



HAL
open science

Birational automorphisms of varieties

Alexandra Kusnetsova

► **To cite this version:**

Alexandra Kusnetsova. Birational automorphisms of varieties. Algebraic Geometry [math.AG]. Institut Polytechnique de Paris; Nacional nyj issledovatel skij universitet "Vysšaâ Škola Ékonomiki". fakul tet filosofii (Moscou), 2022. English. NNT : 2022IPPAX054 . tel-04094707

HAL Id: tel-04094707

<https://theses.hal.science/tel-04094707v1>

Submitted on 11 May 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NNT : 2022IPPAX054

Thèse de doctorat



Automorphismes birationnels des variétés

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à L'École polytechnique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques fondamentales

Thèse présentée et soutenue à Palaiseau, le 1 juillet 2022, par

ALEXANDRA KUZNETSOVA

Composition du Jury :

M. Sébastien BOUCKSOM Professeur, École Polytechnique (CMLS)	Président
M. JérémY BLANC Professeur, Universität Basel	Rapporteur
M. Yuri PROKHOROV Professeur, Steklov Mathematical Institute	Rapporteur
M. Misha VERBITSKY Professeur, Higher School of Economics	Rapporteur
M. John LESIEUTRE Assistant Professor, Pennsylvania State University	Examineur
M. Charles FAVRE Professeur, École Polytechnique (CMLS)	Directeur de thèse
M. Constantin SHRAMOV Professeur, Steklov Mathematical Institute	Co-directeur de thèse

École Polytechnique Centre de Mathématiques Laurent Schwartz
National Research University Higher School of Economics
Faculty of Mathematics

as a manuscript

Alexandra Kuznetsova

Birational automorphisms of varieties

PhD thesis for the purpose of obtaining academic degree Doctor of
Philosophy in Mathematics

Academic supervisors:
Charles Favre, PhD
Constantin Shramov, Doctor of Sciences

Paris - 2022

Эколь Политехник Математический Центр Лорана Шварца
Федеральное государственное автономное образовательное
учреждение высшего образования «Национальный
исследовательский университет «Высшая школа экономики»
Факультет математики

на правах рукописи

Александра Александровна Кузнецова

Бирациональные автоморфизмы многообразий

Диссертация на соискание ученой степени кандидата
математических наук

Научные руководители:
Шарль Фавр, PhD
Константин Шрамов, д.ф.-м.н.

Париж - 2022

Résumé

Soit X une variété algébrique projective définie sur un corps k . L'ensemble des applications bijectives algébriques $f: X \rightarrow X$ forme le groupe $\text{Aut}(X)$ des automorphismes réguliers. Étant donné un automorphisme régulier, on peut considérer son graphe:

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times X.$$

C'est une sous-variété de $X \times X$ telle que les projections $\text{pr}_i: \Gamma_f \rightarrow X$ sont des isomorphismes pour $i = 1$ et 2 . Plus généralement, on peut considérer l'ensemble de toutes les automorphismes birationnels $f: X \dashrightarrow X$. Ces automorphismes sont déterminées par une sous-variété Γ_f de $X \times X$ telle que les deux projections $\text{pr}_i: \Gamma_f \rightarrow X$ induisent un isomorphisme entre un sous-ensemble ouvert pour topologie de Zariski dans Γ_f sur un sous-ensemble ouvert pour topologie de Zariski dans X . La composition de deux automorphismes birationnels est birationnel donc l'ensemble de tous les automorphismes birationnels $\text{Bir}(X)$ forme un groupe.

Par construction $\text{Aut}(X)$ est un sous-groupe de $\text{Bir}(X)$. De plus, le groupe $\text{Aut}(X)$ admet une structure de schéma en groupes de type localement fini. Au contraire, le group $\text{Bir}(X)$ ne peut pas en général être doté d'une structure de schéma en groupes. Par exemple, il peut avoir une infinité de composants et leurs dimensions ne sont pas bornées. La complexité des groupes $\text{Aut}(X)$ et $\text{Bir}(X)$ dépend fortement de la géométrie de la variété algébrique. La thèse couvre divers problèmes sur des groupes d'automorphismes réguliers et birationnels de variétés algébriques et complexes, et explore les différences et les similitudes entre ces deux groupes.

Dans un premier temps, nous nous intéressons à la description des sous-groupes finis du groupe des automorphismes birationnels $\text{Bir}(X)$ d'une variété complexe rationnellement connexe X de dimension 3. Nous prouvons que tout 3-sous-groupe peut être généré par au plus 5 éléments et dans tous les cas sauf deux, il peut être généré par au plus 4 éléments. Nous décrivons également le groupe des automorphismes réguliers $\text{Aut}(S)$ d'une surface quasi-projective S définie sur un corps k tel que $\text{char}(k) = p > 0$. Nous montrons que ce groupe satisfait la propriété p -Jordan. L'un des principaux ingrédients ici est le programme du modèle minimal (MMP), qui nous permet de réduire des questions sur les groupes finis de transformations birationnelles à la classification de groupes d'automorphismes réguliers de variétés algébriques très particulières obtenues après application du MMP.

Ensuite, nous considérons des automorphismes birationnels d'ordre infini, et essayons de comprendre quand il est possible de construire un modèle birationnel où l'automorphisme induit est régulier. Nous nous intéressons principalement à l'exemple d'automorphismes birationnels d'une variété rationnelle de dimension 3 introduits par J. Blanc. Cet automorphisme birationnel induit

un pseudo-automorphisme sur un modèle birationnel spécial de \mathbb{P}^3 , c'est-à-dire qu'il a un petit lieu extrémal ; par conséquent, il est proche d'être régulier. Cependant, le résultat principal de la deuxième partie de la thèse est que ces applications ne sont pas conjuguées à des automorphismes réguliers. L'approche que nous adoptons dans cette partie est de nature dynamique, et l'action des automorphismes birationnels sur les espaces de Néron-Severi joue un rôle important.

La dernière partie de la thèse porte sur la description du groupe d'automorphismes d'une classe de variétés non-Kählériennes introduite par D. Guan et étudiée plus en détails par F. Bogomolov. Ces variétés sont des analogues non-Kähler des variétés hyperkählériennes, et leurs propriétés sont relativement similaires. Selon la construction de Bogomolov, ces variétés fibrent sur l'espace projectif avec des variétés abéliennes comme fibres génériques; ainsi, des outils algébriques peuvent être utilisés pour étudier leur géométrie. Plus précisément, nous prouvons que la réduction algébrique de la variété Bogomolov-Guan de dimension $2n$ est l'espace projectif de dimension n , puis nous étudions le lieu des fibres singulières d'une réduction algébrique et concluons que le groupe des automorphismes réguliers des variétés Bogomolov-Guan est Jordan. Dans le cas de la dimension 4 il en est de même pour le groupe des automorphismes biméromorphes.

Contents

Introduction	8
1 Finite groups of automorphisms	22
1.1 Introduction	22
1.2 Preliminaries	24
1.2.1 Finite groups	24
1.2.2 3-subgroups in $GL_n(\mathbb{C})$ and $PGL_n(\mathbb{C})$	26
1.2.3 Group action on algebraic varieties	29
1.2.4 Examples	33
1.2.5 p -Jordan property	33
1.2.6 Fano threefolds	34
1.3 Automorphisms of quasi-projective surfaces	38
1.3.1 Automorphisms of open subsets of $E \times \mathbb{P}^1$	38
1.3.2 Proof of Theorem 1.1.3	40
1.4 Automorphisms of 3-dimensional Mori fiber spaces	41
1.4.1 Singular Fano threefolds	41
1.4.2 Fano threefolds with $\rho > 1$	45
1.4.3 Fano threefolds with $\rho = 1$	47
1.4.4 Proof of Theorem 1.1.5	51
2 Regularization of birational automorphisms	53
2.1 Introduction	53
2.2 Preliminaries	55
2.2.1 Birational maps acting on the divisor class group	55
2.2.2 A special construction of a flop	59
2.2.3 Dynamical degrees	61
2.2.4 Example	62
2.3 Proof of Theorem 2.1.3	63
2.3.1 Regularizations	63
2.3.2 Fibrations over surfaces	66
2.4 Blanc's pseudo-automorphism	67
2.4.1 Construction of a pseudo-automorphism	67
2.4.2 Formulas for involutions	70

2.4.3	Composition of three involutions	72
2.4.4	Sage computations for Lemma 2.4.17	77
2.4.5	Proof of Theorem 2.1.2	81
3	Automorphisms of Bogomolov–Guan manifolds	82
3.1	Introduction	82
3.2	Preliminaries	83
3.2.1	Holomorphic symplectic manifolds	83
3.2.2	Complex spaces and meromorphic maps	85
3.3	Non-Kähler holomorphic symplectic manifolds	87
3.3.1	Kodaira surfaces	87
3.3.2	Bogomolov–Guan example	88
3.3.3	The algebraic reduction of BG-manifolds	91
3.4	Submanifolds in BG-manifolds	95
3.4.1	Kodaira-type manifolds	95
3.4.2	Submanifolds in S^n/F	97
3.4.3	Fibers of Φ and Π	100
3.4.4	Proof of Theorem 3.1.3	103
3.5	Automorphisms of BG-manifolds	104
3.5.1	Divisor D as a dual variety	104
3.5.2	Regular automorphisms of a BG-manifold	107
3.5.3	Bimeromorphic automorphisms of a BG-fourfold	108
	Conclusion	112
	Bibliography	113

Introduction

Let X be a projective algebraic variety defined over a field k . The set of invertible algebraic self-maps $f: X \rightarrow X$ forms the group $\text{Aut}(X)$ of regular automorphisms. Given a regular automorphism $f: X \rightarrow X$ one can consider its graph:

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times X.$$

It is a subvariety of $X \times X$ such that the projections $\text{pr}_i: \Gamma_f \rightarrow X$ are isomorphisms for $i = 1$ and 2 . More generally, we may consider the set of all birational self-maps $f: X \dashrightarrow X$. Such maps are determined by a subvariety Γ_f of $X \times X$ such that both projections $\text{pr}_i: \Gamma_f \rightarrow X$ induce an isomorphism between a Zariski open subset of Γ_f onto a Zariski open subset of X . Then for any point $x \in X$ one can define its total image: it is the set $f(x) = \text{pr}_2(\Gamma_f \cap \text{pr}_1^{-1}(x))$. A birational automorphism may have *indeterminacy locus*: it is an algebraic subvariety $\text{Ind}(f) \subset X$ such that the total image of any point $p \in \text{Ind}(f)$ has positive dimension. If X is normal then one has $\text{codim}(\text{Ind}(f)) \geq 2$. The union of all irreducible subvarieties $Z \subset X$ such that $\dim(f(Z)) < \dim(Z)$ is called the *exceptional locus* of f and we denote it by $\text{Exc}(f) \subset X$. When X is smooth, then $\text{Exc}(f)$ is of pure codimension 1. The composition of two birational automorphisms remains birational hence the set of all birational automorphisms $\text{Bir}(X)$ also forms a group.

By [Han87] the group $\text{Bir}(X)$ has a natural structure of a k -scheme. Namely, any birational automorphism f of X is determined by its graph Γ_f which is a subscheme in $X \times X$. Thus, the group $\text{Bir}(X)$ can be considered as a subscheme in the Hilbert scheme $\text{Hilb}(X \times X)$.

By construction $\text{Aut}(X)$ is a subgroup in $\text{Bir}(X)$. Moreover, by [MO67, Theorem 3.7] the group $\text{Aut}(X)$ admits a structure of a group scheme of locally finite type over k . Denote by $\text{Aut}(X)^0$ the connected component of the identity map $\text{id}_X: X \rightarrow X$. Then it is a group scheme of finite type (see e.g. [BSU13, Section 4.2]) and $\text{Aut}(X)$ fits into the following exact sequence of groups:

$$1 \rightarrow \text{Aut}(X)^0 \rightarrow \text{Aut}(X) \rightarrow \text{Aut}(X)/\text{Aut}(X)^0 \rightarrow 1.$$

The quotient group $\text{Aut}(X)/\text{Aut}(X)^0$ is a constant group scheme over k associated to at most countable abstract group with an action of the Galois group of k . Thus, the study of automorphisms of X can be reduced to the understanding of $\text{Aut}(X)^0$ and of the quotient group $\text{Aut}(X)/\text{Aut}(X)^0$.

It is a fact that $\text{Bir}(X)$ cannot be in general endowed with a structure of a group scheme, see [Han87, Remark 2.9]. Complexity of groups $\text{Aut}(X)$ and $\text{Bir}(X)$ highly depends on the geometry of the algebraic variety X defined over the field k . We will cover various problems on groups $\text{Aut}(X)$ and $\text{Bir}(X)$ in this thesis, and roughly explore the differences and similarities between these two groups.

If X is a smooth curve, then $\text{Aut}(X) = \text{Bir}(X)$ and one can understand the structure of this group very well. For simplicity let us here discuss the case where k is algebraically closed. First, if the genus of X is zero, then $X = \mathbb{P}^1$ and $\text{Aut}(X) = \text{PGL}_2(k)$. If the genus of X equals 1 then X is an elliptic curve and the group of points of $X(k)$ form a subgroup of finite index in the group $\text{Aut}(X)$, see [HKT08, Theorem 11.94]. If the genus of X equals $g \geq 2$ then by Hurwitz theorem $\text{Aut}(X)$ is a finite group, see [HKT08, Theorem 11.50], and its cardinality is at most $84(g-1)$ when the characteristic of k equals 0. In the case when the characteristic of k is positive there are also uniform bounds for the cardinality of $\text{Aut}(X)$ depending only of the genus of X , see [HKT08, Theorem 11.127].

If $\dim(X) \geq 2$ then $\text{Aut}(X)$ does not coincide with $\text{Bir}(X)$ in general. For instance, if X is a projective space \mathbb{P}_k^n over a field k , then $\text{Aut}(X) = \text{PGL}_{n+1}(k)$. The group $\text{Cr}_n(k)$ of birational automorphisms of \mathbb{P}_k^n , so-called *Cremona group of rank n* , is much bigger than the group of regular automorphisms $\text{Aut}(\mathbb{P}_k^n)$ if $n \geq 2$. In particular, if we consider $\text{Cr}_n(k)$ as a k -scheme then it has infinitely many components and their dimensions are not bounded, see [BF13].

On the other hand there are many varieties X for which the groups $\text{Aut}(X)$ and $\text{Bir}(X)$ coincide. Varieties satisfying this condition are said to be *birationally super-rigid*. If we assume that X is a *minimal model*, i.e. that the canonical class K_X of X is nef, and if there are no other minimal models in the birational class of X then one has $\text{Aut}(X) = \text{Bir}(X)$. Important classes of such varieties include abelian varieties and minimal surfaces of non-negative Kodaira dimensions, see [BHPVdV04, Theorem VI.1.1]. If $\dim(X) \geq 3$ we know fewer examples of birationally super-rigid varieties. If X is a variety of general type then by [HMX13] the cardinality of $\text{Bir}(X)$ can be bounded solely in terms of the dimension and volume of the canonical class, thereby generalizing Hurwitz theorem. Therefore, there exists a birational model \tilde{X} of X such that $\text{Aut}(\tilde{X}) = \text{Bir}(\tilde{X})$. In [BCHM10] there is a construction of such variety \tilde{X} , it is the canonical model of X . If the Kodaira dimension of a variety X is negative, then X does not admit a minimal model. However, among these varieties, some of them are still birationally super-rigid. It is a deep fact due to [IM71] that any smooth quartic hypersurface in \mathbb{P}^4 is a birationally super-rigid Fano threefold. More examples of such phenomenon have been subsequently found, see, e.g., [Puk98], [dF13], [CP17]; see also [Puk13].

Given a variety X , the construction of its minimal model is the subject of the *minimal model program (MMP)*; see, e.g., [CKM88]. The idea of this method is to single out curves intersecting negatively K_X , and to contract them (then maybe compose this with a small birational transformation). The result of MMP is a model X_0 of one of the following types:

- X_0 is a minimal model of X i.e. the canonical class K_{X_0} is nef;
- there exists a dominant morphism $\pi: X_0 \rightarrow B$ where $\dim(B) < \dim(X)$, the rank of the relative Picard group $\text{Pic}(X_0)/\pi^* \text{Pic}(B)$ equals 1 and the relative anticanonical class $-K_{X_0/B}$ is ample.

Recall that the variety X_0 as in the second case and with a restriction on its singularities is called a *Mori fiber space*. The case when we get a Mori fiber space corresponds to the situation when X admits no minimal model; nevertheless, MMP produces a model of X with nice properties which can be used in the study of birational and regular automorphisms of X .

Here is a brief list of the main topics of this thesis. Afterwards, they will be discussed in details.

In Chapter 1, we focus on the description of finite subgroups of $\text{Bir}(X)$ when X is a rationally connected complex threefold. We shall also describe $\text{Aut}(X)$ when X is a quasi-projective surface

defined over a field k such that $\text{char}(k) > 0$. One of the main ingredients here is MMP, which allows us to reduce questions about finite subgroups of $\text{Bir}(X)$ to classifying groups of automorphisms of very special algebraic varieties that arise as the final result of MMP.

In Chapter 2 we consider birational automorphisms of infinite order, and try to understand when it is possible to construct a birational model where the induced automorphism is regular. We are mainly interested in the example of a birational automorphism of a rational threefold introduced by J. Blanc in [Bla13]. The main result here is that it is not conjugate to a regular automorphism. Approach which we take in this part is dynamical in nature, and the action of birational maps on the cohomology groups plays an important role.

Finally Chapter 3 is concerned with the description of the automorphism groups of non-Kähler manifolds introduced by D. Guan [Gua94] and further studied by F. Bogomolov [Bog96]. These manifolds are non-Kähler analogues of hyperkähler manifolds; thus, we expect that their properties are similar. By Bogomolov’s construction these manifolds fiber over the projective space with abelian varieties as generic fibers; thus, algebraic tools can be used to study their geometry.

Finite groups of automorphisms

In this section, we summarize the results that will be presented in Chapter 1 of this thesis. Recall that the Cremona group $\text{Cr}_n(k)$ of rank n is the group of birational automorphisms of the projective space \mathbb{P}_k^n . Two striking results about the Cremona group of rank 2 were nearly simultaneously published in 2009. On the one hand, I. Dolgachev and V. Iskovskikh in [DI09] gave a complete classification of all finite subgroups of $\text{Cr}_2(\mathbb{C})$. On the other hand, J.-P. Serre in [Ser09] proved that the group $\text{Cr}_2(k)$ satisfies the Jordan property for any field k of characteristic 0. The following definition was introduced by V. Popov.

Definition 0.0.1. A group Γ is said to satisfy the *Jordan property* if there exists $J > 0$ such that any finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \subset G$ with $[G : A] \leq J$.

Serre subsequently conjectured that the Cremona group of any rank satisfies the Jordan property over a field of characteristic 0. Serre’s conjecture was proved by Yu. Prokhorov and C. Shramov in [PS16]: they established the Jordan property for Cremona groups of all ranks over a field of characteristic 0 assuming the Borisov–Alexeev–Borisov conjecture which was later proved by C. Birkar in [Bir21]. The Serre’s conjecture inspired study of Jordan property for automorphisms groups of different varieties. An interesting statement of this type was proved by V. Popov in [Pop11, Theorem 2.32]: he showed that in characteristic 0 the group of birational automorphisms of a surface satisfies the Jordan property for all but concretely described birational classes of surfaces. Then S. Meng and D.-Q. Zhang in [MZ18] showed that the Jordan property holds for groups of regular automorphisms of all projective varieties over a field of characteristic 0. The Jordan property was also established for groups of regular automorphisms of Kähler manifolds and (possibly singular) manifolds in the Fujiki class \mathcal{C} , see [Kim18] and [MPZ20]. Groups of birational automorphisms of non-projective compact complex surfaces and threefolds were also proved to be Jordan, see [PS21a], [PS20], [PS21b]. Also T. Bandman and Yu. Zarkhin in [BZ15] showed that the group of regular automorphisms of a quasi-projective complex surface always satisfies the Jordan property.

Some of these results are true over a field k of positive characteristic p . For instance, Prokhorov and Shramov managed to show that the Cremona group $\mathrm{Cr}_2(k)$ is Jordan if k is a finite field. However, if k is an algebraically closed field of positive characteristic the situation is much harder. Actually, many Lie groups over such field do not satisfy the Jordan property, see Example 1.2.35. In view of this F. Hu suggested the following analogue of the Jordan property:

Definition 0.0.2 ([Hu20, Definition 1.6]). We say that a group Γ is *p-Jordan*, if there exist constants $J(\Gamma)$ and $e(\Gamma)$ depending only on Γ such that any finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup A with

$$[G : A] \leq J(\Gamma) \cdot |G_p|^{e(\Gamma)},$$

where G_p is a Sylow p -subgroup of G .

This definition was motivated by the work of M. J. Larsen and R. Pink [LP11] in which they established the p -Jordan property for the group $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ for any prime number p and any $n > 0$. Then Hu proved in [Hu20] that the group of regular automorphisms of any projective variety over a field of characteristic $p > 0$ satisfies the p -Jordan property. Moreover, Y. Chen and C. Shramov in [CS21] generalized Popov's result to positive characteristic; namely, they proved that the group of birational automorphisms of an algebraic surface over an algebraically closed field of characteristic $p > 0$ satisfies the p -Jordan property for all but concretely described birational classes of surfaces. The birational type of the surface S when $\mathrm{Bir}(S)$ is not Jordan is if S is the product $\mathbb{P}^1 \times E$ where E is an elliptic curve.

We study finite subgroups in groups of automorphisms of quasi-projective surfaces, thereby extending Bandman and Zarkhin's theorem to positive characteristic. Here is the first result of the thesis:

Theorem 0.0.3. *If S is a quasi-projective surface defined over a field of characteristic $p > 0$, then the group $\mathrm{Aut}(S)$ is p -Jordan.*

The idea of the proof is the following (the complete proof may be found in Chapter 1). Since the subgroup of a p -Jordan group is p -Jordan then in view of Chen and Shramov's result the theorem can be reduced to the case when S is birationally equivalent to the product $\mathbb{P}^1 \times E$ where E is an elliptic curve. We construct a compactification \overline{S} of S and consider the Albanese map $\pi: \overline{S} \rightarrow E$. If there is an irreducible component of $\overline{S} \setminus S$ whose image under π is a point on E then we prove that $\mathrm{Aut}(S)$ is Jordan. If $\overline{S} \setminus S$ consists of multisections of π then we show that the action of any element of the group $\mathrm{Aut}(S)$ induces a regular automorphism of \overline{S} . The proof of Theorem 0.0.3 highly relies on the fact that any unirational curve is rational. Note that this fact is not true in higher dimensions in positive characteristic, there exist many examples of unirational non-rational surfaces; see, for instance, [Shi74], [Kat81], [Miy76].

We now turn to a more precise discussion of finite groups of birational automorphisms of projective varieties. For $n \geq 3$ a complete description of all finite subgroups of $\mathrm{Cr}_n(\mathbb{C})$ is out of reach. We shall thus focus on bounding the cardinality of the generating sets of p -subgroups in $\mathrm{Cr}_3(\mathbb{C})$. Recall that if p is a prime number then a p -group is a finite group of order p^m for some $m \geq 0$.

The idea of considering such groups comes from the work [BB00] by L. Bayle and A. Beauville where they classified all birational involutions of \mathbb{P}^2 . Then T. de Fernex in [dF04] studied birational automorphisms of \mathbb{P}^2 of prime order, and Blanc in [Bla09] described all conjugacy classes of finite abelian subgroups in $\mathrm{Cr}_2(\mathbb{C})$. Beauville in [Bea07] proved sharp bounds on ranks of abelian p -subgroups of $\mathrm{Cr}_2(\mathbb{C})$ for all prime numbers p . Prokhorov in [Pro11] and [Pro14] extended this result

to dimension 3 and to a wider class of varieties; he proved bounds on the rank of abelian p -subgroups in the group $\text{Bir}(X)$ of birational automorphisms of any rationally connected threefold X .

Prokhorov and Shramov in [PS18] proved that if $p \geq 17$ is a prime number and X is a rationally connected threefold, then a p -subgroup in $\text{Bir}(X)$ is necessarily abelian, its rank is at most 3 and this bound is sharp. In the work by J. Xu [Xu20] this result was generalized to all prime numbers $p \geq 5$; this result in fact gives an estimation on the rank of p -subgroups of $\text{Bir}(X)$ for a rationally connected variety X of any dimension. Prokhorov in [Pro14] gave a sharp bound on the number of generators of any 2-subgroup in $\text{Bir}(X)$ for a rationally connected threefold X . Thus, we have a sharp bound on the number of generators of a p -subgroup in $\text{Bir}(X)$ for a rationally connected threefold X and all prime numbers except $p = 3$.

In our work we study the last remaining case of 3-subgroups in the group $\text{Bir}(X)$ for a rationally connected threefold X . Prokhorov in [Pro11] proved that any abelian 3-subgroup can be generated by at most 5 elements. We extend this result to not necessarily abelian groups and prove the following theorem:

Theorem 0.0.4. *Let X be a projective rationally connected complex threefold and let G be a 3-subgroup in $\text{Bir}(X)$. Then the following is true:*

1. *The group G can be generated by at most 5 elements.*
2. *If G cannot be generated by 4 elements, then $G \subset \text{Aut}(X_0)$ where X_0 satisfies one of the following properties:*
 - (a) *X_0 is a Fano threefold with terminal singularities, the number of its non-Gorenstein singular points is 9 and all these points are cyclic quotient singularities of type $\frac{1}{2}(1, 1, 1)$.*
 - (b) *X_0 is a Fano threefold with terminal Gorenstein singularities with $\text{Pic}(X_0) = \mathbb{Z}K_{X_0}$ of genus 7 or 10 and the number of singular points of X_0 is 9 or 18.*

The second assertion of Theorem 0.0.4 relies heavily on the G -equivariant version of MMP, see [Pro21]. Recall that the G -equivariant MMP which starts with a variety X with a faithful regular action of a finite group G and its result is another variety X_0 with a regular action of G which is G -birational to X . Moreover, X_0 is either a minimal model or it is a G -Mori fiber space i.e. an equivariant analogue of a Mori fiber space. Note that a rationally connected threefold cannot have a minimal model by [KMM92].

If X is a rationally connected threefold and G is a finite subgroup in $\text{Bir}(X)$, then one can construct a birational model \tilde{X} of X such that $G \subset \text{Aut}(\tilde{X})$. Then we apply G -equivariant MMP to \tilde{X} with the action of G and obtain a G -Mori fiber space X_0 . Thus, Theorem 0.0.4 is a consequence of the following proposition.

Proposition 0.0.5. *Let G be a 3-group and let X_0 be a G -Mori fiber space of dimension 3. Then G can be generated by at most 4 elements unless X_0 is a Fano threefold which satisfies properties (a) or (b) in Theorem 0.0.4.*

The equivariant MMP works in the class of complex threefolds with terminal singularities endowed with an action of a finite group. Mori fiber spaces with terminal singularities are relatively well studied, see [Pro13a], [Pro13b], [BCZ04]. Moreover, by [Isk79], [MM82] and other works there exists a complete classification of smooth Fano threefolds. In the proof of Proposition 0.0.5 we use also many results on the geometry of various types of Fano threefolds and on the properties of terminal singularities.

Recently Loginov in [Log21] has studied in more details Fano varieties which satisfy properties (a) and (b) in Theorem 0.0.4. He was able to prove that in both cases the group G can be generated by at most 4 elements. This leads us to the following corollary.

Corollary 0.0.6. *Let G be a 3-subgroup in a group $\text{Bir}(X)$ where X is a complex rationally connected threefold. Then G can be generated by at most 4 elements and this bound is sharp.*

The sharpness of the bound follows from Example 1.2.33.

Regularization of birational automorphisms

In Chapter 2 we shall focus our attention on birational automorphisms of infinite order. The set-up will be as follows. Let X be a normal projective variety defined over an algebraically closed field k of characteristic 0. We say that a birational automorphism $f: X \dashrightarrow X$ is *regularizable on Y* if there exists a birational map $\alpha: X \dashrightarrow Y$ to a projective variety Y and $g \in \text{Aut}(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \alpha \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{g} & Y \end{array}$$

The question whether one can regularize a given birational automorphism $f: X \dashrightarrow X$ becomes increasingly difficult when dimension of X grows. In the curve case the question is trivial: any birational automorphism is already regular. In order to recall the known results in higher dimensions we need the following definitions.

Let X be a smooth variety and let $N^i(X)$ be the \mathbb{R} -vector space generated by classes of irreducible subvarieties of codimension i in X modulo numerical equivalence. Let $H \in N^1(X)$ be an ample divisor class on X and let $\dim(X) = d$. One can define the class $f^*(H^i) \in N^i(X)$ by taking the class of the proper preimage under f^{-1} for a general subvariety in the class of $H^i \in N^i(X)$. Then the *i -th degree of f* for $0 \leq i \leq d$ is defined as the following number:

$$\deg_i(f) = (f)^*(H^i) \cdot H^{d-i}.$$

By [DS05] the growth rate of the sequence $(\deg_i(f^n))_{n>0}$ is a birational invariant of the pair (X, f) . In particular, it does not depend on the choice of the ample divisor H , see also [Tru20]. Moreover, the sequence $(\deg_i(f^n))_{n>0}$ is submultiplicative in n , i.e. there exists the following limit:

$$\lambda_i(f) = \lim_{n \rightarrow \infty} (\deg_i(f^n))^{\frac{1}{n}}.$$

The number $\lambda_i(f)$ is called the *i -th dynamical degree* of the birational automorphism f . By [DN11] and [Tru20] the numbers $\lambda_i(f)$ are real, satisfy $\lambda_i(f) \geq 1$ and they are birational invariants of the pair (X, f) for $0 \leq i \leq d$. In particular, they do not depend on the choice of the ample divisor H . Moreover, $\lambda_1(f) = \lambda_d(f) = 1$ and dynamical degrees are log-concave, i.e. one has the following inequality for all $1 \leq i \leq d-1$:

$$\lambda_{i-1}(f) \cdot \lambda_{i+1}(f) \leq \lambda_i(f)^2.$$

Using the terminology coined in [BV99] we say that a birational automorphism f is of *positive entropy* if for some $0 \leq i \leq d$ one has $\lambda_i(f) > 1$. This terminology is justified by the fact that the topological entropy of f equals $\log(\max_{0 \leq i \leq d}(\lambda_i(f)))$ when f is a regular automorphism of a compact projective complex variety by theorems of Gromov and Yomdin [Gro03] and [Yom87], see also [DS05]. Note that by the log-concavity of dynamical degrees one has f is a positive entropy automorphism if and only if $\lambda_1(f) > 1$.

By [Wei55] (see also [Dés21, Section 3.5]) if the sequence $(\deg_i(f^n))_{n>0}$ is bounded, then there exists a birational model X_0 of X such that f is regularizable on X_0 . Otherwise, one has an additional structure associated with the automorphism f which allows us to understand better its properties.

If $\dim(X) = 2$ and the sequence $(\deg_1(f^n))_{n>0}$ is not bounded then by [BC16] the dynamical degree $\lambda_1(f)$ is an algebraic number with special properties, i.e. it is either a Salem or a Pisot number. Moreover, by [DF01] if the sequence $(\deg_1(f^n))_{n>0}$ is not bounded and $\lambda_1(f) = 1$, then $(\deg_1(f^n))_{n>0}$ grows as n or as n^2 . The birational types of complex surfaces which admit a positive entropy birational automorphism are described in [Can99]; moreover, there exists many examples of surface positive entropy automorphisms, see for instance [McM07], [Bla08] and [BK09]. The first dynamical degree is proved to be lower semi-continuous in families of birational automorphisms of surfaces by [Xie15].

If $\dim(X) = 2$ the growth rate of the sequence $(\deg_i(f^n))_{n>0}$ in many situations determines whether f is regularizable or not. By [DF01] if $\lambda_1(f) = 1$ then f is regularizable if and only if the sequence $(\deg_i(f^n))_{n>0}$ is bounded or grows as n^2 . By [BC16] if $\lambda_1(f)$ is a Salem number then f is regularizable. Moreover, there is a more complicated criterion by [DF01]. It claims that if $\lambda_1(f) > 1$ and there exists a divisor class θ on X such that $f^*\theta = \lambda_1(f)\theta$ and $\theta^2 = 0$, then f is a regularizable automorphism.

Positive entropy automorphisms in higher dimensions are much more complicated. Unlike the case of surfaces a positive entropy automorphism $f: X \dashrightarrow X$ of a smooth projective variety X such that $\dim(X) \geq 3$ can *preserve a fibration* i.e. there can exist a dominant map $\pi: X \dashrightarrow B$ to some variety B and $g \in \text{Bir}(B)$ such that $1 \leq \dim(B) \leq \dim(X) - 1$ and $\pi \circ f = g \circ \pi$. If f preserves a fibration we say that it is *imprimitive*. By J. Lesieutre [Les18] we get that if X is a smooth complex projective threefold and $f: X \dashrightarrow X$ is a positive entropy birational non-regular automorphism which can be regularized on a variety constructed by an iterated blow-up of X in smooth subvarieties, then f is imprimitive.

Note that for any regular automorphism $f: X \rightarrow X$ the canonical class K_X is f^* -invariant. If K_X is an ample or an anti-ample divisor, then one can use it in order to compute the dynamical degree of f . Therefore, if X is a Fano threefold then there is no positive entropy automorphism of X . Moreover, by Lesieutre's result any iterated blow-up of a smooth complex Fano threefold in smooth subvarieties admits no regular primitive positive entropy automorphism. Thus, it is very complicated to construct an example of a primitive positive entropy regular automorphism on a rational threefold. At the moment there is known only one example of such automorphism described in [OT15].

Attempts to generalize regular positive entropy automorphisms of rational surfaces resulted in constructions of birational automorphisms [BK14], [PZ14], [Bla13] which turn out to be *pseudo-automorphisms*. Recall that a birational map $f: X \dashrightarrow X$ of a smooth variety X is called a pseudo-automorphism if neither f nor f^{-1} contract any divisor in X . Note that in the case of surfaces any pseudo-automorphism is a regular automorphism. Thus, pseudo-automorphisms form a class of birational automorphisms which are very close to being regular. One might expect that

any pseudo-automorphism can be regularized and under appropriate assumptions this indeed is true, see Section 2.2.4; however, it may be false. Here we list some known constructions of positive entropy pseudo-automorphisms of rationally connected threefolds:

Example 0.0.7 ([BK14]). This example is obtained as a generalization of a surface automorphism construction from [BK09]. Fix $a \in \mathbb{C} \setminus \{0\}$ and a primitive third root of unity ζ and consider the birational automorphism of \mathbb{P}^3 :

$$f_{a,\zeta}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3; \quad f_{a,\zeta}(x_0 : x_1 : x_2 : x_3) = (x_0x_1 : x_1x_2 : x_1x_3 : ax_0^2 + \zeta x_0x_2 + x_0x_3).$$

Then $\lambda_1(f_{a,\zeta}) = \lambda_2(f_{a,\zeta}) > 1$ and $f_{a,\zeta}$ induces a pseudo-automorphism on a blow-up of \mathbb{P}^3 in several points and curves. Moreover, $f_{a,\zeta}$ is imprimitive and if $a \neq 1$ then $f_{a,\zeta}$ is non-regularizable.

Example 0.0.8 ([BCK14]). Let a, c be complex numbers and let $f_{a,c}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a birational automorphism defined as $f_{a,c} = L_{a,c} \circ J$ where

$$\begin{aligned} L_{a,c}(x_0 : x_1 : x_2 : x_3) &= (x_3 : x_0 + ax_3 : x_1 : x_2 + cx_3); \\ J(x_0 : x_1 : x_2 : x_3) &= (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1}). \end{aligned}$$

Thus, $f_{a,c}$ is the composition of a regular automorphism $L_{a,c}$ of \mathbb{P}^3 and the Cremona involution J . If a and c satisfy a certain quadratic equation then $\lambda_1(f_{a,c}) = \lambda_2(f_{a,c}) > 1$ and $f_{a,c}$ induces a pseudo-automorphism on a blow-up of \mathbb{P}^3 in several points. Moreover, $f_{a,c}$ is primitive and non-regularizable.

Example 0.0.9 ([PZ14], [BDK], [DO88]). This example is obtained as a generalization of a surface automorphism construction from [McM07]. There exists a blow-up $\delta: X \rightarrow \mathbb{P}^3$ of several points p_1, \dots, p_k in \mathbb{P}^3 and a bilinear form $(,)$ on the lattice $H^2(X, \mathbb{Z})$ which induces the structure of the root lattice. Thus, there is a Weyl group W with a natural representation in $H^2(X, \mathbb{Z})$. Moreover, for any element $w \in W$ there is a pseudo-automorphism

$$f_w: X \dashrightarrow X,$$

such that f_w^* acts on $H^2(X, \mathbb{Z})$ as w . We denote by w_0 the Coxeter element in W . If W is an infinite group we get that $\lambda_1(f_{w_0}) = \lambda_2(f_{w_0}) > 1$ and f_{w_0} is imprimitive.

Example 0.0.10 ([BL15]). Let $Y \subset \mathbb{P}^4$ be a smooth cubic threefold and let C_1 be a smooth curve of genus 2 and degree 6 on Y . Denote by Y_1 the blow-up of Y along C_1 , then the linear system $| -K_{Y_1} |$ induces on Y_1 a structure of a surjective morphism $\pi_1: Y_1 \rightarrow \mathbb{P}^3$ with general fiber equal to two points. The involution corresponding to the exchange of these points is a pseudo-automorphism $\tau_1: Y_1 \dashrightarrow Y_1$.

The base locus of a general pencil of hyperquadric sections containing C_1 is the union $C_1 \cup C_2$, where C_2 is a smooth curve of genus 2 and degree 6 on Y . Therefore, if we denote by \tilde{Y} the subsequent blow-up of curves C_1 and C_2 in Y , then the involutions associated to curves C_1 and C_2 as above induce pseudo-automorphisms $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of \tilde{Y} . The composition of these involutions is a pseudo-automorphism:

$$f_{Y,C_1,C_2} = \tilde{\tau}_1 \circ \tilde{\tau}_2: \tilde{Y} \dashrightarrow \tilde{Y},$$

Moreover, one has $\lambda_1(f_{Y,C_1,C_2}) = \lambda_2(f_{Y,C_1,C_2}) > 1$ and f preserves the pencil of quadrics passing through $C_1 \cup C_2$; thus, f is imprimitive.

Example 0.0.11 ([Bla13]). This example is obtained as a generalization of a surface automorphism construction from [Bla08]. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. With each point $p \in S$ one can associate a birational involution $\sigma_p: \mathbb{P}^3 \rightarrow \mathbb{P}^3$. For any collection p_1, \dots, p_k of k points on S consider the following birational automorphism:

$$f_{p_1, \dots, p_k} = \sigma_{p_1} \circ \dots \circ \sigma_{p_k}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3.$$

If the points p_1, \dots, p_k are general and $k \geq 3$ then $\lambda_1(f_{p_1, \dots, p_k}) = \lambda_2(f_{p_1, \dots, p_k}) > 1$ and f_{p_1, \dots, p_k} induces a pseudo-automorphism on a blow-up of \mathbb{P}^3 in several points and curves.

Examples 0.0.7 and 0.0.8 are proved to be non-regularizable. Examples 0.0.9 and 0.0.10 are non-primitive. Thus, we concentrate on the last example and prove that it is non-regularizable under appropriate assumptions. Here is the main result of Chapter 2.

Theorem 0.0.12. *Let $S \subset \mathbb{P}^3$ be a very general smooth complex cubic surface and let p_1, p_2, p_3 be general points on S . Then the birational automorphism $f_{p_1, p_2, p_3}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ described in Example 0.0.11 is non-regularizable and does not preserve a fibration over a surface.*

There are several criteria which allow us to prove that a birational automorphism of a threefold is non-regularizable. By [CDX21] if $\deg_1(f^n)$ grows as n^k where k is odd, then f is non-regularizable; also if $\lambda_1(f) > 1$ is an integer then f is non-regularizable. By [LB19] if X is a threefold and if the sequence $(\deg(f^n))_{n>0}$ grows as n^k where $k > 4$ then f is non-regularizable. Also if the number $\lambda_1(f)$ does not satisfy several conditions given in [LB19, Proposition 4.6.7, 4.7.2, 5.0.1] then f is non-regularizable. Another criterion which does not use dynamical properties of the birational automorphism f was used to prove that Examples 0.0.7 and 0.0.8 are non-regularizable. It is based on [BK14, Corollary 1.6] which says that if $f: X \dashrightarrow X$ is a birational automorphism of a smooth threefold X and if Y is an f -invariant surface in X such that the birational automorphism $f|_Y: Y \dashrightarrow Y$ is non-regularizable then f is non-regularizable. In both Examples 0.0.7 and 0.0.8 one can find an f -invariant surface Y such that $\lambda_1(f|_Y)$ is a Pisot non-quadratic number; thus, by [BC16] we get that $f|_Y$ is non-regularizable.

All these arguments do not work in Example 0.0.11. Thus, in order to prove Theorem 0.0.12 we establish a new criterion. In order to formulate it recall that if $f: X \rightarrow X$ is a pseudo-automorphism such that one has an inequality $\lambda_1(f)^2 > \lambda_2(f)$, then by [Tru14] there exists a unique up to proportionality pseudo-effective divisor class $\theta_1(f)$ such that:

$$f^*\theta_1(f) = \lambda_1(f)\theta_1(f). \tag{0.0.13}$$

Such class was successfully used in [DF01] for the necessary condition on the existence of a regularization of a surface birational automorphism. Now we can formulate our criterion:

Theorem 0.0.14. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth projective threefold X such that*

- (1) $\lambda_1(f)^2 > \lambda_2(f)$; thus, there exists a class $\theta_1(f)$ as in (0.0.13);
- (2) there exists a curve C such that $\theta_1(f) \cdot [C] < 0$;
- (3) there exist infinitely many integers $m > 0$ such that $C \not\subset \text{Ind}(f^{-m})$.

Then f is non-regularizable and it does not preserve a fibration over a surface.

Some comments are in order. By log-concavity of dynamical degrees one has $\lambda_1(f)^2 \geq \lambda_2(f)$. Thus, the condition (1) is always true either for f or for f^{-1} since $\lambda_2(f) = \lambda_1(f^{-1})$. Condition (2) implies that the class $\theta_1(f)$ is not nef and the last condition is required to avoid situations which we describe in Section 2.2.4.

To prove Theorem 0.0.12 we show that the pseudo-automorphism model $\tilde{f}_{p_1, p_2, p_3} : X \dashrightarrow X$ of f_{p_1, p_2, p_3} constructed in [Bla13] satisfies the condition of Theorem 0.0.14. The first condition is obviously true. To show the second condition we consider a curve of indeterminacy of $\tilde{f}_{p_1, p_2, p_3}$ and prove that it intersects $\theta_1(\tilde{f}_{p_1, p_2, p_3})$ negatively. This curve is the proper transform of a line L from \mathbb{P}^3 to X . The verification of the third condition is quite difficult; we prove that the line L does not lie in the indeterminacy locus of f_{p_1, p_2, p_3}^{-m} using explicit formulas for the involutions σ_{p_i} . Most of our computations were done in Sage. Dealing with three involutions already makes our proof tricky. We expect that our theorem is valid for any composition of at least three involutions associated to general points on S .

Automorphisms of Bogomolov–Guan manifolds

In Chapter 3 we shall use the following convention: by a *manifold* we denote a locally ringed space which is locally isomorphic to an open subset of the vanishing locus of a finite set of holomorphic functions in \mathbb{C}^n . In particular, manifolds are not assumed to be smooth. By a *variety* or an *algebraic variety* we denote the vanishing locus of a finite set of polynomials in the complex projective space \mathbb{P}^n .

Our goal in this chapter is to study a special class of non-Kähler complex compact manifolds. These manifolds are particularly interesting in view of their similarity with hyperkähler manifolds. Recall that a *hyperkähler manifold* is a Riemannian manifold (M, g) which equipped with three Kähler complex structures $I, J, K : TM \rightarrow TM$, satisfying the quaternionic relation:

$$I^2 = J^2 = K^2 = IJK = -\text{id}.$$

Any hyperkähler manifold is holomorphically symplectic i.e. admits a non-degenerate $(2, 0)$ -form. Conversely, a compact holomorphically symplectic manifold is hyperkähler, provided that it is Kähler. This follows from the Calabi-Yau theorem [Yau78], see also [Bea83]. A hyperkähler manifold M is called *irreducible holomorphic symplectic (IHS)* if it is compact, complex, simply connected and the group $H^{2,0}(M)$ is 1-dimensional.

An example of non-Kähler manifolds which are very close to IHS manifolds was constructed in several papers by D. Guan [Gua94], [Gua95a] and [Gua95b]. Later F. Bogomolov in [Bog96] gave a more geometric construction for these manifolds. We recall here the main steps of the Bogomolov’s construction.

Let S be a *primary Kodaira surface*, i.e. a smooth complex compact holomorphic symplectic surface which admits a structure of an isotrivial elliptic fiber space:

$$\pi : S \rightarrow E,$$

over an elliptic curve E such that any submanifold in S which admits a structure of an algebraic variety is either a point or a fiber of π . All fibers of π are isomorphic to an elliptic curve F . The map π induces a structure of F -torsor on S . Denote by $\pi^{[n]} : S^{[n]} \rightarrow E^{[n]}$ the induced map between

Douady spaces of length n of S and E respectively. Denote by $\text{Alb}: E^{[n]} \rightarrow E$ the Albanese morphism of the algebraic variety $E^{[n]}$ which is isomorphic to the symmetric power $\text{Sym}^n(E)$. The following variety

$$W = (\pi^{[n]})^{-1}(\text{Alb}^{-1}(0))$$

is a complex manifold with an action of F induced from the diagonal action of F on $S^{[n]}$. Then by [Bog96, Corollary 4.10] under appropriate conditions on the Kodaira surface S there exists a smooth, compact, complex, simply connected manifold Q such that the group $H^{2,0}(Q)$ is generated by a non-degenerate holomorphic symplectic form and which is a finite cover of W/F :

$$p: Q \rightarrow W/F.$$

If $n = 2$ then Q is a K3-surface; if $n \geq 3$ then Q is a non-Kähler $(2n - 2)$ -dimensional manifold.

The manifold Q constructed as described above for $n \geq 3$ is the main object of study in Chapter 3, we call it the *BG-manifold* (for Bogomolov–Guan). Since the fiber $\text{Alb}^{-1}(0)$ is isomorphic to the $(n - 1)$ -dimensional projective space then the map $\pi^{[n]}$ induces the map $\Pi: W/F \rightarrow \mathbb{P}^{n-1}$. Thus, the BG-manifold Q admits a surjective map to a projective space:

$$\Phi = \Pi \circ p: Q \rightarrow \mathbb{P}^{n-1}. \tag{0.0.15}$$

Since many properties of BG-manifolds are similar to those of IHS manifolds we expect that many results about hyperkähler manifolds can be extended to the case of BG-manifolds. Recall here several significant results about hyperkähler manifolds. First, if M is a hyperkähler manifold then by [Fuj87] there exists an important non-degenerate symmetric quadratic form on the cohomology group $H^2(M, \mathbb{Z})$. This form is called the *Beauville–Bogomolov–Fujiki form* or *BBF-form* and it is very useful in the study of the geometry of hyperkähler manifolds and their moduli spaces. Another important result which later led to the proof of the Torelli theorem for hyperkähler manifolds is the Bogomolov–Tian–Todorov theorem [Bog78], which says that the deformation theory of a Kähler manifold with trivial canonical class is unobstructed. Groups of biholomorphic $\text{Aut}(M)$ and bimeromorphic $\text{Bim}(M)$ automorphisms of a hyperkähler manifold M were studied by N. Kurnosov and E. Yasinsky in [KY19] and by A. Cattaneo and L. Fu in [CF19]. They proved in particular that the order of finite subgroups in the groups $\text{Aut}(M)$ and $\text{Bim}(M)$ are bounded. Moreover, there are only finitely many conjugacy classes of finite subgroups in $\text{Aut}(M)$ and $\text{Bim}(M)$.

There are some partial extensions of these results to BG-manifolds. In [KV19] N. Kurnosov and M. Verbitsky proved the existence of a symmetric quadratic form on the cohomology group $H^2(Q, \mathbb{Z})$ on a BG-manifold Q analogous to the BBF-form. They conjectured that this form is non-degenerate and that it satisfies all properties of a BBF-form. Moreover, the study of holomorphic symplectic deformations and this symmetric form led them to the generalization of the Bogomolov–Tian–Todorov theorem to the class of BG-manifolds.

