost surely transient and recurrent otherwise. It turns out that recurrent cases can be themselves classified, this can be found in the works of G. Faraud

[Far11] and equivalently for transient cases in E. Aidekon [Aid08].

Here we consider recurrent cases and more particularly the regime where the random walk is particularly slow (see [HS16b]), that is to say we put ourselves in the boundary case for which

$$\psi(1) = \psi'(1) = 0. \quad (\text{I.2})$$

In this paper, we are interested in the trace of  $\mathbb{X}$  which is the set of vertices visited by this random walk until a given instant. The literature on the subject initially started with the study of the range, that is to say the volume of the trace of the simple random walk on  $\mathbb{Z}^d$ , where  $d \geq 2$  is the dimension. In particular P. Erdős and S. Taylor [ET60] prove that the asymptotic in time of the trace depends on the dimension  $d$ . If we put ourselves in the present context of random walk in random environment on trees then the trace naturally depends on the hypothesis on the environment  $\mathcal{E}$ , see for example [AC18], [AdR17] and [dR16]. A first step in the extension of the notion of the range is to count, for example, the number of vertices visited a large number of time (instead of at least one time). This aspect has been studied for the simple random walk in [?] and in our context by [AD20] and [Che22], about which we will give some details later in the paper. A second step in the study of the trace, especially in the case of random walk in random environment, is to select certain vertices not only with criteria on the trajectory of the walk but also on the underlying potential  $V$ . With this in mind we introduce a generalization of the range : for any  $n$ , let  $\mathbf{f}^n = \{f^{n,k} : \mathbb{R}^k \rightarrow \mathbb{R}^+; k \in \mathbb{N}^*\}$  be a collection of bounded functions. Also, let  $g_n : \mathbb{R}^+ \rightarrow \mathbb{R}$  be a positive function. Then, the generalized range  $\mathcal{R}_n(g_n, \mathbf{f}^n)$  is given by

$$\begin{aligned} \mathcal{R}_n(g_n, \mathbf{f}^n) &:= \sum_{x \in \mathbb{T}} g_n(\mathcal{L}_x^n) f^{n,|x|}(V(x_1), V(x_2), \dots, V(x)), \text{ with} \\ \mathcal{L}_x^n &:= \sum_{k=1}^n \mathbb{1}_{\{X_k=x\}}, \end{aligned} \quad (\text{I.3})$$

$(x_i, i \leq |x|)$  being the sequence of vertices of the unique path from the root (excluded) to vertex  $x$  and  $\mathcal{L}_x^n$  is the usual local time of the walk at  $x$  before the instant  $n$ . As we may see,  $\mathcal{R}_n(g_n, \mathbf{f}^n)$  is quite general and can not be treated in this form, at once for every of these functions  $g_n$  and  $\mathbf{f}^n$ , so additional assumptions (involving  $\mathbf{f}^n$ ,  $g_n$  and distribution  $\mathbf{P}$ ) will be introduced in Section 1.3.

The aim of studying this extended range is twofold, first it allows to understand the interactions between the trajectories of the main process  $\mathbb{X}$  and of the underlying branching potential  $\mathbb{V}$ , second we develop a general tool allowing to treat many examples (for chosen  $\mathbf{f}^n$  and  $g_n$ ). Note that if we take, for example,  $f^{n,k} = 1$  and  $g_n = \mathbb{1}_{[1,\infty)}$  for any integer  $n$  and  $k$ , then we get the regular range (treated in [AC18]), and if  $g_n = \mathbb{1}_{[n^b,\infty)}$  with  $0 < b < 1$ , then we get the heavy range (see [AD20] and [Che22]).

The presentation of the results is divided into three subsections. In the first one below, we detail

and comment particular examples showing a large variety of behaviors for the range for different  $f^n$  and  $g_n$ . In a second subsection, we present an informal statement of the general result, the aim of which is to give the main ideas without introducing too much technical material. Finally, in the last section, we introduce assumptions which leads to the full statement of the main theorem.

### 1.1 First results : examples

The first two theorems (Theorems 1.1 and 1.2) we present in this section derive from three other works : in the first one [HS16a], it is proved that, during its first  $n$  steps, the walk can reach height of potential of order  $(\log n)^2$ . More precisely, it is proved that the random variable  $(\max_{1 \leq k \leq n} V(X_k)/(\log n)^2)_{n \geq 2}$  converges almost surely to one half. Note that this behavior can be quite disappointing if we have in mind the intuitive behavior of Sinai's one dimensional random walk in random environment [Sin82] for which the highest height of potential reached by the walk is of order  $\log n$ . Of course the fact that the walk evolves on a tree instead of a one dimensional lattice changes the deal but at the same time it is also proved in [HS16b] that this walk has a similar behavior than Sinai's one (they are both at a distance of order  $(\log n)^2$  from the origin at a given instant  $n$ ). In both cases, the potential plays a crucial role. In the two other papers ([AC18] and [AD20]), the range is studied : in [AC18], it is proved that regular range (the number of visited vertices up to the instant  $n$ ) is of order  $n/\log n$ , whereas in [AD20], it is proved that the number of edges visited more than  $n^b$  (with  $0 < b < 1$ ) times is typically of order  $n^{1-b}$  (this particular range is called «heavy range» in that paper, see also [Che22] for a refinement of this work).

Our first theorem below mixes the two approaches, showing the influence of a strong constraint on the trajectories of  $V$  on both regular or heavy range. What we mean by strong constraint here is a condition of the form  $V \geq (\log n)^\alpha$  with  $1 < \alpha < 2$ , that is to say when the potential is larger than what we can call regular height of potential for this walk (in the slow regime, a regular height is of order  $\log n$  since it can be proved that  $(V(X_n)/\log n)_{n \geq 2}$  converges weakly, see [HS16b]) but smaller than the extreme value  $(\log n)^2$  of [HS16a].

Before stating this result, let us introduce the following hypothesis on the distribution of the branching random walk : there exists  $\theta > 0$  and  $\delta_1 \in (0, 1/2)$  such that

$$\mathbf{E} \left[ \sum_{|z|=1} e^{-(1+\theta)V(z)} \right] + \mathbf{E} \left[ \sum_{|z|=1} e^{\theta V(z)} \right] < \infty, \quad (\text{I.4})$$

$$\mathbf{E} \left[ \left( \sum_{|z|=1} (1 + |V(z)|) e^{-V(z)} \right)^2 \right] + \mathbf{E} \left[ \left( \sum_{|z|=1} e^{-(1-\delta_1)V(z)} \right)^2 \right] < \infty, \quad (\text{I.5})$$

these are common hypothesis used for example in [AC18].

**Theorem 1.1.** *Assume (I.2), (I.4) and (I.5) hold. If for any  $n$  and  $k$ ,  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbb{1}_{\{t_k \geq (\log n)^\alpha\}}$  with  $\alpha \in (1, 2)$  and if  $g_n(t) = \mathbb{1}_{\{t \geq n^b\}}$  with  $b \in [0, 1)$ , then*

$$\left(\frac{\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - (1-b) \log n}{(\log n)^{\alpha-1}}\right)_{n \geq 2} \text{ converges in } \mathbb{P}^* \text{-probability to } -1,$$

where  $\log^+ x = \log(\max(1, x))$ .

This result shows that the number of vertices with high potential visited at least once (resp. strongly visited, with  $b > 0$ ) is of the same order, though smaller, than the regular range (resp. heavy-range). So visiting high potential is not just an accident appearing a couple of times on very specific paths of the tree. Far from that in fact, as the constraint of high potential creates a decrease of order  $e^{-(\log n)^{\alpha-1+o(1)}}$  and therefore appears as a second order correction comparing to ranges without constraint on the environment.

In the second theorem below, we add a slight different constraint which force the random walk to reach a high level of potential far from the ultimate visited vertices of given paths :

**Theorem 1.2.** *Assume (I.2), (I.4) and (I.5) hold. If for any  $n$  and  $k$ ,  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbb{1}_{\{t_{\lfloor k/\beta \rfloor} \geq (\log n)^\alpha\}}$  with  $\beta > 1$ ,  $\alpha \in (1, 2)$  ( $\lfloor x \rfloor$  stands for the integer part of  $x$ ) and for any  $b \in [0, 1)$ ,  $g_n(t) = \mathbb{1}_{\{t \geq n^b\}}$  then*

$$\left(\frac{\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - (1-b) \log n}{(\log n)^{\alpha-1}}\right)_{n \geq 2} \text{ converges in } \mathbb{P}^* \text{-probability to } -1 - \frac{\pi}{2} \sqrt{\beta - 1} + \rho((\beta - 1) \frac{\pi^2}{4}),$$

where for any  $c > 0$ ,

$$\rho(c) := \frac{c\sigma}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{c\sigma^2}{2}u} \left[ \frac{2}{u^{1/2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{u\sigma^2}) - \frac{1}{2} \int_u^{+\infty} \frac{1}{y^{3/2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{y\sigma^2}) dy \right] du,$$

and  $\mathbf{m}$  is a Brownian meander,  $\bar{\mathbf{m}}_1 := \sup_{s \leq 1} \mathbf{m}_s$  and  $\sigma^2 := \mathbf{E}[\sum_{|x|=1} V^2(x) e^{-V(x)}]$ .

As we may see, a slight change in function  $\mathbf{f}^n$  (comparing to previous theorem) makes appear something new, as the constant in the limit is very different than in Theorem 1.1. Note that  $\rho$  can be explicitly calculated : for any  $c > 0$

$$\rho(c) = 2\sqrt{c} \left( \frac{1 - e^{-\sqrt{c}}}{\sinh(\sqrt{c})} \right) - 2 \left( \sqrt{c} - \log((e^{\sqrt{c}} + 1)/2) \right), \quad (\text{I.6})$$

so we clearly obtain continuity when  $\beta$  converges to 1, getting back to the previous theorem. At this point, we also would like to discuss the appearance of the Brownian meander distribution in  $\rho$ . First, note that a Brownian meander appears in the asymptotic distribution of the (correctly normalized) generation of  $X_n$  (see [HS16b]) which is the consequence of the positivity of  $V$  (see Fact 4 below, page 11) together with an induced constraint on the largest downfall of  $V$  (we call maximal downfall, for a given  $x \in \mathbb{T}$ , the quantity  $\max_{y \leq x} (\bar{V}(y) - V(y))$ , where  $\bar{V}(y) := \max_{z \leq y} V(z)$ ) visited by the walk before the instant  $n$ . Also in [AC18], the distribution of two independent Brownian meanders ( $\mathbf{m}^1$  and  $\mathbf{m}^2$ ) appears in the result for the regular range  $\mathcal{R}_n$  (that is when  $f^{n,k} = 1$  and  $g_n = \mathbb{1}_{[1, \infty)}$ ) : in  $\mathbb{P}^*$ -probability

$$\lim_{n \rightarrow +\infty} \mathcal{R}_n \frac{\log n}{n} = C(\mathcal{D}_{\mathbf{m}^1}, \mathcal{D}_{\mathbf{m}^2}), \quad (\text{I.7})$$

one of these Brownian meanders also coming from the positivity of  $V$  and the other one coming from the fact that for a given visited vertex  $x$ , the maximum of  $V$  (on the unique path from the root to  $x$ ) is attained pretty near the generation of  $x$ .

Here, the Brownian meander appears as we ask a visited vertex  $x$  to have reached a high level of potential in an early generation before the one of  $x$  and it turns out that the constraint of low downfall of  $V$  appearing in [HS16b] ( $\max_{y \leq x} (\bar{V}(y) - V(y)) \leq \log n$ ) along this kind of path produces this appearance of the Brownian meander. However, contrarily to (I.7), the Brownian meander is involved in the correction of the main fluctuation ( $e^{-C(\mathcal{D}_m)(\log n)^{\alpha-1}}$ ) and not just in the constant of the limit ( $C(\mathcal{D}_{m^1}, \mathcal{D}_{m^2})$ ).

In the third example below, we choose  $f^{n,k}$  in such a way that an interaction appears between the trajectory of  $\mathbb{X}$  and the downfalls of  $V$ , which have an important role in the behavior of these walks. More particularly, let us introduce, for a given  $\mathbf{t} = (t_1, t_2, \dots, t_k)$  with  $k$  a positive integer, the following quantity

$$H_k(\mathbf{t}) := \sum_{j=1}^k e^{t_j - t_k},$$

then we call *sum of exponential downfalls* of  $V$  at  $x \in \mathbb{T}$  with  $|x| = k$  the quantity

$$H_{|x|}(\mathbf{V}_x) := H_{|x|}(V(x_1), \dots, V(x_k)) = \sum_{i=1}^{|x|} e^{V(x_i) - V(x_k)}. \quad (\text{I.8})$$

In order to simplify the notation and when there is no possible confusion, we will simply write  $H_x$  instead of  $H_{|x|}(\mathbf{V}_x)$  in the sequel.

**Theorem 1.3.** *Assume (I.2), (I.4) and (I.5) hold.*

*For any  $n$  and  $k$  let  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbb{1}_{\{t_k \geq a(\log n)^\alpha\}} (\sum_{j=1}^k H_j(\mathbf{t}))^{-d}$  with  $\alpha \in [1, 2)$ ,  $a \in \mathbb{R}$ ,  $d \in \{0, 1\}$  and  $g_n(t) = \mathbb{1}_{\{t \geq n^b\}}$  with  $b \geq 0$ .*

*If  $b \in [0, 1/(1+d))$  and  $\alpha = 1$  (with  $a > 1/\delta_1$  when  $d = 1$ ) then*

$$\left( \frac{\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n)}{\log n} \right)_{n \geq 2} \text{ converges in } \mathbb{P}^* \text{-probability to } 1 - (1+d)b,$$

*otherwise if  $a = 1$ ,  $b = 0$ ,  $d = 1$  and  $1 < \alpha < 2$*

$$\left( \frac{\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - \log n}{(\log n)^{\alpha/2}} \right)_{n \geq 2} \text{ converges in } \mathbb{P}^* \text{-probability to } -2,$$

*finally if  $a = 1$ ,  $0 < b < 1/2$ ,  $d = 1$  and  $1 < \alpha < 2$*

$$\left( \frac{\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - (1-2b) \log n}{(\log n)^{\alpha-1}} \right)_{n \geq 2} \text{ converges in } \mathbb{P}^* \text{-probability to } -1/b.$$

For the first limit (when  $\alpha = 1$ , implying that we have set a common height of potential - see Fact 1), by taking  $d = 0$ , we obtain the limit  $(1 - b)$  of the usual heavy range of [AD20]. Otherwise, if we add the penalization with the cumulative exponential downfalls ( $\sum_{y \leq x} H_y$ ), that is when  $d = 1$ , then an extra cost  $d * b = b$  appears.

The second case (with  $b = 0$  but  $1 < \alpha < 2$ ) has two constraints on the environment so the normalization  $(\log n)^{\alpha/2}$  appears as a compromise between the fact that high level of potential is asked ( $\mathbb{1}_{\{t_k \geq (\log n)^\alpha\}}$ ), which alone yields by Theorem 1.1 a normalization  $(\log n)^{\alpha-1}$ , and the fact that cumulative exponential downfall fluctuations ( $\sum_{m \leq k} H_m(\mathbf{t})$ ) can not be too large as it appears in the denominator of the range. This yields the  $(\log n)^{\alpha/2}$  (note that as  $\alpha < 2$ ,  $\alpha/2 > \alpha - 1$ ).

For the last case ( $0 < b < 1/2$  and  $1 < \alpha < 2$ ), the range is of order  $n^{1-2b} e^{-(\log n)^{\alpha-1/b}}$  comparing to  $n e^{-2(\log n)^{\alpha/2}}$  when  $b = 0$  of the previous case. In particular, the parameter  $b$  of the heavy range appears in both the main normalization  $n^{1-2b}$  and in the correction  $e^{-(\log n)^{\alpha-1/b}}$ . This can be intuitively understood as follows : first  $n^{1-2b} = n * n^{-b} * n^{-b}$ , one  $n^{-b}$  is classical from the heavy range when asking for a local time to be larger than  $n^b$  (which already appears in the first part of the Theorem), the second  $n^{-b}$  comes from the fact that a local time at a given vertex  $x$  can be larger than  $n^b$  only if  $\sum_{j=1}^{|x|} e^{V(x_j) - V(x)} \geq n^b$  and as this quantity appears in the normalization of the range (via  $f^{n,k}(t_1, t_2, \dots, t_k)$ ) this produced this second  $n^{-b}$ . So this part ( $n^{1-2b}$ ) appears as a first interaction between the constraints on the trajectory of  $\mathbb{X}$  and the one of  $\mathbb{V}$ . Let us now discuss about  $e^{-(\log n)^{\alpha-1/b}} = e^{-(\log n)^\alpha / (b \log n)}$ . For this term, we see intuitively the constraints for the walk to reach height of potential of order  $(\log n)^\alpha$  but at the same time, in order to keep the denominator  $\sum_{j \leq k} H_j(\mathbf{t})$  as low as possible, the maximal downfall has to remain smaller than  $b \log n$ , thus producing the ratio  $(\log n)^\alpha / (b \log n)$ .

In the ultimate example below, we ask similar constraints for the environment than above but only in the early visited generations :

**Theorem 1.4.** *Assume (I.2), (I.4) and (I.5) hold. For any  $\beta > 1$ ,  $n$  and  $k$ , let  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbb{1}_{\{t_{\lfloor k/\beta \rfloor} \geq (\log n)^\alpha\}} (\sum_{j=1}^{\lfloor k/\beta \rfloor} H_j(\mathbf{t}))^{-1}$ ,  $\alpha \in (1, 2)$  and if  $g_n(t) = \mathbb{1}_{\{t \geq n^b\}}$  with  $b \in (0, 1)$ , then*

$$\left( \frac{\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - (1-b) \log n}{(\log n)^{\alpha/2}} \right)_{n \geq 2} \text{ converges in } \mathbb{P}^* \text{-probability to } -2.$$

This last theorem just prove that if the factor  $(\sum_{j=1}^{\lfloor k/\beta \rfloor} H_j(\mathbf{t}))^{-1}$  only concerns the beginning of the trajectory, that is the sites at a distance  $\lfloor |x|/\beta \rfloor$  of the root (if  $x$  is a visited vertex), then things go back to normal : there is no more multiple interactions between  $\mathbb{X}$  and  $\mathbb{V}$ .

We can imagine more examples like the ones we present above (by acting more on the function  $g_n$  as we did for example) but for now, let us introduce a more general result with general hypothesis on  $g_n$  and  $\mathbf{f}^n$ .

## 1.2 A general result (informal statement)

In this section, we present an informal statement for the asymptotic in  $n$  of  $\mathcal{R}_n(g_n, \mathbf{f}^n)$  for general  $g_n$  and  $\mathbf{f}^n$  (including, in particular, the results of the preceding section). The aim, in a first step, is to introduce the result and the main ideas but to minimize the technical materials.

First recall the expression of the generalized range (I.3)

$$\mathcal{R}_n(g_n, \mathbf{f}^n) = \sum_{x \in \mathbb{T}} g_n(\mathcal{L}_x^n) f^{n, |x|}(V(x_1), V(x_2), \dots, V(x)),$$

with  $\mathcal{L}_x^n$  the local time of  $\mathbb{X}$  at  $x$  before the instant  $n$ .

We assume that  $g_n$  can be written as the product of an indicator function and a function  $\varphi$  which is positive non-decreasing : for any  $b \geq 0$  and  $t \geq 1$ ,  $g_n(t) := \mathbf{1}_{\{t \geq n^b\}} \varphi(t)$ . The indicator function is here to include all types of range (regular or heavy). Also, we ask the function  $t \mapsto \varphi(t)/t$  to be non-increasing, so that  $\varphi(\mathcal{L}_x^n)$  remains reasonable (at most of the order of the local time itself).

Let us introduce the branching object  $\Psi$  as follows : let  $0 \vee \lambda' < \lambda$  be two real numbers and  $k \geq 1$  an integer, also let  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}$  be a bounded function.  $\Psi_{\lambda, \lambda'}^k(\phi)$  is then defined as a mean of  $\phi$  along the trajectory of  $V$  (with constraints) until generation  $k$ , that is

$$\Psi_{\lambda, \lambda'}^k(\phi) := \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} \phi(V(x_1), \dots, V(x)) \mathbf{1}_{\mathcal{O}_{\lambda, \lambda'}(x)} \right], \quad (\text{I.9})$$

where  $\mathcal{O}_{\lambda, \lambda'}$  is the set of  $(\lambda, \lambda')$ -regular lines

$$\mathcal{O}_{\lambda, \lambda'} := \left\{ x \in \mathbb{T}; \max_{j \leq |x|} H_{x_j} \leq \lambda, H_x > \lambda' \right\}, \quad \text{with } H_{x_j} = \sum_{i=1}^j e^{V(x_i) - V(x_j)}, \quad (\text{I.10})$$

also we denote

$$\mathcal{O}_\lambda := \left\{ x \in \mathbb{T}; \max_{j \leq |x|} H_{x_j} \leq \lambda \right\}, \quad \text{and } \Psi_\lambda^k(\phi) := \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} \phi(V(x_1), \dots, V(x)) \mathbf{1}_{\mathcal{O}_\lambda(x)} \right].$$

Note that since  $H_x \geq 1$  ( $H_x > 1$  when  $|x| > 1$ ), we have, for all  $\lambda' < 1$ ,  $\mathcal{O}_{\lambda, \lambda'} = \mathcal{O}_\lambda$  and  $\Psi_{\lambda, \lambda'}^k(\phi) = \Psi_\lambda^k(\phi)$ .

The appearance of this set of regular lines  $\mathcal{O}_{\lambda, \lambda'}$  is partly inspired from the works of [HS16b] ( $\lambda$  representing extreme exponential downfalls of  $V$  related to a reflecting barrier for the walk  $(X_k, k \leq n)$ ), and also (for  $\lambda'$ ) from the constraint on the local time appearing in the function  $g_n$ . It turns out indeed that constraints on the value of the local time at a site  $x$  imply constraints on  $H_x$ . In other words, there are constraints on the branching potential  $\mathbb{V}$  induced by constraints on the random walk  $\mathbb{X}$  and sometimes, these constraints have a major impact on the range. We call this type of contribution «contribution of type one», that is of order  $n^\theta$  where  $\theta \in (0, 1]$  (this actually appears for example in Theorem 1.3). To be more specific, let us introduce the following notations : first  $C_\infty := C_\infty(\{\mathbf{f}^n; n \geq 1\})$  stands for the supremum of  $\{\mathbf{f}^n; n \geq 1\}$  that



is  $C_\infty := \sup_{m,\ell} \|f^{m,\ell}\|_\infty$ . Then, define the set

$$\mathcal{U}_b := \left\{ \kappa \in [0, 1]; \text{ for all } k \geq 1, \mathbf{t} \in \mathbb{R}^k, n \geq 1 : \mathbb{1}_{\{H_k(\mathbf{t}) > n^b\}} f^{n,k}(\mathbf{t}) \leq C_\infty n^{-\kappa} \right\}, \quad (\text{I.11})$$

note that  $\mathcal{U}_b \neq \emptyset$  because  $0 \in \mathcal{U}_b$  and as the supremum is attained, let

$$\kappa_b =: \max \mathcal{U}_b. \quad (\text{I.12})$$

When  $\kappa_b > 0$ , we say that a mixing between the constraints on trajectories of the random walk  $\mathbb{X}$  and on those of the branching potential  $\mathbb{V}$  produce a contribution of type one.

To introduce a second type of contribution, which can be seen as the second order comparing to the contribution of type one, we present an important quantity which is the sum over all the generations of  $\Psi_{n,n^b}(f^{n,\cdot}) : \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$ . First, let us give an heuristic about the way it appears in the asymptotic of the range.

For any  $k \geq 1$ , introduce the  $k^{\text{th}}$  return time  $T^k := \inf\{k > T^{k-1}, X_k = e\}$  to  $e$  and take  $T^0 = 0$ . Recall the definition of  $\bar{V}$  before (I.7) and let  $\bar{\mathcal{R}}_{T^n}(g_n, \mathbf{f}^n)$  be the range on  $\{\bar{V}(\cdot) \geq A \log n\} : \bar{\mathcal{R}}_{T^n}(g_n, \mathbf{f}^n) := \sum_{x \in \mathbb{T}} g_n(\mathcal{L}_x^{T^n}) f^{n,|x|}(V_x) \mathbb{1}_{\{\bar{V}(x) \geq A \log n\}}$  with  $A > 0$ .  $\bar{\mathcal{R}}_{T^n}(g_n, \mathbf{f}^n)$  is a version of the generalized range where we have replaced the instant  $n$  by  $T^n$  and we have made appear the additional constraint  $\bar{V}(x) \geq A \log n$ . Note that it is known (following Lemma 2.1 in [AC18] and its proof at the beginning of Section 4.2) that this additional condition  $\mathbb{1}_{\{\bar{V}(x) \geq A \log n\}}$  has no effect on the normalization of the range, that is

**Fact 1** : There exists  $0 < c_1 = c_1(A) \leq 1$  such that  $\lim_{n \rightarrow +\infty} \mathbb{P}^* \left( \frac{\bar{\mathcal{R}}_{T^n}}{\mathcal{R}_{T^n}} = c_1 \right) = 1$ .

So here, we typically consider collections of functions  $\mathbf{f}^n$  such that  $\bar{\mathcal{R}}_{T^n}(g_n, \mathbf{f}^n) / \mathcal{R}_{T^n}(g_n, \mathbf{f}^n) \rightarrow Cte > 0$ . One of the main gain of this consideration is the fact that relatively high potential yields interesting quasi-independence in the trajectory of  $(X_n, n)$ .

With this fact, we have (see Section 3.1) something like  $\bar{\mathcal{R}}_{T^n}(g_n, \mathbf{f}^n) \gtrsim n \mathbb{E}^\varepsilon[\bar{\mathcal{R}}_{T^1}(g_n, \mathbf{f}^n)]$  in probability and thanks to the fact that  $\varphi$  is non-decreasing and to the expression of the quenched mean of  $\bar{\mathcal{R}}_{T^1}(g_n, \mathbf{f}^n)$ , in probability, for large  $n$

$$\bar{\mathcal{R}}_{T^n}(g_n, \mathbf{f}^n) \gtrsim n \mathbb{E}^\varepsilon[\bar{\mathcal{R}}_{T^1}(g_n, \mathbf{f}^n)] \gtrsim \frac{\varphi(n^b)}{n^b} n \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}),$$

which makes appear  $\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$ . It turns out that this lower bound is exactly the good quantity which leads to our main result.

The following assumption ensures that  $\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$  is not too small, which would correspond to an exaggerate penalization on the potential  $V$  :

*Assumption 1.*

For all  $b \in [0, 1)$ ,  $\varepsilon > 0$  and  $n$  large enough

$$\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \geq \frac{1}{n^{(\kappa_b + \varepsilon) \wedge 1}}. \quad (\text{A1})$$

The second type of contribution that we call «contribution of type two» strongly involves the term  $n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$ . It is negligible with respect to  $n^\varepsilon$  for all  $\varepsilon > 0$  and comes also from a mixing between the constraints on  $\mathbb{X}$  and the constraints on  $\mathbb{V}$ . So finally introduce  $(h_n, n)$  which is certainly the most important sequence of the paper : for any  $n \geq 2$

$$h_n := \begin{cases} |\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}))| & \text{if } \exists \gamma \in (0, 1) : \frac{(\log n)^\gamma}{\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}))} \rightarrow 0 \\ \log n & \text{otherwise} \end{cases}. \quad (\text{I.13})$$

Let us start by a discussion about  $(h_n, n)$  with the following remark in which we note that either  $h_n = o(\log n)$  or  $h_n = \log n$ .

**Remark 1.** *By definition of  $\kappa_b$ ,*

$$n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \leq C_\infty \sum_{k \geq 1} \Psi_n^k(1) = C_\infty \mathbf{E} \left[ \sum_{x \in \mathbb{T}} e^{-V(x)} \mathbb{1}_{\{x \in \mathcal{O}_n\}} \right] \leq C_\infty (\log n)^3,$$

where the last inequality is a quite elementary fact that will be proved later (see **Remark 2**). This implies, in particular, that if there exists  $0 < \gamma < 1$  such that  $(\log n)^\gamma / \log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}))$  goes to 0 when  $n$  goes to  $\infty$ , then necessarily, we have  $\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})) < 0$  and  $\lim_{n \rightarrow +\infty} \log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})) = -\infty$ . Moreover, in this case, there exists  $0 < \gamma < 1$  such that  $h_n \geq (\log n)^\gamma$ . Also assumption **(A1)** above ensures that

$$\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})) \geq \log\left(\frac{n^{\kappa_b}}{n^{(\kappa_b + \varepsilon) \wedge 1}}\right) \geq -((\kappa_b + \varepsilon) \wedge 1 - \kappa_b) \log n \geq -\varepsilon \log n,$$

overall, definition of  $h_n$  implies, under **(A1)**, that

$$(\log n)^\gamma \leq h_n \leq \log n.$$

The sequence  $(h_n, n)$  is the quantity which gives the contribution of type two and produces the second order in our result. It is important to note that we carefully assign an expression to  $h_n$  depending on whether constraints are penalizing or not. According to the asymptotic behavior of the term  $n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$ , we assign  $h_n$  two possible expressions : if  $(\log n)^\gamma$  is negligible with respect to  $|\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}))|$  for some  $\gamma \in (0, 1)$  (which then remains smaller than  $\varepsilon \log n$  by Remark 1), constraints are considered penalizing and we set  $h_n := |\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}))|$ , see Theorem 1.1 for example. Otherwise, constraints are not penalizing enough and we set  $h_n := \log n$ , see Theorem 1.3 with  $\alpha = 1$  for instance. In this latter case, the choice is significant since  $\log n$  is the right order for the logarithm of the regular range, that is to say the range without any constraint on the trajectories of the branching random potential  $\mathbb{V}$ .

We are now almost ready to state a result. But first introduce two last values :  $L$  (with  $L = \pm\infty$

possibly) and  $\xi \in \{-1, 0\}$  defined as follows

$$L := \liminf_{n \rightarrow \infty} h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b)), \text{ and} \quad (\text{I.14})$$

$$\xi := \lim_{n \rightarrow \infty} h_n^{-1} \log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})), \quad (\text{I.15})$$

and note that, following Remark 1,  $\xi$  necessarily exists.

The full statement of our main result below need additional quite complex assumptions, involving  $\mathbf{f}^n$  in particular, they are described precisely in the next section (see (A2), (A3) and (A4)). The interesting point is the fact that all of these assumptions concern  $\sum_{k \geq 1} \Psi^k$ . And more than that, we can resume the actions of (A2), (A3) and (A4) by saying that  $\Psi$  has to be stable for small perturbations of its parameters. In the informal statement below, we will say that  $\Psi$  should have controlled fluctuations.

**Theorem 1.5** (Informal statement). *Assume (I.2), (I.4) and (I.5) hold,  $b \in [0, 1]$ , assume also that (A1) is satisfied and  $\Psi$  has controlled fluctuations. If  $L \in (-\xi, +\infty]$ , then in  $\mathbb{P}^*$ -probability*

$$h_n^{-1} (\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - \log(n^{1-b-\kappa_b} \varphi(n^b))) \xrightarrow[n \rightarrow \infty]{} \xi,$$

if  $L = -\xi$ , with  $\underline{\Delta}_n := h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b)) - \inf_{\ell \geq n} h_\ell^{-1} \log(\ell^{1-b-\kappa_b} \varphi(\ell^b))$ , then in  $\mathbb{P}^*$ -probability

$$h_n^{-1} \log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - \underline{\Delta}_n \xrightarrow[n \rightarrow \infty]{} 0,$$

otherwise  $L \in [-\infty, -\xi[$  and in  $\mathbb{P}^*$ -probability

$$\mathcal{R}_{n_\ell}(g_{n_\ell}, \mathbf{f}^{n_\ell}) \xrightarrow[\ell \rightarrow \infty]{} 0,$$

for some increasing sequence  $(n_\ell)_\ell$  of positive integers. Note that when  $\lim_{n \rightarrow \infty} h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b))$  is equal to  $L$ ,  $n_\ell = \ell$ .

We now present particular examples which lead to different values of  $L$  and  $\xi$ . First, note that all theorems presented in the previous section satisfy  $L = +\infty$  and  $\xi = -1$ , corresponding, from our point of view, to the most interesting case. Let us take, for example,  $g_n(t) = \mathbb{1}_{\{t \geq n^b\}}$  and  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbb{1}_{\{t_k \geq a(\log n)^\alpha\}} (\sum_{l \leq k} H_l(\mathbf{t}))^{-1}$  as in Theorem 1.3, with  $a > 0$ ,  $\alpha \in [1, 2)$ , but  $b \in [1/2, 1)$ . When  $\alpha > 1$  and  $b > 1/2$ , we can prove that  $h_n \sim a(\log n)^{\alpha-1}/b$  (with the usual notation  $t_n \sim s_n$  if and only if  $t_n/s_n \rightarrow 1$ ) and  $n^{1-b-\kappa_b} \varphi(n^b) = n^{1-2b}$  so we obtain  $\lim_{n \rightarrow \infty} h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b)) = L = -\infty$ . However, when  $\alpha = 1$  and  $a > 1/\delta_1$ , we can prove that for all  $b \in [1/2, 1)$ ,  $\kappa_b = b$  and  $h_n = \log n$  thus giving  $L = 1 - 2b$  and  $\xi = 0$ . In other words,  $L \in (-\infty, -\xi]$  (with  $L = -\xi$  if and only if  $b = 1/2$ ).

Let us finally take the simple example  $g_n(t) = t \mathbb{1}_{\{t \geq n^b\}}$  and  $f^{n,k} = 1$ . We can prove that for all  $b \in (0, 1)$ ,  $h_n = \log n$ ,  $\xi = 0$  and  $n^{1-b-\kappa_b} \varphi(n^b) = n$  so  $\lim_{n \rightarrow \infty} h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b)) = L = 1$  and

we are in the case  $L \in (-\xi, +\infty)$ .

To finish, we present an example for which  $f^{n,k}$  is quite general but with a simple form. Assume

- $f^{n,k} = \mathbf{1}_{\mathbf{A}^{n,k}}$  with  $\mathbf{A}^{n,k} \subset \mathbb{R}^k$  and  $\mathbf{A}_b^{n,k} := \mathbf{A}^{n,k} \cap \{\mathbf{t} \in \mathbb{R}^k; \max_{1 \leq j \leq k} H_j(\mathbf{t}) \leq n, H_k(\mathbf{t}) > n^b\}$ ;
- $(\mathbf{A}_b^{n,k} \times \mathbb{R}^{k'-k}) \cap \mathbf{A}_b^{n,k'} = \emptyset$  for all  $k < k'$ ;
- $\kappa_b = 0$ .

We obtain the following simple expression for  $n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) = \mathbf{P}(\cup_{k \geq 1} \{(S_1, \dots, S_k) \in \mathbf{A}_b^{n,k}\})$ , where  $(S_i, i)$  is a sum of i.i.d random variables with mean 0 and variance  $\psi''(1)$  (this comes from the so-called many-to-one Lemma, see Lemma 2.1). So

$$\left| \log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})) \right| = -\log \mathbf{P}(\cup_{k \geq 1} \{(S_1, \dots, S_k) \in \mathbf{A}_b^{n,k}\}).$$

Consequently, if the probability  $\mathbf{P}(\cup_{k \geq 1} \{(S_1, \dots, S_k) \in \mathbf{A}_b^{n,k}\})$  is small enough, that is to say such that  $(\log n)^\gamma$  is negligible comparing to  $-\log \mathbf{P}(\cup_{k \geq 1} \{(S_1, \dots, S_k) \in \mathbf{A}_b^{n,k}\})$  for a certain  $\gamma \in (0, 1)$ , then the constraint is penalizing enough and  $h_n = -\log \mathbf{P}(\cup_{k \geq 1} \{(S_1, \dots, S_k) \in \mathbf{A}_b^{n,k}\})$ . Otherwise,  $h_n = \log n$ . For example, take  $\mathbf{A}^{n,k} = \{\mathbf{t} \in \mathbb{R}^k; \inf\{j \leq k; t_j \geq (\log n)^\alpha\} = k\}$  which leads to an example similar to Theorem 1.1.

### 1.3 A general result (full statement)

In this section, we explain precisely what « $\Psi$  has controlled fluctuations» means. For that, we present the assumptions (A2), (A3) and (A4) mentioned in the previous section. We start with (A2), and then state a preliminary result (Proposition 1) of the main theorem (Theorem 1.5). This proposition is quite technical especially in its statement. However, it stresses on the fact that all the expressions involved depend deeply on  $\sum_k \Psi_{\cdot, \cdot}^k(f^{n,k})$  and therefore justify the last two Assumptions (A3) and (A4) which leads to the formal statement of Theorem 1.5.

*Assumption 2.*

Assumption (A2) below is an upper bound for a conditional version of  $\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$  actually requiring in order to be introduced two facts and additional notations.

**Fact 2 :** By Lemma 2.3 in [AD20], there exists two real numbers  $c_2, \tilde{c}_2 > 0$  such that for any  $h > 0$

$$\mathbf{P}^* \left( \max_{|w| \leq \lceil h/c_2 \rceil} |V(w)| > h \right) \leq h e^{-\tilde{c}_2 h}. \quad (\text{I.16})$$

This fact, that will be useful when cutting on early generations of the tree, justifies the introduction of the following notation : for any  $n$  and  $k$ ,  $f_h^{n,k}$  is the function defined by

$$f_h^{n,k}(t_1, \dots, t_k) := \inf_{\mathbf{s} \in [-h, h]^m} f^{n,k+m}(s_1, \dots, s_m, t_1 + s_m, \dots, t_k + s_m), \quad (\text{I.17})$$

with  $m = \lceil h/c_2 \rceil$  and  $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{R}^m$ .

The second fact is about the largest generation visited by the walk before the instant  $n$  or before  $n$  excursions to the vertex  $e$ .

**Fact 3** : Let  $(\ell_n = (\log n)^3, n \geq 2)$ , by Lemma 3.2 in [AD20], there exists  $A > 0$  such that :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\max_{k \leq T^n} |X_k| \leq A\ell_n) = 1.$$

This fact is here essentially to justify the introduction of the sequence  $(\ell_n, n)$  which appears in our second assumption and all along the paper. Note that a very precise result on the largest generation visited by the walk before the instant  $n$  can be found in [FHS11].

A last notation we need to introduce is a conditional and translated version of  $\Psi_{\lambda, \lambda'}^k(F)$  for a given bounded function  $F$ . For all  $k \in \mathbb{N}^*$ ,  $l \in \mathbb{N}^*$ ,  $F : \mathbb{R}^{l+k} \rightarrow \mathbb{R}$  bounded and  $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$

$$\Psi_{\lambda, \lambda'}^k(F|\mathbf{t}) := \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} F(t_1, \dots, t_l, V(x_1) + t_1, \dots, V(x) + t_l) \mathbb{1}_{\mathcal{O}_{\lambda, \lambda'}(x)} \right], \quad (\text{I.18})$$

where  $\sum_{|x|=k}$  is the sum over all the individuals  $x$  of generation  $k$ . Otherwise, if  $l = 0$ , then  $\Psi_{\lambda, \lambda'}^k(F|\mathbf{t}) := \Psi_{\lambda, \lambda'}^k(F)$ .

We are now ready to introduce the second assumption : for all  $\delta, \varepsilon, A, B > 0$  and  $b \in [0, 1)$ , there exists  $n_0 \in \mathbb{N}^*$  such that for any  $n \geq n_0$ ,  $l \leq \lfloor A\ell_n \rfloor$  and any  $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$  with  $t_l \geq -B$  and  $H_l(\mathbf{t}) \leq n$

$$\sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k(f_{\varepsilon h_n}^{n, l+k}|\mathbf{t}) \leq e^{\delta t_l + \frac{\varepsilon}{A} h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k}). \quad (\text{A2})$$

Let us comment this inequality which plays two roles. A first one ensures that the fluctuations of  $V$  in the early generations of the tree have minor influence, this yields the presence of  $e^{\frac{\varepsilon}{A} h_n}$ .

The second point is technical and aims to show that in probability, we have  $\mathbb{E}^{\mathcal{E}}[\overline{\mathcal{R}}_{T^1}(g_n, \mathbf{f}^n)] \gtrsim n^{-b} \varphi(n^b) \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k})$ . For that, the second moment of

$$Z_n := \sum_{x \in \mathcal{O}_{n, n^b}} e^{-V(x)} f_{\varepsilon h_n}^{n, k}(\mathbf{V}_x) \mathbb{1}_{\{\overline{V}(x) \geq A \log n, \underline{V}(x) \geq -B, \overline{V}(x) = V(x)\}}$$

has to be controlled, with  $\mathbf{V}_x := (V(x_1), \dots, V(x))$  and  $\underline{V}(x) := \min_{v \leq x} V(v)$ . We first observe that

$$Z_n^2 \approx \sum_{z \in \mathbb{T}} \sum_{x, y > z} \prod_{u \in \{x, y\}} \mathbb{1}_{\{u \in \mathcal{O}_{n, n^b}\}} e^{-V(u)} f_{\varepsilon h_n}^{n, |u|}(\mathbf{V}_u) \mathbb{1}_{\{V(u) \geq A \log n, \underline{V}(u) \geq -B, \overline{V}(u) = V(u)\}}.$$

Then taking the expectation of  $Z_n^2$ ,  $\sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k(f_{\varepsilon h_n}^{n, l+k}|\mathbf{t})$  in (A2) actually appears as the conditional expectation of a well chosen function of the translated potential  $(V_z(u) := V(u) - V(z))_{u > z}$ . Indeed, note that  $u \in \mathcal{O}_{n, n^b}$  together with  $\overline{V}(u) = V(u)$  implies that  $u \in \mathcal{O}_{n, n^b - H_z}^z :=$

$\{u > z : \max_{z < v \leq u} H_{z,v} \leq n, H_{z,u} > n^b - H_z\}$  with  $H_{z,v} := \sum_{z < w \leq v} e^{V_z(w) - V_z(v)}$ . Hence, for all  $\delta \in (0, 1/2)$ , by independence of the increments of the branching random walk  $(\mathbb{T}, V(u); u \in \mathbb{T})$

$$\begin{aligned} \mathbf{E}[Z_n^2] &\lesssim e^{(1-2\delta)B} \mathbf{E} \left[ \sum_{z \in \mathcal{O}_n} e^{-V(z)} \sum_{x,y > z} \prod_{u \in \{x,y\}} \mathbb{1}_{\{u \in \mathcal{O}_{n, n^b - H_z}^z\}} e^{-V_z(u)} F_{V_z}^{n, |u|}(V_z(u_{|z|+1}), \dots, V_z(u)) \right] \\ &\approx e^{(1-2\delta)B} \mathbf{E} \left[ \sum_{z \in \mathcal{O}_n} e^{-V(z)} \left( e^{\delta V(z)} \sum_{k \geq 1} \Psi_{n, n^b - H_z}^k(f^{n, l+k} | V_z) \right)^2 \right], \end{aligned}$$

where, for  $|z| = l$  and any  $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$

$$F_{\mathbf{t}}^{n, |u|}(V_z(u_{l+1}), \dots, V_z(u)) := e^{\delta t_l} f_{\varepsilon h_n}^{n, |u|}(t_1, \dots, t_l, V_z(u_{l+1}) + t_l, \dots, V_z(u) + t_l).$$

Assumption **(A2)** finally allows to say that  $\mathbf{E}[Z_n^2] \lesssim e^{\varepsilon h_n} (\sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k}))^2$  for all  $\varepsilon > 0$  and  $n$  large enough.

We are now almost ready to state an intermediate result which is a proposition giving a lower and an upper bound for the generalized range stopped at  $T^n$ . This proposition is followed by the theorem, much easier to read, but requiring extra assumptions. First, let us introduce for any  $\mathfrak{z} > 0$

$$\mathcal{H}_{\mathfrak{z}}^k := \left\{ (t_1, \dots, t_k) \in \mathbb{R}^k; t_k \geq \mathfrak{z} \right\}, \quad \mathcal{H}_{B, \mathfrak{z}}^k := \left\{ (t_1, \dots, t_k) \in \mathbb{R}^k; t_k \geq \mathfrak{z}, \min_{1 \leq i \leq k} t_i \geq -B \right\}, \quad (\text{I.19})$$

respectively the set of vectors such that its last coordinate is larger than  $\mathfrak{z}$  and additionally with all coordinates larger than  $-B$ . The introduction of these last two objects is justified by

**Fact 4 :** for any  $\varepsilon > 0$ , there exists  $a > 0$  such that (see [Aïd13])

$$\mathbf{P}(\min_{u \in \mathbb{T}} V(u) \geq -a) \geq 1 - \varepsilon,$$

and Fact 1 we have already talked about saying that, in  $\mathbb{P}^*$ -probability,  $\ell_n^{1/3}$  is a height of potential usually reached by the walk.

**Proposition 1.** Recall (I.12), let  $\varepsilon_b := \min(b + \mathbb{1}_{\{b=0\}}, 1 - b)/13$  and  $W := \sum_{|z|=1} e^{-V(z)}$ . Assume (I.2), (I.4) and (I.5) hold as well as (A1) and (A2).

Lower bound : there exists  $c_5 > 0$  such that for all  $b \in [0, 1)$ ,  $\varepsilon \in (0, \varepsilon_b)$ ,  $B > 0$  and  $n$  large enough

$$\mathbb{P} \left( \frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b} \varphi(n^b) u_{1,n}} < e^{-5\varepsilon h_n} \right) \leq \frac{e^{-c_5 \varepsilon h_n}}{(u_{1,n})^2} \left( \sum_{k \geq 1} \Psi_{n, n^b}^k(f_{\varepsilon h_n}^{n, k}) \right)^2 + h_n e^{-\varepsilon \tilde{c}_2 h_n} + \frac{e^{-\min(\varepsilon \log n, 3h_n)}}{(n^{\varepsilon_b} u_{1,n})^2}, \quad (\text{I.20})$$

with

$$u_{1,n} = u_{1,n}(\varepsilon) := \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k(f^{n,k} \mathbb{1}_{\Upsilon_n^k}),$$

$\Upsilon_n^k := \{\mathbf{t} \in \mathbb{R}^k; H_k(\mathbf{t}) \leq n^b e^{\varepsilon h_n}\} \cap \mathcal{H}_{B, 2\ell_n^{1/3}/\delta_1}^k$  and  $\lambda_n = ne^{-\min(10\varepsilon \log n, 5h_n)}$ .

Upper bound : for any  $\varepsilon > 0$  and  $n$  large enough

$$\mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b} \varphi(n^b) u_{2,n}} > e^{\varepsilon h_n}\right) \leq e^{-\frac{\varepsilon}{2} h_n} + o(1) \quad (\text{I.21})$$

with

$$u_{2,n} := \sum_{k \geq 1} \left( \Psi_n^k(f^{n,k} \mathbb{1}_{\mathbb{R}^k \setminus \mathcal{H}_{\ell_n^{1/3}/\delta_1}^k}) + \Psi_{n, n^b/(\log n)^2}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n, n^b/(W(\log n)^2)}^k(f^{n,k})] \right).$$

Note that (I.20) and (I.21) remain true replacing  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  by  $\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)$  with  $k_n$  equal to  $\lfloor n/(\log n)^p \rfloor$ ,  $p > 0$ .

This proposition is technical and difficult to read, we present it here however because it shows that all the estimations depend deeply on  $\Psi_{\cdot, \cdot}(f)$  and  $g_n$ , recall indeed that the key sequence  $(h_n, n)$  defined in (I.13) depends both on  $\Psi_{\cdot, \cdot}(f)$  and  $\kappa_b$  (with  $b$  coming from the function  $g_n$ ). This also means that without any more information on  $\Psi_{\cdot, \cdot}(f)$ , it is difficult to state a more explicit result. Finally, note that the exact role of (A1) and (A2) will appear clearly in the proof of the lower bound (Section 3.2).

We now present two new assumptions (A3) and (A4) which lead to the formal statement of the result. These assumptions tell essentially that quantities  $u_{1,n}$  and  $u_{2,n}$ , which appear in the previous proposition, are actually very similar. Now introduce (A3) and (A4) :

*Assumption 3* : for all  $b \in [0, 1)$ ,  $\varepsilon \in (0, \varepsilon_b)$ ,  $\varepsilon_1 \in (0, \varepsilon)$  and  $n$  large enough

$$u_{1,n} \geq e^{-\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}). \quad (\text{A3})$$

*Assumption 4* : for all  $\varepsilon_1 > 0$ ,  $b \in [0, 1)$  and  $n$  large enough

$$u_{2,n} \leq e^{\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}). \quad (\text{A4})$$

The full statement of Theorem 1.5 then writes as follows :

**Theorem 1.5** (Full statement). *Assume (I.2), (I.4) and (I.5) hold,  $b \in [0, 1)$  and (A1), (A2),*

(A3) and (A4) are satisfied. If  $L \in (-\xi, +\infty]$ , then in  $\mathbb{P}^*$ -probability

$$h_n^{-1} (\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - \log(n^{1-b-\kappa_b} \varphi(n^b))) \xrightarrow[n \rightarrow \infty]{} \xi,$$

if  $L = -\xi$ , with  $\underline{\Delta}_n := h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b)) - \inf_{\ell \geq n} h_\ell^{-1} \log(\ell^{1-b-\kappa_b} \varphi(\ell^b))$ , then in  $\mathbb{P}^*$ -probability

$$h_n^{-1} \log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - \underline{\Delta}_n \xrightarrow[n \rightarrow \infty]{} 0,$$

otherwise  $L \in [-\infty, -\xi[$  and in  $\mathbb{P}^*$ -probability

$$\mathcal{R}_{n_\ell}(g_{n_\ell}, \mathbf{f}^{n_\ell}) \xrightarrow[\ell \rightarrow \infty]{} 0,$$

for some increasing sequence  $(n_\ell)_\ell$  of positive integers. Note that when  $\lim h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b))$  is equal to  $L$ ,  $n_\ell = \ell$ .

The rest of the paper is decomposed as follows : in Section 2, after short preliminaries (Section 2.1), we prove the theorems of Section 1.1. For these proofs (Section 2.2), we check that the four assumptions (A1-A4) of Theorem 1.5 are realized, obtaining simultaneously the asymptotic of  $h_n$ . In section 2.3, we prove Theorem 1.5 : essentially, Proposition 1 is assumed to be true and we only check that if Assumptions (A3) and (A4) are true then the theorem comes.

We prove Proposition 1 in section 3, this is the most technical part of the paper which can be read independently of the other parts : in Section 3.1, we summarize usual facts, in a second sub-section we prove a lower bound for stopped generalized range  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  and finally in a last one an upper bound.

In section 4 we present some estimates on sums of i.i.d. random variables useful for the proof of the examples of Section 1.1. Finally, we resume in a last section (page 106) the notations which are transversal along the paper.

## 2 Proof of the theorems

This section is decomposed in three parts : in the first section below, one can find preliminaries that are useful all along the rest of the paper. In the second sub-section, we prove the four theorems presented as examples. Finally, the last section is devoted to the proof of Theorem 1.5.

### 2.1 Preliminary material

We recall the many-to-one formula (see [Shi15] Chapter 1, and [FHS11] equation 2.1) which will be used several times in the paper to compute expectations related to the environment. Note that the identity below comes from a change of probability measure (see references above), however we still keep  $\mathbf{P}$  and  $\mathbf{E}$  for simplicity.



**Lemma 2.1** (Many-to-one Lemma). *Recall the definition of  $\psi$  in (I.1). For any  $t > 0$ ,*

$$\mathbf{E} \left[ \sum_{|x|=m} f(V(x_i), 1 \leq i \leq m) \right] = \mathbf{E}(e^{tS_m + \psi(t)m} f(S_i, 1 \leq i \leq m)),$$

where  $(S_n)_{n \in \mathbb{N}}$  is the random walk starting at 0, such that the increments  $(S_{n+1} - S_n)_{n \in \mathbb{N}}$  are i.i.d. and for any measurable function  $h : \mathbb{R}^m \rightarrow [0, \infty)$ ,

$$\mathbf{E}[h(S_1)] = e^{-\psi(t)} \mathbf{E} \left( \sum_{|x|=1} e^{-tV(x)} h(V(x)) \right).$$

A second very useful fact is contained in the following remark, it tells essentially that, in probability, the  $e^{-V(x)}$ -weighted number of vertices  $x$  such that  $x \in \mathcal{O}_n$  (recall (I.10)) can be found in a quite small quantity when  $|x| \leq A\ell_n$  and can not be found when  $|x| > A\ell_n$ . This remark is not precise at all but will be enough for our purpose.

**Remark 2.** *There exists  $c_3 \in (0, 1)$  such that for any  $A > 0$  and  $n$  large enough*

$$\mathbf{E} \left[ \sum_{|x| > \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \leq n^{-Ac_3} \quad \text{and} \quad \mathbf{E} \left[ \sum_{|x| \leq \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \leq \ell_n/2,$$

which implies  $\mathbf{E} \left[ \sum_{x \in \mathbb{T}} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \leq \ell_n$ .

*Démonstration.* We give a proof here which essentially use technical Lemma 4.6 (for the second inequality below), indeed by Lemma 2.1 above

$$\begin{aligned} \mathbf{E} \left[ \sum_{|x| > \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] &\leq \sum_{k > \lfloor A\ell_n \rfloor} \mathbb{P}(\sup_{i \leq k} (\bar{S}_i - S_i) \leq \log n) \\ &\leq \sum_{k > \lfloor A\ell_n \rfloor} \exp\left(-\frac{k\pi^2\sigma^2(1-\varepsilon)}{8 \log n}\right) \leq n^{-Ac_3}. \end{aligned}$$

A similar computation gives the second fact and both of them the last one.  $\square$

## 2.2 Proofs of Theorems 1.1 to 1.4

The pattern of the proofs of each theorem is the following : we first prove two facts (an upper and a lower bound) about the sum  $\sum_{k \geq 1} \Psi_{\cdot, \cdot}^k(F)$  with specific  $F$ , depending on the considered function  $f^{n,k}$  and on a slightly different version of the latter whether we are looking for an upper or a lower bound. Then we use this two facts to prove that (A1), (A2), (A3) and (A4) are satisfied.

In these proofs, we use several times the notation  $\varepsilon_b = \min(b + \mathbf{1}_{\{b=0\}}, 1 - b)/13$  which was introduced in Proposition 1.

*Proof of Theorem 1.1.* Recall that  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbf{1}_{\{t_k \geq (\log n)^\alpha\}}$ ,  $\alpha \in (1, 2)$  and see (I.10)

for the definition of  $\mathcal{O}_{\lambda, \lambda'}$ . All along the proof, we assume that  $B, \delta > 0$ ,  $\varepsilon \in (0, \varepsilon_b)$ ,  $n$  is large enough and  $t \geq -B$ . Let us start with the proof of the following two facts :

$$\mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \right] \leq e^{\delta t - (\log n)^{\alpha-1}(1-\varepsilon)}, \quad (\text{I.22})$$

and for any  $0 \leq m \leq \log n$

$$\mathbf{E} \left[ \sum_{x \in \mathcal{O}_{\lambda_n, n^b}} e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha + m, H_x \leq n^b e^{\varepsilon(\log n)^{\alpha-1}}, \underline{V}(x) \geq -B\}} \right] \geq e^{-(\log n)^{\alpha-1}(1+\varepsilon)}, \quad (\text{I.23})$$

with  $\lambda_n = n e^{-6(\log n)^{\alpha-1}}$  and recall  $\underline{V}(x) = \min_{u \leq x} V(u)$ . We first deal with the upper bound (I.22). Recall  $\ell_n = (\log n)^3$ ,

$$\begin{aligned} \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \right] &\leq \sum_{k \leq \lfloor A \ell_n \rfloor} \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \mathbb{1}_{\{x \in \mathcal{O}_n\}} \right] \\ &\quad + \mathbf{E} \left[ \sum_{|x| > \lfloor A \ell_n \rfloor} e^{-V(x)} \mathbb{1}_{\{x \in \mathcal{O}_n\}} \right], \end{aligned}$$

where  $A > 0$  is chosen such that  $\mathbf{E}[\sum_{|x| > \lfloor A \ell_n \rfloor} e^{-V(x)} \mathbb{1}_{\{x \in \mathcal{O}_n\}}] \leq 1/n$  (see Remark 2). This yields, as  $\alpha \in (1, 2)$ ,  $\mathbf{E}[\sum_{|x| > \lfloor A \ell_n \rfloor} e^{-V(x)} \mathbb{1}_{\{x \in \mathcal{O}_n\}}] \leq e^{\delta(t+B)} \frac{1}{n} \leq \frac{1}{2} e^{\delta t - (\log n)^{\alpha-1}(1-\varepsilon)}$  for  $n$  large enough and any  $t \geq -B$ . Thanks to many-to-one Lemma 2.1, the first sum in the above inequality is smaller than

$$\sum_{k \leq \lfloor A \ell_n \rfloor} \mathbf{P}(S_k \geq (\log n)^\alpha - t, \max_{j \leq k} \bar{S}_j - S_j \leq \log n) \leq \lfloor A \ell_n \rfloor \mathbf{P}\left(\max_{j \leq \tau_{(\log n)^\alpha - t}} \bar{S}_j - S_j \leq \log n\right),$$

with  $\tau_r = \inf\{i \geq 1; S_i \geq r\}$ . Then, thanks to Lemma A.3 in [HS16a], and as  $t \geq -B$

$$\begin{aligned} \lfloor A \ell_n \rfloor \mathbf{P}\left(\max_{j \leq \tau_{(\log n)^\alpha - t}} \bar{S}_j - S_j \leq \log n\right) &\leq \lfloor A \ell_n \rfloor e^{\frac{t}{\log n} - (\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})} \leq \lfloor A \ell_n \rfloor e^{\frac{t+B}{\log n} - (\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})} \\ &\leq e^{\delta t + \delta B - (\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})} \\ &\leq \frac{1}{2} e^{\delta t - (\log n)^{\alpha-1}(1-\varepsilon)}, \end{aligned}$$

so we get exactly (I.22).

We now turn to the lower bound (I.23). Let  $\ell'_n = (\log n)^4$  and  $\alpha_n = (\log n)^\alpha + \log n$ . By the many-to-one Lemma, for any  $m \leq \log n$ , the expectation in (I.23) is larger than

$$\sum_{k \geq 1} \mathbf{P}(S_k \geq \alpha_n, \max_{j \leq k} H_j^S \leq \lambda_n, n^b < H_k^S \leq n^b e^{\varepsilon(\log n)^{\alpha-1}}, \underline{S}_k \geq -B),$$

with  $H_j^S := \sum_{i=1}^j e^{S_i - S_j}$ . For any  $b \in (0, 1)$ , by Lemma 4.3 (I.77) (with  $\ell = (\log n)^2$ ,  $t_\ell = \alpha_n$ ,  $q = 1$ ,  $a_b = a = 6$ ,  $d = (\alpha - 1)/2$  and  $c = \varepsilon$ ), above sum is larger than  $e^{-(\log n)^{\alpha-1}(1+\varepsilon)}$ . Otherwise,

if  $b = 0$ , observe that for all  $k \leq \ell'_n$ ,  $\bar{S}_k = S_k$  implies  $H_k^S \leq k \leq \ell'_n$  so the sum is larger than  $\sum_{k \leq \ell'_n} \mathbf{P}(S_k \geq \alpha_n, \max_{j \leq k} H_j^S \leq \lambda_n, \bar{S}_k = S_k, \underline{S}_k \geq -B)$ . Lemma 4.5 (with  $\ell = (\log n)^2$ ,  $t_\ell = \alpha_n$ ,  $d = 1/2$ ,  $a = 6$  and  $d' = (\alpha - 1)/2$ ) leads to (I.23) also for  $b = 0$ .

We are now ready to prove that  $\mathbf{f}^n$  satisfies assumptions (A1), (A2), (A3) and (A4). Recall that  $\Psi_{n,n^b}^k(f^{n,k}) = \mathbf{E}[\sum_{|x|=k} e^{-V(x)} f^{n,k}(V(x_1), \dots, V(x)) \mathbf{1}_{\{x \in \mathcal{O}_{n,n^b}\}}]$  where  $x \in \mathcal{O}_{n,n^b}$  if and only if  $\max_{j \leq |x|} H_{x_j} \leq n$  and  $H_x > n^b$ , also  $\mathcal{U}_b = \{\kappa \in [0, 1]; \text{ for all } k \geq 1, \mathbf{t} \in \mathbb{R}^k, n \geq 1 : \mathbf{1}_{\{H_k(\mathbf{t}) > n^b\}} f^{n,k}(\mathbf{t}) \leq C_\infty n^{-\kappa}\}$  with  $C_\infty = \sup_{n,\ell} \|f^{n,\ell}\|_\infty$ .

• Check of (A1) and asymptotic of  $h_n$ . We obtain from (I.23) with  $m = 0$  that for any  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough,  $\mathbf{E}[\sum_{x \in \mathcal{O}_{n,n^b}} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}}]$  is larger than (as  $\lambda_n \leq n$ )

$$\mathbf{E}\left[\sum_{x \in \mathcal{O}_{\lambda_n, n^b}} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \mathbf{1}_{\{H_x \leq n^b e^{\varepsilon(\log n)^{\alpha-1}}, \underline{V}(x) \geq -B\}}\right] \geq e^{-(\log n)^{\alpha-1}(1+\varepsilon)}.$$

Note that above inequality implies that for all  $b \in [0, 1)$ ,  $\kappa_b = \max \mathcal{U}_b = 0$ . Indeed, if we had  $\kappa_b > 0$ , then this should imply that for any  $x \in \mathbb{T}$

$$e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_{n,n^b}\}} f^{n,k}(V(x_1), \dots, V(x)) \leq C_\infty n^{-\kappa_b} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_{n,n^b}\}},$$

which gives that  $\mathbf{E}[\sum_{x \in \mathcal{O}_{n,n^b}} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \mathbf{1}_{\{H_x \leq n^b e^{\varepsilon(\log n)^{\alpha-1}}, \underline{V}(x) \geq -B\}}]$  is smaller than  $C_\infty n^{-\kappa_b} \mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}] \leq C_\infty \ell_n n^{-\kappa_b}$  by Remark 2, but this contradicts the above lower bound (I.23) as  $\alpha \in (1, 2)$ .

Then, by definition of  $\Psi_{n,n^b}^k$ ,

$$\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) = \mathbf{E}\left[\sum_{x \in \mathcal{O}_{n,n^b}} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}}\right] \geq e^{-(\log n)^{\alpha-1}(1+\varepsilon)},$$

and additionally with (I.22) (taking  $t = 0$ ), asymptotic of  $h_n$  is given by

$$h_n = \left| n^{\kappa_b} \log \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right) \right| = \left| \log \mathbf{E}\left[ \sum_{x \in \mathcal{O}_{n,n^b}} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \right] \right| \sim (\log n)^{\alpha-1}.$$

We also deduce from the previous lower bound that (A1) is satisfied, indeed, as  $\alpha \in (1, 2)$ ,  $\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \geq n^{-(\kappa_b + \varepsilon_1) \wedge 1}$  for any  $\varepsilon_1 > 0$  and  $n$  large enough.

• For (A2), recalling  $m_n = \lceil \varepsilon h_n / c_2 \rceil$  (see (I.16)), then by definition

$$\begin{aligned} f_{\varepsilon h_n}^{n,j}(t_1, \dots, t_j) &= \inf_{s \in [-\varepsilon h_n, \varepsilon h_n]^{m_n}} f^{n, m_n + j}(s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_j + s_{m_n}) \\ &= \inf_{s_{m_n} \in [-\varepsilon h_n, \varepsilon h_n]} \mathbf{1}_{\{t_j + s_{m_n} \geq (\log n)^\alpha\}} = \mathbf{1}_{\{t_j \geq (\log n)^\alpha + \varepsilon h_n\}}. \end{aligned}$$

Observe that for  $A > 0$ ,  $n$  large enough, any  $l \in \mathbb{N}$  and  $\mathbf{t} = (t_1, \dots, t_l)$ , by definition of  $\Psi_n^k(F|\mathbf{t})$

(see (I.18)) and (I.22) with  $\varepsilon/3A$  instead of  $\varepsilon$

$$\begin{aligned} \sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k (f_{\varepsilon h_n}^{n, l+k} | \mathbf{t}) &= \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbf{1}_{\{V(x) + t_l \geq (\log n)^\alpha + \varepsilon h_n\}} \mathbf{1}_{\{H_x > n^b - H_l(\mathbf{t})\}} \right] \\ &\leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha - t_l\}} \right] \leq e^{\delta t_l - (\log n)^{\alpha-1} (1 - \frac{\varepsilon}{3A})}. \end{aligned}$$

Moreover,  $e^{-(\log n)^{\alpha-1} (1 - \frac{\varepsilon}{3A})} = e^{\frac{2\varepsilon}{3A} (\log n)^{\alpha-1}} e^{-(\log n)^{\alpha-1} (1 + \frac{\varepsilon}{3A})} \leq e^{\frac{\varepsilon}{A} h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k (f^{n, k})$ , the last inequality coming from the fact that  $h_n \sim (\log n)^{\alpha-1}$  and (I.23) with  $m = 0$  and as above  $\frac{\varepsilon}{3A}$  instead of  $\varepsilon$ . So (A2) is satisfied.

We are left to prove that technical assumptions (A3) and (A4) are realized.

• For (A3), recall first, from Proposition 1, that for all  $b \in [0, 1)$ ,  $\Upsilon_n^k$  is the set

$$\{\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k; H_k(\mathbf{t}) \leq n^b e^{\varepsilon h_n}, t_k \geq 2\ell_n^{1/3}/\delta_1, \min_{j \leq k} t_j \geq -B\},$$

with  $\lambda_n = n e^{\min(10\varepsilon \log n, -5h_n)} = n e^{-5h_n}$  for large  $n$ . Let  $0 < \varepsilon_1 < \varepsilon$ , note that  $\lambda_n/2 \geq \boldsymbol{\lambda}_n = n e^{-6(\log n)^{\alpha-1}}$  so for  $n$  large enough

$$\begin{aligned} u_{1, n} &= \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k (f_{\varepsilon h_n}^{n, k} \mathbf{1}_{\Upsilon_n^k}) = \mathbf{E} \left[ \sum_{x \in \mathcal{O}_{\lambda_n/2, n^b}} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha + \varepsilon h_n, H_x \leq n^b e^{\varepsilon h_n}, V(x) \geq -B\}} \right] \\ &\geq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_{\boldsymbol{\lambda}_n, n^b}} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha + h_n, H_x \leq n^b e^{\frac{\varepsilon_1}{3} (\log n)^{\alpha-1}}, V(x) \geq -B\}} \right] \\ &\geq e^{-(\log n)^{\alpha-1} (1 + \frac{\varepsilon_1}{3})}, \end{aligned}$$

where we use that  $(\log n)^\alpha > 2\ell_n^{1/3}/\delta_1$  for the second equality and the last inequality comes from (I.23), with  $m = h_n$  and  $\varepsilon_1/3$  instead of  $\varepsilon$ .

Moreover,  $e^{-(\log n)^{\alpha-1} (1 + \frac{\varepsilon_1}{3})} = e^{-\frac{2\varepsilon_1}{3} (\log n)^{\alpha-1}} e^{-(\log n)^{\alpha-1} (1 - \frac{\varepsilon_1}{3})} \geq e^{-\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k (f^{n, k})$  which comes from the fact that  $h_n \sim (\log n)^{\alpha-1}$  and (I.22) with  $t = 0$ ,  $\frac{\varepsilon_1}{3}$  instead of  $\varepsilon$ .

• Finally for (A4), recall the definition of  $u_{2, n}$  just below (I.21). First observe that as  $\alpha \in (1, 2)$ , for  $n$  large enough,  $(\log n)^\alpha > \ell_n^{1/3}/\delta_1$  so for any  $k$

$$\Psi_n^k (f^{n, k} \mathbf{1}_{\mathbb{R} \setminus \mathcal{H}_{\ell_n^{1/3}/\delta_1}^k}) = \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha, V(x) < \ell_n^{1/3}/\delta_1\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] = 0.$$

Recall that  $\mathbf{E}[W] = e^{\psi(1)} = 1$  so

$$\begin{aligned} \sum_{k \geq 1} (\Psi_{n, n^b / (\log n)^2}^k (f^{n, k}) + \mathbf{E}[W \Psi_{n, n^b / (W(\log n)^2)}^k (f^{n, k})]) &\leq \sum_{k \geq 1} (\Psi_n^k (f^{n, k}) + \mathbf{E}[W \Psi_n^k (f^{n, k})]) \\ &= 2 \sum_{k \geq 1} \Psi_n^k (f^{n, k}), \end{aligned}$$

$2 \sum_{k \geq 1} \Psi_n^k(f^{n,k}) = 2\mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}}] \leq 2e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{6})} \leq e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})}$   
thanks to (I.22) with  $t = 0$ ,  $\varepsilon_1/6$  instead of  $\varepsilon$ .

Moreover,  $e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})} = e^{\frac{2\varepsilon_1}{3}(\log n)^{\alpha-1}} e^{-(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})} \leq e^{\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$ . The last inequality comes from the fact that  $h_n \sim (\log n)^{\alpha-1}$  and (I.23) with  $m = 0$  and  $\frac{\varepsilon_1}{3}$  instead of  $\varepsilon$ .  $\square$

*Proof of Theorem 1.2.* Here  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbf{1}_{\{t_{\lfloor k/\beta \rfloor} \geq (\log n)^\alpha\}}$  with  $\beta > 1$  and  $\alpha \in (1, 2)$ , let us start with the proof of the two following facts, for all  $B, \delta > 0$ ,  $\varepsilon \in (0, \varepsilon_b)$ ,  $n$  large enough, any  $t \geq -B$  and  $i \in \mathbb{N}$

$$\mathbf{E} \left[ \sum_{x, \lfloor (|x|+i)/\beta \rfloor > i} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta \rfloor - i}) \geq (\log n)^{\alpha-t}\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \leq e^{\delta t - c_\beta (\log n)^{\alpha-1}(1-\varepsilon)}, \quad (\text{I.24})$$

and for any  $m \leq \log n$

$$\mathbf{E} \left[ \sum_{x, \lfloor (|x|+i)/\beta \rfloor > i} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta \rfloor - i}) \geq (\log n)^{\alpha+m}\}} \mathbf{1}_{\{x \in \Upsilon_n \cap \mathcal{O}_{\lambda_n, n^b}\}} \right] \geq e^{-c_\beta (\log n)^{\alpha-1}(1+\varepsilon)}, \quad (\text{I.25})$$

with  $\lambda_n = ne^{-6c_\beta (\log n)^{\alpha-1}}$ , for any  $a > \frac{1}{\delta_1}$

$$\Upsilon_n = \Upsilon_n(\varepsilon) := \{x \in \mathbb{T}; H_x \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}}, V(x) \geq a \log n, \underline{V}(x) \geq -B\},$$

and  $c_\beta = -1 - \pi\sqrt{\beta-1}/2 + \rho((\beta-1)\pi^2/4)$  (for  $\rho$  see (I.6)). Recall  $\ell_n = (\log n)^3$  and introduce  $L_n := \lfloor (\log n)^{2+\varepsilon_\alpha} \rfloor$  with  $\varepsilon_\alpha \in (0, \alpha-1)$ .

