Non-archimedean aspects of the SYZ conjecture

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Résumé

Nous définissons une classe de métriques plurisousharmoniques sur l'espace hybride X^{hyb} associé à une dégénérescence polarisée (X, L) de variétés complexes. Une telle métrique hybride induit une métrique psh sur L au sens usuel, ainsi qu'une métrique psh sur l'analytification non archimédienne X^{an} de X par rapport à la valeur absolue t-adique sur $\mathbb{C}((t))$. Nous démontrons que toute métrique plurisousharmonique complexe sur (X, L) admet une extension canonique à l'espace hybride X^{hyb}. Nous étudions en particulier le cas où (Z, L) est une variété torique polarisée, où nous donnons une description combinatoire des métriques toriques hybrides continues psh sur L.
Nous étudions ensuite les dégénérescences maximales $X/\mathbb{D}^*$ de variétés de Calabi-Yau, dans le but de produire une incarnation non archimédienne $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ de la fibration SYZ conjecturale. Pour ce faire, nous tentons dans un premier temps de comprendre la structure affine entière induite sur le squelette $\mathrm{Sk}(\mathscr{X})$ par la rétraction de Berkovich $\rho_{\mathscr{X}}$ associée à un bon R-modèle dlt de X. Cela nous permet de construire, dans le cas des dégénérescences d'hypersurfaces, une fibration SYZ non archimédienne induisant sur $\mathrm{Sk}(X)$ la structure affine entière prédite par le programme de Gross-Siebert. Lorsque la famille est celle des hypersurfaces de Fermat, nous montrons en outre que la métrique de Calabi-Yau non archimédienne est invariante par cette rétraction.
Enfin, nous considérons une dégénérescence de variétés canoniquement polarisées et calculons la limite non archimédienne de la famille des métriques de Kähler-Einstein dans l'espace hybride associé.

Abstract

We define a class of plurisubharmonic metrics on the hybrid space X^{hyb} associated to a polarized degeneration (X, L) of complex manifolds over the punctured disk. Such a psh metric induces by restriction a psh metric on L in the usual sense, as well as a psh metric on the non-archimedean analytification X^{an} of X with respect to the t-adic absolute value on $\mathbb{C}((t))$. We prove that any complex psh metric on (X, L) admits a canonical plurisubharmonic extension to the hybrid space X^{hyb}. We also focus on the case of a complex polarized toric variety (Z, L), where we provide a combinatorial description of continuous plurisubharmonic hybrid toric metrics on L.

We then study maximal degenerations $X/\mathbb{D}^*$ of Calabi-Yau manifolds, with the goal of constructing a non-archimedean avatar $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ of the conjectural SYZ fibration on the complex fibers. To that extend, we study the integral affine structures induced on the skeleton $\mathrm{Sk}(\mathscr{X})$ by the Berkovich retraction $\rho_{\mathscr{X}}$ associated to a good dlt R-model of X. This allows us to construct, in the case of degenerations of hypersurfaces, a non-archimedean SYZ fibration inducing on $\mathrm{Sk}(X)$ the integral affine structure predicted by the Gross-Siebert program. When the family of hypersurfaces is the Fermat one, we furthermore prove that the non-archimedean Calabi-Yau metric is invariant under this retraction.

Finally, we consider degenerations of canonically polarized manifolds and compute the non-archimedean limit of the family of Kähler-Einstein metrics inside the associated hybrid space.

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Introduction en français

Le sujet principal de cette thèse est l'étude des dégénérescences de variétés algébriques complexes - et plus particulièrement les variétés de Calabi-Yau - au moyen d'outils provenant de la géométrie analytique non archimédienne.
La première partie de ce texte fournit une tentative de généralisation de la théorie du pluripotentiel non archimédien, développée dans [BFJ16], [BBJ18], [BE21], [Reb21] dans le cas d'une variété X sur un corps non archimédien, au cas des espaces hybrides. Pour ce faire, nous définissons une classe de métriques (singulières) semi-positives sur un fibré en droites sur l'analytification au sens de Berkovich d'un schéma X sur un anneau de Banach A. Dans le cas où $A = A_r$ est l'anneau des séries convergentes - pour la norme dite hybride sur $\mathbb{C}$ - ceci induit une notion de métrique semi-positive sur la réunion d'une dégénérescence de variétés complexes avec l'espace analytique non-archimédien associé. Nous faisons le lien entre notre définition et diverses constructions apparaissant dans la littérature [BBJ18], [Reb21], [Fav20]. Nous étudions ensuite plus en détail le cas des métriques toriques sur une variété torique, où celles-ci sont encodées par certaines fonctions sur un espace vectoriel réel de dimension finie, et où la condition de semi-positivité de la métrique se traduit par la convexité au sens usuel de la fonction associée.
La seconde partie est consacrée à l'étude du problème suivant: étant donnée $X \longrightarrow \mathbb{D}^*$ une dégénéréscence maximale de variétés de Calabi-Yau, on souhaiterait construire une incarnation non archimédienne $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ de la fibration SYZ conjecturale sur les fibres X_t, $|t| \ll 1$, de la famille. Une telle rétraction devant induire sur $\mathrm{Sk}(X)$ une structure affine entière, singulière en codimension 2, on espère alors que cette rétraction permette d'identifier la métrique de Calabi-Yau non-archimédienne à une solution de l'équation de Monge-Ampère réelle sur le squelette essentiel $\mathrm{Sk}(X)$. Ceci nécessite en particulier de comprendre les singularités des rétractions de Berkovich $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ associée aux R-modèles de X, où $R = \mathbb{C}[[t]]$, étant donné que notre approche pour construire une rétraction SYZ dans les exemples explicites s'appuie sur des recollements de plusieurs rétractions de Berkovich, associée à différents modèles de X. Nous étudions en particulier une certaine classe de dégénérescences d'hypersurfaces dans l'espace projectif, où, en nous appuyant sur les résultats de la première partie et ceux de [Li22], nous parvenons à identifier de manière satisfaisante les solutions des équations de Monge-Ampère réelle et non archimédienne, lorsque le membre de droite est la mesure de Lebesgue sur le squelette essentiel $\mathrm{Sk}(X)$.

Géométrie non archimédienne

Soit $(K, |\cdot|)$ un corps non archimédien complet, c'est-à-dire un corps muni d'une valeur absolue satisfaisant l'inégalité dite ultramétrique:

$$|x + y| \leq \max\{|x|, |y|\},$$

pour tous $x, y \in K$, et telle que K soit en outre complet par rapport à la distance induite. Parmi les exemples de tels corps, on note le corps $\mathbb{Q}_p$ des nombres p-adiques, qui est la complétion de $\mathbb{Q}$ muni de la valeur absolue p-adique $|\cdot|_p = p^{-v_p}$, où v_p est la valuation p-adique; le corps $\mathbb{C}((t))$ des séries de Laurent complexes muni de la valuation t-adique $|\cdot| = e^{-\operatorname{ord}_0}$; ou encore n'importe quel corps k muni de la valeur absolue triviale $|\cdot|_0$, qui est telle que $|x|_0 = 1$ pour tout $x \in k$ non nul.
On pourrait mentionner diverses tentatives (dues par exemple à Tate, Raynaud ou Huber), au cours du 20ème siècle, de produire une théorie satisfaisante de la géométrie analytique sur un corps non archimédien: étant donné une variété projective X/K, on souhaiterait lui associer de manière naturelle un espace analytique X^{an}, jouissant par exemple d'une topologie plus agréable que celle du schéma X/K; de façon analogue à la manière selon laquelle on associe à un schéma de type fini $X/\mathbb{C}$ une variété complexe. Notons notamment que l'approche naïve, qui consisterait à prendre K muni de la distance induite par $|\cdot|$, comme espace topologique sous-jacent à l'analytification de la droite affine $\mathbb{A}^1_K$, n'est pas très fructueuse: l'inégalité ultramétrique implique qu'un tel espace topologique est totalement discontinu. Ainsi, il est nécessaire d'élaborer une théorie plus sophistiquée de la géométrie analytique sur K pour pouvoir obtenir des espaces topologiques aux propriétés plus raisonnables.
Dans ce texte, nous nous en tiendrons à la théorie des espaces K-analytiques développée par Berkovich, qui est celle qui est la plus fidèle à l'intuition provenant des espaces analytiques complexes. De manière naïve, l'idée est la suivante: si X est une variété projective complexe, on peut voir un point fermé $x \in X$ comme une valeur absolue (dégénérée) $f \mapsto |f(x)|$ sur le corps des fonctions $\mathcal{K}(X)$, étendant la valeur absolue euclidienne sur le corps de base $\mathbb{C} \subset \mathcal{K}(X)$, vu comme les fonctions constantes. Supposons maintenant que K est un corps valué complet et X/K une variété quasi-projective. Son analytification X^{an} au sens de Berkovich est alors définie comme une certaine compactification de l'ensemble des valeurs absolues sur son corps de fonctions $\mathcal{K}(X)$ étendant la valeur absolue de base sur K, ce dernier étant muni de la topologie de la convergence simple. Ceci entraîne en particulier que toute fonction méromorphe f sur X, régulière sur un ouvert de Zariski U, induit une fonction continue $x \mapsto |f(x)| := |f|_x$ sur l'ouvert $U^{\mathrm{an}} \subset X^{\mathrm{an}}$. Comme dans le cas complexe, l'analytification X^{an} d'une variété quasi-projective X/K jouit de propriétés topologiques agréables: celle-ci est séparée, localement connexe par arcs, localement contractile et est compacte si et seulement si X/K est propre. Dans le cas où $K = \mathbb{C}$ est le corps des nombres complexes muni de la valeur absolue euclidienne, l'analytification de Berkovich d'une variété algébrique complexe se trouve être son analytification holomorphe au sens usuel.
La théorie de Berkovich s'applique aussi au-dessus d'une base plus générale: si $(A, |\cdot|)$ est un anneau de Banach, il est possible de définir son spectre analytique $\mathscr{M}(A)$ au sens de Berkovich comme l'ensemble des semi-normes multiplicatives sur

A qui sont majorées par la norme de référence. Si $x \in \mathscr{M}(A)$, le noyau $\mathfrak{p}_x$ de la semi-norme $|\cdot|_x$ est un idéal premier de A, et l'on dénote $\mathscr{H}(x)$ la complétion du corps des fraction de l'anneau quotient $A/\mathfrak{p}_x$ par rapport à la valeur absolue $|\cdot|_x$ - muni de cette dernière, il s'agit d'un corps valué complet, appelé corps résiduel de $\mathscr{M}(A)$ en x.
L'analytifié X^{an} d'un A-schéma de type fini est ensuite défini d'une manière analogue au cas où A est un corps, celui-ci est de plus muni d'un morphisme structural $\pi : X^{\mathrm{an}} \longrightarrow \mathscr{M}(A)$, qui est en particulier continu. Sous certaines conditions sur l'anneau de base A, la fibre $X_x = \pi^{-1}(x)$ s'identifie à l'analytifié du changement de base $X_{\mathscr{H}(x)} = X \times_A \mathscr{H}(x)$, par rapport à la valeur absolue $|\cdot|_x$ sur $\mathscr{H}(x)$, si bien que l'on peut penser l'espace analytique X^{an} comme la famille d'espaces analytiques $(X_{\mathscr{H}(x)})_{x\in\mathscr{M}(A)}$ sur différents corps valués - de manière analogue à la façon dont on peut voir un schéma sur A comme une famille de schémas sur les corps résiduels de A, et paramétrée par $\operatorname{Spec} A$.
Supposons par exemple que notre anneau de Banach est $\mathbb{C}^{\mathrm{hyb}} = (\mathbb{C}, |\cdot|_{\mathrm{hyb}})$, où $|z|_{\mathrm{hyb}} = \max\{|z|_0, |z\ |\}$ est la norme *hybride* sur $\mathbb{C}$, à savoir le maximum entre les valeurs absolues triviale et euclidienne. On peut alors montrer que $\mathscr{M}(\mathbb{C}^{\mathrm{hyb}}) = \{|\cdot|^\lambda/\lambda \in [0,1]\}$, où l'on note abusivement $|\cdot|^0 = |\cdot|_0$. Il s'ensuit que si X est une variété algébrique complexe, son analytification X^{hyb} par rapport à la norme hybride - appelée l'espace hybride associé - fibre sur l'intervalle $[0,1]$, et la fibre X^{hyb}_λ au-dessus de $\lambda > 0$ s'identifie naturellement, après renormalisation de la valeur absolue, à l'analytifié holomorphe de X, tandis que la fibre au-dessus de 0 s'identifie à l'analytifié X_0^{an} par rapport à la valeur absolue triviale sur $\mathbb{C}$. Ceci fournit donc une manière naturelle de voir des renormalisations de la variété analytique complexe X^{hol} dégénérer vers l'espace analytique non archimédien X_0^{an}. Ceci justifie la terminologie, l'espace X^{hyb} étant de nature hybride, encodant à la fois la géométrie analytique holomorphe et non archimédienne de X.

Dégénérescences de variétés complexes et espaces hybrides

Soit désormais $X \longrightarrow \mathbb{D}^*$ une dégénérescence de variétés projectives complexes, qu'on suppose telle que les équations des fibres X_t soient méromorphes en $t = 0$. Ceci nous permet de voir X comme une variété algébrique X_K sur le corps des séries de Laurent $K = \mathbb{C}((t))$ qui, comme mentionné précédemment, est muni de la valeur absolue t-adique qui en fait un corps non archimédien complet. Nous allons maintenant expliquer dans quelle mesure l'espace analytique au sens de Berkovich X_K^{an} encode de diverses manières des informations sur le comportement asymptotique en $t = 0$ de la famille X.
Soit $\pi : \mathscr{X} \longrightarrow \mathbb{D}$ un modèle à croisements normaux simples de X - dont l'existence est assurée par notre hypothèse de méromorphie et par le théorème d'Hironaka. Ecrivant la fibre spéciale $\mathscr{X}_0 := \pi^{-1}(0) = \sum_{i\in I} a_i D_i$ comme la somme de ses composantes irréductibles, la condition de croisements normaux simples signifie que les D_i sont des diviseurs premiers, lisses, et dont les diverses intersections sont toutes transverses. Ceci nous permet d'associer à $\mathscr{X}_0$ un complexe cellulaire $\mathscr{D}(\mathscr{X}_0)$, encodant la combinatoire des intersections des D_i. Le complexe dual est défini de la manière suivante: à toute composante D_i de $\mathscr{X}_0$ est associé un sommet v_i; puis pour

toute composante connexe de l'intersection $D_i \cap D_j$ l'on trace une arête e_{ij} entre v_i et v_j. De la même manière, à chaque composante connexe d'une intersection $D_i \cap D_j \cap D_k$ on associe un 2-simplexe, délimité par des arêtes e_{ij}, e_{ik}, e_{jk}, et ainsi de suite pour aboutir à un complexe cellulaire de dimension majorée par la dimension relative de X. L'un des intêrets de cette construction est que le complexe dual $\mathscr{D}(\mathscr{X}_0)$ peut naturellement être plongé dans l'espace analytique X_K^{an}; en d'autres termes il est possible de voir les points de $\mathscr{D}(\mathscr{X}_0)$ comme des valuations sur le corps des fonctions de X. Par exemple, si v_i est un sommet de $\mathscr{D}(\mathscr{X}_0)$ correspondant à une composante D_i, on lui associe la valuation $v_{D_i} := f \mapsto a_i^{-1} \operatorname{ord}_{D_i}(f)$ pour $f \in \mathcal{K}(\mathscr{X}) \simeq \mathcal{K}(X)$, la normalisation assurant que $v_{D_i}(t) = 1$. Les faces de dimension supérieure de $\mathscr{D}(\mathscr{X}_0)$ sont ensuite à leur tour plongées dans X^{an} en interpolant entre les v_{D_i}, l'image de ce plongement est notée $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ et appelée *squelette* du modèle $\mathscr{X}$.
Berkovich définit en outre une rétraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ pour l'inclusion $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$, induisant une équivalence d'homotopie entre X^{an} et $\mathrm{Sk}(\mathscr{X})$. Ceci permet de penser le squelette $\mathrm{Sk}(\mathscr{X})$ comme une approximation finie de X^{an}, ce dernier étant reconstruit (topologiquement) comme la limite inverse des $\mathrm{Sk}(\mathscr{X})$, prise sur l'ensemble des modèles à croisements normaux simples de X.
Fixons désormais un rayon $r \in (0,1)$. Il est possible de définir, d'une manière similaire à la construction du paragraphe précédent, un anneau A_r de séries convergentes pour la norme hybride sur $\mathbb{C}$, dont le spectre de Berkovich $\mathscr{M}(A_r) \simeq \bar{\mathbb{D}}_r$ est homéomorphe au disque fermé de rayon r. On peut voir la dégénérescence $X \longrightarrow \mathbb{D}^*$ comme un schéma sur A_r, et son analytification est appellée l'espace hybride:

$$X^{\mathrm{hyb}} \xrightarrow{\pi} \bar{\mathbb{D}}_r.$$

La fibre $\pi^{-1}(0) = X_K^{\mathrm{an}}$ est alors l'analytification de X par rapport à la valeur absolue t-adique, tandis qu'en dehors de 0, la pré-image $\pi^{-1}(\bar{D}_r^*)$ s'identifie naturellement avec la restriction de la dégénérescence X à $\bar{D}_r^*$ - après avoir renormalisé la valeur absolue sur X_t par une puissance $\frac{\log r}{\log|t|}$. A nouveau, ceci fournit une manière naturelle de voir les variétés analytiques complexes X_t (avec leur valeur absolue renormalisée) dégénérer vers l'espace analytique non-archimédien X_K lorsque $t \to 0$. On s'attend alors à ce que diverses familles d'objets issus de la géométrie complexe vivant sur les X_t admettent une limite sur l'espace hybride - la limite étant plutôt une pente logarithmique en 0, au vu de la renormalisation de la valeur absolue.

Théorie du pluripotentiel non-archimédienne

La théorie du pluripotentiel classique consiste en l'étude des fonctions plurisousharmoniques: si $\Omega \subset \mathbb{C}^n$ est un domaine, une fonction semi-continue inférieurement $\phi : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ est plurisousharmonique (ou psh) si et seulement si pour toute droite affine complexe ℓ, sa restriction ϕ_ℓ à $\ell \cap \Omega$ est sous-harmonique au sens faible, i.e. son laplacien $\Delta\phi_\ell$ au sens des distributions est positif. De manière équivalente, ceci signifie que la Hessienne complexe $i\partial\bar{\partial}\phi \geq 0$ est une $(1,1)$-forme positive au sens des courants. Par exemple, si $f \in \mathcal{O}(\Omega)$ est une fonction holomorphe, la formule de Lelong-Poincaré implique que la fonction $\phi = \log|f|$ est plurisousharmonique sur Ω, et est singulière précisément sur le lieu $\{f = 0\}$. Les fonctions plurisousharmoniques

peuvent alternativement être définies à l'aide d'une inégalité de la moyenne, ce qui implique que celles-ci sont stables par limites décroissantes et maxima finis.
Cette notion se généralise alors au cadre global, de la manière suivante. Soit X une variété complexe, L un fibré en droites ample sur X et h une métrique hermitienne sur L. On identifie cette dernière à la collection ϕ de ses poids locaux logarithmiques $\phi_i = -\log|s_i|_h$ contre les trivialisations locales s_i de L, et l'on dit que la métrique ϕ est plurisousharmonique si et seulement si ses poids locaux le sont. Comme dans l'exemple qui précède, si $s \in H^0(X, mL)$ est une section globale non-nulle de mL pour un $m \geq 1$, la métrique singulière $\phi = m^{-1}\log|s|$ est plurishousharmonique, à nouveau par la formule de Lelong-Poincaré.
Un des théorèmes centraux de la théorie du pluripotentiel est le résultat de régularisation suivant, dû à Demailly [Dem92]: toute métrique psh ϕ sur un fibré en droites ample peut être écrite comme limite décroissante de métriques de la forme:

$$\phi_{\mathrm{FS}} = \frac{1}{2m}\log(|s_0|^2 + ... + |s_N|^2),$$

où $m \geq 1$ et $(s_0, ..., s_N)$ est une famille de sections globales de mL sans zéros communs. Ces métriques sont nommées métriques de Fubini-Study sur L, pour la raison suivante; si $f : X \longrightarrow \mathbb{CP}^N$ est l'application holomorphe donnée par $f(x) = [s_0(x) : ... : s_N(x)]$, alors ϕ est à un facteur près le tiré en arrière par f de la métrique de Fubini-Study sur $\mathbb{CP}^N$.
Revenons maintenant au cas où K est un corps non archimédien complet. Si (X, L) est une variété projective sur K, on peut définir de manière analogue à ce qui précède des métriques continues sur l'analytification L^{an}: étant donnée une famille (U_i, s_i) de trivialisations locales algébriques de L, une métrique continue ϕ est la donnée d'une famille de poids locaux $\phi_i = -\log|s_i|_\phi$ continus sur les U_i^{an}, compatibles sur les intersections $(U_i \cap U_j)^{\mathrm{an}}$ et compatibles avec la multiplication des sections par des fonctions.
On peut de même définir une notion de métrique singulière sur L, de telle sorte que si s est une section globale non-nulle de mL, alors $m^{-1}\log|s|$ définit une métrique sur L, singulière le long du lieu des zéros de s. Poussant l'analogie plus loin, on dit qu'une métrique de la forme:

$$\phi_{\mathrm{FS}} = m^{-1}\log\max(|s_0|, ..., |s_N|),$$

pour une famille $(s_0, ..., s_N)$ de sections globales de mL sans zéros communs, est une métrique de Fubini-Study (tropicale) sur L. Le théorème de régularisation de Demailly est alors pris comme définition des métriques plurisousharmoniques sur L:

Définition 0.0.1. *La classe* $\mathrm{PSH}(X, L)$ *des métriques plurisousharmoniques sur* L *est la plus petite classe de métriques singulières qui:*

- *contienne les métriques de Fubini-Study tropicales,*
- *soit stable par addition de constantes,*
- *soit stable par maxima finis,*
- *soit stable par limites décroissantes.*

Lorsque le corps K est non archimédien, de valuation v_K que l'on suppose non triviale pour simplifier, une classe naturelle de métriques sur L est définie de la manière suivante. Soit $(\mathscr{X}, \mathscr{L})$ un modèle de (X, L) sur l'anneau de valuation $R = \{v_K \geq 0\}$. De manière simpliste, la métrique modèle $\phi_{\mathscr{L}}$ est la métrique telle que les sections locales de L de norme ≤ 1 soient précisément celles qui s'étendent à $\mathscr{X}$ en tant que sections régulières de $\mathscr{L}$. En particulier, le réseau $H^0(\mathscr{X}, \mathscr{L}) \subset H^0(X, L)$ est la boule unité pour la norme du sup associée à $\phi_{\mathscr{L}}$. Il se trouve alors que les métriques de Fubini-Study sur L ne sont autres que les métriques modèles sur (les puissances positives) de L, associées à des modèles semi-amples $\mathscr{L}$, c'est-à-dire tels que $m\mathscr{L}$ soit globablement engendré pour m suffisamment divisible.

L'opérateur de Monge-Ampère

Soient $\Omega \subset \mathbb{C}^n$ un domaine, et $\phi_i \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ pour $i = 1, ..., n$ une collection de n fonctions psh lisses. Leur mesure de Monge-Ampère (mixte) est définie comme:

$$\mathrm{MA}(\phi_1, ..., \phi_n) = i\partial\bar{\partial}\phi_1 \wedge ... \wedge i\partial\bar{\partial}\phi_n,$$

et constitue l'un des objets centraux de la théorie du pluripotentiel. On souhaiterait alors étendre l'opérateur de Monge-Ampère aux fonctions psh qui ne sont plus nécéssairement lisses; cependant même si $i\partial\bar{\partial}\phi$ est défini pour une fonction psh quelconque, il s'agit désormais d'un courant et non plus d'une forme différentielle, si bien qu'il n'est *a priori* pas possible de considérer des produits d'expressions de la forme $i\partial\bar{\partial}\phi_i$. Cependant, les travaux fondateurs de Bedford-Taylor [BT76] ont montré que l'opérateur de Monge-Ampère pouvait être étendu de manière unique aux fonctions psh bornées - et en particulier, aux fonctions psh continues. Il peut en outre s'étendre plus généralement à la classe des fonctions psh d'énergie finie par [BEGZ10], que nous ne définirons pas ici.
Dans un cadre global, soit X une variété projective complexe de dimension n, et $L_1, ..., L_n$ des fibrés en droites amples sur X. Etant donné des métriques continues psh ϕ_i sur les L_i, l'on définit leur mesure de Monge-Ampère comme ci-dessus; nous continuerons de la dénoter $i\partial\bar{\partial}\phi_1 \wedge ... \wedge i\partial\bar{\partial}\phi_n$. Il s'agit d'une mesure positive sur X, dont la masse totale est (à normalisation près) le nombre d'intersection $(L_1 \cdot ... \cdot L_n)$.
Le théorème suivant, connu initialement sous le nom de conjecture de Calabi, est dû à Yau [Yau78] dans le cas lisse et à Kołodziej [Kol98] dans le cas singulier:

Théorème 0.0.2. *Soit X une variété projective complexe lisse, et L un fibré en droites ample sur X. Soit μ une mesure de probabilité sur X, à densité L^p pour un $p > 1$.*
Alors il existe une unique (à une constante additive près) métrique $\phi \in \mathrm{PSH}(X, L)$ solution de l'équation de Monge-Ampère complexe:

$$(i\partial\bar{\partial}\phi)^n = C\mu,$$

où $C = (2\pi)^n L^n$.

Notons que le résultat de Yau garantit que si la mesure μ est une forme volume lisse, alors la solution ϕ de l'équation ci-dessus est lisse et strictement psh.

La motivation originelle pour ce théorème était la question de l'existence de métriques kähleriennes Ricci-plates sur les variétés de Calabi-Yau: si (X, L) est une variété de Calabi-Yau polarisée, c'est-à-dire telle que $K_X = \mathcal{O}_X$, alors la solution lisse $\phi \in \mathrm{PSH}(X, L)$ de l'équation de Monge-Ampère:

$$(i\partial\bar{\partial}\phi)^n = Ci^{n^2}\Omega \wedge \bar{\Omega},$$

où $\Omega \in H^0(X, K_X)$ est une n-forme holomorphe ne s'annulant pas, est telle que la form de courbure $\omega = i\partial\bar{\partial}\phi$ est une métrique de Kähler sur X de courbure de Ricci nulle.
Revenons maintenant au cas non archimédien, et supposons que $K = k((t))$ pour simplifier l'exposition; notons $R = k[[t]]$ et supposons en outre que k est de caractéristique zéro. Étant donnée une variété polarisée (X, L) sur K, l'analogue des métriques lisses sur L est fourni par les métriques de Fubini-Study qui, comme mentionné précédemment, sont aussi les métriques modèles $\phi_{\mathscr{L}}$ associées aux R-modèles nef $(\mathscr{X}, \mathscr{L})$ de (X, L). La mesure de Monge-Ampère mixte d'une collection $(\phi_{\mathscr{L}_1}, ..., \phi_{\mathscr{L}_n})$ de métriques modèles est alors définie comme une somme explicite de masses de Dirac, dont les coefficients sont divers nombres d'intersection des $\mathscr{L}_i$ sur $\mathscr{X}$; bien que cela ne soit pas apparent à première vue, cette définition imite la définition de l'opérateur de Monge-Ampère réel sur les fonctions convexes, affine par morceaux - un fait notable étant que la mesure atomique $\mathrm{MA}(\phi_{\mathscr{L}_1}, ..., \phi_{\mathscr{L}_n})$ est supportée sur le squelette $\mathrm{Sk}(\mathscr{X})$, et est de masse totale $(L_1 \cdot ... \cdot L_n)$, comme dans le cas complexe.
L'opérateur de Monge-Ampère est ensuite étendu aux métriques continues psh dans [BFJ15], où il est établi que l'analogue non archimédien de la conjecture de Calabi est vrai:

Théorème 0.0.3. *Soit (X, L) une variété polarisée lisse sur K. Soit μ une mesure de probabilité sur X^{an}, supportée sur le squelette d'un R-modèle à croisements normaux simples de X. Alors il existe une unique (à une constante additive près) métrique plurisousharmonique continue ϕ sur L satisfaisant l'équation de Monge-Ampère non archimédienne:*

$$\mathrm{MA}(\phi) = \mu.$$

Ici nous avons écrit $\mathrm{MA}(\phi) := \mathrm{MA}(\phi, ..., \phi)$.
La preuve du théorème ci-dessus repose cependant sur un argument de nature variationelle, si bien qu'elle ne donne pas d'informations sur la nature de la solution, hormis sa continuité. Il serait notamment souhaitable, comme nous le verrons par la suite, de traduire cette équation en une équation de Monge-Ampère réelle sur le squelette $\mathrm{Sk}(\mathscr{X})$ sur lequel la mesure μ est supportée, ce qui s'avère être une question difficile.