In this thesis we are going to explore the groups of biholomorphic and bimeromorphic automorphisms of BG-manifolds. In order to do this we find several structures on a BG-manifold Q which should be preserved under automorphisms. Since a BG-manifold Q is non-algebraic one can consider algebraic submanifolds in Q . The image of an algebraic submanifold under an automorphism is necessarily an algebraic submanifold. One can consider an *algebraic reduction* of Q i.e. a meromorphic map $f: Q \dashrightarrow X$ to an algebraic variety X such that any meromorphic map from Q to an algebraic variety factors through f . A map f with this property is unique up to birational conjugations. Our first result is the following description of the algebraic reduction of Q :

Theorem 0.0.16. *Let $n \geq 3$ be an integer and let Q be a BG-manifold of dimension $2n - 2$. Then the map $\Phi: Q \rightarrow \mathbb{P}^{n-1}$ described in (0.0.15) is an algebraic reduction of Q .*

Then we study subvarieties of BG-manifolds. Recall that a manifold X is called *Moishezon* if its algebraic reduction is a generically finite map. In particular, any algebraic variety is Moishezon. We prove the following result:

Theorem 0.0.17. *Let Q be a BG-manifold of dimension $2n - 2$ and let $\Phi: Q \rightarrow \mathbb{P}^{n-1}$ be its algebraic reduction as in (0.0.15). There exists a divisor $D \subset \mathbb{P}^{n-1}$ of degree $2n$ such that for any point $x \in \mathbb{P}^{n-1}$ one has:*

- (1) *If $x \in \mathbb{P}^{n-1} \setminus D$, then the fiber $\Phi^{-1}(x)$ is an abelian variety.*
- (2) *If $x \in D$, then the fiber $\Phi^{-1}(x)$ is a uniruled Moishezon manifold.*

Moreover, if $X \subset Q$ is a submanifold such that $\dim(\Phi(X)) \geq 2$ then X is not Moishezon.

By this theorem if $X \subset Q$ is a submanifold of a BG-manifold and $\Phi(X)$ is a point then X is Moishezon; in the case where $\dim(\Phi(X)) \geq 2$ one has X is not Moishezon. We also consider the case where $\dim(\Phi(X)) = 1$ and obtain that X may or may not be a Moishezon manifold depending on the curve $\Phi(X)$, see Lemma 3.4.11 and Theorem 3.4.20 for more precise statements.

By definition of an algebraic reduction any bimeromorphic or biholomorphic automorphism of a complex manifold is compatible with algebraic reduction. Therefore, we conclude that the group $\text{Aut}(Q)$ fits into the following exact sequence:

$$1 \rightarrow G'' \rightarrow \text{Aut}(Q) \rightarrow G' \rightarrow 1, \quad (0.0.18)$$

where G' is a subgroup of $\text{Aut}(\mathbb{P}^{n-1})$ and G'' is a subgroup of $\text{Aut}(A)$ where A is an abelian variety $\Phi^{-1}(x)$ and x is a point in $\mathbb{P}^{n-1} \setminus D$.

Theorem 0.0.17 implies that a biholomorphic automorphism of Q induces a regular automorphism of the projective space \mathbb{P}^{n-1} which preserves the divisor D . Thus, we study the geometry of D in details and prove that the group $\text{Aut}(Q)$ satisfies the Jordan property, see Definition 0.0.1. Here is the main result of Chapter 3:

Theorem 0.0.19. *Let Q be a BG-manifold of dimension $2n - 2$ and let $\Phi: Q \rightarrow \mathbb{P}^{n-1}$ be its algebraic reduction as in (0.0.15). Then the group $\text{Aut}(Q)$ fits into the exact sequence (0.0.18) where G' is a finite group, $G'' \subset \text{Aut}(A)$ where A is an $(n - 1)$ -dimensional abelian variety. In particular, $\text{Aut}(Q)$ is a Jordan group.*

This result follows from a description of the divisor D and its singular locus. We prove that D contains a finite set \mathcal{Z} of n^2 points of multiplicity $n - 1$ and that \mathcal{Z} does not lie in a hyperplane in \mathbb{P}^{n-1} . Thus, the group of automorphisms of \mathbb{P}^{n-1} fixing D should fix also \mathcal{Z} ; therefore, it is finite.

It would be extremely interesting to prove a similar result for the group of bimeromorphic automorphisms of a BG-manifold Q . By the same reasons as in the case of biholomorphic automorphisms the group $\text{Bim}(Q)$ fits into the following exact sequence:

$$1 \rightarrow H'' \rightarrow \text{Bim}(Q) \rightarrow H' \rightarrow 1,$$

where $H'' \subset \text{Aut}(A)$ and A is an $(n - 1)$ -dimensional abelian variety isomorphic to a general fiber of Φ and H' is a subgroup in the group $\text{Cr}_{n-1}(\mathbb{C})$ of birational automorphisms g of \mathbb{P}^{n-1} such that either D lies in $\text{Exc}(g)$ or D is g -invariant.

In the simplest case when $n = 3$ and $\dim(Q) = 4$ we managed to establish the Jordan property for the group $\text{Bim}(Q)$. However, in higher dimensions it is still unclear whether the group $\text{Bim}(Q)$ is Jordan or not.

Notation

Algebraic varieties and maps

$\text{Aut}(X)$: The group of biregular automorphisms if X is an algebraic variety or the group of biholomorphic automorphisms if X is a complex manifold.

$\text{Bir}(X)$: The group of birational automorphisms of the algebraic variety X .

$\text{Bir}(X)_\pi$: The group of automorphisms of X which preserve the fiber space $\pi: X \rightarrow B$.

$\text{Cr}_n(k)$: The group of birational automorphisms of the projective space \mathbb{P}_k^n over a field k .

$\text{Ind}(f)$: The indeterminacy locus of the birational map f .

$\text{Exc}(f)$: The exceptional locus of the birational map f .

$\text{Bs}(|D|)$: The base locus of the linear system $|D|$.

$\text{Jac}(C)$: The Jacobian variety of the smooth curve C .

$\text{Gr}(k, V)$: The Grassmannian of k -planes in the vector space V .

$\text{Pic}(X)$: The Picard group of the normal variety X ; i.e. the group of isomorphism classes of Cartier divisors on X modulo the linear equivalence relation.

$\text{Pic}^0(X)$: The connected component of the trivial element in $\text{Pic}(X)$.

$\rho(X)$: The rank of the group $\text{Pic}(X)/\text{Pic}^0(X)$.

$\text{Pic}(X/B)$: The relative Picard group of the fiber space $\pi: X \rightarrow B$; i.e. the group $\text{Pic}(X)/\pi^*\text{Pic}(B)$.

$\text{Sing}(X)$: The singular locus of the normal variety X .

$N^i(X)$: The group of classes of subvarieties of codimension i in the variety X modulo numerical equivalence.

$\lambda_i(f)$: The i -th dynamical degree of the birational automorphism f .

$\theta_1(f)$: The pseudo-effective class in $N^1(X)$ such that $f^*(\theta_1(f)) = \lambda_1(f)\theta_1(f)$ where $f \in \text{Bir}(X)$.

$T_{X,x}$: The Zariski tangent space to the algebraic variety X in the point x .

$r(X)$: The index of the Gorenstein Fano threefold X with $\rho(X) = 1$.

$g(X)$: The genus of the Gorenstein Fano threefold X with $\rho(X) = 1$ and $r(X) = 1$.

Complex manifolds and maps

$\text{Bim}(M)$: The group of bimeromorphic automorphisms of the complex manifold M .

$\mathcal{M}(M)$: The field of meromorphic functions of the complex manifold M .

$a(M)$: The algebraic dimension of the complex manifold M .

$\pi_1(M)$: The fundamental group of the complex manifold M .

$\text{Sym}(M)$: The symmetric space of the complex manifold M i.e. the quotient space M^n / \mathfrak{S}_n where the action of \mathfrak{S}_n on M^n is natural.

$M^{[n]}$: The Douady space of the complex manifold M .

Groups

$[G : H]$: The index of the subgroup H in the group G .

C_n : The cyclic group $\mathbb{Z}/n\mathbb{Z}$.

C_n^\times : The group of automorphisms of C_n .

\mathfrak{S}_n : The group of permutations of n elements.

$\text{GL}_n(A)$: The general linear group of degree n over the algebra A .

$\text{SL}_n(A)$: The special linear group of degree n over the algebra A .

$\text{PGL}_n(A)$: The projective linear group of degree n over the algebra A .

$\text{PSO}_n(A)$: The projective special orthogonal group of degree n over the algebra A with a fixed quadratic form.

\mathcal{H}_3 : The Heisenberg group modulo 3, i.e. the non-abelian group of 27 elements which is non isomorphic to $C_3 \times C_9$.

$\tilde{\mathcal{H}}_3$: The normalizer of the image of the Heisenberg group modulo 3 under the standard embedding to $\text{SL}_3(\mathbb{C})$.

1. Finite groups of automorphisms

1.1. Introduction

In this section we study the structure of finite subgroups in groups of automorphisms of some varieties. We consider two different classes of finite subgroups and deduce that finite subgroups there are bounded in some sense.

First we discuss groups of automorphisms of varieties over a field of positive characteristic. Finite subgroups of such groups have a substantially more complicated structure than in case of characteristic zero. We shall see for instance that the group $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ does not enjoy the Jordan property, see Example 1.2.35.

To remedy this issue Hu in [Hu20] has introduced a class of groups which he called p -Jordan and is a generalization of the Jordan property to characteristic $p > 0$. He also obtained the following theorem:

Theorem 1.1.1 ([Hu20, Theorems 1.7, 1.10]). *Let k be a field of characteristic $p > 0$. Then for any projective variety X defined over k , the group $\mathrm{Aut}(X)$ is p -Jordan.*

Theorem 1.1.1 drew the attention of many researches that tried to extend it for groups of birational transformations of varieties in positive characteristic. One of the most interesting results in this area is the following analogue of Popov's result [Pop11] in positive characteristic:

Theorem 1.1.2 ([CS21, Theorem 1.7]). *Let S be an irreducible algebraic surface defined over a field of characteristic $p > 0$. Then the group $\mathrm{Bir}(S)$ is p -Jordan unless S is birationally equivalent to the product $E \times \mathbb{P}^1$ of an elliptic curve E and a projective line.*

Our goal is to prove an analogue of T. Bandman and Yu. Zarhin theorem [BZ15] in positive characteristic:

Theorem 1.1.3. *Let S be a quasi-projective surface defined over a field of characteristic $p > 0$. Then the group $\mathrm{Aut}(S)$ is p -Jordan.*

We now turn to finite subgroups of the Cremona group $\mathrm{Cr}_n(\mathbb{C})$ of rank n . Finite subgroups in the Cremona group of rank 2 were studied in many works, see e.g. [dF04], [Bla09]. In [DI09] a

complete classification of finite subgroups in $\mathrm{Cr}_2(\mathbb{C})$ was proved. However, since the list of these finite subgroups is very big it is more reasonable to study embeddings of finite groups of some type to the Cremona group and estimate size of this groups. This approach was used by Beauville in [Bea07] where he studied embeddings of groups C_p^r , where p is a prime number, to the Cremona group of rank two and proved bounds on the rank r depending on p . This result was extended by Prokhorov to the case of Cremona group of rank 3. Namely in works [Pro14] and [Pro11] he considered abelian p -subgroups (i.e. finite subgroups of order p^n for an integer n) in groups of birational automorphisms of rationally connected threefolds. Note that the class of rationally connected threefolds includes \mathbb{P}^3 .

Theorem 1.1.4 ([Pro11, Theorem 1.2], [Pro14, Theorem 1.2]). *Let X be a projective rationally connected complex threefold, let G be an abelian p -subgroup of $\mathrm{Bir}(X)$. Then G can be generated by r elements, with*

$$r \leq \begin{cases} 6, & \text{if } p = 2 \text{ and this bound is sharp;} \\ 5, & \text{if } p = 3; \\ 4, & \text{if } p = 5, 7, 11 \text{ or } 13; \\ 3, & \text{if } p \geq 17 \text{ and this bound is sharp.} \end{cases}$$

Our next theorem focuses on the case of 3-subgroups and deals with not necessarily abelian groups.

Theorem 1.1.5. *Let X be a projective rationally connected complex threefold and let G be a 3-subgroup in $\mathrm{Bir}(X)$. Then the following is true:*

1. *The group G can be generated by at most 5 elements.*
2. *If G can not be generated by at most 4 elements, then $G \subset \mathrm{Aut}(X_0)$ where X_0 satisfies one of the following properties:*
 - (a) *X_0 is a Fano threefold with terminal singularities, the number of its non-Gorenstein singular points is 9 and all these points are cyclic quotient singularities of type $\frac{1}{2}(1, 1, 1)$.*
 - (b) *X_0 is a Fano threefold with terminal Gorenstein singularities with $\mathrm{Pic}(X_0) = \mathbb{Z}K_{X_0}$ of genus 7 or 10 and the number of singular points of X_0 is 9 or 18.*

The proof of Theorem 1.1.5 uses MMP. This allows us to prove that for any finite subgroup in $\mathrm{Bir}(X)$ one can construct a terminal complex threefold X_0 with a faithful action of G such that X_0 is a G -Mori fiber space. Thus, Theorem 1.1.5 follows from the following key assertion:

Proposition 1.1.6. *Let G be a 3-group and X_0 be a G -Mori fiber space of dimension 3. Then G can be generated by at most 4 elements unless X_0 satisfies properties (a) or (b) in Theorem 1.1.5.*

Terminal G -Mori fiber spaces are not classified; however, there are several classes of these varieties and in each case we prove a bound on the number of generators of G . We use various methods to deal with Mori fiber spaces which base have positive dimension, Fano threefolds admitting a terminal non-Gorenstein singular point, Gorenstein Fano threefolds which Picard number is greater than 1; then we consider Gorenstein Fano threefolds with Picard number 1 and use the classification of smooth Fano threefolds [Isk79] and the Namikawa smoothing [Nam97] and many other results. Proceeding case by case we prove the proposition.

1.2. Preliminaries

1.2.1. Finite groups

Here we list well-known facts about generators of finite groups and discuss in more details the case of the Heisenberg group.

Recall that a finite group G is a p -group if its order equals p^n for some integer n . If one can estimate the number of generators of a subgroup of a p -group this gives us the information about the number of generators of the whole group.

Lemma 1.2.1. *Let G be a p -group and H be a subgroup of G of index p^n . If H can be generated by m elements, then G can be generated by $n + m$ elements.*

Proof. Choose an element $g \in G$ outside H and consider a subgroup $H' \subset G$ generated by m generators of H and g . We have

$$[G : H'] < [G : H].$$

Thus, H' is a subgroup of index at most p^{n-1} . We conclude by induction that G can be generated by $n + m$ elements. \square

Now consider the situation where the group G is an extension of two groups and we can estimate numbers of generators of them.

Lemma 1.2.2. *Assume that a finite group G is an extension of a group G_2 by a group G_1 and each group G_i can be generated by n_i elements for $i = 1, 2$. Then the group G can be generated by $n_1 + n_2$ elements.*

Proof. Choose sets of generators x_1, \dots, x_{n_1} and y_1, \dots, y_{n_2} of groups G_1 and G_2 . Denote by \tilde{x}_i the image of x_i in G and choose a preimage \tilde{y}_j of y_j in G .

Take any element $z \in G$. Then the image of z in G_2 equals to a product of generators $\prod y_i^{d_i}$. Therefore, the product $z \cdot (\prod \tilde{y}_i^{d_i})^{-1}$ equals to $\prod \tilde{x}_j^{c_j}$. This concludes the proof. \square

Note however that if H is a subgroup of G the knowledge about the number of generators of G gives almost no information about the number of generators of H . For example, any finite group can be embedded to a group of permutation \mathfrak{S}_N which can be generated by two its elements: a transposition and a cycle of length N . In the case of subgroups of a direct product, we can still extract some information.

Lemma 1.2.3. *Assume that $H \subset \prod_{i=1}^N G_i$ and for each i any subgroup of G_i can be generated by n_i elements. Then H can be generated by $\sum_{i=1}^N n_i$ elements.*

Proof. We prove this by induction on N . If $N = 1$, then the assertion is trivial. Assume that the assertion is true for $N - 1$ and show it for N . Denote by pr_N the projection from the product to the group G_N :

$$\text{pr}_N : \prod_{i=1}^N G_i \rightarrow G_N.$$

The groups $\text{pr}_N(H) \subset G_N$ and $\text{Ker}(\text{pr}_N) \cap H \subset \prod_{i=1}^{N-1} G_i$ can be generated by n_N and $\sum_{i=1}^{N-1} n_i$ respectively. The group H fits into the following short exact sequence:

$$1 \rightarrow \text{Ker}(\text{pr}_N) \cap H \rightarrow H \rightarrow \text{pr}_N(H) \rightarrow 1$$

By Lemma 1.2.2 we conclude that H can be generated by $\sum_{i=1}^N n_i$ elements. \square

Consider the subgroup of strictly upper-triangular matrices of $\text{GL}(3, \mathbb{F}_3)$. This group is called the *Heisenberg group modulo 3* we will refer to it as to *Heisenberg group*. The following assertion gives us several equivalent definitions of this group:

Lemma 1.2.4. *Let \mathcal{H}_3 be a Heisenberg group. Then its order equals 27 and*

- (1) *The group \mathcal{H}_3 is generated by 3 elements x, y and z with the following relations:*

$$x^3 = y^3 = z^3 = 1; \quad xyx^{-1}y^{-1} = z; \quad xz = zx; \quad yz = zy.$$

- (2) *The following matrices X and Y generate a subgroup isomorphic to \mathcal{H}_3 in $\text{SL}_3(\mathbb{C})$:*

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix};$$

here ζ is a primitive cube root of unity.

By the first assertion of Lemma 1.2.4 we get that the Heisenberg group can be generated by two its elements x and y . Also we can see that the center of the Heisenberg group is generated by the element z . Recall that for simplicity we denote by C_n the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Thus, we have the following exact sequence.

$$1 \rightarrow C_3\langle z \rangle \rightarrow \mathcal{H}_3 \rightarrow C_3\langle x, y \rangle \rightarrow 1.$$

Here the group $C_3\langle x, y \rangle$ is isomorphic to a direct product of cyclic groups $C_3 \times C_3$.

The embedding of \mathcal{H}_3 to the group $\text{SL}_3(\mathbb{C})$ in the second assertion of Lemma 1.2.4 is called the *standard embedding* of the Heisenberg group. Denote by $N(\mathcal{H}_3)$ the normalizer of the image of the standard embedding of \mathcal{H}_3 to $\text{SL}_3(\mathbb{C})$ and by $\tilde{\mathcal{H}}_3$ the maximal 3-subgroup of $N(\mathcal{H}_3)$.

Lemma 1.2.5. *The group $\tilde{\mathcal{H}}_3 \subset \text{SL}_3(\mathbb{C})$ fits into the following short exact sequence:*

$$1 \rightarrow \mathcal{H}_3 \rightarrow \tilde{\mathcal{H}}_3 \rightarrow C_3 \rightarrow 1$$

Moreover, the image of $\tilde{\mathcal{H}}_3$ under the projection $\pi: \text{SL}_3(\mathbb{C}) \rightarrow \text{PGL}_3(\mathbb{C})$ is isomorphic to \mathcal{H}_3 .

Proof. By [Fei71, 8.5] we get that the quotient group $N(\mathcal{H}_3)/\mathcal{H}_3$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. Since the maximal 3-subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$ is isomorphic to a cyclic group C_3 we get the exact sequence.

The image of the generator z of \mathcal{H}_3 under the standard embedding to $\mathrm{SL}_3(\mathbb{C})$ is a scalar matrix. Thus, $\pi(z) = 1$ and the group $W = \pi(\mathcal{H}_3)$ is generated by $\pi(x)$ and $\pi(y)$. Therefore, $W \cong C_3^2$.

Denote by m the lift of the generator of the Sylow 3-subgroup of $\mathrm{SL}_2(\mathbb{F}_3)$ under the map

$$\tilde{\mathcal{H}}_3 \rightarrow \tilde{\mathcal{H}}_3/\mathcal{H}_3 \subset \mathrm{SL}_2(\mathbb{F}_3)$$

By [Fei71, 8.5] there is a basis w_1, w_2 of W such that the conjugation by $\pi(m)$ acts on W by the following matrix:

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus, the group $\pi(\tilde{\mathcal{H}}_3)$ is a non-abelian group of order 27 and all its elements are of order 3. Since all groups of order 27 are either abelian, a semidirect product of groups C_9 and C_3 or a Heisenberg group we conclude the result. □

1.2.2. 3-subgroups in $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{PGL}_n(\mathbb{C})$

In this section we give some bounds for the number of generators of any 3-subgroup of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{PGL}_n(\mathbb{C})$ where $n \leq 8$. Let us start with a description of p -subgroups of $\mathrm{GL}_2(\mathbb{C})$.

Lemma 1.2.6. *If $p \neq 2$ is a prime number, then any p -subgroup G in $\mathrm{GL}_2(\mathbb{C})$ is abelian and can be generated by 2 elements and we have:*

$$G \cong C_{p^n} \times C_{p^m}.$$

Moreover, any p -subgroup of $\mathrm{PGL}_2(\mathbb{C})$ is cyclic and its action on a projective line preserves a point.

Proof. Any embedding of a finite group G to $\mathrm{GL}_2(\mathbb{C})$ induces a faithful 2-dimensional representation of the group G . The dimension of any irreducible representation of the group G divides the order of G . Thus, if $p \neq 2$, a representation of dimension 2 of G is a sum of linear representations which implies the result. □

We establish bounds on the number of generators of any 3-subgroup of $\mathrm{GL}_3(\mathbb{C})$ and $\mathrm{PGL}_3(\mathbb{C})$.

Lemma 1.2.7. *If G is a 3-subgroup of $\mathrm{PGL}_3(\mathbb{C})$, then it can be generated by 2 elements. If G is a 3-subgroup of $\mathrm{GL}_3(\mathbb{C})$, then it can be generated by 3 elements.*

Proof. The first assertion can be proved as in [Bor61, Section 6.4]. To estimate the number of generators of a 3-subgroup G of $\mathrm{GL}_3(\mathbb{C})$ we consider the following exact sequence:

$$1 \rightarrow \mathbb{C}^* \rightarrow \mathrm{GL}_3(\mathbb{C}) \xrightarrow{\pi} \mathrm{PGL}_3(\mathbb{C}) \rightarrow 1.$$

The kernel of the map $\pi|_G$ is a subgroup of \mathbb{C}^* ; thus, it is a cyclic group. The image $\pi(G)$ is a 3-subgroup of $\mathrm{PGL}_3(\mathbb{C})$, thus, it can be generated by 2 elements. Lemma 1.2.2 implies the result. □

Recall that a torus in a compact Lie group is a maximal connected abelian subgroup. The following assertion describes p -subgroups of a compact Lie group for all prime numbers p .

Theorem 1.2.8 ([BS53, Theorem 1], [Bor61, Section 6.4]). *Let \mathcal{G} be a compact Lie group and let G be a p -subgroup in \mathcal{G} . Then there exists a torus $T \subset \mathcal{G}$ such that the group G is a subgroup of a normalizer $N(T)$.*

In particular, if \mathcal{G} is equal to $\mathrm{SL}_p(\mathbb{C})$ or $\mathrm{PGL}_p(\mathbb{C})$ where p is a prime number, then the quotient group $G/(G \cap T)$ can be embedded to the group $\mathbb{Z}/p\mathbb{Z}$. Moreover, $G/(G \cap T)$ can be generated by the images of permutation matrices in \mathcal{G} .

This theorem allows us to describe the structure of a 3-subgroup of the group $\mathrm{GL}_3(\mathbb{C})$.

Corollary 1.2.9. *Assume that a 3-subgroup G of $\mathrm{GL}_3(\mathbb{C})$ can not be generated by 2 elements. Then either G contains an abelian group \mathbb{C}_3^3 , or in some basis it contains the following matrices*

$$\sigma = \begin{pmatrix} 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \\ \lambda_1 & 0 & 0 \end{pmatrix}; \quad t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}; \quad (1.2.10)$$

where ζ is a primitive cube root of unity and λ_i are roots of unity of degree 3^{k_i} .

Proof. Since G can not be generated by 2 elements, by Theorem 1.2.8 the rank of the group $G \cap T$ is at least 2. If it equals at least 3, then G contains \mathbb{C}_3^3 . Assume, that it equals 2.

By Theorem 1.2.8 in some basis G is a subgroup of a normalizer of the torus $T \subset \mathrm{GL}_3(\mathbb{C})$ of diagonal matrices. Thus, there is an element $\sigma \in G$ such that the conjugation by σ acts on $G \cap T$ as a permutation of coordinates.

Consider t a non trivial element of $G \cap T$. If elements t and $\sigma t \sigma^{-1}$ are not proportional, conjugations of t by σ generate an abelian subgroup of rank 3 in G and this contradicts our assumption. Thus, we can assume that for any non-trivial element t in $G \cap T$ elements $\sigma t \sigma^{-1}$ and t are proportional.

However, if the element t is proportional to $\sigma t \sigma^{-1}$ then t is either a scalar matrix or the second matrix in (1.2.10). Since the rank of the group $G \cap T$ equals 2 there are two non-proportional elements t_1 and t_2 of $G \cap T$. Thus, without loss of generalities we get that t_1 is a scalar matrix and t_2 is the second matrix in (1.2.10). This implies the result. \square

Now we can estimate the number of generators of a 3-subgroup of $\mathrm{GL}_n(\mathbb{C})$ for n greater than 3.

Proposition 1.2.11. *Assume that $3 < n < 9$ and G is a 3-subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then G can be generated by n elements. If G is a 3-subgroup of $\mathrm{PGL}_n(\mathbb{C})$, then G can be generated by $n - 1$ elements.*

Proof. An embedding of the group G to $\mathrm{GL}_n(\mathbb{C})$ induces the n -dimensional representation of the group G . Since by assumption $n < 9$ this representation is isomorphic to a direct sum of irreducible representations of dimensions 1 and 3. Thus, we have

$$G \subset \prod_{i=1}^{n-3k} G'_i \times \prod_{j=0}^k G''_j. \quad (1.2.12)$$

Here G'_i are cyclic 3-groups and G''_j are 3-subgroups of $\mathrm{GL}_3(\mathbb{C})$. Then by Lemmas 1.2.3 and 1.2.7 the group G can be generated by n elements.

Now consider a 3-subgroup G of $\mathrm{PGL}_n(\mathbb{C})$. Denote by $\pi: \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{PGL}_n(\mathbb{C})$ the quotient map by the group of scalar matrices. Let \tilde{G} be a 3-subgroup of $\mathrm{GL}_n(\mathbb{C})$ such that $\pi(\tilde{G}) = G$. If \tilde{G} can be generated by $n - 1$ elements, then G also can. Assume that \tilde{G} can not be generated by less than n elements.

By what precedes and Corollary 1.2.9, the group G either contains an abelian subgroup of rank n or

$$\tilde{G} = H' \times H'', \quad (1.2.13)$$

where $H' \subset \mathrm{GL}_{n-3}(\mathbb{C})$, $H'' \subset \mathrm{GL}_3(\mathbb{C})$ and H'' contains matrices of the form (1.2.10). Moreover, the rank of the group $H'' \cap T$ is at least 2 where T is a torus in $\mathrm{GL}_3(\mathbb{C})$.

In the case \tilde{G} contains an abelian subgroup of rank n we get that its image in $\mathrm{PGL}_n(\mathbb{C})$ can be generated by $n - 1$ element. Otherwise, we have a decomposition (1.2.13). Note that the group H'' is generated by at most 2 elements unless $\mathrm{rk}(H'' \cap T) = 2$ and H'' is generated by a scalar matrix and matrices (1.2.10). In both these situations we see that $\pi(H'')$ can be generated by 2 elements. By induction, we can assume that $\pi(H')$ can be generated by $n - 3$ elements. Then Lemma 1.3.2 implies the result. \square

Corollary 1.2.14. *If $G \subset \mathrm{PGL}_n(\mathbb{C})$ is a 3-subgroup which can not be generated by less than k elements and $n < 9$, then G contains an abelian subgroup A of rank k .*

Proof. Denote by $\pi: \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{PGL}_n(\mathbb{C})$ the quotient map by the subgroup of scalar matrices. Let \tilde{G} be the 3-subgroup of $\mathrm{GL}_n(\mathbb{C})$ such that $\pi(\tilde{G}) = G$. Then, by Proposition 1.2.11 one has that either \tilde{G} is a subgroup of the torus in $\mathrm{GL}_n(\mathbb{C})$ or admits the decomposition (1.2.13).

If \tilde{G} is a subgroup of the torus in $\mathrm{GL}_n(\mathbb{C})$, then we immediately get the result. If \tilde{G} admits the decomposition (1.2.13), then the group $\pi(H'')$ can be generated by two elements and it contains a subgroup \mathbb{C}_3^2 . This implies the result. \square

This bounds on the number of generators of 3-subgroups in $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{PGL}_n(\mathbb{C})$ implies several useful consequences.

Lemma 1.2.15. *Any 3-subgroup of the group $\mathrm{GL}_2(\mathbb{Z})$ or $\mathrm{GL}_3(\mathbb{Z})$ is cyclic.*

It follows from the above assertions, see [Tah71].

Lemma 1.2.16. *Consider a 3-group G acting on a lattice $\Lambda \cong \mathbb{Z}^2$. Then the invariant lattice Λ^G can not be of rank 1.*

Proof. By Lemma 1.2.15 the image of group G is cyclic and it is isomorphic to \mathbb{C}_{3^n} . Denote the generator of this group by γ .

The group G maps to the group of automorphisms of Λ which is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. This action induces an embedding $\iota: \mathrm{SL}_2(\mathbb{C})$. Assume that eigenvalues of $\iota(\gamma)$ are λ_1 and λ_2 .

If the rank of the invariant sublattice Λ^G is non-zero, then one eigenvalue should equal 1. Without loss of generalities we can assume that $\lambda_1 = 1$. Since $\lambda_1 \cdot \lambda_2 = 1$, we get that λ_2 also equals 1. Thus, the rank of the lattice Λ^G is either 0 or 2. \square

1.2.3. Group action on algebraic varieties

We study actions of groups on algebraic varieties over a field k . All varieties are assumed to be irreducible and reduced if not stated otherwise.

Definition 1.2.17. Let X and Y be algebraic varieties, let $\pi: X \rightarrow Y$ be an algebraic map and let G be a group.

- The *action of the group G on X* is a homomorphism of algebraic groups $j_X: G \rightarrow \text{Aut}(X)$. In this situation we say that X is a G -variety.
- The action of G on X is *faithful* if the map j_X is an embedding.
- The map π is a *fiber space* if it is surjective, proper and its general fiber is irreducible and of positive dimension.
- If G acts on both varieties X and Y , then we say that π is *G -equivariant* if for any $g \in G$ one has $j_Y(g) \circ \pi = \pi \circ j_X(g)$. If there are actions of G on varieties X and Y then we say that the map π is *G -fiber space* if it is a G -equivariant fiber space.

In those situations which do not lead to any confusion we will consider elements of G themselves as automorphisms of X .

We list several very well-known assertions which describes properties of groups acting on varieties.

Lemma 1.2.18. *If X and B are G -varieties, $\pi: X \rightarrow B$ is a G -fiber space and the fiber of π over a general point of B is isomorphic to an irreducible algebraic variety F , then G is a following extension of groups:*

$$1 \rightarrow G_F \rightarrow G \rightarrow G_B \rightarrow 1. \quad (1.2.19)$$

Here $G_F \subset \text{Aut}(F)$ is the subgroup consisting of all elements of G inducing the trivial automorphism on the base and $G_B \subset \text{Aut}(B)$ is a quotient group G/G_F .

In the case of a 3-group, this implies the following restriction on the number of its generators.

Lemma 1.2.20. *Assume that $X \rightarrow B$ is a G -fiber space with general fiber F and G is a 3-group. If any 3-subgroup of $\text{Aut}(F)$ and $\text{Aut}(B)$ can be generated by n or m elements respectively, then the group G can be generated by $n + m$ elements.*

In particular, if $X \rightarrow B$ is a double cover, then $G \subset \text{Aut}(B)$.

Proof. Since G is finite and preserves the fiber space structure, we have an exact sequence (1.2.19). By Lemma 1.2.2 since groups G_F and G_B can be generated by n and m elements respectively G can be generated by $n + m$ elements.

If $X \rightarrow B$ is a finite n -cover, then $\text{Aut}(F) \cong \mathfrak{S}_n$. In particular, if $n = 2$, then $G_F \subset C_2$. Since G and G_F are both 3-groups, then $G_F = 1$ and $G \subset \text{Aut}(B)$. \square

Next assertion is useful for actions of finite groups which fix a point on the variety. Given an algebraic variety X and a point x on X denote by $T_{X,x}$ the Zariski tangent space to X at x .

Proposition 1.2.21 ([Pop14, Lemma 4], [CS21, Theorem 3.7]). *Assume that X is an algebraic variety over the field k and a finite group G acts on X faithfully with a fixed point x . If the characteristic of k equals 0 or if it is coprime with $|G|$ one has $G \subset \mathrm{GL}(T_{X,x})$.*

Using the next assertion we can construct for a variety X with a birational action of a finite group G its birational model with a regular action of G . This assertion is well-known in characteristic 0, in positive characteristic the assertion is known only for surfaces, it was proved by Chen and Shramov.

Proposition 1.2.22 (e.g., [PS14, Lemma-Definition 3.1]; [CS21, Lemma 3.6]). *Let X be an algebraic variety over a field k , a group G be a finite subgroup of $\mathrm{Bir}(X)$ and one of the following is true:*

- *the characteristic of k equals 0;*
- *k is perfect and $\dim(X) \leq 2$.*

Then there exists a smooth G -variety \tilde{X} and a G -equivariant birational map $\tilde{X} \dashrightarrow X$.

Now we study groups of automorphisms and their finite subgroups of several concrete algebraic varieties. In the next assertion we recall the description the group of automorphisms of a Grassmannian.

Proposition 1.2.23 ([Cho49, Theorem I]). *Let V be a complex vector space of dimension n ; consider the Grassmannian $\mathrm{Gr}(k, V)$ of k -planes in V . If $n \neq 2k$, then $\mathrm{Aut}(\mathrm{Gr}(k, V))$ is isomorphic to $\mathrm{PGL}(V)$. If $n = 2k$ then the group $\mathrm{PGL}(V)$ is a subgroup of $\mathrm{Aut}(\mathrm{Gr}(k, V))$ of index 2.*

If C is a curve of genus g greater than 1, then the group $\mathrm{Aut}(C)$ is finite; moreover, we have a bound $|\mathrm{Aut}(C)| \leq 84(g-1)$. Since p -subgroups are always nilpotent we have a better bound for them.

Lemma 1.2.24 ([Sch16, Theorem 1.2]). *Suppose G is a complex algebraic curve of genus $g \geq 2$. Then a p -subgroup G of $\mathrm{Aut}(C)$ satisfies*

$$|G| \leq 16(g-1). \tag{1.2.25}$$

Using the formula (1.2.25) we can estimate number of generators of 3-groups acting on the Jacobian variety of a curve.

Lemma 1.2.26. *Assume that $\mathrm{Jac}(C)$ is the Jacobian of a curve C of genus 2 and G is a 3-group with a faithful action on $\mathrm{Jac}(C)$ which preserves its polarization. Then G can be generated by 2 elements.*

Proof. By Torelli theorem for curves one has that the group of automorphisms of $\mathrm{Jac}(C)$ which preserve its polarization is a subgroup of the group $\mathbb{Z}/2\mathbb{Z} \times \mathrm{Aut}(C)$. Thus, the group G is a subgroup of $\mathrm{Aut}(C)$. By the formula (1.2.25) we get

$$|G| \leq |\mathrm{Aut}(C)| \leq 16(g(C)-1) = 16.$$

Since the order $|G|$ is a power of 3, by Lemma 1.2.1 we conclude that G can be generated by two elements. \square

The formula (1.2.25) implies also a bound on the number of generators of a 3-subgroup acting on a fiber space over a curve.

Lemma 1.2.27. *Assume that S is a projectivization of a vector bundle of rank 2 over a complex curve C of genus 3 and G is a 3-subgroup of $\text{Aut}(S)$. Then G can be generated by 4 elements.*

Proof. Any automorphism of a projectivization S of a vector bundle of rank 2 over a curve C of genus 3 commutes with the projection to C . By (1.2.25) any 3-subgroup of $\text{Aut}(C)$ has an order less than or equal to 32; thus, it can be generated by 3 elements. The fiber of the projection to C is isomorphic to a projective line; thus, by Lemma 1.2.6 any 3-group with a faithful action on fiber is cyclic. By Lemma 1.2.20 this implies that G can be generated by 4 elements. \square

Consider a quadric hypersurface X ; in the next assertion we give a bound on the number of generators of a 3-subgroup of $\text{Aut}(X)$.

Lemma 1.2.28. *Assume that X is a quadric hypersurface in the complex projective space \mathbb{P}^{d+1} and $d < 7$. If G is a 3-subgroup of $\text{Aut}(X)$, then G can be generated by d elements.*

Proof. If X is a smooth quadric hypersurface in \mathbb{P}^{d+1} , then G is a subgroup of $\text{PSO}_{d+2}(\mathbb{C})$. By Corollary 1.2.14 if G can not be generated by less than m elements, then there is an abelian subgroup $A \subset G$ of rank m .

However, any abelian subgroup of $\text{PSO}_{d+2}(\mathbb{C})$ is a subgroup of the torus of this group and the dimension of torus of $\text{PSO}_{d+2}(\mathbb{C})$ equals $\lfloor \frac{d+2}{2} \rfloor$. Thus, we get the result:

$$m \leq \left\lfloor \frac{d+2}{2} \right\rfloor \leq d.$$

If X is a singular quadratic hypersurface, it is a cone with a vertex Z over a smooth quadric hypersurface. After a blow up of this vertex we get a G -invariant projective bundle over a smooth quadric hypersurface with a faithful action of G . Lemma 1.2.20 and Proposition 1.2.11 implies the result. \square

If we consider the intersection of several quadric hypersurfaces than a 3-group with a faithful action on this variety should preserve a point.

Lemma 1.2.29. *Consider a non-empty proper intersection Y of 3 quadric surfaces in \mathbb{P}^3 . Assume that G is a 3-group with an action on Y . Then there exists a subgroup H in G of index at most 3 such that H fixes a point y in Y .*

Proof. The dimension of Y can be equal to 0, 1 or 2. If $\dim(Y) = 0$, then Y is a union of less than or equal to 8 points. One concludes observing that the length of the orbit of a 3-group is a power of 3.

If $\dim(Y) = 1$, then Y is a union of curves and points $C_1 \cup \dots \cup C_k \cup \{p_1, \dots, p_m\}$ in \mathbb{P}^3 and

$$\deg(C_1) + \dots + \deg(C_k) \leq 4.$$

If Y contains an isolated point, then its orbit has length at most 8. Thus, the stabilizer of this point is the necessary group H . If Y is singular, then there are at most 6 singular points on it. The group G preserves the singular locus of C ; thus the stabilizer of a singular point is a subgroup of index at most 3.

If Y is a smooth irreducible curve lying on a quadric surface such that $\deg(Y) \leq 3$ then either Y is isomorphic to \mathbb{P}^1 or it is a plane cubic curve lying in a hyperplane section H of \mathbb{P}^3 . In the first case we get the result by Lemma 1.2.6. In the case Y is a smooth cubic curve in H we get that Y is a complete intersection of 2 surfaces; namely, the hyperplane H and a cubic surface. Denote by \mathcal{Q}_C the family of quadric surfaces passing through C . By Koszul exact sequence we get that $\dim(\mathcal{Q}_C) = 1$.

If $\deg(Y) = 4$ we get that Y is a complete intersection of two of three quadric surfaces Q_1, Q_2 and Q_3 . Then by Koszul exact sequence the dimension of the family \mathcal{Q}_C of quadric surfaces passing through C equals 1.

If the dimension of the family \mathcal{Q}_C of quadric surfaces passing through C equals 1, then G acts on this family and preserves its singular fibers. Since in a 1-dimensional family \mathcal{Q}_S of quadric surfaces there are at most 4 singular fibers we get that the index of a stabilizer H of one of singular fibers is at most 3. Since the singular quadric surface contains a unique singular point, then H fixes it.

If $\dim(Y) = 2$, then Y contains a quadric surface Y_0 in \mathbb{P}^3 . If Y_0 is singular, then it is a cone and its vertex is either a point or a line. In any case by Lemma 1.2.6 the group G fixes a point on Y_0 . If Y is smooth, then $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) = C_2 \times (\text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C}))$ by Lemma 1.2.6 there exists a G -fixed point on Y_0 . \square

We need to study 3-groups with a faithful action on varieties of minimal degree. Let us first recall the necessary definitions. Consider a set of integers a_1, \dots, a_k such that $a_i < 0$ for all $1 \leq i \leq k$. The projective bundle $\mathbb{P}(\bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i))$ can be embedded to \mathbb{P}^n by the linear system $|\mathcal{O}_{\mathbb{P}(\bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i))}(1)|$. The image of this map Y is called a *rational normal scroll*. By the *Veronese surface* we denote the image of the following embedding of \mathbb{P}^2 to \mathbb{P}^5 :

$$\iota: \mathbb{P}^2 \rightarrow \mathbb{P}^5; \quad \iota(x_0 : x_1 : x_2) = (x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2).$$

Recall that for a projective variety Y and an embedding $j: Y \rightarrow \mathbb{P}^n$ such that $j(Y)$ is not contained in a hyperplane one has:

$$\text{codim}_{\mathbb{P}^n}(Y) + 1 \leq \deg(Y) \tag{1.2.30}$$

In case we have an equality Y is called a *variety of minimal degree*. Such varieties are classified by del Pezzo and Bertini, see, [EH87]. Namely, a variety of minimal degree Y is either a projective space, a quadric hypersurface, a rational normal scroll, a Veronese surface or a cone over a rational normal scroll or a Veronese surface. In the next lemma we study 3-groups having a faithful action on a variety of minimal degree.

Lemma 1.2.31. *Assume that Y is a variety of minimal degree and G is a 3-subgroup of $\text{Aut}(Y)$. If $\dim(Y) = d$ and $d < 7$, then G can be generated by d elements.*

Proof. If $Y \cong \mathbb{P}^d$, then by Proposition 1.2.11 the group G can be generated by d elements.

If Y is a rational normal scroll, then the rank of the Picard group of Y equals 2 since it is isomorphic to a projective bundle over \mathbb{P}^1 . Thus, by Lemma 1.2.16 the group G acts trivially on the Picard group $\text{Pic}(Y)$. Then it preserves the structure of the map $\mathbb{P}(\bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)) \rightarrow \mathbb{P}^1$. Therefore, by Lemma 1.2.20 and Proposition 1.2.11 the group G can be generated by d elements.

Assume that $Y \subset \mathbb{P}^n$ is a cone over a variety B which is either a rational normal scroll or a Veronese surface. Then the vertex of the cone $Z \subset Y$ is a projective space of codimension c in Y . Consider the projection from \mathbb{P}^n to B with a center in Z :

$$\pi_Z: \mathbb{P}^n \dashrightarrow B.$$

Denote by \tilde{Y} the blow-up of Y in Z . Since Z is an invariant subvariety under the action of G it acts regularly on \tilde{Y} . Moreover, the rational map π_Z induces on \tilde{Y} the structure of a regular map to B such that its general fiber is a projective space of the dimension $d - \dim(B)$. Then G can be generated by at most d elements by Lemmas 1.2.11 and 1.2.20.

If Y is a d -dimensional quadratic hypersurface in \mathbb{P}^{d+1} , then by Lemma 1.2.28 the group G can be generated by d elements. \square

1.2.4. Examples

We give here an example which shows that there exists a 3-subgroup of $\text{Bir}(\mathbb{P}^2)$ which can not be generated by less than 3 elements.

Example 1.2.32. Fix a projectivization $\mathbb{P}^3 \cong \mathbb{P}(V)$ of the vector space V with coordinates x_1, x_2, x_3 and x_4 . Consider the following action on $\mathbb{P}(V)$ of the abelian 3-group $(C_3)^3$ with generators γ_1, γ_2 and γ_3 . Each γ_i acts non-trivially only on x_i by multiplication by the primitive cube root of unity; and $\gamma_i \cdot x_j = x_j$ for all $i \neq j$. This action is faithful and it preserves the Fermat cubic surface S in \mathbb{P}^3 :

$$S = \{(x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^3 \mid x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\}.$$

Note that S is a smooth cubic surface; thus, it is rational. Thus, the 3-group C_3^3 acts faithfully on a rational surface S , though it can not be generated by less than 3 elements.

The following example describes the action of an abelian 3-group of rank 4 on a rationally connected threefold.

Example 1.2.33. Consider a product $X = \mathbb{P}^1 \times S$ of \mathbb{P}^1 and a Fermat cubic surface. It is a product of a rational surface and a rational curve; thus, it is rational itself. Let the group C_3^3 acts on S as in Example 1.2.32 and choose some faithful action of the group C_3 on \mathbb{P}^1 . Then the group $C_3^3 \times C_3$ acts faithfully on the the product $X = \mathbb{P}^1 \times S$ of \mathbb{P}^1 and a Fermat cubic surface. It is a product of a rational surface and a rational curve; thus, it is rational itself. The group $(C_3)^4$ is a subgroup of $\text{Cr}_3(\mathbb{C})$ which can not be generated by less than 4 elements.

1.2.5. p -Jordan property

We start with the following definitions:

Definition 1.2.34. Assume that Γ is a group, then

- Γ is *Jordan* if there exists a constant $J = J(\Gamma)$ such that any finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup A and

$$[G : A] \leq J.$$

- Γ is *p-Jordan* if there exist two constants $J = J(\Gamma)$ and $e = e(\Gamma)$ such that any finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup A and

$$[G : A] \leq J \cdot |G_p|^e,$$

where G_p is a Sylow p -subgroup of G .

C. Jordan proved that the group $\mathrm{GL}_n(\mathbb{C})$ is Jordan for all $n \geq 1$, see [CR62, Theorem 36.13]. However, some groups do not enjoy the Jordan property.

Example 1.2.35. Consider the group $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ and its finite subgroup $G = \mathrm{SL}_2(\mathbb{F}_q)$ where $q = p^n$. Since the quotient group $\mathrm{PSL}_2(\mathbb{F}_q)$ of the group G is simple we get that the greatest abelian subgroup A in G is isomorphic to the multiplicative group \mathbb{C}_q^\times . Thus, we can compute the index of the subgroup A in G :

$$[G : A] = \frac{(p^{2n} - 1)(p^{2n} - p^n)}{p^n - p^{n-1}} \sim p^{3n}$$

Moreover, one can see that the Sylow p -subgroup G_p of G is of the order $|G_p| = p^n$. Note that the group $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is not Jordan, but it is *p-Jordan*.

The notion of being *p-Jordan* was introduced by F. Hu in [Hu20]. The group $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is *p-Jordan* for any integer n ; this was proved in [LP11, Theorem 0.4].

Lemma 1.2.36 ([Pop11, Lemma 2.11], [CS21, Lemma 2.8]). *Consider a short exact sequence of groups:*

$$1 \rightarrow \Gamma' \rightarrow \Gamma \rightarrow \Gamma'' \rightarrow 1.$$

*Assume there exists a number C such that for any finite subgroup G of Γ'' one has $|G| \leq C$. If the group Γ' is Jordan (resp. *p-Jordan*), then the group Γ is Jordan (resp. *p-Jordan*).*

1.2.6. Fano threefolds

Let X be a normal complex variety of dimension n and let $\iota: X_0 \hookrightarrow X$ be the embedding of the non-singular locus of X to X . Recall that Ω_X^n is the sheaf of differential n -forms on X which are regular on X_0 and the canonical class K_X of X is the Chern class of the sheaf $\iota_*(\iota^*\Omega_X)$. The canonical class is a very important invariant of an algebraic variety. In general case the canonical class is a class of a Weil divisor; if X is smooth then it is a Cartier class.

In this section we discuss varieties which arise when we apply MMP to a rationally connected threefold. These varieties can be singular so we have to deal with different types of singularities. Recall definitions of singularities we will need in following sections, the main reference here is [Rei87].

Definition 1.2.37. Let X be a normal variety and let K_X be the canonical class of X .

- A variety X has *terminal singularities* if the following conditions are true:
 - for some $r \geq 1$ the Weil divisor rK_X is Cartier;

- if $f: Y \rightarrow X$ is a resolution of X and $\{E_i\}_{i \in I}$ is a family of prime exceptional divisors of f , then

$$rK_Y = f^*(rK_X) + \sum_{i \in I} a_i E_i,$$

with $a_i > 0$.

- A point p in X is a *Gorenstein singularity* if the ring $\mathcal{O}_{X,p}$ is a Noetherian local ring with finite injective dimension as an $\mathcal{O}_{X,p}$ -module.
- A point p in X is a *cDV-singularity* if $\dim(X) = 3$ and there exists a Zariski open subset $U \subset X$ such that $p \in U$ and a polynomial $F(x, y, z, t)$ in the ring $k[x, y, z, t]$ such that one has an algebraic isomorphism with a hypersurface singularity:

$$(U, p) \cong (\{F = 0\}, 0) \subset (\mathbb{A}^4, 0),$$

where F has the following form:

$$F(x, y, z, t) = f(x, y, z) + tg(x, y, z, t)$$

where f is the equation of a Du Val singularity and g is an arbitrary polynomial.

- We say that C_r acts on \mathbb{A}^3 with weights a, b and c if there exist coordinates x, y, z of \mathbb{A}^3 such that for some generator g of C_r one has:

$$g(x, y, z) = (\zeta^a x, \zeta^b y, \zeta^c z),$$

where ζ is a primitive r -th root of unity.

- A point p in X is a *quotient threefold singularity of index r and of type $\frac{1}{r}(a, b, c)$* if locally X is isomorphic to \mathbb{A}^3/C_r where C_r acts on \mathbb{A}^3 with weights a, b and c and the action is free in codimension 1.
- if the variety X admits only terminal (resp. only Gorenstein) singularities, then we say that X is a *terminal* (resp. *Gorenstein*) variety.

Recall here that by [Rei87, Corollary 3.12] any isolated cDV-singularity is both Gorenstein and terminal. Also by [Rei87, Theorem 5.2] a quotient singularity is terminal if and only if it is of the type $\frac{1}{r}(a, -a, 1)$ where r is coprime to a .

