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Etude de certains comportements asymptotiques pour des équations aux dérivées partielles elliptiques

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*À ma Mère pour tous les sacrifices qu'elle a déployés pour mon éducation et ma formation.
À ma femme qui m'a accompagné tout au long de cette aventure. À mes enfants, Myriam, Ilan et Adam, que j'aime infiniment. À mes sœurs, mes neveux et nièces, et toute ma famille. À mon beau-père et toute ma belle-famille...*

Je dédie ce modeste travail

Résumé

Cette thèse porte sur l'étude des perturbations singulières anisotropiques d'une certaine classe de problèmes elliptiques linéaires et non linéaires. Ces problèmes modélisent des phénomènes de diffusion dont les coefficients de diffusion tendent vers zéro dans certaines directions de l'espace. Nous étudions le comportement asymptotique de la solution dans certains espaces pseudo-Sobolev.

Dans un premier temps, nous nous intéressons à la question d'approximation par la méthode de Galerkin et à la question de sa préservation asymptotique pour un problème semi-linéaire monotone. A l'aide de la technique du produit tensoriel, des estimations d'erreurs des problèmes continus et discrets sont obtenues. La question du comportement asymptotique du semi-groupe de la chaleur est également abordée.

Dans un second temps, nous étudions la conservation de la régularité elliptique pour le comportement asymptotique de la solution du problème semi-linéaire, des convergences locales et globales sont étudiées.

Dans un troisième temps, nous étudions un problème non monotone avec un terme non linéaire abstrait. Des applications à des problèmes intégral-différentiels non standards sont envisagées, notamment à l'équation du transport des neutrons.

Finalement, nous étudions le cas des problèmes avec données irrégulières, nous nous limitons au cas L^p avec $1 < p < 2$. Des estimations du taux de convergence sont obtenues dans différents cas pour le problème linéaire. Des questions de convergence pour des problèmes non linéaires sont également traitées.

Mots-clefs: Perturbations singulières anisotropiques, Problèmes elliptiques, Comportement asymptotique, Méthode de Galerkin, Semi-groupe de la chaleur, Taux de convergence, Données L^p , solution entropique.

Abstract

This thesis deals with the study of anisotropic singular perturbations of some class of linear and nonlinear elliptic problems. These problems model diffusion phenomena whose diffusion coefficients go to zero in certain directions of the space. We study the asymptotic behaviour of the solution in some pseudo-Sobolev spaces.

First, we are interested in the question of approximation by Galerkin's method and the question of its asymptotic preservation for a monotone semilinear problem. Using the tensor product technique, error estimates of continuous and discrete problems are proved. The question of the asymptotic behaviour of the heat semigroup is also discussed .

In a second time, we study the conservation of the elliptic regularity for the asymptotic behaviour of the solution of the semi-linear problem, local and global convergences are studied.

Thirdly, we study a non-monotonic problem with a nonlinear abstract term. Applications to non-standard integro-differential problems are discussed, in particular to the neutron transport equation.

Finally, we study the case of problems with irregular data, we limit ourselves to the case L^p with $1 < p < 2$. Estimates of the rate of convergence are obtained in different cases for the linear problem. Convergence questions for nonlinear problems are also considered.

Key-words: Anisotropic singular perturbations, Elliptic problems, Asymptotic behaviour, Galerkin's method, Heat semi-group, Convergence rate, L^p data, entropy solution.

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Introduction

Les problèmes de perturbations singulières occupent une place importante parmi les problèmes mathématiques étudiés actuellement et dans le passé. Ces problèmes apparaissent souvent dans la nature, ils sont caractérisés par la présence d'un paramètre physique ϵ très petit. On les rencontre dans les équations algébriques, les systèmes linéaires, les équations différentielles ordinaires, et dans les équations aux dérivées partielles. Dans le cas de ces dernières, et en se basant sur les problèmes étudiés dans la littérature, on peut les classer selon la position du petit paramètre. Dans certains problèmes, la perturbation se produit dans le domaine, et dans d'autres, elle se produit dans les coefficients, par exemple dans des coefficients de diffusion, ou de viscosité, etc.... Pour les problèmes elliptiques, on peut classer les perturbations qui affectent les coefficients en deux catégories :

- Des perturbations singulières "classiques" dont les coefficients de diffusion tendent vers 0 dans toutes les directions de l'espace. Des prototypes sont donnés par

$$-\epsilon \Delta u_\epsilon + u_\epsilon = f, \text{ +conditions aux bords} \quad (1)$$

$$\epsilon \Delta^2 u_\epsilon - \Delta u_\epsilon = f, \text{ +conditions aux bords} \quad (2)$$

L'étude des problèmes ci-dessus et d'autres est présentée dans le livre de Lions [30], le lecteur pourra également consulter les travaux de Denise Huet [1], [2], [3].

- Des perturbations singulières "anisotropiques" dont les coefficients de diffusion tendent vers 0, seulement dans certaines directions de l'espace. Un prototype est donné par

$$-\epsilon \Delta_{X_1} u_\epsilon - \Delta_{X_2} u_\epsilon = f, u_\epsilon = 0, \text{ sur } \partial\Omega, \quad (3)$$

où Δ_{X_1} et Δ_{X_2} sont les laplaciens de \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, N est la dimension de l'espace, et Ω est un ouvert borné de \mathbb{R}^N . Un problème elliptique linéaire général, donné sous sa formulation variationnelle, est de la forme

$$u_\epsilon \in H_0^1(\Omega) \text{ et } \int_{\Omega} A_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega). \quad (4)$$

Ces dernières ont été introduites récemment par Chipot dans [6], une étude de (4) quand $f \in L^2(\Omega)$ a été réalisée dans [8]. Des problèmes elliptiques impliquant des p-laplaciens, un problème semi-linéaire, problème de l'obstacle, et un problème intégral-différentiel ont également été étudiés dans [9] et [10]. Les résultats démontrés concernent, d'une manière générale, la convergence forte de u_ϵ et $\nabla_{X_2} u_\epsilon$ dans $L^2(\Omega)$ ainsi que l'estimation du taux de convergence. Donnons une brève comparaison entre les difficultés rencontrées lors de l'étude de ces deux types de perturbations. Prenons (2) et (3) mais avec des seconds membres non-linéaires. D'une manière générale, la convergence faible de u_ϵ est facile à prouver dans les deux cas, mais le passage à la limite dans le terme non-linéaire serait plus simple dans le cas de (2) car la convergence faible de ∇u_ϵ est facile à obtenir ce qui implique une convergence forte (d'une sous-suite) de u_ϵ et le passage à la limite dans le terme non-linéaire est obtenu, tandis que dans le cas de (3), l'espace dans lequel u_ϵ et sa limite faible vivent, ne s'injecte d'une manière compacte dans aucun espace de Lebesgue.

Théoriquement, les perturbations singulières peuvent être étudiées de deux manières. Dans la première, le petit paramètre ϵ est une donnée physique du problème P_ϵ . Dans ce cas, on cherche le lien entre u_ϵ , la solution de P_ϵ , et la solution u du problème limite P_0 qui correspond à $\epsilon = 0$. Dans la deuxième, on cherche à étudier l'existence d'une solution à un problème qui présente certaines singularités. Dans ce cas, on introduit un terme affecté d'un petit paramètre afin de rendre le problème plus régulier. Ainsi, dans [22] par exemple, Coclite a étudié l'existence d'une solution pour une équation intégrale non linéaire de type Hammerstien avec un noyau singulier en régularisant le problème avec le terme $-\epsilon \Delta u_\epsilon$. Dans le cadre des perturbations singulières anisotropiques, un problème elliptique simple est donné initialement sous la forme

$$-\Delta_{X_2} u = \text{non-linéarités},$$

où u est définie sur un ouvert de \mathbb{R}^N . Remarquons que Δ_{X_2} n'est pas elliptique sur \mathbb{R}^N , il l'est seulement sur \mathbb{R}^{N-q} . Prouver l'existence d'une solution pour ce problème elliptique "non-standard" à l'aide du théorème de Schauder (par exemple) n'est pas possible, par défaut de compacité. Pour le résoudre, on régularise avec le terme supplémentaire $-\epsilon\Delta_{X_1}$. Comme problème "non standard" on rencontre par exemple le modèle du transport des neutrons [9].

Dans ce mémoire, nous nous intéressons à l'étude de perturbations singulières anisotropiques pour des problèmes elliptiques linéaires et non linéaires selon la régularité des données des problèmes. Plus précisément, nous étudions le comportement asymptotique de la solution d'après la régularité du second membre et de la matrice de diffusion A . Nous examinons également des questions d'approximation. Avant de donner le résumé de chaque chapitre, nous allons définir la matrice de diffusion A_ϵ impliquée dans tous les problèmes étudiés dans ce mémoire.

Soit Ω un ouvert borné de \mathbb{R}^N . Soit $N > q \geq 1$, on pose $x = (x_1, \dots, x_N) = (X_1, X_2) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$ i.e. on décompose les coordonnées en deux parties. Avec cette notation on pose

$$\nabla = (\partial_{x_1}, \dots, \partial_{x_N})^T = \begin{pmatrix} \nabla_{X_1} \\ \nabla_{X_2} \end{pmatrix},$$

où

$$\nabla_{X_1} = (\partial_{x_1}, \dots, \partial_{x_q})^T \text{ et } \nabla_{X_2} = (\partial_{x_{q+1}}, \dots, \partial_{x_N})^T.$$

Soit $A = (a_{ij})_{1 \leq i, j \leq N} : \Omega \rightarrow \mathcal{M}_N(\mathbb{R})$ une fonction qui satisfait les hypothèses d'ellipticité

- Il existe $\lambda > 0$ tel que pour p.p. $x \in \Omega$

$$A\xi \cdot \xi \geq \lambda |\xi|^2 \text{ pour tout } \xi \in \mathbb{R}^N. \tag{5}$$

- Les coefficients de A sont bornés, c-à-d

$$a_{ij} \in L^\infty(\Omega) \text{ pour tout } (i, j) \in \{1, 2, \dots, N\}^2. \tag{6}$$

On décompose A en quatre blocs

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{7}$$

où A_{11} , A_{22} sont des matrices $q \times q$ et $(N - q) \times (N - q)$ respectivement. Pour $\epsilon \in (0, 1]$ on pose

$$A_\epsilon = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix}.$$

0.1 Résumé du chapitre 1

Ce chapitre est consacré à l'étude d'une méthode de Galerkin générale pour un problème semi-linéaire monotone. On s'intéresse à la préservation asymptotique de la méthode et à l'estimation de l'erreur. A l'aide d'une approximation de Galerkin particulière on démontre un nouveau résultat sur le taux de convergence $\|u_\epsilon - u\|$ pour le problème linéaire (4). On démontre également un nouveau résultat sur le comportement asymptotique du semi-groupe généré par l'opérateur $-\text{div}(A_\epsilon \nabla \cdot)$, un tel résultat permet d'étudier des problèmes de perturbations singulières anisotropiques d'évolution du point de vue fort (solutions fortes). Pour un problème parabolique linéaire on obtient une convergence forte du type $\sup_{t \in [0, T]} \|u_\epsilon(t, \cdot) - u(t, \cdot)\|_{L^2(\Omega)} \rightarrow 0$. Pour des problèmes paraboliques, et hyperboliques du point de vue faible (solutions faibles), le lecteur pourra consulter [7], [12], et [13]. Dans [7] les auteurs obtiennent une convergence de la forme $\|u_\epsilon - u\|_{L^2([0, T] \times \Omega)} \rightarrow 0$ pour un problème parabolique non local. Certains résultats de ce chapitre constituent la base d'un travail (en cours) en collaboration avec David Maltese sur l'étude numérique de ces problèmes. Pour les perturbations singulières "classiques", des méthodes d'éléments finis conformes et non-conformes ont déjà été étudiées depuis longtemps. Voir par exemple [23] et [24] et les références citées dans ces deux articles pour une étude de problèmes de type (1). Les résultats de ce chapitre ont fait l'objet d'une publication [18].

Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés de \mathbb{R}^q et \mathbb{R}^{N-q} , avec $N > q \geq 1$. Soit $f \in L^2(\Omega)$. On considère le problème

$$\beta(u_\epsilon) - \text{div}(A_\epsilon \nabla u_\epsilon) = f \text{ dans } \Omega, \tag{8}$$

complété par la condition aux limites

$$u_\epsilon = 0 \text{ sur } \partial\Omega. \tag{9}$$

la fonction $A = (a_{ij})_{1 \leq i, j \leq N} : \Omega \rightarrow \mathcal{M}_N(\mathbb{R})$ vérifie les hypothèses d'ellipticité (5), (6). La fonction

$\beta : \mathbb{R} \rightarrow \mathbb{R}$ satisfait les hypothèses:

$$\beta \text{ est continue et croissante et } \beta(0) = 0. \quad (10)$$

$$\exists M \geq 0 : \forall s \in \mathbb{R}, |\beta(s)| \leq M(1 + |s|). \quad (11)$$

La formulation faible de (8)-(9) est

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon}) \varphi dx + \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_{\epsilon} \in H_0^1(\Omega), \end{cases} \quad (12)$$

où l'existence découle du théorème de Schauder grâce à (5), (6), (10), (11). Le problème limite est donné par

$$\beta(u) - \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u) = f \text{ dans } \Omega, \quad (13)$$

complété par la condition aux limites

$$u(X_1, \cdot) = 0 \text{ sur } \partial\omega_2, \text{ pour } X_1 \in \omega_1. \quad (14)$$

On introduit l'espace fonctionnel

$$H_0^1(\Omega; \omega_2) = \left\{ v \in L^2(\Omega) \text{ tel que } \nabla_{X_2} v \in L^2(\Omega)^{N-q} \text{ et pour p.p. } X_1 \in \omega_1, v(X_1, \cdot) \in H_0^1(\omega_2) \right\},$$

équipé de la norme Hilbertienne $\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}$. Notons que cette norme est équivalente à

$$\left(\|(\cdot)\|_{L^2(\Omega)}^2 + \|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}^2 \right)^{1/2},$$

grâce à l'inégalité de Poincaré

$$\|v\|_{L^2(\Omega)} \leq C_{\omega_2} \|\nabla_{X_2} v\|_{L^2(\Omega)^{N-q}}, \text{ pour tout } v \in H_0^1(\Omega; \omega_2). \quad (15)$$

L'espace $H_0^1(\Omega)$ est normé par $\|\nabla(\cdot)\|_{L^2(\Omega)^N}$. On vérifie immédiatement que l'injection $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega; \omega_2)$ est continue. La formulation faible du problème limite (13) – (14) est donnée par

$$\begin{cases} \int_{\omega_2} \beta(u)(X_1, \cdot) \psi dX_2 + \int_{\omega_2} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 \\ \quad = \int_{\omega_2} f(X_1, \cdot) \psi dX_2, \forall \psi \in H_0^1(\omega_2) \\ u(X_1, \cdot) \in H_0^1(\omega_2), \text{ pour p.p. } X_1 \in \omega_1 \end{cases} \quad (16)$$

Un autre problème semi-linéaire avec une certaine croissance polynomiale a été étudié dans [10] dans le cas $A = Id$.

Le concept de préservation asymptotique a été introduit par S. Jin in [19] pour des Schémas numériques quelconques. Dans [25], par exemple, le lecteur trouvera une étude assez vaste de ce concept pour des modèles de la mécanique des fluides. L'un des aspects de cette notion peut être modélisé par le diagramme commutatif

$$\begin{array}{ccc} P_{\epsilon,n} & \xrightarrow{n \rightarrow \infty} & P_{\epsilon} \\ \epsilon \rightarrow 0 \downarrow & & \downarrow \epsilon \rightarrow 0 \\ P_n & \xrightarrow{n \rightarrow \infty} & P_0 \end{array} ,$$

Ici, $P_{\epsilon,n}$ est l'approximation de Galerkin du problème perturbé P_{ϵ} , et P_n est l'approximation de Galerkin du problème limite P_0 . L'approximation des espaces dans la méthode de Galerkin est prise au sens suivant [26]

Définition 0.1. Soit (V_n) une suite de sous-espaces de dimensions finies d'un espace de Hilbert H . On dit que (V_n) approche H , si pour tout $w \in H$.

$$\inf_{v \in V_n} \|w - v\|_H \longrightarrow 0 \text{ quand } n \rightarrow \infty.$$

Pour une suite (V_n) de sous-espaces de dimensions finies de $H_0^1(\Omega)$, et pour tout $\epsilon \in (0, 1]$ et $n \in \mathbb{N}$, on note $u_{\epsilon,n}$ l'unique solution de

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon,n}) \varphi dx + \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon,n} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V_n. \\ u_{\epsilon,n} \in V_n. \end{cases} \quad (17)$$

On suppose que

$$\partial_{x_i} a_{ij} \in L^{\infty}(\Omega), \partial_{x_j} a_{ij} \in L^{\infty}(\Omega) \text{ pour } i = 1, \dots, q \text{ et } j = q + 1, \dots, N. \quad (18)$$

Nous avons le théorème suivant :

Théorème 0.2. Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, où $N > q \geq 1$. Supposons que $f \in L^2(\Omega)$ et que (5), (6), (10), (11), et (18) sont satisfaites. Soit (V_n) une suite de sous-espaces de dimensions finies de $H_0^1(\Omega)$ qui l'approche au sens de la Définition 0.1. Soit $(u_{\epsilon,n})$ la suite de solutions de (17), on a :

$$\lim_{\epsilon} (\lim_n u_{\epsilon,n}) = \lim_n (\lim_{\epsilon} u_{\epsilon,n}) = u, \text{ dans } H_0^1(\Omega; \omega_2),$$

où u est la solution unique de (16) dans $H_0^1(\Omega; \omega_2)$.

La preuve de ce théorème est basée sur un lemme de type Céa qui donne les deux estimations suivantes

$$\begin{aligned}\|\nabla_{X_2}(u_n - u)\|_{L^2(\Omega)^{N-q}} &\leq C_{\text{céa}} \left(\inf_{v \in V_n} \|\nabla_{X_2}(v - u)\|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}, \\ \|\nabla(u_{\epsilon,n} - u_\epsilon)\|_{L^2(\Omega)^N} &\leq \frac{C'_{\text{céa}}}{\epsilon^2} \left(\inf_{v \in V_n} \|\nabla(v - u_\epsilon)\|_{L^2(\Omega)^N} \right)^{\frac{1}{2}},\end{aligned}$$

et sur une autre estimation de la forme:

$$\|\nabla_{X_2}(u_{\epsilon,n} - u_n)\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 \|\nabla_{X_1} u_n\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right), \quad (19)$$

Ces estimations combinées avec l'inégalité triangulaire fournissent également une estimation de l'erreur globale $\|\nabla_{X_2}(u_{\epsilon,n} - u)\|_{L^2(\Omega)^{N-q}}$. Passons maintenant à l'estimation du taux de convergence globale pour le problème continu. Dans le cas où $\beta = 0$, Chipot et Guesmia ont démontré l'estimation suivante (Théorème 3.1, [8])

$$\forall \omega'_1 \subset\subset \omega_1 \text{ ouvert} : \|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(\epsilon), \text{ et } \|\nabla_{X_1}(u_\epsilon - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(1), \quad (20)$$

où ils ont supposé que

$$\nabla_{X_1} f \in L^2(\Omega)^q, \quad (21)$$

$$\nabla_{X_1} A_{22} \in L^\infty(\Omega), \quad (22)$$

et l'hypothèse (18). Notre contribution consiste en l'extension de (20) sur tout le domaine Ω . Pour obtenir un tel résultat nous allons rajouter des hypothèses supplémentaires sur A et f , à savoir:

$$\text{Pour p.p. } X_2 \in \omega_2 : f(\cdot, X_2) \in H_0^1(\omega_1), \quad (23)$$

et

$$\text{Le bloc } A_{22} \text{ ne dépend que de } X_2. \quad (24)$$

Dans ce cas, nous avons le théorème suivant:

Théorème 0.3. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés de \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, avec $N > q \geq 1$. Supposons que $\beta = 0$, et que A vérifie (5), (6), (18) et (24). Soit $f \in L^2(\Omega)$ telle que (21) et (23), alors il existe une constante $C > 0$ dépendant de f , λ , C_{ω_2} et A telle que:*

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \leq C\epsilon,$$

où u_ϵ est la solution unique de (12) dans $H_0^1(\Omega)$ et u est l'unique solution de (16) dans $H_0^1(\Omega; \omega_2)$.

De plus, nous avons:

$$u \in H_0^1(\Omega) \text{ et } \nabla_{X_1}(u_\epsilon - u) \rightharpoonup 0 \text{ faiblement dans } L^2(\Omega)^q, \text{ quand } \epsilon \rightarrow 0.$$

La constante C est de la forme $C_1 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)}$ où C_1, C_2 dépendent de A, λ, C_{ω_2} .

L'hypothèse (23) est nécessaire pour obtenir une telle extension. En effet, pour un f général il y a un phénomène de couche limite qui se produit lors du passage à la limite quand $\epsilon \rightarrow 0$. Ce dernier se traduit par la perte des conditions au bord i.e. la fonction limite u n'est pas forcément dans $H_0^1(\Omega)$. Pour mieux comprendre ce qui se passe, observons cet exemple en dimension 2:

$$A = id_2, f : (x_1, x_2) \mapsto \cos(x_1) \sin(x_2), \text{ et } \Omega = (0, \pi) \times (0, \pi).$$

Dans ce cas, nous avons $u(x_1, x_2) = \cos(x_1) \sin(x_2)$. La quantité $\|\nabla_{X_1}(u_\epsilon - u)\|_{L^2(\Omega)^q}$ ne peut pas être bornée. En effet, si nous supposons l'inverse alors le Théorème 1.1 affirme qu'il existe une sous-suite notée également (u_ϵ) telle que $\nabla_{X_1}(u_\epsilon - u) \rightharpoonup 0$ faiblement dans $L^2(\Omega)^q$, et $\|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \rightarrow 0$. Donc $u \in H_0^1(\Omega)$ ce qui est absurde. Pour y remédier à cela, il faudrait introduire un terme supplémentaire dans l'estimation de l'erreur, ce terme s'appelle "correcteur". Le calcul de ce dernier n'est pas toujours une tâche facile, pour plus de détails sur la théorie de ces correcteurs le lecteur pourra consulter le livre de J. L. Lions [30], ainsi que [11] pour une étude du problème (3) dans un domaine Ω particulier.

Présentons maintenant les grandes lignes de la démonstration du Théorème 0.3. La preuve se fait en deux étapes. Nous démontrons le théorème dans le cas $f \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ et puis on conclut par un argument de densité. Dans le premier cas, nous construisons une suite d'approximation particulière dont les termes sont donnés par $V_n = V_n^{(1)} \otimes V_n^{(2)}$, à l'aide de cette suite nous arrivons à contrôler la quantité $\|\nabla_{X_1} u_n\|_{L^2(\Omega)^q}$ dans (19) indépendamment de n , sous l'hypothèse supplémentaire (24). Nous terminons la preuve par un passage à la limite $n \rightarrow \infty$. Pour l'argument de densité nous démontrons (Annexe A) que toute fonction vérifiant (21) et (23) peut être approchée, par rapport à $\|\nabla_{X_1}(\cdot)\|_{L^2(\Omega)^q}$, par une fonction de $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$.

Finissons ce résumé par la donnée du résultat concernant le comportement asymptotique du semi-groupe généré. Pour $\epsilon \in (0, 1]$, on introduit l'opérateur \mathcal{A}_ϵ agissant sur $L^2(\Omega)$ et donné par la formule

$$\mathcal{A}_\epsilon u = \operatorname{div}(A_\epsilon \nabla u).$$

Le domaine de \mathcal{A}_ϵ est donné par

$$D(\mathcal{A}_\epsilon) = \{u \in H_0^1(\Omega) \mid \operatorname{div}(A_\epsilon \nabla u) \in L^2(\Omega)\},$$

où $\operatorname{div}(A_\epsilon \nabla u) \in L^2(\Omega)$ est pris au sens des distributions. On introduit également l'opérateur \mathcal{A}_0 défini sur

$$D(\mathcal{A}_0) = \{u \in H_0^1(\Omega; \omega_2) \mid \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u) \in L^2(\Omega)\},$$

par la formule

$$\mathcal{A}_0 u = \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u).$$

Nous avons besoin d'une hypothèse de régularité supplémentaire sur le bloc A_{12} , à savoir

$$\partial_{x_i x_j}^2 a_{ij} \in L^2(\Omega), \text{ pour } i = 1, \dots, q, j = q + 1, \dots, N, \quad (25)$$

et d'une hypothèse de régularité lipschitzienne sur A i.e.

$$A \in W^{1,\infty}(\Omega)^{N^2}. \quad (26)$$

Notons que (26) montre que pour tout $\epsilon \in (0, 1]$, $H_0^1(\Omega) \cap H^2(\Omega) \subset D(\mathcal{A}_0) \cap D(\mathcal{A}_\epsilon)$. Remarquons aussi que (26) implique (18). Nous avons le Théorème suivant:

Théorème 0.4. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés de \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, avec $N > q \geq 1$. Supposons (5), (6), (24), (25) et (26). Soient $g \in L^2(\Omega)$ et $T \geq 0$, On a :*

$$\sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \rightarrow 0 \text{ quand } \epsilon \rightarrow 0.$$

En particulier, pour $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ il existe $C_{g,A,\Omega} > 0$ tel que :

$$\forall \epsilon \in (0, 1] : \sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times T \times \epsilon.$$

Ici, $(S_\epsilon(t))_{t \geq 0}$ et $(S_0(t))_{t \geq 0}$ sont les semi-groupes générés par \mathcal{A}_ϵ et \mathcal{A}_0 respectivement.

Pour démontrer ce théorème, nous supposons que $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$, ensuite nous dérivons cette estimation uniforme sur les semi-groupes associés aux approximations de Yosida $\mathcal{A}_{\epsilon,\mu}$, $\mathcal{A}_{0,\mu}$ des opérateurs \mathcal{A}_ϵ et \mathcal{A}_0 :

$$\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \left\| e^{t\mathcal{A}_{\epsilon,\mu}} g - e^{t\mathcal{A}_{0,\mu}} g \right\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon.$$

Un passage à la limite quand $\mu \rightarrow +\infty$ nous donne l'estimation donnée dans le théorème, ensuite à l'aide d'un argument de densité nous prouvons la convergence quand $g \in L^2(\Omega)$.

0.2 Résumé du chapitre 2

Ce chapitre est consacré à l'étude de la régularité du comportement asymptotique de u_ϵ solution de (4). Nous supposons que la matrice de diffusion est suffisamment régulière (de classe C^1) et nous démontrons un résultat de convergence dans un espace pseudo-Sobolev local d'ordre 2. On obtient la convergence dans l'espace global dans le cas où le domaine est un pavé de dimension N . Les résultats de ce chapitre, excepté le résultat de la convergence globale, ont fait l'objet d'une publication [16]. Soit Ω un ouvert borné quelconque (non nécessairement cylindrique) de \mathbb{R}^N , et nous considérons (4).

Les coefficients a_{ij}^ϵ de A_ϵ sont donnés par

$$a_{ij}^\epsilon = \begin{cases} \epsilon^2 a_{ij} & \text{pour } i, j \in \{1, \dots, q\} \\ a_{ij} & \text{pour } i, j \in \{q+1, \dots, N\} \\ \epsilon a_{ij} & \text{pour } i \in \{1, \dots, q\}, j \in \{q+1, \dots, N\} \\ \epsilon a_{ij} & \text{pour } i \in \{q+1, \dots, N\}, j \in \{1, \dots, q\}. \end{cases}$$

Nous introduisons l'espace fonctionnel

$$H_0^1(\Omega; \Pi_2\Omega) = \left\{ u \in L^2(\Omega) \mid \nabla_{X_2} u \in L^2(\Omega)^{N-q} \text{ et } u(X_1, \cdot) \in H_0^1(\Omega_{X_1}) \text{ p.p. } X_1 \in \Pi_1\Omega \right\},$$

équipé de la norme Hilbertienne

$$\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}.$$

cette dernière est équivalente à la norme

$$\left(\|\cdot\|_{L^2(\Omega)}^2 + \|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}^2 \right)^{\frac{1}{2}},$$

grâce à l'inégalité de Poincaré

$$\forall v \in H_0^1(\Omega; \Pi_2\Omega) : \|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla_{X_2} v\|_{L^2(\Omega)^{N-q}} \quad (27)$$

Ici, Ω_{X_1} est l'ouvert de \mathbb{R}^{N-q} donné par

$$\Omega_{X_1} = \left\{ X_2 \in \mathbb{R}^{N-q} : (X_1, X_2) \in \Omega \right\}.$$

$\Pi_1\Omega$ et $\Pi_2\Omega$ sont les images de Ω par les projections canoniques $\Pi_1: \mathbb{R}^N \rightarrow \mathbb{R}^q$ et $\Pi_2: \mathbb{R}^N \rightarrow \mathbb{R}^{N-q}$ respectivement. On introduit l'espace local

$$H_{loc}^2(\Omega; \Pi_2\Omega) = \left\{ u \in H_0^1(\Omega; \Pi_2\Omega) \mid \nabla_{X_2}^2 u \in L_{loc}^2(\Omega)^{(N-q)^2} \right\},$$

équipé de la famille des normes

$$\left\{ \left(\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}^2 + \|\nabla_{X_2}^2(\cdot)\|_{L^2(\omega)^{(N-q)^2}}^2 \right)^{\frac{1}{2}} \right\}_{\omega \subset\subset \Omega \text{ ouvert}}$$

où $\nabla_{X_2}^2(\cdot)$ est la matrice Hessienne dans la direction X_2 . Le terme $\|\nabla_{X_2}^2(\cdot)\|_{L^2(\omega)}^2$ est donné par

$$\|\nabla_{X_2}^2(\cdot)\|_{L^2(\omega)^{(N-q)^2}}^2 = \sum_{i,j=q+1}^N \|\partial_{ij}^2(\cdot)\|_{L^2(\omega)}^2.$$

On peut prouver aisément que $H_{loc}^2(\Omega; \Pi_2\Omega)$ est un espace de Fréchet i.e. localement convexe, métrisable, et complet. Nous définissons également

$$\|\nabla_{X_1}^2(\cdot)\|_{L^2(\omega)^{q^2}}^2 = \sum_{i,j=1}^q \|\partial_{ij}^2(\cdot)\|_{L^2(\omega)}^2,$$

et

$$\|\nabla_{X_1 X_2}^2(\cdot)\|_{L^2(\omega)^{q(N-q)}}^2 = \sum_{i=1}^q \sum_{j=q+1}^N \|\partial_{ij}^2(\cdot)\|_{L^2(\omega)}^2.$$

La formulation faible du problème limite est donnée par:

$$u \in H_0^1(\Omega; \Pi_2\Omega) \text{ et } \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega; \Pi_2\Omega). \quad (28)$$

L'existence et l'unicité d'une solution de (28) dans $H_0^1(\Omega; \Pi_2\Omega)$ découlent du théorème de Lax-Milgram. Dans [8] les auteurs ont montré que $u \in H_0^1(\Omega; \Pi_2\Omega)$ ainsi que les convergences suivantes ([8], Théorème 2.1):

$$u_\epsilon \rightarrow u \text{ dans } H_0^1(\Omega; \Pi_2\Omega) \text{ et } \epsilon \nabla_{X_1} u_\epsilon \rightarrow 0 \text{ in } L^2(\Omega)^q. \quad (29)$$

Pour obtenir la convergence des dérivées secondes, nous avons besoin de l'hypothèse de régularité suivante :

$$A \in C^1(\Omega)^{N^2}. \quad (30)$$

Dans ce cas, nous avons le théorème suivant :

Théorème 0.5. *Soit Ω un ouvert borné de \mathbb{R}^N . Supposons (5), (6) et (30). Supposons que $f \in L^2(\Omega)$, alors $u \in H_{loc}^2(\Omega; \Pi_2\Omega)$ et $u_\epsilon \rightarrow u$ dans $H_{loc}^2(\Omega; \Pi_2\Omega)$, où $u_\epsilon \in H_0^1(\Omega) \cap H_{loc}^2(\Omega)$ et u sont les solutions uniques de (4) et (28) respectivement. De plus, on a les convergences $\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0$, $\epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0$ dans $L_{loc}^2(\Omega)^{q^2}$ et $L_{loc}^2(\Omega)^{q(N-q)}$ respectivement.*

Le théorème 0.5 peut être étendu au cas semi-linéaire monotone. Considérons (12) et son problème limite associé

$$u \in H_0^1(\Omega; \Pi_2\Omega) \text{ et } \int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega; \Pi_2\Omega). \quad (31)$$

Nous avons le résultat suivant :

Théorème 0.6. *Soit Ω un ouvert borné de \mathbb{R}^N . Supposons (5), (6) et (30). Supposons que β vérifie (10) et (11). Supposons que $f \in L^2(\Omega)$, alors $u \in H_{loc}^2(\Omega; \Pi_2\Omega)$ et $u_\epsilon \rightarrow u$ dans $H_{loc}^2(\Omega; \Pi_2\Omega)$, où $u_\epsilon \in H_0^1(\Omega) \cap H_{loc}^2(\Omega)$ est la solution unique de (12) et u est la solution unique de (31). De plus, on a les convergences $\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0$, $\epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0$ dans $L_{loc}^2(\Omega)^{q^2}$ et $L_{loc}^2(\Omega)^{q(N-q)}$ respectivement.*

Les démonstrations de ces théorèmes sont similaires. Elles sont basées sur le théorème de compacité de Fréchet-Kolmogorov dans les espaces de Lebesgue. Notons que l'hypothèse $A \in C^1(\Omega)^{N^2}$ est assez forte pour la régularité elliptique des problèmes perturbés. Il est bien connu que si A est lipschitzienne alors $u_\epsilon \in H_{loc}^2(\Omega)$ [20]. Cependant, cette hypothèse ne nous permet pas, suivant notre Schéma de démonstration, de prouver les convergences voulues. En effet, Si A est lipschitzienne et s'il existe un sous-ensemble $S \subset \Omega$ de mesure nulle tel que les dérivées $\partial_i a_{ij}$, $i, j = 1, \dots, N$ vérifient l'hypothèse

$$(\text{Unif}) : \lim_{h \rightarrow 0} \max_{i,j} \sup_{x \in \omega \setminus S} |\partial_i a_{ij}(x) - \tau_h \partial_i a_{ij}(x)| = 0, \text{ pour tout } \omega \subset\subset \Omega \text{ ouvert,}$$

alors nous aurons les mêmes résultats des Théorèmes 0.5 et 0.6. Pour plus de détails, voir Remarque 2.6 du Chapitre 2. Signalons que les convergences données dans les théorèmes ci-dessus sont locales. Pour la convergence globale i.e. sur tout le domaine Ω , nous aurons besoin, tout d'abord, de la régularité H^2 globale de u_ϵ . Il est classique que pour obtenir une telle régularité, la régularité de A doit être étendue jusqu'au bord de Ω . Nous obtenons dans ce cas la régularité globale du comportement asymptotique des dérivées secondes de u_ϵ lorsque Ω est un pavé de dimension N . Pour un ouvert de classe C^1 ou convexe polygonal quelconque, la question reste ouverte. Supposons que Ω est de la forme

$$\Omega = \prod_{k=1}^N (0, l_k), \quad (32)$$

où $l_i > 0$, $i = 1, \dots, N$. Introduisons l'espace

$$H^2(\Omega; \Pi_2\Omega) = \left\{ u \in H_0^1(\Omega; \Pi_2\Omega) \mid \nabla_{X_2}^2 u \in L^2(\Omega)^{(N-q)^2} \right\},$$

normé par la norme Hilbertienne $(\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}^2 + \|\nabla_{X_2}^2(\cdot)\|_{L^2(\Omega)^{(N-q)^2}}^2)^{\frac{1}{2}}$. Supposons que A soit suffisamment régulière jusqu'au bord i.e.

$$A \in C^1(\bar{\Omega})^{N^2}, \quad (33)$$

où les éléments non diagonaux sont nuls sur le bord, c.à.d.

$$\text{Pour tout, } i \neq j : a_{ij} = 0 \text{ sur } \partial\Omega. \quad (34)$$

Remarquons que (33) implique (6). Ainsi, nous avons le Théorème suivant :

Théorème 0.7. *Soit Ω tel que (32). Supposons (5), (33), et (34). Supposons que $f \in L^2(\Omega)$, alors $u \in H^2(\Omega; \Pi_2\Omega)$ et $u_\epsilon \rightarrow u$ dans $H^2(\Omega; \Pi_2\Omega)$, où $u_\epsilon \in H_0^1(\Omega) \cap H^2(\Omega)$ et u sont les solutions uniques de (4) et (28) respectivement. De plus, on a les convergences $\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0$, $\epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0$ dans $L^2(\Omega)^{q^2}$ et $L^2(\Omega)^{q(N-q)}$ respectivement.*

Pour démontrer ce résultat, nous allons prolonger (par symétrie) le problème (4) sur un pavé $\tilde{\Omega}$ contenant Ω , la solution \tilde{u}_ϵ du problème elliptique obtenu prolonge ainsi u_ϵ . Nous appliquons ensuite

le Théorème 0.5 où la nouvelle matrice de diffusion du problème prolongé vérifie l'hypothèse **(Unif)**. Ainsi, nous obtenons la convergence à l'intérieur de $\tilde{\Omega}$ ce qui entraîne la convergence au voisinage de chaque point de $\partial\Omega$. Finalement, on conclut par la compacité de $\overline{\Omega}$.

0.3 Résumé du chapitre 3

Ce chapitre est dédié à l'étude d'un problème elliptique non linéaire non monotone. On donnera des applications à l'étude de certains problèmes elliptiques "non standards" inspirés d'une équation modélisant le transport des neutrons [9] et d'une équation intégrale étudiée dans [22]. Le théorème principal de ce chapitre a fait l'objet d'une publication [17] dans le cas où Ω est convexe. Nous nous sommes rendu compte que cette hypothèse de convexité est en excédent. La version présentée dans ce mémoire ne suppose aucune hypothèse de régularité sur le domaine et la démonstration reste identique à celle de la version publiée.

La distribution des neutrons dans un réacteur nucléaire est modélisée par l'équation de Boltzmann

$$\hat{\Omega} \cdot \nabla \psi(r, \hat{\Omega}) + \sigma(r)\psi(r, \hat{\Omega}) = \sigma_s(r)\phi(r) + s(r, \hat{\Omega}). \quad (35)$$

L'inconnue de l'équation est le flux angulaire des neutrons $\psi(r, \hat{\Omega})$. Le vecteur $\hat{\Omega}$ représente la direction du mouvement d'un neutron. Le gradient ∇ agit seulement sur la variable spatiale r . Le terme $\sigma(r)$ représente la section efficace macroscopique totale, et $\sigma_s(r)$ représente la section efficace macroscopique de diffusion. Le terme source est donné par $s(r, \hat{\Omega})$. Enfin, le flux scalaire $\phi(r)$ est donné par

$$\phi(r) = \int \psi(r, \hat{\Omega}) d\hat{\Omega}$$

En considérant $\psi^+(r, \hat{\Omega}) = \frac{1}{2} (\psi(r, \hat{\Omega}) + \psi(r, -\hat{\Omega}))$ la composante paire du flux $\psi(r, \Omega)$, une équation du second ordre est dérivée, à savoir

$$\hat{\Omega} \cdot \nabla \left(\frac{1}{\sigma(r)} \hat{\Omega} \cdot \nabla \psi^+(r, \hat{\Omega}) \right) + \sigma(r)\psi^+(r, \hat{\Omega}) = \sigma_s(r) \int \psi^+(r, \hat{\Omega}) d\hat{\Omega} + \dots \quad (36)$$

Cette équation est de second ordre en r avec un terme non local donné par une intégrale partielle en $\hat{\Omega}$. Pour plus de détails, nous référons le lecteur à [9] et aux références citées dans cet article.