La conjecture de Kontsevich-Soibelman

Nous allons maintenant présenter un cercle d'idées, initié au début des années 2000 par Kontsevich et Soibelman comme une tentative d'explication du phénomène de symétrie miroir, et qui prédit notamment le comportement asymptotique de la

famille des métriques kähleriennes Ricci-plates sur une famille de variétés de Calabi-Yau dont la structure complexe dégénère de la pire manière possible.

L'heuristique SYZ classique

Soit $X \longrightarrow \mathbb{D}^*$ une famille relativement projective, méromorphe en 0 de variétés de Calabi-Yau, et rappelons que cela signifie pour nous que le fibré canonique K_{X_t} de chaque fibre X_t est trivial, pour $t \in \mathbb{D}^*$. Fixons une polarisation relative L sur la famille X, et supposons en outre que la famille est maximalement dégénérée en $t = 0$, ce qui signifie que les fibres X_t se cassent en le plus de morceaux possibles lorsque $t \longrightarrow 0$. Plus formellement, cela signifie que le complexe dual $\mathscr{D}(\mathscr{X}_0)$ de tout modèle à croisements normaux de X est de dimension maximale $n = \dim X_t$. Dans ce contexte, l'approche de Kontsevich-Soibelman [KS06] de la conjecture de Strominger-Yau-Zaslow [SYZ96] prédit que les variétés de Calabi-Yau (X_t, ω_t) - où ω_t est la métrique kählerienne Ricci-plate sur X_t, normalisée par la condition $\omega_t \in c_1(L_t)$ - ont un diamètre d_t dont l'ordre de grandeur est $\big|\log|t|\big|^{1/2}$, ce qui a été confirmé récemment par les résultats de Li-Tosatti [LT20].
De plus, il est prédit que la fibre X_t munie de la métrique de Kähler $\tilde{\omega}_t := d_t^{-2}\omega_t$ devrait ressembler à l'espace total d'une fibration lagrangienne dont les fibres sont des tores réels $\mathbb{T}^n$ de taille d_t^{-1}, au-dessus d'une certaine base B de dimension réelle n. En particulier, la conjecture de Kontsevich-Soibelman prédit que la famille d'espaces métriques $(X_t, \tilde{\omega}_t)$ converge au sens de Gromov-Hausdorff vers la base B, munie d'une métrique de Monge-Ampère réelle en dehors d'un lieu de codimension de Hausdorff 2.
Plus précisement, la construction classique des coordonnées action-angle en géométrie symplectique implique que la fibration spéciale lagrangienne $f_t : X_t \longrightarrow B$ munit l'ouvert $B^{\mathrm{sm}} \subset B$ d'une structure affine entière $\nabla_t^{\mathbb{Z}}$, celui-ci étant l'ouvert de B au-dessus duquel la fibration f_t est submersive. La structure affine entière $\nabla_t^{\mathbb{Z}}$ consiste en la donnée d'un atlas de cartes dont les fonctions de transition sont des fonctions affines dont la partie linéaire est entière, autrement dit des éléments du groupe $\mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$, et celle-ci permet notamment de parler de fonctions affines entières sur B^{sm}. Les fibres de f_t étant asymptotiquement très petites, le potentiel ϕ_t de la métrique de Calabi-Yau devrait être proche asymptotiquement d'un potentiel $\psi_t \circ f_t$ provenant de la base, où ψ_t est une fonction strictement convexe (multivaluée) en cordonnées affines sur B^{sm}.
On s'attend alors à ce qu'à la limite $t \to 0$, les fonctions ψ_t convergent vers une fonction convexe ψ sur B^{sm}, ce lieu étant muni d'une structure affine entière que l'on peut penser comme la limite tropicale de la structure complexe sur X_t. Enfin, l'équation de Monge-Ampère complexe sur X_t se traduit à la limite par l'équation de Monge-Ampère réelle:

$$\det\Big(\frac{\partial^2\phi}{\partial y_i \partial y_j}\Big) = C,$$

où C est une constante et les y_i sont des coordonnées affines locales. Il s'ensuit que la métrique limite g_B sur B^{sm} est localement donnée par la Hessienne $g_{ij} = \partial^2_{ij}\psi$ du potential local ψ.
Faisons maintenant le lien avec la géométrie non archimédienne.

Le squelette essentiel

Les métriques de Calabi-Yau $\omega_t \in c_1(L_t)$ étant obtenues en résolvant l'équation de Monge-Ampère complexe:

$$\omega_t^n = C_t i^{n^2} \Omega_t \wedge \bar{\Omega}_t,$$

où $\Omega_t \in H^0(X_t, K_{X_t})$ est une n-forme holomorphe sur X_t ne s'annulant pas, une première étape naturelle pour tenter de comprendre leur comportement asymptotique lorsque $t \to 0$ est d'étudier le comportement de la famille de mesures de probabilité $\mu_t = C_t i^{n^2} \Omega_t \wedge \bar{\Omega}_t$, qui sont de nature plus algébrique. Il se trouve alors que ces mesures convergent faiblement dans l'espace hybride X^{hyb} associé à la dégénérescence [BJ17]:

Théorème 0.0.4. *Soit $(X, L) \longrightarrow \mathbb{D}^*$ une dégénérescence maximale polarisée de variétés de Calabi-Yau , et soit $(\mu_t)_{t\in\mathbb{D}^*}$ la famille des mesures (de probabilité) de Calabi-Yau. Alors les μ_t convergent faiblement sur X^{hyb} vers une mesure μ_0, qui est une mesure de type Lebesgue sur un sous-complexe $\mathrm{Sk}(X)$ de dimension n de $\mathrm{Sk}(\mathscr{X})$, pour tout modèle $\mathscr{X}/\mathbb{D}$ à croisements normaux simples.*

Le sous-complexe $\mathrm{Sk}(X)$ dans l'énoncé ci-dessus est appelé *squelette essentiel* de X, et peut-être calculé explicitement dans $\mathrm{Sk}(\mathscr{X})$, étant donné un modèle $\mathscr{X}/\mathbb{D}$ [MN15]. Comme la notation le suggère, il est en outre indépendant du choix d'un modèle $\mathscr{X}$ à croisements normaux.
Ce résultat suggère alors que la base B de la fibration SYZ devrait être le squelette essentiel $\mathrm{Sk}(X) \subset X^{\mathrm{an}}$, et l'on aimerait alors produire une incarnation non archimédienne:

$$\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X),$$

de la fibration SYZ, qui soit de manière naturelle la limite dans X^{hyb} des fibrations SYZ archimédiennes $f_t : X_t \longrightarrow B$; cela est établi par exemple dans le cas où X est une variété abélienne maximalement dégénérée par [Oda18], [GO22]. Poursuivant notre analogie avec la fibration SYZ classique, il est naturel de souhaiter que l'application continue $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ soit une *fibration en tores affinoïdes* hors d'un lieu affine par morceaux $\Gamma \subset \mathrm{Sk}(X)$ de codimension au moins 2 - sans donner de définition précise, il s'agit de l'analogue non-archimédien d'une fibration en tores réels lisse; par exemple, une telle fibration induit sur $\mathrm{Sk}(X) \setminus \Gamma$ une structure affine entière, comme dans le cas classique.
Une première tentative est fournie par les travaux de Nicaise-Xu-Yu [NX16], [NXY19]: si $\mathscr{X}/\mathbb{D}$ est un modèle minimal de la famille X, au sens du programme du modèle minimal, alors on a l'égalité $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$, et il est établi dans [NXY19] que la rétraction de Berkovich:

$$\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$$

est bien une fibration en tores affinoïdes en dehors des faces de codimension ≥ 2 de $\mathrm{Sk}(\mathscr{X}) = \mathrm{Sk}(X)$.

La propriété de comparaison pour Monge-Ampère

In fine, l'un des objectifs centraux de la théorie est de produire la métrique de Monge-Ampère réelle ψ sur $\mathrm{Sk}(X)$ via la géométrie non archimédienne. La convergence

sur X^{hyb} des mesures de Calabi-Yau vers μ_0 suggère fortement que la solution de l'équation de Monge-Ampère non archimédienne:

$$\mathrm{MA}(\phi) = \mu_0$$

devrait fournir la limite non archimédienne des métriques de Calabi-Yau. La difficulté consiste maintenant à essayer d'identifier ϕ de manière naturelle à une fonction convexe multivaluée sur $\mathrm{Sk}(X)$, solution de l'équation de Monge-Ampère réelle. Comme dans le cas archimédien, on cherche alors à produire une fibration SYZ $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ telle que la métrique ϕ s'identifie via ρ avec le tiré en arrière d'une fonction convexe multivaluée sur la base, puis à montrer que ceci identifie l'opérateur de Monge-Ampère non archimédien avec l'opérateur de Monge-Ampère réel sur $\mathrm{Sk}(X) \setminus \Gamma$.
Le théorème suivant, dû à Vilsmeier [Vil20], fournit un progrès partiel dans cette direction:

Théorème 0.0.5. *Soit $(\mathscr{X}, \mathscr{L})$ un modèle semi-stable de (X, L), et écrivons la métrique de Calabi-Yau non archimédienne ϕ comme:*

$$\phi = \phi_{\mathscr{L}} + \psi,$$

où $\phi_{\mathscr{L}}$ est la métrique modèle associée à $\mathscr{L}$ et $\psi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ est une fonction continue. Supposons $\psi = \psi \circ \rho_{\mathscr{X}}$ au-dessus de l'intérieur d'une face $\tau \subset \mathrm{Sk}(\mathscr{X})$ de dimension n. Alors ψ est convexe sur $\mathrm{Int}(\tau)$, et on a l'égalité de mesures suivante:

$$\mathbf{1}_{\mathrm{Int}(\tau)}\, \mathrm{MA}(\phi) = n! \mathscr{M}(\psi),$$

sur $\mathrm{Int}(\tau)$, où $\mathscr{M}(\psi)$ est la mesure de Monge-Ampère réelle de la fonction convexe ψ.

S'appuyant sur ce résultat, les travaux récents de Yang Li [Li20a] réduisent l'existence de la fibration SYZ classique au problème suivant: trouver une fibration SYZ non archimédienne ρ, telle que la métrique de Calabi-Yau non-archimédienne soit constante le long des fibres de ρ:

Théorème 0.0.6. *Soit $X/\mathbb{D}^*$ une dégénérescence maximale polarisée de variétés de Calabi-Yau, et écrivons la solution de l'équation de Monge-Ampère non archimédienne $\mathrm{MA}(\phi) = \mu_0$ sous la forme:*

$$\phi = \phi_{\mathscr{L}} + \psi,$$

pour un modèle à croisements normaux $(\mathscr{X}, \mathscr{L})$ de (X, L), avec $\psi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ une fonction continue. Supposons que la propriété de comparaison pour Monge-Ampère:

$$\psi = \psi \circ \rho_{\mathscr{X}}$$

soit vérifiée au-dessus de l'intérieur des faces de dimension maximale de $\mathrm{Sk}(X)$. Alors pour $|t| \ll 1$, il existe une fibration spéciale lagrangienne $f_t : U_t \longrightarrow \mathrm{Sk}(X)$, où $U_t \subset X_t$ est un ouvert de mesure de Calabi-Yau arbitrairement proche de 1.

Ce résultat réduit la conjecture SYZ classique à un problème de géométrie non archimédienne; cependant il ne traite pas la quesion de la convergence de Gromov-Hausdorff globale des métriques de Calabi-Yau; en effet l'union des faces maximales de $\mathrm{Sk}(X)$ n'étant pas connexe, pour pouvoir accéder à la distance entre deux faces différentes, il semble nécessaire d'obtenir une propriété de comparaison entre les opérateurs de Monge-Ampère réels et non archimédiens en codimension 1- toujours sous une certaine hypothèse d'invariance par rétraction de la métrique de Calabi-Yau non archimédienne sur X^{an}.

Résumé des résultats obtenus

Théorie du pluripotentiel sur les espaces hybrides

Soit $X \longrightarrow \mathbb{D}^*$ une dégénérescence projective de variétés complexes, et L un fibré en droites relativement ample sur X. Etant donné une métrique plurisousharmonique $\phi \in \mathrm{PSH}(X, L)$ satisfaisant une condition de croissance raisonnable en $t = 0$, on peut lui associer une métrique plurisousharmonique $\phi^{\mathrm{NA}} \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ sur l'espace analytique non archimédien X^{an}, encodant le taux de divergence logarithmique de ϕ le long de la fibre spéciale des modèles de (X, L). Dans le cas où $X = Y \times \mathbb{D}^*$ est un produit et ϕ est $\mathbb{S}^1$-invariante, on dit que ϕ est un rayon psh sur Y, et la métrique non archimédienne associée a été définie dans [BBJ18]; tandis que le cas général est traité dans [Reb21].
On aimerait alors dire que ϕ^{NA} est d'une manière naturelle la limite des $\phi_t = (\phi)_{|X_t}$ dans l'espace hybride. Il serait cependant trop naïf de prendre cette limite au sens topologique, étant donné que l'on voudrait qu'un tel énoncé demeure vrai pour des métriques singulières. Il est alors pertinent de définir une classe de métriques singulières sur l'espace hybride, qui encode à la fois les métriques complexes sur la dégénéréscence X, ainsi que les métriques non archimédiennes qui leur sont associées.
Dans le premier chapitre de cette thèse, nous définissons une classe $\mathrm{PSH}(X, L)$ de métriques plurisousharmoniques sur l'analytification de Berkovich X^{an} d'un schéma polarisé (X, L) sur un anneau de Banach intègre A; de manière naïve, une métrique singulière $\phi \in \mathrm{PSH}(X, L)$ peut être pensée comme une famille de métriques psh $\phi_x \in \mathrm{PSH}(X_{\mathscr{H}(x)}, L_{\mathscr{H}(x)})$, variant de manière plurisousharmonique par rapport à $x \in \mathscr{M}(A)$. Il convient cependant de noter qu'il est possible que $\phi_x \equiv -\infty$ pour certains $x \in \mathscr{M}(A)$.
La classe $\mathrm{PSH}(X, L)$ est définie suivant l'approche de [BE21] : il s'agit de la plus petite classe de métriques singulières sur L contenant les métriques de la forme $m^{-1} \log|s|$ lorsque $s \in H^0(X, mL)$ est une section non-nulle, et qui soit stable par limites décroissantes et maxima finis.
Dans le cas où (X, L) est une dégénérescence de variétés complexes polarisées, une métrique psh sur l'espace analytique X^{hyb} induit par restriction et renormalisation une métrique plurisousharmonique $\phi \in \mathrm{PSH}(X, L)$ à croissance logarithmique en 0, ainsi qu'une métrique psh $\phi_0 \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$, et est en outre déterminée de manière unique par ces deux restrictions. Réciproquement, étant donné une métrique psh ϕ sur la dégénérescence X, nous montrons que la métrique non archimédienne associée ϕ^{NA} induit une extension psh canonique de ϕ à l'espace hybride:

Théorème A. *[PS22a, thm. A] Soit $\phi \in \mathrm{PSH}(X, L)$ une métrique psh à croissance logarithmique en $t = 0$. La métrique singulière ϕ^{hyb} sur $(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$ telle que:*

$$\phi_0^{\mathrm{hyb}} = \phi^{\mathrm{NA}},$$

$$\phi^{\mathrm{hyb}}_{|X_t} = \phi_t$$

est plurishousharmonique.

Notons toutefois que toutes les métriques psh sur l'espace hybride ne sont pas obtenues de cette manière: le point $0 \in \bar{\mathbb{D}}_r$ est pluripolaire - non-négligeable au sens de la théorie du pluripotentiel hybride - si bien que les métriques psh hybrides ne sont pas déterminées de manière unique par leur restriction au-dessus du disque épointé. Cependant, cette subtilité disparaît lorsqu'on se restreint aux métriques psh continues, puisque le point $0 \in \bar{\mathbb{D}}_r$ est d'intérieur vide.
Nous étudions ensuite le cas d'une variété torique polarisée (Z, L) sur $\mathbb{C}$. Dans ce cas précis, les métriques psh sur L peuvent être décrites explicitement en termes de certaines fonctions convexes sur l'espace vectoriel réel $N_{\mathbb{R}}$ contenant l'éventail Σ de Z - et ce, indépendamment de la valeur absolue sur $\mathbb{C}$, euclidienne ou triviale. Plus précisement, la polarisation L est encodée au niveau combinatoire par une fonction convexe, linéaire par morceaux $\Psi_L : N_{\mathbb{R}} \longrightarrow \mathbb{R}$, et les métriques toriques sur L sont en correspondance bijective avec les fonctions convexes sur $N_{\mathbb{R}}$ ayant la même croissance que Ψ_L en l'infini.
Nous démontrons alors un énoncé similaire pour les métriques toriques continues et plurisousharmoniques: la positivité dans la direction horizontale est elle aussi encodée par de la convexité dans la direction de la base $\mathscr{M}(\mathbb{C}^{\mathrm{hyb}}) = [0, 1]$.

Théorème B. *Soit (Z, L) une variété torique complexe polarisée. Il existe une correspondance bijective entre les métriques toriques continues psh sur L^{hyb} et les fonctions convexes continues:*

$$\Phi : N_{\mathbb{R}} \times [0, 1] \longrightarrow \mathbb{R}$$

telles que $(\Phi - \Psi_L)$ s'étende continûment à la compactification $N_\Sigma \times [0, 1]$ de $N_{\mathbb{R}} \times [0, 1]$.

La fibration SYZ non-archimédienne

La seconde partie de cette thèse se consacre à l'étude du problème suivant: étant donnée une variété de Calabi-Yau polarisée (X, L) sur $K = \mathbb{C}((t))$, tenter de produire une incarnation non archimédienne $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ de la fibration SYZ - dans l'espoir de trouver une rétraction telle que la métrique de Calabi-Yau non archimédienne sur L^{an} soit constante le long des fibres de ρ.
L'un des objectifs étant d'identifier la solution de l'équation de Monge-Ampère non-archimédienne à une solution de l'équation de Monge-Ampère réelle, il est souhaitable d'avoir à notre disposition une fibration qui soit "lisse" hors d'un lieu de codimension 2 de la base, celle-ci induisant ainsi une structure affine entière sur $\mathrm{Sk}(X)$ dont les singularités sont de codimension 2.
Une première étape consiste à comprendre les singularités de la rétraction de Berkovich

associée à un modèle $\mathscr{X}$, ou plutôt leur absence. Le résultat suivant, établi en collaboration avec Enrica Mazzon, affirme que si Z est une strate torique de la fibre spéciale d'une dégénérescence de variétés algébriques, alors sous une certaine hypothèse de positivité sur le fibré conormal de Z, la rétraction de Berkovich $\rho_{\mathscr{X}}$ n'a pas de singularités sur l'intérieur de l'étoile ouverte $\mathrm{Star}(\tau_Z)$ de la face τ_Z, i.e. la réunion des intérieurs des faces contenant τ_Z:

Théorème C. *[MPS21, thm. B] Soit X/K une variété projective lisse de dimension n, et $\mathscr{X}/R$ un modèle dlt de X à fibre spéciale réduite $\mathscr{X}_0 = \sum_\alpha D_\alpha$, tel que chaque D_α soit un diviseur de Cartier.*
Soit $Z = D_0 \cap D_1 \cap \ldots \cap D_{n-r}$ une strate de dimension r de $\mathscr{X}_0$, telle que:

- *$\mathring{Z} \subset Z$ est le plongement d'un tore dans une variété torique, où $\mathring{Z} = Z \setminus \cup_{\alpha \neq 0,1,\ldots,n-r} D_\alpha$;*
- *le fibré conormal $\nu^*_{Z/\mathscr{X}}$ est un fibré vectoriel nef sur Z;*
- *pour tout $\alpha \notin \{0, ..., n-r\}$, l'intersection $D_\alpha \cap Z$ est connexe ou vide.*

Alors le complété formel $\widehat{\mathscr{X}_{/Z}}$ de $\mathscr{X}$ le long de Z est isomorphe au complété formel du fibré normal $\mathcal{N} = \nu_{Z/\mathscr{X}}$ le long de la section nulle, qui est une variété torique. En particulier, la rétraction de Berkovich $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$ est une fibration en tores affinoïdes au-dessus de l'étoile ouverte $\mathrm{Star}(\tau_Z)$.

Nous étudions ensuite des dégénérescences d'hypersurfaces de la forme:

$$X = \{z_0 ... z_{n+1} + tF_{n+2} = 0\} \subset \mathbb{P}^{n+1} \times \mathbb{D}^*,$$

où F_{n+2} est un polynôme homogène de degré $(n+2)$ général. Le squelette essentiel $\mathrm{Sk}(X)$ est ici donné explicitement comme le complexe dual du bord torique de $\mathbb{P}^{n+1}$, et peut en fait être réalisé canoniquement dans l'espace vectoriel $N_{\mathbb{R}}$ dans lequel l'éventail de $\mathbb{P}^{n+1}$ vit. Plus précisément, écrivant le tore $\mathbb{T} \subset \mathbb{P}^{n+1}$, l'application de tropicalisation:

$$\mathrm{val} : \mathbb{T}^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}$$

induit un homéomorphisme entre $\mathrm{Sk}(X) \subset X^{\mathrm{an}}$ et le sous-complexe borné maximal de la tropicalisation $\mathrm{Trop}(X) := \mathrm{val}(X^{\mathrm{an}} \cap \mathbb{T}^{\mathrm{an}}) \subset N_{\mathbb{R}}$ de l'hypersurface - celle-ci étant munie d'une décomposition cellulaire canonique.
Après un choix adapté de discriminant $\Gamma \subset \mathrm{Sk}(X)$ de codimension 2 (aussi appelé *choix de points de branchements*), il est possible de définir sur $\mathrm{Sk}(X) \setminus \Gamma$ une structure affine entière encodant la nature torique de la dégénérescence, suivant [GS06]. Cette structure affine entière n'a en particulier pas de singularité au voisinage de chaque sommet $v_i \in \mathrm{Sk}(X)$, et le germe de variété affine en v_i n'est autre que le germe en 0 de l'éventail Σ_i de la composante D_i correspondante - mais n'est pas canonique puisque dépendant du choix de points de branchement.
Nous nous appuyons ensuite sur le théorème C et sur la construction par Yamamoto [Yam21] de contractions tropicales:

$$\delta : \mathrm{Trop}(X) \longrightarrow \mathrm{Sk}(X),$$

pour démontrer l'énoncé suivant:

Théorème D. *Soit $\underline{a} = (a_\tau)_\tau$ un choix de points de branchement dans $\mathrm{Sk}(X)$, et soit*
$\delta_{\underline{a}} : \overline{\mathrm{Trop}}(X) \longrightarrow \mathrm{Sk}(X)$ *la contraction tropicale associée [Yam21, thm. 5.1]. Alors la composée:*

$$\rho_{\underline{a}} := \delta_{\underline{a}} \circ \mathrm{val}_\Sigma : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$$

est une fibration en tores affinoïdes au-dessus de $\mathrm{Sk}(X) \setminus \Gamma$. De plus, la structure affine entière induite sur $\mathrm{Sk}(X) \setminus \Gamma$ par $\rho_{\underline{a}}$ coïncide avec celle mentionnée ci-dessus.

En dimension 2, la rétraction $\rho_{\underline{a}}$ a été construite sans utiliser la contraction tropicale dans un travail joint avec E. Mazzon [MPS21], qui contient aussi une construction similaire en dimension 3. Cette construction alternative permet aussi, en dimension 2, de raffiner le discriminant afin d'obtenir des structures affines entières sur la 2-sphère ayant 24 points singuliers de type de Kodaira I_1; nous renvoyons le lecteur à la section 4.2.1.
Enfin, nous étudions l'exemple de la famille des hypersurfaces de Fermat:

$$X = \{z_0 ... z_{n+1} + t(z_0^{n+2} + ... + z_{n+1}^{n+2}) = 0\} \subset \mathbb{P}^{n+1} \times \mathbb{D}^*,$$

qui est munie d'une action du groupe symétrique $\mathfrak{S}_{n+2}$. Dans cet exemple, le comportement asymptotique des métriques de Calabi-Yau (archimédiennes) est bien compris [Li22]: leur potentiel est proche asymptotiquement d'un potentiel torique, qui est lui défini sur l'espace projectif ambiant. En utilisant les résultats de [Li22] et le théorème B, nous démontrons:

Théorème E. *Soit $X \longrightarrow \mathbb{D}^*$ la famille des hypersurfaces de Fermat, polarisée par $L = \mathcal{O}_{\mathbb{P}}(1)$. Il existe une métrique torique continue psh $\phi \in \mathrm{PSH}(\mathbb{P}^{n+1,\mathrm{an}}_K, L^{\mathrm{an}})$, dont la restriction à X^{an} est solution de l'équation de Monge-Ampère non archimédienne:*

$$\mathrm{MA}(\phi) = \mu_0,$$

où μ_0 est la mesure de Lebesgue sur $\mathrm{Sk}(X)$. De plus, en écrivant $\phi = \phi_{\mathrm{FS}} + \psi$, la fonction continue ψ sur X^{an} satisfait la propriété de comparaison:

$$\psi = \psi \circ \rho_{\underline{a}}$$

au-dessus de $\mathrm{Sk}(X) \setminus \Gamma$, où $\rho_{\underline{a}}$ est la rétraction du théorème D, avec a_τ le barycentre de τ pour chaque face $\tau \subset \mathrm{Sk}(\mathscr{X})$, et $\Gamma \subset \mathrm{Sk}(X)$ le discriminant associé.

La fonction ψ est aussi en un sens naturel la limite des potentiels de Calabi-Yau archimédiens.
Un énoncé similaire a été obtenu récemment pour une classe plus large d'hypersurfaces dans [HJMM22].