Given a complex normal variety with a regular action of a finite group G , one can apply G -equivariant MMP. Here is a definition of types of varieties which can be obtained as an end result of running this program.

Definition 1.2.38. Let G be a group, X be a G -variety, K_X be the canonical class on X , then

- X is a *G -Fano variety* if it is terminal, the class $-K_X$ is ample and the rank of $\text{Pic}(X)^G$ equals 1.
- X is a *GQ-factorial variety*, if any G -invariant Weil divisor D in X is a \mathbb{Q} -Cartier divisor, i.e. there exist n such that nD is Cartier.
- X is a *GQ-Fano variety*, if it is a GQ-factorial G -Fano variety.

- The map $\pi: X \rightarrow B$ is a G -Mori fiber space, if B is a G -variety, π is a G -fiber space, the relative class $-K_{X/B} = -(K_X - \pi^*(K_B))$ is relatively ample and the rank of the relative G -invariant Picard group $\text{Pic}(X/B)^G$ equals 1.

If X is a projective complex normal Fano variety, its Picard group is finitely generated and torsion-free. We denote by *Picard number* $\rho(X)$ the rank of the Picard group $\text{Pic}(X)$. Its *Hodge number* $h^{p,q}(X)$ is the real dimension of the group $H^{p,q}(X)$.

The *index* of a Gorenstein Fano variety X is the maximal number $r(X)$ such that there exists an element H of the Picard group of X and

$$-K_X \sim rH.$$

Any regular automorphism f of X preserves the line bundle $\mathcal{O}(H)$ since $f^*K_X \sim K_X$. The Picard group $\text{Pic}(X)$ of a Gorenstein Fano variety X is torsion-free by [IP99, Proposition 2.1.2].

Another invariant of a Gorenstein Fano threefold of index 1 is its *genus*:

$$g(X) = \dim(H^0(X, \mathcal{O}(H))) - 2. \quad (1.2.39)$$

Since X is a Fano variety, the line bundle $\mathcal{O}(H)$ is ample and its linear system induces the following map:

$$\phi_{|H|}: X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}(H)))^\vee \cong \mathbb{P}^{g+1}.$$

We say that a Gorenstein Fano threefold X is a *del Pezzo threefold of degree d* if the index of X equals 2 and $H^3 = d$ where H is a primitive class in $\text{Pic}(X)$ such that $K_X = -2H$.

Smooth Fano threefolds of Picard rank 1 were classified in the series of works by Iskovskikh. There exists 17 deformation classes of smooth Fano threefolds:

Proposition 1.2.40 ([IP99, Table 12.4]). *Assume that X is a smooth Fano threefold of index $r(X)$ and $\rho(X) = 1$, then X lies in one of the following families:*

- (1) if $r(X) = 1$, there are ten families of smooth Fano threefolds and $g(X) \in \{2, 3, 4, \dots, 10, 12\}$;
- (2) if $r(X) = 2$, there are five families of smooth Fano threefolds; namely, del Pezzo threefolds of degrees $1, \dots, 5$;
- (3) if $r(X) = 3$ there is a unique family of smooth Fano threefolds: namely, smooth quadric hypersurfaces in \mathbb{P}^4 ;
- (4) if $r(X) = 4$ there is a unique smooth Fano threefold: \mathbb{P}^3 .

The full classification of smooth Fano threefolds and their invariants (Picard ranks and Hodge numbers $h^{1,2}$) can be found in [IP99, Table 12.4].

Assume that X is a smooth Fano threefold and its anticanonical linear system is very ample. Denote by $S(X)$ the Hilbert scheme of conics on X in its anticanonical embedding. If the index of X is at most 2, then $S(X)$ is a surface and it encodes many informations on X . Here we recall the results from [KPS18, Theorem 1.1.1, Lemma 4.2.1, Lemma 4.3.4, Corollary 4.3.5].

Proposition 1.2.41. *Assume that X is a smooth Fano threefold such that the linear system $|-K_X|$ induces an embedding and $\rho(X) = 1, r(X) = 1, g(X) \geq 6$. Then $S(X)$ is a smooth surface and the group $\text{Aut}(X)$ acts faithfully on $S(X)$. Moreover,*

- (1) if $g(X) = 7$, then $S(X)$ is a symmetric square of a smooth curve C of genus 7 and $\text{Aut}(X)$ acts faithfully on C ;
- (2) if $g(X) = 9$, then $S(X)$ is a projectivization of a vector bundle of rank 2 over a curve C of genus 3;
- (3) if $g(X) = 10$, then $S(X)$ is the Jacobian $\text{Jac}(C)$ of a smooth curve C of genus 2 ;
- (4) if $g(X) = 12$, then $S(X) \cong \mathbb{P}^2$.

Recall an important construction of Fano varieties. Let V_5 and K be a complex vector spaces of dimension 5 and 1 respectively. The Plücker embedding maps the Grassmannian $\text{Gr}(2, V_5)$ of planes in V_5 to the projective space $\mathbb{P}(\Lambda^2 V_5)$. Let W be a vector subspace in $\Lambda^2 V_5$ and Q be a quadric surface in $\mathbb{P}(W)$. We say that (X, H) is a *Gushel–Mukai variety* if X is the following intersection:

$$X = C_K \text{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q,$$

where $C_K \text{Gr}(2, V_5)$ is a cone over the Grassmannian $\text{Gr}(2, 5)$ with a vertex in K in the projective space $\mathbb{P}(\Lambda^2 V_5 \oplus K)$ and H is an inverse image of a hyperplane section in the projective space $\mathbb{P}(W)$. By [IP99, Table 12.4] if X is the smooth Fano threefold of index 1 and genus 6 then $(X, -K_X)$ is a Gushel–Mukai variety.

Theorem 1.2.42 ([DK18, Theorem 2.3, Proposition 2.12 and 2.15]). *Let (X, H) be a normal complex variety. Then the following assertions are true:*

- (1) *If (X, K_X) is a Gushel–Mukai variety and $\dim(X) = 1$, then $\dim(H^0(X, \mathcal{O}_X(K_X))) = 7$ and the canonical model of X is an intersection of 6 quadrics in $\mathbb{P}((H^0(X, \mathcal{O}_X(K_X)))$.*
- (2) *Assume that $\dim(X) = n$ and H is an ample class in X such that $K_X = -(n-2)H$ and $H^1(X, \mathcal{O}_X) = 0$. If $X' \subset X$ is a hypersurface in the linear system of H and $(X', H|_{X'})$ is a normal Gushel–Mukai variety, then (X, H) is a normal Gushel–Mukai variety.*
- (3) *If X is a Gushel–Mukai variety then the natural map $\iota: X \rightarrow \text{Gr}(2, V_5)$ is $\text{Aut}(X)$ -equivariant. Moreover, ι is either an embedding or a double cover onto $\iota(X)$.*

If X is a Gorenstein terminal Fano threefold, then by results of Namikawa it is a deformation of a smooth Fano threefold. Moreover, the number of singularities on X can be estimated in terms of some invariants of the smoothing.

Theorem 1.2.43 ([Nam97]). *If X is a Gorenstein terminal Fano threefold, then there exist normal algebraic varieties \mathcal{X} and B and a proper flat map*

$$\pi: \mathcal{X} \rightarrow B \ni 0,$$

such that the fiber \mathcal{X}_b of π over a point b is a smooth Fano threefold if $b \neq 0$ and the fiber \mathcal{X}_0 over 0 is isomorphic to X . Moreover, if singularities of X are isolated, one has

$$|\text{Sing}(X)| \leq 20 - \rho(\mathcal{X}_b) + h^{1,2}(\mathcal{X}_b), \tag{1.2.44}$$

where \mathcal{X}_b is a smooth fiber of \mathcal{X} .

The family $\pi: \mathcal{X} \rightarrow B$ is called then a *Namikawa smoothing*. Since the number of families of smooth Fano threefolds is bounded, one can estimate the number of isolated singularities on a Gorenstein terminal Fano threefold.

If the terminal Fano threefold X has a terminal singularity x , then by [Rei87, Theorem 10.2(3)] one can associate a set of integers $r_i \geq 2$. In fact, there exists a flat deformation of X such that the singularity x deforms to a set (basket) of quotient singularities of types $\frac{1}{r_i}(a_i, -a_i, 1)$.

Theorem 1.2.45 ([Rei87, Corollary 10.3],[Kaw92]). *Let X be a terminal Fano threefold with N isolated singularities $\text{Sing}(X) = \{x_1, \dots, x_N\}$ and let r_{i1}, \dots, r_{il_i} be set of integers associated to each singular point x_i as in [Rei87, Theorem 10.2(3)]. Then one has the following inequality:*

$$\sum_{i=1}^N \sum_{j=1}^{l_i} \left(r_{ij} - \frac{1}{r_{ij}} \right) < 24, \quad (1.2.46)$$

The formula (1.2.46) follows from the Riemann–Roch formula in the case of a rationally connected variety with terminal singularities.

1.3. Automorphisms of quasi-projective surfaces

Our goal is to prove that the group of automorphisms of a quasi-projective surface defined over a field k of positive characteristic satisfies the p -Jordan property. In the first part of this section, we study the group of automorphisms of surfaces birational to a product of an elliptic curve and a projective line. Then we prove Theorem 1.1.3.

1.3.1. Automorphisms of open subsets of $E \times \mathbb{P}^1$

In this section we assume that S is a quasi-projective surface which is birational to a product $\mathbb{P}^1 \times E$ where E is an elliptic curve. Then the projection from $\mathbb{P}^1 \times E$ to the second component of the product induces the following map:

$$\pi: S \rightarrow E. \quad (1.3.1)$$

Note that the map π is regular since it is a dominant map to an abelian variety. Then finite subgroups of automorphisms of S can be represented as extensions of two groups.

Lemma 1.3.2. *Let S be a smooth quasi-projective surface over any field k birational to $\mathbb{P}^1 \times E$, where E is a smooth curve of positive genus. If $\pi: S \rightarrow E$ is the map (1.3.1) then there exists an exact sequence of groups:*

$$1 \rightarrow \text{Bir}(S)_\pi \rightarrow \text{Bir}(S) \rightarrow \Gamma, \quad (1.3.3)$$

where $\text{Bir}(S)_\pi \subset \text{PGL}(2, k(C))$ and $\Gamma \subset \text{Aut}(E)$. Moreover, the group $\text{Bir}(S)_\pi$ is p -Jordan.

Proof. The exact sequence exists by [CS21, Lemma 4.16]. It follows from the fact that if $g \in \text{Bir}(S)$, then the image of a rational curve F is a fiber of π under the composition $\pi \circ g$ should be a point in the elliptic curve E . The group $\text{Bir}(S)_\pi$ is p -Jordan since it is a subgroup of $\text{PGL}(2, k(C))$ which is p -Jordan by Theorem 1.1.1. \square

Assume that the group $\text{Aut}(S)$ fixes a finite subset of fibers of π , then one has the following assertion.

Lemma 1.3.4. *Let S be a smooth quasi-projective surface birational to $\mathbb{P}^1 \times E$ where E is an elliptic curve, $\pi: S \rightarrow E$ be the map as in (1.3.1) and let $\Gamma \subset \text{Aut}(S)$ be a map and a group as in Lemma 1.3.2. If there exists a non-empty finite Γ -invariant subset $Z \subset E$, then the group $\text{Aut}(S)$ is p -Jordan.*

Proof. Denote by H the stabilizer of a point $z \in Z$ under the action of the group Γ . By the assumption H is a subgroup of Γ of index at most $|Z|$. The group H acts faithfully on the elliptic curve E and preserves a point in it. Thus, $|H| \leq 24$ by [HKT08, Theorem 11.94]. Therefore, $|\Gamma| \leq 24|Z|$.

Thus, if we denote by $\text{Aut}(S)_\pi$ the subgroup of elements of $\text{Aut}(S)$ which induce the trivial automorphism on E , then one can see that $\text{Aut}(S)_\pi$ is a subgroup of finite index in $\text{Aut}(S)$. By Lemma 1.2.18 the group $\text{Aut}(S)_\pi$ can be embedded into the group of automorphisms of a general fiber of π ; thus, it is p -Jordan by Theorem 1.1.1. Therefore, $\text{Aut}(S)$ is also a p -Jordan group by Lemma 1.2.36. \square

Let S be a quasi-projective surface. Consider a projective surface \bar{S} such that S can be embedded into \bar{S} as a Zariski open subset. We call \bar{S} a *projective closure* of S . Starting with a smooth quasi-projective surface S birational to $\mathbb{P}^1 \times E$ we can choose a good projective closure of S as follows:

Lemma 1.3.5. *Assume that S is a smooth quasi-projective surface which is birational to $\mathbb{P}^1 \times E$ over a perfect field k where E is an elliptic curve and $\pi: S \rightarrow E$ is the map as in (1.3.1). Then there exists a smooth projective variety \bar{S} and an open embedding $\iota: S \hookrightarrow \bar{S}$ such that π induces a morphism $\bar{\pi}$ from \bar{S} to E :*

$$\begin{array}{ccc} S & \xrightarrow{\iota} & \bar{S} \\ \pi \downarrow & \searrow \bar{\pi} & \\ E & & \end{array}$$

Proof. By assumption the surface S is quasi-projective; thus, there is an embedding of S to a projective space \mathbb{P}^N . Denote by \bar{S}' the minimal Zariski closed subset in \mathbb{P}^N which contains S . By the construction \bar{S}' is of locally finite type over k ; thus, \bar{S}' is an excellent scheme and by [Lip78] and [Lip69] one can construct its minimal resolution \bar{S} . Since S is a smooth dense subset in \bar{S}' it does not intersect the indeterminacy locus of $\bar{S}' \dashrightarrow \bar{S}$. Thus, \bar{S} is a projective closure of S .

The induced map $\bar{\pi}: \bar{S} \rightarrow E$ is regular since $\bar{\pi}$ is the Albanese map. \square

Note that $\bar{S} \setminus S$ can contain multisections of $\bar{\pi}$; i.e. $\bar{\pi}(C)$ is a dense subset of E for some curve C in $\bar{S} \setminus S$.

Lemma 1.3.6. *Let \bar{S} be a projective surface birational to $\mathbb{P}^1 \times E$ over an algebraically closed field, where E is an elliptic curve and $\bar{\pi}: \bar{S} \rightarrow E$ is the map as in (1.3.1). If $C \subset \bar{S}$ is an irreducible curve such that $\bar{\pi}(C) = E$, then $C \not\subset \text{Exc}(g)$ for any $g \in \text{Bir}(\bar{S})$.*

Proof. The curve C maps dominantly to an elliptic curve. Thus, by Riemann-Hurwitz formula we get that the curve C cannot be rational. Therefore, the curve C is not contractible. \square

There is many different ways to choose a smooth projective closure of S . We need a projective closure with the following properties.

Definition 1.3.7. Assume that \bar{S} is a smooth quasi-projective surface birational to $\mathbb{P}^1 \times E$ over an algebraically closed field k , where E is an elliptic curve and $\pi: \bar{S} \rightarrow E$ is a projection. Let \bar{S} be a smooth projective closure of S . We say that it is *relatively minimal* if there is no (-1) -curve C in the complement $\bar{S} \setminus S$ lying in fiber of $\bar{\pi}$.

By the following lemma any smooth ruled quasi-projective surface S admits a relatively minimal projective closure.

Lemma 1.3.8. *Assume that S is a smooth quasi-projective surface birational to $\mathbb{P}^1 \times E$ over an algebraically closed field k , where E is an elliptic curve and $\pi: S \rightarrow E$ is the map as in (1.3.1). Then there exists a relatively minimal projective closure of S . Moreover, if all fibers of S are smooth, then the same is true for \bar{S} .*

Proof. The same assertion was proved in characteristic 0 in [Fuj82, Section 4]. In fact all arguments from that paper work in positive characteristic. We recall for the convenience of the reader the main step of the argument.

By Lemma 1.3.5 there is a smooth projective closure $\bar{\pi}: \bar{S} \rightarrow E$ of S . If the complement $\bar{S} \setminus S$ contains a (-1) -curve lying in fiber of $\bar{\pi}$, then it can be blown down. Thus, we obtain a new smooth projective closure $\bar{\pi}: \bar{S}' \rightarrow E$ of S and the number of codimension 1 components of $\bar{S}' \setminus S$ is less than one of $\bar{S} \setminus S$. Repeating this process several times we get a smooth projective closure $\bar{\pi}_0: \bar{S}_0 \rightarrow E$ of S such that $\bar{S}_0 \setminus S$ does not contain (-1) -curves lying in fibers of $\bar{\pi}_0$.

By the semi-continuity theorem geometric genus of any fiber of $\bar{\pi}$ vanishes. Thus, if some fiber F of $\bar{\pi}_0$ is singular, then F is reducible, its components are rational curves and the dual graph is a tree. In particular it contains two (-1) -curves corresponding to leaf vertexes of the dual graph of F . If we assume that all fibers of S are smooth irreducible and reduced this implies that $\bar{S}_0 \setminus S$ contains a (-1) -curve lying in the fiber of $\bar{\pi}_0$. Thus, all fibers of $\bar{\pi}_0$ are smooth. \square

1.3.2. Proof of Theorem 1.1.3

Consider a quasi-projective surface S and let \bar{S} be a projective closure of S . Then any automorphism of S induces a birational automorphism of \bar{S} . By [CS21] we are reduced to the situation where \bar{S} is birational to a product of the projective line and an elliptic curve.

Lemma 1.3.9. *Assume that S is a quasi-projective surface over a field k of characteristic $p > 0$. If S is not birational to a product $\mathbb{P}^1 \times E$ of a projective line and an elliptic curve E , then the group $\text{Aut}(S)$ is p -Jordan.*

Proof. Any automorphism of S induces a birational automorphism of \bar{S} . By Theorem 1.1.2 the group $\text{Bir}(\bar{S})$ is p -Jordan; thus, we get the result. \square

Now we are ready to prove Theorem 1.1.3.

Proof of Theorem 1.1.3. We can assume that k is algebraically closed. If the quasi-projective surface S is not irreducible, we consider its irreducible component S_0 . The group $\text{Aut}(S_0)$ is a subgroup of finite index in $\text{Aut}(S)$. Thus, by Lemma 1.2.36 it suffices to prove that the group $\text{Aut}(S_0)$ is p -Jordan. Therefore, from now on we assume that S is irreducible.

Any regular automorphism of S can be lifted to its normalization; thus, we can assume that S is normal. By [Lip78] and [Lip69] one can construct the minimal resolution of the surface S . The group of automorphisms of S embeds to the group of automorphism of the resolution. Thus, we can assume that S is smooth.

By Lemma 1.3.9 we are reduced to the case where S is birational to $\mathbb{P}^1 \times E$ since otherwise $\text{Aut}(S)$ is a p -Jordan group. Thus, we can consider the projection $\pi: S \rightarrow E$. It is a regular map since it is the Albanese map from a smooth variety.

If $\pi(S)$ is a proper subset of E or if some fibers of π are non-reduced or reducible, then the group $\text{Aut}(S)$ preserves a finite union of fibers of π . Thus, by Lemma 1.3.4 we get that $\text{Aut}(S)$ is a p -Jordan group. From now on we assume that all fibers of S are smooth and $\pi(S) = E$.

By Lemma 1.3.8 there exists a smooth closure \bar{S} of S such that all fibers of \bar{S} are smooth and irreducible. Since \bar{S} is relatively minimal and $\pi(S) = E$, then the complement $\bar{S} \setminus S$ is a set of curves which do not lie in fibers of $\bar{\pi}$. If $g \in \text{Aut}(S)$, then it induces a birational automorphism $\bar{g} \in \text{Bir}(\bar{S})$. The exceptional locus of \bar{g} lies in $\bar{S} \setminus S$. Thus, by Lemma 1.3.6 we get that the exceptional locus of \bar{g} is empty. Then g induces a regular automorphism of \bar{S} .

We get that $\text{Aut}(S)$ is a subgroup of $\text{Aut}(\bar{S})$ where \bar{S} is a relatively minimal projective closure of the quasi-projective surface S . By Theorem 1.1.1 this implies that $\text{Aut}(S)$ is p -Jordan. \square

1.4. Automorphisms of 3-dimensional Mori fiber spaces

In this section we study 3-groups having a faithful action on 3-dimensional Mori fiber spaces and construct bounds on number of generators of these groups. All varieties in this section are defined over the field \mathbb{C} .

1.4.1. Singular Fano threefolds

We study complex Fano threefolds with singularities and show that any 3-group having a faithful action on such threefold can be generated by at most 5 elements. First, we recall a very useful description of terminal singularities in the case of threefolds.

Theorem 1.4.1 ([Rei80, Corollary 1.9][Rei87, Theorem 3.2]). *Let X be a complex threefold, $x \in X$ be an isolated terminal singularity and let $r > 0$ be the minimal integer such that rK_X is a Cartier divisor in a neighborhood U of x . Then there exists a complex threefold Y with an isolated cDV -singularity y , an action of the group C_r on Y and an isomorphism $Y/C_r \cong U$. If we denote by π the following map:*

$$\pi: Y \rightarrow Y/C_r \cong U,$$

then $\pi^{-1}(x) = y$ and C_r acts freely on $Y \setminus \{y\}$. Moreover, Y is the spectrum of the following sheaf of algebras on U :

$$\mathcal{A}_U = \mathcal{O}_U \oplus \mathcal{O}_U(K_U) \oplus \cdots \oplus \mathcal{O}_U((r-1)K_U).$$

Now we consider a 3-group with an action on a terminal threefold which preserves a point. This allows us to restrict the number of generators of the group.

Proposition 1.4.2 ([Pro11, Lemma 2.4]). *Let X be a complex quasi-projective threefold with terminal singularities, let G be a 3-subgroup of $\text{Aut}(X)$ which stabilizes a point $x \in X$. Then G can be generated by 3 elements.*

Proof. Since G is finite then by [SGA03, Proposition V.1.8] there is a G -invariant open affine chart of the point x in X which does not contain singular points except x . Replace X by the intersection of this affine chart with the affine chart U described in Theorem 1.4.1.

Note that the algebra \mathcal{A}_X defined in Theorem 1.4.1 is invariant under the action of G . Thus, there is an action of a group \tilde{G} on the projectivization Y of \mathcal{A}_X fitting the following exact sequence:

$$1 \rightarrow C_r \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

where $r > 0$ is the minimal integer such that rK_X is a Cartier divisor. Thus, by Theorem 1.4.1 the variety Y contains a cDV -singularity Y_x , it is smooth on the complement to y and the group \tilde{G} stabilizes the point y . If any 3-subgroup of \tilde{G} can be generated by 3 elements then so can G . Thus, it suffices to prove the assertion in the case where x is cDV -singularity.

Assume that x is a cDV -singularity. Thus, X is a subvariety of \mathbb{A}^4 and the Zariski tangent space $T_{X,x}$ to X in the point x is a vector space of dimension at most 4. Thus, by Proposition 1.2.21 the group G is a subgroup of $\text{GL}_4(\mathbb{C})$ and by Proposition 1.2.11 can be generated by 4 elements.

If the group G can not be generated by less than 4 elements then by Corollary 1.2.9 and Proposition 1.2.11 one gets that G either contains an abelian subgroup of rank 4 or it is a product of a cyclic 3-group and a group H which contains matrices described in (1.2.10).

Assume that G contains an abelian subgroup of rank 4. Denote by G' the subgroup of G isomorphic to C_3^4 and choose coordinates z_1, z_2, z_3 and z_4 of $T_{X,x}$ such that the action of G' is diagonal.

$$g_i(z_j) = \begin{cases} \zeta z_i, & \text{if } i = j; \\ z_j, & \text{otherwise.} \end{cases}$$

Here ζ is a cube root of unity. Since the point x is a cDV -singularity, it is given by an equation $F(z_1, z_2, z_3, z_4)$ from the list [Rei87, Theorem 6.1] and F is an irreducible polynomial with a non-trivial quadratic part. Since the surface X is invariant under the action of G' , the polynomial F is semi-invariant under the action of elements g_i ; i.e. there are complex numbers λ_i such that one has the following equality:

$$g_i(F(z_1, z_2, z_3, z_4)) = \lambda_i F(z_1, z_2, z_3, z_4).$$

After permutation of the coordinates we can assume that either z_1^2 or $z_1 z_2$ has non-zero coefficient in F . Then g_1 acts on F with a non-trivial eigenvalue. Since g_1 acts on z_j trivially for all indices $j \neq 1$, all monomials of f are divisible by F_1 . This contradicts the fact that F is irreducible. Therefore, G cannot contain such an abelian group of rank 4.

Assume that G contains the product $\langle g_1 \rangle \times H$, where the order of the element g_1 equals 3 and H is isomorphic to a subgroup of $\mathrm{GL}_3(\mathbb{C})$ which contains the matrices σ and t described in (1.2.10). Choose coordinates z_1, z_2, z_3 and z_4 of $T_{X,x}$ such that g_1, σ and t acts in the following way:

$$g_1 = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 & 0 \end{pmatrix}; \quad t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix};$$

where ζ is a cube root of unity and λ_i is roots of unity of degree 3^{k_i} for some integer k_i and $i = 1, 2$ and 3. Denote by F the equation of the threefold X in \mathbb{A}^4 in these coordinates. As before, F is an irreducible polynomial with a non-trivial quadratic part and F is semi-invariant under the action of G .

Consider the quadratic part F_2 of F . It is a semi-invariant polynomial under the action of G . A quadratic polynomial F_2 which is semi-invariant under the action of g_1 is proportional to the one of the following form:

$$F_2 = Q(z_2, z_3, z_4); \quad F_2 = z_1 L(z_2, z_3, z_4); \quad F_2 = z_1^2.$$

Here Q and L are respectively quadratic and linear polynomials depending only on z_2, z_3 and z_4 .

A quadratic polynomial $Q(z_2, z_3, z_4)$ which is semi-invariant under the action of the matrix t is of the following form:

$$Q = az_i^2 + bz_j z_k,$$

where $\{i, j, k\} = \{2, 3, 4\}$ and a and b are complex numbers. We deduce that the polynomial Q is semi-invariant under the action of σ only if a and b vanish.

A linear polynomial $L(z_2, z_3, z_4)$ which is semi-invariant under the action of t is proportional to a monomial z_2, z_3 or z_4 . Thus, it is not semi-invariant under the action of σ .

Therefore, we proved that F_2 is proportional to z_1^2 . By [Rei87, Theorem 6.1] one can see that in this case the cubic part F_3 of F is also non-trivial. The polynomial $F_2 + F_3$ is semi-invariant under the action of G . Since this sum is semi-invariant under the action of g_1 we get that

$$F_3 = z_1^2 L(z_2, z_3, z_4),$$

where L is a linear polynomial depending only on z_2, z_3 and z_4 . By the above argument L can be semi-invariant under the action of t and σ only if it vanishes. Thus, we get a contradiction. Therefore, the group G can be generated by 3 elements. \square

Remark 1.4.3. Proposition 1.4.2 shows that if x is a cDV -singularity which is invariant under a faithful action of a 3-group, then this group is generated by at most 3 elements. This result can not be improved in the case x is a smooth point. Moreover, it can not be improved for certain singularities.

For instance, consider the subvariety X in the affine space \mathbb{A}^4 with coordinates z_1, z_2, z_3 and z_4 defined as a zero locus of the polynomial F , where F is as follows:

$$F = z_1^2 + z_2 z_3 + z_1^2 z_4^3.$$

The point $0 \in \mathbb{A}^4$ lies on X and it is a cA_1 -singularity. Consider the following action of the abelian group C_3^3 on \mathbb{A}^4 in coordinates z_1, z_2, z_3 and z_4 :

$$C_3^3 \cong \langle \mathrm{diag}(\zeta, \zeta^{-1}, 1, 1), \mathrm{diag}(\zeta, 1, \zeta^{-1}, 1), \mathrm{diag}(1, 1, 1, \zeta) \rangle \subset \mathrm{GL}_4(\mathbb{C})$$

Here ζ is a primitive cube root of unity. This action preserves X ; thus, it is an abelian 3-group of rank 3 with a faithful action on a cA_1 -singularity.

Proposition 1.4.2 implies the following corollary

Lemma 1.4.4. *Assume that X is a terminal Fano threefold and G is a 3-subgroup of $\text{Aut}(X)$. If the length of the G -orbit of a point $x \in X$ equals $N < 27$, then G can be generated by 5 elements. If $N < 9$, then G can be generated by 4 elements.*

Proof. Consider the stabilizer $\text{Stab}(x)$ of x . It is a 3-subgroup of G of index $N = 3^k$. If $N < 27$ then $k \leq 2$ and if $N < 9$ then $k \leq 1$. By Proposition 1.4.2 the group $\text{Stab}(x)$ is generated by 3 elements. Then Lemma 1.2.1 implies the result. \square

Proposition 1.4.2 and Namikawa's bound in Theorem 1.2.43 on the number of isolated singularities on a Gorenstein terminal Fano threefolds allows estimate the size of 3-groups acting on these threefolds.

Corollary 1.4.5. *Assume that X is a singular Gorenstein terminal Fano threefold, $\rho(X) = 1$, the index and the genus of a general fiber \mathcal{X}_b in Namikawa smoothing are as follows:*

$$r(\mathcal{X}_b) = 1; \quad g(\mathcal{X}_b) \geq 7.$$

If G is a 3-subgroup in $\text{Aut}(X)$ then G can be generated by at most 5 elements. Moreover, if the number of singularities of X differs from 9 and 18 then G can be generated by at most 4 elements.

Proof. Consider a singular point x on X . The bound (1.2.44) and [IP99, Tables 12.6–12.7], implies that the length of the orbit of x is less than 27. Then by Lemma 1.4.4 the group G can be generated by 5 elements.

If the number of singularities of X differs from 9 and 18 and less than 27, then there exists a singularity x on X whose orbit has at most 9 points. In this situation by Lemma 1.4.4 the group G can be generated by 4 elements. \square

By Proposition 1.4.2 and Reid's bound on the number of terminal singularities on Fano threefolds we estimate number of generators of 3-groups acting on terminal threefolds.

Corollary 1.4.6. *Assume that X is a terminal non-Gorenstein Fano threefold with isolated singularities and G is a 3-subgroup in $\text{Aut}(X)$. Then G can be generated by at most 5 elements.*

Moreover, if there is a non-Gorenstein singularity on X then G can be generated by at most 4 elements unless $|\text{Sing}(X)| = 9$ and all singularities of X are quotient-singularities of type $\frac{1}{2}(1, 1, 1)$.

Proof. Assume that N is the number of terminal singularities on X . Then the formula (1.2.46) implies that $N < 16$. Thus, by Lemma 1.4.4 the group G can be generated by 5 elements.

If the number of singularities on X differs from 9, we can find a non-Gorenstein point x on X whose orbit has length 1 or 3. Then by Lemma 1.4.4 the group G can be generated by 4 elements.

If $N = 9$ and G is transitive on the set of singularities, then numbers $l_i = l$ in formula (1.2.46) coincide and $r_{ij} = r_j$ do not depend on i . Thus, we can rewrite the inequality (1.2.46) in a following way:

$$\sum_{j=1}^l \left(r_j - \frac{1}{r_j} \right) < \frac{8}{3}.$$

Then $l = 1$ and $r_1 = 2$ and all non-Gorenstein singularities of X are cyclic quotient-singularities, one can get it by arguments in proofs of [Rei87, Theorem 9.1, Theorem 10.2]. Since $r_1 = 2$, they are of type $\frac{1}{2}(1, 1, 1)$. \square

1.4.2. Fano threefolds with $\rho > 1$

In this section we study G -Fano threefolds with Picard lattice of rank greater than 1. Recall that the Picard group of a Fano threefold is necessarily discrete, finitely generate and torsion-free. The rank of the group $\text{Pic}(X)^G$ equals 1 if X is G -Fano variety. The following assertion is useful in this situation.

Theorem 1.4.7 ([Pro13b, Theorem 1.2, Theorem 6.6]). *Assume that X is a Gorenstein Fano threefold, $\rho(X) > 1$ and G is a finite group with an action on $\text{Pic}(X)$ preserving the intersection form and the canonical class. If we have $\text{rk}(\text{Pic}(X)^G) = 1$, then X is one of the following list:*

- (1) $\rho(X) = 2$;
- (2) $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
- (3) *There is a double cover $f : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along an element of the linear system $|-K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|$;*
- (4) *X is the blow up of a divisor W of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ along a curve C of bidegree $(2, 2)$ and the composition $C \hookrightarrow W \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{pr_i} \mathbb{P}^2$ is an embedding for both projections pr_1 and pr_2 ;*
- (5) *X is a divisor of multidegree $(1, 1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

This theorem implies the following.

Corollary 1.4.8. *Assume that G is 3-group and X is a Gorenstein G -Fano threefold with $\rho(X) > 1$. Then G can be generated by at most 4 elements.*

Proof. Assume that G is a 3-group with a faithful action on a terminal threefold X . Let X is a Gorenstein G -Fano threefold with $\rho(X) > 1$. The action of the group G on X induces the action of G on $\text{Pic}(X)$. Moreover, since X is G -Fano one has $\text{rk}(\text{Pic}(X)^G) = 1$. Thus, X is one of varieties listed in Theorem 1.4.7. We argue by a case-by-case analysis.

(1) The case where $\rho(X) = 2$ is excluded by Lemma 1.2.16.

(2) Assume that $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The group G acts on the Picard lattice $\text{Pic}(X)$. Denote by H the kernel of the action of G on $\text{Pic}(X)$. By construction H is a normal subgroup of G . Since H acts trivially on $\text{Pic}(X)$ it is a subgroup of a product $\text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C})$. Therefore, the group H can be generated by 3 elements by Lemma 1.2.6.

The quotient group G/H acts faithfully on the lattice $\text{Pic}(X) \cong \mathbb{Z}^3$. Then it is a subgroup of $\text{GL}_3(\mathbb{Z})$ and by Lemma 1.2.15 it is cyclic. Therefore, by Lemma 1.2.2 the group G can be generated by 4 elements.

(3) Assume that X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along an element of the linear system $|-K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|$.

Denote by f the map from X to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$f: X \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

By the classification of Fano threefolds one has $\text{Pic}(X) = f^*(\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)) \cong \mathbb{Z}^3$. Let H be the kernel of action of G on $\text{Pic}(X)$. The group G/H acts faithfully on the Picard group $\text{Pic}(X) \cong \mathbb{Z}^3$. Thus, G/H is cyclic by Lemma 1.2.15.

Denote by $\text{pr}_i: \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ the projection onto the i -th component. All line bundles on the threefold X are invariant under the action of the group H . In particular, this implies that the linear system $|(\text{pr}_i \circ f)^*(\mathcal{O}_{\mathbb{P}^1}(1))|$ is H -invariant for all $i = 1, 2, 3$. Since $f = (\text{pr}_1 \circ f) \times (\text{pr}_2 \circ f) \times (\text{pr}_3 \circ f)$ we get that H induces the action on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the map f is H -equivariant.

Thus, by Lemmas 1.2.20 and 1.2.6 the group H can be generated by three elements. Therefore, by Lemma 1.2.2 the group G can be generated by 4 elements.

(4) Assume that W is a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$. Denote by $\text{pr}_i: \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ the projection to the i -th component. The divisor W is isomorphic to the projective bundle $\mathbb{P}(T_{\mathbb{P}^2})$. Here by the projective bundle we mean the bundle of hyperplanes in fibers of the bundle $T_{\mathbb{P}^2}$. By construction of W the following diagrams commutes:

$$\begin{array}{ccc} W & \xrightarrow{\cong} & \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}) \\ \text{pr}_i \downarrow & & \downarrow \pi \\ \mathbb{P}^2 & \xlongequal{\quad} & \mathbb{P}^2 \end{array}$$

where $\pi: \mathbb{P}(T_{\mathbb{P}^2}) \rightarrow \mathbb{P}^2$ is the natural map to the base of the projective bundle. Assume that C is a curve in W of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ such that pr_i induces an isomorphism of the curve C with its image $\text{pr}_i(C) \subset \mathbb{P}^2$. Let $\delta: X \rightarrow W$ be the blow-up of W along C .

By [IP99, Table 12.4] there exist three maps $\delta_i: X \rightarrow W$, each δ_i is a contraction of an exceptional divisor E_i in X and $i = 1, 2, 3$ and $\delta_1 = \delta$. Thus, divisors E_1, E_2, E_3 generate a G -invariant sublattice $\mathcal{E} \subset \text{Pic}(X)$ of rank 3.

Denote by H the kernel of the action of G on the lattice \mathcal{E} . Then the index $[G : H]$ is at most 3. Then the action of H on X induces a faithful action of H on W such that the map δ_1 is H -equivariant. By the classification of Gorenstein Fano threefold we have $\text{Pic}(X) = \mathbb{Z}^2$; thus, the 3-group H acts trivially on $\text{Pic}(W)$. Therefore, H descends to an action on \mathbb{P}^2 such that the following map is G -equivariant.

$$W \cong \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}) \xrightarrow{\pi} \mathbb{P}^2.$$

Since π is a fiber bundle whose general fiber is isomorphic to \mathbb{P}^1 , Lemma 1.2.20 implies that the group H can be generated by at most 3 elements. Therefore, by Lemma 1.2.2 G can be generated by at most 4 elements.

(5) Assume that X is a divisor of multidegree $(1, 1, 1, 1)$ in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then by [IP99, Table 12.5] there exist exactly 4 distinct maps

$$\delta_i: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

where $i = 1, 2, 3$ and 4 such that each δ_i is a contraction of an exceptional divisor E_i in X . The group G permutes divisors E_1, E_2, E_3 and E_4 . Since 4 elements can not form one orbit under the action of the 3-group G we get that the rank of the lattice $\text{Pic}(X)^G$ can not be equal to 1. Thus, we get a contradiction with the assumption that X is a G -Fano threefold. \square

1.4.3. Fano threefolds with $\rho = 1$

In this section we consider different types of terminal Gorenstein G -Fano threefolds with a faithful action of a 3-group G and estimate the number of elements which generate G .

We start by considering a Fano threefold whose anticanonical linear system has a non-trivial base locus.

Lemma 1.4.9. *Assume that X is a terminal Gorenstein Fano threefold such that the linear system $| -K_X |$ has non-empty base locus. Then a 3-subgroup $G \subset \text{Aut}(X)$ can be generated by at most 3 elements.*

Proof. By [Shi89, Theorem 0.5] the base locus $B = \text{Bs}(| -K_X |)$ of a Fano threefold is either a point or isomorphic to a projective line \mathbb{P}^1 .

If B is a point, then it is invariant under the action of G . Thus, by Proposition 1.4.2 the group G can be generated by 3 elements.

If B is isomorphic to \mathbb{P}^1 , then by Lemma 1.2.6 there is a point x on B which is G -invariant. Again, by Proposition 1.4.2 the group G can be generated by 3 elements. \square

We next consider the case where the anticanonical linear system of a G -Fano threefold has no base points but does not induce an embedding.

Lemma 1.4.10. *Assume that X is terminal Gorenstein Fano threefold and the linear system $| -K_X |$ induces a regular map from X to a projective space which is not an embedding. Then a 3-subgroup G of $\text{Aut}(X)$ can be generated by at most 3 elements.*

Proof. Denote by $\phi_{|-K_X|}$ the map induced by the anticanonical linear system:

$$\phi_{|-K_X|}: X \rightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(-K_X)^\vee)).$$

Since the linear system $\phi_{|-K_X|}$ is base point free then by [IP99, Proposition 2.1.15] its degree equals 1 or 2. Since $-K_X$ is an ample class on X its linear system does not contract any divisor. Thus, by our assumption the map $\phi_{|-K_X|}$ is a double cover. Denote by Y the variety $\phi_{|-K_X|}(X)$.

If the index of X equals 1, then by (1.2.39) the codimension of Y in $\mathbb{P}(H^0(X, \mathcal{O}_X(-K_X)^\vee))$ equals $g - 2$. Moreover, by [IP99, Corollary 2.1.14] and since $\phi_{|-K_X|}$ is a double cover and its image we get that the degree of Y equals $g - 1$. Thus, Y is a variety of minimal degree. Then by Lemmas 1.2.20 and 1.2.31 the group G can be generated by 3 elements.

We can a priori exclude the projective space and the quadric hypersurfaces since their anticanonical systems induce embeddings. Thus, the index of X equals 2 and by [IP99, Corollary 2.1.14] one gets that the codimension of Y in $\mathbb{P}(H^0(X, \mathcal{O}_X(-K_X)^\vee))$ equals $2 \deg(Y) - 2$. Thus, the inequality (1.2.30) is satisfied only if $\deg(Y) = 1$; thus, X is a double cover of \mathbb{P}^3 . Then by Lemmas 1.2.7 and 1.2.20 the group G can be generated by 3 elements. \square

If X is a Gorenstein del Pezzo threefold then by the following assertion one can estimate the number of generators of a 3-group acting on it. Recall that a del Pezzo threefold of degree d is a Fano threefold of index 2 such that $K_X = -2H$ and $H^3 = d$.

Lemma 1.4.11. *Assume that X is a terminal Gorenstein del Pezzo threefold and $\rho(X) = 1$. Then a 3-subgroup G of $\text{Aut}(X)$ can be generated by at most 4 elements.*

Proof. If the anticanonical system $|-K_X|$ does not induce an embedding to the anticanonical linear system, then by Lemmas 1.4.9 and 1.4.10 the group G can be generated by at most 3 elements. Thus, by [IP99, Table 12.5] if the degree of X equals 1 or 2 then G can be generated by at most 3 elements.

If the degree of X equals 3 and $|-K_X|$ induces an embedding, then G is a subgroup of $\mathrm{PGL}_5(\mathbb{C})$. Thus by Proposition 1.2.11 the group G can be generated by 4 elements.

If X is a terminal Gorenstein del Pezzo threefold of degree 4, then by [IP99, Theorem 3.2.5] it is isomorphic to the complete intersection of two quadric hypersurfaces. Thus, there is a 1-dimensional pencil of quadrics containing X . Since the group G acts on the anticanonical linear system, then it acts on the pencil. Moreover, by Lemma 1.2.6 the group G preserves one hypersurface in this pencil. Thus, G acts faithfully on a quadric hypersurface of dimension 4. Then Lemma 1.2.28 implies that G can be generated by 4 elements.

Finally if X is a terminal Gorenstein del Pezzo threefold of degree 5, then by [Pro13a, Theorem 1.7] it is smooth. By [Muk88, Proposition 4.4] we get that the group $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$. Thus, Lemma 1.2.6 the group G is cyclic. \square

In the next assertion we compute a bound on a number of generators of a Fano threefold of small genus.

Lemma 1.4.12. *Assume that X is a terminal Gorenstein Fano threefold having the following invariants:*

$$\rho(X) = 1; \quad r(X) = 1; \quad g(X) \leq 4.$$

Any 3-subgroup of $\mathrm{Aut}(X)$ can be generated by at most 4 elements.

Proof. If the linear system $|-K_X|$ does not induce an embedding, then by Lemmas 1.4.9 and 1.4.10 the group G can be generated by 3 elements. From now on we assume, that $|-K_X|$ induces an embedding to a projective space \mathbb{P}^{g+1} . Thus, by [IP99, Table 12.5] one gets that $g(X) \geq 3$.

If $g = 3$ and $|-K_X|$ induces an embedding to \mathbb{P}^4 , then G is a subgroup of $\mathrm{PGL}_5(\mathbb{C})$. Therefore, Proposition 1.2.11 implies that G can be generated by 4 elements.

If $g = 4$ then $|-K_X|$ induces an embedding of X to \mathbb{P}^5 , then by [IP99, Proposition 4.1.12] the variety X is a complete intersection of a cubic and a quadric hypersurfaces. By Koszul resolution of the sheaf of ideals \mathcal{I}_X we get that there is a unique quadric hypersurface Q containing X . Thus, the quadric fourfold Q is invariant under the action of G . Then by Lemma 1.2.28 the group G can be generated by 4 elements. \square

The next assertion gives a bound on the number of generators of a 3-group with an action of a Fano threefold of the genus 5.

Lemma 1.4.13. *Assume that X is a terminal Gorenstein Fano threefold having the following invariants:*

$$\rho(X) = 1; \quad r(X) = 1; \quad g(X) = 5.$$

If G is a 3-subgroup of $\mathrm{Aut}(X)$ then it can be generated by at most 4 elements.

Proof. By Lemmas 1.4.9 and 1.4.10 we are reduced to the situation where the linear system $|-K_X|$ induces an embedding. Thus, $|-K_X|$ embeds X into \mathbb{P}^6 and by [IP99, Proposition 4.1.12] one gets that the image of X in \mathbb{P}^6 is an intersection of 3 quadric hypersurfaces Q_1, Q_2 and Q_3 .

Consider the action of G on $\mathbb{P}^6 = \mathbb{P}(V)$, where $V = H^0(X, \mathcal{O}_X(-K_X))$. This action induces an embedding of G to $\mathrm{PGL}_7(\mathbb{C})$. Let G' be the preimage of the group G in $\mathrm{GL}_7(\mathbb{C})$; denote by \tilde{G} the Sylow 3-subgroup of G' . Thus, there is a surjective map:

$$f: \tilde{G} \rightarrow G.$$

The embedding of \tilde{G} to $\mathrm{GL}_7(\mathbb{C})$ induces a 7-dimensional representation V of the 3-group \tilde{G} . Thus, there exists a 4-dimensional subrepresentation $W \subset V$ of \tilde{G} . Since the map f is surjective then the subspace $\mathbb{P}(W)$ is G -invariant in $\mathbb{P}(V)$.

Denote by Y the intersection $\mathbb{P}(W) \cap X$. Then Y is non-empty intersection of 3 quadratic surfaces in \mathbb{P}^3 . By Lemma 1.2.29 there exists a point $y \in Y$ and a subgroup H of G such that y is invariant under the action of H and the index of H in G is at most 3. Therefore, Proposition 1.4.2 and Lemma 1.2.1 imply that G can be generated by 4 elements. \square

The next assertion proves the bound on the number of generators of a 3-group with an action of a Fano threefold of genus 6.

Lemma 1.4.14. *Assume that X is a terminal Gorenstein Fano threefold having the following invariants:*

$$\rho(X) = 1; \quad r(X) = 1; \quad g(X) = 6.$$

If G is a 3-subgroup of $\mathrm{Aut}(X)$ then it can be generated by at most 4 elements.

Proof. By Lemmas 1.4.9 and 1.4.10 we are reduced to the case where the linear system $| -K_X |$ induces an embedding of X to \mathbb{P}^7 . Denote by Y the intersection of all quadric hypersurfaces in \mathbb{P}^7 which pass through the image of X . Then Y contains X and it is invariant under the action of G ; thus, $G \subset \mathrm{Aut}(Y)$.

If the variety Y does not coincide with X , then by [Isk78, Proposition 2.3] dimension of Y equals 4 and it is a variety of minimal degree. Thus, by Lemma 1.2.31 the group G can be generated by at most 4 elements.

If Y coincides with X denote by C a curve in the intersection of X and two general hyperplanes in \mathbb{P}^7 . By construction the curve C is an intersection of quadric hypersurfaces in \mathbb{P}^5 and one can compute that the genus of C equals 6. By the first assertion of Theorem 1.2.42 the pair (C, K_C) is a Gushel–Mukai curve and K_C equals the inverse image of the hyperplane section from \mathbb{P}^5 to C .

Therefore, by the second assertion of Theorem 1.2.42 the variety $(X, -K_X)$ is a Gushel–Mukai variety. Thus, there exists a map:

$$\iota: X \rightarrow \mathrm{Gr}(2, 5).$$

Then by the third assertion of Theorem 1.2.42 the map ι is either an embedding or a double cover onto its image and it is $\mathrm{Aut}(X)$ -equivariant. Thus, by Propositions 1.2.23, 1.2.11 and Lemma 1.2.20 the group G can be generated by at most 4 elements. \square

By the next assertion some GQ -Fano threefolds happen to be \mathbb{Q} -factorial. We will need this assertion further.

Lemma 1.4.15 (see [PS18, Lemma 7.6]). *Assume that G is a 3-group and X is a terminal Gorenstein GQ -Fano threefold having the following invariants:*

$$\rho(X) = 1; \quad r(X) = 1; \quad g(X) = 8, 9 \text{ or } 12.$$

Then X is a \mathbb{Q} -factorial variety.

Proof. As in proof of [PS18, Lemma 7.6] we can show that if X is not a \mathbb{Q} -factorial variety, then it is birational to a terminal Gorenstein Fano threefold Y such that:

$$-K_Y^3 \geq 2g(X) - 2 + (3 - 1)(4g(X) - 6) \geq 66.$$

By [IP99, Tables 12.6–12.7] and Theorem 1.2.43 the degree of any terminal Gorenstein Fano threefold less than 65. Thus, X is a \mathbb{Q} -factorial variety. \square

Thus, we get the following bound on the number of generators of a 3-group having an action on Gorenstein Fano threefolds with big genus.

Lemma 1.4.16. *Assume that X is a terminal Gorenstein G - \mathbb{Q} -Fano threefold having the following invariants:*

$$\rho(X) = 1; \quad r(X) = 1; \quad g(X) = 8, 9 \text{ or } 12.$$

Then any 3-subgroup of $\text{Aut}(X)$ can be generated by at most 4 elements.

Proof. By Lemma 1.4.15 the variety X is \mathbb{Q} -factorial. By [Pro17, Theorem 1] there are at least 1 and at most 5 singular points on X . Thus, Lemma 1.4.4 implies the result. \square

Lemma 1.4.17. *Assume that X is a smooth Fano threefold having the following invariants:*

$$\rho(X) = 1; \quad r(X) = 1; \quad g(X) = 7, 9, 10 \text{ or } 12.$$

Then any 3-subgroup of $\text{Aut}(X)$ can be generated by at most 4 elements.