Motivé par ce modèle nous considérons des problèmes elliptiques "non standards", des problèmes dont l'opérateur n'est elliptique que sur un sous-espace vectoriel de l'espace \mathbb{R}^N .

Nous traitons, dans un premier temps, un problème de perturbations singulières non monotone, dont le terme non linéaire est donné par un opérateur abstrait. Nous donnons, ensuite, des applications à des problèmes inspirés de (36).

Soit $\Omega = \omega_1 \times \omega_2$. Considérons le problème suivant :

$$\begin{cases} \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx + \gamma \int_{\Omega} u_{\epsilon} \varphi dx = \int_{\Omega} B(u_{\epsilon}) \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_{\epsilon} \in H_0^1(\Omega), \end{cases} \quad (37)$$

où $\gamma > 0$ est une constante. On suppose que l'opérateur B est tel que

$$B : L^1(\Omega) \rightarrow L^2(\Omega) \text{ continu.} \quad (38)$$

On suppose que sa restriction sur $L^2(\Omega)$ vérifie

$$\exists M \geq 0, \forall u \in L^2(\Omega) : \|B(u)\|_{L^2(\Omega)} \leq M(1 + \|u\|_{L^2(\Omega)}). \quad (39)$$

On introduit l'espace vectoriel

$$W = \{u \in L^2(\Omega) : \nabla_{X_1} u \in L^2(\Omega)\}.$$

Nous supposons que B vérifie l'hypothèse géométrique

$$\text{Pour tout } E \subset W \text{ borné dans } L^2(\Omega) : \overline{\text{conv}(B(E))} \subset W. \quad (40)$$

Ici, $\overline{\text{conv}(B(E))}$ est l'enveloppe convexe fermée de $B(E)$ dans $L^2(\Omega)$. Le problème limite associé à (37) est donné par

$$\begin{cases} \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx + \gamma \int_{\Omega} u \varphi dx = \int_{\Omega} B(u) \varphi dx, \forall \varphi \in H_0^1(\Omega; \omega_2) \\ u \in H_0^1(\Omega; \omega_2). \end{cases} \quad (41)$$

Comme l'opérateur B est non monotone, les solutions des problèmes (37) et (41) ne sont pas nécessairement uniques, voir par exemple [29] pour le cas des équations elliptiques semi-linéaires.

Nous avons le théorème principal suivant :

Théorème 0.8. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, où $N > q \geq 1$. Supposons (5), (6), (18), (22) (38), (39), et (40). Supposons que $\gamma > M$. Soit $(u_\epsilon)_{0 < \epsilon \leq 1}$ une suite de solutions de (37), alors il existe $u \in H_0^1(\Omega; \omega_2)$ solution de (41) et une sous-suite (u_{ϵ_k}) telles que*

$$\|\nabla_{X_2}(u_{\epsilon_k} - u)\|_{L^2(\Omega)^{N-q}} \longrightarrow 0 \text{ quand } k \rightarrow +\infty$$

En particulier, à l'aide d'un raisonnement par l'absurde immédiat nous pouvons donner le corollaire suivant :

Corollaire 0.9. *Supposons que les hypothèses du Théorème 0.8 sont satisfaites. Supposons que (41) admet une solution unique u , alors pour toute suite $(u_\epsilon)_{0 < \epsilon \leq 1}$ de solutions de (37) nous avons :*

$$\|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \longrightarrow 0 \text{ quand } \epsilon \rightarrow 0.$$

Remarque 0.10. *Le Théorème 0.8 reste vrai si on remplace l'hypothèse (38) par une autre plus générale, à savoir*

$$\text{Il existe } 1 \leq r < 2 \text{ tel que } B : L^r(\Omega) \longrightarrow L^2(\Omega) \text{ est continu.} \quad (42)$$

Pour plus de détails, voir Remarque 3.7).

Remarque 0.11. *Le Théorème 0.8 reste vrai si on remplace l'hypothèse (39) par*

$$\sup_{u \in L^2(\Omega)} \|B(u)\|_{L^2(\Omega)} < \infty.$$

Dans ce cas on peut prendre $\gamma \geq 0$.

La principale difficulté dans la démonstration du Théorème 0.8 réside dans le passage à la limite dans le terme non linéaire tout en ne disposant que de la convergence faible $u_\epsilon \rightharpoonup u$ dans $H_0^1(\Omega; \omega_2)$. De plus, cet espace ne s'injecte de manière compacte dans aucun espace de Lebesgue. Dans le cas monotone, la difficulté est surmontée à l'aide de la monotonie (Chapitre 1, Théorème 1.1). Pour le présent problème, l'hypothèse (40) va jouer un rôle crucial dans l'obtention des estimations L^2 locales du gradient global de u_ϵ , ainsi une convergence forte dans $L_{loc}^2(\Omega)$ est obtenue ainsi que le passage à la limite dans le terme non linéaire.

0.3.1 Applications 1

Soit $\mathbb{A} : \Omega \longrightarrow \mathcal{M}_{N-q}(\mathbb{R})$ telle qu'il existe $\lambda > 0$ tel que pour p.p. $x \in \Omega$

$$\mathbb{A}\xi \cdot \xi \geq \lambda |\xi|^2 \text{ pour tout } \xi \in \mathbb{R}^{N-q}. \quad (43)$$

Nous supposons que

$$\mathbb{A} \in L^\infty(\Omega)^{(N-q)^2}, \quad (44)$$

avec l'hypothèse de régularité

$$\nabla_{X_1} \mathbb{A} \in L^\infty(\Omega)^{q(N-q)^2}, \quad (45)$$

Nous considérons le problème elliptique "non standard" suivant

$$\begin{cases} \int_{\Omega} \mathbb{A} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx + \gamma \int_{\Omega} u \varphi dx = \int_{\Omega} \int_{\Omega} K(x, y) b(y, u(y)) \varphi dx dy, \forall \varphi \in H_0^1(\Omega; \omega_2) \\ u \in H_0^1(\Omega; \omega_2), \end{cases} \quad (46)$$

où $b : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ est une fonction de Carathéodory telle que

$$\text{Pour p.p. } x \in \Omega : |b(x, t)| \leq g(x) + c |t|^{\frac{1}{2}}, \text{ pour tout } t \in \mathbb{R}. \quad (47)$$

La fonction $g \in L^2(\Omega)$ est positive, et $c \geq 0$ est une constante. Nous supposons que le noyau $K : \Omega^2 \rightarrow \mathbb{R}$ est régulier i.e.

$$K \in L^2(\Omega^2), \text{ (rappelons } (x, y) = (X_1, X_2, Y_1, Y_2)). \quad (48)$$

et

$$\nabla_{X_1} K \in L^2(\Omega^2)^q. \quad (49)$$

Nous avons le théorème suivant:

Théorème 0.12. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, où $N > q \geq 1$. Supposons (43), (44), (45). Soit b une fonction de Carathéodory telle que (47). Supposons (48) et (49). Supposons que $\gamma > 2\|K\|_{L^2(\Omega^2)} \max\left(\|g\|_{L^2(\Omega)}, c|\Omega|^{\frac{1}{4}}\right)$, alors (46) admet au moins une solution.*

0.3.2 Application 2

En se basant sur le modèle physique (36), Chipot et Guesmia ont considéré le modèle mathématique suivant [9]

$$\begin{cases} \int_{\Omega} \mathbb{A} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx + \gamma \int_{\Omega} u \varphi dx = \int_{\Omega} a(l(u)) \varphi(x) dx, \forall \varphi \in H_0^1(\Omega; \omega_2) \\ u \in H_0^1(\Omega; \omega_2), \end{cases}$$

où $a : \mathbb{R} \rightarrow \mathbb{R}$ est sous-linéaire et continue et $l(u)(x) = \int_{\omega_1} \tilde{K}(X'_1, X_1, X_2) u(X'_1, X_2) dX'_1$, où le noyau \tilde{K} est tel que

$$\tilde{K} \in L^\infty(\omega_1 \times \Omega), \quad (50)$$

avec l'hypothèse de régularité $\nabla \tilde{K} \in L^\infty(\omega_2; L^2(\omega_1 \times \omega_1))$. Remarquons que le terme non local est donné par une intégration partielle sur le domaine Ω . Dans le problème perturbé associé, les auteurs ont utilisé le résultat suivant pour le passage à limite dans le terme non-linéaire, à savoir si (v_n) de $H_0^1(\Omega; \omega_2)$ est une suite telle que $v_n \rightharpoonup v$ faiblement dans $H_0^1(\Omega; \omega_2)$, alors $l(v_n) \rightarrow l(v)$ dans $L^2(\Omega)$ fortement. La linéarité de l et la régularité de \tilde{K} permettent de montrer que la suite (∇u_n) est bornée dans $L^2(\Omega)$ ([9], Lemme 3.1). En d'autres termes, l'opérateur $l : H_0^1(\Omega; \omega_2) \rightarrow L^2(\Omega)$ est compact. Ensuite, le passage à la limite dans la non-linéarité a est assurée par convergence dominée. Notons que l a une forme proche d'une certaine classe d'opérateurs de Hilbert-Schmidt intégrales, cependant on peut vérifier que ce n'en est pas un à cause de l'intégration partielle. Un opérateur de la forme $\int_{\Omega} \tilde{K}(x, y) dy$ avec un noyau $\tilde{K} \in L^2(\Omega)$, et sans aucune hypothèse de régularité supplémentaire, est de Hilbert Schmidt de $L^2(\Omega)$ dans $L^2(\Omega)$, et donc il envoie $L^2(\Omega)$ dans $L^2(\Omega)$ de manière compacte.

Revenons à notre deuxième application. Nous avons fait le choix de faire rentrer la non linéarité à l'intérieur de l'intégrale pour la rendre plus forte. La méthode de démonstration donnée dans ([9], Lemme 3.1) devient inefficace dans ce cas, et nous devons faire appel au Théorème 0.8 pour résoudre le problème suivant

$$\begin{cases} \int_{\Omega} \mathbb{A} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx + \gamma \int_{\Omega} u \varphi dx = \int_{\Omega} B_2(u) \varphi dx, \forall \varphi \in H_0^1(\Omega; \omega_2) \\ u \in H_0^1(\Omega; \omega_2), \end{cases} \quad (51)$$

avec

$$B_2(u)(x) = l(a(u)) = \int_{\omega_1} \tilde{K}(X'_1, X_1, X_2) a(u(X'_1, X_2)) dX'_1 \quad (52)$$

où $a : \mathbb{R} \rightarrow \mathbb{R}$ est continue telle que

$$\exists c' > 0, \forall t \in \mathbb{R} : |a(t)| \leq c'(1 + |t|^{\frac{1}{2}}). \quad (53)$$

Nous supposons que le noyau \tilde{K} vérifie l'hypothèse de régularité

$$\nabla_{x_1} \tilde{K} \in L^\infty(\omega_1 \times \Omega)^q. \quad (54)$$

Nous avons le théorème

Théorème 0.13. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, où $N > q \geq 1$. Supposons (43), (44), (45). Soit a continue telle que (53). Supposons (50) et (54). Supposons que $\gamma > 2c'|\omega_1|\|\tilde{K}\|_{L^\infty(\omega_1 \times \Omega)} \max(|\Omega|^{\frac{1}{2}}, |\Omega|^{\frac{1}{4}})$, alors (51) admet au moins une solution.*

Remarque 0.14. *Dans les Théorèmes 0.12 et 0.13 les hypothèses (47) et (53) peuvent être remplacées respectivement par*

$$\text{Pour p.p. } x \in \Omega : |b(x, t)| \leq g(x) + c|t|^{\frac{r}{2}}, \text{ pour tout } t \in \mathbb{R}, \quad (55)$$

$$\exists c' > 0 : \forall t \in \mathbb{R} |a(t)| \leq c'(1 + |t|^{\frac{r}{2}}), \quad (56)$$

pour un certain $1 \leq r < 2$. Pour plus de détails voir Remarques 3.7 et 3.5).

0.4 Résumé du chapitre 4

Ce dernier chapitre est consacré à l'étude des problèmes à données peu régulières. Nous nous intéressons au problème (4) dans le cas $f \in L^p$ avec $1 < p < 2$. Par défaut d'unicité des solutions faibles, nous considérons des solutions entropiques dans le sens précisé par Bénéilan, Boccardo, Gallouët et al. [31]. Nous étudions la convergence de u_ϵ dans un espace pseudo-Sobolev de degré p , nous étendons des résultats sur le taux de convergence déjà prouvés dans le cas L^2 dans [8]. Nous donnons également des extensions à des cas non-linéaires. Cette partie a fait l'objet d'une publication [15].

Soit Ω un ouvert borné de \mathbb{R}^N et considérons le problème (8 – 9) avec $f \in L^p(\Omega)$. La solution faible (i.e. au sens des distributions) de (8 – 9) n'est pas toujours unique, dans [39] Serrin a donné un contre exemple. Dans [34] Gallouët et Herbin ont démontré l'unicité de la solution faible en

dimension 2 d'un problème elliptique linéaire avec second membre mesure. Afin de résoudre ce problème d'unicité, des notions de solutions ont été introduites : solutions renormalisées, solutions entropiques,..etc. Dans [38] nous trouvons une étude comparative de certaines de ces notions.

Pour $k > 0$ on définit la troncature $T_k : \mathbb{R} \rightarrow \mathbb{R}$ par:

$$T_k(s) = \begin{cases} s & |s| \leq k \\ k \operatorname{sgn}(s) & |s| \geq k \end{cases}$$

Soit $\mathcal{T}_0^{1,2}$ l'espace défini par:

$$\mathcal{T}_0^{1,2}(\Omega) = \left\{ \begin{array}{l} u : \Omega \rightarrow \mathbb{R} \text{ mesurable telle que pour tout } k > 0 \text{ il existe } (\phi_n) \in H_0^1(\Omega)^{\mathbb{N}} \\ \text{telle que } : \phi_n \rightarrow T_k(u) \text{ p.p. dans } \Omega, \text{ et } (\nabla \phi_n)_{n \in \mathbb{N}} \text{ est bornée dans } L^2(\Omega)^N \end{array} \right\}.$$

Donnons la définition d'une solution entropique

Définition 0.15. Une fonction $u_\epsilon \in \mathcal{T}_0^{1,2}(\Omega)$ est dite solution entropique de (8) – (9) si

$$\int_{\Omega} \beta(u_\epsilon) T_k(u_\epsilon - \varphi) dx + \int_{\Omega} A \nabla u_\epsilon \cdot \nabla T_k(u_\epsilon - \varphi) dx \leq \int_{\Omega} f T_k(u_\epsilon - \varphi) dx, \forall \varphi \in \mathcal{D}(\Omega), \forall k > 0$$

Le résultat principal de [31] montre que (8)-(9) admet une unique solution entropique qui est également une solution faible. De plus, comme Ω est borné alors cette solution appartient à

$$\bigcap_{1 \leq r < \frac{N}{N-1}} W_0^{1,r}(\Omega).$$

De façon analogue, la solution entropique du problème limite (13)-(14) (où $\partial\omega_2$ est remplacé par $\partial\Omega_{X_1}$) existe dans $\mathcal{T}_0^{1,2}(\Omega_{X_1})$ pour p.p. $X_1 \in \Pi_1\Omega$ et elle est unique. Maintenant, introduisons l'espace fonctionnel.

$$W_0^{1,p}(\Omega; \Pi_2\Omega) = \left\{ u \in L^p(\Omega) \mid \nabla_{X_2} u \in L^p(\Omega)^{N-q} \text{ et } u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}) \text{ p.p. } X_1 \in \Pi_1\Omega \right\}, .$$

équipé de la norme banachique

$$\|\nabla_{X_2}(\cdot)\|_{L^p(\Omega)^{N-q}}.$$

cette dernière est équivalente à la norme

$$\left(\|(\cdot)\|_{L^p(\Omega)}^p + \|\nabla_{X_2}(\cdot)\|_{L^p(\Omega)^{N-q}}^p \right)^{\frac{1}{p}},$$

grâce à l'inégalité de Poincaré

$$\forall v \in W_0^{1,p}(\Omega; \Pi_2\Omega) : \|v\|_{L^p(\Omega)} \leq C_{\Omega} \|\nabla_{X_2} v\|_{L^p(\Omega)^{N-q}}, \quad (57)$$

où C_Ω de dépend que de $\Pi_2\Omega$.

Nous avons le théorème de convergence suivant :

Théorème 0.16. *Soit Ω un ouvert borné de \mathbb{R}^N . Supposons (5), (6), (10), (11). Supposons que $f \in L^p(\Omega)$ avec $1 < p < 2$, alors il existe un unique $u \in W_0^{1,p}(\Omega; \Pi_2\Omega)$ tel que $u(X_1, \cdot)$ est la solution entropique de (13) – (14) (où $\partial\omega_2$ est remplacé par $\partial\Omega_{X_1}$) pour p.p. $X_1 \in \Pi_1\Omega$. De plus, la suite $(u_\epsilon)_{0 < \epsilon \leq 1}$ des solutions entropiques de (8) – (9) converge vers u dans $W_0^{1,p}(\Omega; \Pi_2\Omega)$ et $\epsilon \nabla_{X_1} u_\epsilon \rightarrow 0$ dans $L^p(\Omega)^q$.*

Ce théorème est une extension du Théorème 1.1 (Chapitre 1), prouvé dans le cas variationnel (i.e. $f \in L^2(\Omega)$). Maintenant, supposons que $\beta = 0$. Le deuxième résultat concerne l'estimation du taux de convergence locale dans le cas où le domaine Ω est cylindrique. nous référons le lecteur à [6], [8] pour les différentes estimations (taux de convergence polynomial, exponentiel..etc) pour le cas $f \in L^2(\Omega)$. Nous allons supposer des hypothèses supplémentaires à savoir

$$\text{Le bloc } A_{12} \text{ ne dépend pas de } X_1, \tag{58}$$

$$\text{La fonction } f \text{ ne dépend pas de } X_1 \text{ et } f \in L^p(\omega_2). \tag{59}$$

Nous avons l'estimation suivante du taux de convergence :

Théorème 0.17. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés de \mathbb{R}^q et \mathbb{R}^{N-q} respectivement. Supposons (5), (6), (24), et (58). Supposons que $\beta = 0$ et que f vérifie (59). Alors, pour tout ouvert $\omega'_1 \subset\subset \omega_1$ et $d \in \mathbb{N}^*$ il existe $C > 0$ indépendant de ϵ tel que*

$$\forall \epsilon \in (0, 1] : \|u_\epsilon - u\|_{W^{1,p}(\omega'_1 \times \omega_2)} \leq C\epsilon^d,$$

où u_ϵ et u sont les solutions entropiques de (8) – (9) et (13) – (14) respectivement.

Lorsque ω_1 est convexe contenant 0, nous pouvons obtenir une vitesse de convergence plus rapide.

Théorème 0.18. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés de \mathbb{R}^q et \mathbb{R}^{N-q} respectivement, avec ω_1 convexe contenant 0. Supposons (5), (6), (24), et (58). Supposons que $\beta = 0$ et que f vérifie*

(59). Alors, pour tout $\alpha \in (0, 1)$ il existe $C, c > 0$ indépendants de ϵ tel que

$$\forall \epsilon \in (0, 1] : \|u_\epsilon - u\|_{W^{1,p}(\alpha\omega_1 \times \omega_2)} \leq C e^{-\frac{c}{\epsilon}},$$

où u_ϵ et u sont les solutions entropiques de (8) – (9) et (13) – (14) respectivement.

Nous finissons par l'étude du problème intégral-différentiel du transport de neutrons

$$\begin{cases} \int_{\Omega} \mathbb{A} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = f + \int_{\Omega} B_2(u) \varphi dx, \forall \varphi \in \mathcal{D}(\Omega) \\ u \in W_0^{1,p}(\Omega; \omega_2), \end{cases} \quad (60)$$

La fonction matricielle \mathbb{A} est définie comme dans Application 1 (Introduction 0.3.1), et l'opérateur B_2 est donné par (52). Nous avons le résultat d'existence suivant

Théorème 0.19. *Soit $\Omega = \omega_1 \times \omega_2$ où ω_1 et ω_2 sont deux ouverts bornés de \mathbb{R}^q et \mathbb{R}^{N-q} respectivement. Supposons que \mathbb{A} vérifie (43), (44). Supposons que \mathbb{A} est indépendante de X_1 . Soit f telle que (59). Supposons que a est bornée et continue. Supposons que le noyau \tilde{K} vérifie (50) et (54). Alors, (60) admet au moins une solution.*

Chapter 1

Galerkin Approximation in Anisotropic Singular Perturbations

1 The Analysis of a general Galerkin method

1.1 Preliminaries

In this section we deal with problem (12) that we recall here

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon}) \varphi dx + \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_{\epsilon} \in H_0^1(\Omega). \end{cases} \quad (1.1)$$

The associated limit problem is given by

$$\begin{cases} \int_{\omega_2} \beta(u)(X_1, \cdot) \psi dX_2 + \int_{\omega_2} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 \\ \quad = \int_{\omega_2} f(X_1, \cdot) \psi dX_2, \forall \psi \in H_0^1(\omega_2) \\ u(X_1, \cdot) \in H_0^1(\omega_2), \text{ for a.e. } X_1 \in \omega_1. \end{cases} \quad (1.2)$$

Recall that A satisfies the ellipticity assumptions (5),(6), and that β is nondecreasing continuous and sublinear (assumptions (10), (11)).

Before the analysis of the Galerkin method, let us study the asymptotic behaviour of the solution of problem (1.1). We have the following

Theorem 1.1. *Let $\Omega = \omega_1 \times \omega_2$ where ω_1 and ω_2 are two bounded open sets of \mathbb{R}^q and \mathbb{R}^{N-q} respectively, with $N > q \geq 1$. Under assumptions (5), (6), (10) and (11) we have:*

$$u_{\epsilon} \rightarrow u \text{ in } L^2(\Omega), \quad \epsilon \nabla_{X_1} u_{\epsilon} \rightarrow 0 \text{ in } L^2(\Omega)^q \text{ and } \nabla_{X_2} u_{\epsilon} \rightarrow \nabla_{X_2} u \text{ in } L^2(\Omega)^{N-q},$$

where u_{ϵ} is the unique solution to (1.1) in $H_0^1(\Omega)$ and u is the unique solution to (1.2) in $H_0^1(\Omega; \omega_2)$.

Proof. The existence of u_ϵ follows the Schauder fixed point theorem (see Appendix C). The uniqueness follows from assumption (10) and Poincaré's inequality. Taking u_ϵ as a test function in (1.1) and using (5), (10), and (15). We obtain

$$\epsilon \|\nabla_{X_1} u_\epsilon\|_{L^2(\Omega)}, \|\nabla_{X_2} u_\epsilon\|_{L^2(\Omega)}, \|u_\epsilon\|_{L^2(\Omega)} \leq c, \quad (1.3)$$

where $c > 0$ is independent of ϵ . By using (11) and (1.3) we get

$$\|\beta(u_\epsilon)\|_{L^2(\Omega)} \leq M(|\Omega|^{\frac{1}{2}} + c) \quad (1.4)$$

The bounds (1.3) and (1.4) show, by using reflexivity of $L^2(\Omega)$ and continuity of the derivation on $\mathcal{D}(\Omega)$, that there exist $v \in L^2(\Omega)$, $u \in L^2(\Omega)$, $\nabla_{X_2} u \in L^2(\Omega)$, and a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that

$$\beta(u_{\epsilon_k}) \rightharpoonup v, \epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightharpoonup 0, \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u, u_{\epsilon_k} \rightharpoonup u \text{ in } L^2(\Omega) \text{ weakly.} \quad (1.5)$$

Passing to the limit in (1.1), by using (1.5), we get

$$\int_{\Omega} v \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega). \quad (1.6)$$

Taking $\varphi = u_{\epsilon_k}$ in (1.6) and passing to the limit by using (1.5), we obtain

$$\int_{\Omega} v u dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u dx = \int_{\Omega} f u dx. \quad (1.7)$$

Now, consider the quantity

$$\begin{aligned} I_k &= \int_{\Omega} A_{\epsilon_k} \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2}(u_{\epsilon_k} - u) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2}(u_{\epsilon_k} - u) \end{pmatrix} dx \\ &\quad + \int_{\Omega} (\beta(u_{\epsilon_k}) - \beta(u))(u_{\epsilon_k} - u) dx \\ &= \int_{\Omega} f u_{\epsilon_k} dx - \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u \cdot \nabla_{X_1} u_{\epsilon_k} dx - \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{\epsilon_k} \cdot \nabla_{X_2} u dx \\ &\quad - \int_{\Omega} A_{22} \nabla_{X_2} u_{\epsilon_k} \cdot \nabla_{X_2} u dx - \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u_{\epsilon_k} dx + \int_{\Omega} f u dx \\ &\quad - \int_{\Omega} v u dx - \int_{\Omega} \beta(u) u_{\epsilon_k} dx - \int_{\Omega} \beta(u_{\epsilon_k}) u dx + \int_{\Omega} \beta(u) u dx. \end{aligned}$$

Passing to the limit in the above equality by using (1.5), (1.6), (1.7) we get

$$\lim I_k = 0. \quad (1.8)$$

By using (5) and (10) we get

$$\lambda \|\epsilon_k \nabla_{X_1} u_{\epsilon_k}\|_{L^2(\Omega)}^2 + \lambda \|\nabla_{X_2}(u_{\epsilon_k} - u)\|_{L^2(\Omega)}^2 \leq I_k \quad (1.9)$$

Passing to the limit in (1.9) by using (1.8) and (15). We obtain that

$$\|\epsilon_k \nabla_{X_1} u_{\epsilon_k}\|_{L^2(\Omega)}, \|\nabla_{X_2}(u_{\epsilon_k} - u)\|_{L^2(\Omega)}, \|u_{\epsilon_k} - u\|_{L^2(\Omega)} \rightarrow 0, \quad (1.10)$$

and by a contradiction argument, using the continuity of β and (11), we get

$$\|\beta(u_{\epsilon_k}) - \beta(u)\|_{L^2(\Omega)} \rightarrow 0 \quad (1.11)$$

Whence, by using (1.11) the equality (1.6) becomes

$$\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega). \quad (1.12)$$

For any $\phi \in H_0^1(\omega_1)$ and $\psi \in H_0^1(\omega_2)$ testing with $\phi\psi$ in (1.12). We obtain for a.e. $X_1 \in \omega_1$

$$\int_{\omega_2} \beta(u(X_1, \cdot)) \psi dX_2 + \int_{\omega_2} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 = \int_{\omega_2} f(X_1, \cdot) \psi dX_2, \quad \forall \psi \in H_0^1(\omega_2).$$

Whence for a.e. $X_1 \in \omega_1$, $u(X_1, \cdot)$ is a solution of (1.2). Furthermore, the convergence

$$\|\nabla_{X_2}(u_{\epsilon_k} - u)\|_{L^2(\Omega)} \rightarrow 0$$

of (1.10) shows that $u \in H_0^1(\Omega; \omega_2)$ (recall that $H_0^1(\Omega; \omega_2)$ is a Hilbert space). Notice that (1.2) has at most one solution, thanks to (5), (10), and the Poincaré inequality (15). Therefore, u is the unique solution of (1.2) in $H_0^1(\Omega; \omega_2)$. Finally, the convergence of the global sequence follows by an immediate contradiction argument by using the uniqueness of the solution of (1.2) in $H_0^1(\Omega; \omega_2)$.

At the end, remark that for $\varphi \in H_0^1(\Omega; \omega_2)$, and for a.e. X_1 in ω_1 we have $\varphi(X_1, \cdot) \in H_0^1(\omega_2)$. By testing with $\varphi(X_1, \cdot)$ in (1.2) and by integrating over ω_1 we get

$$\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega; \omega_2). \quad (1.13)$$

□

Now, we give the setting of a general Galerkin method. Let $V \subset H_0^1(\Omega)$ be a closed subspace of $H_0^1(\Omega, \omega_2)$. Notice that V is closed in $H_0^1(\Omega)$, thanks to the continuous embedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2)$. Let $f \in L^2(\Omega)$, we denote by $u_{\epsilon, V, f}$ the unique solution of

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon, V, f}) \varphi dx + \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon, V, f} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u_{\epsilon, V, f} \in V. \end{cases} \quad (1.14)$$

We denote by $u_{V, f}$ the unique solution of

$$\begin{cases} \int_{\Omega} \beta(u_{V, f}) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u_{V, f} \in V. \end{cases} \quad (1.15)$$

Under assumptions of Theorem 1.1, one can prove by using the Schauder fixed point theorem that $u_{\epsilon, V, f}$ exists. For the existence of $u_{V, f}$ see Appendix C. The uniqueness, for the two problems, follows immediately from (5) and (10). Now, let us begin by some preliminary lemmas.

Lemma 1.2. *Under assumptions of Theorem 1.1 and for any $\epsilon \in (0, 1]$, we have the following bounds:*

$$\|\nabla u_{\epsilon, V, f}\|_{L^2(\Omega)^N} \leq \frac{C_{\Omega} \|f\|_{L^2(\Omega)}}{\lambda \epsilon^2}, \text{ and } \|\nabla u_{\epsilon, f}\|_{L^2(\Omega)^N} \leq \frac{C_{\Omega} \|f\|_{L^2(\Omega)}}{\lambda \epsilon^2}. \quad (1.16)$$

$$\|\nabla_{X_2} u_{V, f}\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda}, \text{ and } \|\nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda}. \quad (1.17)$$

$$\|\beta(u_{\epsilon, V, f})\|_{L^2(\Omega)} \leq \frac{M}{\epsilon^2} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\Omega}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right), \text{ and } \|\beta(u_{\epsilon, f})\|_{L^2(\Omega)} \leq \frac{M}{\epsilon^2} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\Omega}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right). \quad (1.18)$$

$$\|\beta(u_{V, f})\|_{L^2(\Omega)} \leq M \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right), \text{ and } \|\beta(u_f)\|_{L^2(\Omega)} \leq M \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right). \quad (1.19)$$

Here, C_{Ω} is the Poincaré constant of Ω , and $u_{\epsilon, f}$, u_f are the unique solutions of (1.1) and (1.2) respectively.

Proof. These bounds follow easily from a suitable choice of the test functions, monotonicity and ellipticity assumptions. Let us prove, for example, the second inequality in (1.17) and the second inequality in (1.19). According to Theorem 1.1 one can take $\varphi = u_f$ in (1.13), using (5) and the fact that $\int_{\Omega} \beta(u_f) u_f dx \geq 0$ (thanks to (10)) we obtain

$$\lambda \int_{\Omega} |\nabla_{X_2} u_f|^2 dx \leq \int_{\Omega} f u_f dx.$$

By the Cauchy-Schwarz inequality and Poincaré's inequality (15), we obtain the second inequality of (1.17). Now, by using assumption (11), we obtain

$$|\beta(u_f)|^2 \leq M^2 (1 + |u_f|)^2.$$

Integrating over Ω and applying Minkowski inequality, (15), and (1.17) we obtain the second inequality of (1.19). \square

By using the above lemma, one can prove the following Céa type lemma

Lemma 1.3. *Under assumptions of Theorem 1.1 we have:*

$$\|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}} \leq C_{c\acute{e}a} \left(\inf_{v \in V} \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}, \quad (1.20)$$

and for any $\epsilon \in (0, 1]$:

$$\|\nabla(u_{\epsilon,V,f} - u_{\epsilon,f})\|_{L^2(\Omega)^N} \leq \frac{C'_{c\acute{e}a}}{\epsilon^2} \left(\inf_{v \in V} \|\nabla(v - u_{\epsilon,f})\|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}, \quad (1.21)$$

where

$$C_{c\acute{e}a}^2 = \frac{1}{\lambda} \left[2MC_{\omega_2} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right) + \|A_{22}\|_{L^\infty(\Omega)} \frac{2C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right],$$

and

$$C'_{c\acute{e}a} = \frac{1}{\lambda} \left[2MC_{\Omega} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\Omega}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right) + \|A\|_{L^\infty(\Omega)} \frac{2C_{\Omega} \|f\|_{L^2(\Omega)}}{\lambda} \right].$$

Proof. The proofs of these two inequalities are similar. So, let us only prove the first one. Using the Galerkin orthogonality one has, for $v \in V$:

$$\begin{aligned} & \int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(u_{V,f} - u_f) dx + \lambda \|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}}^2 \\ & \leq \int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(v - u_f) dx + \int_{\Omega} A_{22} \nabla_{X_2}(u_{V,f} - u_f) \cdot \nabla_{X_2}(v - u_f) dx. \end{aligned}$$

Using the fact that $\int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(u_{V,f} - u_f) dx \geq 0$, then by the Cauchy-Schwarz and Poincaré's inequalities we derive

$$\begin{aligned} \lambda \|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}}^2 & \leq \left[C_{\omega_2} \|\beta(u_{V,f}) - \beta(u_f)\|_{L^2(\Omega)} + \|A_{22}\|_{L^\infty(\Omega)} \|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}} \right] \\ & \quad \times \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}}. \end{aligned}$$

Now, by using (1.17), (1.19) and the triangle inequality we obtain

$$\lambda \|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}}^2 \leq 2 \left[MC_{\omega_2} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right) + \|A_{22}\|_{L^\infty(\Omega)} \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right] \times \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}},$$

and (1.20) follows. \square

Remark 1.4. 1) If $\beta = 0$ (the linear case), then we have for any $\epsilon \in (0, 1]$:

$$\begin{aligned} \|\nabla u_{\epsilon,V,f} - \nabla u_{\epsilon,f}\|_{L^2(\Omega)^N} &\leq \frac{\|A\|_{L^\infty(\Omega)}}{\lambda \epsilon^2} \inf_{v \in V} \|\nabla v - \nabla u_{\epsilon,f}\|_{L^2(\Omega)^N}. \\ \|\nabla_{X_2} u_{V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} &\leq \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}. \end{aligned}$$

2) If β is lipschitz continuous, then we can obtain estimations similar to those of the linear case.

1.2 Error estimates for the Galerkin method

Lemma 1.5. Under assumptions of Theorem 1.1, suppose in addition that (18) holds, then we have for every $\epsilon \in (0, 1]$:

$$\|\nabla_{X_2}(u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

and

$$\|\nabla_{X_1}(u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

where

$$C_1 = \left(\frac{4(C + C')}{\lambda} \right)^{\frac{1}{2}} \quad \text{and} \quad C_2 = \frac{2\sqrt{C''}C_{\omega_2}}{\lambda^{3/2}}.$$

Here, C, C' , and C'' are given by (1.23), (1.25) and (1.26). Notice that these constants are independent of ϵ, V and f .

Proof. By subtracting (1.15) from (1.14) we get, for every $v \in V$:

$$\begin{aligned} \int_{\Omega} (\beta(u_{\epsilon,V,f}) - \beta(u_{V,f})) v dx + \epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{\epsilon,V,f} \cdot \nabla_{X_1} v dx \\ + \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{\epsilon,V,f} \cdot \nabla_{X_1} v dx + \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{\epsilon,V,f} \cdot \nabla_{X_2} v dx \\ + \int_{\Omega} A_{22} \nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f}) \cdot \nabla_{X_2} v dx = 0. \end{aligned}$$

Testing with $v = u_{\epsilon, V, f} - u_{V, f}$, we obtain

$$\begin{aligned} & \int_{\Omega} (\beta(u_{\epsilon, V, f}) - \beta(u_{V, f})) (u_{\epsilon, V, f} - u_{V, f}) dx + \int_{\Omega} A_{\epsilon} \nabla(u_{\epsilon, V, f} - u_{V, f}) \cdot \nabla(u_{\epsilon, V, f} - u_{V, f}) \\ &= -\epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx - \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \quad - \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f}) dx. \end{aligned}$$

Whence, by using (10) and the ellipticity assumption we get

$$\begin{aligned} & \epsilon^2 \lambda \int_{\Omega} |\nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f})|^2 dx + \lambda \int_{\Omega} |\nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f})|^2 dx \leq \\ & \quad - \epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx - \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \quad - \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f}) dx. \end{aligned}$$

Let us estimate the first and the last term of the second member in the above inequality. By using Young's inequality we obtain

$$\begin{aligned} & - \epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \quad \leq \frac{\epsilon^2 \lambda}{2} \int_{\Omega} |\nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f})|^2 dx + \epsilon^2 \frac{\|A_{11}\|_{L^{\infty}(\Omega)}^2}{2\lambda} \int_{\Omega} |\nabla_{X_1} u_{V, f}|^2 dx, \end{aligned}$$

and

$$\begin{aligned} & - \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \quad \leq \epsilon^2 \frac{\|A_{21}\|_{L^{\infty}(\Omega)}^2}{2\lambda} \int_{\Omega} |\nabla_{X_1} u_{V, f}|^2 dx + \frac{\lambda}{2} \int_{\Omega} |\nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f})|^2 dx, \end{aligned}$$

thus

$$\begin{aligned} & \frac{\epsilon^2 \lambda}{2} \|\nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f})\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|\nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f})\|_{L^2(\Omega)^{N-q}}^2 \\ & \quad \leq C \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V, f}|^2 dx - \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx, \end{aligned} \tag{1.22}$$

where

$$C = \frac{\|A_{21}\|_{L^{\infty}(\Omega)}^2 + \|A_{11}\|_{L^{\infty}(\Omega)}^2}{2\lambda}. \tag{1.23}$$

Now, we estimate $-\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f}) dx$. Since $u_{\epsilon,V,f} - u_{V,f} \in H_0^1(\Omega)$ and $\partial_{x_i} a_{ij} \in L^\infty(\Omega)$, $\partial_{x_j} a_{ij} \in L^\infty(\Omega)$ for $i = 1, \dots, q$ and $j = q+1, \dots, N$, (Assumption (18)) then we can show by a density argument that for $i = 1, \dots, q$ and $j = q+1, \dots, N$, $\partial_{x_k} (a_{ij} (u_{\epsilon,V,f} - u_{V,f})) \in L^2(\Omega)$ and:

$$\partial_{x_k} (a_{ij} (u_{\epsilon,V,f} - u_{V,f})) = (u_{\epsilon,V,f} - u_{V,f}) \partial_{x_k} a_{ij} + a_{ij} \partial_{x_k} (u_{\epsilon,V,f} - u_{V,f}), \text{ for } k = i, j.$$

Whence

$$\begin{aligned} -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f}) dx &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} a_{ij} \partial_{x_j} u_{V,f} \partial_{x_i} (u_{\epsilon,V,f} - u_{V,f}) dx \\ &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} \partial_{x_i} (a_{ij} (u_{\epsilon,V,f} - u_{V,f})) \partial_{x_j} u_{V,f} dx \\ &\quad + \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon,V,f} - u_{V,f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V,f} dx \\ &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} \partial_{x_j} (a_{ij} (u_{\epsilon,V,f} - u_{V,f})) \partial_{x_i} u_{V,f} dx \\ &\quad + \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon,V,f} - u_{V,f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V,f} dx, \end{aligned}$$

where we have used $\int_{\Omega} \partial_{x_i} (a_{ij} (u_{\epsilon,V,f} - u_{V,f})) \partial_{x_j} u_{V,f} dx = \int_{\Omega} \partial_{x_j} (a_{ij} (u_{\epsilon,V,f} - u_{V,f})) \partial_{x_i} u_{V,f} dx$ which follows by a density argument (recall that $u_{V,f} \in H_0^1(\Omega)$). Therefore

$$\begin{aligned} -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f}) dx &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon,V,f} - u_{V,f}) \partial_{x_j} a_{ij} \partial_{x_i} u_{V,f} dx \quad (1.24) \\ &\quad - \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} a_{ij} \partial_{x_j} (u_{\epsilon,V,f} - u_{V,f}) \partial_{x_i} u_{V,f} dx \\ &\quad + \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon,V,f} - u_{V,f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V,f} dx. \end{aligned}$$

By Young's and Poincaré's inequalities we obtain

$$\begin{aligned} -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f}) dx &\leq \frac{\lambda}{4} \int_{\Omega} |\nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f})|^2 dx \\ &\quad + C' \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 dx + C'' \epsilon^2 \int_{\Omega} |\nabla_{X_2} u_{V,f}|^2 dx, \end{aligned}$$

where

$$C' = \frac{3 \left[C_{\omega_2} \max_{1 \leq i \leq q, q+1 \leq j \leq N} \|\partial_{x_j} a_{ij}\|_{L^\infty(\Omega)} (N - q) \right]^2 + 3 \left(\max_{1 \leq i \leq q, q+1 \leq j \leq N} \|a_{ij}\|_{L^\infty(\Omega)} (N - q) \right)^2}{\lambda}. \quad (1.25)$$

and

$$C'' = \frac{3 \left[q C_{\omega_2} \max_{1 \leq i \leq q, q+1 \leq j \leq N} \|\partial_{x_i} a_{ij}\|_{L^\infty(\Omega)} \right]^2}{\lambda}. \quad (1.26)$$

By using (1.17) we obtain

$$\begin{aligned} -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f}) dx &\leq \\ &\frac{\lambda}{4} \int_{\Omega} |\nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f})|^2 dx + C' \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 dx + \epsilon^2 C'' \left(\frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right)^2. \end{aligned} \quad (1.27)$$

Combining (1.22) and (1.27) we get

$$\begin{aligned} \frac{\epsilon^2 \lambda}{2} \|\nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^q}^2 + \frac{\lambda}{4} \|\nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^{N-q}}^2 \\ \leq \epsilon^2 \left((C + C') \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 dx + C'' \left(\frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right)^2 \right), \end{aligned}$$

and the proof is complete. \square

Using the triangle inequality, the above Lemma and (1.20) we obtain the following estimation of the global error between $u_{\epsilon,V,f}$ and u_f .