Dégénérescences de variétés canoniquement polarisées

Soit $X \longrightarrow \mathbb{D}^*$ une dégénérescence de variétés canoniquement polarisées, et posons $L = K_{X/\mathbb{D}^*}$. D'après le théorème d'Aubin-Yau, chaque fibre X_t admet une unique

métrique de Kähler-Einstein ϕ_t sur L_t, dont la forme de courbure $\omega_t = dd^c\phi_t$ est de courbure de Ricci constante et négative:

$$\mathrm{Ric}(\omega_t) = -\omega_t.$$

En outre, il découle des travaux de Schumacher [Sch12] que la famille de métriques de Kähler-Einstein $\phi = (\phi_t)_{t\in\mathbb{D}^*}$ est aussi à courbure positive dans la direction de la base, et est à croissance logarithmique en $t = 0$, de telle sorte que $\phi \in \mathrm{PSH}(X, L)$ satisfait les hypothèses du théorème A.
Les résultats généraux issus du programme du modèle minimal entraînent en outre que la famille X admet un unique modèle canonique $\mathscr{X}_c/\mathbb{D}$, dont le fibré canonique $K_{\mathscr{X}_c/\mathbb{D}}$ est relativement ample. Les travaux de J. Song [Son17] montrent alors que les métriques de Kähler-Einstein convergent en un sens naturel vers l'unique courant de Kähler-Einstein $\omega_{KE,0}$ sur la fibre spéciale $\mathscr{X}_{c,0}$, et même si les potentiels locaux de ce dernier ne sont pas bornés, leurs singularités sont moindres que n'importe quels pôles logarithmiques. Nous sommes alors en mesure de montrer que la limite non archimédienne des métriques de Kähler-Einstein est la métrique modèle $\phi_{K_{\mathscr{X}_c/R}}$ associée au modèle canonique $(\mathscr{X}_c, K_{\mathscr{X}_c/\mathbb{D}})$:

Théorème F. *[PS22a, thm. B] Soit $X \xrightarrow{\pi} \mathbb{D}^*$ une dégénérescence de variétés canoniquement polarisées, $L = K_{X/\ \mathbb{D}^*}$, et $\phi_{\mathrm{KE}} \in \mathrm{PSH}(X, L)$ la famille des métriques de Kähler-Einstein. Supposons que la famille X a réduction stable sur $\mathbb{D}$. Alors la métrique sur L^{hyb} définie par:*

$$\phi_{|X} = \phi_{\mathrm{KE}},$$

$$\phi_0 = \phi_{K_{\mathscr{X}_c/R}}$$

est continue et plurisousharmonique sur X^{hyb}.

Plan du manuscrit

Ce texte est organisé comme suit.
Le premier chapitre est consacré à la définition et à l'étude de la classe des métriques plurisousharmoniques sur l'analytification de Berkovich d'un schéma polarisé (X, L) sur un anneau de Banach A - en pratique, nous nous intéressons surtout au cas de l'anneau des séries convergentes pour la norme hybride sur $\mathbb{C}$, de telle sorte que les espaces analytiques associés seront des espaces hybrides. Dans la section 1.1, nous rappelons des faits généraux sur les espaces analytiques au sens de Berkovich - en particulier sur un corps discrètement valué - et sur les espaces hybrides. Dans la section 1.2, nous définissons de manière globale une classe de métriques plurisousharmoniques sur l'analytification de Berkovich d'un A-schéma X, et énoncons ses propriétés de base. La section 1.3 est consacrée à l'étude du cas où X est une dégénérescence projective de variétés complexes sur le disque épointé $\mathbb{D}^*$, et son analytification au sens de Berkovich est l'espace hybride X^{hyb}. Nous comparons en particulier notre formalisme aux travaux [BBJ18], [Reb21], et nous démontrons le théorème A (théorème 1.3.13). Enfin, la section 1.4 contient une discussion sur les mesures de Monge-Ampère associées aux métriques psh sur un espace analytique sur un corps valué, et les familles de mesures induites sur une base plus générale. En particulier, dans le cas d'un espace hybride, nous énonçons le résulat suivant, essentiellement dû à Favre [Fav20]: la famille de mesures de Monge-Ampère sur X^{hyb} associée à une métrique continue psh sur un fibré ample est faiblement continue.
Dans le deuxième chapitre, nous considérons une variété torique projective Z sur $\mathbb{C}$ munie d'un fibré en droites semi-ample L, ainsi que son analytification Z^{hyb} par rapport à la norme hybride sur $\mathbb{C}$. Dans ce cadre, nous prouvons le théorème B, qui fournit une description combinatoire des métriques continues psh sur L^{hyb} qui sont invariantes par l'action du tore fibre-à-fibre: elles sont données par des fonctions convexes continues $\Phi : N_{\mathbb{R}} \times [0,1] \longrightarrow \mathbb{R}$ satisfaisant une certaine condition de croissance en l'infini (cf. 2.2.12). La section 2.1 est consacrée à divers rappels de géométrie convexe et torique, nécessaires à la section 2.2, où nous introduisons les métriques toriques fibre-à-fibre sur Z^{hyb} et prouvons le théorème B.
Fixons pour l'ensemble du troisième chapitre une dégénérescence maximale polarisée $(X, L) \longrightarrow \mathbb{D}^*$ de variétés de Calabi-Yau. Dans la section 3.1, nous motivons les conjectures SYZ et de Kontsevich-Soibelman, et étudions le modèle local non-archimédien des fibrations en tores lagrangiens. La section 3.2 est dédiée à la preuve du théorème C, qui fournit une première étape pour la construction de la fibration SYZ non archimédienne. Nous incluons aussi, dans la section 3.3, des exemples de calculs de structures affines entières construites via le théorème C.
Dans le quatrième chapitre, nous étudions certaines dégénérescences maximales d'hypersurfaces de Calabi-Yau dans l'espace projectif. Nous produisons, dans la section 4.1, une fibration SYZ non archimédienne ρ pour de telles hypersurfaces, et prouvons le théorème D. Nous présentons aussi une construction similaire en dimensions 2 and 3, issue du travail [MPS21], il s'agit du contenu de la section 4.2. La section 4.3, est consacrée à l'étude de la famille des hypersurfaces de Fermat où nous démontrons le théorème E en nous appuyant sur les résulats des chapitres 1 et 2 et de l'article [Li22]: la métrique de Calabi-Yau non archimédienne est la restriction d'une métrique torique sur l'espace projectif ambiant.

Enfin, dans un bref cinquième chapitre, nous étudions les dégénérescences de variétés de Kähler-Einstein à courbure négative, dans le formalisme du chapitre 1. Nous décrivons le cadre du chapitre dans la section 5.1, où nous énoncons le théorème F, et nous rappelons les résultats d'existence de modèles canoniques pour de telles dégénérescences dans la section 5.2. Dans la section 5.3, nous décrivons les résultats de [Son17] concernant la convergence des métriques de Kähler-Einstein sur le modèle canonique, et nous démontrons le théorème F. Nous concluons ce chapitre avec une brève discussion sur le cas des variétés de Fano, c'est l'objet de la section 5.4.

Notations et conventions

Tous les anneaux sont supposés commutatifs et unitaires.
Nous utiliserons la notation additive pour les fibrés en droites: si L, M sont deux fibrés en droites sur une variété X, nous écrirons $L + M := L \otimes M$, et $kL := L^{\otimes k}$ pour $k \in \mathbb{Z}$.
Si X est une variété complexe et ϕ une fonction lisse sur X, nous posons $dd^c\phi = \frac{i}{2\pi}\partial\bar{\partial}\phi$. Cette notation est étendue aux métriques hermitiennes sur les fibrés en droites: si L est un fibré en droites holomorphe sur X et ϕ une métrique lisse sur L, sa forme de courbure sera dénotée par $dd^c\phi \in c_1(L)$ - et idem pour les métriques singulières.
Tout au long de ce texte, lorsque nous dirons que $X \xrightarrow{\pi} \mathbb{D}^*$ est une dégénérescence de variétés complexes, cela signifiera que X est une variété complexe lisse et π une submersion holomorphe (qui sera généralement omise). Nous supposerons de plus toujours que la dégénérescence est méromorphe en 0, i.e. que X est définie sur le corps des fractions de l'anneau des germes en l'origine de fonctions holomorphes. De manière équivalente, cela signifie qu'il existe un espace analytique complexe normal $\mathscr{X} \xrightarrow{\pi} \mathbb{D}$ tel que $\mathscr{X}_{|\mathbb{D}^*} = X$.

Introduction

The main subject of this thesis is the study of degenerations of complex algebraic varieties - and more specifically, Calabi-Yau manifolds - through means of non-archimedean geometry.
The first part of this text provides an attempt to generalize the non-archimedean pluripotential theory that has been developed in [BFJ16], [BBJ18], [BE21], [Reb21] for a variety X over a non-archimedean field K to the case of hybrid spaces. To that purpose, we define a class of (singular) semi-positive metrics on line bundles on Berkovich analytifications of schemes X over a Banach ring A. In the case where $A = A_r$ is the ring of convergent power series - with respect to the hybrid norm on $\mathbb{C}$ - this yields a notion of semi-positive metrics on the union of a degeneration of complex manifolds with the associated non-archimedean analytic space. We also relate our definition with various constructions appearing in the literature [BBJ18], [Reb21], [Fav20]. We focus in particular on the case of toric metrics on a toric variety, where the metrics are encoded by functions on a finite-dimensional real vector space, and the semi-positivity of the metric translates into convexity of the corresponding function.
The second part is devoted to the following problem: given a maximal degeneration of Calabi-Yau manifolds $X \longrightarrow \mathbb{D}^*$, trying to construct a non-archimedean avatar $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ of the conjectural SYZ fibration on the fibers X_t for $|t| \ll 1$. The hope is that this retraction induces an integral affine structure singular in codimension 2, and that the retraction allows us to identify the non-archimedean Calabi-Yau metric with a solution of the real Monge-Ampère metric on the essential skeleton $\mathrm{Sk}(X)$. This involves in particular understanding the singularities of Berkovich retractions $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ associated to R-models of X, where $R = \mathbb{C}[[t]]$, as our approach to construct the non-archimedean SYZ fibration on explicit examples involves patching several of these retractions together. We focus in particular on a certain class of degenerating hypersurfaces in projective space, where building on the results from [Li22], we obtain a satisfactory picture regarding the comparison between the non-archimedean and the real Monge-Ampère equation on $\mathrm{Sk}(X)$, with right-hand side the Lebesgue measure on the essential skeleton $\mathrm{Sk}(X)$.

Non-archimedean geometry

Berkovich analytic spaces

Let $(K, |\cdot|)$ be a complete non-archimedean field, i.e. a field endowed with an absolute value satisfying the ultrametric inequality:

$$|x + y| \leq \max\{|x|, |y|\}$$

for all $x, y \in K$, and such that K is complete with respect to the induced distance. Examples of such fields include the field $\mathbb{Q}_p$ of p-adic numbers, which is the metric completion of $\mathbb{Q}$ endowed with the absolute value $|\cdot|_p = p^{-v_p}$, where v_p is the p-adic valuation; the field $\mathbb{C}((t))$ of Laurent series endowed with the t-adic absolute value $|\cdot| = e^{-\operatorname{ord}_0}$; or simply any field k endowed with the *trivial* absolute value $|\cdot|_0$, which is such that $|x|_0 = 1$ for any non-zero $x \in k$.

There is a long history, throughout the 20th century, of attempts to produce a satisfying theory of analytic geometry over non-archimedean fields (due for instance to Tate, Raynaud or Huber): given a projective variety X/K, one would like to attach to it in a natural way an analytic space X^{an}, satisfying for instance better topological properties than the scheme X; in a similar spirit to how one attaches a complex manifold to a quasi-projective scheme of finite type over $\mathbb{C}$. Let us point out that the naive attempt of taking K endowed with the distance induced by the non-archimedean absolute value, as the underlying topological space of the analytification of the affine line $\mathbb{A}^1_K$, works out rather poorly, as the ultrametric inequality implies that K is a totally disconnected topological space. As a result, there is a need for a more sophisticated theory of analytic spaces over K.

Throughout this text, we will only be using Berkovich's theory of K-analytic spaces [Ber90]. The very basic idea is the following: if X is a complex projective variety, then one can view a closed point $x \in X$ as the (degenerate) absolute value $f \mapsto |f(x)|$ on the function field $\mathcal{K}(X)$ of X, which extends the Euclidean absolute value on the base field $\mathbb{C} \subset \mathcal{K}(X)$, viewed as the constant functions on X.

We now let K be a complete valued field and X/K is a quasi-projective variety. Its Berkovich analytification X^{an} is defined as a certain compactification of the space of real valuations on the function field of X that extend the given absolute value on K, and is endowed with the topology of pointwise convergence. In particular, if f is a meromorphic function on X, defined on a Zariski open $U \subset X$, it induces a continuous function $x \mapsto |f(x)| := |f|_x$ on the open subset $U^{\mathrm{an}} \subset X^{\mathrm{an}}$. The analytification X^{an} of a quasi-projective variety X/K then satisfies nice topological properties: it is Hausdorff, locally contractible and pathwise-connected, and is compact if and only if X/K is proper. Notably, if $K = \mathbb{C}$ endowed with the archimedean absolute value, the Berkovich analytification of a scheme over $\mathbb{C}$ is its usual holomorphic analytification.

The theory of Berkovich also works over a more general base: given a Banach ring $(A, |\cdot|)$, one can define its Berkovich spectrum $\mathscr{M}(A)$ as the set of multiplicative semi-norms on A which are bounded by the reference norm $|\cdot|$. Given a point $x \in \mathscr{M}(A)$, the kernel $\mathfrak{p}_x$ of the semi-norm $|\cdot|_x$ is a prime ideal of A, and we denote by $\mathscr{H}(x)$ the completion of the fraction field of the quotient $A/\mathfrak{p}_x$ with respect to $|\cdot|_x$, and call it the residue field of $\mathscr{M}(A)$ at x - equipped with $|\cdot|_x$, it is a complete

valued field.
Berkovich then defines the analytification X^{an} of an A-scheme of finite type in a manner similar to the previous construction; it comes with a continuous structure map $X^{\mathrm{an}} \xrightarrow{\pi} \mathscr{M}(A)$. When the base ring A is reasonable enough, the fiber $X_x = \pi^{-1}(x)$ can be identified with the analytification of the base change $X_{\mathscr{H}(x)} = X \times_A \mathscr{H}(x)$, with respect to the absolute value on $\mathscr{H}(x)$, so that one can roughly think of X^{an} as the family of analytic spaces $(X^{\mathrm{an}}_{\mathscr{H}(x)})_{x \in \mathscr{M}(A)}$ over different fields; this is similar to viewing a scheme over a ring A as a family of varieties over the residue fields of A, parametrized by $\operatorname{Spec} A$.
For instance, assume that our base Banach ring is $\mathbb{C}^{\mathrm{hyb}} = (\mathbb{C}, |\cdot|_{\mathrm{hyb}})$, where $|z|_{\mathrm{hyb}} = \max\{|z|_0, |z|\}$ is the *hybrid* norm on $\mathbb{C}$, i.e. the maximum of the trivial and Euclidean absolute values on $\mathbb{C}$. Then $\mathscr{M}(\mathbb{C}^{\mathrm{hyb}}) = \{|\cdot|^\lambda / \lambda \in [0,1]\}$, where we abusively denote $|\cdot|^0 = |\cdot|_0$ the trivial absolute value. As a result, given a complex variety X, its analytification X^{hyb} over the ring $\mathbb{C}^{\mathrm{hyb}}$ - called the associated hybrid space - fibers over the unit interval $[0,1]$, and the fiber X^{hyb}_λ for $\lambda > 0$ is (up to rescaling the absolute value) the usual complex analytification of X, while the fiber over 0 is its analytification X^{an}_0 with respect to the trivial absolute value on $\mathbb{C}$. The upshot of this construction is that it provides a natural way to see suitable rescalings of the complex manifold X^{hol} degenerate to the non-archimedean analytic space X^{an}_0. This justifies the name, as the space X^{hyb} is of hybrid nature, somewhere between non-archimedean and usual Euclidean geometry.

Degenerations of complex manifolds and hybrid spaces

We let $X \longrightarrow \mathbb{D}^*$ be a projective degeneration of complex manifolds, and assume that the equations of X inside projective space have meromorphic singularity at $t = 0$. This allows us to view X as an algebraic variety X_K over the field $K = \mathbb{C}((t))$ of complex Laurent series, which is endowed with the t-adic absolute value, making it a complete non-archimedean field. As we shall explain, the Berkovich analytification X^{an} of X_K encodes in various ways asymptotic information attached to the degeneration $X \longrightarrow \mathbb{D}^*$.
Let $\pi : \mathscr{X} \longrightarrow \mathbb{D}$ be a simple normal crossing model of X - which exists by the meromorphic assumption and Hironaka's theorem. Writing the special fiber $\mathscr{X}_0 = \pi^{-1}(0) = \sum_{i \in I} a_i D_i$ as the sum of its irreducible components, the simple normal crossing condition means that these are smooth prime divisors, each having transverse intersection with any intersection of other components of $\mathscr{X}_0$. One can then attach to $\mathscr{X}_0$ a cell complex $\mathscr{D}(\mathscr{X}_0)$ encoding the combinatorics of the intersections of the D_i. The dual complex $\mathscr{D}(\mathscr{X}_0)$ is defined roughly as follows: its vertices v_i are in one-to-one correspondence with the components D_i of $\mathscr{X}_0$; then for each connected component of $D_i \cap D_j$, we attach an edge e_{ij} between v_i and v_j. Similarly, each connected component Y of $D_i \cap D_j \cap D_k$ yields a triangle, bounded by three edges of the form e_{ij}, e_{ik}, e_{jk}; and the final outcome of this construction is a cell complex $\mathscr{D}(\mathscr{X}_0)$ of dimension less than n, where n is the relative dimension of X. The upshot of this is that the dual complex can be continuously embedded inside X^{an}, i.e. we may view the points of $\mathscr{D}(\mathscr{X}_0)$ as valuations on the function field of X. For instance, if v_i is a vertex of $\mathscr{D}(\mathscr{X}_0)$, we view it as the valuation $f \mapsto a_i^{-1} \operatorname{ord}_{D_i}(f)$

for $f \in \mathcal{K}(X) \simeq \mathcal{K}(\mathscr{X})$, and the higher-dimensional faces of $\mathscr{D}(\mathscr{X}_0)$ are embedded in X^{an} by interpolating between these. The image of the embedding is called the *skeleton* of $\mathscr{X}$, denoted by $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$.
Moreover, Berkovich defines a retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ for the inclusion $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$, which in fact induces a homotopy equivalence between X^{an} and $\mathrm{Sk}(\mathscr{X})$. One can think of the skeleton $\mathrm{Sk}(\mathscr{X})$ as a sort of approximation of X^{an}, as X^{an} can be recovered (topologically) as the inverse limit of the skeleta $\mathrm{Sk}(\mathscr{X})$ of all possible snc models of X.
Let us now fix a radius $r \in (0,1)$. One can define a certain hybrid ring A_r of convergent power series in a similar spirit to the construction from the previous paragraph, such that the Berkovich spectrum $\mathscr{M}(A_r) \simeq \bar{\mathbb{D}}_r$. We may view X as a scheme over A_r, and its analytification is called the *hybrid space*:

$$X^{\mathrm{hyb}} \xrightarrow{\pi} \bar{\mathbb{D}}_r,$$

which is itself a Berkovich analytic space. Moreover, $\pi^{-1}(0) = X_K^{\mathrm{an}}$ is the Berkovich analytification of X with respect to the non-archimedean t-adic absolute value on K, while $\pi^{-1}(\bar{\mathbb{D}}_r^*)$ is naturally identified with the restriction of the degeneration X to the closed punctured disk $\bar{\mathbb{D}}_r^*$ - up to rescaling the absolute value on the fiber X_t by a factor $\frac{\log r}{\log|t|}$. Once again, this provides a natural way to see the (rescaled) complex manifolds $(X_t)_{t\in\mathbb{D}^*}$ degenerate to a non-archimedean space X^{an} as $t \to 0$. As a result, one would expect various families of objects from complex geometry living on the X_t's to have a natural limit living on the space X^{an} - with the caveat that the limit is taken with respect to the rescaled absolute value, so that it rather corresponds to a logarithmic slope at zero.

Non-archimedean pluripotential theory

Roughly speaking, classical pluripotential theory is the study of plurisubharmonic functions: if $\Omega \subset \mathbb{C}^n$ is a domain, an upper semi-continuous function $\phi : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic (psh for short) if and only for any complex line $\ell \subset \mathbb{C}^n$, its restriction ϕ_ℓ to $\ell \cap \Omega$ is a subharmonic function in the weak sense - that is, the Laplacian $\Delta\phi_\ell \geq 0$ is the sense of distributions. Concretely, this means that the mixed Hessian $i\partial\bar{\partial}\phi$ is a semi-positive $(1,1)$-form on Ω - in the sense of currents, as we are allowing ϕ to be singular. For instance, if $f \in \mathcal{O}(\Omega)$ is a holomorphic function, then it follows from the Lelong-Poincaré formula that $\phi = \log|f|$ is a psh function on Ω, singular precisely along the locus $\{f = 0\}$. Alternatively, psh functions can be defined through a mean value inequality, which implies that they are stable under finite maxima and decreasing limits.
In a more global setting, let (X, L) be a polarized complex manifold, and h a Hermitian metric on L. The metric h is identified with the collection ϕ of its logarithmic local weights $\phi_i = -\log|s_i|_h$, where the s_i are local sections of L, so that using additive notation for metrics on L, we say that a (singular) metric ϕ on L is plurisubharmonic if and only if its local weights ϕ_i are. In a way similar to the previous example, if $s \in H^0(X, mL)$ is a non-zero global section of some positive power of L, then the singular metric $\phi = m^{-1}\log|s|$ is psh, by the Lelong-Poincaré formula.

One of the central theorems in pluripotential theory is the following regularization result, due to Demailly [Dem92]: any psh metric ϕ on an ample line bundle L can be written as the decreasing limit of metrics of the form:

$$\phi_{\mathrm{FS}} = \frac{1}{2m}\log(|s_0|^2 + ... + |s_N|^2),$$

where $m \geq 1$ and $(s_0, ..., s_N)$ are global sections of mL without common zeroes. These metrics are called Fubini-Study metrics, for the following reason: if $f : X \longrightarrow \mathbb{CP}^N$ is the holomorphic map given by $f(x) = [s_0(x) : ... : s_N(x)]$, then ϕ_{FS} is the pull-back via f of the standard Fubini-Study metric on $\mathbb{CP}^N$.
We now move to the non-archimedean world, and let K be a complete non-archimedean field. If (X, L) is a polarized variety over K, then one can define metrics on the analytification L^{an} as in the complex case: given local algebraic trivializations (U_i, s_i) of L, a continuous metric ϕ on L^{an} is a collection of continuous local weights $\phi_i = -\log|s_i|_\phi$ on the U_i^{an}, compatible on the overlaps $(U_i \cap U_j)^{\mathrm{an}}$.
We may then define singular metrics as in the complex case, the upshot being that if $s \in H^0(X, mL)$ is a non-zero section, then $\phi = m^{-1}\log|s|$ defines a singular metric on L^{an}. Pushing the analogy with complex pluripotential theory even further, we say that a metric on L^{an} of the form:

$$\phi_{\mathrm{FS}} = m^{-1}\log\max(|s_0|, ..., |s_N|),$$

where $(s_0, ..., s_N)$ are global sections of mL without common zeroes, is a (tropical) Fubini-Study metric. Demailly's regularization theorem is now taken as the definition of the class of semi-positive metrics:

Definition 0.0.7. *The class* $\mathrm{PSH}(X, L)$ *of plurisubharmonic metrics is the smallest class of singular metrics on* L *that:*

- *contains all tropical Fubini-Study metrics,*
- *is stable under addition of constants,*
- *is stable under finite maxima,*
- *is stable under decreasing limits.*

When the field K is non-archimedean, a natural class of continuous metrics on L arises as follows. We assume that K is non-trivially valued for convenience, with valuation v_K. Let $(\mathscr{X}, \mathscr{L})$ be a model of (X, mL) over the valuation ring $R = \{v_K \geq 0\}$, and assume that $\mathscr{L}$ is semi-ample. Loosely speaking, the model metric $\phi_{\mathscr{L}}$ is the continuous metric on L such that local sections of L have $\phi_{\mathscr{L}}$-norm ≤ 1 if and only they extend as regular sections of $\mathscr{L}$ - as a result, the lattice $H^0(\mathscr{X}, \mathscr{L}) \subset H^0(X, L)$ is the unit ball for the sup-norm of the model metric $\phi_{\mathscr{L}}$, following the correspondence between norms and lattices on vector spaces over non-archimedean fields. It turns out that in this setting, Fubini-Study metrics on L are nothing but model metrics associated to semi-ample models $\mathscr{L}$ of (positive multiples) L - that is, such that $m\mathscr{L}$ is globally generated for some $m \geq 1$.

The Monge-Ampère operator

Let $\Omega \subset \mathbb{C}^n$ a domain, and $\phi_i \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ for $i = 1, ..., n$ be n smooth psh functions. Then their (mixed) Monge-Ampère measure is defined as:

$$\mathrm{MA}(\phi_1, ..., \phi_n) = i\partial\bar{\partial}\phi_1 \wedge ... \wedge i\partial\bar{\partial}\phi_n,$$

and is one of the central objects of pluripotential theory. One would naturally want to extend the Monge-Ampère operator to psh functions that are not necessarily smooth; unfortunately while $i\partial\bar{\partial}\phi$ makes sense for an arbitrary psh function, it is now a current and no longer a differential form, so that it is *a priori* not possible to consider products of the $i\partial\bar{\partial}\phi_i$. However, the fundamental work of Bedford-Taylor [BT76] showed that the Monge-Ampère operator can be uniquely extended to tuples of locally bounded psh functions - and in particular, to continuous psh functions. Let us mention that it was further extended in [BEGZ10] to the class of psh functions of finite energy, which we will not define here.
In the global setting, let X be a smooth projective complex variety, and $L_1, ..., L_n$ ample line bundles on X. Then given bounded semi-positive metrics ϕ_i on the L_i, one can define the mixed Monge-Ampère measure of the ϕ_i's as above; we still denote it by $i\partial\bar{\partial}\phi_1 \wedge ... \wedge i\partial\bar{\partial}\phi_n$. It is a positive measure on X, whose total mass is (up to a normalizing factor) the intersection number $(L_1 \cdot ... \cdot L_n)$. The following theorem, known as the Calabi conjecture, is due to Yau [Yau78] in the smooth case and to Kołodziej [Kol98] in the singular case:

Theorem 0.0.8. *Let X be a smooth projective complex manifold, and L an ample line bundle on X. Let μ be a probability measure on X with L^p-density for a $p > 1$. Then there exists a unique (up to an additive constant) metric $\phi \in \mathrm{PSH}(X, L)$ solving the complex Monge-Ampère equation:*

$$(i\partial\bar{\partial}\phi)^n = C\mu,$$

where $C = (2\pi)^n L^n$.

Note that Yau's result also ensures that if μ is a smooth volume form on X, then ϕ is smooth and strictly psh.
The original motivation for this theorem was the existence of Kähler Ricci-flat metric on Calabi-Yau manifolds: if (X, L) is a polarized Calabi-Yau manifold, i.e. such that $K_X = \mathcal{O}_X$, then the (smooth) solution $\phi \in \mathrm{PSH}(X, L)$ to the Monge-Ampère equation:

$$(i\partial\bar{\partial}\phi)^n = Ci^{n^2}\Omega \wedge \bar{\Omega},$$

where $\Omega \in H^0(X, K_X)$ is a nowhere-vanishing holomorphic n-form, is such that the curvature form $\omega = i\partial\bar{\partial}\phi$ is a Kähler metric on X with zero Ricci curvature.
We now move on the non-archimedean case, and we let $K = k((t))$ to simplify the exposition; we write $R = k[[t]]$ and assume that k has characteristic zero. Given a polarized variety (X, L) over K, the analog of smooth metrics on L are Fubini-Study metrics which, as alluded to above, are the same as model metrics $\phi_{\mathscr{L}}$ for R-models $(\mathscr{X}, \mathscr{L})$ of (X, L). The mixed Monge-Ampère measure of a collection $(\phi_{\mathscr{L}_1}, ..., \phi_{\mathscr{L}_n})$ of model metrics is defined as an explicit sum of Dirac masses, using various intersection numbers of the $\mathscr{L}_i$ on $\mathscr{X}$; while not transparent at first sight the definition

is designed to mimic the real Monge-Ampère operator on convex, piecewise-affine functions - one important point being that $\mathrm{MA}(\phi_{\mathscr{L}_1}, ..., \phi_{\mathscr{L}_n})$ is supported on the skeleton $\mathrm{Sk}(\mathscr{X})$, and has total mass $(L_1 \cdot ... \cdot L_n)$, as in the complex case.
The Monge-Ampère operator is then extended canonically to continuous psh metrics in [BFJ15], where it is proven that the non-archimedean analog of the Calabi conjecture holds:

Theorem 0.0.9. *Let (X, L) be a smooth polarized variety over K. Let μ be a probability measure on X^{an}, supported on the skeleton of some snc R-model of X. Then there exists a unique (up to an additive constant) continuous, semi-positive metric ϕ on L satisfying the non-archimedean Monge-Ampère equation:*

$$\mathrm{MA}(\phi) = \mu.$$

Here we write $\mathrm{MA}(\phi) := \mathrm{MA}(\phi, ..., \phi)$.
The proof of the above theorem is however of variational nature, so that it yields very little information on the solution, besides its continuity. As we shall see later on, it would be in particular desirable to translate this equation into a real Monge-Ampère equation on the skeleton $\mathrm{Sk}(\mathscr{X})$ on which the measure μ is supported, which turns out to be a rather delicate question.