Proof. By Lemmas 1.4.9 and 1.4.10 we are reduced to the case where the anticanonical linear system induces an embedding. Denote by $S(X)$ the Hilbert scheme of conics on the anticanonical embedding of X . By Proposition 1.2.41 we see that $S(X)$ is a smooth surface and the group G can be realized as a subgroup of $\text{Aut}(S(X))$.

If $g = 7$ then by Proposition 1.2.41 the group G is a subgroup of $\text{Aut}(C)$ where C is a curve of genus 7. The bound (1.2.25) implies

$$|G| \leq 16 \cdot (g(C) - 1) = 96 < 3^5.$$

Thus, the group G can be generated by at most 4 elements.

If $g = 9$ then by Lemma 1.2.27 the group G can be generated by at most 4 elements.

If $g = 10$ then by Lemma 1.2.26 the group G can be generated by at most 2 elements.

If $g = 12$ then the group $\text{Aut}(S(X))$ is isomorphic to $\text{PGL}_3(\mathbb{C})$. Thus, Proposition 1.2.11 implies that G can be generated by at most 2 elements. \square

The next assertion describes 3-groups having an action on smooth Fano threefolds of genus 8.

Lemma 1.4.18. *Assume that X is a smooth Fano threefold having the following invariants*

$$\rho(X) = 1; \quad r(X) = 1; \quad g(X) = 8.$$

Then a 3-subgroup G in $\text{Aut}(X)$ can be generated by at most 4 elements.

Proof. By Lemmas 1.4.9 and 1.4.10 we are reduced to the case where $|-K_X|$ induces an embedding. As was shown in [KPS18, Section B.6] one can associate a smooth del Pezzo threefold Y of degree 3 to any smooth Fano threefold X of index 1 and genus 8 and the group G can be realized as a subgroup of $\text{Aut}(Y)$.

The anticanonical system of Y induces an embedding by [IP99, Proposition 3.2.4]. Denote by $\Sigma(Y)$ the Hilbert scheme of lines on Y in this anticanonical embedding.

By [KPS18, Proposition B.6.3] the scheme $\Sigma(Y)$ is a smooth surface and there is a canonical isomorphism of $\Sigma(Y)$ and the Hilbert scheme of conics $S(X)$ on X in the anticanonical embedding. The action of G on $\Sigma(X)$ is faithful by Proposition 1.2.41. Thus, the action of G on $\Sigma(Y)$ is also faithful. Therefore, the action of G on Y is faithful. By Lemma 1.4.11 we conclude that G can be generated by at most 4 elements. \square

1.4.4. Proof of Theorem 1.1.5

We start with an assertion which establishes the relation between ranks of abelian p -subgroups in the group of birational automorphisms of a variety and numbers of generators of non-abelian p -subgroups there.

Proposition 1.4.19. *Assume that for any complex rationally connected variety Y of dimension n the rank of any abelian p -subgroup of $\text{Bir}(Y)$ equals at most r . Then any p -subgroup of $\text{Bir}(X)$ can be generated by at most n elements for any complex rationally connected variety X .*

Proof. By Proposition 1.2.22 there exists a birational model \tilde{X} of X with a regular action of G . Consider a Frattini subgroup $\Phi(G) \subset G$ the intersection of all maximal subgroups of G . The quotient group $A = G/\Phi(G)$ is an abelian group and it acts on a quotient variety $Y = \tilde{X}/\Phi(G)$. This quotient variety is rationally connected and of dimension n ; thus, the rank of the group A is less than or equal to r . Therefore, by Burnside theorem [Hal76, Theorem 12.2.1] the group G can be generated by r elements. \square

Proposition 1.4.19 and the bounds on the ranks of abelian p -subgroups in $\text{Cr}_2(\mathbb{C})$ given in [Bea07] imply a bound on the number of generators of a non-abelian 3-subgroup of $\text{Cr}_2(\mathbb{C})$. This result can be deduced from the Dolgachev and Iskovskikh classification of finite subgroups of $\text{Cr}_2(\mathbb{C})$, but we give another proof.

Proposition 1.4.20. *Any 3-subgroup G of $\text{Cr}_2(\mathbb{C})$ can be generated by at most 3 elements and this bound is sharp.*

Proof. By [Bea07] the rank of any abelian 3-subgroup of $\text{Cr}_2(\mathbb{C})$ is at most 3 and the result follows from Proposition 1.4.19. The bound is sharp in view of Example 1.2.32. \square

Proposition 1.4.19 implies in particular that a 3-group with a faithful action on a rational surface can be generated by at most 3 elements. This implies the following bound on the number of generators of a 3-groups with a regular faithful action on a G -Mori fiber space.

Lemma 1.4.21. *Assume that X and B are normal G -varieties and $\pi: X \rightarrow B$ is a G -Mori fiber space. If $\dim(X) = 3$ and $\dim(B) > 0$, then any 3-subgroup G of $\text{Aut}(X)$ can be generated by at most 4 elements.*

Proof. The base B is rationally connected as well as a general fiber of the map to B . By Proposition 1.4.20 a 3-subgroups in the group of birational automorphisms of a rational surface can be generated by at most 3 elements. Thus, by Lemmas 1.2.6 and 1.2.20 we get the result. \square

Now we are ready to prove Proposition 1.1.6.

Proof of Proposition 1.1.6. Let $\pi: X_0 \rightarrow B$ be a G -Mori fiber space with terminal singularities where G be a 3-group. To prove Proposition 1.1.6 we argue by considering all possible constructions of equivariant Mori fiber spaces.

If $\dim(B) > 0$, then by Lemma 1.4.21 the group G can be generated by at most 4 elements. From now on we assume that X_0 is a terminal G -Fano threefold.

If there is a non-Gorenstein terminal singularity on X_0 and it is not the variety described in point (a) of Corollary 1.4.6, then by Corollary 1.4.6 the group G can be generated by at most 4 elements. From now on we assume that X_0 is a Gorenstein terminal G -Fano threefold.

If X_0 is a Gorenstein Fano threefold and $\rho(X) > 1$, then by Corollary 1.4.8 the group G can be generated by at most 4 elements.

If X_0 is a Gorenstein Fano threefold, $\rho(X) = 1$ and index of X is at least 3, then X_0 is either a quadric hypersurface or the projective space. By Proposition 1.2.11 and 1.2.31 the group G can be generated by at most 3 elements.

If X_0 is a Gorenstein del Pezzo threefold, then by Lemma 1.4.11 the group G can be generated by at most 4 elements.

If X_0 is a Gorenstein Fano threefold of index 1 and genus of X_0 equals at most 6, then by Lemmas 1.4.12, 1.4.13 and 1.4.14 the group G can be generated by at most 4 elements.

If X_0 is a Gorenstein singular Fano threefold of index 1 and genera 8, 9 or 12, then by Lemma 1.4.16 the group G can be generated by at most 4 elements.

If X_0 is a smooth Fano threefold of index 1 and genus greater than 6, then by Lemmas 1.4.17 and 1.4.18 the group G can be generated by at most 4 elements.

If X_0 is a singular Gorenstein Fano threefold of index 1 and genera 7 and 10 and the number of singularities on X_0 differs from 9 or 18, then by Corollary 1.4.5 the group G can be generated by at most 4 elements. This finishes the proof of Proposition 1.1.6. \square

This leads us to the proof of Theorem 1.1.5.

Proof of Theorem 1.1.5. Let X be a rationally connected complex threefold and let G be a 3-subgroup of $\text{Bir}(X)$. Recall that by Theorem 1.1.4 the rank of an abelian 3-subgroup of $\text{Bir}(X)$ is at most 5. Thus, Proposition 1.4.19 implies the first assertion of Theorem 1.1.5.

To prove the second assertion of Theorem 1.1.5 we assume that $G \subset \text{Bir}(X)$ can not be generated by 4 elements. Then by Proposition 1.2.22 there exists a regularization \tilde{X} of the action of G on X ; i.e. \tilde{X} is a smooth variety birational to X and $G \subset \text{Aut}(\tilde{X})$. Applying G -equivariant MMP to \tilde{X} we get a terminal G -Mori fiber space X_0 . Since G can not be generated by less than 5 elements, X_0 is a threefold described in points (a) or (b) of Proposition 1.1.6. \square

Note that the argument in the proof of the second point of Theorem 1.1.5 gives another proof of the first point of this theorem since by Corollaries 1.4.5 and 1.4.6 any 3-group with a faithful action on threefolds described in points (a) or (b) can be generated by 5 elements.

2. Regularization of birational automorphisms

2.1. Introduction

Let X be any normal projective variety defined over an algebraically closed field k of characteristic 0. Recall that a birational automorphism $f: X \dashrightarrow X$ is regularizable if there exists a birational map $\alpha: X \dashrightarrow Y$ to a variety Y and an automorphism $g \in \text{Aut}(Y)$ such that one has:

$$\alpha \circ f = g \circ \alpha.$$

Understanding whether a given birational self-map is regularizable or not is a delicate problem.

In this chapter we explore the regularization properties for the following class of birational automorphisms described in [Bla13].

Example 2.1.1. Fix a smooth cubic surface $S \subset \mathbb{P}^3$. Then to each point $p \in S$ one can associate a birational involution

$$\sigma_p: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3,$$

such that for any line $L \subset \mathbb{P}^3$ if one has $L \cap S = \{p, p_1, p_2\}$ where p, p_1 and p_2 are distinct points then the restriction $\sigma_p|_L$ is a non-trivial involution which fixes p_i for $i = 1$ and 2 . Such involution of L is unique; thus, σ_p is a correctly defined birational involution of \mathbb{P}^3 . Choose k points p_1, \dots, p_k and consider the following composition:

$$f_{p_1, \dots, p_k} = \sigma_{p_1} \circ \dots \circ \sigma_{p_k}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3.$$

Then f_{p_1, \dots, p_k} is a birational automorphism of \mathbb{P}^3 ; moreover, by [Bla13] if $k \geq 2$ and if p_1, \dots, p_k are sufficiently general, then the order of f_{p_1, \dots, p_k} is infinite.

We are interested whether we can construct a regularization of f_{p_1, \dots, p_k} . Note that if $k = 1$ i.e. if f_{p_1} is an involution, then the regularization exists, see, e.g., [PS14, Lemma-Definition 3.1]. If dimension of X is greater than 1 and $f: X \dashrightarrow X$ is an infinite order birational automorphism then the question whether one can regularize f can be very hard. In the case of surfaces there are criteria devised in [DF01] and [BC16], in these works regularization properties of f are related with properties of the dynamical degree $\lambda_1(f)$.

If f_{p_1, \dots, p_k} is a birational automorphism defined in Example 2.1.1 and if points p_1, \dots, p_k are sufficiently general and $k \geq 3$ then by [Bla13] one has $\lambda_1(f_{p_1, \dots, p_k}) = \lambda_2(f_{p_1, \dots, p_k}) > 1$. We prove the following theorem:

Theorem 2.1.2. *Let $S \subset \mathbb{P}^3$ be a very general complex smooth cubic surface and let p_1, p_2, p_3 be general points on S . Then the birational automorphism $f_{p_1, p_2, p_3}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ described in Example 2.1.1 is non-regularizable and does not preserve a fibration over a surface.*

Recall that a birational automorphism $f: X \dashrightarrow X$ of a smooth threefold X is called *primitive* if it does not preserve a fibration neither over a surface nor over a curve. Thus, the second assertion of Theorem 2.1.2 allows us to expect that f_{p_1, p_2, p_3} maybe primitive.

To prove Theorem 2.1.2 we use the following nice model of the birational automorphism f_{p_1, \dots, p_k} . We denote by X the successive blow-ups of the points p_1, \dots, p_k followed by the blow-up of the proper transforms of all irreducible curves in $\text{Ind}(f_{p_1, \dots, p_k})$. Then f_{p_1, \dots, p_k} induces a birational automorphism

$$\tilde{f}_{p_1, \dots, p_k}: X \dashrightarrow X,$$

such that neither $\tilde{f}_{p_1, \dots, p_k}$ nor $\tilde{f}_{p_1, \dots, p_k}^{-1}$ contracts divisors in X ; i.e. $\tilde{f}_{p_1, \dots, p_k}$ is a pseudo-automorphism. Pseudo-automorphisms have many good properties; in particular, if $f: X \dashrightarrow X$ is a pseudo-automorphism such that $\lambda_1(f)^2 > \lambda_2(f)$ then by [Tru14] there exists a pseudo-effective divisor class $\theta_1(f)$ on X such that

$$f^*(\theta_1(f)) = \lambda_1(f)\theta_1(f).$$

In the case of surface birational automorphisms the considering of such class in [DF01] was extremely useful. We use this class to prove the following criterion:

Theorem 2.1.3. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth projective threefold X such that*

- (1) $\lambda_1(f)^2 > \lambda_2(f)$;
- (2) *there exists a curve C such that $\theta_1(f) \cdot [C] < 0$;*
- (3) *there exists infinitely many integers $m > 0$ such that $C \not\subset \text{Ind}(f^{-m})$.*

Then f is non-regularizable and it does not preserve a fibration over a surface.

Thus, to prove Theorem 2.1.2 we need to show that the pseudo-automorphism $\tilde{f}_{p_1, p_2, p_3}$ satisfies conditions (1–3). Note that the condition (1) is satisfied by Blanc’s computation in [Bla13]. Thus, one can consider the class $\theta_1(f_{p_1, p_2, p_3})$. We prove that the condition (2) is satisfied i.e. that the class $\theta_1(f_{p_1, p_2, p_3})$ is not nef; it intersects negatively a curve lying in $\text{Ind}(f_{p_1, p_2, p_3})$. The hardest part of the proof is to prove the condition (3) i.e. to show that this curve does not lie in $\text{Ind}(f_{p_1, p_2, p_3}^{-m})$ for infinitely many integers $m > 0$. We prove this fact computing the orbit of a general point on this curve.

2.2. Preliminaries

2.2.1. Birational maps acting on the divisor class group

Throughout this chapter we consider smooth algebraic varieties over an algebraically closed field of characteristic 0. Recall that by $N^i(X)$ we denote the \mathbb{R} -space of codimension i cycles modulo the numerical equivalence. If X is a smooth variety, then $N^i(X)$ is a finite dimensional space for any $0 \leq i \leq \dim(X)$. For simplicity we denote by $N_i(X)$ the group $N_{\dim(X)-i}(X)$. Then by Chow's moving lemma on a smooth projective variety X there is a natural multiplication

$$N^i(X) \times N^j(X) \rightarrow N^{i+j}(X); \quad (Z, W) \mapsto Z \cdot W.$$

Since $N_0(X) = N^{\dim(X)}(X)$ is a 1-dimensional space by Chow's moving lemma one gets that this multiplication induces the perfect pairing between vector spaces $N^i(X)$ and $N_i(X)$.

Definition 2.2.1. Let X and Y be smooth varieties such that $\dim(X), \dim(Y) \geq 2$, then

- If D is a reduced codimension i subvariety in X , then we denote by $[D]$ the equivalence class of D in $N^i(X)$.
- The class $Z \in N^i(X)$ is *effective* if $Z = \sum_{i=1}^n a_i [D_i]$ where $a_i \geq 0$ and D_i is a codimension i subvariety in X for all $1 \leq i \leq n$.
- The class $Z \in N^1(X)$ is *nef* (resp. *ample*) if one has $Z \cdot [C] > 0$ (resp. $Z \cdot [C] \geq 0$) for any curve C on X .
- If $f: X \rightarrow Y$ is an algebraic map then the class $D \in N^1(X)$ is *f-nef* (resp. *f-ample*) if one has $D \cdot [C] \geq 0$ (resp. $D \cdot [C] > 0$) for any curve $C \subset X$ such that $f(C)$ is a point.

Let $f: X \dashrightarrow Y$ be a rational map between two varieties. Consider a smooth variety V and two regular morphisms δ_X and δ_Y to X and Y respectively such that the diagram commutes and δ_X is birational:

$$\begin{array}{ccc} & V & \\ \delta_X \swarrow & & \searrow \delta_Y \\ X & \overset{f}{\dashrightarrow} & Y \end{array} \quad (2.2.2)$$

Such a diagram always exists (take, e.g., the resolution of indeterminacy of the graph of the map f in $X \times Y$). The variety V fitting the diagram of the form (2.2.2) is not unique, in the following assertion we describe the case when V has additional useful properties.

Lemma 2.2.3 ([Kol07, Corollary 3.18, Theorem 3.21]). *Let $f: X \dashrightarrow Y$ be a rational map between smooth varieties X and Y . Then there exists a diagram of the form (2.2.2) such that δ_X is the following composition:*

$$\delta_X = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_N,$$

where $X_0 = X$, $X_N = V$ and $\delta_i: X_i \rightarrow X_{i-1}$ is the blow-up in a smooth center $Z_i \subset X_{i-1}$ for all $1 \leq i \leq N$. Moreover, $\delta_1(\dots(\delta_{i-1}(Z_i))\dots)$ lies in $\text{Ind}(\delta_X^{-1})$.

Definition 2.2.4. Let $f: X \dashrightarrow Y$ be a rational map and let V be some variety fitting the diagram of the form (2.2.2).

- The *total image* of a subvariety D in X is the subvariety $f(D) = \delta_Y(\delta_X^{-1}(D))$ in Y .
- The *proper transform* of a subvariety D which does not lie in $\text{Ind}(f)$ is the following subvariety:

$$\check{f}(D) = \overline{f(D \setminus \text{Ind}(f))}.$$

- If f is regular and $Z \in N_i(X)$, then $f_*(Z) \in N_i(Y)$ is the class of $f(Z)$; in particular, if Z is contracted to a subvariety of less dimension then its direct image vanishes.
- If f is regular and $W \in N^i(Y)$, then $f^*(W) \in N^i(X)$ is the class such that

$$f^*(W) \cdot Z = W \cdot f_*(Z),$$

for all classes $Z \in N_i(X)$.

- If f is not regular, $Z \in N_i(X)$ and $W \in N^j(Y)$, then

$$f_*(Z) = \delta_{Y*}(\delta_X^*(Z)) \in N_i(Y); \quad f^*(W) = \delta_{X*}(\delta_Y^*(W)) \in N^j(X).$$

There are many choices for the smooth variety V as in the diagram (2.2.2). However, constructions of the total image, proper transform, inverse and direct images do not depend on this choice.

The following assertion known as Mori negativity lemma, it is very useful in many situations.

Lemma 2.2.5 ([KM98, Lemma 3.39]). *Let $f: X \rightarrow Y$ be a proper birational morphism between smooth varieties X and Y and let $-D$ be an f -nef divisor class on X . The class D is effective if and only if so is $f_*(D)$.*

The inverse image of a nef class under a regular map is nef. In case of rational maps it is not true, but we have the following generalization of this fact.

Lemma 2.2.6. *Let $f: X \dashrightarrow Y$ be a birational map between smooth varieties X and Y . If D is a nef class of a divisor on Y and C is a curve on X such that $f^*D \cdot [C] < 0$, then C lies in $\text{Ind}(f)$.*

Proof. Consider a diagram as in (2.2.2). If D is a nef divisor in Y , then its pullback $\tilde{D} = \delta_Y^*D$ is also nef. Let E be the following difference:

$$E = \delta_X^*(\delta_{X*}(\tilde{D})) - \tilde{D}.$$

By construction the class E is equivalent to a linear combination of effective divisors with support on $\text{Exc}(\delta_X)$. Moreover, since \tilde{D} is nef and $\delta_X^*(\delta_{X*}(\tilde{D}))$ is trivial on classes of curves contracting by δ_X . Thus, E is δ_X -antinef and $\delta_{X*}(E) = 0$. Therefore, by Lemma 2.2.5 the class E is effective.

Let C be a reduced irreducible curve on X which does not lie in $\delta_X(\text{Exc}(\delta_X))$. Denote by \tilde{C} the proper transform of the curve C in V . Then \tilde{C} is a reduced curve such that $\delta_{X*}[\tilde{C}] = [C]$. Then we have

$$\delta_{X*}(\tilde{D}) \cdot [C] = \delta_{X*}(\tilde{D}) \cdot \delta_{X*}[\tilde{C}] = \delta_X^*(\delta_{X*}(\tilde{D})) \cdot [\tilde{C}] = (\tilde{D} + E) \cdot [\tilde{C}] \geq 0.$$

The last inequality is true since \tilde{D} is nef, E is effective and \tilde{C} lies outside the support of E . Thus, we get the result. \square

The next assertion shows the difference between the inverse image of a composition of two birational maps and the composition of inverse images of these maps.

Lemma 2.2.7. *Let X, Y and Z be smooth varieties and $f: Y \dashrightarrow X$ and $g: Z \dashrightarrow Y$ be rational maps such that $g(Z)$ does not lie in $\text{Ind}(f)$. Then the composition $f \circ g: Z \dashrightarrow X$ is well-defined and for each class $D \in N^1(X)$ we have the following equality:*

$$g^*(f^*(D)) - (f \circ g)^*(D) = E, \quad (2.2.8)$$

where E is a class of a divisor on Z supported in $g^{-1}(\text{Ind}(f))$. If D is nef, then E is effective.

Proof. Denote by Γ_f and Γ_g resolutions of singularities of graphs of maps f and g respectively. Denote by p_f, q_f projections to X and Y from Γ_f and by p_g and q_g projection from Γ_g to Y and Z . Both maps q_f and q_g are birational and $q_f(\text{Exc}(q_f^{-1})) = \text{Ind}(f)$. The composition $q_f^{-1} \circ p_g$ induces the rational map $\gamma: \Gamma_g \dashrightarrow \Gamma_f$. Denote by Γ the resolution of the graph of γ and by p and q projections to Γ_f and Γ_g . The morphism q is birational and the following diagram commutes:

$$\begin{array}{ccccc}
 & & \Gamma & & \\
 & & \swarrow q & & \searrow p \\
 & \Gamma_g & & \xrightarrow{\gamma} & \Gamma_f \\
 & \swarrow q_g & & \searrow p_g & \swarrow q_f \\
 Z & & & \xrightarrow{g} & Y & & \xrightarrow{f} & X \\
 & & & \swarrow p_g & \searrow q_f & & \searrow p_f & \\
 & & & & & & &
 \end{array}$$

By definition $(f \circ g)^*D = q_{g*}(q_*(p^*(p_f^*D)))$ and $g^*(f^*D) = q_{g*}(p_g^*(q_f^*(p_f^*D)))$. Denote by \tilde{D} the class p_f^*D in $N^1(\Gamma_f)$ and denote by \tilde{E} the following difference:

$$\tilde{E} = p_g^*(q_f^*\tilde{D}) - q_*(p^*\tilde{D}).$$

Then the class E we are interested in equals $q_{g*}\tilde{E}$. Thus, it suffices to prove that \tilde{E} is the class of a divisor supported on $p_g^{-1}(\text{Ind}(f))$. Denote by E' the class $q_f^*(q_f^*\tilde{D}) - \tilde{D}$, then one has the following equality:

$$\begin{aligned}
 \tilde{E} &= q_*(q^*\tilde{E}) = q_*(q^*(p_g^*(q_f^*\tilde{D}))) - q_*(p^*\tilde{D}) = \\
 &= q_*(p^*(q_f^*(q_f^*\tilde{D}))) - q_*(p^*\tilde{D}) = q_*(p^*(\tilde{D} + E')) - q_*(p^*\tilde{D}) = q_*(p^*E').
 \end{aligned}$$

By construction of E' it is a class of a divisor in Γ_f with the support in $\text{Exc}(q_f)$. This implies that $p_{g*}(\tilde{E}) = p_{g*}(q_*(p^*(E')))$ is a class of a divisor supported in $\text{Ind}(f)$. This implies that E is a class of a divisor in Z supported on $g^{-1}(\text{Ind}(f))$.

Now assume that D is a nef divisor. Then so is \tilde{D} ; thus, the class E' is q_f -antinef by construction and $q_f^*E' = 0$ is an effective divisor. By Lemma 2.2.5 we get that the class E' is effective.

The class p^*E' is q -nef and $p_*(p^*E') = E'$ is the class of an effective divisor by construction. Thus, by Lemma 2.2.5 we get that p^*E' is effective. Therefore, the class $E = q_{g*}(q_*(p^*E'))$ is also effective. \square

The next assertion describes the inverse image of a class of curves under the blow-up.

Lemma 2.2.9. *Let $\delta: Y \rightarrow X$ be a blow-up of a smooth variety X in smooth center Z and $C \not\subset Z$ be an irreducible reduced curve in X , Then there exists a curve T such that $\delta(T)$ is a point such that*

$$\delta^*[C] = [\tilde{C}] + \mu[T] \in N_1(X),$$

where \tilde{C} is the proper transform of C and μ is a non-negative number.

Proof. Let \tilde{C} be a proper transform of the curve C to Y . Denote by ξ the following class in $N_1(Y)$:

$$\xi = \delta^*[C] - [\tilde{C}].$$

Then by the projection formula we get that $\delta_*\xi = 0$. Since δ is a blow-up along a smooth center, the class ξ is proportional to the class of the curve T in the exceptional divisor of δ such that $\delta(T)$ is a point. By the projection formula we have

$$0 = E \cdot \delta^*[C] = E \cdot ([\tilde{C}] + \xi) = E \cdot [\tilde{C}] + E \cdot \xi$$

Since \tilde{C} does not lie inside E the product $E \cdot [\tilde{C}]$ is non-negative. Thus, $E \cdot \xi \leq 0$, the class $\xi = \mu[T]$ is effective and $\mu \geq 0$. \square

Lemma 2.2.9 implies the following assertion describing the inverse image of a curve under any birational map.

Lemma 2.2.10. *Let $f: X \dashrightarrow Y$ be a birational map between smooth varieties X and Y and let C be an irreducible curve on X such that $C \not\subset \text{Ind}(f)$. If \tilde{C} is the proper image of C under f , then there exist effective curves T_1, \dots, T_N in $f(\text{Ind}(f))$ and non-negative numbers $\mu_1, \dots, \mu_N \geq 0$ such that*

$$f_*[C] = [\tilde{C}] + \sum_{i=1}^M \mu_i [T_i] \in N_1(X).$$

In particular, if C does not lie in $\text{Ind}(f)$, then the class $f_[C]$ is effective.*

Proof. By Lemma 2.2.3 one can construct a variety V fitting the diagram of the form (2.2.2) and such that $\delta_X = \delta_N \circ \dots \circ \delta_1$ is a composition of several blow-ups $\delta_i: X_i \rightarrow X_{i-1}$ along smooth centers Z_i with $X = X_0$, $V = X_N$ and $\delta_1(\dots(\delta_{i-1}(Z_i))\dots)$ lies in $\text{Ind}(f)$ for all $1 \leq i \leq N$. Then, applying Lemma 2.2.9 several times we get the result. \square

We need to discuss birational automorphisms and their properties. We start with several definitions.

Definition 2.2.11. Let X and X' be smooth varieties, let $f: X \dashrightarrow X'$ be a birational map and let $\tilde{f}: X \dashrightarrow X$ be a birational automorphism, then

- f is a *pseudo-isomorphism* if codimensions of $\text{Exc}(f)$ in X and of $\text{Exc}(f^{-1})$ in X' are greater than 1.
- f is a *pseudo-automorphism* if it is a pseudo-isomorphism from X to X .

- f is *regularizable*, if there exists a smooth variety Y , a birational map α and a regular automorphism $g: Y \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \alpha \downarrow & & \alpha \downarrow \\ Y & \xrightarrow{g} & Y \end{array}$$

In this situation we call the triple (Y, g, α) is a *regularization* of f .

Note that on surfaces the notion of pseudo-automorphisms coincides with the one of regular automorphisms, while in higher dimensions the notions differ. Nevertheless, in higher dimension there are restrictions on the indeterminacy locus of a pseudo-automorphism.

Lemma 2.2.12 ([BK14]). *If $f: X \dashrightarrow X$ is a pseudo-automorphism of a smooth variety X of dimension 3 or greater, then $\text{Ind}(f)$ has no isolated points.*

Observe that pseudo-automorphisms induce invertible maps on the group of classes of divisors on the variety:

Lemma 2.2.13. *If $f: X \dashrightarrow X$ is a pseudo-automorphism of a smooth variety X , then*

$$f^*: N^1(X) \rightarrow N^1(X)$$

is an isomorphism of vector spaces and $(f^n)^ = (f^*)^n$.*

Proof. This follows from Lemma 2.2.7. □

2.2.2. A special construction of a flop

In this section we consider the construction of a concrete pseudo-isomorphism and construct the resolution of its graph.

Let X be a smooth threefold and let Γ_1 and Γ_2 be smooth curves on X such that $\Gamma_1 \cap \Gamma_2$ is a finite set of points and the union $\Gamma_1 \cup \Gamma_2$ is a nodal curve. In this section we consider the blow-up of curves Γ_1 and Γ_2 in one order and in another and compare the resulting varieties.

Denote by $\delta_1: Y_1 \rightarrow X$ the blow-up of X along the curve Γ_1 . Let $\tilde{\Gamma}_2 \subset Y_1$ be the proper transform of the curve Γ_2 under δ_1 . Let $\delta_{12}: Y_{12} \rightarrow Y_1$ be the blow-up of Y_1 along $\tilde{\Gamma}_2$.

Denote by $\delta_{2+}: Y_{2+} \rightarrow X$ the blow-up of X along the curve Γ_2 . Let $\tilde{\Gamma}_1 \subset Y_{2+}$ be the proper transform of the curve Γ_1 under δ_{2+} . Let $\delta_{12+}: Y_{21+} \rightarrow Y_{2+}$ be the blow-up of Y_{2+} in $\tilde{\Gamma}_1$.

Denote by τ the composition of maps $\tau = \delta_{21+}^{-1} \circ \delta_{2+}^{-1} \circ \delta_1 \circ \delta_{12}: Y_{12} \rightarrow Y_{21+}$. Then the following diagram commutes:

$$\begin{array}{ccc} Y_{12} & \dashrightarrow^{\tau} & Y_{21+} \\ \delta_{12} \downarrow & & \downarrow \delta_{21+} \\ Y_1 & & Y_{2+} \\ \delta_1 \swarrow & & \swarrow \delta_{2+} \\ & X & \end{array}$$

Denote by U the open subset $(\delta_1 \circ \delta_{12})^{-1}(X \setminus (\Gamma_1 \cap \Gamma_2))$ of Y_{12} . From the construction of this diagram we can deduce the following lemma:

Lemma 2.2.14. *The indeterminacy locus of τ lies in $Y_{12} \setminus (\delta_1 \circ \delta_{12})^{-1}(X \setminus (\Gamma_1 \cap \Gamma_2))$. The indeterminacy locus of τ^{-1} lies in $Y_{21+} \setminus (\delta_{2+} \circ \delta_{21+})^{-1}(X \setminus (\Gamma_1 \cap \Gamma_2))$. In particular, τ is a pseudo-isomorphism.*

Assume that $\Gamma_1 \cap \Gamma_2 = \{p_1, \dots, p_m\}$. Denote by C_i the irreducible component of $(\delta_1 \circ \delta_{12})^{-1}(p_i)$ which does not lie in the exceptional divisor of δ_{12} for all $1 \leq i \leq m$. Denote by C_i^+ the irreducible component of $(\delta_{2+} \circ \delta_{21+})^{-1}(p_i)$ which does not lie in the exceptional divisor of δ_{21+} . Then C_i and C_i^+ are smooth rational curves.

Moreover, if $p_i, p_j \in \Gamma_1 \cap \Gamma_2$ are two distinct points then curves C_i and C_j do not intersect in Y_{12} and curves C_{i+} and C_{j+} do not intersect in Y_{21+} .

In fact, locally in a neighborhood of a curve C_i the map τ is a flop, see [HM10, Example 1.12]. Denote by f_1 and $f_2 \in N_1(Y_{12})$ the classes of the fibers of the morphisms δ_1 and δ_{12} over Γ_1 and Γ_2 respectively. Then the following assertion is true:

Proposition 2.2.15 ([HM10, Example 1.12]). *Let $\Gamma_1 \cap \Gamma_2 = \{p_1, \dots, p_m\}$. Then the sets of indeterminacy of τ and τ^{-1} consist of disjoint unions of m curves:*

$$\text{Ind}(\tau) = C_1 \sqcup C_2 \sqcup \dots \sqcup C_m; \quad \text{Ind}(\tau^{-1}) = C_1^+ \sqcup C_2^+ \sqcup \dots \sqcup C_m^+.$$

The blow-up $\Delta: \text{Bl}_{C_1 \sqcup \dots \sqcup C_m} Y_{12} \rightarrow Y_{12}$ is an elimination of the map τ , i.e. the map $\tau \circ \Delta$ is regular. Moreover, the class of each curve C_i and of its direct image under τ satisfy:

$$[C_i] = f_1 - f_2 \quad \tau_*[C_i] = -[C_i^+].$$

Sketch of proof. Denote by V be the blow-up $\text{Bl}_{C_1 \sqcup \dots \sqcup C_m} Y_{12} \rightarrow Y_{12}$ and by $\Delta: V \rightarrow Y_{12}$ birational morphism from V to Y_{12} . Since $C_1 \sqcup \dots \sqcup C_m$ is a disjoint union of m smooth curves the variety V is smooth.

Consider the map $\delta_1 \circ \delta_{12} \circ \Delta: V \rightarrow X$. Since the preimage of the curve Γ_2 is of pure dimension 2 and since V is smooth then by the universal property of blow-ups (see [Har77, Proposition II.7.14]) we get that the following map is regular:

$$(\delta_{2+}^{-1} \circ \delta_1 \circ \delta_{12} \circ \Delta): V \rightarrow Y_{2+}.$$

Repeating this argument we can show that there exists a regular map $f: V \rightarrow V_+$, where V_+ is a blow-up of the variety Y_{21+} in the smooth curve $C_{1+} \sqcup \dots \sqcup C_{m+}$ such that the following diagram commutes:

$$\begin{array}{ccc} & V & \xrightarrow{f} V_+ \\ \Delta \swarrow & & \searrow \Delta_+ \\ Y_{12} & \xrightarrow{\tau} & Y_{21+} \end{array}$$

Since Picard numbers of V and V_+ are same, this implies that the map f is an isomorphism.

Considering the Neron-Severi group of Y_{12} we prove that $[C_i] = f_1 - f_2$. Since τ is a pseudo-isomorphism the groups $N^1(Y_{12})$ and $N^1(Y_{21+})$ are isomorphic. Since Y_{12} is a threefold then by projection formula we get that τ induces an isomorphism between groups $N_1(Y_{12})$ and $N_1(Y_{21+})$. Under this identification we get $[C_i^+] = f_2 - f_1$, thus, $\tau_*[C_i] = -[C_i^+]$. \square

2.2.3. Dynamical degrees

One can associate the following invariant to a birational automorphism of a smooth projective variety.

Definition 2.2.16. Let $f: X \rightarrow X$ be a birational automorphism of a smooth projective variety X of dimension n and let H be an ample class on X . The i -th dynamical degree of f is the following number:

$$\lambda_i(f) = \lim_{n \rightarrow \infty} \left((f^n)^*(H^i) \cdot H^{\dim(X)-i} \right)^{\frac{1}{n}}.$$

Dynamical degrees of birational automorphisms are correctly defined and do not depend on the choice of the ample class H by [Tru20].

Theorem 2.2.17 ([Tru20]). *Let $f: X \dashrightarrow X$ be a birational automorphism of a smooth variety X of dimension n . Then dynamical degrees of f are well-defined and*

- (1) $\lambda_i(f)$ is a real number and $\lambda_i(f) \geq 1$ for all $0 \leq i \leq n$. Moreover, $\lambda_0(f) = \lambda_n(f) = 1$;
- (2) $\lambda_i(f)$ is a birational invariant for all $0 \leq i \leq n$.
- (3) The dynamical degrees are log-concave, that is

$$\lambda_i(f)^2 \geq \lambda_{i+1}(f)\lambda_{i-1}(f),$$

for all $1 \leq i \leq n-1$.

The projection formula implies the relation between dynamical degrees of an automorphism and of its inverse:

$$\lambda_i(f) = \lambda_{n-i}(f^{-1}).$$

Note that the log-concavity implies the inequality $\lambda_1(f)^2 \geq \lambda_2(f)$. If this inequality is strict and f is a pseudo-automorphism, then the action of f^* on the group of classes of divisors has the following property:

Theorem 2.2.18 ([Tru14, Theorem 1, Corollary 3]). *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth projective variety X satisfying $\lambda_1^2(f) > \lambda_2(f)$. Then there exists a non-zero pseudo-effective class $\theta_1(f)$ in $N^1(X)$ such that:*

- (1) For any ample class H the limit $\lim_{n \rightarrow \infty} \frac{(f^n)^*(H)}{\lambda_1(f)^n}$ exists, is non-zero and proportional to $\theta_1(f)$;
- (2) $f^*(\theta_1(f)) = \lambda_1(f)\theta_1(f)$;
- (3) The eigenvalue $\lambda_1(f)$ is simple, i.e. there is a decomposition $N^1(X) = \langle \theta_1(f) \rangle \oplus V$ which is f^* -invariant and absolute values of all eigenvalues of $f|_V$ are less than $\lambda_1(f)$.

In case when the birational automorphism f is regular the class $\theta_1(f)$ is nef. However, even in the case of a pseudo-automorphisms of threefolds this class can intersect some curves negatively.

Lemma 2.2.19. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth projective threefold X such that $\lambda_1(f)^2 > \lambda_2(f)$. If C is an irreducible curve and $\theta_1(f) \cdot [C] < 0$, then there exists an integer N such that*

$$C \subset \bigcap_{n>N} \text{Ind}(f^n).$$

Proof. Let C be a curve on X such that $\theta_1(f) \cdot C < 0$ and let H be an ample divisor on X . Then by Theorem 2.2.18 there exists N such that for all $n > N$ we have the following inequality

$$f^{n*}(H) \cdot C < 0.$$

By Lemma 2.2.6 this is possible only if C lies in $\text{Ind}(f^n)$ for all $n > N$. □

The set of curves C such that $\theta_1(f) \cdot [C] < 0$ is finite hence $\theta_1(f)$ is a movable class in the sense of [BDPP13].

2.2.4. Example

We recall here the construction from [OT15] and [Les18, Section 7] in order to give an example of a regularizable (not regular) pseudo-automorphism f_+ such that the class $\theta_1(f_+)$ exists and is not nef.

Consider the lattice of Eisenstein integers $\Lambda = \mathbb{Z}[j]$ in \mathbb{C} , here j is a primitive cube root of unity. Denote by E the elliptic curve \mathbb{C}/Λ . The group $G = \langle -j \rangle$ acts on E . Then the group G is isomorphic to C_6 . Consider the diagonal action of G on abelian variety $A = E \times E \times E$:

$$G \times A \rightarrow A \quad (g, (x_1, x_2, x_3)) \mapsto (g(x_1), g(x_2), g(x_3)),$$

where $g \in G$ and (x_1, x_2, x_3) is a point in $A = E \times E \times E$. Denote by $q: A \rightarrow A/G$ the quotient map by the action of G on A .

Theorem 2.2.20 ([OT15]). *The variety A/G is rational.*

The variety A/G contains 27 isolated quotient singular points. Let $\delta: X \rightarrow A/G$ be the blow-up of all these points. It is a resolution of singularities of A/G . Denote by \tilde{q} the rational map from A to X :

$$\tilde{q} = \delta \circ q: A \dashrightarrow X.$$

There is a natural action of the group $\text{SL}(3, \mathbb{Z})$ on the abelian threefold $A = E \times E \times E$. Moreover, since the actions of groups $\text{SL}(3, \mathbb{Z})$ and G on A commute, there is an induced action of the group $\text{SL}(3, \mathbb{Z})$ on A/G and on X . Let M be the following matrix in $\text{SL}(3, \mathbb{Z})$:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 0 \end{pmatrix}$$

Then M induces a regular automorphism f_A of A and it extends to a regular automorphism f of X such that:

$$\lambda_1(f)^2 > \lambda_2(f).$$

Thus, by Theorem 2.2.18 there is a unique up to proportionality pseudo-effective eigenclass $\theta_1(f)$ in $N^1(X)$ with eigenvalue $\lambda_1(f)$.

Consider a curve $C_E = E \times \{0\} \times \{0\}$ in the abelian threefold A . Let $C \subset X$ be a proper image of C_E under the rational map \tilde{q} . Then C is a smooth rational curve on X and by construction one can check that $\theta_1(f) \cdot [C] > 0$. By [Les18, Section 7] there exists a pseudo-isomorphism

$$\alpha: X \dashrightarrow X_+,$$

such that $\text{Ind}(\alpha) = C$ and the total image of C under α is a smooth rational curve C_+ . In fact, this map is a standard Atiyah flop in the curve C .

Lemma 2.2.21. *Let $\alpha: X \dashrightarrow X^+$ be a flop in a curve C , let C_+ be the total image of C in X_+ and D be a class in $N^1(X)$. If $D \cdot [C] > 0$ then $(\alpha^{-1})^*(D) \cdot [C_+] < 0$.*

As a consequence of Lemma 2.2.21 we get that $(\alpha^{-1})^*(\theta_1(f)) \cdot [C_+] < 0$. Denote by f_+ the following composition:

$$f_+ = \alpha \circ f \circ \alpha^{-1}: X_+ \dashrightarrow X_+.$$

Since f is a composition of flops and a regular automorphism it is a pseudo-automorphism. Thus, by Theorem 2.2.18 one can define the pseudo-effective class $\theta_1(f_+)$. Moreover, since α is a pseudo-isomorphism, then Lemma 2.2.7 implies that $\theta_1(f_+)$ is proportional to $(\alpha^{-1})^*(\theta_1(f))$. Therefore, we conclude that

$$\theta_1(f_+) \cdot [C_+] < 0.$$

Note that by construction the curve C_+ lies in the indeterminacy locus $\text{Ind}(f^n)$ for all non-zero integers n . Thus, the pseudo-automorphism f_+ satisfies all but last condition in Theorem 2.1.3.

2.3. Proof of Theorem 2.1.3

2.3.1. Regularizations

Let X be a smooth threefold, let $f: X \dashrightarrow X$ be a pseudo-automorphism of X and let (Y, g, α) be a regularization of f . Namely, Y is a smooth threefold, g is an automorphism of Y and α is a birational map from X to Y such that

$$\alpha \circ f = g \circ \alpha.$$

Since α is a birational map then by Theorem 2.2.18 one has

$$\lambda_1(f) = \lambda_1(g).$$

Moreover, if one has an inequality $\lambda_1(f)^2 > \lambda_2(f)$ then by Theorem 2.2.18 one can define pseudo-effective classes $\theta_1(f) \in N^1(X)$ and $\theta_1(g) \in N^1(Y)$ which are eigenvectors with eigenvalue $\lambda_1(f)$ under the action of f^* and g^* respectively.

Lemma 2.3.1. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth threefold X and let (Y, g, α) be a regularization of f . If $\lambda_1(f)^2 > \lambda_2(f)$, then the class $\alpha^*(\theta_1(g))$ is non-zero.*

Proof. Let V, δ_X, δ_Y be a smooth threefold and algebraic maps $\delta_X: V \rightarrow X$ and $\delta_Y: V \rightarrow Y$ fitting the diagram (2.2.2). We can assume moreover, that V, δ_X, δ_Y are as in Lemma 2.2.3. Suppose by contradiction that we have

$$\alpha^*\theta_1(g) = \delta_{X*}(\delta_Y^*(\theta_1(g))) = 0.$$

Denote by $E \in N^1(V)$ the class $\delta_Y^*(\theta_1(g))$. Since the map δ_X is a composition of blow-ups along smooth centers this implies that the class E can be represented as a linear combination of divisors supported on $\text{Exc}(\delta_X)$.

The class $\theta_1(g)$ is nef, then so is E . Thus, by Lemma 2.2.5 we conclude that E is an anti-effective class. On the other hand, since $\theta_1(g)$ is a pseudo-effective class, then so is E . Thus, we get a contradiction. \square

Lemma 2.3.2. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth threefold X and let (Y, g, α) be a regularization of f . If $\lambda_1(f)^2 > \lambda_2(f)$, then $\alpha^*\theta_1(g)$ is proportional to $\theta_1(f)$.*

Proof. Since f is a pseudo-automorphism it does not contract divisors. Thus, by Lemma 2.2.7 we get the following:

$$f^*(\alpha^*\theta_1(g)) = (\alpha \circ f)^*\theta_1(g).$$

Since g is regular its indeterminacy locus is empty; thus, by Lemma 2.2.7 we get the following:

$$(g \circ \alpha)^*\theta_1(g) = \alpha^*(g^*\theta_1(g)).$$

Therefore, we get that $f^*(\alpha^*\theta_1(g)) = \lambda_1(g)\alpha^*\theta_1(g)$. Since $\lambda_1(f) = \lambda_1(g)$ by Theorem 2.2.18 we get the result. \square

Lemma 2.3.3. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth threefold X and let (Y, g, α) be a regularization of f . If $\lambda_1(f)^2 > \lambda_2(f)$ and C is an irreducible curve such that $\theta_1(f) \cdot [C] < 0$, then C lies in $\text{Ind}(\alpha)$.*

Proof. Let H be an ample divisor on X . By Theorem 2.2.18 the class $\theta_1(f)$ is proportional to the limit the sequence of classes $(g^n)^*(H)/\lambda_1(g)^n$ in $N^1(Y)$. Since g is regular one has that $\theta_1(g)$ is a limit of a sequence of ample classes. Thus, $\theta_1(g)$ is nef.

By Lemmas 2.3.1 and 2.3.2 the class $\alpha^*(\theta_1(g))$ is non-zero and proportional to $\theta_1(f)$. Since both this classes are pseudo-effective the coefficient of the proportionality is strictly positive; thus, we get

$$\alpha^*(\theta_1(g)) \cdot [C] < 0.$$

Thus, by Lemma 2.2.6 we get that the curve C lies in $\text{Ind}(\alpha)$. \square

Lemma 2.3.4. *Let X be a smooth threefold and let $f: X \dashrightarrow X$ be a pseudo-automorphism such that $\lambda_1(f)^2 > \lambda_2(f)$. If C is an irreducible curve such that $\theta_1(f) \cdot [C] < 0$, then there exists N such that C lies in $\text{Ind}(f^n)$ for all $n \geq N$.*

Proof. By Theorem 2.2.18 if H is an ample class on X then the class $\theta_1(f)$ is proportional to the following:

$$\lim_{n \rightarrow \infty} \frac{(f^n)^*(H)}{\lambda_1(f)^n}.$$

Since the class $\theta_1(f)$ is pseudo-effective the coefficient of the proportionality is strictly positive. Since $\theta_1(f) \cdot [C] < 0$ there exists N such that for each $n \geq N$ one has $(f^n)^*H \cdot [C] < 0$. Thus, Lemma 2.2.6 implies the result. \square

Lemma 2.3.5. *Let X be a smooth threefold and $f: X \dashrightarrow X$ is a pseudo-automorphism and C is an irreducible curve on X . Assume that the following conditions are satisfied:*

- (1) *there exists $N > 0$ such that $C \subset \text{Ind}(f^n)$ for all $n \geq N$;*
- (2) *C does not lie in $\text{Ind}(f^{-i}) \cup \text{Ind}(f^{-j})$ for two distinct integers $i \neq j$.*

Then the proper transforms of C under f^{-i} and under f^{-j} do not coincide.

Proof. Denote by C_i and C_j the proper transforms of C under f^{-i} and f^{-j} respectively. They are irreducible curves on X . Since f is a pseudo-automorphism the curve C_j does not lie in $\text{Ind}(f^j)$.

Since $f^{j-i} = f^{-i} \circ f^j$ the indeterminacy locus of f^{j-i} contains in the union of $\text{Ind}(f^j)$ and subvarieties in $X \setminus \text{Ind}(f^j)$ which proper images under f^j lies in $\text{Ind}(f^{-i})$.

Thus, the curve C_j does not lie in the indeterminacy locus of f^{j-i} . Therefore, if we assume that curves C_i and C_j coincide then C_j does not lie $\text{Ind}(f^{m(j-i)})$ for any $m > 0$. This implies that the curve C does not lie in $\text{Ind}(f^{m(j-i)+j})$ for all $m > 0$ and we get a contradiction with the first condition of the lemma. Thus, we conclude the result. \square

Now we are ready to prove the criterion for non-regularizable automorphisms.

Proposition 2.3.6. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth threefold X . If f satisfies conditions (1 – 3) in Theorem 2.1.3, then f is non-regularizable.*

Proof. Assume by contradiction that there exists a regularization (Y, g, α) of the pseudo-automorphism $f: X \dashrightarrow X$. By Theorem 2.2.18 we see that $\lambda_1(f) = \lambda_1(g) = \lambda$. By the condition (1) in Theorem 2.1.3 and Theorem 2.2.18 we can consider pseudo-effective classes $\theta_1(f)$ and $\theta_1(g)$ which are unique up to proportionality.

Consider a curve C as in the condition (2) in Theorem 2.1.3. By Lemma 2.3.3 the curve C lies in the indeterminacy locus $\text{Ind}(\alpha)$. Moreover, by Lemma 2.3.4 there exists N such that $C \subset \text{Ind}(f^n)$ for any $n \geq N$.

Denote by I the set $I \subset \mathbb{Z}$ such that for each $i \in I$ the curve C does not lie in $\text{Ind}(f^{-i})$. Denote by C_i the proper image of C under f^{-i} for any $i \in I$.