*Proof of (I.24)* : first note that if  $t > (\log n)^\alpha/2$ , (I.24) is obviously satisfied, indeed

$$\mathbf{E} \left[ \sum_{x, \lfloor (|x|+i)/\beta \rfloor > i} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta \rfloor - i}) \geq (\log n)^{\alpha-t}\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \right],$$

and by Remark 2,  $\mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}] = \mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}] e^{\delta t - \delta t} \leq \ell_n e^{\delta t - \frac{\delta}{2}(\log n)^\alpha} \leq e^{\delta t - c_\beta (\log n)^{\alpha-1}}$   
for  $n$  large enough. Now assume  $t \leq (\log n)^\alpha/2$ . The expectation in (I.24) is smaller than

$$\sum_{k \leq \lfloor A\ell_n \rfloor} \sum_{p \geq 1} \mathbf{1}_{\{p = \lfloor \frac{k+i}{\beta} \rfloor - i\}} \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} \mathbf{1}_{\{V(x_p) \geq (\log n)^{\alpha-t}\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] + \mathbf{E} \left[ \sum_{|x| > \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right],$$

with  $A > 0$  such that the last term is smaller than  $1/n$  (Remark 2). Note that  $p = \lfloor \frac{k+i}{\beta} \rfloor - i$  implies  $k \geq \lceil \beta p \rceil$  and as  $\sum_{k \leq \lfloor A\ell_n \rfloor} \mathbf{1}_{\{p = \lfloor \frac{k+i}{\beta} \rfloor - i\}} \leq \beta$  for any  $p \geq 1$ , the above sum is smaller, by the many-to-one Lemma, than

$$\beta \sum_{p \leq \lfloor A\ell_n \rfloor} \mathbf{P}(S_p \geq (\log n)^\alpha - t, \max_{j \leq \lfloor p\beta \rfloor} H_j^S \leq n) + \frac{1}{n} \leq \beta \sum_{p=L_n}^{\lfloor A\ell_n \rfloor} \mathbf{P}(S_p \geq (\log n)^\alpha - t, \max_{j \leq \lfloor p\beta \rfloor} H_j^S \leq n) \quad (\text{I.26})$$

$$+ \beta \sum_{p < L_n} \mathbf{P}(S_p \geq (\log n)^\alpha - t) + \frac{1}{n}.$$

For the second sum in (I.26), by the exponential Markov inequality, for  $n$  large enough, all  $p < L_n$  and  $t \geq -B$

$$\mathbf{P}(S_p \geq (\log n)^\alpha - t) \leq e^{\delta_n t - \delta_n (\log n)^\alpha + p\psi(1-\delta_n)} \leq e^{\delta_n(t+B) - \frac{(\log n)^{2\alpha}}{2\sigma^2 L_n} + L_n\psi(1-\delta_n)} \leq e^{\delta t - (1-\varepsilon)\frac{(\log n)^{2\alpha}}{2\sigma^2 L_n}},$$

with  $\delta_n := (\log n)^\alpha / \sigma^2 L_n$ , and we have used that  $\psi(1-\delta_n) \in \mathbb{R}^+$  for the second inequality and that  $\delta_n \rightarrow 0$  ( $\alpha \in (1, 2)$ ) together with  $\psi(1) = \psi'(1) = 0$  and  $\psi''(1) = \sigma^2$  for the last one.

For the first sum in (I.26), which gives the main contribution, by the Markov property at time  $p$ ,  $\mathbf{P}(S_p \geq (\log n)^\alpha - t, \max_{j \leq \lceil \beta p \rceil} H_j^S \leq n)$  is smaller than  $\mathbf{P}(S_p \geq (\log n)^\alpha - t, \max_{j \leq p} H_j^S \leq n) \mathbf{P}(\max_{j \leq \lceil (\beta-1)p \rceil} H_j^S \leq n)$ . Then thanks to Lemma 4.6 (I.79) (with  $\ell = (\log n)^2$ ,  $\lceil (\beta-1)p \rceil$  and  $\varepsilon/2$  in place of, respectively,  $k$  and  $\varepsilon$ ), for  $n$  large enough and any  $p \in \{L_n, \dots, \lfloor A\ell_n \rfloor\}$

$$\mathbf{P}\left(\max_{j \leq \lceil (\beta-1)p \rceil} H_j^S \leq n\right) \leq e^{-p \frac{\pi^2 \sigma^2 (\beta-1)}{8(\log n)^2} (1-\frac{\varepsilon}{2})} = e^{-p \frac{\pi^2 \sigma^2 (\beta-1)}{8((1-\varepsilon/2)^{-1/2} \log n)^2}}.$$

Hence, as  $\log n \leq (1-\varepsilon/2)^{-1/2} \log n$ ,  $\sum_{p=L_n}^{\lfloor A\ell_n \rfloor} \mathbf{P}(S_p \geq (\log n)^\alpha - t, \max_{j \leq \lceil \beta p \rceil} H_j^S \leq n)$  is smaller than

$$\begin{aligned} & \sum_{p=L_n}^{\lfloor A\ell_n \rfloor} \mathbf{E} \left[ \mathbf{1}_{\{\tau_{(\log n)^\alpha - t} \leq p, \max_{j \leq k} \bar{S}_j - S_j \leq (1-\varepsilon/2)^{-1/2} \log n\}} e^{-p \frac{\pi^2 \sigma^2 (\beta-1)}{8((1-\varepsilon/2)^{-1/2} \log n)^2}} \right] \\ & \leq A\ell_n \mathbf{E} \left[ \mathbf{1}_{\{\max_{j \leq \tau_{(\log n)^\alpha - t}} \bar{S}_j - S_j \leq (1-\varepsilon/2)^{-1/2} \log n\}} e^{-\tau_{(\log n)^\alpha - t} \frac{\pi^2 \sigma^2 (\beta-1)}{8((1-\varepsilon/2)^{-1/2} \log n)^2}} \right] \\ & \leq A\ell_n e^{\sqrt{1-\frac{\varepsilon}{2}} \frac{c_\beta t}{\log n} - c_\beta (\log n)^{\alpha-1} (1-\frac{\varepsilon}{2})} \leq A\ell_n e^{\frac{c_\beta(t+B)}{\log n} - c_\beta (\log n)^{\alpha-1} (1-\frac{\varepsilon}{2})} \leq \frac{1}{3} e^{\delta t - c_\beta (\log n)^{\alpha-1} (1-\varepsilon)}, \end{aligned}$$

where Lemma 4.1 (with  $\ell = ((1-\varepsilon/2)^{-1/2} \log n)^2$ ,  $r(\ell) = (\log n)^\alpha - t$ ,  $c = \pi^2(\beta-1)/4$  and  $1 - \sqrt{1-\varepsilon/2}$  instead of  $\varepsilon$ ) provides the second inequality. Finally collecting all the upper bounds of the three sums in (I.26), for  $n$  large enough

$$\begin{aligned} & \mathbf{E} \left[ \sum_{x: \lfloor (|x|+i)/\beta \rfloor > i} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta \rfloor - i}) \geq (\log n)^\alpha - t\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \\ & \leq \frac{1}{3} e^{\delta t - c_\beta (\log n)^{\alpha-1} (1-\varepsilon)} + \beta e^{\delta t - (1-\varepsilon)\frac{(\log n)^{2\alpha}}{2\sigma^2 L_n}} + \frac{1}{n} \leq \frac{2}{3} e^{\delta t - c_\beta (\log n)^{\alpha-1} (1-\varepsilon)} + \frac{e^{\delta(t+B)}}{n}, \end{aligned}$$

which is smaller than  $e^{\delta t - c_\beta (\log n)^{\alpha-1} (1-\varepsilon)}$  (we have used that  $(\log n)^{2\alpha} / L_n \geq (\log n)^{2(\alpha-1) - \varepsilon_\alpha}$  and  $(\log n)^{\alpha-1} = o((\log n)^{2(\alpha-1) - \varepsilon_\alpha})$ ). This yields the upper bound in (I.24).

*Proof of (I.25).* Let  $\alpha_n := (\log n)^\alpha + \log n$ . For all  $m \leq \log n$ , by the many-to-one Lemma, the

expectation in (I.25) is larger than

$$\sum_{p,k \geq 1} \mathbf{1}_{\{p = \lfloor (k+i)/\beta \rfloor - i\}} \mathbf{P}(S_p \geq \alpha_n, n^b < H_k^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}}, \max_{j \leq k} H_j^S \leq \lambda_n, S_k \geq \frac{2\ell_n^{\frac{1}{3}}}{\delta_1}, \underline{S}_k \geq -B).$$

The above probability is larger than (as  $\alpha_n > a \log n$  for all  $a > \frac{1}{\delta_1}$ )

$$\mathbf{P}(S_p \geq \alpha_n, \underline{S}_p \geq -B, \bar{S}_p = S_p, n^b < H_k^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}}, \max_{j \leq k} H_j^S \leq \lambda_n, \min_{p < j \leq k} S_j \geq S_p).$$

Recall that  $H_j^S = \sum_{i=1}^j e^{S_i - S_j}$  so we have, for any  $p < j \leq k$ ,  $H_j^S = e^{S_p - S_j} H_p^S + H_{p,j}^S$  where  $H_{p,j}^S = \sum_{i=p+1}^j e^{S_i - S_j}$ . Note that  $\bar{S}_p = S_p$  and  $\min_{p < j \leq k} S_j \geq S_p$  implies  $H_j^S \leq p + H_{p,j}^S$  so the previous probability is larger than

$$\mathbf{P}(S_p \geq \alpha_n, \underline{S}_p \geq -B, \bar{S}_p = S_p, \max_{j \leq p} H_j^S \leq \lambda_n, n^b < H_{p,k}^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}} - p, \max_{p < j \leq k} H_{p,j}^S \leq \lambda_n - p, \min_{p < j \leq k} S_j \geq S_p),$$

which, thanks to the Markov property at time  $p$ , is nothing but the product of  $\mathbf{P}(S_p \geq \alpha_n, \underline{S}_p \geq -B, \bar{S}_p = S_p, \max_{j \leq p} H_j^S \leq \lambda_n)$  and  $\mathbf{P}(n^b < H_{k-p}^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}} - p, \max_{j \leq k-p} H_j^S \leq \lambda_n - p = n e^{-6(\log n)^{\alpha-1}} - p, \underline{S}_{k-p} \geq 0)$ . From now, let  $p \in \{L_n, \dots, \ell'_n = (\log n)^4\}$ . We first deal with the second probability. Observe that for all  $i \geq 0$ ,  $p = \lfloor (k+i)/\beta \rfloor - i$  implies  $k-p \geq \lceil (\beta-1)L_n \rceil$ . It follows that for all  $\varepsilon \in (0, \varepsilon_b)$ ,  $n$  large enough, for all  $L_n \leq p \leq \ell'_n$ ,  $k \geq 1$ ,  $i \geq 0$  such that  $p = \lfloor (k+i)/\beta \rfloor - i$ ,  $\mathbf{P}(n^b < H_{k-p}^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}} - p, \max_{j \leq k-p} H_j^S \leq \lambda_n - p, \underline{S}_{k-p} \geq 0)$  is larger than (as  $\lambda_n - p \geq \lambda_n - \ell'_n \geq n e^{-7c_\beta (\log n)^{\alpha-1}}$ )

$$\mathbf{P}(n^b < H_{k-p}^S \leq n^b e^{\frac{\varepsilon}{2} c_\beta (\log n)^{\alpha-1}}, \max_{j \leq k-p} H_j^S \leq n e^{-7c_\beta (\log n)^{\alpha-1}}, \underline{S}_{k-p} \geq 0) \geq e^{-\frac{\pi^2 \sigma^2}{8} \frac{(k-p)}{(\log \lambda'_n)^2}},$$

with  $\lambda'_n := n^{(1+\varepsilon/2)^{-1/2}}$ . The last inequality comes from Lemma 4.6 (I.80) (with  $\ell = (\log n)^2$ ,  $a = 7$ ,  $c = \frac{\varepsilon c_\beta}{2}$ ,  $d = \frac{\alpha-1}{2}$ ,  $k-p$  and  $\varepsilon/2$  instead respectively of  $k$  and  $\varepsilon$ ). The equality  $p = \lfloor (k+i)/\beta \rfloor - i$  also implies, for any  $0 \leq i \leq \log n$  that  $k-p \leq (p + \log n)(\beta-1) + \beta$  so it follows that the above probability is larger than  $C \exp(\frac{\pi^2 \sigma^2 (\beta-1)}{8(\log \lambda'_n)^2} p)$  for some positive constant  $C \in (0, 1)$ . Collecting the previous inequalities together with Lemma 4.4 gives, as  $\sum_{k \geq 1} \mathbf{1}_{\{p = \lfloor (k+i)/\beta \rfloor - i\}} \geq 1$ , that for  $n$  large enough, the mean in (I.25) is larger than

$$\begin{aligned} & C \sum_{p=L_n}^{\ell'_n} \mathbf{E} \left[ e^{-\frac{\pi^2 \sigma^2 (\beta-1)}{8(\log \lambda'_n)^2} p} \mathbf{1}_{\{S_p \geq \alpha_n, \underline{S}_p \geq -B, \bar{S}_p = S_p, \max_{j \leq p} H_j^S \leq n e^{-7c_\beta (\log n)^{\alpha-1}}\}} \right] \sum_{k \geq 1} \mathbf{1}_{\{p = \lfloor (k+i)/\beta \rfloor - i\}} \\ & \geq C \mathbf{P}(\underline{S}_{\ell'_n} \geq 0)^2 \mathbf{E} \left[ e^{-\frac{\pi^2 \sigma^2 (\beta-1)}{8(\log \lambda'_n)^2} \tau_{\alpha_n}} \mathbf{1}_{\{L_n \leq \tau_{\alpha_n} \leq \ell'_n, \forall j \leq \tau_{\alpha_n}: \bar{S}_j - S_j \leq \log \lambda'_n\}} \right] \\ & \geq C \mathbf{P}(\underline{S}_{\ell'_n} \geq 0)^2 \mathbf{P}(\bar{S}_{\ell'_n} \geq \alpha_n) \mathbf{E} \left[ e^{-\frac{\pi^2 \sigma^2 (\beta-1)}{8(\log \lambda'_n)^2} \tau_{\alpha_n}} \mathbf{1}_{\{\forall j \leq \tau_{\alpha_n}: \bar{S}_j - S_j \leq \log \lambda'_n\}} \right] - \mathbf{P}(\bar{S}_{L_n} \geq \alpha_n). \end{aligned}$$

Note that thanks to (I.69) and the fact that  $\alpha \in (1, 2)$ , we can find a constant  $c_{(1.2)} > 0$  such that  $\mathbf{CP}(S_{\ell'_n} \geq 0)^2 \mathbf{P}(\bar{S}_{\ell'_n} \geq \alpha_n) \geq c_{(1.2)}(\ell'_n)^{-1} \geq 2e^{-\frac{\varepsilon}{2}(\log n)^{\alpha-1}}$ . Then applying Lemma 4.1 (with  $\ell = \log \lambda'_n$ ,  $r = \alpha_n$ ,  $c = \pi^2(\beta - 1)/4$  and  $\sqrt{1 + \varepsilon/2} - 1$  instead of  $\varepsilon$ ), for  $n$  large enough

$$\mathbf{E} \left[ e^{-\frac{\pi^2 \sigma^2 (\beta-1)}{8(\log \lambda'_n)^2} \tau \alpha_n} \mathbf{1}_{\{\forall j \leq \tau \alpha_n : \bar{S}_j - S_j \leq \log \lambda'_n\}} \right] \geq e^{-c_\beta (\log n)^{\alpha-1} (1 + \frac{\varepsilon}{2})}.$$

Finally, by Markov inequality,  $\mathbf{P}(\bar{S}_{L_n} \geq \alpha_n) \leq L_n e^{-c'_{(1.2)} \alpha_n^2 / L_n}$  for some constant  $c'_{(1.2)} > 0$ . Since  $\alpha_n^2 / L_n \geq (\log n)^{2(\alpha-1) - \varepsilon \alpha}$  and  $(\log n)^{\alpha-1} = o((\log n)^{2(\alpha-1) - \varepsilon \alpha})$ , we get that  $\mathbf{P}(\bar{S}_{L_n} \geq \alpha_n) \leq e^{-c_\beta (\log n)^{\alpha-1} (1 + \varepsilon)}$ . Collecting the different estimates yields (I.25).

We are ready to prove that  $\mathbf{f}^n$  satisfies assumptions (A1), (A2), (A3) and (A4). Recall that  $\Psi_{n,n^b}^k(f^{n,k}) = \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} f^{n,k}(V(x_1), \dots, V(x)) \mathbf{1}_{\{x \in \mathcal{O}_{n,n^b}\}} \right]$  where  $x \in \mathcal{O}_{n,n^b}$  if and only if  $\max_{j \leq |x|} H_{x_j} \leq n$  and  $H_x > n^b$ ,  $\mathcal{U}_b = \{\kappa \in [0, 1]; \text{ for all } k \geq 1, \mathbf{t} \in \mathbb{R}^k, n \geq 1 : \mathbf{1}_{\{H_k(\mathbf{t}) > n^b\}} f^{n,k}(\mathbf{t}) \leq C_\infty n^{-\kappa}\}$  with  $C_\infty = \sup_{n,\ell} \|f^{n,\ell}\|_\infty$ .

• Check of (A1) and asymptotic of  $h_n$ . We obtain from (I.25) with  $i = m = 0$  and  $n$  large enough

$$\begin{aligned} \mathbf{E} \left[ \sum_{x \in \mathcal{O}_{n,n^b}} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor |x|/\beta \rfloor}) \geq (\log n)^\alpha\}} \right] &\geq \mathbf{E} \left[ \sum_{x \in \mathbb{T}} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor |x|/\beta \rfloor}) \geq (\log n)^\alpha\}} \mathbf{1}_{\{x \in \Upsilon_n \cap \mathcal{O}_{\lambda_n, n^b}\}} \right] \\ &\geq e^{-c_\beta (\log n)^{\alpha-1} (1 + \varepsilon)}. \end{aligned}$$

This implies that for all  $b \in [0, 1)$ ,  $\kappa_b = \max \mathcal{U}_b = 0$  (we use a similar argument than in the proof of Theorem 1.1) and additionally with (I.24), gives, taking  $i = t = 0$

$$h_n = \left| n^{\kappa_b} \log \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right) \right| = \left| \log \mathbf{E} \left[ \sum_{x \in \mathcal{O}_{n,n^b}} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor |x|/\beta \rfloor}) \geq (\log n)^\alpha\}} \right] \right| \sim c_\beta (\log n)^{\alpha-1}.$$

We also deduce from the previous lower bound that (A1) is satisfied.

• For (A2), recalling  $m_n = \lceil \varepsilon h_n / c_2 \rceil$  ( $c_2$  is defined in (I.16)), by definition, for any  $j > 0$

$$\begin{aligned} f_{\varepsilon h_n}^{n,j}(t_1, \dots, t_j) &= \inf_{\mathbf{s} \in [-\varepsilon h_n, \varepsilon h_n]^{m_n}} f^{n,m_n+j}(s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_j + s_{m_n}) \\ &= \inf_{s_{m_n} \in [-\varepsilon h_n, \varepsilon h_n]} \mathbf{1}_{\{t_{\lfloor (m_n+j)/\beta \rfloor - m_n} \geq (\log n)^\alpha - s_{m_n}\}} \\ &= \mathbf{1}_{\{\lfloor (m_n+j)/\beta \rfloor > m_n\}} \mathbf{1}_{\{t_{\lfloor (m_n+j)/\beta \rfloor - m_n} \geq (\log n)^\alpha + \varepsilon h_n\}}. \end{aligned}$$

Then for any  $l \in \mathbb{N}^*$  and all  $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$ ,  $f_{\varepsilon h_n}^{n,l+k}(t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l)$ , with  $|x| = k$ , is equal to

$$\mathbf{1}_{\{m_n < \lfloor (k+i)/\beta \rfloor \leq i\}} \mathbf{1}_{\{t_{\lfloor (k+i)/\beta \rfloor - m_n} \geq (\log n)^\alpha + \varepsilon h_n\}} + \mathbf{1}_{\{\lfloor (k+i)/\beta \rfloor > i\}} \mathbf{1}_{\{V(x_{\lfloor (k+i)/\beta \rfloor - i}) + t_l \geq (\log n)^\alpha + \varepsilon h_n\}},$$



with  $i = m_n + l$ . Recall the definition of  $\Psi_{\cdot, \cdot}(F|\mathbf{t})$  in (I.18), we have

$$\begin{aligned} \sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k(f_{\varepsilon h_n}^{n, l+k} | \mathbf{t}) &\leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} f_{\varepsilon h_n}^{n, l+k}(t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l) \right] \\ &\leq \sum_{k \geq 1} \mathbf{1}_{\{m_n < \lfloor (i+k)/\beta \rfloor \leq i\}} \mathbf{1}_{\{t_{\lfloor (i+k)/\beta} - m_n \geq (\log n)^\alpha\}} \Psi_n^k(1) \\ &\quad + \mathbf{E} \left[ \sum_{x: \lfloor (|x|+i)/\beta \rfloor > i} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta} - i}) \geq (\log n)^\alpha - t_l\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right]. \end{aligned}$$

$\sum_{k \geq 1} \mathbf{1}_{\{m_n < \lfloor (i+k)/\beta \rfloor \leq i\}} \mathbf{1}_{\{t_{\lfloor (i+k)/\beta} - m_n \geq (\log n)^\alpha\}} \Psi_n^k(1)$  is equal to

$$\sum_{p=1}^l \mathbf{1}_{\{t_p \geq (\log n)^\alpha\}} \sum_{k \geq 1} \Psi_n^k(1) \mathbf{1}_{\{p = \lfloor \frac{i+k}{\beta} \rfloor - m_n\}} \leq \beta \sum_{p=1}^l \mathbf{1}_{\{t_p \geq (\log n)^\alpha\}},$$

where we have used that  $\sum_{k \geq 1} \Psi_n^k(1) \mathbf{1}_{\{p = \lfloor \frac{i+k}{\beta} \rfloor - m_n\}} \leq e^{\psi(1)} \sum_{k \geq 1} \mathbf{1}_{\{p = \lfloor \frac{i+k}{\beta} \rfloor - m_n\}} \leq \beta$ . Also by (I.24) with  $i = m_n + l$ ,  $t = t_l$  and  $\frac{\varepsilon}{4A}$  instead of  $\varepsilon$ ,

$$\mathbf{E} \left[ \sum_{x: \lfloor (|x|+i)/\beta \rfloor > i} e^{-V(x)} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta} - i}) \geq (\log n)^\alpha - t_l\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \leq e^{\delta t_l - c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon}{4A})},$$

so

$$\sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k(f_{\varepsilon h_n}^{n, l+k} | \mathbf{t}) \leq \beta \sum_{p=1}^l \mathbf{1}_{\{t_p \geq (\log n)^\alpha\}} + \frac{1}{2} e^{\delta t_l - c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon}{3A})}.$$

Note that  $\beta \sum_{p=1}^l \mathbf{1}_{\{t_p \geq (\log n)^\alpha\}}$  is very small for  $n$  large enough, any  $l < \lfloor Al_n \rfloor$  and  $H_l(\mathbf{t}) \leq n$ . Indeed,  $\sum_{p=1}^l e^{\delta(t_p - t_l)} \leq l H_l(\mathbf{t})^\delta \leq Al_n n^\delta$  so

$$\beta \sum_{p=1}^l \mathbf{1}_{\{t_p \geq (\log n)^\alpha\}} = e^{\delta t_l} \beta \sum_{p=1}^l e^{\delta(t_p - t_l)} e^{-\delta t_p} \mathbf{1}_{\{t_p \geq (\log n)^\alpha\}} \leq e^{\delta t_l} \beta Al_n n^\delta e^{-\delta (\log n)^\alpha},$$

which, as  $\alpha \in (1, 2)$ , is smaller than  $\frac{1}{2} e^{\delta t_l - c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon}{3A})}$ . Finally observe that

$$e^{-c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon}{3A})} = e^{c_\beta (\log n)^{\alpha-1} \frac{2\varepsilon}{3A}} e^{-c_\beta (\log n)^{\alpha-1} (1 + \frac{\varepsilon}{3A})} \leq e^{\frac{\varepsilon}{A} h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k}),$$

where we have used that  $h_n \sim c_\beta (\log n)^{\alpha-1}$  and (I.25) with  $i = m = 0$ .

We are left to prove that the technical assumptions (A3) and (A4) are realized. The ideas are very similar to those of the proof of these two assumptions in the previous theorem, we give details here however to keep the proofs independent from one another.

- For (A3), recall that  $\Upsilon_n^k$  is the set

$$\{\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k; H_k(\mathbf{t}) \leq n^b e^{\varepsilon h_n}, t_k \geq 2\ell_n^{1/3}/\delta_1, \min_{j \leq k} t_j \geq -B\}.$$

Let  $0 < \varepsilon_1 < \varepsilon$  and recall that  $\lambda_n = ne^{-5h_n}$ . Note that  $\lambda_n/2 \geq \boldsymbol{\lambda}_n = ne^{-6(\log n)^{\alpha-1}}$  so the sum  $\sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k(f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k})$  is larger than  $\sum_{k \geq 1} \Psi_{\boldsymbol{\lambda}_n, n^b}^k(f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k})$  which is nothing but

$$\begin{aligned} & \mathbf{E} \left[ \sum_{x \in \mathcal{O}_{\boldsymbol{\lambda}_n, n^b}} e^{-V(x)} \mathbb{1}_{\{\lfloor \frac{|x|+m_n}{\beta} \rfloor > m_n\}} \mathbb{1}_{\{V(x_{\lfloor (|x|+m_n)/\beta \rfloor - m_n}) \geq (\log n)^\alpha + \varepsilon h_n, H_x \leq n^b e^{\varepsilon h_n}, V(x) \geq \frac{2}{\delta_1} \ell_n^{1/3}, \underline{V}(x) \geq -B\}} \right] \\ & \geq \mathbf{E} \left[ \sum_{x, \lfloor (|x|+m_n)/\beta \rfloor > m_n} e^{-V(x)} \mathbb{1}_{\{V(x_{\lfloor (|x|+m_n)/\beta \rfloor - m_n}) \geq (\log n)^\alpha + h_n\}} \mathbb{1}_{\{x \in \Upsilon_n(\frac{\varepsilon_1}{3}) \cap \mathcal{O}_{\boldsymbol{\lambda}_n, n^b}\}} \right] \\ & \geq e^{-c_\beta (\log n)^{\alpha-1} (1 + \frac{\varepsilon_1}{3})}, \end{aligned}$$

where this last inequality comes from (I.25) with  $i = m = m_n$  and  $\varepsilon_1/3$  instead of  $\varepsilon$ . Moreover,  $e^{-c_\beta (\log n)^{\alpha-1} (1 + \frac{\varepsilon_1}{3})} = e^{-\frac{2\varepsilon_1}{3} c_\beta (\log n)^{\alpha-1}} e^{-c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon_1}{3})} \geq e^{-\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})$ , the last inequality comes from the fact that  $h_n \sim c_\beta (\log n)^{\alpha-1}$  and (I.24) with  $i = t = 0$ .

- For (A4), first observe that for all  $k \in \mathbb{N}^*$  and  $\alpha \in (1, 2)$ ,  $(\log n)^\alpha - \ell_n^{1/3}/\delta_1 > \log n$  for  $n$  large enough so

$$\begin{aligned} \Psi_n^k(f^{n,k} \mathbb{1}_{\mathbb{R} \setminus \mathcal{R}_{\ell_n^{1/3}/\delta_1}^k}) &= \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} \mathbb{1}_{\{V(x_{\lfloor |x|/\beta \rfloor}) \geq (\log n)^\alpha, V(x) < \ell_n^{1/3}/\delta_1\}} \mathbb{1}_{\{x \in \mathcal{O}_n\}} \right] \\ &\leq \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} \mathbb{1}_{\{\bar{V}(x) \geq (\log n)^\alpha, V(x) < \ell_n^{1/3}/\delta_1\}} \mathbb{1}_{\{\bar{V}(x) - V(x) \leq \log n\}} \right] = 0. \end{aligned}$$

Recall that  $W = \sum_{|z|=1} e^{-V(z)}$  and

$$\sum_{k \geq 1} (\Psi_{n, n^b / (\log n)^2}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n, n^b / (W(\log n)^2)}^k(f^{n,k})]) \leq \sum_{k \geq 1} (\Psi_n^k(f^{n,k}) + \mathbf{E}[W \Psi_n^k(f^{n,k})]),$$

which is equal to  $2 \sum_{k \geq 1} \Psi_n^k(f^{n,k})$  since  $\mathbf{E}[W] = e^{\psi(1)} = 1$  and thanks to (I.24) with  $i = t = 0$  and  $\frac{\varepsilon_1}{4}$  in place of  $\varepsilon$

$$\begin{aligned} 2 \sum_{k \geq 1} \Psi_n^k(f^{n,k}) &= 2 \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbb{1}_{\{V(x_{\lfloor |x|/\beta \rfloor}) \geq (\log n)^\alpha\}} \right] \leq 2e^{-c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon_1}{4})} \\ &\leq e^{-c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon_1}{3})}. \end{aligned}$$

Moreover,  $e^{-c_\beta (\log n)^{\alpha-1} (1 - \frac{\varepsilon_1}{3})} = e^{\frac{2\varepsilon_1}{3} c_\beta (\log n)^{\alpha-1}} e^{-c_\beta (\log n)^{\alpha-1} (1 + \frac{\varepsilon_1}{3})} \leq e^{\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})$ , the last inequality comes from the fact that  $h_n \sim c_\beta (\log n)^{\alpha-1}$  and (I.25) with  $i = m = 0$ .  $\square$

*Proof of Theorem 1.3.*

**Assume first that  $a = d = 1$  and  $\alpha \in (1, 2)$**  which corresponds to the second and third case of

the theorem. Let us start with the proof of the two facts, note that we distinguish whether  $b = 0$  or  $b \in (0, 1/2)$ .

Facts for the case  $b = 0$  : for all  $B, \delta > 0$ ,  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough, for any  $t \geq -B$ ,

$$\mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)}}{\sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \right] \leq e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}, \quad (\text{I.27})$$

and for all  $0 \leq m \leq \log n$ ,  $0 \leq M \leq e^{(\log n)^{\alpha/2}}$

$$\mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)} \mathbf{1}_{\{x \in \Upsilon_{n,1} \cap \mathcal{O}_{\lambda_{n,1}}\}}}{M|x| + \sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha + m\}} \right] \geq e^{-2(\log n)^{\alpha/2}(1+\varepsilon)}, \quad (\text{I.28})$$

with  $\lambda_{n,1} = ne^{-12(\log n)^{\alpha/2}}$  and

$$\Upsilon_{n,1} = \Upsilon_{n,1}(\varepsilon) := \{x \in \mathbb{T}; H_x \leq e^{2\varepsilon(\log n)^{\alpha/2}}, \underline{V}(x) \geq -B\}.$$

We first deal with the upper bound (I.27). Note that if  $t > (\log n)^\alpha/2$ , then (I.27) is obviously satisfied. Indeed,  $(\sum_{j \leq |x|} H_{x_j})^{-1} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \leq 1$  so for  $n$  large enough

$$\begin{aligned} \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)}}{\sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \right] &\leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \right] e^{-\delta t} e^{\delta t} \leq \ell_n e^{\delta t - \frac{\delta}{2}(\log n)^\alpha} \\ &\leq e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}, \end{aligned}$$

where we have used Remark 2. Now assume  $t \leq (\log n)^\alpha/2$ , by the many-to-one Lemma, the expectation in (I.27) is smaller than

$$\sum_{k \leq \lfloor A\ell_n \rfloor} \mathbf{E} \left[ \frac{1}{\sum_{j=1}^k H_j^S} \mathbf{1}_{\{\tau_{(\log n)^\alpha - t} \leq k, \max_{j \leq k} H_j^S \leq n\}} \right] + \mathbf{E} \left[ \sum_{|x| > \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right], \quad (\text{I.29})$$

the second sum is treated as usual : Remark 2 with a chosen  $A$ , together with the fact that  $\alpha \in (1, 2)$  and  $t \geq -B$  implies that  $\mathbf{E} \left[ \sum_{|x| > \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] \leq 1/n \leq \frac{1}{2} e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}$ .

Also using that  $(\sum_{j=1}^k H_j^S)^{-1} \leq e^{-\max_{j \leq k} \bar{S}_j - S_j}$  leads to

$$\sum_{k \leq \lfloor A\ell_n \rfloor} \mathbf{E} \left[ \frac{1}{\sum_{j=1}^k H_j^S} \mathbf{1}_{\{\tau_{(\log n)^\alpha - t} \leq k, \max_{j \leq k} H_j^S \leq n\}} \right] \leq \lfloor A\ell_n \rfloor \mathbf{E} \left[ e^{-\max_{j \leq \tau_{(\log n)^\alpha - t}} \bar{S}_j - S_j} \right].$$

Since  $t < (\log n)^\alpha/2$ ,  $(\log n)^\alpha - t > (\log n)^\alpha/2$  so by Lemma 4.2 with  $\frac{\varepsilon}{2}$  instead of  $\varepsilon$  and any  $t \geq -B$

$$\mathbf{E} \left[ e^{-\max_{j \leq \tau_{(\log n)^\alpha - t}} \bar{S}_j - S_j} \right] \leq e^{-2(1-\frac{\varepsilon}{2})\sqrt{(\log n)^\alpha - t}} \leq e^{-2(1-\frac{\varepsilon}{2})\frac{(\log n)^\alpha - (t+B)}{\sqrt{(\log n)^\alpha + B}}} \leq \frac{1}{2} e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}.$$

This treats the first sum in (I.29) and yields (I.27).

We now turn to the lower bound (I.28). Recall  $\ell'_n = (\log n)^4$ , using that  $\sum_{j=1}^k H_j^S \leq k \max_{j \leq k} H_j^S$  and the fact that  $m \leq \log n$ ,  $0 \leq M \leq e^{(\log n)^{\alpha/2}}$  and  $\lambda_{n,1} > e^{(\log n)^{\alpha/2}}$ , we obtain thanks to the many-to-one Lemma

$$\begin{aligned} & \mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)} \mathbf{1}_{\{x \in \Upsilon_{n,1} \cap \mathcal{O}_{\lambda_{n,1}}\}}}{M|x| + \sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha + m\}} \right] \\ & \geq \sum_{k \leq \ell'_n} \mathbf{E} \left[ \frac{1}{2k e^{(\log n)^{\alpha/2}}} \mathbf{1}_{\{S_k \geq \alpha_n, \max_{j \leq k} H_j^S \leq e^{(\log n)^{\alpha/2}}, \underline{S}_k \geq -B, \bar{S}_k = S_k\}} \right] \\ & \geq \frac{e^{-(\log n)^{\alpha/2}}}{2\ell'_n} \sum_{k \leq \ell'_n} \mathbf{P}(S_k \geq \alpha_n, \max_{j \leq k} H_j^S \leq e^{(\log n)^{\alpha/2}}, \underline{S}_k \geq -B, \bar{S}_k = S_k), \end{aligned}$$

where  $\alpha_n = (\log n)^\alpha + \log n$ . By Lemma 4.5 (with  $\ell = (\log n)^2$ ,  $t_\ell = \alpha_n$ ,  $d = \alpha/4$  and  $a = 0$ ), the previous probability is larger than  $e^{-(\log n)^{\alpha/2}(1+\frac{\varepsilon}{2})}$ . Finally collecting the inequalities, we get (I.28).

Facts for the case  $b \in (0, 1/2)$  : for any  $t \geq -B$ ,  $r \geq 0$  and  $w > 0$

$$\mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)} \mathbf{1}_{\{r+H_x > n^b/(w(\log n)^2)\}}}{r + \sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \right] \leq (w+1)n^{-b} e^{\delta t - \frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}. \quad (\text{I.30})$$

Also for all  $0 \leq m \leq \log n$ ,  $0 \leq M \leq n^b$

$$\mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)} \mathbf{1}_{\{x \in \Upsilon_{n,2} \cap \mathcal{O}_{\lambda_{n,2}, n^b}\}}}{M|x| + \sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha + m\}} \right] \geq n^{-b} e^{-\frac{1+\varepsilon}{b}(\log n)^{\alpha-1}}, \quad (\text{I.31})$$

with  $\lambda_{n,2} = n e^{-\frac{6}{b}(\log n)^{\alpha-1}}$  and

$$\Upsilon_{n,2} = \Upsilon_{n,2}(\varepsilon) := \{x \in \mathbb{T}; H_x \leq n^b e^{\frac{\varepsilon}{b}(\log n)^{\alpha-1}}, V(x) \geq -B\}.$$

We first deal with the upper bound (I.30). We split the sum according to the generation of  $x$  : when  $|x| > \lfloor A\ell_n \rfloor$ , we use that  $\mathbf{1}_{\{r+H_x > n^b/(w(\log n)^2), V(x) \geq (\log n)^\alpha - t\}} (r + \sum_{j \leq |x|} H_{x_j})^{-1} \leq 1$  so the expectation in (I.30) is smaller than

$$\mathbf{E} \left[ \sum_{|x| > \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] + \mathbf{E} \left[ \sum_{|x| \leq \lfloor A\ell_n \rfloor} \frac{e^{-V(x)} \mathbf{1}_{\{r+H_x > n^b/(w(\log n)^2)\}}}{r + \sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha - t\}} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right].$$

Then, when  $|x| \leq \lfloor A\ell_n \rfloor$ , we again split the sum but this time according to  $\max_{j \leq |x|} H_{x_j}$  : when  $\max_{j \leq |x|} H_{x_j} > n^b e^{\frac{1}{b}(\log n)^{\alpha-1}}$ , we use that  $\mathbf{1}_{\{r+H_x > n^b/(w(\log n)^2), V(x) \geq (\log n)^\alpha - t\}} (r + \sum_{j \leq |x|} H_{x_j})^{-1} \leq (\max_{j \leq |x|} H_{x_j})^{-1} \leq n^{-b} e^{-\frac{1}{b}(\log n)^{\alpha-1}}$ . Otherwise, one can observe that  $\mathbf{1}_{\{r+H_x > n^b/(w(\log n)^2)\}} (r + \sum_{j \leq |x|} H_{x_j})^{-1} \leq \mathbf{1}_{\{r+H_x > n^b/(w(\log n)^2)\}} (r + H_x)^{-1} \leq w n^{-b} (\log n)^2$ . Therefore, the expectation in

(I.30) is smaller than

$$\begin{aligned} & \mathbf{E} \left[ \sum_{|x| > \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] + \mathbf{E} \left[ \sum_{|x| \leq \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}} \right] n^{-b} e^{-\frac{1}{b}(\log n)^{\alpha-1}} \\ & + wn^{-b}(\log n)^2 \mathbf{E} \left[ \sum_{|x| \leq \lfloor A\ell_n \rfloor} e^{-V(x)} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha - t, \max_{j \leq |x|} H_{x_j} \leq n^b e^{\frac{1}{b}(\log n)^{\alpha-1}}\}} \right], \end{aligned}$$

which, by Remark 2 and the many-to-one Lemma, is smaller, for  $n$  large enough, than

$$\frac{1}{n} + \ell_n n^{-b} e^{-\frac{1}{b}(\log n)^{\alpha-1}} + wn^{-b}(\log n)^2 \sum_{k \leq \lfloor A\ell_n \rfloor} \mathbf{P}(S_k \geq (\log n)^\alpha - t, \max_{j \leq k} H_j^S \leq n^b e^{\frac{1}{b}(\log n)^{\alpha-1}}).$$

Also,  $\sum_{k \leq \lfloor A\ell_n \rfloor} \mathbf{P}(S_k \geq (\log n)^\alpha - t, \max_{j \leq k} H_j^S \leq n^b e^{-\frac{1}{b}(\log n)^{\alpha-1}})$  is smaller than

$$\lfloor A\ell_n \rfloor \mathbf{P} \left( \max_{j \leq \tau(\log n)^{\alpha-t}} \bar{S}_j - S_j \leq b \log n + \frac{1}{b}(\log n)^{\alpha-1} \right) \leq e^{\frac{t}{\log n} - \frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})},$$

where Lemma A.3 in [HS16a] provides us the last inequality for  $n$  large enough and any  $t$ . Finally, note that for any  $\delta > 0$ ,  $n$  large enough, any  $w > 0$  and any  $t \geq -B$ ,  $1/n \leq \frac{1}{3}n^{-b}e^{-\delta B - \frac{1-\varepsilon}{b}(\log n)^{\alpha-1}} \leq \frac{w+1}{3}n^{-b}e^{\delta t - \frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}$ ,  $\ell_n n^{-b}e^{-\frac{1}{b}(\log n)^{\alpha-1}} \leq \frac{1}{3}n^{-b}e^{-\delta B - \frac{1-\varepsilon}{b}(\log n)^{\alpha-1}} \leq \frac{w+1}{3}n^{-b}e^{\delta t - \frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}$ ,  $wn^{-b}(\log n)^2 e^{\frac{t}{\log n} - \frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})} \leq \frac{w+1}{n^b}(\log n)^2 e^{\frac{t+B}{\log n} - \frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})} \leq \frac{w+1}{3n^b}e^{\delta t - \frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}$  and this finish the proof of the first fact. We now turn to the lower bound (I.31). By the many-to-one Lemma, for any  $m \leq \log n$ ,  $0 \leq M \leq n^b$  and  $A > 0$ , the mean in (I.31) is larger than (as  $\lambda_{n,2} > n^b e^{\frac{\varepsilon}{3b}(\log n)^{\alpha-1}}$ )

$$\begin{aligned} & \sum_{k \leq \lfloor A\ell_n \rfloor} \mathbf{E} \left[ \frac{1}{kn^b + \sum_{j=1}^k H_j^S} \mathbf{1}_{\{S_k \geq \alpha_n, \max_{1 \leq j \leq k} H_j^S \leq n^b e^{\frac{\varepsilon}{3b}(\log n)^{\alpha-1}}, H_k^S > n^b, \underline{S}_k \geq -B\}} \right] \\ & \geq \frac{n^{-b}}{2A\ell_n} e^{-\frac{\varepsilon}{3b}(\log n)^{\alpha-1}} \sum_{k \leq \lfloor A\ell_n \rfloor} \mathbf{P}(S_k \geq \alpha_n, \max_{1 \leq j \leq k} H_j^S \leq n^b e^{\frac{\varepsilon}{3b}(\log n)^{\alpha-1}}, H_k^S > n^b, \underline{S}_k \geq -B), \end{aligned}$$

with  $\alpha_n := (\log n)^\alpha + \log n$ . By Lemma 4.3 (I.77) (with  $\ell = (\log n)^2$ ,  $t_\ell = \alpha_n$ ,  $q = b$ ,  $a_b = -a = -\frac{\varepsilon}{3b}$ ,  $d = \frac{\alpha-1}{2}$  and  $c = \frac{\varepsilon}{3b}$ ) the above sum is larger, for  $n$  large enough, than  $e^{-\frac{1}{b}(\log n)^{\alpha-1}(1+\frac{\varepsilon}{2})} \geq 2A\ell_n e^{-\frac{1}{b}(\log n)^{\alpha-1}(1+\varepsilon)}$ , which completes the proof of the upper bound.

We are ready to prove that  $\mathbf{f}^n$  satisfies assumptions (A1), (A2), (A3) and (A4).

• Check of (A1) and asymptotic of  $h_n$ . (I.28) with  $m = M = 0$  implies, for  $b = 0$  and  $n$  large enough

$$\sum_{k \geq 1} \Psi_n^k(f^{n,k}) \geq \mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)}}{\sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \mathbf{1}_{\{x \in \Upsilon_{n,1} \cap \mathcal{O}_{\lambda_{n,1}}\}} \right] \geq e^{-2(\log n)^{\alpha/2}(1+\varepsilon)}.$$

This implies that  $\kappa_0 = \max \mathcal{U}_0 = 0$  (see the part concerning  $\kappa_b$  in the proof of Theorem 1.1 for

details) and additionally with (I.27) and  $t = 0$

$$h_n = \left| n^{\kappa_b} \log \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right) \right| \sim 2(\log n)^{\alpha/2}.$$

We also deduce from the previous lower bound that (A1) is satisfied.

From (I.30) with  $r = t = 0$ ,  $w = 1$  and  $\frac{\varepsilon}{2}$  instead of  $\varepsilon$ , we get for all  $b \in (0, 1)$  and  $n$  large enough

$$\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)} \mathbf{1}_{\{H_x > n^b / (\log n)^2\}}}{\sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \right] \leq n^{-b} e^{-\frac{1-\varepsilon}{b} (\log n)^{\alpha-1}}.$$

This implies that for all  $b \in (0, 1)$ ,  $\kappa_b \geq b$ . From (I.31) with  $m = M = 0$ , we get that for all  $b \in (0, 1)$

$$\sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \geq \mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)}}{\sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \mathbf{1}_{\{x \in \Upsilon_{n,2} \cap \mathcal{O}_{\lambda_{n,2}, n^b}\}} \right] \geq n^{-b} e^{-\frac{1+\varepsilon}{b} (\log n)^{\alpha-1}}.$$

This implies that for all  $b \in (0, 1/2)$ ,  $\kappa_b \leq b$ . Finally, for any  $b \in (0, 1/2)$ ,  $\kappa_b = b$  and

$$h_n = \left| n^{\kappa_b} \log \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right) \right| \sim \frac{1}{b} (\log n)^{\alpha-1}.$$

We also deduce from the previous lower bound that (A1) is satisfied.

• For (A2), recalling  $m_n = \lceil \varepsilon h_n / c_2 \rceil$  (see (I.16)) and for all  $\mathbf{s} = (s_1, \dots, s_{m_n}) \in \mathbb{R}^{m_n}$ ,  $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$ , with  $\mathbf{u} = (s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_k + s_{m_n})$

$$f^{n, m_n+k}(s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_k + s_{m_n}) = \mathbf{1}_{\{t_k + s_{m_n} \geq (\log n)^\alpha\}} \frac{1}{\sum_{j=1}^{m_n+k} H_j(\mathbf{u})}. \quad (\text{I.32})$$

Note that  $\sum_{j=1}^{m_n+k} H_j(\mathbf{u}) = \sum_{j=1}^{m_n} H_j(\mathbf{s}) + \sum_{j=1}^k (e^{-t_j} H_{m_n}(\mathbf{s}) + H_j(\mathbf{t})) \geq \sum_{j=1}^k H_j(\mathbf{t})$  so

$$\begin{aligned} f_{\varepsilon h_n}^{n,k}(t_1, \dots, t_k) &= \inf_{\mathbf{s} \in [-\varepsilon h_n, \varepsilon h_n]^{m_n}} f^{n, m_n+k}(s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_k + s_{m_n}) \\ &\leq \inf_{s_{m_n} \in [-\varepsilon h_n, \varepsilon h_n]} \mathbf{1}_{\{t_k + s_{m_n} \geq (\log n)^\alpha\}} \frac{1}{\sum_{j=1}^k H_j(\mathbf{t})} = \frac{\mathbf{1}_{\{t_k \geq (\log n)^\alpha + \varepsilon h_n\}}}{\sum_{j=1}^k H_j(\mathbf{t})}. \end{aligned}$$

It follows that  $f_{\varepsilon h_n}^{n,k}(t_1, \dots, t_k) \leq \mathbf{1}_{\{t_k \geq (\log n)^\alpha\}} (\sum_{j=1}^k H_j(\mathbf{t}_j))^{-1}$  and for  $|x| = k$  with  $\mathbf{u}_x = (t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l)$

$$f_{\varepsilon h_n}^{n, l+k}(t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l) \leq \mathbf{1}_{\{V(x) \geq (\log n)^{\alpha-t_l}\}} \frac{1}{\sum_{j=1}^{l+k} H_j(\mathbf{u}_x)}.$$

Assume  $b = 0$ . Observe again that  $\sum_{j=1}^{l+k} H_j(\mathbf{u}_x) = \sum_{j=1}^l H_j(\mathbf{t}) + \sum_{j=1}^k (e^{-V(x_j)} H_l(\mathbf{t}) + H_{x_j}) \geq \sum_{j \leq k} H_{x_j}$ . Then, by definition of  $\Psi_n^k(F|\mathbf{t})$  (see (I.18)), for all  $A, B, \varepsilon, \delta > 0$ ,  $n$  large enough, for

any  $l \in \mathbb{N}^*$  and all  $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$  with  $t_l \geq -B$

$$\begin{aligned} \sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k (f_{\varepsilon h_n}^{n, l+k} | \mathbf{t}) &\leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} f_{\varepsilon h_n}^{n, l+k}(t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l) \right] \\ &\leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^{\alpha - t_l}\}} \frac{1}{\sum_{j \leq k} H_{x_j}} \right] \leq e^{\delta t_l - 2(\log n)^{\alpha/2}(1 - \frac{\varepsilon}{3A})}, \end{aligned}$$

where we have used (I.27) with  $t = t_l$  and replaced  $\varepsilon$  by  $\frac{\varepsilon}{3A}$  for the last inequality. Finally, observe that

$$e^{-2(\log n)^{\alpha/2}(1 - \frac{\varepsilon}{3A})} = e^{\frac{4\varepsilon}{3A}(\log n)^{\alpha/2}} e^{-2(\log n)^{\alpha/2}(1 + \frac{\varepsilon}{3A})} \leq e^{\frac{\varepsilon}{A}h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k (f^{n, k}),$$

where we have used that  $h_n \sim 2(\log n)^{\alpha/2}$  and (I.28) with  $m = M = 0$ .

Assume  $b \in (0, 1/2)$ . Note that  $\sum_{j=1}^{l+k} H_j(\mathbf{u}_x) \geq H_l(\mathbf{t}) + \sum_{j \leq k} H_{x_j}$ . Then for all  $A, B, \varepsilon, \delta > 0$ ,  $n$  large enough, for any  $l \in \mathbb{N}^*$  and all  $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$  with  $t_l \geq -B$

$$\begin{aligned} \sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k (f_{\varepsilon h_n}^{n, l+k} | \mathbf{t}) &\leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \frac{\mathbb{1}_{\{V(x) \geq (\log n)^{\alpha - t_l}\}}}{H_l(\mathbf{t}) + \sum_{j \leq |x|} H_{x_j}} \mathbb{1}_{\{H_l(\mathbf{t}) + H_x > n^b / (\log n)^2\}} \right] \\ &\leq 2n^{-b} e^{\delta t_l - \frac{1}{b}(\log n)^{\alpha-1}(1 - \frac{\varepsilon}{4A})} \leq n^{-b} e^{\delta t_l - \frac{1}{b}(\log n)^{\alpha-1}(1 - \frac{\varepsilon}{3A})}, \end{aligned}$$

where we have used (I.30) with  $r = H_l(\mathbf{t})$ ,  $w = 1$ ,  $t = t_l$  and  $\frac{\varepsilon}{4A}$  instead of  $\varepsilon$  for the last inequality. Finally, observe that

$$n^{-b} e^{\delta t_l - \frac{1}{b}(\log n)^{\alpha-1}(1 - \frac{\varepsilon}{3A})} = e^{\frac{2\varepsilon}{3bA}(\log n)^{\alpha-1}} n^{-b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1 + \frac{\varepsilon}{3A})} \leq e^{\frac{\varepsilon}{A}h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k (f^{n, k}),$$

where we have used that  $h_n \sim \frac{1}{b}(\log n)^{\alpha-1}$  and (I.31) with  $m = M = 0$ .

We are left to prove that technical assumptions (A3) and (A4) are realized.

• For (A3), recall that  $\Upsilon_n^k = \{\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k; H_k(\mathbf{t}) \leq n^b e^{\varepsilon h_n}, V(x) \geq 2\ell_n^{1/3} / \delta_1, t_k \geq -B\}$ . By (I.32), for  $|x| = k$  with  $\mathbf{v}_x = (s_1, \dots, s_{m_n}, V(x_1) + s_{m_n}, \dots, V(x) + s_{m_n})$

$$f_{\varepsilon h_n}^{n, k}(V(x_1), \dots, V(x)) = \inf_{\mathbf{s} \in [-\varepsilon h_n, \varepsilon h_n]^{m_n}} \mathbb{1}_{\{V(x) + s_{m_n} \geq (\log n)^\alpha\}} \frac{1}{\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x)},$$

and recall that  $\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x) = \sum_{j=1}^{m_n} H_j(\mathbf{s}) + \sum_{j=1}^k (e^{-V(x_j)} H_{m_n}(\mathbf{s}) + H_{x_j})$ . For  $|x| = k$  such that  $V(x) \geq -B$ , observe, as  $\mathbf{s} \in [-\varepsilon h_n, \varepsilon h_n]^{m_n}$ , that  $\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x) \leq m_n e^{2\varepsilon h_n} + k m_n^2 e^{2\varepsilon h_n + B} + \sum_{j=1}^k H_{x_j}$ . Also recall, by definition, that  $h_n \geq (\log n)^\gamma$  for  $\gamma \in (0, 1)$  so  $\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x) \leq$

$2km_n^2 e^{2\varepsilon h_n + B} + \sum_{j=1}^k H_{x_j} \leq ke^{3\varepsilon h_n} + \sum_{j=1}^k H_{x_j}$ . It follows that

$$f_{\varepsilon h_n}^{n,k}(V(x_1), \dots, V(x)) \geq \mathbb{1}_{\{V(x) \geq (\log n)^\alpha + \varepsilon h_n\}} \left( ke^{3\varepsilon h_n} + \sum_{j=1}^k H_{x_j} \right)^{-1}.$$

Let  $0 < \varepsilon_1 < \varepsilon$  and recall  $\lambda_n = ne^{-5h_n} \geq 2\lambda_{n,i}$ ,  $i \in \{1, 2\}$ . Thanks to the previous inequality and the fact that  $(\log n)^\alpha > 2\ell_n^{1/3}/\delta_1$ , we have

$$\sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k(f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}) \geq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_{\lambda_{n,i}, n^b}} \frac{e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha + \varepsilon h_n\}}}{|x| e^{3\varepsilon h_n} + \sum_{j \leq |x|} H_{x_j}} \mathbb{1}_{\{H_x \leq n^b e^{\varepsilon h_n}, V(x) \geq -B\}} \right].$$

Assume  $b = 0$ . By (L.28) with  $m = h_n$ ,  $M = e^{(\log n)^{\alpha/2}}$  and  $\frac{\varepsilon_1}{3}$  instead of  $\varepsilon$ , together with the fact that  $h_n \sim 2(\log n)^{\alpha/2}$ , for  $n$  large enough

$$\begin{aligned} \sum_{k \geq 1} \Psi_{\lambda_n/2}^k(f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}) &\geq \mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha + h_n\}}}{|x| e^{(\log n)^{\alpha/2}} + \sum_{j \leq |x|} H_{x_j}} \mathbb{1}_{\{x \in \Upsilon_{n,1}(\frac{\varepsilon_1}{3}) \cap \mathcal{O}_{\lambda_{n,1}}\}} \right] \\ &\geq e^{-2(\log n)^{\alpha/2}(1 + \frac{\varepsilon_1}{3})}. \end{aligned}$$

Moreover,  $e^{-2(\log n)^{\alpha/2}(1 + \frac{\varepsilon_1}{3})} = e^{-\frac{4\varepsilon_1}{3}(\log n)^{\alpha/2}} e^{-2(\log n)^{\alpha/2}(1 - \frac{\varepsilon_1}{3})} \geq e^{-\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})$ , the last inequality comes from the fact that  $h_n \sim 2(\log n)^{\alpha/2}$  and (L.27) with  $t = 0$ .

Assume  $b \in (0, 1/2)$ . By (L.31) with  $m = h_n$  and  $M = n^b$ , together with the fact that  $h_n \sim \frac{1}{b}(\log n)^{\alpha-1}$ , for  $n$  large enough

$$\begin{aligned} \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k(f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}) &\geq \mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha + h_n\}}}{|x| n^b + \sum_{j \leq |x|} H_{x_j}} \mathbb{1}_{\{x \in \Upsilon_{n,2}(\frac{\varepsilon_1}{3}) \cap \mathcal{O}_{\lambda_{n,2}}\}} \right] \\ &\geq n^{-b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1 + \frac{\varepsilon_1}{3})}. \end{aligned}$$

Moreover,  $e^{-\frac{1}{b}(\log n)^{\alpha-1}(1 + \frac{\varepsilon_1}{3})} = e^{-\frac{2\varepsilon_1}{3b}(\log n)^{\alpha-1}} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1 - \frac{\varepsilon_1}{3})} \geq n^b e^{-\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})$ , the last inequality comes from the fact that  $h_n \sim \frac{1}{b}(\log n)^{\alpha-1}$  and (L.30) with  $r = t = 0$ ,  $w = 1$  and we have used that  $n^b(\log n)^{-2} < n^b$ .

• Finally for (A4), we first observe that for all  $k \in \mathbb{N}^*$  and  $\alpha \in (1, 2)$ ,  $(\log n)^\alpha > 2\ell_n^{1/3}/\delta_1$  for  $n$  large enough so

$$\Psi_n^k(f^{n,k} \mathbb{1}_{\mathbb{R} \setminus \mathcal{R}_{\frac{\ell_n^{1/3}}{\delta_1}}^k}) = \mathbf{E} \left[ \sum_{|x|=k} \frac{e^{-V(x)}}{\sum_{j=1}^k H_{x_j}} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha, V(x) < \ell_n^{1/3}/\delta_1\}} \mathbb{1}_{\{x \in \mathcal{O}_n\}} \right] = 0.$$

Recall that  $W = \sum_{|z|=1} e^{-V(z)}$  and  $\mathbf{E}[W] = e^{\psi(1)} = 1$  so when  $b = 0$

$$\sum_{k \geq 1} (\Psi_{n, n^b/(\log n)^2}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n, n^b/(W(\log n)^2)}^k(f^{n,k})]) \leq \sum_{k \geq 1} (\Psi_n^k(f^{n,k}) + \mathbf{E}[W \Psi_n^k(f^{n,k})])$$



$$= 2 \sum_{k \geq 1} \Psi_n^k(f^{n,k}),$$

and thanks to (I.27) for  $n$  large enough with  $t = 0$

$$\begin{aligned} 2 \sum_{k \geq 1} \Psi_n^k(f^{n,k}) &= 2 \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)}}{\sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \right] \leq 2e^{-2(\log n)^{\alpha/2}(1-\frac{\varepsilon_1}{3})} \\ &\leq e^{-2(\log n)^{\alpha/2}(1-\frac{\varepsilon_1}{3})}. \end{aligned}$$

Moreover,  $e^{-2(\log n)^{\alpha/2}(1-\frac{\varepsilon_1}{3})} = e^{\frac{4\varepsilon_1}{3}(\log n)^{\alpha/2}} e^{-2(\log n)^{\alpha/2}(1+\frac{\varepsilon_1}{3})} \leq e^{\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$ , the last inequality comes from the fact that  $h_n \sim 2(\log n)^{\alpha/2}$  and (I.28) with  $m = M = 0$ .

Otherwise,  $b \in (0, 1/2)$  and thanks to (I.30) for  $n$  large enough with  $r = t = 0$ ,  $w = 1$  and  $\frac{\varepsilon_1}{4}$  instead of  $\varepsilon$

$$\sum_{k \geq 1} \Psi_{n,n^b/(\log n)^2}^k(f^{n,k}) = \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)} \mathbf{1}_{\{H_x > n^b/(\log n)^2\}}}{r + \sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \right] \leq \frac{1}{n^b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})},$$

and we also get from (I.30) with  $r = t = 0$  and  $w = W$  that for  $n$  large enough

$$\begin{aligned} \Psi_{n,n^b/(W(\log n)^2)}^k(f^{n,k}) &= \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)} \mathbf{1}_{\{H_x > n^b/(W(\log n)^2)\}}}{r + \sum_{j \leq |x|} H_{x_j}} \mathbf{1}_{\{V(x) \geq (\log n)^\alpha\}} \right] \\ &\leq \frac{W+1}{n^b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{4})}. \end{aligned}$$

By (I.5), telling that  $\mathbf{E}[W^2] < \infty$ , we have  $C_4 := \mathbf{E}[W(W+1)+1] = \mathbf{E}[W^2+2] < \infty$  and then

$$\begin{aligned} \sum_{k \geq 1} (\Psi_{n,n^b/(\log n)^2}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n,n^b/(W(\log n)^2)}^k(f^{n,k})]) &\leq \frac{C_4}{n^b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{4})} \\ &\leq \frac{1}{2n^b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})}. \end{aligned}$$

Moreover,  $e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})} = e^{\frac{\varepsilon_1}{3b}(\log n)^{\alpha-1}} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})} \leq n^b e^{\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})$ , the last inequality comes from the fact that  $h_n \sim \frac{1}{b}(\log n)^{\alpha-1}$  and (I.31) with  $m = M = 0$ . This completes the proof for these two cases.

**Assume now  $\alpha = 1$  and  $a \in \mathbb{R}$  (with  $a > 1/\delta_1$  when  $d = 1$ ),** which corresponds to the first case of the theorem. As usual, let us first state the following two facts :

for all  $b \in [0, 1/(d+1))$ ,  $B, \delta > 0$ ,  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough, for any  $t \geq -B$ ,  $r \geq 0$  and  $w > 0$

$$\mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)} \mathbf{1}_{\{r+H_x > n^b/(w(\log n)^2)\}}}{(r + \sum_{j \leq |x|} H_{x_j})^d} \mathbf{1}_{\{V(x) \geq a \log n - t\}} \right] \leq (w+1) \ell_n^2 e^{\delta t} n^{-bd}, \quad (\text{I.33})$$

For any  $0 \leq M \leq n^b$ ,  $\varepsilon < b/3$  (when  $b > 0$ )

$$\mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)} \mathbb{1}_{\{x \in \Upsilon_n \cap \mathcal{O}_{\lambda_n, n^b}\}}}{(M|x| + \sum_{j \leq |x|} H_{x_j})^d} \mathbb{1}_{\{V(x) \geq a \log n\}} \right] \geq \frac{1}{\ell_n^2} n^{-bd}, \quad (\text{I.34})$$

with  $\lambda_n = n^{1-11\varepsilon}$  and for any  $a' > 1/\delta_1$

$$\Upsilon_n = \Upsilon_n(\varepsilon) := \{x \in \mathbb{T}; H_x \leq n^{b+\varepsilon}, V(x) \geq a' \log n, \underline{V}(x) \geq -B\}.$$

These facts ensure that  $\mathbf{f}^n$  satisfies assumptions **(A1)**, **(A2)**, **(A3)** and **(A4)** for  $b \in (0, 1/(d+1))$ . **(A3)** does not hold exactly when  $b = 0$  so we use **(I.38)** (which appears in the proof of Theorem 1.5) together with the result when  $b > 0$  to conclude this case.

- Check of **(A1)** and asymptotic of  $h_n$ . We get from **(I.34)** that  $\kappa_b = \max \mathcal{U}_b \leq bd$  and **(I.33)** gives  $\kappa_b \geq bd$ . It follows that for all  $b \in [0, 1/(d+1))$ ,  $\kappa_b = bd$  and for any  $n \geq 2$ ,  $h_n = \log n$ . Indeed, on the one hand, **(I.33)** with  $r = t = 0$  and  $w = 1$  gives, for  $n$  large enough

$$n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}) = n^{bd} \mathbf{E} \left[ \sum_{x \in \mathcal{O}_{n, n^b}} \frac{e^{-V(x)}}{(\sum_{j=1}^k H_{x_j})^d} \mathbb{1}_{\{V(x) \geq a \log n\}} \right] \leq 2^d \ell_n^2,$$

and on the other hand, we get from **(I.34)**, for  $n$  large enough that

$$n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}) \geq \frac{1}{\ell_n^2}.$$

From these inequalities, we get that for any  $\gamma \in (0, 1)$ ,  $|\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}))| \leq 3 \log \ell_n = o((\log n)^\gamma)$ . Then  $h_n = \log n$  and we also deduce that **(A1)** is satisfied.

- For **(A2)**, let  $|x| = k$  and observe that  $f_{\varepsilon h_n}^{n, l+k}(t_1, \dots, t_l, V(x_1), \dots, V(x)) \leq (H_l(\mathbf{t}) + H_x)^{-d}$  so it follows, for all  $\varepsilon \in (0, \varepsilon_b)$ ,  $A, \delta, B > 0$ ,  $n$  large enough, any  $l \in \mathbb{N}^*$ ,  $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$  and  $t_l \geq -B$ , by **(I.33)** with  $r = H_l(\mathbf{t})$ ,  $t = t_l$  and  $w = 1$

$$\sum_{k \geq 1} \Psi_{n, n^b - H_l(\mathbf{t})}^k(f_{\varepsilon h_n}^{n, l+k} | \mathbf{t}) \leq 2^d \ell_n^2 e^{\delta t_l} n^{-bd} \leq e^{\delta t_l + \frac{\varepsilon}{A} h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}),$$

where the last inequality comes from **(I.34)**.

- For **(A3)**, recall that  $\Upsilon_n^k = \{\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k; H_k(\mathbf{t}) \leq n^{b+\varepsilon}, V(x) \geq 2\ell_n^{1/3}/\delta_1, t_k \geq -B\}$ . For  $|x| = k$ , we have

$$f_{\varepsilon h_n}^{n,k}(V(x_1), \dots, V(x)) \geq \mathbb{1}_{\{V(x) \geq (a+\varepsilon)(\log n)\}} \left( kn^{3\varepsilon} + \sum_{j=1}^k H_{x_j} \right)^{-d},$$

and thanks to (I.34) with  $M = n^b$ ,  $b \in (0, 1/(d+1))$

$$\sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k(f_{\varepsilon_1 h_n}^{n,k} \mathbb{1}_{\Gamma_n^k}) \geq \mathbf{E} \left[ \sum_{x \in \mathbb{T}} \frac{e^{-V(x)} \mathbb{1}_{\{\Gamma_n \cap \mathcal{O}_{\lambda_n, n^b}\}}}{(|x|n^b + \sum_{j \leq |x|} H_{x_j})^d} \right] \geq \frac{1}{\ell_n^2} n^{-bd} \geq e^{-\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}),$$

where we recall  $\lambda_n = n^{1-10\varepsilon}$ .

• Finally, for (A4) with  $d = 1$  (and then  $a > 1/\delta_1$ )

$$\Psi_n^k(f^{n,k} \mathbb{1}_{\mathbb{R} \setminus \mathcal{H}_{\ell_n^{1/3}/\delta_1}^k}) = \mathbf{E} \left[ \sum_{|x|=k} \frac{e^{-V(x)} \mathbb{1}_{\{x \in \mathcal{O}_n\}}}{(\sum_{j=1}^k H_{x_j})^d} \mathbb{1}_{\{V(x) \geq a \log n, V(x) < \ell_n^{1/3}/\delta_1\}} \right] = 0.$$

Otherwise,  $d = 0$  and for any  $a \in \mathbb{R}$ , thanks to Remark 2

$$\sum_{k \geq 1} \Psi_n^k(f^{n,k} \mathbb{1}_{\mathbb{R} \setminus \mathcal{H}_{\ell_n^{1/3}/\delta_1}^k}) = \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbb{1}_{\{V(x) \geq a \log n, V(x) < \ell_n^{1/3}/\delta_1\}} \right] \leq \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \right] \leq \ell_n,$$

which, thanks to (I.34), is smaller than  $e^{\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})$  for all  $\varepsilon_1 > 0$ . We get from (I.33) with  $r = t = 0$  and  $w = W$  that for  $n$  large enough

$$\Psi_{n, n^b/(W(\log n)^2)}^k(f^{n,k}) = \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \frac{e^{-V(x)} \mathbb{1}_{\{H_x > n^b/(W(\log n)^2)\}}}{(\sum_{j \leq |x|} H_{x_j})^d} \mathbb{1}_{\{V(x) \geq a \log n\}} \right] \leq (W+1) \ell_n^2 n^{-bd}.$$

By (I.5), telling that  $\mathbf{E}[W^2] < \infty$ , we have  $C_4 := \mathbf{E}[W(W+1)+1] = \mathbf{E}[W^2+2] < \infty$  and then

$$\sum_{k \geq 1} (\Psi_{n, n^b/(W(\log n)^2)}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n, n^b/(W(\log n)^2)}^k(f^{n,k})]) \leq 2C_4 \ell_n n^{-bd} \leq e^{\varepsilon_1 h_n} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}),$$

where, again, the last inequality comes from (I.34). This finishes the proof of the result of the theorem for  $b \in (0, 1/(d+1))$ .

Now assume  $b = 0$  and let  $\varepsilon > 0$ . Using the result of the theorem with  $b_\varepsilon = \varepsilon/(2+d)$  and the fact that  $\mathcal{R}_n(\mathbb{1}_{[n^{b_\varepsilon}, \infty)}, \mathbf{f}^n) \leq \mathcal{R}_n(\mathbb{1}_{[1, \infty)}, \mathbf{f}^n)$ , we get the following lower bound for  $\mathcal{R}_n(\mathbb{1}_{[1, \infty)}, \mathbf{f}^n)$ :  $\mathbb{P}(\log^+ \mathcal{R}_n(\mathbb{1}_{[1, \infty)}, \mathbf{f}^n) < (1-\varepsilon) \log n)$  is smaller than

$$\mathbb{P}(\log^+ \mathcal{R}_n(\mathbb{1}_{[n^{b_\varepsilon}, \infty)}, \mathbf{f}^n) < (1 - (1+d)b_\varepsilon - \varepsilon/(2+d)) \log n) \rightarrow 0,$$

where we have used the case  $b > 0$ . For the upper bound, we use an intermediate result in the proof of Theorem 1.5: recall that  $\kappa_0 = 0$  and  $h_n = \log n$ .