The Kontsevich-Soibelman conjecture

We now move on to explain a circle of ideas, initiated in the early 2000's by Kontsevich-Soibelman as a temptative explanation to mirror symmetry, that predict the asymptotic behavior of a family of Calabi-Yau metrics when the complex structure degenerates in the worst possible way.

The classical SYZ picture

We let $X \longrightarrow \mathbb{D}^*$ be a projective, meromorphic at zero, family of Calabi-Yau manifolds - recall that by Calabi-Yau we mean that the canonical bundle K_{X_t} is trivial for every $t \in \mathbb{D}^*$ - and fix a relative polarization L on X. We furthermore assume that the family is maximally degenerate at $t = 0$, which means that the fibers X_t break into as many pieces as possible as $t \to 0$ - more precisely, the dual complex $\mathscr{D}(\mathscr{X}_0)$ of any snc model of X has the maximal possible dimension, which is $n = \dim X_t$.
In this setting, the Kontsevich-Soibelman approach to the Strominger-Yau-Zaslow [SYZ96] conjecture stipulates that the Calabi-Yau manifolds (X_t, ω_t) - where ω_t is the unique Kähler Ricci-flat metric on X_t, normalized by the condition $\omega_t \in c_1(L_t)$ - have diameters of magnitude $d_t := \big|\log|t|\big|^{1/2}$, which was indeed confirmed by the results of Li-Tosatti [LT20].
Moreover, the fiber X_t endowed with the rescaled Calabi-Yau metric $\tilde{\omega}_t = d_t^{-2}\omega_t$ should asymptotically look like the total space of a Lagrangian $\mathbb{T}^n$-fibration, whose fibers are of size d_t^{-1}, over a certain real, n-dimensional base B. In particular, the metric spaces $(X_t, \tilde{\omega}_t)$ are expected to converge in the Gromov-Hausdorff sense to the base B, endowed with a real Monge-Ampère metric g_B away from a Hausdorff codimension 2 subset.

More precisely, it follows from the classical construction of action-angle coordinates in symplectic geometry that the special Lagrangian fibration $f_t : X_t \longrightarrow B$ induces an integral affine structure $\nabla_t^{\mathbb{Z}}$ on the subset $B^{\mathrm{sm}} \subset B$ over which the fibration f_t is submersive, i.e. an atlas of charts whose transition functions lie in the group $\mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$ of integral affine transformations. Since at the limit $t \to 0$ the torus fibers are very small, the Calabi-Yau potential ϕ_t should be very close to a potential $\psi_t \circ f_t$ pulled back from the base, where ψ_t is a (multivalued) convex function (in affine coordinates) on B^{sm}.
As a result, heuristically at the limit $t \to 0$, the ψ_t should converge to a convex function ψ on B^{sm}. The latter should thus be equipped with an integral affine structure, which should be thought as the 'tropical limit' of the complex structure of X_t. Additionally, the complex Monge-Ampère equation defining the Calabi-Yau metric translates into the real Monge-Ampère equation:

$$\det\big(\frac{\partial^2 \psi}{\partial y_i \partial y_j}\big) = C,$$

where the y_i are local affine coordinates. As a result, the limiting metric g_B should be given on B^{sm} as the Hessian $g_{ij} = \partial^2_{ij}\psi$ of the convex potential locally in affine coordinates.
We now let non-archimedean geometry enter the picture.

The essential skeleton

Since the Calabi-Yau metrics $\omega_t \in c_1(L_t)$ are obtained by solving the complex Monge-Ampère equation:

$$\omega_t^n = C_t i^{n^2} \Omega_t \wedge \bar{\Omega}_t,$$

where $\Omega_t \in H^0(X_t, K_{X_t})$ is a nowhere-vanishing holomorphic n-form, a natural first step towards understanding their behaviour as $t \to 0$ is to understand the behaviour of the probability measures $\mu_t = C_t i^{n^2} \Omega_t \wedge \bar{\Omega}_t$, which are of algebraic origin. It turns out that these measures converge weakly inside the hybrid space X^{hyb} associated to the degeneration [BJ17]:

Theorem 0.0.10. *Let $(X, L) \longrightarrow \mathbb{D}^*$ be a polarized maximal degeneration of Calabi-Yau manifolds, and let $(\mu_t)_{t \in \mathbb{D}^*}$ be the family of Calabi-Yau probability measures. Then the μ_t converge weakly on X^{hyb} to a measure μ_0, which is a Lebesgue-type measure on an n-dimensional subcomplex $\mathrm{Sk}(X)$ of $\mathrm{Sk}(\mathscr{X})$, for any snc model $\mathscr{X}/\mathbb{D}$.*

The subcomplex $\mathrm{Sk}(X)$ in the statement above is called the *essential skeleton* of X, and can be computed explicitly inside $\mathrm{Sk}(\mathscr{X})$, given a model $\mathscr{X}/\mathbb{D}$ [MN15]. As suggested by the notation, it is furthermore independent of the choice of snc model $\mathscr{X}$.
The above theorem then suggests that the base B of the SYZ fibration should be the essential skeleton $\mathrm{Sk}(X) \subset X^{\mathrm{an}}$, and one would like to produce a non-archimedean avatar of the SYZ fibration:

$$\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X),$$

that is in a natural way the limit inside X^{hyb} of the archimedean SYZ fibrations $f_t : X_t \longrightarrow B$; this is know for instance when X is a maximally degenerate abelian variety by [Oda18], [GO22]. Pushing the analogy with the classical SYZ fibration further, we require for the continuous map $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ to be an *affinoid torus fibration* away from a piecewise-affine locus $\Gamma \subset \mathrm{Sk}(X)$ of codimension greater than 2 - without giving the precise definition here, this is the non-archimedean counterpart to smooth torus fibrations; for instance, such a fibration induces an integral affine structure over $\mathrm{Sk}(X) \setminus \Gamma$, as in the classical case.
A first attempt was provided by Nicaise-Xu-Yu [NX16], [NXY19]: if $\mathscr{X}/\mathbb{D}$ is a minimal model of the family X, in the sense of the Minimal Model Program, then the equality $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$ holds, and it is proven in [NXY19] that the Berkovich retraction:

$$\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$$

is indeed an affinoid torus fibration away from the interior of the faces of codimension ≥ 2 of $\mathrm{Sk}(\mathscr{X}) = \mathrm{Sk}(X)$.

The Monge-Ampère comparison property

Ultimately, one of the goals is to produce the limiting real Monge-Ampère metric ψ on $\mathrm{Sk}(X)$ through non-archimedean means. The convergence of the Calabi-Yau measures on X^{hyb} towards μ_0 strongly suggests that the solution to the non-archimedean Monge-Ampère equation:

$$\mathrm{MA}(\phi) = \mu_0$$

should be the appropriate non-archimedean limit to the Calabi-Yau metrics. The difficulty is now to try and find a way to identify ϕ in a way to the multivalued convex function on $\mathrm{Sk}(X)$ from the previous paragraph. As in the archimedean case, one needs to produce an SYZ fibration $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ such that the metric ϕ may be identified via ρ with the pullback of a (multivalued) convex function on the base, and show that this identifies the non-archimedean Monge-Ampère operator with the real Monge-Ampère operator on $\mathrm{Sk}(X) \setminus \Gamma$.
A first step is provided by the following theorem, due to Vilsmeier [Vil20]:

Theorem 0.0.11. *Let $(\mathscr{X}, \mathscr{L})$ be a semi-stable model of (X, L), and write the non-archimedean Calabi-Yau metric ϕ as:*

$$\phi = \phi_{\mathscr{L}} + \psi,$$

where $\phi_{\mathscr{L}}$ is the model metric associated to $\mathscr{L}$ and $\psi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ is a continuous function. Assume that $\psi = \psi \circ \rho_{\mathscr{X}}$ over the interior of an n-dimensional face $\tau \subset \mathrm{Sk}(\mathscr{X})$. Then ψ is convex on $\mathrm{Int}(\tau)$, and the equality of measures:

$$\mathbf{1}_{\mathrm{Int}(\tau)}\, \mathrm{MA}(\phi) = n! \mathscr{M}(\psi),$$

holds on $\mathrm{Int}(\tau)$, where $\mathscr{M}(\psi)$ is the real Monge-Ampère measure of the convex function ψ.

Building on this, the recent results of Yang Li [Li20a] reduce the existence of the classical SYZ fibration to the question of finding a non-archimedean SYZ fibration ρ, such that the NA Calabi-Yau metric is constant on the fibers of ρ:

Theorem 0.0.12. *Let $X/\mathbb{D}^*$ be a polarized, maximal degeneration of Calabi-Yau manifolds, and write the solution to the non-archimedean Monge-Ampère equation* $\mathrm{MA}(\phi) = \mu_0$ *as:*

$$\phi = \phi_{\mathscr{L}} + \psi,$$

for an snc model $(\mathscr{X}, \mathscr{L})$ of (X, L) and $\psi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ a continuous function. Assume that the NA/real Monge-Ampère comparison property:

$$\psi = \psi \circ \rho_{\mathscr{X}}$$

holds on the interior of the maximal faces of $\mathrm{Sk}(X)$.
Then for $|t| \ll 1$, there exists a special Lagrangian fibration $f_t : U_t \longrightarrow \mathrm{Sk}(X)$, where $U_t \subset X_t$ is an open subset of Calabi-Yau measure arbitrarily close to 1.

The above theorem reduces the classical SYZ conjecture to a problem in non-archimedean geometry, but however does not answer the question of global Gromov-Hausdorff convergence of the Calabi-Yau metrics; indeed the union of open maximal faces of $\mathrm{Sk}(X)$ is disconnected, so that to be able to access the distance between two different faces, it seems necessary to establish a connection between the real and non-archimedean Monge-Ampère operator in codimension 1 - still under a suitable invariance property for the Calabi-Yau potential on X^{an}.

Summary of the main results

Pluripotential theory on hybrid spaces

Let $X \longrightarrow \mathbb{D}^*$ be a projective degeneration of complex manifolds, and let L be a relatively ample line bundle on X. Given a plurisubharmonic metric $\phi \in \mathrm{PSH}(X, L)$ satisfying a reasonable growth condition at $t = 0$, one can associate to it a plurisubharmonic metric $\phi^{\mathrm{NA}} \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ on the non-archimedean analytic space X^{an}, encoding the logarithmic blow-up rate of ϕ along the special fibers of models of (X, L). In the case where $X = Y \times \mathbb{D}^*$ is a product and ϕ is $\mathbb{S}^1$-invariant, one says that ϕ is a psh ray on Y, and the associated non-archimedean metric was defined in [BBJ18]; while the general case was treated in [Reb21].
One would like to say that the metric ϕ^{NA} is in a natural way the limit of the $\phi_t = (\phi)_{|X_t}$ on the hybrid space. However taking the limit in the topological sense of the word would be too naive, as it is desirable to be able to work with singular metrics. It is thus natural to try and define a class of singular metrics on the hybrid space, which recover both complex singular metrics on the degeneration X, as well as their associated non-archimedean metrics.
In the first chapter of this thesis, we define a class of plurisubharmonic metrics $\mathrm{PSH}(X, L)$ on the Berkovich analytification X^{an} of a polarized scheme X over an integral Banach ring A; roughly speaking, a singular metric $\phi \in \mathrm{PSH}(X, L)$ can be seen as a family of psh metrics $\phi_x \in \mathrm{PSH}(X_{\mathscr{H}(x)}, L_{\mathscr{H}(x)})$, varying in a plurisubharmonic way with respect to $x \in \mathscr{M}(A)$. Note however that it can happen that $\phi_x \equiv -\infty$ for some $x \in \mathscr{M}(A)$.
The class $\mathrm{PSH}(X, L)$ is defined following the approach of [BE21] : it is the smallest

class of singular metrics on L that contains metrics of the form $m^{-1}\log|s|$ whenever $s \in H^0(X, mL)$ is a non-zero section, and is closed under finite maxima and decreasing limits.
In the case where (X, L) is a degeneration of polarized complex manifolds, a psh metric on the analytic space X^{hyb} induces by restriction and rescaling a semi-positive metric $\phi \in \mathrm{PSH}(X, L)$ with logarithmic growth, as well as a psh metric $\phi_0 \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$, and is uniquely recovered by those two restrictions. Conversely, given a psh metric ϕ with logarithmic growth on the degeneration X, we prove that the associated non-archimedean metric ϕ^{NA} induces a canonical semi-positive extension of ϕ to the hybrid space:

Theorem A. *[PS22a, thm. A] Let $\phi \in \mathrm{PSH}(X, L)$ be a psh metric with logarithmic growth at $t = 0$. Then the singular metric ϕ^{hyb} on $(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$ such that:*

$$\phi_0^{\mathrm{hyb}} = \phi^{\mathrm{NA}},$$

$$\phi^{\mathrm{hyb}}_{|X_t} = \phi_t$$

is semi-positive.

Note that not all psh metrics on the hybrid space arise in this way: the point $0 \in \bar{\mathbb{D}}_r$ is pluripolar - non-negligible in the sense of hybrid pluripotential theory - so that psh hybrid metrics are not uniquely recovered by their restriction to the punctured disk. This subtlety however disappears when restricting our attention to continuous psh metrics, as $0 \in \bar{\mathbb{D}}_r$ has empty interior.
We then focus on the case of a polarized toric variety (Z, L) over $\mathbb{C}$. In that case, continuous toric psh metrics on L can be described explicitly in terms of certain convex functions on the vector space $N_{\mathbb{R}}$ where the fan Σ of Z lives - and this, regardless of the choice of absolute value on $\mathbb{C}$, Euclidean or trivial. More precisely, the polarization L is encoded combinatorially by a piecewise-linear convex function $\Psi_L : N_{\mathbb{R}} \longrightarrow \mathbb{R}$, and toric metrics on L are in one-to-one correspondance with convex functions on $N_{\mathbb{R}}$ which have the same growth as Ψ_L at infinity.
We then prove that a similar statement holds for toric continuous psh hybrid metrics: the semi-positivity in the horizontal direction is naturally encoded by convexity along the direction of the base $\mathscr{M}(\mathbb{C}^{\mathrm{hyb}}) = [0, 1]$.

Theorem B. *Let (Z, L) be a polarized complex toric variety. There is a one-to-one correspondence between continuous toric psh metrics on L^{hyb} and convex continuous functions:*

$$\Phi : N_{\mathbb{R}} \times [0, 1] \longrightarrow \mathbb{R}$$

such that $(\Phi - \Psi_L)$ extends continuously to the compactification $N_\Sigma \times [0, 1]$ of $N_{\mathbb{R}} \times [0, 1]$.

The non-archimedean SYZ fibration

The second part of this thesis is devoted to the following problem: given a maximally degenerate Calabi-Yau manifold X/K, polarized by an ample line bundle L, trying to produce a non-archimedean avatar $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ of the conjectural

SYZ fibration - with the hope of finding a retraction such that the non-archimedean Calabi-Yau metric on L^{an} is constant along the fibers of the retraction ρ.
Since one of the main goals is to identify the solution to the non-archimedean Monge-Ampère equation with a solution of the real Monge-Ampère equation, it is desirable to have at our disposal a fibration which is "smooth" away from a codimension 2 locus of the base, as it yields this way an integral affine structure on $\mathrm{Sk}(X)$ which has singularities in codimension 2.
A first step was to understand the singularities of the Berkovich retraction associated to a model $\mathscr{X}$, or rather the absence thereof. The following theorem, joint with Enrica Mazzon, states that if a Z is a toric stratum of the special fiber of a degeneration of algebraic varieties, then under a certain positivity assumption on the conormal bundle of Z, the Berkovich retraction $\rho_{\mathscr{X}}$ has no singularities over the open star $\mathrm{Star}(\tau_Z)$ of the face $\tau_Z \subset \mathrm{Sk}(\mathscr{X})$, i.e. the union of the interiors of the faces containing τ_Z:

Theorem C. *[MPS21, thm. B] Let X/K be a smooth projective variety of dimension n, and $\mathscr{X}/R$ be a good dlt model of X with reduced special fiber $\mathscr{X}_0 = \sum_\alpha D_\alpha$, such that every D_α is a Cartier divisor.*
Let $Z = D_0 \cap D_1 \cap \ldots \cap D_{n-r}$ be an r-dimensional stratum of $\mathscr{X}_0$, such that:

- *$\mathring{Z} \subset Z$ is a torus embedding, where $\mathring{Z} = Z \setminus \cup_{\alpha \neq 0,1,\ldots,n-r} D_\alpha$;*
- *the conormal bundle $\nu^*_{Z/\mathscr{X}}$ is a nef vector bundle on Z;*
- *for each $\alpha \notin \{0,\ldots,n-r\}$, the intersection $D_\alpha \cap Z$ is either empty or connected.*

Then the formal completion $\widehat{\mathscr{X}_{/Z}}$ is isomorphic to the formal completion of the normal bundle $\mathcal{N} = \nu_{Z/\mathscr{X}}$ along the zero section, which is a toric variety. In particular, the Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$ is an n-dimensional affinoid torus fibration over $\mathrm{Star}(\tau_Z)$.

We then focus on degenerations of hypersurfaces of the form:

$$X = \{z_0 \ldots z_{n+1} + tF_{n+2} = 0\} \subset \mathbb{P}^{n+1} \times \mathbb{D}^*,$$

where F_{n+2} is a generic homogeneous polynomial of degree $(n+2)$. The essential skeleton $\mathrm{Sk}(X)$ is explicitly given as the dual intersection complex of the toric boundary of $\mathbb{P}^{n+1}$, and can be in fact realized canonically inside the vector space $N_{\mathbb{R}}$ where the fan of $\mathbb{P}^{n+1}$ lives. More precisely, writing $\mathbb{T} \subset \mathbb{P}^{n+1}$ the open torus, the usual tropicalization map:

$$\mathrm{val} : \mathbb{T}^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}$$

maps $\mathrm{Sk}(X) \subset X^{\mathrm{an}}$ homeomorphically onto the maximal bounded subcomplex of the tropicalization $\mathrm{Trop}(X) := \mathrm{val}(X^{\mathrm{an}} \cap \mathbb{T}^{\mathrm{an}}) \subset N_{\mathbb{R}}$ of the hypersurface - which is endowed with a canonical polyhedral decomposition.
After suitably choosing a discriminant locus $\Gamma \subset \mathrm{Sk}(X)$ of codimension 2 (also called *choice of branch cuts*), we may put an integral affine structure on $\mathrm{Sk}(X) \setminus \Gamma$ that encodes the toric nature of the degeneration, following [GS06]. This integral affine structure has for instance no singularity near each vertex $v_i \in \mathrm{Sk}(X)$, and

the germ of affine manifold at v_i is nothing but the germ at 0 of the fan Σ_i of the corresponding component D_i. It does however depend on the choice of branch cuts, and is thus not canonical.
Building on theorem C and on the construction by Yamamoto [Yam21] of certain tropical contractions:

$$\delta : \mathrm{Trop}(X) \longrightarrow \mathrm{Sk}(X),$$

we prove the following:

Theorem D. *Let $\underline{a} = (a_\tau)_\tau$ be a choice of branch cuts in $\mathrm{Sk}(X)$, and let $\delta_{\underline{a}} : \overline{\mathrm{Trop}}(X) \longrightarrow \mathrm{Sk}(X)$ be the associated tropical contraction [Yam21, thm. 5.1]. Then the composition:*

$$\rho_{\underline{a}} := \delta_{\underline{a}} \circ \mathrm{val}_\Sigma : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$$

is an affinoid torus fibration over $\mathrm{Sk}(X) \setminus \Gamma$. Moreover, the induced integral affine structure on $\mathrm{Sk}(X) \setminus \Gamma$ coincides with the one mentioned above.

In dimension 2 and 3, the retraction $\rho_{\underline{a}}$ was constructed without using the tropical contraction in a joint work with E. Mazzon [MPS21]. This alternative construction also allowed us to refine further the discriminant locus in dimension 2, and obtain integral affine structures on the 2-sphere with 24 singular points of Kodaira type I_1; we refer to section 4.2.1 for details.
Finally, we focus on the particular example of the Fermat family of hypersurfaces:

$$X = \{z_0 ... z_{n+1} + t(z_0^{n+2} + ... + z_{n+1}^{n+2}) = 0\} \subset \mathbb{P}^{n+1} \times \mathbb{D}^*,$$

which enjoys the $\mathfrak{S}_{n+2}$ symmetry. In that example, the asymptotic behaviour of the (archimedean) Calabi-Yau metrics is well-understood [Li22]: the Calabi-Yau potential with respect to the ambient Fubini-Study metric is close to a toric potential, which is defined on the ambient projective space. Using the results of [Li22] and theorem B, we are able prove the following:

Theorem E. *Let $X \longrightarrow \mathbb{D}^*$ be the Fermat family of hypersurfaces, polarized by $L = \mathcal{O}_{\mathbb{P}}(1)$. Then there exists a semi-positive toric metric $\phi \in \mathrm{CPSH}(\mathbb{P}_K^{n+1,\mathrm{an}}, L^{\mathrm{an}})$ whose restriction to X^{an} solves the non-archimedean Monge-Ampère equation:*

$$\mathrm{MA}(\phi) = \mu_0,$$

where μ_0 is the Lebesgue measure on $\mathrm{Sk}(X)$. Moreover, writing $\phi = \phi_{\mathrm{FS}} + \psi$, the continuous function ψ on X^{an} satisfies the comparison property:

$$\psi = \psi \circ \rho_{\underline{a}}$$

over $\mathrm{Sk}(X) \setminus \Gamma$, where $\rho_{\underline{a}}$ is the retraction from theorem D, a_τ is the barycenter of τ for each face $\tau \subset \mathrm{Sk}(\mathscr{X})$, and $\Gamma \subset \mathrm{Sk}(X)$ is the associated discriminant locus of codimension 2.

The function ψ is also in a natural sense the limit of the archimedean Calabi-Yau potentials.
A similar statement for more general hypersurfaces was obtained independently in [HJMM22].

Degenerations of canonically polarized manifolds

Let $X \longrightarrow \mathbb{D}^*$ be a degeneration of canonically polarized manifolds, and write $L = K_{X/\mathbb{D}^*}$. By the classical Aubin-Yau theorem, each fiber X_t admits a unique Kähler-Einstein metric ϕ_t on L_t, whose curvature form $\omega_t = dd^c\phi_t$ has negative constant Ricci curvature:

$$\mathrm{Ric}(\omega_t) = -\omega_t.$$

Moreover, it follows from the results of Schumacher [Sch12] that the family of Kähler-Einstein metrics $\phi = (\phi_t)_{t\in\mathbb{D}^*}$ also has positive curvature in the direction of the base and has logarithmic growth at $t = 0$, so that $\phi \in \mathrm{PSH}(X, L)$ satisfies the assumptions of theorem A.
The machinery of the Minimal Model Program furthermore implies that after a finite base change, the family admits a unique canonical model $\mathscr{X}_c/\mathbb{D}$, which has ample relative canonical bundle $K_{\mathscr{X}_c/\mathbb{D}}$. Moreover, by the results of J. Song [Son17], the Kähler-Einstein metrics converge in a natural sense to a unique Kähler-Einstein current $\omega_{KE,0}$ on the special fiber $\mathscr{X}_{c,0}$, and even though this current does not have bounded potentials, its singularities are milder than any log poles. We are thus able to show that the non-archimedean limit of the Kähler-Einstein metrics is the model metric $\phi_{K_{\mathscr{X}_c/R}}$ associated to the canonical model $(\mathscr{X}_c, K_{\mathscr{X}_c/\mathbb{D}})$:

Theorem F. *[PS22a, thm. B] Let $X \xrightarrow{\pi} \mathbb{D}^*$ be a degeneration of canonically polarized manifolds, $L = K_{X/\,\mathbb{D}^*}$, and let $\phi_{\mathrm{KE}} \in \mathrm{PSH}(X, L)$ be the family of Kähler-Einstein metrics. We assume that the family X has semi-stable reduction over $\mathbb{D}$. Then the metric on L^{hyb} defined by:*

$$\phi_{|X} = \phi_{\mathrm{KE}},$$

$$\phi_0 = \phi_{K_{\mathscr{X}_c/R}}$$

is continuous and plurisubharmonic on X^{hyb}.

Organization of the manuscript

This text is organized as follows.
The first chapter is devoted to the definition and study of a class of global semi-positive metrics on Berkovich analytifications of schemes over a Banach ring A - in practice, we will be mostly interested about the ring of convergent power series for the hybrid norm on $\mathbb{C}$, so that the analytic spaces we are considering are hybrid spaces. Section 1.1 recalls general facts about Berkovich analytic spaces - in particular over a discretely-valued field - and about hybrid spaces. In section 1.2, we define in a global way a class of plurisubharmonic metrics on the Berkovich analytification of an A-scheme X, and study its basic properties. In section 1.3 we focus on the case where X is a projective degeneration of complex varieties over the punctured disk $\mathbb{D}^*$, and its Berkovich analytification is the associated hybrid space X^{hyb}. We compare in particular our framework to the work of [BBJ18], [Reb21], and prove theorem A (see theorem 1.3.13). In section 1.4, we discuss Monge-Ampère measures associated to psh metrics on analytic spaces over a valued field, and families there of in the case of a more general base. In particular, we state that the family of Monge-Ampère measures associated to a continuous psh metric on a hybrid space X^{hyb} is weakly continuous, this result is essentially due to Favre [Fav20].
The second chapter focuses on the following setting: Z is a complex, projective toric variety, Z^{hyb} is its analytification with respect to the hybrid norm $\max\{|\cdot|_0, |\cdot|\}$ on $\mathbb{C}$, and L is a semi-ample line bundle on Z. In this setting, we prove theorem B which provide a combinatorial description of semi-positive metrics on L^{hyb} that are fiberwise-invariant under the torus action: they are given by continuous convex functions $\Phi : N_{\mathbb{R}} \times [0,1] \longrightarrow \mathbb{R}$ satisfying a suitable growth condition at infinity (see 2.2.12). In section 2.1, we recall standard facts from convex and toric geometry that will be used in section 2.2, where we discuss fiberwise-toric metric on hybrid spaces and prove theorem B.
Throughout the third chapter, we let $(X, L) \longrightarrow \mathbb{D}^*$ be a polarized, maximal degeneration of Calabi-Yau manifolds. We devote section 3.1 to a discussion on the SYZ and Kontsevich-Soibelman conjecture, and study the local model of the non-archimedean analog of a Lagrangian torus fibration. Section 3.2 is devoted to the proof of theorem C, which is a first step toward producing more global examples of NA torus fibrations. We also include, in section 3.3, various computations of the singular, integral affine structures constructed via theorem C.
The fourth chapter focuses on maximally degenerate Calabi-Yau hypersurfaces in projective space. In section 4.1, we construct a non-archimedean SYZ fibration ρ for such hypersurfaces, and prove theorem D. We also present an alternative construction of ρ in dimensions 2 and 3 that was performed in [MPS21], this is the content of section 4.2. In section 4.3, we focus on the Fermat family of hypersurfaces, and prove theorem E building on the results from chapters 1 and 2 and [Li22]: the non-archimedean Calabi-Yau metric in this setting is the restriction of a toric metric living on the ambient projective space.
Finally, in a fifth short chapter, we investigate degenerations of Kähler-Einstein manifolds with negative curvature, in the framework from chapter 1. We set the scene in section 5.1, and state theorem F, and then recall some facts about canonical models of such degenerations in section 5.2. In section 5.3, we describe the results

from [Son17] regarding convergence of the Kähler-Einstein metrics on the canonical model, and prove theorem F. We conclude with a short discussion on the Fano case in section 5.4.