Corollary 1.6. *Under assumptions of Lemma 1.5 we have for any $\epsilon \in (0, 1]$:*

$$\|\nabla_{X_2} (u_{\epsilon,V,f} - u_f)\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_{c\acute{e}a} \left(\inf_{v \in V} \|\nabla_{X_2} (v - u_f)\|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}.$$

Now, we give an important remark which will be used to prove the inequality given in Remark 1.13 .

Remark 1.7. *When $\beta(s) = \mu s$ for some $\mu > 0$ and when the block A_{12} satisfies assumption (25), then by performing some integration by parts in the last term of (1.24), and by using the fact that*

$$\|u_{V,f}\|_{L^2(\Omega)} \leq \frac{1}{\mu} \|f\|_{L^2(\Omega)},$$

we can obtain the following estimation:

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C'_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + \frac{C'_2}{\mu} \|f\|_{L^2(\Omega)} \right),$$

where $C'_1, C'_2 > 0$ are independent of f, V, μ and ϵ .

1.3 The Asymptotic Preserving property

In this subsection we will prove Theorem 0.2. Let us recall this density rule, which will be used throughout this chapter: If (E, τ) and (F, τ') are two topological spaces such that $E \subset F$, and E is dense in F and the canonical injection $E \rightarrow F$ is continuous, then every dense subset in (E, τ) is dense in (F, τ') .

Let (V_n) be a sequence of finite dimensional subspaces which approximates $H_0^1(\Omega)$ in the sense of Definition 0.1. Using the density of $H_0^1(\Omega)$ in $H_0^1(\Omega, \omega_2)$ (Lemma A.1, Appendix A), one can check easily that (V_n) approximates $H_0^1(\Omega, \omega_2)$ in the same sense. Therefore, one has:

$$\text{For every } \epsilon \in (0, 1] : \inf_{v \in V_n} \|\nabla(v - u_{\epsilon, f})\|_{L^2(\Omega)^N} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (1.28)$$

and

$$\inf_{v \in V_n} \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.29)$$

According to Lemma 1.5, (1.20) and (1.21) we have, for every $n \in \mathbb{N}$ and $\epsilon \in (0, 1]$:

$$\|\nabla_{X_2}(u_{\epsilon, V_n, f} - u_{V_n, f})\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 \|\nabla_{X_1} u_{V_n, f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right), \quad (1.30)$$

$$\|\nabla_{X_2}(u_{V_n, f} - u_f)\|_{L^2(\Omega)^{N-q}} \leq C_{c\acute{e}a} \left(\inf_{v \in V_n} \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}, \quad (1.31)$$

and

$$\|\nabla(u_{\epsilon, V_n, f} - u_{\epsilon, f})\|_{L^2(\Omega)^N} \leq \frac{C'_{c\acute{e}a}}{\epsilon^2} \left(\inf_{v \in V_n} \|\nabla(v - u_{\epsilon, f})\|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}. \quad (1.32)$$

- Fix ϵ and pass to the limit in (1.32) by using (1.28). We get

$$u_{\epsilon, V_n, f} \rightarrow u_{\epsilon, f} \text{ as } n \rightarrow \infty \text{ in } H_0^1(\Omega),$$

in particular, by using the continuous embedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2)$ we deduce

$$u_{\epsilon, V_n, f} \rightarrow u_{\epsilon, f} \text{ as } n \rightarrow \infty \text{ in } H_0^1(\Omega, \omega_2).$$

Now, passing to the limit as $\epsilon \rightarrow 0$ by using Theorem 1.1, we get

$$\lim_{\epsilon} (\lim_n u_{\epsilon, V_n, f}) = u_f \text{ in } H_0^1(\Omega, \omega_2). \quad (1.33)$$

- Fix n and pass to the limit as $\epsilon \rightarrow 0$ in (1.30). We get

$$u_{\epsilon, V_n, f} \rightarrow u_{V_n, f} \text{ as } \epsilon \rightarrow 0 \text{ in } H_0^1(\Omega, \omega_2).$$

Now, passing to the limit as $n \rightarrow \infty$ in (1.31) by using (1.29), we get

$$\lim_n (\lim_\epsilon u_{\epsilon, V_n, f}) = u_f \text{ in } H_0^1(\Omega, \omega_2). \quad (1.34)$$

Finally, Theorem 0.2 follows from (1.33) and (1.34).

1.4 The Rate of convergence on the whole domain for the continuous problem

In this subsection we prove Theorem 0.3. Throughout this subsection, we suppose that $\beta = 0$. The key of the proof of Theorem 0.3 is based on the control of the quantity $\|\nabla_{X_1} u_{V, f}\|_{L^2(\Omega)^q}$ independently of V . In fact, we use the following lemma.

Lemma 1.8. *Let us assume that A satisfies (5), (6), and that A_{22} satisfies (24). Let V_1 and V_2 be two finite dimensional subspaces of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively. Let $f \in V_1 \otimes V_2$, and let $u_{V, f}$ be the unique solution in $V = V_1 \otimes V_2$ to:*

$$\int_{\Omega} A_{22}(X_2) \nabla_{X_2} u_{V, f} \cdot \nabla_{X_2} v dx = \int_{\Omega} f v dx, \quad \forall v \in V_1 \otimes V_2, \quad (1.35)$$

then we have:

$$\|\nabla_{X_1} u_{V, f}\|_{L^2(\Omega)^q} \leq C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q},$$

where C_3 is given by $C_3 = \frac{\sqrt{q} C_{\omega_2}}{\lambda}$.

Proof. The proof is based on the difference quotient method (see for instance [20] page 168). Let $v = \varphi \otimes \psi \in V_1 \otimes V_2$. The function $X_1 \mapsto \int_{\omega_2} A_{22}(X_2) \nabla_{X_2} u_{V, f}(X_1, X_2) \cdot \nabla_{X_2} \psi dX_2$ belongs to V_1 . In fact $u_{V, f} = \sum_{finite} \varphi_i \otimes \psi_i$, and whence $\int_{\omega_2} A_{22}(X_2) \nabla_{X_2} u_{V, f} \cdot \nabla_{X_2} \psi dX_2$ is a linear combination of φ_i , thanks to the linearity of the integral. Similarly, the function $X_1 \mapsto \int_{\omega_2} f(X_1, X_2) \psi dX_2$ belongs to V_1 . Now, testing with v in (1.35), we derive:

$$\int_{\omega_1} \left(\int_{\omega_2} \{A_{22}(X_2) \nabla_{X_2} u_{V, f} \cdot \nabla_{X_2} \psi - f \psi\} dX_2 \right) \varphi dX_1 = 0,$$

thus, when φ run through a set of an orthogonal basis of the euclidean space V_1 equipped with the $L^2(\omega_1)$ -scalar product, one can deduce that for a.e. $X_1 \in \omega_1$:

$$\int_{\omega_2} A_{22}(X_2) \nabla_{X_2} u_{V,f}(X_1, X_2) \cdot \nabla_{X_2} \psi dX_2 = \int_{\omega_2} f(X_1, X_2) \psi dX_2, \quad \forall \psi \in V_2.$$

Now, fix $i \in \{1, \dots, q\}$. Let $\omega'_1 \subset\subset \omega_1$ open, for any $0 < h < d(\omega'_1, \partial\omega_1)$ and for any $(X_1, X_2) \in \omega'_1 \times \omega_2$ we denote $\tau_h^i u_{V,f}(x) = u_{V,f}(x_1, \dots, x_i + h, \dots, x_q, X_2)$. According to the above equality, we get for a.e. $X_1 \in \omega'_1$ and for every $\psi \in V_2$:

$$\int_{\omega_2} A_{22}(X_2) \nabla_{X_2} \left\{ \tau_h^i u_{V,f}(X_1, X_2) - u_{V,f}(X_1, X_2) \right\} \nabla_{X_2} \psi dX_2 = \int_{\omega_2} \left\{ \tau_h^i f(X_1, X_2) - f(X_1, X_2) \right\} \psi dX_2.$$

For every $w \in V_1 \otimes V_2$, and for every X_1 fixed the function $w(X_1, \cdot)$ belongs to V_2 , so one can take $\psi = \tau_h^i u_{V,f}(X_1, \cdot) - u_{V,f}(X_1, \cdot)$ as a test function in the above equality. Therefore, by using the Cauchy-Schwarz inequality, the ellipticity assumption, and Poincaré's inequality (15), we obtain:

$$\int_{\omega_2} \left| \tau_h^i u_{V,f}(X_1, \cdot) - u_{V,f}(X_1, \cdot) \right|^2 dX_2 \leq \frac{C_{\omega_2}^4}{\lambda^2} \int_{\omega_2} \left| \tau_h^i f(X_1, \cdot) - f(X_1, \cdot) \right|^2 dX_2.$$

Now, integrating the above inequality over ω'_1 , yields

$$\int_{\omega'_1 \times \omega_2} \left| \tau_h^i u_{V,f} - u_{V,f} \right|^2 dx \leq \frac{C_{\omega_2}^4}{\lambda^2} \int_{\omega'_1 \times \omega_2} \left| \tau_h^i f - f \right|^2 dx.$$

Since $\nabla_{X_1} f \in L^2(\Omega)^q$, then

$$\int_{\omega'_1 \times \omega_2} \left| \tau_h^i f - f \right|^2 dx \leq \|\nabla_{X_1} f\|_{L^2(\Omega)^q}^2 h^2.$$

Finally, we obtain

$$\int_{\omega'_1 \times \omega_2} \left| \frac{\tau_h^i u_{V,f} - u_{V,f}}{h} \right|^2 dx \leq \frac{C_{\omega_2}^4 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}^2}{\lambda^2}.$$

Therefore

$$\|D_{x_i} u_{V,f}\|_{L^2(\Omega)} \leq \frac{C_{\omega_2}^2}{\lambda} \|\nabla_{X_1} f\|_{L^2(\Omega)^q},$$

and hence

$$\|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} \leq C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q},$$

with $C_3 = \frac{\sqrt{q} C_{\omega_2}^2}{\lambda}$. □

Remark 1.9. We have a similar result when (1.35) is replaced by

$$\mu \int_{\Omega} u_{V,f} v dx + \int_{\Omega} A_{22}(X_2) \nabla_{X_2} u_{V,f} \cdot \nabla_{X_2} v dx = \int_{\Omega} f v dx, \quad \forall v \in V_1 \otimes V_2,$$

where $\mu > 0$. In this case, we obtain the following:

$$\|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} \leq \frac{\sqrt{q}}{\mu} \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Now, we can refine the estimations of Lemma 1.5 as follows.

Lemma 1.10. Under assumptions of Lemmas 1.5 and 1.8 we have:

$$\begin{aligned} \|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} &\leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) \\ &\quad + \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}, \end{aligned}$$

and

$$\|\nabla_{X_1} u_{\epsilon,V,f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Proof. We have

$$\|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_{V,f}\|_{L^2(\Omega)^{N-q}} + \|\nabla_{X_2} u_{V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}.$$

By using Lemma 1.5 and Lemma 1.8 we obtain that

$$\|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_{V,f}\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

and by using Remark 1.4, we deduce

$$\|\nabla_{X_2} u_{V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}.$$

By using the above inequalities, we get the expected result. The second inequality follows from the triangle inequality and Lemmas 1.5 and 1.8. \square

Remark 1.11. Let $\beta(s) = \mu s$, for some $\mu > 0$. Under the assumptions of the above Lemma and (25) we obtain, by combining Remarks 1.7 and 1.9, the estimation:

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2}(u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)} \leq \frac{\epsilon}{\mu} \left(\sqrt{q} C'_1 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C'_2 \|f\|_{L^2(\Omega)} \right).$$

Now, we are able to give the following convergence result.

Lemma 1.12. *Suppose that the assumptions of Lemmas 1.5 and 1.8 hold. Let $f \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$.*

Then we have for any $\epsilon \in (0, 1]$:

$$\|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

and

$$\|\nabla_{X_1} u_{\epsilon, f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Proof. Let $(V_n^{(1)})_{n \geq 0}$ and $(V_n^{(2)})_{n \geq 0}$ be two nondecreasing sequences of finite dimensional subspaces of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, such that $\cup V_n^{(1)}$ and $\cup V_n^{(2)}$ are dense in $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, and such that $f \in V_0^{(1)} \otimes V_0^{(2)}$, such sequences always exist. Indeed, let $\{e_i^{(1)}\}_{i \in \mathbb{N}}$ and $\{e_i^{(2)}\}_{i \in \mathbb{N}}$ be Hilbert bases of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, then $\cup_{n \geq 0} \text{span}(e_0^{(1)}, \dots, e_n^{(1)})$ and $\cup_{n \geq 0} \text{span}(e_0^{(2)}, \dots, e_n^{(2)})$ are dense in $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, in the other hand we have $f = \sum_{i=0}^m f_i^{(1)} \times f_i^{(2)}$ for some $m \in \mathbb{N}$ and $f_i^{(1)} \in H_0^1(\omega_1)$, $f_i^{(2)} \in H_0^1(\omega_2)$ for $i = 0, \dots, m$, then we set, for every $n \in \mathbb{N}$:

$$V_n^{(1)} := \text{span}(e_0^{(1)}, \dots, e_n^{(1)}, f_0^{(1)}, \dots, f_m^{(1)}),$$

$$V_n^{(2)} := \text{span}(e_0^{(2)}, \dots, e_n^{(2)}, f_0^{(2)}, \dots, f_m^{(2)}).$$

Now, since f belongs to each $V_n^{(1)} \otimes V_n^{(2)}$ then according to Lemma 1.10 one has, for every $\epsilon \in (0, 1]$, $n \in \mathbb{N}$:

$$\begin{aligned} \|\nabla_{X_2} u_{\epsilon, V_n, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} &\leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) \\ &\quad + \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_n} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}, \end{aligned}$$

where $V_n := V_n^{(1)} \otimes V_n^{(2)}$. According to Corollary A.5 in Appendix A, $\cup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)})$ is dense in $H_0^1(\Omega)$. Using the fact that the sequence $(V_n)_{n \geq 0}$ is nondecreasing, then we obtain that

$$\forall \epsilon \in (0, 1] : \lim_{n \rightarrow \infty} \inf_{v \in V_n} \|\nabla v - \nabla u_{\epsilon, f}\|_{L^2(\Omega)^N} = 0,$$

and therefore, by using (1.21) we get

$$\forall \epsilon \in (0, 1] : \lim_{n \rightarrow \infty} \|\nabla u_{\epsilon, V_n, f} - \nabla u_{\epsilon, f}\|_{L^2(\Omega)^N} = 0,$$

and thus

$$\forall \epsilon \in (0, 1] : \lim_{n \rightarrow \infty} \|\nabla_{X_2} u_{\epsilon, V_n, f} - \nabla_{X_2} u_{\epsilon, f}\|_{L^2(\Omega)^{N-q}} = 0, \text{ and } \lim_{n \rightarrow \infty} \|\nabla_{X_1} u_{\epsilon, V_n, f} - \nabla_{X_1} u_{\epsilon, f}\|_{L^2(\Omega)^q} = 0.$$

Using the fact that $H_0^1(\Omega)$ is dense in $H_0^1(\Omega, \omega_2)$ (Lemma A.1, Appendix A) and that the embedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2)$ is continuous then $\cup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)})$ is dense in $H_0^1(\Omega, \omega_2)$. Using the fact that the sequence $(V_n)_{n \geq 0}$ is nondecreasing, then we obtain that

$$\lim_{n \rightarrow \infty} \inf_{v \in V_n} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} = 0.$$

Now, passing to the limit, as $n \rightarrow \infty$, in the above inequality we deduce

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right).$$

Finally, by using the second inequality of Lemma 1.10 we get

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_1} u_{\epsilon, V_n, f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q},$$

and the passage to limit as $n \rightarrow \infty$ shows the second estimation of the lemma. \square

Now, we are able to give the proof of Theorem 0.3. Let us introduce the space

$$H_0^1(\Omega; \omega_1) = \left\{ v \in L^2(\Omega) \text{ such that } \nabla_{X_1} v \in L^2(\Omega)^q \text{ and for a.e. } X_2 \in \omega_2, v(\cdot, X_2) \in H_0^1(\omega_1) \right\},$$

normed by the Hilbertian norm $\|\nabla_{X_1}(\cdot)\|_{L^2(\Omega)^q}$. We have the Poincaré's inequality

$$\|v\|_{L^2(\Omega)} \leq C_{\omega_1} \|\nabla_{X_1} v\|_{L^2(\Omega)^q} \text{ for any } v \in H_0^1(\Omega; \omega_1). \quad (1.36)$$

Let $f \in L^2(\Omega)$ such that (21) and (23), thus $f \in H_0^1(\Omega; \omega_1)$. According to Lemma A.3 in Appendix A, $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ is dense in $H_0^1(\Omega)$, and according to Remark A.2 in Appendix A, $H_0^1(\Omega)$ is dense in $H_0^1(\Omega; \omega_1)$, then it follows that $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ is dense in $H_0^1(\Omega; \omega_1)$, thanks to the continuous embedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega; \omega_1)$. Therefore, for $\delta > 0$ there exists $g_\delta \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ such that

$$\|\nabla_{X_1}(f - g_\delta)\|_{L^2(\Omega)^q} \leq \delta. \quad (1.37)$$

Let u_{ϵ, g_δ} be the unique solution of (1.1) with f replaced by g_δ . Testing with $u_{\epsilon, f} - u_{\epsilon, g_\delta}$ in the difference of the weak formulations (recall that $\beta = 0$)

$$\int_{\Omega} A_\epsilon \nabla(u_{\epsilon, f} - u_{\epsilon, g_\delta}) \cdot \nabla \varphi dx = \int_{\Omega} (f - g_\delta) \varphi dx, \quad \forall \varphi \in H_0^1(\Omega),$$

we obtain

$$\|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_{\epsilon, g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta, \quad \text{and} \quad \|\nabla_{X_1} u_{\epsilon, f} - \nabla_{X_1} u_{\epsilon, g_\delta}\|_{L^2(\Omega)^q} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda \epsilon} \delta,$$

where we have used the ellipticity assumption, Poincaré's inequalities (15), (1.36), and (1.37). By passing to the limit as $\epsilon \rightarrow 0$ in the first inequality above, using Theorem 1.1, we get

$$\|\nabla_{X_2} u_f - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta.$$

Applying Lemma 1.12 on u_{ϵ, g_δ} and u_{g_δ} we obtain

$$\|\nabla_{X_2} u_{\epsilon, g_\delta} - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} g_\delta\|_{L^2(\Omega)^q} + C_2 \|g_\delta\|_{L^2(\Omega)} \right),$$

and from (1.37) we derive

$$\|\nabla_{X_2} u_{\epsilon, g_\delta} - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 (\|\nabla_{X_1} f\|_{L^2(\Omega)^q} + \delta) + C_2 \|g_\delta\|_{L^2(\Omega)} \right).$$

Notice that $\|g_\delta\|_{L^2(\Omega)} \rightarrow \|f\|_{L^2(\Omega)}$ as $\delta \rightarrow 0$, thanks to (1.37) and Poincaré's inequality (1.36). Finally, the triangle inequality gives

$$\begin{aligned} \|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} &\leq \|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_{\epsilon, g_\delta}\|_{L^2(\Omega)^{N-q}} \\ &\quad + \|\nabla_{X_2} u_{\epsilon, g_\delta} - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} + \|\nabla_{X_2} u_{g_\delta} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \\ &\leq \epsilon \left(C_1 C_3 (\|\nabla_{X_1} f\|_{L^2(\Omega)^q} + \delta) + C_2 \|g_\delta\|_{L^2(\Omega)} \right) + 2 \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta. \end{aligned}$$

Passing to the limit as $\delta \rightarrow 0$ we obtain

$$\|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

which is the estimation given in Theorem 0.3.

For the estimation in the first direction, we have

$$\begin{aligned} \forall \epsilon \in (0, 1] : \|\nabla_{X_1} u_{\epsilon, f}\|_{L^2(\Omega)^q} &\leq \|\nabla_{X_1} u_{\epsilon, f} - \nabla_{X_1} u_{\epsilon, g\delta}\|_{L^2(\Omega)^q} + \|\nabla_{X_1} u_{\epsilon, g\delta}\|_{L^2(\Omega)^q} \\ &\leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda \epsilon} \delta + \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} g\delta\|_{L^2(\Omega)^q} + C_2 \|g\delta\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} g\delta\|_{L^2(\Omega)^q}, \end{aligned}$$

where we have applied the triangle inequality and Lemma 1.12. Passing to the limit as $\delta \rightarrow 0$ by using (1.37), we obtain

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_1} u_{\epsilon, f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Hence, passing to the limit in $L^2(\Omega)$ -weak as $\epsilon \rightarrow 0$, up to a subsequence, we show that u_f belongs to $H_0^1(\Omega)$, and by a contradiction argument, using the metrizable (for the weak topology) of weakly compact subsets in separable Hilbert spaces, one can show that the global sequence $(\nabla_{X_1} u_{\epsilon, f})_\epsilon$ converges weakly to $\nabla_{X_1} u_f$ in $L^2(\Omega)^q$, and this completes the proof of Theorem 0.3.

Remark 1.13. *Suppose that $\beta : s \mapsto \mu s$, for some $\mu > 0$, and suppose that assumptions of Theorem 0.3 hold, then we have the same results of Theorem 0.3 with the same constants. Assume in addition that (25) holds, then we have:*

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \leq \frac{\epsilon}{\mu} \left(C'_1 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C'_2 \|f\|_{L^2(\Omega)} \right),$$

where C'_1, C'_2 are only dependent on A, λ, C_{ω_2} .

To obtain the above estimation, we repeat the same arguments of this section by using Remark 1.11. This estimation shows the asymptotic behaviour of the resolvent of the perturbed elliptic operator $-\operatorname{div}(A_\epsilon \nabla(\cdot))$. It will be used in the next section to study the asymptotic behaviour of the semigroup generated by the above elliptic operator.

2 Anisotropic Perturbations of the heat semigroup

2.1 Preliminaries

For the standard basic theory of semigroups of bounded linear operators, we refer the reader to [21]. Let us begin by some reminders. Let E be a real Banach space. An unbounded linear operator

$\mathcal{A} : D(\mathcal{A}) \subset E \rightarrow E$ is said to be closed if for every sequence (x_n) of $D(\mathcal{A})$ such that (x_n) and $(\mathcal{A}(x_n))$ converge in E , we have $\lim x_n \in D(\mathcal{A})$ and $\lim \mathcal{A}(x_n) = \mathcal{A}(\lim x_n)$. An operator is said to be densely defined on E if its domain $D(\mathcal{A})$ is dense in E . Let $\mu \in \mathbb{R}$, we said that μ belongs to the resolvent set of \mathcal{A} if $(\mu I - \mathcal{A}) : D(\mathcal{A}) \rightarrow E$ is one-to-one and onto and such that $R_\mu = (\mu I - \mathcal{A})^{-1} : E \rightarrow D(\mathcal{A}) \subset E$ is a bounded operator on E . Notice that R_μ and \mathcal{A} commute on $D(\mathcal{A})$, that is $\forall x \in D(\mathcal{A}) : R_\mu \mathcal{A}x = \mathcal{A}R_\mu x$. Let \mathcal{A} be a densely defined closed operator. The bounded operator

$$\mathcal{A}_\mu = \mu \mathcal{A}(\mu I - \mathcal{A})^{-1} = \mu \mathcal{A}R_\mu = \mu^2 R_\mu - \mu I,$$

is called the Yosida approximation of \mathcal{A} . We check immediately that \mathcal{A}_μ and \mathcal{A} commute on $D(\mathcal{A})$ that is for every $x \in D(\mathcal{A})$ we have $\mathcal{A}_\mu x \in D(\mathcal{A})$ and $\mathcal{A}\mathcal{A}_\mu x = \mathcal{A}_\mu \mathcal{A}x$. Furthermore, since \mathcal{A} is closed then $e^{t\mathcal{A}_\mu}$ and \mathcal{A} commute on $D(\mathcal{A})$, that is

$$\forall t \in \mathbb{R}, \forall x \in D(\mathcal{A}), e^{t\mathcal{A}_\mu} x \in D(\mathcal{A}), \quad (1.38)$$

and

$$\mathcal{A}e^{t\mathcal{A}_\mu} x = e^{t\mathcal{A}_\mu} \mathcal{A}x = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\mathcal{A}_\mu)^k \mathcal{A}x,$$

indeed, we can check by induction that if $x \in D(\mathcal{A})$ then $(\mathcal{A}_\mu)^k x \in D(\mathcal{A})$, and that $(\mathcal{A}_\mu)^k$ and \mathcal{A} commute on $D(\mathcal{A})$. Recall also that if $(\mu I - \mathcal{A})^{-1}$ exists for $\mu > 0$ and such that $\|(\mu I - \mathcal{A})^{-1}\| \leq \frac{1}{\mu}$ then

$$\forall t \geq 0 : \|e^{t\mathcal{A}_\mu}\| = \|e^{t\mu^2 R_\mu}\| \times \|e^{-\mu t I}\| \leq e^{t\mu^2 \|R_\mu\|} \times e^{-\mu t} \leq 1,$$

where $\|\cdot\|$ is the operator norm of $\mathcal{L}(E)$. A C_0 semigroup of bounded linear operators on E is a family of bounded operators $(S(t))_{t \geq 0}$ of $\mathcal{L}(E)$ such that: $S(0) = I$, for every $t, s \geq 0 : S(t+s) = S(t)S(s)$, and for every $x \in E : \|S(t)x - x\|_E \rightarrow 0$ as $t \rightarrow 0$. $(S(t))_{t \geq 0}$ is called a semigroup of contractions if for every $t \geq 0 : \|S(t)\|_E \leq 1$. Now, let us recall the well-known Hille-Yosida theorem in its Hilbertian (real) version: An unbounded operator \mathcal{A} is the infinitesimal generator of a C_0 semigroup of contractions $(S(t))_{t \geq 0}$ if and only if \mathcal{A} is maximal dissipative, that is when $\mu I - \mathcal{A}$ is surjective for every $\mu > 0$ and for every $x \in D(\mathcal{A}) : \langle \mathcal{A}x, x \rangle \leq 0$. Recall that, in this case $D(\mathcal{A})$ is dense and

\mathcal{A} is closed and its resolvent set contains $]0, +\infty[$. Furthermore, for every $t \geq 0$, $e^{t\mathcal{A}\mu}$ converges, in the strong operator topology, to $S(t)$, as $\mu \rightarrow +\infty$ i.e. $\forall x \in E : e^{t\mathcal{A}\mu}x \rightarrow S(t)x$ in E as $\mu \rightarrow +\infty$.

The basic Hilbert space in the sequel is $E = L^2(\Omega)$. We check immediately, by using assumptions (5–6), that \mathcal{A}_ϵ and \mathcal{A}_0 are maximal dissipative and therefore, they are the infinitesimal generators of C_0 semigroups of contractions on $L^2(\Omega)$, denoted $(S_\epsilon(t))_{t \geq 0}$ and $(S_0(t))_{t \geq 0}$ respectively. For $\mu > 0$ we denote by $R_{\epsilon,\mu}$ the resolvent of \mathcal{A}_ϵ . Similarly, we denote by $R_{0,\mu}$ the resolvent of \mathcal{A}_0 . For $f \in L^2(\Omega)$, we denote $u_{\epsilon,\mu}$ the unique solution in $H_0^1(\Omega)$ to

$$\mu \int_{\Omega} u_{\epsilon,\mu} \varphi dx + \int_{\Omega} A_\epsilon \nabla u_{\epsilon,\mu} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega),$$

we have $R_{\epsilon,\mu}f = u_{\epsilon,\mu}$ and $\|R_{\epsilon,\mu}\| \leq \frac{1}{\mu}$, where $\|\cdot\|$ is the operator norm of $\mathcal{L}(L^2(\Omega))$. Similarly, let $u_{0,\mu}$ be the unique solution in $H_0^1(\Omega; \omega_2)$ to

$$\mu \int_{\Omega} u_{0,\mu} \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u_{0,\mu} \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega; \omega_2), \quad (1.39)$$

we have $R_{0,\mu}f = u_{0,\mu}$ and $\|R_{0,\mu}\| \leq \frac{1}{\mu}$. According to Remark 1.13, we have the following

Lemma 1.1. *Assume (5), (6), (18), (24) and (25). Let $f \in H_0^1(\Omega; \omega_1)$, then there exists $C_{A,\Omega} > 0$ depending only on A and Ω . such that:*

$$\forall \epsilon \in (0, 1], \forall \mu > 0 : \|R_{\epsilon,\mu}f - R_{0,\mu}f\|_{L^2(\Omega)} \leq C_{A,\Omega} \times \frac{\epsilon}{\mu} \times \left(\|\nabla_{X_1} f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right). \quad (1.40)$$

2.2 The asymptotic behaviour of the perturbed semigroup

In this subsection, we will prove Theorem 0.4.

Let us begin by the following lemma

Lemma 1.2. *Suppose that the assumptions of Theorem 0.4 hold. Let $f \in H_0^1(\Omega) \cap D(\mathcal{A}_0)$ such that*

$$\operatorname{div}_{X_1}(A_{11} \nabla_{X_1} f), \operatorname{div}_{X_1}(A_{12} \nabla_{X_2} f), \operatorname{div}_{X_2}(A_{21} \nabla_{X_1} f) \in L^2(\Omega), \text{ and } \mathcal{A}_0 f \in H_0^1(\Omega; \omega_1), \quad (1.41)$$

then there exists a constant $C_{f,A,\Omega} > 0$ such that for every $\mu > 0, \epsilon \in (0, 1]$ we have:

$$\|\mathcal{A}_{\epsilon,\mu}f - \mathcal{A}_{0,\mu}f\|_{L^2(\Omega)} \leq C_{f,A,\Omega} \times \epsilon,$$

where $\mathcal{A}_{\epsilon,\mu}$ and $\mathcal{A}_{0,\mu}$ are the Yosida approximations of \mathcal{A}_ϵ and \mathcal{A}_0 respectively. The constant $C_{f,A,\Omega}$ is given by:

$$\begin{aligned} C_{f,A,\Omega} &= \|\operatorname{div}_{X_1}(A_{11}\nabla_{X_1}f)\|_{L^2(\Omega)} + \|\operatorname{div}_{X_1}(A_{12}\nabla_{X_2}f)\|_{L^2(\Omega)} \\ &\quad + \|\operatorname{div}_{X_2}(A_{21}\nabla_{X_1}f)\|_{L^2(\Omega)} + C_{A,\Omega} \left(\|\nabla_{X_1}\mathcal{A}_0f\|_{L^2(\Omega)} + \|\mathcal{A}_0f\|_{L^2(\Omega)} \right). \end{aligned}$$

Proof. Let $\epsilon \in (0, 1]$ and $\mu > 0$. The bounded operators $\mathcal{A}_{\epsilon,\mu}, \mathcal{A}_{0,\mu}$ of $\mathcal{L}(L^2(\Omega))$ are given by:

$$\mathcal{A}_{\epsilon,\mu} = \mu\mathcal{A}_\epsilon R_{\epsilon,\mu} \text{ and } \mathcal{A}_{0,\mu} = \mu\mathcal{A}_0 R_{0,\mu}.$$

Now, under the above hypothesis we obtain that $f \in D(\mathcal{A}_\epsilon) \cap D(\mathcal{A}_0)$. We have:

$$\begin{aligned} \|\mathcal{A}_{\epsilon,\mu}f - \mathcal{A}_{0,\mu}f\|_{L^2(\Omega)} &= \mu \|\mathcal{A}_\epsilon R_{\epsilon,\mu}f - \mathcal{A}_0 R_{0,\mu}f\|_{L^2(\Omega)} = \mu \|R_{\epsilon,\mu}\mathcal{A}_\epsilon f - R_{0,\mu}\mathcal{A}_0f\|_{L^2(\Omega)} \\ &\leq \mu \|R_{\epsilon,\mu}\mathcal{A}_\epsilon f - R_{\epsilon,\mu}\mathcal{A}_0f\|_{L^2(\Omega)} + \mu \|R_{\epsilon,\mu}\mathcal{A}_0f - R_{0,\mu}\mathcal{A}_0f\|_{L^2(\Omega)} \\ &\leq \mu \|R_{\epsilon,\mu}\| \times \|\mathcal{A}_\epsilon f - \mathcal{A}_0f\|_{L^2(\Omega)} + \mu \|R_{\epsilon,\mu}\mathcal{A}_0f - R_{0,\mu}\mathcal{A}_0f\|_{L^2(\Omega)}. \end{aligned}$$

Since $\mathcal{A}_0f \in H_0^1(\Omega; \omega_1)$ by hypothesis, then by using (1.40) (where we replace f by \mathcal{A}_0f) and the fact that $\|R_{\epsilon,\mu}\| \leq \frac{1}{\mu}$, we obtain

$$\begin{aligned} \|\mathcal{A}_{\epsilon,\mu}f - \mathcal{A}_{0,\mu}f\|_{L^2(\Omega)} &\leq \|\mathcal{A}_\epsilon f - \mathcal{A}_0f\|_{L^2(\Omega)} + \epsilon C_{A,\Omega} \left(\|\nabla_{X_1}\mathcal{A}_0f\|_{L^2(\Omega)} + \|\mathcal{A}_0f\|_{L^2(\Omega)} \right) \\ &= \epsilon \left(\begin{aligned} &\epsilon \|\operatorname{div}_{X_1}(A_{11}\nabla_{X_1}f)\|_{L^2(\Omega)} + \|\operatorname{div}_{X_1}(A_{12}\nabla_{X_2}f)\|_{L^2(\Omega)} \\ &+ \|\operatorname{div}_{X_2}(A_{21}\nabla_{X_1}f)\|_{L^2(\Omega)} + C_{A,\Omega} \left(\|\nabla_{X_1}\mathcal{A}_0f\|_{L^2(\Omega)} + \|\mathcal{A}_0f\|_{L^2(\Omega)} \right) \end{aligned} \right) \\ &\leq C_{f,A,\Omega} \times \epsilon, \end{aligned}$$

where we have used the identity:

$$\mathcal{A}_\epsilon f - \mathcal{A}_0f = \epsilon^2 \operatorname{div}_{X_1}(A_{11}\nabla_{X_1}f) + \epsilon \operatorname{div}_{X_1}(A_{12}\nabla_{X_2}f) + \epsilon \operatorname{div}_{X_2}(A_{21}\nabla_{X_1}f),$$

and the proof of the lemma is concluded. \square

Lemma 1.3. *Under assumptions of Theorem 0.4, we have for any $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$:*

$$\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \left\| e^{t\mathcal{A}_{\epsilon,\mu}}g - e^{t\mathcal{A}_{0,\mu}}g \right\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon,$$

where $C_{g,A,\Omega}$ is independent of μ and ϵ .

Proof. Let $\mu > 0$ and $t \geq 0$ and $\epsilon \in (0, 1]$, we have

$$\begin{aligned} e^{t\mathcal{A}_{0,\mu}} - e^{t\mathcal{A}_{\epsilon,\mu}} &= \int_0^t \frac{d}{ds} \left(e^{(t-s)\mathcal{A}_{\epsilon,\mu}} e^{s\mathcal{A}_{0,\mu}} \right) ds \\ &= \int_0^t e^{(t-s)\mathcal{A}_{\epsilon,\mu}} (\mathcal{A}_{0,\mu} - \mathcal{A}_{\epsilon,\mu}) e^{s\mathcal{A}_{0,\mu}} ds. \end{aligned}$$

Hence, for $g \in L^2(\Omega)$ we have

$$\left\| e^{t\mathcal{A}_{\epsilon,\mu}} g - e^{t\mathcal{A}_{0,\mu}} g \right\|_{L^2(\Omega)} \leq \int_0^t \left\| \mathcal{A}_{0,\mu} e^{s\mathcal{A}_{0,\mu}} g - \mathcal{A}_{\epsilon,\mu} e^{s\mathcal{A}_{0,\mu}} g \right\|_{L^2(\Omega)} ds, \quad (1.42)$$

where we have used $\left\| e^{(t-s)\mathcal{A}_{\epsilon,\mu}} \right\| \leq 1$, since $t - s \geq 0$.

Now, we suppose that $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ (remark that $g \in D(\mathcal{A}_0)$). For $s \geq 0$ and $\mu > 0$ we set:

$$f_{g,s,\mu} := e^{s\mathcal{A}_{0,\mu}} g.$$

We can prove that $f_{g,s,\mu} \in H_0^1(\Omega) \cap D(\mathcal{A}_0)$ and that it fulfils the same hypothesis (1.41) satisfied by the function f of Lemma 1.2. Moreover, for every $i, j = 1, \dots, q$ we have $D_{x_i x_j}^2 f_{g,s,\mu} \in L^2(\Omega)$ with:

$$\left\| D_{x_i x_j}^2 f_{g,s,\mu} \right\|_{L^2(\Omega)} \leq \left\| D_{x_i x_j}^2 g \right\|_{L^2(\Omega)}, \quad \left\| D_{x_i} f_{g,s,\mu} \right\|_{L^2(\Omega)} \leq \left\| D_{x_i} g \right\|_{L^2(\Omega)}, \quad (1.43)$$

and

$$\left\| \mathcal{A}_0 f_{g,s,\mu} \right\|_{L^2(\Omega)} \leq \left\| \mathcal{A}_0 g \right\|_{L^2(\Omega)}, \quad \left\| D_{x_i} (\mathcal{A}_0 f_{g,s,\mu}) \right\|_{L^2(\Omega)} \leq \left\| D_{x_i} (\mathcal{A}_0 g) \right\|_{L^2(\Omega)}, \quad (1.44)$$

also for every $i = 1, \dots, q, j = q + 1, \dots, N$ we have $D_{x_i x_j}^2 f_{g,s,\mu} \in L^2(\Omega)$ with :

$$\left\| D_{x_j} f_{g,s,\mu} \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\| \mathcal{A}_0 g \right\|_{L^2(\Omega)} \left\| g \right\|_{L^2(\Omega)} \quad \text{and} \quad \left\| D_{x_i x_j}^2 f_{g,s,\mu} \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\| D_{x_i} \mathcal{A}_0 g \right\|_{L^2(\Omega)} \left\| D_{x_i} g \right\|_{L^2(\Omega)}. \quad (1.45)$$

The proof of these assertions follows from the identity $e^{s\mathcal{A}_{0,\mu}}(g_1 \otimes g_2) = g_1 \otimes e^{s\mathcal{A}_{0,\mu}} g_2$ (see Appendix B). Notice that the above bounds are independent of s, ϵ , and μ .