Since C_i does not lie in the indeterminacy locus of f^i for any $i \in I$ and since the proper image of C_i is the curve C which lies in $\text{Ind}(\alpha)$ we get that C_i lies in $\text{Ind}(\alpha \circ f^i)$. Moreover, since g is regular and $g^i \circ \alpha = \alpha \circ f^i$ the following indeterminacy loci coincide:

$$\text{Ind}(\alpha) = \text{Ind}(g^i \circ \alpha) = \text{Ind}(\alpha \circ f^i).$$

Thus, $\text{Ind}(\alpha)$ contains curves C_i for all $i \in I$. The condition (3) in Theorem 2.1.3 and Lemma 2.3.5 implies that $\text{Ind}(\alpha)$ contains infinitely many distinct curves. This leads us to a contraction; thus, we get the result. \square

2.3.2. Fibrations over surfaces

Let X be a smooth threefold and $f: X \dashrightarrow X$ be a birational automorphism of X . We say that f preserves a fibration over a surface (S, g, π) if there exist a smooth surface S , its birational automorphism g and a dominant rational map $\pi: X \dashrightarrow S$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi \downarrow & & \downarrow \pi \\ S & \xrightarrow{g} & S \end{array}$$

Lemma 2.3.7. *Let $f: X \dashrightarrow X$ be a pseudo-automorphism of a smooth threefold X . If f preserves a fibration over a surface (S, g, π) then it also preserves a fibration over a surface $(\tilde{S}, \tilde{g}, \tilde{\pi})$ such that \tilde{g} is algebraically stable and $(\tilde{g} \circ \tilde{\pi})^* = \tilde{\pi}^* \circ \tilde{g}^*$.*

Proof. By [DF01, Theorem 0.1] there exists a smooth surface \tilde{S} and a birational morphism $\delta: \tilde{S} \rightarrow S$ such that the birational automorphism

$$\tilde{g} = \delta^{-1} \circ g \circ \delta: \tilde{S} \dashrightarrow \tilde{S}$$

is algebraically stable. Denote by $\tilde{\pi}$ the composition $\delta^{-1} \circ \pi$. Then f preserves the fibration over a surface $(\tilde{S}, \tilde{g}, \tilde{\pi})$. It remains to show that $(\tilde{g} \circ \tilde{\pi})^* = \tilde{\pi}^* \circ \tilde{g}^*$. In order to prove it we show that there is no divisor in X which proper image under $\tilde{\pi}$ is a point of indeterminacy of \tilde{g} .

Assume by contradiction that E is a divisor in X such that its proper transform under $\tilde{\pi}$ is a point $p \in \tilde{S}$ which lie in the indeterminacy locus of \tilde{g} . Since \tilde{g} is algebraically stable the point p does not lie in $\text{Ind}(\tilde{g}^{-n})$ for all $n > 0$ by [DF01, Theorem 1.14]. Denote by p_n the proper image of the point p under \tilde{g}^{-n} for $n > 0$. Denote by E_n the proper image of E under f^n . Since f is a pseudo-automorphism E_n is a divisor for all $n \in \mathbb{Z}$. Then the proper image of E_n under $\tilde{\pi}$ is a point $p_n \in \tilde{S}$ for all $n < 0$.

Let $(V, \delta_X, \delta_{\tilde{S}})$ be an elimination of indeterminacy of $\tilde{\pi}$ as in Lemma 2.2.3. Since V is irreducible the set of points in \tilde{S} such that the dimension of fiber of $\delta_{\tilde{S}}$ over this point equals 2 is finite. Thus, the set of divisors E_n such that their proper image under $\tilde{\pi}$ are points finite. Therefore, the divisor E is f -periodic and the point p is \tilde{g} -periodic. This contradicts our assumption that \tilde{g} is algebraically stable and p lies in $\text{Ind}(\tilde{g})$.

Thus, no divisors in X are contracted to indeterminacy locus of \tilde{g} and by Lemma 2.2.7 we get the result. \square

Lemma 2.3.8. *Let $f: X \dashrightarrow X$ be a birational automorphism of a smooth variety X and let f preserve a fibration over a surface (S, g, π) . If g is algebraically stable and $\lambda_1(g) > 1$, then $\pi^*(\theta_1(g))$ is a non-zero class in $N^1(X)$.*

Proof. Since g is algebraically stable then by [DF01, Theorem 0.3] one can construct a non-zero nef pseudo-effective eigenclass $\theta_1(g)$ with eigenvalue $\lambda_1(f)$. Let $V, \delta_X: V \rightarrow X$ and $\delta_S: V \rightarrow S$ be as in Lemma 2.2.3. Then we get the following:

$$\pi^*(\theta_1(g)) = \delta_{X*}(\delta_S^*(\theta_1(g))).$$

The class $\delta_S^*(\theta_1(g))$ is non-zero, nef and pseudo-effective. Thus, by Lemma 2.2.5 we get that the class $\pi^*(\theta_1(g))$ is non-zero. \square

This lemma allows us to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. By Proposition 2.3.6 we get that f is non-regularizable.

Then assume by contradiction that the pseudo-automorphism $f: X \dashrightarrow X$ preserves a fibration over a surface (S, g, π) . By Lemma 2.3.7 one can assume that g is algebraically stable and that $(g \circ \pi)^* = \pi^* \circ g^*$. Moreover, by [Tru20, Theorem 1.1] since the relative dimension of π equals 1 we have the following equality:

$$1 < \lambda_1(f) = \lambda_1(g).$$

By the condition (1) in Theorem 2.1.3, Theorem 2.2.18 and [DF01, Theorem 0.3] there are pseudo-effective eigenclasses $\theta_1(f) \in N^1(X)$ and $\theta_1(g) \in N^1(S)$ with eigenvalue $\lambda_1(f)$. Since f does not contract divisors by Lemma 2.2.7 we get the following:

$$f^*(\pi^*(\theta_1(g))) = (\pi \circ f)^*(\theta_1(g)).$$

By construction of (S, g, π) we has also

$$(g \circ \pi)^*(\theta_1(g)) = \lambda_1(g)\pi^*(\theta_1(g)).$$

Therefore, the class $\pi^*\theta_1(g)$ is proportional to $\theta_1(f)$. Moreover, by Lemma 2.3.8 we get that the class $\theta_1(f)$ is proportional to $\pi^*(\theta_1(g))$ with a strictly positive coefficient.

By the condition (2) in Theorem 2.1.3 we get that $\pi^*(\theta_1(g)) \cdot [C] < 0$. Therefore, by Lemma 2.2.6 the curve C lies in $\text{Ind}(\pi)$.

By the condition (3) in Theorem 2.1.3 and by Lemma 2.3.5 we get that there is an infinite set of distinct curves which lie in $\text{Ind}(\pi)$. Thus, we get a contradiction. Therefore, f do not preserve any structure of a fibration over a surface. \square

2.4. Blanc's pseudo-automorphism

In this section we study the family of positive entropy birational automorphisms introduced by Blanc in [Bla13]. We recall the construction of these automorphisms and then apply Theorem 2.1.3 to show that they are not regularizable and do not preserve any fibration over a surface.

2.4.1. Construction of a pseudo-automorphism

Let S be a smooth cubic surface in \mathbb{P}^3 and let p be a point in S . One can associate to p a birational involution of \mathbb{P}^3 :

$$\sigma_p: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3,$$

such that for any line $L \subset \mathbb{P}^3$ passing through p and intersecting the surface S in three distinct points p, p_1, p_2 the restriction $\sigma_p|_L: L \rightarrow L$ is a unique involution of $L \cong \mathbb{P}^1$ such that $\sigma_p|_L(p_i) = p_i$ for $i = 1, 2$.

Proposition 2.4.1 ([Bla13, Proposition 2.1]). *The birational involution $\sigma_p: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ preserves pointwise an open subset of S . The indeterminacy locus of σ_p consists of an irreducible curve $\Gamma \subset S$ of degree 6 containing the point p . Moreover, if $\delta: \text{Bl}_{p,\Gamma}(\mathbb{P}^3) \rightarrow \mathbb{P}^3$ is the blow-up of the point p and the proper preimage of the curve Γ , then σ_p induces a regular automorphism of $\text{Bl}_{p,\Gamma}(\mathbb{P}^3)$:*

$$\delta^{-1} \circ \sigma_p \circ \delta: \text{Bl}_{p,\Gamma}(\mathbb{P}^3) \rightarrow \text{Bl}_{p,\Gamma}(\mathbb{P}^3).$$

Let p_1, \dots, p_k be k points on the cubic surface S ; let σ_{p_i} be an involution associated to the point p_i for all $1 \leq i \leq k$. Denote by Γ_i the indeterminacy locus of the involution σ_{p_i} , then Γ_i is a curve in \mathbb{P}^3 of degree 6 lying in the surface S . Let $\delta: X \rightarrow \mathbb{P}^3$ be the successive blow-ups of the points p_1, \dots, p_k followed by the blow-up of the proper transforms of the curves $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ in this order. Denote by $\hat{\sigma}_{p_i}$ the birational involution of X induced by σ_{p_i} :

$$\hat{\sigma}_{p_i} = \delta^{-1} \circ \sigma_{p_i} \circ \delta: X \dashrightarrow X.$$

The composition of several involutions of this type gives a positive entropy birational automorphism.

Theorem 2.4.2 ([Bla13, Theorem 1.2]). *Let p_1, \dots, p_k be k distinct points on a smooth cubic surface S in \mathbb{P}^3 , and let Γ_i be the curve of indeterminacy of σ_{p_i} for all $1 \leq i \leq k$. Then the following composition is a pseudo-automorphism:*

$$f = \hat{\sigma}_{p_1} \circ \dots \circ \hat{\sigma}_{p_k}: X \dashrightarrow X \quad (2.4.3)$$

Moreover, if $k \geq 3$, then we have $\lambda_1(f) = \lambda_2(f) > 1$.

Let $\delta_i: X_i \rightarrow \mathbb{P}^3$ be the successive blow-ups of the points p_1, \dots, p_k , followed by the blow-up of the proper transforms of the curves $\Gamma_i, \Gamma_1, \dots, \Gamma_{i-1}, \Gamma_{i+1}, \dots, \Gamma_k$ in this order. By construction we get $X_1 = X$ and $\delta_1 = \delta$. Moreover, an involution σ_{p_i} induces a regular involution on X_i for all $1 \leq i \leq k$:

$$\tilde{\sigma}_{p_i}: X_i \rightarrow X_i.$$

Let $\tau_{i,j}: X_i \rightarrow X_j$ be the birational map induced by the identity map on \mathbb{P}^3 . Note that by Proposition 2.2.15 the map $\tau_{i,j}$ is a pseudo-isomorphism.

Lemma 2.4.4. *The pseudo-automorphism f is the composition of the regular involutions $\tilde{\sigma}_{p_i}: X_i \rightarrow X_i$ and the pseudo-isomorphisms $\tau_{i-1,i}: X_{i-1} \dashrightarrow X_i$, i.e. the following diagram commutes:*

$$\begin{array}{ccccccccccccccc} X_1 & \xrightarrow{\tilde{\sigma}_{p_1}} & X_1 & \xrightarrow{\tau_{1,2}} & X_2 & \xrightarrow{\tilde{\sigma}_{p_2}} & \dots & \xrightarrow{\tilde{\sigma}_{p_{k-1}}} & X_{k-1} & \xrightarrow{\tau_{k-1,k}} & X_k & \xrightarrow{\tilde{\sigma}_{p_k}} & X_k & \xrightarrow{\tau_{k,1}} & X_1 \\ \delta_1 \downarrow & & \delta_1 \downarrow & & \delta_2 \downarrow & & & & \delta_{k-1} \downarrow & & \delta_k \downarrow & & \delta_k \downarrow & & \delta_1 \downarrow \\ \mathbb{P}^3 & \xrightarrow{\sigma_{p_1}} & \mathbb{P}^3 & \xrightarrow{\text{id}_{\mathbb{P}^3}} & \mathbb{P}^3 & \xrightarrow{\sigma_{p_2}} & \dots & \xrightarrow{\sigma_{p_{k-1}}} & \mathbb{P}^3 & \xrightarrow{\text{id}_{\mathbb{P}^3}} & \mathbb{P}^3 & \xrightarrow{\sigma_{p_k}} & \mathbb{P}^3 & \xrightarrow{\text{id}_{\mathbb{P}^3}} & \mathbb{P}^3 \end{array} \quad (2.4.5)$$

Let $H \in N^1(X)$ be the inverse image of the hyperplane section from \mathbb{P}^3 to X , let E_j be the exceptional divisor over the points p_j and let F_j be the exceptional divisor over Γ_j for all $1 \leq j \leq k$. Let $e_j, f_j \in N^2(X)$ be the classes of irreducible curves in E_j and F_j respectively which are contracted under δ . Let $h \in N^2(X)$ be the class of the proper transform of a line in \mathbb{P}^3 to X .

Lemma 2.4.6. *Vector spaces $N^1(X)$ and $N^2(X)$ have the following bases:*

$$\begin{aligned} N^1(X) &= \langle H, E_1, \dots, E_k, F_1, \dots, F_k \rangle; \\ N^2(X) &= \langle h, e_1, \dots, e_k, f_1, \dots, f_k \rangle. \end{aligned}$$

Moreover, the intersection form on X is as follows:

$$\begin{array}{lll}
H \cdot h = 1; & E_i \cdot h = 0; & F_i \cdot h = 0 \text{ for all } i; \\
H \cdot e_i = 0; & E_i \cdot e_i = -1; & F_i \cdot e_i = 0 \text{ for all } i; \\
& E_i \cdot e_j = 0; & F_i \cdot e_j = 0 \text{ for all } i \neq j; \\
H \cdot f_i = 0; & E_i \cdot f_i = 0; & F_i \cdot f_i = -1 \text{ for all } i; \\
& E_i \cdot f_j = 0; & F_i \cdot f_j = 0 \text{ for all } i \neq j.
\end{array}$$

Lemma 2.4.7 ([Bla13, Proposition 2.3]). *Under the assumptions and notation of Lemma 2.4.6 the involution $\hat{\sigma}_{p_i}$ acts on $N^1(X)$ as follows for any $1 \leq i \leq k$:*

$$\begin{aligned}
\hat{\sigma}_{p_i}^*(D) &= D + \nu_i, \text{ if } D = H \text{ or } E_i; \\
\hat{\sigma}_{p_i}^*(F_i) &= F_i + 2\nu_i; \\
\hat{\sigma}_{p_i}^*(D) &= D, \text{ if } D = E_j \text{ or } F_j \text{ for } j \neq i; \\
\hat{\sigma}_{p_i}^*(\nu_i) &= -\nu_i; \\
\hat{\sigma}_{p_i}^*(\nu_j) &= \nu_j + 2\nu_i.
\end{aligned}$$

Here $\nu_i = 2H - 2E_i - F_i \in N^1(X)$ for all $1 \leq i \leq k$.

Note that by Theorem 2.4.2 one has $\lambda_1(f)^2 > \lambda_2(f)$. Thus, by Theorem 2.2.18 one can find a unique up to multiplication by a positive real number pseudo-effective class $\theta_1(f) \in N^1(X)$ such that $f^*\theta_1(f) = \lambda_1(f)\theta_1(f)$.

Lemma 2.4.8 ([Bla13, Proof of Proposition 2.3]). *We have:*

$$\theta_1(f) = \sum_{i=1}^k \alpha_i \nu_i \in N^1(X), \tag{2.4.9}$$

where $\alpha_1 > \alpha_2 > \dots > \alpha_k$ are strictly positive numbers and $\alpha_1 > \alpha_2 + 2 \left(\sum_{i=3}^k \alpha_i \right)$.

One can show that $\theta_1(f)$ is not a nef class in $N^1(X)$.

Corollary 2.4.10. *Let q be any point lying in the intersection of Γ_1 and Γ_2 . Assume that the intersection of Γ_1 and Γ_2 is transversal in q and let \tilde{L} be the proper transform of a line in \mathbb{P}^3 passing through p_1 and q to X . Then one has the following inequality:*

$$\theta_1(f) \cdot [\tilde{L}] < 0.$$

Note that if points p_1 and p_2 are sufficiently general then curves Γ_1 and Γ_2 intersect transversally; thus, the condition of Corollary 2.4.10 is satisfied.

Proof of Corollary 2.4.10. By construction of the birational involution σ_{p_1} we get that $\tilde{\sigma}_{p_1}(\tilde{L})$ is the fiber of the exceptional divisor F_1 over the point q on Γ_1 . Thus, by Proposition 2.2.15 we get that $\sigma_{p_1}(\tilde{L})$ is a curve of indeterminacy of τ_{12} . Then by Proposition 2.2.15 and since $\tilde{\sigma}_{p_1}$ is a regular involution we get:

$$\tilde{\sigma}_{p_1}^*[\tilde{L}] = \tilde{\sigma}_{p_1^*}[L] = f_1 - f_2.$$

Then by Lemma 2.4.7 we have the following equalities:

$$\begin{aligned} H \cdot [\tilde{L}] &= \tilde{\sigma}_{p_1}^*(H) \cdot \tilde{\sigma}_{p_1}^*[\tilde{L}] = (3H - 2E_1 - F_1) \cdot (f_1 - f_2) = 1; \\ E_1 \cdot [\tilde{L}] &= \tilde{\sigma}_{p_1}^*(E_1) \cdot \tilde{\sigma}_{p_1}^*[\tilde{L}] = (2H - E_1 - F_1) \cdot (f_1 - f_2) = 1; \\ F_1 \cdot [\tilde{L}] &= \tilde{\sigma}_{p_1}^*(F_1) \cdot \tilde{\sigma}_{p_1}^*[\tilde{L}] = (4H - 4E_1 - F_1) \cdot (f_1 - f_2) = 1; \\ F_2 \cdot [\tilde{L}] &= \tilde{\sigma}_{p_1}^*(F_2) \cdot \tilde{\sigma}_{p_1}^*[\tilde{L}] = F_2 \cdot (f_1 - f_2) = 1. \end{aligned}$$

By Lemma 2.4.6 this implies that $[\tilde{L}] = h - e_1 - f_1 - f_2$. Then by Lemma 2.4.8 we get

$$\begin{aligned} \theta_1(f) \cdot [\tilde{L}] &= \left(\sum_{i=1}^k \alpha_i \nu_i \right) \cdot (h - e_1 - f_1 - f_2) = \\ &= \sum_{i=1}^k \alpha_i (2H - 2E_j - F_j) \cdot (h - e_1 - f_1 - f_2) = 2 \sum_{i=1}^k \alpha_i - 3\alpha_1 - \alpha_2 < 0. \end{aligned}$$

Thus, \tilde{L} is an effective curve which intersects the class $\theta_1(f)$ negatively. \square

2.4.2. Formulas for involutions

Let p_1, p_2 and p_3 be three points on a smooth cubic surface S in \mathbb{P}^3 . Recall that the indeterminacy locus of σ_{p_i} is a curve Γ_i . By the following assertion if points p_1, p_2 and p_3 are sufficiently general then curves Γ_i intersect each other transversally for all $i = 1, 2, 3$.

Proposition 2.4.11. *Let p_1, p_2, p_3 be non-collinear points on a smooth cubic surface S in \mathbb{P}^3 such that a line passing through p_1 and p_2 is not tangent to S . Let Γ_i be the curve of indeterminacy of the involution $\sigma_{p_i}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ associated to the point p_i for all $i = 1, 2, 3$. Then the following assertions are true.*

- (1) *If q is a point in $\Gamma_1 \cap \Gamma_2$, then points q, p_1, p_2 are non-collinear.*
- (2) *Assume that p_3 does not lie in a hyperplane passing through q, p_1, p_2 and that x_0, x_1, x_2, x_3 are homogeneous coordinates of \mathbb{P}^3 such that*

$$q = (1 : 0 : 0 : 0); \quad p_1 = (0 : 1 : 0 : 0); \quad p_2 = (0 : 0 : 1 : 0); \quad p_3 = (0 : 0 : 0 : 1).$$

- (3) *If x_0, x_1, x_2, x_3 are homogeneous coordinates of \mathbb{P}^3 as in point (2) and $\varphi(x) = \sum_{|I|=3} a_I x^I$ is the equation of the surface S in these coordinates, then*

$$a_{3000} = a_{0300} = a_{0030} = a_{0003} = a_{2100} = a_{2010} = 0.$$

Moreover, the intersection of the curves Γ_1 and Γ_2 in the point q is transversal if one has the following inequality:

$$a_{2001}(4a_{1200}a_{1020} - a_{1110}^2) \neq 0.$$

Proof. (1) The assertion follows from the fact that q is a point on S such that the line passing through p_i and q is tangent to S in q for all $i = 1$ and 2 .

(2) By assumption and the point (1) we get there is no hyperplane in \mathbb{P}^3 which passes through points q, p_1, p_2 and p_3 . Thus, one can construct the necessary coordinates.

(3) By construction the surface S passes through points $(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 0 : 1)$ and $(0 : 0 : 0 : 1)$. Thus, one gets

$$a_{3000} = a_{0300} = a_{0030} = a_{0003} = 0.$$

Moreover, the line passing through q and p_1 is tangent to the surface S in q . The equation of this line is $\{x_2 = x_3 = 0\}$; thus, we get that $a_{2100} = 0$. Analogously, since the line passing through q and p_2 is tangent to S in q we get that $a_{2010} = 0$.

Consider the affine chart $U_0 = \{x_0 \neq 0\}$ of \mathbb{P}^3 . This chart is isomorphic to an affine space \mathbb{A}^3 . Consider coordinates t_1, t_2, t_3 of \mathbb{A}^3 such that $t_i = \frac{x_i}{x_0}$ for all $i = 1, 2, 3$. In these coordinates the point q corresponds to the origin $(0, 0, 0)$. By [Bla13, Section 2] the curve Γ_i is the intersection of the surfaces $\{\varphi = 0\}$ and $\{\frac{\partial \varphi}{\partial x_i} = 0\}$ for $i = 1, 2$. Denote by $\psi(t_1, t_2, t_3)$ the equation induced by φ on the chart U_0 . Then the tangent line to Γ_i in the point $q = (0, 0, 0) \in U_0$ defined by the following equations:

$$T_{\Gamma_i, q} = \left\{ \sum_{j=1}^3 \frac{\partial \psi}{\partial t_j}(0, 0, 0) dt_j = \sum_{l=1}^3 \frac{\partial^2 \psi}{\partial t_i \partial t_l}(0, 0, 0) dt_l = 0 \right\} \subset T_{U_0, q} = \langle dt_1, dt_2, dt_3 \rangle.$$

The intersection of the curves Γ_1 and Γ_2 at the point q is transversal if the tangent lines $T_{L_1, q}$ and $T_{L_2, q}$ do not coincide. This happens if the following determinant is non-zero:

$$\det \begin{pmatrix} \frac{\partial \psi}{\partial t_1}(0, 0, 0) & \frac{\partial \psi}{\partial t_2}(0, 0, 0) & \frac{\partial \psi}{\partial t_3}(0, 0, 0) \\ \frac{\partial^2 \psi}{\partial t_1 \partial t_1}(0, 0, 0) & \frac{\partial^2 \psi}{\partial t_1 \partial t_2}(0, 0, 0) & \frac{\partial^2 \psi}{\partial t_1 \partial t_3}(0, 0, 0) \\ \frac{\partial^2 \psi}{\partial t_2 \partial t_1}(0, 0, 0) & \frac{\partial^2 \psi}{\partial t_2 \partial t_2}(0, 0, 0) & \frac{\partial^2 \psi}{\partial t_2 \partial t_3}(0, 0, 0) \end{pmatrix} = a_{2001}(4a_{1200}a_{1020} - a_{1110}^2).$$

This implies the result. \square

In the coordinate which we described in Proposition 2.4.11 write formulas defining the involutions $\sigma_{p_1}, \sigma_{p_2}$ and σ_{p_3} .

Lemma 2.4.12 ([Bla13, Section 2]). *Let x_0, x_1, x_2, x_3 be homogeneous coordinates in \mathbb{P}^3 and let $\varphi(x) = 0$ be an equation of a smooth cubic surface S containing the points $p_1 = (0 : 1 : 0 : 0), p_2 = (0 : 0 : 1 : 0)$ and $p_3 = (0 : 0 : 0 : 1)$. Then the involutions associated with points p_1, p_2 and p_3 are given by the following formulas:*

$$\begin{aligned} \sigma_{p_1}(x_0, x_1, x_2, x_3) &= \left(x_0 \frac{\partial \varphi}{\partial x_1}(x) : x_1 \frac{\partial \varphi}{\partial x_1}(x) - 2\varphi(x) : x_2 \frac{\partial \varphi}{\partial x_1}(x) : x_3 \frac{\partial \varphi}{\partial x_1}(x) \right); \\ \sigma_{p_2}(x_0, x_1, x_2, x_3) &= \left(x_0 \frac{\partial \varphi}{\partial x_2}(x) : x_1 \frac{\partial \varphi}{\partial x_2}(x) : x_2 \frac{\partial \varphi}{\partial x_2}(x) - 2\varphi(x) : x_3 \frac{\partial \varphi}{\partial x_2}(x) \right); \\ \sigma_{p_3}(x_0, x_1, x_2, x_3) &= \left(x_0 \frac{\partial \varphi}{\partial x_3}(x) : x_1 \frac{\partial \varphi}{\partial x_3}(x) : x_2 \frac{\partial \varphi}{\partial x_3}(x) : x_3 \frac{\partial \varphi}{\partial x_3}(x) - 2\varphi(x) \right). \end{aligned}$$

Proof. It suffices to prove that σ_{p_1} is the involution associated with the point p_1 . Let l be a line in \mathbb{P}^3 passing through the point $p_1 = (0 : 1 : 0 : 0)$. This line intersects the plane $\{x_1 = 0\}$ in a point $(a : 0 : b : c)$. Then we get that l is as follows:

$$l = \{(as : t : bs : cs) \mid (s : t) \in \mathbb{P}^1\}.$$

Therefore, one can compute the restriction of φ and $\frac{\partial\varphi}{\partial x_1}$ to l :

$$\begin{aligned}\varphi(as, t, bs, cs) &= \mu_1 st^2 + \mu_2 s^2 t + \mu_3 s^3 \\ \frac{\partial\varphi}{\partial x_1}(as, t, bs, cs) &= 2\mu_1 st + \mu_2 s^2.\end{aligned}$$

Here μ_1, μ_2 and μ_3 are some numbers in \mathbb{C} depending only on the line l and the cubic surface S . Then if we apply the formula of σ_{p_1} to the point $(as : t : bs : cs)$, we get the following point on \mathbb{P}^3 :

$$\begin{aligned}\sigma_{p_1}(as, t, bs, cs) &= \\ (as^2(2\mu_1 t + \mu_2 s) : st(2\mu_1 t + \mu_2 s) - 2(\mu_1 st^2 + \mu_2 s^2 t + \mu_3 s^3) : bs^2(2\mu_1 t + \mu_2 s) : cs^2(2\mu_1 t + \mu_2 s)) &= \\ = (a(2\mu_1 t + \mu_2 s) : -\mu_2 t - 2\mu_3 s : b(2\mu_1 t + \mu_2 s) : c(2\mu_1 t + \mu_2 s)).\end{aligned}$$

Then the restriction of the map σ_{p_1} to the line l is an automorphism of l . In coordinates s, t it corresponds to the following matrix in $\text{PGL}(2, \mathbb{C})$:

$$\sigma_{p_1}|_l = \begin{pmatrix} \mu_2 & -2\mu_3 \\ 2\mu_1 & -\mu_2 \end{pmatrix}$$

Thus, $\sigma_{p_1}|_l$ is a non-trivial involution of l and it preserves points $(s : t)$ such that $\varphi(as, t, bs, cs) = 0$ and $(s : t) \neq (0 : 1)$. By the definition of σ_{p_1} this implies the result. \square

2.4.3. Composition of three involutions

Our goal in this section is to prove the following assertion, it is the key to the proof of Theorem 2.1.2.

Proposition 2.4.13. *Let p_1, p_2 and p_3 be three general points on a very general cubic surface S and let σ_{p_i} be the involution associated to the point p_i for all $i = 1, 2, 3$. If L is the line in \mathbb{P}^3 passing through the point p_1 and a point in $\text{Ind}(\sigma_{p_1}) \cap \text{Ind}(\sigma_{p_2})$ and F is the following composition:*

$$F = \sigma_{p_3} \circ \sigma_{p_2} \circ \sigma_{p_1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3,$$

then L does not lie in $\text{Ind}((F)^{-n})$ for all $n > 0$.

To prove Proposition 2.4.13 we fix a general point x on the line L and consider coordinates of the image $F^{-n}(x)$ as polynomials of coordinates of x . The condition that x does not lie in $\text{Ind}(F^{-(n-1)})$ is equivalent to the fact that these polynomials do not vanish. Computing modulo some ideal and using Sage we get the result.

Set $B = \{(2001), (1200), (1110), (1101), (0210)\}$. Denote by R the following ring:

$$R = \mathbb{Z}[a_I]_{I \in B} = \mathbb{Z}[a_{2001}, a_{1200}, a_{1110}, a_{1101}, a_{0210}].$$

Consider a free commutative ring $R[X, Y]$. Set $w = a_{2001}a_{0210} + a_{1110}a_{1101}$ and denote by \mathcal{I} the ideal $(4, 2w)$ in $R[X, Y]$, i.e., the ideal of polynomials in X and Y whose coefficients lie in $(4, 2w) \subset R$. If P is an element in $R[X, Y]$ we consider it as a polynomial of X and Y with coefficients in R .

By $\deg_Y(P)$ we denote the degree of the polynomial P in Y . Let $r(X)$ be the leading term in Y of P viewed as a polynomial in the variable Y , so that $\deg(P - r(X)Y^{\deg(P)})$ is strictly less than $\deg(P)$. We need the following useful property of degrees of polynomials:

Lemma 2.4.14. *Let P_1 and P_2 be two polynomials such that:*

$$P_i = r_i(X)Y^{d_i} + Q_i(X, Y);$$

Here the degree in Y of all monomials in Q_i is strictly less than d_i . If $d_1 > d_2$ and $r_1(X)$ is a non-zero polynomial which does not lie in \mathcal{I} , then $P_1 + P_2$ also does not lie in \mathcal{I} and $\deg_Y(P_1 + P_2) = d_1$.

Proof. Since X and Y are free variables, then the polynomial $\sum_{i,j} r_{i,j} X^i Y^j$ lies in \mathcal{I} only if all coefficients $r_{i,j}$ lie in \mathcal{I} . Since $\deg(P_1) > \deg(P_2)$, then the leading term in Y of $P_1 + P_2$ coincides with the leading term of P_1 . \square

Let $\mathbb{P}_{R[X,Y]}^3$ be a projective space defined over the ring $R[X, Y]$. Let x_0, x_1, x_2, x_3 be homogeneous coordinates of $\mathbb{P}_{R[X,Y]}^3$ and let $\varphi(x)$ be an equation of a cubic surface in $\mathbb{P}_{R[X,Y]}^3$:

$$\varphi(x) = a_{2001}x_0^2x_3 + a_{1200}x_0x_1^2 + a_{1110}x_0x_1x_2 + a_{1101}x_0x_1x_3 + a_{0210}x_1^2x_2.$$

Let $\sigma_{p_1}, \sigma_{p_2}$ and σ_{p_3} be birational involutions of $\mathbb{P}_{R[X,Y]}^3$ given by formulas from Lemma 2.4.12. Denote by F_R the composition of these involutions:

$$F_R = \sigma_{p_3} \circ \sigma_{p_2} \circ \sigma_{p_1} : \mathbb{P}_{R[X,Y]}^3 \dashrightarrow \mathbb{P}_{R[X,Y]}^3 \quad (2.4.15)$$

Our goal is to compute the orbit of a point under the action of F_R^{-1} . We will consider points of the following form.

Notation 2.4.16. Let $p = (M_0 + 2N_0 : M_1 + 2N_1 : \tilde{g}_2 + 2g_2 : \tilde{g}_3 + 2g_3)$ be a point in $\mathbb{P}_{R[X,Y]}^3$, such that $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 are elements of $R[X, Y]$ of the following form:

- (B1) Polynomials \tilde{g}_2 and \tilde{g}_3 are elements of \mathcal{I} ; also g_2 and g_3 lie in $2R[X, Y]$ and the leading term in Y of \tilde{g}_3 do not lie in $4R[X, Y]$;
- (B2) The leading terms in Y of polynomials $M_i(X, Y)$ do not lie in $2R[X]$ for $i = 0$ and $i = 1$.
- (B3) We have the following conditions on degrees of polynomials:

$$\deg_Y(\tilde{g}_2) \leq \deg_Y(M_0) < \deg_Y(\tilde{g}_3) < \deg_Y(M_1).$$

In the next lemma we show that the image of a point satisfying conditions (B1 – B3) under F_R^{-1} still satisfies these conditions.

Lemma 2.4.17. *Let $p = (M_0 + 2N_0 : M_1 + 2N_1 : \tilde{g}_2 + 2g_2 : \tilde{g}_3 + 2g_3)$ be a point in the projective space $\mathbb{P}_{R[X,Y]}^3$ such that polynomials $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 satisfy conditions (B1 – B3). Then the point p does not lie in $\text{Ind}(F_R^{-1})$. Moreover, one has*

$$F_R^{-1}(p) = (M'_0 + 2N'_0 : M'_1 + 2N'_1 : \tilde{g}'_2 + 2g'_2 : \tilde{g}'_3 + 2g'_3),$$

where polynomials $M'_0, N'_0, M'_1, N'_1, g'_2, \tilde{g}'_2, g'_3$ and \tilde{g}'_3 satisfy conditions (B1 – B3).

Proof. In view of condition (B3) one has $\deg_Y(M_0) = d_0$, $\deg_Y(M_1) = d_1$, $\deg_Y(\tilde{g}_2) = d_2$ and $\deg_Y(\tilde{g}_3) = d_3$ and we have the following inequality:

$$d_2 \leq d_0 < d_3 < d_1. \quad (2.4.18)$$

In order to compute $F_R^{-1}(p)$ we use the formulas from Proposition 2.4.11 associated with the cubic surface given by the equation $\sum_{i \in B} a_I x^I$. Then $F_R^{-1}(p) = (q_0 : q_1 : q_2 : q_3)$, where the expressions q_i can be computed in Sage explicitly, see Section 2.4.4.

First thing we note is that all polynomials q_0, q_1, q_2 and q_3 lie in $2R[X, Y]$. Our goal is to construct polynomials $M'_0, N'_0, M'_1, N'_1, g'_2, \tilde{g}'_2, g'_3$ and \tilde{g}'_3 such that

$$\begin{aligned} q_0 &= 2(M'_0 + 2N'_0); \\ q_1 &= 2(M'_1 + 2N'_1); \\ q_2 &= 2(\tilde{g}'_2 + 2g'_2); \\ q_3 &= 2(\tilde{g}'_3 + 2g'_3). \end{aligned}$$

We are going to do it considering all components q_0, q_1, q_2 and q_3 one by one.

Consider the component q_0 . By formulas in Lemma 2.4.12 we can see that this is a homogeneous polynomial of $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 of degree 27. We define M'_0 to be the sum of all monomials in $\frac{1}{2}q_0$ except those divisible by $2, g_2, \tilde{g}_2, g_3, \tilde{g}_3$ or $\frac{1}{2}m$ where m is a degree 2 monomial of g_2, \tilde{g}_2, g_3 and \tilde{g}_3 . Note that by construction $\frac{1}{2}q_0 - M'_0$ lies in $2R[X, Y]$ since so do g_2, \tilde{g}_2, g_3 and \tilde{g}_3 . Denote $N'_0 = \frac{1}{2}(\frac{1}{2}q_0 - M'_0)$. The computation in Sage shows:

$$M'_0 = a_{1101}^8 a_{0210}^2 M_0^{11} M_1^{15} (w a_{1200} M_0 + \frac{1}{2} a_{1110} a_{1101} a_{0210} \tilde{g}_2 + \frac{1}{2} a_{1101}^2 a_{0210} \tilde{g}_3) + h_{M'_0}, \quad (2.4.19)$$

where $h_{M'_0}$ is a homogeneous polynomial of $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 of degree 27 and the degree in M_1 of all monomials is strictly less than 15. By Lemma 2.4.14 and (2.4.18) we get that the leading term in Y of M'_0 does not lie in $2R[X, Y]$ and

$$\deg_Y(M'_0) = 11d_0 + 15d_1 + d_3. \quad (2.4.20)$$

Consider the component q_1 . This is a homogeneous polynomial of $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 of degree 27. We define M'_1 to be the sum of all monomials in $\frac{1}{2}q_1$ except those divisible by $2, g_2, \tilde{g}_2, g_3, \tilde{g}_3$ or $\frac{1}{2}m$ where m is a degree 2 monomial of g_2, \tilde{g}_2, g_3 and \tilde{g}_3 . Note that by construction $\frac{1}{2}q_1 - M'_1$ lies in $2R[X, Y]$. Denote $N'_1 = \frac{1}{2}(\frac{1}{2}q_1 - M'_1)$. The computation in Sage shows:

$$M'_1 = a_{1200} a_{1101}^9 a_{0210}^3 M_0^{10} M_1^{17} + h_{M'_1}, \quad (2.4.21)$$

where $h_{M'_1}$ is a homogeneous polynomial of $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 of degree 27 and the degree in M_1 of all monomials is strictly less than 17. By Lemma 2.4.14 and (2.4.18) we get that the leading term in Y of M'_1 does not lie in $2R[X, Y]$ and

$$\deg_Y(M'_1) = 10d_0 + 17d_1. \quad (2.4.22)$$

Consider the component q_2 . This is a homogeneous polynomial of $M_0, N_0, M_1, N_1, p_2 = \tilde{g}_2 + 2g_2$ and $p_3 = \tilde{g}_3 + 2g_3$ of degree 27. We define q'_2 to be the sum of all monomials of q_2 except those divisible by 8, $2p_2, 2p_3$ or a degree 2 monomial of p_2 and p_3 . Note that by construction $q_2 - q'_2$ lies in $2\mathcal{I}$. The computation in Sage shows:

$$q'_2 = 4wa_{1200}^2 M_0^{12} M_1^6 (a_{1110} M_0 + a_{0210} M_1) (a_{2001}^8 M_0^8 + a_{1101}^8 M_1^8) \quad (2.4.23)$$

Since q'_2 is divisible by $4w$ it lies in $2\mathcal{I}$. Therefore, by construction q_2 lies in $2\mathcal{I}$. Define \tilde{g}'_2 to be the sum of all monomials of $\frac{1}{2}q_2$ except those divisible by 4, $2g_2, 2\tilde{g}_2, 2g_3, 2\tilde{g}_3$ or a degree 2 monomial of g_2, \tilde{g}_2, g_3 and \tilde{g}_3 . Note that by construction $g_2 = \frac{1}{2}(\frac{1}{2}q_2 - \tilde{g}'_2)$ lies in $2R[X, Y]$. The computation in Sage shows:

$$\begin{aligned} \tilde{g}'_2 = & a_{0210} a_{1101}^8 M_0^{10} M_1^{15} (2wa_{1200}^2 M_0^2 + a_{2001} a_{1200} a_{0210}^2 M_0 \tilde{g}_2 + \\ & + \frac{1}{2} a_{1110} a_{1101} a_{0210}^2 \tilde{g}_2^2 + a_{1200} a_{1101}^2 a_{0210} M_0 \tilde{g}_3 + \frac{1}{2} a_{1101}^2 a_{0210}^2 \tilde{g}_2 \tilde{g}_3) + h_{\tilde{g}'_2}, \end{aligned} \quad (2.4.24)$$

where $h_{\tilde{g}'_2}$ is a homogeneous polynomial of $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 of degree 27 and the degree in M_1 of all monomials is strictly less than 15. By (2.4.18) and we get that

$$\deg_Y(\tilde{g}'_2) \leq 11d_0 + 15d_1 + d_3. \quad (2.4.25)$$

Consider the component q_3 . This is a homogeneous polynomial of $M_0, N_0, M_1, N_1, p_2 = \tilde{g}_2 + 2g_2$ and $p_3 = \tilde{g}_3 + 2g_3$ of degree 27. We define q'_3 to be the sum of all monomials of q_3 except those divisible by 8, $2p_2, 2p_3$ or a degree 2 monomial of p_2 and p_3 . Note that by construction $q_3 - q'_3$ lies in $2\mathcal{I}$. The computation in Sage shows:

$$\begin{aligned} q'_3 = & 4wa_{1200}^2 M_0^{11} M_1^7 (a_{2001} M_0 + a_{1101} M_1) (a_{1110}^2 M_0^2 + \\ & + a_{0210}^2 M_1^2) (a_{2001}^2 M_0^2 + a_{1101}^2 M_1^2) (a_{2001}^4 M_0^4 + a_{1101}^4 M_1^4). \end{aligned} \quad (2.4.26)$$

Since q'_3 is divisible by $4w$ it lies in $2\mathcal{I}$. Therefore, by construction q_3 lies in $2\mathcal{I}$.

Define \tilde{g}'_3 to be the sum of all monomials of $\frac{1}{2}q_3$ except those divisible by 4, $2g_2, 2\tilde{g}_2, 2g_3, 2\tilde{g}_3$ or degree 2 monomial of g_2, \tilde{g}_2, g_3 and \tilde{g}_3 . Note that by construction $g_3 = \frac{1}{2}(\frac{1}{2}q_3 - \tilde{g}'_3)$ lies in $2R[X, Y]$. Moreover, the Sage computation shows:

$$\tilde{g}'_3 = a_{1200} a_{0210}^2 a_{1101}^7 M_0^{10} M_1^{16} (2wa_{1200} M_0 + a_{1110} a_{1101} a_{0210} \tilde{g}_2 + a_{1101}^2 a_{0210} \tilde{g}_3) + h_{\tilde{g}'_3}. \quad (2.4.27)$$

Here $h_{\tilde{g}'_3}$ is a homogeneous polynomial of $M_0, N_0, M_1, N_1, g_2, \tilde{g}_2, g_3$ and \tilde{g}_3 of degree 27 and the degree in M_1 of all monomials is strictly less than 16. By Lemma 2.4.14 and (2.4.18) we get that the leading term in Y of \tilde{g}'_3 does not lie in $4R[X, Y]$ and

$$\deg_Y(\tilde{g}'_3) = 10d_0 + 16d_1 + d_3. \quad (2.4.28)$$

We have defined polynomials $M'_0, N'_0, M'_1, N'_1, g'_2, \tilde{g}'_2, g'_3$ and \tilde{g}'_3 ; thus, it remains to show conditions (B1 – B3) for these polynomials.

Polynomials $\tilde{g}'_2 + 2g'_2$ and $\tilde{g}'_3 + 2g'_3$ lie in \mathcal{I} by equations (2.4.23) and (2.4.26). Polynomials g'_2 and g'_3 lie in $2R[X, Y]$ by the construction. Also by (2.4.24) and (2.4.27) we get that the leading terms of \tilde{g}_2 and \tilde{g}_3 are not divisible by 4. Thus, the condition (B1) is satisfied.

By (2.4.19) and (2.4.21) we get that the leading monomials of M'_0 and M'_1 do not lie in $2R[X, Y]$. Thus, the condition (B2) is satisfied.

Finally the condition (B3) follows from the formulas (2.4.20), (2.4.22), (2.4.25), (2.4.28) and the assumption (2.4.18).

Since by our computation polynomials q_0, q_1, q_2 and q_3 are non-zero, we get that the point p does not lie in $\text{Ind}(F_R^{-1})$. \square

Corollary 2.4.29. *Let $p = (X : Y : 0 : 0)$ be a point in $\mathbb{P}^3_{R[X, Y]}$ and F_R be a birational automorphism as in (2.4.15). Then p does not lie in $\text{Ind}(F_R^{-n})$ for all $n > 0$.*

Proof. Let $q = (q_0 : q_1 : q_2 : q_3)$ be the point $F_R^{-1}(p)$ in $\mathbb{P}^3_{R[X, Y]}$. Below we define set of polynomials $M'_0, N'_0, M'_1, N'_1, \tilde{g}'_2, g'_2, \tilde{g}'_3$ and g'_3 such that they satisfy conditions (B1 – B3) and

$$\begin{aligned} q_0 &= 2(M'_0 + 2N'_0); \\ q_1 &= 2(M'_1 + 2N'_1); \\ q_2 &= 2(\tilde{g}'_2 + 2g'_2); \\ q_3 &= 2(\tilde{g}'_3 + 2g'_3). \end{aligned} \tag{2.4.30}$$

Consider the component q_0 , it is a homogeneous polynomial in X and Y of degree 27. Define M'_0 to be the sum of all monomials of $\frac{1}{2}q_0$ whose coefficients are not divisible by 4. Set $N'_0 = \frac{1}{2}(\frac{1}{2}q_0 - M'_0)$. Using the formula (2.4.19) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in Y of M'_0 is equal to $wa_{1200}a_{1101}^8a_{0210}^2X^{12}Y_1^{15}$. It does not lie in $2R[X, Y]$ and the degree in Y of M'_0 is

$$\deg_Y(M'_0) = 15.$$

Consider the component q_1 , it is a homogeneous polynomial in X and Y of degree 27. Define M'_1 to be the sum of all monomials of $\frac{1}{2}q_1$ whose coefficients are not divisible by 4. Set $N'_1 = \frac{1}{2}(\frac{1}{2}q_1 - M'_1)$. Using the formula (2.4.21) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in Y of M'_1 is equal to $a_{1200}a_{1101}^9a_{0210}^3X^{10}Y^{17}$. It does not lie in $2R[X, Y]$ and the degree in Y of M'_1 is

$$\deg_Y(M'_1) = 17.$$

Consider the component q_2 . By the formula (2.4.23) we see that q_2 lies in the ideal \mathcal{I} . Define \tilde{g}'_2 to be the sum of all monomials of $\frac{1}{2}q_2$ which are not divisible by 4. Set $g'_2 = \frac{1}{2}(\frac{1}{2}q_2 - \tilde{g}'_2)$. Using the formula (2.4.24) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in Y of \tilde{g}'_2 is equal to $2wa_{1200}^2a_{0210}a_{1101}^8X^{12}Y^{15}$. Then the degree in Y of \tilde{g}'_2 is

$$\deg_Y(\tilde{g}'_2) = 15.$$

Consider the component q_3 . By the formula (2.4.26) we see that q_3 lies in the ideal \mathcal{I} . Define \tilde{g}'_3 to be the sum of all monomials of $\frac{1}{2}q_3$ which are not divisible by 4. Set $g'_3 = \frac{1}{2}(\frac{1}{2}q_3 - \tilde{g}'_3)$. Using the formula (2.4.27) and substituting there $M_0 = X, M_1 = Y, \tilde{g}_2 = \tilde{g}_3 = 0$ we can see that the leading term in Y of \tilde{g}'_3 is equal to $2wa_{1200}^2a_{0210}^2a_{1101}^7X^{11}Y^{16}$. It does not lie in $4R[X, Y]$ and the degree in Y of \tilde{g}'_3 is

$$\deg_Y(\tilde{g}'_3) = 16.$$

Thus, one gets that q has the form (2.4.30) and polynomials $M'_0, N'_0, M'_1, N'_1, \tilde{g}'_2, g'_2, \tilde{g}'_3$ and g'_3 satisfy conditions (B1 – B3). Therefore, the point p does not lie in the indeterminacy locus of F_R^{-1} . Moreover, by Lemma 2.4.17 the point $q = F_R^{-1}(p)$ does not lie in $\text{Ind}(F_R^{-n})$ for all $n > 0$. This finishes the proof. \square

Proof of Proposition 2.4.13. Consider a 4-dimensional complex vector space V and its associated projective space $\mathbb{P}(V) = \mathbb{P}^3$. A cubic surface in $\mathbb{P}(V)$ is given by an equation in $\mathbb{P}(S^3V^\vee)$. The coordinates of the space $\mathbb{P}(S^3V^\vee)$ are coefficients a_I of the cubic equation $\sum a_I x^I$. Denote by W the following subspace of S^3V^\vee :

$$W = \langle a_I \mid I \neq (3000), (0300), (0030), (0003), (2100), (2010) \rangle \subset S^3V^\vee.$$

Consider a universal cubic surface $\mathcal{S} \subset \mathbb{P}(V) \times \mathbb{P}(W)$. Denote by $\Pi: \mathcal{S} \rightarrow \mathbb{P}(W)$ the projection onto the second factor. The fiber of Π over a point $\varphi \in \mathbb{P}(W)$ is a cubic surface $\mathcal{S}_\varphi = \{\varphi = 0\}$. Moreover, by construction the surface \mathcal{S}_φ contains the points $q = (1 : 0 : 0 : 0)$, $p_1 = (0 : 1 : 0 : 0)$, $p_2 = (0 : 0 : 1 : 0)$ and $p_3 = (0 : 0 : 0 : 1)$ and the lines passing through p_i and q are tangent to \mathcal{S}_φ in q for $i = 1$ and 2 . Note that for any point $\varphi \in \mathbb{P}(W)$ the point q lies in $\text{Ind}(\sigma_{p_1}) \cap \text{Ind}(\sigma_{p_2})$ where σ_{p_1} and σ_{p_2} are involutions associated with the cubic equation φ and points p_1 and p_2 .