Also recall  $\xi = \lim_{n \rightarrow \infty} h_n^{-1} \log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}))$ . It's easy to see that  $\xi = 0$  and by (I.38)

$$\mathbb{P}(\log^+ \mathcal{R}_n(\mathbb{1}_{[1, \infty)}, \mathbf{f}^n) > (1+\varepsilon) \log n) \leq \mathbb{P}\left(\frac{1}{n} \mathcal{R}_n(\mathbb{1}_{[1, \infty)}, \mathbf{f}^n) > e^{\varepsilon h_n}\right) \rightarrow 0,$$

this ends the proof of the theorem for all  $b \in [0, 1/(d+1))$ .  $\square$

*Proof of Theorem 1.4.* Here  $f^{n,k}(t_1, t_2, \dots, t_k) = \mathbf{1}_{\{t_{\lfloor k/\beta \rfloor} \geq (\log n)^\alpha\}} (\sum_{j=1}^{\lfloor k/\beta \rfloor} H_j(\mathbf{t}))^{-1}$  with  $\beta > 1$  and  $\alpha \in (1, 2)$ . We state the following facts : for all  $B, \delta > 0$ ,  $\varepsilon \in (0, \varepsilon_b)$ ,  $n$  large enough, any  $t \geq -B$  and  $i \in \mathbb{N}$

$$\mathbf{E} \left[ \sum_{x; \lfloor |x|+i/\beta \rfloor > i} \frac{e^{-V(x)} \mathbf{1}_{\{x \in \mathcal{O}_n\}}}{\sum_{j=1}^{\lfloor |x|/\beta \rfloor} H_{x_j}} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta \rfloor - i}) \geq (\log n)^\alpha - t\}} \right] \leq e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}, \quad (\text{I.35})$$

and for all  $0 \leq i, m \leq \log n$ ,  $0 \leq M \leq e^{(\log n)^{\alpha/2}}$

$$\mathbf{E} \left[ \sum_{x; \lfloor |x|+i/\beta \rfloor > i} \frac{e^{-V(x)} \mathbf{1}_{\{x \in \Upsilon_n \cap \mathcal{O}_{\lambda_n}\}}}{M|x| + \sum_{j=1}^{\lfloor |x|/\beta \rfloor} H_{x_j}} \mathbf{1}_{\{V(x_{\lfloor (|x|+i)/\beta \rfloor - i}) \geq (\log n)^\alpha + m\}} \right] \geq e^{-2(\log n)^{\alpha/2}(1+\varepsilon)}, \quad (\text{I.36})$$

with  $\lambda_n = ne^{-12(\log n)^{\alpha/2}}$  and for any  $a > \frac{1}{\delta_1}$

$$\Upsilon_n = \Upsilon_n(\varepsilon) := \{x \in \mathbb{T}; H_x \leq e^{2\varepsilon(\log n)^{\alpha/2}}, V(x) \geq a \log n, \underline{V}(x) \geq -B\}.$$

Using these two facts, we follow the same lines as in the previous theorem to prove that  $h_n \sim 2(\log n)^{\alpha/2}$  and that (A1) to (A4) are satisfied.  $\square$

### 2.3 Proof of Theorem 1.5

First, note that Remark 1 implies that  $\xi = \lim_{n \rightarrow \infty} h_n^{-1} \log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}))$  well exists. To prove Theorem 1.5, we first show that Assumptions (A3) and (A4) yield a simpler statement for both lower and upper bound of Proposition 1. This implies a convergence in probability for stopped ranges  $\mathcal{R}_{T^{k_n}}$  with  $k_n = \lceil n/(\log n)^{3/2} \rceil$  and  $\mathcal{R}_{T^n}$ . Then, we use a result of [HS16b] (Proposition 2.4) implying that  $T_n/(n \log n)$  converges in probability to a positive limit in order to obtain the result for  $\mathcal{R}_n$ . Let us start with the

*Lower bound :* Recalling the expression of  $u_{1,n} = \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k(f_{\varepsilon h_n}^{n,k} \mathbf{1}_{\Upsilon_n^k})$  (see below (I.20)), together with (A3) choosing  $\varepsilon_1 = \min(1, c_5) \frac{\varepsilon}{4}$  (see Proposition 1 for  $c_5$ ), we get

$$u_{1,n} \geq e^{-\min(1, c_5) \frac{\varepsilon}{4}} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}).$$

This, together with the fact that, by definition of  $\xi$ ,  $n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}) > e^{(\xi - \varepsilon)h_n}$  for  $n$  large enough, implies

$$\begin{aligned} \mathbb{P} \left( \frac{\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b} \varphi(n^b)} < e^{(\xi - 7\varepsilon)h_n} \right) &\leq \mathbb{P} \left( \frac{\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)}{n^{1-b} \varphi(n^b) \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})} < e^{-6\varepsilon h_n} \right) \\ &\leq \mathbb{P} \left( \frac{\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)}{n^{1-b} \varphi(n^b) u_{1,n}} < e^{-5\varepsilon h_n} \right). \end{aligned}$$

Also considering (I.20),  $\mathbb{P}\left(\frac{\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{1,n}} < e^{-5\varepsilon h_n}\right)$  is smaller than

$$\begin{aligned} & e^{(-c_5 + \frac{\min(1, c_5)}{2})\varepsilon h_n} + h_n e^{-\varepsilon h_n} + \frac{e^{-\min(\varepsilon \log n, 3h_n) + \min(1, c_5)\frac{\varepsilon}{2}h_n}}{\left(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k})\right)^2} \\ & \leq e^{-\frac{\varepsilon c_5}{2}h_n} + h_n e^{-\varepsilon h_n} + e^{-\min(\varepsilon \log n, 3h_n) + \frac{\varepsilon}{2}h_n + 2|\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k}))|}. \end{aligned}$$

Now, thanks to Remark 1, for  $n$  large enough,  $|\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k}))|$  is smaller than  $\frac{\varepsilon}{8} \log n \leq -\min(-\frac{\varepsilon}{8} \log n, -h_n)$  and  $\frac{\varepsilon}{2}h_n$  is smaller than  $\leq -\frac{1}{2} \min(-\varepsilon \log n, -h_n)$  so we have that  $-\min(\varepsilon \log n, 3h_n) + \frac{\varepsilon}{2}h_n + 2|\log(n^{\kappa_b} \sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n, k}))|$  is smaller than  $-\frac{1}{2} \min(\varepsilon \log n, h_n)$ . Finally, for all  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough

$$\mathbb{P}\left(\frac{\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-7\varepsilon)h_n}\right) \leq e^{-\frac{\varepsilon c_5}{2}h_n} + h_n e^{-\varepsilon \tilde{c}_2 h_n} + e^{-\frac{1}{4} \min(\varepsilon \log n, h_n)},$$

then switching  $\varepsilon$  by  $\varepsilon/7$  in the above probability, we obtain as  $h_n \rightarrow +\infty$ , the desired expression : for all  $\varepsilon \in (0, 7\varepsilon_b)$

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\right) = 0.$$

We are now ready to move from  $\mathcal{R}_{T^{k_n}}$  to  $\mathcal{R}_n$ . First note that

$$\mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\right) \leq \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}, T^{k_n} \leq n\right) + \mathbb{P}(T^{k_n} > n),$$

recalling that  $\mathcal{R}_n(g_n, \mathbf{f}^n) = \sum_{x \in \mathbb{T}} g_n(\mathcal{L}_x^n) f^{n, |x|}(V(x_1), V(x_2), \dots, V(x))$  and  $g_n(t) = \varphi(t) \mathbf{1}_{\{t \geq n^b\}}$  with  $b \in [0, 1)$ . Then, as  $\varphi$  is non-decreasing and positive, so is  $g_n$ , hence  $T^{k_n} \leq n$  implies  $g_n(\mathcal{L}_x^{T^{k_n}}) \leq g_n(\mathcal{L}_x^n)$  and therefore  $\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n) \leq \mathcal{R}_n(g_n, \mathbf{f}^n)$  since  $f^{n, k} \geq 0$ . It follows that

$$\mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\right) \leq \mathbb{P}\left(\frac{\mathcal{R}_{T^{k_n}}(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\right) + \mathbb{P}(T^{k_n} > n),$$

and thanks to the above convergence, together with the fact that  $(T^n/(n \log n))_n$  converges in  $\mathbb{P}$ -probability to an almost surely finite and positive random variable, we obtain the desired expression : for all  $\varepsilon \in (0, 7\varepsilon_b)$  :

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\right) = 0. \quad (\text{I.37})$$

*Upper bound* : we prove the following statement, for all  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\right) = 0. \quad (\text{I.38})$$

Recall that  $u_{2,n} = \sum_{k \geq 1} (\Psi_n^k(f^{n,k} \mathbf{1}_{\mathbb{R}^k \setminus \mathcal{H}_n^k} + \Psi_{n,n^b/(W(\log n)^2)}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n,n^b/(W(\log n)^2)}^k(f^{n,k})])$ . Assumption (A4) with  $\varepsilon_1 = \frac{\varepsilon}{4}$  gives that

$$u_{2,n} \leq e^{\frac{\varepsilon}{4} h_n} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}),$$

so for  $n$  large enough, as  $n^{\kappa_b} \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \leq e^{(\xi + \frac{\varepsilon}{2}) h_n}$  and  $T^n \geq n$

$$\begin{aligned} \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b} \varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\right) &\leq \mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b} \varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\right) \\ &\leq \mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b} \varphi(n^b) \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})} > e^{\frac{\varepsilon}{2} h_n}\right) \\ &\leq \mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b} \varphi(n^b) u_{2,n}} > e^{\frac{\varepsilon}{4} h_n}\right) \leq e^{-\frac{\varepsilon}{8} h_n} + o(1), \end{aligned}$$

where the last inequality comes from (I.21) replacing  $\varepsilon$  by  $\frac{\varepsilon}{4}$ . Then, taking the limit, we get (I.38).

We are now ready to prove the theorem. We split this proof in three parts depending on the values of (recall)  $L = \liminf_{n \rightarrow \infty} h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b))$ .

- Assume  $L \in (-\xi, +\infty]$ . For any  $t \in \mathbb{R}$ ,  $e^{\log^+ t} = e^{\log(t \vee 1)} \geq t$  so for any  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough,  $\mathbb{P}(\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - \log(n^{1-b-\kappa_b} \varphi(n^b)) < (\xi - \varepsilon) h_n)$  is smaller than

$$\mathbb{P}(e^{\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n)} < n^{1-b-\kappa_b} \varphi(n^b) e^{(\xi-\varepsilon)h_n}) \leq \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b} \varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\right) \rightarrow 0,$$

where the limit comes from (I.37). Note that this lower bound remains true even when  $L \notin (-\xi, +\infty]$ . However, we need that  $L \in (-\xi, +\infty]$  for the upper bound. Indeed, in this case, for  $n$  large enough,  $n^{1-b-\kappa_b} \varphi(n^b) > e^{-\xi h_n}$  and for any  $\varepsilon > 0$ ,  $n^{1-b-\kappa_b} \varphi(n^b) e^{(\xi+\varepsilon)h_n} > e^{\varepsilon h_n} > 1$  so for  $n$  large enough  $\mathbb{P}(\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) - \log(n^{1-b-\kappa_b} \varphi(n^b)) > (\xi + \varepsilon) h_n) = \mathbb{P}(\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) > \log(n^{1-b-\kappa_b} \varphi(n^b) e^{(\xi+\varepsilon)h_n}), \mathcal{R}_n(g_n, \mathbf{f}^n) > 1)$ . Moreover, when  $\mathcal{R}_n(g_n, \mathbf{f}^n) > 1$ ,  $\log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) = \log \mathcal{R}_n(g_n, \mathbf{f}^n)$  so the previous probability is equal to

$$\mathbb{P}(\log \mathcal{R}_n(g_n, \mathbf{f}^n) > \log(n^{1-b-\kappa_b} \varphi(n^b) e^{(\xi+\varepsilon)h_n}), \mathcal{R}_n(g_n, \mathbf{f}^n) > 1) \leq \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b} \varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\right).$$

Then, taking the limit, we get the result thanks to (I.38).

- Assume  $L = -\xi$ . Recall that  $\underline{\Delta}_n = h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b)) - \inf_{\ell \geq n} h_\ell^{-1} \log(\ell^{1-b-\kappa_b} \varphi(\ell^b))$ .  $L = -\xi$  implies that for any  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough,  $\inf_{\ell \geq n} h_\ell^{-1} \log(\ell^{1-b-\kappa_b} \varphi(\ell^b)) > -\xi - \frac{\varepsilon}{2}$  so  $h_n \underline{\Delta}_n < \log(n^{1-b-\kappa_b} \varphi(n^b)) + (\xi + \frac{\varepsilon}{2}) h_n$  and as  $e^{\log^+ t} \geq t$

$$\begin{aligned} \mathbb{P}(h_n^{-1} \log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) < -\varepsilon + \underline{\Delta}_n) &\leq \mathbb{P}(\mathcal{R}_n(g_n, \mathbf{f}^n) < e^{-\varepsilon h_n + h_n \underline{\Delta}_n}) \\ &\leq \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b} \varphi(n^b)} < e^{(\xi - \frac{\varepsilon}{2}) h_n}\right) \rightarrow 0, \end{aligned}$$

where the limit comes from (I.37). Also,  $L = -\xi$  implies that for any  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough,  $\inf_{\ell \geq n} h_\ell^{-1} \log(\ell^{1-b-\kappa_b} \varphi(\ell^b)) < -\xi + \frac{\varepsilon}{2}$  so  $h_n \underline{\Delta}_n > \log(n^{1-b-\kappa_b} \varphi(n^b)) + (\xi - \frac{\varepsilon}{2})h_n$  and as  $h_n(\varepsilon + \underline{\Delta}_n) > 0$ ,  $\mathbb{P}(h_n^{-1} \log^+ \mathcal{R}_n(g_n, \mathbf{f}^n) > \varepsilon + \underline{\Delta}_n) = \mathbb{P}(\log \mathcal{R}_n(g_n, \mathbf{f}^n) > h_n(\varepsilon + \underline{\Delta}_n), \mathcal{R}_n(g_n, \mathbf{f}^n) > 1)$  which is smaller than

$$\mathbb{P}(\mathcal{R}_n(g_n, \mathbf{f}^n) > e^{\varepsilon h_n + h_n \underline{\Delta}_n}) \leq \mathbb{P}\left(\frac{\mathcal{R}_n(g_n, \mathbf{f}^n)}{n^{1-b-\kappa_b} \varphi(n^b)} > e^{(\xi + \frac{\varepsilon}{2})h_n}\right) \rightarrow 0,$$

where the limit comes from (I.38).

• Assume  $L \in [-\infty, -\xi)$ . In this case, there exists an increasing sequence  $(n_\ell)_\ell$  of positive integers (with  $n_\ell = \ell$  when  $\lim h_n^{-1} \log(n^{1-b-\kappa_b} \varphi(n^b)) = L$ ) and  $\varepsilon_L > 0$  such that for any  $\ell \in \mathbb{N}^*$ ,  $n_\ell^{1-b-\kappa_b} \varphi(n_\ell^b) < e^{-(\xi+2\varepsilon_L)h_n}$  and for any  $\varepsilon' > 0$

$$\mathbb{P}(\mathcal{R}_{n_\ell}(g_{n_\ell}, \mathbf{f}^{n_\ell}) > \varepsilon') \leq \mathbb{P}(\mathcal{R}_{n_\ell}(g_{n_\ell}, \mathbf{f}^{n_\ell}) > e^{-\varepsilon_L h_n}) \leq \mathbb{P}\left(\frac{\mathcal{R}_{n_\ell}(g_{n_\ell}, \mathbf{f}^{n_\ell})}{n_\ell^{1-b-\kappa_b} \varphi(n_\ell^b)} > e^{(\xi + \varepsilon_L)h_n}\right) \rightarrow 0,$$

where the limit comes from (I.38) with  $\varepsilon = \varepsilon_L$ , which ends the proof.  $\square$

### 3 Proof of Proposition 1

The proof of Proposition 1 is decomposed as follows. In the first short section below, we present the expression of the generating function with constraint of edge local time. In a second sub-section, we prove the lower bound (I.20), this section is itself decomposed in different steps treating successively the random walk at fixed environment and then an important quantity of the environment. Finally, in a third section, we obtain the upper bound (I.21). Note that the fact that the upper and the lower bounds are robust when replacing  $T^n$  by  $T^{k_n}$  with  $k_n = \lfloor n/(\log n)^p \rfloor$ , with  $p > 0$ , does not need extra arguments than the ones that follow.

#### 3.1 Preliminary

We first introduce the edge local time  $N_x^n$  of a vertex  $x \in \mathbb{T}$ , that is the number of times the random walk  $\mathbb{X}$  visits the edge  $(x^*, x)$  before the instant  $n$  :

$$N_x^n := \sum_{i=1}^n \mathbb{1}_{\{X_{i-1}=x^*, X_i=x\}}, \quad (\text{I.39})$$

the law of  $N_x^{T^1}$  (recall that  $T^1$  is the instant of the first return to the root  $e$ ) and  $\sum_{y; y^*=x} N_y^{T^1}$  at fixed environment, that is under  $\mathbb{P}^\mathcal{E}$ , are given by

**Lemma 3.1.** *Let  $x \in \mathbb{T}$ , and  $T_x := \inf\{k > 0, X_k = x\}$ , then  $\mathbb{P}^\mathcal{E}(T_x < T^1) = e^{-V(x)}/H_x$  and for any  $i \in \mathbb{N}^*$ ,  $s \in [0, 1]$  and  $\nu \geq 0$ ,*

*i) The distribution of  $N_x^{T^1}$  under  $\mathbb{P}_x^\mathcal{E}(\cdot) = \mathbb{P}^\mathcal{E}(\cdot | X_0 = x)$  is geometrical on  $\mathbb{N}$  with mean*

$H_x - 1 = \sum_{1 \leq j < |x|} e^{V(x_j) - V(x)}$ . In particular

$$\mathbb{E}^\mathcal{E} \left[ s^\nu N_x^{T^1} \mathbf{1}_{\{N_x^{T^1} \geq i\}} \right] = \frac{e^{-V(x)}}{H_x^2} \left( 1 - \frac{1}{H_x} \right)^{i-1} \frac{s^{i\nu}}{1 - s^\nu \left( 1 - \frac{1}{H_x} \right)}.$$

ii) For any  $z \in \mathbb{T}$  such that  $z^* = x$ , the distribution of  $\sum_{y; y^*=x} N_y^{T^1}$  under  $\mathbb{P}_z^\mathcal{E}$  is geometrical on  $\mathbb{N}$  with mean  $\tilde{H}_x := H_x \sum_{y; y^*=x} e^{-V_x(y)}$  with  $V_x(y) = V(y) - V(x)$ . In particular

$$\mathbb{E}^\mathcal{E} \left[ s^\nu \sum_{y; y^*=x} N_y^{T^1} \mathbf{1}_{\{\sum_{y; y^*=x} N_y^{T^1} \geq i\}} \right] = \frac{e^{-V(x)}}{H_x} \frac{\tilde{H}_x}{(1 + \tilde{H}_x)^2} \left( 1 - \frac{1}{1 + \tilde{H}_x} \right)^{i-1} \frac{s^{i\nu}}{1 - s^\nu \left( 1 - \frac{1}{1 + \tilde{H}_x} \right)}.$$

*Démonstration.* The fact that  $\mathbb{P}^\mathcal{E}(T_x < T^1) = e^{-V(x)}/H_x$  comes from a standard result for one-dimensional random walks in random environment, see for example [Gol84]. The proofs of points i) and ii) are very similar and elements for the first one can be found in [AD20] so we will only deal with the second one.

For any  $x \in \mathbb{T}$ ,  $\min_{y; y^*=x} T_y$  is the first hitting time of the set  $\{y \in \mathbb{T}; y^* = x\}$  of children of  $x$  and let  $\beta_x := \mathbb{P}_x^\mathcal{E}(\min_{y; y^*=x} T_y < T^1)$  be the quenched probability, starting from  $x$ , to reach a children of  $x$  before hitting the root  $e$ . Hence,  $\sum_{y; y^*=x} N_y^{T^1}$  is the number of times the random walk  $\mathbb{X}$  visits the «edge»  $(x, \{y \in \mathbb{T}; y^* = x\})$  before the instant  $T^1$ . It follows, thanks to the strong Markov property, that for all  $z \in \mathbb{T}$  such that  $x^* = z$  and  $k \in \mathbb{N}$

$$\mathbb{P}_z^\mathcal{E} \left( \sum_{y; y^*=x} N_y^{T^1} = k \right) = \beta_x^k (1 - \beta_x). \quad (\text{I.40})$$

Note that the right part above does not depend on  $z$ . We now compute  $\beta_x$ . On the one hand, thanks to (I.40), we have  $\mathbb{E}_z^\mathcal{E}[\sum_{y; y^*=x} N_y^{T^1}] = \beta_x/(1 - \beta_x)$  and on the other hand, thanks to the first point,  $\mathbb{E}_z^\mathcal{E}[\sum_{y; y^*=x} N_y^{T^1}] = \sum_{y; y^*=x} \mathbb{E}_z^\mathcal{E}[N_y^{T^1}] = \sum_{y; y^*=x} (H_y - 1) = H_x \sum_{y; y^*=x} e^{-V_x(y)} = \tilde{H}_x$ .  $\sum_{y; y^*=x} N_y^{T^1}$  is finally geometrical on  $\mathbb{N}$  under  $\mathbb{P}_z^\mathcal{E}$  with mean  $\tilde{H}_x$  and  $\beta_x = \tilde{H}_x/(1 + \tilde{H}_x)$ . Introduce  $\alpha_x := \mathbb{P}^\mathcal{E}(\min_{y; y^*=x} T_y < T^1)$ , the quenched probability to reach the set  $\{y \in \mathbb{T}; y^* = x\}$  during the first excursion. Thanks to (I.40), we have for all  $k \in \mathbb{N}^*$

$$\mathbb{P}^\mathcal{E} \left( \sum_{y; y^*=x} N_y^{T^1} = k \right) = \alpha_x \beta_x^{k-1} (1 - \beta_x) \quad \text{and} \quad \mathbb{P}^\mathcal{E} \left( \sum_{y; y^*=x} N_y^{T^1} = 0 \right) = 1 - \alpha_x,$$

so on the one hand,  $\mathbb{E}^\mathcal{E}[\sum_{y; y^*=x} N_y^{T^1}] = \alpha_x/(1 - \beta_x)$  and on the other hand, thanks to the first point,  $\mathbb{E}^\mathcal{E}[\sum_{y; y^*=x} N_y^{T^1}] = \sum_{y; y^*=x} \mathbb{E}^\mathcal{E}[N_y^{T^1}] = \sum_{y; y^*=x} e^{-V(y)}$ . It follows that  $\alpha_x$  is equal to  $\sum_{y; y^*=x} e^{-V(y)}/(1 + \tilde{H}_x)$  and the result is proved.  $\square$



### 3.2 Lower bound for $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$

Let us first introduce two key random variables denoted  $\mathcal{R}_{T^n}(\mathbf{f}^n)$  and  $R(\mathbf{f}^n)$ .  $\mathcal{R}_{T^n}(\mathbf{f}^n)$  is a simplified version of  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  which does not depend on the function  $g_n$  and with a constraint on  $V$  : recall  $\lambda_n = ne^{-\min(10\varepsilon \log n, 5h_n)}$  and  $\mathcal{H}_{\mathfrak{z}_n}^k = \{(t_1, \dots, t_k) \in \mathbb{R}^k; t_k \geq \mathfrak{z}_n\}$  where we set for convenience  $\mathfrak{z}_n := \ell_n^{1/3}/\delta_1$  with  $\ell_n = (\log n)^3$  and  $\delta_1 \in (0, 1/2)$  (see (I.5)), then

$$\mathcal{R}_{T^n}(\mathbf{f}^n) := \sum_{i=1}^n \sum_{x \in \mathcal{O}_{\lambda_n, n^b}} \mathbb{1}_{\{N_x^{T^i} - N_x^{T^{i-1}} \geq n^b\}} \mathbb{1}_{\{\forall j \neq i: N_x^{T^j} - N_x^{T^{j-1}} = 0\}} f^{n, |x|} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x),$$

where we use the notation  $F(\mathbf{V}_x) = F(V(x_1), \dots, V(x))$ . Note that the local time until  $T^n$  which appears in  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  is replaced in  $\mathcal{R}_{T^n}(\mathbf{f}^n)$  by edge local times excursion by excursion. Also, visited vertices are restricted to some  $V$ -regular lines  $\mathcal{O}_{\lambda_n, n^b}$ .  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  and  $\mathcal{R}_{T^n}(\mathbf{f}^n)$  are related as follows, first since  $\varphi$  is non-decreasing

$$\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) \geq \varphi(n^b) \sum_{x \in \mathbb{T}} \mathbb{1}_{\{\mathcal{L}_x^{T^n} \geq n^b\}} f^{n, |x|} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x).$$

Then, introduce  $E_x^n = \sum_{i=1}^n \mathbb{1}_{\{\mathcal{L}_x^{T^i} - \mathcal{L}_x^{T^{i-1}} \geq 1\}}$ , the number of excursions to the root where the walk hits vertex  $x$ . Notice that  $E_x^n = 1$  if and only if there exists  $i \in \{1, \dots, n\}$  such that  $\mathcal{L}_x^{T^i} - \mathcal{L}_x^{T^{i-1}} \geq 1$  and for any  $j \in \{1, \dots, n\}, j \neq i$ ,  $\mathcal{L}_x^{T^j} - \mathcal{L}_x^{T^{j-1}} = 0$  that is  $N_x^{T^j} - N_x^{T^{j-1}} = 0$ . Thus

$$\begin{aligned} \sum_{x \in \mathbb{T}} \mathbb{1}_{\{\mathcal{L}_x^{T^n} \geq n^b\}} f^{n, |x|} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) &\geq \sum_{x \in \mathcal{O}_{\lambda_n, n^b}} \mathbb{1}_{\{\mathcal{L}_x^{T^n} \geq n^b, E_x^n = 1\}} f^{n, |x|} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) \\ &\geq \sum_{i=1}^n \sum_{x \in \mathcal{O}_{\lambda_n, n^b}} \mathbb{1}_{\{\mathcal{L}_x^{T^i} - \mathcal{L}_x^{T^{i-1}} \geq n^b\}} \mathbb{1}_{\{\forall j \neq i: N_x^{T^j} - N_x^{T^{j-1}} = 0\}} f^{n, |x|} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x), \end{aligned}$$

so finally, as  $\mathcal{L}_x^{T^i} - \mathcal{L}_x^{T^{i-1}} \geq N_x^{T^i} - N_x^{T^{i-1}}$ , we have the following relation

$$\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) \geq \varphi(n^b) \mathcal{R}_{T^n}(\mathbf{f}^n). \quad (\text{I.41})$$

The second random variable  $R(\mathbf{f}^n)$  depends only on the environment :

$$R(\mathbf{f}^n) := \sum_{x \in \mathcal{O}_{\lambda_n, n^b}} e^{-V(x)} \frac{1}{H_x} \left(1 - \frac{1}{H_x}\right)^{n^b-1} f^{n, |x|} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x),$$

it can be related to the quenched mean of  $\mathcal{R}_{T^n}(\mathbf{f}^n)$  as follows

$$1 \leq \frac{nR(\mathbf{f}^n)}{\mathbb{E}^\varepsilon[\mathcal{R}_{T^n}(\mathbf{f}^n)]} \leq (1 - e^{-\mathfrak{z}_n})^{-(n-1)}. \quad (\text{I.42})$$

Indeed, the random variables  $N_x^{T^i} - N_x^{T^{i-1}}, i \in \{1, \dots, n\}$ , are i.i.d under  $\mathbb{P}^\varepsilon$  so,

$$\mathbb{E}^\varepsilon [\mathcal{R}_{T^n}(\mathbf{f}^n)] = n \sum_{x \in \mathcal{O}_{\lambda_n, n^b}} \mathbb{P}^\varepsilon(N_x^{T^1} \geq n^b) \mathbb{P}^\varepsilon(N_x^{T^1} = 0)^{n-1} f^{n,|x|} \mathbf{1}_{\mathcal{H}_{3n}^{|x|}}(\mathbf{V}_x).$$

Moreover, on the event  $\{V(x) \geq 3n\}$ , thanks to Lemma 3.1,  $\mathbb{P}^\varepsilon(N_x^{T^1} = 0)^{n-1} = \mathbb{P}^\varepsilon(T_x > T^1)^{n-1} = (1 - e^{-V(x)/H_x})^{n-1} \geq (1 - e^{-V(x)})^{n-1} \geq (1 - e^{-3n})^{n-1}$  since  $H_x \geq 1$ , and thanks to Lemma 3.1 i) with  $\nu = 0$ ,  $\mathbb{P}^\varepsilon(N_x^{T^1} \geq n^b) = e^{-V(x)}(1 - 1/H_x)^{n^b-1}/H_x$  which gives (I.42). We are now ready to obtain a relation between a lower bound for  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  and a lower bound for  $R(\mathbf{f}^n)$ .

**Lemma 3.2.** *Recall  $\varepsilon_b = \min(b + \mathbf{1}_{\{b=0\}}, 1 - b)/13$  and let  $(a_n)$  be a sequence of positive numbers. For all  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough*

$$\mathbb{P}^*(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b) \leq \mathbf{P}^*(R(\mathbf{f}^n) < a_n/n^b) + \frac{ne^{-\min(9\varepsilon \log n, 4h_n)}}{n^{2\kappa_b}a_n^2}. \quad (\text{I.43})$$

*Démonstration.* Note that thanks to (I.42), for  $n$  large enough,  $nR(\mathbf{f}^n) \leq 2\mathbb{E}^\varepsilon[\mathcal{R}_{T^n}(\mathbf{f}^n)]$ , so by (I.41), on the event  $\{R(\mathbf{f}^n) \geq a_n/n^b\}$

$$\mathbb{P}^\varepsilon(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b) \leq \mathbb{P}^\varepsilon(\mathcal{R}_{T^n}(\mathbf{f}^n) < \mathbb{E}^\varepsilon[\mathcal{R}_{T^n}(\mathbf{f}^n)]/2).$$

Using Bienaymé-Tchebychev inequality and the fact that  $N_x^{T^i} - N_x^{T^{i-1}}, i \in \{1, \dots, n\}$ , are i.i.d under  $\mathbb{P}^\varepsilon$  implies, on the event  $\{R(\mathbf{f}^n) \geq a_n/n^b\}$

$$\begin{aligned} \mathbb{P}^\varepsilon(\mathcal{R}_{T^n}(\mathbf{f}^n) < \mathbb{E}^\varepsilon[\mathcal{R}_{T^n}(\mathbf{f}^n)]/2) &\leq \frac{4}{\mathbb{E}^\varepsilon[\mathcal{R}_{T^n}(\mathbf{f}^n)]^2} n \text{Var}^\varepsilon(\mathcal{R}_{T^1}(\mathbf{f}^n)) \\ &\leq \frac{16n^{2b}}{a_n^2 n} \sum_{x, y \in \mathcal{O}_{\lambda_n, n^b}} \mathbb{P}^\varepsilon(N_x^{T^1} \wedge N_y^{T^1} \geq n^b) f^{n,|x|} \mathbf{1}_{\mathcal{H}_{3n}^{|x|}}(\mathbf{V}_x) f^{n,|y|} \mathbf{1}_{\mathcal{H}_{3n}^{|y|}}(\mathbf{V}_y). \end{aligned} \quad (\text{I.44})$$

The last inequality coming from the fact that, on  $\{R(\mathbf{f}^n) \geq a_n/n^b\}$ , thanks to (I.42), we have  $\mathbb{E}^\varepsilon[\mathcal{R}_{T^n}(\mathbf{f}^n)]^2 \geq n^2 R(\mathbf{f}^n)^2/4 \geq n^2 a_n^2/4n^{2b}$ . Markov inequality in (I.44) yields  $\mathbb{P}^\varepsilon(N_x^{T^1} \wedge N_y^{T^1} \geq n^b) \leq \mathbb{E}^\varepsilon[N_x^{T^1} N_y^{T^1}]/n^{2b}$ ,

so finally, on the event  $\{R(\mathbf{f}^n) \geq a_n/n^b\}$

$$\mathbb{P}^\varepsilon(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b) \leq \frac{16}{na_n^2} \sum_{x, y \in \mathcal{O}_{\lambda_n, n^b}} \mathbb{E}^\varepsilon[N_x^{T^1} N_y^{T^1}] f^{n,|x|} \mathbf{1}_{\mathcal{H}_{3n}^{|x|}}(\mathbf{V}_x) f^{n,|y|} \mathbf{1}_{\mathcal{H}_{3n}^{|y|}}(\mathbf{V}_y). \quad (\text{I.45})$$

To treat the above sum, we first make a simplification by using the uniform upper bound of the

set  $\mathcal{U}_b$ , see (I.11)

$$\sum_{x,y \in \mathcal{O}_{\lambda_n, n^b}} \mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}] f^{n,|x|} \mathbf{1}_{\mathcal{H}_{3n}^{|x|}}(\mathbf{V}_x) f^{n,|y|} \mathbf{1}_{\mathcal{H}_{3n}^{|y|}}(\mathbf{V}_y) \leq \frac{C_\infty^2}{n^{2\kappa_b}} \sum_{x,y \in \mathcal{O}_{\lambda_n}} \mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}]. \quad (\text{I.46})$$

We then split the computations in two distinct steps : the first step is dedicated to the cases  $x \leq y$  or  $y \leq x$  and the second one to the cases nor  $x \leq y$  neither  $y \leq x$ . The key here is to take into account that we are only interested in vertices belonging to  $\lambda_n$ -regular lines  $\mathcal{O}_{\lambda_n}$  with  $\lambda_n = ne^{-\min(10\varepsilon \log n, 5h_n)}$  for  $\varepsilon \in (0, \varepsilon_b)$ .

We start with the cases  $x \leq y$  and  $y \leq x$  and as they are symmetrical, we only deal with the first one. First note that as  $\mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}] \leq 2e^{-V(y)} H_x = 2H_x e^{-V(x)} e^{-V_x(y)}$  (see [AD20] Lemma 3.6)

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{x \leq y \\ x,y \in \mathcal{O}_{\lambda_n}}} \mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}] \right] &\leq 2\mathbf{E} \left[ \sum_{x \in \mathcal{O}_{\lambda_n}} e^{-V(x)} H_x \sum_{\substack{y \geq x \\ y \in \mathcal{O}_{\lambda_n}^x}} e^{-V_x(y)} \right] \leq 2\mathbf{E} \left[ \sum_{x \in \mathcal{O}_{\lambda_n}} e^{-V(x)} \right]^2 \lambda_n \\ &\leq 2\ell_n^2 \lambda_n, \end{aligned}$$

where for all  $\lambda > 0$ ,  $\mathcal{O}_\lambda^x$  is translated set of  $\lambda$ -regular lines

$$\mathcal{O}_\lambda^x = \{y \in \mathbb{T}, y > x; \max_{|x| < j \leq |y|} H_{x,y_j} \leq \lambda\}, \quad H_{x,y_j} = \sum_{x < w \leq y_j} e^{V_x(w) - V_x(y_j)},$$

also, the second inequality is obtained thanks to the regular line which yields  $H_x \mathbf{1}_{\mathcal{O}_{\lambda_n}}(x) \leq \lambda_n$  and the last one comes from Remark 2.

We then move to the second case, neither  $x \leq y$  nor  $y \leq x$ , that we denote  $x \not\leq y$ . In this case,  $\mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}] = 2H_{x \wedge y} e^{V(x \wedge y) - V(x) - V(y)}$  (see [AD20] Lemma 3.6). Thus

$$\mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}] \leq 2\lambda_n \sum_{l \geq 1} \sum_{|z|=l} e^{-V(z)} \mathbf{1}_{\{z \in \mathcal{O}_{\lambda_n}\}} \sum_{\substack{u \neq v \\ u^* = v^* = z}} e^{-V_z(u)} e^{-V_z(v)} \sum_{\substack{x \geq u \\ x \in \mathcal{O}_{\lambda_n}^u}} e^{-V_u(x)} \sum_{\substack{y \geq v \\ y \in \mathcal{O}_{\lambda_n}^v}} e^{-V_v(y)},$$

where we have used again the regular line  $\mathcal{O}_{\lambda_n}$  which gives an upper bound for  $H_{x \wedge y}$ . Finally, independence of the increments of  $V$  conditionally to  $(\mathbb{T}, V(w); w \in \mathbb{T}, |w| \leq l+1)$  and Remark 2 yields

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{x \not\leq y \\ x,y \in \mathcal{O}_{\lambda_n}}} \mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}] \right] &\leq 2\lambda_n \mathbf{E} \left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^2 \right] \mathbf{E} \left[ \sum_{z \in \mathcal{O}_{\lambda_n}} e^{-V(z)} \right]^3 \\ &\leq 2\lambda_n \mathbf{E} \left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^2 \right] (\ell_n)^3, \end{aligned}$$

and thanks to (I.5), the second moment above is finite. Collecting the upper bounds for the two

cases and moving back to (I.46), we get for  $n$  large enough

$$\mathbf{E} \left[ \sum_{x,y \in \mathcal{C}_{\lambda_n, n^b}} \mathbb{E}^\varepsilon [N_x^{T^1} N_y^{T^1}] f^{n,|x|} \mathbf{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(V_x) f^{n,|y|} \mathbf{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|y|}}(V_y) \right] \leq \frac{(\ell_n)^4 \lambda_n}{n^{2\kappa_b}} \leq \frac{ne^{-\min(9\varepsilon \log n, 4h_n)}}{n^{2\kappa_b}}, \quad (\text{I.47})$$

the last inequality is justified by the fact (see Remark 1) that  $(\ell_n)^4 = o(e^{h_n})$  and  $(\ell_n)^4 = o(e^{\varepsilon \log n})$ . We are now ready to conclude the proof of the lemma :  $\mathbb{P}^*(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b)$  is smaller than

$$\mathbb{P}^*(R(\mathbf{f}^n) < a_n/n^b) + \mathbb{P}^*(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b, R(\mathbf{f}^n) \geq a_n/n^b),$$

then, as the second term in the above inequality is nothing but

$$\mathbf{E}^* [\mathbb{P}^\varepsilon(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b) \mathbf{1}_{\{R(\mathbf{f}^n) \geq a_n/n^b\}}],$$

the proof ends thanks to (I.45) and (I.47).  $\square$

### Lower bound for $R(\mathbf{f}^n)$

This is the most technical part of the proof of Proposition 1. For any  $n \geq 2$  and  $\varepsilon \in (0, \varepsilon_b)$ , recall that  $\lambda_n = ne^{-\min(10\varepsilon \log n, 5h_n)}$  and  $\mathfrak{z}_n = \ell_n^{1/3}/\delta_1$ ,  $\delta_1 \in (0, 1/2)$  (see (I.5)) with  $\ell_n = (\log n)^3$ . For any  $\varepsilon > 0$ , let us choose  $(a_n)$  as follows

$$a_n := e^{-4\varepsilon h_n} \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k (f_{\varepsilon h_n}^{n,k} \mathbf{1}_{\Upsilon_n^k}) \quad (\text{I.48})$$

with  $\Upsilon_n^k = \{\mathbf{t} \in \mathbb{R}^k; H_k(\mathbf{t}) \leq n^b e^{\varepsilon h_n}\} \cap \mathcal{H}_{B, 2\mathfrak{z}_n}^k$ . Recall that  $\Psi_{\lambda, \lambda'}^k$ ,  $h_n$ ,  $\mathcal{H}_{B, 2\mathfrak{z}_n}^k$  and  $f_{\varepsilon h_n}^{n,k}$  can be found respectively in (I.9), (I.13), (I.19) and (I.17).

**Lemma 3.3.** *There exists  $c_4 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_b)$  and  $n$  large enough*

$$\mathbf{P}^* \left( R(\mathbf{f}^n) < a_n/n^b \right) \leq \frac{e^{-\varepsilon \frac{c_4}{c_2} h_n} \mathbf{E}[Z_n^2]}{\left( \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k (f_{\varepsilon h_n}^{n,k} \mathbf{1}_{\Upsilon_n^k}) \right)^2} + h_n e^{-\varepsilon \tilde{c}_2 h_n}, \quad (\text{I.49})$$

with, recall,  $m_n = \lceil \varepsilon h_n / c_2 \rceil$  (see (I.16)).

*Démonstration.* Recall the expression of  $R(\mathbf{f}^n)$  :

$$R(\mathbf{f}^n) = \sum_{x \in \mathcal{C}_{\lambda_n, n^b}} e^{-V(x)} \frac{1}{H_x} \left( 1 - \frac{1}{H_x} \right)^{n^b - 1} f^{n,|x|} \mathbf{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(V(x_1), \dots, V(x)),$$

with  $H_x$  and  $\mathcal{H}_{\mathfrak{z}_n}^{|x|}$  respectively defined in (I.8) and (I.19). The main idea here is to cut the tree

at the generation  $m_n$  to introduce independence between generations. First note that

$$R(\mathbf{f}^n) \geq \sum_{|u|=m_n} \sum_{k \geq 1} \sum_{\substack{|x|=k+m_n \\ x > u; x \in \mathcal{O}_{\lambda_n, n^b}^u}} \frac{e^{-V(x)}}{H_x} \left(1 - \frac{1}{H_x}\right)^{n^b} f^{n, k+m_n} \mathbb{1}_{\mathcal{H}_{3n}^{k+m_n}}(V(x_1), \dots, V(x)),$$

from here we would like to make a translation to decompose the trajectories of  $V$  before and after the generation  $m_n$  and to do that, we have in particular to re-write  $H_{x_j}$  for  $j \leq |x|$ . Let  $u < x$  with  $|u| = m_n$ . For all  $m_n < j \leq |x|$ , we have  $H_{x_j} = H_u e^{-V_u(x_j)} + H_{u, x_j}$  where, for any  $z < v$ ,  $H_{z, v} := \sum_{z < w \leq v} e^{V_z(w) - V_z(v)}$ .

So on the events  $\{\max_{|w| \leq m_n} |V(w)| \leq \varepsilon h_n\}$  and  $\{V_u(x) := \min_{u < w \leq x} (V(w) - V(u)) \geq -B\}$ , for any  $B > 0$ :

$$\forall i \leq m_n : H_{x_i} \leq m_n e^{2\varepsilon h_n} \quad \text{and} \quad \forall m_n < j \leq |x| : H_{x_j} \leq m_n e^{2\varepsilon h_n + B} + H_{u, x_j}.$$

Assume  $n^b < H_{u, x} \leq n^b e^{\varepsilon h_n}$ . Then,  $H_x > n^b$  and for  $n$  large enough (recall  $h_n \leq \log n$  for  $n$  large enough,  $h_n \rightarrow \infty$  and  $\varepsilon \in (0, \varepsilon_b)$ )

$$\frac{1}{H_x} \left(1 - \frac{1}{H_x}\right)^{n^b} \geq \frac{(1 - 1/n^b)^{n^b}}{m_n e^{2\varepsilon h_n + B} + H_{u, x}} \geq \frac{(1 - 1/n^b)^{n^b}}{m_n e^{2\varepsilon h_n + B} + n^b e^{\varepsilon h_n}} \geq \frac{e^{-3\varepsilon h_n}}{n^b}.$$

Now introduce the translated  $(\lambda, \lambda')$ -regular lines

$$\mathcal{O}_{\lambda, \lambda'}^v := \{y \in \mathbb{T}, y > v; \max_{|v| < j \leq |y|} H_{v, y_j} \leq \lambda, H_{v, y} > \lambda'\}.$$

Note that for  $n$  large enough,  $\mathcal{O}_{\lambda_n/2, n^b}^u \subset \mathcal{O}_{\lambda_n, n^b}$ . Indeed, if  $|u| = m_n$  and  $m_n < j \leq |x|$ , then  $H_{x_j} \leq m_n e^{2\varepsilon h_n + B} + H_{u, x_j}$ . Moreover,  $m_n e^{2\varepsilon h_n + B} \leq e^{3\varepsilon h_n} \leq \lambda_n/2$  for  $n$  large since  $\varepsilon \in (0, 1/13)$ , so  $H_{u, x_j} \leq \lambda_n/2$  implies  $H_{x_j} \leq \lambda_n$ .

For  $f^{n, m_n+k}$ , we simply write (still on the event  $\{\max_{|w| \leq m_n} |V(w)| \leq \varepsilon h_n\}$ )

$$f^{n, m_n+k}(V(x_1), \dots, V(x)) \geq f_{\varepsilon h_n}^{n, k}(V_u(x_{m_n+1}), \dots, V_u(x)),$$

where we recall that  $f_h^{n, k}(t_1, \dots, t_k) = \inf_{s \in [-h, h]^m} f^{n, m+k}(s_1, \dots, s_m, t_1 + s_m, \dots, t_k + s_m)$  with  $m = \lceil h/c_2 \rceil$ . In the same way, if  $|V(u)| \leq \varepsilon h_n$  then  $\mathbb{1}_{\{V(x) \geq 3n\}} \geq \mathbb{1}_{\{V_u(x) \geq 23n\}}$  since  $\varepsilon < 1$  and  $h_n \leq \ell_n^{1/3}$ . We finally obtain, for  $n$  large enough (independently of the environment) on  $\{\max_{|w| \leq m_n} |V(w)| \leq \varepsilon h_n\}$  that  $R(\mathbf{f}^n)$  is larger than

$$\frac{e^{-3\varepsilon h_n}}{n^b} \sum_{|u|=m_n} e^{-V(u)} \sum_{k \geq 1} \sum_{\substack{|x|=k+m_n \\ x > u; x \in \mathcal{O}_{\lambda_n/2, n^b}^u}} e^{-V_u(x)} \mathbb{1}_{\{H_{u, x} \leq n^b e^{\varepsilon h_n}\}} f_{\varepsilon h_n}^{n, k} \mathbb{1}_{\mathcal{H}_{B, 23n}^k}(V_u(x_{m_n+1}), \dots, V_u(x))$$

$$\geq \frac{e^{-4\varepsilon h_n}}{n^b} \sum_{|u|=m_n} \sum_{k \geq 1} \sum_{\substack{|x|=k+m_n \\ x > u; x \in \mathcal{O}_{\lambda_n/2, n^b}^u}} e^{-V_u(x)} f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}(V_u(x_{m_n+1}), \dots, V_u(x)). \quad (\text{I.50})$$

Now, introduce the random variable  $Z_n^u$

$$Z_n^u := \sum_{k \geq 1} \sum_{\substack{|x|=k+m_n \\ x > u; x \in \mathcal{O}_{\lambda_n/2, n^b}^u}} e^{-V_u(x)} f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}(V_u(x_{m_n+1}), \dots, V_u(x)),$$

we obtain

$$\mathbf{P}(R(\mathbf{f}^n) < e^{-4\varepsilon h_n} \mathbf{E}[Z_n]/n^b, \max_{|w| \leq m_n} |V(w)| \leq \varepsilon h_n) \leq \mathbf{P}\left(\sum_{|u|=m_n} Z_n^u < \mathbf{E}[Z_n]\right),$$

with

$$Z_n := \sum_{x \in \mathcal{O}_{\lambda_n/2, n^b}} e^{-V(x)} f_{\varepsilon h_n}^{n,|x|} \mathbb{1}_{\Upsilon_n^{|x|}}(V(x_1), \dots, V(x)). \quad (\text{I.51})$$

Hence, by Lemma 2.4 in [AD20], there exists  $c_4 > 0$  such that for  $n$  large enough

$$\mathbf{P}^*(R(\mathbf{f}^n) < e^{-4\varepsilon h_n} \mathbf{E}[Z_n]/n^b, \max_{|w| \leq m_n} |V(w)| \leq \varepsilon h_n) \leq e^{-c_4 m_n} \frac{\mathbf{E}[Z_n^2]}{\mathbf{E}[Z_n]^2}, \quad (\text{I.52})$$

and finally, (I.48) yields

$$\mathbf{P}^*(R(\mathbf{f}^n) < a_n/n^b, \max_{|w| \leq m_n} |V(w)| \leq \varepsilon h_n) \leq \frac{e^{-\varepsilon \frac{c_4}{c_2} h_n} \mathbf{E}[Z_n^2]}{\left(\sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k (f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k})\right)^2},$$

we have used that  $\mathbf{E}[Z_n] = \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k (f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k})$  and  $m_n = \lceil \varepsilon h_n / c_2 \rceil$ . Finally, (I.16) finishes the proof.  $\square$

The next step is to give a lower bound for  $\mathbf{E}[Z_n^2]$ , we do that in the dedicated section below.

### Control of the second moment $\mathbf{E}[Z_n^2]$

In this section we prove the following lemma,

**Lemma 3.4.** *Assume (A1) and (A2) hold. For all  $\varepsilon \in (0, \varepsilon_b)$ ,  $A > 2/c_3$  and  $n$  large enough*

$$\mathbf{E}[Z_n^2] \leq e^{\frac{6\varepsilon}{A} h_n} \left( \sum_{k \geq 1} \Psi_{n, n^b}^k (f^{n,k}) \right)^2,$$

recall also that  $c_3$  comes from Remark 2.

*Démonstration.* The expression of  $Z_n^2$  is given by the following sum

$$\sum_{x,y \in \mathcal{O}_{\lambda_n/2, n^b}} e^{-V(x)-V(y)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\Upsilon_n^{|x|}}(\mathbf{V}_x) f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\Upsilon_n^{|y|}}(\mathbf{V}_y)$$

(see (I.51)) and  $\lambda_n \leq n$  so

$$Z_n^2 \leq \sum_{x,y \in \mathcal{O}_{n, n^b}} e^{-V(x)} e^{-V(y)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|y|}}(\mathbf{V}_y), \quad (\text{I.53})$$

with (recall)  $F(\mathbf{V}_w) = F(V(w_1), \dots, V(w))$ . Let us split the computations of the upper bound of the mean of  $Z_n^2$  into two main cases : the first one is when  $x$  and  $y$  in the sum (I.53) are directly related in the tree and the second one when it is not :

*Cases 1* ( $x \leq y$  or  $y \leq x$ ) : recall  $\mathfrak{z}_n = \ell_n^{1/3}/\delta_1$  with  $\ell_n = (\log n)^3$ ,  $\delta_1 \in (0, 1/2)$  (see (I.5)). For this case, we simply use the fact that  $f_{\varepsilon h_n}^{n,i} \leq C_\infty$  and  $e^{-2V(w)} \mathbf{1}_{\{V(w) \geq 2\mathfrak{z}_n\}} \leq e^{-V(w)}/n^2$  so by symmetry

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{x \leq y \text{ or } y \leq x \\ x,y \in \mathcal{O}_{n, n^b}}} e^{-V(x)-V(y)} \mathbf{1}_{\{V(x) \geq 2\mathfrak{z}_n\}} \right] &\leq 2\mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-2V(x)} \mathbf{1}_{\{V(x) \geq 2\mathfrak{z}_n\}} \sum_{\substack{y \geq x \\ y \in \mathcal{O}_n^x}} e^{-V_x(y)} \right] \\ &\leq \frac{2}{n^2} \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \sum_{\substack{y \geq x \\ y \in \mathcal{O}_n^x}} e^{-V_x(y)} \right], \end{aligned}$$

which is equal, by using that the increments of  $V$  are conditionally independent and stationary, to  $2\mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}]^2/n^2$ . Then, thanks to Remark 2 and the fact that  $h_n \geq (\log n)^\gamma$  with  $0 < \gamma \leq 1$ ,  $2\mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}]^2 \leq \ell_n \leq e^{\varepsilon h_n/A}$ . In addition with assumption (A1), the part  $\{x \leq y \text{ or } y \leq x\}$  in the sum (I.53) is smaller than  $e^{\frac{\varepsilon}{A} h_n} (\sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k}))^2$ .

*Cases 2* ( $x \not\leq y$ ) : recall that  $x \not\leq y$  if and only if neither  $x \leq y$  nor  $y \leq x$ . First let

$$\Sigma_0(z) := \sum_{\substack{x \not\leq y \\ x,y \in \mathcal{O}_{n, n^b}}} \mathbf{1}_{\{x \wedge y = z\}} e^{-V(x)} e^{-V(y)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|y|}}(\mathbf{V}_y).$$

We decompose  $\Sigma_0(z)$  as follows : for all  $A > 2/c_3$

$$\sum_{z \in \mathbb{T}} \Sigma_0(z) = \sum_{|z| \geq \lfloor A\ell_n \rfloor} \Sigma_0(z) + \sum_{|z| < \lfloor A\ell_n \rfloor} (\Sigma_1(z) + \Sigma_2(z)), \quad (\text{I.54})$$

and for any  $i \in \{1, 2\}$ ,

$$\Sigma_i(z) := \sum_{\substack{x \not\leq y \\ x,y \in \mathcal{O}_{n, n^b}}} \mathbf{1}_{\{x \wedge y = z\}} e^{-V(x)} e^{-V(y)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|y|}}(\mathbf{V}_y) \mathbf{1}_{\{(x,y) \in \mathcal{C}_{i,z}\}},$$

with  $\mathcal{C}_{1,z} := \{(x, y) \in \mathbb{T}^2; x^* > z \text{ and } y^* > z\}$  and  $\mathcal{C}_{2,z} := \{(x, y) \in \mathbb{T}^2; x^* = z \text{ or } y^* = z\}$ .

Let us start with the easiest part :  $\sum_{|z| \geq \lfloor Al_n \rfloor} \Sigma_0(z)$ . Observe that

$$\sum_{|z| \geq \lfloor Al_n \rfloor} \Sigma_0(z) \leq C_\infty^2 \sum_{l \geq \lfloor Al_n \rfloor} \sum_{|z|=l} \mathbf{1}_{\{V(z) \geq -B, z \in \mathcal{O}_n\}} \sum_{\substack{u \neq v \\ u^* = v^* = z}} \sum_{\substack{x \geq u \\ x \in \mathcal{O}_n}} e^{-V(x)} \sum_{\substack{y \geq v \\ y \in \mathcal{O}_n}} e^{-V(y)}.$$

By conditional independence of the increments of  $V$  and Remark 2, for any  $n$  large enough

$$\begin{aligned} \mathbf{E} \left[ \sum_{|z| \geq \lfloor Al_n \rfloor} \Sigma_0(z) \right] &\leq C_\infty^2 e^B \mathbf{E} \left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^2 \right] \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} e^{-V(x)} \right]^2 \sum_{l \geq \lfloor Al_n \rfloor} \mathbf{E} \left[ \sum_{|z|=l} e^{-V(z)} \mathbf{1}_{\{z \in \mathcal{O}_n\}} \right] \\ &\leq C_\infty^2 e^B \mathbf{E} \left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^2 \right] \ell_n^2 n^{-2} \leq \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}), \end{aligned} \quad (\text{I.55})$$

where we have used (A1) and (I.5) for the last inequality.

For  $\Sigma_1(z)$ ,  $|z| < \lfloor Al_n \rfloor$ , we decompose according to the value of  $V(w)$  with  $w \in \{u, v\}$  :  $\Sigma_1(z) = \Sigma_{1,1}(z) + \Sigma_{1,2}(z)$  with

$$\Sigma_{1,1}(z) := \sum_{\substack{u \neq v \\ u^* = v^* = z}} \mathbf{1}_{\{V(u) \vee V(v) < 2\mathfrak{z}_n\}} \sum_{\substack{x > u \\ x \in \mathcal{O}_{n,n^b}}} e^{-V(x)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) \sum_{\substack{y > v \\ y \in \mathcal{O}_{n,n^b}}} e^{-V(y)} f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|y|}}(\mathbf{V}_y),$$

and

$$\Sigma_{1,2}(z) := \sum_{\substack{u \neq v \\ u^* = v^* = z}} \mathbf{1}_{\{V(u) \vee V(v) \geq 2\mathfrak{z}_n\}} \sum_{\substack{x > u \\ x \in \mathcal{O}_{n,n^b}}} e^{-V(x)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) \sum_{\substack{y > v \\ y \in \mathcal{O}_{n,n^b}}} e^{-V(y)} f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|y|}}(\mathbf{V}_y).$$

We first deal with  $\Sigma_{1,1}(z)$ . Observe that  $x \in \mathcal{O}_{n,n^b}$  (resp.  $y \in \mathcal{O}_{n,n^b}$ ) means  $H_u \leq n$  (resp.  $H_v \leq n$ ),  $x \in \mathcal{O}_n^u$  (resp.  $y \in \mathcal{O}_n^v$ ) and  $n^b - H_u e^{-V_u(x)} < H_{u,x}$  (resp.  $n^b - H_v e^{-V_v(y)} < H_{v,y}$ ). Besides,  $V(u) < 2\mathfrak{z}_n$  and  $V(x) > 2\mathfrak{z}_n$  (resp.  $V(v) < 2\mathfrak{z}_n$  and  $V(y) > 2\mathfrak{z}_n$ ) implies  $V_u(x) > 0$  (resp.  $V_v(y) > 0$ ) that is  $n^b - H_u < H_{u,x}$  (resp.  $n^b - H_v < H_{v,y}$ ), so  $\Sigma_{1,1}(z)$  is smaller than

$$\sum_{\substack{u \neq v \\ u^* = v^* = z}} \mathbf{1}_{\{V(u) \wedge V(v) \geq -B, H_u \vee H_v \leq n\}} \sum_{\substack{x > u \\ x \in \mathcal{O}_{n,n^b - H_u}^u}} e^{-V(x)} f_{\varepsilon h_n}^{n,|x|}(\mathbf{V}_x) \sum_{\substack{y > v \\ y \in \mathcal{O}_{n,n^b - H_v}^v}} e^{-V(y)} f_{\varepsilon h_n}^{n,|y|}(\mathbf{V}_y). \quad (\text{I.56})$$

We now move to  $\Sigma_{1,2}(z)$ . Note that  $\{V(u) \vee V(v) \geq 2\mathfrak{z}_n\} = \{V(u) \geq 2\mathfrak{z}_n, V(v) < 2\mathfrak{z}_n\} \cup \{V(v) \geq 2\mathfrak{z}_n, V(u) < 2\mathfrak{z}_n\} \cup \{V(u) \wedge V(v) \geq 2\mathfrak{z}_n\}$ . By symmetry,  $\Sigma_{1,2}(z)$  is equal to

$$2 \sum_{\substack{u \neq v \\ u^* = v^* = z}} \mathbf{1}_{\{V(u) \geq 2\mathfrak{z}_n, V(v) < 2\mathfrak{z}_n\}} \sum_{\substack{x > u \\ x \in \mathcal{O}_{n,n^b}}} e^{-V(x)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) \sum_{\substack{y > v \\ y \in \mathcal{O}_{n,n^b}}} e^{-V(y)} f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|y|}}(\mathbf{V}_y)$$



$$+ \sum_{\substack{u \neq v \\ u^* = v^* = z}} \mathbf{1}_{\{V(u) \wedge V(v) \geq 2\mathfrak{z}_n\}} \sum_{\substack{x > u \\ x \in \mathcal{O}_{n,n^b}}} e^{-V(x)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) \sum_{\substack{y > v \\ y \in \mathcal{O}_{n,n^b}}} e^{-V(y)} f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\mathfrak{z}_n}^{|y|}}(\mathbf{V}_y).$$

The same decomposition of  $H_y$  we used for  $\Sigma_{1,1}(z)$  also works for the part  $\{V(v) < 2\mathfrak{z}_n\}$  in the above sum, so as in (I.56) and first using that on  $\{V(u) \geq 2\mathfrak{z}_n\} \cap \{V(y) \geq -B\}$ ,  $V(u) \geq (1-\delta_1)V(u) + 2 \log n$  and  $\delta_1 V(v) \geq -\delta_1 B$ , then using that on  $\{V(u) \wedge V(v) \geq 2\mathfrak{z}_n\}$ ,  $V(u) + V(v) \geq (1-\delta_1)(V(u) + V(v)) + 4 \log n$ ,  $\Sigma_{1,2}(z)$  is smaller than

$$\begin{aligned} & \mathbf{1}_{\{V(z) \geq -B, z \in \mathcal{O}_n\}} 2e^{\delta_1 B} \frac{C_\infty^2}{n^2} \sum_{\substack{u \neq v \\ u^* = v^* = z}} e^{-(1-\delta_1)(V(u)+V(v))} \sum_{\substack{x > u \\ x \in \mathcal{O}_n^u}} e^{-V_u(x)} \sum_{\substack{y > v \\ y \in \mathcal{O}_n^v}} e^{-V_v(y)} \\ & + \mathbf{1}_{\{V(z) \geq -B, z \in \mathcal{O}_n\}} \frac{C_\infty^2}{n^4} \sum_{\substack{u \neq v \\ u^* = v^* = z}} e^{-(1-\delta_1)(V(u)+V(v))} \sum_{\substack{x > u \\ x \in \mathcal{O}_n^u}} e^{-V_u(x)} \sum_{\substack{y > v \\ y \in \mathcal{O}_n^v}} e^{-V_v(y)} \\ & \leq \mathbf{1}_{\{V(z) \geq -B, z \in \mathcal{O}_n\}} 3e^{\delta_1 B} \frac{C_\infty^2}{n^2} \sum_{\substack{u \neq v \\ u^* = v^* = z}} e^{-(1-\delta_1)(V(u)+V(v))} \sum_{\substack{x > u \\ x \in \mathcal{O}_n^u}} e^{-V_u(x)} \sum_{\substack{y > v \\ y \in \mathcal{O}_n^v}} e^{-V_v(y)}. \end{aligned}$$

Note that the genealogical common line between  $x$  and  $y$  is the common line of individuals before  $u$  and  $v$  so for any  $p \leq |z|$ ,  $x_p = y_p = u_p = v_p$  and

$$f_{\varepsilon h_n}^{n,|x|}(\mathbf{V}_x) = f_{\varepsilon h_n}^{n,|x|}(V(u_1), \dots, V(u), V_u(x_{|u|+1}) + V(u), \dots, V_u(x) + V(u)),$$

and

$$f_{\varepsilon h_n}^{n,|y|}(\mathbf{V}_y) = f_{\varepsilon h_n}^{n,|y|}(V(v_1), \dots, V(v), V_v(y_{|v|+1}) + V(v), \dots, V_v(y) + V(v)).$$

Recall that for all  $q \geq 1$  and  $\mathbf{t}_q = (t_1, \dots, t_q) \in \mathbb{R}^q$ ,

$$\Psi_n^k(F|\mathbf{t}_p) = \mathbf{E} \left[ \sum_{|x|=k} e^{-V(x)} F(t_1, \dots, t_p, V(x_1) + t_p, \dots, V(x) + t_p) \mathbf{1}_{\mathcal{O}_n}(x) \right].$$

We naturally note  $\Psi_n^k(F|\mathbf{V}_w)$  when we evaluate the function  $\Psi_n^k(F|\cdot)$  at  $(V(w_1), \dots, V(w))$ .

By conditional independence of the increments of  $V$ ,  $\mathbf{E}[\sum_{|z|=l} \Sigma_1(z)] = \mathbf{E}[\sum_{|z|=l} \Sigma_{1,1}(z) + \Sigma_{1,2}(z)]$  is smaller, for  $n$  large enough with  $l < \lfloor A\ell_n \rfloor$ , than

$$\begin{aligned} & \mathbf{E} \left[ \sum_{|z|=l} \sum_{\substack{u \neq v \\ u^* = v^* = z}} \mathbf{1}_{\{V(u) \wedge V(v) \geq -B, H_u \vee H_v \leq n\}} \sum_{i,j \geq 1} \prod_{(i,u);(j,v)} e^{-V(w)} \Psi_{n,n^b-H_w}^k(f_{\varepsilon h_n}^{n,|w|+k}|\mathbf{V}_w) \right] \\ & + 3e^{\delta_1 B} \ell_n^2 \frac{C_\infty^2}{n^2} \mathbf{E} \left[ \sum_{|z|=l} \mathbf{1}_{\{V(z) \geq -B, z \in \mathcal{O}_n\}} \sum_{\substack{u \neq v \\ u^* = v^* = z}} e^{-(1-\delta_1)(V(u)+V(v))} \right], \end{aligned}$$

where we have used that  $\mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}] \leq \ell_n$ . Then, by assumption **(A2)** with  $\delta = \delta_1$  (see **(I.5)** for the definition of  $\delta_1$ ), for all  $l < \lfloor A\ell_n \rfloor$  ( $|u| = |v| = l + 1$ ) and  $n$  large enough, on the event  $\{V(u) \wedge V(v) \geq -B, H_u \vee H_v \leq n\}$

$$\sum_{i,j \geq 1} \prod_{(k,w) \in \{(i,u);(j,v)\}} \Psi_{n,n^b-H_w}^k(f_{\varepsilon h_n}^{n,|w|+k} | \mathbf{V}_w) \leq e^{\delta_1 V(u) + \delta_1 V(v) + \frac{2\varepsilon}{A} h_n} \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right)^2.$$

Hence,  $\mathbf{E}[\sum_{|z| < \lfloor A\ell_n \rfloor} \Sigma_1]$  is smaller, for  $n$  large enough, than

$$\begin{aligned} & e^{\frac{2\varepsilon}{A} h_n} \mathbf{E} \left[ \left( \sum_{|w|=1} e^{-(1-\delta_1)V(w)} \right)^2 \right] \mathbf{E} \left[ \sum_{|z| < \lfloor A\ell_n \rfloor} e^{-V(z) - (1-2\delta_1)V(z)} \mathbf{1}_{\{V(z) \geq -B\}} \right] \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right)^2 \\ & + 3e^{\delta_1 B} \ell_n^2 \frac{C_\infty^2}{n^2} \mathbf{E} \left[ \left( \sum_{|w|=1} e^{-(1-\delta_1)V(w)} \right)^2 \right] \mathbf{E} \left[ \sum_{z \in \mathcal{O}_n} e^{-V(z) - (1-2\delta_1)V(z)} \mathbf{1}_{\{V(z) \geq -B\}} \right]. \end{aligned}$$

Finally, thanks to assumption **(A1)**, **(I.5)** and by Remark 2, for  $n$  large enough

$$\mathbf{E} \left[ \sum_{|z| < \lfloor A\ell_n \rfloor} \Sigma_1(z) \right] \leq e^{\frac{5\varepsilon}{A} h_n} \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right)^2. \quad (\text{I.57})$$

We now turn to  $\Sigma_2(z)$ , that is the sum

$$\sum_{\substack{x \neq y \\ x, y \in \mathcal{O}_{n,n^b}}} \mathbf{1}_{\{x \wedge y = z\}} e^{-V(x)} e^{-V(y)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\delta_1 n}^{|x|}}(\mathbf{V}_x) f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\delta_1 n}^{|y|}}(\mathbf{V}_y) \mathbf{1}_{\{(x,y) \in \mathcal{C}_{2,z}\}},$$

with  $\mathcal{C}_{2,z} := \{(x,y) \in \mathbb{T}^2; x^* = z \text{ or } y^* = z\}$ . The first step is to split the set  $\{x^* = z \text{ or } y^* = z\}$  into three disjoint sets :  $\{x^* = z \text{ and } y^* > z\}$ ,  $\{x^* > z \text{ and } y^* = z\}$  and  $\{x^* = z \text{ and } y^* = z\}$ . By symmetry, the previous sum is equal to

$$\begin{aligned} & 2 \sum_{\substack{x \neq v \\ x^* = v^* = z}} \mathbf{1}_{\{x \in \mathcal{O}_{n,n^b}\}} e^{-V(x)} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\delta_1 n}^{|x|}}(\mathbf{V}_x) \sum_{\substack{y > v \\ y \in \mathcal{O}_{n,n^b}}} e^{-V(y)} f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\delta_1 n}^{|y|}}(\mathbf{V}_y) \\ & + \sum_{\substack{x \neq y \\ x^* = y^* = z}} e^{-V(x)} e^{-V(y)} \mathbf{1}_{\{x, y \in \mathcal{O}_{n,n^b}\}} f_{\varepsilon h_n}^{n,|x|} \mathbf{1}_{\mathcal{H}_{B,2\delta_1 n}^{|x|}}(\mathbf{V}_x) f_{\varepsilon h_n}^{n,|y|} \mathbf{1}_{\mathcal{H}_{B,2\delta_1 n}^{|y|}}(\mathbf{V}_y). \end{aligned}$$

We then use a similar approach as the one we used for  $\Sigma_1(z)$  to obtain

$$\begin{aligned} \sum_{|z|=l} \Sigma_2(z) & \leq \frac{2C_\infty^2}{n^2} \sum_{|z|=l} \sum_{\substack{x \neq v \\ x^* = v^* = z}} e^{-(1-\delta_1)V(x)} \mathbf{1}_{\{V(v) \geq -B\}} \sum_{\substack{y > v \\ y \in \mathcal{O}_n^v}} e^{-V(y)} \\ & + \frac{C_\infty^2}{n^4} \sum_{|z|=l} \mathbf{1}_{\{V(z) \geq -B\}} \sum_{\substack{x \neq y \\ x^* = y^* = z}} e^{-(1-\delta_1)(V(x) + V(y))}. \end{aligned}$$

Hence, by using conditional independence of the increments of  $V$ ,  $\mathbf{E}[\sum_{|z|=l} \Sigma_2(z)]$  is smaller, for  $n$  large enough, than

$$\begin{aligned} & 2e^{\delta_1 B} \ell_n \frac{C_\infty^2}{n^2} \mathbf{E} \left[ \sum_{|z|=l} \mathbb{1}_{\{V(z) \geq -B\}} \sum_{\substack{x \neq v \\ x^* = v^* = z}} e^{-(1-\delta_1)(V(x)+V(v))} \right] \\ & + \frac{C_\infty^2}{n^4} \mathbf{E} \left[ \sum_{|z|=l} \mathbb{1}_{\{V(z) \geq -B\}} \sum_{\substack{x \neq v \\ x^* = y^* = z}} e^{-(1-\delta_1)(V(x)+V(y))} \right], \end{aligned}$$

where we used as usual  $\mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}] \leq \ell_n$ . Hence, thanks to assumption (A1) and (I.5), for  $n$  large enough

$$\mathbf{E} \left[ \sum_{|z| < \lfloor A\ell_n \rfloor} \Sigma_2(z) \right] \leq e^{\frac{3\varepsilon}{A} h_n} \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k}) \right)^2. \quad (\text{I.58})$$

Collecting Case 1, Case 2 ((I.54), inequalities (I.55), (I.57) and (I.58)) and considering (I.53) give the lemma.  $\square$

We are now ready to prove the lower bound of  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  in Proposition 1. Recall  $u_{1,n} = \sum_{k \geq 1} \Psi_{\lambda_n/2, n^b}^k(f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k})$  where  $\Upsilon_n^k = \{\mathbf{t} \in \mathbb{R}^k; H_k(\mathbf{t}) \leq n^b e^{\varepsilon h_n}\} \cap \mathcal{H}_{B, 2\mathfrak{z}_n}^k$ ,  $\mathcal{H}_{B, 2\mathfrak{z}_n}^k$  is defined in (I.19) and  $\mathfrak{z}_n = 2\ell_n^{1/3}/\delta_1$ . Thanks to Lemmata 3.2, 3.3 and the expression of  $a_n$  (I.48), for  $n$  large enough, as  $e^{-\varepsilon h_n} \leq \frac{1}{4}$ , the probability  $\mathbb{P}(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n^{1-b} \varphi(n^b) e^{-5\varepsilon h_n} u_{1,n})$  is smaller than

$$\mathbb{P}(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b) e^{-4\varepsilon h_n} u_{1,n}/4n^b) \leq e^{-\varepsilon \frac{c_4}{c_2} h_n} \frac{\mathbf{E}[Z_n^2]}{u_{1,n}^2} + h_n e^{-\varepsilon \tilde{c}_2 h_n} + \frac{e^{8\varepsilon h_n - \min(9\varepsilon \log n, 4h_n)}}{n^{2\kappa_b} u_{1,n}^2}.$$

Then, Lemma 3.4 provides the upper bound of  $\mathbf{E}[Z_n^2]$  so  $\mathbb{P}(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b) e^{-4\varepsilon h_n} u_{1,n}/4n^b)$  is smaller, for  $n$  large enough, than (recall that  $h_n \leq \log n$ )

$$e^{-\left(\frac{c_4}{c_2} - \frac{6}{A}\right)\varepsilon h_n} \left( \sum_{k \geq 1} \Psi_{n,n^b}^k(f^{n,k})/u_{1,n} \right)^2 + h_n e^{-\varepsilon \tilde{c}_2 h_n} + \frac{e^{-\min(\varepsilon \log n, 3h_n)}}{n^{2\kappa_b} u_{1,n}^2},$$

which yields the lower bound of Proposition 1.