Now, applying Lemma 1.2, we get

$$\left\| \mathcal{A}_{0,\mu} e^{s\mathcal{A}_{0,\mu}} g - \mathcal{A}_{\epsilon,\mu} e^{s\mathcal{A}_{0,\mu}} g \right\|_{L^2(\Omega)} \leq \epsilon \left(\begin{array}{c} \left\| \operatorname{div}_{X_1} (\mathcal{A}_{11} \nabla_{X_1} f_{g,s,\mu}) \right\|_{L^2(\Omega)} + \left\| \operatorname{div}_{X_1} (\mathcal{A}_{12} \nabla_{X_2} f_{g,s,\mu}) \right\|_{L^2(\Omega)} \\ + \left\| \operatorname{div}_{X_2} (\mathcal{A}_{21} \nabla_{X_1} f_{g,s,\mu}) \right\|_{L^2(\Omega)} + \\ C_{A,\Omega} \left(\left\| \nabla_{X_1} \mathcal{A}_0 f_{g,s,\mu} \right\|_{L^2(\Omega)} + \left\| \mathcal{A}_0 f_{g,s,\mu} \right\|_{L^2(\Omega)} \right) \end{array} \right).$$

By using (1.43–1.45) with (26), one can show that the quantity in parentheses in the above inequality is bounded by some $C_{g,A,\Omega} > 0$ independent of s , ϵ , and μ , thus

$$\left\| \mathcal{A}_{0,\mu} e^{s\mathcal{A}_{0,\mu}} g - \mathcal{A}_{\epsilon,\mu} e^{s\mathcal{A}_{0,\mu}} g \right\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times \epsilon.$$

Finally, integrating the above inequality in s over $[0, t]$, and using (1.42), we get the desired result. \square

Now, we are able to prove Theorem 0.4. First we prove the case when $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ and we conclude by a density argument. So, let $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$, by Lemma 1.3 we have

$$\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \left\| e^{t\mathcal{A}_{\epsilon,\mu}} g - e^{t\mathcal{A}_{0,\mu}} g \right\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon. \quad (1.46)$$

Passing to the limit in (1.46) as $\mu \rightarrow +\infty$ we get (see the preliminaries 2.1)

$$\forall t \geq 0, \forall \epsilon \in (0, 1] : \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon,$$

whence for $T \geq 0$ fixed we obtain

$$\forall \epsilon \in (0, 1] : \sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times T \times \epsilon. \quad (1.47)$$

Whence

$$\sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (1.48)$$

Now, let $g \in L^2(\Omega)$ and let $\delta > 0$, by density there exists $g_\delta \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ such that

$$\|g - g_\delta\|_{L^2(\Omega)} \leq \frac{\delta}{4}.$$

According to (1.48) there exists $\epsilon_\delta > 0$ such that

$$\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \|S_\epsilon(t)g_\delta - S_0(t)g_\delta\|_{L^2(\Omega)} \leq \frac{\delta}{2}.$$

Whence, by the triangle inequality we get

$$\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq \frac{\delta}{2} + \sup_{t \in [0, T]} (\|S_\epsilon(t)\| + \|S_0(t)\|) \times \|g_\delta - g\|_{L^2(\Omega)}.$$

Using the fact that the semigroups $(S_\epsilon(t))_{t \geq 0}$ and $(S_0(t))_{t \geq 0}$ are of contractions, we get

$$\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq \delta.$$

So, $\sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$. The second assertion of the theorem is given by (1.47) and the proof of Theorem 0.4 is completed.

2.3 An application to linear parabolic equations

Theorem 0.4 gives an opening for the study of anisotropic singular perturbations of evolution partial differential equations from the semigroup point of view. In this subsection, we give a simple and short application to the linear parabolic equation

$$\frac{\partial u_\epsilon}{\partial t} - \operatorname{div}(A_\epsilon \nabla u_\epsilon) = 0, \quad (1.49)$$

supplemented with the boundary and the initial conditions

$$u_\epsilon(t, \cdot) = 0 \text{ in } \partial\Omega \text{ for } t \in (0, +\infty) \quad (1.50)$$

$$u_\epsilon(0, \cdot) = u_{\epsilon,0}. \quad (1.51)$$

The limit problem is

$$\frac{\partial u}{\partial t} - \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u) = 0, \quad (1.52)$$

supplemented with the boundary and the initial conditions

$$u(t, \cdot) = 0 \text{ in } \omega_1 \times \partial\omega_2 \text{ for } t \in (0, +\infty) \quad (1.53)$$

$$u(0, \cdot) = u_0. \quad (1.54)$$

The operator forms of (1.49 – 1.51) and (1.52 – 1.54) read

$$\frac{du_\epsilon}{dt} - \mathcal{A}_\epsilon u_\epsilon = 0, \text{ with } u_\epsilon(0) = u_{\epsilon,0}, \quad (1.55)$$

and

$$\frac{du}{dt} - \mathcal{A}_0 u = 0, \text{ with } u(0) = u_0. \quad (1.56)$$

Suppose that $u_0 \in D(\mathcal{A}_0)$ and $u_{\epsilon,0} \in D(\mathcal{A}_\epsilon)$. Assume that (5), (6) hold, then it follows that (1.55), (1.56) have unique classical solutions

$$u_\epsilon \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(\mathcal{A}_\epsilon)), \text{ and } u \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(\mathcal{A}_0))$$

We have the following convergence result.

Proposition 1.4. *Suppose that $u_0 \in D(\mathcal{A}_0)$ and $u_{\epsilon,0} \in D(\mathcal{A}_\epsilon)$ such that $u_{\epsilon,0} \rightarrow u_0$ in $L^2(\Omega)$, then under assumptions of Theorem 0.4, we have for any $T \geq 0$:*

$$\sup_{t \in [0, T]} \|u_\epsilon(t) - u(t)\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (1.57)$$

Moreover, if $u_{\epsilon,0}$ and u_0 are in $H^2(\Omega)$ such that $(u_{\epsilon,0})$ is bounded in $H^2(\Omega)$ and $\|\nabla_{X_2}(u_{\epsilon,0} - u_0)\|_{L^2(\Omega)} \rightarrow 0$, $\|\nabla_{X_2}^2(u_{\epsilon,0} - u_0)\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$, then:

$$\sup_{t \in [0, T]} \left\| \frac{d}{dt}(u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} \rightarrow 0.$$

Proof. It is well known that the solutions u_ϵ , u are given by

$$u_\epsilon(t) = S_\epsilon(t)u_{\epsilon,0} \text{ and } u_0(t) = S_0(t)u_0, \text{ for every } t \geq 0.$$

Let $T \geq 0$, we have

$$\begin{aligned} \sup_{t \in [0, T]} \|u_\epsilon(t) - u(t)\|_{L^2(\Omega)} &\leq \sup_{t \in [0, T]} \|S_\epsilon(t)u_{\epsilon,0} - S_\epsilon(t)u_0\|_{L^2(\Omega)} + \sup_{t \in [0, T]} \|S_\epsilon(t)u_0 - S_0(t)u_0\|_{L^2(\Omega)} \\ &\leq \|u_{\epsilon,0} - u_0\|_{L^2(\Omega)} + \sup_{t \in [0, T]} \|S_\epsilon(t)u_0 - S_0(t)u_0\|_{L^2(\Omega)}. \end{aligned}$$

Passing to the limit as $\epsilon \rightarrow 0$ by using Theorem 0.4, we get $\sup_{t \in [0, T]} \|u_\epsilon(t) - u(t)\|_{L^2(\Omega)} \rightarrow 0$.

For the second affirmation, we have:

$$\begin{aligned} \left\| \frac{d}{dt}(u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} &= \|S_\epsilon(t)\mathcal{A}_\epsilon u_{\epsilon,0} - S_0(t)\mathcal{A}_0 u_0\|_{L^2(\Omega)} \\ &\leq \|\mathcal{A}_\epsilon u_{\epsilon,0} - \mathcal{A}_0 u_0\|_{L^2(\Omega)} + \sup_{t \in [0, T]} \|S_\epsilon(t)\mathcal{A}_0 u_0 - S_0(t)\mathcal{A}_0 u_0\|_{L^2(\Omega)}. \end{aligned}$$

As $(u_{\epsilon,0})$ is bounded in $H^2(\Omega)$, $u_0 \in H^2(\Omega)$ and $\|\nabla_{X_2}(u_{\epsilon,0}-u_0)\|_{L^2(\Omega)} \rightarrow 0$, $\|\nabla_{X_2}^2(u_{\epsilon,0}-u_0)\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$, then by using (26) we get immediately $\|\mathcal{A}_\epsilon u_{\epsilon,0} - \mathcal{A}_0 u_0\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$, and we conclude by applying Theorem 0.4. \square

Remark 1.5. Consider the nonhomogeneous parabolic equations associated to (1.49) and (1.52) with second member $f(t,x)$. Suppose that f is regular enough, for example $f \in Lip([0,T]; L^2(\Omega))$, then the associated classical solutions u_ϵ and u exist and they are unique. In this case, we have the same convergence result (1.57). The proof follows immediately from the use of the following integral representation formulas

$$u_\epsilon(t) = S_\epsilon(t)u_{\epsilon,0} + \int_0^t S_\epsilon(t-r)f(r)dr, \quad u(t) = S_0(t)u_0 + \int_0^t S_0(t-r)f(r)dr, \quad t \in [0,T],$$

Theorem 0.4, and Lebesgue's theorem.

Chapter 2

Convergence in high-order pseudo-Sobolev spaces

In this chapter, we deal with problem (12) that we recall here

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon}) \varphi dx + \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_{\epsilon} \in H_0^1(\Omega), \end{cases}$$

where Ω is a bounded open set of \mathbb{R}^N , not necessarily cylindrical. Recall that the associated limit problem (31) is given by

$$u \in H_0^1(\Omega; \Pi_2\Omega) \text{ and } \int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega; \Pi_2\Omega).$$

Recall that A satisfies the ellipticity assumptions (5), (6), and that β is nondecreasing continuous and sublinear (assumptions (10), (11)).

As mentioned in the introduction, the proof of the main theorem (Theorem 0.5) is a bit technical so, let us begin by studying the simple case, that of the Laplacian.

1 The perturbed Poisson equation

Let us start by the following important Lemma

Lemma 2.1. *Let $f \in L^2(\mathbb{R}^N)$. Let $\epsilon \in (0, 1]$ and let $u_{\epsilon} \in H^2(\mathbb{R}^N)$ such that*

$$-\epsilon^2 \Delta_{X_1} u_{\epsilon}(x) - \Delta_{X_2} u_{\epsilon}(x) = f(x) \text{ a.e. } x \in \mathbb{R}^N, \quad (2.1)$$

then we have the bounds:

$$\left\| \nabla_{X_2}^2 u_{\epsilon} \right\|_{L^2(\mathbb{R}^N)^{(N-g)^2}} \leq \|f\|_{L^2(\mathbb{R}^N)},$$

$$\epsilon^2 \left\| \nabla_{X_1}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)^{q^2}} \leq \|f\|_{L^2(\mathbb{R}^N)},$$

$$\sqrt{2}\epsilon \left\| \nabla_{X_1 X_2}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)^{N(N-q)}} \leq \|f\|_{L^2(\mathbb{R}^N)}.$$

Proof. Let \mathcal{F} be the Fourier transform defined on $L^2(\mathbb{R}^N)$ as the extension, by density, of the Fourier transform defined on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ by

$$\mathcal{F}(u)(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u(x) e^{-ix \cdot \xi} dx, \quad u \in \mathcal{S}(\mathbb{R}^N)$$

where \cdot is the standard scalar product of \mathbb{R}^N . Applying \mathcal{F} on (2.1) we obtain

$$\left(\epsilon^2 \sum_{i=1}^q \xi_i^2 + \sum_{i=q+1}^N \xi_i^2 \right) \mathcal{F}(u_\epsilon)(\xi) = \mathcal{F}(f)(\xi),$$

then

$$\left(\epsilon^4 \sum_{i,j=1}^q \xi_i^2 \xi_j^2 + \sum_{i,j=q+1}^N \xi_i^2 \xi_j^2 + 2\epsilon^2 \sum_{j=q+1}^N \sum_{i=1}^q \xi_i^2 \xi_j^2 \right) |\mathcal{F}(u_\epsilon)(\xi)|^2 = |\mathcal{F}(f)(\xi)|^2, \quad (2.2)$$

thus

$$\sum_{i,j=q+1}^N \xi_i^2 \xi_j^2 |\mathcal{F}(u_\epsilon)(\xi)|^2 \leq |\mathcal{F}(f)(\xi)|^2,$$

hence

$$\sum_{i,j=q+1}^N \left| \mathcal{F}(\partial_{ij}^2 u_\epsilon)(\xi) \right|^2 \leq |\mathcal{F}(f)(\xi)|^2,$$

then

$$\sum_{i,j=q+1}^N \left\| \mathcal{F}(\partial_{ij}^2 u_\epsilon) \right\|_{L^2(\mathbb{R}^N)}^2 \leq \|\mathcal{F}(f)\|_{L^2(\mathbb{R}^N)}^2,$$

and the Parseval identity gives

$$\sum_{i,j=q+1}^N \left\| \partial_{ij}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)}^2 \leq \|f\|_{L^2(\mathbb{R}^N)}^2.$$

Hence

$$\left\| \nabla_{X_2}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)^{(N-q)^2}} \leq \|f\|_{L^2(\mathbb{R}^N)}.$$

Similarly we obtain from (2.2) the bounds

$$\epsilon^2 \left\| \nabla_{X_1}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)^{q^2}} \leq \|f\|_{L^2(\mathbb{R}^N)},$$

$$\sqrt{2}\epsilon \left\| \nabla_{X_1 X_2}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)}.$$

□

Now, let Ω be an open bounded set of \mathbb{R}^N . Suppose $A = Id$ and $f \in L^2(\Omega)$, and let $u_\epsilon \in H_0^1(\Omega)$ be the unique solution to (4). Notice that the elliptic regularity [20] shows that $u_\epsilon \in H_{loc}^2(\Omega)$. Let $(\epsilon_k)_{k \in \mathbb{N}}$ be a sequence in $(0, 1]$ with $\lim \epsilon_k = 0$, and let $u_k := u_{\epsilon_k}$ be the unique solution of (4) with ϵ replaced by ϵ_k . For a function v and for $h \in \mathbb{R}^N$ we denote $\tau_h v$ the function defined by $x \rightarrow v(x + h)$. Under the above notations one can prove the following

Proposition 2.2. 1) *Let $\omega \subset\subset \Omega$ open then:*

$$\begin{aligned} \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k \right\|_{L^2(\omega)^{(N-q)^2}} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k) \right\|_{L^2(\omega)^{q^2}} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k) \right\|_{L^2(\omega)^{q(N-q)}} &= 0. \end{aligned}$$

2) *The sequences $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$, $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L_{loc}^2(\Omega)$ i.e. for every $\omega \subset\subset \Omega$ open there exists $C_\omega > 0$ such that*

$$\sup_{k \in \mathbb{N}} \left\| \epsilon_k^2 \nabla_{X_1}^2 u_k \right\|_{L^2(\omega)^{q^2}}, \sup_{k \in \mathbb{N}} \left\| \nabla_{X_2}^2 u_k \right\|_{L^2(\omega)^{(N-q)^2}}, \sup_{k \in \mathbb{N}} \left\| \epsilon_k \nabla_{X_1 X_2}^2 u_k \right\|_{L^2(\omega)^{q(N-q)}} \leq C_\omega.$$

Proof. 1) Let $\omega \subset\subset \Omega$ open, then one can choose ω' open such that $\omega \subset\subset \omega' \subset\subset \Omega$, let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω , $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega'$. Let $h \in \mathbb{R}^N$, $0 < |h| < dist(\omega', \partial\Omega)$, and let $k \in \mathbb{N}$. To make the notations less heavy we set $U_k^h = \tau_h u_k - u_k$, then $U_k^h \in H^2(\omega')$. Notice that translation and derivation commute then we have

$$-\epsilon_k^2 \Delta_{X_1} U_k^h(x) - \Delta_{X_2} U_k^h(x) = F^h(x), \quad \text{a.e } x \in \omega',$$

with $F^h = \tau_h f - f$.

We set $\mathcal{W}_k^h = \rho U_k^h$ then we get

$$\begin{aligned} -\epsilon_k^2 \Delta_{X_1} \mathcal{W}_k^h(x) - \Delta_{X_2} \mathcal{W}_k^h(x) &= \rho(x) F^h(x) - 2\epsilon_k^2 \nabla_{X_1} \rho(x) \cdot \nabla_{X_1} U_k^h(x) \\ &\quad - 2\nabla_{X_2} \rho(x) \cdot \nabla_{X_2} U_k^h(x) - U_k^h(x) (\epsilon_k^2 \Delta_{X_1} \rho(x) - \Delta_{X_2} \rho(x)), \end{aligned}$$

for a.e $x \in \omega'$.

Since $U_k^h \in W^{2,2}(\omega')$ then $\mathcal{W}_k^h \in H_0^2(\omega')$, so we can extend \mathcal{W}_k^h by 0 on the outside of ω' then $\mathcal{W}_k^h \in H^2(\mathbb{R}^N)$. The right hand side of the above equality is extended by 0 outside of ω' , hence the

equation is satisfied in \mathbb{R}^N , and thus by Lemma 2.1 we get

$$\begin{aligned} \left\| \nabla_{X_2}^2 \mathcal{W}_k^h \right\|_{L^2(\mathbb{R}^N)^{(N-q)^2}} &\leq \left\| \rho F^h \right\|_{L^2(\mathbb{R}^N)} + 2\epsilon_k^2 \left\| \nabla_{X_1} \rho \cdot \nabla_{X_1} U_k^h \right\|_{L^2(\mathbb{R}^N)} \\ &\quad + 2 \left\| \nabla_{X_2} \rho \cdot \nabla_{X_2} U_k^h \right\|_{L^2(\mathbb{R}^N)} + \left\| U_k^h (\epsilon_k^2 \Delta_{X_1} \rho - \Delta_{X_2} \rho) \right\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Whence

$$\begin{aligned} \left\| \nabla_{X_2}^2 U_k^h \right\|_{L^2(\omega)^{(N-q)^2}} &\leq \left\| F^h \right\|_{L^2(\omega')} + 2\epsilon_k \left\| \nabla_{X_1} \rho \right\|_{\infty} \left\| \epsilon_k \nabla_{X_1} U_k^h \right\|_{L^2(\omega')} \\ &\quad + 2 \left\| \nabla_{X_2} \rho \right\|_{\infty} \left\| \nabla_{X_2} U_k^h \right\|_{L^2(\omega')} + \left\| (\epsilon_k^2 \Delta_{X_1} \rho - \Delta_{X_2} \rho) \right\|_{\infty} \left\| U_k^h \right\|_{L^2(\omega')}. \end{aligned}$$

Notice that by (29) we have $u_k \rightarrow u$ in $H_0^1(\Omega; \Pi_2\Omega)$ and $\epsilon_k \nabla_{X_1} u_k \rightarrow 0$ in $L^2(\Omega)$, then by the Fréchet-Kolmogorov theorem we deduce that

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k \nabla_{X_1} U_k^h \right\|_{L^2(\omega')} = \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k (\tau_h \nabla_{X_1} u_k - \nabla_{X_1} u_k) \right\|_{L^2(\omega')} = 0,$$

and similarly we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \nabla_{X_2} U_k^h \right\|_{L^2(\omega')} &= 0, \quad \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| F^h \right\|_{L^2(\omega')} = 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| U_k^h \right\|_{L^2(\omega')} &= 0. \end{aligned}$$

Hence, by passing to the limit in the above inequality we get

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k \right\|_{L^2(\omega)^{(N-q)^2}} = \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \nabla_{X_2}^2 U_k^h \right\|_{L^2(\omega)^{(N-q)^2}} = 0.$$

Similarly we obtain

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k) \right\|_{L^2(\omega)^{q^2}} = 0,$$

and

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k) \right\|_{L^2(\omega)^{q(N-q)}} = 0.$$

2) Following the same arguments, we get the estimation

$$\begin{aligned} \epsilon_k^2 \left\| \nabla_{X_1}^2 u_k \right\|_{L^2(\omega)^{q^2}} + \left\| \nabla_{X_2}^2 u_k \right\|_{L^2(\omega)^{(N-q)^2}} + \sqrt{2} \epsilon_k \left\| \nabla_{X_1 X_2}^2 u_k \right\|_{L^2(\omega)^{q(N-q)}} &\leq \\ 3 \left\| f \right\|_{L^2(\omega')} + 6\epsilon_k \left\| \nabla_{X_1} \rho \right\|_{\infty} \left\| \epsilon_k \nabla_{X_1} u_k \right\|_{L^2(\omega')} & \\ + 6 \left\| \nabla_{X_2} \rho \right\|_{\infty} \left\| \nabla_{X_2} u_k \right\|_{L^2(\omega')} + 3 \left\| (\epsilon_k^2 \Delta_{X_1} \rho - \Delta_{X_2} \rho) \right\|_{\infty} \left\| u_k \right\|_{L^2(\omega')} &. \end{aligned}$$

The convergences $u_k \rightarrow u$ in $H_0^1(\Omega; \Pi_2\Omega)$, and $\epsilon_k \nabla_{X_1} u_k \rightarrow 0$ in $L^2(\Omega)$ and boundedness of ρ and its derivatives show that the right hand side of the above inequality is bounded in k , i.e. for some $C_\omega > 0$ independent of k we have

$$\forall k \in \mathbb{N} : \epsilon_k^2 \left\| \nabla_{X_1}^2 u_k \right\|_{L^2(\omega)^{q^2}} + \left\| \nabla_{X_2}^2 u_k \right\|_{L^2(\omega)^{(N-q)^2}} + \sqrt{2} \epsilon_k \left\| \nabla_{X_1 X_2}^2 u_k \right\|_{L^2(\omega)^{q(N-q)}} \leq C_\omega,$$

and therefore, the sequences $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$, $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L_{loc}^2(\Omega)^{(N-q)^2, q^2, q(N-q)}$.

□

Now, we are ready to prove the following

Theorem 2.3. *Let Ω a bounded open set of \mathbb{R}^N . Suppose $A = Id$. Let $f \in L^2(\Omega)$. Let $u_\epsilon \in H_0^1(\Omega) \cap H_{loc}^2(\Omega)$ be the unique solution of (4), then $u_\epsilon \rightarrow u$ strongly in $H_{loc}^2(\Omega; \Pi_2\Omega)$ where u is the unique solution of the limit problem (28). In addition, we have*

$$\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0 \text{ and } \epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0, \text{ strongly in } L_{loc}^2(\Omega)^{q^2} \text{ and } L_{loc}^2(\Omega)^{q(N-q)} \text{ respectively.}$$

Proof. Let $u \in H_0^1(\Omega; \Pi_2\Omega)$ be the solution of the limit problem (28) and let $(u_k)_{k \in \mathbb{N}}$, $u_k = u_{\epsilon_k} \in H_0^1(\Omega) \cap H_{loc}^2(\Omega)$ be a sequence of solutions to (4) with ϵ replaced by ϵ_k with $\epsilon_k \rightarrow 0$, then Proposition 2.2 shows that the hypothesis of the Fréchet-Kolmogorov theorem are fulfilled (For the statement of the theorem, see for instance [27]). Whence, it follows that $\{\nabla_{X_2}^2 u_k\}_{k \in \mathbb{N}}$ is relatively compact in $L^2(\omega)^{(N-q)^2}$ for every $\omega \subset\subset \Omega$ open. Now, for $\omega \subset\subset \Omega$ fixed there exists $U^\omega \in L^2(\omega)^{(N-q)^2}$ and a subsequence still labelled $(\nabla_{X_2}^2 u_k)_{k \in \mathbb{N}}$ such that $\nabla_{X_2}^2 u_k \rightarrow U^\omega$ in $L^2(\omega)^{(N-q)^2}$ strongly. Since $u_k \rightarrow u$ in $L^2(\omega)$ and the second order differential operators ∂_{ij}^2 are continuous on $\mathcal{D}'(\omega)$ then $U^\omega = \nabla_{X_2}^2 u$ on ω . Whence, since ω is arbitrary we get $\nabla_{X_2}^2 u \in L_{loc}^2(\Omega)^{(N-q)^2}$, i.e. $u \in H_{loc}^2(\Omega; \Pi_2\Omega)$. Now, let (ω_n) be a countable covering of Ω with $\omega_n \subset\subset \Omega$, $\omega_n \subset \omega_{n+1}, \forall n \in \mathbb{N}$. By the diagonal process one can construct a subsequence still labelled (u_k) such that

$$\nabla_{X_2}^2 u_k \rightarrow \nabla_{X_2}^2 u \text{ in } L_{loc}^2(\Omega)^{(N-q)^2} \text{ strongly.}$$

Combining this with the convergence $u_k \rightarrow u$ of (29) we get

$$u_k \rightarrow u \text{ strongly in } H_{loc}^2(\Omega; \Pi_2\Omega), \text{ i.e. } d(u_k, u) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

where d is the distance of the Fréchet space $H_{loc}^2(\Omega; \Pi_2\Omega)$.

Finally, to prove the convergence of the whole sequence $(u_\epsilon)_{0 < \epsilon \leq 1}$ we can reason by contradiction. Suppose that there exists $\delta > 0$ and a subsequence (u_k) such that $d(u_k, u) > \delta$. It follows by the first part of this proof that there exists a subsequence still labelled (u_k) such that $d(u_k, u) \rightarrow 0$, which is a contradiction. By using the same arguments we show similarly that

$$\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0 \text{ and } \epsilon \nabla_{X_1 X_2}^2 u_\epsilon \rightarrow 0 \text{ strongly in } L_{loc}^2(\Omega)^{q^2} \text{ and } L_{loc}^2(\Omega)^{q(N-q)} \text{ respectively.}$$

□

2 General elliptic problems

2.1 Interior convergence on general bounded domain for the linear case

In this subsection we will prove Theorems 0.5, 0.6. At first, we suppose that the coefficients of A are constants. We have the following lemma

Lemma 2.1. *Suppose that the coefficients of A are constants and assume (5). Let $(u_\epsilon)_{0 < \epsilon \leq 1}$ be a sequence in $H^2(\mathbb{R}^N)$ such that $-\sum_{i,j=1}^N a_{ij}^\epsilon \partial_{ij}^2 u_\epsilon = f$, with $f \in L^2(\mathbb{R}^N)$, then we have for every $\epsilon \in (0, 1]$:*

$$\begin{aligned} \lambda \left\| \nabla_{X_2}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)} \\ \lambda \epsilon^2 \left\| \nabla_{X_1}^2 u_\epsilon \right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}, \\ \sqrt{2} \lambda \epsilon \left\| \nabla_{X_1 X_2}^2 u \right\|_{L^2(\mathbb{R}^N)} &\leq \|f\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Proof. As in proof of Lemma 2.1 we use the Fourier transform and we obtain

$$\left(\sum_{i,j} a_{ij}^\epsilon \xi_i \xi_j \right) \mathcal{F}(u_\epsilon)(\xi) = \mathcal{F}(f)(\xi), \quad \xi \in \mathbb{R}^N.$$

From the ellipticity assumption (5) we deduce

$$\lambda^2 \left(\epsilon^2 \sum_{i=1}^q \xi_i^2 + \sum_{i=q+1}^N \xi_i^2 \right)^2 |\mathcal{F}(u_\epsilon)(\xi)|^2 \leq |\mathcal{F}(f)(\xi)|^2$$

Thus, similarly we obtain the desired bounds. □

Now, suppose (5), (6), and (30). Let u_ϵ be the unique solution of (4), then it follows by the elliptic regularity that $u_\epsilon \in H_{loc}^2(\Omega)$. We denote $u_k := u_{\epsilon_k}$ where (ϵ_k) is a sequence in $(0, 1]$ such that $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Under the above notation we have the following

Proposition 2.2. *Let Ω be an open bounded set of \mathbb{R}^N . Suppose (5), (6), and (30). Let $f \in L^2(\Omega)$. Let $z_0 \in \Omega$ fixed then there exists $\omega_0 \subset\subset \Omega$ open with $z_0 \in \omega_0$ such that the sequences $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$ and $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L^2(\omega_0)$.*

Proof. Let $k \in \mathbb{N}$. Since $u_k \in H_0^1(\Omega) \cap H_{loc}^2(\Omega)$ and $A \in C^1(\Omega)$ then u_k satisfies

$$-\sum_{i,j=1}^N a_{ij}^k(x) \partial_{ij}^2 u_k(x) - \sum_{i,j=1}^N \partial_i a_{ij}^k(x) \partial_j u_k(x) = f(x), \text{ for a.e } x \in \Omega \quad (2.3)$$

where we have set $a_{ij}^k := a_{ij}^{\epsilon_k}$.

Let $z_0 \in \Omega$ fixed, and let $\theta > 0$ such that

$$\min \left\{ [\lambda - 3\theta(N - q)], [\lambda - 3\theta q], [\sqrt{2}\lambda - 6(N - q)q\theta] \right\} \geq \frac{\lambda}{2}. \quad (2.4)$$

By using the continuity of the a_{ij} one can choose $\omega'_0 \subset\subset \Omega$, $z_0 \in \omega'_0$ such that

$$\max_{i,j} \sup_{x \in \omega'_0} |a_{ij}(x) - a_{ij}(z_0)| \leq \theta \quad (2.5)$$

Let $\omega_0 \subset\subset \omega'_0$ open with $z_0 \in \omega_0$ and let $\rho \in \mathcal{D}(\mathbb{R}^N)$ such that $\rho = 1$ on ω_0 , $0 \leq \rho \leq 1$ and $Supp(\rho) \subset \omega'_0$. We set $U_k = \rho u_k$, and we extend it by 0 on the outside of ω'_0 then $U_k \in H^2(\mathbb{R}^N)$.

Therefore from (2.3) we obtain

$$-\sum_{i,j} a_{ij}^k(z_0) \partial_{ij}^2 U_k(x) = \sum_{i,j} (a_{ij}^k(x) - a_{ij}^k(z_0)) \partial_{ij}^2 U_k(x) + g_k(x), \text{ for a.e } x \in \mathbb{R}^N,$$

where we have set

$$g_k := \rho f + \rho \sum_{i,j} \partial_i a_{ij}^k \partial_j u_k - u_k \sum_{i,j} a_{ij}^k \partial_{ij}^2 \rho - \sum_{i,j} a_{ij}^k \partial_i \rho \partial_j u_k - \sum_{i,j} a_{ij}^k \partial_j \rho \partial_i u_k, \quad (2.6)$$

and we have extended it by 0 on the outside of ω'_0 . Now, applying Lemma 2.1 to the above differential equality we get

$$\begin{aligned} \lambda \left\| \nabla_{X_2}^2 U_k \right\|_{L^2(\omega'_0)} + \lambda \epsilon_k^2 \left\| \nabla_{X_1}^2 U_k \right\|_{L^2(\omega'_0)} + \sqrt{2} \lambda \epsilon_k \left\| \nabla_{X_1 X_2}^2 U_k \right\|_{L^2(\omega'_0)} \\ \leq 3 \left\| \sum_{i,j} (a_{ij}^k - a_{ij}^k(z_0)) \partial_{ij}^2 U_k \right\|_{L^2(\omega'_0)} + 3 \|g_k\|_{L^2(\omega'_0)} \end{aligned}$$

Whence, by using (2.5) we get

$$\begin{aligned} & \lambda \left\| \nabla_{X_2}^2 U_k \right\|_{L^2(\omega'_0)} + \lambda \epsilon_k^2 \left\| \nabla_{X_1}^2 U_k \right\|_{L^2(\omega'_0)} + \sqrt{2} \lambda \epsilon_k \left\| \nabla_{X_1 X_2}^2 U_k \right\|_{L^2(\omega'_0)} \\ & \leq 3\theta \epsilon_k^2 \sum_{i,j=1}^q \left\| \partial_{ij}^2 U_k \right\|_{L^2(\omega'_0)} + 3\theta \sum_{i,j=q+1}^N \left\| \partial_{ij}^2 U_k \right\|_{L^2(\omega'_0)} \\ & \quad + 6\theta \epsilon_k \sum_{i=1}^q \sum_{j=q+1}^N \left\| \partial_{ij}^2 U_k \right\|_{L^2(\omega'_0)} + 3 \|g_k\|_{L^2(\omega'_0)}. \end{aligned}$$

Therefore, by the discrete Cauchy-Schwarz inequality we deduce

$$\begin{aligned} & \lambda \left\| \nabla_{X_2}^2 U_k \right\|_{L^2(\omega'_0)} + \lambda \epsilon_k^2 \left\| \nabla_{X_1}^2 U_k \right\|_{L^2(\omega'_0)} + \sqrt{2} \lambda \epsilon_k \left\| \nabla_{X_1 X_2}^2 U_k \right\|_{L^2(\omega'_0)} \\ & \leq 3\theta(N-q) \left\| \nabla_{X_2}^2 U_k \right\|_{L^2(\omega'_0)} + \epsilon_k^2 3\theta q \left\| \nabla_{X_1}^2 U_k \right\|_{L^2(\omega'_0)} \\ & \quad + \epsilon_k 6(N-q)q\theta \left\| \nabla_{X_1 X_2}^2 U_k \right\|_{L^2(\omega'_0)}^2 + 3 \|g_k\|_{L^2(\omega'_0)}, \end{aligned}$$

and thus

$$\begin{aligned} & [\lambda - 3\theta(N-q)] \left\| \nabla_{X_2}^2 U_k \right\|_{L^2(\omega'_0)} + \epsilon_k^2 [\lambda - 3\theta q] \left\| \nabla_{X_1}^2 U_k \right\|_{L^2(\omega'_0)} + \\ & \quad \epsilon_k [\sqrt{2}\lambda - 6(N-q)q\theta] \left\| \nabla_{X_1 X_2}^2 U_k \right\|_{L^2(\omega'_0)} \leq 3 \|g_k\|_{L^2(\omega'_0)}. \end{aligned}$$

Hence, by (2.4) we get

$$\left\| \nabla_{X_2}^2 u_k \right\|_{L^2(\omega_0)^{(N-q)^2}} + \epsilon_k^2 \left\| \nabla_{X_1}^2 u_k \right\|_{L^2(\omega_0)^{q^2}} + \epsilon_k \left\| \nabla_{X_1 X_2}^2 u_k \right\|_{L^2(\omega_0)^{q(N-q)}} \leq \frac{6}{\lambda} \|g_k\|_{L^2(\omega'_0)}.$$

To complete the proof, we will show the boundedness of (g_k) in $L^2(\omega'_0)$. Indeed, ρ and its derivatives, a_{ij} and their first derivatives are bounded on ω'_0 , moreover (29) shows that the sequences $(\epsilon_k \nabla_{X_1} u_k)$, $(\nabla_{X_2} u_k)$ and (u_k) are bounded in $L^2(\Omega)$, and therefore from (2.6) the boundedness of (g_k) in $L^2(\omega'_0)$ follows. \square

Corollary 2.3. *Under assumptions of Proposition 2.2 the sequences $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$, $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L_{loc}^2(\Omega)^{(N-q)^2, q^2, q(N-q)}$.*

Proof. Let $\omega \subset\subset \Omega$ open, for every $z \in \bar{\omega}$ there exists $\omega_z \subset\subset \Omega$, $z \in \omega_z$ which satisfies the affirmations of Proposition 2.2 in $L^2(\omega_z)$. By using the compactness of $\bar{\omega}$, one can extract a finite cover (ω_{z_i}) of ω , and hence the sequences $(\nabla_{X_2}^2 u_k)$, $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$, $(\epsilon_k \nabla_{X_1 X_2}^2 u_k)$ are bounded in $L^2(\omega)$. \square

Proposition 2.4. *Assume the same hypothesis of Proposition 2.2. Let $z_0 \in \Omega$ then there exists $\omega_0 \subset\subset \Omega$ with $z_0 \in \omega_0$ such that*

$$\begin{aligned} \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k \right\|_{L^2(\omega_0)^{(N-q)^2}} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k) \right\|_{L^2(\omega_0)^{q^2}} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k) \right\|_{L^2(\omega_0)^{q(N-q)}} &= 0. \end{aligned}$$

Proof. Let $z_0 \in \Omega$ fixed and let $\theta > 0$ such that (2.4). By using the continuity of the a_{ij} one can choose $\omega'_0 \subset\subset \Omega$ with $z_0 \in \omega'_0$ such that we have (2.5). Let $\omega_0 \subset\subset \omega'_0$, with $z_0 \in \omega_0$, and let $\rho \in \mathcal{D}(\mathbb{R}^N)$ with $\rho = 1$ on ω_0 , $0 \leq \rho \leq 1$, and $\text{Supp}(\rho) \subset \omega'_0$. Let $h \in \mathbb{R}^N$, $0 < |h| < \text{dist}(\omega'_0, \partial\Omega)$. Let $k \in \mathbb{N}$. We set $\mathcal{W}_k^h = \rho U_k^h$, with $U_k^h = (\tau_h u_k - u_k)$ and we extend it by 0 on the outside of ω'_0 then $\mathcal{W}_k^h \in H^2(\mathbb{R}^N)$. By using (2.3) we get:

$$-\sum_{i,j} a_{ij}^k(z_0) \partial_{ij}^2 \mathcal{W}_k^h(x) = \sum_{i,j} (a_{ij}^k(x) - a_{ij}^k(z_0)) \partial_{ij}^2 \mathcal{W}_k^h(x) + G_k^h(x), \text{ a.e } x \in \mathbb{R}^N$$

where we have set

$$\begin{aligned} G_k^h &= -U_k^h \sum_{i,j} a_{ij}^k \partial_{ij}^2 \rho - \sum_{i,j} a_{ij}^k \partial_i \rho \partial_j U_k^h - \sum_{i,j} a_{ij}^k \partial_j \rho \partial_i U_k^h \\ &\quad - \rho \sum_{i,j} (a_{ij}^k - \tau_h a_{ij}^k) \tau_h \partial_{ij}^2 u_k - \rho (f - \tau_h f) \\ &\quad - \rho \sum_{i,j} [\partial_i a_{ij}^k \partial_j u_k - \partial_i \tau_h a_{ij}^k \partial_j \tau_h u_k], \end{aligned} \tag{2.7}$$

and we have extended it by 0 on the outside of ω'_0 .