Then we can define the involutions $\sigma_{p_1}, \sigma_{p_2}$ and σ_{p_3} of $\mathbb{P}(V) \times \mathbb{P}(W)$ by the formulas in Lemma 2.4.12. Denote by $F_{\mathcal{S}}$ the following composition:

$$F_{\mathcal{S}} = \sigma_{p_1} \circ \sigma_{p_2} \circ \sigma_{p_3}: \mathbb{P}(V) \times \mathbb{P}(W) \dashrightarrow \mathbb{P}(V) \times \mathbb{P}(W).$$

Consider the following cubic equation $\varphi_0 \in W$:

$$\varphi_0 = a_{2001}x_0^2x_3 + a_{1200}x_0x_1^2 + a_{1110}x_0x_1x_2 + a_{1101}x_0x_1x_3 + a_{0210}x_1^2x_2,$$

here $a_{2001}, a_{1200}, a_{1110}, a_{1101}, a_{0210}$ are non-zero complex numbers. By Corollary 2.4.29 we get that if X and Y are very general complex numbers then the point $(X : Y : 0 : 0)$ does not lie in the indeterminacy locus of F^{-n} for all $n > 0$.

This implies that the point $((X : Y : 0 : 0), \varphi_0)$ does not lie in $\text{Ind}(F_{\mathcal{S}}^{-n})$ for all $n > 0$. Denote by L the line connecting q and p_1 . Then the point $(X : Y : 0 : 0)$ lies on L in \mathbb{P}^3 . Therefore, the subvariety $L \times \mathbb{P}(W)$ does not lie in $\text{Ind}(F_{\mathcal{S}}^{-n})$ for all $n > 0$.

Since $\bigcup_{n>0} \text{Ind}(F_{\mathcal{S}}^{-n})$ is a countable union of closed codimension 2 subvarieties of $\mathbb{P}(V) \times \mathbb{P}(W)$, then for a very general point φ in $\mathbb{P}(W)$ the line L passing through p_1 and q does not lie in $\text{Ind}(F^{-n})$ for all $n > 0$.

Finally, by Proposition 2.4.11 if S is a smooth cubic surface and p_1, p_2 and p_3 are three general points on S , then there exists an isomorphism between S and \mathcal{S}_φ which maps points p_1, p_2 and p_3 to $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$. Thus, we get the result. \square

2.4.4. Sage computations for Lemma 2.4.17

In order to prove Lemma 2.4.17 we have to compute the preimage of the point in the projective space $\mathbb{P}_{R[X,Y]}^3$ described in Notation 2.4.16. Here is the Sage code we used to perform these computations. Below we define birational involutions $\sigma_{p_j} = \text{I}_j$ for $j = 1, 2$ and 3 and their composition $F_R^{-1} = \text{F_inverse}$. Variables $m_0, n_0, m_1, n_1, g_2, gg_2, g_3$ and gg_3 there play the roles of the

polynomials $M_0, N_0, M_1, N_1, \tilde{g}_2, g_2, \tilde{g}_3$ and g_3 respectively. Variables p_2 and p_3 represent the elements $\tilde{g}_2 + 2g_2$ and $\tilde{g}_3 + 2g_3$ of the ideal \mathcal{I} . Afterwards, we apply the function `F_inverse` to the point $p = (M_0+2*N_0, M_1+2*N_1, g_2+ 2*gg_2, g_3+2*gg_3)$ and get set of four polynomials γ_1 . Then we apply the function `F_inverse` to the point $p = (M_0+2*N_0, M_1+2*N_1, p_2, p_3)$ and get Q_2 .

```
K.<a_2001, a_1200, a_1110, a_1101, a_0210, M0, M1, N0, N1, g2, gg2, g3, gg3, x0,
  x1, x2, x3, p2,p3>=ZZ []

cubic = a_2001*x0^2*x3 + a_1200*x0*x1^2 + a_1110*x0*x1*x2 + a_1101*x0*x1*x3 +
  a_0210*x1^2*x2

dcubicdx1 =cubic.derivative(x1)
dcubicdx2 =cubic.derivative(x2)
dcubicdx3 =cubic.derivative(x3)

def I1(x0,x1,x2,x3):
  return x0*dcubicdx1(x0=x0,x1=x1,x2=x2,x3=x3),
  x1*dcubicdx1(x0=x0,x1=x1,x2=x2,x3=x3) - 2*cubic(x0=x0,x1=x1,x2=x2,x3=x3),
  x2*dcubicdx1(x0=x0,x1=x1,x2=x2,x3=x3), x3*dcubicdx1(x0=x0,x1=x1,x2=x2,x3=x3)
def I2(x0,x1,x2,x3):
  return x0*dcubicdx2(x0=x0,x1=x1,x2=x2,x3=x3),
  x1*dcubicdx2(x0=x0,x1=x1,x2=x2,x3=x3),
  x2*dcubicdx2(x0=x0,x1=x1,x2=x2,x3=x3) - 2*cubic(x0=x0,x1=x1,x2=x2,x3=x3),
  x3*dcubicdx2(x0=x0,x1=x1,x2=x2,x3=x3)
def I3(x0,x1,x2,x3):
  return x0*dcubicdx3(x0=x0,x1=x1,x2=x2,x3=x3),
  x1*dcubicdx3(x0=x0,x1=x1,x2=x2,x3=x3),
  x2*dcubicdx3(x0=x0,x1=x1,x2=x2,x3=x3),
  x3*dcubicdx3(x0=x0,x1=x1,x2=x2,x3=x3) - 2*cubic(x0=x0,x1=x1,x2=x2,x3=x3)

def F_inverse(x0,x1,x2,x3):
  return(I1(*I2(*I3(x0,x1,x2,x3))))

Q1 = F_inverse(M0+2*N0,M1+2*N1,g2+ 2*gg2,g3+2*gg3)
Q2 = F_inverse(M0+2*N0,M1+2*N1,p2,p3)
```

The set of polynomials q_1 and q_2 is exactly the set of polynomials (q_0, q_1, q_2, q_3) from the proof of Lemma 2.4.17. To analyze the output we need several additional functions. The following function `mod_2I` deletes all monomials in a polynomial which lie in the ideal $2\mathcal{I}$.

```
def mod_2I(P):
  Q = 0
  for m in P.monomials():
    c= P.monomial_coefficient(m)
    d2 = m.degree(p2)
    d3 = m.degree(p3)
    if d2+d3==2:
      # here we factor mod (pi*pj) \subset 2I
      Q = Q
    if d2+d3 ==1:
      # here we factor mod (2pi) \subset 2I
      c2 = c % 2
      Q = Q + c2*m
    if d2+d3 ==0:
      # here we factor mod (8) \subset 2I
      c8 = c % 8
      Q = Q+c8*m
  return Q
```

The function `mod_4` deletes all monomials in a polynomial which lie in the ideal $4R[X, Y]$.

```
def mod_4(P):
    Q = 0
    for m in P.monomials():
        c = P.monomial_coefficient(m)
        d2 = m.degree(g2)
        d3 = m.degree(g3)
        dd2 = m.degree(gg2)
        dd3 = m.degree(gg3)
        if d2+d3+dd2+dd3 ==1:
            # here we factor mod (2gi) \subset (4)
            c2 = c % 2
            Q = Q + c2*m
        if d2+d3+dd2+dd3 ==0:
            # here we factor mod (4) \subset (4)
            c4 = c % 4
            Q = Q+c4*m
    return Q
```

The function `mod_8` deletes all monomials in a polynomial which lie in the ideal $8R[X, Y]$.

```
def mod_8(P):
    Q = 0
    for m in P.monomials():
        c = P.monomial_coefficient(m)
        d2 = m.degree(g2)
        d3 = m.degree(g3)
        dd2 = m.degree(gg2)
        dd3 = m.degree(gg3)
        if d2+d3+dd2+dd3==2:
            # here we factor mod (2*gi*gj) \subset (8)
            c2 = c % 2
            Q = Q +c2*m
        if d2+d3+dd2+dd3 ==1:
            # here we factor mod (4*gi) \subset (8)
            c4 = c % 4
            Q = Q + c4*m
        if d2+d3+dd2+dd3 ==0:
            # here we factor mod (8) \subset (8)
            c8 = c % 8
            Q = Q+c8*m
    return Q
```

The function `leading_term_M1` returns the leading term in M_1 of the polynomial.

```
def leading_term_M1(P):
    Q=0
    d = P.degree(M1)
    for m in P.monomials():
        if d == m.degree(M1):
            c = P.monomial_coefficient(m)
            Q = Q + c*m
    return Q
```

The last thing we need is a function `factorization` which factors polynomials into a product of irreducible polynomials. Unfortunately the computation above results in so-called symbolic functions, so to factor them we transform the symbolic function into a polynomial and then use the factorization in polynomials.


```

b_2001, b_1200, b_1110, b_1101, b_0210, Z0, Z1, W0, W1, f0, f1, f2, f3, ff2, ff3,
q2,q3 = PolynomialRing(RationalField(), 17, ['a_2001', 'a_1200', 'a_1110',
'a_1101', 'a_0210', 'M0', 'M1', 'N0', 'N1', 'g0', 'g1', 'g2', 'g3', 'gg2',
'gg3', 'p2', 'p3']).gens()

def factorization(P):
    Q=0
    for m in P.monomials():
        c= P.monomial_coefficient(m)
        d2001 = m.degree(a_2001)
        d1200 = m.degree(a_1200)
        d1110 = m.degree(a_1110)
        d1101 = m.degree(a_1101)
        d0210 = m.degree(a_0210)
        dx0 = m.degree(M0)
        dx1 = m.degree(M1)
        dy0 = m.degree(N0)
        dy1 = m.degree(N1)
        d2 = m.degree(g2)
        d3 = m.degree(g3)
        dd2 = m.degree(gg2)
        dd3 = m.degree(gg3)
        dp2 = m.degree(p2)
        dp3 = m.degree(p3)
        Q = Q+ c * b_2001^d2001 * b_1200^d1200 * b_1110^d1110 * b_1101^d1101 *
            b_0210^d0210 * f2^d2 * f3^d3 * ff2^dd2 * ff3^dd3 * Z0^dx0 * Z1^dx1 *
            W0^dy0 * W1^dy1 * q2^dp2 * q3^dp3
    return Q.factor()

```

Now we are ready to get results:

```

M0new = mod_4(Q1[0])
M1new = mod_4(Q1[1])
Q2_new = mod_2I(Q2[2])
Q3_new = mod_2I(Q2[3])
G2new = mod_8(Q1[2])
G3new = mod_8(Q1[3])

```

```

print(factorization(leading_term_M1(M0new)))
print(factorization(leading_term_M1(M1new)))
print(factorization(Q2_new))
print(factorization(Q3_new))
print(factorization(leading_term_M1(G2new)))
print(factorization(leading_term_M1(G3new)))

```

After running a program we get six lines of results:

```

a_0210^2 * a_1101^8 * M0^11 * M1^15 *(2*a_1200*a_1110*a_1101*M0 +
2*a_2001*a_1200*a_0210*M0 +a_1110*a_1101*a_0210*g2 + a_1101^2*a_0210*g3)

(2) * a_1200 * a_0210^3 * a_1101^9 * M0^10 * M1^17

(4) * a_1200^2 * M1^6 * M0^12 * (a_1110*M0 + a_0210*M1) * (a_1110*a_1101 +
a_2001*a_0210) * (a_2001^8*M0^8 + a_1101^8*M1^8)

(4) * a_1200^2 * M1^7 * M0^11 * (a_2001*M0 + a_1101*M1) * (a_1110*a_1101 +
a_2001*a_0210) * (a_1110^2*M0^2 + a_0210^2*M1^2) * (a_2001^2*M0^2 +
a_1101^2*M1^2) * (a_2001^4*M0^4 + a_1101^4*M1^4)

```

$$a_{0210} * a_{1101}^8 * M_0^{10} * M_1^{15} * (4 * a_{1200}^2 * a_{1110} * a_{1101} * M_0^2 + 4 * a_{2001} * a_{1200}^2 * a_{0210} * M_0^2 + 2 * a_{2001} * a_{1200} * a_{0210}^2 * M_0 * g_2 + a_{1110} * a_{1101} * a_{0210}^2 * g_2^2 + 2 * a_{1200} * a_{1101}^2 * a_{0210} * M_0 * g_3 + a_{1101}^2 * a_{0210}^2 * g_2 * g_3)$$

$$(2) * a_{1200} * a_{0210}^2 * a_{1101}^7 * M_0^{10} * M_1^{16} * (2 * a_{1200} * a_{1110} * a_{1101} * M_0 + 2 * a_{2001} * a_{1200} * a_{0210} * M_0 + a_{1110} * a_{1101} * a_{0210} * g_2 + a_{1101}^2 * a_{0210} * g_3)$$

The first line produces the leading term in M_1 for $2M'_0$, we use this formula in (2.4.19). The second line produces the leading term in M_1 for $2M'_1$, we use this formula in (2.4.21). The third line produces the expression for q'_2 , we use it in (2.4.23). The fourth line produces the expression for q'_3 , we use it in (2.4.26). The fifth and sixth lines produce the leading terms in M_1 of \tilde{g}'_2 and \tilde{g}'_3 respectively, we use them in (2.4.24) and (2.4.27).

2.4.5. Proof of Theorem 2.1.2

Let p_1, p_2 and p_3 be three points on a smooth cubic surface S , let $\sigma_{p_i} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the involution associated with these points for all $i = 1, 2, 3$ and let Γ_i be the curve of indeterminacy of σ_{p_i} . By the generality of points p_1, p_2 and p_3 and by Proposition 2.4.11 we can assume that the line passing through p_1 and p_2 is not tangent to S , that the curves Γ_1 and Γ_2 intersect transversally in the point q and that points q, p_1, p_2 and p_3 do not lie in one hyperplane.

Recall that X is the successive blow-ups of points p_1, p_2 and p_3 , then the proper transforms of curves Γ_1, Γ_2 and Γ_3 . Consider the composition f of three involutions:

$$f = \sigma_{p_3} \circ \sigma_{p_2} \circ \sigma_{p_1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3.$$

By Theorem 2.4.2 one gets that f is a positive entropy automorphism and

$$\lambda_1(f) = \lambda_2(f) > 1.$$

Therefore, by Theorem 2.2.18 one can define a pseudo-effective class $\theta_1(f)$ in $N^1(X)$ such that

$$f^*(\theta_1(f)) = \lambda_1(f)\theta_1(f).$$

Let L be a line passing through points p_1 and q . Denote by \tilde{L} the proper transform of the line L to X . By Corollary 2.4.10 one has the following inequality:

$$\theta_1(f) \cdot [\tilde{L}] < 0.$$

By Proposition 2.4.13 since the cubic surface S is very general and points p_1, p_2 and p_3 are general the curve \tilde{L} does not lie in the indeterminacy locus of f^{-n} for all $n > 0$. Thus, the pseudo-automorphism f satisfy the condition of Theorem 2.1.3. Thus, we get that the pseudo-automorphism f is not regularizable and it does not preserve the structure of a fibration over a surface.

3. Automorphisms of Bogomolov–Guan manifolds

3.1. Introduction

Our goal is to shed some light on the geometry of a partial class of non-Kähler manifolds, described in the works of Guan [Gua94], [Gua95a], [Gua95b] and Bogomolov [Bog96]. We call these objects BG-manifolds and we recall Bogomolov’s construction of a BG-manifold in Section 3.3.2. Our first result describes an algebraic reduction of a BG-manifold.

Theorem 3.1.1 (Corollary 3.3.17). *Let $n \geq 2$ be an integer and Q be a BG-manifold of dimension $2n$. Then Q has a structure of a fiber space*

$$\Phi: Q \rightarrow \mathbb{P}^n \tag{3.1.2}$$

with typical fiber being an abelian variety. This map is an algebraic reduction of Q .

Our second result describes properties of submanifolds in BG-manifolds. Recall here that a manifold M is Moishezon if it is a proper modification of a projective manifold.

Theorem 3.1.3 (Theorem 3.4.20). *Let Q be a BG-manifold of dimension $2n$ where $n \geq 2$ and let $\Phi: Q \rightarrow \mathbb{P}^n$ be its algebraic reduction as in (3.1.2). Then there exists a divisor $D \subset \mathbb{P}^n$ such that for any point $x \in \mathbb{P}^n$ one has:*

- (1) *If $x \in \mathbb{P}^n \setminus D$, then the fiber $\Phi^{-1}(x)$ is an abelian variety.*
- (2) *If $x \in D$, then the fiber $\Phi^{-1}(x)$ is a uniruled Moishezon manifold.*

Moreover, if $X \subset Q$ is a submanifold such that $\dim(\Phi(X)) \geq 2$ then X is not Moishezon.

This theorem says nothing about subvarieties $X \subset Q$ such that $\Phi(X)$ is a curve. In this case X may or may not be Moishezon depending on the curve $\Phi(X) \subset \mathbb{P}^n$. See Theorem 3.4.20 for a more precise statement.

We next turn to the description of the groups of biholomorphic and bimeromorphic automorphisms of BG-manifolds Q . We denote these groups $\text{Aut}(Q)$ and $\text{Bim}(Q)$ respectively. Note that

a biholomorphic or bimeromorphic automorphism of Q factors through an algebraic reduction. Thus, if Q is a BG-manifold of dimension $2n$ then there is a map from the group $\text{Aut}(Q)$ (respectively, $\text{Bim}(Q)$) to $\text{Aut}(\mathbb{P}^n)$ (respectively, $\text{Bir}(\mathbb{P}^n)$).

We establish the Jordan property for the group of biholomorphic automorphisms of BG-manifold Q and give some restrictions for the group of bimeromorphic automorphisms of Q .

Theorem 3.1.4 (Proposition 3.5.12 and Theorem 3.5.14). *Assume that Q is a BG-manifold of dimension $2n$ and $\Phi: Q \rightarrow \mathbb{P}^n$ is the algebraic reduction of Q . A typical fiber of Φ is an abelian variety A .*

- (1) *The group $\text{Aut}(Q)$ is Jordan. More precisely, it fits into the following exact sequence:*

$$1 \rightarrow G' \rightarrow \text{Aut}(Q) \rightarrow \Delta_{\text{Aut}} \rightarrow 1,$$

where G' is a subgroup of $\text{Aut}(A)$ and Δ_{Aut} is a finite group.

- (2) *The group $\text{Bim}(Q)$ fits into the following exact sequence:*

$$1 \rightarrow G'' \rightarrow \text{Bim}(Q) \rightarrow \Delta \rightarrow 1,$$

where G'' is a subgroup of $\text{Aut}(A)$ and Δ is the subgroup of $\text{Bir}(\mathbb{P}^n)$. Moreover, if $D \subset \mathbb{P}^n$ is the divisor as in Theorem 3.1.3, then for any $g \in \Delta$ one has either $D \subset \text{Exc}(g)$ or $g(D) = D$.

Moreover, in Theorem 3.5.18 we prove that the group of bimeromorphic automorphisms of BG-manifold Q of dimension 4 is Jordan. We find out that the divisor D in this situation is a Halphen curve: a plane curve with interesting geometric properties studied in [Hal82]. Then using the description of the group of birational automorphisms of a plane which preserve a Halphen curve we get the result.

Our proof of the Jordan property for a BG-fourfold uses a deep results in the geometry of plane curves. We believe that in higher dimensions groups of bimeromorphic automorphisms of BG-manifolds should also be Jordan.

3.2. Preliminaries

In this section we recall important definitions and some basic facts on complex manifolds and their bimeromorphic geometry.

3.2.1. Holomorphic symplectic manifolds

We say that M a compact complex manifold if it admits a complex form I . The space of differential forms of degree n on a compact complex manifold M admits a decomposition as a direct sum of spaces of (p, q) -forms where $p + q = n$. The Dolbeault cohomology $H^{p,q}(M)$ is defined to be the quotient of $\bar{\partial}$ -closed forms modulo $\bar{\partial}$ -exact forms. This is a finite dimensional \mathbb{C} -vector space. We denote $h^{p,q}(M) = \dim_{\mathbb{C}}(H^{p,q}(M))$.

Definition 3.2.1. Assume that M is a smooth compact manifold with a Riemannian metric g . Then

- An *almost complex structure* on M is a vector bundle isomorphism $I: T_M \rightarrow T_M$ on the tangent bundle of M such that $I^2 = -\text{Id}_{T_M}$. A *complex structure* on M is an almost complex structure I which is integrable i.e. such that there exists a holomorphic atlas for M such that in coordinates of each chart I takes the canonical form. We say that M is a *compact complex manifold* if it admits a complex form.
- A $(1, 1)$ -form α on a compact complex manifold M with a complex form I is called *Kähler* if

$$\alpha(z_1, z_2) = g(z_1, I(z_2)).$$

In this case the the manifold M , the metric g and the complex structure I are called *Kähler manifold*, *Kähler metric* and *Kähler complex form* with respect to α .

- A $(2, 0)$ -form ω is *holomorphic symplectic* if it is non-degenerate i.e. $\omega^{\dim(M)}(p)$ has maximal rank at every point p in M . A compact complex manifold M is *holomorphic symplectic* if it can be equipped with a holomorphic symplectic form ω .

If M is a complex Kähler manifold then by the conjugacy one has $h^{p,q}(M) = h^{q,p}(M)$ and by Hodge theorem:

$$b_i(M) = \sum_{p+q=i} h^{p,q}(M) \tag{3.2.2}$$

where $b_i = \dim(\mathbb{C})(H^i(M, \mathbb{C}))$ is the *Betti* number of M . In case of non-Kähler complex compact manifolds in general $h^{p,q}(M) \neq h^{q,p}(M)$ and (3.2.2) is not true. The counterexamples for (3.2.2) can be found in [PSU20]. However, if M is a compact complex surface, then (3.2.2) is satisfied for all $0 \leq i \leq 4$ by [BHPVdV04, Theorem IV.2.10].

In dimension 2, we have 3 types of holomorphic symplectic surfaces: complex tori, K3 surfaces and Kodaira surfaces. The latter is of central importance for the present paper. We recall its main properties in Section 3.3.1. Holomorphic symplectic manifolds are highly connected to the following notion.

Definition 3.2.3. A *hyperkähler manifold* M is a compact smooth manifold with a Riemannian metric g equipped with three Kähler complex structures $I, J, K: T_M \rightarrow T_M$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\text{id}.$$

The metric g in this situation is called *hyperkähler metric*.

Since we do not assume our manifolds to be Kähler, in this paper we have to distinguish between the terms “hyperkähler” and “holomorphic symplectic”, which are often used as synonyms in the literature. Any hyperkähler manifold is automatically holomorphic symplectic. This is true because if α_J and α_K are Kähler forms on M with respect to the Riemannian metric g and to the complex structures J and K respectively, then

$$\omega = \alpha_J + \sqrt{-1}\alpha_K$$

is a holomorphic symplectic form on a hyperkähler manifold M with respect to the complex structure I . Now recall the fundamental result which proves the inverse assertion.

Theorem 3.2.4 ([Yau78], [Bea83, Proposition 4]). *Let M be a compact holomorphic symplectic Kähler manifold. Then M admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form.*

Therefore, every Kähler holomorphic symplectic manifold is actually hyperkähler.

Remark 3.2.5. A hyperkähler manifold M is called *irreducible holomorphic symplectic (IHS)* if $\pi_1(M) = 0$ and $H^{2,0}(M)$ is generated by one holomorphic symplectic form. IHS hyperkähler manifolds are building blocks of all compact hyperkähler manifolds: according to Beauville-Bogomolov decomposition theorem, any hyperkähler manifold admits a finite covering which is a product of a torus and several IHS manifolds. The IHS hyperkähler components of this decomposition are defined uniquely.

At the moment, there are only few available examples of simple hyperkähler manifolds: two sporadic examples of O’Grady in dimensions six and ten; the Hilbert schemes $Z^{[n]}$ of 0-dimensional closed subschemes of length n of a K3 surface Z ; and generalized *Kummer varieties*, i.e. the kernels of the composition

$$T^{[n]} \rightarrow \mathrm{Sym}^n T \xrightarrow{\Sigma} T,$$

where T is a complex torus and Σ is the sum morphism. As we will see in Section 3.3.2, the construction of BG-manifolds actually reminds that of generalized Kummer varieties (but starts with the Kodaira surface, rather than a complex torus).

3.2.2. Complex spaces and meromorphic maps

Our main references for this paragraph are [GPR94] and [Uen75]. All complex spaces are assumed to be reduced, irreducible and compact, if not stated otherwise. The term “variety” is reserved for algebraic ones (which differs from Ueno’s book).

Definition 3.2.6. Let X and Y be complex spaces.

- A *fiber space* is a proper surjective morphism $f : X \rightarrow Y$ with irreducible general fiber.
- A morphism $f : X \rightarrow Y$ is called a (*proper*) *modification*, if f is proper, surjective, and there exist closed analytic subsets $Z_X \subsetneq X$ and $Z_Y \subsetneq Y$ such that f induces an isomorphism $X \setminus Z_X \cong Y \setminus Z_Y$.
- A *meromorphic map* $f : X \dashrightarrow Y$ is a map $X \rightarrow 2^Y$ such that its graph Γ_f is an irreducible closed analytic subset of $X \times Y$, and the projection $p_X : \Gamma_f \rightarrow X$ is a proper modification. A meromorphic map is called *bimeromorphic* if the projection $p_Y : \Gamma_f \rightarrow Y$ is a modification as well.
- A *meromorphic fiber space* is a meromorphic map $f : X \dashrightarrow Y$ such that p_Y is a fiber space.

The group of all bimeromorphic maps $X \dashrightarrow X$ will be denoted $\mathrm{Bim}(X)$. By GAGA For a smooth complex projective variety X one has the group of bimeromorphic automorphisms $\mathrm{Bim}(X)$ of X is isomorphic to the group of birational automorphisms $\mathrm{Bir}(X)$ of X .

Given a complex space X , its field of meromorphic functions will be denoted $\mathcal{M}(X)$. This is a finitely generated extension over \mathbb{C} which satisfies

$$0 \leq \operatorname{tr} \deg_{\mathbb{C}} \mathcal{M}(X) \leq \dim(X), \quad (3.2.7)$$

see [Uen75, Theorem 3.1]. The integer number $a(X) = \operatorname{tr} \deg_{\mathbb{C}} \mathcal{M}(X)$ is called the *algebraic dimension* of X . Further, X is said to be a *Moishezon space* if $a(X) = \dim(X)$, see [Moi66].

Remark 3.2.8. Any bimeromorphic map $f : X \dashrightarrow Y$ induces an isomorphism of the fields of meromorphic functions

$$f^* : \mathcal{M}(Y) \xrightarrow{\cong} \mathcal{M}(X),$$

so $a(X)$ is a bimeromorphic invariant. Conversely suppose that X and Y are Moishezon spaces and we have a field isomorphism $\mathcal{M}(X) \cong \mathcal{M}(Y)$. Then X and Y are bimeromorphic by [GPR94, VII, Corollary 6.8].

Let us also mention the behavior of the algebraic dimension under some maps.

Lemma 3.2.9 ([Uen75, Theorem 3.8]). *Let $f : X \dashrightarrow Y$ be a surjective meromorphic map of irreducible compact complex spaces. Set*

$$a(f) = \inf_{y \in Y} a(f^{-1}(y)), \quad \dim f = \dim X - \dim Y.$$

Then one has

$$a(Y) \leq a(X) \leq a(Y) + a(f) \leq a(Y) + \dim f.$$

Definition 3.2.10. Assume that X is a compact complex space. Then an *algebraic reduction* of X is a meromorphic fiber space $f : X \dashrightarrow X_0$ to a projective variety X_0 , such that f induces an isomorphism $\mathcal{M}(X_0) \cong \mathcal{M}(X)$.

An algebraic reduction is unique up to bimeromorphic equivalence and we have $a(X) = \dim(X_0)$.

Example 3.2.11. The algebraic dimension of a Hopf surface can be equal 0 or 1. In case it equals 1 the algebraic reduction is an elliptic fiber space over \mathbb{P}^1 , see [BHPVdV04, V.18.5].

The algebraic dimension and reduction of a Kodaira surface are discussed in Section 3.3.1.

Example 3.2.12. If X is a surface of algebraic dimension 1, then its algebraic reduction is a holomorphic map, whose typical fiber is an elliptic curve [BHPVdV04, VI.5.1].

We shall need the following result in the future.

Lemma 3.2.13. *Let $f_X : X \rightarrow X_0$ and $f_Y : Y \rightarrow Y_0$ be algebraic reductions of compact complex manifolds X and Y . Then*

$$(f_X \times f_Y) : X \times Y \rightarrow X_0 \times Y_0$$

is an algebraic reduction of $X \times Y$. In particular, we have $a(X \times Y) = a(X) + a(Y)$.

Proof. It is clear that $f_X \times f_Y$ is a fiber space. Consider the map

$$(f_X \times \operatorname{id}_Y) : X \times Y \rightarrow X_0 \times Y.$$

Fix a generic point $y \in Y$ and restrict this map to a submanifold $X \times \{y\}$:

$$(f_X \times \text{id}_Y)|_{X \times \{y\}} : X \times \{y\} \rightarrow X_0 \times \{y\}. \quad (3.2.14)$$

Now take a meromorphic function f on $X \times Y$. The map (3.2.14) is an algebraic reduction, so the restriction of the function $f|_{X \times \{y\}}$ is constant on fibers of $(f_X \times \text{id}_Y)|_{X \times \{y\}}$. Then, the function f is constant on all fibers of the map $f_X \times \text{id}_Y$. Therefore, one can define a map

$$\varphi : \mathcal{M}(X \times Y) \rightarrow \mathcal{M}(X_0 \times Y), \quad \varphi(f)(x_0, y) = f(x, y), \quad \text{where } x \in f_X^{-1}(x_0),$$

which is easily checked to be an isomorphism. By repeating this argument for the map $\text{id}_{X_0} \times f_Y$ we get the following chain of isomorphisms:

$$\mathcal{M}(X \times Y) \cong \mathcal{M}(X_0 \times Y) \cong \mathcal{M}(X_0 \times Y_0).$$

Since $X_0 \times Y_0$ is a projective variety, it is an algebraic reduction of $X \times Y$. □

Also we will use the following properties of non-Moishezon manifolds:

Lemma 3.2.15. *Let X and Y be two complex manifolds. Then the following assertions hold:*

- (1) *if Y is not Moishezon and $f : X \rightarrow Y$ is a surjective morphism, then X is not Moishezon;*
- (2) *if X is not Moishezon and $f : X \rightarrow Y$ is a proper finite morphism, then Y is not Moishezon;*
- (3) *if X contains a non-Moishezon submanifold then X is not Moishezon.*

Proof. Let Y be not Moishezon, i.e. $\dim(Y) - a(Y) \geq 1$. Then by Lemma 3.2.9 we have

$$a(X) \leq a(Y) + \dim(X/Y) \leq \dim X - 1,$$

which implies that X is not Moishezon. This proves the assertion (1).

The assertion (2) follows from the definition of Moishezon manifolds.

The assertion (3) follows from [Uen75, Corollary 3.9]. □

3.3. Non-Kähler holomorphic symplectic manifolds

In this section we recall the construction of BG-manifolds introduced by Bogomolov in [Bog96]. Then we prove that an algebraic reduction of a BG-manifold is a fiber space over a projective space.

3.3.1. Kodaira surfaces

The known examples of non-Kähler irreducible holomorphic symplectic manifolds are constructed from Kodaira surfaces, so we first recall main properties of the latter ones. Most of the statements in this paragraph can be found in Kodaira's original paper [Kod64].

A *Kodaira surface* is a compact complex surface of Kodaira dimension 0 with $b_1(S) = 3$. As $b_1(S)$ is odd then S cannot be Kähler. There are two kinds of Kodaira surfaces: primary and secondary ones. Every secondary surface is obtained as a quotient of a primary Kodaira surface S by a finite cyclic group acting freely on S . Any primary Kodaira surface S can be constructed as follows. Let E be an elliptic curve and \mathcal{L} be a line bundle on E with the first Chern class $c_1(\mathcal{L}) \neq 0$. Denote by \mathcal{L}^* the complement of the zero section in the total space of \mathcal{L} . Fix $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, and let $g_\lambda : \mathcal{L}^* \rightarrow \mathcal{L}^*$ be the corresponding homothety. Then the quotient space $S = \mathcal{L}^*/\mathbb{Z}\langle g_\lambda \rangle$ is a Kodaira surface. The projection from the total space of \mathcal{L} to E induces the following map:

$$\pi : S \rightarrow E. \tag{3.3.1}$$

All fibers of π are isomorphic to the smooth curve $F = \mathbb{C}^\times / \langle \lambda \rangle$ of genus 1. Thus, π carries a structure of a principal elliptic fibration though it does not admit a section. This map is an algebraic reduction of the Kodaira surface S , see [BHPVdV04, Theorem VI.1.1].

Remark 3.3.2. The standard fiberwise action of the circle \mathbb{S}^1 on \mathcal{L}^* induces on the Kodaira surface S the structure of a principal \mathbb{S}^1 -fibration over the torus $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$.

One can show that S has the following invariants [BHPVdV04, Table 10]:

$$K_S \sim 0, \quad a(S) = 1, \quad b_1(S) = 3, \quad b_2(S) = 4, \quad \chi(S) = 0, \quad h^{0,1}(S) = 2, \quad h^{0,2}(S) = 1.$$

Lemma 3.3.3. *Assume that $f : M \dashrightarrow S$ is a bimeromorphic map from a smooth surface M to a Kodaira surface S . Then M is non-Kähler.*

Proof. Indeed by [BHPVdV04, III.4.2] the map f is a σ -process. Since by [BHPVdV04, I.9.1] the σ -process does not change the Betti number b_1 , we get $b_1(M) = b_1(S) = 3$. Thus, M is non-Kähler. \square

3.3.2. Bogomolov–Guan example

Let us recall the first example of simply-connected non-Kähler compact complex manifolds, which we call *BG-manifolds* as in [KV19]. This example was introduced by D. Guan in [Gua94] and further studied in [Bog96]. Here we mainly follow Bogomolov’s work.

Let M be a compact complex manifold, then there is a natural action of the permutation group \mathfrak{S}_n on the M^n . Set $\text{Sym}^n(M) = M^n / \mathfrak{S}_n$ and let $M^{[n]}$ be the Douady space of length n of M . Then $M^{[n]}$ is a smooth complex manifold and there is a canonical holomorphic map:

$$\delta : M^{[n]} \rightarrow \text{Sym}^n(M).$$

This map is called *Douady–Barlet morphism*. Note that in general situation the manifold $\text{Sym}^n(M)$ is singular; however, if M is a smooth complex curve, then $\text{Sym}^n(M)$ is also smooth and the Douady–Barlet morphism is identity.

Consider a Kodaira surface S ; as we discussed in Section 3.3.1 there is the structure of a principal elliptic fibration $\pi : S \rightarrow E$ with fiber F on S . Choose a point $0 \in E$ and consider the morphism:

$$\text{Alb} : \text{Sym}^n E \rightarrow E, \quad x = \{x_1, \dots, x_n\} \mapsto x_1 + \dots + x_n.$$

It is the Albanese map of the manifold $\text{Sym}^n E$, fibers of the map Alb are projective spaces.

Denote by $S^{[n]}$ the Douady space of length n zero-dimensional analytic subspaces of S . Then we have the induced projection:

$$\pi^{[n]} = (\text{Sym}^n(\pi) \circ \delta) : S^{[n]} \rightarrow \text{Sym}^n E.$$

Here δ is the Douady–Barlet morphism $\delta : S^{[n]} \rightarrow \text{Sym}^n S$, then we can consider the following composition:

$$\pi_n = (\text{Alb} \circ \pi^{[n]}) : S^{[n]} \rightarrow E.$$

Denote by W the fiber of π_n over 0:

$$W = \pi_n^{-1}(0) \xrightarrow{\pi^{[n]}} \text{Alb}^{-1}(0).$$

The fiber of the Albanese map Alb containing a point $D \in \text{Sym}^n E$ (e.g. $D = n(0)$) is the projective space isomorphic to the linear system $|D| = \mathbb{P}H^0(\mathcal{O}_E(D))$, i.e. $\text{Alb}^{-1}(0) \cong \mathbb{P}^{n-1}$ by Riemann-Roch formula.

Since the fibration π is principle the action of F on S induces a fiberwise diagonal action of F on $S^{[n]}$, and, therefore, on the fiber W . Thus, we have a projection

$$\Pi : W/F \rightarrow \mathbb{P}^{n-1}.$$

One can understand the geometry of fibers of the map Π .

Lemma 3.3.4. *Let Π be a map as in (3.3.2). A general fiber of Π is isomorphic to F^n/F .*

A more precise description of fibers of Π is given in Lemma 3.4.17.

Proof. Let $\alpha_E : E^n \rightarrow \text{Sym}^n(E)$ be the map of symmetrization. Let $\bar{x} = \{x_1, \dots, x_n\}$ be a point on the symmetric space $\text{Sym}^n(E)$ such that $\text{Alb}(\bar{x}) = 0$ and $x_i \neq x_j$ for all $1 \leq i < j \leq n$. By construction the group \mathfrak{S}_n acts faithfully on the set $\alpha_E^{-1}(\bar{x})$. Thus, it acts faithfully on the set of fibers of π^n over $\alpha_E^{-1}(\bar{x})$.

Thus, the fiber of $\text{Sym}^n(\pi)$ over the point \bar{x} is isomorphic to F^n . Moreover, this fiber does not intersect the indeterminacy locus of the Douady–Barlet map δ . Thus, we get the result. \square

Finally, the BG-manifold is constructed as an appropriate finite cover of W/F .

Theorem 3.3.5 ([Bog96, Section 3]). *Assume that E is an elliptic curve, \mathcal{L} is a line bundle over E with the Chern class $c_1(\mathcal{L}) \neq 0$ and $\pi : S \rightarrow E$ is a Kodaira surface associated with \mathcal{L} . For any integer $n \geq 3$ dividing $c_1(\mathcal{L})$ there exists a simply connected non-Kähler compact complex smooth holomorphic symplectic manifold Q of dimension $2n - 2$ and a finite morphism of degree n^2 :*

$$p : Q \rightarrow W/F.$$

The following diagram summarizes the construction of Q .

$$\begin{array}{ccccccc}
Q & \xrightarrow{p} & W/F & \xleftarrow{q} & W \hookrightarrow & S^{[n]} & \xrightarrow{\delta} & \text{Sym}^n(S) & \xleftarrow{\alpha} & S \times \dots \times S \\
& \searrow & \Phi & \searrow & \Pi & \downarrow \pi_0 & \downarrow \pi^{[n]} & \downarrow \text{Sym}^n(\pi) & \downarrow \pi \times \dots \times \pi & \\
& & & & & \mathbb{P}^{n-1} \hookrightarrow & E^{[n]} & \xrightarrow{\cong} & \text{Sym}^n(E) & \xleftarrow{\alpha_E} & E \times \dots \times E \\
& & & & & \downarrow & \downarrow \text{Alb} & & & & \\
& & & & & 0 \hookrightarrow & E & & & &
\end{array} \tag{3.3.6}$$

Remark 3.3.7. Guan actually proposed two different methods to build non-Kähler compact holomorphic symplectic manifold.

- (i) Consider a compact holomorphic symplectic manifold M with the Albanese map $\text{alb}: M \rightarrow A$. Let V be a subspace of the inverse images of holomorphic 1-forms from A such that the holomorphic symplectic structure vanishes on them. Let G be a Lie group of holomorphic vector field dual to V . Fix a point x on A . Then a special smooth cover of $\text{alb}^{-1}(x)/G$ is a compact holomorphic symplectic manifold.
- (ii) Given a holomorphic symplectic nilmanifold \mathcal{N} corresponding to a Lie group G , one can try to find a faithful representation of a finite subgroup S of $\text{Aut}(G)$. Suppose that this representation preserves the complex structure as well as the symplectic structure and all irreducible components of the set

$$D = \{ n \in \mathcal{N} : s(n) = n \text{ for some } s \in S \}$$

are of codimension 2 and $\pi_1(\mathcal{N})^S = 1$. Then a desingularization of \mathcal{N}/S might be a simply-connected non-Kähler holomorphic symplectic manifold.

The first method was used by Guan in [Gua95a] for the Hilbert scheme $S^{[n]}$ of length n of any Kodaira surface S . The cover of the orbifold $\text{alb}^{-1}(x)/G$ was obtained using its isomorphism to a quotient of a nilmanifold by a some group. This method is close to the Bogomolov's one described above.

The second method was applied in [Gua95b]. The author started with a concrete nilmanifold $M_{n,t}$ given by structure equations, where $n \in \mathbb{N}, t \in \mathbb{Z}$. Then he obtained a deformation family of simply-connected holomorphic symplectic manifolds $Q_{n+1,t}$ which are non-Kähler of dimension $2n - 2$ parametrized by an integer parameter t .

In particular, if we fix an elliptic curve E , a line bundle \mathcal{L} such that $c_1(\mathcal{L}) = t \neq 0$ and construct the BG-manifold Q of dimension $2n$ as in Theorem 3.3.5 then Q is isomorphic to the manifold $Q_{n,t}$, see [Gua95b, Theorem 1].

Example 3.3.8. Consider the manifold constructed as in Theorem 3.3.5, but in the case $n = 2$. Then the manifold W/F , where $W := (\text{Sym}^2 \pi)^{-1}(0)$, is the fiber of a Kähler torus of dimension 2 (see Lemma 3.4.11) and the induced action of \mathfrak{S}_2 is an involution on this torus. Therefore, W/F is bimeromorphic to a Kummer K3 and Q is its cover of degree 4. By construction Q is smooth simply-connected Kähler holomorphic symplectic manifold; thus, it should be also a K3 surface.

Proposition 3.3.9. *Assume that Q is the BG-manifold of dimension $2n - 2$ and Φ is a composition of the finite cover p and the map $\Pi: W/F \rightarrow \mathbb{P}^{n-1}$. Then a generic fiber of $\Phi: Q \rightarrow \mathbb{P}^{n-1}$ is an abelian variety; in particular, it is connected.*

Proof. Recall some details of the construction of the finite cover $p: Q \rightarrow W/F$. Fix the Kodaira surface S associated with a line bundle \mathcal{L} over an elliptic curve E . There is an action of the group \mathbb{C}^* on the complement \mathcal{L}^* of the zero section in the total space of \mathcal{L} . Consider the Kodaira surface S as a real manifold of dimension 4 and fix a subgroup $\mathbb{S}^1 \subset \mathbb{C}^*$. The action of \mathbb{S}^1 action on \mathcal{L}^* induces a free action on the Kodaira surface S . This action gives us a structure of a fiber space:

$$\psi: S \rightarrow T,$$

where $T = S/\mathbb{S}^1$ is a compact real 3-dimensional torus. This map induces a map from the Douady space of S to the symmetric space of T :

$$S^{[n]} \xrightarrow{\text{Sym}^n(\psi) \circ \delta} \text{Sym}^n T.$$

Here $\text{Sym}^n T$ is the quotient of the scheme $T \times T \times \cdots \times T$ by the action of the symmetric group \mathfrak{S}_n . Moreover, since T has a structure of an abelian group one can construct the following map:

$$\text{Sym}^n T \xrightarrow{\text{Alb}_T} T.$$

Here Alb_T maps a set of n points in T to their sum. By [Bog96, Proposition 2.1] the map Alb_T is well-defined and fibers of the composition $\text{Alb}_T \circ \text{Sym}^n(\psi): S^{[n]} \rightarrow T$ are smooth and irreducible. Denote by M the fiber of the map $\text{Alb}_T \circ \text{Sym}^n(\psi)$ over $0 \in T$. Then there is an action of the group \mathbb{S}^1 on M induced from the diagonal action of \mathbb{S}^1 on $\text{Sym}^n T$ and M/\mathbb{S}^1 is a real manifold of dimension $4n - 4$. Denote by Π_T the induced map from M/\mathbb{S}^1 to $\text{Alb}_T^{-1}(0)$. Also the map $T \rightarrow E$ induces the map r_T from $\text{Alb}_T^{-1}(0)$ to $\text{Alb}^{-1}(0) = \mathbb{P}^{n-1}$.

By [Bog96, Lemma 3.8] there are finite maps of real manifolds $\tau: Q \rightarrow M/\mathbb{S}^1$ and $r: M/\mathbb{S}^1 \rightarrow W/F$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & p & & \\ & & \curvearrowright & & \\ Q & \xrightarrow{\tau} & M/\mathbb{S}^1 & \xrightarrow{r} & W/F \\ & & \downarrow \Pi_T & \searrow \Psi & \downarrow \Pi \\ & & \text{Alb}_T^{-1}(0) & \xrightarrow{r_T} & \mathbb{P}^{n-1} \end{array}$$

Moreover, by [Bog96, Lemma 3.8] the degrees of both maps τ and r are equal to n .

Denote by Ψ the composition of maps r and Π . Then by construction, a general fiber of Π_T is isomorphic to $(\mathbb{S}^1)^n/\mathbb{S}^1$; in particular, it is connected. Moreover, the fibers of the map r_T are also connected. Thus, we get that the map Ψ has connected fibers.

Now we fix a general point $x \in \mathbb{P}^{n-1}$ and consider fibers Q_x and $(M/\mathbb{S}^1)_x$ of maps $\Phi = \Pi \circ p$ and Ψ . By [Bog96, Remark 3.5] we see that the generator of the fundamental group $\pi_1(M/\mathbb{S}^1)$ is induced by an element in $\pi_1((M/\mathbb{S}^1)_x)$. Moreover, by [Bog96, Corollary 3.7] we see

$$\pi_1(Q_x) \subset \pi_1((M/\mathbb{S}^1)_x)$$

and the index of the subgroup equals n . Since the degree of the map $\tau|_{Q_x}: Q_x \rightarrow (M/\mathbb{S}^1)_x$ equals n we deduce that Q_x is a connected finite unramified cover of $(M/\mathbb{S}^1)_x$. Therefore, fibers of the map Φ are connected.

Moreover, by [Bog96, Remark 3.9] we see that the connected manifold Q_x is a finite unramified cover of a generic fiber of Π which is isomorphic to an abelian variety F^n/F by Lemma 3.3.4. By the Serre-Lang theorem [Mum74, Chapter IV], we conclude that Q_x is an abelian variety as well. \square

3.3.3. The algebraic reduction of BG-manifolds

The goal of this section is to prove that an algebraic reduction of a BG-manifold Q is the map Φ described in Section 3.3.

We consider a Kodaira surface S as in (3.3.1) and denote by F the fiber of π . Denote by π^n the power of map π and by ϵ the map which sends a point on E^n to the sum of its components in E :

$$\begin{aligned}\pi^n: S^n &\rightarrow E^n; & \pi^n(s_1, \dots, s_n) &= (\pi(s_1), \dots, \pi(s_n)); \\ \epsilon: E^n &\rightarrow E; & \epsilon(x_1, \dots, x_n) &= x_1 + \dots + x_n.\end{aligned}$$

Denote by A the kernel of the map ϵ . Consider the preimage $(\pi^n)^{-1}(A)$ of A in S^n ; there is a diagonal action of F on $(\pi^n)^{-1}(A)$. Denote by X the quotient space $(\pi^n)^{-1}(A)/F$ and by η the map from X to A :

$$\begin{array}{ccc} X = (\pi^n)^{-1}(A)/F & \hookrightarrow & S^n/F \\ \eta \downarrow & & \downarrow \pi_F^n \\ A = \epsilon^{-1}(0) & \hookrightarrow & E^n \end{array}$$

Denote the general fiber of η by B . By construction the variety A is isomorphic to E^{n-1} and B is isomorphic to F^n/F . The action of F^n on S^n induces the action of B on X and, therefore, on $\mathcal{M}(X)$.

There is a natural action of the symmetric group \mathfrak{S}_n on X and the map $\delta^{-1} \circ \alpha: S^n \dashrightarrow S^{[n]}$ induces the following bimeromorphic map:

$$X/\mathfrak{S}_n \dashrightarrow W/F$$

Thus, our plan of the proof of Theorem 3.1.1 is first to show that an algebraic reduction of X is the map η to the abelian variety A . As a consequence we will get that an algebraic reduction of the manifold W/F is the map Π . Then using properties of the map from the BG-manifold Q to W/F we deduce the result.

Lemma 3.3.10. *Let B_0 be the maximal subgroup of B such that:*

$$\mathcal{M}(X)^{B_0} = \mathcal{M}(X).$$

Then either $B_0 \neq 0$ or $\text{tr deg}_{\mathcal{M}(A)}(\mathcal{M}(X)) = n - 1$.

Proof. Assume that $B_0 = 0$ and $\text{tr deg}_{\mathcal{M}(A)}(\mathcal{M}(X)) < n - 1$. Consider the module \mathcal{D} of $\mathcal{M}(A)$ -linear derivations on $\mathcal{M}(X)$:

$$\mathcal{D} = \text{Der}_{\mathcal{M}(A)}(\mathcal{M}(X)). \tag{3.3.11}$$

By Noether normalization lemma, $\dim_{\mathcal{M}(A)}(\mathcal{D}) = \text{tr deg}_{\mathcal{M}(A)}(\mathcal{M}(X)) < n - 1$.

One can construct $n - 1$ derivations on $\mathcal{M}(X)$. Namely, any 1-parameter subgroup $\Gamma = \{\gamma_t\}$ in B defines a derivation in \mathcal{D} . The dimension of the abelian variety B equals $n - 1$; denote its generators by $\gamma_1, \dots, \gamma_{n-1}$. To each generator γ_i one can associate a 1-parameter subgroup in B :

$$\Gamma_i = \langle t \cdot \gamma_i \rangle.$$

Denote by D_1, \dots, D_{n-1} derivations of the $\mathcal{M}(A)$ -module $\mathcal{M}(X)$ corresponding to 1-parameter subgroups $\Gamma_1, \dots, \Gamma_{n-1}$.