### 3.3 Upper bound for $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$

For all  $n \geq 1$  and  $x \in \mathbb{T}$ , recall that  $E_x^n$  is the number of excursions, among the first  $n$  excursions to the root, for which the edge  $(x^*, x)$  is reached. In a similar way,  $\tilde{E}_x^n$  is the number of excursions such that the vertex  $x$  is reached more often from above than from below :

$$E_x^n = \sum_{i=1}^n \mathbb{1}_{\{N_x^{T^i} - N_x^{T^{i-1}} \geq 1\}} \quad \text{and} \quad \tilde{E}_x^n := \sum_{i=1}^n \mathbb{1}_{\{\sum_{y: y^* = x} N_y^{T^i} - N_y^{T^{i-1}} > N_x^{T^i} - N_x^{T^{i-1}}\}}.$$

Also introduce the event  $\mathcal{A}_n$  such that all vertices of the trace of  $\{X_k, k \leq T^n\}$  have exponential downfall fluctuation lower than  $n$ , potential larger than  $\mathfrak{z}_n = \ell_n^{1/3}/\delta_1$  and which are visited during a single excursion to the root

$$\mathcal{A}_n := \left\{ \forall j \leq T^n, X_j \in \mathcal{O}_n, \sum_{x \in \mathcal{O}_n} (\mathbb{1}_{\{E_x^n \geq 2\}} + \mathbb{1}_{\{\tilde{E}_x^n \geq 2\}}) \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) = 0 \right\}. \quad (\text{I.59})$$

Note that  $\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{A}_n) = 1$ . Indeed,  $\tilde{E}_x^n \geq 2$  implies  $E_x^n \geq 2$  so

$$1 - \mathbb{P}(\mathcal{A}_n) \leq \mathbb{P}(\exists j \leq T^n : X_j \notin \mathcal{O}_n) + \mathbb{P}\left(\sum_{x \in \mathcal{O}_n} \mathbb{1}_{\{E_x^n \geq 2\}} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) > 0\right).$$

By [AC18] (equation 2.2),  $\mathbb{P}(\exists j \leq T^n : X_j \notin \mathcal{O}_n) \rightarrow 0$ . Moreover,  $\mathbb{P}(\sum_{x \in \mathcal{O}_n} \mathbb{1}_{\{E_x^n \geq 2\}} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) > 0)$  is smaller than

$$\mathbf{E}\left[\sum_{x \in \mathcal{O}_n} \mathbb{P}^\mathcal{E}(E_x^n \geq 2) \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x)\right] = \mathbf{E}\left[\sum_{x \in \mathcal{O}_n} (\mathbb{P}^\mathcal{E}(E_x^n \geq 1) - \mathbb{P}^\mathcal{E}(E_x^n = 1)) \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x)\right].$$

Thanks to the strong Markov property,  $N_x^{T^i} - N_x^{T^{i-1}}, i \in \{1, \dots, n\}$ , are i.i.d under  $\mathbb{P}^\mathcal{E}$  so  $\mathbb{P}^\mathcal{E}(E_x^n \geq 1) - \mathbb{P}^\mathcal{E}(E_x^n = 1) \leq \mathbb{E}^\mathcal{E}[E_x^n] - \mathbb{P}^\mathcal{E}(E_x^n = 1) = n\mathbb{P}^\mathcal{E}(N_x^{T^1} \geq 1)(1 - \mathbb{P}^\mathcal{E}(N_x^{T^1} = 0)^{n-1}) \leq n^2\mathbb{P}^\mathcal{E}(N_x^{T^1} \geq 1)^2$  and by Lemma 3.1, for all  $x$  with  $V(x) \geq \mathfrak{z}_n$ ,  $n^2\mathbb{P}^\mathcal{E}(N_x^{T^1} \geq 1)^2 \leq n^2 e^{-2V(x)} \leq n^{2-1/\delta_1} e^{-V(x)}/\ell_n^{1/\delta_1}$ .  $\delta_1 \in (0, 1/2)$ , hence, by Remark 2

$$\mathbb{P}\left(\sum_{x \in \mathcal{O}_n} \mathbb{1}_{\{E_x^n \geq 2\}} \mathbb{1}_{\mathcal{H}_{\mathfrak{z}_n}^{|x|}}(\mathbf{V}_x) > 0\right) \leq \frac{n^{2-1/\delta_1}}{\ell_n^{1/\delta_1}} \mathbf{E}\left[\sum_{x \in \mathcal{O}_n} e^{-V(x)}\right] \leq \frac{n^{2-1/\delta_1}}{\ell_n^{1/\delta_1-1}} \rightarrow 0.$$

**Lemma 3.5.** *Let  $(u_n, n)$  be a sequence of positive numbers, then*

$$\mathbb{P}^\mathcal{E}(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) > u_n, \mathcal{A}_n) \leq \frac{2n^{1-b}\varphi(n^b)}{u_n} (\mathcal{X}_{1,n} + \mathcal{X}_{2,n} + \mathcal{X}_{3,n}),$$

where

$$\mathcal{X}_{1,n} := \sum_{x \in \mathcal{O}_n} \mathbb{1}_{\{V(x) < \mathfrak{z}_n\}} \left( e^{-V(x)} + \sum_{y: y^* = x} e^{-V(y)} \right) f^{n,|x|}(\mathbf{V}_x), \quad (\text{I.60})$$

$$\mathcal{X}_{2,n} := \sum_{x \in \mathcal{O}_n} \mathbb{1}_{\{V(x) \geq \mathfrak{z}_n\}} \frac{e^{-V(x)}}{H_x} \left( 1 - \frac{1}{H_x} \right)^{\lceil n^b/2 \rceil - 1} (n^b + H_x) f^{n,|x|}(\mathbf{V}_x), \quad (\text{I.61})$$

and

$$\mathcal{X}_{3,n} := \sum_{x \in \mathcal{O}_n} \mathbb{1}_{\{V(x) \geq \mathfrak{z}_n\}} \frac{e^{-V(x)}}{H_x} \frac{\tilde{H}_x}{1 + \tilde{H}_x} \left( 1 - \frac{1}{1 + \tilde{H}_x} \right)^{\lceil n^b/2 \rceil - 1} (n^b + 1 + \tilde{H}_x) f^{n,|x|}(\mathbf{V}_x), \quad (\text{I.62})$$

recall the definition of  $\tilde{H}_x$  in Lemma 3.1.

*Démonstration.* Since  $g_n(0) = 0$ , we have, by Markov inequality, that  $\mathbb{P}^{\mathcal{E}}(\mathcal{R}_{T^n}(g_n, \mathbf{f}^n) > u_n, \mathcal{A}_n)$  is smaller than

$$\begin{aligned} & \frac{2}{u_n} \left( \sum_{x \in \mathcal{O}_n} \mathbf{1}_{\{V(x) < \delta_n\}} \mathbb{E}^{\mathcal{E}} [g_n(\mathcal{L}_x^{T^n})] f^{n,|x|}(\mathbf{V}_x) \right. \\ & \quad \left. + \sum_{x \in \mathcal{O}_n} \mathbf{1}_{\{V(x) \geq \delta_n\}} \mathbb{E}^{\mathcal{E}} [g_n(\mathcal{L}_x^{T^n}) \mathbf{1}_{\{E_x^n, \tilde{E}_x^n \in \{0,1\}\}}] f^{n,|x|}(\mathbf{V}_x) \right). \end{aligned}$$

The first part in the above sum is the easiest to deal with. Indeed, the application  $t \in [1, \infty) \mapsto \varphi(t)/t$  is non-increasing so  $g_n(t) \leq tn^{-b}\varphi(n^b)$  and we have

$$\begin{aligned} \sum_{x \in \mathcal{O}_n} \mathbf{1}_{\{V(x) < \delta_n\}} \mathbb{E}^{\mathcal{E}} [g_n(\mathcal{L}_x^{T^n})] f^{n,|x|}(\mathbf{V}_x) & \leq n^{1-b}\varphi(n^b) \sum_{x \in \mathcal{O}_n} \mathbf{1}_{\{V(x) < \delta_n\}} \mathbb{E}^{\mathcal{E}} [\mathcal{L}_x^{T^1}] f^{n,|x|}(\mathbf{V}_x) \\ & = n^{1-b}\varphi(n^b) \mathcal{X}_{1,n}. \end{aligned}$$

We have used that for all  $1 \leq i \leq n$ ,  $\mathcal{L}_x^{T^i} - \mathcal{L}_x^{T^{i-1}}$  is distributed as  $\mathcal{L}_x^{T^1}$  under  $\mathbb{P}^{\mathcal{E}}$  with mean  $e^{-V(x)} + \sum_{y:y^*=x} e^{-V(y)}$  by Lemma 3.1.

We then move to the high potential part. Assume  $E_x^n \in \{0, 1\}$  and  $\tilde{E}_x^n \in \{0, 1\}$ . If  $E_x^n = 0$ , then the vertex  $x$  is never visited during any of the first  $n$  excursions and  $\tilde{E}_x^n = 0$ . Thus,  $g_n(\mathcal{L}_x^{T^n}) = g_n(0) = 0$ . If  $E_x^n = 1$  and  $\tilde{E}_x^n = 0$ , then there exists  $i \in \{1, \dots, n\}$  such that  $N_x^{T^i} - N_x^{T^{i-1}} \geq 1$  and  $\forall j \neq i$ ,  $N_x^{T^j} - N_x^{T^{j-1}} = 0$  and  $\forall m \in \{1, \dots, n\}$ ,  $\sum_{y:y^*=x} N_y^{T^m} - N_y^{T^{m-1}} \leq N_x^{T^m} - N_x^{T^{m-1}}$ . In particular, since, starting from the root  $e$ ,  $\mathcal{L}_x^{T^n} = \sum_{j=1}^n (N_x^{T^j} - N_x^{T^{j-1}} + \sum_{y:y^*=x} N_y^{T^j} - N_y^{T^{j-1}})$ , we have, on  $\{E_x^n = 1, \tilde{E}_x^n = 0\}$

$$\mathcal{L}_x^{T^n} = N_x^{T^i} - N_x^{T^{i-1}} + \sum_{y:y^*=x} N_y^{T^i} - N_y^{T^{i-1}} \leq 2(N_x^{T^i} - N_x^{T^{i-1}}). \quad (\text{I.63})$$

Otherwise, if  $E_x^n = 1$  and  $\tilde{E}_x^n = 1$ , then there exists  $i \in \{1, \dots, n\}$  such that  $N_x^{T^i} - N_x^{T^{i-1}} \geq 1$  and  $\forall j \neq i$ ,  $N_x^{T^j} - N_x^{T^{j-1}} = 0$  and  $\exists m' \in \{1, \dots, n\}$  such that  $\sum_{y:y^*=x} N_y^{T^{m'}} - N_y^{T^{m'-1}} > N_x^{T^{m'}} - N_x^{T^{m'-1}}$  and  $\forall m \neq m'$ ,  $\sum_{y:y^*=x} N_y^{T^m} - N_y^{T^{m-1}} \leq N_x^{T^m} - N_x^{T^{m-1}}$ . So we have necessarily  $m' = i$  and, on  $\{E_x^n = 1, \tilde{E}_x^n = 1\}$

$$\mathcal{L}_x^{T^n} = N_x^{T^i} - N_x^{T^{i-1}} + \sum_{y:y^*=x} N_y^{T^i} - N_y^{T^{i-1}} \leq 2 \sum_{y:y^*=x} N_y^{T^i} - N_y^{T^{i-1}}. \quad (\text{I.64})$$

$g_n$  is non-decreasing so (I.63) and (I.64) give, when  $E_x^n \in \{0, 1\}$  and  $\tilde{E}_x^n \in \{0, 1\}$

$$g_n(\mathcal{L}_x^{T^n}) \leq \sum_{i=1}^n g_n(2(N_x^{T^i} - N_x^{T^{i-1}})) + \sum_{i=1}^n g_n(2 \sum_{y:y^*=x} N_y^{T^i} - N_y^{T^{i-1}}).$$

From this inequality, it follows that  $\mathbb{E}^\mathcal{E} [g_n(\mathcal{L}_x^{T^n}) \mathbf{1}_{\{E_x^n, \tilde{E}_x^n \in \{0,1\}\}}]$  is smaller than

$$\begin{aligned} n\mathbb{E}^\mathcal{E} [g_n(2N_x^{T^1})] + n\mathbb{E}^\mathcal{E} [g_n(2 \sum_{y;y^*=x} N_y^{T^1})] &\leq n^{1-b}\varphi(n^b)\mathbb{E}^\mathcal{E} [N_x^{T^1} \mathbf{1}_{\{N_x^{T^1} \geq \lceil n^b/2 \rceil\}}] \\ &\quad + n^{1-b}\varphi(n^b)\mathbb{E}^\mathcal{E} \left[ \sum_{y;y^*=x} N_y^{T^1} \mathbf{1}_{\{\sum_{y;y^*=x} N_y^{T^1} \geq \lceil n^b/2 \rceil\}} \right]. \end{aligned}$$

We have used that for all  $1 \leq i \leq n$ ,  $N_x^{T^i} - N_x^{T^{i-1}}$  (resp.  $\sum_{y;y^*=x} N_y^{T^i} - N_y^{T^{i-1}}$ ) is distributed as  $N_x^{T^1}$  (resp.  $\sum_{y;y^*=x} N_y^{T^1}$ ) under  $\mathbb{P}^\mathcal{E}$  and the fact that the application  $t \in [1, \infty) \mapsto \varphi(t)/t$  is non-increasing. Then, by Lemma 3.1

$$\mathbb{E}^\mathcal{E} [N_x^{T^1} \mathbf{1}_{\{N_x^{T^1} \geq \lceil n^b/2 \rceil\}}] \leq \frac{e^{-V(x)}}{H_x} \left(1 - \frac{1}{H_x}\right)^{\lceil n^b/2 \rceil - 1} (n^b + H_x),$$

and

$$\mathbb{E}^\mathcal{E} \left[ \sum_{y;y^*=x} N_y^{T^1} \mathbf{1}_{\{\sum_{y;y^*=x} N_y^{T^1} \geq \lceil n^b/2 \rceil\}} \right] \leq \frac{e^{-V(x)}}{H_x} \frac{\tilde{H}_x}{1 + \tilde{H}_x} \left(1 - \frac{1}{1 + \tilde{H}_x}\right)^{\lceil n^b/2 \rceil - 1} (n^b + 1 + \tilde{H}_x),$$

which ends the proof.  $\square$

**Lemma 3.6.** *Let  $b \in [0, 1)$ . For  $n$  large enough*

$$\mathbf{E}[\mathcal{X}_{1,n} + \mathcal{X}_{2,n} + \mathcal{X}_{3,n}] \leq 3(\log n)^2 u_{2,n}.$$

where we recall  $u_{2,n} = \sum_{k \geq 1} (\Psi_n^k(f^{n,k} \mathbf{1}_{\mathbb{R}^k \setminus \mathcal{A}_{3n}^k}) + \Psi_{n,n^b/(\log n)^2}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n,n^b/(W(\log n)^2)}^k(f^{n,k})])$ , with  $W = \sum_{|z|=1} e^{-V(z)}$ .

*Démonstration.* We start with the easiest part, that is the expression of  $\mathbf{E}[\mathcal{X}_{1,n}]$ . Thanks to hypothesis (I.2)

$$\begin{aligned} \mathbf{E}[\mathcal{X}_{1,n}] &= \mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \mathbf{1}_{\{V(x) < \mathfrak{J}_n\}} \left( e^{-V(x)} + e^{-V(x)} \sum_{y;y^*=x} e^{-V_x(y)} \right) f^{n,|x|}(\mathbf{V}_x) \right] \\ &= 2\mathbf{E} \left[ \sum_{x \in \mathcal{O}_n} \mathbf{1}_{\{V(x) < \mathfrak{J}_n\}} e^{-V(x)} f^{n,|x|}(\mathbf{V}_x) \right] = 2 \sum_{k \geq 1} \Psi_n^k(f^{n,k} \mathbf{1}_{\mathbb{R}^k \setminus \mathcal{A}_{3n}^k}). \end{aligned}$$

Let

$$\tilde{\lambda}_n := \frac{\lceil n^b/2 \rceil - 1}{\log q_n} \quad \text{with} \quad q_n := \frac{4C_\infty \ell_n n^b}{\sum_{k \geq 1} \Psi_{n,n^b/(\log n)^2}^k(f^{n,k})},$$

and let us find an upper bound for  $\mathbf{E}[\mathcal{X}_{2,n}]$ . For that, we decompose  $\mathcal{X}_{2,n}$  into two parts according

to the value of  $H_x$  :

$$\begin{aligned} \mathcal{X}_{2,n} &\leq \sum_{x \in \mathcal{O}_n} (\mathbf{1}_{\{H_x \leq \tilde{\lambda}_n\}} + \mathbf{1}_{\{H_x > \tilde{\lambda}_n\}}) \frac{e^{-V(x)}}{H_x} \left(1 - \frac{1}{H_x}\right)^{\lceil n^b/2 \rceil - 1} (n^b + H_x) f^{n,|x|}(\mathbf{V}_x) \\ &\leq C_\infty (n^b + \tilde{\lambda}_n) \left(1 - \frac{1}{\tilde{\lambda}_n}\right)^{\lceil n^b/2 \rceil - 1} \sum_{x \in \mathcal{O}_n} e^{-V(x)} + \left(1 + \frac{n^b}{\tilde{\lambda}_n}\right) \sum_{x \in \mathcal{O}_{n, \tilde{\lambda}_n}} e^{-V(x)} f^{n,|x|}(\mathbf{V}_x). \end{aligned}$$

By definition of  $\tilde{\lambda}_n$  and  $q_n$  (see above),  $(1 - 1/\tilde{\lambda}_n)^{\lceil n^b/2 \rceil - 1} \leq 1/q_n$ . Moreover, by Remark 2,  $\mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}] \leq \ell_n$  and  $\mathbf{E}[\sum_{k \geq 1} \Psi_{n, n^b}^k(f^{n,k})] \leq C_\infty \mathbf{E}[\sum_{x \in \mathcal{O}_n} e^{-V(x)}] \leq C_\infty \ell_n$  so for  $n$  large enough ( $q_n \geq 4n^b$  implying  $\tilde{\lambda}_n \leq n^b$ ), we obtain

$$\mathbf{E}[\mathcal{X}_{2,n}] \leq \frac{1}{2} \sum_{k \geq 1} \Psi_{n, n^b / (\log n)^2}^k(f^{n,k}) + \left(1 + \frac{n^b}{\tilde{\lambda}_n}\right) \sum_{k \geq 1} \Psi_{n, \tilde{\lambda}_n}^k(f^{n,k}).$$

For  $\mathbf{E}[\mathcal{X}_{3,n}]$ , we decompose  $\mathcal{X}_{3,n}$  into two parts according to the value of  $\tilde{H}_x$  :  $\mathcal{X}_{3,n}$  is smaller than

$$\begin{aligned} &\sum_{x \in \mathcal{O}_n} (\mathbf{1}_{\{1 + \tilde{H}_x \leq \tilde{\lambda}_n\}} + \mathbf{1}_{\{1 + \tilde{H}_x > \tilde{\lambda}_n\}}) \frac{e^{-V(x)}}{H_x} \frac{\tilde{H}_x}{1 + \tilde{H}_x} \left(1 - \frac{1}{1 + \tilde{H}_x}\right)^{\lceil n^b/2 \rceil - 1} (n^b + 1 + \tilde{H}_x) f^{n,|x|}(\mathbf{V}_x) \\ &\leq C_\infty (n^b + \tilde{\lambda}_n) \left(1 - \frac{1}{\tilde{\lambda}_n}\right)^{\lceil n^b/2 \rceil - 1} \sum_{x \in \mathcal{O}_n} e^{-V(x)} + \left(1 + \frac{n^b}{\tilde{\lambda}_n}\right) \sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbf{1}_{\{1 + \tilde{H}_x > \tilde{\lambda}_n\}} \\ &\times \sum_{y; y^* = x} e^{-V_x(y)} f^{n,|x|}(\mathbf{V}_x). \end{aligned}$$

Then, as above,  $C_\infty (n^b + \tilde{\lambda}_n) \left(1 - 1/\tilde{\lambda}_n\right)^{\lceil n^b/2 \rceil - 1} \leq \sum_{k \geq 1} \Psi_{n, n^b / (\log n)^2}^k(f^{n,k})/2$ , also recall that  $\tilde{H}_x = H_x \sum_{y; y^* = x} e^{-V_x(y)}$  so by conditional independence of  $H_x$  and  $\sum_{y; y^* = x} e^{-V_x(y)}$  together with the fact that this random variable has the same law as  $W = \sum_{|x|=1} e^{-V(x)}$ ,

$$\mathbf{E}\left[\sum_{x \in \mathcal{O}_n} e^{-V(x)} \mathbf{1}_{\{1 + \tilde{H}_x > \tilde{\lambda}_n\}} \sum_{y; y^* = x} e^{-V_x(y)} f^{n,|x|}(\mathbf{V}_x)\right] = \sum_{k \geq 1} \mathbf{E}\left[W \Psi_{n, (\tilde{\lambda}_n - 1)/W}^k(f^{n,k})\right].$$

Hence

$$\mathbf{E}[\mathcal{X}_{3,n}] \leq \frac{1}{2} \sum_{k \geq 1} \Psi_{n, n^b / (\log n)^2}^k(f^{n,k}) + \left(1 + \frac{n^b}{\tilde{\lambda}_n}\right) \sum_{k \geq 1} \mathbf{E}\left[W \Psi_{n, (\tilde{\lambda}_n - 1)/W}^k(f^{n,k})\right].$$

Finally, note that  $\Psi_{n, n^b}^k(f^{n,k}) \leq \Psi_{n, n^b / (\log n)^2}^k(f^{n,k})$  so using assumption (A1), we get  $q_n \leq 4C_\infty \ell_n n^{1+b}$  thus giving  $\tilde{\lambda}_n - 1 \geq n^b (\log n)^{-2}$  for all  $b \in (0, 1)$  and  $n$  large enough. Hence, for all  $b \in [0, 1)$  and  $n$  large enough,  $(1 + n^b/\tilde{\lambda}_n) \leq 2(\log n)^2$  and  $\Psi_{n, (\tilde{\lambda}_n - 1)/W}^k(f^{n,k})$  (resp.  $\Psi_{n, \tilde{\lambda}_n}^k(f^{n,k})$ ) is smaller than  $\Psi_{n, n^b / (W(\log n)^2)}^k(f^{n,k})$  (resp.  $\Psi_{n, n^b / (\log n)^2}^k(f^{n,k})$ ) so we obtain the result.  $\square$

We are now ready to prove the upper bound in Proposition 1. Recall (I.59) and let  $\varepsilon > 0$

$$\mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}} > e^{\varepsilon h_n}\right) \leq \mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}} > e^{\varepsilon h_n}, \mathcal{A}_n\right) + 1 - \mathbb{P}(\mathcal{A}_n),$$

where  $u_{2,n} = \sum_{k \geq 1} (\Psi_n^k(f^{n,k} \mathbf{1}_{\mathbb{R}^k \setminus \mathcal{R}_{3^n}^k}) + \Psi_{n, n^b/(\log n)^2}^k(f^{n,k}) + \mathbf{E}[W \Psi_{n, n^b/(\log n)^2}^k(f^{n,k})])$ . By Lemma 3.5 with  $u_n = e^{\varepsilon h_n} n^{1-b} \varphi(n^b) u_{2,n}$  and Lemma 3.6, for  $n$  large enough

$$\mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}} > e^{\varepsilon h_n}, \mathcal{A}_n\right) \leq \frac{2e^{-\varepsilon h_n}}{u_{2,n}} \mathbf{E}[\mathcal{X}_{1,n} + \mathcal{X}_{2,n} + \mathcal{X}_{3,n}] \leq 6(\log n)^2 e^{-\varepsilon h_n},$$

and then for  $n$  large enough

$$\mathbb{P}\left(\frac{\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}} > e^{\varepsilon h_n}\right) \leq 6(\log n)^2 e^{-\varepsilon h_n} + 1 - \mathbb{P}(\mathcal{A}_n).$$

Finally, observe (see Remark 1) that  $(\log n)^2 = o(e^{\varepsilon h_n})$  and we complete the proof of the upper bound recalling (see below (I.59)) that  $1 - \mathbb{P}(\mathcal{A}_n) = o(1)$ .

## 4 Technical estimates for one-dimensional random walk

In this section, we prove some technical expressions involving sums of i.i.d. random variables introduced via the many-to-one Lemma at the beginning of Section 2. Recall that  $(S_i - S_{i-1}, i \geq 1)$  is a sequence of i.i.d. random variables such that  $\mathbb{E}(S_1) = 0$ , there exists  $\eta > 0$  for which  $\mathbb{E}(e^{\eta S_1}) < +\infty$ . Also we denote  $\sigma^2 = \psi''(1) = \mathbb{E}(S_1^2)$ . We also use the following notations : for any  $a$ ,  $\tau_a := \inf\{k > 0, S_k \geq a\}$ ,  $\tau_a^- := \inf\{k > 0, S_k \leq a\}$  and  $\tau_a^{\bar{S}-S} := \inf\{k > 0, \bar{S}_k - S_k \geq a\}$  with  $\bar{S}_k := \max_{1 \leq m \leq k} S_m$  and  $H_j^S := \sum_{i=1}^j e^{S_i - S_j}$ .

### 4.1 Two Laplace transforms

In this section, we deal with Laplace transforms which appear when we study the range with underlying constraint on  $V$ .

**Lemma 4.1.** *Let  $r := r(\ell)$  such that  $\lim_{\ell \rightarrow +\infty} r(\ell)/\ell = +\infty$ , then for any  $\varepsilon > 0$*

$$e^{-(1+\sqrt{c}-\rho(c))\frac{r}{\ell}(1+\varepsilon)} \leq \mathbb{E}\left[e^{-\frac{c\sigma^2}{2\ell^2}\tau_r \mathbf{1}_{\tau_r \leq \tau_\ell^{\bar{S}-S}}}\right] \leq e^{-(1+\sqrt{c}-\rho(c))\frac{r}{\ell}(1-\varepsilon)},$$

with  $\rho(c) = \frac{c\sigma}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{c\sigma^2}{2}u} f(u) du$ , and  $f(u) = \frac{2}{u^{1/2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{u\sigma^2}) - \frac{1}{2} \int_u^{+\infty} \frac{1}{y^{3/2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{y\sigma^2}) dy$ . Note that  $\rho$  can be explicitly calculated : for any  $c > 0$

$$\rho(c) = 2\sqrt{c} \left( \frac{1 - e^{-\sqrt{c}}}{\sinh(\sqrt{c})} \right) - 2 \left( \sqrt{c} - \log((e^{\sqrt{c}} + 1)/2) \right).$$



*Démonstration.* We start with the *upper bound*.

Let us introduce the usual strict ladder epoch sequence  $(T_k := \inf\{i > T_{k-1}, S_i > S_{T_{k-1}}\}, k; T_0 = 0)$ . Then for any  $k$

$$\begin{aligned} \mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2}\tau_r} \mathbf{1}_{\tau_r \leq \tau_\ell^-} \right] &\leq \mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2}\tau_r} \mathbf{1}_{S_{T_k} < r} \mathbf{1}_{\tau_r \leq \tau_\ell^-} \right] + \mathbb{P}(S_{T_k} \geq r) \\ &\leq \left( \mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2}\tau_0} \mathbf{1}_{\tau_0 \leq \tau_\ell^-} \right] \right)^k + \mathbb{P}(S_{T_k} \geq r), \end{aligned} \quad (\text{I.65})$$

where the last equality comes from the strong Markov property and equality  $T_1 = \tau_0 := \inf\{m > 0, S_m > 0\}$ . From here we need the asymptotic in  $\ell$  of  $\mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2}\tau_0} \mathbf{1}_{\tau_0 \leq \tau_\ell^-} \right]$ . First we use following identity

$$\mathbb{E} \left[ e^{-\frac{\lambda}{\ell^2}\tau_0} \mathbf{1}_{\tau_0 \leq \tau_\ell^-} \right] = \mathbb{E}[e^{-\frac{\lambda}{\ell^2}\tau_0}] - \mathbb{P}(\tau_0 > \tau_\ell^-) + \mathbb{E} \left( (1 - e^{-\frac{\lambda}{\ell^2}\tau_0}) \mathbf{1}_{\tau_0 > \tau_\ell^-} \right), \quad (\text{I.66})$$

and then give an upper bound for each of the three terms. Lemma 2.2 in [Aïd10] gives for  $m$  large enough

$$\mathbb{P}(\tau_0 > \tau_\ell^-) = \frac{\mathbb{E}(S_{\tau_0})}{\ell} + o\left(\frac{1}{\ell}\right), \quad (\text{I.67})$$

Both of the other terms can be obtained with a Tauberian theorem, we give here some details for the third one which is more delicate. Let  $dH_\ell(u)$  the measure defined by  $\mathbb{P}(\tau_0 > z\ell^2, \tau_0 > \tau_\ell^-) = \int_z^\infty dH_\ell(u)$ , integration by part gives  $\mathbb{E} \left( (1 - e^{-\frac{\lambda}{\ell^2}\tau_0}) \mathbf{1}_{\tau_0 > \tau_\ell^-} \right) = \int_0^{+\infty} (1 - e^{-\lambda u}) dH_\ell(u) = \lambda \int_0^{+\infty} e^{-\lambda u} \mathbb{P}(\tau_0 > u\ell^2, \tau_0 > \tau_\ell^-) du$ . So we need an asymptotic in  $\ell$  of the tail probability  $\mathbb{P}(\tau_0 > u\ell^2, \tau_0 > \tau_\ell^-)$ . Let us decompose this probability as follows

$$\begin{aligned} \mathbb{P}(\tau_0 > z\ell^2, \tau_0 < \tau_\ell^-) &= \mathbb{P}(\tau_0 > \tau_\ell^- > z\ell^2) + \mathbb{P}(\tau_0 > z\ell^2, \tau_\ell^- \leq z\ell^2) \\ &= \mathbb{P}(\boldsymbol{\tau}_0^- > \boldsymbol{\tau}_\ell > z\ell^2) + \mathbb{P}(\boldsymbol{\tau}_0 > z\ell^2, \boldsymbol{\tau}_\ell \leq z\ell^2) =: P_1 + P_2. \end{aligned} \quad (\text{I.68})$$

where  $\boldsymbol{\tau}_0^- := \inf\{k > 0, \mathbf{S}_k < 0\}$  with for any  $k$ ,  $\mathbf{S}_k = -S_k$  and similarly  $\boldsymbol{\tau}_\ell := \inf\{k > 0, \mathbf{S}_k \geq \ell\}$ .

For  $P_2$ , we just use Donsker's theorem for conditioned random walk to remain positive obtain in [Bol76] which gives  $\lim_{\ell \rightarrow +\infty} \mathbb{P}(\boldsymbol{\tau}_\ell \leq z\ell^2 | \boldsymbol{\tau}_0 > z\ell^2) = \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sigma\sqrt{z})$ , where  $\mathbf{m}$  is the Brownian meander and  $\bar{\mathbf{m}}_1 = \sup_{s \leq 1} \mathbf{m}_s$ . Also we know from Feller [Fel68] (see the first equivalence page 514 of Caravenna [Car05] for the expression we use here) that for any  $z > 0$  :

$$\lim_{\ell \rightarrow \infty} \ell \mathbb{P}(\boldsymbol{\tau}_0 > z\ell^2) = \sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\sqrt{z\sigma^2}}, \quad (\text{I.69})$$

so

$$\lim_{\ell \rightarrow \infty} \ell P_2 = \sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\sqrt{z\sigma^2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sigma\sqrt{z}). \quad (\text{I.70})$$

For  $P_1$  we use a similar strategy, for any  $A > x$ ,  $\varepsilon > 0$  and  $\ell$  large enough

$$\begin{aligned} P_1 &\leq \mathbb{P}(z\ell^2 \leq \tau_\ell \leq A\ell^2, \tau_0 > \tau_\ell) + \mathbb{P}(\tau_0 > A\ell^2) \\ &\leq \sum_{k=z\ell^2}^{A\ell^2} \mathbb{P}(\bar{\mathbf{S}}_{k-1} \leq \ell, \mathbf{S}_k > \ell \mid \tau_0 > k) \mathbb{P}(\tau_0 > k) + \mathbb{P}(\tau_0 > A\ell^2) \\ &\leq (1 + \varepsilon) \sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\ell\sigma} \sum_{k=z\ell^2}^{A\ell^2} \mathbb{P}(\bar{\mathbf{S}}_{k-1} \leq \ell, \mathbf{S}_k > \ell \mid \tau_0 > k) \frac{\ell}{k^{1/2}} + \frac{C}{\ell A^{1/2}}, \end{aligned}$$

where we have used (I.69) for the last inequality and  $C > 0$  is a constant. Also functional limit theorem [Bol76] implies that  $\lim_{\ell \rightarrow +\infty} \sum_{k=z\ell^2}^{A\ell^2} \mathbb{P}(\bar{\mathbf{S}}_{k-1} \leq \ell, \mathbf{S}_k > \ell \mid \tau_0 > k) \frac{\ell}{k^{1/2}} = -\int_z^A \frac{1}{y^{1/2}} d\mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{y\sigma^2})$ . We deduce from that, taking limits  $A \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} &\lim_{\ell \rightarrow \infty} \ell * P_1 \\ &\leq -\sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\sigma} \int_z^{+\infty} \frac{1}{y^{1/2}} d\mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{y\sigma^2}) \\ &= \sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\sigma} \left( \frac{1}{z^{1/2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{z\sigma^2}) - \frac{1}{2} \int_z^{+\infty} \frac{1}{y^{3/2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{y\sigma^2}) dy \right). \end{aligned}$$

Note that just by noticing that  $P_1 \geq \mathbb{P}(z\ell^2 \leq \tau_\ell \leq A\ell^2, \tau_0 > \tau_\ell)$ , above expression is also a lower bound for  $\lim_{\ell \rightarrow \infty} \ell * P_1$ . Considering this, (I.70) and (I.68), we obtain

$$\lim_{\ell \rightarrow \infty} \ell \mathbb{P}(\tau_0 > z\ell^2, \tau_0 > \tau_{-\ell}^-) = \sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\sigma} f(z) \quad (\text{I.71})$$

where  $f$  is the function given in the statement of the Lemma. Note that this convergence is uniform on any compact set in  $(0, \infty)$  by monotonicity of  $z \rightarrow \ell \mathbb{P}(\tau_0 > z\ell^2, \tau_0 < \tau_{-\ell}^-)$ , continuity of the limit and Dini's theorem. From here we follow the same lines of the proof of a Tauberian theorem (Feller [Fel68]) for completion we recall the main lines for our particular case. For any  $\varepsilon > 0$ , by the uniform convergence we have talked about just above,

$$\lim_{\ell \rightarrow +\infty} \ell \int_\varepsilon^{1/\varepsilon} e^{-\lambda u} \mathbb{P}(\tau_0 > u\ell^2, \tau_0 > \tau_{-\ell}^-) du = \sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\sigma} \int_\varepsilon^{1/\varepsilon} e^{-\lambda u} f(u) du.$$

By (I.69), we also have for any  $\ell$  and  $z > 0$ ,  $\mathbb{P}(\tau_0 > z\ell^2, \tau_0 > \tau_{-\ell}^-) \leq \frac{\text{Const}}{z^{1/2}\ell}$  and since we have  $\int_0^{+\infty} e^{-\lambda u} u^{-1/2} du < +\infty$ , we get  $\lim_{\varepsilon \rightarrow 0} \lim_{\ell \rightarrow +\infty} \int_0^\varepsilon e^{-\lambda u} \ell \mathbb{P}(\tau_0/\ell^2 > u) = 0$ .

Similarly  $\lim_{\varepsilon \rightarrow 0} \lim_{\ell \rightarrow +\infty} \int_{1/\varepsilon}^{+\infty} e^{-\lambda u} \ell \mathbb{P}(\tau_0/\ell^2 > u, \tau_0 > \tau_{-\ell}^-) du = 0$ . Finally

$$\begin{aligned} \lim_{\ell \rightarrow +\infty} \ell \int_0^{+\infty} (1 - e^{-\lambda u}) dH_\ell(u) &= \lim_{\ell \rightarrow +\infty} \ell \mathbb{E} \left( (1 - e^{-\frac{\lambda}{\ell^2} \tau_0}) \mathbf{1}_{\tau_0 > \tau_{-\ell}} \right) \\ &= \lambda \sqrt{\frac{2}{\pi}} \frac{\mathbb{E}(S_{\tau_0})}{\sigma} \int_0^{+\infty} e^{-\lambda u} f(u) du. \end{aligned} \quad (\text{I.72})$$

Note also that just by using (I.69) we also have  $\lim_{\ell \rightarrow +\infty} \ell \mathbb{E}[1 - e^{-\frac{\lambda}{\ell^2} \tau_0}] = \sqrt{2\lambda} \mathbb{E}(S_{\tau_0}) \sigma^{-1}$ . Then collecting (I.66), (I.67) and (I.72) and taking  $\lambda = c\sigma^2/2$  we obtain for  $\ell$  large enough

$$\mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2} \tau_0} \mathbf{1}_{\tau_0 \leq \tau_{-\ell}^-} \right] = 1 - \frac{\mathbb{E}(S_{\tau_0})}{\ell} \left( 1 + \sqrt{c} - \frac{c\sigma}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{c\sigma^2 u}{2}} f(u) du \right) + o\left(\frac{1}{\ell}\right). \quad (\text{I.73})$$

To obtain an explicit expression for the above integral, we integrate by parts

$$\begin{aligned} &\int_0^{+\infty} e^{-\lambda u} f(u) du \\ &= 2 \int_0^{+\infty} \frac{e^{-\lambda u}}{u^{1/2}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{u\sigma^2}) du - \frac{1}{2\lambda} \int_0^{+\infty} \frac{1}{u^{3/2}} (1 - e^{-\lambda u}) \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{u\sigma^2}) du, \end{aligned}$$

then using the expression of  $\mathbb{P}(\bar{\mathbf{m}}_1 > u) := -2 \sum_{k=1}^{\infty} (-1)^k \exp(-(ku)^2/2)$ ,  $\forall u > 0$ , and elementary computations

$$\int_0^{+\infty} e^{-\lambda u} f(u) du = 2\sqrt{\frac{\pi}{\lambda}} \left( \frac{1}{\sinh(\sqrt{2\lambda}/\sigma)} - \frac{e^{-\sqrt{2\lambda}/\sigma}}{\sinh(\sqrt{2\lambda}/\sigma)} \right) - \frac{\sigma\sqrt{2\pi}}{\lambda} \left( \frac{\sqrt{2\lambda}}{\sigma} - \log((e^{\sqrt{2\lambda}/\sigma} + 1)/2) \right). \quad (\text{I.74})$$

Now we deal with the probability  $\mathbb{P}(S_{T_k} \geq r)$  in the same way as [HS16a]. As  $T_k$  can be written as a sum of i.i.d random variables with common law given by  $\tau_0$ , the exponential Markov property gives for any  $\eta > 0$ ,  $\mathbb{P}(S_{T_k} \geq r) \leq e^{-\eta r} (\mathbb{E}(e^{\eta S_{\tau_0}}))^k$ . Taking  $k = (1 - \varepsilon)r/\mathbb{E}(S_{\tau_0})$  we can find constants  $c'$  and  $c''$  such that  $\mathbb{P}(S_{T_k} \geq r) \leq c' e^{-c'' r}$  for any  $r \geq 1$ . So replacing this and (I.73) in (I.65), we finally get for any  $m$  large enough

$$\begin{aligned} \mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2} \tau_r} \mathbf{1}_{\tau_r \leq \tau_{\ell}^{\bar{S}-S}} \right] &\leq \left( \mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2} \tau_0} \mathbf{1}_{\tau_0 \leq \tau_{-\ell}} \right] \right)^k + \mathbb{P}(S_{T_k} \geq r) \\ &\leq \left( 1 - \frac{\mathbb{E}(S_{\tau_0})}{\ell} \left( 1 + \sqrt{c} - \frac{c\sigma}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{c\sigma^2 u}{2}} f(u) du \right) \right)^{(1-\varepsilon)r/\mathbb{E}(S_{\tau_0})} + c' e^{-c'' r}, \end{aligned}$$

which gives the upper bound.

For the *lower bound* the very beginning starts with the same spirit as the proof of Lemma A.2 in [HS16a] : let  $r_k = a * k$  for  $0 \leq k \leq N := \frac{r}{a}$  and  $a > 0$  (chosen later) then

$$\cap_{k=0}^N \{ \inf\{i > \tau_{r_k}, S_i \geq r_{k+1}\} < \inf\{i > \tau_{r_k}, S_i \leq r_k - \ell\} \} \subset \{ \tau_r \leq \tau_{\ell}^{\bar{S}-S} \},$$

then, the strong Markov property gives

$$\begin{aligned} \mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2} \tau_r} \mathbf{1}_{\tau_r \leq \tau_\ell^-} \right] &\geq \prod_{k=0}^N \mathbb{E}_{r_k} \left( e^{-\frac{c\sigma^2}{2\ell^2} \tau_{r_{k+1}}} \mathbf{1}_{\tau_{r_{k+1}} < \tau_{r_k}^-} \right) \\ &= \prod_{k=0}^N \mathbb{E} \left( e^{-\frac{c\sigma^2}{2\ell^2} \tau_{r_{k+1} - r_k}} \mathbf{1}_{\tau_{r_{k+1} - r_k} < \tau_{-\ell}^-} \right) \\ &= \left( \mathbb{E} \left( e^{-\frac{c\sigma^2}{2\ell^2} \tau_a} \mathbf{1}_{\tau_a < \tau_{-\ell}^-} \right) \right)^{N+1}. \end{aligned}$$

So we only need a lower bound for Laplace transform of the form  $\mathbb{E}(e^{-h\tau_a} \mathbf{1}_{\tau_a < \tau_{-\ell}^-})$ , with  $h = h(\ell) \rightarrow 0$ . From here we follow the same lines as for the upper bound with following differences,  $\tau_0$  (resp.  $\tau_0^-$ ) is replaced by  $\tau_a$  (resp. by  $\tau_{-a}^-$ ), also estimation (I.69) should be replaced by following one that can be found in [AS14b] : there exists  $0 < \theta < +\infty$  such that uniformly in  $a \in [0, a_\ell]$  with  $a_\ell = o(\ell^{1/2})$

$$\ell \mathbb{P}(\tau_{-a}^- \geq z\ell^2) \sim \frac{\theta R(a)}{\sqrt{z}},$$

for large  $\ell$ , where  $R$  is the usual renewal function (see (2.3) in [AS14b]) with following property (see (2.6) together with Lemma 2.1 in [AS14b])

$$\lim_{a \rightarrow \infty} \frac{R(a)}{a} = \frac{1}{\theta} \left( \frac{2}{\pi\sigma^2} \right)^{1/2}. \quad (\text{I.75})$$

Now considering (I.68), with the change we have just talked above, as for any  $a > 0$ , the limit  $\lim_{\ell \rightarrow +\infty} \mathbb{P}(\tau_\ell \leq z\ell^2 | \tau_{-a}^- > z\ell^2)$  is equal to  $\mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sigma\sqrt{z})$ , we obtain

$$\lim_{m \rightarrow \infty} \ell P_2 = \lim_{\ell \rightarrow \infty} \ell \mathbb{P}(\tau_{-a}^- > z\ell^2, \tau_\ell \leq z\ell^2) = \frac{\theta R(a)}{\sqrt{z}} \mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sigma\sqrt{z}),$$

similarly for  $P_1 = \mathbb{P}(\tau_a^- > \tau_\ell > z\ell^2)$ , for  $\ell$  large enough and then taking the limit  $A \rightarrow +\infty$

$$\begin{aligned} P_1 &\geq (1 - \varepsilon) \frac{\theta R(a)}{\ell} \sum_{k=z\ell^2}^{A\ell^2} \mathbb{P}(\bar{\mathbf{S}}_{k-1} \leq \ell, \mathbf{S}_k > \ell | \tau_a^- > k) \frac{\ell}{k^{1/2}} \\ &\geq (1 - 2\varepsilon) \frac{\theta R(a)}{\ell} \int_z^{+\infty} \frac{1}{y^{1/2}} d\mathbb{P}(\bar{\mathbf{m}}_1 > 1/\sqrt{y\sigma^2}). \end{aligned}$$

We then obtain the equivalent of (I.71), that is  $\lim_{\ell \rightarrow \infty} \ell \mathbb{P}(\tau_a > z\ell^2, \tau_a > \tau_{-\ell}^-) = \theta R(a) f(z)$  from which we deduce following lower bound for associated Laplace transform :

$$\lim_{\ell \rightarrow +\infty} m \mathbb{E} \left( (1 - e^{-\frac{\lambda}{\ell^2} \tau_a}) \mathbf{1}_{\tau_a > \tau_{-\ell}^-} \right) = \lambda \theta R(a) \int_0^{+\infty} e^{-\lambda u} f(u) du.$$

In the same spirit  $\lim_{\ell \rightarrow +\infty} \ell \mathbb{E}[1 - e^{-\frac{\lambda}{\ell^2} \tau_{-a}}] = \sqrt{\lambda\pi} \theta R(a)$ . Also first Lemma 2.2 in [Aïd10] gives

for any  $a > 0$  and any  $\ell$  large  $\mathbb{P}(\tau_{-a} > \tau_{-\ell}^-) = \mathbb{P}_{-a}(\tau_0 > \tau_{-\ell-a}^-) \sim \mathbb{E}(-S_{\tau_{-a}})/\ell$ . So finally collecting these estimates and taking  $\lambda = \sigma^2 c/2$ , for any  $\varepsilon > 0$  and  $\ell$  large enough

$$\begin{aligned} & \mathbb{E} \left[ e^{-\frac{c\sigma^2}{2\ell^2}\tau_r} \mathbb{1}_{\tau_r \leq \tau_{\ell}^-} \bar{S}_{-S} \right] \\ & \geq \left( 1 - \left( \frac{\mathbb{E}(-S_{\tau_{-a}})}{\ell} + \frac{\theta R(a)}{\ell} \left( \sqrt{\frac{\pi}{2}} \sigma \sqrt{c} - \frac{c\sigma^2}{2} \int_0^{+\infty} e^{-\frac{c\sigma^2}{2}u} f(u) du \right) \right) (1 + \varepsilon) \right)^{N+1}. \end{aligned}$$

Now recall that  $N = r/a$ , so let us take  $a$  large enough in such a way that (using (1.75))  $R(a)/a \leq \frac{1}{\theta} \left( \frac{2}{\pi\sigma^2} \right)^{1/2} (1 + \varepsilon)$ . Also for large  $a$ ,  $\mathbb{E}(-S_{\tau_{-a}})/a \leq (1 + \varepsilon)$  (this can be seen easily, noticing that undershoot  $S_{\tau_{-a}} - a$  has a second moment). This finishes the proof.  $\square$

**Lemma 4.2.** *For any  $\varepsilon > 0$ ,  $\beta > 0$ , any  $r$  large enough uniformly in  $t = t(r)$  with  $\lim_{r \rightarrow +\infty} r - t = +\infty$ ,*

$$\mathbb{E} \left( e^{-\max_{1 \leq j \leq \tau_{r-t}} \bar{S}_j - S_j} \right) \leq e^{-2\sqrt{r-t}(1-\varepsilon)}.$$

*Démonstration.* Like in the proof of Lemma 4.1 we use strict ladder epoch sequence  $(T_k := \inf\{s > T_{k-1}, S_s > S_{T_{k-1}}\}, k; T_0 = 0)$ , also let us introduce the following random variable :  $Y_k := \max_{T_{k-1} \leq j \leq T_k} \bar{S}_j - S_j$  for any  $k \geq 1$ . Let  $m$  a positive integer to be chosen later, by the strong Markov property

$$\begin{aligned} \mathbb{E} \left( e^{-\max_{1 \leq k \leq m} Y_k} \right) &= \sum_{k=1}^m \mathbb{E} \left( e^{-Y_k} \mathbb{1}_{Y_k > \max_{i \leq k-1} Y_i, Y_k \geq \max_{k+1 \leq i \leq m} Y_i} \right) \\ &\leq m \mathbb{E} \left( e^{-Y_2} (1 - \mathbb{P}(Y_1 > Y_2 | Y_2))^{m-1} \right). \end{aligned}$$

At this point we need an asymptotic in  $y$  of  $M(y) := \mathbb{P}(Y_1 > y) = \mathbb{P}(\max_{0 \leq s \leq T_0} S_s < -y) = \mathbb{P}(\tau_0 > \tau_{-y})$ , for that we use following equality (see for example [Aïd10] Lemma 2.2) : for large  $y$ ,  $\mathbb{P}(\tau_0 > \tau_{-y}) = \mathbb{E}(S_{\tau_0})/y + o(1/y)$ . So for any large  $A$ , and  $\varepsilon > 0$

$$\begin{aligned} & e^{-Y_2} (1 - \mathbb{P}(Y_1 > Y_2 | Y_2))^{m-1} \\ &= e^{-Y_2} (1 - \mathbb{P}(Y_1 > Y_2 | Y_2))^{m-1} \mathbb{1}_{Y_2 > A} + e^{-Y_2} (1 - \mathbb{P}(Y_1 > Y_2 | Y_2))^{m-1} \mathbb{1}_{Y_2 \leq A} \\ &\leq e^{-Y_2} (1 - \mathbb{E}(S_{\tau_0})(1 - \varepsilon)(Y_2)^{-1})^{m-1} \mathbb{1}_{Y_2 > A} + (1 - \mathbb{P}(Y_1 > A))^{m-1}, \end{aligned}$$

For the second term above we can find constant  $c = c(A)$  such that  $(1 - \mathbb{P}(Y_1 > A))^{m-1} \leq e^{-cm}$ . For the first term, let us introduce measure  $dM$  defined as  $M(x) = \int_x^{+\infty} dM(z) dz$ , then integrating by parts

$$\begin{aligned} \mathbb{E} \left( e^{-Y_2} (1 - \mathbb{E}(S_{\tau_0})(1 - \varepsilon)(Y_2)^{-1})^{m-1} \mathbb{1}_{Y_2 > A} \right) &= - \int_A^{+\infty} e^{-x} \left( 1 - \frac{\mathbb{E}(S_{\tau_0})(1 - \varepsilon)}{x} \right)^{m-1} dR(x) \\ &\leq e^{-A} \left( 1 - \frac{\mathbb{E}(S_{\tau_0})(1 - \varepsilon)}{A} \right)^{m-1} - \int_A^{+\infty} e^{-x} \left( 1 - \frac{\mathbb{E}(S_{\tau_0})(1 - \varepsilon)}{x} \right)^{m-1} R(x) dx \end{aligned}$$

$$\begin{aligned}
& - (m-1)S_{\tau_0}(1-\varepsilon) \int_A^{+\infty} \frac{e^{-x}}{x^2} \left(1 - \frac{\mathbb{E}(S_{\tau_0})(1-\varepsilon)}{x}\right)^{m-2} R(x) dx \\
& \leq e^{-2(1-4\varepsilon)\sqrt{\mathbb{E}(S_{\tau_0})m}},
\end{aligned}$$

the last inequality is definitely not optimal but enough for what we need, we can obtain it easily decomposing the interval  $(A, +\infty)$  on the intervals  $(A, \sqrt{\mathbb{E}(S_{\tau_0})m}(1-\varepsilon))$ ,  $(\sqrt{\mathbb{E}(S_{\tau_0})m}(1-\varepsilon), \sqrt{\mathbb{E}(S_{\tau_0})m}(1+\varepsilon))$  and  $(\sqrt{\mathbb{E}(S_{\tau_0})m}(1+\varepsilon), +\infty)$ . Collecting the above inequalities, we obtain that for any  $\varepsilon > 0$  and  $m$  large enough

$$\mathbb{E}\left(e^{-\max_{1 \leq k \leq m} Y_k}\right) \leq 2me^{-2(1-4\varepsilon)\sqrt{\mathbb{E}(S_{\tau_0})m}}.$$

To finish the proof we follow the same lines as the end of the proof of Lemma 4.1 (below (I.74)), that is saying that  $\mathbb{E}\left(e^{-\max_{1 \leq j \leq \tau_r-t} \bar{S}_j - S_j}\right) \leq \mathbb{E}\left(e^{-\max_{1 \leq k \leq m} Y_k}\right) + \mathbb{P}(S_{T_k} \geq r-t)$  then taking  $k = (1-\varepsilon)(r-t)/\mathbb{E}(S_{\tau_0})$ .  $\square$

## 4.2 Additional technical estimates

**Lemma 4.3.** *Let  $(t_\ell)$  a positive increasing sequence such that  $t_\ell \ell^{-1/2} \rightarrow +\infty$  but  $t_\ell \ell^{-1} \rightarrow 0$ . For any  $B > 0$  and  $\ell$  large enough*

$$\mathbb{P}(\tau_{\ell^{3/2}}^{\bar{S}-S} \vee \tau_{-B}^- > \tau_{t_\ell}) \geq e^{-\frac{t_\ell}{\sqrt{\ell}}(1+o(1))}. \quad (\text{I.76})$$

Let  $A > 0$  large,  $d \in (0, 1/2)$ ,  $a > 0$ ,  $0 < b < 1$ ,  $q \in [b, 1]$ ,  $a_b := a(2\mathbf{1}_{q>b} - 1)$  and  $c > 0$

$$\sum_{j \leq A\ell^{3/2}} \mathbb{P}(S_j \geq t_\ell, \sup_{m \leq j} H_m^S \leq e^{q\sqrt{\ell}-a_b\ell^d}, e^{b\sqrt{\ell}} \leq H_j^S \leq e^{b\sqrt{\ell}+c\ell^d}, \underline{S}_j \geq -B) \geq e^{-\frac{t_\ell}{q\sqrt{\ell}}(1+o(1))}. \quad (\text{I.77})$$

*Démonstration.* The proof of (I.76) follows the same lines as the proof of Lemma A.2 in [HS16a]. For (I.77), as  $j \leq A\ell^{3/2}$ , for any  $(d, e)$  and any  $m \leq j$ ,  $A\ell^{3/2} \exp(\bar{S}_m - S_m) \leq e^{d\sqrt{\ell}+e\ell^d}$  implies  $H_m^S \leq e^{d\sqrt{\ell}+e\ell^d}$  then

$$\begin{aligned}
& \mathbb{P}\left[S_j \geq t_\ell, e^{b\sqrt{\ell}} \leq H_j^S \leq e^{b\sqrt{\ell}+c\ell^d}, \sup_{m \leq j} H_m^S \leq e^{q\sqrt{\ell}-a_b\ell^d}, \underline{S}_j \geq -B\right] \\
& \geq \mathbb{P}\left[S_j \geq t_\ell, b\sqrt{\ell} \leq \bar{S}_j - S_j \leq b\sqrt{\ell} + c\ell^d, \sup_{m \leq j} \bar{S}_m - S_m \leq q\sqrt{\ell} - a'\ell^d, \underline{S}_j \geq -B\right]
\end{aligned}$$

with  $c' = c/2$  and  $a' = a_b + 1$ . To obtain a lower bound for the above probability, the idea is to say that maximum of  $S$  is obtained at a certain instant  $k \leq j$  and that this maximum is larger than  $t_\ell + b\sqrt{\ell} + c'\ell^d + r$  for a certain  $r > 0$  to be chosen latter, then above probability is larger

than :

$$\sum_{k \leq j} \mathbb{P}(\bar{S}_{k-1} < S_k, S_k \geq t_\ell + b\sqrt{\ell} + c'\ell^d + r, \sup_{m \leq k} \bar{S}_m - S_m \leq \sqrt{\ell} - a'\ell^d, \underline{S}_k \geq -B; S_j - S_k \geq t_\ell - S_k, b\sqrt{\ell} \leq S_k - S_j \leq b\sqrt{\ell} + c'\ell^d, \forall m \geq k+1, S_m \leq S_k, S_k - S_m \leq \sqrt{\ell} - a'\ell^d, S_m - S_k \geq -B - S_k).$$

Now, the events  $\{S_m - S_k \geq -B - x\}$ , as well as  $\{S_j - S_k \geq t_\ell - x\}$  increases in  $x$  and as  $S_k \geq t_\ell + b\sqrt{\ell} + c'\ell^d + r$  so we can replace, in the two events of the above probability, « $-S_k$ » by  $-(t_\ell + b\sqrt{\ell} + c'\ell^d + r)$ . This makes appear two independent events, so above probability is larger than

$$\begin{aligned} & \mathbb{P}(\bar{S}_{k-1} < S_k, S_k \geq t_\ell + b\sqrt{\ell} + c'\ell^d + r, \sup_{m \leq k} \bar{S}_m - S_m \leq \sqrt{\ell} - a'\ell^d, \underline{S}_k \geq -B) \times \\ & \mathbb{P}(S_j - S_k \geq -b\sqrt{\ell} - c'\ell^d - r, b\sqrt{\ell} \leq S_k - S_j \leq b\sqrt{\ell} + c'\ell^d, \forall m \geq k+1, \\ & -B - t_\ell - b\sqrt{\ell} + c'\ell^d - r \leq S_m - S_k \leq 0, S_m - S_k \geq -\sqrt{\ell} + a'\ell^d) =: p_1(k) * p_2(k, j). \quad (\text{I.78}) \end{aligned}$$

Probability  $p_2$  can be easily simplified, indeed as  $\lim_{\ell \rightarrow +\infty} t_\ell/\sqrt{\ell} = +\infty$  and  $\ell$  large,  $-B - t_\ell - b\sqrt{\ell} + c'\ell^d - r \leq -\sqrt{\ell}$  and by taking  $r = c'\ell^d$ ,  $p_2$  is smaller than

$$\begin{aligned} & \mathbb{P}(-b\sqrt{\ell} - c'\ell^d \leq S_j - S_k \leq -b\sqrt{\ell}, \forall m \geq k+1, -\sqrt{\ell} + a'\ell^d \leq S_m - S_k \leq 0) \\ & = \mathbb{P}(\forall m \leq j-k, -\sqrt{\ell} + a'\ell^d \leq S_m \leq 0, -b\sqrt{\ell} - c'\ell^d \leq S_{j-k} \leq -b\sqrt{\ell}) \\ & = \mathbb{P}(\forall m \leq j-k, \mathbf{S}_m \leq \sqrt{\ell} - a'\ell^d | \underline{\mathbf{S}}_{j-k} \geq 0, \mathbf{S}_{j-k} \in [b\sqrt{\ell}, b\sqrt{\ell} + c'\ell^d]) \times \\ & \mathbb{P}(\underline{\mathbf{S}}_{j-k} \geq 0, \mathbf{S}_{j-k} \in [b\sqrt{\ell}, b\sqrt{\ell} + c'\ell^d]), \end{aligned}$$

with  $\mathbf{S}_m = -S_m$  for any  $m$ . For the conditional probability we can use a similar result proved by Caravenna and Chaumont [CC13] telling that the distribution  $\mathbb{P}_x(\cdot | \forall m \leq n, S_m \geq 0, S_n \in [0, h])$  converges. Note that they need in their work additional hypothesis on the distribution of  $S_1$  (more especially absolute continuity of the distribution of  $S_1$ ) which is not necessary here as the size of interval  $[b\sqrt{\ell}, b\sqrt{\ell} + c'\ell^d]$  equals  $c'\ell^d \rightarrow +\infty$ . So as  $a'\ell^d = o(\sqrt{\ell})$

$$\lim_{\ell \rightarrow +\infty} \mathbb{P}(\forall m \leq \ell, \mathbf{S}_m \leq c\sqrt{\ell} - a'\ell^d | \underline{\mathbf{S}}_\ell \geq 0, \mathbf{S}_\ell \in [b\sqrt{\ell}, b\sqrt{\ell} + c'\ell^d]) = Cte > 0.$$

Moreover another work of Caravenna ([Car05] Theorem 1) gives for large  $\ell$ ,  $\mathbb{P}(\underline{\mathbf{S}}_\ell \geq 0, \mathbf{S}_\ell \in [b\sqrt{\ell}, b\sqrt{\ell} + c'\ell^d]) \geq b/\ell$ . So finally when  $j-k$  is of the order of  $\ell$ , there exists a constante  $Cte > 0$  such that  $p_2(k, j) \geq Cte * \ell^{-1}$ . Turning back to (I.78) and summing over  $k$  and  $j$ , we obtain

$$\begin{aligned} & \sum_{j \leq A\ell^{3/2}} \sum_{k \leq j} p_1(k) p_2(k, j) \\ & = \sum_{k \leq A\ell^{3/2}} p_1(k) \sum_{j \geq k} p_2(k, j) \geq \sum_k p_1(k) \sum_{j, j-k \sim \ell} p_2(k, j) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{Cte}{\ell} \sum_{k \leq A\ell^{3/2}} \mathbb{P}(\bar{S}_{k-1} < S_k, S_k \geq t_\ell + b(1 + 2\varepsilon_\ell)\sqrt{\ell}, \sup_{m \leq k} \bar{S}_m - S_m \leq \sqrt{\ell}, \underline{S}_k \geq -B) \\
&\geq \frac{Cte}{\ell} \left( \mathbb{P}(\tau_{\sqrt{\ell}}^{\bar{S}-S} \vee \underline{\tau}_{-B} > \tau_{t_\ell + b\sqrt{\ell} + c'\ell^d}) - \sum_{k > A\ell^{3/2}} \mathbb{P}(\sup_{m \leq k} \bar{S}_m - S_m \leq \sqrt{\ell}) \right)
\end{aligned}$$

Now we can check that above sum  $\sum_{k > A\ell^{3/2}} \dots$  as a negligible contribution, indeed the probability  $\mathbb{P}(\sup_{m \leq k} \bar{S}_m - S_m \leq \sqrt{\ell})$  is smaller, thanks to Proposition 3.1 in [FHS11], to  $e^{-\pi^2 \sigma^2 j / 4\ell}$  this implies that  $\sum_{k > A\ell^{3/2}} \mathbb{P}(\sup_{m \leq k} \bar{S}_m - S_m \leq \sqrt{\ell}) \leq e^{-\pi^2 \sigma^2 A\ell^{1/2} / 2}$ . Now if we apply (I.76) to the first probability above as  $b\sqrt{\ell} + c'\ell^d = o(t_\ell)$ , this finishes the proof.  $\square$

Lemma below is a simple extension of FKG inequality.

In the following, a function  $F : \mathbb{R}^k \rightarrow \mathbb{R}$  is said to be non-decreasing if : for all  $\mathbf{s} = (s_1, \dots, s_k) \in \mathbb{R}^k$  and  $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$ ,  $\mathbf{s} \leq_k \mathbf{t}$  implies  $F(\mathbf{s}) \leq F(\mathbf{t})$  where  $\mathbf{s} \leq_k \mathbf{t}$  if and only if  $s_j \leq t_j$  for all  $j \in \{1, \dots, k\}$ .

**Lemma 4.4.** *Let  $r > 0$ ,  $k \in \mathbb{N}^*$ ,  $f_1, f_2 : \mathbb{R}^k \rightarrow \mathbb{R}^+$ . For any  $i \in \{1, 2\}$ , introduce  $\tilde{f}_i(u_1, \dots, u_k) := f_i(u_1, u_1 + u_2, \dots, u_1 + u_2 + \dots + u_k)$ . If  $\tilde{f}_1$  and  $\tilde{f}_2$  are non-decreasing then*

$$\mathbb{E}[f_1(S_1, S_2, \dots, S_k) f_2(S_1, S_2, \dots, S_k)] \geq \mathbb{E}[f_1(S_1, S_2, \dots, S_k)] \mathbb{E}[f_2(S_1, S_2, \dots, S_k)].$$

*Démonstration.* When  $\mathbb{R}^k$  is a totally order set, the first inequality above is the well known regular FKG inequality. Here, we can easily extend it to the partial order  $\leq_k$ . Indeed, since  $\tilde{f}_i$  is non-decreasing for any  $i \in \{1, 2\}$ , we have, by independence of increments of  $S$

$$\prod_{i \in \{1, 2\}} \mathbb{E}[f_i(S_1, S_2, \dots, S_k)] = \prod_{i \in \{1, 2\}} \mathbb{E}[\tilde{f}_i(S_1, S_2 - S_1, \dots, S_k - S_{k-1})] = \mathbb{E}[\mathbf{F}_1(S_1)] \mathbb{E}[\mathbf{F}_2(S_1)],$$

with  $\mathbf{F}_i(u_1) := \mathbb{E}[\tilde{f}_i(u_1, S_2 - S_1, \dots, S_k - S_{k-1})]$  for any  $i \in \{1, 2\}$ . Since  $\tilde{f}_i$  is non-decreasing,  $\mathbf{F}_i$  is also non-decreasing so thanks to the regular FKG inequality,  $\mathbb{E}[\mathbf{F}_1(S_1)] \mathbb{E}[\mathbf{F}_2(S_1)] \leq \mathbb{E}[\mathbf{F}_1 \mathbf{F}_2(S_1)]$ . Again, using that the increments of  $S$  are independent and stationary, the result follows by induction.  $\square$

**Lemma 4.5.** *Let  $(t_\ell)$  a sequence of positive numbers such that  $t_\ell / \ell \rightarrow 0$ . For all  $d \in (0, 1/2]$  such that  $t_\ell / \ell^d \rightarrow +\infty$  and all  $\varepsilon, B > 0$ ,  $a \geq 0$  and  $0 \leq d' < d$  for  $n$  large enough*

$$\sum_{k \leq \ell^2} \mathbb{P}(S_k \geq t_\ell, \max_{j \leq k} H_j^S \leq e^{\ell^d - a\ell^{d'}}, \underline{S}_k \geq -B, \bar{S}_k = S_k) \geq e^{-\frac{t_\ell}{\ell^d} (1 + \varepsilon)}.$$

*Démonstration.* Recall that  $\tau_r = \inf\{i \geq 1; S_i \geq r\}$ . First, observe that for all  $j \leq k \leq \ell^2$ ,  $H_j^S \leq \ell^2 e^{\bar{S}_j - S_j}$  so

$$\sum_{k \leq \ell^2} \mathbb{P}(S_k \geq t_\ell, \max_{j \leq k} H_j^S \leq e^{\ell^d - a\ell^{d'}}, \underline{S}_k \geq -B, \bar{S}_k = S_k)$$



$$\geq \sum_{k \leq \ell^2} \mathbb{P}(k = \tau_{t_\ell}, \max_{j \leq k} \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, \underline{S}_k \geq -B),$$

which is equal to  $\mathbb{P}(\bar{S}_{\ell^2} \geq t_\ell, \forall j \leq \tau_{t_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B)$ .