As in proof of Proposition 2.2, we obtain

$$\begin{aligned} \left\| \tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k \right\|_{L^2(\omega_0)} + \epsilon_k^2 \left\| \tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k \right\|_{L^2(\omega_0)} \\ + \epsilon_k \left\| \tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k \right\|_{L^2(\omega_0)} \leq \frac{6}{\lambda} \left\| G_k^h \right\|_{L^2(\omega'_0)}. \end{aligned}$$

To complete the proof, we have to show that $\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| G_k^h \right\|_{L^2(\omega'_0)} = 0$. Using the boundedness of the

a_{ij} and the boundedness of ρ and its derivatives on ω'_0 we get from (2.7) the following

$$\begin{aligned}
 \|G_k^h\|_{L^2(\omega'_0)} &\leq C_{\omega'_0} \|U_k^h\|_{L^2(\omega'_0)} + C_{\omega'_0} \epsilon_k \|\nabla_{X_1} U_k^h\|_{L^2(\omega'_0)} \\
 &\quad + C_{\omega'_0} \|\nabla_{X_2} U_k^h\|_{L^2(\omega'_0)} + \|\tau_h f - f\|_{L^2(\omega'_0)} \\
 &\quad + \sum_{i,j} \left\| \left(a_{ij}^k - \tau_h a_{ij}^k \right) \tau_h \partial_{ij}^2 u_k \right\|_{L^2(\omega'_0)} \\
 &\quad + \sum_{i,j} \left\| \partial_i a_{ij}^k \partial_j u_k - \tau_h \partial_i a_{ij}^k \tau_h \partial_j u_k \right\|_{L^2(\omega'_0)},
 \end{aligned} \tag{2.8}$$

where $C_{\omega'_0} > 0$ is independent of h and k . Now, estimating the fifth term of the right hand side of the above inequality we obtain

$$\begin{aligned}
 \sum_{i,j} \left\| \left(a_{ij}^k - \tau_h a_{ij}^k \right) \tau_h \partial_{ij}^2 u_k \right\|_{L^2(\omega'_0)} &\leq C_{N,q} \max_{i,j} \sup_{x \in \omega'_0} |a_{ij}(x) - \tau_h a_{ij}(x)| \times \\
 &\quad \left(\|\nabla_{X_2}^2 u_k\|_{L^2(\omega'_0+h)} + \epsilon_k^2 \|\nabla_{X_1}^2 u_k\|_{L^2(\omega'_0+h)} + \epsilon_k \|\nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega'_0+h)} \right),
 \end{aligned}$$

with $C_{N,q} = \max(q, N - q, \sqrt{2q(N - q)})$. Let $\delta > 0$ be fixed and small enough such that for every $h \in \mathbb{R}^N$, $0 < |h| \leq \delta$ we have $\omega'_0 + h \subset\subset \Omega$. We define the open set $\omega_\delta := \bigcup_{0 < |h| \leq \delta} (\omega'_0 + h)$. Notice that $\omega_\delta \subset\subset \Omega$ then follows by Corollary 2.3, that there exists C_δ independent of k such that:

$$\|\nabla_{X_2}^2 u_k\|_{L^2(\omega_\delta)^{(N-q)^2}} + \epsilon_k^2 \|\nabla_{X_1}^2 u_k\|_{L^2(\omega_\delta)^{q^2}} + \epsilon_k \|\nabla_{X_1 X_2}^2 u_k\|_{L^2(\omega_\delta)^{q(N-q)}} \leq C_\delta.$$

Since the a_{ij} are uniformly continuous on every $\omega \subset\subset \Omega$ open (thanks to (30)), then

$$\lim_{h \rightarrow 0} \max_{i,j} \sup_{x \in \omega'_0} |a_{ij}(x) - \tau_h a_{ij}(x)| = 0,$$

and hence

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \sum_{i,j} \left\| \left(a_{ij}^k - \tau_h a_{ij}^k \right) \tau_h \partial_{ij}^2 u_k \right\|_{L^2(\omega'_0)} = 0. \tag{2.9}$$

Now, estimating the last term of (2.8). By the triangular inequality we obtain

$$\begin{aligned}
 \sum_{i,j} \left\| \partial_i a_{ij}^k \partial_j u_k - \tau_h \partial_i a_{ij}^k \tau_h \partial_j u_k \right\|_{L^2(\omega'_0)} &\leq \sum_{i,j} \left\| \partial_i a_{ij}^k \partial_j u_k - \tau_h \partial_i a_{ij}^k \partial_j u_k \right\|_{L^2(\omega'_0)} \\
 &\quad + \sum_{i,j} \left\| \tau_h \partial_i a_{ij}^k \partial_j u_k - \partial_i \tau_h a_{ij}^k \tau_h \partial_j u_k \right\|_{L^2(\omega'_0)}.
 \end{aligned}$$

By using the boundedness of the first derivatives of the a_{ij} on ω'_0 we get

$$\begin{aligned} \sum_{i,j} \left\| \partial_i a_{ij}^k \partial_j u_k - \partial_i \tau_h a_{ij}^k \partial_j \tau_h u_k \right\|_{L^2(\omega'_0)} \\ \leq C'_{N,q} \max_{i,j} \sup_{x \in \omega'_0} |\partial_i a_{ij}(x) - \partial_i \tau_h a_{ij}(x)| \left(\epsilon_k \|\nabla_{X_1} u_k\|_{L^2(\omega'_0)} + \|\nabla_{X_2} u_k\|_{L^2(\omega'_0)} \right) \\ + C'_{\omega'_0} \left(\epsilon_k \|\nabla_{X_1} U_k^h\|_{L^2(\omega'_0)} + \|\nabla_{X_2} U_k^h\|_{L^2(\omega'_0)} \right), \end{aligned}$$

where $C'_{N,q} = \max(N\sqrt{q}, N\sqrt{N-q})$ and $C'_{\omega'_0} > 0$ is independent of h and k . Now, since $\partial_i a_{ij}$ are uniformly continuous on every $\omega \subset\subset \Omega$ ((thanks to (30)), then

$$\lim_{h \rightarrow 0} \max_{i,j} \sup_{x \in \omega'_0} |\partial_i a_{ij}(x) - \tau_h \partial_i a_{ij}(x)| = 0. \quad (2.10)$$

In the other hand by using (29) and the Fréchet-Kolmogorov theorem we get

$$\sup_{k \in \mathbb{N}} \left(\epsilon_k \|\nabla_{X_1} U_k^h\|_{L^2(\omega'_0)} + \|\nabla_{X_2} U_k^h\|_{L^2(\omega'_0)} + \|U_k^h\|_{L^2(\omega'_0)} \right) \longrightarrow 0, \quad (2.11)$$

therefore by using (2.10) and (2.11) in the above inequality we get

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \sum_{i,j} \left\| \partial_i a_{ij}^k \partial_j u_k - \partial_i \tau_h a_{ij}^k \partial_j \tau_h u_k \right\|_{L^2(\omega'_0)} = 0. \quad (2.12)$$

Passing to the limit in (2.8) by using (2.9), (2.11), and (2.12) we get

$$\lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|G_k^h\|_{L^2(\omega'_0)} = 0,$$

and the proposition follows. \square

Corollary 2.5. *Under assumptions Proposition 2.4 we have, for every $\omega \subset\subset \Omega$ open:*

$$\begin{aligned} \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \tau_h \nabla_{X_2}^2 u_k - \nabla_{X_2}^2 u_k \right\|_{L^2(\omega)^{(N-q)^2}} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k^2 (\tau_h \nabla_{X_1}^2 u_k - \nabla_{X_1}^2 u_k) \right\|_{L^2(\omega)^{q^2}} &= 0, \\ \lim_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \left\| \epsilon_k (\tau_h \nabla_{X_1 X_2}^2 u_k - \nabla_{X_1 X_2}^2 u_k) \right\|_{L^2(\omega)^{q(N-q)}} &= 0. \end{aligned}$$

Proof. Similar to proof of Corollary 2.3, where we use the compacity of $\bar{\omega}$ and Proposition 2.4. \square

Now, we are able to prove Theorem 0.5. Indeed, it is similar to the proof of Theorem 2.3. For example, let us show the convergence

$$\epsilon^2 \nabla_{X_1}^2 u_\epsilon \rightarrow 0 \text{ in } L_{loc}^2(\Omega)^{q^2}.$$

Fix $\omega \subset\subset \Omega$ open, and let (u_k) be a sequence defined as in this subsection. It follows from Corollaries 2.3, 2.5 that the subset $\{\epsilon_k^2 \nabla_{X_1}^2 u_k\}_{k \in \mathbb{N}}$ is relatively compact in $L^2(\omega)^{q^2}$, then there exists $V^\omega \in L^2(\omega)^{q^2}$ and a subsequence still labelled $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$ such that

$$\epsilon_k^2 \nabla_{X_1}^2 u_k \rightarrow V^\omega \text{ in } L^2(\omega)^{q^2}.$$

Since $\epsilon_k^2 u_k \rightarrow 0$ in $L^2(\omega)$ (thanks to (29)) then $V^\omega = 0$ (we used the continuity of $\nabla_{X_1}^2$ on $\mathcal{D}'(\omega)^{q^2}$).

Whence, by the diagonal process one can construct a sequence still labelled $(\epsilon_k^2 \nabla_{X_1}^2 u_k)$ such that

$$\epsilon_k^2 \nabla_{X_1}^2 u_k \rightarrow 0 \text{ in } L_{loc}^2(\Omega)^{q^2}.$$

To prove the convergence for the whole sequence $(\epsilon^2 \nabla_{X_1}^2 u_\epsilon)_{0 < \epsilon \leq 1}$, we reason by contradiction as in Section 1, recall that $L_{loc}^2(\Omega)$ is metrizable (that is a Fréchet space!), and the proof finished.

Remark 2.6. *Remark that the Lipschitz regularity of A is sufficient for all the arguments of this subsection except for (2.12). Following the same path we remark that we can obtain (2.12) if we assume that there exists a zero measure subset S of Ω such that:*

$$(\mathbf{Unif}) : \lim_{h \rightarrow 0} \max_{i,j} \sup_{x \in \omega \setminus S} |\partial_i a_{ij}(x) - \tau_h \partial_i a_{ij}(x)| = 0, \text{ for every } \omega \subset\subset \Omega, \text{ open.}$$

Therefore, we obtain the same affirmations of Theorems 0.5, 0.6 if we suppose that A is Lipschitz which satisfies (\mathbf{Unif}) .

2.2 Interior convergence on general bounded domain for the semilinear case

The proof of Theorem 0.6 is similar to that of Theorem 0.5. At first, the convergences of (29) are given in Proposition 2.7 below. For the rest of the arguments we will have additional terms of the form

$$\|\tau_h \beta(u_k) - \beta(u_k)\|_{L^2(\omega'_0)}, \quad \|\beta(u_k)\|_{L^2(\omega'_0)},$$

when we show Propositions 2.2 and 2.4. The boundedness of (u_k) in $L^2(\Omega)$ and (11) show that $(\|\beta(u_k)\|_{L^2(\Omega)})_k$ is bounded. Similarly, since $u_k \rightarrow u$ in $L^2(\Omega)$, then by using the continuity of β and

(11) we show that $\beta(u_k) \rightarrow \beta(u)$ in $L^2(\Omega)$, and therefore we get

$$\limsup_{h \rightarrow 0} \sup_{k \in \mathbb{N}} \|\tau_h \beta(u_k) - \beta(u_k)\|_{L^2(\omega'_0)} = 0.$$

Proposition 2.7. *Let Ω be an open bounded set of \mathbb{R}^N . Assume (5), (6). Assume that β satisfies (10) and (11). Suppose that $f \in L^2(\Omega)$, then*

$$u_\epsilon \rightarrow u \text{ in } L^2(\Omega), \quad \epsilon \nabla_{X_1} u_\epsilon \rightarrow 0 \text{ in } L^2(\Omega)^q \text{ and } \nabla_{X_2} u_\epsilon \rightarrow \nabla_{X_2} u \text{ in } L^2(\Omega)^{N-q},$$

where u_ϵ is the unique solution to (12) in $H_0^1(\Omega)$ and u is the unique solution to (31) in $H_0^1(\Omega; \Pi_2\Omega)$.

Proof. This proposition is a generalization of Theorem 1.1 for general domains. Following the same arguments we get the convergences (1.10) and

$$\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega). \quad (2.13)$$

Therefore, since $H_0^1(\Omega; \Pi_2\Omega)$ is a Hilbert space, then (1.10) implies that $u \in H_0^1(\Omega; \Pi_2\Omega)$. By using some Hilbert basis and a partition of the unity as in the linear case ([8], Theorem 2.1), We can show that, for a.e. $X_1 \in \Pi_1(\Omega)$

$$\int_{\Omega_{X_1}} \beta(u(X_1, \cdot)) \psi dX_2 + \int_{\Omega_{X_1}} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 = \int_{\Omega_{X_1}} f(X_1, \cdot) \psi dX_2, \quad \forall \psi \in H_0^1(\Omega_{X_1}). \quad (2.14)$$

Finally, for $\varphi \in H_0^1(\Omega; \Pi_2\Omega)$ we use (2.14) to get (31). The convergence of the global sequence follows as in proof of Theorem 1.1. \square

2.3 High-order global convergence in a cuboid

In this section, we will prove Theorem 0.7. Let $\tilde{\Omega} = \prod_{i=1}^N (-l_i, 2l_i)$. We denote $sub(\tilde{\Omega})$ the set of all disjoint subsets of $\tilde{\Omega}$ of the form $I = \prod_{i=1}^N I_i$ where each I_i has one of the forms $(-l_i, 0)$, $(0, l_i)$, $(l_i, 2l_i)$. For $0 \leq j \leq N$ fixed we denote $sub(\tilde{\Omega})_{-1,j}$, $sub(\tilde{\Omega})_{1,j}$, $sub(\tilde{\Omega})_{2,j}$ the subsets of $sub(\tilde{\Omega})$ such that I_j is of the form $(-l_j, 0)$, $(0, l_j)$, $(l_j, 2l_j)$ respectively. It is clear that these three subsets define a partition of $sub(\tilde{\Omega})$. For $x = (x_i)_{1 \leq i \leq N} \in I$, we denote

$$y = (s_i(I) + r_i(I)x_i)_{1 \leq i \leq N} \in \Omega$$

where r_i and s_i , $1 \leq i \leq N$, are defined on $sub(\tilde{\Omega})$ by :

$$r_i(I) = \begin{cases} 1 & \text{if } I \in sub(\tilde{\Omega})_{1,i} \\ -1 & \text{else} \end{cases}$$

$$s_i(I) = \begin{cases} 0 & \text{if } I \in sub(\tilde{\Omega})_{-1,i} \cup sub(\tilde{\Omega})_{1,i} \\ 2l_i & \text{if } I \in sub(\tilde{\Omega})_{2,i} \end{cases}$$

Now, we define the function $\tilde{f} \in L^2(\tilde{\Omega})$ by

$$\text{For } x \in I \in sub(\tilde{\Omega}) : \tilde{f}(x) = \left(\prod_{i=1}^N r_i(I) \right) \times f(y)_{1 \leq i \leq N}, \text{ and } \tilde{f} = 0 \text{ else.}$$

Similarly, we define the extension \tilde{u}_ϵ of u_ϵ by:

$$\text{For } x \in I \in sub(\tilde{\Omega}) : \tilde{u}_\epsilon(x) = \left(\prod_{i=1}^N r_i(I) \right) \times u_\epsilon(y), \text{ and } \tilde{u}_\epsilon = 0 \text{ else.}$$

We define $\tilde{A} = (\tilde{a}_{ij})$ the extension of A as follows: For $x \in \tilde{\Omega}$, there exists $I \in sub(\tilde{\Omega})$ such that $x \in \bar{I}$, in this case we set

$$\tilde{a}_{ij}(x) = r_i(I)r_j(I)a_{ij}(y), \quad i, j = 1, \dots, N.$$

Notice that assumption (34) implies that the value of each $\tilde{A}(x)$ does not depend in the choice of I so \tilde{A} is well-defined. Notice also that (33) implies that \tilde{A} is Lipschitz on $\tilde{\Omega}$ and that it satisfies hypothesis (Unif) of Remark 2.6 on $\tilde{\Omega}$, where the zero measure subset S is the finite union of all the subsets of the form $\prod_{i=1}^N I_i$ whose there is at least some i_0 such that $I_{i_0} = \{0\}$ or $\{l_{i_0}\}$. We can check immediately that \tilde{A} satisfies the ellipticity assumption (5) with the same constant. Finally, we define \tilde{A}_ϵ as we have defined A_ϵ (see the introduction of the this thesis).

Under the above notations we have the following Lemma:

Lemma 2.8. *Assume (5), (33), and (34). Let Ω such that (32). Let f and u_ϵ as in Theorem 0.5, then \tilde{u}_ϵ is the unique weak solution in $H_0^1(\tilde{\Omega})$ to the elliptic equation*

$$-div(\tilde{A}_\epsilon \nabla \tilde{u}_\epsilon) = \tilde{f}.$$

Moreover, we have $\tilde{u}_\epsilon \in H_{loc}^2(\tilde{\Omega})$.

Proof. At first, One can check immediately that the restriction of \tilde{u}_ϵ to each $I \in \text{Sub}(\tilde{\Omega})$ belongs to $H_0^1(I)$ and hence $\tilde{u}_\epsilon \in H_0^1(\tilde{\Omega})$, and moreover for $1 \leq j \leq N$ and $I \in \text{sub}(\tilde{\Omega})$, we have :

$$\text{For a.e. } x \in I : \partial_j \tilde{u}_\epsilon(x) = r_j(I) \left(\prod_{i=1}^N r_i(I) \right) \partial_j u_\epsilon(y)$$

Now, let $\varphi \in \mathcal{D}(\tilde{\Omega})$ we have

$$\int_{\tilde{\Omega}} \tilde{A}_\epsilon \nabla \tilde{u}_\epsilon \cdot \nabla \varphi dx = \sum_{I \in \text{Sub}(\tilde{\Omega})} \int_I \tilde{A}_\epsilon \nabla \tilde{u}_\epsilon \cdot \nabla \varphi dx,$$

by a change of variables we get

$$\int_{\tilde{\Omega}} \tilde{A}_\epsilon \nabla \tilde{u}_\epsilon \cdot \nabla \varphi dx = \sum_{I \in \text{Sub}(\tilde{\Omega})} \int_{\Omega} A_\epsilon \nabla u_\epsilon \cdot \nabla \tilde{\varphi}_I dx, \quad (2.15)$$

where $\tilde{\varphi}_I$ is defined on $\bar{\Omega}$ by

$$\tilde{\varphi}_I(x) = \left(\prod_{i=1}^N r_i(I) \right) \varphi_I((s_i(I) + r_i(I)x_i)_i),$$

and φ_I is the restriction of φ on I . Let us show that

$$\sum_{I \in \text{Sub}(\tilde{\Omega})} \tilde{\varphi}_I \in H_0^1(\Omega) \quad (2.16)$$

It is clear that $\sum_{I \in \text{sub}(\tilde{\Omega})} \tilde{\varphi}_I \in H^1(\Omega) \cap C(\bar{\Omega})$, it is enough to show that it vanishes on $\partial\Omega$. So, let $x^0 = (x_i^0)$ be un élément of $\partial\Omega$, then there exists at least j such that $x_j^0 = 0$ or $x_j^0 = l_j$.

1) If $x_j^0 = 0$: For any $I \in \text{sub}(\tilde{\Omega})_{2,j}$, we have $y_j^0 = s_j(I) + r_j(I)x_j^0 = 2l_j$, then $y^0 \in \partial\tilde{\Omega}$ therefore $\tilde{\varphi}_I(x) = 0$. Now, for any $I \in \text{sub}(\tilde{\Omega})_{-1,j}$, we have $y_j^0 = s_j(I) + r_j(I)x_j^0 = -x_j^0 = 0$, and any $I \in \text{sub}(\tilde{\Omega})_{1,j}$, we have : $y_j^0 = s_j(I) + r_j(I)x_j^0 = x_j^0 = 0$, notice that there is a bijection from $\text{sub}(\tilde{\Omega})_{1,j}$ onto $\text{sub}(\tilde{\Omega})_{-1,j}$ defined by : $I \mapsto I'$ such that I and I' have the same intervals except for the j^{th} one we have $I_j = (0, l_j)$ and $I'_j = (-l_j, 0)$. For such I and I' we have $r_j(I) = 1$ and $r_j(I') = -1$, then $\tilde{\varphi}_I(x^0) + \tilde{\varphi}_{I'}(x^0) = 0$. Finally, we get $\sum_{I \in \text{Sub}(\tilde{\Omega})} \tilde{\varphi}_I(x^0) = 0$.

2) If $x_j^0 = l_j$: For any $I \in \text{sub}(\tilde{\Omega})_{-1,j}$, $y_j^0 = s_j(I) + r_j(I)x_j^0 = -l_j$, then $y^0 \in \partial\tilde{\Omega}$ therefore $\tilde{\varphi}_I(x) = 0$. Now, for any $I \in \text{sub}(\tilde{\Omega})_{1,j}$, we have $y_j^0 = s_j(I) + r_j(I)x_j^0 = l_j$, and any $I \in \text{sub}(\tilde{\Omega})_{2,j}$, we have : $y_j^0 = 2l_j - l_j = l_j$, notice that there is a bijection from $\text{sub}(\tilde{\Omega})_{1,j}$ onto $\text{sub}(\tilde{\Omega})_{2,j}$ defined

by : $I \mapsto I'$ such that I and I' have the same intervals except for the j^{th} one we have $I_j = (0, l_j)$ and $I'_j = (l_j, 2l_j)$. For such I and I' we have $r_j(I) = 1$ and $r_j(I') = -1$, then $\tilde{\varphi}_I(x^0) + \tilde{\varphi}_{I'}(x^0) = 0$. Finally, we get $\sum_{I \in \text{Sub}(\tilde{\Omega})} \tilde{\varphi}_I(x^0) = 0$. At the end, (2.16) follows from the two points above.

Now, since u_ϵ is the solution of (4) in $H_0^1(\Omega)$ then (2.15) and (2.16) give

$$\int_{\tilde{\Omega}} \tilde{A}_\epsilon \nabla \tilde{u}_\epsilon \cdot \nabla \varphi dx = \int_{\Omega} f \sum_{I \in \text{Sub}(\tilde{\Omega})} \tilde{\varphi}_I dx.$$

By using an other variables change in the second member of the above equality we get the first affirmation of the Lemma. Finally, as we have mentioned above, the function \tilde{A}_ϵ is Lipschitz on $\tilde{\Omega}$ (thanks to (33)), then the H^2 interior elliptic regularity gives the second affirmation of the Lemma. \square

As a consequence of the above lemma, we will prove the convergence of second derivatives of u_ϵ near each point of the boundary $\partial\Omega$. Explicitly we have the following

Proposition 2.9. *Suppose the same assumptions of Theorem 0.7. Let $x_{\partial\Omega}$ a point of $\partial\Omega$, then there exists an open neighbourhood \mathcal{O} of $x_{\partial\Omega}$ such that:*

$$\begin{aligned} \nabla_{X_2}^2 u_\epsilon &\rightarrow \nabla_{X_2}^2 u \text{ in } L^2(\Omega \cap \mathcal{O})^{(N-q)^2}, \\ \epsilon^2 \nabla_{X_1}^2 u_\epsilon &\rightarrow 0 \text{ in } L^2(\Omega \cap \mathcal{O})^{q^2}, \\ \epsilon \nabla_{X_1 X_2}^2 u_\epsilon &\rightarrow 0 \text{ in } L^2(\Omega \cap \mathcal{O})^{(N-q)^2}. \end{aligned}$$

Proof. The proofs of these convergences are similar, so let us show the first one, for example. Let \tilde{u}_ϵ be as in Lemma 2.8, and let $\tilde{u} \in H_0^1(\tilde{\Omega}; \tilde{\Omega}_{X_1})$ be the solution of the associated limit problem. As we have mentioned just before Lemma 2.8, the function \tilde{A} satisfies the hypothesis **(Unif)**, therefore it follows by Remark 2.6 that the convergences of Theorem 0.5 hold for \tilde{u}_ϵ , therefore

$$\nabla_{X_2}^2 \tilde{u}_\epsilon \rightarrow \nabla_{X_2}^2 \tilde{u} \text{ in } L_{loc}^2(\tilde{\Omega})^{(N-q)^2}.$$

Now, let \mathcal{O} be an open set containing $x_{\partial\Omega}$ such that $\mathcal{O} \subset\subset \tilde{\Omega}$ (notice that $\mathcal{O} \cap \Omega$ is non empty), then according to the above convergence we have, in particular,

$$\nabla_{X_2}^2 \tilde{u}_\epsilon \rightarrow \nabla_{X_2}^2 \tilde{u} \text{ in } L^2(\mathcal{O} \cap \Omega)^{(N-q)^2}.$$

In the other hand we know that $u_\epsilon \rightarrow u$ in $L^2(\Omega)$, and that $\tilde{u}_\epsilon = u_\epsilon$ on Ω , then the continuity of $\nabla_{X_2}^2$ on $\mathcal{D}(\mathcal{O} \cap \Omega)^{(N-q)^2}$ gives

$$\nabla_{X_2}^2 u_\epsilon \rightarrow \nabla_{X_2}^2 u \text{ in } L^2(\mathcal{O} \cap \Omega)^{(N-q)^2}.$$

and the Proposition follows. □

Now, using the compactness of $\overline{\Omega}$ we can cover Ω by a finite family of open sets, a boundary ones like \mathcal{O} and interior ones i.e. like $\omega \subset\subset \Omega$, we obtain the desired convergences in each kind by applying the above proposition for the \mathcal{O} -type of sets, and Theorem 0.5 to the ω -type ones, and Theorem 0.7 follows.

Chapter 3

Singular perturbations for some class of non monotone elliptic problems and application to integro-differential problems

1 Singular perturbation for some class of nonlinear problem

1.1 Preliminaries

Let $\Omega = \omega_1 \times \omega_2$, and let $\gamma > 0$. In this chapter, we deal with the following problem:

$$\begin{cases} \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx + \gamma \int_{\Omega} u_{\epsilon} \varphi dx = \int_{\Omega} B(u_{\epsilon}) \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_{\epsilon} \in H_0^1(\Omega), \end{cases} \quad (3.1)$$

where the abstract operator B maps $L^1(\Omega)$ to $L^2(\Omega)$ continuously, and it is sublinear on $L^2(\Omega)$ (assumption (39)). Recall also that A satisfies the ellipticity assumptions (5), (6).

In this section, we prove Theorem 0.8. At first, let us prove the existence of solutions of (3.1). We have the following

Proposition 3.1. *Let Ω be a bounded open set of \mathbb{R}^N . Assume (5), (6), (38), and (39). Suppose that $\gamma > M$, then (3.1) has at least a solution.*

Proof. The proof follows from the Schauder fixed point theorem. Let $v \in L^2(\Omega)$ and $v_{\epsilon} \in H_0^1(\Omega)$ be the unique solution to the linearised problem

$$\begin{cases} \int_{\Omega} A_{\epsilon} \nabla v_{\epsilon} \cdot \nabla \varphi dx + \gamma \int_{\Omega} v_{\epsilon} \varphi dx = \int_{\Omega} B(v) \varphi dx, \forall \varphi \in H_0^1(\Omega), \\ v_{\epsilon} \in H_0^1(\Omega). \end{cases} \quad (3.2)$$

The existence of and the uniqueness of v_ϵ follows from the Lax-Milgram theorem (thanks to assumptions (5), (6)). Let $\mathcal{T} : L^2(\Omega) \rightarrow L^2(\Omega)$ be the mapping defined by $\mathcal{T}(v) = v_\epsilon$. Since B is continuous from $L^1(\Omega)$ to $L^2(\Omega)$ and that the injection $L^2(\Omega) \hookrightarrow L^1(\Omega)$ is continuous then B is continuous from $L^2(\Omega)$ to $L^2(\Omega)$. Therefore, one can check immediately that \mathcal{T} is continuous from $L^2(\Omega)$ to $L^2(\Omega)$.

We define the set

$$\mathcal{B} = \left\{ v \in H_0^1(\Omega) : \|\nabla v\|_{L^2(\Omega)} \leq \frac{\sqrt{\gamma}}{\epsilon\sqrt{\lambda}} \left(\frac{M}{\gamma - M} \right) \text{ and } \|v\|_{L^2(\Omega)} \leq \frac{M}{\gamma - M} \right\}$$

It is clear that \mathcal{B} is non-empty and it is convex and bounded in $H_0^1(\Omega)$ and it is closed in $L^2(\Omega)$, then \mathcal{B} is compact in $L^2(\Omega)$.

Let us show that \mathcal{B} is stable under \mathcal{T} . Let $v \in L^2(\Omega)$, testing with v_ϵ in (3.2) and using assumptions (5) and (39) we get

$$\lambda\epsilon^2 \|\nabla v_\epsilon\|_{L^2(\Omega)^N}^2 + \gamma \|v_\epsilon\|_{L^2(\Omega)}^2 \leq M(1 + \|v\|_{L^2(\Omega)}) \|v_\epsilon\|_{L^2(\Omega)}.$$

Whence

$$\|v_\epsilon\|_{L^2(\Omega)} \leq \frac{M(1 + \|v\|_{L^2(\Omega)})}{\gamma} \text{ and } \|\nabla v_\epsilon\|_{L^2(\Omega)^N} \leq \frac{M(1 + \|v\|_{L^2(\Omega)})}{\epsilon\sqrt{\gamma\lambda}}.$$

Now, when $v \in \mathcal{B}$ we get

$$\|v_\epsilon\|_{L^2(\Omega)} \leq \frac{M}{\gamma - M} \text{ and } \|\nabla v_\epsilon\|_{L^2(\Omega)^N} \leq \frac{\sqrt{\gamma}}{\epsilon\sqrt{\lambda}} \frac{M}{\gamma - M}.$$

and the proposition follows. □

Now, we establish some bounds on the solutions

Proposition 3.2. *Suppose that assumptions of Proposition 3.1 hold. Let u_ϵ , $0 < \epsilon \leq 1$ be a solution to (3.1), then we have*

$$\|\epsilon \nabla_{X_1} u_\epsilon\|_{L^2(\Omega)^q} \leq \frac{M}{\gamma - M} \sqrt{\frac{\gamma}{\lambda}}, \quad \|\nabla_{X_2} u_\epsilon\|_{L^2(\Omega)^{N-q}} \leq \frac{M}{\gamma - M} \sqrt{\frac{\gamma}{\lambda}}, \quad \|u_\epsilon\|_{L^2(\Omega)} \leq \frac{M}{\gamma - M}.$$

Proof. Testing with u_ϵ in (3.1) and using (5), the Cauchy-Schwarz inequality, and (39), we get

$$\lambda\epsilon^2 \|\nabla_{X_1} u_\epsilon\|_{L^2(\Omega)^q}^2 + \lambda \|\nabla_{X_2} u_\epsilon\|_{L^2(\Omega)^{N-q}}^2 + \gamma \|u_\epsilon\|_{L^2(\Omega)}^2 \leq M(1 + \|u_\epsilon\|_{L^2(\Omega)}) \|u_\epsilon\|_{L^2(\Omega)}.$$

Thus

$$\|u_\epsilon\|_{L^2(\Omega)} \leq \frac{M}{\gamma - M},$$

and therefore

$$\lambda \epsilon^2 \|\nabla_{X_1} u_\epsilon\|_{L^2(\Omega)}^2 + \lambda \|\nabla_{X_2} u_\epsilon\|_{L^2(\Omega)}^2 + \gamma \|u_\epsilon\|_{L^2(\Omega)}^2 \leq \gamma \left(\frac{M}{\gamma - M} \right)^2,$$

and we get the desired bounds. \square

Corollary 3.3. *Suppose that assumptions of Proposition 3.1 hold. Let $(u_\epsilon)_{0 < \epsilon \leq 1}$, be a sequence of solutions to (3.1), then there exist $u \in L^2(\Omega)$ such that $\nabla_{X_2} u \in L^2(\Omega)$, and a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that*

$$u_{\epsilon_k} \rightharpoonup u; \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u \text{ and } \epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightharpoonup 0 \text{ weakly in } L^2(\Omega). \quad (3.3)$$

Proof. Follows from Proposition 3.2 by using reflexivity of $L^2(\Omega)$ and continuity of the derivation on $\mathcal{D}(\Omega)$. \square

1.2 The asymptotic behaviour of the solution

1.2.1 Local H^1 bounds on the solution:

The convergences of (3.3) show that u is the potential candidate to be a solution to (41). The main difficult to prove such a result, is the passage to the limit in the non linear term $B(u_\epsilon)$. Recall that the injection $H_0^1(\Omega; \omega_2) \hookrightarrow L^2(\Omega)$ is not compact, therefore the passage to the limit in $B(u_\epsilon)$ by using the weak convergences (3.3) is not trivial. In proof of Theorem 1.1, we have used a monotonicity to obtain the strong convergence from the weak ones. In the present case, we will show that u_ϵ is bounded in $H_{loc}^1(\Omega)$, and then we use Sobolev embeddings to show the strong convergence of subsequence in $L_{loc}^2(\Omega)$ and we will use this strong convergence to pass to the limit in the nonlinear term. We require the following classical Banach-Steinhaus theorem [28].

Theorem 3.4. *Let \mathcal{X} and \mathcal{Y} be two topological vector spaces. Let \mathcal{K} be a convex compact subset of \mathcal{X} . Let $(\mathcal{T}_\iota)_{\iota \in \mathcal{I}}$ be a family of continuous linear applications from \mathcal{X} to \mathcal{Y} . Suppose that for every $\mathbf{x} \in \mathcal{K}$ the subset $\{\mathcal{T}_\iota(\mathbf{x})\}_{\iota \in \mathcal{I}}$ is bounded in \mathcal{Y} . Then there exists F bounded in \mathcal{Y} such that, for every $\iota \in \mathcal{I}$ we have $\mathcal{T}_\iota(\mathcal{K}) \subset F$. In other words, the family $\{\mathcal{T}_\iota\}_{\iota \in \mathcal{I}}$ is uniformly bounded on \mathcal{K}*

To obtain the desired H^1 bounds. We need some additional assumptions on B and A , namely assumptions (40), (18), and (22) that we recall here

$$\text{Pour tout } E \subset W \text{ borné dans } L^2(\Omega) : \overline{\text{conv}(B(E))} \subset W,$$

$$\partial_{x_i} a_{ij} \in L^\infty(\Omega), \partial_{x_j} a_{ij} \in L^\infty(\Omega) \text{ pour } i = 1, \dots, q \text{ et } j = q + 1, \dots, N,$$

$$\nabla_{X_1} A_{22} \in L^\infty(\Omega).$$

Now, we are ready to prove the following principal lemma

Lemma 3.5. *Assume (5), (6), (18), (22), (38), (39), (40), and $\gamma > M$. Let $(u_\epsilon)_{0 < \epsilon \leq 1}$ be a sequence of solutions to (3.1), then $(u_\epsilon)_{0 < \epsilon \leq 1}$ is bounded in $H_{loc}^1(\Omega)$. That is for every open $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} > 0$ independent of ϵ such that:*

$$\forall 0 < \epsilon \leq 1 : \|u_\epsilon\|_{H^1(\Omega')} \leq C_{\Omega'}.$$

Proof. Let $(\Omega_j)_{j \in \mathbb{N}}$ be an open cover of Ω such that for every $j \in \mathbb{N} : \Omega_j \subset\subset \Omega_{j+1}$. Recall that the topology of $H_{loc}^1(\Omega)$ is completely determined by the family of the semi-norms $(p_j)_{j \in \mathbb{N}}$ defined by

$$p_j(v) = \|v\|_{H^1(\Omega_j)} \text{ for every } v \in H_{loc}^1(\Omega).$$

We denote $H_{loc}^1(\Omega)^{weak}$ the topological vector space $H_{loc}^1(\Omega)$ equipped with its weak topology. We denote $L^2(\Omega)^{weak}$ the Hilbert space $L^2(\Omega)$ equipped with its weak topology. We define the family $(\mathcal{T}_\epsilon)_{0 < \epsilon \leq 1} : L^2(\Omega) \longrightarrow H_{loc}^1(\Omega)$ of linear mappings by

$$\forall f \in L^2(\Omega) : \mathcal{T}_\epsilon(f) = v_\epsilon,$$

where v_ϵ is the unique weak solution to the linear problem

$$\begin{cases} -\text{div}(A_\epsilon \nabla v_\epsilon) + \gamma v_\epsilon = f, \\ v_\epsilon \in H_0^1(\Omega). \end{cases} \quad (3.4)$$

$(\mathcal{T}_\epsilon)_{0 < \epsilon \leq 1}$ is well defined thanks to assumptions (5), (6), and the Lax-Milgram theorem.

For every $0 < \epsilon \leq 1$, $\mathcal{T}_\epsilon : L^2(\Omega) \rightarrow H_{loc}^1(\Omega)$ is continuous as a composition of the continuous application

$$L^2(\Omega) \longrightarrow H^1(\Omega), f \longmapsto v_\epsilon,$$

and the canonical injection $H^1(\Omega) \hookrightarrow H_{loc}^1(\Omega)$. Therefore, since $(\mathcal{T}_\epsilon)_{0 < \epsilon \leq 1}$ are linear, then

$$\forall 0 < \epsilon \leq 1, \mathcal{T}_\epsilon : L^2(\Omega)^{weak} \longrightarrow H_{loc}^1(\Omega)^{weak} \text{ is continuous.} \quad (3.5)$$

We introduce the subset $E_0 = \left\{ v \in W : \|v\|_{L^2(\Omega)} \leq \frac{M}{\gamma-M} \right\}$, then $B(E_0)$ is bounded in $L^2(\Omega)$, thanks to (39). Whence, the convex hull of $B(E_0)$ is bounded in $L^2(\Omega)$. Hence, its closure (in $L^2(\Omega)$)

$$\mathcal{K} = \overline{\text{conv}(B(E_0))} \text{ is convex, and it is compact in } L^2(\Omega)^{weak} \quad (3.6)$$

Now, by assumption (40) we get $\mathcal{K} \subset W$. Let $f \in \mathcal{K}$, then $f \in W$ so f satisfies (21), combining that with assumptions (18), (22) we get the estimation (20) (Theorem 3.1, [8]) i.e.

$$\forall \omega'_1 \subset\subset \omega_1 \text{ open} : \|\nabla_{X_2}(v_\epsilon - v)\|_{L^2(\omega'_1 \times \omega_2)^{N-q}} = O(\epsilon), \text{ and } \|\nabla_{X_1}(v_\epsilon - v)\|_{L^2(\omega'_1 \times \omega_2)^q} = O(1). \quad (3.7)$$

Here, the function v is the weak solution to the limit problem associated to (3.4). Notice that the problem treated in [8] did not contain the term γv_ϵ . Since γ is positive, then we can check, by following the proof of (Theorem 3.1, [8]), that the estimations (3.7) hold. It follows from (3.7) that

$$\forall j \in \mathbb{N}, \exists C_{j,f} > 0, \forall \epsilon \in (0, 1] : p_j(\mathcal{T}_\epsilon(f)) = \|v_\epsilon\|_{H^1(\Omega_j)} \leq C_{j,f}.$$

In other words, the set $\{\mathcal{T}_\epsilon(f)\}_{\epsilon \in (0,1]}$ is bounded in $H_{loc}^1(\Omega)$ for any f fixed in \mathcal{K} . Therefore,

$$\forall f \in \mathcal{K} : \{\mathcal{T}_\epsilon(f)\}_{\epsilon \in (0,1]} \text{ is bounded in } H_{loc}^1(\Omega)^{weak}. \quad (3.8)$$

Now, the affirmations (3.5), (3.6), and (3.8) show that the assumptions of Theorem 3.4 are fulfilled (Here, we set $\mathcal{X} = L^2(\Omega)^{weak}$ and $\mathcal{Y} = H_{loc}^1(\Omega)^{weak}$). Whence, there exists a bounded set F in $H_{loc}^1(\Omega)^{weak}$ such that

$$\text{For every } 0 < \epsilon \leq 1 : \mathcal{T}_\epsilon(\mathcal{K}) \subset F.$$

The boundedness of F in $H_{loc}^1(\Omega)^{weak}$ implies its boundedness in $H_{loc}^1(\Omega)$, whence

$$\forall j \in \mathbb{N}, \exists C_j > 0, \forall \epsilon \in (0, 1], \forall f \in \mathcal{K} : p_j(\mathcal{T}_\epsilon(f)) \leq C_j. \quad (3.9)$$

Now, let (u_ϵ) be a sequence of solutions of (3.1), then from the third inequality in Proposition 3.2 we have

$$\{u_\epsilon\}_{\epsilon \in (0,1]} \subset E_0.$$

Hence, according to assumption (40) we have

$$\forall \epsilon \in (0, 1] : B(u_\epsilon) \in \mathcal{K}. \quad (3.10)$$

Remark that for every $\epsilon \in (0, 1]$, $\mathcal{T}_\epsilon(B(u_\epsilon)) = u_\epsilon$. Therefore, by using (3.9) and (3.10) we get

$$\forall \epsilon \in (0, 1], \forall j \in \mathbb{N} : \|u_\epsilon\|_{H^1(\Omega_j)} = \|\mathcal{T}_\epsilon(B(u_\epsilon))\|_{H^1(\Omega_j)} \leq C_j. \quad (3.11)$$

Finally, for $\Omega' \subset\subset \Omega$ arbitrary there exists $j_{\Omega'} \in \mathbb{N}$ such that $\Omega' \subset \Omega_{j_{\Omega'}}$. Whence, from (3.11) we get

$$\forall 0 < \epsilon \leq 1 : \|u_\epsilon\|_{H^1(\Omega')} \leq C_{j_{\Omega'}},$$

and the proof of the lemma is complete. \square

1.2.2 The convergence of a subsequence and proof of Theorem 0.8:

Let us prove the following

Proposition 3.6. *Let us suppose that the assumptions of Lemma 3.5 hold. Let $(u_\epsilon)_{0 < \epsilon \leq 1}$ be a sequence of solutions of (3.1), then there exists a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that we have (3.3) and*

$$u_{\epsilon_k} \longrightarrow u \text{ in } L^1(\Omega) \text{ strongly,} \quad (3.12)$$

where u is the function given in Corollary 3.3.