By our assumption derivations D_1, \dots, D_{n-1} are linearly dependent over the field $\mathcal{M}(A)$. Thus, there exist meromorphic functions f_1, \dots, f_{n-1} in $\mathcal{M}(A)$ such that

$$\sum_{i=1}^{n-1} f_i D_i = 0. \tag{3.3.12}$$

By construction of D_i for any function $g \in \mathcal{M}(X)$ and any $a \in A$ such that the fiber $\eta^{-1}(a)$ is not a pole of g we have:

$$(D_i(g))|_{\eta^{-1}(a)} = D_i(g|_{\eta^{-1}(a)}).$$

Denote by \mathcal{U} the following open subset in A :

$$\mathcal{U} = \left\{ a \in A \left| \begin{array}{l} a \text{ is not a pole of } f_i \quad \forall i \in \{1, \dots, n-1\} \\ f_j(a) \neq 0 \quad \exists j \in \{1, \dots, n-1\} \end{array} \right. \right\}$$

By construction for any $a \in \mathcal{U}$ and any $g \in \mathcal{M}(X)$ such that $\eta^{-1}(a)$ is not a pole of g we have

$$\sum_{i=1}^{n-1} f_i(a) D_i(g|_{\eta^{-1}(a)}) = \left(\sum_{i=1}^{n-1} (f_i D_i)(g) \right) \Big|_{\eta^{-1}(a)} = 0.$$

Fix an element $a \in \mathcal{U}$ and consider the following element of the group B :

$$\gamma_a = \sum_{i=1}^{n-1} f_i(a) \gamma_i.$$

Denote by D_a the derivation which corresponds to the 1-parameter group $\Gamma_a = \langle t \cdot \gamma_a \rangle$. By construction $D_a(g|_{\eta^{-1}(a)})$ vanishes. Then we get $\gamma_a \cdot g|_{\eta^{-1}(a)} = g|_{\eta^{-1}(a)}$.

Denote by B'_a a minimal abelian subvariety of B which contains Γ . Since Γ is Zariski dense subset of B'_a we get that for each $b \in B'_a$

$$b \cdot g|_{\eta^{-1}(a)} = g|_{\eta^{-1}(a)}. \tag{3.3.13}$$

For each $a \in \mathcal{U}$ we have constructed an abelian subvariety B'_a with the property (3.3.13). Since a choice of an abelian subvariety is a choice of a sublattice, i.e. of discrete data, all these subgroups coincide, denote by B' this subgroup of B . By construction the subgroup B' is non-zero and $\mathcal{M}(X)^{B'} = \mathcal{M}(X)$. Thus, we get a contradiction. \square

Corollary 3.3.14. *If B_0 is the maximal subgroup of B such that $\mathcal{M}(X)^{B_0} = \mathcal{M}(X)$ then $B_0 \neq 0$.*

Proof. Assume that $B_0 = 0$, then by Lemma 3.3.10 we get that X is a Moishezon manifold. This implies in particular that X carries an algebraic space structure by [Art70]. Therefore, X does not contain a projective line since any map from \mathbb{P}^1 to the abelian varieties A and B is constant. Therefore, [BCHM10, Corollary, 1.4.6] implies that X is projective and, consequently, Kähler.

On the other hand, consider the following embedding of the curve E to the abelian variety A :

$$\iota: E \hookrightarrow A \subset E^n; \quad \iota(x) = (-x, x, 0, \dots, 0).$$

Denote $(\pi^n)^{-1}(E)/F$ by X_E . The manifold X_E is isomorphic to $S \times_E S \times F^{n-3}$. Then X_E maps surjectively to a Kodaira surface S by the projection to the first component. Since S is non-Kähler by [Fuj83, Remark 1.1] so is X_E . \square

The map η induces the map between fields of meromorphic functions of A and X . Now we can show that this map is an isomorphism.

Proposition 3.3.15. *The map $\eta: X \rightarrow A$ is an algebraic reduction of X .*

Proof. By construction the abelian variety B is the quotient of the variety F^n by the diagonal action of F ; denote by q_F the map from F^n to B :

$$q_F: F^n \rightarrow F^n/F = B$$

There is the natural action of the symmetric group \mathfrak{S}_n on S^n and, therefore, on F^n . The map q_F induces the action of \mathfrak{S}_n on B . Denote by B_0 the maximal subgroup of B such that

$$\mathcal{M}(X)^{B_0} = \mathcal{M}(X).$$

The action of \mathfrak{S}_n on S^n preserves X . Moreover, by construction actions of B_0 and of \mathfrak{S}_n on X commute. Thus, B_0 is an \mathfrak{S}_n -invariant abelian subvariety in B . Consider the following subvariety of F^n :

$$\tilde{B}_0 = q_F^{-1}(B_0).$$

Then \tilde{B}_0 is an \mathfrak{S}_n -invariant abelian subvariety in F^n . Consider the embedding of the tangent subspace of \tilde{B}_0 :

$$T_{\tilde{B}_0,0} \hookrightarrow T_{F^n,0}.$$

If \tilde{B}_0 is \mathfrak{S}_n -invariant, then the same is true for $T_{\tilde{B}_0,0}$.

There are only two irreducible subrepresentations of the representation $T_{F^n,0} \cong \mathbb{C}^n$ of the group \mathfrak{S}_n . Namely, these are a trivial subrepresentation of \mathbb{C} and a standard $(n-1)$ -dimensional subrepresentation Δ . The embedding $\iota_{\text{diag}}: \mathbb{C} \rightarrow \mathbb{C}^n$ is diagonal; i.e. $\iota_{\text{diag}}(z) = (z, \dots, z)$. The image of the embedding $\iota_\Delta: \Delta \rightarrow \mathbb{C}^n$ is the hyperplane $\{\sum_{i=1}^n z_i = 0\}$ where z_i are standard coordinates of \mathbb{C}^n .

If $T_{\tilde{B}_0,0} = \iota_{\text{diag}}(\mathbb{C})$, then $B_0 = 0$. This is impossible by Corollary 3.3.14.

Thus, $T_{\tilde{B}_0,0} = \iota_\Delta(\Delta)$ and then $B_0 = B$. This implies that B acts transitively on fibers of η and all meromorphic functions on X are constant on fibers of η . Therefore, $\mathcal{M}(X)$ is isomorphic to $\mathcal{M}(A)$. \square

Proposition 3.3.16. *The map $\Pi: W/F \rightarrow \mathbb{P}^{n-1}$ is an algebraic reduction.*

Proof. The manifold W/F is a resolution of singularities of X/\mathfrak{S}_n . Then Proposition 3.3.15 implies

$$\mathcal{M}(W/F) \cong \mathcal{M}(X)^{\mathfrak{S}_n} \cong \mathcal{M}(A)^{\mathfrak{S}_n}$$

Since by Abel theorem $A^{\mathfrak{S}_n} = \mathbb{P}^{n-1}$, we get the result. \square

Corollary 3.3.17. *The map $\Phi: Q \rightarrow \mathbb{P}^{n-1}$ is an algebraic reduction, and $\mathcal{M}(Q) = \mathcal{M}(W/F)$.*

Proof. Since Q is a finite cover of W/F their algebraic dimensions are equal

$$a(W/F) = a(Q).$$

Denote by $f: Q \rightarrow Q_0$ an algebraic reduction of Q ; by definition of the algebraic reduction there exists a finite rational map:

$$p_0: Q_0 \rightarrow \mathbb{P}^{n-1}$$

By Lemma 3.3.9 for a dense set of points $x \in \mathbb{P}^{n-1}$ we know that the fiber $\Phi^{-1}(x) = (p_0 \circ f)^{-1}(x)$ is a connected manifold. Therefore, p_0 is a map of degree 1 and consequently Φ is an algebraic reduction of Q . \square

3.4. Submanifolds in BG-manifolds

In this section we are going to give a partial description of subvarieties of BG-manifolds. We give a construction of manifolds which arises as submanifolds in the quotient space S^n/F and show in which situations they are not Moishezon and not Kähler. This lead us to the partial description of submanifolds of S^n/F and using it we deduce properties of submanifolds of a BG-manifold Q .

3.4.1. Kodaira-type manifolds

We are going to present a construction of a big class of submanifolds in S^n/F . Frequently this construction leads to a non-Moishezon manifold.

Fix a complex number $q \in \mathbb{C}^*$ with $|q| > 1$, and denote by F the elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$.

Definition 3.4.1. Consider a manifold X and a line bundle \mathcal{L} on X . The group $q^{\mathbb{Z}}$ acts on the complement \mathcal{L}^* to the zero section in the total space of line bundle \mathcal{L} . Then we call

$$M_{\mathcal{L}} = \mathcal{L}^*/q^{\mathbb{Z}}.$$

a *Kodaira-type manifold with the base X* .

Remark 3.4.2. L. Ornea and M. Verbitsky describe a very similar construction of manifolds in [OV05, Section 1.4]. They called these manifolds quasiregular Vaisman manifolds. However, here we will stick to the terminology we used in [BKKY21].

By construction there is a canonical projection from $M_{\mathcal{L}}$ to X :

$$\pi_{\mathcal{L}}: M_{\mathcal{L}} \rightarrow X.$$

Note that if X is an elliptic curve and $c_1(\mathcal{L}) \neq 0$, then the Kodaira-type manifold $M_{\mathcal{L}}$ is a (primary) Kodaira surface (see Subsection 3.3.1). More generally, if X is a smooth curve then the Kodaira-type manifold over X is non-Kähler:

Lemma 3.4.3. *Let C be a smooth curve and \mathcal{L} be a line bundle on C with $c_1(\mathcal{L}) \neq 0$. Then $M_{\mathcal{L}}$ is neither Kähler nor Moishezon.*

Proof. The Enriques-Kodaira classification implies then that the surface $M_{\mathcal{L}}$ is neither Kähler nor Moishezon. \square

Given a complex manifold X , consider two line bundles $\mathcal{L}_1, \mathcal{L}_2$ on X , a fixed complex number q in \mathbb{C}^* and the fibered product

$$Y = M_{\mathcal{L}_1} \times_X M_{\mathcal{L}_2}.$$

There is a standard diagonal action of the elliptic curve $F = \mathbb{C}^*/q^{\mathbb{Z}}$ on Y . If we consider the quotient space under this action, we will get a Kodaira-type manifold.

Lemma 3.4.4. *Assume that X is a manifold, \mathcal{L}_1 and \mathcal{L}_2 are line bundles on X . The quotient of the product $M_{\mathcal{L}_1} \times_X M_{\mathcal{L}_2}$ by the diagonal action of $F = \mathbb{C}^* / q^{\mathbb{Z}}$ is a smooth manifold which is biholomorphic to $M_{\mathcal{L}_1 \otimes \mathcal{L}_2^\vee}$.*

Proof. We construct a map from Y/F to $M_{\mathcal{L}_1 \otimes \mathcal{L}_2^\vee}$, starting with an auxiliary map

$$\begin{aligned} \tilde{\Theta}: Y &= (\mathcal{L}_1^*/q^{\mathbb{Z}}) \times_X (\mathcal{L}_2^*/q^{\mathbb{Z}}) \rightarrow (\mathcal{L}_1 \otimes \mathcal{L}_2^\vee)^* / q^{\mathbb{Z}}; \\ \tilde{\Theta}(p, l_1, l_2) &= l_1 \otimes l_2^\vee. \end{aligned}$$

Here $p \in X$ is a point, l_1 and l_2 are points on the fibers $(\mathcal{L}_1^*)_p$ and $(\mathcal{L}_2^*)_p$ modulo action of the group $q^{\mathbb{Z}}$ and by l_2^\vee we denote the unique linear functional on $(\mathcal{L}_2)_p$ which maps l_2 to 1. The map $\tilde{\Theta}$ is well-defined. If we change l_1 and l_2 by another representatives $q^{d_1} \cdot l_1$ and $q^{d_2} \cdot l_2$ in fibers $(\mathcal{L}_1^*)_p$ and $(\mathcal{L}_2^*)_p$ of the quotient-sets $(\mathcal{L}_1^*)_p/q^{\mathbb{Z}}$ and $(\mathcal{L}_2^*)_p/q^{\mathbb{Z}}$, then

$$\tilde{\Theta}(p, q^{d_1} \cdot l_1, q^{d_2} \cdot l_2) = q^{d_1 - d_2} \cdot (l_1 \otimes l_2^\vee) = l_1 \otimes l_2^\vee.$$

The last equality holds, since we consider $l_1 \otimes l_2^\vee$ as an element in the quotient-set $(\mathcal{L}_1 \otimes \mathcal{L}_2^\vee)^*_p / q^{\mathbb{Z}}$.

Moreover, the map $\tilde{\Theta}$ is invariant by the action of F on Y . Since the action of $q^{\mathbb{Z}}$ commutes with the map $\tilde{\Theta}$, it suffices to show that for any representative $\lambda \in \mathbb{C}$ of an element of F the action of λ commutes with $\tilde{\Theta}$:

$$\tilde{\Theta}(\lambda \cdot (p, l_1, l_2)) = \tilde{\Theta}(p, \lambda \cdot l_1, \lambda \cdot l_2) = \lambda \cdot l_1 \otimes (\lambda \cdot l_2)^\vee = \lambda \cdot l_1 \otimes \lambda^{-1} \cdot l_2^\vee = l_1 \otimes l_2^\vee.$$

Therefore, the map $\tilde{\Theta}$ factors through the quotient-manifold Y/F and we get the following map:

$$\Theta: Y/F \rightarrow M_{\mathcal{L}_1 \otimes \mathcal{L}_2^\vee}.$$

Note that any element of $(\mathcal{L}_1 \otimes \mathcal{L}_2^\vee)_p$ can be written as a decomposable tensor $l_1 \otimes l_2^\vee$. Then we can define the map

$$\Theta': M_{\mathcal{L}_1 \otimes \mathcal{L}_2^\vee} \rightarrow Y/F, \quad \Theta'(l_1 \otimes l_2^\vee) = (p, l_1, l_2).$$

As above, we can easily see that this is a well-defined map and that $\Theta' \circ \Theta$ and $\Theta \circ \Theta'$ are identity maps of $M_{\mathcal{L}_1 \otimes \mathcal{L}_2^\vee}$ and Y/F . Thus, Θ is a biholomorphism. \square

Remark 3.4.5. Lemma 3.4.4 implies in particular that $M_{\mathcal{L}} \cong M_{\mathcal{L}^{-1}}$.

We can generalize Lemma 3.4.4 to the fibered product of n Kodaira-type manifolds.

Lemma 3.4.6. *If $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ are line bundles on X , then we have the following isomorphism:*

$$(M_{\mathcal{L}_1} \times_X M_{\mathcal{L}_2} \times_X \dots \times_X M_{\mathcal{L}_n}) / F \cong M_{\mathcal{L}_2 \otimes \mathcal{L}_1^\vee} \times_X \dots \times_X M_{\mathcal{L}_{n-1} \otimes \mathcal{L}_1^\vee} \times_X M_{\mathcal{L}_n \otimes \mathcal{L}_1^\vee}.$$

Here we consider the diagonal action of $F = \mathbb{C}^* / q^{\mathbb{Z}}$ on $M_{\mathcal{L}_1} \times_X M_{\mathcal{L}_2} \times_X \dots \times_X M_{\mathcal{L}_n}$.

Proof. Denote $M_{\mathcal{L}_i}$ by M_i and consider the following map:

$$\begin{aligned} \Upsilon: (M_1 \times_X \dots \times_X M_n) / F &\rightarrow (M_2 \times_X M_1) / F \times_X \dots \times_X (M_n \times_X M_1) / F; \\ \Upsilon(x, m_1, \dots, m_n) &= ((x, m_2, m_1), \dots, (x, m_n, m_1)). \end{aligned}$$

If we fix a point $((x, m_2, m_{12}), \dots, (x, m_n, m_{1n})) \in (M_2 \times_X M_1) / F \times_X \dots \times_X (M_n \times_X M_1) / F$ then there exists a unique set of elements f_3, \dots, f_n in F such that $f_i \cdot m_{i1} = m_{21}$ for all $i = 3, \dots, n$. Thus, we can define the following map:

$$\Psi((x, m_2, m_{12}), \dots, (x, m_n, m_{1n})) = (x, m_{12}, m_2, f_3 \cdot m_3, \dots, f_n \cdot m_n).$$

One can deduce that $\Psi \circ \Upsilon = \text{Id}$ hence Υ is a biholomorphism. We conclude observing that by Lemma 3.4.4 one has $(M_i \times_X M_1) / F = M_{\mathcal{L}_i \otimes \mathcal{L}_1^\vee}$. \square

3.4.2. Submanifolds in S^n / F

Recall that a Kodaira surface S is a principal elliptic bundle $\pi: S \rightarrow E$ with typical fiber F . Consider the diagonal action of F on S^n . In this section we give a description of submanifolds in the quotient space S^n / F .

The power π^n of the map π induces the following morphism

$$\pi_F^n: S^n / F \rightarrow E^n.$$

Denote by $p_i: E^n \rightarrow E$ and $P_i: S^n \rightarrow S$ the projections from the products E^n and S^n to their i -th components. The goal of this section is to prove the following assertion.

Proposition 3.4.7. *Let X be a submanifold of S^n / F , write $Z = \pi_F^n(X) \subset E^n$. Then we are in one of the following situations:*

- (1) *If Z is a point and X is a subvariety of the fiber $F^n / F \cong (\pi_F^n)^{-1}(Z)$ which is an abelian variety.*
- (2) *Suppose Z is a smooth curve.*
 - (a) *If there exists i and j such that $\deg(p_i|_Z) \neq \deg(p_j|_Z)$, then X is neither Kähler nor Moishezon.*
 - (b) *If for all i and j we have $\deg(p_i|_Z) = \deg(p_j|_Z)$, then X is Kähler.*
 - (c) *Let \mathcal{L} be the line bundle on E appearing in the construction of the Kodaira surface. If for all i and j we have $(p_i|_Z)^* \mathcal{L} \cong (p_j|_Z)^* \mathcal{L}$, then X is algebraic. In particular, any variety in the preimage of $Z = \{(x, \dots, x) \mid x \in E\} \subset E^n$ is algebraic.*
- (3) *If $\dim(Z) \geq 2$ then X is a neither Kähler nor Moishezon manifold.*

Lemmas 3.4.4 and 3.4.6 provides us with a description of the fibers of S^n / F over any subvariety.

Lemma 3.4.8. *Assume that Z is a subvariety in E^n , then $L_Z = (\pi^{[n]})^{-1}(Z)$ is a fiber product of Kodaira type surfaces:*

$$L_Z = M_{(p_1|_Z)^* \mathcal{L}} \times_Z M_{(p_2|_Z)^* \mathcal{L}} \times_Z \dots \times_Z M_{(p_n|_Z)^* \mathcal{L}},$$

where \mathcal{L} is a line bundle over E such that $S = M_{\mathcal{L}}$.

Proof. By definition the manifold L_Z is the following pullback:

$$\begin{array}{ccc}
L_Z = (\pi^n)^{-1}(Z) & \hookrightarrow & S^n \\
\downarrow \pi^n|_{L_Z} & & \downarrow \pi \\
Z & \hookrightarrow & E^n
\end{array}$$

Denote by S_i the fibered product of S and Z over E^n . Then, L_Z is isomorphic to the quotient of the product $S_1 \times \cdots \times S_n$ by the action of F . Thus, by Lemma 3.4.6 we get the result. \square

We shall need the following result in the future.

Theorem 3.4.9 ([Var86, Corollaire 2.9, 2.10]). *Let X be a Kähler manifold, Y be a complex analytic space and $\pi : X \rightarrow Y$ be a proper surjective morphism. Assume that one of the following conditions hold:*

- *The fibers of π have the same dimension and either Y is normal, or π is flat;*
- *Y has no non-trivial analytic subsets different from divisors (e.g. Y is a surface).*

Then Y is Kähler.

Sketch of proof. Given an analytic space X one can associate to it a space $\mathcal{C}_m(X)$ of m -cycles in X . Points of this space are finite linear combinations $\sum_{i=1}^k n_i [X_i]$ where n_i is a positive integer and X_i is an m -dimensional analytic subspace of X for all $1 \leq i \leq k$. Then $\mathcal{C}_m(X)$ carries a structure of an analytic space by [Bar75]. Moreover, by [Var86, Théorème 2] if X is a Kähler manifold, then $\mathcal{C}_m(X)$ is also a Kähler manifold.

If all fibers of π have the same dimension m one can construct a morphism $c : Y \rightarrow \mathcal{C}_m(X)$ which maps a point of y to the m -cycle $[\pi^{-1}(y)]$ in $\mathcal{C}_m(X)$. This map is an embedding; thus, Y is a Kähler manifold.

Assume that Y has no non-trivial analytic subsets different from divisors. Denote by Z the set of points such that outside Z the map π is equidimensional. The codimension of Z is at least 2; thus, Z is a union of several points. By the previous argument one can show that $Y \setminus Z$ is a Kähler manifold. By [Miy74] one can extend the Kähler metric to a point; thus, Y is Kähler. \square

Lemma 3.4.10. *Assume that $X \subset S^n/F$ and $Z = \pi^n(X)$ is a smooth curve in E^n . If there exists two indices i and j such that $\deg(p_i|_Z) \neq \deg(p_j|_Z)$, then X is non-Kähler and non-Moishezon.*

Proof. Consider the fiber $L_Z = (\pi^{[n]})^{-1}(Z)$, denote by \mathcal{L}_i the pull-back $p_i|_Z^*(\mathcal{L})$ for $i = 1, \dots, n$. Without loss of generality we can assume that $\deg \mathcal{L}_i \neq \deg \mathcal{L}_j$. By Lemmas 3.4.6 and 3.4.8 we have $L_Z/F \cong M_{(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)} \times_Z \cdots \times_Z M_{(\mathcal{L}_n \otimes \mathcal{L}_1^\vee)}$. Consider the following composition:

$$X \subset L_Z/F \cong M_{(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)} \times_Z \cdots \times_Z M_{(\mathcal{L}_n \otimes \mathcal{L}_1^\vee)} \xrightarrow{\text{pr}_1} M_{(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)},$$

where pr_1 is a projection to the first component of the product. By Lemma 3.4.3 we know that Z is an algebraic reduction of $M_{(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)}$. Since X maps surjectively to Z the image of X in $M_{(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)}$ coincides with it. Since $M_{(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)}$ is not Kähler by Lemma 3.4.3, we get by Theorem 3.4.9 that X is not Kähler either. By Lemma 3.2.15 we get that X is not Moishezon also. \square

Lemma 3.4.10 describes a big class of non-algebraic manifolds inside S^n/F . Now we are going to show that there are some algebraic varieties in S^n/F which map to curves in E^n .

Lemma 3.4.11. *If Z is a smooth curve in E^n such that $\deg(p_i|Z) = \deg(p_j|Z)$ for all i and j , then L_Z/F is Kähler. Moreover, if all line bundles $\mathcal{L}_i = (p_i|Z)^*\mathcal{L}$ are isomorphic, then L_Z/F is an algebraic variety.*

Proof. Consider the fiber $L_Z = (\pi^{[n]})^{-1}(Z)$, denote by \mathcal{L}_i the pull-back $p_i|_Z^*(\mathcal{L})$ for $i = 1, \dots, n$. By the assumption $\deg \mathcal{L}_i = \deg \mathcal{L}_j$ for $1 \leq i \leq j \leq n$. By Lemmas 3.4.6 and 3.4.8 we have the following isomorphism:

$$L_Z/F \cong M_{(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee)} \times_Z \cdots \times_Z M_{(\mathcal{L}_n \otimes \mathcal{L}_1^\vee)}.$$

For any i we have $\deg(\mathcal{L}_i \otimes \mathcal{L}_1^\vee) = 0$. Hence the Kodaira type surface $M_{(\mathcal{L}_i \otimes \mathcal{L}_1^\vee)}$ is a compact complex torus by [BHPVdV04, Section V.5]; thus, it is Kähler. Therefore, L_Z/F is Kähler.

If all line bundles $\mathcal{L}_i \otimes \mathcal{L}_1^\vee$ are trivial, then by construction we get

$$M_{\mathcal{L}_i \otimes \mathcal{L}_1^\vee} = M_{\mathcal{O}_Z} \cong Z \times F,$$

and Lemma 3.4.6 implies that $L_Z/F \cong Z \times (F^n/F)$ is an algebraic manifold. □

Remark 3.4.12. Lemma 3.4.11 proves in particular that if we consider a diagonal in E^n

$$\Delta = \{(x, x, \dots, x) \mid x \in E\} \in E^n,$$

then the fiber $L_\Delta/F \subset S^n/F$ is an algebraic manifold.

Denote by $t_x: E^n \rightarrow E^n$ the translation of the abelian variety E^n by $x = (x_1, \dots, x_n) \in E^n$. Consider the antidiagonal $\nabla = \{(x, -x) \mid x \in E\}$ in the product E^2 . Then depending on the Kodaira surface the fiber over $t_x(\nabla)$ in S^2/F may or may not be algebraic.

Lemma 3.4.13. *If the line bundle \mathcal{L} associated to the Kodaira surface S admits a section vanishing in exactly one point then there exists $x = (x_1, x_2) \in E^2$ such that $L_{t_x(\nabla)}/F \subset S^2/F$ is an algebraic manifold.*

Proof. By construction the fiber L_Z is isomorphic to the following fibered product of the Kodaira-type manifolds:

$$L_Z = M_{t_{x_1}^* \mathcal{L}} \times_Z M_{t_{x_2}^* ((-1)^* \mathcal{L})}.$$

By assumption, there exists a point p on E such that $\mathcal{L} \sim \mathcal{O}_E(N \cdot [p])$ for some integer N . We have

$$(t_{x_1}^* \mathcal{L}) \otimes (t_{x_2}^* ((-1)^* \mathcal{L}))^\vee = \mathcal{O}_Z(N \cdot [p - x_1] - N \cdot [-x_2 - p]) \cong \mathcal{O}_Z(N \cdot [2p - x_1 + x_2] - N \cdot [0]).$$

If we fix x_1 and x_2 such that $x_1 - x_2 = 2p$, then this bundle will be trivial. By Lemma 3.4.4 the fiber L_Z/F is isomorphic to an algebraic manifold:

$$L_Z/F = M_{\mathcal{O}_Z} = Z \times F.$$

This concludes the proof. □

Now we study submanifolds in S^2/F which map surjectively to the base.

Lemma 3.4.14. *Let S^2/F be the quotient space of S^2 by the diagonal action of F and consider the commutative diagram*

$$\begin{array}{ccc} S^2 & \xrightarrow{\alpha} & S^2/F \\ \pi^2 \downarrow & \searrow & \swarrow \pi_F^2 \\ E^2 & & \end{array}$$

If $X \subset S^2/F$ and $\pi_F^2(X) = E^2$, then X coincides with S^2/F and X is not Moishezon.

Proof. The dimension of X equals either 2 or 3. Note that by Lemma 3.2.15 the complex space S^2/F is not Moishezon since the Kodaira surface S is not Moishezon. Since π^2 is an algebraic reduction of S^2 (see Lemma 3.2.13) we get that π_F^2 is an algebraic reduction of S^2/F , which implies that any divisor X in S^2/F is a preimage of a divisor on E^2 . But this contradicts to our assumption $\pi_F^2(X) = E^2$, so $\dim(X) = 3$ and X coincides with S^2/F . \square

Here is the proof of Proposition 3.4.7.

Proof of Proposition 3.4.7. If Z is a smooth curve in E^n and $\deg(p_i|_Z) \neq \deg(p_j|_Z)$ for some i and j , then by Lemma 3.4.10 the manifold X is non-Kähler. Otherwise, by Lemma 3.4.11 the manifold X is Kähler. Moreover, if for all i and j we have $(p_i|_Z)^*\mathcal{L} \cong (p_j|_Z)^*\mathcal{L}$ then by Lemma 3.4.11 the manifold X is algebraic. We conclude using Remark 3.4.12 and Lemma 3.4.13.

If $\dim(Z) \geq 2$, then there exists two indices, say 1 and 2, such that $(p_1 \times p_2)(Z) = E \times E$. Projections to the i -th component of the product $P_i: S^n \rightarrow S$ induce a morphism

$$(P_1 \times P_2)_F: S^n/F \rightarrow S^2/F,$$

which maps X to a submanifold Y on S^2/F . By the choice of p_1 and p_2 we get that $\pi_F^2(Y) = E^2$. By Lemma 3.4.14 we get that $Y = S^2/F$ (in particular, it is not Kähler) and Y is not Moishezon. Then Lemma 3.2.15 imply that X is not Moishezon. Take a smooth curve $Z' \subset Z$ such that $\deg(p_1|_{Z'}) \neq \deg(p_2|_{Z'})$. As we showed above the intersection $X' = X \cap (\pi^n)^{-1}(Z')$ is non-Kähler. Since it is a submanifold of X this implies that X is non-Kähler. \square

3.4.3. Fibers of Φ and Π

In this section we study the special fibers of algebraic reductions of a BG-manifold Q as well as of the manifold W/F which appears in Bogomolov's construction. Recall that these manifolds fit the following diagram:

$$\begin{array}{ccccccc} Q & \xrightarrow{p} & W/F & \xleftarrow{q} & W & \xrightarrow{\delta|_W} & \text{Sym}^n(S) \\ & \searrow & & \searrow \Pi & \downarrow \pi^{[n]} & & \downarrow \text{Sym}^n(\pi) \\ & & & & \mathbb{P}^{n-1} \cong \text{Alb}^{-1}(0) & \hookrightarrow & \text{Sym}^n(E) \xrightarrow{\text{Alb}} E \end{array}$$

The next lemma follows from the construction of the symmetric power of a manifold.

Lemma 3.4.15. *Let E be an elliptic curve and let y_1, \dots, y_l be l pairwise distinct points on E . Consider a point $\bar{y} = \{y_1, \dots, y_1, y_2, \dots, y_2, \dots, y_l, \dots, y_l\}$ in $\text{Sym}^n(E)$ such that we have exactly m_i copies of y_i for each $i = 1, \dots, l$. Then the fiber of $\text{Sym}^n(\pi)$ over \bar{y} is as follows:*

$$\text{Sym}^n(\pi)^{-1}(\bar{y}) \cong F^{[m_1]} \times F^{[m_2]} \times \dots \times F^{[m_l]}.$$

Moreover, the diagonal action of F in $\text{Sym}^n(S)$ preserves the structure of the map $\text{Sym}^n(\pi)$ and induces the action of F on a fiber $\text{Sym}^n(\pi)^{-1}(\bar{y})$ for any point \bar{y} .

Fix the following subvariety in $\text{Alb}^{-1}(0) \cong \mathbb{P}^{n-1}$:

$$D = \{ \{y_1, \dots, y_n\} \in \text{Alb}^{-1}(0) \mid \text{there exist } i \neq j \text{ such that } y_i = y_j \} \subset \text{Sym}^n(E). \quad (3.4.16)$$

By definition D is a divisor in \mathbb{P}^{n-1} ; by the following assertion D is the locus of special fibers of Π :

Lemma 3.4.17. *Let $\Pi: W/F \rightarrow \mathbb{P}^{n-1}$ be an algebraic reduction of W/F and $D \subset \mathbb{P}^{n-1}$ be the divisor defined in (3.4.16). Consider a point $\bar{y} = \{y_1, \dots, y_n\}$ in $\text{Alb}^{-1}(0) \cong \mathbb{P}^{n-1}$.*

- (1) *If $\bar{y} \notin D$ then $\Pi^{-1}(\bar{y})$ is isomorphic to an abelian variety F^n/F .*
- (2) *Assume that $\bar{y} \in D$. Then all $(n-1)$ -dimensional components of $\Pi^{-1}(\bar{y})$ are uniruled manifolds of Albanese dimension at most $n-2$. Moreover, if one has $y = \{y_1, \dots, y_n\}$ such that there exist $l < n$ integers k_1, \dots, k_l such that $\sum_{i=1}^l k_i = n$ and*

$$y_1 = \dots = y_{k_1} \neq y_{k_1+1} = \dots = y_{k_1+k_2} \neq \dots \neq y_{k_1+\dots+k_{l-1}+1} = \dots = y_n.$$

then one component of $\Pi^{-1}(\bar{y})$ is birational to $(F^{[k_1]} \times F^{[k_2]} \times \dots \times F^{[k_l]})/F$.

Proof. Denote by Δ_n the diagonal in the symmetric power of S :

$$\Delta_n = \{ \{x_1, \dots, x_n\} \mid \exists i, j : x_i = x_j \} \subset \text{Sym}^n(S).$$

By construction of the Douady–Barlet morphism $\delta: S^{[n]} \rightarrow \text{Sym}^n(S)$ we see that the following map is an isomorphism:

$$\delta|_{S^{[n]} \setminus \delta^{-1}(D)}: S^{[n]} \setminus (\delta^{-1}(\Delta_n)) \xrightarrow{\cong} \text{Sym}^n(S) \setminus \Delta_n$$

Fix a point $\bar{y} = \{y_1, \dots, y_n\}$ in $\text{Alb}^{-1}(0) \cong \mathbb{P}^{n-1}$ and consider a fiber of $\text{Sym}^n(\pi)$ over this point. By construction the diagonal Δ_n intersect the fiber $(\text{Sym}^n(\pi))^{-1}(\bar{y})$ along a proper submanifold. Denote by $X_1 \cup \dots \cup X_m$ all $(n-1)$ -dimensional components of $(\text{Sym}^n(\pi))^{-1}(\bar{y})$. Renumerating components we get that the following morphism is birational:

$$\delta|_{X_1}: X_1 \rightarrow \text{Sym}^n(\pi)^{-1}(\bar{y}),$$

and X_i is a rationally connected fibration over subvarieties of $\Delta_n \cap (\text{Sym}^n(\pi))^{-1}(\bar{y})$ for all $2 \leq i \leq m$.

If \bar{y} does not lie in D , then $k_1 = \dots = k_n = 1$ and $\text{Sym}^n(\pi)^{-1}(\bar{y}) \cong F^n$. Moreover, in this case the fiber $\text{Sym}^n(\pi)^{-1}(\bar{y})$ does not intersect the submanifold Δ_n in $\text{Sym}^n(S)$. Therefore, the map $\delta|_{(\text{Sym}^n(\pi))^{-1}(\bar{y})}$ is an isomorphism and $(\text{Sym}^n(\pi))^{-1}(\bar{y}) \cong F^n$. Since $\Pi^{-1}(\bar{y}) = \text{Sym}^n(\pi)^{-1}(\bar{y})/F$ and the action of F induced from the diagonal action on $\text{Sym}^n(S)$ one gets that the fiber $\Pi^{-1}(\bar{y})$ is actually isomorphic to the abelian variety F^n/F .

If \bar{y} lies in D then irreducible components X_2, \dots, X_m are uniruled by construction. Moreover, by Lemma 3.4.15 we get that X_1 is bimeromorphic to $(F^{[k_1]} \times F^{[k_2]} \times \dots \times F^{[k_l]})/F$ and $k_1 > 1$ and $l \leq n - 1$. Then the Albanese map for $(F^{[k_1]} \times F^{[k_2]} \times \dots \times F^{[k_l]})/F$ is as follows:

$$\text{Alb}_{\bar{y}}: (F^{[k_1]} \times F^{[k_2]} \times \dots \times F^{[k_l]})/F \rightarrow F^l/F.$$

The fibers of the morphism $\text{Alb}_{\bar{y}}$ are rational. Thus, X_1 is bimeromorphic to a uniruled variety. Therefore, the same is true for X_1 and we get that all irreducible components of the variety $\Pi^{-1}(\bar{y})$ are uniruled. Thus, their Albanese dimension are less than their complex dimension. \square

Lemma 3.3.4 implies the following description of fibers of an algebraic reduction of a BG-manifold.

Lemma 3.4.18. *Let $\Phi: Q \rightarrow \mathbb{P}^{n-1}$ be an algebraic reduction of a BG-manifold Q , D be a divisor defined in (3.4.16) and \bar{y} be a point in \mathbb{P}^{n-1} . Then*

- (1) *if $\bar{y} \notin D$, then $\Phi^{-1}(\bar{y})$ is an abelian variety;*
- (2) *if \bar{y} is a general point on D , then all $(n - 1)$ -dimensional components of $\Phi^{-1}(\bar{y})$ are uniruled and their Albanese dimensions are at most $n - 2$.*

Proof. Recall that by [Bog96, Corollary 3.7, Remark 3.9] the branching locus of $p: Q \rightarrow W/F$ is the union of subvarieties B_γ of W/F , where γ is a torsion element in F such that its order is a divisor of n . If the order of γ equals $\frac{n}{l}$ then the dimension of B_γ equals $2l - 2$ and B_γ consists of points $\bar{y} \in W/F$ of the following form:

$$\bar{y} = \left\{ y_1, (y_1 + \gamma), \dots, \left(y_1 + \left(\frac{n}{l} - 1 \right) \gamma \right), \dots, y_l, (y_l + \gamma), \dots, \left(y_l + \left(\frac{n}{l} - 1 \right) \gamma \right) \right\}.$$

If $\bar{y} \in \mathbb{P}^{n-1}$ is not a point on D , then by Proposition 3.3.9 one gets that $\Phi^{-1}(\bar{y})$ is an abelian variety.

Let $\bar{y} = \{y_1, y_1, y_2, y_3, \dots, y_{n-1}\}$ be a point in D such that all y_i are distinct points for distinct indices i . Then $\Pi^{-1}(x)$ does not intersect the branching locus of p . Thus fiber $\Phi^{-1}(\bar{y})$ is an unramified finite cover of $\Pi^{-1}(\bar{y})$. Since all $(n - 1)$ -dimensional components of $\Pi^{-1}(\bar{y})$ are uniruled and their Albanese dimension equal $n - 2$ by Lemma 3.4.17 then so are $(n - 1)$ -dimensional components of $\Phi^{-1}(\bar{y})$. \square

3.4.4. Proof of Theorem 3.1.3

In this section we are going to prove Theorem 3.1.3. Recall that BG-manifolds Q fit into the following commutative diagram:

$$\begin{array}{ccccccc}
 & & W^{\mathbb{C}} & \longrightarrow & S^{[n]} & \xrightarrow{\delta} & \mathrm{Sym}^n S & \xleftarrow{\alpha} & S^n & & \\
 & & \downarrow q & & \downarrow & & \downarrow & & \downarrow r & & \\
 Q & \xrightarrow{p} & W/F^{\mathbb{C}} & \longrightarrow & S^{[n]}/F & \xrightarrow{\delta_F} & (\mathrm{Sym}^n S)/F & \xleftarrow{\alpha_F} & S^n/F & & \\
 & \searrow \Phi & \downarrow \Pi & & \downarrow & & \downarrow & & \downarrow \pi_F^n & & \\
 & & \mathbb{P}^{n-1} & \longrightarrow & E^{[n]} & \xrightarrow{\cong} & \mathrm{Sym}^n(E) & \xleftarrow{\alpha_E} & E^n & &
 \end{array} \tag{3.4.19}$$

The upper vertical arrows are quotients by the action of F and the horizontal maps in the middle row are induced by the corresponding maps from above. Then the proof of Theorem 3.1.3 follows from the following assertion.

Theorem 3.4.20. *Let Q be a BG-manifold of dimension $2n - 2$, let $\Phi: Q \rightarrow \mathbb{P}^{n-1}$ be its algebraic reduction and let D be the divisor defined in (3.4.16). If X is a submanifold of Q , set $Z = \Phi(X)$, then the following properties are satisfied:*

- (1) *If Z is a point, then X is a subvariety of the Moishezon manifold $\Phi^{-1}(Z)$; moreover,*
 - (a) *if $Z \in \mathbb{P}^{n-1} \setminus D$, then $\Phi^{-1}(Z)$ is an abelian variety;*
 - (b) *if $Z \in D$ then $\Phi^{-1}(Z)$ is a uniruled Moishezon manifold.*
- (2) *Let Z be a curve which does not lie in D and a connected component \tilde{Z} of $\alpha_F^{-1}(Z)$ is smooth*
 - (a) *if there exist i and j such that $\deg(p_i|_{\tilde{Z}}) \neq \deg(p_j|_{\tilde{Z}})$, then X is non-Moishezon;*
 - (b) *if for all i and j we have $(p_i|_{\tilde{Z}})^* \mathcal{L} \cong (p_j|_{\tilde{Z}})^* \mathcal{L}$, then X is Moishezon.*
- (3) *If $\dim(Z) \geq 2$ and Z does not lie in D then X is not Moishezon.*

Proof. Irreducible components of $\alpha_F^{-1}(\delta_F(p(X)))$ are isomorphic to each other. Denote by \tilde{X} one of them. Note that if Z does not lie in D , then $\dim(X) = \dim(\tilde{X})$. Set $\tilde{Z} = \pi_F^n(\tilde{X})$.

If Z is a point, then X is a submanifold of the fiber $\Phi^{-1}(Z)$ which is a finite cover of the Moishezon manifold $\Pi^{-1}(Z)$. Lemma 3.4.18 and the fact that the degeneration of uniruled manifolds is uniruled imply points (a) and (b) of the first assertion.

If $\dim(Z) \geq 2$, then $\dim(\tilde{Z}) \geq 2$ and \tilde{X} is a non-Moishezon manifold. Therefore, X is also non-Moishezon by Lemma 3.2.15.

Finally, assume that Z is a curve such that its preimage \tilde{Z} in E^n is smooth. Then the result follows from Proposition 3.4.7 (2) and Lemma 3.2.15. \square

3.5. Automorphisms of BG-manifolds

Our goal in this section is to establish some properties of the group of biholomorphic and bimeromorphic automorphisms of BG-manifolds. We prove that the locus of degeneracy is a divisor, describe its geometric properties and use them to prove the Jordan property for the group of biholomorphic automorphisms.

3.5.1. Divisor D as a dual variety

In this section we study properties of the variety D defined in (3.4.16). By construction D is a subvariety in the symmetric power $\text{Sym}^n(E)$ of an elliptic curve E .

Let x_0 be the identity element for the group structure on E . The line bundle $\mathcal{O}_E(nx_0)$ is very ample for $n \geq 3$; denote by V the vector space $H^0(E, \mathcal{O}_E(nx_0))^\vee$. Then by Riemann–Roch theorem $\dim(V) = n$. The linear system $|nx_0|$ defines the following embedding:

$$E \hookrightarrow \mathbb{P}(V) \cong \mathbb{P}^{n-1}$$

Denote by E^\vee the projectively dual variety to E ; i.e. E^\vee is a subvariety in $\mathbb{P}(V^\vee)$ whose points correspond to hyperplanes in $\mathbb{P}(V)$ tangent to E :

$$E^\vee = \{H \subset \mathbb{P}(V^\vee) \mid H \text{ is tangent to } E \text{ in a point } x\} \subset \mathbb{P}(V^\vee).$$

There is a map from the dual projective space $\mathbb{P}(V)$ to the Hilbert scheme of n points of E .

Lemma 3.5.1. *If $n \geq 3$, then the curve E is not contained in any hyperplane in \mathbb{P}^{n-1} . Each hyperplane $H \subset \mathbb{P}^{n-1}$ intersects E by n points counted with multiplicities and we have a natural map from $\mathbb{P}(V^\vee)$ to the Hilbert scheme of E :*

$$\begin{aligned} \rho: \mathbb{P}(V^\vee) &\rightarrow E^{[n]} \\ \rho(H) &= E \cap H. \end{aligned}$$

The map ρ induces an isomorphism between $\mathbb{P}(V^\vee)$ and $\text{Alb}^{-1}(0)$ and $\rho(E^\vee)$ equals D .

Proof. By construction the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)|_E$ equals $\mathcal{O}_E(nx_0)$. Thus, for any hyperplane H in $\mathbb{P}(V)$ the divisor $H|_E$ is effective and of degree n . Therefore, the map ρ is well-defined.

Consider a point $\bar{y} = \{y_1, \dots, y_n\}$ on $\text{Alb}^{-1}(0)$ and associate a divisor $H_{\bar{y}}$ on E to this set of points. Since $\sum_{i=1}^n y_i = 0$ in E we get that $H_{\bar{y}}$ is linearly equivalent to nx_0 . Since $\mathbb{P}(V)$ is the linear system of $\mathcal{O}_E(nx_0)$ one gets that there exists a hyperplane H in $\mathbb{P}(V)$ such that $H|_E = H_{\bar{y}}$. Thus, we construct the following map:

$$\tau: \text{Alb}^{-1}(0) \rightarrow \mathbb{P}(V^\vee).$$

By construction the maps ρ and τ are inverse one to another. Thus, ρ maps $\mathbb{P}(V)$ isomorphically to $\text{Alb}^{-1}(0)$.

Consider a hyperplane H which lies in $E^\vee \subset \mathbb{P}(V^\vee)$. Then H is tangent to E in a point y_1 and intersects E in points y_2, \dots, y_{n-1} counted with multiplicities. Thus, we get

$$H \cap E = \{y_1, y_1, y_2, \dots, y_{n-1}\}.$$

Therefore, $\rho(H)$ lies in D . Since E^\vee and D have the same dimension and D is irreducible we get the result. \square

Example 3.5.2. If $n = 3$, then by Lemma 3.5.1 we see that $D \subset \mathbb{P}(V^\vee)$ is a dual curve to a plane cubic curve $E \subset \mathbb{P}^2$. Thus, D is a plane curve of degree 6 with 9 cusps. Such curves are called Halphen curves of index 2, see [Hal82].

In case when $n \geq 4$ the variety D is a natural generalisation of the dual variety to a curve E in the projective space.

We need the following well-known facts.

Lemma 3.5.3. *Let $U \subset \mathbb{P}(V)$ be a linear subspace of codimension 2 which does not intersect E . If l is a line in $\mathbb{P}(V)$ which does not intersect U , then the projection from U to l induces a regular finite map of degree n :*

$$\pi_U: E \rightarrow l.$$

Lemma 3.5.4. *Let $U \subset \mathbb{P}(V)$ be a linear subspace of codimension 2 which does not intersect E . If l is a line in $\mathbb{P}(V)$ which does not intersect U , then there is an embedding:*

$$\varphi_U: l \hookrightarrow \mathbb{P}(V^\vee).$$

Moreover, $\varphi_U(l)$ is the line U^\vee in $\mathbb{P}(V^\vee)$.

Definition 3.5.5. Denote by Z the following subvariety of $E^{[n]}$:

$$Z = \{\bar{y} = \{y_1, \dots, y_n\} \in \text{Alb}^{-1}(0) \mid \exists i < j \leq k < l \text{ such that } y_i = y_j \text{ and } y_k = y_l\} \subset \text{Alb}^{-1}(0).$$

By definition Z lies in the divisor $D \subset \text{Alb}^{-1}(0)$.

Lemma 3.5.6. *The set Z is a proper subvariety of D .*

Proof. Take $n - 3$ pairwise distinct points $y_1, \dots, y_{n-3} \in E$, choose y such that $y \notin \{y_1, \dots, y_{n-3}\}$ and set

$$z = \left(-2y - \sum_{i=1}^{n-3} y_i \right) \notin \{y_1, \dots, y_{n-3}, y\}$$

Then $\{y_1, \dots, y_{n-3}, y, y, z\}$ is a point on the divisor D , but it does not lie in Z . Thus, Z does not coincide with D . \square

Recall that if D is a divisor in \mathbb{P}^N of degree d then by definition a point $x \in D$ has multiplicity m , if for a general line $l \subset \mathbb{P}^N$ passing through x we have:

$$|l \cap D| = d - m + 1.$$

Here we count points in $l \cap D$ without their multiplicities.

Proposition 3.5.7. *The degree of the divisor D in $\text{Alb}^{-1}(0) \cong \mathbb{P}(V^\vee)$ equals $2n$. If x is a point in D such that $x = \{x_1, \dots, x_n\}$ and $|\{x_1, \dots, x_n\}| = r$ then x is a point on D of multiplicity $n - r$.*

Proof. Choose a general linear subspace U in $\mathbb{P}(V)$ of codimension 2, then $L = U^\vee$ is a general line in $\mathbb{P}(V^\vee)$. By Lemma 3.5.6 we can assume that $L \cap Z$ is empty. Moreover, without loss of generality we can assume that U does not intersect $E \subset \mathbb{P}(V)$.

Choose a line $l \subset \mathbb{P}(V)$ which does not intersect U and consider the projection $\pi_U: E \rightarrow l$ as in Lemma 3.5.3. By Lemma 3.5.4 any point x on L corresponds to the hyperplane H which contains $\varphi_U^{-1}(x)$ and U .

Then by Lemma 3.5.1 the hyperplane H containing $\varphi_U^{-1}(x)$ and U corresponds to a point on D only if the point $\varphi_U^{-1}(x)$ is a ramification point of π_U .

Since π_U is a map of degree n , the ramification divisor is of degree $2n$. Moreover, since L does not intersect in Z the ramification divisor is a union of exactly $2n$ distinct points. Thus, $\deg(D)$ equals $2n$.

Consider a point $x \in D$ such that $|\{x_1, \dots, x_n\}| = r$. If $r = n - 1$, then a general line L through x does not intersect Z by Lemma 3.5.6 and the multiplicity of x equals 1.