Now let  $k_\ell = \lfloor (e^\ell t_\ell)^2 \rfloor + \ell^2$ . First note that, since  $\ell^2 \leq k_\ell$ , we have, on  $\{\bar{S}_{\ell^2} \geq t_\ell\}$ ,  $\tau_{t_\ell} = \tau_{t_\ell}^{k_\ell}$  with  $\tau_{t_\ell}^{k_\ell} := k_\ell \wedge \inf\{i \leq k_\ell; S_i \geq t_\ell\}$  so

$$\begin{aligned} & \mathbb{P}(\bar{S}_{\ell^2} \geq t_\ell, \forall j \leq \tau_{t_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B) \\ &= \mathbb{P}(\bar{S}_{\ell^2} \geq t_\ell, \forall j \leq \tau_{t_\ell}^{k_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B). \end{aligned}$$

For any  $k \in \mathbb{N}^*$  and  $r > 0$ , let  $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$  and define the  $\mathbf{t}$ -version  $\tau_r^{k, \mathbf{t}}$  of  $\tau_r^k$  that is

$$\tau_r^{k, \mathbf{t}} := k \wedge \inf\{i \leq k; t_i \geq r\},$$

with the usual convention  $\inf \emptyset = +\infty$ . Then

$$\mathbb{P}(\bar{S}_{\ell^2} \geq t_n, \forall j \leq \tau_{t_\ell}^{k_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B) = \mathbb{E}[f_1 f_2(S_1, S_2, \dots, S_{k_\ell})],$$

with for all  $i \in \{1, 2\}$ ,  $f_i := \mathbf{1}_{A_i^\ell}$ ,  $f_1 f_2(\mathbf{u}) = f_1(\mathbf{u}) f_2(\mathbf{u})$  and

$$A_1^\ell := \{\mathbf{u} = (u_1, \dots, u_{k_\ell}) \in \mathbb{R}^{k_\ell}; \exists j \leq \ell^2 : u_j \geq t_\ell\},$$

and

$$A_2^\ell := \{\mathbf{u} = (u_1, \dots, u_{k_\ell}) \in \mathbb{R}^{k_\ell}; \forall j \leq \tau_{t_\ell}^{k_\ell, \mathbf{t}}, \forall i < j : u_j - u_i \geq -\ell^d + a\ell^{d'} + 2 \log \ell, u_j \geq -B\}.$$

Then, it is easy to see that for all  $i \in \{1, 2\}$ ,  $\tilde{f}_i$  (see Lemma 4.4 for the definition) is non-decreasing according to the partial order  $\leq_{k_\ell}$  defined above. Then, thanks to Lemma 4.4,  $\mathbb{E}[f_1 f_2(S_1, S_2, \dots, S_{k_\ell})]$  is larger than

$$\begin{aligned} & \geq \mathbb{P}((S_1, S_2, \dots, S_{k_\ell}) \in A_1^\ell) \mathbb{P}((S_1, S_2, \dots, S_{k_\ell}) \in A_2^\ell) \\ & \geq \mathbb{P}(\bar{S}_{\ell^2} \geq t_\ell) \mathbb{P}(\forall j \leq \tau_{t_\ell}^{k_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B, \tau_{t_\ell} \leq k_\ell). \end{aligned}$$

Again, on  $\{\tau_{t_\ell} \leq k_\ell\}$ ,  $\tau_{t_\ell}^{k_\ell} = \tau_{t_\ell}$  and thanks to [Koz76] (Theorem A), there exists  $C_K > 0$  such that for  $\ell$  large enough

$$\begin{aligned} & \mathbb{P}(\forall j \leq \tau_{t_\ell}^{k_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B, \tau_{t_\ell} \leq k_\ell) \\ & \geq \mathbb{P}(\forall j \leq \tau_{t_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B) - \mathbb{P}(\tau_{t_\ell} > k_\ell) \\ & \geq \mathbb{P}(\forall j \leq \tau_{t_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2 \log \ell, S_j \geq -B) - C_K e^{-\ell}. \end{aligned}$$

Moreover,  $t_\ell/\ell \rightarrow 0$  so  $\mathbb{P}(\bar{S}_{\ell^2} \geq t_\ell) \rightarrow 1$ . Finally, by (I.76) together with the fact that  $\ell^d \sim$

$\ell^d - a\ell^{d'} - 2\log \ell$  (as  $d > d'$ ) for  $\ell$  large enough,  $\mathbb{P}(\forall j \leq \tau_{t_\ell} : \bar{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2\log \ell, S_j \geq -B) \geq 2e^{-t_\ell \ell^{-d(1+\varepsilon)}}$  and since  $t_{\ell_n}/\ell^d = o(\ell)$ ,  $C_K e^{-\ell} \leq e^{-t_\ell \ell^{-d(1+\varepsilon)}}$ , the result follows.  $\square$

**Lemma 4.6.** *Let  $\alpha \in (1, 2)$  and  $\varepsilon_\alpha \in [0, \alpha - 1)$  and introduce  $\mathbf{L}_\ell := \lfloor \chi \ell^{1+\frac{\varepsilon_\alpha}{2}} \rfloor$ ,  $\chi > 0$ . For all  $\varepsilon > 0$ ,  $\ell$  large enough and any  $k \in \{\mathbf{L}_\ell, \dots, \ell^2\}$*

$$\mathbb{P}(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell}}) \leq e^{-\frac{k\pi^2\sigma^2}{8\ell}(1-\varepsilon)}, \quad (\text{I.79})$$

and for any  $a, d, c > 0$ ,  $b \in (0, 1)$ ,  $\ell$  large enough and any  $k \in \{\mathbf{L}_\ell, \dots, \ell^2\}$

$$\mathbb{P}(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell} - a\ell^d}, e^{b\sqrt{\ell}} < H_k^S \leq e^{b\sqrt{\ell} + c\ell^d}, \underline{S}_k \geq 0) \geq e^{-\frac{k\pi^2\sigma^2}{8\ell}(1+\varepsilon)}. \quad (\text{I.80})$$

*Démonstration.* Let us start with the upper bound. Thanks to the Markov property, for any  $k \in \mathbb{N}$ ,  $k > \mathbf{L}_\ell$

$$\mathbb{P}(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell}}) \leq \mathbb{P}(\max_{j \leq k} \bar{S}_j - S_j \leq \sqrt{\ell}) \leq \mathbb{P}(\max_{j \leq \mathbf{L}_\ell} \bar{S}_j - S_j \leq \sqrt{\ell})^{\lfloor \frac{k}{\mathbf{L}_\ell} \rfloor},$$

and thanks to [FHS11], for  $\ell$  large enough,  $\mathbb{P}(\max_{j \leq \mathbf{L}_\ell} \bar{S}_j - S_j \leq \sqrt{\ell}) \leq e^{-\frac{\pi^2\sigma^2\mathbf{L}_\ell}{8\ell}(1-\frac{\varepsilon}{2})}$ , so for any  $\varepsilon$ ,  $\ell$  large enough and any  $k > \mathbf{L}_\ell$

$$\mathbb{P}(\max_{j \leq k} \bar{S}_j - S_j \leq \sqrt{\ell}) \leq e^{-(1-\frac{\varepsilon}{2})\frac{\pi^2\sigma^2\mathbf{L}_\ell}{8\ell} \lfloor \frac{k}{\mathbf{L}_\ell} \rfloor} \leq e^{-(1-\varepsilon)\frac{k\pi^2\sigma^2}{8\ell}}.$$

For the lower bound, observe that for any  $k \leq \ell^2$ ,  $\mathbb{P}(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell} - a\ell^d}, e^{b\sqrt{\ell}} < H_k^S \leq e^{b\sqrt{\ell} + c\ell^d}, \underline{S}_k \geq 0)$  is larger than  $\mathbb{P}(\max_{j \leq k} \bar{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, b\sqrt{\ell} < \bar{S}_k - S_k \leq b\sqrt{\ell} + c\ell^d - \log \ell^2, \underline{S}_k \geq 0)$ , where  $\boldsymbol{\lambda}'_\ell := \sqrt{\ell} - a\ell^d - \log \ell^2$ . As  $\frac{\varepsilon}{2}\ell^d \geq \log \ell^2$  ( $d > 0$ ), the previous probability is larger than  $\mathbb{P}(\max_{j \leq k} \bar{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, b\sqrt{\ell} < \bar{S}_k - S_k \leq b\sqrt{\ell} + \frac{\varepsilon}{2}\ell^d, \underline{S}_k \geq 0)$ . We need independence to compute this probability so for all  $k \in \mathbb{N}^*$ ,  $\mathbf{L}_\ell < k \leq \ell^2$ , we say that  $\bar{S}_k = S_{k-\ell} \geq \boldsymbol{\lambda}'_\ell$  which gives that for all  $k - \ell < j \leq k$ ,  $\bar{S}_j \leq S_{k-\ell}$  and then,  $\max_{k-\ell < j \leq k} S_{k-\ell} - S_j \leq \boldsymbol{\lambda}'_\ell$  implies that  $S_j \geq S_{k-\ell} - \boldsymbol{\lambda}'_\ell \geq 0$  for all  $k - \ell < j \leq k$ . Hence

$$\mathbb{P}(\max_{j \leq k} \bar{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, b\sqrt{\ell} < \bar{S}_k - S_k \leq b\sqrt{\ell} + \frac{c}{2}\ell^d, \underline{S}_k \geq 0) \geq \mathbb{P}(A_{k,\ell} \cap B_{k,\ell}) = \mathbb{P}(A_{k,\ell})\mathbb{P}(B_{k,\ell}),$$

with

$$A_{k,\ell} := \left\{ \max_{j \leq k-\ell} \bar{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \underline{S}_{k-\ell} \geq 0, S_{k-\ell} = \bar{S}_{k-\ell} \geq \boldsymbol{\lambda}'_\ell \right\},$$

and

$$B_{k,\ell} := \left\{ \forall k - \ell < j \leq k, S_{k-\ell} - S_j \leq \boldsymbol{\lambda}'_\ell, S_j \leq S_{k-\ell}, b\sqrt{\ell} < S_{k-\ell} - S_k \leq b\sqrt{\ell} + \frac{c}{2}\ell^d \right\}.$$

Let  $\mathbf{S} := -S$ .  $\mathbb{P}(B_{k,\ell})$  is nothing but

$$\begin{aligned} \mathbb{P}(\overline{\mathbf{S}}_\ell \leq \boldsymbol{\lambda}'_\ell, \underline{\mathbf{S}}_\ell \geq 0, \mathbf{S}_\ell \in (b\sqrt{\ell}, b\sqrt{\ell} + \frac{c}{2}\ell^d]) &= \mathbb{P}(\underline{\mathbf{S}}_\ell > 0) \mathbb{P}(\mathbf{S}_\ell \in (b\sqrt{\ell}, b\sqrt{\ell} + \frac{c}{2}\ell^d] | \underline{\mathbf{S}}_\ell \geq 0) \\ &\quad \times \mathbb{P}(\overline{\mathbf{S}}_\ell \leq \boldsymbol{\lambda}'_\ell | \underline{\mathbf{S}}_\ell > 0, \mathbf{S}_\ell \in (b\sqrt{\ell}, b\sqrt{\ell} + \frac{c}{2}\ell^d]), \end{aligned}$$

which is larger than  $C/\ell$  for  $\ell$  large enough (see Lemma 4.3).

We then deal with  $\mathbb{P}(A_{k,\ell})$ . Thanks to Lemma 4.4, this probability is larger than

$$\mathbb{P}\left(\max_{j \leq k-\ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell\right) \mathbb{P}(\overline{S}_{k-\ell} \geq \boldsymbol{\lambda}'_\ell) \mathbb{P}(\underline{S}_{k-\ell} \geq 0)^2,$$

and again, using [Koz76] together with the fact that  $\mathbb{P}(\overline{S}_{\mathbf{L}_\ell} \geq \boldsymbol{\lambda}'_\ell) \rightarrow 1$ , there exists  $C > 0$  such that for  $\ell$  large enough and any  $k \in \{\mathbf{L}_\ell, \dots, \ell^2\}$ ,

$$\mathbb{P}(\overline{S}_{k-\ell} \geq \sqrt{\ell}) \mathbb{P}(\underline{S}_{k-\ell} \geq 0) \geq \mathbb{P}(\overline{S}_{\mathbf{L}_\ell} \geq \sqrt{\ell}) \mathbb{P}(\underline{S}_{\ell^2} \geq 0)^2 \geq \frac{C}{\ell^2}.$$

We now turn to the most important part :  $\mathbb{P}(\max_{j \leq k-\ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell)$ . We follow the same lines as the proof of (I.79) : for any  $k \in \{\mathbf{L}_\ell, \dots, \ell^2\}$ ,  $k - \ell > \mathbf{L}_\ell - \ell$  so  $\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell$  together with  $\overline{S}_{\mathbf{L}_\ell - \ell} = S_{\mathbf{L}_\ell - \ell} \leq S_j$  and  $\max_{\mathbf{L}_\ell - \ell < i \leq j} S_i - S_j \leq \boldsymbol{\lambda}'_\ell$  for all  $\mathbf{L}_\ell - \ell < j \leq k - \ell$  implies that  $\max_{j \leq k-\ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell$ . It follows that  $\mathbb{P}(\max_{j \leq k-\ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell)$  is larger than

$$\begin{aligned} &\mathbb{P}\left(\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \overline{S}_{\mathbf{L}_\ell - \ell} = S_{\mathbf{L}_\ell - \ell}, \max_{\mathbf{L}_\ell - \ell < i \leq j} S_i - S_j \leq \boldsymbol{\lambda}'_\ell, S_j \geq S_{\mathbf{L}_\ell - \ell} \forall \mathbf{L}_\ell - \ell < j \leq k - \ell\right) \\ &= \mathbb{P}\left(\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \overline{S}_{\mathbf{L}_\ell - \ell} = S_{\mathbf{L}_\ell - \ell}\right) \mathbb{P}\left(\max_{j \leq k-\ell-(\mathbf{L}_\ell-\ell)} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \underline{S}_{k-\ell-(\mathbf{L}_\ell-\ell)} \geq 0\right). \end{aligned}$$

Moreover, by Lemma 4.4,  $\mathbb{P}(\max_{j \leq k-\ell-(\mathbf{L}_\ell-\ell)} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \underline{S}_{k-\ell-(\mathbf{L}_\ell-\ell)} \geq 0)$  is larger than  $\mathbb{P}(\max_{j \leq k-\ell-(\mathbf{L}_\ell-\ell)} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell) \mathbb{P}(\underline{S}_{k-\ell-(\mathbf{L}_\ell-\ell)} \geq 0)$ . By induction, we get that  $\mathbb{P}(\max_{j \leq k-\ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell)$  is larger than

$$\mathbb{P}\left(\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \overline{S}_{\mathbf{L}_\ell - \ell} = S_{\mathbf{L}_\ell - \ell}\right)^{\mathbf{L}_\ell(k)} \prod_{i \leq \mathbf{L}_\ell(k)} \mathbb{P}(\underline{S}_{k-\ell-i(\mathbf{L}_\ell-\ell)} \geq 0)$$

with  $\mathbf{L}_\ell(k) := \lfloor (k - \ell) / (\mathbf{L}_\ell - \ell) \rfloor$ . Again, by Lemma 4.4,  $\mathbb{P}(\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \overline{S}_{\mathbf{L}_\ell - \ell} = S_{\mathbf{L}_\ell - \ell}) \geq \mathbb{P}(\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell) \mathbb{P}(\underline{S}_{\mathbf{L}_\ell - \ell} \geq 0)$  and as  $k \leq \ell^2$ ,  $\mathbb{P}(\underline{S}_{k-\ell-i(\mathbf{L}_\ell-\ell)} \geq 0) \geq \mathbb{P}(\underline{S}_k \geq 0) \geq \mathbb{P}(\underline{S}_{\ell^2} \geq 0)$ . Hence, by [Koz76]

$$\mathbb{P}\left(\max_{j \leq k-\ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell\right) \geq \left(\frac{C}{\ell\sqrt{\mathbf{L}_\ell} - \ell} \mathbb{P}\left(\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell\right)\right)^{\mathbf{L}_\ell(k)}$$

for some  $C > 0$ . Then, thanks to [FHS11], for all  $\varepsilon > 0$  and  $\ell$  large enough  $\mathbb{P}(\max_{j \leq \mathbf{L}_\ell - \ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell) \geq e^{-(1+\frac{\varepsilon}{4})\frac{\pi^2\sigma^2(\mathbf{L}_\ell-\ell)}{8}}(\boldsymbol{\lambda}'_\ell)^{-2}$  so for  $\ell$  large enough and any  $k \in \{\mathbf{L}_\ell, \dots, \ell^2\}$ ,  $\mathbb{P}(\max_{j \leq k-\ell} \overline{S}_j -$

$S_j \leq \lambda'_\ell$ ) is larger than

$$\left( \frac{C}{\ell \sqrt{\mathbf{L}_\ell - \ell}} e^{-(1+\frac{\varepsilon}{4}) \frac{\pi^2 \sigma^2 (\mathbf{L}_\ell - \ell)}{8(\lambda'_\ell)^2}} \right)^{\mathbf{L}_\ell(k)} \geq e^{-(1+\frac{\varepsilon}{3}) \frac{\pi^2 \sigma^2 (\mathbf{L}_\ell - \ell)}{8(\lambda'_\ell)^2} \mathbf{L}_\ell(k)} \geq e^{-(k-\ell)(1+\frac{\varepsilon}{2}) \frac{\pi^2 \sigma^2}{8(\lambda'_\ell)^2}},$$

where we have used for the first inequality that  $e^{-\eta \frac{\pi^2 \sigma^2 (\mathbf{L}_\ell - \ell)}{8(\lambda'_\ell)^2}}$  is smaller than  $\frac{1}{\ell^{\eta'}}$  for any  $\eta, \eta' > 0$ . Collecting previous inequalities, we obtain

$$\mathbb{P}(A_{k,\ell}) \geq \frac{C}{\ell^2} e^{-(k-\ell)(1+\frac{\varepsilon}{2}) \frac{\pi^2 \sigma^2}{8(\lambda'_\ell)^2}}.$$

Finally, observe that  $\lambda'_\ell \sim \sqrt{\ell}$  and then for any  $k \in \{\mathbf{L}_\ell, \dots, \ell^2\}$

$$\mathbb{P}(A_{k,\ell}) \geq e^{-\frac{k\pi^2 \sigma^2}{8\ell}(1+\varepsilon)},$$

which completes the proof.  $\square$

## Notations

In this section, we have summarized the transversal notations, give a short description of them when it is possible and the page or equation where they are introduced.

*Sequences and constants in the statement of the main theorem*

$\kappa_b$  (equation (I.12)), critical exponent.

$h_n$  (equation (I.13)), resume the constraint on  $V$  and second order for  $\mathcal{R}_n(g_n, \mathbf{f}^n)$ .

$L$  (equation (I.14)).

$\xi$  (equation (I.15)).

*Different form of the cumulative exponential drop of  $V$*

$H_x$  (below (I.8)), variable appearing in the distribution under  $\mathbb{P}^\mathcal{E}$  of the edge local time

at  $x$  before the instant  $T^1$ .

$\tilde{H}_x$  (Lemma 3.1), variable appearing in the distribution under  $\mathbb{P}^\mathcal{E}$  of the sum of edge local times of the descendants of  $x$  before the instant  $T^1$ .

$H^S$  (page 56) version of above  $H$  after the many to one Lemma is applied.

*The regular lines and their possible parameters* :  $\mathcal{O}_{\lambda, \lambda'} := \{x \in \mathbb{T}; \max_{j \leq |x|} H_{x_j} \leq \lambda, H_x > \lambda'\}$

$\lambda = \lambda_n$  (above I.21),  $\lambda = \lambda_n$  (below (I.23)),  $\lambda = \lambda_{n,1}$  and  $\lambda = \lambda_{n,2}$  (in the proof of Theorem 1.3),  $\lambda = \tilde{\lambda}_n$  in the proof of Lemma 3.6. All along the paper  $\lambda'$  is typically of order  $n^b$ .

*Secondary constraints on the environment* :  $\mathcal{H}_{B,\mathfrak{z}}^k = \{(t_1, \dots, t_k) \in \mathbb{R}^k; t_k \geq \mathfrak{z}, \min_{i \leq k} t_i \geq -B\}$

$\mathfrak{z} = \mathfrak{z}_n = \ell_n^{1/3}/\delta_1$  (beginning of Section 3.2).

$\Upsilon$ : (Proposition 1) : various intersections of conditions on  $H$  and  $\mathcal{H}_{B,\cdot}^k$ .

*The branching function*  $\Psi$

$\Psi_{\lambda,\lambda'}^k$  (equation (I.9)),  $k$  is a generation,  $\lambda$  an upper bound for  $H$ ,  $\lambda'$  a lower bound for  $H$ .

$\Psi_{\lambda,\lambda'}^k(\cdot|\cdot)$  (equation (I.18)) a conditional version of  $\Psi_{\lambda,\lambda'}^k$ .

*Elementary random variables related to the random walk*  $\mathbb{X}$

$N_x^n$  Edge local time at  $(x^*, x)$  before  $n$  (equation (I.39)).

$T^n$  (page 47)  $n$ -th instant of return to the root  $e$ .

$E_x^n, \tilde{E}_x^n$  (above I.59).

*Different ranges*

$\mathcal{R}_n(g_n, \mathbf{f}^n)$  the generalized range (equation (I.3)) with

$g_n$  function of constraints on the trajectory of  $(X_n, n)$ ,

$\mathbf{f}^n$  function of constraints on the potential  $V$ .

$\overline{\mathcal{R}}_{T^n}(g_n, \mathbf{f}^n)$  variant of  $\mathcal{R}_{T^n}(g_n, \mathbf{f}^n)$  with additional condition on  $\overline{V}$  (page 47).



# Chapitre II

## Coalescence in small generations for the diffusive randomly biased walk on Galton-Watson tree

**Abstract :** We investigate the range  $\mathcal{R}_T$  of the diffusive biased walk  $\mathbb{X}$  on a Galton-Watson tree  $\mathbb{T}$  in random environment, that is to say the sub-tree of  $\mathbb{T}$  of all distinct vertices visited by this walk up to the time  $T$ . We study the volume of the range with constraints and more precisely the number of  $k$ -tuples ( $k \geq 2$ ) of distinct vertices in this sub-tree, in small generations and satisfying an hereditary condition. A special attention is paid to the vertices visited during distinct excursions of  $\mathbb{X}$  above the root of the Galton-Watson tree as we observe they give the major contribution to this range. As an application, we study the genealogy of  $k \geq 2$  distinct vertices of the tree  $\mathcal{R}_T$  picked uniformly from those in small generations. It turns out that two or more vertices among them share a common ancestor for the last time in the remote past. We also point out an hereditary character in their genealogical tree due to the random environment.

### Contents

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<b>1</b>	<b>Introduction</b> . . . . .	<b>110</b>
1.1	Randomly biased random walk on trees . . . . .	110
1.2	Genealogy of uniformly chosen vertices in the range . . . . .	112
1.3	The tiny generations . . . . .	113
1.4	The small generations : examples . . . . .	114
1.5	The small generations : a general result . . . . .	121
1.6	Further discussion : the critical generations . . . . .	123
<b>2</b>	<b>Proofs of the theorems</b> . . . . .	<b>124</b>
2.1	Proofs of Theorems 1.2 to 1.5 . . . . .	125
2.2	Proof of Theorem 1.6 and Theorem 1.7 . . . . .	131
<b>3</b>	<b>Proofs of Propositions 2 and 3</b> . . . . .	<b>135</b>
3.1	The range on $\mathfrak{C}^{k,\cdot}$ . . . . .	138

# 1 Introduction

## 1.1 Randomly biased random walk on trees

Let  $(N, A)$  be a random variable under a probability measure  $\mathbf{P}$  taking values in  $\mathbb{N} \times \mathbb{R}$  and consider the following Galton-Waston marked tree  $(\mathbb{T}, (A_x; x \in \mathbb{T}))$  rooted at  $e$  : the generation 0 contains one marked individual  $(e, A_e) = (e, 0)$ . For any  $n \in \mathbb{N}^*$ , assume the generation  $n - 1$  has been built. If it is empty, then the generation  $n$  is also empty. Otherwise, any vertex  $x$  in the generation  $n - 1$  gives progeny to  $N_x$  marked children  $(x^1, A_{x^1}), \dots, (x^{N_x}, A_{x^{N_x}})$  independently of other vertices in generation  $n - 1$  according to the law of  $(N, A)$ , thus forming the generation  $n$ , denoted by  $\mathbb{T}_n$ . We assume  $\mathbf{E}[N] > 1$  so that  $\mathbb{T}$  is a super-critical Galton-Watson tree, that is  $\mathbf{P}(\text{non-extinction of } \mathbb{T}) > 0$  and we define  $\mathbf{P}^*(\cdot) := \mathbb{P}(\cdot | \text{non-extinction of } \mathbb{T})$ .

For any vertex  $x \in \mathbb{T}$ , we denote by  $|x|$  the generation of  $x$ , by  $x_i$  its ancestor in generation  $i \in \{0, \dots, |x|\}$  and  $x^* := x_{|x|-1}$  stands for the parent of  $x$ . In particular,  $x_0 = e$  and  $x_{|x|} = x$ . For any  $x, y \in \mathbb{T}$ , we write  $x \leq y$  if  $x$  is an ancestor of  $y$  ( $y$  is said to be a descendent of  $x$ ) and  $x < y$  if  $x \leq y$  and  $x \neq y$ . We then write  $\llbracket x_i, x \rrbracket := \{x_j; j \in \{i, \dots, |x|\}\}$ . Finally, for any  $x, y \in \mathbb{T}$ , we denote by  $x \wedge y$  the most recent common ancestor of  $x$  and  $y$ , that is the ancestor  $u$  of  $x$  and  $y$  such that  $\max\{|z|; z \in \llbracket e, x \rrbracket \cap \llbracket e, y \rrbracket\} = |u|$ .

Let us introduce the branching potential  $V$  : let  $V(e) = A_e = 0$  and for any  $x \in \mathbb{T} \setminus \{e\}$

$$V(x) := \sum_{e < z \leq x} A_z = \sum_{i=1}^{|x|} A_{x_i}.$$

Under  $\mathbf{P}$ ,  $\mathcal{E} := (\mathbb{T}, (V(x); x \in \mathbb{T}))$  is a real valued branching random walk such that  $(V(x) - V(x^*))_{x \in \mathbb{T} \setminus \{e\}}$  is distributed as  $A$ . We will then refer to  $\mathcal{E}$  as the random environment.

For convenience, we add a parent  $e^*$  to the root  $e$  and we introduce the  $\mathbb{T} \cup \{e^*\}$ -valued random walk  $\mathbb{X} := (X_j)_{j \in \mathbb{N}}$  reflected in  $e^*$  such that under the quenched probabilities  $\{\mathbb{P}_z^\mathcal{E}; z \in \mathbb{T} \cup \{e^*\}\}$ , the transition probabilities are given by : for any  $x \in \mathbb{T}$

$$p^\mathcal{E}(x, x^*) = \frac{e^{-V(x)}}{e^{-V(x)} + \sum_{i=1}^{N_x} e^{V(x^i)}} \quad \text{and for all } 1 \leq i \leq N_x, \quad p^\mathcal{E}(x, x^i) = \frac{e^{-V(x^i)}}{e^{-V(x)} + \sum_{i=1}^{N_x} e^{V(x^i)}}.$$

Otherwise,  $p^\mathcal{E}(x, u) = 0$  and  $p^\mathcal{E}(e^*, e) = 1$ . Let  $\mathbb{P}^\mathcal{E} := \mathbb{P}_e^\mathcal{E}$  and we finally define the following annealed probabilities

$$\mathbb{P}(\cdot) := \mathbf{E}[\mathbb{P}^\mathcal{E}(\cdot)] \quad \text{and} \quad \mathbb{P}^*(\cdot) := \mathbf{E}^*[\mathbb{P}^\mathcal{E}(\cdot)].$$

R. Lyons and R. Pemantle [LP92] initiated the study of the randomly biased random walk  $\mathbb{X}$ . When, for all  $x \in \mathbb{T}$ ,  $V(x) = \log \lambda$  for a some constant  $\lambda > 0$ , the walk  $\mathbb{X}$  is known as the  $\lambda$ -biased random walk on  $\mathbb{T} \cup \{e^*\}$  and was first introduced by R. Lyons (see [Lyo90] and [Lyo92]). The



$\lambda$ -biased random walk is transient unless the bias is strong enough : if  $\lambda \geq \mathbf{E}[N]$  then,  $\mathbf{P}^*$ -almost surely,  $\mathbb{X}$  is recurrent (positive recurrent if  $\lambda > \mathbf{E}[N]$ ). It is known since Y. Peres and O. Zeitouni [PZ06] that when  $\lambda = \mathbf{E}[N]$ ,  $\mathbb{X}$  is diffusive : there exists  $\sigma^2 \in (0, \infty)$  such that  $(|X_{[nt]}|/\sqrt{\sigma^2 n})_{t \geq 0}$  converges in law to a standard reflected brownian motion. R. Lyons, R. Pemantle and Y. Peres (see [LPP96b] and [LPP96a]), later joined by E. Aïdékon [Aïd14] studied the transient case and showed that  $\mathbb{X}$  has a deterministic and positive speed  $v_\lambda := \lim_{n \rightarrow \infty} |X_n|/n$ . Moreover, the expression of  $v_\lambda$  is explicit.

When the bias is random, the behavior of  $\mathbb{X}$  depends on the fluctuations of the following log-Laplace transform  $\psi(t) := \log \mathbf{E}[\sum_{|x|=1} e^{-tV(x)}]$  which we assume to be well defined on  $[0, 1]$  : if  $\inf_{t \in [0,1]} \psi(t) > 0$ , then  $\mathbf{P}^*$ -almost surely,  $\mathbb{X}$  is transient and we refer to the work of E. Aïdékon [Aïd08] for this case. Otherwise, it is recurrent. More specifically, G. Faraud [Far11] proved that the random walk  $\mathbb{X}$  is  $\mathbf{P}^*$ -almost surely positive recurrent either if  $\inf_{t \in [0,1]} \psi(t) < 0$  or if  $\inf_{t \in [0,1]} \psi(t) = 0$  and  $\psi'(1) > 0$ . It is null recurrent if  $\inf_{t \in [0,1]} \psi(t) = 0$  and  $\psi'(1) \leq 0$ . When  $\psi'(1) = 0$ , the largest generation reached by the walk  $\mathbb{X}$  is of order  $(\log n)^3$  and the walk is in the slow regime (see [HS07a] and [FHS11])

In the present paper, we focus on the null recurrent randomly biased walk  $\mathbb{X}$  and assume

**Assumption 1.**

$$\inf_{t \in [0,1]} \psi(t) = \psi(1) = 0 \quad \text{and} \quad \psi'(1) < 0. \quad (\text{II.1})$$

Let us introduce

$$\kappa := \inf\{t > 1; \psi(t) = 0\}, \quad (\text{II.2})$$

and assume  $\kappa \in (1, \infty)$ . Under (II.1) and some integrability conditions, it has been proven that  $|X_n|$  and  $\max_{1 \leq j \leq n} |X_j|$  is of order  $n^{1-1/\min(\kappa, 2)}$  (see [HS07b], [Far11], [AdR17] and [dR22]). In other words, the random walk  $\mathbb{X}$  is sub-diffusive for  $\kappa \in (1, 2]$  and diffusive for  $\kappa > 2$ . In this paper, we put ourselves in the latter case.

We now define the range of the random walk  $\mathbb{X}$ . Let  $T \in \mathbb{N}^*$ . The range  $\mathcal{R}_T$  of the random walk  $\mathbb{X}$  is the set of distinct vertices of  $\mathbb{T}$  visited by  $\mathbb{X}$  up to the time  $T$  : if  $\mathcal{L}_u^T := \sum_{j=1}^T \mathbf{1}_{\{X_j=u\}}$  denotes the local time of a vertex  $u \in \mathbb{T}$  at time  $T$  then

$$\mathcal{R}_T = \{u \in \mathbb{T}; \mathcal{L}_u^T \geq 1\}, \quad (\text{II.3})$$

its cardinal is denoted by  $R_T$  and we also called it range. It has been proved by E. Aïdékon and L. de Raphélis that  $R_n$  is of order  $n$  (see [AdR17]). Moreover,  $\mathcal{R}_n$  is a finite sub-tree of  $\mathbb{T}$  and properly renormalized, it converges in law to a random real tree when  $n$  goes to infinity.

Introduce  $T^j$ , the  $j$ -th return time to  $e^*$  :  $T^0 = 0$  and for any  $j \geq 1$ ,  $T^j = \inf\{i > T^{j-1}; X_i = e^*\}$ . Thanks to (II.31), we know that  $T^{n^{1/2}}$  is of order  $n$ . We will be focusing our attention on the range  $\mathcal{R}_{T^{n^{1/2}}}$  and we shall finally present an extension of the range  $\mathcal{R}_n$ .

Let us now define an extension of the volume  $R_{T^{n1/2}}$  : for any integer  $k \geq 2$  and any subset  $\mathfrak{D}$  of  $\mathbb{T}$  with cardinal  $|\mathfrak{D}| \geq k$ , let  $\mathfrak{D}^{\times k} := \mathfrak{D} \times \cdots \times \mathfrak{D}$ , introduce  $\Delta^k := \{\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \mathbb{T}^{\times k}; \forall i_1 \neq i_2, x^{(i_1)} \neq x^{(i_2)}\}$  which is the set of  $k$ -tuples of distinct vertices of  $\mathbb{T}$ . Also introduce the set (assumed to be nonempty)  $\Delta^k(\mathfrak{D}) := \Delta^k \cap \mathfrak{D}^{\times k}$  of  $k$ -tuples of distinct vertices of  $\mathfrak{D}$ . For any  $n \in \mathbb{N}^*$ , any subset  $\mathcal{D}_n$  of  $\mathcal{R}_{T^{n1/2}}$  with cardinal  $D_n$  and for any function  $f : \Delta^k \rightarrow \mathbb{R}^+$ , if  $D_n \geq k$ , we define the range  $\mathcal{A}^k(\mathcal{D}_n, f)$  by

$$\mathcal{A}^k(\mathcal{D}_n, f) := \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} f(\mathbf{x}). \quad (\text{II.4})$$

Otherwise,  $\mathcal{A}^k(\mathcal{D}_n, f)$  is equal to 0. The aim of studying the range  $\mathcal{A}^k(\mathcal{D}_n, f)$  is to understand the interactions between the vertices in the tree  $\mathcal{R}_{T^{n1/2}}$  and most of all to give a description of the genealogy of the vertices in  $\mathcal{R}_{T^{n1/2}}$ . Note that the range we investigate here differs from the range studied in [AK21], where authors focus on the interactions between the trajectories of the random walk  $\mathbb{X}$  and on the trajectories of the underlying branching potential  $V$ .

## 1.2 Genealogy of uniformly chosen vertices in the range

For a nonempty subset  $\mathcal{D}_n$  of  $\mathcal{R}_{T^{n1/2}}$ , introduce the random variable  $\mathcal{X}^n = (\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)})$  taking values in  $\Delta^k$  with law defined by : for any  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ , if  $\mathbb{P}^*(D_n \geq k) > 0$ , then

$$\mathbb{P}^*(\mathcal{X}^n = \mathbf{x}) = \frac{1}{\mathbb{P}^*(D_n \geq k)} \mathbb{E}^* \left[ \frac{\mathbb{1}_{\{\mathbf{x} \in \Delta^k(\mathcal{D}_n)\}}}{|\Delta^k(\mathcal{D}_n)|} \mathbb{1}_{\{D_n \geq k\}} \right], \quad (\text{II.5})$$

and  $\mathbb{P}^*(\mathcal{X}^n = \mathbf{x}) = 0$  otherwise. Note that  $|\Delta^k(\mathcal{D}_n)| = D_n(D_n - 1) \times \cdots \times (D_n - k + 1)$  so the vertices  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  are nothing but  $k$  vertices picked uniformly and without replacement in the set  $\mathcal{D}_n$ . For any bounded function  $f : \Delta^k \rightarrow \mathbb{R}^+$ , we actually have

$$\mathbb{E}^*[f(\mathcal{X}^n)] = \frac{1}{\mathbb{P}^*(D_n \geq k)} \mathbb{E}^* \left[ \frac{\mathcal{A}^k(\mathcal{D}_n, f)}{\mathcal{A}^k(\mathcal{D}_n, 1)} \mathbb{1}_{\{D_n \geq k\}} \right], \quad (\text{II.6})$$

thus making a link between the generalized range  $\mathcal{A}^k(\mathcal{D}_n, \cdot)$  and the law of  $\mathcal{X}^n$ . Recall that our main interest is the genealogy of the  $k$  vertices  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  so let us define the genealogical tree of these  $k$  vertices. First, introduce the largest generation  $M_n := \max_{x \in \mathcal{D}_n} |x|$  of the set  $\mathcal{D}_n$ . Recall that in the diffusive regime (see (II.1) and (II.2) with  $\kappa > 2$ ),  $\max_{x \in \mathcal{R}_{T^{n1/2}}} |x|$ , the largest generation of the tree  $\mathcal{R}_{T^{n1/2}}$ , is of order  $n^{1/2}$  when  $n \rightarrow \infty$ .

If  $D_n \geq k$ , we then define for any  $m \in \{0, \dots, M_n\}$  the equivalence relation  $\sim_m$  on  $\{1, \dots, k\}$  by :  $i_1 \sim_m i_2$  if and only if  $\mathcal{X}^{(i_1,n)}$  and  $\mathcal{X}^{(i_2,n)}$  share a common ancestor in generation  $m$ . We denote by  $\pi_m^{k,n}$  the partition of  $\{1, \dots, k\}$  whose blocks are given by equivalent classes of the relation  $\sim_m$ . The process  $(\pi_m^{k,n})_{0 \leq m \leq M_n}$  is called the genealogical tree of  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$ . Let

$\mathcal{G}^{(i,n)} = |\mathcal{X}^{(i,n)}|$  be the generation of  $\mathcal{X}^{(i,n)}$ . By definition,

$$\pi_0^{k,n} = \{\{1, \dots, k\}\} \text{ and } \pi_m^{k,n} = \{\{1\}, \dots, \{k\}\} \text{ for any } m \in \{\max_{1 \leq i \leq k} \mathcal{G}^{(i,n)}, \dots, M_n\}.$$

Replacing  $\mathcal{R}_{T^{n^{1/2}}}$  by a regular Galton-Watson tree  $\mathfrak{T}$  and  $\mathcal{D}_n$  by  $\{x \in \mathfrak{T}; |x| = T\}$  (the  $T$ -th generation of  $\mathfrak{T}$ ), the genealogy of  $k$  vertices  $\mathcal{X}_T^{(1)}, \dots, \mathcal{X}_T^{(k)}$  uniformly chosen in  $\{x \in \mathfrak{T}; |x| = T\}$  has been deeply studied for fixed  $T$  as well as for  $T \rightarrow \infty$ . First, when  $k = 2$ , K.B. Athreya [Ath12b] proved that when  $\mathfrak{T}$  is super-critical (the mean of the reproduction law is larger than 1)  $\mathcal{X}_T^{(1)}$  and  $\mathcal{X}_T^{(2)}$  share a common ancestor for the last time in the remote past : if  $\mathcal{M}_T := |\mathcal{X}_T^{(1)} \wedge \mathcal{X}_T^{(2)}|$  denotes the generation of the most recent common ancestor of  $\mathcal{X}_T^{(1)}$  and  $\mathcal{X}_T^{(2)}$  then  $(\mathcal{M}_T)$  converges in law to a non-negative random variable depending on the reproduction law  $N$  when  $T$  goes to  $\infty$ . However, when  $\mathfrak{T}$  is critical (the mean of the reproduction law is equal to 1),  $\mathcal{X}_T^{(1)}$  and  $\mathcal{X}_T^{(2)}$  share a common ancestor for the last time in the recent past :  $(\mathcal{M}_T/T)$  converges in law to a  $[0, 1]$ -valued random variable which doesn't depend on the reproduction law  $N$  when  $T$  goes to  $\infty$ , see [Ath12a]. K.B. Athreya also dealt with the sub-critical case (the mean of the reproduction law is smaller than 1) in the latter paper and it is quite similar to the critical case. More recently S. Harris, S. Johnston and M. Roberts gave a full description of the genealogy of the vertices  $\mathcal{X}_T^{(1)}, \dots, \mathcal{X}_T^{(k)}$  for a given integer  $k \geq 2$  for both fixed  $T$  and  $T \rightarrow \infty$ , when the underlying process is a continuous-time Galton-Watson process (see [HJR17] and [Joh19]). See also [AD21] for a study of the genealogy of randomly chosen individuals when the underlying process is a continuous-state branching process.

Let us return to the case of the random walk in random environment. The generations at which the vertices  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  are chosen have a major influence on their genealogical structure. The next four subsections are dedicated to the three regimes we observe : the tiny generations, the small generations, on which we spend most of our time and the critical generations. For the second regime, we are able to give a quite full description of the genealogy of  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$ , displaying five examples we believe to be relevant, see subsection 1.4, and a general result is proven in 1.5. Finally, we show that we can easily extend our results on  $\mathcal{R}_{T^{n^{1/2}}}$  to the range up to the time  $n$ .

### 1.3 The tiny generations

Recall that  $\psi(t) = \log \mathbf{E}[\sum_{|x|=1} e^{-tV(x)}]$  and introduce  $\tilde{\gamma} := \sup\{a \in \mathbb{R}; \inf_{t \geq 0} (\psi(-t) - at) > 0\}$ . By tiny generations, we mean those of order  $\ell_n$  where  $\ell_n \rightarrow \infty$  when  $n \rightarrow \infty$  and  $\ell_n \leq G \log n$  with  $G \in (0, (2\tilde{\gamma})^{-1})$ . The fact is that for these generations, the random environment has a uniform impact. Indeed, P. Andreatti and P. Debs proved in [AD14a] that with high probability,  $\mathcal{R}_n = \{x \in \mathbb{T}; |x| \leq G \log n\}$  for all  $G \in (0, (2\tilde{\gamma})^{-1})$ . Moreover, the value  $(2\tilde{\gamma})^{-1}$  is optimal : if  $G_n$  denotes the largest generation entirely visited by the random walk  $\mathbb{X}$  up to the time  $n$ , then

$\mathbb{P}^*$ -almost surely

$$\frac{G_n}{\log n} \xrightarrow{n \rightarrow \infty} \frac{1}{2\tilde{\gamma}}.$$

For this case, we are therefore capable of giving a description of the genealogy of  $k \geq 2$  vertices uniformly chosen by adapting the works of S. Harris, S. Johnston and M. Roberts to discrete super-critical Galton-Watson trees.

#### 1.4 The small generations : examples

In this subsection, we focus on the small generations of the tree  $\mathcal{R}_{n^{1/2}}$  where we recall that  $T^j$  is the  $j$ -th return time to  $e^* : T^0 = 0$  and for any  $j \geq 1$ ,  $T^j = \inf\{i > T^{j-1}; X_i = e^*\}$ . Let  $(\mathfrak{L}_n)$  be a sequence of positive integers such that  $\mathfrak{L}_n \geq \delta_0^{-1} \log n$  (see Lemma 3.4 for the definition of  $\delta_0$ ) but smaller than the typical generations of the diffusive random walk  $\mathbb{X}$  :

**Assumption 2** (The small generations). *Let  $(\Lambda_i)_{i \in \mathbb{N}}$  be the sequence of functions defined recursively by : for all  $t > 0$ ,  $\Lambda_0(t) = t$  and for any  $i \in \mathbb{N}^*$ ,  $\Lambda_{i-1}(t) = e^{\Lambda_i(t)}$ . There exists  $l_0 \in \mathbb{N}$  such that*

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{L}_n}{n^{1/2}} \Lambda_{l_0}(\mathfrak{L}_n) = 0. \quad (\text{II.7})$$

Assumption 2 ensures that  $\mathfrak{L}_n/n^{1/2}$ , renormalized by a sequence that grows very slowly, goes to 0 when  $n$  goes to  $\infty$ . Note that when  $k = 2$ , it is enough to assume that  $\mathfrak{L}_n = o(n^{1/2})$ .

Let  $(\ell_n)$  be a sequence of positive integers such that  $\delta_0^{-1} \log n \leq \ell_n \leq \mathfrak{L}_n$  and introduce the set

$$\mathcal{D}_n := \{x \in \mathcal{R}_{T^{n^{1/2}}}; \ell_n \leq |x| \leq \mathfrak{L}_n\},$$

with high  $L_n - 1$  where  $L_n := \mathfrak{L}_n - \ell_n + 1$ . Recall that  $D_n$  is the cardinal of  $\mathcal{D}_n$ . Note that  $\lim_{n \rightarrow \infty} \mathbb{P}^*(D_n \geq k) = 1$  so we will refer to the set  $\{D_n \geq k\}$  only if necessary.

For any  $m \in \mathbb{N}$ , recall that  $\mathbb{T}_m = \{x \in \mathbb{T}; |x| = m\}$  be the  $m$ -th generation of the tree  $\mathbb{T}$  and let  $\Delta_m^j := \Delta^j(\mathbb{T}_m)$ . In addition, we also require the following technical assumption.

**Assumption 3.** *There exists  $\delta_1 > 0$  such that  $\psi(t) < \infty$  for all  $t \in [1 - \delta_1, \lceil \kappa + \delta_1 \rceil]$  and for all  $1 \leq j \leq \lceil \kappa + \delta_1 \rceil$ , for all  $\beta = (\beta_1, \dots, \beta_j) \in (\mathbb{N}^*)^{\times j}$  such that  $\sum_{i=1}^j \beta_i \leq \lceil \kappa + \delta_1 \rceil$*

$$c_j(\beta) := \mathbf{E} \left[ \sum_{x \in \Delta_1^j} e^{-\langle \beta, V(x) \rangle_j} \right] < \infty, \quad (\text{II.8})$$

where  $\langle \beta, V(x) \rangle_j := \sum_{i=1}^j \beta_i V(x^{(i)})$ .

The next assumption is an ellipticity condition.

**Assumption 4.** *There exists  $\mathfrak{h} > 0$  such that*

$$\mathbf{P} \left( \inf_{x \in \mathbb{T}} (V(x) - V(x^*)) \geq -\mathfrak{h} \right) = 1. \quad (\text{II.9})$$

**Remark 3.** Although we assumed  $k \geq 2$ , the case  $k = 1$ , that is to say the volume  $D_n = \sum_{\ell_n \leq |z| \leq \mathfrak{L}_n} \mathbb{1}_{\{z \in \mathcal{R}_{T_n^{1/2}}\}}$  of the regular range  $\mathcal{D}_n$  is interesting. The convergence of  $(D_n/(n^{1/2}\mathbf{L}_n))_n$  does not require all the previous assumptions and holds for  $\kappa > 2$ . However, since it is an easy consequence of Theorem 1.6 with  $k = 2$  and  $f = 1$ , we state the following result :

**Theorem 1.1.** Let  $\kappa > 4$ . Under the assumptions 1, 2, 3 and 4, in  $\mathbb{P}^*$ -probability

$$\frac{1}{n^{1/2}\mathbf{L}_n} D_n \xrightarrow[n \rightarrow \infty]{} c_\infty W_\infty,$$

where  $c_\infty$  is defined in (II.26),  $W_\infty$  is the limit of the  $\mathcal{F}_n := \sigma((\mathbb{T}; V(x), |x| \leq n))$  additive martingale  $W_n = \sum_{|x|=n} e^{-V(x)}$  and according to [Big77],  $\mathbf{P}^*(W_\infty > 0) = 0$ . Moreover,  $\lim_{t \rightarrow \infty} t^\kappa \mathbf{P}^*(W_\infty > t)$  exists (see [Liu00]).

In particular, if  $R_n(\ell) = \sum_{|z|=\ell} \mathbb{1}_{\{z \in \mathcal{R}_{T_n^{1/2}}\}}$  denotes the volume of the  $\ell$ -th generation of the range  $\mathcal{R}_{T_n^{1/2}}$  and  $\log n = o(\mathfrak{L}_n)$ , then both  $(R_n(\mathfrak{L}_n)/n^{1/2})$  and  $(\sum_{\ell=\delta_0^{-1} \log n}^{\mathfrak{L}_n} R_n(\ell)/(n^{1/2}\mathfrak{L}_n))$  converge in  $\mathbb{P}^*$ -probability to  $c_\infty W_\infty$ .

In view of Theorem 1.1, we deduce that whenever  $\mathfrak{L}_n$  is large enough but not too close to the largest generation of the tree  $\mathcal{R}_{T_n^{1/2}}$ , the range  $R_n(\mathfrak{L}_n)$  is of order  $n^{1/2}$ . Moreover,  $\mathbf{L}_n - 1$  denotes the height of the set  $\mathcal{D}_n$  in the tree  $\mathcal{R}_{T_n^{1/2}}$  and the volume of  $\mathcal{D}_n$  behaves like  $\mathbf{L}_n \times R_n(\mathfrak{L}_n)$ .

The following theorems are composed of two parts : the first part will be a convergence of the range  $\mathcal{A}_n^k(f)$  for a given function  $f$  and the second part will be an application of this convergence to the genealogy of the vertices  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$ .

In the second example, we present a range such that for a  $k$ -tuple  $\mathbf{x} \in \Delta^k$ , some of the vertices are free while others are obliged to interact with each other. Let  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in (\mathbb{N}^*)^{\times k}$  and introduce

$$f_{\boldsymbol{\lambda}}(x^{(1)}, \dots, x^{(k)}) := \prod_{i=2}^k \mathbb{1}_{\{|x^{(i-1)} \wedge x^{(i)}| < \lambda_i\}}.$$

Note that there is no constraint between  $x^{(i_1)}$  and  $x^{(i_2)}$  if  $i_1 \notin \{i_2 - 1, i_2 + 1\}$ .

**Theorem 1.2.** Let  $k \geq 2$  and assume  $\kappa > 2k$ . Under the assumptions 1, 2, 3 and 4, in  $\mathbb{P}^*$ -probability

$$\frac{1}{(n^{1/2}\mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_n, f_{\boldsymbol{\lambda}}) \xrightarrow[n \rightarrow \infty]{} (c_\infty)^k \mathcal{A}_\infty^k(f_{\boldsymbol{\lambda}}),$$

where  $\mathcal{A}_\infty^k(f_{\boldsymbol{\lambda}}) = \lim_{l \rightarrow \infty} \sum_{\mathbf{x} \in \Delta_l^k} e^{-V(x^{(1)})} \prod_{i=2}^k e^{-V(x^{(i)})} \mathbb{1}_{\{|x^{(i-1)} \wedge x^{(i)}| < \lambda_i\}}$  and this limit holds in  $L^2(\mathbf{P}^*)$ .

In the next example, we are interested in the number of vertices of  $\mathcal{D}_n$  sharing a common ancestor for the last time at the bottom of the tree  $\mathcal{R}_{T_n^{1/2}}$ . Let  $k \geq 2$  be an integer. For any

$\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ , let  $\mathcal{S}^k(\mathbf{x})$  be the first generation at which none of  $x^{(1)}, \dots, x^{(k)}$  share a common ancestor :

$$\mathcal{S}^k(\mathbf{x}) := \min\{m \geq 1; \forall i_1 \neq i_2, |x^{(i_1)} \wedge x^{(i_2)}| < m\}, \quad (\text{II.10})$$

where we recall that  $|x^{(i_1)} \wedge x^{(i_2)}|$  is the most recent common ancestor of  $x^{(i_1)}$  and  $x^{(i_2)}$ . For any  $m \in \mathbb{N}^*$ , introduce  $\mathcal{C}_m^k := \{\mathbf{x} \in \Delta^k; \mathcal{S}^k(\mathbf{x}) \leq m\}$  (see Figure II.1).

Let us also introduce the coalescent times (or split times) of the vertices  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$ , uniformly chosen in the set  $\mathcal{D}_n$ . For a given partition  $\boldsymbol{\pi}$  of  $\{1, \dots, k\}$ , we denote by  $|\boldsymbol{\pi}|$  the total number of blocks of  $\boldsymbol{\pi}$ . Define the coalescent times by :  $\mathcal{S}_0^{k,n} := 0$  and for all  $j \in \mathbb{N}^*$ ,  $k \geq 2$

$$\mathcal{S}_j^{k,n} := \min\{m \geq \mathcal{S}_{j-1}^{k,n}; |\pi_m^{k,n}| > |\pi_{\mathcal{S}_{j-1}^{k,n}}^{k,n}| \wedge (k-1)\}. \quad (\text{II.11})$$

Note that there exists  $\mathcal{J}^{k,n} \in \mathbb{N}$  such that for any  $j \geq \mathcal{J}^{k,n}$ ,  $\mathcal{S}_j^{k,n} = \mathcal{S}^k(\mathcal{X}^{(n)})$  and by definition,  $2 \leq |\{\mathcal{S}_j^{k,n}; j \in \mathbb{N}\}| \leq k$ . One can notice that seen backwards in time, each random time  $\mathcal{S}_j^{k,n} - 1$  with  $0 < j \leq \mathcal{J}^{k,n}$  corresponds to a generation at which two or more vertices among  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  share a common ancestor for the first time.  $\mathcal{S}_j^{k,n}$  is usually referred to as the  $j$ -th split time while  $\mathcal{S}_{\mathcal{J}^{k,n}-j+1}^{k,n}$  is the  $j$ -th coalescent time.

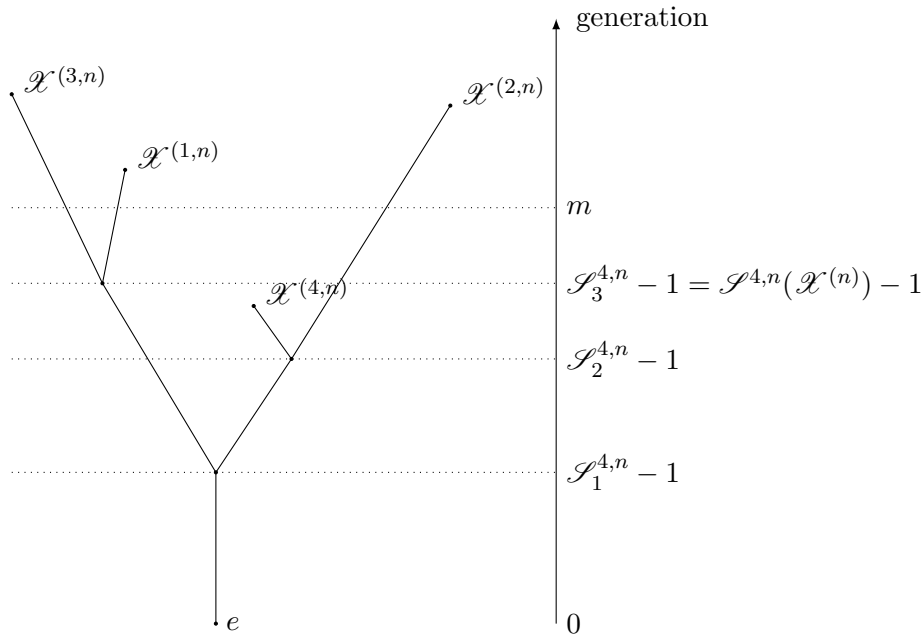


FIGURE II.1 – An example of four vertices belonging to  $\mathcal{C}_m^4$  together with their three coalescent times.

It appears that the number of vertices visited by the random walk  $\mathbb{X}$  belonging to  $\mathcal{C}_m^k$  is large and as a consequence, the sequence of random times  $(\mathcal{S}^k(\mathcal{X}^{(n)}) = \inf\{m \geq 1; \pi_m^{k,n} = \{\{1\}, \dots, \{k\}\}\})_n$  converges in law.

**Theorem 1.3.** *Let  $k \geq 2$ . Assume that  $\kappa > 2k$  and for any  $m \in \mathbb{N}^*$ ,  $\mathbf{x} \in \Delta^k$ ,  $f_m(\mathbf{x}) = \mathbb{1}_{\mathcal{C}_m^k}(\mathbf{x})$ . Recall that  $\mathcal{A}^k(\mathcal{D}_n, f_m)$  is the number of  $k$ -tuples  $\mathbf{x}$  of distinct vertices of  $\mathcal{D}_n$  such that  $\mathcal{S}^k(\mathbf{x}) \leq m$ . Under the assumptions 1, 2, 3 and 4*

1. in  $\mathbb{P}^*$ -probability

$$\frac{1}{(n^{1/2} \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_n, f_m) \xrightarrow{n \rightarrow \infty} (c_\infty)^k \mathcal{A}_\infty^k(f_m),$$

where  $\mathcal{A}_\infty^k(f_m)$  is defined in (II.23). Note that  $\lim_{m \rightarrow \infty} \mathcal{A}_\infty^k(f_m) = (W_\infty)^k$  in  $L^2(\mathbf{P}^*)$ .

2. Moreover, the sequence of random times  $(\mathcal{S}^k(\mathcal{X}^{(n)}))$  converges in law, under  $\mathbb{P}^*$  : for any  $m \in \mathbb{N}^*$

$$\mathbb{P}^*(\mathcal{S}^k(\mathcal{X}^{(n)}) \leq m) \xrightarrow{n \rightarrow \infty} \mathbf{E}^* \left[ \frac{\mathcal{A}_\infty^k(f_m)}{(W_\infty)^k} \right]. \quad (\text{II.12})$$

The convergence in (II.12) is somewhat reminiscent of the result of K.B Athreya ([Ath12b], Theorem 2) for a super-critical Galton-Watson tree stated earlier : each coalescence occurs in a generation close to the root.

In the following result, we compute the law of  $\pi^{k,n}$ . Before that, we add, for convenience, a collection  $\{e^{(i)}; i \in \mathbb{N}^*\}$  of distinct leafs in the generation 0. Let  $q \geq 2$  be an integer and  $\pi$  be a partition of  $\{1, \dots, q\}$ . For any  $m \in \mathbb{N}^*$ , define the set  $\Upsilon_{m,\pi}$  by :  $\mathbf{x} = (x^{(1)}, \dots, x^{(q)}) \in \Upsilon_{m,\pi}$  if and only if  $\mathbf{x} \in \Delta^q$  and

$$\forall \mathbf{B} \in \pi, \forall i_1, i_2 \in \mathbf{B} : (x^{(i_1)})_m = (x^{(i_2)})_m,$$

and if  $|\pi| \geq 2$

$$\forall \mathbf{B} \neq \tilde{\mathbf{B}} \in \pi, \forall i_1 \in \mathbf{B}, i_2 \in \tilde{\mathbf{B}} : (x^{(i_1)})_m \neq (x^{(i_2)})_m,$$

where we recall that, when  $|x^{(i)}| \geq m$ ,  $(x^{(i)})_m$  denotes the ancestor of  $x^{(i)}$  in generation  $m$ . Otherwise, if  $|x^{(i)}| < m$ , we set  $(x^{(i)})_m := e^{(i)}$  so  $\Upsilon_{m,\pi}$  is well defined.

Now, let  $1 \leq d < q$  be two integers. A collection  $(\Xi_i)_{0 \leq i \leq d}$  of partitions of  $\{1, \dots, q\}$  is said to be increasing if it satisfies  $\Xi_0 = \{\{1, \dots, q\}\}$ ,  $\Xi_d = \{\{1\}, \dots, \{q\}\}$  and for all  $i \in \{1, \dots, d\}$ ,  $|\Xi_{i-1}| < |\Xi_i|$ , where we recall that  $|\Xi_i|$  is the total number of blocks of the partition  $\Xi_i$ . For  $p \in \{1, \dots, d\}$ , the  $j$ -th block  $\mathbf{B}_j^{p-1}$  of the partition  $\Xi_{p-1}$  (blocks are ordered by their least element) is the union of  $b_{p-1}(\mathbf{B}_j^{p-1}) \geq 1$  (we will write  $b_{p-1}(\mathbf{B}_j)$  instead) block(s)  $\mathbf{B}_{l_1}^p, \dots, \mathbf{B}_{l_{b_{p-1}(\mathbf{B}_j)}}^p$ ,  $1 \leq l_1 < \dots < l_{b_{p-1}(\mathbf{B}_j)} \leq |\Xi_p|$ , of the partition  $\Xi_p$  and for any  $i \in \{1, \dots, l_{b_{p-1}(\mathbf{B}_j)}\}$ , define

$$\beta_{j,i}^{p-1} := |\mathbf{B}_{l_i}^p|, \quad (\text{II.13})$$

be the cardinal of the block  $\mathbf{B}_{l_i}^p$ .

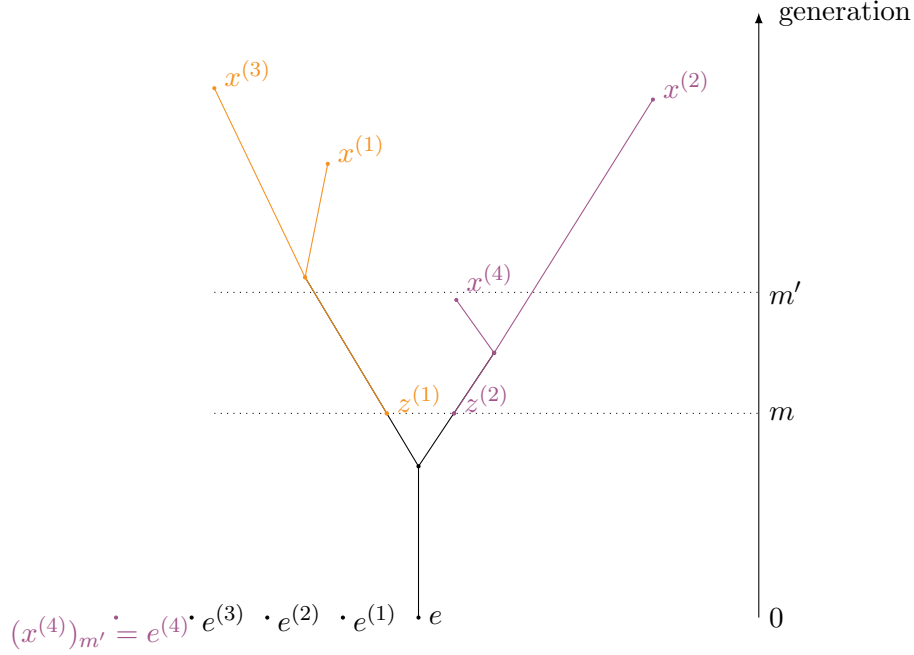


FIGURE II.2 – In the present illustration, the 4-tuple of vertices  $(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)})$  belongs to  $\Upsilon_{m, \pi}$  with  $\pi = \{\{1, 3\}, \{2, 4\}\}$ , since  $z^{(1)} = (x^{(1)})_m = (x^{(3)})_m$ ,  $z^{(2)} = (x^{(2)})_m = (x^{(4)})_m$  and  $z^{(1)} \neq z^{(2)}$ . However, it doesn't belong to  $\Upsilon_{m', \pi}$ .

Let  $(\Xi_i)_{0 \leq i \leq d}$  be an increasing collection of partitions of  $\{1, \dots, q\}$  and let  $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{N}^{\times d}$  such that  $t_1 < \dots < t_d$ . Introduce the set  $\Gamma_{\mathbf{t}, \Xi}^i := \Upsilon_{t_i - 1, \Xi_{i-1}} \cap \Upsilon_{t_i, \Xi_i}$ . We then define the function  $f_{\mathbf{t}, \Xi}^d$  by : for all  $\mathbf{x} \in \Delta^q$

$$f_{\mathbf{t}, \Xi}^d(\mathbf{x}) = \prod_{i=1}^d \mathbb{1}_{\Gamma_{\mathbf{t}, \Xi}^i}(\mathbf{x}). \quad (\text{II.14})$$

The function defined in (II.14) plays a key role in our study :  $f_{\mathbf{t}, \Xi}^d(\mathbf{x})$  characterizes the genealogy of  $\mathbf{x} := (x^{(1)}, \dots, x^{(q)})$ . Indeed, for any  $i \in \{1, \dots, d\}$ , the partition  $\Xi_i$  corresponds to the  $i$ -th generation of the genealogical tree of  $x^{(1)}, \dots, x^{(q)}$  while  $t_i - 1$  denotes the  $i$ -th generation at which at least two branches of this genealogical tree split ( $t_i - 1$  therefore corresponds to a coalescent/split time, see Figure II.3 for instance). We are now ready to state our result :

**Theorem 1.4.** *Let  $k \geq 2$  and assume that  $\kappa > 2k$ . Under the assumptions 1, 2, 3 and 4, for any  $\ell \in \mathbb{N}^*$  such that  $\ell < k$ , any  $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{N}^{\times \ell}$  such that  $s_1 < \dots < s_\ell$  and any increasing collection  $\Pi = (\pi_i)_{0 \leq i \leq \ell}$  of partitions of  $\{1, \dots, k\}$*

1. in  $\mathbb{P}^*$ -probability

$$\frac{1}{(n^{1/2} \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_n, f_{\mathbf{s}, \Pi}^\ell) \xrightarrow{n \rightarrow \infty} (c_\infty)^k \mathcal{A}_\infty^k(f_{\mathbf{s}, \Pi}^\ell). \quad (\text{II.15})$$



2. Moreover, for any non-negative integers  $m_0 < m_1 < \dots < m_\ell$

$$\mathbb{P}^*(\pi_{m_0}^{k,n} = \boldsymbol{\pi}_0, \dots, \pi_{m_\ell}^{k,n} = \boldsymbol{\pi}_\ell) \xrightarrow{n \rightarrow \infty} \mathbf{E}^* \left[ \frac{1}{(W_\infty)^k} \sum_{\substack{\mathbf{s}=(s_1, \dots, s_\ell) \\ m_{i-1} < s_i \leq m_i}} \mathcal{A}_\infty^k(f_{\mathbf{s}, \Pi}^\ell) \right], \quad (\text{II.16})$$

where  $\mathcal{A}_\infty^k(f_{\mathbf{s}, \Pi}^\ell)$  is the limit in  $L^2(\mathbf{P}^*)$  of the martingale  $(\mathcal{A}_l^k(f_{\mathbf{s}, \Pi}^\ell))_l$ , satisfying

$$\sum_{\Pi \text{ increasing}} \sum_{\substack{\mathbf{s}=(s_1, \dots, s_\ell) \\ m_{i-1} < s_i \leq m_i}} \mathcal{A}_\infty^k(f_{\mathbf{s}, \Pi}^\ell) = (W_\infty)^k,$$

and

$$\mathbf{E}^* [\mathcal{A}_\infty^k(f_{\mathbf{s}, \Pi}^\ell)] = e^{\psi(k)} \prod_{i=1}^{\ell} \prod_{j=1}^{|\boldsymbol{\pi}_{i-1}|} c_{b_{i-1}(\mathbf{B}_j)}(\boldsymbol{\beta}_j^{i-1}) \prod_{\substack{\mathfrak{B} \in \boldsymbol{\pi}_i \\ |\mathfrak{B}| \geq 2}} e^{s_{i+1}^* \psi(|\mathfrak{B}|)}, \quad (\text{II.17})$$

with  $s_{i+1}^* = s_{i+1} - s_i - 1$ ,  $s_{\ell+1}^* = 1$ ,  $\boldsymbol{\beta}_j^p := (\beta_{j,1}^p, \dots, \beta_{j,b_p(\mathbf{B}_j)}^p)$  (see (II.13)). We also use the convention  $\prod_{\emptyset} = 1$  and see the assumption 3 for the definition of  $c_l(\boldsymbol{\beta})$ .

**Remark 4** (An hereditary character). *There is an hereditary character hidden in the previous formula (II.17) due to the random environment. The fact is, unlike the case of regular supercritical Galton-Watson trees depending on  $(b_i(\mathbf{B}); \mathbf{B} \in \boldsymbol{\pi}_i, 0 \leq i \leq \ell - 1)$  (see [Joh19], Theorem 3.5), the limit law of the present genealogical tree depends on the collection  $(\boldsymbol{\beta}_j^i; 1 \leq i \leq \ell, 1 \leq j \leq |\boldsymbol{\pi}_{i-1}|)$  and on  $(|\mathfrak{B}|; \mathfrak{B} \in \boldsymbol{\pi}_i, 1 \leq i \leq \ell)$ , making a huge difference. Indeed, by definition, the latter take more account of the genealogical structure than  $(b_i(\mathbf{B}); \mathbf{B} \in \boldsymbol{\pi}_i, 0 \leq i \leq \ell - 1)$ . For instance, let  $k = 4$ ,  $\ell = 3$  and define the increasing collection of partitions  $\Pi = (\boldsymbol{\pi}_i)_{1 \leq i \leq \ell}$  by  $\boldsymbol{\pi}_3 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ ,  $\boldsymbol{\pi}_2 = \{\{1, 3\}, \{2\}, \{4\}\}$ ,  $\boldsymbol{\pi}_1 = \{\{1, 3\}, \{2, 4\}\}$  and  $\boldsymbol{\pi}_0 = \{1, 2, 3, 4\}$ . We have  $\boldsymbol{\beta}_1^3 = (1, 1)$ ,  $\boldsymbol{\beta}_2^3 = 1$ ,  $\boldsymbol{\beta}_3^3 = 1$ ;  $\boldsymbol{\beta}_1^2 = 2$ ,  $\boldsymbol{\beta}_2^2 = (1, 1)$ ;  $\boldsymbol{\beta}_1^1 = (2, 2)$  and thanks to (II.17), for any  $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{N}^{\times 3}$  such that  $t_1 < t_2 < t_3$*

$$\begin{aligned} \mathbf{E}^* [\mathcal{A}_\infty^4(f_{\mathbf{t}, \Pi}^3)] &= \mathbf{E} \left[ \sum_{|x|=1} e^{-2V(x)} \right] \mathbf{E} \left[ \sum_{\substack{x \neq y \\ |x|=|y|=1}} e^{-V(x)-V(y)} \right]^2 \mathbf{E} \left[ \sum_{\substack{x \neq y \\ |x|=|y|=1}} e^{-2V(x)-2V(y)} \right] \\ &\quad \times e^{t_3^* \psi(2) + 2t_2^* \psi(2) + \psi(4)}. \end{aligned}$$

Also introduce the increasing collection of partitions  $\Pi' = (\boldsymbol{\pi}'_i)_{1 \leq i \leq \ell}$  such that  $\boldsymbol{\pi}'_3 = \boldsymbol{\pi}_3$ ,  $\boldsymbol{\pi}'_2 = \boldsymbol{\pi}_2$ ,  $\boldsymbol{\pi}'_1 = \{\{1, 3, 4\}, \{2\}\}$  and  $\boldsymbol{\pi}'_0 = \boldsymbol{\pi}_0$ . We have  $\boldsymbol{\beta}_1^3 = (1, 1)$ ,  $\boldsymbol{\beta}_2^3 = 1$ ,  $\boldsymbol{\beta}_3^3 = 1$ ;  $\boldsymbol{\beta}_1^2 = (2, 1)$ ,  $\boldsymbol{\beta}_2^2 = 1$ ;  $\boldsymbol{\beta}_1^1 = (3, 1)$  and thanks to (II.17), for any  $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{N}^{\times 3}$  such that  $t_1 < t_2 < t_3$

$$\mathbf{E}^* [\mathcal{A}_\infty^4(f_{\mathbf{s}, \Pi'}^3)] = \mathbf{E} \left[ \sum_{\substack{x \neq y \\ |x|=|y|=1}} e^{-V(x)-V(y)} \right] \mathbf{E} \left[ \sum_{\substack{x \neq y \\ |x|=|y|=1}} e^{-2V(x)-V(y)} \right] \mathbf{E} \left[ \sum_{\substack{x \neq y \\ |x|=|y|=1}} e^{-3V(x)-V(y)} \right]$$

$$\times e^{s_3^* \psi(2) + s_2^* \psi(3) + \psi(4)}.$$

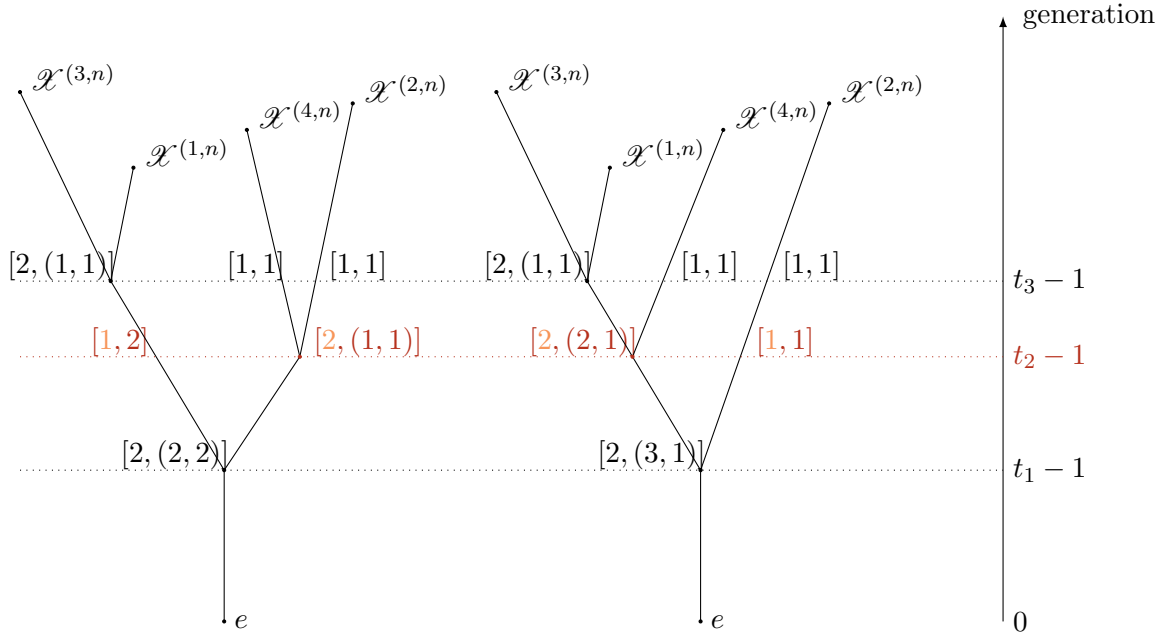


FIGURE II.3 – An example of a genealogical tree of the four vertices  $\mathcal{X}^{(1,n)}$ ,  $\mathcal{X}^{(2,n)}$ ,  $\mathcal{X}^{(3,n)}$ ,  $\mathcal{X}^{(4,n)}$  associated to  $\Pi$  (left) and associated to  $\Pi'$  (right).  $[1, 2]$  means that  $b_{1,\Pi}(\{1, 3\}) = 1$  and  $\beta_1^{2,\Pi} = 2$ ,  $[2, (1, 1)]$  means that  $b_{2,\Pi}(\{2, 4\}) = 2$  and  $\beta_2^{2,\Pi} = (1, 1)$ . In the same way,  $[2, (2, 1)]$  means that  $b_{1,\Pi'}(\{1, 3\}) = 2$  and  $\beta_1^{2,\Pi'} = (2, 1)$ ,  $[1, 1]$  means that  $b_{2,\Pi'}(\{2\}) = 1$  and  $\beta_2^{2,\Pi'} = 1$ .

The difference between these two examples is that in the second one, we ask  $(\mathcal{X}^{(4,n)})_{t_1-1}$  (the ancestor of  $\mathcal{X}^{(4,n)}$  of in generation  $t_1-1$ ) to belong to both genealogical line  $\llbracket (\mathcal{X}^{(1,n)})_{t_1-1}, \mathcal{X}^{(1,n)} \rrbracket$  and  $\llbracket (\mathcal{X}^{(3,n)})_{t_1-1}, \mathcal{X}^{(3,n)} \rrbracket$ . This constraint can be satisfied only if the vertex  $(\mathcal{X}^{(4,n)})_{t_1-1}$  is often visited by the random walk  $\mathbb{X}$ , inducing more dependence in the trajectories of  $\mathbb{X}$  thus giving the factor  $t_2^* \psi(3)$  instead of  $2t_2^* \psi(2) = t_2^* \psi(2) + t_2^* \psi(2)$ .