Notation and conventions

All rings are assumed to be unitary and commutative.
We will use additive notation for line bundles: if L, M are two line bundles on a variety X, we write $L + M := L \otimes M$, and $kL := L^{\otimes k}$ for $k \in \mathbb{Z}$.
If X is a complex manifold and ϕ a smooth function on X, we set $dd^c\phi = \frac{i}{2\pi}\partial\bar{\partial}\phi$.
We extend the notation to Hermitian metrics on line bundles, so that if L is a holomorphic line bundle on X and ϕ a smooth metric on L, the curvature form $dd^c\phi \in c_1(L)$ - and similarly for singular semi-positive metrics.
Throughout this text, whenever we say that $X \xrightarrow{\pi} \mathbb{D}^*$ is a degeneration of complex manifolds, we mean that X is a smooth complex manifold and π a holomorphic submersion (which will often be omitted from notation). We will furthermore always assume that the degeneration is meromorphic at 0, i.e. that X is defined over the ring of germs at zero of holomorphic functions. Equivalently, this means that there exists a normal complex analytic space $\mathscr{X} \xrightarrow{\pi} \mathbb{D}$ such that $\mathscr{X}_{|\mathbb{D}^*} = X$.

Chapter 1

Non-archimedean pluripotential theory

The study of plurisubharmonic functions and positive currents on complex manifolds - referred to as *pluripotential theory* - has proven itself over the past decades to be a central tool in complex Kähler and algebraic geometry. The heuristic idea that one should be able to develop a pluripotential theory on Berkovich analytic spaces over a non-archimedean field K, similar to the classical one over the complex numbers, is by now well-established, see for instance [BR10], [CLD12] [BFJ16], [BE21] in chronological order.

To be more precise, let K be a complete non-archimedean field, and X/K a proper algebraic variety. In this setting, Berkovich's theory of K-analytic spaces [Ber90] associates to the variety X/K its Berkovich analytification X^{an}, in a similar spirit to how one associates to a variety over $\mathbb{C}$ its complex analytification. One may then define in this setting a class $\mathrm{PSH}(X, L)$ of plurisubharmonic (singular) metrics on L^{an}, whose properties mimic those of plurisubharmonic metrics in the complex case. While the work of Chambert-Loir - Ducros [CLD12] provides a local definition of the semi-positivity condition, we will adopt a global point of view throughout this paper, following the approach initiated by Boucksom-Favre-Jonsson in [BFJ16] and further developped in [BJ18], [BJ22]. The basic idea is as follows: for $m \geq 1$, given a non-zero global section $s \in H^0(X, mL)$, the singular metric (using additive notation for metrics) $\phi = m^{-1} \log|s|$ should be plurisubharmonic, as follows in the complex world from the Lelong-Poincaré formula. It moreover follows from Demailly's seminal work [Dem92] on regularization of plurisubharmonic functions that on a smooth polarized complex variety (X, L), the class $\mathrm{PSH}(X, L)$ is the smallest class of singular metrics containing the metrics of the form $\phi = m^{-1} \log|s|$ as above, and that is furthermore stable by addition of constants, finite maxima and decreasing limits (see thm. 1.2.24). It is thus natural to take this characterization as the definition of $\mathrm{PSH}(X, L)$ in the non-archimedean setting, which turns out to be consistent with the more local, Chambert-Loir - Ducros approach, by the results from [BE21].

One of the upsides of Berkovich's construction of analytic spaces is that it works over more general bases than non-archimedean fields: given a Banach ring $(A, |\cdot|)$, one can define its Berkovich spectrum $\mathscr{M}(A)$, and for any reasonable scheme X/A, a Berkovich analytic space $X^{\mathrm{an}} \xrightarrow{\pi} \mathscr{M}(A)$ equipped with a continuous structure map

to $\mathscr{M}(A)$. Each point $x \in \mathscr{M}(A)$ has a residue field $\mathscr{H}(x)$ in a natural sense, which is a complete valued field, and when A is a geometric base ring (which will always be true of the rings considered in this chapter, see section 1.1.1) the fiber $\pi^{-1}(x)$ of the structure map is naturally homeomorphic to Berkovich analytification of the base change $X_{\mathscr{H}(x)}$. One may thus view the A-analytic space X^{an} as the family of analytic spaces $(X^{\mathrm{an}}_{\mathscr{H}(x)})_{x\in\mathscr{M}(A)}$ over different base fields, in a similar manner to which one views a scheme over a ring A as a family of varieties over different base fields, parametrized by $\operatorname{Spec} A$. More general Berkovich analytic spaces over Banach rings were studied more extensively recently in [LP20], where it is for instance proved that A-analytic spaces form a category in a natural way, under certain assumptions on A.

At this point, pluripotential theory on Berkovich spaces over a Banach ring remains a vastly unexplored territory. In this paper, under suitable assumptions on the base ring, we define a class $\mathrm{PSH}(X, L)$ of semi-positive singular metrics on a scheme X/A of finite type, endowed with a semi-ample line bundle L. Roughly speaking, a singular metric $\phi \in \mathrm{PSH}(X, L)$ can be seen as a family of semi-positive singular metrics $\phi_x \in \mathrm{PSH}(X_{\mathscr{H}(x)}, L_{\mathscr{H}(x)})$ on the fibers of the structure map, varying in a plurisubharmonic way with respect to $x \in \mathscr{M}(A)$. Note however that it can happen that $\phi_x \equiv -\infty$ for some $x \in \mathscr{M}(A)$, as one would expect in the complex world. The class $\mathrm{PSH}(X, L)$ is defined following the global approach of [BFJ16], [BE21] : it is the smallest class of singular metrics on L that contains metrics of the form $m^{-1}\log|s|$ whenever $s \in H^0(X, mL)$ is a non-zero global section, and is stable under addition of constants, finite maxima and decreasing limits.

Our main concern is the case where A is a *hybrid ring* (see section 1.1.3 for the definitions) and $X \longrightarrow \mathbb{D}^*$ is a projective degeneration of complex manifolds - we will always assume the degeneration to be meromorphic at $t = 0$, which means that we may view X as a projective scheme over the field $K = \mathbb{C}((t))$ of complex Laurent series - so that $X^{\mathrm{an}} = X^{\mathrm{hyb}}$ is the associated hybrid space, as studied for instance in [BJ17], [Fav20]. The hybrid space $X^{\mathrm{hyb}} \xrightarrow{\pi} \bar{\mathbb{D}}_r$ comes with a continuous structure map to the closed disk of radius $r \in (0, 1)$, such that $\pi^{-1}(0) = X^{\mathrm{an}}_K$ is the Berkovich analytification of X with respect to the non-archimedean t-adic absolute value on K, while $\pi^{-1}(\bar{\mathbb{D}}^*_r)$ can be naturally identified with the restriction of the degeneration X to the closed punctured disk $\bar{\mathbb{D}}^*_r$ - up to rescaling the absolute value on the fiber X_t by a factor $\frac{\log r}{\log|t|}$. As a result, this provides a natural way to see the (rescaled) complex manifolds $(X_t)_{t\in\mathbb{D}^*}$ degenerate to a non-archimedean space X^{an}_K as $t \to 0$. In this setting, if L is a semi-ample line bundle on X, then a psh metric on $(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$ corresponds to the data of a family of psh metrics $\phi_t \in \mathrm{PSH}(X_t, L_t)$ varying in a subharmonic way with respect to t, together with a non-archimedean metric $\phi_0 \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$.

In the case where the line bundle L is ample on X, given a semi-positive metric $\phi \in \mathrm{PSH}(X, L)$ which has logarithmic growth at $t = 0$ (see definition 1.3.9), one can associate to it a semi-positive metric $\phi^{\mathrm{NA}} \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ on the non-archimedean analytic space X^{an}, encoding the generic Lelong numbers of ϕ along the centrals fibers of models of (X, L) over the disk. In the case where $X = Y \times \mathbb{D}^*$ is a product and ϕ is an $\mathbb{S}^1$-invariant metric on p_1^*L for an ample line bundle L on Y, one can view $\phi : \mathbb{R}_{\geq 0} \longrightarrow \mathrm{PSH}(Y, L)$ as a psh ray on Y, and the non-archimedean limit was

defined by Berman-Boucksom-Jonsson [BBJ18] in the context of their proof of the Yau-Tian-Donaldson conjecture; while the general case was treated in [Reb21].
The main result of this chapter is theorem 1.3.13, which states that if (X, L) is a polarized degeneration of complex manifolds over $\mathbb{D}^*$, and $\phi \in \mathrm{PSH}(X, L)$ is a psh family of metrics on L, then ϕ extends in a canonical way to X^{hyb} to a metric ϕ^{hyb}, whose restriction to the non-archimedean fiber ϕ^{NA} is the metric mentioned. We then compare our formalism with the set-up of [Fav20] for continuous psh metrics on the hybrid space, which yields a continuity result for the family of fiberwise Monge-Ampère measures on the hybrid space associated to a continuous psh hybrid metric (theorem 1.4.11).
This chapter is organized as follows. In section 1.1, we recall some general facts about Berkovich analytic spaces over Banach rings, and in section 1.2 we define and prove basic properties of the class $\mathrm{PSH}(X, L)$ of semi-positive singular metrics on a polarized scheme $(X, L)/A$ over a geometric base ring; we also give some explicit examples along the way. We then move on to the case of hybrid spaces: section 1.3 is devoted to the statement and the proof of theorem 1.3.13, and section 1.4 is a discussion on families of Monge-Ampère measures, where we prove theorem 1.4.11.

1.1 Berkovich analytic spaces

1.1.1 Definitions

Definition 1.1.1. *Let $A \neq 0$ be a ring. A (submultiplicative) semi-norm $\|\cdot\|$ on A is a map $\|\cdot\| : A \longrightarrow \mathbb{R}_{\geq 0}$ such that :*

- $\|1\| = 1$ *and* $\|0\| = 0$,
- $\forall a, b \in A,\ \|a + b\| \leq \|a\| + \|b\|$,
- $\forall a, b \in A,\ \|ab\| \leq \|a\|\|b\|$.

Its kernel $\mathrm{Ker}\|\cdot\| = \{a \in A / \|a\| = 0\}$ *is an ideal of A, which is prime when $\|\cdot\|$ is multiplicative. A submultiplicative semi-norm on A whose kernel is reduced to zero is called a norm on A.*
Finally, a Banach ring A is a non-zero ring equipped with a submultiplicative norm $\|\cdot\|$ such that A is complete with respect to $\|\cdot\|$.

For example, any non-zero ring A endowed with the trivial norm $\|\cdot\|_0$ (such that $\|a\|_0 = 1$ for any non-zero $a \in A$) is a Banach ring.

Definition 1.1.2. *Let A be a Banach ring.*
The Berkovich spectrum $\mathscr{M}(A)$ is the set whose points $x \in \mathscr{M}(A)$ are multiplicative semi-norms $|\cdot|_x : A \longrightarrow \mathbb{R}_{\geq 0}$ satisfying $|\cdot|_x \leq \|\cdot\|$.
It is equipped with the topology of pointwise convergence on A, which makes it into a non-empty Hausdorff compact topological space by [Ber90, thm. 1.2.1], and with a map $q : |\cdot|_x \mapsto \mathfrak{p}_x = \mathrm{Ker}(|\cdot|_x)$ to $\mathrm{Spec}(A)$ which is continuous.

For instance, if $A = k$ is a complete valued field, then $\mathscr{M}(k)$ is reduced to the point $\|\cdot\|$.

Example 1.1.3. *Let $A = \mathbb{Z}$ be the ring of integers, endowed with the usual archimedean absolute value $|\cdot|_\infty$. Let $x_0 = |\cdot|_0 \in \mathscr{M}(\mathbb{Z})$ be the trivial absolute value on $\mathbb{Z}$, and let $x_p = |\cdot|_p$ be the p-adic absolute value, normalized setting $|p|_p = p^{-1}$.*
It follows from Ostrowski's theorem that any point $x \in \mathscr{M}(\mathbb{Z})$ is of the following form: either there exists $\varepsilon \in [0, +\infty]$ and p a prime number such that $x = |\cdot|_p^\varepsilon$, or there exists $\varepsilon \in [0, 1]$ such that $x = |\cdot|_\infty^\varepsilon$; where we denote $|\cdot|_x^0 = |\cdot|_0$ the trivial absolute value for any $x \in \mathscr{M}(\mathbb{Z})$, while $|\cdot|_p^\infty$ is the absolute value such that $|n|_p^\infty = 0$ if p divides n, and $|n|_p^\infty = 1$ otherwise.
Topologically, the Berkovich spectrum $\mathscr{M}(\mathbb{Z})$ is thus an infinite wedge of segments parametrized by the prime numbers and ∞, glued together at the trivial absolute value x_0. Note that for each neighbourhood V of x_0 in $\mathscr{M}(\mathbb{Z})$, the set of branches not contained in V is finite.
The map $\mathscr{M}(\mathbb{Z}) \xrightarrow{q} \mathrm{Spec}(\mathbb{Z})$ maps the outer end $|\cdot|_p^\infty$ of the p-adic branch to the prime ideal (p), and any other point to the generic point of $\mathrm{Spec}(\mathbb{Z})$.

Fix A a Banach ring, and let B be a finitely generated A-algebra. The analytification Y^{an} of $Y = \mathrm{Spec}(B)$ is defined as the set of multiplicative semi-norms $|\cdot|_x$ on B whose restriction to A belong to $\mathscr{M}(A)$. It is endowed with the coarsest topology making the maps $x \mapsto |f|_x$ continuous, for $f \in B$, and comes with a continuous structure morphism $Y^{\mathrm{an}} \longrightarrow \mathscr{M}(A)$ sending a semi-norm to its restriction to A.
If X is a scheme of finite type over A, one can then glue the analytifications of affine charts of X in order to obtain an *analytification functor* $X \mapsto X^{\mathrm{an}}$ from the category of A-schemes of finite type to the category of A-analytic spaces [LP20]. Any A-analytic space comes with a sheaf of analytic functions, as defined in [Ber90, def. 1.5.3].
When the base ring A is a geometric base ring (see below), the space X^{an} satisfies nice topological properties by [LP20]: if X/A is separated, then X^{an} is Hausdorff; and X^{an} is compact when X/A is projective.

Example 1.1.4. *Let $A = \mathbb{C}$ endowed with the Euclidean absolute value, and B be a complex Banach algebra of finite type. Then the classical Gelfan'd-Mazur theorem implies that $Y^{\mathrm{an}} = (\mathrm{Spec}\, B)^{\mathrm{an}}$ is the set of maximal ideals of B. As a result, Y^{an} is the set $Y(\mathbb{C})$ of closed points of the affine complex algebraic variety Y, and the induced topology on Y is the Euclidean one. More generally, if Y is a reduced scheme of finite type over $\mathbb{C}$, then the analytification Y^{an} of Y with respect to the Euclidean absolute value on $\mathbb{C}$ is homeomorphic to the complex variety Y endowed with the Euclidean topology.*

Example 1.1.5. *Let $A = K$ be a complete non-archimedean field, and let X/K be a separated scheme of finite type. Then the Berkovich space X^{an} can be described more explicitly as the set of couples $x = (\xi, v_x)$, where $\xi = \xi_x$ is a scheme-theoretic point of X and v_x is a real valuation on the function field of $\overline{\xi}_x$, extending the valuation on K.*
Moreover, the map $x \mapsto \xi_x$ from X^{an} to X^{sch} is continuous, surjective, and induces a bijection between the respective sets of connected components; the closed subscheme $Y := \overline{\xi}_x$ will be called the support of $x \in X^{\mathrm{an}}$.

Definition 1.1.6. *Let A be a Banach ring, and $x \in \mathscr{M}(A)$. Write $\mathfrak{p}_x = q(x) = \mathrm{Ker}(|\cdot|_x)$, $\kappa(x) = \mathrm{Frac}(A/\mathfrak{p}_x)$ the schematic residue field of A at $\mathfrak{p}_x$. The semi-*

norm $|\cdot|_x$ descends to an absolute value on $\kappa(x)$; we write $\mathscr{H}(x)$ for the associated completion of $\kappa(x)$ and call it the residue field of $\mathscr{M}(A)$ at x.

Let A and B be two Banach rings, and assume from now on that A and B are geometric base rings. We will not give the definition here, and rather refer the reader to [LP20, def. 3.3.8]. Note however that this includes all Banach rings relevant for us, in particular discrete valuation rings, hybrid rings (see section 1.1.3 for the definition) or the ring of integers of a number field. In this setting, one can define [LP20] a well-behaved category of analytic spaces over A, which in particular contains the analytifications of A-schemes of finite type (and similarly for B).
Assume given a bounded ring homomorphism $A \longrightarrow B$. This induces a continuous map:

$$\rho : \mathscr{M}(B) \longrightarrow \mathscr{M}(A),$$

which simply sends a semi-norm on B to its pull-back to A. Given an analytic space $\mathfrak{X}$ over A, its base change $\mathfrak{X}_B$ to B exists by [LP20, thm. 4.2.4]; when $\mathfrak{X} = X^{\mathrm{an}}$ is the analytification of an A-scheme X of locally finite presentation, the base change $(X^{\mathrm{an}})_B$ is canonically isomorphic to the analytification of the B-scheme X_B by [LP20, §4.1, 4.2], so that we simply denote it by X_B^{an}, and we have a commutative diagram:

$$\begin{array}{ccc} X_B^{\mathrm{an}} & \xrightarrow{F} & X^{\mathrm{an}} \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_A} \\ \mathscr{M}(B) & \xrightarrow{\rho} & \mathscr{M}(A). \end{array}$$

The following proposition allows us, still assuming that A is a geometric base ring, to view a Berkovich space over a Banach ring A as a family of Berkovich spaces over complete valued fields, parametrized by $\mathscr{M}(A)$:

Proposition 1.1.7. *[LP20, prop. 4.4.8]*
Let A be a geometric base ring, X a scheme of finite type over A, and $\pi : X^{\mathrm{an}} \longrightarrow \mathscr{M}(A)$ the associated analytic space. If $x \in \mathscr{M}(A)$, then $\pi^{-1}(x)$ is canonically homeomorphic to the analytification of the base change $X_{\mathscr{H}(x)} = X \times_A \mathscr{H}(x)$ with respect to the absolute value $|\cdot|_x$ on $\mathscr{H}(x)$.

More precisely, the base change morphism $F_x : X^{\mathrm{an}}_{\mathscr{H}(x)} \longrightarrow X^{\mathrm{an}}$ is a topological embedding, and induces a homeomorphism between $X^{\mathrm{an}}_{\mathscr{H}(x)}$ and $\pi^{-1}(x) \subset X^{\mathrm{an}}$.

Definition 1.1.8. *Let A be an integral Banach ring. We say a point $x \in \mathscr{M}(A)$ is Zariski-dense if and only the kernel of $|\cdot|_x$ is reduced to zero.*
We write $\mathscr{M}(A)^\eta = q^{-1}(\eta_A) \subset \mathscr{M}(A)$ for the subset of Zariski-dense points.

As the name suggests, those are indeed the points of $\mathscr{M}(A)$ which are dense for the Zariski topology, where the Zariski topology on $\mathscr{M}(A)$ is the coarsest making the map $q : \mathscr{M}(A) \longrightarrow \mathrm{Spec}(A)$ continuous.

1.1.2 Discretely-valued fields

Let K be a complete, discretely-valued field, with valuation v. We denote by $R = \{v \geq 0\}$ its valuation ring, $\mathfrak{m} = \{v > 0\}$ its maximal ideal and $k = R/\mathfrak{m}$ the residue field. If k and K have the same characteristic, it follows from Cohen's structure theorem that K is isomorphic to the field $K = k((t))$ of Laurent series over its residue field, endowed with the valuation $v = \mathrm{ord}_0$.
Otherwise, K has mixed characteristic, i.e. K has characteristic 0 and k has characteristic $p > 0$. The most common example of such a field is $K = \mathbb{Q}_p$ with the standard p-adic valuation, which has residue field $\mathbb{F}_p$. More generally, any finite extension K of $\mathbb{Q}_p$ is a discretely-valued field of mixed characteristic.
From now on, we will assume K has equicharacteristic zero, so that $K = k((t))$. We let X be a separated K-scheme of finite type, and write $n = \dim(X)$. The purpose of this section is to explain how one can understand the topological space X^{an} more concretely using piecewise-affine geometry. The basic idea is that for a large enough class of integral R-models $\mathscr{X}$ of X, there exists a finite-dimensional cell complex $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$, which we may view as a tropicalization of the model $\mathscr{X}$. As shown in thm. 1.1.19, the space X^{an} is then realized as the inverse limit of all such $\mathrm{Sk}(\mathscr{X})$, so that X^{an} is homeomorphic to a tower of simplicial complexes.
We start with a definition:

Definition 1.1.9. *A model of X is a flat, separated R-scheme $\mathscr{X}$, together with an isomorphism of K-schemes $\mathscr{X} \times_R K \simeq X$.*
We will denote $\mathscr{X}_0 := \mathscr{X} \times_R k$ the special fiber of $\mathscr{X}$, and by $\mathrm{Div}_0(\mathscr{X})$ the group of Weil divisors on $\mathscr{X}$ supported on the special fiber.

If $\mathscr{X}$, $\mathscr{X}'$ are two models of X, a morphism of models $f : \mathscr{X}' \longrightarrow \mathscr{X}$ is an R-morphism whose base change to K induces the identity on X. We will say that $\mathscr{X}'$ dominates $\mathscr{X}$ if there exists such a morphism, in which case it is unique.
Assume that X/K is proper, and let $\mathscr{X}/R$ be a proper model of X. By the valuative criterion of properness, for any $x = (\xi_x, v_x) \in X^{\mathrm{an}}$, the K-morphism $\mathrm{Spec}\,\mathscr{H}(x) \longrightarrow X$ - whose image is the point ξ_x - lifts in a unique way to an R-morphism from the valuation ring $\mathscr{H}(x)^\circ$ to $\mathscr{X}$:

$$\begin{array}{ccc} \mathrm{Spec}\,\mathscr{H}(x) & \xrightarrow{\xi_x} & \mathscr{X} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}\,\mathscr{H}(x)^\circ & \longrightarrow & \mathrm{Spec}\,R. \end{array}$$

The image of the closed point of $\mathrm{Spec}\,\mathscr{H}(x)^\circ$ under the extended morphism is called the center of x and denoted by $c_{\mathscr{X}}(x)$. The map $c_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathscr{X}_0$ turns out to be surjective and anticontinuous, i.e. the preimage of a closed subset of $\mathscr{X}_0$ by $c_{\mathscr{X}}$ is open in X^{an}.
In the case where X/K is smooth, we say a model $\mathscr{X}/R$ has simple normal crossing singularities if $\mathscr{X}/R$ is smooth, and the special fiber $\mathscr{X}_0$ is a simple normal crossing divisor inside $\mathscr{X}$. Such models always exists when K has equicharacteristic zero and X/K is projective, by Hironaka's theorem on resolution of singularities. More precisely, any model $\mathscr{X}/R$ can be dominated by an snc model.

To every snc model $\mathscr{X}$ of X, with special fiber $\mathscr{X}_0 = \sum_{i\in I} a_i D_i$, we can associate a cell complex encoding the combinatorics of the intersections of the irreducible components of $\mathscr{X}_0$. We say $Y \subset \mathscr{X}_0$ is a *stratum* if there exists a non-empty $J \subset I$ such that Y is a connected component of $D_J := \cap_{j\in J} D_j$. The dual complex of $\mathscr{X}$ is now defined as follows:

Definition 1.1.10. *Let $\mathscr{X}$ be an snc model of X. To each stratum Y of $\mathscr{X}_0$ which is a connected component of D_J, we associate a simplex:*

$$\tau_Y = \{w \in \mathbb{R}^{|J|}_{\geqslant 0} \,|\, \sum_{j\in J} a_j w_j = 1\}.$$

We define the cell complex $\mathcal{D}(\mathscr{X}_0)$ by the following incidence relations: τ_Y is a face of $\tau_{Y'}$ if and only if $Y' \subset Y$.

For reasons that will appear clearer later on, we will sometimes need to work with a broader class of models, those are divisorially log terminal models:

Definition 1.1.11. *Let $\mathscr{X}/R$ be a model of X. We say that $\mathscr{X}$ is a* dlt *(divisorially log terminal) model of X if the following conditions hold:*

- *the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$ is log canonical in the sense of the Minimal Model Program (see [KM98]);*
- *the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$ is simple normal crossing at the generic points of log canonical centers of $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$.*

Moreover, a dlt model $\mathscr{X}$ is good *if each irreducible component of $\mathscr{X}_{0,\mathrm{red}}$ is $\mathbb{Q}$-Cartier.*

See [NXY19, §1.12-1.14] for an overview on existence results of such models.
We will not give a precise definition of log canonical centers here, and refer the reader to [KM98]. As a matter of fact, in the sequel, we will assume the following:

Condition 1.1.12. *The log canonical centers of $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$ are precisely the strata of $\mathscr{X}_0$.*

Whenever X is defined over an algebraic curve - which is the most relevant case for applications - the above condition is satisfied by [Kol13, 4.16], so that a dlt model $\mathscr{X}$ is simple normal crossing at the generic points of the strata of $\mathscr{X}_0$. If $\mathscr{X}_0$ is reduced, it follows from the approximation arguments of [NXY19, Corollary 4.4] that this holds in the general case as well. It is in fact expected that condition 1.1.12 holds in general.
In that case, the second assumption in the definition means that the singularities of the pair $(\mathscr{X}, \mathscr{X}_{0,\mathrm{red}})$ do not interact with the combinatorics of the intersections of the components of the special fiber, so that def. 1.1.10 extends in a straightforward way to dlt models satisfying condition 1.1.12.
Given any dlt model $\mathscr{X}$ of X over R, there exists a natural embedding $i_{\mathscr{X}}$ of the dual complex $\mathcal{D}(\mathscr{X}_0)$ into X^{an}, given as follows. The vertices v_i of $\mathcal{D}(\mathscr{X}_0)$ are

in one-to-one correspondence with irreducible components D_i of the special fiber $\mathscr{X}_0 = \sum_{i\in I} a_i D_i$, so that we set

$$i_{\mathscr{X}}(v_i) = v_{D_i} := a_i^{-1}\,\mathrm{ord}_{D_i},$$

where the valuation ord_{D_i} associates to a meromorphic function $f \in K(X) \simeq K(\mathscr{X})$ its vanishing order along D_i - the normalisation by a_i^{-1} ensuring that $v_{D_i}(t) = 1$.