Proof. Let \mathcal{C}_Ω be a countable open cover of Ω by balls, such that for any $\mathbb{B} \in \mathcal{C}$ we have $\mathbb{B} \subset\subset \Omega$, such a cover always exists. Fix a ball $\mathbb{B} \in \mathcal{C}_\Omega$, and let $(u_{\epsilon_k})_{k \in \mathbb{N}}$ be the sequence of Corollary 3.3. It follows by Lemma 3.5 that $(u_{\epsilon_k})_{k \in \mathbb{N}}$ is bounded in $H^1(\mathbb{B})$. Therefore, by the Sobolev compact embedding $H^1(\mathbb{B}) \hookrightarrow L^2(\mathbb{B})$ there exists a subsequence still labelled $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that

$$u_{\epsilon_k} \longrightarrow u \text{ in } L^2(\mathbb{B}) \text{ strongly.}$$

We repeat this operation in each ball and we use the diagonal process, then we get a unique subsequence still labelled $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that

$$u_{\epsilon_k} \longrightarrow u \text{ in } L^2(\mathbb{B}) \text{ for any } \mathbb{B} \in \mathcal{C}_\Omega \text{ strongly.}$$

Therefore,

$$u_{\epsilon_k} \longrightarrow u \text{ in } L^2_{loc}(\Omega) \text{ strongly.} \quad (3.13)$$

Now, by using the regularity of the Lebesgue measure, we have for $\delta > 0$ there exists $Q_\delta \subset \Omega$ compact such that $mes(\Omega \setminus Q_\delta) \leq \delta$, so

$$\int_{\Omega} |u_{\epsilon_k} - u| dx \leq \sqrt{mes(Q_\delta)} \|u_{\epsilon_k} - u\|_{L^2(Q_\delta)} + \sqrt{\delta} \|u_{\epsilon_k} - u\|_{L^2(\Omega)}, \quad (3.14)$$

where we have used the Cauchy-Schwarz inequality two times. By using Proposition 3.2 we obtain that

$$\int_{\Omega} |u_{\epsilon_k} - u| dx \leq \sqrt{mes(Q_\delta)} \|u_{\epsilon_k} - u\|_{L^2(Q_\delta)} + \sqrt{\delta} \left(\frac{M}{\gamma - M} + \|u\|_{L^2(\Omega)} \right). \quad (3.15)$$

Passing to the limit as $k \rightarrow +\infty$ we get

$$\limsup \int_{\Omega} |u_{\epsilon_k} - u| dx \leq \sqrt{\delta} \left(\frac{M}{\gamma - M} + \|u\|_{L^2(\Omega)} \right).$$

Passing to the limit as $\delta \rightarrow 0$ we get (3.12). □

Now, we are able ready to finish the proof of Theorem 0.8. Let $(u_{\epsilon_k})_{k \in \mathbb{N}}$ be the subsequence of Proposition 3.6, then we have (3.12). Combining that with (38) we get

$$B(u_{\epsilon_k}) \longrightarrow B(u) \text{ in } L^2(\Omega) \text{ strongly.} \quad (3.16)$$

Passing to the limit in (3.1) by using (3.3) and (3.16) we get

$$\int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx + \gamma \int_{\Omega} u \varphi dx = \int_{\Omega} B(u) \varphi dx, \forall \varphi \in H_0^1(\Omega). \quad (3.17)$$

Taking $\varphi = u_{\epsilon_k}$ in (3.17) and passing to the limit by using (3.3) again. We derive

$$\int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u dx + \gamma \int_{\Omega} u^2 dx = \int_{\Omega} B(u) u dx. \quad (3.18)$$

Now, consider the quantity

$$\begin{aligned} J_k &= \int_{\Omega} A_{\epsilon_k} \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2} (u_{\epsilon_k} - u) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2} (u_{\epsilon_k} - u) \end{pmatrix} dx + \gamma \int_{\Omega} (u_{\epsilon_k} - u)^2 dx \\ &= -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u \cdot \nabla_{X_1} u_{\epsilon_k} dx - \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{\epsilon_k} \cdot \nabla_{X_2} u dx - \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u_{\epsilon_k} dx \\ &\quad - \int_{\Omega} A_{22} \nabla_{X_2} u_{\epsilon_k} \cdot \nabla_{X_2} u dx - 2\gamma \int_{\Omega} u u_{\epsilon_k} dx \\ &\quad + \int_{\Omega} B(u_{\epsilon_k}) u_{\epsilon_k} dx + \int_{\Omega} B(u) u dx. \end{aligned}$$

Passing to the limit by using (3.3), (3.16) and (3.18) we get

$$\lim J_k = 0$$

Finally, by using the ellipticity assumption (5) we obtain

$$u_{\epsilon_k} \longrightarrow u; \nabla_{X_2} u_{\epsilon_k} \longrightarrow \nabla_{X_2} u \text{ and } \epsilon_k \nabla_{X_1} u_{\epsilon_k} \longrightarrow 0 \text{ Strongly in } L^2(\Omega). \quad (3.19)$$

The second convergence of (3.19) means that $u_{\epsilon_k} \longrightarrow u$ in the Hilbert space $H_0^1(\Omega; \omega_2)$, therefore $u \in H_0^1(\Omega; \omega_2)$. Finally, we achieve the proof as in proof of Theorem 1.1, therefore we obtain that u satisfies (41), and the proof of Theorem 0.8 is complete.

Remark 3.7. *Theorem 0.8 still true when we replace assumption (38) by (42). The only changes in the arguments are done in (3.14) and (3.15). In fact by the Hölder inequality we get, for $1 \leq r < 2$*

$$\int_{\Omega} |u_{\epsilon_k} - u|^r dx \leq \text{mes}(Q_\delta)^{\frac{2-r}{2}} \|u_{\epsilon_k} - u\|_{L^2(Q_\delta)} + \delta^{\frac{2-r}{2}} \|u_{\epsilon_k} - u\|_{L^2(\Omega)}, \quad (3.20)$$

and

$$\int_{\Omega} |u_{\epsilon_k} - u|^r dx \leq \text{mes}(Q_\delta)^{\frac{2-r}{2}} \|u_{\epsilon_k} - u\|_{L^2(Q_\delta)} + \delta^{\frac{2-r}{2}} \left(\frac{M}{\gamma - M} + \|u\|_{L^2(\Omega)} \right). \quad (3.21)$$

Therefore we obtain $u_{\epsilon_k} \longrightarrow u$ in $L^r(\Omega)$, and hence from (42) we get (3.16).

2 Applications to integro-differential problems

2.1 A non standard Hammerstein elliptic equation

In this subsection we prove Theorem 0.12. We introduce the perturbed problem associated to (46)

$$\begin{cases} \int_{\Omega} \mathbb{A}_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx + \gamma \int_{\Omega} u_\epsilon \varphi dx = \int_{\Omega} B_1(u_\epsilon) \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_\epsilon \in H_0^1(\Omega), \end{cases} \quad (3.22)$$

where

$$\tilde{\mathbb{A}} = \begin{pmatrix} \lambda Id_q & 0 \\ 0 & \mathbb{A} \end{pmatrix}, \text{ and } \tilde{\mathbb{A}}_\epsilon = \begin{pmatrix} \lambda \epsilon^2 Id_q & 0 \\ 0 & \mathbb{A} \end{pmatrix} \quad (3.23)$$

$$B_1(u_\epsilon) = \int_{\Omega} K(x, y) b(y, u_\epsilon(y)) dy$$

We will check that $\tilde{\mathbb{A}}$ and B_1 satisfy the same hypothesis of Theorem 0.8.

At first, It is clear that $\tilde{\mathbb{A}}$ satisfies (5) and (6). The hypothesis (18) is satisfied since the non diagonal blocks of $\tilde{\mathbb{A}}$ are zero. The hypothesis (22) follows from (45).

Now, for the operator B_1 . The hypothesis (38) and (39) follow from the following proposition.

Proposition 3.1. *Under the hypothesis (47), (48) the operator B_1 maps $L^1(\Omega)$ to $L^2(\Omega)$ continuously, and we have :*

$$\forall v \in L^2(\Omega) : \|B_1(v)\|_{L^2(\Omega)} \leq M_1(1 + \|v\|_{L^2(\Omega)}),$$

with

$$M_1 = 2\|K\|_{L^2(\Omega^2)} \max\left(\|g\|_{L^2(\Omega)}, c|\Omega|^{\frac{1}{4}}\right)$$

Proof. The operator B_1 is well defined from $L^1(\Omega)$ to $L^2(\Omega)$, and by using (47), (48), and the Cauchy-Schwarz inequality we get

$$\forall v \in L^1(\Omega) : \|B_1(v)\|_{L^2(\Omega)} \leq \left(\|g\|_{L^2(\Omega)} + c\|v\|_{L^1(\Omega)}^{\frac{1}{2}}\right) \|K\|_{L^2(\Omega^2)}.$$

In particular,

$$\forall v \in L^2(\Omega) : \|B_1(v)\|_{L^2(\Omega)} \leq \left(\|g\|_{L^2(\Omega)} + c|\Omega|^{\frac{1}{4}}\|v\|_{L^2(\Omega)}^{\frac{1}{2}}\right) \|K\|_{L^2(\Omega^2)}.$$

Whence,

$$\forall v \in L^2(\Omega) : \|B_1(v)\|_{L^2(\Omega)} \leq \|K\|_{L^2(\Omega^2)} \max\left(\|g\|_{L^2(\Omega)}, c|\Omega|^{\frac{1}{4}}\right) \left(1 + \|v\|_{L^2(\Omega)}^{\frac{1}{2}}\right).$$

Finally, we obtain

$$\forall v \in L^2(\Omega) : \|B_1(v)\|_{L^2(\Omega)} \leq 2\|K\|_{L^2(\Omega^2)} \max\left(\|g\|_{L^2(\Omega)}, c|\Omega|^{\frac{1}{4}}\right) \left(1 + \|v\|_{L^2(\Omega)}\right),$$

and the inequality of the proposition follows.

Now, let us prove that B_1 is continuous from $L^1(\Omega)$ to $L^2(\Omega)$. Let $v_n \rightarrow v$ in $L^1(\Omega)$, then there exists a subsequence still labelled (v_n) and a function $w \in L^1(\Omega)$ such that

$$v_n(x) \rightarrow v(x) \text{ and } |v_n| \leq w(x), \text{ for a.e. } x \in \Omega. \quad (3.24)$$

For a.e. $y \in \Omega$, we have

$$b(y, v_n(y)) \longrightarrow b(y, v(y)),$$

thanks to the continuity of b in its second variable (recall that b is a Caratheodory function). and for a.e. $x \in \Omega$, and for a.e. $y \in \Omega$ we get by using (47)

$$\forall n : |K(x, y)b(y, v_n(y))| \leq |K(x, y)| \left(g(y) + c|w(y)|^{\frac{1}{2}} \right) \quad (3.25)$$

The function (in y) of the second member of (3.25). Therefore by using the convergence dominated theorem (in y) we obtain

$$\text{For a.e. } x \in \Omega : B_1(v_n)(x) \longrightarrow B_1(v)(x).$$

In the other hand, by using the Cauchy-Schwarz inequality we obtain that

$$\text{For a.e. } x \in \Omega, \forall n : |B_1(v_n)(x)| \leq \left(\|g\|_{L^2(\Omega)} + c\|w\|_{L^2(\Omega)} \right) \|K(x, \cdot)\|_{L^2(\Omega)}.$$

Notice that the second member of the previous inequality belongs to $L^2(\Omega)$, therefore by the convergence dominated theorem we obtain that

$$B_1(v_n) \longrightarrow B_1(v) \text{ in } L^2(\Omega) \quad (3.26)$$

Notice that (3.26) holds for the subsequence of (v_n) given in (3.24). For the whole sequence we obtain (3.26) by an immediate reasoning by contradiction, and the continuity of B_1 follows, and the proof is complete. \square

Now, let us check that the hypothesis (40) of Theorem 0.8 holds for the operator B_1 .

Proposition 3.2. *Under the hypothesis (47), (48), and (49) the operator B_1 satisfies (40).*

Proof. Let $v \in W$, $\varphi \in \mathcal{D}(\Omega)$ and $1 \leq j \leq q$ we have:

$$\left| \int_{\Omega} B_1(v) \partial_j \varphi(x) dx \right| \leq \int_{\Omega} \left| \int_{\Omega} K(x, y) \partial_j \varphi(x) b(y, v(y)) dx \right| dy.$$

Since $\partial_j K \in L^2(\Omega^2)$ then integrating by parts we get

$$\left| \int_{\Omega} B_1(v) \partial_j \varphi(x) dx \right| \leq \int_{\Omega} \left(\int_{\Omega} |\partial_j K(x, y) b(y, v(y))| dy \right) |\varphi(x)| dx.$$

Therefore, by using assumption (47) we get

$$\left| \int_{\Omega} B_1(v) \partial_j \varphi(x) dx \right| \leq (\|g\|_{L^2(\Omega)} + c|\Omega|^{\frac{1}{4}} \|v\|_{L^2(\Omega)}^{\frac{1}{2}}) \|\partial_j K\|_{L^2(\Omega^2)} \|\varphi\|_{L^2}.$$

Whence, $\partial_j B_1(v) \in L^2(\Omega)$, and

$$\|\partial_j B_1(v)\|_{L^2(\Omega)} \leq (\|g\|_{L^2(\Omega)} + c|\Omega|^{\frac{1}{4}} \|v\|_{L^2(\Omega)}^{\frac{1}{2}}) \|\partial_j K\|_{L^2(\Omega^2)}. \quad (3.27)$$

Since j is arbitrary then $B_1(v) \in W$.

Now, let $E \subset W$ bounded in $L^2(\Omega)$. Let us show that $\overline{\text{conv}(B_1(E))} \subset W$ (here, the closure is taken in $L^2(\Omega)$). Let $v_0 \in \overline{\text{conv}(B(E))}$ fixed, then there exists a sequence $(v_n)_n$ in $\text{conv}(B(E))$ which converges to v_0 in $L^2(\Omega)$. By using (3.27) and the convexity of the norm we obtain that the sequences $(\partial_j v_n)_n$, $j = 1, \dots, q$ are bounded in $L^2(\Omega)$, hence one can extract a subsequence still labeled $(v_n)_n$ such that $(\nabla_{X_1} v_n)_n$ converges weakly to some V_0 in $L^2(\Omega)^q$. Since the operators ∂_j are continuous on $\mathcal{D}'(\Omega)$ then we get $V_0 = \nabla_{X_1} v_0$, and therefore $v_0 \in w$, whence (40) follows. \square

2.2 A nonlinear second order neutron transport equation

In this subsection we show Theorem 0.13. We introduce the perturbed problem (51)

$$\begin{cases} \int_{\Omega} \mathbb{A}_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx + \gamma \int_{\Omega} u_{\epsilon} \varphi dx = \int_{\Omega} B_2(u_{\epsilon}) \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_{\epsilon} \in H_0^1(\Omega), \end{cases} \quad (3.28)$$

where $\tilde{\mathbb{A}}_{\epsilon}$ is given as in (3.23). The hypothesis (38) and (39) follow from the following proposition.

Proposition 3.3. *Under the hypothesis (50) and (53) the operator B_2 maps $L^1(\Omega)$ to $L^2(\Omega)$ continuously, and we have :*

$$\forall v \in L^2(\Omega) : \|B_2(v)\|_{L^2(\Omega)} \leq M_2(1 + \|v\|_{L^2(\Omega)}),$$

with

$$M_2 = 2c'|\omega_1| \|\tilde{K}\|_{L^{\infty}(\omega_1 \times \Omega)} \max\left(|\Omega|^{\frac{1}{2}}, |\Omega|^{\frac{1}{4}}\right)$$

Proof. The operator B_2 is well defined from $L^1(\Omega)$ to $L^2(\Omega)$, and by using (50), (53), and the Cauchy-Schwarz inequality we get

$$\forall v \in L^1(\Omega) : \|B_2(v)\|_{L^2(\Omega)} \leq c'|\omega_1| \|\tilde{K}\|_{L^{\infty}(\omega_1 \times \Omega)} \left(|\Omega|^{\frac{1}{2}} + \|v\|_{L^1(\Omega)}^{\frac{1}{2}} \right).$$

In particular,

$$\forall v \in L^2(\Omega) : \|B_2(v)\|_{L^2(\Omega)} \leq c'|\omega_1| \|\tilde{K}\|_{L^\infty(\omega_1 \times \Omega)} \left(|\Omega|^{\frac{1}{2}} + |\Omega|^{\frac{1}{4}} \|v\|_{L^2(\Omega)}^{\frac{1}{2}} \right).$$

Whence,

$$\forall v \in L^2(\Omega) : \|B_2(v)\|_{L^2(\Omega)} \leq c'|\omega_1| \|\tilde{K}\|_{L^\infty(\omega_1 \times \Omega)} \max \left(|\Omega|^{\frac{1}{2}}, |\Omega|^{\frac{1}{4}} \right) \left(1 + \|v\|_{L^2(\Omega)}^{\frac{1}{2}} \right).$$

Finally, we obtain

$$\forall v \in L^2(\Omega) : \|B_2(v)\|_{L^2(\Omega)} \leq 2c'|\omega_1| \|\tilde{K}\|_{L^\infty(\omega_1 \times \Omega)} \max \left(|\Omega|^{\frac{1}{2}}, |\Omega|^{\frac{1}{4}} \right) \left(1 + \|v\|_{L^2(\Omega)} \right).$$

and the inequality of the proposition follows.

Now, let us prove that B_2 is continuous from $L^1(\Omega)$ to $L^2(\Omega)$. Let $v_n \rightarrow v$ in $L^1(\Omega)$, then there exists a subsequence still labelled (v_n) and a function $w \in L^1(\Omega)$ such that

$$v_n(x) \rightarrow v(x) \text{ and } |v_n(x)| \leq w(x), \text{ for a.e. } x \in \Omega. \quad (3.29)$$

Let us show that

$$a(v_n) \rightarrow a(v) \text{ in } L^2(\Omega) \quad (3.30)$$

In fact, by using (3.29) and the continuity of a we obtain that for a.e. $x \in \Omega : a(v_n)(x) \rightarrow a(v)(x)$.

In the other hand, we have for a.e. $x \in \Omega, \forall n : |a(v_n)(x)| \leq c'(1 + |w(x)|^{\frac{1}{2}})$. Therefore, by the Lebesgue theorem we get (3.30). Now, by the Cauchy-Schwarz inequality and (50) we get

$$\|B_2(v_n) - B_2(v)\| \leq |\omega_1| \|\tilde{K}\|_{L^\infty(\omega_1 \times \Omega)} \|a(v_n) - a(v)\|_{L^2(\Omega)}. \quad (3.31)$$

Whence, by using (3.30) we get

$$B_2(v_n) \rightarrow B_2(v) \text{ in } L^2(\Omega) \quad (3.32)$$

Notice that (3.32) holds for the subsequence (v_n) given in (3.29). For the whole sequence we obtain (3.32) by immediate reasoning by contradiction, and the continuity of B_2 follows, and the proof is complete. \square

Now, let us check that the hypothesis (40) of Theorem 0.8 holds for the operator B_2 .

Proposition 3.4. *Under the hypothesis (50), (53), and (54) the operator B_2 satisfies (40).*

Proof. Let $v \in W$, $\varphi \in \mathcal{D}(\Omega)$ and $1 \leq j \leq q$ we have:

$$\left| \int_{\Omega} B_2(v) \partial_{x_j} \varphi(x) dx \right| \leq \int_{\Omega} \left| \int_{\omega_1} \tilde{K}(X'_1, X_1, X_2) \partial_j \varphi(X_1, X_2) a(u(X'_1, X_2)) dX_1 \right| dX'_1 dX_2.$$

Since $\partial_{x_j} K \in L^\infty(\omega_1 \times \Omega)$ then integrating by parts, and using Cauchy-Schwarz and (53) we get

$$\left| \int_{\Omega} B_2(v) \partial_j \varphi(x) dx \right| \leq c' |\omega_1| \|\tilde{K}\|_{L^\infty(\omega_1 \times \Omega)} \left(|\Omega|^{\frac{1}{2}} + |\Omega|^{\frac{1}{4}} \|v\|_{L^2(\Omega)}^{\frac{1}{2}} \right) \|\varphi\|_{L^2}.$$

Whence, $\partial_{x_j} B_2(v) \in L^2(\Omega)$, and

$$\|\partial_j B_1(v)\|_{L^2(\Omega)} \leq c' |\omega_1| \|\tilde{K}\|_{L^\infty(\omega_1 \times \Omega)} \left(|\Omega|^{\frac{1}{2}} + |\Omega|^{\frac{1}{4}} \|v\|_{L^2(\Omega)}^{\frac{1}{2}} \right)$$

Since j is arbitrary then $B_1(v) \in W$, and we conclude as at the end of the proof of Proposition 3.2 □

Remark 3.5. *According to Remark 3.7, we can deduce that Theorems 0.12 and 0.13 hold when we replace assumptions (47) and (53) by (55) and (56) respectively. In fact, under these assumptions, we prove that B_1 and B_2 satisfy assumption (42).*

Chapter 4

Anisotropic singular perturbations in L^p

In this chapter, we will first deal with the following problem

$$\beta(u_\epsilon) - \operatorname{div}(A_\epsilon \nabla u_\epsilon) = f, \quad (4.1)$$

complemented by the boundary condition

$$u_\epsilon = 0 \text{ on } \partial\Omega. \quad (4.2)$$

The associated limit problem is

$$\beta(u) - \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u) = f, \quad (4.3)$$

complemented by the boundary condition

$$u(X_1, \cdot) = 0 \text{ on } \partial\Omega_{X_1}, \text{ for } X_1 \in \Pi_1\Omega. \quad (4.4)$$

The function f is supposed to be in $L^p(\Omega)$ and the matrix valued function A satisfies the ellipticity assumptions (5),(6). The function β is nondecreasing continuous and sublinear (assumptions (10), (11)).

At a second time, we will study the problem of the neutron transport

$$\begin{cases} \int_\Omega \mathbb{A} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = f + \int_\Omega B_2(u) \varphi dx, \forall \varphi \in \mathcal{D}(\Omega) \\ u \in W_0^{1,p}(\Omega; \omega_2), \end{cases}$$

1 The convergence theorem

Let Ω be a bounded open set of \mathbb{R}^N . Let $f \in L^p(\Omega)$ with $1 < p < 2$. By density let $(f_n)_{n \in \mathbb{N}} \in L^2(\Omega)^\mathbb{N}$ be a sequence such that $f_n \rightarrow f$ in $L^p(\Omega)$. Consider the regularized problem:

$$u_\epsilon^n \in H_0^1(\Omega), \quad \int_{\Omega} \beta(u_\epsilon^n) \varphi dx + \int_{\Omega} A_\epsilon \nabla u_\epsilon^n \cdot \nabla \varphi dx = \int_{\Omega} f_n \varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega). \quad (4.5)$$

The existence and the uniqueness of u_ϵ^n follow thanks to assumptions (5), (6), (10), (11).

Let us begin by this primary analysis of the linear problem.

1.1 Weak convergence of a subsequence of weak solutions

Proposition 4.1. *Assume (5), (6). Suppose that $\beta = 0$ and $f \in L^p(\Omega)$ with $1 < p < 2$, then there exist a sequence $(u_{\epsilon_k})_{k \in \mathbb{N}} \in W_0^{1,p}(\Omega)^\mathbb{N}$ of weak solutions to (4.1) – (4.2), ($\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$), and $u \in W_0^{1,p}(\Omega; \Pi_2 \Omega)$ such that $\nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u$, $\epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightarrow 0$, $u_{\epsilon_k} \rightharpoonup u$ in $L^p(\Omega)$ – weak, and u is a weak solution to (4.3) – (4.4) for a.e $X_1 \in \Pi_1 \Omega$.*

Proof. Throughout the proof of this proposition C denotes a generic positive constant which independent of ϵ , n , and k . Notice that u_ϵ^n belongs to $W_0^{1,p}(\Omega)$. We introduce the function

$$\theta(t) = \int_0^t (1 + |s|)^{p-2} ds, \quad t \in \mathbb{R}.$$

This kind of function has been used in [34]. We have $\forall t \in \mathbb{R} : \theta'(t) = (1 + |t|)^{p-2} \leq 1$ and $\theta(0) = 0$, therefore we have $\theta(v) \in H_0^1(\Omega)$ for every $v \in H_0^1(\Omega)$. Testing with $\theta(u_\epsilon^n)$ in (4.5) and using the ellipticity assumption (5) we deduce

$$\begin{aligned} \lambda \epsilon^2 \int_{\Omega} (1 + |u_\epsilon^n|)^{p-2} |\nabla_{X_1} u_\epsilon^n|^2 dx + \lambda \int_{\Omega} (1 + |u_\epsilon^n|)^{p-2} |\nabla_{X_2} u_\epsilon^n|^2 dx \\ \leq \int_{\Omega} f_n \theta(u_\epsilon^n) dx \leq \frac{2}{p-1} \int_{\Omega} |f_n| (1 + |u_\epsilon^n|)^{p-1} dx, \end{aligned}$$

where we have used the fact that $|\theta(t)| \leq \frac{2(1+|t|)^{p-1}}{p-1}$. In the other hand, by Hölder's inequality we have

$$\int_{\Omega} |\nabla_{X_2} u_\epsilon^n|^p dx \leq \left(\int_{\Omega} (1 + |u_\epsilon^n|)^{p-2} |\nabla_{X_2} u_\epsilon^n|^2 dx \right)^{\frac{p}{2}} \left(\int_{\Omega} (1 + |u_\epsilon^n|)^p dx \right)^{1-\frac{p}{2}}.$$

From the two above integral inequalities we get

$$\int_{\Omega} |\nabla_{X_2} u_{\epsilon}^n|^p dx \leq \left(\frac{2}{\lambda(p-1)} \int_{\Omega} |f_n| (1 + |u_{\epsilon}^n|)^{p-1} dx \right)^{\frac{p}{2}} \times \left(\int_{\Omega} (1 + |u_{\epsilon}^n|)^p dx \right)^{1-\frac{p}{2}}.$$

By the Hölder inequality we get

$$\|\nabla_{X_2} u_{\epsilon}^n\|_{L^p(\Omega)} \leq \left(\frac{2 \|f_n\|_{L^p}}{\lambda(p-1)} \right)^{\frac{1}{2}} \left(\int_{\Omega} (1 + |u_{\epsilon}^n|)^p dx \right)^{\frac{1}{2p}}. \quad (4.6)$$

Using the Minkowski inequality we get

$$\|\nabla_{X_2} u_{\epsilon}^n\|_{L^p(\Omega)}^2 \leq C(1 + \|u_{\epsilon}^n\|_{L^p(\Omega)}),$$

By using the Poincaré inequality (57) we obtain that

$$\|\nabla_{X_2} u_{\epsilon}^n\|_{L^p(\Omega)}^2 \leq C(1 + \|\nabla_{X_2} u_{\epsilon}^n\|_{L^p(\Omega)}),$$

whence

$$\|u_{\epsilon}^n\|_{L^p(\Omega)}, \|\nabla_{X_2} u_{\epsilon}^n\|_{L^p(\Omega)} \leq C. \quad (4.7)$$

Similarly, we obtain

$$\|\epsilon \nabla_{X_1} u_{\epsilon}^n\|_{L^p(\Omega)} \leq C, \quad (4.8)$$

and

$$\|u_{\epsilon}^n\|_{W^{1,p}(\Omega)} \leq \frac{C}{\epsilon}. \quad (4.9)$$

Fix ϵ , since $W^{1,p}(\Omega)$ is reflexive then (4.9) implies that there exist a subsequence $(u_{\epsilon}^{n_l(\epsilon)})_{l \in \mathbb{N}}$ and $u_{\epsilon} \in W_0^{1,p}(\Omega)$ such that $u_{\epsilon}^{n_l(\epsilon)} \rightharpoonup u_{\epsilon} \in W_0^{1,p}(\Omega)$ (as $l \rightarrow \infty$) in $W^{1,p}(\Omega)$ -weak. Now, passing to the limit in (4.5) as $l \rightarrow \infty$ we deduce

$$\int_{\Omega} A_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad \varphi \in \mathcal{D}(\Omega). \quad (4.10)$$

Whence u_{ϵ} is a weak solution of (4.1)-(4.2) (recall that $\beta = 0$). Now, from (4.7) we deduce

$$\|u_{\epsilon}\|_{L^p(\Omega)} \leq \liminf_{l \rightarrow \infty} \|u_{\epsilon}^{n_l(\epsilon)}\|_{L^p(\Omega)} \leq C,$$

and

$$\|\nabla_{X_2} u_{\epsilon}\|_{L^p(\Omega)} \leq \liminf_{l \rightarrow \infty} \|\nabla_{X_2} u_{\epsilon}^{n_l(\epsilon)}\|_{L^p(\Omega)} \leq C,$$

and similarly from (4.8) we obtain

$$\|\epsilon \nabla_{X_1} u_\epsilon\|_{L^p(\Omega)} \leq C.$$

By using the three above inequalities, and the reflexivity of $L^p(\Omega)$ and continuity of the derivation operator on $\mathcal{D}'(\Omega)$, one can extract a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that $\nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u$, $\epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightharpoonup 0$, $u_{\epsilon_k} \rightharpoonup u$ in $L^p(\Omega)$ – weak, where $u \in L^p(\Omega)$ such that $\nabla_{X_2} u \in L^p(\Omega)^{N-q}$. Passing to the limit in (4.10) we get

$$\int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega). \quad (4.11)$$

Now, we will prove that $u \in W_0^{1,p}(\Omega; \Pi_2 \Omega)$. Since $\nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u$ and $u_{\epsilon_k} \rightharpoonup u$ in $L^p(\Omega)$ – weak then there exists a sequence $(U_n)_{n \in \mathbb{N}} \subset \text{conv}(\{u_{\epsilon_k}\}_{k \in \mathbb{N}})$ such that $\nabla_{X_2} U_n \rightarrow \nabla_{X_2} u$ in $L^p(\Omega)$ – strong, where $\text{conv}(\{u_{\epsilon_k}\}_{k \in \mathbb{N}})$ is the convex hull of the set $\{u_{\epsilon_k}\}_{k \in \mathbb{N}}$. Notice that we have $U_n \in W_0^{1,p}(\Omega)$ then we have $U_n(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1})$, a.e. $X_1 \in \Pi_1 \Omega$. Therefore, up to a subsequence, $\nabla_{X_2} U_n(X_1, \cdot) \rightarrow \nabla_{X_2} u(X_1, \cdot)$ in $L^p(\Omega_{X_1})$ – strong a.e. $X_1 \in \Pi_1 \Omega$. Whence $u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1})$ for a.e. $X_1 \in \Pi_1 \Omega$, so $u \in W_0^{1,p}(\Omega; \Pi_2 \Omega)$.

To complete the proof of the proposition, we will prove that u is a weak solution of (4.3)-(4.4) (where $\beta = 0$). Let E be a Banach space, a family of vectors $\{e_n\}_{n \in \mathbb{N}}$ in E is said to be a Banach basis or a Schauder basis of E if for every $x \in E$ there exists a family of scalars $(\alpha_n)_{n \in \mathbb{N}}$ such that $x = \sum_{n=0}^{\infty} \alpha_n e_n$, where the series converges in the norm of E . Notice that Schauder basis does not always exist. In [36] P. Enflo has constructed a separable reflexive Banach space without Schauder basis! However, the Sobolev space $W_0^{1,r}$ ($1 < r < \infty$) has a Schauder basis whenever the boundary of the domain is sufficiently smooth [37]. Let $(U_i \times V_i)_{i \in \mathbb{N}}$ be a countable covering of Ω such that $U_i \times V_i \subset \Omega$ where $U_i \subset \mathbb{R}^q, V_i \subset \mathbb{R}^{N-q}$ are two bounded open domains, where ∂V_i is smooth (V_i are Euclidean balls for example), such a covering always exists. Now, fix $\psi \in W_0^{1,p'}(V_i)$ then it follows from (4.11) that for every $\phi \in W_0^{1,p'}(U_i)$ we have

$$\int_{U_i} \phi dX_1 \int_{V_i} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \psi dX_2 = \int_{U_i} \phi dX_1 \int_{V_i} f \psi dX_2.$$

Whence for a.e. $X_1 \in U_i$ we have

$$\int_{V_i} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 = \int_{V_i} f(X_1, \cdot) \psi dX_2. \quad (4.12)$$

Here, p' is the Hölder conjugate of p . When ψ run through a Schauder basis of $W_0^{1,p'}(V_i)$ we obtain that for a.e. $X_1 \in U_i$

$$\int_{V_i} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 = \int_{V_i} f(X_1, \cdot) \psi dX_2, \quad \forall \psi \in W_0^{1,p'}(V_i). \quad (4.13)$$

Since $\Pi_1 \Omega$ could be covered by a countable set of U_i then from (4.13) we get

$$\text{For a.e. } X_1 \in \Pi_1 \Omega : \int_{V_i} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 = \int_{V_i} f(X_1, \cdot) \psi dX_2, \quad \forall \psi \in W_0^{1,p'}(V_i). \quad (4.14)$$

Now, let $\psi \in \mathcal{D}(\Omega_{X_1})$ then the $\text{Supp} \psi$ could be covered by a finite set of V_i , by using a partition of unity subordinate to such a cover, and by using (4.14) we get

$$\int_{\Omega_{X_1}} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 = \int_{\Omega_{X_1}} f(X_1, \cdot) \psi dX_2, \quad \psi \in \mathcal{D}(\Omega_{X_1}),$$

for a.e $X_1 \in \Pi_1 \Omega$. Finally, since $u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1})$ (as we proved above), then $u(X_1, \cdot)$ is a weak solution of (4.3)-(4.4). \square

1.2 Strong convergence of the entropy solutions

Theorem 0.16 will be proved in three steps. It is well known that the entropy solution are the solution constructed by approximation [31, 38]. The proof is based on the use of the approximation problem (4.5). Throughout this subsection C denotes a positive generic constant independent of n and ϵ .

Step 1

In this step we construct u , the entropy solution of the limit problem (4.3). Let $u_\epsilon^n \in H_0^1(\Omega)$ be the unique solution to (4.5), we have the following proposition.