However, in the case of regular super-critical Galton-Watson trees, the events  $\cap_{i=0}^3 \{\pi_i\}$  and  $\cap_{i=0}^3 \{\tilde{\pi}_i\}$  have the same probability under the limit law of the genealogical tree. Indeed, one can notice (see Figure II.3) that for all  $i \in \{1, 2, 3\}$  and all  $j \in \{1, \dots, |\pi_i|\}$  ( $|\pi_i| = |\tilde{\pi}_i|$  by definition),  $b_{i,\Pi}(\mathbf{B}_j) = b_{i,\Pi'}(\mathbf{B}_{\mathfrak{p}(j)})$  for some permutation  $\mathfrak{p}$  on  $\llbracket 1, |\pi_i| \rrbracket$ , but this not the case when replacing  $b_{i,\Pi}(\mathbf{B}_\cdot)$  by  $\beta^{i,\Pi}$  and  $b_{i,\Pi'}(\mathbf{B}_\cdot)$  by  $\beta^{i,\Pi'}$ .

Since all coalescences of the genealogical lines of  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$  occur in the remote past with large probability, one could focus on this particular vertices of the tree  $\mathcal{R}_{T_n^{1/2}}$ . To do that, we pick a  $k$ -tuple  $\mathcal{Y}^{(n)} = (\mathcal{Y}^{(1,n)}, \dots, \mathcal{Y}^{(k,n)})$  uniformly in the set  $\mathcal{D}_n^{\times k} \cap \mathcal{C}_s^k$  for  $s \in \mathbb{N}^*$ . In other words, the law of  $\mathcal{Y}^{(n)}$  is given in (II.5) by replacing  $\Delta^k(\mathcal{D}_n)$  with  $\Delta^k(\mathcal{D}_n) \cap \mathcal{C}_s^k$ . We keep the same notations for  $\mathcal{Y}^{(n)}$  as for  $\mathcal{X}^{(n)}$ .

The last example gives the law of the coalescent times  $(\mathcal{J}^{k,n})_{1 \leq j \leq \mathcal{J}^{k,n}}$  of  $\mathcal{Y}^{(1,n)}, \dots, \mathcal{Y}^{(k,n)}$  :

**Theorem 1.5.** *Let  $k \geq 2$  and assume that  $\kappa > 2k$ . Let  $1 \leq \ell < k$ ,  $\mathfrak{s} \in \mathbb{N}^*$  be two integers, and  $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{N}^{\times \ell}$  such that  $s_1 < \dots < s_\ell \leq \mathfrak{s}$ . Assume that for all  $\mathbf{x} \in \Delta^k$ ,*

$$F_{\mathbf{s}}^\ell(\mathbf{x}) = \sum_{\Xi \text{ increasing}} f_{\mathbf{s}, \Xi}^\ell(\mathbf{x}),$$

where  $\Xi$  increasing means here that  $\Xi = (\Xi_i)_{0 \leq i \leq \ell}$  is an increasing collection of partitions of  $\{1, \dots, k\}$ . Under the assumptions 1, 2, 3 and 4,

1. in  $\mathbb{P}^*$ -probability

$$\frac{\mathcal{A}^k(\mathcal{D}_n, F_{\mathbf{s}}^\ell)}{\mathcal{A}^k(\mathcal{D}_n, \mathbb{1}_{\mathcal{C}_s^k})} \mathbb{1}_{\{D_n \geq k\}} \xrightarrow{n \rightarrow \infty} \frac{\mathcal{A}_\infty^k(F_{\mathbf{s}}^\ell)}{\mathcal{A}_\infty^k(\mathbb{1}_{\mathcal{C}_s^k})}. \quad (\text{II.18})$$

2. Moreover

$$\mathbb{P}^*(\mathcal{S}_1^{k,n} = s_1, \dots, \mathcal{S}_\ell^{k,n} = s_\ell, \mathcal{J}^{k,n} = \ell) \xrightarrow{n \rightarrow \infty} \mathbf{E}^* \left[ \frac{\mathcal{A}_\infty^k(F_{\mathbf{s}, \Xi}^\ell)}{\mathcal{A}_\infty^k(\mathbb{1}_{\mathcal{C}_s^k})} \right], \quad (\text{II.19})$$

where  $\mathcal{A}_\infty^k(F_{\mathbf{s}, \Xi}^\ell)$  is the limit in  $L^2(\mathbb{P}^*)$  of the martingale  $(\mathcal{A}_l^k(F_{\mathbf{s}, \Xi}^\ell))_l$  and satisfying

$$\sum_{\ell=1}^{k-1} \sum_{\substack{\mathbf{s}=(s_1, \dots, s_\ell) \\ s_1 < \dots < s_\ell \leq \mathfrak{s}}} \mathcal{A}_\infty^k(F_{\mathbf{s}}^\ell) = \mathcal{A}_\infty^k(\mathbb{1}_{\mathcal{C}_s^k}).$$

## 1.5 The small generations : a general result

In this section, we present results for the range  $\mathcal{A}^k(\mathcal{D}_n, f)$  with  $f$  satisfying a very natural heredity condition we will discuss later and including previous examples. First, recall that  $\mathcal{D}_n = \{x \in \mathcal{R}_{T_n^{1/2}}; \ell_n \leq |x| \leq \mathfrak{L}_n\}$  with  $(\mathfrak{L}_n)$  a sequence of positive integers such that  $\delta_0^{-1} \log n \leq \mathfrak{L}_n \leq n^{1/2}$  (see Lemma 3.4 for the definition of  $\delta_0$ ) and  $(\ell_n)$  is a sequence of positive integers such that  $\delta_0^{-1} \log n \leq \ell_n \leq \mathfrak{L}_n$ . Then recall the definition of  $\mathcal{A}^k(\mathcal{D}_n, f)$  defined in (II.4) : if  $D_n \geq k$

$$\mathcal{A}^k(\mathcal{D}_n, f) := \sum_{\mathbf{x} \in \Delta^k(\mathcal{D}_n)} f(\mathbf{x}),$$

with  $\Delta^k(\mathcal{D}_n) = \{\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \mathcal{D}_n^{\times k}; \forall i_1 \neq i_2, x^{(i_1)} \neq x^{(i_2)}\}$  and equal to 0 otherwise. Although we obtain quite general results, we however require the following assumption on  $f$  : recall that for all  $k \geq 2$ ,  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ ,  $\mathcal{C}_m^k = \{\mathbf{x} \in \Delta^k; \mathcal{S}^k(\mathbf{x}) \leq m\}$  where  $\mathcal{S}^k(\mathbf{x}) - 1$  denotes the last generation at which two or more vertices among  $x^{(1)}, \dots, x^{(k)}$  share a common ancestor (see (II.10)). Assume

**Assumption 5.** *there exists  $\mathfrak{g} \in \mathbb{N}^*$  such that for all integer  $p \geq \mathfrak{g}$  and all  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in$*

$\Delta^k$ , if  $\min_{1 \leq i \leq k} |x^{(i)}| \geq p$  and  $\mathbf{x} \in \mathcal{C}_p^k$  then

$$f((x^{(1)}, \dots, x^{(k)})) = f((x^{(1)})_p, \dots, (x^{(k)})_p), \quad (\text{II.20})$$

where we recall that  $(x^{(i)})_p$  is the ancestor of  $x^{(i)}$  in the generation  $p$ . In other words, we ask the constraint  $f$  to be hereditary from a given generation  $\mathbf{g}$ .

Introduce the local time  $\mathcal{L}^n := \sum_{j=1}^n \mathbf{1}_{\{X_j=e^*\}}$  of the parent  $e^*$  of the root  $e$  at time  $n$ . Recall that  $T^j$  is the  $j$ -th return time to  $e^*$ :  $T^0 = 0$  and for any  $j \geq 1$ ,  $T^j = \inf\{i > T^{j-1}; X_i = e^*\}$ . Let  $s \in \mathbb{N}^*$  and introduce  $\mathcal{D}_{n,T^s} := \{x \in \mathcal{B}_{T^s}; \ell_n \leq |x| \leq \mathfrak{L}_n\}$ . We denote by  $\mathfrak{E}^{k,s}$  the set defined by : for a given  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ ,  $\mathbf{x} \in \mathfrak{E}^{k,s}$  if and only if the vertices of  $x^{(1)}, \dots, x^{(k)}$  are visited during  $k$  distinct excursions before the instant  $T^s$  :

$$\mathfrak{E}^{k,s} := \bigcup_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \bigcap_{i=1}^k \{x = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k; \mathcal{L}_{x^{(i)}}^{T^{j_i}} - \mathcal{L}_{x^{(i)}}^{T^{j_i-1}} \geq 1\}, \quad (\text{II.21})$$

where we denote by  $\llbracket 1, s \rrbracket_k$  the set of  $k$ -tuples  $\mathbf{j}$  of  $\{1, \dots, s\}$  such that for all  $i_1 \neq i_2 \in \{1, \dots, s\}$ ,  $j_{i_1} \neq j_{i_2}$ . Our first proposition is a convergence of the range  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{E}^{k,s}})$  for any  $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$ ,  $\varepsilon_1 \in (0, 1)$ .

**Proposition 2.** *Let  $k \geq 2$  and assume  $\kappa > 2k$ . Under the assumptions 1, 2, 3 and 4, if  $f$  satisfies the hereditary assumption 5 then for all  $\varepsilon, \varepsilon_1 \in (0, 1)$ ,  $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$*

$$\mathbb{P}^* \left( \left| \frac{1}{(s\mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{E}^{k,s}}) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon \right) \xrightarrow{n \rightarrow \infty} 0,$$

where  $(S_i - S_{i-1})_{i \in \mathbb{N}^*}$  is a sequence of i.i.d real valued random variables such that  $S_0 = 0$  and

$$\mathbf{E}[\mathbf{h}(S_1)] = \mathbf{E} \left[ \sum_{|x|=1} \mathbf{h}(V(x)) e^{-V(x)} \right], \quad (\text{II.22})$$

and, in  $L^2(\mathbf{P}^*)$

$$\mathcal{A}_\infty^k(f) := \lim_{l \rightarrow \infty} \mathcal{A}_l^k(f), \quad (\text{II.23})$$

with  $\mathcal{A}_l^k(f, \beta) := \sum_{\mathbf{x} \in \Delta_l^k} f(\mathbf{x}) e^{-\langle \beta, V(\mathbf{x}) \rangle_k}$ ,  $\mathcal{A}_l(f) := \mathcal{A}_l(f, \mathbf{1})$  and  $\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^{\times k}$ .

In the next proposition, we claim  $k$ -tuples in  $\Delta^k \setminus \mathfrak{E}^{k,s}$  with  $n^{1/2}/\varepsilon_1$  and  $\varepsilon_1 \in (0, 1)$ , that is  $k$ -tuples of vertices such that at least two among them are visited during the same excursion above  $e^*$  and before  $T^s$ , have a minor contribution to the range  $\mathcal{A}^k(\mathcal{D}_n, \mathbf{1})$ .

**Proposition 3.** *Let  $\varepsilon \in (0, 1)$ ,  $k \geq 2$  and assume  $\kappa > 2k$ . Under the assumptions 1, 2, 3 and 4*

$$\mathbb{P}^* \left( \sup_{s \leq n^{1/2}/\varepsilon_1} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbf{1}_{\Delta^k \setminus \mathfrak{E}^{k,s}}) > \varepsilon (n^{1/2} \mathbf{L}_n)^k \right) \xrightarrow{n \rightarrow \infty} 0 \quad (\text{II.24})$$

We are now ready to state our main result :

**Theorem 1.6.** *Let  $k \geq 2$  and assume  $\kappa > 2k$ . Under the assumptions 1, 2, 3 and 4, if  $f$  satisfies the hereditary assumption 5 then, in  $\mathbb{P}^*$ -probability*

$$\frac{\mathcal{A}^k(\mathcal{D}_n, f)}{(n^{1/2} \mathbf{L}_n)^k} \xrightarrow[n \rightarrow \infty]{} (c_\infty)^k \mathcal{A}_\infty^k(f), \quad (\text{II.25})$$

and if  $g \neq 0$  also satisfies assumption 5 then in  $\mathbb{P}^*$ -probability

$$\frac{\mathcal{A}^k(\mathcal{D}_n, f)}{\mathcal{A}^k(\mathcal{D}_n, g)} \mathbb{1}_{\{D_n \geq k\}} \xrightarrow[n \rightarrow \infty]{} \frac{\mathcal{A}_\infty^k(f)}{\mathcal{A}_\infty^k(g)}, \quad (\text{II.26})$$

where  $\mathbf{L}_n = \mathfrak{L}_n - \ell_n + 1$ ,  $c_\infty := \mathbf{E}[(\sum_{j \geq 0} e^{-S_j})^{-1}]$  and  $\mathcal{A}_\infty^k(f)$  is defined in (II.23). Note that a constraint satisfying assumption 5 doesn't have any influence on the normalization of the range. Moreover,  $\mathcal{A}^k(\mathcal{D}_n, f)$  behaves like  $(\mathbf{L}_n \max_{x \in \mathcal{R}_{T_n^{1/2}}} |x|)^k$  and the limiting value  $\mathcal{A}_\infty(f)$  contains all the information about the interactions between the vertices of the tree.

Since  $\psi(2) < 0$ ,  $c_\infty$  and  $c_0$  are well defined in  $(0, \infty)$ . Indeed, the sequence  $((\sum_{j=0}^l e^{-S_j})^{-1})_{l \in \mathbb{N}}$  is bounded and non-increasing and by Jensen inequality,  $1 \geq \mathbf{E}[(\sum_{j=0}^l e^{-S_j})^{-1}] \geq \mathbf{E}[(\sum_{j=0}^l e^{-S_j})]^{-1} = (\sum_{j=0}^l e^{j\psi(2)})^{-1} \geq 1 - e^{\psi(2)} > 0$ .

We end this subsection by stating an extension of Theorem 1.6 to the range  $\mathcal{R}_n$ . Before that, introduce  $\tilde{\mathcal{D}}_n := \{x \in \mathcal{R}_n; \ell_n \leq |x| \leq \mathfrak{L}_n\}$  with cardinal  $\tilde{D}_n$ .

**Theorem 1.7.** *Let  $k \geq 2$ . There exists a non-increasing sequence of positive integers  $(q_j)_j$ , satisfying  $q_j \in (0, 1/2)$  and  $q_j \rightarrow 0$  when  $j \rightarrow \infty$  such that if  $\kappa > 2\xi k$  for some integer  $\xi \geq 2$  and  $\mathfrak{L}_n = o(n^{1/2 - q_\xi})$ , then, in law, under  $\mathbb{P}^*$*

$$\frac{\mathcal{A}^k(\tilde{\mathcal{D}}_n, f)}{(n^{1/2} \mathbf{L}_n)^k} \xrightarrow[n \rightarrow \infty]{} \frac{\mathcal{A}_\infty^k(f)}{(W_\infty)^k} (c_\infty c_0^{1/2} |\mathcal{N}|)^k, \quad (\text{II.27})$$

and if  $g \neq 0$  also satisfies assumption 5 then in  $\mathbb{P}^*$ -probability

$$\frac{\mathcal{A}^k(\tilde{\mathcal{D}}_n, f)}{\mathcal{A}^k(\tilde{\mathcal{D}}_n, g)} \mathbb{1}_{\{D_n \geq k\}} \xrightarrow[n \rightarrow \infty]{} \frac{\mathcal{A}_\infty^k(f)}{\mathcal{A}_\infty^k(g)}, \quad (\text{II.28})$$

where  $c_0 := \mathbf{E}[\sum_{x \neq y; |x|=|y|=1} e^{-V(x)-V(y)}] / (1 - e^{\psi(2)})$  and  $\mathcal{N}$  is a standard Gaussian random variable.

In particular, all the previous results on  $\mathcal{D}_n$  hold for  $\tilde{\mathcal{D}}_n$  with  $\mathfrak{L}_n = o(n^{1/2 - q_\xi})$ .

## 1.6 Further discussion : the critical generations

The critical generations, that is to say of order  $n^{1/2}$ , correspond to the typical generations but also to the largest reached by the diffusive random walk  $\mathbb{X}$  up to the time  $n$ . E. Aïdékon and L.

de Raphélis [AdR17] showed that  $n^{1/2}$  is also the right normalisation for the tree  $\mathcal{R}_n$  : in law, under  $\mathbb{P}^*$

$$\frac{c_0^{1/2}}{n^{1/2}} \mathcal{R}_n \xrightarrow[n \rightarrow \infty]{} \mathcal{T}_{|B|},$$

where for any  $c > 0$ ,  $c\mathcal{R}_T$  is tree  $\mathcal{R}_T$  with edge lengths equal to  $c$  and  $\mathcal{T}_{|B|}$  is the real tree coded by the standard reflected Brownian motion  $|B| = (|B_t|)_{t \in [0,1]}$  on  $[0, 1]$  (see [Gal06]).  $\mathcal{T}_{|B|}$  is what we can call a Brownian forest thus suggesting that two vertices  $\mathcal{X}^{(1,n)}$  and  $\mathcal{X}^{(2,n)}$  chosen uniformly in the range  $\mathcal{R}_n$  at a generation of order  $n^{1/2}$  can share a common ancestor in both remote past and recent past. That is actually what is happening when considering two vertices  $\tilde{\mathcal{X}}^{(1,n)}$  and  $\tilde{\mathcal{X}}^{(2,n)}$  picked uniformly at generation  $n^{1/2}$  in the tree  $\mathcal{R}_{t_n}$ , where  $t_n$  is the  $n^{1/2}$ -th return time of  $\mathbb{X}$  to  $e^*$  (which is quite similar to  $\mathcal{R}_n$ ) : let  $\tilde{\mathcal{M}}_n$  be the most recent common ancestor of  $\tilde{\mathcal{X}}^{(1,n)}$  and  $\tilde{\mathcal{X}}^{(2,n)}$ . First observe that

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}^*(\tilde{\mathcal{M}}_n < 1/\varepsilon) > 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^*(\varepsilon n^{1/2} \leq \tilde{\mathcal{M}}_n < n^{1/2}) > 0. \quad (\text{II.29})$$

Moreover, coalescence can't occur anywhere else :

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}^* \left[ \frac{1}{n} \sum_{\substack{x \neq y \\ |x|=|y|=n^{1/2}}} \mathbb{1}_{\{x,y \in \mathcal{R}_{t_n}, 1/\varepsilon \leq |x \wedge y| < \varepsilon n^{1/2}\}} \right] = 0.$$

Although  $\mathbb{T}$  is a super-critical Galton-Watson tree, the genealogy of  $\mathcal{R}_{t_n}$  (or  $\mathcal{R}_n$ ) is a mix of the super-critical case and the critical case for a regular Galton-Watson trees (see subsection 1.2).

The fact is using standard techniques for randomly biased random walks and branching random walks, we are able to deal with the quenched mean of  $(D_{t_n})^{p_1}$  for  $p_1 \leq \lfloor \kappa \rfloor$  and  $(\mathcal{A}^2(\mathcal{D}_{t_n}, f))^{p_2}$  with  $p_2 \leq \lfloor \kappa/2 \rfloor$  but not with the actual random variables.

The computation for any  $m > 0$  and any  $0 < a < b < 1$  of  $\mathbb{P}^*(\tilde{\mathcal{M}}_n < m)$  and  $\mathbb{P}^*(an^{1/2} \leq \tilde{\mathcal{M}}_n < bn^{1/2})$  is part of an ongoing work with P. Androletti and L. de Raphélis.

The present paper aims in some way to describe the interaction between the vertices of the tree  $\mathcal{R}_{Tn^{1/2}}$  in the set of generations «squashed» when rescaling the tree by  $n^{1/2}$ .

**Remark 5.** *The curiosity here is the fact that critical generations and small generations equally contributed to the range. Indeed, whether  $\mathfrak{L}_n$  is negligible with respect to  $n^{1/2}$  (with  $\mathfrak{L}_n \geq \delta_0^{-1} \log n$ ) or not,  $\sum_{|u|=\mathfrak{L}_n} \mathbb{1}_{\{u \in \mathcal{R}_{t_n}\}}$  is of order  $n^{1/2}$ . This fact makes a deep difference with the slow regime in which only the critical generations (that is typical generations, of order  $(\log n)^2$ ) contribute significantly to the range (see [AC18], Theorem 1.2 and Proposition 1.4).*

## 2 Proofs of the theorems

In this section, we prove theorems presented as examples and end it with the proof of Theorem 1.6.

### 2.1 Proofs of Theorems 1.2 to 1.5

In this subsection, we give a proof of each example stated above except for the Theorem 1.1 which is the simplest application of Theorem 1.6, taking  $f = 1$ . For each example, the procedure is as follows : we first prove the function  $f$  we consider satisfies the hereditary assumption 5 and we then give useful precisions on  $\mathcal{A}_\infty^k(f)$  for the description of the genealogy of the vertices  $\mathcal{X}^{(1,n)}, \dots, \mathcal{X}^{(k,n)}$ .

*Proof of Theorem 1.2.* Recall that for  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$  and  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in (\mathbb{N}^*)^{\times k}$

$$f_{\boldsymbol{\lambda}}(x^{(1)}, \dots, x^{(k)}) := \prod_{i=2}^k \mathbf{1}_{\{|x^{(i-1)} \wedge x^{(i)}| < \lambda_i\}}.$$

Let us prove that the hereditary assumption 5 is satisfied by  $f_{\boldsymbol{\lambda}}$ . Recall that for  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ ,  $\mathcal{S}^k(\mathbf{x}) - 1$  denotes the last generation at which two or more vertices among  $x^{(1)}, \dots, x^{(k)}$  share a common ancestor. If  $p \geq \max_{2 \leq i \leq k} \lambda_i$  and  $\mathbf{x} \in \Delta^k$  such that  $p \leq \min_{1 \leq i \leq k} |x^{(i)}|$ , then  $\mathcal{S}^k(\mathbf{x}) \leq p$  implies that for any  $\mathbf{z} \in \llbracket (x^{(1)})_p, x^{(1)} \rrbracket \times \dots \times \llbracket (x^{(k)})_p, x^{(k)} \rrbracket$ , we have  $\mathcal{S}^k(\mathbf{z}) = \mathcal{S}^k(\mathbf{x})$  since, by definition of  $f_{\boldsymbol{\lambda}}$ , the highest obliged coalescence between at least two vertices among  $(x^{(1)}, \dots, x^{(k)})$  must append at generation  $\max_{2 \leq i \leq k} \lambda_i$ . Thus,  $\mathcal{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq m$ . Moreover, by definition,  $\mathcal{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq \max_{2 \leq i \leq k} \lambda_i$  implies  $\mathcal{S}^k(\mathbf{x}) \leq \max_{2 \leq i \leq k} \lambda_i$ . Consequently, assumption 5 holds for  $\mathfrak{g} = \max_{2 \leq i \leq k} \lambda_i$ . We conclude using Theorem 1.6.  $\square$

We now prove Theorem 1.3 :

*Proof of Theorem 1.3.* Recall that for  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ ,  $\mathcal{S}^k(\mathbf{x}) - 1$  denotes the last generation at which two or more vertices among  $x^{(1)}, \dots, x^{(k)}$  share a common ancestor and for  $m \in \mathbb{N}^*$ , recall that

$$f_m(\mathbf{x}) = \mathbf{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq m\}}.$$

First, note that the hereditary assumption 5 is satisfied by  $f_m$ . Indeed, if  $p \geq m$  and  $\mathbf{x} \in \Delta^k$  such that  $p \leq \min_{1 \leq i \leq k} |x^{(i)}|$ , then  $\mathcal{S}^k(\mathbf{x}) \leq p$  implies that for any  $\mathbf{z} \in \llbracket (x^{(1)})_p, x^{(1)} \rrbracket \times \dots \times \llbracket (x^{(k)})_p, x^{(k)} \rrbracket$ , we have  $\mathcal{S}^k(\mathbf{z}) = \mathcal{S}^k(\mathbf{x})$ . Thus,  $\mathcal{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq m$ . Moreover, by definition,  $\mathcal{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq m$  implies  $\mathcal{S}^k(\mathbf{x}) \leq m$ . Consequently, assumption 5 holds for  $\mathfrak{g} = m$ .

We then deduce the converge of the trace in (II.15) by using Theorem 1.6.

We now move to the limit law of  $(\mathcal{S}^k(\mathcal{X}^{(n)}))$  in (II.12). Note, by definition, that

$$\mathbb{P}^*(\mathcal{S}^k(\mathcal{X}^{(n)}) \leq m) = \frac{1}{\mathbb{P}^*(D_n \geq k)} \mathbb{E}^* \left[ \frac{\mathcal{A}^k(\mathcal{D}_n, f_m)}{\mathcal{A}^k(\mathcal{D}_n, 1)} \mathbf{1}_{\{D_n \geq k\}} \right],$$

so  $\mathbb{P}^*(\mathcal{S}^k(\mathcal{X}^n) \leq m)$  goes to  $\mathbf{E}^*[\mathcal{A}_\infty^k(f_m)/(W_\infty)^k]$  when  $n$  goes to  $\infty$  thanks to Theorem 1.6 with  $f = f_m$  and  $g = 1$  together with the fact that  $\lim_{n \rightarrow \infty} \mathbb{P}^*(D_n \geq k) = 1$ . It is left to show that  $\lim_{m \rightarrow \infty} \mathcal{A}_\infty^k(f_m) = (W_\infty)^k$ . For that, we use Lemma 3.9 with  $f = 1$  and  $\mathbf{p} = (l, \dots, l) \in (\mathbb{N}^*)^{\times k}$

$$\sup_{l > m} \mathbf{E}^* [|\mathcal{A}_l^k(f_m) - \mathcal{A}_l^k(1)|^2] \xrightarrow{m \rightarrow \infty} 0.$$

Moreover,  $\lim_{l \rightarrow \infty} \mathcal{A}_l^k(1) = (W_\infty)^k$  and  $\lim_{l \rightarrow \infty} \mathcal{A}_l^k(f_m) = \mathcal{A}_\infty^k(f_m)$  so  $(\mathcal{A}_\infty^k(f_m))_m$  converges to  $(W_\infty)^k$  in  $L^2(\mathbf{P}^*)$ , which allows to end the proof.  $\square$

We now turn to the proof of Theorem 1.4.

*Proof of Theorem 1.4.* Recall that for any  $1 \leq d < q \in \mathbb{N}^*$ , for an increasing collection  $\Xi = (\Xi_i)_{0 \leq i \leq d}$  of partitions of  $\{1, \dots, q\}$ , for all  $\mathbf{x} = (x^{(1)}, \dots, x^{(q)}) \in \Delta^q$  and all  $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{N}^{\times d}$  such that  $t_1 < t_2 < \dots < t_d$ ,

$$f_{\mathbf{t}, \Xi}^d(\mathbf{x}) = \prod_{i=1}^d \mathbb{1}_{\Gamma_{\mathbf{t}, \Xi}^i}(\mathbf{x}),$$

where  $\Gamma_{\mathbf{t}, \Xi}^i = \Upsilon_{t_i-1, \Xi_{i-1}} \cap \Upsilon_{t_i, \Xi_i}$  and for any  $r \in \{1, \dots, d\}$  and any  $m \in \mathbb{N}^*$ ,  $\mathbf{x}$  belongs to  $\Upsilon_{m, \Xi_r}$  if and only if

$$\forall \mathbf{B} \in \Xi_r, \forall i_1, i_2 \in \mathbf{B} : (x^{(i_1)})_m = (x^{(i_2)})_m,$$

and for  $r \neq 0$

$$\forall \mathbf{B} \neq \tilde{\mathbf{B}} \in \Xi_r, \forall i_1 \in \mathbf{B}, i_2 \in \tilde{\mathbf{B}} : (x^{(i_1)})_m \neq (x^{(i_2)})_m,$$

where we recall that  $(x^{(i)})_m$  denotes the ancestor of  $x^{(i)}$  in generation  $m$  if exists,  $(x^{(i)})_m = e^{(i)}$  otherwise. Recall that  $\mathcal{C}_g^k = \{\mathbf{y} \in \Delta^q; \mathcal{S}^q(\mathbf{y}) \leq \mathbf{g}\}$  where  $\mathcal{S}^q(\mathbf{y}) - 1$  is the last generation at which two or more vertices among  $y^{(1)}, \dots, y^{(q)}$  share a common ancestor. Let  $p \geq t_d$  such that  $\min_{1 \leq i \leq q} x^{(i)} \geq p$  and  $\mathbf{x} \in \mathcal{C}_p^k$ . If  $\mathbf{x} \in \cap_{j=1}^d \Gamma_{\mathbf{t}, \Xi}^j$ , then  $(z^{(i)})_t = (x^{(i)})_t$  for all  $\mathbf{z} \in [(x^{(1)})_p, x^{(1)}] \times \dots \times [(x^{(q)})_p, x^{(q)}]$ ,  $1 \leq i \leq q$  and  $t \in \{0, \dots, p\}$  thus giving  $((x^{(1)})_p, \dots, (x^{(q)})_p) \in \cap_{j=1}^d \Gamma_{\mathbf{t}, \Xi}^j$ . Moreover, by definition,  $((x^{(1)})_p, \dots, (x^{(q)})_p) \in \cap_{j=1}^d \Gamma_{\mathbf{t}, \Xi}^j$  implies  $\mathbf{x} \in \cap_{j=1}^d \Gamma_{\mathbf{t}, \Xi}^j$ , then  $(z^{(i)})_t = (x^{(i)})_t$ . Consequently,  $f_{\mathbf{t}, \Xi}^d$  satisfies assumption 5 with  $\mathbf{g} = t_d$  and this prove that the convergence in (II.15) holds.

We move to the limit law of  $(\pi^{k,n})$  in (II.16). Recall the definition of  $\mathcal{S}_i^{k,n}$  in (II.11). First, note that

$$\mathbb{P}^*(\pi_{m_0}^{k,n} = \boldsymbol{\pi}_0, \dots, \pi_{m_\ell}^{k,n} = \boldsymbol{\pi}_\ell) = \mathbb{P}^*\left(\bigcap_{i=1}^{\ell} \{\pi_{m_{i-1}}^{k,n} = \boldsymbol{\pi}_{i-1}, \pi_{m_i}^{k,n} = \boldsymbol{\pi}_i, m_{i-1} < \mathcal{S}_i^{k,n} \leq m_i\}\right).$$

Indeed, for all  $1 \leq i \leq \ell$ ,  $|\boldsymbol{\pi}_{i-1}| < |\boldsymbol{\pi}_i|$  so the interval  $(m_{i-1}, m_i]$  necessarily contains at least one coalescent time. But since  $\boldsymbol{\pi}_0 = \{\{1, \dots, k\}\}$  and  $\boldsymbol{\pi}_\ell = \{\{1\}, \dots, \{k\}\}$ ,  $\cup_{i=1}^{\ell} (m_{i-1}, m_i]$  can't



contain more than  $\ell$  coalescent times so  $\mathcal{S}_i^{k,n}$  is the only one belonging to  $(m_{i-1}, m_i]$ . We now write

$$\begin{aligned} \mathbb{P}^* \left( \bigcap_{i=1}^{\ell} \{ \pi_{m_{i-1}}^{k,n} = \boldsymbol{\pi}_{i-1}, m_{i-1} < \mathcal{S}_i^{k,n} \leq m_i \} \right) \\ &= \sum_{s_1=m_0+1}^{m_1} \cdots \sum_{s_{\ell}=m_{\ell-1}+1}^{m_{\ell}} \mathbb{P}^* \left( \bigcap_{i=1}^{\ell} \{ \pi_{m_{i-1}}^{k,n} = \boldsymbol{\pi}_{i-1}, \pi_{m_i}^{k,n} = \boldsymbol{\pi}_i, \mathcal{S}_i^{k,n} = s_i \} \right) \\ &= \sum_{s_1=m_0+1}^{m_1} \cdots \sum_{s_{\ell}=m_{\ell-1}+1}^{m_{\ell}} \mathbb{P}^* \left( \bigcap_{i=1}^{\ell} \{ \pi_{s_{i-1}}^{k,n} = \boldsymbol{\pi}_{i-1}, \pi_{s_i}^{k,n} = \boldsymbol{\pi}_i \} \right), \end{aligned}$$

Moreover,  $\pi_{s_{i-1}}^{k,n} = \boldsymbol{\pi}_{i-1}, \pi_{s_i}^{k,n} = \boldsymbol{\pi}_i$  means nothing but  $\mathcal{X}^{(n)} \in \Gamma_{\mathbf{s}, \Pi}^i$  and it follows that

$$\begin{aligned} \mathbb{P}^* \left( \bigcap_{i=1}^{\ell} \{ \pi_{m_{i-1}}^{k,n} = \boldsymbol{\pi}_{i-1}, \pi_{m_i}^{k,n} = \boldsymbol{\pi}_i, \mathcal{S}_i^{k,n} = s_i \} \right) &= \mathbb{E}^* [f_{\mathbf{s}, \Pi}^{\ell}(\mathcal{X}^{(n)})] \\ &= \mathbb{E}^* \left[ \frac{\mathcal{A}^k(\mathcal{D}_n, f_{\mathbf{s}, \Pi}^{\ell})}{\mathcal{A}^k(\mathcal{D}_n, \mathbf{1})} \mathbb{1}_{\{D_n \geq k\}} \right], \end{aligned}$$

where we have used the definition of  $\mathcal{X}^{(n)}$  (see (II.6)) in the last equation. Since  $f_{\mathbf{s}, \Pi}^{\ell}$  satisfies the hereditary assumption 5, we finally get (II.15) from (II.25) with  $f = f_{\mathbf{s}, \Pi}^{\ell}$  and by (II.26) with  $g = 1$

$$\lim_{n \rightarrow \infty} \mathbb{P}^* (\pi_{m_0}^{k,n} = \boldsymbol{\pi}_0, \dots, \pi_{m_{\ell}}^{k,n} = \boldsymbol{\pi}_{\ell}) = \sum_{s_1=m_0+1}^{m_1} \cdots \sum_{s_{\ell}=m_{\ell-1}+1}^{m_{\ell}} \mathbf{E}^* \left[ \frac{\mathcal{A}_{\infty}^k(f_{\mathbf{s}, \Pi}^{\ell})}{(W_{\infty})^k} \right].$$

We now compute the conditional expectation of  $\mathcal{A}_{\infty}^k(f_{\mathbf{s}, \Pi}^{\ell})$  conditionally given the sigma-algebra  $\mathcal{F}_{s_p-1} = \sigma(\mathbb{T}; (V(x); |x| < s_p))$ . Start with  $p = \ell$ . Let  $s_i \in \{m_{i-1}+1, \dots, m_i\}$  for all  $i \in \{1, \dots, \ell\}$ . Using the definition of  $\mathcal{A}_{\infty}^k(f_{\mathbf{s}, \Pi}^{\ell})$  and the fact that  $\mathbf{x} \in \Delta_{\ell}^k \cap \Gamma_{\mathbf{s}, \Pi}^{\ell}$  for  $l > s_{\ell}$  implies  $\mathcal{S}^k(\mathbf{x}) \leq s_{\ell}$ , we obtain, on the set of non-extinction

$$\mathbf{E}^* [\mathcal{A}_{\infty}^k(f_{\mathbf{s}, \Pi}^{\ell}) | \mathcal{F}_{s_{\ell}}] = \lim_{l \rightarrow \infty} \mathbf{E}^* \left[ \sum_{\mathbf{x} \in \Delta_{\ell}^k} f_{\mathbf{s}, \Pi}^{\ell}(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} | \mathcal{F}_{s_{\ell}} \right] = \sum_{\mathbf{x} \in \Delta_{s_{\ell}}^k} f_{\mathbf{s}, \Pi}^{\ell}(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k},$$

since  $s_{\ell} - 1$  corresponds to the last generation at which two or more vertices among  $x^{(1)}, \dots, x^{(\ell)}$  share a common ancestor and we recall that  $\langle \mathbf{1}, V(\mathbf{x}) \rangle_k = \sum_{i=1}^k V(x^{(i)})$ . In particular, these vertices don't share any common ancestor in generation  $s_{\ell}$  and last inequality comes from independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$  together with the fact that  $\psi(1) = 0$ . Before going any further, let us define a transformation of the collection of partitions  $\Pi$ . We build from  $\Pi$ , which is a collection of partitions of the set  $\{1, \dots, k\}$ , a new collection  $\Pi^{\ell-1} = (\tilde{\boldsymbol{\pi}}_i)_{0 \leq i \leq \ell-1}$  of partitions of the set  $\{1, \dots, |\boldsymbol{\pi}_{\ell-1}|\}$  as follows :

- $\tilde{\boldsymbol{\pi}}_{\ell-1} = \{\{1\}, \dots, \{|\boldsymbol{\pi}_{\ell-1}|\}\}$ ;

- for any  $1 \leq i \leq \ell - 2$  and any  $1 \leq j \leq |\pi_i|$ , the  $j$ -th block  $\mathbf{B}_j^i$  of the partition  $\pi_i$  is the union of  $\mathfrak{b}_{\ell-1}(\mathbf{B}_j^i) \geq 1$  block(s) of the partition  $\pi_{\ell-1}$ . We then denote by  $\tilde{\mathbf{B}}_j^i$  the subset of  $\{1, \dots, |\pi_{\ell-1}|\}$  composed of all indices of these  $\mathfrak{b}_{\ell-1}(\mathbf{B}_j^i)$  block(s) and let  $\tilde{\pi}_i = \{\tilde{\mathbf{B}}_1^i, \dots, \tilde{\mathbf{B}}_{|\pi_i|}^i\}$ . By definition,  $\tilde{\pi}_0$  remains a one-block partition :  $\tilde{\pi}_0 = \{\{1, \dots, |\pi_{\ell-1}|\}\}$ .

Note that for any  $0 \leq i \leq \ell - 1$ ,  $|\tilde{\pi}_i| = |\pi_i|$  and for any  $0 \leq i \leq \ell - 2$ ,  $1 \leq j \leq |\pi_i|$ ,  $b_i(\mathbf{B}_j) = \tilde{b}_i(\tilde{\mathbf{B}}_j)$ , where  $\tilde{\mathbf{B}}_j \in \tilde{\pi}_i$  is the union of  $\tilde{b}_i(\tilde{\mathbf{B}}_j) \geq 1$  block(s) of  $\tilde{\pi}_{i+1}$ .

**Example 6.** If  $\Pi$  is defined by  $\pi_4 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ ,  $\pi_3 = \{\{1, 3\}, \{2\}, \{4\}, \{5\}\}$ ,  $\pi_2 = \{\{1, 3\}, \{2, 5\}, \{4\}\}$ ,  $\pi_1 = \{\{1, 3, 4\}, \{2, 5\}\}$  and  $\pi_0 = \{\{1, 2, 3, 4, 5\}\}$  then we have :  
 $\tilde{\pi}_3 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ ,  $\tilde{\pi}_2 = \{\{1\}, \{2, 4\}, \{3\}\}$ ,  $\tilde{\pi}_1 = \{\{1, 3\}, \{2, 4\}\}$ , and  $\tilde{\pi}_0 = \{\{1, 2, 3, 4\}\}$ .

If  $\Pi^\ell := \Pi$ , then for any  $i \in \{0, \dots, \ell - 1\}$ , let  $\Pi^i$  be the collection of partitions of  $\{1, \dots, |\pi_i|\}$  resulting from the previous procedure applied to  $\Pi^{i+1}$ . Note that  $\Pi^i$  is an increasing collection of partitions of  $\{1, \dots, |\pi_i|\}$ . This construction is a way of preserving the genealogical information through the generations.

Let  $\mathbf{s}^{\ell-1} = (s_1, \dots, s_{\ell-1})$  and recall the definitions regarding partitions in (II.13). One can now notice that, since that the number of vertices of the  $k$ -tuple  $\mathbf{x} \in \Delta_{s_\ell}^k$  sharing the same parent  $u^{(j)}$  is  $b_{\ell-1}(\mathbf{B}_j)$  (where we recall that  $b_{\ell-1}(\mathbf{B}_j)$  stands for  $b_{\ell-1}(\mathbf{B}_j^{\ell-1})$ ), we have

$$\sum_{\mathbf{x} \in \Delta_{s_\ell}^k} f_{\mathbf{s}, \Pi}^\ell(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} = \sum_{\mathbf{u} \in \Delta_{s_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{u}) \prod_{j=1}^{|\pi_{\ell-1}|} \sum_{\mathbf{x}^{(j)} \in \Delta_{s_\ell}^{b_{\ell-1}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-1}(\mathbf{B}_j)} \mathbb{1}_{\{(x^{(j,i)})^* = u^{(j)}\}} \times e^{-V(x^{(j,i)})},$$

where  $\mathbf{x}^{(j)} = (x^{(j,1)}, \dots, x^{(j, b_{\ell-1}(\mathbf{B}_j))})$  and  $(x^{(j,i)})^*$  is the parent of  $x^{(j,i)}$ . Moreover, by definition,  $b_{\ell-1}(\mathbf{B}_j) = |\mathbf{B}_j^{\ell-1}|$  (it comes from the fact that  $\pi_\ell = \{\{1\}, \dots, \{k\}\}$ ) so

$$\prod_{j=1}^{|\pi_{\ell-1}|} \sum_{\mathbf{x}^{(j)} \in \Delta_{s_\ell}^{b_{\ell-1}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-1}(\mathbf{B}_j)} \mathbb{1}_{\{(x^{(j,i)})^* = u^{(j)}\}} e^{-V(x^{(j,i)})} = e^{-\langle \beta^{\ell-1}, V(\mathbf{u}) \rangle_{|\pi_{\ell-1}|}} \prod_{j=1}^{|\pi_{\ell-1}|} \sum_{\mathbf{x}^{(j)} \in \Delta_{s_\ell}^{b_{\ell-1}(\mathbf{B}_j)}} \times \prod_{i=1}^{b_{\ell-1}(\mathbf{B}_j)} \mathbb{1}_{\{(x^{(j,i)})^* = u^{(j)}\}} e^{-V_{u^{(j)}}(x^{(j,i)})},$$

where  $\beta^{\ell-1} = (|\mathbf{B}_1^{\ell-1}|, \dots, |\mathbf{B}_{|\pi_{\ell-1}|}^{\ell-1}|)$  and  $V_{u^{(j)}}(x^{(j,i)}) = V(x^{(j,i)}) - V(u^{(j)})$ . By independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ , since  $\psi(1) = 0$

$$\mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_{s_\ell}^k} f_{\mathbf{s}, \Pi}^\ell(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} \middle| \mathcal{F}_{s_{\ell-1}} \right] = \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \prod_{j=1}^{|\pi_{\ell-1}|} c_{b_{\ell-1}(\mathbf{B}_j)}(\mathbf{1}) \prod_{\substack{\mathfrak{B} \in \pi_\ell \\ |\mathfrak{B}| \geq 2}} e^{\psi(|\mathfrak{B}|)}$$

$$= \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}) \prod_{j=1}^{|\pi_{\ell-1}|} c_{b_{\ell-1}(\mathbf{B}_j)}(\boldsymbol{\beta}_j^{\ell-1}),$$

where  $\boldsymbol{\beta}_j^{\ell-1} := (\beta_{j,1}^{\ell-1}, \dots, \beta_{j,b_{\ell-1}(\mathbf{B}_j)}^{\ell-1}) = (1, \dots, 1)$ , see (II.13). We also recall that  $\mathcal{A}_l^m(g, \boldsymbol{\beta}) = \sum_{\mathbf{x} \in \Delta_l^m} g(\mathbf{x}) e^{-\langle \boldsymbol{\beta}, V(\mathbf{x}) \rangle_m}$  and see assumption 3 for the definition of  $c_l(\boldsymbol{\beta})$ . Now recall that  $\Pi^{\ell-2}$  is the collection of partitions of  $\{1, \dots, |\pi_{\ell-2}|\}$  obtain from  $\Pi^{\ell-1}$  with the same procedure as above (see Example 6). Let  $\mathbf{s}^{\ell-2} = (s_1, \dots, s_{\ell-2})$ . Again, exactly  $b_{\ell-2}(\mathbf{B}_j)$  vertices in generation  $s_{\ell-1}$  are sharing the same parent  $z^{(j)}$  so

$$\begin{aligned} \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}) &= \sum_{\mathbf{z} \in \Delta_{s_{\ell-1}-1}^{|\pi_{\ell-2}|}} f_{\mathbf{s}^{\ell-2}, \Pi^{\ell-2}}^{\ell-2}(\mathbf{z}) \prod_{j=1}^{|\pi_{\ell-2}|} \sum_{\mathbf{u}^{(j)} \in \Delta_{s_{\ell-1}}^{b_{\ell-2}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} \\ &\times e^{-\beta_{j,i}^{\ell-2} V(u^{(j,i)})} \sum_{\mathbf{x}^{(j)} \in \Delta_{s_{\ell-1}}^{b_{\ell-2}(\mathbf{B}_j)}} \mathbb{1}_{\{x^{(j,i)} \geq u^{(j,i)}\}} e^{-\beta_{j,i}^{\ell-2} V_{u^{(j,i)}}(x^{(j,i)})}, \end{aligned}$$

where  $\mathbf{u}^{(j)} = (u^{(j,1)}, \dots, u^{(j,b_{\ell-2}(\mathbf{B}_j))})$ ,  $\mathbf{x}^{(j)} = (x^{(j,1)}, \dots, x^{(j,b_{\ell-2}(\mathbf{B}_j))})$  and  $V_{u^{(j,i)}}(x^{(j,i)})$  is the increment  $V(x^{(j,i)}) - V(u^{(j,i)})$ . Then, by independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ ,

$$\begin{aligned} \mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}) \middle| \mathcal{F}_{s_{\ell-1}} \right] &= \sum_{\mathbf{z} \in \Delta_{s_{\ell-1}-1}^{|\pi_{\ell-2}|}} f_{\mathbf{s}^{\ell-2}, \Pi^{\ell-2}}^{\ell-2}(\mathbf{z}) \prod_{j=1}^{|\pi_{\ell-2}|} \sum_{\mathbf{u}^{(j)} \in \Delta_{s_{\ell-1}}^{b_{\ell-2}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \\ &\times \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} e^{-\beta_{j,i}^{\ell-2} V(u^{(j,i)})} e^{s_{\ell}^* \psi(\beta_{j,i}^{\ell-2})}, \end{aligned}$$

with  $s_{\ell}^* = s_{\ell} - s_{\ell-1} - 1$ . Moreover, since  $\sum_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \beta_{j,i}^{\ell-2} = |\mathbf{B}_j^{\ell-2}|$  (see (II.13)), we have

$$\begin{aligned} &\prod_{j=1}^{|\pi_{\ell-2}|} \sum_{\mathbf{u}^{(j)} \in \Delta_{s_{\ell-1}}^{b_{\ell-2}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} e^{-\beta_{j,i}^{\ell-2} V(u^{(j,i)})} \\ &= e^{-\langle \boldsymbol{\beta}^{\ell-2}, V(\mathbf{z}) \rangle_{|\pi_{\ell-2}|}} \prod_{j=1}^{|\pi_{\ell-2}|} \sum_{\mathbf{u}^{(j)} \in \Delta_{s_{\ell-1}}^{b_{\ell-2}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} e^{-\beta_{j,i}^{\ell-2} V_{z^{(j)}}(u^{(j,i)})}, \end{aligned}$$

with  $\boldsymbol{\beta}^{\ell-2} = (|\mathbf{B}_1^{\ell-2}|, \dots, |\mathbf{B}_{|\pi_{\ell-2}|}^{\ell-2}|)$  and again, by independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ , using again that  $\sum_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \beta_{j,i}^{\ell-2} = |\mathbf{B}_j^{\ell-2}|$

$$\mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}) \middle| \mathcal{F}_{s_{\ell-1}-1} \right] = \mathcal{A}_{s_{\ell-1}-1}^{|\pi_{\ell-2}|} (f_{\mathbf{s}^{\ell-2}, \Pi^{\ell-2}}^{\ell-2}, \boldsymbol{\beta}^{\ell-2}) \prod_{j=1}^{|\pi_{\ell-2}|} c_{b_{\ell-2}(\mathbf{B}_j)}(\boldsymbol{\beta}_j^{\ell-2})$$

$$\times \prod_{\substack{\mathfrak{B} \in \pi_{\ell-1} \\ |\mathfrak{B}| \geq 2}} e^{s_{\ell}^* \psi(|\mathfrak{B}|)},$$

where  $\beta_j^{\ell-2} = (\beta_{j,1}^{\ell-2}, \dots, \beta_{j,b_{\ell-2}(\mathbf{B}_j)}^{\ell-2})$ . Thus, we obtain

$$\begin{aligned} \mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_{s_{\ell}}^k} f_{\mathbf{s}, \Pi}^{\ell}(\mathbf{x}) e^{-\langle 1, V(\mathbf{x}) \rangle_k} | \mathcal{F}_{s_{\ell-1}-1} \right] &= \mathcal{A}_{s_{\ell-1}-1}^{|\pi_{\ell-2}|} (f_{\mathbf{s}^{\ell-2}, \Pi^{\ell-2}}^{\ell-2}, \beta^{\ell-2}) \prod_{i=\ell-1}^{\ell} \prod_{j=1}^{|\pi_{i-1}|} c_{b_{i-1}(\mathbf{B}_j)}(\beta_j^{i-1}) \\ &\quad \times \prod_{\substack{\mathfrak{B} \in \pi_i \\ |\mathfrak{B}| \geq 2}} e^{s_{i+1}^* \psi(|\mathfrak{B}|)}. \end{aligned}$$

By induction on  $2 \leq p \leq \ell$ , we finally get, on the set of non-extinction

$$\mathbf{E}^* [\mathcal{A}_{\infty}^k (f_{\mathbf{s}, \Pi}^{\ell}) | \mathcal{F}_{s_p-1}] = \mathcal{A}_{s_p-1}^{|\pi_{p-1}|} (f_{\mathbf{s}^{p-1}, \Pi^{p-1}}^{p-1}, \beta^{p-1}) \prod_{i=p}^{\ell} \prod_{j=1}^{|\pi_{i-1}|} c_{b_{i-1}(\mathbf{B}_j)}(\beta_j^{i-1}) \prod_{\substack{\mathfrak{B} \in \pi_i \\ |\mathfrak{B}| \geq 2}} e^{s_{i+1}^* \psi(|\mathfrak{B}|)}.$$

Taking  $p = 2$  in the above formula, we have, on the set of non-extinction

$$\mathbf{E}^* [\mathcal{A}_{\infty}^k (f_{\mathbf{s}, \Pi}^{\ell}) | \mathcal{F}_{s_2-1}] = \mathcal{A}_{s_2-1}^{|\pi_1|} (f_{\mathbf{s}^1, \Pi^1}^1, \beta^1) \prod_{i=2}^{\ell} \prod_{j=1}^{|\pi_{i-1}|} c_{b_{i-1}(\mathbf{B}_j)}(\beta_j^{i-1}) \prod_{\substack{\mathfrak{B} \in \pi_i \\ |\mathfrak{B}| \geq 2}} e^{s_{i+1}^* \psi(|\mathfrak{B}|)},$$

where for any  $i \in \{2, \dots, \ell\}$ ,  $s_i^* = s_i - s_{i-1} - 1$  and  $s_{\ell+1}^* = 1$ . Since  $\sum_{j=1}^{b_0(\mathbf{B}_1)} |\mathbf{B}_j^1| = k$  (it comes from the fact that  $\pi_0 = \{\{1, \dots, k\}\}$ ), we have

$$\begin{aligned} \mathbf{E}^* [\mathcal{A}_{s_2-1}^{|\pi_1|} (f_{\mathbf{s}^1, \Pi^1}^1, \beta^1) | \mathcal{F}_{s_1-1}] &= \sum_{|z|=s_1-1} e^{-kV(z)} c_{b_0(\mathbf{B}_1)}(\beta^1) \prod_{\substack{\mathfrak{B} \in \pi_1 \\ |\mathfrak{B}| \geq 2}} e^{s_2^* \psi(|\mathfrak{B}|)} \\ &= \sum_{|z|=s_1-1} e^{-kV(z)} \prod_{j=1}^{|\pi_0|} c_{b_0(\mathbf{B}_j)}(\beta_j^0) \prod_{\substack{\mathfrak{B} \in \pi_1 \\ |\mathfrak{B}| \geq 2}} e^{s_2^* \psi(|\mathfrak{B}|)}, \end{aligned}$$

the last equality coming from the fact  $\beta_j^0 = \beta^1 = (|\mathbf{B}_1^1|, \dots, |\mathbf{B}_{|\pi_1|}^1|)$ . Finally,

$$\mathbf{E}^* [\mathcal{A}_{\infty}^k (f_{\mathbf{s}, \Pi}^{\ell})] = e^{\psi(k)} \prod_{i=1}^{\ell} \prod_{j=1}^{|\pi_{i-1}|} c_{b_{i-1}(\mathbf{B}_j)}(\beta_j^{i-1}) \prod_{\substack{\mathfrak{B} \in \pi_i \\ |\mathfrak{B}| \geq 2}} e^{s_{i+1}^* \psi(|\mathfrak{B}|)},$$

thus completing to proof. □

We end this subsection with the proof of Theorem 1.5.

*Proof of Theorem 1.5.* First recall that for  $1 \leq \ell < k$ ,  $\mathfrak{s} \in \mathbb{N}^*$  and  $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{N}^{\times \ell}$  such that  $s_1 < \dots < s_\ell \leq \mathfrak{s}$ , for all  $\mathbf{x} \in \Delta^k$  such that  $\min_{1 \leq j \leq k} |x^{(j)}| \geq \mathfrak{s}$ ,

$$F_{\mathfrak{s}}^\ell(\mathbf{x}) = \sum_{\Xi \text{ increasing}} f_{\mathfrak{s}, \Xi}^\ell(\mathbf{x}).$$

By  $\Xi$  increasing, we mean here that  $\Xi = (\Xi_i)_{0 \leq i \leq \ell}$  is an increasing collection of partitions of  $\{1, \dots, k\}$ . Since  $f_{\mathfrak{s}, \Xi}^\ell$  satisfies the hereditary assumption 5, the same goes for  $F_{\mathfrak{s}}^\ell$  by taking  $\mathfrak{g} = \mathfrak{s}$ . Using the linearity of  $g \mapsto \mathcal{A}_1^k(g)$ , we get (II.18) thanks to Theorem 1.6.

First note that

$$\{\mathcal{J}^{k,n} = \ell, \mathcal{J}^k(\mathcal{Y}^{(n)}) \leq \mathfrak{s}\} = \bigcup_{\mathbf{m}; m_1 < \dots < m_\ell \leq \mathfrak{s}} \bigcup_{\Pi \text{ increasing}} \bigcap_{i=1}^{\ell} \{\pi_{m_{i-1}}^{k,n} = \pi_{i-1}, \pi_{m_i}^{k,n} = \pi_i\},$$

where  $\Pi$  increasing means here that  $\Pi = (\pi_i)_{0 \leq i \leq \ell}$  is an increasing collection of partitions of  $\{1, \dots, k\}$ . It follows that

$$\mathbb{P}^*(\mathcal{J}_1^{k,n} = s_1, \dots, \mathcal{J}_\ell^{k,n} = s_\ell, \mathcal{J}^{k,n} = \ell) = \mathbb{E}^* \left[ \frac{\mathcal{A}^k(\mathcal{D}_n, F_{\mathfrak{s}}^\ell)}{\mathcal{A}^k(\mathcal{D}_n, \mathbf{1}_{\mathcal{E}_s^k})} \mathbf{1}_{\{D_n \geq k\}} \right],$$

and we conclude using Theorem 1.6.  $\square$

## 2.2 Proof of Theorem 1.6 and Theorem 1.7

*Proof of Theorem 1.6.* First,  $\mathcal{A}^k(\mathcal{D}_{n, Tn^{1/2}}, f) = \mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{E}^{k, n^{1/2}}}) + \mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\Delta^k \setminus \mathcal{E}^{k, n^{1/2}}})$  and then for any  $\varepsilon \in (0, 1)$

$$\begin{aligned} & \mathbb{P}^* \left( \left| \frac{1}{(n^{1/2} \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_n, f) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon \right) \\ & \leq \mathbb{P}^* \left( \left| \frac{1}{(n^{1/2} \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_n, f) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \frac{\varepsilon}{2} \right) + \mathbb{P}^* \left( \mathcal{A}^k(\mathcal{D}_n, \mathbf{1}_{\Delta^k \setminus \mathcal{E}^{k, n^{1/2}}}) > \frac{\varepsilon}{2} (n^{1/2} \mathbf{L}_n)^k \right). \end{aligned}$$

Noticing that  $\mathcal{D}_n = \mathcal{D}_{n, Tn^{1/2}}$ , the first probability in this sum goes to 0 when  $n \rightarrow \infty$  thanks to Proposition 2 with  $s = n^{1/2}$  and the second one also goes to 0 thanks to Proposition 3 thus giving (II.25). For the convergence in  $\mathbb{P}^*$ -probability (II.26), note that

$$\begin{aligned} & \mathbb{P}^* \left( \left| \frac{\mathcal{A}^k(\mathcal{D}_n, f)}{\mathcal{A}^k(\mathcal{D}_n, f)} \mathbf{1}_{\{D_n \geq k\}} - \frac{\mathcal{A}_\infty^k(f)}{\mathcal{A}_\infty^k(g)} \right| > \varepsilon \right) \\ & \leq \mathbb{P}^* \left( \left| \frac{\mathcal{A}^k(\mathcal{D}_n, f)}{\mathcal{A}^k(\mathcal{D}_n, f)} - \frac{\mathcal{A}_\infty^k(f)}{\mathcal{A}_\infty^k(g)} \right| > \varepsilon, D_n \geq k \right) + \mathbb{P}^*(D_n < k), \end{aligned}$$

these two probabilities go to 0 when  $n \rightarrow \infty$  and the proof is completed.  $\square$

We now prove Theorem 1.7. Recall that  $\tilde{\mathcal{D}}_n = \{x \in \mathcal{R}_n; \ell_n \leq |x| \leq \mathfrak{L}_n\}$ . The main idea of the proof is to show that, when  $\kappa > 2\xi k$ ,  $\xi \geq 2$ , and  $\mathfrak{L}_n = o(n^{1/2 - \mathfrak{q}\xi})$  for some non-increasing

sequence  $\mathfrak{q}$  such that  $\mathfrak{q}_j \rightarrow 0$  when  $j \rightarrow \infty$ , the volume  $\tilde{D}_n$  of the range  $\tilde{\mathcal{D}}_n$  behaves like the volume of the range up to the last complete excursion of  $(X)_{j \leq n}$  above the parent  $e^*$  of the root  $e$ .

For that, one can notice that for this choice of  $\kappa$ , Proposition 2 holds uniformly in  $s$  : there exists a non-increasing sequence of positive integers  $(\mathfrak{q}_j)_j$ , satisfying  $\mathfrak{q}_j \in (0, 1/2)$  and  $\mathfrak{q}_j \rightarrow 0$  when  $j \rightarrow \infty$  such that if  $\kappa > 2\xi k$  for some integer  $\xi \geq 2$  and  $\mathfrak{L}_n = o(n^{1/2 - \mathfrak{q}_\xi})$  then, for any  $\varepsilon_1 \in (0, 1)$

$$\mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left| \frac{1}{(s\mathfrak{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n, T^s}, f \mathbf{1}_{\mathfrak{C}^{k,s}}) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon \right) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{II.30})$$

The proof of (II.30) is the same as the proof of Proposition 2 but for any  $\varepsilon, \varepsilon_1 \in (0, 1)$ , by Markov inequality

$$\begin{aligned} & \mathbb{P} \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left\{ \left| \sum_{j \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(j, f \mathbf{1}_{\mathfrak{C}_{a_n}^k}) - \mathbb{E}^\mathfrak{E} \left[ \sum_{j \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(j, f \mathbf{1}_{\mathfrak{C}_{a_n}^k}) \right] \right| > \varepsilon (s\mathfrak{L}_n)^k / 16 \right\} \right) \\ & \leq \sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \frac{16^{2\xi k}}{\varepsilon^{2\xi k} (s\mathfrak{L}_n)^{2\xi k}} \mathbb{E} \left[ \left( \sum_{j \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(j, f \mathbf{1}_{\mathfrak{C}_{a_n}^k}) - \mathbb{E}^\mathfrak{E} \left[ \sum_{j \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(j, f \mathbf{1}_{\mathfrak{C}_{a_n}^k}) \right] \right)^{2\xi k} \right] \\ & \leq 16^{2\xi k} \mathfrak{C}_{3.7} \sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left( \frac{\mathfrak{L}_n}{s} \right)^{\tilde{\mathfrak{q}}_\xi} \leq \mathfrak{C}_2 \frac{(\mathfrak{L}_n)^{\tilde{\mathfrak{q}}_\xi}}{n^{(\tilde{\mathfrak{q}}_\xi - 1)/2}}, \end{aligned}$$

where we have used Lemma 3.7 with  $\mathfrak{a} = \xi$  for second inequality. Note that  $\tilde{\mathfrak{q}}_\xi \geq 2$  since  $\xi \geq 2$  so, as in the proof of Proposition 2, we obtain (II.30) by taking  $\mathfrak{q}_j := (2\tilde{\mathfrak{q}}_j)^{-1}$ .

*Proof of Theorem 1.7.* First, let us state the following fact, proved by Y. Hu ([Hu17], Corollary 1.2) : in law, under  $\mathbb{P}^*$

$$\frac{1}{n^{1/2}} \sum_{j=1}^n \mathbf{1}_{\{X_k=e\}} \xrightarrow{n \rightarrow \infty} \frac{1}{p^\mathfrak{E}(e, e^*)} \frac{c_0^{1/2}}{W_\infty} |\mathcal{N}|.$$

We can actually adapt this result to the local time  $\mathcal{L}^n$  of the parent  $e^*$  of the root  $e$  : in law, under  $\mathbb{P}^*$

$$\frac{1}{n^{1/2}} \mathcal{L}^n \xrightarrow{n \rightarrow \infty} \frac{c_0^{1/2}}{W_\infty} |\mathcal{N}|, \quad (\text{II.31})$$

where  $c_0$  is defined in (II.26). Moreover, recall that  $\mathcal{N}$  denotes, under  $\mathbb{P}^\mathfrak{E}$ , a standard Gaussian variable. Then, we show that  $\mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f)$  and  $\mathcal{A}^k(\mathcal{D}_n, f)$  are close in the following :

$$\mathbb{P}^* \left( \frac{1}{(\mathcal{L}^n \mathfrak{L}_n)^k} \left| \mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f) - \mathcal{A}^k(\mathcal{D}_n, f) \right| > \varepsilon \right) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{II.32})$$

Indeed, by (II.31),  $\lim_{\varepsilon_1 \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^*(\varepsilon_1 n^{1/2} \leq \mathcal{L}^n \leq n^{1/2}/\varepsilon_1) = 1$  so thanks to Proposition 3, together with Lemma 3.6 taking  $\mathfrak{R} = \Delta^k \setminus \mathcal{C}_{a_n}^k$  and Lemma 3.10,  $(\mathcal{A}^k(\mathcal{D}_{n,T^{\mathcal{L}^n}}, f\mathbb{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k}) + \mathcal{A}^k(\mathcal{D}_n, f\mathbb{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k})) / (\mathcal{L}^n \mathbf{L}_n)^k \rightarrow 0$  when  $n \rightarrow \infty$ , in  $\mathbb{P}^*$ -probability where we recall that  $a_n = (2\delta_0)^{-1} \log n$ . Therefore, it is enough to show that

$$\mathbb{P}^* \left( \frac{1}{(\mathcal{L}^n \mathbf{L}_n)^k} \left| \mathcal{A}^k(\mathcal{D}_{n,T^{\mathcal{L}^n}}, f\mathbb{1}_{\mathcal{C}_{a_n}^k}) - \mathcal{A}^k(\mathcal{D}_n, f\mathbb{1}_{\mathcal{C}_{a_n}^k}) \right| > \varepsilon \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

For that, assume  $\mathcal{L}^n = s \in \{\varepsilon_1 n^{1/2}, \dots, n^{1/2}/\varepsilon_1\}$ . By definition,  $\mathcal{L}^n = \sup\{j \geq 1; T^j \leq n\}$  so on the set  $\{D_{n,T^s} \geq k\}$ , where  $D_{n,T^s}$  is the cardinal of  $\mathcal{D}_{n,T^s}$ , both  $\mathcal{D}_{n,T^s}$  and  $\mathcal{D}_n$  are nonempty and note that

$$\begin{aligned} \mathcal{A}^k(\mathcal{D}_n, f\mathbb{1}_{\mathcal{C}_{a_n}^k}) - \mathcal{A}^k(\mathcal{D}_{n,T^s}, f\mathbb{1}_{\mathcal{C}_{a_n}^k}) &= \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathcal{L}_n\}^{\times k}} \sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{1}_{\{T^s < T_{\mathbf{x}} \leq n\}} \\ &\leq \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathcal{L}_n\}^{\times k}} \sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{1}_{\{T^s < T_{\mathbf{x}} < T^{s+1}\}}, \end{aligned}$$

where  $|\mathbf{x}| = |\mathbf{p}|$  means that for all  $1 \leq i \leq k$ ,  $x^{(i)} = p_i$ . Using the strong Markov property at time  $T^s$  first and Markov inequality then, we have

$$\begin{aligned} &\mathbb{P}^\varepsilon \left( \frac{1}{(s\mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_n, f\mathbb{1}_{\mathcal{C}_{a_n}^k}) - \mathcal{A}^k(\mathcal{D}_{n,T^s}, f\mathbb{1}_{\mathcal{C}_{a_n}^k}) > \varepsilon, D_{n,T^s} \geq k, \mathcal{L}^n = s \right) \\ &\leq \mathbb{P}^\varepsilon \left( \frac{1}{((s\mathbf{L}_n)^k} \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathcal{L}_n\}^{\times k}} \sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{1}_{\{T_{\mathbf{x}} < T^1\}} > \varepsilon \right) \\ &\leq \frac{1}{(\varepsilon s \mathbf{L}_n)^{2k}} \mathbb{E}^\varepsilon \left[ \left( \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathcal{L}_n\}^{\times k}} \sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{1}_{\{T_{\mathbf{x}} < T^1\}} \right)^2 \right]. \end{aligned}$$

Moreover  $\mathbb{E}^\varepsilon [(\sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{1}_{\{T_{\mathbf{x}} < T^1\}})^2]$  is equal to

$$\sum_{\substack{\mathbf{x}, \mathbf{y} \in \Delta^k \\ \mathbf{x} \neq \mathbf{y}}} \mathbb{1}_{\{|\mathbf{x}|=|\mathbf{y}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \vee \mathcal{S}^k(\mathbf{y}) \leq a_n\}} \mathbb{P}^\varepsilon(T_{\mathbf{x}} \vee T_{\mathbf{y}} < T^1) + \sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{P}^\varepsilon(T_{\mathbf{x}} < T^1),$$

where  $t \vee s = \max(t, s)$  and thanks to Lemma 3.2

$$\mathbb{E}^\varepsilon \left[ \left( \sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{1}_{\{T_{\mathbf{x}} < T^1\}} \right)^2 \right] \leq \mathfrak{C}' (a_n \mathbf{L}_n)^{2k},$$

for some constant  $\mathfrak{C}' > 0$  that doesn't depend on  $\mathbf{p}$  thus giving

$$\mathbb{P}^* \left( \frac{1}{(s\mathbf{L}_n)^k} \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathcal{L}_n\}^{\times k}} \sum_{\mathbf{x} \in \Delta^k} \mathbb{1}_{\{|\mathbf{x}|=\mathbf{p}, \mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbb{1}_{\{T^s < T_{\mathbf{x}} \leq n\}} > \varepsilon, \mathcal{L}^n = s \right) \leq \frac{\tilde{\mathfrak{C}}(a_n)^{2k}}{(\varepsilon s)^2},$$

for some constant  $\tilde{\mathfrak{C}} > 0$ . Finally, for all  $\varepsilon_1 \in (0, 1)$

$$\begin{aligned} & \mathbb{P}^* \left( \left\{ \frac{1}{(\mathcal{L}^n \mathbf{L}_n)^k} \left| \mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) - \mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) \right| > \varepsilon \right\} \right) \leq \mathbb{P}^*(D_{n, T^{\mathcal{L}^n}} < k) \\ & + \sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \mathbb{P}^* \left( \frac{1}{(s \mathbf{L}_n)^k} \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} \mathbf{1}_{\{\mathcal{S}^k(\mathbf{x}) \leq a_n\}} \mathbf{1}_{\{T^s < T_{\mathbf{x}} \leq n\}} > \varepsilon, D_{n, T^s} \geq k, \mathcal{L}^n = s \right) \\ & + \mathbb{P}^*(\mathcal{L}^n < \varepsilon_1 n^{1/2}) + \mathbb{P}^*(\mathcal{L}^n > n^{1/2}/\varepsilon_1) \leq \frac{\tilde{\mathfrak{C}}(a_n)^{2k}}{\varepsilon^2 \varepsilon_1 n^{1/2}} + \mathbb{P}^*(\mathcal{L}^n < \varepsilon_1 n^{1/2}) + \mathbb{P}^*(\mathcal{L}^n > n^{1/2}/\varepsilon_1). \end{aligned}$$

Note that  $\lim_{n \rightarrow \infty} \mathbb{P}^*(D_{n, T^{\mathcal{L}^n}} < k) = 0$  and using (II.31) with the definition of  $a_n = (2\delta_0)^{-1} \log n$ , we have  $\lim_{\varepsilon_1 \rightarrow 0} \limsup_{n \rightarrow \infty} (\tilde{\mathfrak{C}}(a_n)^{2k} / (\varepsilon^2 \varepsilon_1 n^{1/2}) + \mathbb{P}^*(\mathcal{L}^n < \varepsilon_1 n^{1/2}) + \mathbb{P}^*(\mathcal{L}^n > n^{1/2}/\varepsilon_1)) = 0$ , which yields (II.32).

Now, since  $\mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f \mathbf{1}_{\Delta^k \setminus \mathfrak{C}^k, \mathcal{L}^n}) / (\mathcal{L}^n \mathbf{L}_n)^k \rightarrow 0$  when  $n \rightarrow \infty$ , in  $\mathbb{P}^*$ -probability and thanks to (II.32), we can focus our attention on  $\mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f \mathbf{1}_{\mathfrak{C}^k, \mathcal{L}^n}) / (\mathcal{L}^n \mathbf{L}_n)^k$ .