If $r < n - 1$, then $x \in Z$. Since Z is of codimension 2, a general line L passing through x does not intersect Z in any point except x . Then we conclude that the multiplicity of x equals $n - r$. \square

Denote by \mathcal{Z}_k the set of singular points on $D \subset \mathbb{P}(V^\vee)$ of multiplicity $\geq k$ by the previous proposition. These sets give a stratification of D :

$$\emptyset = \mathcal{Z}_n \subsetneq \mathcal{Z}_{n-1} \subsetneq \dots \subsetneq \mathcal{Z}_2 \subsetneq \mathcal{Z}_1 = D.$$

Proposition 3.5.8. *The set $\mathcal{Z}_{n-1} \subset D$ is a finite set of singular points of maximal multiplicity. It does not lie on any hyperplane in $\mathbb{P}(V^\vee) \cong \text{Alb}^{-1}(0)$.*

Proof. By Lemma 3.5.1 and Proposition 3.5.7 each point of \mathcal{Z}_{n-1} corresponds to a point $H_{x, \dots, x}$ of $\mathbb{P}(V^\vee)$, where x is a point in the group of n -torsion $E[n] \cong C_n \times C_n$. Thus,

$$|\mathcal{Z}_{n-1}| = n^2.$$

There is a standard action of a group of n -torsion $E[n]$ on the projective space $\mathbb{P}(V^\vee)$. For each $\xi \in E[n]$ and a point H_{x_1, \dots, x_n} in $\mathbb{P}(V^\vee)$ we have:

$$\xi \cdot H_{x_1, \dots, x_n} = H_{x_1 + \xi, \dots, x_n + \xi}. \quad (3.5.9)$$

This action induces the action of $E[n]$ on $\mathbb{P}(V)$ which leaves the curve $E \subset \mathbb{P}(V)$ invariant and acts on it by translation with n -torsion points. By [Hul86, Theorem I.2.5.] this projective action is induced by the linear action of the standard representation of the finite Heisenberg group \mathcal{H}_n on the n -dimensional vector space V (for the definition see [Hul86, Page 11]).

Denote by L_x a 1-dimensional subspace of V^\vee associated to a point $H_{x, \dots, x} \in \mathbb{P}(V^\vee)$ for some $x \in E[n]$. By (3.5.9) the action of each element of \mathcal{H}_n maps L_x to L_y for some $y \in E[n]$. Thus, there is a subrepresentation of V^\vee :

$$L = \langle L_x \rangle_{x \in E[n]} \subset V^\vee.$$

However the standard representation V of \mathcal{H}_n and its dual V^\vee are irreducible. Thus, $L = V^\vee$ and there is no hyperplane in $\mathbb{P}(V^\vee)$ containing all points $H_{x, \dots, x}$ for $x \in E[n]$. \square

Denote by $\text{Aut}_D(\mathbb{P}^{n-1})$ the group of biholomorphic automorphisms of \mathbb{P}^{n-1} which preserves D ; i.e. for any $\varphi \in \text{Aut}_D(\mathbb{P}^{n-1})$ one has $\varphi(D) = D$.

Corollary 3.5.10. *If $D \subset \mathbb{P}^{n-1}$ is the divisor defined in (3.4.16) then $\text{Aut}_D(\mathbb{P}^{n-1})$ is a subgroup of the permutation group \mathfrak{S}_{n^2} .*

Proof. By Proposition 3.5.8 the group of biholomorphic automorphisms $\text{Aut}_D(\mathbb{P}^{n-1})$ of \mathbb{P}^{n-1} which preserves D should also preserve a finite subset of n^2 points $Z_{n-1} \subset D$, which spans \mathbb{P}^{n-1} . Therefore, the group $\text{Aut}_D(\mathbb{P}^{n-1})$ is a subgroup of the permutation group \mathfrak{S}_{n^2} . \square

3.5.2. Regular automorphisms of a BG-manifold

In this section properties of the automorphism groups of BG-manifolds are studied. We start with a description of fibers of the algebraic reduction Φ of a BG-manifold.

Given a surjective morphism $\phi : X \rightarrow Y$ between complex spaces we denote by $\text{Bim}(X)_\phi$ the subgroup of $\text{Bim}(X)$ which consists of automorphisms $g : X \dashrightarrow X$ such that $\phi = \phi \circ g$. Recall that the group $\text{Bim}(X)_\phi$ can be embedded in the group of bimeromorphic automorphisms of a very general fiber of ϕ .

Lemma 3.5.11 ([PS20, Lemma 4.1]). *Let $\phi : X \rightarrow Y$ be a surjective morphism with connected fibers between compact complex manifolds. Let $\{G_i\}$ be a countable family of finite subgroups in the group $\text{Bim}(X)_\phi$. Then there exists a smooth, irreducible and reduced fiber F of the map ϕ of dimension $\dim X - \dim Y$ such that all groups G_i can be embedded into $\text{Bim}(F)$.*

Moreover, given a countable family Ξ of proper closed analytic subsets in Y the fiber F can be chosen so that $\phi(F)$ does not lie in Ξ .

In the next assertion we show that groups of automorphisms of a BG-manifold Q and the manifold W/F which arises in the construction of Q are very similar.

Proposition 3.5.12. *Let Q be a BG-manifold of dimension $2n-2$, let $\Phi : Q \rightarrow \mathbb{P}^{n-1}$ be its algebraic reduction and let A be a general fiber of Φ . If D is a divisor (3.4.16) in \mathbb{P}^{n-1} , then $\text{Bim}(Q)$ fits into the following short exact sequence:*

$$1 \rightarrow \text{Bim}(Q)_\Phi \rightarrow \text{Bim}(Q) \rightarrow \Delta \rightarrow 1, \quad (3.5.13)$$

where $\text{Bim}(Q)_\Phi \subset \text{Aut}(A)$ and $\Delta \subset \text{Bim}(\mathbb{P}^{n-1})$. Moreover, for any $\varphi_0 \in \Delta$, either D lies in the exceptional locus of φ_0 , or the proper image of D under φ_0 coincides with D .

Proof. Let φ be a bimeromorphic automorphism of Q . Then by the definition of an algebraic reduction, there exists a bimeromorphic automorphism φ_0 of \mathbb{P}^{n-1} such that $\Phi \circ \varphi = \varphi_0 \circ \Phi$. It remains to show that the divisor D is either exceptional or invariant under φ .

Since φ is bimeromorphic, there exists maximal open subsets \mathcal{U} and \mathcal{V} in Q , such that $\varphi|_{\mathcal{U}}$ is an isomorphism onto \mathcal{V} :

$$\varphi|_{\mathcal{U}} : \mathcal{U} \xrightarrow{\sim} \mathcal{V}.$$

Denote by \mathcal{U}_0 and \mathcal{V}_0 the images of \mathcal{U} and \mathcal{V} in \mathbb{P}^{n-1} ; these are open subsets in \mathbb{P}^{n-1} . If x is a point in \mathbb{P}^{n-1} , set $\mathcal{U}_x = \Phi^{-1}(x) \cap \mathcal{U}$ and $\mathcal{V}_x = \Phi^{-1}(x) \cap \mathcal{V}$. By construction for any $x \in \mathcal{U}_0$ the set \mathcal{U}_x is dense in $\Phi^{-1}(x)$ and for any x in \mathcal{V}_0 the set \mathcal{V}_x is dense in $\Phi^{-1}(x)$.

Assume that the divisor D does not lie in $\text{Exc}(\varphi_0)$, then $\mathcal{U}_0 \cap D$ is not empty. Let x be a point in $\mathcal{U}_0 \cap D$. Then by construction one has $\varphi(\mathcal{U}_x) \cong \mathcal{V}_{\varphi_0(x)}$. Thus, by Lemma 3.4.18 we get that the fiber $\mathcal{V}_{\varphi_0(x)}$ is not birational to an abelian variety and $\varphi_0(x)$ lies in D . Therefore, the proper image of D coincides with D . \square

Now we can prove that the group of biholomorphic automorphisms of BG-manifold is Jordan.

Theorem 3.5.14. *Let Q be a BG-manifold. Then the group $\text{Aut}(Q)$ is Jordan.*

Proof. Assume that $\dim(Q) = 2n - 2$. Denote by $\text{Aut}(Q)_\Phi$ the intersection of the groups $\text{Bim}(Q)_\Phi$ and $\text{Aut}(Q)$. By Proposition 3.5.12, there is a short exact sequence of groups

$$1 \rightarrow \text{Aut}(Q)_\Phi \rightarrow \text{Aut}(Q) \rightarrow \Delta_{\text{Aut}} \rightarrow 1.$$

Here Δ_{Aut} is a subgroup of $\Delta \subset \text{Bim}(\mathbb{P}^{n-1})$ consisting of the bimeromorphic automorphisms of \mathbb{P}^{n-1} induced by biholomorphic automorphisms of Q . By construction, no element φ_0 of Δ_{Aut} contracts divisors in \mathbb{P}^{n-1} . This implies that φ_0 preserves the linear system of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Thus, the automorphism φ_0 is a biholomorphism and Δ_{Aut} lies in the intersection $\Delta \cap \text{Aut}(\mathbb{P}^{n-1})$.

By Proposition 3.5.12 we get that for any element φ_0 in Δ_{Aut} one has $\varphi_0(D) = D$. Thus, by Corollary 3.5.10 the group Δ_{Aut} is a subgroup of a permutation group \mathfrak{S}_{n^2} ; in particular it is finite.

By Lemma 3.5.11 the group $\text{Aut}(Q)_\Phi$ is a subgroup of the group $\text{Bim}(Q_x)$ where $Q_x = \Phi^{-1}(x)$ and x is a very general point on \mathbb{P}^{n-1} . By Lemma 3.3.9 this implies that the group $\text{Aut}(Q)_\Phi$ is a subgroup of the group of bimeromorphic automorphisms of an abelian variety of dimension $n - 1$. Since the group of bimeromorphic automorphisms of the abelian variety coincides with the group of biholomorphic automorphisms of it and by [MZ18] we get that the group $\text{Aut}(Q)_\Phi$ is Jordan.

Finally, the group $\text{Aut}(Q)$ is an extension of a finite group Δ_{Aut} by a Jordan group $\text{Aut}(Q)_\Phi$. Then by [PS20, Lemma 2.1] the group $\text{Aut}(Q)$ is also Jordan. \square

3.5.3. Bimeromorphic automorphisms of a BG-fourfold

In this section we prove the Jordan property for a BG-manifold of dimension 4. In order to do it we study in details the divisor $D \subset \mathbb{P}^2$ defined in (3.4.16) and groups of birational automorphisms which preserve such pair.

Let D be a reduced divisor in an algebraic variety X . By $\text{Bir}_D(X)$ we denote the group of all birational automorphisms $f: X \dashrightarrow X$ such that the proper transform of D under f coincides with D . Recall that an *embedded resolution of a pair* (X, D) is a birational morphism $\delta: \tilde{X} \rightarrow X$ such that the proper transform of D is a smooth subvariety. In the case where X is a surface the embedded resolution exists by [Har77, Proposition V.3.8]; moreover, there is a canonical construction of an embedded resolution $\delta_0: \tilde{X}_0 \rightarrow X$ such that any embedded resolution of D factors through δ_0 . We call such morphism δ_0 the *minimal embedded resolution of* (X, D) .

Here we recall a theorem which describes the group $\text{Bir}_D(\mathbb{P}^2)$ where D is a singular curve.

Theorem 3.5.15 ([BPV09, Corollary 2.3.6]). *Let $n > 0$ be an integer, let $D \subset \mathbb{P}^2$ be a curve of degree $3n$ and let $\delta_0: X_0 \rightarrow \mathbb{P}^2$ be the minimal embedded resolution of D . If $\text{Sing}(D) = \{p_1, \dots, p_k\}$ and the multiplicity of D in the point p_i equals n for all $1 \leq i \leq k$, then for any $f \in \text{Bir}_D(\mathbb{P}^2)$ there exists $f_0 \in \text{Aut}(X_0)$ such that:*

$$\delta_0 \circ f_0 = f \circ \delta_0.$$

Proof. Let H be the class of hyperplane section on \mathbb{P}^2 . Then the divisor $D \subset \mathbb{P}^2$ is numerically equivalent to $6H$. Let d be an integer such that $f^*H = dH$. If $d = 1$ then f is regular; thus, we can assume that $d \geq 2$. By [Kol07, Corollary 3.18] there exists a smooth variety X and birational morphisms $\delta: X \rightarrow \mathbb{P}^2$ and $g: X \rightarrow \mathbb{P}^2$ such that the following diagram commutes:

$$\begin{array}{ccc} & X & \\ \delta \swarrow & & \searrow g \\ \mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^2 \end{array}$$

Moreover, we can assume that δ is an iterated blow-up in points. Let $X_{00} = \mathbb{P}^2$ and $X_{k,m_k} = X$, then δ is the following composition:

$$X = X_{km_k} \xrightarrow{\delta_{km_k}} X_{km_k-1} \xrightarrow{\delta_{km_k-1}} \dots \xrightarrow{\delta_{22}} X_{21} \xrightarrow{\delta_{21}} X_{1m_1} \xrightarrow{\delta_{1m_1}} \dots \xrightarrow{\delta_{12}} X_{11} \xrightarrow{\delta_{11}} X_{00} = \mathbb{P}^2$$

Namely, we set $m_0 = 0$ then there exists integers $m_1, \dots, m_k \geq 0$ and δ is the composition of birational morphisms:

$$\delta_{ij}: X_{ij} \rightarrow X_{i'j'},$$

where $i' = i$ and $j' = j - 1$ if $j \geq 2$ and $i' = i - 1$, $j' = m_{i-1}$ if $j = 1$. The morphism δ_{ij} is the blow-up of a point $p_{i'j'} \in X_{i'j'}$. Denote by D_{ij} the proper preimage of D in X_{ij} and denote by $E_{ij} \subset X$ the inverse images of the exceptional divisor of δ_{ij} to X . Then one has $E_{ij}^2 = -1$ and $E_{i_1j_1} \cdot E_{i_2j_2} = 0$ for all pairs $(i_1, j_1) \neq (i_2, j_2)$. Moreover, the canonical class of X is as follows:

$$K_X = -3\delta^*H + \sum_{i=1}^k \sum_{j=1}^{m_i} E_{ij}.$$

Denote by $\tilde{H} \in N^1(X)$ the divisor class g^*H . Then by construction there exist integers $a_{ij} \geq 0$ where $1 \leq i \leq k$ and $1 \leq j \leq m_i$ such that

$$[\tilde{H}] = d\delta^*H + \sum_{i=1}^k \sum_{j=1}^{m_i} a_{ij}E_{ij}.$$

By construction $a_{ij} = 0$ if and only if the point $p_{i'j'} = \delta_{ij}(E_{ij})$ does not lie in the indeterminacy locus of the composition $f \circ \delta_{11} \circ \delta_{12} \circ \dots \circ \delta_{i'j'}: X_{i'j'} \rightarrow \mathbb{P}^2$, where $i' = i$ and $j' = j - 1$ if $j \geq 2$ and $i' = i - 1$, $j' = m_{i-1}$ if $j = 1$. Therefore, by projection formula one gets:

$$\begin{aligned} -3d - \sum_{i=1}^k \sum_{j=1}^{m_i} a_{ij} &= \left(-3\delta^*H + \sum_{i=1}^k \sum_{j=1}^{m_i} E_{ij} \right) \cdot \left(d\delta^*H + \sum_{i=1}^k \sum_{j=1}^{m_i} a_{ij}E_{ij} \right) = \\ &= K_X \cdot [\tilde{H}] = K_X \cdot g^*H = -3. \end{aligned} \quad (3.5.16)$$

Denote by $\tilde{D} \subset X$ the proper preimage of the divisor D under g . On the one hand, since the linear system of D in \mathbb{P}^2 is free one has $[\tilde{D}] = 3ng^*H$. On the other hand, since the proper preimage of D under f coincides with D one has:

$$[\tilde{D}] = \delta^*[D] + \sum_{i=1}^k \sum_{j=1}^{m_i} M_{ij}E_{ij},$$

where M_{ij} is the multiplicity of the curve $D_{ij} \subset X_{ij}$ in the point $p_{ij} \in X$. By assumption of the theorem one has $M_{ij} \leq n$ for all $1 \leq i \leq k$ and $1 \leq j \leq m_i$. The computation of the product of classes \tilde{H} and $[\tilde{D}]$ leads us to the following equality:

$$\begin{aligned} 3nd - \sum_{i=1}^k \sum_{j=1}^{m_i} M_{ij} a_{ij} &= \\ &= \left(\delta^*[D] + \sum_{i=1}^k \sum_{j=1}^{m_i} M_{ij} E_{ij} \right) \cdot \left(d\delta^*H + \sum_{i=1}^k \sum_{j=1}^{m_i} a_{ij} E_{ij} \right) = [\tilde{D}] \cdot \tilde{H} = 3n. \end{aligned} \quad (3.5.17)$$

The formulas (3.5.16) and (3.5.17) implies the following equality:

$$\sum_{i=1}^k \sum_{j=1}^{m_i} (n - M_{ij}) a_{ij} = 0.$$

Since one has $M_{ij} \leq n$ and $a_{ij} \geq 0$ for all $1 \leq i \leq k$ and $1 \leq j \leq m_i$ we get that for any i and j one has either $a_{ij} = 0$ or $M_{ij} = 2$. Note that if $M_{ij} \neq 2$ then the point p_{ij} either does not lie on D_{ij} or it is a smooth point of D_{ij} and in this case p_{ij} does not lie in the $\text{Ind}(f \circ \delta_{11} \circ \delta_{12} \circ \dots \circ \delta_{ij})$.

Therefore, if $\tilde{\delta}: \tilde{X} \rightarrow \mathbb{P}^2$ is an embedded resolution of the pair (\mathbb{P}^2, D) then the map $f \circ \tilde{\delta}$ is regular. Let $\delta_0 \circ X_0 \rightarrow \mathbb{P}^2$ be the minimal embedded resolution of (X, D) ; then $f \circ \delta_0$ is an embedded resolution of D . Thus, the composition $f \circ \delta_0$ factors through δ_0 i.e. there exists $f_0 \in \text{Aut}(X_0)$ such that $f \circ \delta_0 = \delta_0 \circ f_0$. Thus, we get the result. \square

Moreover, by the next assertion we get that the group of bimeromorphic automorphisms of a 4-dimensional BG-manifold is also Jordan.

Theorem 3.5.18. *If Q is a BG-manifold of dimension 4 then the group $\text{Bim}(Q)$ is Jordan.*

Proof. Let $\Phi: Q \rightarrow \mathbb{P}^2$ be an algebraic reduction of Q defined in Theorem 3.1.1 and let $D \subset \mathbb{P}^2$ be the divisor defined in (3.4.16). By Proposition 3.5.12 the group $\text{Bim}(Q)$ fits into the exact sequence (3.5.13) and for any element $g \in \Delta$ one has $D \subset \text{Exc}(g)$ or $g(D) = D$.

By Example 3.5.2 we get that D is a curve of degree 6 and geometric genus 1 with 9 cusps.

Since the geometric genus of D equals 1 it cannot lie in $\text{Exc}(g)$. Thus, Δ is a subgroup of the group $\text{Bir}_D(\mathbb{P}^2)$. Denote by $\delta: X \rightarrow \mathbb{P}^2$ the minimal embedded resolution of the pair (\mathbb{P}^2, D) . Then by Theorem 3.5.15 one has the following equality:

$$\Delta \subset \text{Bir}_D(\mathbb{P}^2) = \text{Bir}_{D_X}(X) = \text{Aut}_{D_X}(X),$$

where D_X is the proper transform of D to X and $\text{Aut}_{D_X}(X) = \text{Bir}_{D_X}(X) \cap \text{Aut}(X)$.

There exists at least one cubic curve $C \subset \mathbb{P}^2$ which passes through 9 points $\text{Sing}(D)$. The curve $2C$ does not coincide with the curve D since D is reduced. Thus, there is a pencil of sextic surfaces passing through $\text{Sing}(D)$ and it induces the rational map:

$$\pi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1,$$

and curves D and $2C$ are fibers of π . Then there exists a birational morphism $\tilde{\delta}: \tilde{X} \rightarrow X$ such that the following composition is a regular map:

$$\tilde{\pi}: \tilde{X} \rightarrow \mathbb{P}^1,$$

is a regular map. Then \tilde{X} is a smooth elliptic surface. Moreover, the curve $D \in |-2K_{\tilde{X}}|$ is a fiber of the map $\tilde{\pi}$. Thus, $\tilde{\pi}$ is the map given by linear system $|-2K_{\tilde{X}}|$. By construction $K_{\tilde{X}}^2 = 0$. Therefore, there is no sextic curves in \mathbb{P}^2 passing through $\text{Sing}(D)$ with multiplicities 2 except linear combinations of D and $2C$, see also [Hal82].

Thus, we get $\text{Aut}_{D_X}(X) = \text{Aut}_{\tilde{D}}(\tilde{X})$ where \tilde{D} is the proper preimage of D to \tilde{D} . Then by construction $\text{Aut}_{\tilde{D}}(\tilde{X})$ fits into the following exact sequence:

$$1 \rightarrow G' \rightarrow \text{Aut}_{\tilde{D}}(\tilde{X}) \rightarrow G'' \rightarrow 1,$$

where G' is a subgroup of $\text{Aut}(F)$ for a general fiber F of $\varphi_{|-2K_X|}$ and G'' is a subgroup of the group $\text{Aut}(\mathbb{P}^1)$ induced by elements of $\text{Aut}_{\tilde{D}}(\tilde{X})$.

Fibers of smooth elliptic surface are either smooth elliptic curve or unions of several rational curves. The action of the group $\text{Aut}_{\tilde{D}}(\tilde{X})$ preserves the smooth fiber \tilde{D} and the set of singular fibers. By [CF03, Lemma 3.7] there exist at least two singular fibers of $\varphi_{|-2K_{\tilde{X}}|}$. Thus, the group G'' is finite.

Since for any finite subgroup of $\text{Aut}(F)$ for a smooth elliptic curve F its order is at most 6, we get that there exists a number C such that for any finite subgroup $G \subset \text{Aut}_{\tilde{D}}(\tilde{X})$ one has $|G| \leq C$. Then the group Δ also satisfies this property. Thus, by Proposition 3.5.12 and by Lemma 1.2.36 we get that the group $\text{Bim}(Q)$ is Jordan. \square

Conclusion

We studied 3-subgroups in groups of birational automorphisms of rationally connected threefolds. Taken together, the results [Pro11], [Pro14], [PS18], [Xu20], [Log21] and Theorem 1.1.5 give a sharp bound on the minimal cardinality of generating sets of p -subgroups in these groups for all prime numbers p . Thus, this project is finished.

Also we proved that groups of regular automorphisms of quasi-projective surfaces over fields of characteristic $p > 0$ are p -Jordan. It is natural to try to generalize this result to dimension 3. However, Lemma 1.3.6 cannot be generalized to higher dimension since in positive characteristic there exist non-rational unirational varieties. In particular, there is a long list of non-rational unirational surfaces, see [Kat81], [KS20], [KS79], [Miy76], [Ohh92], [RS78], [Shi74], etc. Thus, it would be very interesting to study these surfaces and try to construct an example of quasi-projective threefold which automorphisms group is not p -Jordan.

The criterion which we construct in Theorem 2.1.3 works in the case of Example 0.0.11. However, it would be very interesting to see if it is applicable for other examples of positive entropy automorphisms: for instance it is not clear whether Examples 0.0.9 and 0.0.10 are regularizable.

Another natural question is whether the automorphism $f_{p_1, \dots, p_k} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ in Example 0.0.11 is primitive or not. In Theorem 2.1.3 we managed to prove that f_{p_1, \dots, p_k} does not preserve the structure of a fibration over a surface. The idea of our proof was as following. Assume that there exists a dominant map $\pi : \mathbb{P}^3 \dashrightarrow B$ and $g \in \text{Bir}(B)$ such that $1 < \dim(B) < \dim(X)$ and $\pi \circ f_{p_1, \dots, p_k} = g \circ \pi$. In the case where B is a surface by [Tru20] one has $1 < \lambda_1(f_{p_1, \dots, p_k}) = \lambda_1(g)$; i.e. g has non-trivial dynamics which allows us to show that this situation is impossible. If we want to show that f_{p_1, \dots, p_k} is primitive it remains to consider the case where B is a curve; this situation is more complicated since g has no interesting dynamics. Thus, one should use some other approach.

There are also many natural questions about BG-manifolds. In Theorem 3.1.4 we proved that the group of biholomorphic automorphisms of such manifold is Jordan; however, we constructed no concrete examples of an automorphism of a BG-manifold. Thus, it would be very interesting to find constructions of biholomorphic or bimeromorphic automorphism of a BG-manifold. One can use constructions of automorphisms of Kodaira surfaces described in [FN05] and understand whether they induce an automorphism of a BG-manifold. Another direction which looks curious is to prove that the group of bimeromorphic automorphisms of a BG-manifold of dimension $2n$ is Jordan for $n \geq 3$. By Theorem 3.1.4 this is equivalent to the study of the decomposition group of the divisor D is the projective space \mathbb{P}^n where D is the dual variety to a twisted cubic curve. Thus, it is a purely algebraic problem which seems to be very natural and interesting.

Bibliography

- [Art70] M. Artin. Algebraization of formal moduli. II. Existence of modifications. *Ann. of Math. (2)*, 91:88–135, 1970.
- [Bar75] D. Barlet. Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie. In *Fonctions de plusieurs variables complexes, II (Sém. François Norguet, 1974–1975)*, pages 1–158. Lecture Notes in Math., Vol. 482. 1975.
- [BB00] L. Bayle and A. Beauville. Birational involutions of \mathbf{P}^2 . volume 4, pages 11–17. 2000. Kodaira's issue.
- [BC16] J. Blanc and S. Cantat. Dynamical degrees of birational transformations of projective surfaces. *J. Amer. Math. Soc.*, 29(2):415–471, 2016.
- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.
- [BCK14] E. Bedford, S. Cantat, and K. Kim. Pseudo-automorphisms with no invariant foliation. *J. Mod. Dyn.*, 8(2):221–250, 2014.
- [BCZ04] G. Brown, A. Corti, and F. Zucconi. Birational geometry of 3-fold Mori fibre spaces. In *The Fano Conference*, pages 235–275. Univ. Torino, Turin, 2004.
- [BDK] E. Bedford, J. Diller, and K. Kim. Pseudoautomorphisms with invariant curves. In *Complex geometry and dynamics*, volume 10 of *Abel Symp.*, pages 1–27.
- [BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.*, 22(2):201–248, 2013.
- [Bea83] A. Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.
- [Bea07] Arnaud Beauville. p -elementary subgroups of the Cremona group. *J. Algebra*, 314(2):553–564, 2007.
- [BF13] J. Blanc and J.-P. Furter. Topologies and structures of the Cremona groups. *Ann. of Math. (2)*, 178(3):1173–1198, 2013.

- [BHPVdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.
- [Bir21] C. Birkar. Singularities of linear systems and boundedness of Fano varieties. *Ann. of Math. (2)*, 193(2):347–405, 2021.
- [BK09] E. Bedford and K. Kim. Dynamics of rational surface automorphisms: linear fractional recurrences. *J. Geom. Anal.*, 19(3):553–583, 2009.
- [BK14] E. Bedford and K. Kim. Dynamics of (pseudo) automorphisms of 3-space: periodicity versus positive entropy. *Publ. Mat.*, 58(1):65–119, 2014.
- [BKKY21] F. Bogomolov, N. Kurnosov, A. Kuznetsova, and E. Yasinsky. Geometry and automorphisms of non-Kähler holomorphic symplectic manifolds. *International Mathematics Research Notices*, 2021. doi:10.1093/imrn/rnab043.
- [BL15] J. Blanc and S. Lamy. On birational maps from cubic threefolds. *North-West. Eur. J. Math.*, 1:55–84, 2015.
- [Bla08] J. Blanc. On the inertia group of elliptic curves in the Cremona group of the plane. *Michigan Math. J.*, 56(2):315–330, 2008.
- [Bla09] J. Blanc. Linearisation of finite abelian subgroups of the Cremona group of the plane. *Groups Geom. Dyn.*, 3(2):215–266, 2009.
- [Bla13] J. Blanc. Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces. *Indiana Univ. Math. J.*, 62(4):1143–1164, 2013.
- [Bog78] F. Bogomolov. Hamiltonian Kählerian manifolds. *Dokl. Akad. Nauk SSSR*, 243(5):1101–1104, 1978.
- [Bog96] F. Bogomolov. On Guan’s examples of simply connected non-Kähler compact complex manifolds. *Amer. J. Math.*, 118(5):1037–1046, 1996.
- [Bor61] A. Borel. Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. *Tohoku Math. J. (2)*, 13:216–240, 1961.
- [BPV09] J. Blanc, I. Pan, and T. Vust. On birational transformations of pairs in the complex plane. *Geom. Dedicata*, 139:57–73, 2009.
- [BS53] A. Borel and J.-P. Serre. Sur certains sous-groupes des groupes de Lie compacts. *Comment. Math. Helv.*, 27:128–139, 1953.
- [BSU13] M. Brion, P. Samuel, and V. Uma. *Lectures on the structure of algebraic groups and geometric applications*, volume 1 of *CMI Lecture Series in Mathematics*. Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013.
- [BV99] M. P. Bellon and C.-M. Viallet. Algebraic entropy. *Comm. Math. Phys.*, 204(2):425–437, 1999.

- [BZ15] T. Bandman and Yu. Zarhin. Jordan groups and algebraic surfaces. *Transform. Groups*, 20(2):327–334, 2015.
- [Can99] S. Cantat. Dynamique des automorphismes des surfaces projectives complexes. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(10):901–906, 1999.
- [CDX21] S. Cantat, J. Déserti, and J. Xie. Three chapters on Cremona groups. *Indiana Univ. Math. J.*, 70(5):2011–2064, 2021.
- [CF03] S. Cantat and C. Favre. Symétries birationnelles des surfaces feuilletées. *J. Reine Angew. Math.*, 561:199–235, 2003.
- [CF19] A. Cattaneo and L. Fu. Finiteness of klein actions and real structures on compact hyperkähler manifolds. *Math. Ann.*, 375(3-4):1783–1822, 2019.
- [Cho49] W.-L. Chow. On the geometry of algebraic homogeneous spaces. *Ann. of Math. (2)*, 50:32–67, 1949.
- [CKM88] H. Clemens, J. Kollár, and S. Mori. Higher-dimensional complex geometry. *Astérisque*, (166):144 pp. (1989), 1988.
- [CP17] I. Cheltsov and J. Park. Birationally rigid Fano threefold hypersurfaces. *Mem. Amer. Math. Soc.*, 246(1167):v+117, 2017.
- [CR62] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. AMS Chelsea Publishing, Providence, RI, 1962.
- [CS21] Y. Chen and C. Shramov. Automorphisms of surfaces over fields of finite characteristic, 2021. [arXiv:2106.15906](https://arxiv.org/abs/2106.15906).
- [Dés21] J. Déserti. *The Cremona group and its subgroups*, volume 252 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, [2021] ©2021.
- [DF01] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.*, 123(6):1135–1169, 2001.
- [dF04] T. de Fernex. On planar Cremona maps of prime order. *Nagoya Math. J.*, 174:1–28, 2004.
- [dF13] T. de Fernex. Birationally rigid hypersurfaces. *Invent. Math.*, 192(3):533–566, 2013.
- [DI09] I. Dolgachev and V. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston, Inc., Boston, MA, 2009.
- [DK18] O. Debarre and A. Kuznetsov. Gushel-Mukai varieties: classification and birationalities. *Algebr. Geom.*, 5(1):15–76, 2018.
- [DN11] T.-C. Dinh and V.-A. Nguyễn. Comparison of dynamical degrees for semi-conjugate meromorphic maps. *Comment. Math. Helv.*, 86(4):817–840, 2011.

- [DO88] I. Dolgachev and D. Ortland. Point sets in projective spaces and theta functions. *Astérisque*, (165):210 pp. (1989), 1988.
- [DS05] T.-C. Dinh and N. Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)*, 161(3):1637–1644, 2005.
- [EH87] D. Eisenbud and J. Harris. On varieties of minimal degree (a centennial account). In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 3–13. Amer. Math. Soc., Providence, RI, 1987.
- [Fei71] W. Feit. The current situation in the theory of finite simple groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 55–93. 1971.
- [FN05] Yo. Fujimoto and N. Nakayama. Compact complex surfaces admitting non-trivial surjective endomorphisms. *Tohoku Math. J. (2)*, 57(3):395–426, 2005.
- [Fuj82] T. Fujita. On the topology of noncomplete algebraic surfaces. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 29(3):503–566, 1982.
- [Fuj83] A. Fujiki. On the structure of compact complex manifolds in \mathcal{C} . In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 231–302. North-Holland, Amsterdam, 1983.
- [Fuj87] A. Fujiki. On the de Rham cohomology group of a compact Kähler symplectic manifold. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 105–165. North-Holland, Amsterdam, 1987.
- [GPR94] H. Grauert, T. Peternell, and R. Remmert, editors. *Several complex variables. VII*, volume 74 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1994. Sheaf-theoretical methods in complex analysis, A reprint of it Current problems in mathematics. Fundamental directions. Vol. 74 (Russian), Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow.
- [Gro03] M. Gromov. On the entropy of holomorphic maps. *Enseign. Math. (2)*, 49(3-4):217–235, 2003.
- [Gua94] D. Guan. Examples of compact holomorphic symplectic manifolds which admit no Kähler structure. In *Geometry and analysis on complex manifolds*, pages 63–74. World Sci. Publ., River Edge, NJ, 1994.
- [Gua95a] D. Guan. Examples of compact holomorphic symplectic manifolds which are not Kählerian. II. *Invent. Math.*, 121(1):135–145, 1995.
- [Gua95b] D. Guan. Examples of compact holomorphic symplectic manifolds which are not Kählerian. III. *Internat. J. Math.*, 6(5):709–718, 1995.
- [Hal82] G. Halphén. Sur les courbes planes du sixième degré à neuf points doubles. *Bull. Soc. Math. France*, 10:162–172, 1882.
- [Hal76] M. Hall, Jr. *The theory of groups*. Chelsea Publishing Co., New York, 1976. Reprinting of the 1968 edition.

- [Han87] M. Hanamura. On the birational automorphism groups of algebraic varieties. *Compositio Math.*, 63(1):123–142, 1987.
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [HKT08] J. W. P. Hirschfeld, G. Korchmáros, and F. Torres. *Algebraic curves over a finite field*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2008.
- [HM10] C. D. Hacon and J. McKernan. Flips and flops. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 513–539. Hindustan Book Agency, New Delhi, 2010.
- [HMX13] C. D. Hacon, J. McKernan, and C. Xu. On the birational automorphisms of varieties of general type. *Ann. of Math. (2)*, 177(3):1077–1111, 2013.
- [Hu20] F. Hu. Jordan property for algebraic groups and automorphism groups of projective varieties in arbitrary characteristic. *Indiana Univ. Math. J.*, 69(7), 2020.
- [Hul86] Klaus Hulek. Projective geometry of elliptic curves. *Astérisque*, (137):143, 1986.
- [IM71] V. A. Iskovskih and Ju. I. Manin. Three-dimensional quartics and counterexamples to the Lüroth problem. *Mat. Sb. (N.S.)*, 86(128):140–166, 1971.
- [IP99] V. Iskovskih and Yu. Prokhorov. Fano varieties. In *Algebraic geometry, V*, volume 47 of *Encyclopaedia Math. Sci.*, pages 1–247. Springer, Berlin, 1999.
- [Isk78] V. Iskovskih. Fano 3-folds. ii. *Math. USSR-Izv.*, 12(3):469–506, 1978.
- [Isk79] V. A. Iskovskih. Anticanonical models of three-dimensional algebraic varieties. In *Current problems in mathematics, Vol. 12 (Russian)*, pages 59–157, 239 (loose errata). VINITI, Moscow, 1979.
- [Kat81] T. Katsura. Unirational elliptic surfaces in characteristic p . *Tohoku Math. J. (2)*, 33(4):521–553, 1981.
- [Kaw92] Yujiro Kawamata. Boundedness of \mathbb{Q} -Fano threefolds. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 439–445. Amer. Math. Soc., Providence, RI, 1992.
- [Kim18] J. H. Kim. Jordan property and automorphism groups of normal compact Kähler varieties. *Commun. Contemp. Math.*, 20(3):1750024, 9, 2018.
- [KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
- [KMM92] J. Kollár, Yo. Miyaoka, and Sh. Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.

- [Kod64] K. Kodaira. On the structure of compact complex analytic surfaces. I. *Amer. J. Math.*, 86:751–798, 1964.
- [Kol07] J. Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007.
- [KPS18] A. Kuznetsov, Yu. Prokhorov, and C. Shramov. Hilbert schemes of lines and conics and automorphism groups of Fano threefolds. *Jpn. J. Math.*, 13(1):109–185, 2018.
- [KS79] T. Katsura and T. Shioda. On Fermat varieties. *Tohoku Math. J. (2)*, 31(1):97–115, 1979.
- [KS20] T. Katsura and M. Schütt. Zariski K3 surfaces. *Rev. Mat. Iberoam.*, 36(3):869–894, 2020.
- [KV19] N. Kurnosov and M. Verbitsky. Deformations and BBF-form on non-Kähler holomorphically symplectic manifolds., 2019. [arXiv:1908.05258](#).
- [KY19] N. Kurnosov and E. Yasinsky. Automorphisms of hyperkähler manifolds and groups acting on CAT(0) spaces, 2019. [arXiv:1810.09730](#).
- [LB19] F. Lo Bianco. On the cohomological action of automorphisms of compact Kähler threefolds. *Bull. Soc. Math. France*, 147(3):469–514, 2019.
- [Les18] J. Lesieutre. Some constraints of positive entropy automorphisms of smooth threefolds. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(6):1507–1547, 2018.
- [Lip69] J. Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, (36), 1969.
- [Lip78] J. Lipman. Desingularization of two-dimensional schemes. *Ann. of Math. (2)*, 107(1):151–207, 1978.
- [Log21] K. Loginov. A note on 3-subgroups in the space cremona group, 2021. [arXiv:2102.04522v2](#).
- [LP11] M. J. Larsen and R. Pink. Finite subgroups of algebraic groups. *J. Amer. Math. Soc.*, 24(4):1105–1158, 2011.
- [McM07] C. T. McMullen. Dynamics on blowups of the projective plane. *Publ. Math. Inst. Hautes Études Sci.*, (105):49–89, 2007.
- [Miy74] Y. Miyaoka. Extension theorems for Kähler metrics. *Proc. Japan Acad.*, 50:407–410, 1974.
- [Miy76] M. Miyanishi. Unirational quasi-elliptic surfaces in characteristic 3. *Osaka Math. J.*, 13(3):513–522, 1976.
- [MM82] Sh. Mori and Sh. Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.*, 36(2):147–162, 1981/82.
- [MO67] H. Matsumura and F. Oort. Representability of group functors, and automorphisms of algebraic schemes. *Invent. Math.*, 4:1–25, 1967.

- [Moi66] B. G. Moishezon. On n -dimensional compact varieties with n algebraically independent meromorphic functions, i, ii and iii. *Izv. Ross. Akad. Nauk Ser. Mat.*, 30(2):51–177, 1966.
- [MPZ20] Sh. Meng, F. Perroni, and D.-Q. Zhang. Jordan property for automorphism groups of compact spaces in fujiki’s class \mathcal{C} , 2020. [arXiv:2011.09381](https://arxiv.org/abs/2011.09381).
- [Muk88] Sh. Mukai. Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10 . In *Algebraic geometry and commutative algebra, Vol. I*, pages 357–377. Kinokuniya, Tokyo, 1988.
- [Mum74] D. Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Bombay. OUP, second edition, 1974.
- [MZ18] S. Meng and D.-Q. Zhang. Jordan property for non-linear algebraic groups and projective varieties. *Amer. J. Math.*, 140(4):1133–1145, 2018.
- [Nam97] Yo. Namikawa. Smoothing Fano 3-folds. *J. Algebraic Geom.*, 6(2):307–324, 1997.
- [Ohh92] M. Ohhira. Unirational elliptic surfaces in characteristic 2. *J. Math. Soc. Japan*, 44(4):709–738, 1992.
- [OT15] K. Oguiso and T. T. Truong. Explicit examples of rational and Calabi-Yau threefolds with primitive automorphisms of positive entropy. *J. Math. Sci. Univ. Tokyo*, 22(1):361–385, 2015.
- [OV05] L. Ornea and M. Verbitsky. An immersion theorem for Vaisman manifolds. *Math. Ann.*, 332(1):121–143, 2005.
- [Pop11] V. Popov. On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. In *Affine algebraic geometry*, volume 54 of *CRM Proc. Lecture Notes*, pages 289–311. 2011.
- [Pop14] V. Popov. Jordan groups and automorphism groups of algebraic varieties. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 185–213. Springer, Cham, 2014.
- [Pro11] Yu. Prokhorov. p -elementary subgroups of the Cremona group of rank 3. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep., pages 327–338. Eur. Math. Soc., Zürich, 2011.
- [Pro13a] Yu. Prokhorov. G -Fano threefolds, I. *Adv. Geom.*, 13(3):389–418, 2013.
- [Pro13b] Yu. Prokhorov. G -Fano threefolds, II. *Adv. Geom.*, 13(3):419–434, 2013.
- [Pro14] Yu. Prokhorov. 2-elementary subgroups of the space Cremona group. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 215–229. 2014.
- [Pro17] Yu. Prokhorov. On the number of singular points of terminal factorial Fano threefolds. *Mat. Zametki*, 101(6):949–954, 2017.

- [Pro21] Yu. Prokhorov. Equivariant minimal model program. *Uspekhi Mat. Nauk*, 76(3(459)):93–182, 2021.
- [PS14] Yu. Prokhorov and C. Shramov. Jordan property for groups of birational selfmaps. *Compos. Math.*, 150(12):2054–2072, 2014.
- [PS16] Yu. Prokhorov and C. Shramov. Jordan property for Cremona groups. *Amer. J. Math.*, 138(2):403–418, 2016.
- [PS18] Yu. Prokhorov and C. Shramov. p -subgroups in the space Cremona group. *Math. Nachr.*, 291(8-9):1374–1389, 2018.
- [PS20] Yu. Prokhorov and C. Shramov. Finite groups of bimeromorphic selfmaps of uniruled kähler threefolds. *Izv. Ross. Akad. Nauk Ser. Mat.*, 84(5):169–196, 2020.
- [PS21a] Yu. Prokhorov and C. Shramov. Automorphism groups of compact complex surfaces. *Int. Math. Res. Not. IMRN*, (14):10490–10520, 2021.
- [PS21b] Yu. Prokhorov and C. Shramov. Finite groups of bimeromorphic selfmaps of non-uniruled Kähler threefolds, 2021. [arXiv:2110.05825](https://arxiv.org/abs/2110.05825).
- [PSU20] D. Popovici, J. Stelzig, and L. Ugarte. Higher-page hodge theory of compact complex manifolds, 2020. [arXiv:2001.02313v2](https://arxiv.org/abs/2001.02313v2).
- [Puk98] A. Pukhlikov. Birational automorphisms of Fano hypersurfaces. *Invent. Math.*, 134(2):401–426, 1998.
- [Puk13] A. Pukhlikov. *Birationally rigid varieties*, volume 190 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.
- [PZ14] F. Perroni and D.-Q. Zhang. Pseudo-automorphisms of positive entropy on the blowups of products of projective spaces. *Math. Ann.*, 359(1-2):189–209, 2014.
- [Rei80] M. Reid. Canonical 3-folds. In *Journées de Géométrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.
- [Rei87] M. Reid. Young person’s guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.
- [RS78] A. N. Rudakov and I. R. Shafarevich. Supersingular $K3$ surfaces over fields of characteristic 2. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(4):848–869, 1978.
- [Sch16] A. Schweizer. On the exponent of the automorphism group of a compact Riemann surface. *Arch. Math. (Basel)*, 107(4):329–340, 2016.
- [Ser09] J.-P. Serre. A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field. *Mosc. Math. J.*, 9(1):193–208, back matter, 2009.

- [SGA03] *Revêtements étales et groupe fondamental (SGA 1)*, volume 3 of *Documents Mathématiques (Paris) [Mathematical Documents (Paris)]*. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960–61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].
- [Shi74] T. Shioda. An example of unirational surfaces in characteristic p . *Math. Ann.*, 211:233–236, 1974.
- [Shi89] K.-H. Shin. 3-dimensional Fano varieties with canonical singularities. *Tokyo J. Math.*, 12(2):375–385, 1989.
- [Tah71] K. Tahara. On the finite subgroups of $GL(3, Z)$. *Nagoya Math. J.*, 41:169–209, 1971.
- [Tru14] T. T. Truong. The simplicity of the first spectral radius of a meromorphic map. *Michigan Math. J.*, 63(3):623–633, 2014.
- [Tru20] T. T. Truong. Relative dynamical degrees of correspondences over a field of arbitrary characteristic. *J. Reine Angew. Math.*, 758:139–182, 2020.
- [Uen75] K. Ueno. *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, volume 439 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, first edition, 1975.
- [Var86] J. Varouchas. Sur l’image d’une variété kählérienne compacte. In *Fonctions de Plusieurs Variables Complexes V*, volume 1188 of *Lecture Notes in Mathematics*, pages 245–259. Springer-Verlag, 1986.
- [Wei55] A. Weil. On algebraic groups of transformations. *Amer. J. Math.*, 77:355–391, 1955.
- [Xie15] J. Xie. Periodic points of birational transformations on projective surfaces. *Duke Math. J.*, 164(5):903–932, 2015.
- [Xu20] J. Xu. A remark on the rank of finite p -groups of birational automorphisms. *C. R. Math. Acad. Sci. Paris*, 358(7):827–829, 2020.
- [Yau78] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation I. *Comm. on Pure and Appl. Math.*, 31:339–411, 1978.
- [Yom87] Y. Yomdin. Volume growth and entropy. *Israel J. Math.*, 57(3):285–300, 1987.

Titre : Automorphismes birationnels des variétés

Mots clés : Automorphismes, Degrés dynamiques, Géométrie birationnelle

Résumé : La thèse couvre divers problèmes portant sur les groupes d'automorphismes réguliers et birationnels de variétés algébriques complexes, et explore les différences et les similitudes entre ces deux groupes.

Dans un premier temps, nous nous intéressons à la description des sous-groupes finis du groupe des automorphismes birationnels $\text{Bir}(X)$ d'une variété complexe rationnellement connexe X de dimension 3. Nous prouvons que tout 3-sous-groupe peut être généré par au plus 5 éléments et dans tous les cas sauf deux, il peut être généré par au plus 4 éléments. Nous décrivons également le groupe des automorphismes réguliers $\text{Aut}(S)$ d'une surface quasi-projective S définie sur un corps k tel que $\text{char}(k) = p > 0$. Nous montrons que ce groupe satisfait la propriété p -Jordan. L'un des principaux ingrédients ici est le programme du modèle minimal (MMP), qui nous permet de réduire des questions sur les groupes finis de transformations birationnelles à la classification de groupes d'automorphismes réguliers de variétés algébriques très particulières obtenues après application du MMP.

Ensuite, nous considérons des automorphismes birationnels d'ordre infini, et essayons de comprendre

quand il est possible de construire un modèle birationnel où l'automorphisme induit est régulier. Nous nous intéressons principalement à l'exemple d'automorphismes birationnels d'une variété rationnelle de dimension 3 introduits par J. Blanc. Cet automorphisme birationnel induit un pseudo-automorphisme sur un modèle birationnel spécial de l'espace projectif, c'est-à-dire qu'il a un petit lieu extrémal ; par conséquent, il est proche d'être régulier. Cependant, le résultat principal de la deuxième partie de la thèse est que ces applications ne sont pas conjuguées à des automorphismes réguliers.

La dernière partie de la thèse porte sur la description du groupe d'automorphismes d'une classe de variétés non-Kählériennes introduite par D. Guan et étudiée plus en détails par F. Bogomolov. Nous prouvons que la réduction algébrique de la variété Bogomolov-Guan de dimension $2n$ est l'espace projectif de dimension n , puis nous étudions le lieu des fibres singulières d'une réduction algébrique et concluons que le groupe des automorphismes réguliers des variétés Bogomolov-Guan est Jordan. Dans le cas de la dimension 4 il en est de même pour le groupe des automorphismes biméromorphes.

Title : Birational automorphisms of varieties

Keywords : Automorphisms, Dynamical degrees, Birational geometry

Abstract : The thesis covers various problems on groups of regular and birational automorphisms of algebraic and complex varieties, and roughly explores the differences and similarities between these two groups.

First, we focus on the description of finite subgroups of the group of birational automorphisms $\text{Bir}(X)$ of a rationally connected complex threefold X . We prove that any 3-subgroup in $\text{Bir}(X)$ can be generated by at most 5 elements and in all but two concrete cases it can be generated by at most 4 elements. Also we describe the group of regular automorphisms $\text{Aut}(S)$ of a quasi-projective surface S defined over a field k such that $\text{char}(k) = p > 0$. We show that this group satisfies the p -Jordan property. One of the main ingredients here is the MMP, which allows us to reduce questions about finite subgroups of birational transformations to classifying groups of regular automorphisms of very special algebraic varieties that arise as the end result of MMP.

Then we consider birational automorphisms of infinite order, and try to understand when it is possible

to construct a birational model where the induced automorphism is regular. We are mainly interested in a family of examples of birational automorphisms of the projective space introduced by J. Blanc. These birational automorphisms induce pseudo-automorphisms on a special birational model of the projective space i.e. it has a small extremal locus, and is therefore close to being regular. However, the main result of the second part of the thesis is that it is not conjugate to a regular automorphism.

The last part of the thesis is concerned with the description of the automorphism group of a class of non-Kähler manifolds introduced by D. Guan and further studied by F. Bogomolov. We prove that an algebraic reduction of Bogomolov-Guan manifold of dimension $2n$ is the projective space of dimension n . Then we study the locus of singular fibers of an algebraic reduction and conclude that the group of regular automorphisms of Bogomolov-Guan manifolds is Jordan. In the four-dimensional case the same is true for the group of bimeromorphic automorphisms.