Definition 1.1.13. *A valuation given in this way, for some dlt (or equivalently, snc) model $\mathscr{X}$ of X, is called* divisorial. *We write $X^{\div} \subset X^{\mathrm{an}}$ for the set of divisorial valuations, it is a dense subset of X^{an}.*

One can now interpolate between those divisorial valuations using *quasi-monomial* valuations, in order to embed $\mathcal{D}(\mathscr{X}_0)$ into X^{an}:

Proposition 1.1.14 ([MN15, Proposition 2.4.4])**.** *Let $\mathscr{X}$ be a dlt model of X satisfying condition 1.1.12, with special fiber $\mathscr{X}_0 = \sum_{i\in I} a_i D_i$. Let $J \subset I$ such that $D_J = \cap_{j\in J} D_j$ is non-empty, and Y a connected component of D_J, with generic point η. We furthermore fix a local equation $z_j \in \mathcal{O}_{\mathscr{X},\eta}$ for D_j, for any $j \in J$. Then, for any $w \in \tau_Y = \{w \in \mathbb{R}_{\geqslant 0}^{|J|} \,|\, \sum_{j\in J} a_j w_j = 1\}$, there exists a unique valuation*

$$v_w : \mathcal{O}_{\mathscr{X},\eta} \longrightarrow \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$$

such that for every $f \in \mathcal{O}_{\mathscr{X},\eta}$, with expansion $f = \sum_{\beta\in\mathbb{N}^{|J|}} c_\beta z^\beta$ (with c_β either zero or unit), we have:

$$v_w(f) = \min\{\langle w, \beta\rangle \,|\beta \in \mathbb{N}^{|J|}, c_\beta \neq 0\},$$

where $\langle\ ,\ \rangle$ is the usual scalar product on $\mathbb{R}^{|J|}$.

The above valuation is called the quasi-monomial valuation associated with the data (Y, w). Then:

$$\begin{aligned} i_{\mathscr{X}} :\quad & \mathcal{D}(\mathscr{X}_0) \to X^{\mathrm{an}} \\ & \tau_Y \ni w \mapsto v_w \end{aligned}$$

gives a well-defined continuous injective map from $\mathcal{D}(\mathscr{X}_0)$ to X^{an}.

Definition 1.1.15. *We call the image of $\mathcal{D}(\mathscr{X}_0)$ by $i_{\mathscr{X}}$ the* skeleton *of $\mathscr{X}$, written as $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$. It is a cell complex of dimension at most $\dim X$.*

By compactness of $\mathcal{D}(\mathscr{X}_0)$, $i_{\mathscr{X}}$ induces a homeomorphism between $\mathcal{D}(\mathscr{X}_0)$ and $\mathrm{Sk}(\mathscr{X})$, so that we will sometimes abusively identify $\mathcal{D}(\mathscr{X}_0)$ with $\mathrm{Sk}(\mathscr{X})$.

Definition 1.1.16. *Let Y be a stratum of $\mathscr{X}_0$. We define $\mathrm{Star}(\tau_Y)$ as the union of open faces in $\mathrm{Sk}(\mathscr{X})$ whose closure contains τ_Y.*

Let us now assume that $\mathscr{X}$ is a *good* dlt model of X, i.e. that the irreducible components of $\mathscr{X}_0$ are $\mathbb{Q}$-Cartier. We can now define a retraction for the inclusion $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ as follows: for any $v \in X^{\mathrm{an}}$, there exists a minimal stratum $Y \subseteq \cap_{j\in J} D_j$ of $\mathscr{X}_0$ such that the center $c_{\mathscr{X}}(v)$ of v is contained in Y. Since $\mathscr{X}$ is good, for each $j \in J$, there exists $e_j > 0$ such that $e_j D_j$ is Cartier at $c_{\mathscr{X}}(v)$. We now associate to v the quasi-monomial valuation $\rho_{\mathscr{X}}(v)$ corresponding to the data (Y, w) with $w_j = e_j^{-1} v(z_j)$, where z_j is a local equation of $e_j D_j$ at the generic point of $c_{\mathscr{X}}(v)$. This should be seen as a monomial approximation of the valuation v at the generic point of Y, with respect to the model $\mathscr{X}$ (which is snc there).

Definition 1.1.17. *The above map $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ is the Berkovich retraction associated with the model $\mathscr{X}/R$.*

The Berkovich retraction is continuous, restricts to the identity on $\mathrm{Sk}(\mathscr{X})$, and by [Thu07] [Ber99], $\rho_{\mathscr{X}}$ is a strong deformation retraction, i.e. there exists a homotopy between $\rho_{\mathscr{X}}$ and the identity on X^{an} that fixes the points of $\mathrm{Sk}(\mathscr{X})$. It follows that X^{an} and $\mathrm{Sk}(\mathscr{X})$ are homotopy equivalent.

Remark 1.1.18. *The Berkovich retraction is a local construction on $\mathscr{X}$, in the following sense. Let Y be a stratum of $\mathscr{X}_0$. The formal completion $\mathfrak{X}_Y := \widehat{\mathscr{X}_{/Y}}$ of $\mathscr{X}$ along Y is a formal R-scheme, and admits a generic fiber $\mathfrak{X}_Y^{\eta}$ in the sense of Berkovich, which is a Berkovich analytic space and can be explicitly described as the following open subset of X^{an}:*

$$\mathfrak{X}_Y^{\eta} = \{x \in X^{\mathrm{an}} |\, c_{\mathscr{X}}(v_x) \in Y\} = \rho_{\mathscr{X}}^{-1}(\mathrm{Star}(\tau_Y)).$$

The global construction from above can in fact be made on $\mathfrak{X}_Y^{\eta}$ [Ber99], so that there exists a retraction:

$$\rho_Y : \mathfrak{X}_Y^{\eta} \longrightarrow \mathrm{Star}(\tau_Y),$$

which coincides with the restriction of the retraction $\rho_{\mathscr{X}}$. Thus, the restriction of $\rho_{\mathscr{X}}$ over $\mathrm{Star}(\tau_Y)$ only depends on the formal completion $\widehat{\mathscr{X}_{/Y}}$.

We now want to understand how $\mathrm{Sk}(\mathscr{X})$ changes when we change the model $\mathscr{X}$. Since every dlt model can be dominated by an snc one, we restrict ourselves to snc models for convenience. Let $f : \mathscr{X}' \longrightarrow \mathscr{X}$ be two snc models of X, together with a morphism of models f. Then one can show that we have an inclusion $\mathrm{Sk}(\mathscr{X}) \subset \mathrm{Sk}(\mathscr{X}')$, so that the retraction $\rho_{\mathscr{X}'}$ factors through $\rho_{\mathscr{X}}$: there exists a map:

$$r_{\mathscr{X}'\mathscr{X}} : \mathrm{Sk}(\mathscr{X}') \longrightarrow \mathrm{Sk}(\mathscr{X})$$

which is affine on each face of $\mathrm{Sk}(\mathscr{X}')$, and such that $\rho_{\mathscr{X}} = r_{\mathscr{X}'\mathscr{X}} \circ \rho_{\mathscr{X}'}$. In fact, $r_{\mathscr{X}'\mathscr{X}}$ is none other than the restriction of $\rho_{\mathscr{X}}$ to $\mathrm{Sk}(\mathscr{X}')$, but can also be computed explicitely in terms of coordinates on the simplices, see [BJ17, §4.2]. Moreover, if $\mathscr{X}'' \longrightarrow \mathscr{X}' \longrightarrow \mathscr{X}$ are three snc models with morphisms as depicted, then the equality $r_{\mathscr{X}''\mathscr{X}'} = r_{\mathscr{X}'\mathscr{X}} \circ r_{\mathscr{X}''\mathscr{X}'}$ holds. As a result, the family $(\mathrm{Sk}(\mathscr{X}))_{\mathscr{X}}$ of Berkovich skeleta, indexed by the ordered set of snc models of X, is an inverse system of topological spaces, and the family of $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ defines a continuous map ρ from X^{an} to the inverse limit of all skeleta. We now have the following result:

Theorem 1.1.19. *([KS06, thm. 10]).*
The map $\rho : X^{\mathrm{an}} \longrightarrow \varprojlim_{\mathscr{X}} \mathrm{Sk}(\mathscr{X})$ is a homeomorphism.

Note that by Hironaka's theorem, snc models are cofinal in the category of good dlt models (that is, every good dlt model of X can be dominated by an snc one), so that taking the inverse limit over good dlt models yields the same result.
Let $\mathscr{X}/R$ be a model of X, and let $\mathscr{I}$ be an ideal sheaf on $\mathscr{X}$. It induces a function:

$$\phi_{\mathscr{I}} : X^{\mathrm{an}} \longrightarrow \mathbb{R},$$

sometimes denoted by $\phi_{\mathscr{I}} = \log|\mathscr{I}|$, as follows. For $x \in X^{\mathrm{an}}$, we set:

$$\phi_{\mathscr{I}} = \sup_f \log|f(x)|,$$

where the supremum runs over the $f \in \mathscr{I}_{c_{\mathscr{X}}(x)}$ - or equivalently, over the finitely many generators of $\mathscr{I}$ at $c_{\mathscr{X}}(x)$.
If $\mathscr{I}_1$, $\mathscr{I}_2$ are vertical ideal sheaves on $\mathscr{X}$, $\mathscr{X}'$ respectively (i.e., cosupported on the special fiber), then the equality $\phi_{\mathscr{I}_1} = \phi_{\mathscr{I}_2}$ holds if and only there exists a model $\mathscr{X}''$ dominating both $\mathscr{X}$ and $\mathscr{X}'$ such that the pullbacks:

$$\mathcal{O}_{\mathscr{X}''} \cdot \mathscr{I}_1 = \mathcal{O}_{\mathscr{X}''} \cdot \mathscr{I}_2$$

agree on $\mathscr{X}''$. In particular, when working with a function of the form $\phi_{\mathscr{I}}$ for a vertical ideal sheaf, we may choose a log-resolution of the ideal $\mathscr{I}$, so that we may assume that there exists a model $\mathscr{X}$ such that $\mathscr{I} = \mathcal{O}_{\mathscr{X}}(D)$ for a vertical divisor $D \in \mathrm{Div}_0(\mathscr{X})$. This implies for instance the following:

Lemma 1.1.20. *Let $\mathscr{I}$ be a vertical ideal sheaf on a model $\mathscr{X}$ of X, and $\phi_{\mathscr{I}} : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ the associated function. Then for any snc model $\mathscr{X}'$ of X, the restriction of $\phi_{\mathscr{I}}$ to the skeleton $\mathrm{Sk}(\mathscr{X}')$ is piecewise-affine.*

Note that the converse also holds. This motivates the following terminology:

Definition 1.1.21. *A function $\phi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ is piecewise-affine (PA for short) if there exists an snc model $\mathscr{X}$ and a vertical ideal sheaf $\mathscr{I}$ on $\mathscr{X}$ such that $\phi = \phi_{\mathscr{I}}$.*

1.1.3 Hybrid spaces

Let $(k, |\cdot|)$ be a non-trivially valued field, either archimedean or non-archimedean.

Definition 1.1.22. *Let k^{hyb} be the Banach ring obtained by equipping the field k with the norm $\|\cdot\|_{\mathrm{hyb}}$, defined for non-zero $z \in k$ by:*

$$\|z\|_{\mathrm{hyb}} = \max\{1, |z|\}.$$

One can show [LP20, ex. 1.1.15] that the elements of $\mathscr{M}(k^{\mathrm{hyb}})$ are of the form $|\cdot|^\lambda$, for $\lambda \in [0,1]$, where $|\cdot|^0 = |\cdot|_0$ denotes the trivial absolute value on k. This yields a homeomorphism $\lambda : \mathscr{M}(k^{\mathrm{hyb}}) \xrightarrow{\sim} [0,1]$.
Thus, if Z is a scheme of finite type over k, its analytification with respect to $|\cdot|_{\mathrm{hyb}}$, which we denote by Z^{hyb}, comes with a structure morphism $\pi : Z^{\mathrm{hyb}} \longrightarrow [0,1]$. If $Z = \mathrm{Spec}(A)$ is affine, the fiber over $\lambda \neq 0$ is by definition of π the set of semi-norms extending the absolute value $|\cdot|^\lambda$ on k, so that by rescaling, this is easily seen to be homeomorphic to the analytification Z^{an} of Z with respect to the absolute value $|\cdot|$. One can in fact show that for any Z of finite type, we have a homeomorphism:

$$p : \pi^{-1}((0,1]) \xrightarrow{\sim} (0,1] \times Z^{\mathrm{an}},$$

compatible with the projections to $(0,1]$.
On the other hand, the fiber $\pi^{-1}(0)$ consists of the semi-norms extending the trivial

absolute value on k, so that this is homeomorphic to the analytification Z_0^{an} of Z with respect to the trivial absolute value on k.
Hence, the space Z^{hyb} allows us to see the analytic space Z^{an} degenerate to its trivially-valued counterpart.
In the case where $k = \mathbb{C}$ with the Euclidean absolute value, the analytification Z^{an} is homeomorphic to the usual complex analytification Z^{hol} of Z, by example 1.1.4. Thus, the space Z^{hyb} provides a natural way to degenerate the complex manifold Z^{hol} to the non-archimedean analytic space Z_0^{an}.
We now want to perform a similar construction for degenerations of complex varieties. Let $X \xrightarrow{\pi} \mathbb{D}^*$ be a holomorphic family of n-dimensional complex manifolds, where $\mathbb{D}^* = \{|t| < 1\}$ is the punctured unit disk in $\mathbb{C}$. We will furthermore assume that the family is quasi-projective and meromorphic at zero, i.e. that there exists a relatively algebraic embedding $\iota : X \hookrightarrow \mathbb{P}^N \times \mathbb{D}^*$ such that $\pi = \mathrm{pr}_2 \circ \iota$, and the equations of X have meromorphic singularity at $t = 0$. This allows us to view (the base change of) X as a quasi-projective scheme over the non-archimedean field $K = \mathbb{C}((t))$ of Laurent series; we fix a radius $r \in (0, 1)$ and write X^{an} for the Berkovich analytification of X with respect to the t-adic valuation on K, normalized so that $|t| = r$.
We consider the following Banach ring, which we call the *hybrid ring*:

$$A_r = \{f = \sum_{n\in\mathbb{Z}} a_n t^n \in K \ / \ \|f\|_{\mathrm{hyb}} := \sum_n \|a_n\|_{\mathrm{hyb}}\, r^n < \infty\}.$$

The purpose of the above Banach ring is to provide a presentation of the closed complex disk as an affine non-archimedean analytic space: we denote by $C^{\mathrm{hyb}}(r) := \mathscr{M}(A_r)$ the Berkovich spectrum of A_r and call it the *hybrid circle*, the terminology stems from the fact that $C^{\mathrm{hyb}}(r)$ is homeomorphic to the circle $\{|T| = r\}$ inside the Berkovich affine line over $\mathbb{C}^{\mathrm{hyb}}$ [Poi10]. We now have the following more explicit description of the hybrid circle:

Lemma 1.1.23. *([BJ17, prop. A.4])*
The map $\tau : t \mapsto |\cdot|_t$ defined by:

$$|f|_t = \begin{cases} r^{\mathrm{ord}_0(f)} & \text{if } t = 0, \\ r^{\log|f(t)|/\log|t|} & \text{if } t \neq 0 \end{cases}$$

for $f \in A_r$ induces a homeomorphism from $\bar{\mathbb{D}}_r$ to $\mathscr{M}(A_r)$.

The upshot of this construction is that is f in A_r, then up to a constant $\log|f(\tau(t))| = \frac{\log|f(t)|}{\log|t|}$ for $t \neq 0$ - we are viewing the point t as a rescaling of the Euclidean absolute value composed with the evaluation map at t. Additionally, as $t \to 0$, these rescaled absolute values converge to the non-archimedean t-adic absolute value $e^{-\mathrm{ord}_0}$ on $A_r \subset \mathbb{C}((t))$. This motivates the following definition:

Definition 1.1.24. *Let $X \xrightarrow{\pi} \mathbb{D}^*$ be a quasi-projective degeneration of complex manifolds as above, and view it as a scheme of finite type over the ring of convergent power series. We write X_{A_r} its base change to the ring A_r. We define the hybrid space X_r^{hyb} associated to X as the analytification of X_{A_r} over A_r, which comes with a structure map:*

$$\pi_{\mathrm{hyb}} : X_r^{\mathrm{hyb}} \longrightarrow C^{\mathrm{hyb}}(r).$$

The hybrid space allows us to see the complex space X degenerate to its non-archimedean analytification, as a consequence of the following:

Proposition 1.1.25. *[Fav20, thm. 1.2] Let X be a degeneration of complex manifolds, X_{A_r} the associated A_r-scheme, and denote the associated hybrid space by $\pi_{\mathrm{hyb}} : X_r^{\mathrm{hyb}} \longrightarrow C^{\mathrm{hyb}}(r)$. Then:*

- *$\pi_{\mathrm{hyb}}^{-1}(0)$ can be canonically identified with X^{an},*
- *there exists a homeomorphism $\beta : X_{|\mathbb{D}_r^*} \xrightarrow{\sim} \pi_{\mathrm{hyb}}^{-1}(\tau(\mathbb{D}_r^*))$, satisfying $\pi_{\mathrm{hyb}} \circ \beta = \tau \circ \pi$,*
- *if ϕ is a rational function on X_{A_r}, then $|\phi(\beta(z))| = |\phi(z)|^{\frac{\log r}{\log|t|}}$ for z not in the indeterminacy locus of ϕ.*

Thus, heuristically, the hybrid space allows us to see the scalings of the usual modulus on $\mathbb{C}$ given by $|\cdot|^{\frac{\log r}{\log|t|}}$ degenerate to the non-archimedean absolute value $r^{-\mathrm{ord}_0}$ on K, and hence to see the complex manifolds $\{X_t\}_{t\in\mathbb{D}^*}$ degenerate to the non-archimedean analytification X^{an}.
Assume for instance that $(\mu_t)_{t\in\mathbb{D}^*}$ is a continuous family of probability measures on X, such that μ_t is supported on X_t for each $t \in \mathbb{D}^*$. Since the hybrid space provides a canonical compactification of X over the puncture, it is a natural question to ask whether or not the family of measures converges on X^{hyb}, at least in a weak sense - more concrete examples of such situations will be given in sections 1.4.3 and 5.1.
We now assume that $X \longrightarrow \mathbb{D}^*$ is proper, and will explain how one can adapt the statement of thm. 1.1.19 to the hybrid setting. If $\mathscr{X} \longrightarrow \mathbb{D}$ is a proper snc model of X over the disk, define:

$$\mathscr{X}^{\mathrm{hyb}} := X \sqcup \mathscr{D}(\mathscr{X}),$$

which comes with a map $\pi : \mathscr{X}^{\mathrm{hyb}} \longrightarrow \mathbb{D}$ mapping $\mathscr{D}(\mathscr{X})$ to $0 \in \mathbb{D}$. Following a construction going back to Morgan-Shalen [MS84], Boucksom-Jonsson [BJ17] define a topology on $\mathscr{X}^{\mathrm{hyb}}$ as follows.
Let $(\mathscr{U}, z)$ be a coordinate chart, with $z \in \mathbb{D}^{n+1}$, and write $U = \mathscr{U} \cap X$. We assume that each z_i is either a local equation for some $D_i \subset \mathscr{X}_0$ or invertible, and that if J is the set of $j \in I$ such that $\mathscr{U}$ meets D_j, then $\mathscr{U} \cap D$ meets only one connected component $Y_{\mathscr{U}} \subset D_J$ (those charts are called *adapted* in [BJ17]). Given such an open chart, we define a continuous map:

$$\mathrm{Log}_{\mathscr{U}} : U \longrightarrow \tau_Y,$$

$$z \longmapsto \left(\frac{\log|z_j|}{\log|f_{\mathscr{U}}|}\right)_{j\in J},$$

where $f_{\mathscr{U}} = \prod_{j\in J} z_j^{m_j}$ is a local equation for D on $\mathscr{U}$.
We then extend this map to $\mathscr{U}^{\mathrm{hyb}} := U \sqcup \sigma_Y$ by setting $\mathrm{Log}_{\mathscr{U}} = \mathrm{Id}_\tau$ on $\tau_{\mathscr{U}}$. This construction can then be globalized (although non-canonically), using a partition of unity argument:

Proposition 1.1.26. *([BJ17, prop. 2.1])*
There exists an open neighbourhood $\mathscr{V}$ of $\mathscr{X}_0$ in $\mathscr{X}$, and a map $\mathrm{Log}_{\mathscr{V}} : \mathscr{V}^{\mathrm{hyb}} \longrightarrow \mathscr{D}(\mathscr{X})$, satisfying the following property: for every chart $(\mathscr{U}, z)$ as above, and $\mathscr{U} \subset \mathscr{V}$, we have $\mathrm{Log}_{\mathscr{V}}(\mathscr{U}) \subset \tau_{Y_{\mathscr{U}}}$ and:

$$\mathrm{Log}_{\mathscr{V}} = \mathrm{Log}_{\mathscr{U}} + O((\log|f_{\mathscr{U}}|^{-1})^{-1})$$

uniformly on compacts subsets of $\mathscr{U}$.

The topology on $\mathscr{X}^{\mathrm{hyb}}$ is now defined as the coarsest topology such that:

- X is open inside $\mathscr{X}^{\mathrm{hyb}}$;
- for every open neighbourhood $\mathscr{U}$ of $\mathscr{X}_0$ inside $\mathscr{X}$, the subset $U \sqcup \mathscr{D}(\mathscr{X})$ is open in $\mathscr{X}^{\mathrm{hyb}}$;
- $\mathrm{Log}_{\mathscr{V}} : \mathscr{V}^{\mathrm{hyb}} \longrightarrow \mathscr{D}(\mathscr{X})$ is continuous.

It follows from the above proposition that this topology is independent of the choice of $\mathscr{V}$ and map $\mathrm{Log}_{\mathscr{V}}$.
If $\mathscr{X}' \longrightarrow \mathscr{X}$ is an snc model that dominates $\mathscr{X}$, then the map $r_{\mathscr{X}'\mathscr{X}} : \mathscr{D}(\mathscr{X}') \longrightarrow \mathscr{D}(\mathscr{X})$ induces a continuous map:

$$r^{\mathrm{hyb}}_{\mathscr{X}'\mathscr{X}} : \mathscr{X}'^{\mathrm{hyb}} \longrightarrow \mathscr{X}^{\mathrm{hyb}},$$

by declaring that $r^{\mathrm{hyb}}_{\mathscr{X}'\mathscr{X}}$ induces the identity on X. Similarly, there exists a continuous map:

$$\rho^{\mathrm{hyb}} : X^{\mathrm{hyb}} \longrightarrow \mathscr{X}^{\mathrm{hyb}}_{|\bar{\mathbb{D}}_r},$$

equal to $\rho_{\mathscr{X}}$ on X^{an} and to the homeomorphism β from prop. 1.1.25 over $\bar{\mathbb{D}}^*_r$. The upshot is now the following extension of theorem 1.1.19:

Proposition 1.1.27. *[BJ17, prop. 4.12] The induced map:*

$$\rho : X^{\mathrm{hyb}} \longrightarrow \varprojlim_{\mathscr{X}} \mathscr{X}^{\mathrm{hyb}}_{|\bar{\mathbb{D}}_r}$$

is a homeomorphism, where the inverse limit runs over all proper snc models of X.

In practice, this means that if $(\mu_t)_{t\in\mathbb{D}^*}$ is a family of probability measures on X, then to compute the hybrid limit of the μ_t's, it is enough to study weak convergence of the measures:

$$\nu_t = (\mathrm{Log}_{\mathscr{V}})_*\mu_t$$

on $\mathscr{D}(\mathscr{X})$, where $\mathscr{X}$ ranges over the snc models of X. In the case where the measures μ_t's are given explicitly, this is often more tractable than working directly on X^{hyb}.

1.1.4 The isotrivial hybrid space

We set $K = \mathbb{C}((t))$. Let X be a projective complex variety, and write $\tilde{X} := X \times \mathbb{D}^*$ the associated trivial degeneration of complex varieties, as well as $X_K = X \times_{\mathbb{C}} K$. We thus have two hybrid spaces associated to X: the hybrid space X_0^{hyb} obtained by viewing X as a scheme over $\mathbb{C}^{\mathrm{hyb}}$, and X_K^{hyb} the hybrid space associated to the degeneration $\tilde{X}$. The goal of this section is to compare both hybrid spaces, so that results established for hybrid spaces associated to degenerations will naturally yield similar statements for spaces over $\mathbb{C}^{\mathrm{hyb}}$, simply by specializing to a trivial degeneration.

We start by comparing the non-archimedean fibers. We write X_K^{an} for the analytification of X_K, and X_0^{an} for the analytification of X with respect to the trivial absolute value on $\mathbb{C}$. The t-adic absolute value on $\mathbb{C}((t))$ restricts to the trivial absolute value on $\mathbb{C}$, so that there exists a base change morphism $f : X_K^{\mathrm{an}} \longrightarrow X_0^{\mathrm{an}}$, that can be described as follows. If $\mathcal{K}(X)$ is the function field of X, then the function field of X_K is simply $\mathcal{K}(X_K) = \mathcal{K}(X)((t)) = \mathcal{K}(X) \otimes_{\mathbb{C}} K$. Hence, any valuation v on $\mathcal{K}(X_K)$ induces by restriction a valuation $f(v)$ on $\mathcal{K}(X)$, and similarly for semi-valuations.

We now compare the base rings: for $r \in (0,1)$, the inclusion $\mathbb{C}^{\mathrm{hyb}} \hookrightarrow A_r$ is compatible with the hybrid norms, so that it induces a continuous map $\lambda : \mathscr{M}(A_r) \longrightarrow \mathscr{M}(\mathbb{C}^{\mathrm{hyb}})$, obtained by restricting semi-norms from A_r to $\mathbb{C}^{\mathrm{hyb}}$. It is straightforward to check that under the homeomorphisms $\mathscr{M}(A_r) \simeq \bar{\mathbb{D}}_r$ and $\mathscr{M}(\mathbb{C}^{\mathrm{hyb}}) \simeq [0,1]$, we have $\lambda(t) = \frac{\log r}{\log|t|}$ for $t \in \bar{\mathbb{D}}_r$. We furthermore have the following description of the base change of X from $\mathbb{C}^{\mathrm{hyb}}$ to A_r, which is a straightforward consequence of transitivity of base change:

Proposition 1.1.28. *Let $r \in (0,1)$, and let $\lambda : \bar{\mathbb{D}}_r \longrightarrow [0,1]$ be the map defined by $\lambda(t) = \frac{\log r}{\log|t|}$. We have a commutative diagram:*

$$\begin{array}{ccc} X_K^{\mathrm{hyb}} & \xrightarrow{F} & X_0^{\mathrm{hyb}} \\ \downarrow{\scriptstyle \pi_K} & & \downarrow{\scriptstyle \pi_0} \\ \bar{\mathbb{D}}_r & \xrightarrow{\lambda} & [0,1]. \end{array}$$

where F is the base change of X to $A_r/\mathbb{C}^{\mathrm{hyb}}$, and such that $F_{|\pi_K^{-1}(0)} = f$. Moreover, for any $t \in \bar{\mathbb{D}}_r^*$, $F_{|\pi_K^{-1}(t)}$ induces the identity on X^{hol} under the homeomorphisms from section 1.1.3.