Note the  $\mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f \mathbf{1}_{\mathfrak{C}^k, \mathcal{L}^n})$  concentrates around  $(c_\infty)^k \mathcal{A}_\infty^k(f)$ . Indeed, for any  $\varepsilon, \varepsilon_1 \in (0, 1)$

$$\begin{aligned} & \mathbb{P}^* \left( \left| \frac{1}{(\mathcal{L}^n \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f \mathbf{1}_{\mathfrak{C}^k, \mathcal{L}^n}) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon \right) \\ & \leq \mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left\{ \left| \frac{1}{(s \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n, T^s}, f \mathbf{1}_{\mathfrak{C}^k, s}) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon \right\} \right) + \mathbb{P}^*(\mathcal{L}^n < \varepsilon_1 n^{1/2}) \\ & + \mathbb{P}^*(\mathcal{L}^n > n^{1/2}/\varepsilon_1). \end{aligned}$$

Thanks to equation (II.30), the first probability above goes to 0 when  $n$  goes to  $\infty$  and by (II.31),  $\lim_{\varepsilon_1 \rightarrow 0} \lim_{n \rightarrow \infty} (\mathbb{P}^*(\mathcal{L}^n < \varepsilon_1 n^{1/2}) + \mathbb{P}^*(\mathcal{L}^n > n^{1/2}/\varepsilon_1)) = 0$  thus giving

$$\lim_{n \rightarrow \infty} \mathbb{P}^* \left( \left| \frac{1}{(\mathcal{L}^n \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n, T^{\mathcal{L}^n}}, f \mathbf{1}_{\mathfrak{C}^k, \mathcal{L}^n}) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon \right) = 0. \quad (\text{II.33})$$

We obtain from (II.33), together with (II.32) that

$$\lim_{n \rightarrow \infty} \mathbb{P}^* \left( \left| \frac{1}{(\mathcal{L}^n \mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon \right) = 0,$$

which gives (II.26). We also deduce (II.25) using (II.31). For the convergence in  $\mathbb{P}^*$ -probability (II.26), note that

$$\begin{aligned} & \mathbb{P}^* \left( \left| \frac{\mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k})}{\mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k})} \mathbf{1}_{\{D_n \geq k\}} - \frac{\mathcal{A}_\infty^k(f)}{\mathcal{A}_\infty^k(g)} \right| > \varepsilon \right) \\ & \leq \mathbb{P}^* \left( \left| \frac{\mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k})}{\mathcal{A}^k(\mathcal{D}_n, f \mathbf{1}_{\mathcal{C}_{a_n}^k})} - \frac{\mathcal{A}_\infty^k(f)}{\mathcal{A}_\infty^k(g)} \right| > \varepsilon, D_n \geq k \right) + \mathbb{P}^*(D_n < k), \end{aligned}$$



which goes to 0 when  $n$  goes to  $\infty$  and the proof is completed.  $\square$

### 3 Proofs of Propositions 2 and 3

This section is devoted to the proofs of our two propositions. We show that relevant  $k$ -tuples of visited vertices are those in the set  $\mathfrak{E}^k$ .

Let us recall the well-known many-to-one lemma :

**Lemma 3.1** (many-to-one). *For any  $p \in \mathbb{N}^*$  and any bounded function  $\mathbf{h}$*

$$\mathbf{E}[\mathbf{h}(S_1, \dots, S_p)] = \mathbf{E}\left[\sum_{|\mathbf{x}|=p} e^{-V(\mathbf{x})} \mathbf{h}(V(x_1), \dots, V(x_p))\right],$$

where  $(S_i)_{i \in \mathbb{N}}$  is the real valued random walk defined in (II.22).

We now state and prove a lemma that will be useful all along this section. For any vertex  $x \in \mathbb{T}$ , introduce  $T_x := \inf\{i \geq 1 \mid X_i = x\}$ , the hitting time of  $x$  and for any  $\mathbf{x} = (x^{(1)}, \dots, x^{(q)}) \in \Delta^q$ ,  $T_{\mathbf{x}} := \max_{1 \leq i \leq q} T_{x^{(i)}}$ . Recall that for any  $j \in \mathbb{N}^*$ ,  $T^j$  denotes the  $j$ -th return time to the parent  $e^*$  of the root  $e$ . For  $1 \leq \ell < q$  two integers,  $\mathbf{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^{\times \ell}$  such that  $m_1 < \dots < m_\ell$  and  $\Pi = (\boldsymbol{\pi})_{0 \leq i \leq \ell}$  an increasing collection of partitions of  $\{1, \dots, q\}$  that is to say  $|\boldsymbol{\pi}_{i-1}| < |\boldsymbol{\pi}_i|$  with  $\boldsymbol{\pi}_0 = \{\{1, \dots, q\}\}$  and  $\boldsymbol{\pi}_\ell = \{\{1\}, \dots, \{q\}\}$ , recall the definition of  $f_{\mathbf{m}, \Pi}^\ell$  in (II.14).

**Lemma 3.2.** *Let  $k \geq 2$  and  $\mathbf{a} \geq 1$  be two integers and assume  $\kappa > 2\mathbf{a}k$ . Let  $q \in \{k, \dots, 2\mathbf{a}k\}$  and  $\mathbf{p} = (p_1, \dots, p_q) \in \mathbb{N}^{\times q}$ . Under the assumptions 1 and 3, there exists a constant  $\mathfrak{C} > 0$  doesn't depending neither on  $\mathbf{p}$ , nor on  $\mathbf{m}$  such that*

$$\mathbf{E}\left[\sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}|=\mathbf{p}}} f_{\mathbf{m}, \Pi}^\ell(\mathbf{x}) \mathbb{P}^{\mathcal{E}}(T_{\mathbf{x}} < T^1)\right] \leq \mathfrak{C},$$

where  $|\mathbf{x}| = \mathbf{p}$  means that  $|x^{(i)}| = p_i$  for any  $i \in \{1, \dots, q\}$ . In particular, for any integer  $m \in \mathbb{N}^*$ , there exists a constant  $C_{3.2} > 0$  doesn't depending on  $\mathbf{p}$  such that

$$\mathbf{E}\left[\sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}|=\mathbf{p}}} \mathbf{1}_{\mathcal{C}_m^k}(\mathbf{x}) \mathbb{P}^{\mathcal{E}}(T_{\mathbf{x}} < T^1)\right] \leq C_{3.2} m^{q-1}. \quad (\text{II.34})$$

*Proof in the case  $\cap_{j=1}^\ell \Gamma_{\mathbf{m}, \Pi}^j \subset \{\mathbf{x} \in \Delta^q; \mathcal{C}^q(\mathbf{x}) < \min_{1 \leq i \leq q} p_i\}$ .* First recall that  $\Pi^i$  is the partition of  $\{1, \dots, |\boldsymbol{\pi}_i|\}$  obtained via the procedure defined above Example 6 and for any  $i \in \{1, \dots, \ell\}$ , any  $j \in \{1, \dots, |\boldsymbol{\pi}_{i-1}|\}$ , the  $j$ -th block  $\mathbf{B}_j^i$  of the partition  $\boldsymbol{\pi}_{i-1}$  is the union of

$b_{i-1}(\mathbf{B}_j) \geq 1$  block(s) of the partition  $\pi_j$ . Note (see the proof of Theorem 1.4) that

$$\begin{aligned} \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} f_{\mathbf{m}, \Pi}^\ell(\mathbf{x}) \mathbb{P}^\delta(T_{\mathbf{x}} < T^1) &= \sum_{\mathbf{z} \in \Delta_{m_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{m}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{z}) \prod_{j=1}^{|\pi_{\ell-1}|} \sum_{\mathbf{u}^{(j)} \in \Delta_{m_\ell}^{b_{\ell-1}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-1}(\mathbf{B}_j)} \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} \\ &\quad \times \sum_{\mathbf{x} \in \Delta^q} \mathbb{1}_{\{|\mathbf{x}| = \mathbf{p}, \mathbf{x} \geq \mathbf{u}\}} \mathbb{P}^\delta(T_{\mathbf{x}} < T^1), \end{aligned}$$

where  $\mathbf{m}^{\ell-1} = (m_1, \dots, m_{\ell-1})$ ,  $\mathbf{u}$  is the concatenation of  $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(|\pi_{\ell-1}|)}$  and  $\mathbf{x} \geq \mathbf{u}$  means that  $x^{(p)} \geq u^{(p)}$ . Thanks to the strong Markov property at time  $T_{z^{(j)}}$ , there exists a constant  $C_q \geq 1$  such that

$$\begin{aligned} \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} f_{\mathbf{m}, \Pi}^\ell(\mathbf{x}) \mathbb{P}^\delta(T_{\mathbf{x}} < T^1) &\leq C_q \sum_{\mathbf{z} \in \Delta_{m_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{m}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{z}) \mathbb{P}^\delta(T_{\mathbf{z}} < T^1) \prod_{j=1}^{|\pi_{\ell-1}|} \sum_{\mathbf{u}^{(j)} \in \Delta_{m_\ell}^{b_{\ell-1}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-1}(\mathbf{B}_j)} \\ &\quad \times \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} \sum_{\substack{\mathbf{x}^{(j)} \in \Delta^{b_{\ell-1}(\mathbf{B}_j)} \\ |\mathbf{x}^{(j)}| = \mathbf{p}^{(j)}}} \mathbb{1}_{\{x^{(j,i)} \geq u^{(j,i)}\}} \mathbb{P}_{z^{(j)}}^\delta(T_{x^{(j,i)}} < T^1), \end{aligned}$$

where  $\mathbf{p}$  is now seen as the concatenation of  $\mathbf{p}^{(1)}, \dots, \mathbf{p}^{(|\pi_{\ell-1}|)}$ . Moreover, it is known that for all  $z \leq x$  in  $\mathbb{T}$ ,

$$\mathbb{P}_z^\delta(T_x < T^1) = \frac{\sum_{e \leq w \leq z} e^{V(w)}}{\sum_{e \leq w \leq x} e^{V(w)}} \quad \text{if } z \neq e, \quad \mathbb{P}^\delta(T_x < T^1) = \frac{1}{\sum_{e \leq w \leq x} e^{V(w)}} \quad \text{else,} \quad (\text{II.35})$$

so  $\mathbb{P}_z^\delta(T_x < T^1) \leq e^{-V(x)} \sum_{e \leq w \leq z} e^{V(w)}$ . By independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ , using that  $b_{\ell-1}(\mathbf{B}_j) = |\mathbf{B}_j^{\ell-1}|$  and  $\psi(1) = 0$

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} f_{\mathbf{m}, \Pi}^\ell(\mathbf{x}) \mathbb{P}^\delta(T_{\mathbf{z}} < T^1) \right] &\leq \mathfrak{C}_{\ell-1} \mathbf{E} \left[ \sum_{\mathbf{z} \in \Delta_{m_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{m}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{z}) \mathbb{P}^\delta(T_{\mathbf{z}} < T^1) \prod_{j=1}^{|\pi_{\ell-1}|} (H_{z^{(j)}})^{|\mathbf{B}_j^{\ell-1}|} \right] \\ &\quad \times C_q \prod_{\mathbf{B} \in \pi_{\ell-1}} c_{|\mathbf{B}|}(\mathbf{1}), \end{aligned}$$

with  $H_z = \sum_{e \leq w \leq z} e^{V(w) - V(z)}$  and  $\mathfrak{C}_{\ell-1} = C_q \prod_{\mathbf{B} \in \pi_{\ell-1}} c_{|\mathbf{B}|}(\mathbf{1}) \in (0, \infty)$  thanks to assumption 3 since for any  $\mathbf{B} \in \pi_{\ell-1}$ ,  $|\mathbf{B}| < q \leq 4k < \kappa$ . Again, thanks to the strong Markov property at time  $T_{w^{(i)}}$

$$\sum_{\mathbf{z} \in \Delta_{m_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{m}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{z}) \mathbb{P}^\delta(T_{\mathbf{z}} < T^1) \prod_{j=1}^{|\pi_{\ell-1}|} (H_{z^{(j)}})^{|\mathbf{B}_j^{\ell-1}|}$$

$$\begin{aligned} &\leq C_{\ell-1} \sum_{\mathbf{w} \in \Delta_{m_{\ell-1}-1}^{|\pi_{\ell-2}|}} f_{\mathbf{m}^{\ell-2}, \Pi^{\ell-2}}^{\ell-2}(\mathbf{w}) \mathbb{P}^{\mathcal{E}}(T_{\mathbf{w}} < T^1) \prod_{j=1}^{|\pi_{\ell-2}|} \sum_{\mathbf{v}^{(j)} \in \Delta_{m_{\ell-1}}^{b_{\ell-2}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \mathbb{1}_{\{(v^{(j,i)})^* = w^{(j)}\}} \\ &\times \sum_{z^{(j)} \in \Delta_{s_{\ell-1}}^{|\pi_{\ell-1}|}} \mathbb{1}_{\{z^{(j,i)} \geq v^{(j,i)}\}} (H_{z^{(j,i)}})^{\beta_{j,i}^{\ell-2}} \mathbb{P}_{w^{(j)}}^{\mathcal{E}}(T_{z^{(j,i)}} < T^1), \end{aligned}$$

for some constant  $C_{\ell-1} \geq 1$ , where  $\mathbf{v}^{(j)} = (v^{(j,1)}, \dots, v^{(j, b_{\ell-2}(\mathbf{B}_j))})$  and recall the definition of  $\beta_{j,i}^{\ell-2}$  in (II.13). Thanks to (II.35)

$$(H_{z^{(j,i)}})^{\beta_{j,i}^{\ell-2}} \mathbb{P}_{w^{(j)}}^{\mathcal{E}}(T_{z^{(j,i)}} < T^1) = H_{w^{(j)}} e^{-V_{w^{(j)}}(z^{(j,i)})} (H_{z^{(j,i)}})^{\beta_{j,i}^{\ell-2}-1},$$

and  $H_{z^{(j,i)}} = H_{v^{(j,i)}} e^{-V_{v^{(j,i)}}(z^{(j,i)})} + H_{v^{(j,i)}, z^{(j,i)}}$  where, for any  $u < x$ ,  $H_{u,x} := \sum_{u < w \leq x} e^{V(w)-V(x)}$ . Since  $H_u \geq 1$  for all  $u \in \mathbb{T}$ , we have

$$H_{z^{(j,i)}} \leq H_{w^{(j)}} (e^{-V_{w^{(j)}}(v^{(j,i)})} + 1) (e^{-V_{v^{(j,i)}}(z^{(j,i)})} + H_{v^{(j,i)}, z^{(j,i)}}),$$

thus giving that  $(H_{z^{(j,i)}})^{\beta_{j,i}^{\ell-2}} \mathbb{P}_{w^{(j)}}^{\mathcal{E}}(T_{z^{(j,i)}} < T^1)$  is smaller than

$$\begin{aligned} &(H_{w^{(j)}})^{\beta_{j,i}^{\ell-2}} e^{-V_{w^{(j)}}(v^{(j,i)})} (e^{-V_{w^{(j)}}(v^{(j,i)})} + 1)^{\beta_{j,i}^{\ell-2}-1} e^{-V_{v^{(j,i)}}(z^{(j,i)})} \\ &\quad \times (e^{-V_{v^{(j,i)}}(z^{(j,i)})} + H_{v^{(j,i)}, z^{(j,i)}})^{\beta_{j,i}^{\ell-2}-1}. \end{aligned}$$

By independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ , using that  $\sum_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \beta_{j,i}^{\ell-2} = |\mathbf{B}_j^{\ell-2}|$

$$\begin{aligned} &\mathbf{E} \left[ \sum_{z \in \Delta_{m_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{m}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(z) \mathbb{P}^{\mathcal{E}}(T_z < T^1) \prod_{j=1}^{|\pi_{\ell-1}|} (H_{z^{(j)}})^{|\mathbf{B}_j^{\ell-1}|} \right] \\ &\leq \mathfrak{C}_{\ell-2} \mathbf{E} \left[ \sum_{\mathbf{w} \in \Delta_{m_{\ell-1}-1}^{|\pi_{\ell-2}|}} f_{\mathbf{m}^{\ell-2}, \Pi^{\ell-2}}^{\ell-2}(\mathbf{w}) \mathbb{P}^{\mathcal{E}}(T_{\mathbf{w}} < T^1) \prod_{j=1}^{|\pi_{\ell-2}|} (H_{w^{(j)}})^{|\mathbf{B}_j^{\ell-2}|} \right], \end{aligned}$$

where, thanks to the many-to-one Lemma 3.1

$$\mathfrak{C}_{\ell-2} = \prod_{j=1}^{|\pi_{\ell-2}|} \mathbf{E} \left[ \sum_{\mathbf{v} \in \Delta_1^{b_{\ell-2}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} e^{-V(v^{(i)})} (e^{-V(v^{(i)})} + 1)^{\beta_{j,i}^{\ell-2}} \right] \prod_{\mathbf{B} \in \pi_{\ell-1}} \mathbf{E} [(e^{-S_{m_{\ell}^*}} + H_{m_{\ell}^*}^S)^{|\mathbf{B}|-1}],$$

$m_{\ell}^* = m_{\ell} - m_{\ell-1} - 1$ ,  $H_m^S := \sum_{p=1}^m e^{S_p - S_m}$  (the random walk  $(S_p)$  is defined in (II.22)). Note that  $\mathfrak{C}_{\ell-2} \in (0, \infty)$ . Indeed, the first mean in the definition of  $\mathfrak{C}_{\ell-2}$  belongs to  $(0, \infty)$  thanks to assumption 3 since for any  $1 \leq j \leq |\pi_{\ell-2}|$ ,  $b_{\ell-2}(\mathbf{B}_j) < q \leq 2\mathfrak{a}\mathfrak{k} < \kappa$  and  $\sum_{i=1}^{b_{\ell-2}(\mathbf{B}_j)} \beta_{j,i}^{\ell-2} =$

$|\mathbf{B}_j^{\ell-2}| < q$ . The second one also belongs to  $(0, \infty)$  since for all  $\mathbf{B} \in \pi_{\ell-1}$ ,  $|\mathbf{B}| - 1 \leq q - 2 < \kappa - 2$  and as it is proved in [AD20] that  $\sup_{m \in \mathbb{N}^*} \mathbf{E}[(H_m)^{\kappa-1-\varepsilon}] < \infty$  for any  $\varepsilon > 0$ . We also deduce from this, together with the fact that  $\psi'(1) < 0$  and  $m_\ell^* \geq 0$  that  $\mathfrak{C}_{\ell-2}$  is bounded by a positive constant doesn't depending on  $\mathbf{m}$ . By induction, there exists a constant  $\mathfrak{C}_2 \in (0, \infty)$  (still not depending on  $\mathbf{m}$ ) such that

$$\mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}|=p}} f_{\mathbf{m}, \Pi}^\ell(\mathbf{x}) \mathbb{P}^\mathfrak{E}(T_{\mathbf{x}} < T^1) \right] \leq \mathfrak{C}_2 \mathbf{E} \left[ \sum_{|z|=m_1-1} \sum_{\mathbf{u} \in \Delta_{m_2-1}^{|\pi_1|}} \mathbb{P}^\mathfrak{E}(T_{\mathbf{u}} < T^1) \prod_{i=1}^{|\pi_1|} (H_{u^{(i)}})^{|\mathbf{B}_i^1|} \mathbf{1}_{\{u^{(i)} > z\}} \right].$$

Thanks to the strong Markov property,  $\mathbb{P}^\mathfrak{E}(T_{\mathbf{u}} < T^1) \leq C_{|\pi_1|} \mathbb{P}^\mathfrak{E}(T_z < T^1) \prod_{i=1}^{|\pi_1|} \mathbb{P}_z^\mathfrak{E}(T_{u^{(i)}} < T^1) = C_{|\pi_1|} \mathbb{P}^\mathfrak{E}(T_z < T^1) \prod_{i=1}^{|\pi_1|} e^{-V(u^{(i)})} / H_{u^{(i)}}$  for some constant  $C_{|\pi_1|} \geq 1$  and then, using (II.35) together with the many-to-one Lemma 3.1

$$\mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}|=p}} f_{\mathbf{m}, \Pi}^\ell(\mathbf{x}) \mathbb{P}^\mathfrak{E}(T_{\mathbf{x}} < T^1) \right] \leq \mathfrak{C}_1 \mathbf{E} \left[ \sum_{|z|=m_1-1} e^{-V(z)} (H_z)^{|\pi_1|-1} \right] = \mathfrak{C}_1 \mathbf{E}[(H_{m_1-1}^S)^{|\pi_1|-1}].$$

Again,  $|\pi_1| - 1 \leq q - 1 \leq 2\mathbf{a}k - 1 < \kappa - 1$  so  $\mathbf{E}[(H_{m_1-1}^S)^{|\pi_1|-1}] \leq \sup_{m \in \mathbb{N}^*} \mathbf{E}[(H_{m-1}^S)^{|\pi_1|-1}] \in (0, \infty)$  which ends the proof.  $\square$

### 3.1 The range on $\mathfrak{E}^{k, \cdot}$ .

This section is dedicated to the proof of Proposition 2 in which the range is restricted to the  $k$ -tuples of vertices belonging to the set  $\mathfrak{E}^{k, \cdot}$ , that is such that the vertices are visited during  $k$  distinct excursions, see (II.21) for the definition of  $\mathfrak{E}^{k, \cdot}$ .

#### The relevant vertices : the set $\mathcal{C}_{a_n}^k$

First recall that  $\mathcal{C}_m^k = \{\mathbf{x} \in \Delta^k; \mathcal{S}^k(\mathbf{x}) \leq m\}$  where, for any  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$  and  $\mathcal{S}^k(\mathbf{x}) - 1$  is the last generation at which two or more vertices among  $x^{(1)}, \dots, x^{(k)}$  share a common ancestor (see (II.10)). In this subsection, we focus on the range on  $\mathfrak{E}^{k, \cdot} \cap \mathcal{C}_{a_n}^k$  with  $a_n = (2\delta_0)^{-1} \log n$ , which is the set of relevant  $k$ -tuples of vertices in the case of small generations. Before going any further, let us state and prove the following lemma. Recall that  $H_u = \sum_{e \leq z \leq u} e^{V(z)-V(u)}$ .

**Lemma 3.3.** *Let  $k \geq 2$  and  $\mathbf{a} \geq 1$  be two integers and assume  $\kappa > 2\mathbf{a}k$ . Under the assumptions 1, 3 and 4*

- (i) *for any integer  $q \in \{k, \dots, 2\mathbf{a}k\}$  and any  $\beta = (\beta_1, \dots, \beta_q) \in (\mathbb{N}^*)^{\times q}$  such that  $\sum_{j=1}^q \beta_j \leq 2\mathbf{a}k$ , there exists a constant  $\mathfrak{C}_{3.3,1} > 0$  such that*

$$\sup_{\mathbf{p} \in (\mathbb{N}^*)^{\times q}} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}|=p}} e^{-\langle \beta, V(\mathbf{x}) \rangle_q} \right] \leq \mathfrak{C}_{3.3,1};$$

(ii) for any integer  $q \in \{k, \dots, 2ak\}$  there exists a constant  $\mathfrak{C}_{3.3,2} > 0$  such that for  $n$  large enough and any  $h > 0$

$$\mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_{a_n}^q} \mathbb{1}_{\{\max_{1 \leq i \leq q} H_{x^{(i)}} > h\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \leq \frac{\mathfrak{C}_{3.3,2}}{h^{\kappa-1}} + o(1).$$

*Proof in the case  $\cap_{j=1}^{\ell} \Gamma_{\mathbf{m}, \Pi}^j \subset \{\mathbf{x} \in \Delta^q; \mathcal{C}^q(\mathbf{x}) < \min_{1 \leq i \leq q} p_i\}$ .* Not that, since  $H_u \geq 1$ , we have  $\mathbf{E}[\sum_{\mathbf{x} \in \Delta_{a_n}^q} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q}] = \mathbf{E}[\sum_{\mathbf{x} \in \Delta_{a_n}^q} \mathbb{1}_{\{\max_{1 \leq i \leq q} H_{x^{(i)}} > h\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q}]$  for all  $h \leq 1$ . The proof of (i) is similar to the proof of Theorem 1.4 and Lemma 3.2 so we focus on (ii). In order to avoid unnecessary technical difficulties, we prove it for any  $\mathbf{a} \geq 2$ . Recall the definition of  $f_{\mathbf{s}, \Pi}^\ell$  in (II.14) for  $\ell \in \{1, \dots, q-1\}$ ,  $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{N}^{\times \ell}$  such that  $s_1 < \dots < s_\ell$  and  $\Pi = (\pi_i)_{0 \leq i \leq \ell}$  an increasing collection of partitions of  $\{1, \dots, q\}$ . Note that

$$\begin{aligned} \sum_{\mathbf{x} \in \Delta_{a_n}^q} \sum_{j=1}^q \mathbb{1}_{\{\max_{1 \leq i \leq q} H_{x^{(i)}} > h\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} &= \sum_{\ell=1}^{q-1} \sum_{\mathbf{s}; s_1 < \dots < s_\ell \leq a_n} \sum_{\Pi \text{ increasing}} \sum_{\mathbf{x} \in \Delta_{a_n}^q} f_{\mathbf{s}, \Pi}^\ell(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \\ &\quad \times \mathbb{1}_{\{\max_{1 \leq i \leq q} H_{x^{(i)}} > h\}}, \end{aligned}$$

and  $\sum_{\mathbf{x} \in \Delta_{a_n}^q} f_{\mathbf{s}, \Pi}^\ell(\mathbf{x}) \mathbb{1}_{\{\max_{1 \leq i \leq q} H_{x^{(j)}} > h\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q}$  is equal to

$$\begin{aligned} \sum_{\mathbf{z} \in \Delta_{s_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}(\mathbf{z}) \prod_{j=1}^{|\pi_{\ell-1}|} \sum_{\mathbf{u}^{(j)} \in \Delta_{s_\ell}^{b_{\ell-1}(\mathbf{B}_j)}} \prod_{i=1}^{b_{\ell-1}(\mathbf{B}_j)} \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} \sum_{\mathbf{x}^{(j)} \in \Delta_{a_n}^{b_{\ell-1}(\mathbf{B}_j)}} \mathbb{1}_{\{x^{(j,i)} \geq u^{(j,i)}\}} \\ \times e^{-V(\mathbf{x}^{(j,i)})} \mathbb{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} \max_{1 \leq i' \leq b_{\ell-1}(\mathbf{B}_{j'})} H_{x^{(j',i')}} > h\}}. \end{aligned}$$

Recall that for any  $u < x$ ,  $H_{u,x} = \sum_{u < z \leq x} e^{V(z)-V(x)}$ , thanks to assumption 4 together with the fact that  $H_{z^{(j')}} \geq 1$

$$H_{x^{(j',i')}} \leq H_{z^{(j')}} (1 + e^h) e^{-V_{u^{(j',i')}}(x^{(j',i')})} + H_{u^{(j',i')}, x^{(j',i')}},$$

so  $H_{x^{(j',i')}} > h$  implies that  $H_{z^{(j')}} (1 + e^h) e^{-V_{u^{(j',i')}}(x^{(j',i')})} > h/2$  or  $H_{u^{(j',i')}, x^{(j',i')}} > h/2$ . We also decompose according to the values of  $H_{z^{(j')}} :$

$$\begin{aligned} &\mathbb{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} \max_{1 \leq i' \leq b_{\ell-1}(\mathbf{B}_{j'})} H_{z^{(j')}} (1 + e^h) e^{-V_{u^{(j',i')}}(x^{(j',i')})} > h/2\}} \\ &\leq \mathbb{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} H_{z^{(j')}} > h\}} + \mathbb{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} \max_{1 \leq i' \leq b_{\ell-1}(\mathbf{B}_{j'})} 2(1 + e^h) e^{-V_{u^{(j',i')}}(x^{(j',i')})} > 1\}}. \end{aligned}$$

We therefore deduce that  $\mathbb{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} \max_{1 \leq i' \leq b_{\ell-1}(\mathbf{B}_{j'})} H_{x(j',i')} > h\}}$  is smaller than

$$\mathbb{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} H_{z(j')} > h\}} + \sum_{j'=1}^{|\pi_{\ell-1}|} \sum_{i'=1}^{b_{\ell-1}(\mathbf{B}_{j'})} \left( \mathbb{1}_{\{H_{u(j',i'),x(j',i')} > h/2\}} + \mathbb{1}_{\{2(1+e^h)e^{-V_{u(j',i')}(x(j',i'))} > 1\}} \right).$$

By independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ , since  $\psi(1) = 0$

$$\begin{aligned} & \mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_{a_n}^q} f_{\mathbf{s}, \Pi}^\ell(\mathbf{x}) \mathbb{1}_{\{\max_{1 \leq i \leq q} H_{x(i)} > h\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \\ & \leq \mathbf{E} \left[ \sum_{\mathbf{z} \in \Delta_{s_{\ell-1}}^{|\pi_{\ell-1}|}} f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{z}) \mathbb{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} H_{z(j')} > h\}} e^{-\langle \beta^{\ell-1}, V(\mathbf{z}) \rangle_{|\pi_{\ell-1}|}} \right] \prod_{j=1}^{|\pi_{\ell-1}|} c_{b_{\ell-1}(\mathbf{B}_j)}(\mathbf{1}) \\ & + q(\mathfrak{z}_{1,n} + \mathfrak{z}_{2,n}) \prod_{j=1}^{|\pi_{\ell-1}|} c_{b_{\ell-1}(\mathbf{B}_j)}(\mathbf{1}) \mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \right], \end{aligned}$$

where we recall that  $\mathcal{A}_l^q(g, \beta) = \sum_{\mathbf{x} \in \Delta_l^q} e^{-\langle \beta, V(\mathbf{x}) \rangle_q}$ ,

$$\mathfrak{z}_{1,n} = \mathbf{E} \left[ \sum_{|x|=a_n-s_\ell} e^{-V(x)} \mathbb{1}_{\{H_x > h/2\}} \right] \quad \text{and} \quad \mathfrak{z}_{2,n} = \mathbf{E} \left[ \sum_{|x|=a_n-s_\ell} e^{-V(x)} \mathbb{1}_{\{2(1+e^h)e^{-V(x)} > 1\}} \right].$$

Thanks to the many-to-one Lemma 3.1

$$\mathfrak{z}_{1,n} = \mathbf{P}(H_{a_n-s_\ell}^S > h/2) \leq \mathfrak{C}_{3.3,3}/h^{\kappa-1},$$

for some constant  $\mathfrak{C}_{3.3,3} > 0$ , the last inequality coming from ([AD20], Lemma 2.2). We now turn to  $\mathfrak{z}_{2,n}$ . If  $s_\ell \leq a_n/2$  then, for any  $\rho \in (0, \kappa - 1)$

$$\mathfrak{z}_{2,n} \leq 2^\rho (1 + e^h)^\rho e^{a_n \psi(1+\rho)/2}.$$

Otherwise  $s_\ell - 1 \geq a_n/2$  and thanks to the Cauchy–Schwarz inequality

$$\begin{aligned} \mathbf{E} \left[ \mathcal{A}_{s_\ell-1}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \right] & \leq \mathbf{E} \left[ \mathcal{A}_{s_\ell-1}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \mathbb{1}_{\tilde{\mathcal{Y}}_n} \right] + \\ & + (1 - \mathbf{P}(\tilde{\mathcal{Y}}_n))^{1/2} \mathbf{E} \left[ \left( \mathcal{A}_{s_\ell-1}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \right)^2 \right]^{1/2}, \end{aligned}$$

where  $\tilde{\mathcal{Y}}_n := \{\min_{a_n/2 \leq |z| \leq a_n} V(z) > 3/2 \log n\}$  (recall that  $a_n = (2\delta_0)^{-1} \log n$ ). On the one hand, by definition, there exists  $i_\alpha \in \{1, \dots, |\pi_{\ell-1}|\}$  such that  $|\mathbf{B}_{i_\alpha}^{\ell-1}| \geq 2$ . It follows that

$$\mathbf{E} \left[ \mathcal{A}_{s_\ell-1}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \mathbb{1}_{\tilde{\mathcal{Y}}_n} \right] \leq n^{-3/2} \mathbf{E} \left[ \mathcal{A}_{s_\ell-1}^{|\pi_{\ell-1}|} (f_{\mathbf{s}^{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \tilde{\beta}^{\ell-1}) \right],$$

where  $\tilde{\beta}_j^{\ell-1} = |\mathbf{B}_j^{\ell-1}|$  for all  $j \neq i_\alpha$  and  $\tilde{\beta}_{i_\alpha}^{\ell-1} = |\mathbf{B}_{i_\alpha}^{\ell-1}| - 1 \geq 1$ . On the other hand,  $1 - \mathbf{P}(\tilde{\mathcal{Y}}_n) \leq n^{-\rho_5}$  with  $\rho_5 > 0$  thanks to Lemma 3.4. Moreover, both  $\sum_{j=1}^{|\pi_{\ell-1}|} \beta_j^{\ell-1}$  and  $\sum_{j=1}^{|\pi_{\ell-1}|} \tilde{\beta}_j^{\ell-1}$  are smaller than  $2k$  since  $q \leq 2k$ . Hence, thanks to (i)

$$\mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{s_{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \tilde{\beta}^{\ell-1}) \right] + \mathbf{E} \left[ \left( \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{s_{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \right)^2 \right]^{1/2} \leq \mathfrak{C}_{3.3,4},$$

for some constant  $\mathfrak{C}_{3.3,4} > 0$ . We obtain

$$\begin{aligned} & \mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_{a_n}^q} f_{\mathbf{s}, \Pi}^\ell(\mathbf{x}) \mathbf{1}_{\{\max_{1 \leq i \leq q} H_{x(i)} > h\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \\ & \leq \mathbf{E} \left[ \sum_{\mathbf{z} \in \Delta_{s_{\ell-1}}^{|\pi_{\ell-1}|}} f_{s_{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{z}) \mathbf{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} H_{z(j')} > h\}} e^{-\langle \beta^{\ell-1}, V(\mathbf{z}) \rangle_{|\pi_{\ell-1}|}} \right] \prod_{j=1}^{|\pi_{\ell-1}|} c_{b_{\ell-1}(\mathbf{B}_j)}(\mathbf{1}) \\ & + q \beta_{1,n} \prod_{j=1}^{|\pi_{\ell-1}|} c_{b_{\ell-1}(\mathbf{B}_j)}(\mathbf{1}) \mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{s_{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \right] + n^{-\rho_6}, \end{aligned}$$

thanks to the assumption 3 and for  $\rho_6 > 0$ . Note (see the proof of Theorem 1.4) that

$$\mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{s_{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \right] = \mathcal{A}_{s_{\ell-1}-1}^{|\pi_{\ell-2}|} (f_{s_{\ell-2}, \Pi^{\ell-2}}^{\ell-2}, \beta^{\ell-2}) \prod_{j=1}^{|\pi_{\ell-2}|} c_{b_{\ell-2}(\mathbf{B}_j)}(\beta_j^{\ell-2}) \prod_{\substack{\mathfrak{B} \in \pi_{\ell-1} \\ |\mathfrak{B}| \geq 2}} e^{s_\ell^* \psi(|\mathfrak{B}|)},$$

with  $\beta_j^{\ell-2} = (\beta_{j,1}^{\ell-2}, \dots, \beta_{j, b_{\ell-2}(\mathbf{B}_j)}^{\ell-2})$  and  $s_\ell^* = s_\ell - s_{\ell-1} - 1$ . Since for any  $\mathfrak{B} \in \pi_{\ell-1}$  such that  $|\mathfrak{B}| \geq 2$ ,  $\psi(|\mathfrak{B}|) < 0$ , we have

$$\begin{aligned} \sum_{s_\ell = s_{\ell-1} + 1}^{a_n} \mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}}^{|\pi_{\ell-1}|} (f_{s_{\ell-1}, \Pi^{\ell-1}}^{\ell-1}, \beta^{\ell-1}) \right] & \leq \mathbf{E} \left[ \mathcal{A}_{s_{\ell-1}-1}^{|\pi_{\ell-2}|} (f_{s_{\ell-2}, \Pi^{\ell-2}}^{\ell-2}, \beta^{\ell-2}) \right] \prod_{\mathbf{B} \in \pi_{\ell-2}} c_{b_{\ell-2}(\mathbf{B})}(\beta^{\ell-1}) \\ & \left( 1 - \prod_{\substack{\mathfrak{B} \in \pi_{\ell-1} \\ |\mathfrak{B}| \geq 2}} e^{\psi(|\mathfrak{B}|)} \right)^{-1}. \end{aligned}$$

Doing the same for  $\mathbf{E} \left[ \sum_{\mathbf{z} \in \Delta_{s_{\ell-1}}^{|\pi_{\ell-1}|}} f_{s_{\ell-1}, \Pi^{\ell-1}}^{\ell-1}(\mathbf{z}) \mathbf{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-1}|} H_{z(j')} > h\}} e^{-\langle \beta^{\ell-1}, V(\mathbf{z}) \rangle_{|\pi_{\ell-1}|}} \right]$ , we obtain, thanks to assumption 3

$$\begin{aligned} & \sum_{\mathbf{s}; s_1 < \dots < s_\ell \leq a_n} \mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_{a_n}^q} f_{\mathbf{s}, \Pi}^\ell(\mathbf{x}) \mathbf{1}_{\{\max_{1 \leq i \leq q} H_{x(i)} > h\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \\ & \leq \sum_{s_1 < \dots < s_{\ell-1} \leq a_n} \mathbf{E} \left[ \sum_{\mathbf{z} \in \Delta_{s_{\ell-1}-1}^{|\pi_{\ell-2}|}} f_{s_{\ell-2}, \Pi^{\ell-2}}^{\ell-2}(\mathbf{z}) \mathbf{1}_{\{\max_{1 \leq j' \leq |\pi_{\ell-2}|} H_{z(j')} > h\}} e^{-\langle \beta^{\ell-2}, V(\mathbf{z}) \rangle_{|\pi_{\ell-2}|}} \right] \\ & + \frac{\mathfrak{C}_{3.3,5}}{h^{\kappa-1}} + n^{-\rho_7}, \end{aligned}$$

for some constant  $\mathfrak{C}_{3.3,5} > 0$  and  $\rho_7 > 0$ . We conclude by induction together with assumption 3.  $\square$

We remind the definition of the range  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, g)$

$$\mathcal{A}^k(\mathcal{D}_{n,T^s}, g) = \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} g(\mathbf{x}) \mathbb{1}_{\{T_{\mathbf{x}} < T^s\}},$$

where  $T_{\mathbf{x}} = \max_{1 \leq i \leq k} T_{x^{(i)}}$  and  $\ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n$  means that  $\ell_n \leq |x^{(i)}| \leq \mathfrak{L}_n$  for all  $i \in \{1, \dots, k\}$ . Vertices with high potential have a major contribution to the range. One can note that under the assumption 1, the potential  $V(u)$  of the vertex  $u \in \mathbb{T}$  behaves like  $|u|$  when  $|u|$  is large (see [Big76] and [HS09] for instance). It allows to say that

**Fact 1.** For all  $\varepsilon \in (0, 1)$ , there exists  $a_\varepsilon > 0$  such that

$$\mathbf{P}^* \left( \inf_{z \in \mathbb{T}} V(z) \geq -a_\varepsilon \right) \geq 1 - \varepsilon. \quad (\text{II.36})$$

Moreover,

**Lemma 3.4.** *Under the assumption 1, there exists  $\delta_0 > 0$  and  $\rho_1 > 1/2$  such that for any positive integer  $\zeta$*

$$\mathbf{P} \left( \min_{|z|=\delta_0^{-1}\zeta} V(z) \geq 3\zeta \right) \geq 1 - e^{-\rho_1\zeta},$$

Using Lemma 3.4, we are able to prove that any vertex  $x \in \mathbb{T}$  in a generation between  $\delta_0^{-1} \log n$  and  $n^{1/2}$  is visited during a single excursion above the parent  $e^*$  of the root  $e$ . For that, let us define the edge local time  $N_u^T := \sum_{j=1}^T \mathbb{1}_{\{X_{j-1}=u^*, X_j=u\}}$  of the vertex  $u \in \mathbb{T}$  and introduce

$$E_u^s := \sum_{j=1}^s \mathbb{1}_{\{N_u^{T^j} - N_u^{T^{j-1}} \geq 1\}},$$

the number of excursions during which the vertex  $x$  is visited by the random walk  $\mathbb{X}$ .

**Lemma 3.5.** *Under the assumption 1, for all  $\varepsilon_1 \in (0, 1)$ , there exists  $\rho_2 := \rho_2(\varepsilon_1) > 0$  such that for  $n$  large enough*

$$\mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \bigcup_{|z|=\delta_0^{-1} \log n}^{n^{1/2}} \{E_z^s \geq 2\} \right) \leq n^{-\rho_2}.$$

The proof of Lemma 3.5 is similar to the one of Lemma 3.5 in [AD20].

Introduce the set  $\mathfrak{S}^{k,s}$  of  $k$ -tuples of vertices visited during a single excursion :

$$\mathfrak{S}^{k,s} := \{\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k; \forall 1 \leq i \leq k, E_{x^{(i)}}^s = 1\}. \quad (\text{II.37})$$



In other words, Lemma 3.5 says that we can restrict the study of the range  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, f\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathcal{C}_{a_n}^k})$  to the set  $\mathfrak{S}^{k,s}$ . This restriction allows to get quasi-independence in the trajectory of the random walk  $\mathbb{X}$  and the resulting quasi-independent version of the range  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, f\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathcal{C}_{a_n}^k})$  is easier to deal with. A similar idea is developed in [AD20] and [AK21]. Let  $\mathbf{j} \in \llbracket 1, s \rrbracket_k$ ,  $\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}$  and define

$$\mathcal{A}_{\mathbf{p}}^{k,n}(\mathbf{j}, g) := \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} g(\mathbf{x}) \prod_{i=1}^k \mathbf{1}_{\{N_{x^{(i)}}^{T^{j_i}} - N_{x^{(i)}}^{T^{j_i-1}} \geq 1\}} \quad \text{and} \quad \mathcal{A}^{k,n}(\mathbf{j}, g) := \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \mathcal{A}_{\mathbf{p}}^{k,n}(\mathbf{j}, g), \quad (\text{II.38})$$

where for any  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)})$ ,  $|\mathbf{x}| = \mathbf{p}$  means nothing but  $|x^{(i)}| = p_i$  for all  $i \in \{1, \dots, k\}$ . In the next lemma, we show that  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, f\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathcal{C}^k})$  and  $\sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{C}^k})$  have the same behavior

**Lemma 3.6.** *Let  $k \geq 2$  be an integer and assume  $\kappa > 2k$ . Under the assumptions 1 and 3, for all bounded function  $g$ , any  $\varepsilon, \varepsilon_1 \in (0, 1)$ , there exists  $\rho_4 := \rho_4(\varepsilon, \varepsilon_1) > 0$  such that for  $n$  large enough and any subset  $\mathfrak{R}$  of  $\mathbb{T}^{\times k}$*

$$\mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left\{ \left| \mathcal{A}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{R}}) - \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, g\mathbf{1}_{\mathfrak{R}}) \right| > \varepsilon (s\mathbf{L}_n)^k \right\} \right) \leq n^{-\rho_4}.$$

*Démonstration.* We first decompose as follows

$$\mathcal{A}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{R}}) = \mathcal{A}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s} \cap \mathfrak{R}}) + \mathcal{A}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{R} \setminus \mathfrak{S}^{k,s}}).$$

We first deal with  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s} \cap \mathfrak{R}})$ . Note that  $\mathbf{x} \in \mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s}$  means nothing but there exists  $\mathbf{j} \in \llbracket 1, s \rrbracket_k$  such that for any  $i \in \{1, \dots, k\}$ ,  $N_{x^{(i)}}^{T^{j_i}} - N_{x^{(i)}}^{T^{j_i-1}} \geq 1$  and for all  $j \neq j_i$ ,  $N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} = 0$ , thus giving that  $\mathcal{A}_{\mathbf{p}}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s} \cap \mathfrak{R}})$  is equal to

$$\sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} g\mathbf{1}_{\mathfrak{R}}(\mathbf{x}) \prod_{i=1}^k \mathbf{1}_{\{N_{x^{(i)}}^{T^{j_i}} - N_{x^{(i)}}^{T^{j_i-1}} \geq 1; \forall j \neq j_i, N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} = 0\}},$$

Thanks to the strong Markov property, the random variables  $N_z^{T^l} - N_z^{T^{l-1}}$  are i.i.d under  $\mathbb{P}^{\mathcal{E}}$  and distributed as  $N_z^{T^1}$ . It follows that

$$\mathbb{E}^{\mathcal{E}} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} g\mathbf{1}_{\mathfrak{R}}(\mathbf{x}) \prod_{i=1}^k \mathbf{1}_{\{N_{x^{(i)}}^{T^{j_i}} - N_{x^{(i)}}^{T^{j_i-1}} \geq 1; \forall j \neq j_i, N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} = 0\}} \right]$$

$$= \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} g \mathbf{1}_{\mathfrak{R}}(\mathbf{x}) \prod_{i=1}^k \mathbb{P}^{\mathcal{E}}(T_{x^{(i)}} < T^1) (1 - \mathbb{P}^{\mathcal{E}}(T_{x^{(i)}} < T^1))^{s-1}.$$

Using (II.35),  $\mathbb{P}^{\mathcal{E}}(T_{x^{(i)}} < T^1) \leq e^{-V(x^{(i)})}$  and on  $\mathcal{V}_n = \{\min_{\delta_0^{-1} \log n \leq |x| \leq n^{1/2}} V(z) \geq 3 \log n\}$

$$(1 - \mathbb{P}^{\mathcal{E}}(T_{x^{(i)}} < T^1))^{s-1} \geq (1 - e^{-V(x^{(i)})})^s \geq (1 - n^{-3})^s \geq (1 - n^{-3})^{n^{1/2}/\varepsilon_1},$$

so

$$\begin{aligned} & \mathbb{E}^{\mathcal{E}} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} g \mathbf{1}_{\mathfrak{R}}(\mathbf{x}) \prod_{i=1}^k \mathbf{1}_{\{N_{x^{(i)}}^{T^{\mathbf{j}_i}} - N_{x^{(i)}}^{T^{\mathbf{j}_i-1}} \geq 1\}} \right] \\ & - \mathbb{E}^{\mathcal{E}} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} g \mathbf{1}_{\mathfrak{R}}(\mathbf{x}) \prod_{i=1}^k \mathbf{1}_{\{N_{x^{(i)}}^{T^{\mathbf{j}_i}} - N_{x^{(i)}}^{T^{\mathbf{j}_i-1}} \geq 1; \forall p \neq j_i, N_{x^{(i)}}^{T^p} - N_{x^{(i)}}^{T^{p-1}} = 0\}} \right] \\ & \leq s(s-1) \times \cdots \times (s-k+1) \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} g(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} (1 - (1 - n^{-3})^{kn^{1/2}/\varepsilon_1}). \end{aligned}$$

The next step is to show that the mean of the previous sum goes to 0 when  $n$  goes to  $\infty$ , uniformly in  $\mathbf{p}$ . For that, one can notice that

$$\mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} g(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} (1 - (1 - n^{-3})^{\frac{kn^{1/2}}{\varepsilon_1}}) \right] \leq \frac{2kn^{-5/2}}{\varepsilon_1} \|g\|_{\infty} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} \right],$$

thus giving, thanks to Lemma 3.3 (i) with  $\beta = \mathbf{1}$

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} g \mathbf{1}_{\mathfrak{R}}(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} (1 - (1 - n^{-3})^{\frac{n^{1/2}}{\varepsilon_1}}) \right] & \leq \frac{2kn^{-5/2}}{\varepsilon_1} \|g\|_{\infty} \sup_{\mathbf{p} \in (\mathbb{N}^*)^{\times k}} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta_p^k \\ |\mathbf{x}|=p}} e^{-\langle \beta, V(\mathbf{x}) \rangle_k} \right] \\ & \leq \mathfrak{C}_{3.6} n^{-5/2}, \end{aligned} \tag{II.39}$$

for some constant  $\mathfrak{C}_{3.6} > 0$  doesn't depending on  $\mathbf{p}$ . We then use (II.39) to conclude. Note that

$$\begin{aligned} & \mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left\{ \left| \mathcal{A}^k(\mathcal{D}_{n, T^s}, g \mathbf{1}_{\mathfrak{C}^{k, s} \cap \mathfrak{R}}) - \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k, n}(\mathbf{j}, g \mathbf{1}_{\mathfrak{R}}) \right| > \varepsilon (sL_n)^k \right\} \right) \\ & \leq \mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left\{ \left| \mathcal{A}^k(\mathcal{D}_{n, T^s}, g \mathbf{1}_{\mathfrak{C}^{k, s} \cap \mathfrak{R}}) - \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k, n}(\mathbf{j}, g \mathbf{1}_{\mathfrak{R}}) \right| > \varepsilon (sL_n)^k \right\}, \mathcal{V}_n \right) \\ & + 1 - \mathbf{P}^*(\mathcal{V}_n). \end{aligned}$$

By Lemma 3.4, for  $n$  large enough  $1 - \mathbf{P}^*(\mathcal{V}_n) \leq n^{-\rho'_1}$  for some  $\rho'_1 > 0$  and thanks to Markov inequality

$$\begin{aligned} & \mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left\{ \left| \mathcal{A}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^k, s \cap \mathfrak{R}}) - \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, g\mathbf{1}_{\mathfrak{R}}) \right| > \varepsilon(s\mathbf{L}_n)^k \right\}, \mathcal{V}_n \right) \\ & \leq \mathbb{P}^* \left( \bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \left\{ \mathcal{A}^k(\mathcal{D}_{n,T^s}, g\mathbf{1}_{\mathfrak{E}^k, s \cap \mathfrak{R} \cap \Delta^k \setminus \mathfrak{E}^k, s}) > \varepsilon(s\mathbf{L}_n)^k / 2 \right\} \right) \\ & + \sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \frac{2}{\varepsilon(s\mathbf{L}_n)^k} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=\mathbf{p}}} g\mathbf{1}_{\mathfrak{R}}(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} (1 - (1 - n^{-3})^{\frac{kn^{1/2}}{\varepsilon_1}}) \right], \end{aligned}$$

which, thanks to Lemma 3.5 and (II.39), is smaller than  $n^{-\rho_2} + \frac{2\mathfrak{C}_{3.6}}{n^2 \varepsilon_1 \varepsilon}$  for  $n$  large enough and the proof is completed.  $\square$

The next lemma relates  $\sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{E}_{a_n}^k})$  with its quenched mean and illustrates why this quasi-independent version of the range is easier to deal with.

**Lemma 3.7.** *Let  $k \geq 2$  and  $\mathfrak{a} \geq 1$  be two integers and assume  $\kappa > 2\mathfrak{a}k$ . Under the assumptions 1, 2 and 3, there exists a constant  $\mathfrak{C}_{3.7} > 0$  and a non-decreasing sequence of positive numbers  $(\tilde{\mathfrak{q}}_j)_{j \geq 2}$  satisfying  $\tilde{\mathfrak{q}}_2 = 1$  and  $\tilde{\mathfrak{q}}_j \rightarrow \infty$  when  $j \rightarrow \infty$  such that for  $n$  large enough and any  $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$*

$$\mathbb{E} \left[ \left( \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{E}_{a_n}^k}) - \mathbb{E}^\mathfrak{E} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{E}_{a_n}^k}) \right] \right)^{2\mathfrak{a}} \right] \leq \mathfrak{C}_{3.7}(\mathbf{L}_n)^{2\mathfrak{a}k} (\mathfrak{L}_n)^{\tilde{\mathfrak{q}}_{\mathfrak{a}}} s^{2\mathfrak{a}k - \tilde{\mathfrak{q}}_{\mathfrak{a}}}.$$

*Démonstration.* Recall the definition of  $\mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{E}_{a_n}^k})$  in (II.38). Thanks to the strong Markov property, the random variables  $N_z^{T^i} - N_z^{T^{i-1}}$  are i.i.d under  $\mathbb{P}^\mathfrak{E}$  and distributed as  $N_z^{T^1}$  for any  $z \in \mathbb{T}$ . It follows that

$$\begin{aligned} & \mathbb{E}^\mathfrak{E} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{E}_{a_n}^k}) \right] \\ & = s(s-1) \cdots (s-k+1) \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=\mathbf{p}}} f\mathbf{1}_{\mathcal{E}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \mathbb{P}^\mathfrak{E}(N_{x^{(i)}}^{T^1} \geq 1). \end{aligned}$$

and for  $\mathfrak{a} = 1$ , the term  $s^{2k}$  in the above mean is equal to 0 and using again the strong Markov property leads to

$$\begin{aligned} & \mathbb{E} \left[ \left( \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{E}_{a_n}^k}) - \mathbb{E}^\mathfrak{E} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f\mathbf{1}_{\mathcal{E}_{a_n}^k}) \right] \right)^2 \right] \\ & \leq \mathfrak{C}_{3.7,1}(\mathbf{L}_n)^{2k} ((\mathfrak{L}_n)^2 s^{2k-2} + \mathfrak{L}_n s^{2k-1}) \leq 2\mathfrak{C}_{3.7,1}(\mathbf{L}_n)^{2k} \mathfrak{L}_n s^{2k-1}, \end{aligned}$$

where the constant  $\mathfrak{C}_{3.7,1} > 0$  comes from Lemma 3.2 and the last inequality comes the fact that  $\mathfrak{L}_n \leq s$  for  $n$  large enough.

When  $\mathfrak{a} \geq 2$ , using similar arguments we have

$$\begin{aligned} & \mathbb{E} \left[ \left( \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbb{1}_{\mathcal{C}_{a_n}^k}) - \mathbb{E}^\rho \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbb{1}_{\mathcal{C}_{a_n}^k}) \right] \right)^{2\mathfrak{a}} \right] \\ & \leq \mathfrak{C}_{3.7,2} \leq (\mathbf{L}_n)^{2\mathfrak{a}k} (\mathfrak{L}_n)^{2\lceil \mathfrak{a}/2 \rceil} s^{2\mathfrak{a}k - 2\lceil \mathfrak{a}/2 \rceil}. \end{aligned}$$

We finally obtain the result by taking  $\mathfrak{q}_\mathfrak{a} := \mathfrak{a} \mathbb{1}_{\mathfrak{a}=1} + 2\lceil \mathfrak{a}/2 \rceil \mathbb{1}_{\mathfrak{a} \geq 2}$ .  $\square$

### Convergence of the quenched mean of the range on $\mathcal{C}_{a_n}^k$

We prove that the quenched mean of the quasi-independent version  $\sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^k(\mathbf{j}, f \mathcal{C}_{a_n}^k)$  of the range on the set  $\mathcal{C}_{a_n}^k$  converges in  $\mathbf{P}^*$ -probability by using the hereditary assumption 5.

**Lemma 3.8.** *Let  $k \geq 2$  be an integer and assume  $\kappa > 2k$ . Under the assumptions 1, 3 and 4, if  $f$  satisfies the hereditary assumption 5 then*

$$\lim_{n \rightarrow \infty} \mathbf{E}^* \left[ \left| \frac{1}{(\mathbf{L}_n)^k} \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} f \mathbb{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} - (c_\infty)^k \mathcal{A}_{a_n}^k(f) \right| \right] = 0$$

*Démonstration.* The first step is to decompose  $\sum_{\mathbf{x} \in \Delta^k; \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n} f \mathbb{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k e^{-V(x^{(i)})} / H_{x^{(i)}} :$

$$\begin{aligned} \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} f \mathbb{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} &= \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\mathbf{z} \in \Delta_{a_n}^k} \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}; x^{(i)} > z^{(i)}}} f \mathbb{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \\ &= \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\mathbf{z} \in \Delta_{a_n}^k} f(\mathbf{z}) \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}; x^{(i)} > z^{(i)}}} \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}}, \end{aligned}$$

where the last equality comes from the hereditary assumption 5. As we did above, we decompose  $H_{x^{(i)}} : H_{x^{(i)}} = H_{z^{(i)}} e^{-V_{z^{(i)}}(x^{(i)})} + H_{z^{(i)}, x^{(i)}}$ . By independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$

$$\mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} f \mathbb{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \middle| \mathcal{F}_{a_n} \right] = \sum_{\mathbf{z} \in \Delta_{a_n}^k} f(\mathbf{z}) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n, p_i}(H_{z^{(i)}}), \quad (\text{II.40})$$

where  $\mathcal{F}_{a_n} = \sigma(\mathbb{T}, (V(x); |x| \leq a_n))$  and  $\varphi_{n, p_i}(r) = \mathbf{E}[\sum_{|x|=p_i - a_n} e^{-V(x)} / (r e^{-V(x)} + H_x)]$ .

Let us prove that

$$\lim_{n \rightarrow \infty} \sup_{\ell_n \leq \mathbf{p} \leq \mathcal{L}_n} \mathbf{E} \left[ \left( \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} - \sum_{\mathbf{x} \in \Delta_{a_n}^k} f(\mathbf{x}) \prod_{i=1}^k e^{-V(x^{(i)})} \varphi_{n, p_i}(H_{x^{(i)}}) \right)^2 \right] = 0. \quad (\text{II.41})$$

Thanks to (II.40)

$$\begin{aligned} & \mathbf{E} \left[ \left( \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} - \sum_{\mathbf{x} \in \Delta_{a_n}^k} f(\mathbf{x}) \prod_{i=1}^k e^{-V(x^{(i)})} \varphi_{n, p_i}(H_{x^{(i)}}) \right)^2 \right] \\ &= \mathbf{E} \left[ \left( \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \right)^2 \right] - \mathbf{E} \left[ \left( \sum_{\mathbf{x} \in \Delta_{a_n}^k} f(\mathbf{x}) \prod_{i=1}^k e^{-V(x^{(i)})} \varphi_{n, p_i}(H_{x^{(i)}}) \right)^2 \right]. \end{aligned}$$

For  $\mathbf{x}, \mathbf{y} \in \Delta^k$ , denote by  $\mathbf{xy} = (x^{(1)}, \dots, x^{(k)}, y^{(1)}, \dots, y^{(k)})$  the concatenation of  $\mathbf{x}$  and  $\mathbf{y}$ . Note that

$$\begin{aligned} \left( \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \right)^2 &= \sum_{\substack{\mathbf{x}, \mathbf{y}; \mathbf{xy} \in \Delta^{2k} \\ |\mathbf{x}| = |\mathbf{y}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}} \\ &+ \sum_{\substack{\mathbf{x}, \mathbf{y} \in \Delta^k; \mathbf{xy} \notin \Delta^{2k} \\ |\mathbf{x}| = |\mathbf{y}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}}, \end{aligned}$$

where for any  $\mathbf{x}, \mathbf{y} \in \Delta^k$ ,  $\mathbf{xy} \notin \Delta^{2k}$  means that there exists  $\alpha \in \{1, \dots, k\}$  and  $i_1, \dots, i_\alpha \in \{1, \dots, k\}$  distinct such that  $x^{(i_j)} = y^{(i_j)}$  for all  $j \in \{1, \dots, \alpha\}$ . It follows

$$\lim_{n \rightarrow \infty} \sup_{\ell_n \leq \mathbf{p} \leq \mathcal{L}_n} \mathbf{E} \left[ \sum_{\substack{\mathbf{x}, \mathbf{y} \in \Delta^k; \mathbf{xy} \notin \Delta^{2k} \\ |\mathbf{x}| = |\mathbf{y}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}} \right] = 0,$$

Indeed, by independence of the increments of the branching random walk  $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ , since  $H_x \geq 1$

$$\begin{aligned} & \mathbf{E} \left[ \sum_{\substack{\mathbf{x}, \mathbf{y} \in \Delta^k; \mathbf{xy} \notin \Delta^{2k} \\ |\mathbf{x}| = |\mathbf{y}| = \mathbf{p}}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}} \right] \\ & \leq \|f\|_\infty^2 \sum_{\alpha=1}^k \sum_{i_1 \neq i_2 \dots \neq i_\alpha=1}^k \prod_{j=1}^\alpha e^{(p_{i_j} - a_n) \psi(2)} \mathbf{E} \left[ \sum_{\mathbf{z} \in \Delta_{a_n}^{2k-\alpha}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_{2k-\alpha}} \right]. \end{aligned}$$

By Lemma 3.3,  $\sum_{n \in \mathbb{N}} \mathbf{E}[\sum_{z \in \Delta_{a_n}^{2k-\alpha}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_{2k-\alpha}}] < \infty$  and for any  $p_i \in \{\ell_n, \dots, \mathfrak{L}_n\}$ ,  $i \in \{1, \dots, k\}$ ,  $e^{(p_i - a_n)\psi(2)} \leq e^{(\ell_n - a_n)\psi(2)} \rightarrow 0$  when  $n \rightarrow \infty$  since  $\ell_n \geq \delta_0^{-1} \log n$  and  $a_n = (2\delta_0)^{-1} \log n$ , which gives (II.41). We now prove that

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \left| (c_\infty)^k \sum_{z \in \Delta_{a_n}^k} f(z) \prod_{i=1}^k e^{-V(z^{(i)})} - \sum_{z \in \Delta_{a_n}^k} f(z) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n, p_i}(H_{z^{(i)}}) \right| \right] = 0. \quad (\text{II.42})$$

Let  $h_n = \log n$  (the choice of  $h_n$  is almost arbitrary,  $h_n \rightarrow \infty$  with  $h_n = o(n^\theta)$  for all  $\theta > 0$  should be enough). Note that  $|c_\infty - \varphi_{n, p_i}(H_{z^{(i)}})| \leq 2$  so

$$\begin{aligned} & \mathbf{E} \left[ \left| (c_\infty)^k \sum_{z \in \Delta_{a_n}^k} f(z) \prod_{i=1}^k e^{-V(z^{(i)})} - \sum_{z \in \Delta_{a_n}^k} f(z) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n, p_i}(H_{z^{(i)}}) \right| \right] \\ & \leq \|f\|_\infty \mathbf{E} \left[ \sum_{z \in \Delta_{a_n}^k} \prod_{i=1}^k e^{-V(z^{(i)})} |c_\infty - \varphi_{n, p_i}(H_{z^{(i)}})| \mathbf{1}_{\{H_{z^{(i)}} \leq h_n\}} \right] \\ & \quad + 2\|f\|_\infty \mathbf{E} \left[ \sum_{z \in \Delta_{a_n}^k} \mathbf{1}_{\{\max_{1 \leq i \leq k} H_{z^{(i)}} > h_n\}} e^{-\langle \mathbf{1}, V(\mathbf{z}) \rangle_k} \right]. \end{aligned}$$

We show that  $\lim_{n \rightarrow \infty} \sup_{\{\ell_n \leq p_i \leq \mathfrak{L}_n, 1 \leq r \leq h_n\}} |\varphi_{n, p_i}(r) - c_\infty| = 0$ . For that, on the first hand,  $\varphi_{n, p_i}(r) \leq \mathbf{E}[1/H_{\ell_n - a_n}^S]$  where we recall that  $H_m^S = \sum_{j=1}^m e^{S_j - S_m}$  (see (II.22) for the definition of the random walk  $S$ ). On the other, for any  $\ell_n \leq p_i \leq \mathfrak{L}_n$  and  $1 \leq r \leq h_n$ ,  $\varphi_{n, p_i}(r)$  is larger, for any  $r_1 > 0$ , than

$$\mathbf{E} \left[ \sum_{|x|=p_i - a_n} \frac{e^{-V(x)}}{h_n e^{-V(x)} + H_x} \mathbf{1}_{\{V(x) \geq r_1 \log n\}} \right] \geq \mathbf{E} \left[ \frac{1}{h_n n^{-r_1} + H_{\mathfrak{L}_n - a_n}^S} \right] - \mathbf{P}(S_{p_i - a_n} < r_1 \log n).$$

where we have used the many-to-one Lemma 3.1.

Note that  $\mathbf{P}(S_{p_i - a_n} < 3 \log n) \leq \mathbf{P}(\min_{(2\delta_0)^{-1} \log n \leq j \leq \mathfrak{L}_n} S_j < r_1 \log n) \rightarrow 0$  when  $n \rightarrow \infty$  for some  $r_1 > 0$  since  $a_n = (2\delta_0)^{-1} \log n$  and  $\psi'(1) < 0$ . Moreover, by definition, both  $(\mathbf{E}[1/H_{\ell_n - a_n}^S])$  and  $(\mathbf{E}[1/(h_n n^{-r_1} + H_{\mathfrak{L}_n - a_n}^S)])$  goes to  $c_\infty$  when  $n$  goes to  $\infty$  and we obtain the convergence. Then

$$\begin{aligned} & \mathbf{E} \left[ \left| (c_\infty)^k \sum_{z \in \Delta_{a_n}^k} f(z) \prod_{i=1}^k e^{-V(z^{(i)})} - \sum_{z \in \Delta_{a_n}^k} f(z) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n, p_i}(H_{z^{(i)}}) \right| \right] \\ & \leq \|f\|_\infty \mathbf{E} \left[ \sum_{z \in \Delta_{a_n}^k} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} \right] \max_{1 \leq i \leq k} \sup_{\ell_n \leq p_i \leq \mathfrak{L}_n, 1 \leq r \leq h_n} |\varphi_{n, p_i}(r) - c_\infty|^k \\ & \quad + 2\|f\|_\infty \mathbf{E} \left[ \sum_{z \in \Delta_{a_n}^k} \mathbf{1}_{\{\max_{1 \leq i \leq k} H_{z^{(i)}} > h_n\}} e^{-\langle \mathbf{1}, V(\mathbf{z}) \rangle_k} \right]. \end{aligned}$$

Using Lemma 3.3, first (i), then (ii) with  $h = h_n$ ,  $\sup_{n \in \mathbb{N}} \mathbf{E}[\sum_{z \in \Delta_{a_n}^k} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k}] < \infty$  and

$\lim_{n \rightarrow \infty} \mathbf{E}[\sum_{\mathbf{z} \in \Delta_{a_n}^k} \mathbb{1}_{\{\max_{1 \leq i \leq k} H_{z^{(i)}} > h_n\}} e^{-\langle \mathbf{1}, V(\mathbf{z}) \rangle_k}] = 0$  thus giving (II.42).

Finally, putting together (II.41) and (II.42), we obtain the result.  $\square$

### Convergence of the quasi-martingale $\mathcal{A}_l^k(f)$

Recall that

$$\mathcal{A}_l^k(f, \beta) = \sum_{\mathbf{x} \in \Delta_l^k} f(\mathbf{x}) e^{-\langle \beta, V(\mathbf{x}) \rangle_k} = \sum_{\mathbf{x} \in \Delta_l^k} f(\mathbf{x}) \prod_{i=1}^k e^{-\beta_i V(x^{(i)})} \quad \text{and} \quad \mathcal{A}_l^k(f) = \mathcal{A}_l^k(f, \mathbf{1}).$$

The aim of this subsection is to prove that  $\mathcal{A}_\infty^k := \lim_{l \rightarrow \infty} \mathcal{A}_l^k(f)$  exists when  $f$  satisfies our hereditary assumption 5. For that, let us define for any  $\mathbf{p} \in (\mathbb{N}^*)^{\times k}$

$$\mathcal{A}_{\mathbf{p}}^k(f) := \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} f(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k},$$

where we recall that for any  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ ,  $|\mathbf{x}| = \mathbf{p}$  if and only if  $|x^{(i)}| = p_i$  for all  $i \in \{1, \dots, k\}$ . One can notice that when  $\mathbf{p} = (l, \dots, l) \in (\mathbb{N}^*)^{\times k}$ , we have  $\mathcal{A}_{\mathbf{p}}^k(f) = \mathcal{A}_l^k(f)$ .

**Lemma 3.9.** *Let  $k \geq 2$  be an integer and assume  $\kappa > 2k$ . Under the assumptions 1, 3 and 4, for any bounded function  $f : \Delta^k \rightarrow \mathbb{R}^+$ , there exists two constants  $\mathfrak{C}_{3.9} > 0$  and  $\mathfrak{b} \in (0, 1)$  such that for any  $\mathbf{p} \in (\mathbb{N}^*)^{\times k}$  and any integer  $m \geq 1$  such that  $m \leq \max \mathbf{p} := \max_{1 \leq i \leq k} p_i$*

$$\mathbf{E}^* [|\mathcal{A}_{\mathbf{p}}^k(f \mathbb{1}_{\mathcal{E}_m^k}) - \mathcal{A}_{\mathbf{p}}^k(f)|^2] \leq \mathfrak{C}_{3.9} e^{-\mathfrak{b}m}.$$

*Démonstration.* In order to avoid unnecessary technical difficulties, we prove it for any  $\kappa > 4$ . First note that  $\mathcal{A}_{\mathbf{p}}^k(f) - \mathcal{A}_{\mathbf{p}}^k(f \mathbb{1}_{\mathcal{E}_m^k}) = \sum_{\mathbf{x} \in \Delta^k; |\mathbf{x}| = \mathbf{p}} f(\mathbf{x}) \mathbb{1}_{\{\mathcal{I}^k(\mathbf{x}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k}$  which is smaller than  $\|f\|_\infty \sum_{\mathbf{x} \in \Delta^k; |\mathbf{x}| = \mathbf{p}} \mathbb{1}_{\{\mathcal{I}^k(\mathbf{x}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k}$ . Using a similar argument as we developed in the proof of Lemma 3.8, it is enough to show the following estimation :

$$\mathbf{E}^* \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{\mathcal{I}^q(\mathbf{x}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \leq \mathfrak{C}_{3.9,1} e^{-\mathfrak{b}m}, \quad (\text{II.43})$$

for any  $q \in \{k, \dots, 2k\}$  and some constant  $\mathfrak{C}_{3.9,1} > 0$ . Assume that  $\min \mathbf{p} < \max \mathbf{p}$  (the proof is similar when  $\min \mathbf{p} = \max \mathbf{p}$ ). Note that if  $m < \min \mathbf{p}$ , then

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{\mathcal{I}^q(\mathbf{x}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] &= \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{m < \mathcal{I}^q(\mathbf{x}) \leq \min \mathbf{p}\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \\ &\quad + \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{\mathcal{I}^q(\mathbf{x}) > \min \mathbf{p}\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right]. \end{aligned}$$

One can notice that, if  $|\mathbf{x}| = \mathbf{p}$  and  $\mathcal{S}^q(\mathbf{x}) \leq \min \mathbf{p}$ , then  $\mathcal{S}^q(\mathbf{x}) = \mathcal{S}^q(\mathbf{u})$  for any  $\mathbf{u} \in \Delta^q$  such that  $\max |\mathbf{u}| = \min |\mathbf{u}| = \min \mathbf{p}$ . Hence, as usual

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{m < \mathcal{S}^q(\mathbf{x}) \leq \min \mathbf{p}\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] &= \mathbf{E} \left[ \sum_{\mathbf{u} \in \Delta_{\min \mathbf{p}}^q} \mathbb{1}_{\{\mathcal{S}^q(\mathbf{u}) > m\}} \sum_{\mathbf{x} \in \Delta^q; \mathbf{x} \geq \mathbf{u}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \\ &= \mathbf{E} \left[ \sum_{\mathbf{u} \in \Delta_{\min \mathbf{p}}^q} \mathbb{1}_{\{\mathcal{S}^q(\mathbf{u}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{u}) \rangle_q} \right], \end{aligned}$$

thus giving

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{\mathcal{S}^q(\mathbf{x}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] &= \mathbf{E} \left[ \sum_{\mathbf{u} \in \Delta_{\min \mathbf{p}}^q} \mathbb{1}_{\{\mathcal{S}^q(\mathbf{u}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{u}) \rangle_q} \right] \\ &\quad + \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{\mathcal{S}^q(\mathbf{x}) > \min \mathbf{p}\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right]. \end{aligned}$$

We deduce from this equality that it is enough to prove (II.43) for any  $m \leq \min \mathbf{p}$  with  $q \geq 3$ . Again, we focus on the case  $\min \mathbf{p} < \max \mathbf{p}$ .