Proposition 4.2. *Under assumptions of Theorem 0.16 there exists a unique sequence $(u^n)_{n \in \mathbb{N}} \in H_0^1(\Omega; \Pi_2 \Omega)^\mathbb{N}$ such that $\epsilon \nabla_{X_1} u_\epsilon^n \rightarrow 0$ in $L^2(\Omega)$, $u_\epsilon^n \rightarrow u^n$ in $H_0^1(\Omega; \Pi_2 \Omega)$ as $\epsilon \rightarrow 0$, for every $n \in \mathbb{N}$, in particular the two convergences hold in $L^p(\Omega)$ and $W_0^{1,p}(\Omega; \Pi_2 \Omega)$ respectively, where u^n is the unique solution in $H_0^1(\Omega; \Pi_2 \Omega)$ to the problem*

$$\text{For a.e. } X_1 \in \Pi_1 \Omega, \quad \int_{\Omega_{X_1}} \beta(u^n) \varphi dX_2 + \int_{\Omega_{X_1}} A_{22} \nabla_{X_2} u^n \cdot \nabla_{X_2} \varphi dX_2 = \int_{\Omega_{X_1}} f_n \varphi dX_2, \quad \forall \varphi \in H_0^1(\Omega_{X_1}). \quad (4.15)$$

Proof. This result follows from the L^2 -theory Proposition 2.7. The convergences in $W_0^{1,p}(\Omega; \Pi_2\Omega)$ and $L^p(\Omega)$ follow from the continuous embeddings $H_0^1(\Omega; \Pi_2\Omega) \hookrightarrow W_0^{1,p}(\Omega; \Pi_2\Omega)$, $L^2(\Omega) \hookrightarrow L^p(\Omega)$, $1 < p < 2$. \square

Now, we construct u . Testing with $\varphi = \theta(u^n(X_1, \cdot))$ in (4.15) and estimating like in proof of Proposition 4.1 where we use assumptions (5), (10), we obtain as in (4.6). For a.e. $X_1 \in \Pi_1\Omega$

$$\forall n \in \mathbb{N} : \|\nabla_{X_2} u^n(X_1, \cdot)\|_{L^p(\Omega_{X_1})} \leq \left(\frac{\|f_n(X_1, \cdot)\|_{L^p(\Omega_{X_1})}}{\lambda(p-1)} \right)^{\frac{1}{2}} \times \left(\int_{\Omega_{X_1}} (1 + |u^n(X_1, \cdot)|)^p dX_2 \right)^{\frac{1}{2p}} \quad (4.16)$$

Integrating (4.16) over $\Pi_1\Omega$ and using Cauchy-Schwarz's inequality we deduce

$$\forall n \in \mathbb{N} : \|\nabla_{X_2} u^n\|_{L^p(\Omega)}^p \leq C \|f_n\|_{L^p(\Omega)}^{\frac{p}{2}} \left(\int_{\Omega} (1 + |u^n|)^p dx \right)^{\frac{1}{2}},$$

and therefore

$$\forall n \in \mathbb{N} : \|\nabla_{X_2} u^n\|_{L^p(\Omega)}^2 \leq C(1 + \|u^n\|_{L^p(\Omega)}).$$

Using Poincaré's inequality (57) (which holds since $u^n \in W_0^{1,p}(\Omega; \Pi_2\Omega)$), we obtain that

$$\forall n \in \mathbb{N} : \|u^n\|_{L^p(\Omega)} \leq C \text{ for every } n \in \mathbb{N}, \quad (4.17)$$

Now, using the monotonicity of the problem (assumption (10)) and testing in (4.15) with the test function $\theta(u^n(X_1, \cdot) - u^m(X_1, \cdot))$, $m, n \in \mathbb{N}$, one can obtain like in (4.16)

$$\begin{aligned} & \|\nabla_{X_2} (u^n(X_1, \cdot) - u^m(X_1, \cdot))\|_{L^p(\Omega_{X_1})} \\ & \leq \left(\frac{\|f_n(X_1, \cdot) - f_m(X_1, \cdot)\|_{L^p(\Omega_{X_1})}}{\lambda(p-1)} \right)^{\frac{1}{2}} \times \\ & \quad \left(\int_{\Omega_{X_1}} (1 + |u^n(X_1, \cdot) - u^m(X_1, \cdot)|)^p dX_2 \right)^{\frac{1}{2p}}. \end{aligned}$$

Integrating over $\Pi_1\Omega$ and using the Cauchy-Schwarz inequality and (4.17) we get

$$\forall n, m \in \mathbb{N} : \|\nabla_{X_2} (u^n - u^m)\|_{L^p(\Omega)} \leq C \|f_n - f_m\|_{L^p(\Omega)}^{\frac{1}{2}},$$

where C is independent of m and n . Since $(f_n)_{n \in \mathbb{N}}$ is a converging sequence in $L^p(\Omega)$ then this last inequality shows that $(u^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $W_0^{1,p}(\Omega; \Pi_2\Omega)$, consequently

there exists $u \in W_0^{1,p}(\Omega; \Pi_2\Omega)$ such that $u^n \rightarrow u$ in $W_0^{1,p}(\Omega; \Pi_2\Omega)$. Therefore, there exists a subsequence still labelled (u^n) such that for a.e. $X_1 \in \Pi_1$ we have $u^n(X_1, \cdot) \rightarrow u(X_1, \cdot)$ in $W_0^{1,p}(\Omega_{X_1})$, therefore for a.e. $X_1 \in \Pi_1$, $u(X_1, \cdot)$ is the entropy solution of (4.3) (It is well known that, up to a subsequence, we have for a.e. $X_1 \in \Pi_1\Omega$, $u^n(X_1, \cdot) \rightarrow u(X_1, \cdot)$ in $L^r(\Omega_{X_1})$ for every $1 \leq r < \frac{N}{N-1}$). Whence we have proved the following

Proposition 4.3. *Under assumptions of Proposition 4.2 there exists $u \in W_0^{1,p}(\Omega; \Pi_2\Omega)$ unique such that for a.e. $X_1 \in \Pi_1\Omega$, $u(X_1, \cdot)$ is the entropy solution to (4.3), and $u^n \rightarrow u$ in $W_0^{1,p}(\Omega; \Pi_2\Omega)$ where $(u^n)_{n \in \mathbb{N}}$ is the sequence given in Proposition 4.2.*

1.2.1 Step 2

We construct the sequence $(u_\epsilon)_{0 < \epsilon \leq 1}$ of the entropy solutions to (4.1), one can prove the following

Proposition 4.4. *Let $(u_\epsilon)_{0 < \epsilon \leq 1}$ be a sequence of the entropy solutions to (4.1), then we have $u_\epsilon^n \rightarrow u_\epsilon$ in $W^{1,p}(\Omega)$ as $n \rightarrow \infty$, for every ϵ fixed. Moreover, $u_\epsilon^n \rightarrow u_\epsilon$ in $W_0^{1,p}(\Omega; \Pi_2\Omega)$ and $\epsilon \nabla_{X_1} u_\epsilon^n \rightarrow \nabla_{X_1} u_\epsilon$ in $L^p(\Omega)$, uniformly in ϵ .*

Proof. Testing with $\theta(u_\epsilon^n - u_\epsilon^m)$, $m, n \in \mathbb{N}$ in (4.5), and using assumptions (5) and (10), we obtain as in (4.6)

$$\|\nabla_{X_2}(u_\epsilon^n - u_\epsilon^m)\|_{L^p(\Omega)} \leq \left(\frac{\|f_n - f_m\|_{L^p}}{\lambda(p-1)} \right)^{\frac{1}{2}} \left(\int_{\Omega} (1 + |u_\epsilon^n - u_\epsilon^m|)^p \right)^{\frac{1}{2p}}.$$

Therefore(4.7) gives

$$\forall m, n : \|\nabla_{X_2}(u_\epsilon^n - u_\epsilon^m)\|_{L^p(\Omega)} \leq C \|f_n - f_m\|_{L^p}^{\frac{1}{2}}, \quad (4.18)$$

Similarly, we obtain

$$\forall m, n : \|\epsilon \nabla_{X_1}(u_\epsilon^n - u_\epsilon^m)\|_{L^p(\Omega)} \leq C \|f_n - f_m\|_{L^p}^{\frac{1}{2}}. \quad (4.19)$$

It follows that

$$\forall m, n : \|u_\epsilon^n - u_\epsilon^m\|_{W^{1,p}(\Omega)} \leq \frac{C}{\epsilon} \|f_n - f_m\|_{L^p}^{\frac{1}{2}}.$$

The last inequality implies that for every ϵ fixed $(u_\epsilon^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$, then there exists $u_\epsilon \in W_0^{1,p}(\Omega)$ such that $u_\epsilon^n \rightarrow u_\epsilon$ in $W^{1,p}(\Omega)$, notice that u_ϵ is the entropy solution to (4.1) (it

is well know that, up to a subsequence in n , we have $u_\epsilon^n \rightarrow u_\epsilon$ in $L^r(\Omega)$ for every $1 \leq r < \frac{N}{N-1}$. Finally (4.18) and (4.19) show that $u_\epsilon^n \rightarrow u_\epsilon$ (resp $\epsilon \nabla_{X_1} u_\epsilon^n \rightarrow \epsilon \nabla_{X_1} u_\epsilon$) in $W_0^{1,p}(\Omega; \Pi_2 \Omega)$ (resp in $L^p(\Omega)$), uniformly in ϵ . \square

1.2.2 Step 3

Now, we are ready to conclude. Propositions 4.2, 4.3, 4.4 combined with the triangle inequality show that $u_\epsilon \rightarrow u$ in $W_0^{1,p}(\Omega; \Pi_2 \Omega)$ and $\epsilon \nabla_{X_1} u_\epsilon \rightarrow 0$ in $L^p(\Omega)$, and the proof of Theorem 0.16 is completed.

1.3 A regularity result for the entropy solution of the limit problem

In this subsection, we assume that $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 are two bounded open sets of $\mathbb{R}^q, \mathbb{R}^{N-q}$ respectively. We introduce the space

$$W_p = \{v \in L^p(\Omega) \mid \nabla_{X_1} v \in L^p(\Omega)^q\}$$

We suppose the following

$$f \in W_p. \tag{4.20}$$

Theorem 4.5. *Assume (5), (6), (22), (4.20). Suppose that $\beta = 0$, then $u \in W^{1,p}(\Omega)$, where u is the entropy solution of (4.3) – (4.4).*

Proof. The proof of this theorem is based on the difference quotient method (see for instance [20] page 168). Let (u^n) be the sequence of Proposition 4.3, then we have $u^n \rightarrow u$ in $W_0^{1,p}(\Omega; \omega_2)$, where u is the entropy solution of (4.3).

Let $1 \leq i \leq q$ be fixed. Let $\omega'_1 \subset\subset \omega_1$ be an open subset, for $0 < h < d(\partial\omega_1, \omega'_1)$ and for a.e. $X_1 \in \omega'_1$ we set $\tau_h^i u^n = u^n(X_1 + h e_i, X_2)$ where $e_i = (0, \dots, 1, \dots, 0)$ then we have by (4.15) (with $\beta = 0$)

$$\int_{\omega_2} A_{22} \nabla_{X_2} (\tau_h^i u^n - u^n) \cdot \nabla_{X_2} \varphi dX_2 = \int_{\omega_2} (\tau_h^i f_n - f_n) \varphi dX_2, \forall \varphi \in H_0^1(\omega_2),$$

where we have used the fact that $A_{22}(x) = A_{22}(X_2)$. We introduce the function $\theta_\delta(t) = \int_0^t (\delta + |s|)^{p-2} ds$, $\delta > 0$, $t \in \mathbb{R}$ we have $0 < \theta'_\delta(t) = (\delta + |t|)^{p-2} \leq \delta^{p-2}$ and $|\theta_\delta(t)| \leq \frac{2(\delta+|t|)^{p-1}}{p-1}$ Testing with $\varphi = \frac{1}{h} \theta_\delta(\frac{\tau_h^i u^n - u^n}{h}) \in H_0^1(\omega_2)$. To make the notations less heavy we set

$$U = \frac{\tau_h^i u^n - u^n}{h}, \frac{(\tau_h^i f_n - f_n)}{h} = F$$

Then we get

$$\int_{\omega_2} \theta'_\delta(U) A_{22} \nabla_{X_2} U \cdot \nabla_{X_2} U dX_2 = \int_{\omega_2} F \theta_\delta(U) dX_2 .$$

Using the ellipticity assumption (5) for the left hand side, and Hölder's inequality for the right hand side of the above integral equality we deduce

$$\lambda \int_{\omega_2} \theta'_\delta(U) |\nabla_{X_2} U|^2 dX_2 \leq \frac{2}{p-1} \|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} (\delta + |U|)^p dX_2 \right)^{\frac{p-1}{p}} .$$

Using Hölder's inequality we derive

$$\begin{aligned} \|\nabla_{X_2} U\|_{L^p(\omega_2)}^p &\leq \left(\int_{\omega_2} \theta'_\delta(U) |\nabla_{X_2} U|^2 dX_2 \right)^{\frac{p}{2}} \left(\int_{\omega_2} \theta'_\delta(U)^{\frac{p}{p-2}} dX_2 \right)^{\frac{2-p}{2}} \\ &\leq \left(\frac{2}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} (\delta + |U|)^p dX_2 \right)^{\frac{p-1}{p}} \right)^{\frac{p}{2}} \times \\ &\quad \left(\int_{\omega_2} \theta'_\delta(U)^{\frac{p}{p-2}} dX_2 \right)^{\frac{2-p}{2}} . \end{aligned}$$

Then we deduce

$$\|\nabla_{X_2} U\|_{L^p(\omega_2)}^2 \leq \frac{2}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} (\delta + |U|)^p dX_2 \right)^{\frac{1}{p}} .$$

Now passing to the limit as $\delta \rightarrow 0$ using the Lebesgue theorem we deduce

$$\|\nabla_{X_2} U\|_{L^p(\omega_2)}^2 \leq \frac{2}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} |U|^p dX_2 \right)^{\frac{1}{p}} ,$$

and Poincaré's inequality (57) gives

$$\|\nabla_{X_2} U\|_{L^p(\omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \|F\|_{L^p(\omega_2)} .$$

Now, integrating over ω'_1 yields

$$\left\| \frac{\tau_h^i u^n - u^n}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\| \frac{(\tau_h^i f_n - f_n)}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} .$$

Passing to the limit as $n \rightarrow \infty$ using the invariance of the Lebesgue measure under translations we get

$$\left\| \frac{\tau_h^i u - u}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\| \frac{(\tau_h^i f - f)}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} .$$

Whence, since $f \in W_p$ then

$$\left\| \frac{\tau_h^i u - u}{h} \right\|_{L^p(\omega'_1 \times \omega_2)} \leq C,$$

where C is independent of h , therefore we have $\nabla_{X_1} u \in L^p(\Omega)^q$. Combining this with $u \in W_0^{1,p}(\Omega; \omega_2)$ we get the desired result. \square

2 Estimation of the rate of convergence in cylindrical domains

In this section we prove Theorem 0.17 and Theorem 0.18. We suppose that $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 are two bounded open sets of \mathbb{R}^q and \mathbb{R}^{N-q} respectively. Throughout this section, we suppose that $\beta = 0$ and that A_{22} , A_{12} , and f satisfy assumptions (24), (58), and (59) respectively i.e. $\forall x \in \Omega$, $A_{12}(x) = A_{12}(X_2)$, $A_{22}(x) = A_{22}(X_2)$, and $f(x) = f(X_2) \in L^p(\omega_2)$ ($1 < p < 2$).

2.1 A polynomial rate

Let u_ϵ , u be the entropy solutions of (4.1), (4.3) respectively (recall that $\beta = 0$), we use the approximated sequence $(u_\epsilon^n)_{\epsilon,n}$, $(u^n)_n$ introduced in Section 1. Subtracting (4.15) from (4.5) we obtain

$$\int_{\Omega} A_\epsilon \nabla(u_\epsilon^n - u^n) \cdot \nabla \varphi dx = 0,$$

where we have used that u^n is independent of X_1 (since f and A_{22} are independent of X_1) and that A_{12} is independent of X_1 . Let $\omega'_1 \subset\subset \omega_1$ then there exists $\omega'_1 \subset\subset \omega''_1 \subset\subset \omega_1$. We introduce the function $\rho \in \mathcal{D}(\omega_1)$ such that $Supp(\rho) \subset \omega''_1$ and $\rho = 1$ on ω'_1 (we can choose $0 \leq \rho \leq 1$) Testing with $\varphi = \rho^2 \theta_\delta(u_\epsilon^n - u^n) \in H_0^1(\Omega)$ (we can check that this function belongs to $H_0^1(\Omega)$ by using a density argument) in the above integral equality we get

$$\begin{aligned} & \int_{\Omega} \rho^2 \theta'_\delta(u_\epsilon^n - u^n) A_\epsilon \nabla(u_\epsilon^n - u^n) \cdot \nabla(u_\epsilon^n - u^n) dx \\ &= - \int_{\Omega} \rho \theta_\delta(u_\epsilon^n - u^n) A_\epsilon \nabla(u_\epsilon^n - u^n) \cdot \nabla \rho dx \\ &= -\epsilon^2 \int_{\Omega} \rho \theta_\delta(u_\epsilon^n - u^n) A_{11} \nabla_{X_1}(u_\epsilon^n - u^n) \cdot \nabla_{X_1} \rho dx \\ & \quad - \epsilon \int_{\Omega} \rho \theta_\delta(u_\epsilon^n - u^n) A_{12} \nabla_{X_2}(u_\epsilon^n - u^n) \cdot \nabla_{X_1} \rho dx. \end{aligned}$$

where we have used that ρ is independent of X_2 . Using the ellipticity assumption (5) for the left hand side and assumption (6) for the right hand side of the previous integral equality we deduce

$$\begin{aligned} & \epsilon^2 \lambda \int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_1}(u_\epsilon^n - u^n)|^2 dx + \lambda \int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_2}(u_\epsilon^n - u^n)|^2 dx \\ & \leq \epsilon^2 C \int_{\Omega} \rho |\theta_\delta(u_\epsilon^n - u^n)| |\nabla_{X_1}(u_\epsilon^n - u^n)| dx \\ & \quad + \epsilon C \int_{\Omega} \rho |\theta_\delta(u_\epsilon^n - u^n)| |\nabla_{X_2}(u_\epsilon^n - u^n)| dx. \end{aligned}$$

Using Young's inequality for the two terms in the right hand side of the previous inequality, we obtain

$$\begin{aligned} & \epsilon^2 \frac{\lambda}{2} \int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_1}(u_\epsilon^n - u^n)|^2 dx + \frac{\lambda}{2} \int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_2}(u_\epsilon^n - u^n)|^2 dx \\ & \leq \epsilon^2 C \int_{\omega_1'' \times \omega_2} |\theta_\delta(u_\epsilon^n - u^n)|^2 \theta'_\delta(u_\epsilon^n - u^n)^{-1} dx. \end{aligned}$$

Whence

$$\begin{aligned} & \epsilon^2 \frac{\lambda}{2} \int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_1}(u_\epsilon^n - u^n)|^2 dx + \frac{\lambda}{2} \int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_2}(u_\epsilon^n - u^n)|^2 dx \\ & \leq \epsilon^2 C \int_{\omega_1'' \times \omega_2} (\delta + |u_\epsilon^n - u^n|)^p dx, \end{aligned}$$

Now, using Hölder's inequality and the above inequality we deduce

$$\begin{aligned} & \epsilon^2 \frac{\lambda}{2} \|\rho \nabla_{X_1}(u_\epsilon^n - u^n)\|_{L^p(\Omega)}^2 + \frac{\lambda}{2} \|\rho \nabla_{X_2}(u_\epsilon^n - u^n)\|_{L^p(\Omega)}^2 \\ & \leq \left[\begin{aligned} & \epsilon^2 \frac{\lambda}{2} \left(\int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_1}(u_\epsilon^n - u^n)|^2 dx \right) \\ & + \frac{\lambda}{2} \left(\int_{\Omega} \theta'_\delta(u_\epsilon^n - u^n) |\rho \nabla_{X_2}(u_\epsilon^n - u^n)|^2 dx \right) \end{aligned} \right] \times \\ & \quad \left(\int_{\omega_1'' \times \omega_2} (\delta + |u_\epsilon^n - u^n|)^p dx \right)^{\frac{2-p}{p}} \\ & \leq C \epsilon^2 \left(\int_{\omega_1'' \times \omega_2} (\delta + |u_\epsilon^n - u^n|)^p dx \right)^{\frac{2}{p}}. \end{aligned}$$

Passing to the limit as $\delta \rightarrow 0$ using the Lebesgue theorem. Passing to the limit as $n \rightarrow \infty$ we get

$$\epsilon^2 \|\nabla_{X_1}(u_\epsilon - u)\|_{L^p(\omega_1' \times \omega_2)}^2 + \|\nabla_{X_2}(u_\epsilon - u)\|_{L^p(\omega_1' \times \omega_2)}^2 \leq C \epsilon^2 \|(u_\epsilon - u)\|_{L^p(\omega_1'' \times \omega_2)}^2. \quad (4.21)$$

Using Poincaré's inequality (57) we obtain

$$\epsilon^2 \|\nabla_{X_1}(u_\epsilon - u)\|_{L^p(\omega_1' \times \omega_2)}^2 + \|\nabla_{X_2}(u_\epsilon - u)\|_{L^p(\omega_1' \times \omega_2)}^2 \leq C \epsilon^2 \|\nabla_{X_2}(u_\epsilon - u)\|_{L^p(\omega_1'' \times \omega_2)}^2. \quad (4.22)$$

Let $d \in \mathbb{N}^*$ then there exist $\omega'_1 \subset \subset \omega''_1 \subset \subset \dots \omega_1^{(d+1)} \subset \subset \omega_1$. Iterating (4.22) d -times we deduce

$$\epsilon^2 \|\nabla_{X_1}(u_\epsilon - u)\|_{L^p(\omega'_1 \times \omega_2)}^2 + \|\nabla_{X_2}(u_\epsilon - u)\|_{L^p(\omega'_1 \times \omega_2)}^2 \leq C\epsilon^{2d} \|\nabla_{X_2}(u_\epsilon - u)\|_{L^p(\omega_1^{(d)} \times \omega_2)}^2.$$

Now, from (4.21) (with ω'_1 and ω''_1 replaced by $\omega_1^{(d)}$ and $\omega_1^{(d+1)}$ respectively) we deduce

$$\epsilon^2 \|\nabla_{X_1}(u_\epsilon - u)\|_{L^p(\omega'_1 \times \omega_2)}^2 + \|\nabla_{X_2}(u_\epsilon - u)\|_{L^p(\omega'_1 \times \omega_2)}^2 \leq C\epsilon^{2(d+1)} \|u_\epsilon - u\|_{L^p(\omega_1^{(d+1)} \times \omega_2)}^2.$$

Since $u_\epsilon \rightarrow u$ in $L^p(\Omega)$ then $\|u_\epsilon - u\|_{L^p(\Omega)}$ is bounded and therefore we obtain

$$\|u_\epsilon - u\|_{W^{1,p}(\omega'_1 \times \omega_2)} \leq C\epsilon^d,$$

where the constant C only depends on $d, A, \lambda, f, \Omega, \omega'_1$.

Remark 4.1. *It is not easy to estimate the convergence rate for general data. However, under assumptions of Theorem 0.17 and when $f = f_1 + f_2$ with $f_1 \in L^p(\omega_2)$ and $f_2 \in W_2$ we only have the estimation*

$$\epsilon \|\nabla_{X_1}(u_\epsilon - u)\|_{L^p(\omega'_1 \times \omega_2)} + \|\nabla_{X_2}(u_\epsilon - u)\|_{L^p(\omega'_1 \times \omega_2)} + \|u_\epsilon - u\|_{L^p(\omega'_1 \times \omega_2)} \leq C\epsilon.$$

This follows from the linearity of the equation, Theorem 0.17 and the estimation (20).

2.2 Elliptic problems in a cylinder becoming unbounded

Anisotropic singular perturbations problems could be seen as problems in a cylinder becoming unbounded [4], [5]. Indeed the two category of problems could be connected to each other via a scaling $\epsilon = \frac{1}{\ell}$. Let us consider the problem:

$$\begin{cases} -\nabla \cdot (\tilde{A} \nabla u_\ell) = f \\ u_\ell = 0 \quad \text{on } \partial\Omega_\ell \end{cases} \quad (4.23)$$

where $\tilde{A} = (\tilde{a}_{ij})$ is a $N \times N$ matrix valued function which has the same block structure as in (7), and such that

$$\tilde{a}_{ij} \in L^\infty(\mathbb{R}^q \times \omega_2), \quad (4.24)$$

$$\exists \lambda > 0 : \tilde{A}(x)\xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^N \text{ for a.e } x \in \mathbb{R}^q \times \omega_2. \quad (4.25)$$

For $\ell > 0$ we set $\Omega_\ell = \ell\omega_1 \times \omega_2$ where ω_1, ω_2 are two bounded open sets with ω_1 convex and containing 0. We consider the limit problem:

$$\begin{cases} -\nabla_{X_2} \cdot (\tilde{A}_{22} \nabla_{X_2} u_\infty) = f \\ u_\infty = 0 \quad \text{on } \partial\omega_2 \end{cases} \quad (4.26)$$

We have the following result which gives Theorem 0.18 as a corollary.

Theorem 4.2. *Assume that the blocks $\tilde{A}_{22}, \tilde{A}_{12}$ satisfy (24), (58) respectively. Suppose (59), (4.24), (4.25). Let u_ℓ, u_∞ be the unique entropy solutions to (4.23) and (4.26) then for every $\alpha \in (0, 1)$ there exist $C > 0, c > 0$ independent of ℓ such that*

$$\forall \ell > 0 : \|\nabla(u_\ell - u_\infty)\|_{W^{1,p}(\Omega_{\alpha\ell})} \leq Ce^{-c\ell}$$

Proof. Let u_ℓ, u_∞ be the unique entropy solutions to (4.23) and (4.26) respectively, and let (u_ℓ^n) and (u_∞^n) the approximation sequences (as in Section 1, Chapter 4). we have $u_\ell^n \rightarrow u_\ell$ in $W_0^{1,p}(\Omega_\ell)$ and $u_\infty^n \rightarrow u_\infty$ in $W_0^{1,p}(\omega_2)$. Subtracting the associated approximated problems to (4.23) and (4.26), and taking the weak formulations. We get

$$\int_{\Omega_\ell} \tilde{A} \nabla(u_\ell^n - u_\infty^n) \nabla \varphi dx = 0, \quad \forall \varphi \in H_0^1(\Omega_\ell). \quad (4.27)$$

Where we have used that $\tilde{A}_{22}, \tilde{A}_{12}, u_\infty^n$ are independent of X_1 . Now we will use the iteration technique introduced in [35]. Let $0 < \ell_0 \leq \ell - 1$ fixed, and let $\rho \in \mathcal{D}(\mathbb{R}^q)$ a bump function such that

$$0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } \ell_0\omega_1 \text{ and } \rho = 0 \text{ on } \mathbb{R}^q \setminus (\ell_0 + 1)\omega_1, \quad |\nabla_{X_1} \rho| \leq c_0.$$

where c_0 is a positive constant which only depends in ω_1 ([4], Chapter 6). Testing with $\rho^2 \theta_\delta(u_\ell^n - u_\infty^n) \in H_0^1(\Omega_\ell)$ in (4.27) we get

$$\int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) \tilde{A} \nabla(u_\ell^n - u_\infty^n) \cdot \nabla(u_\ell^n - u_\infty^n) dx + \int_{\Omega_\ell} \rho \theta_\delta(u_\ell^n - u_\infty^n) \tilde{A} \nabla(u_\ell^n - u_\infty^n) \cdot \nabla \rho dx = 0.$$

Using the ellipticity assumption (4.25). We obtain that

$$\int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) |\nabla(u_\ell^n - u_\infty^n)|^2 dx \leq 2 \int_{\Omega_\ell} \rho |\theta_\delta(u_\ell^n - u_\infty^n)| |\tilde{A} \nabla(u_\ell^n - u_\infty^n)| |\nabla \rho| dx.$$

Notice that $\nabla\rho = 0$ on Ω_{ℓ_0} , and $\Omega_{\ell_0} \subset \Omega_{\ell_0+1}$ (since ω_1 is convex and containing 0). Then, by the Cauchy-Schwarz inequality we get

$$\begin{aligned} \int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) |\nabla(u_\ell^n - u_\infty^n)|^2 dx &\leq 2c_0 \|\tilde{A}\|_\infty \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} \rho |\theta_\delta(u_\ell^n - u_\infty^n)| |\nabla(u_\ell^n - u_\infty^n)| dx \\ &\leq 2c_0 \|\tilde{A}\|_\infty \left(\int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) |\nabla(u_\ell^n - u_\infty^n)|^2 dx \right)^{\frac{1}{2}} \times \\ &\quad \left(\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |\theta_\delta(u_\ell^n - u_\infty^n)|^2 \theta'_\delta(u_\ell^n - u_\infty^n)^{-1} dx \right)^{\frac{1}{2}}. \end{aligned}$$

where we have used (4.24). Whence, we get (since $\rho = 1$ on Ω_{ℓ_0})

$$\begin{aligned} \int_{\Omega_{\ell_0}} \theta'_\delta(u_\ell^n - u_\infty^n) |\nabla(u_\ell^n - u_\infty^n)|^2 dx &\leq \int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) |\nabla(u_\ell^n - u_\infty^n)|^2 dx \\ &\leq \left(\frac{4c_0 \|\tilde{A}\|_\infty}{p-1} \right)^2 \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx. \end{aligned}$$

From Hölder's inequality it holds that

$$\begin{aligned} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})}^2 &\leq \left(\int_{\Omega_{\ell_0}} \theta'_\delta(u_\ell^n - u_\infty^n) |\nabla(u_\ell^n - u_\infty^n)|^2 dx \right) \left(\int_{\Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx \right)^{\frac{2-p}{p}} \\ &\leq \left(\frac{4c_0 \|\tilde{A}\|_\infty}{p-1} \right)^2 \left(\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx \right) \left(\int_{\Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx \right)^{\frac{2-p}{p}}. \end{aligned}$$

Passing to the limit as $\delta \rightarrow 0$ (using the Lebesgue theorem) we get

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})}^2 \leq C \left(\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |u_\ell^n - u_\infty^n|^p dx \right) \times \left(\int_{\Omega_{\ell_0}} |u_\ell^n - u_\infty^n|^p dx \right)^{\frac{2-p}{p}},$$

where we have used $0 \leq \rho \leq 1$. Notice that in the rest of this proof C is a generic positive constant independent of n and ℓ . Using the following Poincaré's inequality

$$\|u_\ell^n - u_\infty^n\|_{L^p(\Omega_{\ell_0})} \leq C_{\omega_2} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})},$$

we get

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})}^p \leq C \|u_\ell^n - u_\infty^n\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})}^p.$$

Using the following Poincaré's inequality

$$\|u_\ell^n - u_\infty^n\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})} \leq C_{\omega_2} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})},$$

we get

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})}^p \leq C \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})}^p.$$

Whence

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})}^p \leq \frac{C}{C+1} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0+1})}^p.$$

Let $\alpha \in (0, 1)$ and $\ell \geq \frac{1}{1-\alpha}$. Iterating the previous formula starting from $\alpha\ell$ we get

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell})}^p \leq \left(\frac{C}{C+1}\right)^{[(1-\alpha)\ell]} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell + [(1-\alpha)\ell]})}^p,$$

where $[(1-\alpha)\ell]$ is the integer part of $(1-\alpha)\ell$. Whence

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell})} \leq C e^{-c(1-\alpha)\ell} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_\ell)}, \quad (4.28)$$

where $c > 0$ is independent of ℓ and n . Now we have to estimate the right hand side of (4.28). Testing with $\theta(u_\ell^n)$ in the approximated problem associated to (4.23) one can obtain as in Section 1

$$\|\nabla u_\ell^n\|_{L^p(\Omega_\ell)} \leq C \ell^{\frac{q}{2}}. \quad (4.29)$$

Similarly, testing with $\theta(u_\infty^n)$ in the approximated problem associated to (4.26). we get

$$\|\nabla u_\infty^n\|_{L^p(\Omega_\ell)} \leq C \ell^{\frac{q}{2}}. \quad (4.30)$$

Apply the triangle inequality to the right hand side of (4.28), and use (4.29), (4.30). Passing to the limit as $n \rightarrow \infty$ we obtain that

$$\forall \ell \geq \frac{1}{1-\alpha} : \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell})} \leq C e^{-c(1-\alpha)\ell} \quad (4.31)$$

Since the left hand side of (4.31) is bounded when ℓ goes through the interval $(0, \frac{1}{1-\alpha})$, then we get

$$\forall \ell > 0 : \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell})} \leq C e^{-c\ell},$$

for some positives C and c independent of ℓ . □

3 Non standard neutron transport equation with L^p data

Suppose that $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 are two bounded open sets of $\mathbb{R}^q, \mathbb{R}^{N-q}$ respectively. Consider the following nonlinear problem:

$$\begin{cases} -\nabla \cdot (A_\epsilon \nabla u_\epsilon) = f + B(u_\epsilon) \\ u_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases} \quad (4.32)$$

Where $f \in L^p(\Omega)$, $1 < p < 2$ and $B : L^p(\Omega) \rightarrow L^p(\Omega)$ is a continuous nonlinear operator. We suppose that

$$\exists M \geq 0, \forall u \in L^p(\Omega) : \|B(u)\|_{L^p} \leq M \quad (4.33)$$

Proposition 4.1. *Assume (5), (6). Suppose that $f \in L^p(\Omega)$. Suppose that $B : L^p(\Omega) \rightarrow L^p(\Omega)$ is continuous such that (4.33). Then:*

1) *There exists a sequence $(u_\epsilon)_{0 < \epsilon \leq 1} \in W_0^{1,p}(\Omega)^\mathbb{N}$ of entropy solutions to (4.32) which are also weak solutions such that*

$$\epsilon \|\nabla_{X_1} u_\epsilon\|_{L^p(\Omega)}, \|\nabla_{X_2} u_\epsilon\|_{L^p(\Omega)}, \|u_\epsilon\|_{L^p(\Omega)} \leq C_0,$$

where $C_0 \geq 0$ is independent of ϵ (the constant C_0 depends only on Ω, λ, f and M).

2) *If $(u_\epsilon)_{0 < \epsilon \leq 1}$ is a sequence of entropy solutions to (4.32), then we have the above estimates.*

Proof. 1) The existence of u_ϵ is based on the Schauder fixed point theorem, we define the mapping $\Gamma : L^p(\Omega) \rightarrow L^p(\Omega)$ by

$$v \in L^p(\Omega) \rightarrow \Gamma(v) = v_\epsilon \in W_0^{1,p}(\Omega)$$

where v_ϵ is the entropy solution (which is also a weak solution) of the linearised problem:

$$\begin{cases} -\nabla \cdot (A_\epsilon \nabla v_\epsilon) = f + B(v) \\ v_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases} \quad (4.34)$$

Since the entropy solution is unique then Γ is well defined, we can prove that Γ is continuous, from $L^p(\Omega)$ to $L^p(\Omega)$, as a composition of continuous mappings. As in Section 1, Chapter 4 we can obtain the estimates

$$\epsilon \|\nabla_{X_1} v_\epsilon\|_{L^p(\Omega)}, \|\nabla_{X_2} v_\epsilon\|_{L^p(\Omega)}, \|v_\epsilon\|_{L^p(\Omega)} \leq C_0$$

where C_0 is independent of ϵ and v (thanks to (4.33)) Now, for ϵ fixed we define the subset

$$K_\epsilon = \left\{ v \in W_0^{1,p}(\Omega) : \epsilon \|\nabla_{X_1} v\|_{L^p(\Omega)}, \|\nabla_{X_2} v\|_{L^p(\Omega)}, \|v\|_{L^p(\Omega)} \leq C_0 \right\}$$

The subset K_ϵ is convex and compact in $L^p(\Omega)$ thanks to the Sobolev compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. Since C_0 is independent of v then the subset K_ϵ is stable under Γ . Whence Γ admits at least a fixed point $u_\epsilon \in K_\epsilon$, in other words u_ϵ is an entropy solution to (4.32) which is also a weak solution.

2) Let $(u_\epsilon)_{0 < \epsilon \leq 1}$ be a sequence of entropy solutions to (4.32), u_ϵ is the unique entropy solution (which is also a weak solution) to (4.34) with v replaced by u_ϵ and therefore we obtain the desired estimates as proved in 1). \square

Remark 4.2. *In the general case, the uniqueness theory of [31] could not ensure that the entropy solution u_ϵ of (4.32) is unique.*

Now, we assume that for every $E \subset W_p$ bounded in $L^p(\Omega)$ we have

$$\overline{\text{conv}} \{B(E)\} \subset W_2, \quad (4.35)$$

where $\overline{\text{conv}} \{B(E)\}$ is the closed convex-hull of $B(E)$ in $L^p(\Omega)$. Assumption (4.35) has been introduced in Chapter 3 (see assumption (40)). We have the following

Theorem 4.3. *Assume that Ω, A, f satisfy assumptions of Theorem 0.17. Suppose that $B : L^p(\Omega) \rightarrow L^p(\Omega)$ is continuous such that (4.33) and (4.35). Let $(u_\epsilon)_{0 < \epsilon \leq 1} \subset W_0^{1,p}(\Omega)$ be a sequence of entropy solutions to (4.32), then for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} \geq 0$ independent of ϵ such that*

$$\forall \epsilon \in (0, 1] : \|u_\epsilon\|_{W^{1,p}(\Omega')} \leq C_{\Omega'}.$$

Proof. The proof is similar to the one of Lemma 3.5. Let $(\Omega_i)_{i \in \mathbb{N}}$ be an open cover of Ω such that $\overline{\Omega_j} \subset \Omega_{j+1}$. We equip the space $W_{loc}^{1,p}(\Omega)$ with the topology generated by the family of seminorms $(p_j)_{j \in \mathbb{N}}$ defined by

$$p_j(u) = \|u_\epsilon\|_{W^{1,p}(\Omega_j)}$$

We define the family of linear continuous mappings

$$\Lambda_\epsilon : L^p(\Omega) \rightarrow W_{loc}^{1,p}(\Omega)$$

by: $g \in L^p(\Omega)$, $\Lambda_\epsilon(g) = v_\epsilon$ where v_ϵ is the unique entropy solution to

$$\begin{cases} -\nabla \cdot (A_\epsilon \nabla v_\epsilon) = g \\ v_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases}$$

The continuity of Λ_ϵ follows immediately if we observe Λ_ϵ as a composition of the continuous mapping $\Lambda_\epsilon : L^p(\Omega) \rightarrow W^{1,p}(\Omega)$ and the canonical injection $W^{1,p}(\Omega) \rightarrow W_{loc}^{1,p}(\Omega)$. Now, we denote $L^p(\Omega)^{weak}$, $W_{loc}^{1,p}(\Omega)^{weak}$ the spaces $L^p(\Omega)$, $W_{loc}^{1,p}(\Omega)$ equipped with their weak topology respectively. Therefore, $\Lambda_\epsilon : L^p(\Omega)^{weak} \rightarrow W_{loc}^{1,p}(\Omega)^{weak}$ is also continuous. Consider the bounded (in $L^p(\Omega)$) subset

$$E_0 = \{u \in W_p \mid \|u\|_{L^p(\Omega)} \leq C_0\},$$

where C_0 is the constant of Proposition 4.1. Consider the subset $G = f + \overline{conv} \{B(E_0)\}$ where the closure is taken in the L^p -topology. G is convex closed in $L^p(\Omega)$ and it is bounded thanks to (4.33). According to (4.35) we have $G \subset L^p(\omega_2) + W_2$, it follows from Remark 4.1 that for every $g \in G$ the orbit $\{\Lambda_\epsilon g\}_\epsilon$ is bounded in $W_{loc}^{1,p}(\Omega)$, and therefore $\{\Lambda_\epsilon g\}_\epsilon$ is bounded in $W_{loc}^{1,p}(\Omega)^{weak}$. It is clear that the set G is compact in $L^p(\Omega)^{weak}$. Then, it follows by Theorem 3.4 that there exists a bounded subset F in $W_{loc}^{1,p}(\Omega)^{weak}$ such that

$$\forall \epsilon \in (0, 1] : \Lambda_\epsilon(G) \subset F.$$

The boundedness of F in $W_{loc}^{1,p}(\Omega)^{weak}$ implies its boundedness in $W_{loc}^{1,p}(\Omega)$. i.e For every $j \in \mathbb{N}$ there exists $C_j \geq 0$ independent of ϵ such that

$$\forall \epsilon \in (0, 1] : p_j(\Lambda_\epsilon(G)) \leq C_j. \quad (4.36)$$

Let $(u_\epsilon)_\epsilon$ be a sequence of entropy solutions to (4.32), then we have $(u_\epsilon)_\epsilon \subset E_0$ as proved in Proposition 4.1, then $f + B(u_\epsilon) \in G$ and $\Lambda_\epsilon(f + B(u_\epsilon)) = u_\epsilon \in F$ for every $\epsilon \in (0, 1]$, therefore by (4.36) we get

$$\forall \epsilon \in (0, 1] : \|u_\epsilon\|_{W^{1,p}(\Omega_j)} \leq C_j.$$

Whence, for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} \geq 0$ independent of ϵ such that

$$\forall \epsilon \in (0, 1] : \|u_\epsilon\|_{W^{1,p}(\Omega')} \leq C_{\Omega'}.$$

□

Now, we are ready to prove the convergence theorem. Assume that

$$\exists r \in [1, p) \text{ such that } B : L^r(\Omega) \rightarrow L^p(\Omega) \text{ is continuous.} \quad (4.37)$$

Then, we have the following

Theorem 4.4. *Under assumptions of Theorem 4.3, assume in addition (4.37), then there exist $u \in W_0^{1,p}(\Omega; \omega_2)$ and a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ of entropy solutions to (4.32) such that*

$$\begin{aligned} \epsilon_k \nabla_{X_1} u_{\epsilon_k} &\rightharpoonup 0, \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u_0 \text{ in } L^p(\Omega) - \text{weak} \\ \text{and } u_{\epsilon_k} &\rightarrow u_0 \text{ in } L^r(\Omega) - \text{strong} \end{aligned}$$

Moreover u satisfies in $\mathcal{D}'(\omega_2)$ the equation

$$-\nabla_{X_2} \cdot (A_{22} \nabla_{X_2} u(X_1, \cdot)) = f + B(u)(X_1, \cdot),$$

for a.e $X_1 \in \omega_1$.

Proof. The estimations given in Proposition 4.1 show that there exist $u_0 \in W_0^{1,p}(\Omega; \omega_2)$ and a sequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ solutions to (4.32) such that

$$\epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightharpoonup 0, \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u_0 \text{ and } u_{\epsilon_k} \rightharpoonup u_0 \text{ in } L^p(\Omega) - \text{weak} \quad (4.38)$$

Now, we pass will to the limit in the nonlinear term. As in proof of the convergence (3.12) we show by using Theorem 4.3 and the regularity of the Lebesgue measure that there exists a subsequence still labelled $(u_{\epsilon_k})_k$ such that $u_{\epsilon_k} \rightarrow u$ in $L^r(\Omega)$, therefore by using assumption (4.37) we get

$$B(u_{\epsilon_k}) \rightarrow B(u) \text{ in } L^p(\Omega) - \text{strong.} \quad (4.39)$$

Passing to the limit in the weak formulation of (4.32) by using (4.38), (4.39). We deduce

$$-\nabla_{X_2} \cdot (A_{22} \nabla_{X_2} u_0(X_1, \cdot)) = f + B(u_0)(X_1, \cdot), \text{ in } \mathcal{D}'(\omega_2),$$

and the proof of the theorem is completed. \square

We can check as in Section 2 of Chapter 3 that the operator B_2 of Theorem 0.19 satisfies the same assumptions of the operator B of Theorem 4.4. Therefore, Theorem 0.19 is a direct corollary of Theorem 4.4.