The map $f : X_K^{\mathrm{an}} \longrightarrow X_0^{\mathrm{an}}$ furthermore admits a continuous section, called the Gauss section, defined in the following way. In the terminology of [Poi13, def. 3.2], every point of X_0^{an} is universal (peaked point, in the terminology of Berkovich) as $\mathbb{C}$ is algebraically closed, so that any $x \in X_0^{\mathrm{an}}$ admits a canonical lift to X_K^{an}, denoted by $\gamma(x)$; we call the map $\gamma : X_0^{\mathrm{an}} \longrightarrow X_K^{\mathrm{an}}$ the *Gauss section*. More concretely, if $x = v_x \in X_0^{\mathrm{an}}$ is a valuation on $\mathcal{K}(X)$, it is extended as a valuation:

$$\gamma(v) : \mathcal{K}(X)((t)) \longrightarrow \mathbb{R} \cup \{-\infty\},$$

such that $\gamma(v)(t) = 1$. For instance, if $S = \sum_{n \geq 0} s_n t^n$ an element of $\mathcal{K}(X)[t]$, the canonical extension is defined by the formula:

$$\gamma(v)(S) = \min_n (v(s_n) + n).$$

1.2 Global pluripotential theory

Let A be an integral Banach ring, and X a projective A-scheme of finite type. Following common practice, we will call line bundle on X any locally free $\mathcal{O}_X$-module of rank 1. We will use additive notation for the group law on the set of isomorphism classes of line bundles.
Let L be a semi-ample line bundle on X, and write $X^{\mathrm{an}} \to \mathscr{M}(A)$ the Berkovich analytification of X. The purpose of this section is to define a class of semi-positive (singular) metrics $\mathrm{PSH}(X, L)$ on L, in the similar way to the case where $A = K$ is a complete valued field.
If $K = \mathbb{C}$, and (X, L) is a smooth polarized variety, then the class $\mathrm{PSH}(X, L)$ of plurisubharmonic metrics on L is one of the central objects of pluripotential theory, and has been extensively studied at this point.
When K is a non-archimedean field, a similar class of semi-positive metrics has been defined in increasing order of generality in [Zha95], [BFJ16], [BE21].
In our setting, the basic idea is that a metric $\phi \in \mathrm{PSH}(X, L)$ can be viewed as a family of semi-positive metrics $(\phi_x)_{x \in \mathscr{M}(A)}$, where $\phi_x \in \mathrm{PSH}(X_{\mathscr{H}(x)}, L_{\mathscr{H}(x)})$, that also varies in a plurisubharmonic way with respect to $x \in \mathscr{M}(A)$. This is well-illustrated for instance by thm. 2.2.12 in the toric setting, in which case the semi-positivity translates into a standard convexity condition.
The main case of interest for us will be when $A = A_r$ is a hybrid ring, so that $X^{\mathrm{an}} = X^{\mathrm{hyb}}$ is the hybrid space associated to a degeneration of complex varieties. In this setting, our main result, theorem 1.3.13 states that any psh metric $\phi \in \mathrm{PSH}(X, L)$ on a polarized degeneration of complex manifolds satisfying a certain growth condition, induces naturally a psh metric on $(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$, whose restriction to the non-archimedean fiber X^{an} is the non-archimedean metric ϕ^{NA} constructed in [BBJ18], [Reb21], and encodes the logarithmic singularities of ϕ along the special fibers of models of X.
We also compare our definition with the setting of [Fav20], and obtain the continuity on X^{hyb} of the family of Monge-Ampère measures associated to a continuous, semi-positive metric on L^{hyb}.
Throughout this section, we assume that the Banach ring A is a geometric base ring.

1.2.1 Metrics on Berkovich spaces

We start with some very general definition of metrics on line bundles on Berkovich analytic spaces.

Definition 1.2.1. *Let L be a line bundle on X. A continuous metric ϕ on L^{an} consists of the following data: for any Zariski open subset $U \subset X$ and $s \in H^0(U, L_{|U})$ a trivializing section, a continuous function:*

$$\|s\|_\phi : U^{\mathrm{an}} \longrightarrow \mathbb{R}_{>0}$$

such that $\|fs\|_\phi = |f|\|s\|_\phi$ for any regular function $f \in H^0(U, \mathcal{O}_U)$, and compatible with restriction of sections.

This allows us to define, for any open subset $V \subset X^{\mathrm{an}}$ and any analytic section s of L^{an} on V, a continuous function:

$$\|s\|_\phi : V \longrightarrow \mathbb{R}_{\geq 0},$$

as follows: cover X by Zariski open subsets $U_i, i \in I$ such that $L_{|U_i} = s_i \cdot \mathcal{O}_{U_i}$, and write, on $V \cap U_i$:

$$s = f_i \cdot s_i,$$

with f_i an analytic function on V. Then we set:

$$\|s\|_\phi := |f_i| \|s\|_\phi$$

on $V \cap U_i^{\mathrm{an}}$. It is straightforward to check that this is independent on the choice of trivializations, compatible with restrictions and that the equality:

$$\|fs\|_\phi = |f| \|s\|_\phi$$

holds for any section s and any analytic function f on V.
From now on, we will use additive notation for metrics, i.e. identify the metric $\|\cdot\|_\phi$ with $\phi = -\log\|\cdot\|_\phi$. In particular, if L_1, L_2 are two line bundles on X and ϕ_i is a continuous metric on L_i for $i = 1, 2$, then $\phi_1 + \phi_2$ is a continuous metric on $L_1 + L_2$. Moreover, if $\psi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ is a continuous function and ϕ a continuous metric on L, then $\phi + \psi$ is also a continuous metric on L.

Example 1.2.2. *Let $A = \mathbb{C}$ with the Euclidian absolute value, and $X/\mathbb{C}$ a variety endowed with a line bundle. Then our definition matches the standard definition of a continuous Hermitian metric on L.*

Example 1.2.3. *Let $A = k$ be a trivially-valued field, X/k a variety and L a line bundle on X. Then the trivial metric $\phi^{\mathrm{triv}} = -\log|\cdot|_{\mathrm{triv}}$ on L is the unique metric on L^{an} such that for any pair (U, s), with $U \subset X$ a Zariski open and $s \in H^0(U, L)$ a nowhere-vanishing section of L, the equality:*

$$|s(x)|_{\mathrm{triv}} = 1$$

holds whenever the center $c(v_x)$ is contained in U.

Example 1.2.4. *Let $X = \mathbb{P}^N_A$, and $L = \mathcal{O}(1)$. Then the Fubini-Study metric ϕ_{FS} on L is defined by the formula:*

$$\|s(x)\|_{\phi_{\mathrm{FS}}} = \frac{|s(x)|}{\max(|x_0|, ..., |x_N|)},$$

where the x_i's are standard coordinates on $\mathbb{P}^N_A$. We will write:

$$\phi_{\mathrm{FS}} = \max_{i \leq N} \log|x_i|.$$

Note that while this definition is well-suited for the case when A is a non-archimedean field, it does not recover the usual Fubini-Study metric on $\mathbb{CP}^N$ when $A = \mathbb{C}$, so that we will sometimes call the metric above the tropical Fubini-Study metric.

In order to define a large enough class of semi-positive metrics, we need to allow metrics with some singularities.

Definition 1.2.5. *Let L be a line bundle on X. A singular metric ϕ on L^{an} (or simply on L when the norm is clear from the context) consists of the following data: for any Zariski open subset $U \subset X$ and $s \in H^0(U, L_{|U})$ a trivializing section, an upper semi-continuous (usc) function:*

$$\phi_s = -\log\|s\|_\phi : U^{\mathrm{an}} \longrightarrow \mathbb{R} \cup \{-\infty\}$$

not identically $-\infty$, such that $\|fs\|_\phi = |f| \times \|s\|_\phi$ for any regular function $f \in H^0(U, \mathcal{O}_U)$, and compatible with restriction of sections.

Example 1.2.6. *The following example will be particularly relevant for our purposes. Let $m \geq 1$, and $s_0 \in H^0(X, mL)$ be a global (algebraic) section of some positive power of L, we associate to it a metric on L denoted by:*

$$\phi = m^{-1}\log|s_0|,$$

which is the metric such that for s a local section of L, we have:

$$\|s\|_\phi(x) = \big(\big|\frac{s^m(x)}{s_0(x)}\big|\big)^{1/m}.$$

This metric is singular precisely along the zero locus of s_0.

Definition 1.2.7. *Let X, Y be two A-schemes of finite type, and $f : Y \longrightarrow X$ be an A-morphism. If L is a line bundle on X endowed with a (singular) metric ϕ, we define the pull-back metric $f^*\phi$ on f^*L as follows: cover $X = \cup_{i\in I} U_i$ by Zariski open subsets, and choose a trivialization σ_i of L on each U_i.*
Then we cover $Y = \cup_{i\in I} V_i$ with $V_i = f^{-1}(U_i)$, and trivialize f^*L by the sections f^*s_i on V_i. We now set:

$$\|f^*s_i\|_{f^*\phi} := \|s_i\|_\phi \circ f.$$

It is straightforward to check that this is independent on the choice of open cover and trivializations, and thus defines a metric $f^*\phi$ on f^*L.
We conclude this section with a discussion on the behaviour on metrics under base change. We assume that A and B are two geometric base rings, together with a bounded ring homomorphism $A \longrightarrow B$, so that we have a base change morphism $F : X_B^{\mathrm{an}} \longrightarrow X^{\mathrm{an}}$ for any scheme X/A of finite type.
Now let L be a line bundle on X, and $L_B = L \otimes_{\mathcal{O}_X} \mathcal{O}_{X_B}$ the induced line bundle on X_B. Given a continuous metric ϕ on L^{an}, we want to define a continuous metric ϕ_B on L_B by a base change operation. To that purpose, cover $X = \cup_{i\in I} U_i$ by Zariski open subsets trivializing L, and set $U_{B,i} = U_i \times_A B$, which yields an open cover of X_B. If s_i is a generator of the free $\mathcal{O}_X(U_i)$-module $H^0(U_i, L)$, then $s_i \otimes 1$ is a generator of $H^0(U_{B,i}, L_B)$ over $\mathcal{O}_{X_B}(U_{B,i})$, so that we naturally set, for $x \in U_{B,i}^{\mathrm{an}}$:

$$\|(s \otimes 1)(x)\|_{\phi_B} := \|s(F(x))\|_\phi.$$

It is now a straightforward verification that $\phi \mapsto \phi_B$ defines a base change map from the set of continuous metrics on $(X^{\mathrm{an}}, L^{\mathrm{an}})$ to the set of continuous metrics on $(X_B^{\mathrm{an}}, L_B^{\mathrm{an}})$, which commutes with the usual operations of addition and scaling of metrics, as well as finite maxima.

Example 1.2.8. *Let A be a geometric base ring, and let $x \in \mathscr{M}(A)$. Then we have a canonical morphism of Banach rings $A \longrightarrow \mathscr{H}(x)$, so that any continuous metric ϕ on L induces by base change a continuous metric ϕ_x on $(X^{\mathrm{an}}_{\mathscr{H}(x)}, L^{\mathrm{an}}_{\mathscr{H}(x)})$. The metric ϕ_x can also be seen as the restriction of ϕ to the fiber $\pi^{-1}(x)$ of the structure map $\pi : X^{\mathrm{an}} \longrightarrow \mathscr{M}(A)$, by prop. 1.1.7.*

1.2.2 Pluripotential theory over a field

Let (X, L) be a smooth polarized variety over $\mathbb{C}$. The class $\mathrm{PSH}(X, L)$ of semi-positive metrics on L lies at the heart of (global) complex pluripotential theory; it is the class of singular metrics ϕ on L whose curvature form $dd^c\phi$ is semi-positive in the sense of currents. We refer the reader for instance to [Dem12], [GZ17] for a more thorough introduction.
Let $m \geq 1$ such that mL is globally generated. Given a family $(s_0, ..., s_N)$ of global sections of mL without commons zeroes, one can associate to them the continuous semi-positive metric:

$$\phi = \frac{1}{2m} \log(|s_0|^2 + ... + |s_N|^2),$$

i.e. the continuous metric such that for any local section s of L, we have:

$$|s|_\phi = \frac{|s|}{(|s_0|^2 + ... + |s_N|^2)^{1/2m}}.$$

The metric ϕ is none other than the pull-back of the standard Fubini-Study metric on $\mathbb{CP}^N$ via the holomorphic map:

$$x \mapsto [s_0(x) : ... : s_N(x)].$$

We call such a metric on L a *Fubini-Study* metric. The following theorem, due to Demailly [Dem92] when X is smooth and L ample, highlights the importance of such metrics as basic building blocks of complex pluripotential theory:

Theorem 1.2.9. *[BE21, thm. 7.1]*
Let X be a complex projective variety, L a semi-ample line bundle on X, and $\phi \in \mathrm{PSH}(X, L)$ a semi-positive singular metric on L.
Then there exists a decreasing sequence $(\phi_j)_{j\in\mathbb{N}}$ of Fubini-Study metrics on L, converging pointwise to ϕ.

In particular, the class $\mathrm{PSH}(X, L)$ is the smallest class of singular metrics containing all Fubini-Study metrics, and that is stable under addition of constants, finite maxima and decreasing limits.
We now move to the case of a non-archimedean field $(K, |\cdot|)$, and assume as above that (X, L) is a polarized variety over K, with mL globally generated. Following the general heuristic of replacing sums of squares with maxima in the non-archimedean world, a (tropical) Fubini-Study metric on L is a continuous metric of the form:

$$\phi = m^{-1} \max_{0\leq i\leq N} \big(\log|s_i| + \lambda_i\big),$$

where $(s_0, ..., s_N)$ is a family of global sections of mL without common zeroes and $(\lambda_0, ..., \lambda_N)$ are real constants - unlike in the Archimedean case, the valuation $v_K : K^\times \longrightarrow \mathbb{R}$ need not be surjective, so we are allowing these constants to ensure the class of Fubini-Study metrics is stable by addition of constants.
Continuous, semi-positive metrics ϕ on L can now be defined, as in the complex case, by the positivity of their curvature current $dd^c\phi$, this is the approach taken in [CLD12], where Chambert-Loir and Ducros develop a theory of real differential forms and currents on Berkovich spaces, paralleling the complex case. We will not use this approach in this text, and rather define (singular) semi-positive metrics on L in a way such that Demailly's regularization theorem still holds:

Definition 1.2.10. *Let X be a variety over a non-archimedean field K, and L a semi-ample line bundle on X. A singular metric ϕ on L is semi-positive if and only if it can be written as the pointwise decreasing limit of a net $(\phi_j)_j$ of tropical Fubini-Study metrics.*

This is consistent with the approach of Chambert-Loir - Ducros by [BE21, thm. 7.14].
In the sequel, we will define a class of semi-positive metrics on analytifications of schemes over a Banach ring A, and the Berkovich spectrum $\mathscr{M}(A)$ will have both an (open) Archimedean part and a non-archimedean part - that is , for $x \in \mathscr{M}(A)$, the complete residue field may be Archimedean or not. As a result, it is desirable to have a more uniform definition of Fubini-Study metrics, independent of the nature of the residue field. To that extent, if X is a projective variety over either $\mathbb{R}$ or $\mathbb{C}$ and L a semi-ample line bundle on X, we say a continuous metric ϕ on L is a tropical Fubini-Study metric if it can written as:

$$\phi = m^{-1} \max_{0 \leq i \leq N} \big(\log|s_i| + \lambda_i \big),$$

where $(s_0, ..., s_N)$ is a family of global sections of mL without common zeroes and $(\lambda_0, ..., \lambda_N)$ are real constants - which can always be absorbed in the s_i, so that they are allowed only for convenience. Note that over the complex numbers, any tropical Fubini-Study metric is psh in the usual sense. In fact, Demailly's regularization theorem still holds after replacing Fubini-Study metrics by tropical ones:

Theorem 1.2.11. *Let X be a projective complex variety, and L a semi-ample line bundle on X. Then any semi-positive metric $\phi \in \mathrm{PSH}(X, L)$ can be written as the decreasing limit of a net of tropical Fubini-Study metrics.*

The converse is an obvious consequence of the usual properties of $\mathrm{PSH}(X, L)$: any decreasing limit of tropical Fubini-Study metrics is psh. As a result, given any complete valued field K, we have the following uniform characterization of the class psh metrics on L: it is the smallest class of metrics that contains tropical Fubini-Study metrics, and that is stable under addition of constants, finite maxima and decreasing limits.

Proof. We set $\mathrm{PSH}^\tau(X, L)$ for the class of singular metrics that can be written as the decreasing limit of a net of tropical Fubini-Study metrics. The class $\mathrm{PSH}^\tau(X, L)$

is closed under decreasing limits by the proof of [BJ18, prop. 5.6].
Let $(s_\alpha)_{\alpha\in A}$ be a finite family of sections of mL without common zeroes, and ϕ be the associated L^2-Fubini-Study metric:

$$\phi = \frac{1}{2m}\log(\sum_{\alpha\in A}|s_\alpha|^2),$$

and set $\phi_\alpha = m^{-1}\log|s_\alpha| \in \mathrm{PSH}(X,L)$. Then we have $\phi = \chi((\phi_\alpha)_{\alpha\in A})$, with $\chi(x) = \frac{1}{2m}\log(\sum_{\alpha\in A} e^{2mx_\alpha})$. It now follows from lemma 2.1.8 that ϕ is a decreasing limit of a sequence of tropical Fubini-Study metrics, hence $\phi \in \mathrm{PSH}^\tau(X,L)$. By Demailly's regularization theorem when X is smooth and L ample, and [BE21, thm. 7.1] in the general case, any metric ϕ in $\mathrm{PSH}(X,L)$ can be written as the decreasing limit of metrics in $\mathrm{PSH}^\tau(X,L)$. Since the latter is closed under decreasing limits, we infer that $\phi \in \mathrm{PSH}^\tau(X,L)$, which concludes. □

The following example provides an alternative description of tropical Fubini-Study metrics over a discretely-valued field K of equicharacteristic zero:

Example 1.2.12. *Assume that $A = K$ is a non-trivially valued non-archimedean field, with valuation ring R and residue field k. If (X,L) is a polarized variety over K, one can define the class of* model metrics *on L as follows: for any normal, projective R-model $\mathscr{X}/R$ of X and $\mathscr{L}$ a model of mL on $\mathscr{X}$ for $m \geq 1$, define:*

$$\phi_{\mathscr{L}}(x) = m^{-1}\log|s_{\mathscr{L}}(x)|,$$

where $s_{\mathscr{L}}$ is a trivialization of $\mathscr{L}$ at the center $c_{\mathscr{X}}(v_x)$ of x. One directly checks that this defines a continuous metric on L, such that the lattice $H^0(\mathscr{X}, m\mathscr{L}) \subset H^0(X,mL)$ is the unit ball for the induced supnorm whenever $\mathscr{X}_0$ is reduced.
It then follows from [BE21, thm. 5.14] that model metrics associated to semi-ample models are the same as pure Fubini-Study metrics on L, i.e. Fubini-Study metrics where the constants are taken to be zero in the definition. As an easy consequence, model metrics are the same as differences of pure Fubini-Study metrics.

1.2.3 Tropical Fubini-Study metrics

Throughout this section, X is an A-scheme of finite type over a geometric base ring A, and L is a semi-ample line bundle on X.
The discussion from the previous section motivates the definition of the following class of metrics, that will be the building blocks for our class of semi-positive metrics:

Definition 1.2.13. *Let L be a line bundle on X, and let $m \geq 1$ be an integer. A tropical Fubini-Study metric on L is a (non-singular) metric of the form:*

$$\phi = m^{-1}\max_{j\in J}(\log|s_j| + a_j),$$

where $(s_j)_{j\in J}$ is a finite family of sections of mL without common zeroes and $a_j \in \mathbb{R}$. We write $\mathrm{FS}^\tau(L)$ for the set of tropical Fubini-Study metrics on L.
If $L = \mathcal{O}_X$, we will simply say that ϕ is a Fubini-Study function on X, and write $\mathrm{FS}^\tau(X) = \mathrm{FS}^\tau(\mathcal{O}_X)$.
Finally, if the constants a_j are all zero in the above definition, we will say that ϕ is a pure *Fubini-Study metric.*

It follows from the definition that $\mathrm{FS}^\tau(L)$ is non-empty if and only L is semi-ample.
The following properties of $\mathrm{FS}^\tau(L)$ are straightforward consequences of the definition:

Proposition 1.2.14. *Let X be an A-scheme of finite type and L a line bundle on X. Then:*

1. *if $\phi \in \mathrm{FS}^\tau(L)$ and $c \in \mathbb{R}$, then $\phi + c \in \mathrm{FS}^\tau(L)$;*
2. *if $\phi_1, \phi_2 \in \mathrm{FS}^\tau(L)$, then $\max\{\phi_1, \phi_2\} \in \mathrm{FS}^\tau(L)$;*
3. *if $\phi_i \in \mathrm{FS}^\tau(L_i)$ for $i = 1, 2$ then $\phi_1 + \phi_2 \in \mathrm{FS}^\tau(L_1 + L_2)$;*
4. *if ϕ is a metric on L such that $m\phi \in \mathrm{FS}^\tau(mL)$ for $m \geq 1$, then $\phi \in \mathrm{FS}^\tau(L)$;*
5. *if $\phi_1, \phi_2 \in \mathrm{FS}^\tau(L)$ and $c_1, c_2 \in \mathbb{Q}_{\geq 0}$ with $c_1 + c_2 = 1$, then $c_1\phi_1 + c_2\phi_2 \in \mathrm{FS}^\tau(L)$;*
6. *if $f : Y \longrightarrow X$ is a morphism of A-schemes of finite type and $\phi \in \mathrm{FS}^\tau(L)$, then $f^*\phi \in \mathrm{FS}^\tau(f^*L)$.*
7. *if B is a Banach ring together with a bounded homomorphism $A \longrightarrow B$ and $\phi \in \mathrm{FS}^\tau(X, L)$, then the base change metric $\phi_B \in \mathrm{FS}^\tau(X_B, L_B)$.*

We now introduce the following class of metrics, which usually play the role of smooth metrics in the non-archimedean case:

Definition 1.2.15. *Let L be a line bundle on X.*
A DFS *(difference of Fubini-Study) metric on L is a metric of the form $\phi = \phi_1 - \phi_2$, where $\phi_i \in \mathrm{FS}(L_i)$ for $i = 1, 2$, with $L = L_1 - L_2$.*
We write $\mathrm{DFS}(L)$ for the set of DFS metrics on L, and $\mathrm{DFS}(X) \subset \mathcal{C}^0(X^{\mathrm{an}})$ for the set of DFS functions on $\mathcal{O}_X$, i.e. DFS metrics on $\mathcal{O}_X$.

Theorem 1.2.16. *Assume that X/A is projective. Then the $\mathbb{Q}$-vector space $\mathrm{DFS}(X)$ is dense in $\mathcal{C}^0(X^{\mathrm{an}})$.*

Proof. This is essentially the same proof as in [BJ18, thm. 2.7].
It follows easily from prop. 1.2.14 that $\mathrm{DFS}(X)$ is a $\mathbb{Q}$-subvector space of $\mathcal{C}^0(X)$, containing constant functions. It is furthermore stable under maxima, as if $\phi_i, \psi_i \in \mathrm{FS}(L_i)$ for $i = 1, 2$, then:

$$\max\{\phi_1 - \psi_1, \phi_2 - \psi_2\} = \max\{\phi_1 + \psi_2, \phi_2 + \psi_1\} - (\psi_1 + \psi_2) \in \mathrm{DFS}(X).$$

As a result, since X^{an} is compact, by the Stone-Weierstrass theorem, it is enough to prove that $\mathrm{DFS}(X)$ separates points.
Since DFS is stable by pullback, we may assume that $X = \mathbb{P}^n_A$. Let $x \neq y \in X^{\mathrm{an}}$, then by considering a hyperplane not containing either x or y, we may assume $x, y \in \mathbb{A}^{n,\mathrm{an}}_A = \mathscr{M}\,\mathrm{Spec}(A[t_1, ..., t_n])$. By definition of $\mathscr{M}$ Spec and since $|\cdot|_x \neq |\cdot|_y$, there exists a polynomial $f \in A[t_1, ..., t_n]$ such that $|f(x)| \neq |f(y)|$; we will assume $|f(x)| < |f(y)|$.
Take homogeneous coordinates $z_0, ..., z_n \in H^0(\mathbb{P}^n_A, \mathcal{O}(1))$ on $\mathbb{P}^n_A$, such that $t_i = z_i/z_0$

on $\mathbb{A}^n$. We may write $f(t_1, ..., t_n) = z_0^{-d}s$, with $s \in H^0(\mathbb{P}^n_A, \mathcal{O}(d))$.
Let $N \in \mathbb{Z}$, and $\lambda_0, ..., \lambda_n \in \mathbb{Z}$, and set:

$$\psi = d \max_{0\leq j\leq n}(\log|z_j| - \lambda_j),$$

which is an FS metric on $\mathcal{O}(d)$, so that:

$$u = \max\{\log|s|, \psi - N\} - \psi = \max\{\log|s| - \psi, -N\}$$

is a DFS function on $\mathbb{P}^n_A$.
Then for $\lambda_0 = 0$ and λ_j large enough, we have $\psi(x) = d\log|z_0(x)|$ and $\psi(y) = d\log|z_0(y)|$.
Thus, for $N > -\log|f(y)|$, we have:

$$u(x) = \max\{\log|s(x)| - d\log|t_0(x)|, -N\} = \max\{\log|f(x)|, -N\}$$

$$< \max\{\log|f(y)|, -N\} = u(y),$$

whence the result. □

Example 1.2.17. *Let $Y = \operatorname{Spec} A$, so that $Y^{\mathrm{an}} = \mathscr{M}(A)$. Then a Fubini-Study function on Y^{an} is a continuous function of the following form:*

$$\eta = m^{-1}\max_{\alpha\in B}(\log|f_\alpha| + \lambda_\alpha),$$

where the $f_\alpha \in A$ have no common zeroes and $\lambda_\alpha \in \mathbb{R}$.
Now, if X/A is a scheme of finite type and $X^{\mathrm{an}} \xrightarrow{\pi} \mathscr{M}(A)$ is its analytification, the function $\eta \circ \pi$ (that we will still write as $\eta : X^{\mathrm{an}} \longrightarrow \mathbb{R}$) is also an FS function on X.

1.2.4 Semi-positive metrics

From now on, we will assume that A is an integral Banach ring (in addition to being a geometric base ring). Recall that $\mathscr{M}(A)^\eta$ is the subset of $\mathscr{M}(A)$ whose elements $|\cdot|_x$ have trivial kernel. In particular, the residue field $\mathscr{H}(x)$ of $\mathscr{M}(A)$ at x is the completion of the fraction ring κ of A with respect to $|\cdot|_x$, so that $X_{\mathscr{H}(x)}$ is the flat base change of X to $\mathscr{H}(x)$.

Definition 1.2.18. *Let X be a scheme of finite type over A, and L a semi-ample line bundle on X. A plurisubharmonic (or semi-positive) metric ϕ on L is a singular metric on L that is the pointwise limit of a decreasing net of tropical Fubini-Study metrics on L, and such that $\phi_x \not\equiv -\infty$, for all $x \in \mathscr{M}(A)^\eta$, where ϕ_x is the restriction of ϕ to $X^{\mathrm{an}}_{\mathscr{H}(x)}$.*
We write $\mathrm{PSH}(X, L)$ or $\mathrm{PSH}(L)$ for the set of semi-positive metrics on L, and $\mathrm{PSH}(X)$ for the set of PSH functions on X^{an}.

Note that since our base ring A is arbitrary, the space $\mathrm{PSH}(X)$ could be very large; for instance even for $X = \operatorname{Spec} A$, every non-zero element $a \in A$ induces a PSH

function $\phi_a = \log|a| \in \mathrm{PSH}(X)$ on $X^{\mathrm{an}} = \mathscr{M}(A)$. Indeed, we have $\phi = \lim_j (\phi_j)_j$, with:

$$\phi_j = \max(\log|a|, \log|1-a| - j),$$

and $\phi(x) = \log|a(x)| > -\infty$ whenever $x \in \mathscr{M}(A)^\eta$.
Let us point out that while the condition $\phi_x \not\equiv -\infty$ for all $x \in \mathscr{M}(A)^\eta$ is natural in the setting of hybrid spaces (we will see later that it translates into finiteness of Lelong numbers) and spaces over $\mathscr{M}(\mathbb{Z})$, it might be too strong in general - the analytification of $\mathbb{P}^2$ over a trivially valued field contains points that are pluripolar and Zariski-dense. Such a point lies in an affinoid domain $\mathscr{M}(A)$, which will then admit a psh function in the sense of [BJ18] that is $-\infty$ at a Zariski-dense point.