Assume  $m \leq \min \mathbf{p}$ . Let  $\mathbf{x} \in \Delta^q$  such that  $|\mathbf{x}| = \mathbf{p}$  and  $\mathcal{C}^q(\mathbf{x}) > m$ . There exists an integer  $\mathfrak{f} \in \{m+1, \dots, \max \mathbf{p}\}$  such that, seen backwards in time, at least two vertices among  $x^{(1)}, \dots, x^{(q)}$  share a common ancestor for the first times in the generation  $\mathfrak{f}-1$  and there exists at least one vertex among these vertices in a generation smaller or equal to  $\mathfrak{f}-1$ . Then, one can notice that

$$\sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{\mathcal{S}^q(\mathbf{x}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} = \sum_{\mathfrak{f}=m+1}^{\max \mathbf{p}} \sum_{\pi \text{ partition of } \mathbf{x} \in \Delta^q} \sum_{\substack{\{1, \dots, q\}, |\pi| < q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\Upsilon_{\mathfrak{f}-1, \pi} \cap \Upsilon_{\mathfrak{f}, \eta}(\mathbf{x})} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q},$$

where  $\eta = \{\{1\}, \dots, \{q\}\}$  (recall the definition of  $\Upsilon_{p-1, \pi} \cap \Upsilon_{p, \eta}$  in (II.14)).

By definition, there exists  $\eta \in \{1, \dots, q-2\}$  and  $(i_1, \dots, i_\eta, i_{\eta+1}, \dots, i_q) \in \llbracket 1, q \rrbracket_q$  such that  $\max_{1 \leq l \leq \eta} p_{i_l} \leq \mathfrak{f}-1$  and  $\min_{\eta+1 \leq l \leq q} p_{i_l} \geq \mathfrak{f}-1$ . By definition of the set  $\Upsilon_{\cdot, \cdot}$ , for all  $l \in \{1, \dots, \eta\}$ , if  $i_l$  belongs to the block  $\mathbf{B}$  of the partition  $\pi$ , then  $\mathbf{B} = \{i_l\}$ . Let  $\bar{\pi} := \pi \setminus \{\{i_1\}, \dots, \{i_\eta\}\}$  and for all  $j \in \{1, \dots, |\pi| - \eta\}$ , denote by  $\bar{\mathbf{B}}_j$  the  $j$ -th block (ordered by their least element) of the partition  $\bar{\pi}$  of the set  $\{i_{\eta+1}, \dots, i_q\} = \{1, \dots, q\} \setminus \{i_1, \dots, i_\eta\}$ . We have

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\Upsilon_{\mathfrak{f}-1, \pi} \cap \Upsilon_{\mathfrak{f}, \eta}(\mathbf{x})} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \middle| \mathcal{F}_{\mathfrak{f}} \right] &= \sum_{\substack{\mathbf{u} \in \Delta^{\eta} \\ |\mathbf{u}| = \mathbf{p}}} e^{-\langle \mathbf{1}, V(\mathbf{u}) \rangle_\eta} \sum_{\mathbf{z} \in \Delta_{\mathfrak{f}-1}^{|\pi| - \eta}} \prod_{j=1}^{|\pi| - \eta} \sum_{\mathbf{v}^{(j)} \in \Delta_{\mathfrak{f}}^{|\bar{\mathbf{B}}_j|}} \prod_{i=1}^{|\bar{\mathbf{B}}_j|} \\ &\quad \times \mathbb{1}_{\{(v^{(j,i)})^* = z^{(j)}\}} e^{-V(v^{(j,i)})}, \end{aligned}$$



where  $|\mathbf{u}| = \mathbf{p}$  means that  $u^{(l)} = p_{i_l}$  for all  $l \in \{1, \dots, \eta\}$ ,  $\mathbf{v}^{(j)} = (v^{(j,1)}, \dots, v^{(j,|\bar{B}_j|)})$ . Thus

$$\begin{aligned} \mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\Upsilon_{f-1, \pi} \cap \Upsilon_{f, \eta}}(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \Big| \mathcal{F}_{f-1} \right] &= \sum_{\substack{\mathbf{u} \in \Delta^\eta \\ |\mathbf{u}| = \mathbf{p}}} e^{-\langle \mathbf{1}, V(\mathbf{u}) \rangle_\eta} \sum_{z \in \Delta_{f-1}^{|\pi| - \eta}} e^{-\langle \bar{\beta}, V(z) \rangle_{|\pi| - \eta}} \prod_{\mathbf{B} \in \bar{\pi}} c_{\mathbf{B}}(\mathbf{1}) \\ &= \prod_{\mathbf{B} \in \bar{\pi}} c_{\mathbf{B}}(\mathbf{1}) \sum_{\substack{\mathbf{u} \in \Delta^{|\pi|} \\ |\mathbf{u}| = \tilde{\mathbf{p}}}} e^{-\langle \tilde{\beta}, V(\mathbf{u}) \rangle_{|\pi|}}, \end{aligned}$$

where  $\tilde{\mathbf{p}} = (p_1, \dots, p_\eta, f-1, \dots, f-1) \in (\mathbb{N}^*)^{\times |\pi|}$  and  $\tilde{\beta} = (1, \dots, 1, \bar{B}_1, \dots, \bar{B}_{|\pi| - \eta}) \in (\mathbb{N}^*)^{\times |\pi|}$ .

One can notice that there exists  $r_0 > 0$  such that

$$\mathbf{E} \left[ \sum_{\substack{\mathbf{u} \in \Delta^{|\pi|} \\ |\mathbf{u}| = \tilde{\mathbf{p}}}} e^{-\langle \tilde{\beta}, V(\mathbf{u}) \rangle_{|\pi|}} \mathbb{1}_{\{\min_{|w|=f-1} V(w) < r_0(f-1)\}} \right] \leq \mathfrak{C}_{3.9,2} e^{-(f-1)}, \quad (\text{II.44})$$

for some constant  $\mathfrak{C}_{3.9,2} > 0$ . Indeed, By the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbf{E} \left[ \left( \sum_{\substack{\mathbf{u} \in \Delta^{|\pi|} \\ |\mathbf{u}| = \tilde{\mathbf{p}}}} e^{-\langle \mathbf{1}, V(\mathbf{u}) \rangle_{|\pi|}} \mathbb{1}_{\{\min_{|w|=f-1} V(w) < r_0(f-1)\}} \right)^2 \right] &\leq \mathbf{E} \left[ \sum_{\substack{\mathbf{u} \in \Delta^{|\pi|} \\ |\mathbf{u}| = \tilde{\mathbf{p}}}} e^{-2\langle \mathbf{1}, V(\mathbf{u}) \rangle_{|\pi|}} \right]^{1/2} \\ &\times \mathbf{P} \left( \min_{|w|=f-1} V(w) < r_0(f-1) \right)^{1/2}, \end{aligned}$$

and thanks to Lemma 3.3 (i),  $\mathbf{E}[(\sum_{\mathbf{u} \in \Delta^{|\pi|}; |\mathbf{u}| = \tilde{\mathbf{p}}} e^{-\langle \mathbf{1}, V(\mathbf{u}) \rangle_{|\pi|}})^2] \leq \mathfrak{C}_{3.3,1}$ , where we recall that  $\mathfrak{C}_{3.3,1} > 0$  is a constant doesn't depending on  $\mathbf{p}$  (or  $\tilde{\mathbf{p}}$ ) since  $|\pi| < q \leq 2k$ . Moreover, since  $\psi'(1) < 0$ , we can find  $r_0 > 0$  and a constant  $\mathfrak{C}_{3.9,3} > 0$  such that  $\mathbf{P}(\min_{|w|=f-1} V(w) < r_0(f-1)) \leq \mathfrak{C}_{3.9,3} e^{2(f-1)}$ . This yields (II.44).

Now, note that, since  $|\pi| < q$ , there is at least one block of the partition  $\pi$  with cardinal larger or equal to 2 so  $\langle \tilde{\beta}, V(z) \rangle_{|\pi|} \geq \langle \mathbf{1}, V(z) \rangle_{|\pi|} + \min_{|w|=f-1} V(w)$  thus giving that the mean  $\mathbf{E}[\sum_{\mathbf{x} \in \Delta^q; |\mathbf{x}| = \mathbf{p}} \mathbb{1}_{\Upsilon_{f-1, \pi} \cap \Upsilon_{f, \eta}}(\mathbf{x}) e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q}]$  is smaller than

$$\begin{aligned} &\prod_{\mathbf{B} \in \bar{\pi}} c_{\mathbf{B}}(\mathbf{1}) \left( \mathbf{E} \left[ \left( \sum_{\substack{\mathbf{u} \in \Delta^{|\pi|} \\ |\mathbf{u}| = \tilde{\mathbf{p}}}} e^{-\langle \tilde{\beta}, V(\mathbf{u}) \rangle_{|\pi|}} \mathbb{1}_{\{\min_{|w|=f-1} V(w) < r_0(f-1)\}} \right)^2 \right] \right. \\ &\left. + \mathbf{E} \left[ e^{-\min_{|w|=f-1} V(w)} \sum_{\substack{\mathbf{u} \in \Delta^{|\pi|} \\ |\mathbf{u}| = \tilde{\mathbf{p}}}} e^{-\langle \mathbf{1}, V(\mathbf{u}) \rangle_{|\pi|}} \mathbb{1}_{\{\min_{|w|=f-1} V(w) \geq r_0(f-1)\}} \right] \right), \end{aligned}$$

which, thanks to Lemma 3.3 (i) and (II.44), is smaller than  $\mathfrak{C}_{3.9,4} e^{-(1 \wedge r_0)(f-1)}$  for some constant  $\mathfrak{C}_{3.9,4} > 0$ . Finally

$$\mathbf{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^q \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{\mathcal{S}^q(\mathbf{x}) > m\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_q} \right] \leq \mathfrak{C}_{3.9,5} \sum_{f=m+1}^{\max \mathbf{p}} e^{-(1 \wedge r_0)(f-1)} \leq \mathfrak{C}_{3.9,1} e^{-(1 \wedge r_0)m},$$

for some constant  $\mathfrak{C}_{3.9,5} > 0$  and (II.43) is proved.  $\square$

The convergence of the sequence of random variables  $(\mathcal{A}_l^k(f))_l$  directly follows from Lemma 3.9. Indeed, let  $f$  be a bounded function satisfying the hereditary assumption 5. For any  $l > l' > m > \mathfrak{g}$

$$\mathcal{A}_l^k(f\mathbf{1}_{\mathcal{C}_m^k}) = \sum_{x \in \Delta_l^k} f\mathbf{1}_{\mathcal{C}_m^k}(\mathbf{x})e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} = \sum_{u \in \Delta_{l'}^k} f\mathbf{1}_{\mathcal{C}_m^k}(\mathbf{u}) \sum_{\substack{x \in \Delta_l^k \\ \mathbf{x} > \mathbf{u}}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k},$$

so  $\mathbf{E}[\mathcal{A}_l^k(f\mathbf{1}_{\mathcal{C}_m^k}) | \mathcal{F}_{l'}] = \mathcal{A}_{l'}^k(f\mathbf{1}_{\mathcal{C}_m^k})$  where  $\mathcal{F}_m = \sigma(\mathbb{T}, (V(x); |x| \leq m))$  and  $(\mathcal{A}_l^k(f\mathbf{1}_{\mathcal{C}_m^k}))_{l > m}$  is a martingale bounded in  $L^2(\mathbf{P})$ . In particular, for any integer  $m > \mathfrak{g}$ ,  $(\mathcal{A}_l^k(f\mathbf{1}_{\mathcal{C}_m^k}))_{l > m}$  converges in  $L^2(\mathbf{P}^*)$  and  $\mathbf{P}^*$ -almost surely. Hence, thanks to Lemma 3.9,  $(\mathcal{A}_l(f))_l$  is a Cauchy sequence in  $L^2(\mathbf{P}^*)$  and therefore,  $\mathcal{A}_\infty^k(f)$  exists.

### $k$ -tuples in the set $\Delta^k \setminus \mathcal{C}_{a_n}^k$

Before proving Proposition 2, let us show that the contribution of the  $k$ -tuples in the set  $\mathfrak{C}^{k, \cdot} \cap \Delta^k \setminus \mathcal{C}_{a_n}^k$  is not significant. To do that, the following lemma provides an estimation for the quasi-independent version (II.38) of the range on the set  $\mathcal{C}_{a_n}^k$ :

**Lemma 3.10.** *Let  $\varepsilon_1 \in (0, 1)$ ,  $k \geq 2$  and assume  $\kappa > 2k$ . Under the assumptions 1, 3 and 4, there exist two constants  $C_{3.10} > 0$  and  $\rho_8 > 0$  such that*

$$\mathbf{E}^* \left[ \frac{1}{(n^{1/2} \mathbf{L}_n)^k} \sum_{j \in [1, \mathfrak{s}_n]_k} \mathcal{A}^{k,n}(j, \mathbf{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k}) \right] \leq C_{3.10} n^{-\rho_8}, \quad (\text{II.45})$$

with  $\mathfrak{s}_n = n^{1/2}/\varepsilon_1$ .

*Démonstration.* Recall that, thanks to the strong Markov property together with (II.35)

$$\mathbf{E}^\mathcal{E} \left[ \mathcal{A}^{k,n}(j, \mathbf{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k}) \right] = \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} \mathbf{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}},$$

and since  $H_{x^{(i)}} \geq 1$

$$\begin{aligned} \mathbf{E}^* \left[ \sum_{j \in [1, \mathfrak{s}_n]_k} \mathcal{A}^{k,n}(j, \mathbf{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k}) \right] &\leq (\mathfrak{s}_n)^k \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \mathbf{E}^* \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} \mathbf{1}_{\{\mathcal{A}^k(\mathbf{x}) > a_n\}} e^{-\langle \mathbf{1}, V(\mathbf{x}) \rangle_k} \right] \\ &\leq (\mathfrak{s}_n \mathbf{L}_n)^k \mathfrak{C}_{3.9} e^{-ba_n} = (\mathfrak{s}_n \mathbf{L}_n)^k \mathfrak{C}_{3.9} n^{-b(2\delta_0)^{-1}}, \end{aligned}$$

which ends the proof.  $\square$

We are now ready to prove Proposition 2.

*Proof of Proposition 2.* We have to prove that for any  $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$ ,  $\varepsilon_1 \in (0, 1)$

$$\mathbb{P}^* \left( \left| \frac{1}{(s\mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{C}^{k,s} \cap \mathcal{C}_{a_n}^k}) - (c_\infty)^k \mathcal{A}_\infty^k(f) \right| > \varepsilon/2 \right) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{II.46})$$

We deduce from Lemma 3.7 with  $\mathfrak{a} = 1$  that the range  $\sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k})$  concentrates around its quenched mean. Indeed, for any  $\varepsilon \in (0, 1)$ , by Markov inequality

$$\begin{aligned} & \mathbb{P} \left( \left| \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) - \mathbb{E}^\mathcal{E} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) \right] \right| > \varepsilon (s\mathbf{L}_n)^k / 16 \right) \\ & \leq \frac{16^2}{\varepsilon^1 (s\mathbf{L}_n)^{2k}} \mathbb{E} \left[ \left( \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) - \mathbb{E}^\mathcal{E} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) \right] \right)^2 \right] \\ & \leq 16^2 \mathfrak{C}_{3.7} \frac{\mathfrak{L}_n}{s} \leq \mathfrak{C}_2 \frac{\mathfrak{L}_n}{n^{1/2}} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where the last inequality comes from the fact that  $\mathfrak{L}_n = o(n^{1/2})$ . Then, we know, thanks to Lemma 3.6 with  $g = f$  and  $\mathfrak{R} = \mathcal{C}_{a_n}^k$ , that  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{C}^{k,s} \cap \mathcal{C}_{a_n}^k})$  behaves like its quasi-independent version  $\sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k})$ : for  $n$  large enough

$$\mathbb{P}^* \left( \left| \mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{C}^{k,s} \cap \mathcal{C}_{a_n}^k}) - \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) \right| > \varepsilon (s\mathbf{L}_n)^k / 16 \right) \leq n^{-\rho_4},$$

hence

$$\lim_{n \rightarrow \infty} \mathbb{P}^* \left( \left| \mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{C}^{k,s} \cap \mathcal{C}_{a_n}^k}) - \mathbb{E}^\mathcal{E} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) \right] \right| > \varepsilon (s\mathbf{L}_n)^k / 8 \right) = 0.$$

One can notice that

$$\mathbb{E}^\mathcal{E} \left[ \sum_{\mathbf{j} \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(\mathbf{j}, f \mathbf{1}_{\mathcal{C}_{a_n}^k}) \right] = s(s-1) \cdots (s-k+1) \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} f \mathbf{1}_{\mathcal{C}_{a_n}^k}(\mathbf{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}}.$$

Finally, Lemma 3.8 yields

$$\mathbb{P}^* \left( \left| \frac{1}{(s\mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{C}^{k,s} \cap \mathcal{C}_{a_n}^k}) - (c_\infty)^k \mathcal{A}_{a_n}^k(f) \right| > \varepsilon/4 \right) \xrightarrow{n \rightarrow \infty} 0,$$

and the result of the subsection 3.1 leads to the convergence in (II.46).

Now using Lemma 3.10, we show that

$$\mathbb{P}^* \left( \frac{1}{(s\mathbf{L}_n)^k} \mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{C}^{k,s} \cap \Delta^k \setminus \mathcal{C}_{a_n}^k}) > \varepsilon/2 \right) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{II.47})$$

Indeed, by Lemma 3.6 with  $g = f$  and  $\mathfrak{R} = \Delta^k \setminus \mathcal{C}_{a_n}^k$

$$\mathbb{P}^* \left( \left| \mathcal{A}^k(\mathcal{D}_{n,T^s}, f \mathbf{1}_{\mathfrak{E}^{k,s} \cap \Delta^k \setminus \mathcal{C}_{a_n}^k}) - \sum_{j \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(j, f \mathbf{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k}) \right| > \varepsilon (s \mathbf{L}_n)^k / 4 \right) \leq n^{-\rho_4},$$

with  $\rho_4 = \rho_4(\varepsilon/4, \varepsilon_1)$  instead of  $\rho_4(\varepsilon, \varepsilon_1)$ . Finally, note that for any  $s \leq n^{1/2}/\varepsilon_1$ , the sum  $\sum_{j \in \llbracket 1, s \rrbracket_k} \mathcal{A}^{k,n}(j, f \mathbf{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k})$  is smaller than  $\sum_{j \in \llbracket 1, n^{1/2}/\varepsilon_1 \rrbracket_k} \mathcal{A}^{k,n}(j, f \mathbf{1}_{\Delta^k \setminus \mathcal{C}_{a_n}^k})$  so Lemma 3.10, together with Markov inequality leads to (II.47). We end the proof putting together (II.46) and (II.47).  $\square$

### 3.2 The range on $\Delta^k \setminus \mathfrak{E}^k$ .

Recall

$$\mathcal{A}^k(\mathcal{D}_{n,T^s}, g) = \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} g(\mathbf{x}) \mathbf{1}_{\{T_{\mathbf{x}} < T^s\}},$$

where  $T_{\mathbf{x}} = \max_{1 \leq i \leq k} T_{x^{(i)}}$ ,  $T_z = \min\{j \geq 0; X_j = z\}$ ,  $T^0 = 0$  and  $T^s = \min\{j > T^{s-1}; X_j = e\}$  for  $s \in \mathbb{N}^*$ . Also recall that  $(\ell_n)$  and  $(\mathfrak{L}_n)$  are two sequences of positive integers such that  $\delta_0^{-1} \log n \leq \ell_n \leq \mathfrak{L}_n \leq n^{1/2}$ .

The last step of our study is to show that the contribution of the  $k$ -tuples of vertices in small generations (see (II.7)) and such that at least two of these vertices are visited during the same excursion is not significant. This section is thus devoted to the proof of Proposition 3, claiming that

$$\mathbb{P}^* \left( \sup_{s \leq n^{1/2}/\varepsilon_1} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbf{1}_{\Delta^k \setminus \mathfrak{E}^{k,s}}) > \varepsilon (n^{1/2} \mathbf{L}_n)^k \right) \xrightarrow{n \rightarrow \infty} 0$$

**Lemma 3.11.** *Let  $\varepsilon_1 \in (0, 1)$ ,  $k \geq 2$ , let  $\mathfrak{s}_n = n^{1/2}/\varepsilon_1$  and assume  $\kappa > 2k$ . Assume that the assumptions 1, 3, 4 hold and that  $\mathfrak{L}_n = o(n^{1/2})$ .*

(i) *If*

$$\mathfrak{E}_1^{k,s} := \bigcup_{j=1}^s \bigcap_{i=1}^k \{ \mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k; \mathcal{L}_{x^{(i)}}^{T^j} - \mathcal{L}_{x^{(i)}}^{T^{j-1}} \}$$

*denotes the set of  $k$ -tuples of vertices visited during the same excursion before the instant  $T^s$ , then*

$$\lim_{n \rightarrow \infty} \mathbb{E}^* \left[ \frac{1}{(n^{1/2} \mathbf{L}_n)^k} \sup_{s \leq \mathfrak{s}_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbf{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{E}_1^{k,s}}) \right] = 0.$$

(ii) Let  $\mathfrak{E}_2^{k,s} := \Delta^k \setminus (\mathfrak{E}^{k,s} \cup \mathfrak{E}_1^{k,s})$ . If  $k \geq 3$  and the assumption 2 hold, then, for all  $B > 0$

$$\lim_{n \rightarrow \infty} \mathbb{E}^* \left[ \frac{1}{(n^{1/2} \mathbf{L}_n)^k} \sup_{s \leq \mathfrak{s}_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{E}_2^{k,s}} \mathbb{1}_{\{V(\cdot) \geq -B\}}) \right] = 0,$$

with  $V(\mathbf{x}) \geq -B$  if and only if  $V(x^{(i)}) \geq -B$  for all  $i \in \{1, \dots, k\}$ .

*Démonstration.* In order to avoid unnecessary technical difficulties, we prove it for any  $\kappa > 4$ . Let us start with the proof of (i). By definition,  $\mathbf{x} \in \mathfrak{E}^{k,s} \cap \mathfrak{E}_1^{k,s}$  if and only if there exists  $j \in \{1, \dots, s\}$  such that for all  $1 \leq i \leq k$ ,  $N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} \geq 1$  and for all  $p \neq j$ ,  $N_{x^{(i)}}^{T^p} - N_{x^{(i)}}^{T^{p-1}} = 0$ . Thus, using again the strong Markov property

$$\begin{aligned} \mathbb{E} \left[ \sup_{s \leq \mathfrak{s}_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{E}_1^{k,s}}) \right] &= \mathbb{E} \left[ \sup_{s \leq \mathfrak{s}_n} \sum_{j=1}^s \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} \mathbb{1}_{\cap_{i=1}^k \cap_{p \neq j} \{N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} \geq 1, N_{x^{(i)}}^{T^p} - N_{x^{(i)}}^{T^{p-1}} = 0\}} \right] \\ &\leq \sum_{j=1}^{\mathfrak{s}_n} \mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ \ell_n \leq |\mathbf{x}| \leq \mathfrak{L}_n}} \prod_{i=1}^k \mathbb{1}_{\{N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} \geq 1\}} \right] \\ &\leq \mathfrak{s}_n \sum_{\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{P}^{\mathcal{E}}(T_{\mathbf{x}} < T^1) \right] \\ &\leq \mathfrak{C}_{3,1} \mathfrak{s}_n (\mathbf{L}_n)^k (\mathfrak{L}_n)^{k-1}, \end{aligned}$$

where we have used Lemma 3.2 (II.34) with  $m = \mathfrak{L}_n$  for the last inequality, recalling that the constant  $\mathfrak{C}'_{3,1} > 0$  doesn't depend on  $\mathbf{p}$ . By definition of  $\mathfrak{s}_n$

$$\mathbb{E} \left[ \frac{1}{(n^{1/2} \mathbf{L}_n)^k} \sup_{s \leq \mathfrak{s}_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{E}_1^{k,s}}) \right] \leq \frac{\mathfrak{C}_{3,1}}{\varepsilon_1} \left( \frac{\mathfrak{L}_n}{n^{1/2}} \right)^{k-1},$$

which goes to 0 when  $n$  goes to  $\infty$  since  $\mathfrak{L}_n = o(n^{1/2})$  and this yields (i).

We now focus on (ii). Since  $k \geq 3$ ,  $\mathfrak{E}_2^{k,s}$  is nothing but the set of  $k$ -tuples of distinct vertices of  $\mathbb{T}$  neither visited during  $k$  distinct excursions, nor during the same excursion. Therefore, there exists  $\mathfrak{e} \in \{2, \dots, k-1\}$  and  $\mathfrak{e}$  disjoint subsets  $I_1, \dots, I_{\mathfrak{e}}$  of  $\{1, \dots, k\}$  such that  $\{1, \dots, k\} = I_1 \cup \dots \cup I_{\mathfrak{e}}$  and for any  $j \in \{1, \dots, \mathfrak{e}\}$ ,  $i, i' \in I_j$  if and only if  $x^{(i)}$  and  $x^{(i')}$  are visited during the same excursion before the instant  $T^s$  :

$$\exists j \in \{1, \dots, \mathfrak{e}\} : (\mathcal{L}_{x^{(i)}}^{T^j} - \mathcal{L}_{x^{(i)}}^{T^{j-1}}) \wedge (\mathcal{L}_{x^{(i')}}^{T^j} - \mathcal{L}_{x^{(i')}}^{T^{j-1}}) \geq 1.$$

Let  $m \in \mathbb{N}^*$  and introduce the following subset of  $\Delta^k$

$$\Upsilon_m^{k,s} := \{\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k; \forall j \neq j' \in \{1, \dots, \mathfrak{e}\}, \forall i \in I_j, \forall i' \in I_{j'} : |x^{(i)} \wedge x^{(i')}| < m\},$$

where we recall that  $u \wedge v$  is the most recent common ancestor (MRCA) of  $u$  and  $v$ .  $\Upsilon_m^k$  is the set of  $k$ -tuples of vertices such that the MRCA of two vertices visited during two distinct excursions before the instant  $T^s$  has to be in a generation smaller than  $m$ . Note that the MRCA of two vertices visited during the same excursion can be in a generation larger or equal to  $m$ .

Recall that  $(\Lambda_l)_{l \in \mathbb{N}}$  is the sequence of functions such that for all  $t > 0$ ,  $\Lambda_0(t) = t$  and for any  $l \in \{1, \dots, l_0\}$ ,  $\Lambda_{l-1}(t) = e^{\Lambda_l(t)}$  (see the assumption 2). Introduce  $\mathfrak{g}_{l,n} := 4k\delta_0^{-1}\Lambda_l(\mathfrak{L}_n)$ . Note that  $\mathfrak{g}_{0,n} > \mathfrak{L}_n$  so

$$\mathbb{E} \left[ \sup_{s \leq \mathfrak{s}_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s} \cap \mathfrak{C}_2^{k,s}} \mathbb{1}_{\{V(\cdot) \geq -B\}}) \right] = \mathbb{E} \left[ \sup_{s \leq \mathfrak{s}_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s} \cap \mathfrak{C}_2^{k,s}} \mathbb{1}_{\{V(\cdot) \geq -B\} \cap \mathcal{C}_{\mathfrak{g}_{0,n}}^k}) \right].$$

Recall that for any  $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ , it belongs to  $\mathfrak{S}^{k,s}$  if and only if  $x^{(i)}$  is visited during a single excursion before the instant  $T^s$  for all  $i \in \{1, \dots, k\}$ . Using what we previously said, we have, for any  $s \leq \mathfrak{s}_n$

$$\begin{aligned} \mathbb{1}_{\mathfrak{S}^{k,s} \cap \mathfrak{C}_2^{k,s}}(\mathbf{x}) &\leq \sum_{\mathfrak{e}=2}^{k-1} \sum_{j \in \llbracket 1, s \rrbracket_{\mathfrak{e}}} \sum_{\substack{I_1, \dots, I_{\mathfrak{e}} \text{ sets} \\ \cup_{l=1}^{\mathfrak{e}} I_l = \{1, \dots, k\}}} \prod_{p=1}^{\mathfrak{e}} Y_p \\ &\leq \sum_{\mathfrak{e}=2}^{k-1} \sum_{j \in \llbracket 1, \mathfrak{s}_n \rrbracket_{\mathfrak{e}}} \sum_{\substack{I_1, \dots, I_{\mathfrak{e}} \text{ sets} \\ \cup_{l=1}^{\mathfrak{e}} I_l = \{1, \dots, k\}}} \prod_{p=1}^{\mathfrak{e}} Y_p, \end{aligned}$$

where, for any  $p \in \{1, \dots, \mathfrak{e}\}$ ,  $Y_p := \mathbb{1}_{\cap_{i \in I_p} \{\mathcal{L}_{x^{(i)}}^{T^{j_p}} - \mathcal{L}_{x^{(i)}}^{T^{j_p-1}} \geq 1\}}$ . It follows that

$$\begin{aligned} &\mathbb{E} \left[ \sup_{s \leq \mathfrak{s}_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s} \cap \mathfrak{C}_2^{k,s}} \mathbb{1}_{\{V(\cdot) \geq -B\} \cap \mathcal{C}_{\mathfrak{g}_{0,n}}^k}) \right] \\ &\leq \sum_{\mathbf{p} \in \{\ell, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{l=1}^{l_0} \sum_{\mathfrak{e}=2}^{k-1} \sum_{j \in \llbracket 1, \mathfrak{s}_n \rrbracket_{\mathfrak{e}}} \sum_{\substack{I_1, \dots, I_{\mathfrak{e}} \text{ sets} \\ \cup_{l=1}^{\mathfrak{e}} I_l = \{1, \dots, k\}}} \mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Upsilon_{\mathfrak{g}_{l-1,n}}^{k, \mathfrak{s}_n} \setminus \Upsilon_{\mathfrak{g}_{l,n}}^{k, \mathfrak{s}_n}}(\mathbf{x}) \prod_{p=1}^{\mathfrak{e}} Y_p \right] \\ &+ \sum_{\mathbf{p} \in \{\ell, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\mathfrak{e}=2}^{k-1} \sum_{j \in \llbracket 1, \mathfrak{s}_n \rrbracket_{\mathfrak{e}}} \sum_{\substack{I_1, \dots, I_{\mathfrak{e}} \text{ sets} \\ \cup_{l=1}^{\mathfrak{e}} I_l = \{1, \dots, k\}}} \mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Upsilon_{\mathfrak{g}_{l_0,n}}^{k, \mathfrak{s}_n}}(\mathbf{x}) \prod_{p=1}^{\mathfrak{e}} Y_p \right]. \end{aligned} \quad (\text{II.48})$$

First, let us prove that for any  $\mathbf{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}$ ,

$$\mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}| = \mathbf{p}}} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Upsilon_{\mathfrak{g}_{l-1,n}}^{k, \mathfrak{s}_n} \setminus \Upsilon_{\mathfrak{g}_{l,n}}^{k, \mathfrak{s}_n}}(\mathbf{x}) \prod_{p=1}^{\mathfrak{e}} Y_p \right] \leq \mathfrak{C}_{3.11,1}(\mathfrak{L}_n)^{k-\mathfrak{e}}. \quad (\text{II.49})$$

The proof of (II.49) is quite technical so in order to keep it as clear as possible, as one can notice in the proof of Lemmas 3.3 (i) and 3.2 (II.34) with  $m = \mathfrak{L}_n$ , we can and shall restrict to the case  $\mathbf{p} = (m, \dots, m) \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}$ .

Thanks to the strong Markov property, the random variables  $Y_1, \dots, Y_\epsilon$  are i.i.d under  $\mathbb{P}^\epsilon$  and

$$\begin{aligned} & \mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Gamma_{\mathfrak{gl}-1,n}^{k,s_n} \setminus \Gamma_{\mathfrak{gl},n}^{k,s_n}}(\mathbf{x})} \prod_{p=1}^{\epsilon} Y_p \right] \\ &= \sum_{\mathbf{x} \in \Delta_m^k} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Gamma_{\mathfrak{gl}-1,n}^{k,s_n} \setminus \Gamma_{\mathfrak{gl},n}^{k,s_n}}(\mathbf{x})} \prod_{p=1}^{\epsilon} \mathbb{P}^\epsilon \left( \max_{i \in I_p} T_{x^{(i)}} < T^1 \right). \end{aligned}$$

As usual,  $\sum_{\mathbf{x} \in \Delta_m^k} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Gamma_{\mathfrak{gl}-1,n}^{k,s_n} \setminus \Gamma_{\mathfrak{gl},n}^{k,s_n}}(\mathbf{x})} \prod_{p=1}^{\epsilon} \mathbb{P}^\epsilon (\max_{i \in I_p} T_{x^{(i)}} < T^1)$  is equal to

$$\sum_{\ell=1}^{k-1} \sum_{\Pi \text{ increasing}} \sum_{\mathbf{t}; t_1 < \dots < t_\ell < m} \sum_{\mathbf{x} \in \Delta_m^k} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} f_{\mathbf{t}, \Pi}^\ell \mathbb{1}_{\Gamma_{\mathfrak{gl}-1,n}^{k,s_n} \setminus \Gamma_{\mathfrak{gl},n}^{k,s_n}}(\mathbf{x})} \prod_{p=1}^{\epsilon} \mathbb{P}^\epsilon \left( \max_{i \in I_p} T_{x^{(i)}} < T^1 \right), \quad (\text{II.50})$$

where the genealogical tree function  $f_{\mathbf{t}, \Pi}^\ell$  is defined in (II.14). Recall that  $t_1 - 1, \dots, t_\ell - 1$  correspond to the consecutive coalescent/split times. We then define

$$\tau^\ell := \max\{j \in \{1, \dots, \ell\}; \exists p \neq p' \in \{2, \dots, \epsilon\}, \exists \mathbf{B} \in \pi_{j-1} : \mathbf{B} \cap I_p \neq \emptyset \text{ and } \mathbf{B} \cap I_{p'} \neq \emptyset\},$$

and the  $\mathbf{x}$ -version  $\tau^\ell(\mathbf{x})$  of  $\tau^\ell$  :

$$\tau^\ell(\mathbf{x}) := \max\{j \in \{1, \dots, \ell\}; \exists p \neq p' \in \{2, \dots, \epsilon\}, \exists i \in I_p, i' \in I_{p'} : |x^{(i)} \wedge x^{(i')}| = t_j - 1\}.$$

In other words, if the genealogical tree of  $\mathbf{x} \in \Delta^k$  is given by  $f_{\mathbf{t}, \Pi}^\ell$ , then  $\tau^\ell = \tau^\ell(\mathbf{x})$  and  $t_{\tau^\ell} - 1$  is the last generation at which two or more vertices visited during two distinct excursions share a common ancestor.

By definition of  $\tau^\ell$ , for all  $j \geq \tau^\ell$ , if  $\mathbf{B} \in \pi_j$ , then  $\mathbf{B}$  is necessarily a subset of  $I_{p'}$  for some  $p' \in \{1, \dots, \epsilon\}$ . In other words, each coalescence that occurs between  $t_{\tau^\ell+1}$  and  $t_\ell$  involves exclusively two or more vertices visited during the same excursion. As a consequence, for any  $i \in \{\tau^\ell, \dots, \ell\}$  and  $p \in \{1, \dots, \epsilon\}$ , we can defined the set  $I_p^i$  as follows : we first set  $I_p^\ell := I_p$  so  $I_1^\ell, \dots, I_\epsilon^\ell$  form a partition of  $\{1, \dots, k\}$ . As we said before, by definition of  $\tau^\ell$ , coalescences can only happen between two or more vertices which indexes belong to the same  $I_p^\ell$ . Thus, for any  $p \in \{1, \dots, \epsilon\}$ , there exists an integer  $\epsilon_p^{\ell-1} \geq 1$  and  $\epsilon_p^{\ell-1}$  distinct integers  $k_{p,1}^{\ell-1}, \dots, k_{p,\epsilon_p^{\ell-1}}^{\ell-1}$  in  $\{1, \dots, |\pi_{\ell-1}|\}$  such that for any  $j \in \{k_{p,1}^{\ell-1}, \dots, k_{p,\epsilon_p^{\ell-1}}^{\ell-1}\}$ , the block  $\mathbf{B}_j^{\ell-1}$  of the partition  $|\pi_{\ell-1}|$  is the union of  $b_{\ell-1}(\mathbf{B}_j)$  block(s) of the partition  $\pi_\ell$  of elements of  $F_p^\ell$ . We set  $F_p^{\ell-1} := \{k_{p,1}^{\ell-1}, \dots, k_{p,\epsilon_p^{\ell-1}}^{\ell-1}\}$  so  $I_1^{\ell-1}, \dots, I_\epsilon^{\ell-1}$  form a partition of  $\{1, \dots, |\pi_{\ell-1}|\}$ . Now, let  $i \in \{\tau^\ell + 1, \dots, \ell\}$  and assume that  $F_p^i$  has been built. By definition of  $\tau^\ell$ , for any  $p \in \{1, \dots, \epsilon\}$ , there exists an integer  $\epsilon_p^{i-1} \geq 1$  and  $\epsilon_p^{i-1}$

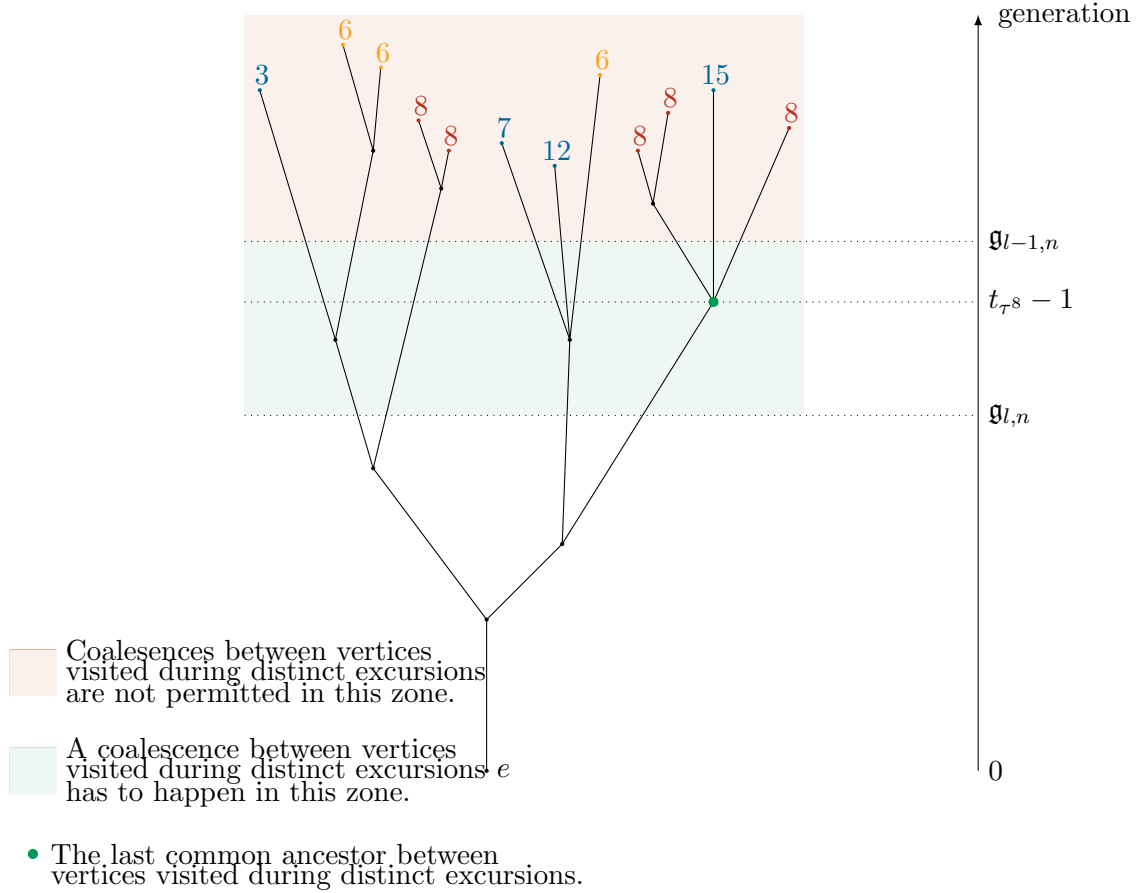


FIGURE II.4 – An example of a 12-tuple belonging to  $\Upsilon_{\mathfrak{g}_{l-1,n}}^{12,\cdot} \setminus \Upsilon_{\mathfrak{g}_{l,n}}^{12,\cdot}$  whose genealogical tree is given by  $f_{\mathbf{t},\Pi}^\ell$ . **6** means that the corresponding vertex is visited during the 6-th excursion above  $e^*$ . In the present example,  $\ell = 8$  and  $\tau^8 = 4$ .

distinct integer  $k_{p,1}^{i-1}, \dots, k_{p,\epsilon_p}^{i-1}$  in  $\{1, \dots, |\pi_{i-1}|\}$  such that for any  $j \in \{k_{p,1}^{i-1}, \dots, k_{p,\epsilon_p}^{i-1}\}$ , the block  $\mathbf{B}_j^{i-1}$  of the partition  $|\pi_{i-1}|$  is the union of  $b_{i-1}(\mathbf{B}_j)$  block(s) of the partition  $\pi_i$  of elements of  $I_p^i$ . We set  $I_p^{i-1} := \{k_{p,1}^{i-1}, \dots, k_{p,\epsilon_p}^{i-1}\}$  so  $I_1^{i-1}, \dots, I_\epsilon^{i-1}$  form a partition of  $\{1, \dots, |\pi_{i-1}|\}$ . Hence, noticing that

$$f_{\mathbf{t},\Pi}^\ell \mathbb{1}_{\Upsilon_{\mathfrak{g}_{l-1,n}}^{k,s_n} \setminus \Upsilon_{\mathfrak{g}_{l,n}}^{k,s_n}}(\mathbf{x}) \leq f_{\mathbf{t},\Pi}^\ell(\mathbf{x}) \mathbb{1}_{\{\mathfrak{g}_{l,n} \leq t_{\tau^\ell} - 1 < \mathfrak{g}_{l-1,n}\}} = f_{\mathbf{t},\Pi}^\ell(\mathbf{x}) \mathbb{1}_{\{\mathfrak{g}_{l,n} \leq t_{\tau^\ell} - 1 < \mathfrak{g}_{l-1,n}\}},$$

it is enough to show (II.49) for  $\mathfrak{g}_{l,n} \leq t_{\tau^\ell} - 1 < \mathfrak{g}_{l-1,n}$ . We then have

$$\mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_m^k} \mathbb{1}_{\{\underline{V}(\mathbf{x}) \geq -B\}} f_{\mathbf{t},\Pi}^\ell \mathbb{1}_{\Upsilon_{\mathfrak{g}_{l-1,n}}^{k,s_n} \setminus \Upsilon_{\mathfrak{g}_{l,n}}^{k,s_n}}(\mathbf{x}) \prod_{p=1}^{\epsilon} \mathbb{P}^{\mathcal{E}}(\max_{i \in I_p} T_{x^{(i)}} < T^1) \mid \mathcal{F}_{t_{\tau^\ell}} \right]$$



$$\leq \mathfrak{C}_{3.11,2} \sum_{\mathbf{u} \in \Delta_{t_{\tau_\ell}}^{|\pi_{\tau_\ell}|}} \mathbb{1}_{\{V(\mathbf{u}) \geq -B\}} f_{\mathbf{t}^{\tau_\ell}, \Pi^{\tau_\ell}}^{\tau_\ell}(\mathbf{u}) \prod_{p=1}^{\mathfrak{c}} \mathbb{P}^{\mathcal{E}} \left( \max_{i \in I_p^{\tau_\ell}} T_{u^{(i)}} < T^1 \right) \prod_{j=1}^{|\pi_{\tau_\ell}|} (H_{u^{(j)}})^{|\mathcal{B}_j^{\tau_\ell}|},$$

for some constant  $\mathfrak{C}_{3.11,2} > 0$  where  $\mathbf{t}^{\tau_\ell}$  and  $\Pi^{\tau_\ell}$  are defined in Example 6.

Note that  $t_{\tau_\ell} - 1$  is the first generation (backwards in time) at which a coalescence between two or more vertices visited during distinct excursions occurs so there exists a subset  $J_\ell$  of  $\{1, \dots, |\pi_{t_{\tau_\ell}-1}|\}$  and a collection  $\{\alpha_i; i \in J_\ell\}$  of  $|J_\ell|$  integers satisfying  $\alpha_i \geq 1$  for all  $i \in J_\ell$  and  $\sum_{i \in J_\ell} \alpha_i \leq k$  such that

$$\begin{aligned} & \mathbf{E} \left[ \sum_{\mathbf{x} \in \Delta_m^k} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} f_{\mathbf{t}, \Pi}^{\ell} \mathbb{1}_{\Upsilon_{\mathfrak{g}_{l-1}, n}^{k, s_n} \setminus \Upsilon_{\mathfrak{g}_{l, n}^{k, s_n}}(\mathbf{x})} \prod_{p=1}^{\mathfrak{c}} \mathbb{P}^{\mathcal{E}} \left( \max_{i \in I_p} T_{x^{(i)}} < T^1 \right) \middle| \mathcal{F}_{t_{\tau_\ell}-1} \right] \\ & \leq \mathfrak{C}_{3.11,3} \sum_{\mathbf{z} \in \Delta_{t_{\tau_\ell}-1}^{|\pi_{\tau_\ell}-1|}} f_{\mathbf{t}^{\tau_\ell-1}, \Pi^{\tau_\ell-1}}^{\tau_\ell-1}(\mathbf{z}) \mathbb{P}^{\mathcal{E}}(T_{\mathbf{z}} < T^1) \prod_{j=1}^{|\pi_{\tau_\ell-1}|} (H_{u^{(j)}})^{|\mathcal{B}_j^{\tau_\ell-1}|} \prod_{i \in J_\ell} e^{-\alpha_i V(z^{(i)})} \mathbb{1}_{\{V(\mathbf{z}) \geq -B\}}. \end{aligned}$$

Note that

$$\begin{aligned} \prod_{i \in J_\ell} e^{-\alpha_i V(z^{(i)})} \mathbb{1}_{\{V(\mathbf{z}) \geq -B\}} & \leq \prod_{i \in J_\ell} e^{-\alpha_i V(z^{(i)})} \mathbb{1}_{\{\min_{i \in J_\ell} V(z^{(i)}) \geq -B, \min_{|z|=t_{\tau_\ell}-1} V(z) < \delta_0(t_{\tau_\ell}-1)\}} \\ & \quad + e^{-\min_{|z|=t_{\tau_\ell}-1} V(z)} \mathbb{1}_{\{\min_{|z|=t_{\tau_\ell}-1} V(z) \geq \delta_0(t_{\tau_\ell}-1)\}}, \end{aligned}$$

so  $\mathbf{E}[\sum_{\mathbf{x} \in \Delta_m^k} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} f_{\mathbf{t}, \Pi}^{\ell} \mathbb{1}_{\Upsilon_{\mathfrak{g}_{l-1}, n}^{k, s_n} \setminus \Upsilon_{\mathfrak{g}_{l, n}^{k, s_n}}(\mathbf{x})} \prod_{p=1}^{\mathfrak{c}} \mathbb{P}^{\mathcal{E}}(\max_{i \in I_p} T_{x^{(i)}} < T^1)]$  is smaller than

$$\begin{aligned} & \mathbf{E} \left[ \mathfrak{C}_{3.11,3} \sum_{\mathbf{z} \in \Delta_{t_{\tau_\ell}-1}^{|\pi_{\tau_\ell}-1|}} f_{\mathbf{t}^{\tau_\ell-1}, \Pi^{\tau_\ell-1}}^{\tau_\ell-1}(\mathbf{z}) \mathbb{P}^{\mathcal{E}}(T_{\mathbf{z}} < T^1) \prod_{j=1}^{|\pi_{\tau_\ell-1}|} (H_{u^{(j)}})^{|\mathcal{B}_j^{\tau_\ell-1}|} \right. \\ & \quad \left. \times \left( e^{kB} \mathbb{1}_{\{\min_{|z|=t_{\tau_\ell}-1} V(z) < \delta_0(t_{\tau_\ell}-1)\}} + e^{-3\delta_0(t_{\tau_\ell}-1)} \right) \right]. \end{aligned}$$

Using the same argument as the one we used in the proof of Lemma 3.2 together with the Cauchy–Schwarz inequality, we obtain that the previous mean is smaller than

$$\begin{aligned} & \mathfrak{C}_{3.11,3} \sup_{d \in \mathbb{N}^*} \mathbf{E}[(H_{d-1}^S)^{4k-1}] \left( e^{kB} \mathbb{P} \left( \min_{|z|=t_{\tau_\ell}-1} V(z) < \delta_0(t_{\tau_\ell}-1) \right)^{1/2} + e^{-3\delta_0(t_{\tau_\ell}-1)} \right) \\ & \leq \mathfrak{C}_{3.11,3} \sup_{d \in \mathbb{N}^*} \mathbf{E}[(H_{d-1}^S)^{4k-1}] (e^{kB} + 1) e^{-k\Lambda_l(\mathfrak{L}_n)}, \end{aligned}$$

where we have used Lemma 3.4 with  $\zeta = \delta_0 t_{\tau_\ell}$  and the fact that  $t_{\tau_\ell} - 1 \geq \mathfrak{g}_{l, n}$ .

Back to (II.50) together with what we have just obtained and the fact that for all  $j \in \{1, \dots, \tau_\ell\}$ ,

$t_j \leq \mathfrak{g}_{l-1,n}$ ,  $\mathbb{E}[\sum_{\mathbf{x} \in \Delta^k, |\mathbf{x}|=p} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Gamma_{\mathfrak{g}_{l-1,n}}^{k,s_n} \setminus \Gamma_{\mathfrak{g}_{l,n}}^{k,s_n}}(\mathbf{x})} \mathbb{1}_{\mathfrak{G}^{k,s_n} \cap \mathfrak{C}_2^{k,s_n}}(\mathbf{x})]$  is smaller than

$$\mathfrak{C}_{3.11,3} \sup_{d \in \mathbb{N}^*} \mathbb{E}[(H_{d-1}^S)^{4k-1}] (e^{kB} + 1) e^{-k\Lambda_l(\mathfrak{L}_n)} \sum_{\ell=1}^{k-1} \sum_{\Pi \text{ increasing}} (\mathfrak{g}_{l-1,n})^{\tau_\ell} (\mathfrak{L}_n)^{\ell - \tau_\ell}.$$

Note that  $\tau_\ell \leq \ell < k$ . Moreover, by definition,  $\ell - \tau_\ell$  is smaller than the total number of coalescences occurring between two or more vertices which indexes belong to the same set  $I_p^\ell$  and this number is smaller than  $\sum_{p=1}^{\mathfrak{c}} (|I_p^\ell| - 1) = k - \mathfrak{c}$  thus giving

$$\mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Gamma_{\mathfrak{g}_{l-1,n}}^{k,s_n} \setminus \Gamma_{\mathfrak{g}_{l,n}}^{k,s_n}}(\mathbf{x})} \prod_{p=1}^{\mathfrak{c}} Y_p \right] \leq \mathfrak{C}_{3.11,1} (\Lambda_{l-1}(\mathfrak{L}_n) e^{-\Lambda_l(\mathfrak{L}_n)})^k (\mathfrak{L}_n)^{k-\mathfrak{c}},$$

which, by definition of  $\Lambda_l(\mathfrak{L}_n)$ , is equal to  $\mathfrak{C}_{3.11,1} (\mathfrak{L}_n)^{k-\mathfrak{c}}$  and it yields (II.49).

In the same way, we can prove that

$$\mathbb{E} \left[ \sum_{\substack{\mathbf{x} \in \Delta^k \\ |\mathbf{x}|=p}} \mathbb{1}_{\{V(\mathbf{x}) \geq -B\}} \mathbb{1}_{\Gamma_{\mathfrak{g}_{l,n}}^{k,s_n}}(\mathbf{x})} \prod_{p=1}^{\mathfrak{c}} Y_p \right] \leq \mathfrak{C}'_{3.11,1} (1 + \Lambda_{l+1}(\mathfrak{L}_n)^k) (\mathfrak{L}_n)^{k-\mathfrak{c}}, \quad (\text{II.51})$$

for some constant  $\mathfrak{C}'_{3.11,1} > 0$ . Putting together (II.48), (II.49) and (II.51), we obtain, for some constant  $\mathfrak{C}_{3.11,4} > 0$

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{(n^{1/2} \mathbf{L}_n)^k} \sup_{s \leq s_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{G}^{k,s} \cap \mathfrak{C}_2^{k,s}} \mathbb{1}_{\{V(\cdot) \geq -B\}} \cap \mathfrak{C}_{\mathfrak{g}_{0,n}}^k) \right] \\ & \leq \mathfrak{C}_{3.11,4} \sum_{\mathfrak{c}=2}^{k-1} \left( \frac{\mathfrak{L}_n}{n^{1/2}} \right)^{k-\mathfrak{c}} (2 + \Lambda_{l_0+1}(\mathfrak{L}_n)^k). \end{aligned}$$

Using the fact that  $\Lambda_{l_0+1}(\mathfrak{L}_n)^k = (\log \Lambda_{l_0}(\mathfrak{L}_n))^k$ , we obtain (ii) thanks to the assumption 2.  $\square$

We are now ready to prove Proposition 3 :

*Proof of Proposition 3.* Let  $\varepsilon' > 0$ . First, note that thanks to Lemma 3.5 and Fact 1 (II.36) there exists  $a_{\varepsilon'} > 0$  such that we can restrict our study to the  $k$ -tuples of vertices in the set  $\mathfrak{G}^{k,s} \cap \{V(\cdot) \geq -a_{\varepsilon'}\}$

$$\lim_{\varepsilon' \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^* \left( \sup_{s \leq s_n} \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\Delta^k \setminus \mathfrak{E}^{k,s}} (1 - \mathbb{1}_{\mathfrak{G}^{k,s} \cap \{V(\cdot) \geq -a_{\varepsilon'}\}})) > \varepsilon (n^{1/2} \mathbf{L}_n)^k \right) = 0,$$

where we recall that  $s_n = n^{1/2}/\varepsilon_1$ . Then, note that  $\mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\Delta^k \setminus \mathfrak{E}^{k,s}} \mathbb{1}_{\mathfrak{G}^{k,s} \cap \{V(\cdot) \geq -a_{\varepsilon'}\}})$  is smaller than

$$\mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{G}^{k,s} \cap \mathfrak{C}_1^{k,s}}) + \mathcal{A}^k(\mathcal{D}_{n,T^s}, \mathbb{1}_{\mathfrak{G}^{k,s} \cap \mathfrak{C}_2^{k,s}} \mathbb{1}_{\{V(\cdot) \geq -a_{\varepsilon'}\}}).$$

---

Hence, by Markov inequality, the result follows using Lemma 3.11 with  $B = a_{\varepsilon'}$ .  $\square$



# Bibliographie

- [AC18] Pierre Andreatti and Xinxin Chen. Range and critical generations of a random walk on Galton-Watson trees. Ann. Inst. Henri Poincaré Probab. Stat., 54(1) :466–513, 2018.
- [AD14a] P. Andreatti and P. Debs. The number of generations entirely visited for recurrent random walks on random environment. J. Theoret. Probab., 27 : 518 – 538, 2014.
- [AD14b] P. Andreatti and P. Debs. Spread of visited sites of a random walk along the generations of a branching process. Electronic Journal of Probab., 19 :1–22, 2014.
- [AD20] Pierre Andreatti and Roland Diel. The heavy range of randomly biased walks on trees. Stochastic Processes and their Applications, 130(2) :962 – 999, 2020.
- [AD21] Romain Abraham and Jean-Francois Delmas. Exact simulation of the genealogical tree for a stationary branching population and application to the asymptotics of its total length. Advances in Applied Probability, 53 :537–574, 06 2021.
- [AdR17] E. Aïdékon and L. de Raphélis. Scaling limit of the recurrent biased random walk on a Galton-Watson tree. Probability Theory and Related Fields, 169(3) :643–666, 2017.
- [Aïd08] E. Aïdékon. Transient random walks in random environment on a Galton-Watson tree. Probability Theory and Related, 142(3) :525–559, 2008.
- [Aïd10] E. Aïdékon. Tail asymptotics for the total progeny of the critical killed branching random walk. Elec. Comm. in Probab., 15 :522–533, 2010.
- [Aïd13] E. Aïdékon. Convergence in law of the minimum of a branching random walk. Ann. probab., 41 :1362–1426, 2013.
- [Aïd14] Elie Aïdékon. Speed of the biased random walk on a galton–watson tree. Probability Theory and Related Fields, 159(3) :597–617, 2014.
- [AK21] Pierre Andreatti and Alexis Kagan. Generalized range of slow random walks on trees. Preprint, [arXiv:2111.05029](https://arxiv.org/abs/2111.05029), 2021.

- [AS14a] E. Aïdékon and Z. Shi. The seneta-heyde scaling for the branching random walk. The Annals of Probability, 42(3) :959–993, 2014.
- [AS14b] Elie Aïdékon and Zhan Shi. The Seneta-Heyde scaling for the branching random walk. The Annals of Probability, 42(3) :959 – 993, 2014.
- [Ath12a] K. Athreya. Coalescence in critical and subcritical Galton-Watson branching processes. Journal of Applied Probability, 49, 09 2012.
- [Ath12b] K.B. Athreya. Coalescence in the recent past in rapidly growing populations. Stochastic Processes and their Applications, 122 :3757–3766, 11 2012.
- [Aue90] Peter Auer. The circle homogeneously covered by random walk on  $\mathbb{Z}^2$ . Statistics & Probability Letters, 9(5) :403–407, 1990.
- [Big76] J.D. Biggins. The first- and last-birth problems for a multitype age-dependent branching process. Adv. Appl. Probab., 8 : 446–459, 1976.
- [Big77] J. D. Biggins. Martingale convergence in the branching random walk. Journal of Applied Probability, 14(1) :25–37, 1977.
- [BK04] J. D. Biggins and A.E. Kyprianou. Measure change in multitype branching. Adv. Appl. Probab., 36 :544–581, 2004.
- [BM19] Pierre Bouteaud and Pascal Maillard. A revisited proof of the Seneta-Heyde norming for branching random walks under optimal assumptions. Electronic Journal of Probability, 24(none) :1 – 22, 2019.
- [Bol76] E. Bolthausen. On a functional central limit theorem for random walks conditioned to stay positive. Annals of Probability, 4 :480–485, 1976.
- [Bü71] Wolfgang Bühler. Generations and degree of relationship in supercritical markov branching processes. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 18 :141–152, 06 1971.
- [Car05] F. Caravenna. A local limit theorem for random walks conditioned to stay positive. Probab. Theory Related Fields, 133 : 508–530, 2005.
- [CC13] F. Caravenna and L. Chaumont. An invariance principle for random walk bridges conditioned to stay positive. Electron. J. Probab., 18 :32 pp., 2013.
- [CdR20] Xinxin Chen and Loïc de Raphélis. Maximal local time of randomly biased random walks on a Galton-Watson tree, 2020.
- [CdRH16] D. Chen, L. de Raphélis, and Y. Hu. Favorite sites of randomly biased walks on a supercritical galton – watson tree. Stochastic Process. Appl., 129 :1525–1557, 2016.

- [Che15] X. Chen. A necessary and sufficient condition for the non- a necessary and sufficient condition for the non-trivial limit of the derivative martingale in a branching random walk. Adv. Appl. Probab., To be published, 2015.
- [Che22] Xinxin Chen. Heavy range of the randomly biased walk on galton–watson trees in the slow movement regime. Stochastic Processes and their Applications, 150 :446–509, 2022.
- [DKLT22] Thomas Duquesne, Robin Khanfir, Shen Lin, and Niccolò Torri. Scaling limits of tree-valued branching random walks. Electronic Journal of Probability, 27(none) :1 – 54, 2022.
- [DLG05] Thomas Duquesne and Jean-Francois Le Gall. Random trees, levy processes and spatial branching processes. Astérisque, 281, 10 2005.
- [dR16] L. de Raphélis. Scaling limit of the random walk in random environment in the subdiffusive case. Arxiv : <https://arxiv.org/abs/1608.07061>, 2016.
- [dR17] Loïc de Raphélis. Scaling limit of multitype Galton–Watson trees with infinitely many types. Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, 53(1) :200 – 225, 2017.
- [dR22] Loïc de Raphélis. Scaling limit of the subdiffusive random walk on a Galton–Watson tree in random environment. The Annals of Probability, 50(1) :339 – 396, 2022.
- [Dur78] Rick Durrett. The genealogy of critical branching processes. Stochastic Processes and their Applications, 8 :101–116, 11 1978.
- [ED51] Paul Erdős and Aryeh Dvoretzky. Some Problems on Random Walk in Space. Berkeley Symposium on Mathematical Statistics and Probability, pages 353 – 357, 1951.
- [ER84] P. Erdős and P. Révész. On the favourite points of a random walk. Mathematical Structures - Computational Mathematics - Mathematical Modelling, 2 :152–157, 01 1984.
- [ET60] P. Erdős and S. Taylor. Some problems concerning the structure of random walk paths. Acta Mathematica Hungarica - ACTA MATH HUNG, 11 :137–162, 03 1960.
- [Far11] G. Faraud. A central limit theorem for random walk in a random environment on marked Galton-Watson trees. Electronic Journal of Probability, 16(6) :174–215, 2011.
- [Fel68] W. Feller. An Introduction to Probability Theory, Vol. 1. Wiley, New York, NY, third edition, 1968.

- [FHS11] G. Faraud, Y. Hu, and Z. Shi. Almost sure convergence for stochastically biased random walks on trees. Probab. Theory Relat. Fields, 154 :621–660, 2011.
- [Gal06] Jean-François Le Gall. Random real trees. Annales de la Faculté des sciences de Toulouse : Mathématiques, pages 35–62, 2006.
- [GL13] Jean-François Le Gall and Shen Lin. The range of tree-indexed random walk. Journal of the Institute of Mathematics of Jussieu, -1, 07 2013.
- [GL15] Jean-François Le Gall and Shen Lin. The range of tree-indexed random walk in low dimensions. The Annals of Probability, 43(5) :2701 – 2728, 2015.
- [Gol84] A. O. Golosov. Localization of random walks in one-dimensional random environments. Communications in Mathematical Physics, 92 : 491–506, 1984.
- [Ham74] J. M. Hammersley. Postulates for Subadditive Processes. The Annals of Probability, 2(4) :652 – 680, 1974.
- [Hey70] C. C. Heyde. Extension of a Result of Seneta for the Super-Critical Galton-Watson Process. The Annals of Mathematical Statistics, 41(2) :739 – 742, 1970.
- [HJR17] Simon Harris, Samuel Johnston, and Matthew Roberts. The coalescent structure of continuous-time Galton-Watson trees. Annals of Applied Probability, 30, 03 2017.
- [HS07a] Y. Hu and Z. Shi. Slow movement of recurrent random walk in random environment on a regular tree. Ann. Probab., 35 :1978–1997, 2007.
- [HS07b] Y. Hu and Z. Shi. A subdiffusive behavior of recurrent random walk in random environment on a regular tree. Probab. Theory Related Fields, 138 :521–549, 2007.
- [HS09] Y. Hu and Z. Shi. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. Ann. Probab., 37 :742–789, 2009.
- [HS16a] Y. Hu and Z. Shi. The potential energy of biased random walks on trees. arXiv : <http://arxiv.org/abs/1403.6799>, 2016.
- [HS16b] Yueyun Hu and Zhan Shi. The slow regime of randomly biased walks on trees. Ann. Probab., 44(6) :3893–3933, 2016.
- [HSY15] Yueyun Hu, Zhan Shi, and Marc Yor. The maximal drawdown of the Brownian meander. Electronic Communications in Probability, 20(none) :1 – 6, 2015.
- [Hu17] Y. Hu. Local times of subdiffusive biased walks on trees. J. of Theoret. Probab., 30(2) :529–550, 2017.



- [Joh19] Samuel Johnston. The genealogy of Galton-Watson trees. Electronic Journal of Probability, 24, 01 2019.
- [Kag22] Alexis Kagan. Coalescence in small generations for the diffusive randomly biased walk on Galton-Watson tree. Preprint, 2022.
- [Kin75] J. F. C. Kingman. The First Birth Problem for an Age-dependent Branching Process. The Annals of Probability, 3(5) :790 – 801, 1975.
- [Koz76] M. V. Kozlov. On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment. Theory Probab. Appl., 21 :791–804, 1976.
- [KS66] H. Kesten and B.P. Stigum. A limit theorem for multidimensional Galton-Watson processes. Ann. Math. Statist., 37 : 1211–1223, 1966.
- [Liu00] Quansheng Liu. On generalized multiplicative cascades. Stochastic Processes and their Applications, 86(2) :263–286, 2000.
- [LP92] Russell Lyons and Robin Pemantle. Random walk in a random environment and first-passage percolation on trees. Annals of Probability, 20 :125–136, 1992.
- [LPP96a] Russell Lyons, Robin Pemantle, and Yuval Peres. Biased random walks on Galton-Watson trees. Probability Theory and Related Fields, 106, 10 1996.
- [LPP96b] Russell Lyons, Robin Pemantle, and Yuval Peres. Ergodic theory on Galton-Watson trees : Speed of random walk and dimension of harmonic measure. Ergodic Theory and Dynamical Systems, 15, 10 1996.
- [Lyo90] R. Lyons. Random walks and percolation on trees. Ann. Probab., 18 :931–958, 1990.
- [Lyo92] R. Lyons. Random walks, capacity and percolation on trees. Ann. Probab., 20 :2043–2088, 1992.
- [Lyo97] Russell Lyons. A Simple Path to Biggins’ Martingale Convergence for Branching Random Walk, pages 217–221. Springer New York, New York, NY, 1997.
- [Mad12] T. Madaule. First order transition for the branching random walk at the critical parameter. arXiv :<http://arxiv.org/abs/1206.3835>, 2012.
- [MP02] M.V. Menshikov and D. Petritis. On random walks in random environment on trees and their relationship with multiplicative chaos. Math. Comput. Sci., 415-422, 2002.
- [Nev86] J. Neveu. Arbres et processus de Galton-Watson. Ann. de l’IHP, 22(2), 1986.
- [PZ06] Yuval Peres and Ofer Zeitouni. A central limit theorem for biased random walks on Galton-Watson trees. Probability Theory and Related Fields, 140, 07 2006.

- [Ros05] Jay Rosen. A random walk proof of the Erdős-Taylor conjecture. Periodica Mathematica Hungarica, 50(1) :223 – 245, 2005.
- [Sen68] E. Seneta. On Recent Theorems Concerning the Supercritical Galton-Watson Process. The Annals of Mathematical Statistics, 39(6) :2098 – 2102, 1968.
- [Shi15] Z. Shi. Branching random walks. École d’été de Probabilités de Saint-Flour XLII - 2012, 2015.
- [Sin82] Ya. G. Sinai. The limit behaviour of a one-dimensional random walk in a random medium. Theory Probab. Appl., 27(2) : 256–268, 1982.
- [Zub76] A. M. Zubkov. Limiting distributions of the distance to the closest common ancestor. Theory of Probability & Its Applications, 20(3) :602–612, 1976.

Alexis KAGAN

## Traces de marches aléatoires en milieux aléatoires sur des arbres

### Résumé :

Si on considère une marche aléatoire à valeurs dans un graphe  $G$ , la trace de cette dernière jusqu'à un instant  $n$  donné est définie comme le nombre de sites distincts visités par la marche jusqu'à l'instant  $n$ . Lorsque  $G$  est un arbre de Bienaymé-Galton-Watson sur-critique marqué, nous introduisons, sur une réalisation de ce processus, une marche aléatoire au plus proche voisin. Ces marches sont, sauf dans certains cas particuliers, transientes. On ajoute donc un biais vers la racine qui, s'il est suffisant, va rendre la marche récurrente. Ces marches biaisées introduites par R. Lyons ont une grande richesse de comportements (voir notamment les travaux de Y. Hu et Z. Shi). En particulier, en prenant un biais aléatoire spécifique et en renormalisant convenablement, la trace converge en probabilité vers une constante explicite et strictement positive (X. Chen et P. Andreoletti, E. Aïdékon et L. de Raphélis).

La première partie de cette thèse est consacrée à une extension de ce résultat dans le régime lent. Nous étudions une trace généralisée: au lieu de regarder seulement la trace usuelle, nous nous intéressons à des traces avec des contraintes à la fois le long des trajectoires de la marche aléatoire et le long de celles du potentiel branchant sous-jacent. Un exemple important est d'imposer que les sommets visités par la marche aient un potentiel supérieur à une valeur donnée. A travers l'étude de cette trace généralisée, nous souhaitons mettre en lumière les interactions entre la marche aléatoire et son potentiel aléatoire branchant.

La deuxième partie de cette thèse est consacrée à la marche diffusive et plus particulièrement à la généalogie de l'ensemble des sommets visités par cette dernière. Nous étudions une trace avec des contraintes sur les sommets visités permettant ainsi de mieux comprendre les interactions entre ces derniers. Nous considérons enfin le fameux problème de généalogie suivant: tirons uniformément et sans remise  $k$  sommets visités par la marche aléatoire. A quoi ressemble l'arbre généalogique de ces sommets? L'étude de la trace contrainte permet d'apporter quelques éléments de réponses.

Mots clés : Arbres de Bienaymé-Galton-Watson, Généalogie, Marches aléatoires branchantes, Marches aléatoires sur des arbres, Traces.

## Ranges of random walks in random environments on trees

### Abstract :

If we consider a random walk on a graph  $G$ , the range of this walk until a given instant  $n$  is defined to be the number of vertices visited by the walk up to the time  $n$ . When  $G$  is a super-critical Bienaymé-Galton-Watson marked tree, we introduce, on a realization of this process, a nearest-neighbour random walk. These walks are in general transient, but it can be added a bias towards the root so that the walk is recurrent. Introduced by R. Lyons, these processes present various behaviours and have been deeply studied by Y. Hu and Z. Shi. Lately, in some cases with a specific bias, it has been proved that the range (well renormalized) converges in probability towards an explicit positive constant (X. Chen and P. Andreoletti, E. Aïdékon and L. de Raphélis).

The first part of this thesis is devoted to an extension of this result in the slow regime. We study a generalized range: instead of looking at the classical range, we are interested in a range with both constraints on the trajectories of the random walk and on the trajectories of the underlying branching random potential. An important example of that is to enforce the vertices visited by the walk to have a potential larger than a given value. By studying this generalized range, we do wish to highlight interactions between the random walk and its random branching potential.

The second part of this thesis is dedicated to the diffusive random walk and most particularly to the genealogy of the set of vertices visited by the walk. We study a range with constraints on visited vertices to better understand interactions between them. We finally consider the following well known genealogy problem: pick  $k$  visited vertices uniformly and without replacement. What does their genealogical tree look like? The study of the previous range with constraints allows to answer this question.

Keywords : Bienaymé-Galton-Watson trees, Branching random walks, Genealogy, Random walks on trees, Ranges