Conclusion

Nous terminons ce mémoire par une courte description de quelques travaux en cours, et par la donnée de quelques questions ouvertes.

Les résultats des Chapitres 1 et 2 constituent la base théorique d'une étude numérique (en cours) du problème linéaire (4). Nous construirons un schéma numérique asymptotiquement préservant au sens de J. Sin [19] et nous dériverons des estimations uniformes de l'erreur de la forme

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2}(u_{\epsilon,h} - u_\epsilon)\|_{L^2(\Omega)^{N-q}} = O(h^\alpha),$$

des questions de régularité elliptique sont également étudiée.

Le Théorème 0.3 donne un développement asymptotique d'ordre 0 de la solution u_ϵ dans l'espace $H_0^1(\Omega; \omega_2)$, avec plus de régularité sur f dans la direction X_1 à savoir $f(\cdot, X_2) \in H_0^d(\omega_1)$ nous pouvons obtenir un développement asymptotique (dans le cas du laplacien perturbé) d'ordre $d-1$ dans l'espace $H_0^1(\Omega; \omega_2)$, les arguments utilisent la technique du produit tensoriel introduite dans le Chapitre 1. Nous pensons qu'avec plus de régularité sur les coefficients de A , nous pouvons obtenir un développement asymptotique d'ordre $d-1$ pour le problème général. L'existence d'un développement asymptotique local (i.e. loin du bord) a été étudiée dans [14].

Concernant les questions ouvertes, commençons par le cas des problèmes à données L^p . Nous avons démontré des convergences fortes dans le cas $1 < p < 2$ pour le problème (8). Dans le cas $p > 2$, nous avons une convergence forte triviale dans l'espace $H_0^1(\Omega; \omega_2)$. Par régularité elliptique de degré p la solution du problème perturbé vit dans l'espace $W^{1,p}(\Omega)$. Peut-on utiliser les techniques de la théorie de Caldéron-Zygmund pour démontrer une convergence dans $W^{1,p}(\Omega, \omega_2)$?

Le cas des problèmes à données L^1 est plus délicat. En effet, d'une manière générale la solution entropique u_ϵ ne peut pas converger dans $W_0^{1,r}(\Omega, \omega_2)$ avec $1 \leq r < \frac{N}{N-1}$. Observons l'exemple

suisant :

On choisit un $f_1 \in L^1(0, 1) \setminus (\cup_{1 < r < 2} L^r(0, 1))$. Soit $f_2 \in L^1(0, 1)$. On note $f = f_1 \otimes f_2$. et soit u_2 l'unique solution faible de

$$-\partial_{x_2}^2 u_2 = f_2, \text{ avec } u_2(0) = u_2(1) = 0 \quad (4.40)$$

Soit u_ϵ l'unique solution faible de

$$-\epsilon^2 \partial_{x_1}^2 u_\epsilon - \partial_{x_2}^2 u_\epsilon = f, \text{ avec } u_\epsilon = 0 \text{ sur } \partial(0, 1)^2.$$

Pour tout $r \in [1, 2)$, au moins l'une des deux suites $(\epsilon \partial_{x_1} u_\epsilon)$, $(\partial_{x_2} u_\epsilon)$ n'est pas bornée dans $L^r(\Omega)$. En effet, supposons le contraire, alors on peut extraire une sous suite (u_{ϵ_k}) faiblement convergente dans $W_0^{1,r}((0, 1)^2; (0, 1))$, notons u sa limite. Nous montrons immédiatement que $u(x_1, \cdot)$ est solution faible de (4.40) pour p.p. $x_1 \in (0, 1)$. Ainsi, Par unicité de la solution faible de (4.40) on obtient $u = f_1 \otimes u_2$ ce qui est absurde car $f_1 \otimes u_2 \notin L^r(\Omega)$. Nous pouvons conclure que pour un problème elliptique avec une donnée L^1 générale, il est impossible de prouver que les suites $(\epsilon \nabla_{X_1} u_\epsilon)$, $(\nabla_{X_2} u_\epsilon)$ sont bornées dans $L^r(\Omega)$ simultanément, et donc prouver la convergence dans $L^r(\Omega)$, vers la solution du problème limite, en d'autres termes il est inutile d'utiliser les techniques d'estimation de la théorie L^1 pour chercher des estimations dans $L^r(\Omega)$. Cependant, il reste une seule issue possible, celle de démontrer la convergence (même faible) directement dans $L^1(\Omega)$ et celle-ci reste une question ouverte.

Maintenant, considérons l'écoulement d'un fluide dans un cylindre dont la longueur tend vers l'infinie. Soit $\Omega = (0, \ell) \times \omega$ un Ouvert borné de \mathbb{R}^3 . L'écoulement est modélisée par l'équation de Navier-Stokes: $\partial_t \vec{u} - \nu \Delta \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + \nabla p = \vec{f}$, avec $\nabla \cdot \vec{u} = 0$, $\vec{u} = \vec{u}_0$, $\vec{u} = 0$ sur $\partial\Omega$. Supposons que f ne dépend pas de x_1 . On souhaite étudier le comportement asymptotique de \vec{u} quand ℓ tend vers $+\infty$. Avec le rescaling $\epsilon = \frac{1}{\ell}$. Nous pouvons considérer le problème de perturbations singulières suivant:

$$\partial_t \vec{u}_\epsilon - \nu \Delta_\epsilon \vec{u}_\epsilon + (\vec{u}_\epsilon \cdot \nabla_\epsilon) \vec{u}_\epsilon + \nabla_\epsilon p_\epsilon = \vec{f},$$

avec $\nabla_\epsilon \cdot \vec{u}_\epsilon = 0$, $\vec{u}_\epsilon = \vec{u}_{0,\epsilon}$, $\vec{u}_\epsilon = 0$ sur $\partial((0, 1) \times \omega)$. Ici, $\nabla_\epsilon := \begin{pmatrix} \epsilon \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_2} \end{pmatrix}$, et $\Delta_\epsilon := \epsilon^2 \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$.

Passons à la limite $\epsilon \rightarrow 0$ formellement, nous pouvons conjecturer que le champs de vecteurs ayant comme composantes la deuxième et la troisième composante de \vec{u}_ϵ , tend vers la solution de l'équation

de Navier-Stokes bidimensionnelle sur ω . Afin d'étudier ce problème de convergence rigoureusement, nous pouvons étudier, dans un premier temps, le problème stationnaire. Nous pouvons étudier aussi, à la lumière du Théorème 0.4, le comportement asymptotique du semi-groupe généré par l'opérateur de Stokes perturbé dont le domaine est donné par $\{\vec{u} \in H_0^1((0, 1) \times \omega))^3 : \nabla_\epsilon \cdot \vec{u} = 0\} \cap H^2((0, 1) \times \omega)^3$.

Appendix A

Density lemmas

Let ω_1 and ω_2 be two open bounded subsets of \mathbb{R}^q and \mathbb{R}^{N-q} respectively. Recall that

$$H_0^1(\Omega; \omega_2) = \left\{ u \in L^2(\Omega) \mid \nabla_{X_2} u \in L^2(\Omega)^{N-q}, \text{ and for a.e. } X_1 \in \omega_1, u(X_1, \cdot) \in H_0^1(\omega_2) \right\},$$

normed by $\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)}$. We have the following

Lemma A.1. *The space $H_0^1(\Omega)$ is dense in $H_0^1(\Omega; \omega_2)$.*

Proof. Let $u \in H_0^1(\Omega; \omega_2)$ fixed. Let l be the linear form defined on $H_0^1(\Omega)$ by

$$\forall \varphi \in H_0^1(\Omega) : l(\varphi) = \int_{\Omega} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx.$$

l is continuous on $H_0^1(\Omega)$, indeed we have

$$\forall \varphi \in H_0^1(\Omega) : |l(\varphi)| \leq \|\nabla_{X_2} u\|_{L^2(\Omega)} \|\nabla_{X_2} \varphi\|_{L^2(\Omega)},$$

and then,

$$\forall \varphi \in H_0^1(\Omega) : |l(\varphi)| \leq \|\nabla_{X_2} u\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}.$$

For every $n \in \mathbb{N}^*$, we denote u_n the unique solution to

$$\begin{cases} \frac{1}{n^2} \int_{\Omega} \nabla_{X_1} u_n \cdot \nabla_{X_1} \varphi dx + \int_{\Omega} \nabla_{X_2} u_n \cdot \nabla_{X_2} \varphi dx = l(\varphi), \forall \varphi \in H_0^1(\Omega) \\ u_n \in H_0^1(\Omega), \end{cases} \quad (\text{A.1})$$

where the existence and the uniqueness follow from the Lax-Milgram theorem. Testing with u_n in (A.1) we get, for every $n \in \mathbb{N}^*$

$$\frac{1}{n^2} \int_{\Omega} |\nabla_{X_1} u_n|^2 dx + \int_{\Omega} |\nabla_{X_2} u_n|^2 dx \leq \|\nabla_{X_2} u\|_{L^2(\Omega)} \|\nabla_{X_2} u_n\|_{L^2(\Omega)},$$

then, we deduce that

$$\forall n \in \mathbb{N}^* : \|\nabla_{X_2} u_n\|_{L^2(\Omega)} \leq \|\nabla_{X_2} u\|_{L^2(\Omega)}, \quad (\text{A.2})$$

and

$$\forall n \in \mathbb{N}^* : \frac{1}{n} \|\nabla_{X_1} u_n\|_{L^2(\Omega)} \leq \|\nabla_{X_2} u\|_{L^2(\Omega)}. \quad (\text{A.3})$$

Using (A.2) and Poincaré's inequality we obtain:

$$\forall n \in \mathbb{N}^* : \|u_n\|_{L^2(\Omega)} \leq C_{\omega_2} \|\nabla_{X_2} u\|_{L^2(\Omega)}. \quad (\text{A.4})$$

Reflexivity of $L^2(\Omega)$ shows that there exists, $u_\infty, u'_\infty, u''_\infty \in L^2(\Omega)$ and a subsequence still labelled (u_n) such that

$$u_n \rightharpoonup u_\infty, \nabla_{X_2} u_n \rightharpoonup u'_\infty \text{ and } \frac{1}{n} \nabla_{X_1} u_n \rightharpoonup u''_\infty \text{ in } L^2(\Omega), \text{ weakly.}$$

Using the continuity of derivation on $\mathcal{D}'(\Omega)$ we get

$$u_n \rightharpoonup u_\infty, \nabla_{X_2} u_n \rightharpoonup \nabla_{X_2} u_\infty \text{ and } \frac{1}{n} \nabla_{X_1} u_n \rightharpoonup 0 \text{ in } L^2(\Omega), \text{ weakly.} \quad (\text{A.5})$$

1) we have $u_\infty \in H_0^1(\Omega; \omega_2)$: By the Mazur Lemma, there exists a sequence (U_n) of convex combinations of $\{u_n\}$ such that

$$\nabla_{X_2} U_n \rightarrow \nabla_{X_2} u_\infty \text{ in } L^2(\Omega) \text{ strongly,} \quad (\text{A.6})$$

then by the Lebesgue theorem there exists a subsequence (U_{n_k}) such that:

$$\text{For a.e. } X_1 \in \omega_1 : \nabla_{X_2} U_{n_k}(X_1, \cdot) \rightarrow \nabla_{X_2} u_\infty(X_1, \cdot) \text{ in } L^2(\omega_2) \text{ strongly.} \quad (\text{A.7})$$

Now, since $(U_{n_k}) \in H_0^1(\Omega)^\mathbb{N}$ then

$$\text{For a.e. } X_1 \in \omega_1 : (U_{n_k}(X_1, \cdot)) \in H_0^1(\omega_2)^\mathbb{N}. \quad (\text{A.8})$$

Combining (A.7) and (A.8) we deduce:

$$\text{For a.e. } X_1 \in \omega_1, u_\infty(X_1, \cdot) \in H_0^1(\omega_2),$$

and the proof of $u_\infty \in H_0^1(\Omega; \omega_2)$ is finished.

2) we have $u_\infty = u$: Passing to the limit in (A.1) by using (A.5) we obtain

$$\int_{\Omega} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx, \forall \varphi \in H_0^1(\Omega). \quad (\text{A.9})$$

For every $\varphi_1 \in H_0^1(\omega_1)$ and $\varphi_2 \in H_0^1(\omega_2)$ take $\varphi = \varphi_1 \otimes \varphi_2$ in (A.9) we obtain, for a.e. $X_1 \in \omega_1$

$$\int_{\omega_2} \nabla_{X_2} u_\infty(X_1, \cdot) \cdot \nabla_{X_2} \varphi_2 dX_2 = \int_{\omega_2} \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \varphi_2 dX_2, \forall \varphi_2 \in H_0^1(\omega_2).$$

For a.e. $X_1 \in \omega_1$, take $\varphi_2 = u_\infty(X_1, \cdot) - u(X_1, \cdot)$ (which belongs to $H_0^1(\omega_2)$) in the above equality, we get:

$$\int_{\omega_2} |\nabla_{X_2} (u_\infty(X_1, \cdot) - u(X_1, \cdot))|^2 dX_2 = 0.$$

Integrating over ω_1 we deduce

$$\int_{\Omega} |\nabla_{X_2} (u_\infty - u)|^2 dx = 0.$$

Finally, since $\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)}$ is a norm on $H_0^1(\Omega; \omega_2)$ we get,

$$u_\infty = u. \quad (\text{A.10})$$

Combining (A.6) and (A.10) we get the desired result. \square

Remark A.2. By symmetry, $H_0^1(\Omega)$ is dense in the space

$$H_0^1(\Omega; \omega_1) = \left\{ u \in L^2(\Omega) \mid \nabla_{X_1} u \in L^2(\Omega), \text{ and for a.e. } X_2 \in \omega_2, u(\cdot, X_2) \in H_0^1(\omega_1) \right\},$$

normed by $\|\nabla_{X_1}(\cdot)\|_{L^2(\Omega)}$.

Lemma A.3. The space $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ is dense in $H_0^1(\Omega)$.

Proof. It is well known that $D(\omega_1) \otimes D(\omega_2)$ is dense in $D(\omega_1 \times \omega_2)$. Here, $D(\omega_1 \times \omega_2)$ is equipped with its natural topology (the inductive limit topology). It is clear that the injection of $D(\omega_1 \times \omega_2)$ in $H_0^1(\omega_1 \times \omega_2)$ is continuous, thanks to the inequality

$$\forall u \in D(\Omega) : \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{N \times \text{mes}(\Omega)} \times \left(\max_{1 \leq i \leq N} \sup_{\text{Support}(u)} |\partial_{x_i} u| \right).$$

Hence, by the density rule we obtain the density of $D(\omega_1) \otimes D(\omega_2)$ in $H_0^1(\Omega)$, and the lemma follows. \square

\square

Lemma A.4. *Let $(V_n^{(1)})$ and $(V_n^{(2)})$ be two sequences of subspaces (not necessarily of finite dimension) of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively. If $\cup V_n^{(1)}$ and $\cup V_n^{(2)}$ are dense in $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, then $\text{vect} \left(\bigcup_{n,m} (V_n^{(1)} \otimes V_m^{(2)}) \right)$ is dense in $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ for the induced topology of $H_0^1(\Omega)$. In particular, if $(V_n^{(1)})$ and $(V_n^{(2)})$ are nondecreasing then $\bigcup_n (V_n^{(1)} \otimes V_n^{(2)})$ is dense in $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$.*

Proof. Let us start by a useful inequality. For $u \otimes v$ in $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ we have :

$$\begin{aligned}
 \|u \otimes v\|_{H_0^1(\Omega)}^2 &= \int_{\Omega} |\nabla_{X_1}(u \otimes v)|^2 dx + \int_{\Omega} |\nabla_{X_2}(u \otimes v)|^2 dx \\
 &= \left(\int_{\omega_2} v^2 dX_2 \right) \times \left(\int_{\omega_1} |\nabla_{X_1} u|^2 dX_1 \right) \\
 &\quad + \left(\int_{\omega_1} u^2 dX_1 \right) \times \left(\int_{\omega_2} |\nabla_{X_2} v|^2 dX_2 \right) \\
 &\leq C \|u\|_{H_0^1(\omega_1)}^2 \times \|v\|_{H_0^1(\omega_2)}^2,
 \end{aligned} \tag{A.11}$$

where we have used Fubini's theorem and Poincaré's inequality. Here, $C = C_{\omega_1}^2 + C_{\omega_2}^2 > 0$. Now, fix $\eta > 0$ and let $\varphi \otimes \psi \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$, by density of $\cup V_n^{(1)}$ in $H_0^1(\omega_1)$ there exists $n \in \mathbb{N}$ and $\varphi_n \in V_n^{(1)}$ such that:

$$\|\psi\|_{H_0^1(\omega_2)} \times \|\varphi_n - \varphi\|_{H_0^1(\omega_1)} \leq \frac{\eta}{2\sqrt{C}}.$$

Similarly by density of $\cup V_n^{(2)}$ in $H_0^1(\omega_2)$, there exists $m \in \mathbb{N}$ (which depends on n and ψ) and $\psi_m \in V_m^{(2)}$ such that

$$\|\varphi_n\|_{H_0^1(\omega_1)} \times \|\psi_m - \psi\|_{H_0^1(\omega_2)} \leq \frac{\eta}{2\sqrt{C}}.$$

Whence, by using the triangle inequality and (A.11) we obtain

$$\|\varphi \otimes \psi - \varphi_n \otimes \psi_m\|_{H_0^1(\Omega)} \leq \eta. \tag{A.12}$$

Now, since every element of $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ could be written as $\sum_{i=1}^l \varphi_i \otimes \psi_i$, then by using the triangle inequality and using (A.12) with η replaced by $\frac{\eta}{l}$, one gets the desired result. \square

Corollary A.5. *$\text{vect} \left(\bigcup_{n,m} (V_n^{(1)} \otimes V_m^{(2)}) \right)$ is dense in $H_0^1(\Omega)$. in particular, if $(V_n^{(1)})$ and $(V_n^{(2)})$ are nondecreasing, then $\bigcup_n (V_n^{(1)} \otimes V_n^{(2)})$ is dense in $H_0^1(\Omega)$.*

Appendix B

Semigroups

Lemma B.1. *Assume (5), (6), (24) and let $f_1 \in L^2(\omega_1)$, $f_2 \in L^2(\omega_2)$, then for every $\mu > 0$ we have:*

$$R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes (R_{0,\mu}f_2).$$

Notice that $R_{0,\mu}f_2 \in H_0^1(\omega_2)$. Moreover, we have

$$\mathcal{A}_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes (\mathcal{A}_{0,\mu}f_2).$$

Notice also that $\mathcal{A}_{0,\mu}f_2 \in L^2(\omega_2)$. Here, $\mathcal{A}_{0,\mu}$ is the Yosida approximation of \mathcal{A}_0 , recall that $\mathcal{A}_{0,\mu} = \mu\mathcal{A}_0R_{0,\mu}$.

Proof. Let $v_2 \in H_0^1(\omega_2)$ be the unique solution in $H_0^1(\omega_2)$ to

$$\mu \int_{\omega_2} v_2 \varphi_2 dX_2 + \int_{\omega_2} A_{22}(X_2) \nabla_{X_2} v_2 \cdot \nabla_{X_2} \varphi_2 dX_2 = \int_{\omega_2} f_2 \varphi_2 dX_2, \quad \forall \varphi_2 \in H_0^1(\omega_2), \quad (\text{B.1})$$

Let $\varphi \in H_0^1(\Omega; \omega_2)$, then $\varphi(X_1, \cdot) \in H_0^1(\omega_2)$ for a.e. $X_1 \in \omega_1$. Let $f_1 \in L^2(\omega_1)$, multiplying (B.1) by f_1 , testing in (B.1) with $\varphi(X_1, \cdot)$ and integrating over ω_1 yields

$$\mu \int_{\Omega} f_1 v_2 \varphi dx + \int_{\Omega} A_{22}(X_2) \nabla_{X_2}(f_1 v_2) \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f_1 f_2 \varphi dx.$$

It is clear that $f_1 v_2 \in H_0^1(\Omega; \omega_2)$ whence, $R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes v_2$, in particular when $f_1 = 1$ we have $R_{0,\mu}(f_2) = v_2$, and therefore $R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes R_{0,\mu}(f_2)$. The second assertion follows immediately from the first one, in fact

$$\mathcal{A}_{0,\mu}(f_1 \otimes f_2) = \mu\mathcal{A}_0R_{0,\mu}(f_1 \otimes f_2) = \mu\mathcal{A}_0(f_1 \otimes R_{0,\mu}f_2).$$

We have $R_{0,\mu}f_2 \in D(\mathcal{A}_0) \cap H_0^1(\omega_2)$ then by using (24) we get

$$\mathcal{A}_0(f_1 \otimes R_{0,\mu}f_2) = f_1 \otimes \mathcal{A}_0(R_{0,\mu}f_2),$$

Notice that the operator \mathcal{A}_0 is independent of the X_1 direction and that $\mathcal{A}_0(R_{0,\mu}f_2) \in L^2(\omega_2)$. Finally we get

$$\mathcal{A}_{0,\mu}(f_1 \otimes f_2) = \mu f_1 \otimes \mathcal{A}_0(R_{0,\mu}f_2) = f_1 \otimes \mathcal{A}_{0,\mu}(f_2).$$

□

Now, let $s \geq 0$, $\mu > 0$ and $g \in L^2(\Omega)$. To simplify the notations, we denote $f_g := e^{s\mathcal{A}_{0,\mu}}g$ instead of $f_{g,s,\mu}$.

Lemma B.2. *Assume (5), (6), (24). Let $g = g_1 \otimes g_2 \in L^2(\omega_1) \otimes L^2(\omega_2)$, then for $s \geq 0$, $\mu > 0$ we have:*

$$f_g = g_1 \otimes e^{s\mathcal{A}_{0,\mu}}g_2.$$

Notice that $e^{s\mathcal{A}_{0,\mu}}g_2 \in L^2(\omega_2)$.

Proof. we have

$$f_g = e^{s\mathcal{A}_{0,\mu}}g = \sum_{k=0}^{\infty} \frac{s^k}{k!} \mathcal{A}_{0,\mu}^k g,$$

where the series converges in $L^2(\Omega)$. By an immediate induction we get by using Lemma B.1

$$\forall k \in \mathbb{N} : \mathcal{A}_{0,\mu}^k g = g_1 \otimes \mathcal{A}_{0,\mu}^k g_2,$$

with $\mathcal{A}_{0,\mu}^k g_2 \in L^2(\omega_2)$ for every $k \in \mathbb{N}$, and the Lemma follows. □

Lemma B.3. *Assume (5), (6), (24). Let $g \in H^2(\omega_1) \otimes L^2(\omega_2)$, then for $s \geq 0$, $\mu > 0$, $i, j = 1, \dots, q$ we have $D_{x_i x_j}^2 f_g, D_{x_i} f_g \in L^2(\Omega)$, with:*

$$D_{x_i x_j}^2 f_g = e^{s\mathcal{A}_{0,\mu}}(D_{x_i x_j}^2 g), \quad D_{x_i} f_g = e^{s\mathcal{A}_{0,\mu}}(D_{x_i} g). \quad (\text{B.2})$$

$$\|D_{x_i x_j}^2 f_g\|_{L^2(\Omega)} \leq \|D_{x_i x_j}^2 g\|_{L^2(\Omega)}, \quad \|D_{x_i} f_g\|_{L^2(\Omega)} \leq \|D_{x_i} g\|_{L^2(\Omega)}. \quad (\text{B.3})$$

Proof. 1) Suppose the simple case when $g = g_1 \otimes g_2$. So, let $g = g_1 \otimes g_2 \in H^2(\omega_1) \otimes L^2(\omega_2)$ and let us prove assertions (B.2). By Lemma B.2 we get

$$f_g = g_1 \otimes e^{s\mathcal{A}_0, \mu}(g_2),$$

with $e^{s\mathcal{A}_0, \mu}g_2 \in L^2(\omega_2)$. Hence, for $i, j = 1, \dots, q$ we get $D_{x_i x_j}^2 f_g \in L^2(\Omega)$ and $D_{x_i x_j}^2 f_g = (D_{x_i x_j}^2 g_1) \otimes e^{s\mathcal{A}_0, \mu}g_2$, and applying Lemma B.2 we get

$$D_{x_i x_j}^2 f_g = e^{s\mathcal{A}_0, \mu}(D_{x_i x_j}^2 g).$$

Similarly we get $D_{x_i} f_g = e^{s\mathcal{A}_0, \mu}(D_{x_i} g)$, and assertion (B.2) follows when $g = g_1 \otimes g_2$.

2) Now, let $g \in H^2(\omega_1) \otimes L^2(\omega_2)$, since g is a finite sum of elements of the form $g_1 \otimes g_2$, then by linearity we get $D_{x_i x_j}^2 f_g, D_{x_i} f_g \in L^2(\Omega)$ and

$$D_{x_i x_j}^2 f_g = e^{s\mathcal{A}_0, \mu}(D_{x_i x_j}^2 g), \quad D_{x_i} f_g = e^{s\mathcal{A}_0, \mu}(D_{x_i} g), \quad \text{for } i, j = 1, \dots, q,$$

therefore

$$\|D_{x_i x_j}^2 f_g\|_{L^2(\Omega)} \leq \|e^{s\mathcal{A}_0, \mu}\| \|D_{x_i x_j}^2 g\|_{L^2(\Omega)} \leq \|D_{x_i x_j}^2 g\|_{L^2(\Omega)}, \quad \text{for } i, j = 1, \dots, q,$$

and similarly we obtain the second inequality of (B.3). \square

Lemma B.4. Assume (5), (6), (24) and (26). Let $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$, then for $s \geq 0, \mu > 0$ we have:

$$f_g \in D(\mathcal{A}_0), \quad \mathcal{A}_0(f_g) \in H_0^1(\Omega; \omega_1), \quad \text{and } D_{x_i}(\mathcal{A}_0 f_g) = e^{s\mathcal{A}_0, \mu}(D_{x_i} \mathcal{A}_0 g), \quad i = 1, \dots, q, \quad (\text{B.4})$$

$$\|(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} \leq \|\mathcal{A}_0 g\|_{L^2(\Omega)} \quad \text{and} \quad \|D_{x_i}(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} \leq \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)}, \quad i = 1, \dots, q. \quad (\text{B.5})$$

Proof. 1) Suppose $g = g_1 \otimes g_2 \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ and let us prove (B.4). Since $g \in D(\mathcal{A}_0)$, thanks to (26), then $f_g = e^{s\mathcal{A}_0, \mu}g \in D(\mathcal{A}_0)$ and $\mathcal{A}_0 f_g = e^{s\mathcal{A}_0, \mu}\mathcal{A}_0 g$ (thanks to (1.38)).

Now, we have

$$\mathcal{A}_0 f_g = \mathcal{A}_0(e^{s\mathcal{A}_0, \mu}g) = \mathcal{A}_0(g_1 \otimes e^{s\mathcal{A}_0, \mu}g_2).$$

Notice that, $g_2 \in D(\mathcal{A}_0)$, thanks to (26), then $e^{s\mathcal{A}_0, \mu}g_2 \in D(\mathcal{A}_0)$ (thanks to (1.38)), hence

$$\mathcal{A}_0 f_g = g_1 \mathcal{A}_0 e^{s\mathcal{A}_0, \mu}g_2,$$

where we have used the fact that \mathcal{A}_0 is independent of the X_1 – *direction*. Since $e^{s\mathcal{A}_0, \mu}$ and \mathcal{A}_0 commute on $D(\mathcal{A}_0)$, then

$$\mathcal{A}_0 f_g = g_1 e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g_2.$$

Now, we have $\mathcal{A}_0 g_2 \in L^2(\omega_2)$ then $e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g_2 \in L^2(\omega_2)$ (thanks to Lemma B.2), however $g_1 \in H_0^1(\omega_1)$, then $\mathcal{A}_0 f_g \in H_0^1(\Omega; \omega_1)$. Whence, for $i = 1, \dots, q$ we have

$$D_{x_i}(\mathcal{A}_0 f_g) = D_{x_i} g_1 \otimes e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g_2,$$

and hence by, Lemma B.2 we get

$$\begin{aligned} D_{x_i}(\mathcal{A}_0 f_g) &= e^{s\mathcal{A}_0, \mu} (D_{x_i} g_1 \otimes \mathcal{A}_0 g_2) \\ &= e^{s\mathcal{A}_0, \mu} (D_{x_i} \mathcal{A}_0 g). \end{aligned}$$

(Remark that $D_{x_i} \mathcal{A}_0 g \in L^2(\Omega)$ since $g_1 \in H_0^1(\omega_1)$ and $\mathcal{A}_0 g_2 \in L^2(\omega_2)$).

2) Now, for a general $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$, assertion (B.4) follows by linearity. Finally, we show (B.5). We have

$$\begin{aligned} \|(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} &= \|e^{s\mathcal{A}_0, \mu}(\mathcal{A}_0 g)\|_{L^2(\Omega)} \leq \|e^{s\mathcal{A}_0, \mu}\| \|\mathcal{A}_0 g\|_{L^2(\Omega)} \\ &\leq \|\mathcal{A}_0 g\|_{L^2(\Omega)}. \end{aligned}$$

For $i = 1, \dots, q$ we get

$$\begin{aligned} \|D_{x_i}(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} &= \|e^{s\mathcal{A}_0, \mu}(D_{x_i} \mathcal{A}_0 g)\|_{L^2(\Omega)} \leq \|e^{s\mathcal{A}_0, \mu}\| \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)} \\ &\leq \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)}. \end{aligned}$$

□

Lemma B.5. Assume (5), (6), (24) and (26). Let $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$, then for $s \geq 0$, $\mu > 0$, $i = 1, \dots, q$, $j = q + 1, \dots, N$ we have $D_{x_j} f_g, D_{x_i x_j}^2 f_g \in L^2(\Omega)$ with:

$$\|D_{x_j} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|\mathcal{A}_0 g\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}, \quad \|D_{x_i x_j}^2 f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)} \|D_{x_i} g\|_{L^2(\Omega)}. \quad (\text{B.6})$$

Proof. 1) Let us show the first inequality of (B.6). Suppose $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$. Notice that $g \in D(\mathcal{A}_0)$, thanks to (26), then according to (1.38) we have $f_g \in D(\mathcal{A}_0) \subset H_0^1(\Omega; \omega_2)$, hence for $j \in \{q+1, \dots, N\}$ the ellipticity assumption gives

$$\begin{aligned} \|D_{x_j} f_g\|_{L^2(\Omega)}^2 &\leq \frac{1}{\lambda} \langle -\mathcal{A}_0 f_g, f_g \rangle_{L^2(\Omega)} \\ &\leq \frac{1}{\lambda} \|\mathcal{A}_0 f_g\|_{L^2(\Omega)} \|f_g\|_{L^2(\Omega)}. \end{aligned}$$

We have, $\|\mathcal{A}_0 f_g\|_{L^2(\Omega)} = \|\mathcal{A}_0 e^{s\mathcal{A}_0, \mu} g\|_{L^2(\Omega)} = \|e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g\|_{L^2(\Omega)} \leq \|\mathcal{A}_0 g\|_{L^2(\Omega)}$, and $\|f_g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}$, therefore

$$\|D_{x_j} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|\mathcal{A}_0 g\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

2) Now, let $1 \leq i \leq q$ fixed, then according to Lemma B.3 we have $D_{x_i} f_g = e^{s\mathcal{A}_0, \mu} (D_{x_i} g)$, notice that $D_{x_i} g \in D(\mathcal{A}_0)$ and hence, $D_{x_i} f_g \in D(\mathcal{A}_0)$, in particular $D_{x_i} f_g \in H_0^1(\Omega; \omega_2)$, and for $q+1 \leq j \leq N$ we get

$$\begin{aligned} \|D_{x_j x_i}^2 f_g\|_{L^2(\Omega)}^2 &\leq \frac{1}{\lambda} \langle -\mathcal{A}_0 D_{x_i} f_g, D_{x_i} f_g \rangle_{L^2(\Omega)} \\ &\leq \frac{1}{\lambda} \|\mathcal{A}_0 D_{x_i} f_g\|_{L^2(\Omega)} \|D_{x_i} f_g\|_{L^2(\Omega)}. \end{aligned}$$

We have,

$$\begin{aligned} \|\mathcal{A}_0 D_{x_i} f_g\|_{L^2(\Omega)} &= \|\mathcal{A}_0 e^{s\mathcal{A}_0, \mu} (D_{x_i} g)\|_{L^2(\Omega)} = \|e^{s\mathcal{A}_0, \mu} (\mathcal{A}_0 D_{x_i} g)\|_{L^2(\Omega)} \\ &\leq \|(D_{x_i} \mathcal{A}_0 g)\|_{L^2(\Omega)}. \end{aligned}$$

Finally, by using (B.3) and the above inequality we obtain

$$\|D_{x_j x_i}^2 f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|(D_{x_i} \mathcal{A}_0 g)\|_{L^2(\Omega)} \|D_{x_i} g\|_{L^2(\Omega)}.$$

□

Lemma B.6. *Under assumptions of Lemma B.5, we have for $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$:*

$$f_g \in H_0^1(\Omega) \cap D(\mathcal{A}_0), \tag{B.7}$$

and

$$\operatorname{div}_{X_1}(A_{11} \nabla_{X_1} f), \operatorname{div}_{X_1}(A_{12} \nabla_{X_2} f), \operatorname{div}_{X_2}(A_{21} \nabla_{X_1} f) \in L^2(\Omega). \tag{B.8}$$

Proof. Let us prove (B.7). In Lemma B.4 we proved that $f_g \in D(\mathcal{A}_0)$. Let us show that $f_g \in H_0^1(\Omega)$. Suppose the simple case $g = g_1 \otimes g_2$, we have $f_g = g_1 \otimes e^{s\mathcal{A}_0, \mu} g_2$. Since $g_2 \in D(\mathcal{A}_0)$, then $e^{s\mathcal{A}_0, \mu} g_2 \in D(\mathcal{A}_0)$, in particular we have $e^{s\mathcal{A}_0, \mu} g_2 \in H_0^1(\Omega; \omega_2)$ however, according to Lemma B.2 $e^{s\mathcal{A}_0, \mu} g_2 \in L^2(\omega_2)$, hence $e^{s\mathcal{A}_0, \mu} g_2 \in H_0^1(\omega_2)$. Finally as $g_1 \in H_0^1(\omega_1)$ we get $f_g \in H_0^1(\Omega)$. For a general g in the tensor product space, the proof follows by linearity.

Now, let us show (B.8). According to Lemmas B.3, B.5 all these derivatives $D_{x_i} f_g$, $D_{x_i x_j}^2 f_g$ for $1 \leq i, j \leq q$, and $D_{x_j} f_g$, $D_{x_i x_j}^2 f_g$ for $1 \leq i \leq q$, $q+1 \leq j \leq N$ belong to $L^2(\Omega)$. Whence, combining this with (26) we get (B.8). \square

Appendix C

Existence theorems

Let $V \subset H_0^1(\Omega)$ be a subspace. We consider the problem

$$\begin{cases} \int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u \in V, \end{cases} \quad (\text{C.1})$$

with A_{22} and β as in the introduction.

Proposition C.1. *If V is closed in $H_0^1(\Omega; \omega_2)$, then there exists a solution to (C.1).*

Proof. We consider the perturbed problem

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon}) \varphi dx + \int_{\Omega} \tilde{A}_{\epsilon} \nabla u_{\epsilon} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u_{\epsilon} \in V, \end{cases} \quad (\text{C.2})$$

with

$$\tilde{A}_{\epsilon} = \begin{pmatrix} \epsilon^2 I_q & 0 \\ 0 & A_{22} \end{pmatrix}.$$

The space V is closed in $H_0^1(\Omega)$, thanks to the continuous embedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega; \omega_2)$. The function \tilde{A}_{ϵ} is bounded and coercive, then by using the Schauder fixed point theorem, one can show the existence of a solution u_{ϵ} to (C.2). This solution is unique in V thanks to (10) and coercivity of \tilde{A}_{ϵ} . Testing with u_{ϵ} in (C.2) we obtain

$$\epsilon \|\nabla_{X_1} u_{\epsilon}\|_{L^2(\Omega)}, \|\nabla_{X_2} u_{\epsilon}\|_{L^2(\Omega)}, \|u_{\epsilon}\|_{L^2(\Omega)} \leq C,$$

where C is independent of ϵ , we have used that $\int_{\Omega} \beta(u_{\epsilon}) u_{\epsilon} dx \geq 0$ (thanks to (10)). By using (11), we get

$$\|\beta(u_{\epsilon})\|_{L^2(\Omega)} \leq M(|\Omega|^{\frac{1}{2}} + C).$$

So, there exist $v \in L^2(\Omega)$, $u \in L^2(\Omega)$ with $\nabla_{X_2} u \in L^2(\Omega)$, and a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that

$$\beta(u_{\epsilon_k}) \rightharpoonup v, \epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightharpoonup 0, \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u, u_{\epsilon_k} \rightharpoonup u \text{ in } L^2(\Omega)\text{-weak.} \quad (\text{C.3})$$

Passing to the limit in (C.2) we get

$$\int_{\Omega} v \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V. \quad (\text{C.4})$$

Take $\varphi = u_{\epsilon_k}$ in (C.4) and passing to the limit we get

$$\int_{\Omega} v u dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u dx = \int_{\Omega} f u dx \quad (\text{C.5})$$

Let us consider the quantity

$$\begin{aligned} 0 \leq I_k &= \int_{\Omega} \epsilon^2 |\nabla_{X_1} u_{\epsilon_k}|^2 dx + \int_{\Omega} A_{22} \nabla_{X_2} (u_{\epsilon_k} - u) \cdot \nabla_{X_2} (u_{\epsilon_k} - u) \\ &\quad + \int_{\Omega} (\beta(u_{\epsilon_k}) - \beta(u))(u_{\epsilon_k} - u) dx \\ &= \int_{\Omega} f u_{\epsilon_k} dx - \int_{\Omega} A_{22} \nabla_{X_2} u_{\epsilon_k} \cdot \nabla_{X_2} u dx - \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u_{\epsilon_k} dx \\ &\quad + \int_{\Omega} f u dx - \int_{\Omega} v u dx - \int_{\Omega} \beta(u) u_{\epsilon_k} dx \\ &\quad \quad \quad - \int_{\Omega} \beta(u_{\epsilon_k}) u dx + \int_{\Omega} \beta(u) u dx \end{aligned}$$

Remark that this quantity is nonnegative, thanks to the ellipticity and monotonicity assumptions.

Passing to the limit as $k \rightarrow \infty$ using (C.3), (C.5) we get

$$\lim I_k = 0.$$

Therefore, the ellipticity assumption shows that

$$\|\epsilon_k \nabla_{X_1} u_{\epsilon_k}\|_{L^2(\Omega)}, \|u_{\epsilon_k} - u\|_{L^2(\Omega)}, \|\nabla_{X_2} (u_{\epsilon_k} - u)\|_{L^2(\Omega)} \rightarrow 0, \quad (\text{C.6})$$

and hence, by a contradiction argument one has

$$\beta(u_{\epsilon_k}) \rightarrow \beta(u) \text{ strongly in } L^2(\Omega).$$

Whence (C.4) becomes

$$\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V.$$

Finally, $\|\nabla_{X_2} (u_{\epsilon_k} - u)\|_{L^2(\Omega)} \rightarrow 0$ shows that $u \in H_0^1(\Omega; \omega_2)$, and therefore as V is closed in $H_0^1(\Omega; \omega_2)$

we obtain that $u \in V$. \square

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