Definition 1.2.19. *We write* $\mathrm{CPSH}(X, L)$ *for the set of continuous, plurisubharmonic metrics on* L*. It is endowed with the topology of uniform convergence on* X*.*

Note that $\mathrm{FS}^\tau(X, L) \subset \mathrm{CPSH}(X, L)$ by definition.

Proposition 1.2.20. *The following properties hold:*

1. *if* $\phi \in \mathrm{PSH}(L)$ *and* $c \in \mathbb{R}$, $\phi + c \in \mathrm{PSH}(L)$*;*
2. *if* $\phi_i \in \mathrm{PSH}(L_i)$ *for* $i = 1, 2$, $\phi_1 + \phi_2 \in \mathrm{PSH}(L_1 + L_2)$*;*
3. *if* $\phi_1, \phi_2 \in \mathrm{PSH}(L)$*, then* $\max\{\phi_1, \phi_2\} \in \mathrm{PSH}(L)$*;*
4. *if* ϕ *is a singular metric on* L *such that* $m\phi \in \mathrm{PSH}(mL)$ *for* $m \geq 1$*, then* $\phi \in \mathrm{PSH}(L)$*;*
5. *if* $\phi_1, \phi_2 \in \mathrm{PSH}(L)$ *and* $c_1, c_2 \in \mathbb{R}_{\geq 0}$ *with* $c_1 + c_2 = 1$*, then* $c_1\phi_1 + c_2\phi_2 \in \mathrm{PSH}(L)$*;*
6. *if* $(\phi_j)_j$ *is a decreasing net in* $\mathrm{PSH}(L)$ *and* $\phi = \lim_j \phi_j$ *is such that* $\phi_x \not\equiv -\infty$ *for all* $x \in \mathscr{M}(A)^\eta$*, then* $\phi \in \mathrm{PSH}(L)$*;*
7. *if* X/A *is proper and* $(\phi_j)_j$ *is a net in* $\mathrm{PSH}(L)$ *converging uniformly to* ϕ*, then* $\phi \in \mathrm{PSH}(L)$*.*

Note that the difference of two singular psh metrics does not make sense as a function in general, so that the last item means the following: if $(\phi_j)_j$ is a net in $\mathrm{PSH}(L)$, such that there exists a net of continuous functions $(f_j)_j$ in $\mathcal{C}^0(X^{\mathrm{an}})$ converging uniformly to zero and such that $\phi_j = \phi + f_j$, then $\phi \in \mathrm{PSH}(L)$.

Proof. The first 5 items are straightforward consequences of the corresponding properties for FS^τ, stated in prop. 1.2.14; while (6) and (7) follows from [BJ18, lem. 4.6], (i) and (ii) respectively. □

The subset $\mathrm{CPSH}(X, L) \subset \mathrm{PSH}(X, L)$ is naturally endowed with the topology of uniform convergence; the next proposition states that for this topology, $\mathrm{FS}^\tau(L)$ is dense in CPSH:

Proposition 1.2.21. *[BJ18, prop. 5.20] Assume that X/A is proper, and let ϕ be a continuous metric on L. Then $\phi \in \mathrm{CPSH}(X,L)$ if and only if there exists a net $(\phi_j)_j$ in $\mathrm{FS}^\tau(L)$ converging uniformly to ϕ.*

Note that the natural topology of uniform convergence on the subset of continuous PSH metrics does not extend to $\mathrm{PSH}(L)$, so that it is unclear in this generality how to put a reasonable topology on $\mathrm{PSH}(L)$.
While the Fubini-Study metrics considered above are always continuous - hence bounded - one can also consider FS metrics with singularities:

Example 1.2.22. *Assume that L is a semi-ample, and let $s \in H^0(X,mL)$ for some $m \geq 1$ be a non-zero global section. Then $\phi = m^{-1}\log|s| \in \mathrm{PSH}(X,L)$, since we may write ϕ as the decreasing limit of the following:*

$$\phi_j = (md)^{-1}\max\big(\log|s^d|, \max_{\alpha\in A}(\log|s_\alpha| - j)\big),$$

where $(s_\alpha)_{\alpha\in A}$ is a family of sections of mdL without common zeroes. Moreover, if $x \in \mathscr{M}(A)$ is a Zariski-dense point, the base change of s to $X_{\mathscr{H}(x)}$ is non-zero by flatness, so that $\phi_x \not\equiv -\infty$.
More generally, by prop. 1.2.20, for semi-ample L and any finite family $(s_\alpha)_{\alpha\in A}$ of sections of mL, the metric:

$$\phi = m^{-1}\max_{\alpha\in A}(\log|s_\alpha| + c_\alpha)$$

is semi-positive, i.e. $\phi \in \mathrm{PSH}(X,L)$.

Remark 1.2.23. *Let us point out that unlike the class FS^τ, the class of plurisubharmonic metrics is not - strictly speaking - stable under base change. Indeed, if the ring homomorphism $A \longrightarrow B$ is not flat and $s \in H^0(X,mL)$ is a non-zero global section, it could very well happen that the section $s_B \in H^0(X_B,mL_B)$ is the zero section. As a result, the base change to X_B of the psh metric $\phi = m^{-1}\log|s|$ satisfies $\phi_B \equiv -\infty$. For instance, if $\phi \in \mathrm{PSH}(X,L)$ and $x \in \mathscr{M}(A)$ is not Zariski-dense, then the restriction ϕ_x to $X^{\mathrm{an}}_{\mathscr{H}(x)}$ may be identically $-\infty$.*
This occurs in the complex world as well, as a psh metric on a the total space of a holomorphic fibration $X \longrightarrow B$ may be identically $-\infty$ on certain fibers X_b. One easy way to remedy this is for instance to allow psh metrics to be identically $-\infty$, we choose not to as this psh metric would have to be treated separately in several proofs, making the exposition more cumbersome.

1.2.5 Examples

We start with the case where $A = \mathbb{C}$ with the usual absolute value. The following statement is a mere reformulation of theorem 1.2.11:

Theorem 1.2.24. *Let X be a projective complex variety, and let L be a semi-ample line bundle on X.*
Then $\mathrm{PSH}(X,L)$ is the space of plurisubharmonic metrics on L in the sense of usual pluripotential theory.

As mentioned above, for any Banach ring A, the above definitions for $X = \operatorname{Spec} A$ and $L = \mathcal{O}_X$ yield a space $\mathrm{PSH}(X)$ of plurisubharmonic functions on $\mathscr{M}(A)$. We will compute this space in simple examples.
We start with the case where $A = k^{\mathrm{hyb}}$ is a hybrid field. Since k is non-trivially valued, we may assume (up to scaling) that $\log|k^\times| \supseteq \mathbb{Z}$. In that case, the homeomorphism:

$$\lambda : \mathscr{M}(A) \xrightarrow{\sim} [0,1]$$

is in fact such that λ is a Fubini-Study function on X, since $\lambda(x) = \log|a|_x$ for any $a \in k^\times$ such that $\log|a| = 1$.
Conversely, any Fubini-Study function on X is of the form:

$$\phi(x) = m^{-1} \max_{j \in J} \big(\log|a_j|_x + c_j \big),$$

for $a_j \in k^\times$ and $c_j \in \mathbb{Z}$. Since $\log|a_j|_x = \lambda(x) \log|a_j|$, ϕ is a finite maximum of affine functions, hence convex, and $\mathrm{FS}(X)$ contains all finite maxima of affine functions with rational coefficients. Taking decreasing limits (which have finite values everywhere since all points in $\mathscr{M}(k^{\mathrm{hyb}})$ are Zariski-dense), we conclude that the homeomorphism λ identifies the space $\mathrm{PSH}(\mathscr{M}(k^{\mathrm{hyb}}))$ with the space of real-valued convex functions on the segment $[0,1]$. In particular, plurisubharmonic functions on X are continuous away from the boundary of the interval. Note however that the function $\phi : \mathscr{M}(k^{\mathrm{hyb}}) \longrightarrow \mathbb{R}$ defined by $\phi(0) = 1$, $\phi(\lambda) = 0$ for $\lambda > 0$ is also psh.
We now move on to the case $A = A_r$, the ring of Laurent series that are convergent for the hybrid norm on $\mathbb{C}$; recall that we have a canonical homeomorphism from the hybrid circle $C^{\mathrm{hyb}}(r) := \mathscr{M}(A_r) \simeq \bar{\mathbb{D}}_r$ to the closed Euclidean disk. The following proposition asserts that away from the boundary of the closed disk, we may, after rescaling, identify psh functions on the hybrid circle with subharmonic functions on the punctured disk that have logarithmic growth at the puncture:

Proposition 1.2.25. *Let:*

$$A_r = \{ f = \sum_{n \in \mathbb{Z}} a_n t^n \in \mathbb{C}((t)) \ / \ \|f\|_{\mathrm{hyb}} = \sum_n \|a_n\|_{\mathrm{hyb}}\, r^n < \infty \},$$

and write $\tau : \bar{\mathbb{D}}_r \xrightarrow{\sim} C^{\mathrm{hyb}}(r)$ the homeomorphism from prop. 1.1.23.
There exists an order-preserving, injective map:

$$\rho_r : \mathrm{PSH}(C^{\mathrm{hyb}}(r)) \longrightarrow \mathrm{SH}(\mathbb{D}_r) + \mathbb{R}\log|t|$$

$$\phi \longmapsto \big(t \ \mapsto \log_r|t| \times (\phi(\tau(t)) \big)$$

for $t \neq 0$. Moreover, if ϕ is continuous, then $\phi(0) = \log r \times \nu_0(\rho_r(\phi))$ is a negative multiple of the (generalized) Lelong number of $\rho_r(\phi)$ at 0.
Conversely, for any $r' > r$, there exists an order-preserving, injective map:

$$\rho_{r',r} : \mathrm{SH}(\mathbb{D}_{r'}) + \mathbb{R}\log|t| \longrightarrow \mathrm{PSH}(C^{\mathrm{hyb}}(r)),$$

$$\phi \longmapsto \big(\tilde{\phi} : t \mapsto \frac{\phi(\tau^{-1}(t))}{\log_r|t|} \big)$$

with $\tilde{\phi}(0) = \frac{\nu_0(\phi)}{\log r}$. Finally, the composition $\rho_r \circ \rho_{r',r}$ is (up to a scaling factor) the usual restriction map.

Proof. Let $\phi \in \mathrm{FS}^\tau(C^{\mathrm{hyb}}(r))$, and write:

$$\phi = m^{-1} \max_{\alpha \in A}(\log|f_\alpha| + c_\alpha),$$

with $f_\alpha \in A_r$, so that in particular the formal series f_α induces a holomorphic function F_α on $\mathbb{D}_r$. Moreover, we have $\log|f_\alpha|(\tau(t)) = \frac{\log r}{\log|t|}\log|F_\alpha(t)|$ for $t \neq 0$.Up to shifting ϕ by a constant, we may assume that $\phi \leq 0$ on $C^{\mathrm{hyb}}(r)$, so that each term in the maximum is nonpositive.
Thus, defining $\rho_r(\phi)(t) = \log_r|t| \times \phi(\tau(t))$ for $t \neq 0$, we have:

$$\rho_r(\phi)(t) = \max_{\alpha \in A}\big(\log|F_\alpha(t)| + c_\alpha \log_r|t|\big),$$

since all the terms in the maximum have the same sign. This implies that $\rho_r(\phi)$ extends at $t = 0$ as the sum of a subharmonic function on $\mathbb{D}_r$ and a multiple of $\log|t|$, and with Lelong number at zero:

$$\nu_0(\rho_r(\phi)) = \max_{\alpha \in A}(\mathrm{ord}_0(f_\alpha) + \frac{c_\alpha}{\log r}) = \frac{\phi(0)}{\log r} \geq 0.$$

Now if $\phi \in \mathrm{PSH}(C^{\mathrm{hyb}}(r))$, it is finitely-valued at the Zariski-dense point 0, so that up to shifting by a constant, we may assume that $\phi(0) = -1$. We now write ϕ as the decreasing limit of a net $(\phi_j)_j$ in FS^τ, with $\phi_j(0) \leq 0$ for all j large enough. By the computations above, the latter condition means precisely that $\rho_r(\phi_j)$ extends over zero as a subharmonic function. We then define the function $\rho_r(\phi) \in \mathrm{SH}(\mathbb{D}_r)$ as the decreasing limit of the $\rho_r(\phi_j)$, which is independent of the choice of decreasing sequence, since $\rho_r(\phi)$ is determined uniquely by ϕ outside 0. If ϕ is furthermore continuous, then:

$$\phi(0) = \lim_{t \to 0}\phi(\tau(t)) = \lim_{t \to 0}\frac{\rho_r(\phi)(t)}{\log_r|t|} = \log r \times \nu_0((\rho_r(\phi)).$$

Conversely, let $r' > r$ and let $\phi \in \mathrm{SH}(\mathbb{D}_{r'})$. The fact that $\frac{\log r}{\log|t|} \cdot \phi \in \mathrm{PSH}(C^{\mathrm{hyb}}(r))$ follows from the more general theorem 1.3.13.
Finally it is clear that from the constructions that if $\phi \in \mathrm{SH}(\mathbb{D}_{r'}) + \mathbb{R}\log|t|$, then $\rho_r(\rho_{r',r}(\phi)) = \phi_{|\mathbb{D}_r}$. □

We furthermore expect that the image of the restriction of ρ_r to $\mathrm{CPSH}(C^{\mathrm{hyb}}(r))$ is the space of continuous subharmonic functions on $\mathbb{D}_r$, extending continuously to the boundary of the disk.

Remark 1.2.26. *Let $\eta : C^{\mathrm{hyb}}(r) \longrightarrow \mathbb{R}$ be such that $\eta(0) = 1$ and $\eta \equiv 0$ outside zero. Then unsatisfyingly, $\eta \in \mathrm{PSH}(C^{\mathrm{hyb}}(r))$; we interpret this as the non-archimedean realization of the following phenomenon. Let:*

$$\psi_j = \max(\log|t|, -j),$$

which decrease to $\psi = \log|t|$. We have $\nu_0(\psi_j) \equiv 0$ since ψ_j is bounded near 0, while $\nu_0(\psi) = 1 > \lim_j \nu_0(\psi_j)$. Writing ψ_j^{hyb} and ψ^{hyb} the associated psh functions

on $C^{\mathrm{hyb}}(r)$ for some $r \in (0,1)$, the jump of Lelong numbers along this decreasing sequence means that the non-archimedean data $\nu_0(\psi)$ attached to ψ differs from the restriction $\psi^{\mathrm{hyb}}(0) := \lim_j \psi_j^{\mathrm{hyb}}(0) = \lim_j \nu_0(\psi_j)$ to the origin of the hybrid data associated to ψ. The point $0 \in C^{\mathrm{hyb}}(r)$ is in fact non-pluripolar (as it is Zariski-dense), and thus is not negligible in the sense of hybrid pluripotential theory.

We also describe subharmonic functions on the Berkovich spectrum $\mathscr{M}(\mathbb{Z})$. We start by introducing some notation: if $I_p \simeq [0,+\infty]$ is the p-adic branch inside $\mathscr{M}(\mathbb{Z})$, and $\phi_p : I_p \longrightarrow \mathbb{R} \cup \{-\infty\}$ is a convex function, its (outgoing) slope at zero is the limit $s_p := \lim_{\varepsilon \to 0} \frac{\phi_p(\varepsilon) - \phi_p(0)}{\varepsilon}$, and its slope at infinity is the (possibly infinite) limit $\tilde{s}_p = \lim_{\varepsilon \to \infty} \frac{\phi(\varepsilon)}{\varepsilon}$. The outgoing slope at zero s_∞ is defined similarly on the archimedean branch.

Proposition 1.2.27. *Let $X = \mathscr{M}(\mathbb{Z})$, and write $X = \bigcup_{p \in \mathcal{P} \cup \infty} I_p$ as the union of the p-adic and archimedean branches.*
Then a continuous function $\phi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ is psh if and only:

- *for every prime number p, its restriction ϕ_p to the branch I_p is convex, with negative slopes $s_p, \tilde{s}_p$ at 0 and $+\infty$ respectively, and value at infinity $\phi_p(|\cdot|_p^\infty) \in \mathbb{R} \cup \{-\infty\}$,*
- *its restriction to the branch I_∞ is convex and increasing, with positive slope at 0,*
- *the sum of slopes at zero $\sum_{p \in \mathcal{P}} s_p + s_\infty \geq 0$; in particular the sum $\sum_{p \in \mathcal{P}} -s_p < +\infty$.*

In other words, the function ϕ is psh on $\mathscr{M}(\mathbb{Z})$ if and only it is subharmonic in the usual sense on the $\mathbb{R}$-tree $\mathscr{M}(\mathbb{Z})$.

As a consequence, a point $x \in \mathscr{M}(\mathbb{Z})$ is polar (that is, contained in $\{\phi = -\infty\}$ for some $\phi \in \mathrm{CPSH}(X)$) if and only if it is the outer end of a p-adic branch.

Proof. Let $\phi \in \mathrm{FS}(X)$ be a Fubini-Study function, then there exists a family of integers $(n_\alpha)_{\alpha \in A}$, with $\min_{\alpha \in A} v_p(n_\alpha) = 0$ for every prime p, such that:

$$\phi = m^{-1} \max_{\alpha \in A} (\log|n_\alpha| + c_\alpha),$$

with $c_j \in \mathbb{R}$. Denoting $0 \in X$ the trivial absolute value, we have $\phi(0) = m^{-1} \max_\alpha c_\alpha$, and we write $A' \subset A$ the set of indices realizing the maximum. Set $n_1 = \gcd_{\alpha \in A'} n_\alpha$, and $n_2 = \mathrm{lcm}_{\alpha \in A'} n_\alpha$.
Under the homeomorphism $I_p \simeq [0,+\infty]$, we have:

$$\phi_p(\varepsilon) = m^{-1} \max_{\alpha \in A} (-v_p(n_\alpha) \log p \cdot \varepsilon + c_\alpha),$$

which shows that ϕ_p is a piecewise-affine convex function, and with slope at zero:

$$s_p = (-\log p) \min_{\alpha \in A'} v_p(n_\alpha) = \log|n_1|_p \leq 0,$$

and constant for $\varepsilon \gg 1$ since there exists a n_α with $v_p(n_\alpha) = 0$, so that the slope at infinity $\tilde{s}_p = 0$. A similar computation on the archimedean branch shows that ϕ_∞ is also convex, with:

$$s_\infty = \sum_{p\in\mathcal{P}} \log p \cdot \big(\max_{\alpha\in A'} v_p(n_\alpha)\big) = -\sum_{p\in\mathcal{P}} \log|n_2|_p = \log|n_2|_\infty \geq 0,$$

so that the sum of slopes:

$$s_\infty + \sum_{p\in\mathcal{P}} s_p = \log\left|\frac{n_2}{n_1}\right|_\infty$$

is positive. Now if ϕ is a continuous psh function on X (with possibly infinite values), then it is the uniform limit of Fubini-Study functions near zero, hence the only if part by taking decreasing - hence locally uniform by Dini's lemma - limits.
Conversely, let $\phi : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a function satisfying the above properties. We assume that $\phi(0) = 0$, so that $\phi_p \leq 0$ for every prime p, and $\phi_\infty \geq 0$. We divide the argument in several steps.
Step 1: Assume that $\phi_p \equiv 0$ for every prime p, and $\phi_\infty : [0,1] \longrightarrow \mathbb{R}$ is a continuous, increasing convex function, with $\phi_\infty(0) = 0$.
For any $a \geq 0$ and $b \leq 0$ two real numbers, the function $\psi = \psi_{a,b}$ on X defined by $\psi_\infty(x) = ax + b$ and $\psi_p \equiv 0$ for every prime p is psh (although not necessarily continuous at 0). Indeed, choosing a prime q and writing $a = \lim_j r_j \log q$ as the decreasing limit of rational multiples of $\log q$, we see that $\psi = \lim_j \max\{r_j \log|q| + b, 0\}$ as a decreasing limit - the max is realized by 0 on every p-adic branch, even when $p = q$ due to the assumption on a, b.
As a result, writing ϕ_∞ as a decreasing limit of piecewise affine convex functions of the form $\max_{\alpha\in A_j}(a_\alpha x + b_\alpha)$ as above (recall that $\min_\alpha a_\alpha \geq 0$, and $\max_\alpha b_\alpha = 0$ since $\phi_\infty(0) = 0$), we get that:

$$\phi = \lim_j \max_{\alpha\in A_j} \psi_{a_\alpha, b_\alpha}$$

as a decreasing limit, and $\phi \in \mathrm{PSH}(X)$.
Step 2: we now regularize the ϕ_p. By our assumptions on the slopes s_p, $\tilde{s}_p$, for every prime p, we may find a decreasing sequence $(\phi_{j,p})$ of convex functions on I_p of the form:

$$\phi_{j,p}(\varepsilon) = m^{-1} \max_{\alpha\in A_p}(-\ell_\alpha \log p \cdot \varepsilon + c_\alpha)$$

converging to ϕ_p, where the ℓ_α's are positive integers and such that $\phi(0) = \phi_p(0) = \max_\alpha c_\alpha = 0$. Write $s_{j,p} = \min_{\alpha, c_\alpha = 0}(-\ell_\alpha \log p)$ the slope at zero of $\phi_{j,p}$; by continuity near zero, the $s_{j,p}$ decrease to s_p. We use the same notation for the (singular) Fubini-Study function:

$$\phi_{j,p} = m^{-1} \max_{\alpha\in A}(\log|p^{\ell_\alpha}| + c_\alpha),$$

by straightforward computation we see that the restriction of $\phi_{j,p}$ to the branch I_∞ is linear:

$$\phi_{j,p}(|\cdot|_\infty^x) = -s_{j,p} x.$$

For $k \in \mathbb{N}$, let $\mathcal{P}_k = \{2, .., p_k\}$ be the k smallest primes. We set:

$$\phi_k = \sum_{p \in \mathcal{P}_k} \phi_{p,k},$$

which is psh on X, and such that $(\phi_k)_k$ decreases to ϕ on each p-adic branch.
Step 3: this does not yield the desired outcome on the archimedean branch: the restrictions of ϕ_k to the archimedean branch are increasing to $x \mapsto sx$, where we have set $s := -\sum_p s_p \leq s_\infty$. However, the convergence is uniform on $I_\infty \simeq [0,1]$, so that (after extraction of a subsequence) we may find a decreasing sequence $(\varepsilon_k)_k$ of constants going to zero, such that on I_∞, the $\phi'_{k,\infty} = \phi_{k,\infty} + \varepsilon_k$ decrease to $x \mapsto sx$. As a result, the psh functions $\phi'_k = \phi_k + \varepsilon_k$ decrease on X, and the limit ϕ' satisifies $\phi'_p = \phi_p$, and $\phi'_\infty(x) = sx$ for $x \in [0,1]$.
Step 4: by step 1, the function $\psi : X \longrightarrow \mathbb{R}$ such that $\psi_p \equiv 0$ for every prime p, and $\psi_\infty(x) = \phi_\infty(x) - sx$ is psh, since $s \leq s_\infty$. As a result, $\phi = \phi' + \psi$ is indeed subharmonic on $\mathscr{M}(\mathbb{Z})$. □

1.3 PSH metrics on hybrid spaces

Throughout this section, we let $X \xrightarrow{\pi} \mathbb{D}^*$ be a degeneration of projective complex manifolds, endowed with a semi-ample line bundle L. We fix $r \in (0,1)$, and write $X^{\mathrm{hyb}} \xrightarrow{\pi_{\mathrm{hyb}}} \bar{\mathbb{D}}_r$ the associated hybrid space, which is the analytification of X viewed as an A_r-scheme, see section 1.1.3.
We will use the t-adic valuation on $K = \mathbb{C}((t))$ normalized so that $|t| = r$, and write:

$$\log_r|t| = \frac{\log|t|}{\log r},$$

which is non-negative on $\bar{\mathbb{D}}_r$.

Definition 1.3.1. *Let $X \xrightarrow{\pi} \mathbb{D}^*$ be a degeneration of projective complex manifolds, and let L be a line bundle on X. A hybrid (continuous) metric ϕ on L is a singular (resp. continuous) metric on L^{hyb} in the sense of the previous section, viewing X as an A_r-scheme.*
We write $\mathrm{PSH}(L^{\mathrm{hyb}})$ for the set of hybrid semi-positive metrics on L.

Using the explicit description of the hybrid space from prop. 1.1.25, we are able to describe more concretely continuous hybrid metrics on L:

Proposition 1.3.2. *Let X be a degeneration of complex manifolds, and L be a line bundle on X. A continuous hybrid metric ϕ on L is equivalent to the data of a continuous family of metrics $(\phi_t)_{t \in \bar{\mathbb{D}}^*_r}$ on the (X_t, L_t), together with a continuous metric ϕ_0 on L^{an}, such that the following holds: for every Zariski open subset $U \subset X$ and any section $s \in H^0(U, L_{|U})$, the function:*

$$z \mapsto \frac{\log\|s(z)\|_{\phi_t}}{\log_r|t|}$$

on U^{hol} (with the Euclidean topology) extends as a continuous function to U^{hyb} via $x \in U^{\mathrm{an}} \mapsto \log\|s(x)\|_{\phi_0}$.

Proof. If $|t| > 0$, we have a canonical homeomorphism $\beta_t : X_t \xrightarrow{\sim} X_t^{\mathrm{hyb}}$, such that for any (local) regular function on X, we have:

$$|f(\beta(z))| = |f(z)|^{\frac{\log r}{\log|t|}},$$

and a homeomorphism $\beta_0 : X^{\mathrm{an}} \xrightarrow{\sim} \pi_{\mathrm{hyb}}^{-1}(0)$.
Hence if ϕ is a continuous hybrid metric on L, it induces a continuous family $\{\phi_t\}_{t\in\bar{\mathbb{D}}_r}$ of metrics ϕ_t on L_t, obtained as follows: if $U \subset X$ is a Zariski open and s a trivialization of L on U, set:

$$\|s(z)\|_{\phi_t} := \|s(\beta_t(z))\|_{\phi}^{\frac{\log|t|}{\log r}},$$

for $z \in U \cap X_t$. The fact that this defines a continuous metric on L_t is an easy consequence of the above equality for functions.
Similarly, the formula:

$$\log\|s(x)\|_{\phi_0} := \log\|s(\beta_0(x))\|_{\phi}$$

defines a continuous metric ϕ_0 on L^{an}.
The fact that the data of ϕ_0 and $\{\phi_t\}_{t\in\bar{\mathbb{D}}_r^*}$ recovers ϕ uniquely is clear. □

We will sometimes write the above relation more loosely as:

$$\phi_t = \frac{\log|t|}{\log r}\phi_{|X_t^{\mathrm{hyb}}},$$

where the left-hand side lives on the complex fiber X_t, and the metric $\phi_{|X_t^{\mathrm{hyb}}}$ lives on the Berkovich space X^{hyb}.

Example 1.3.3. *Let $s \in H^0(X, mL)$ be a global section of mL, and let:*

$$\phi = m^{-1}\log|s| \in \mathrm{PSH}(L^{\mathrm{hyb}})$$

be the associated (singular) hybrid metric. Then one checks directly that for any $t \neq 0$, the metric $\phi_t \in \mathrm{PSH}(X_t, L_t)$ is equal to $m^{-1}\log|s_t|$, where $s_t = s_{|X_t} \in H^0(X_t, mL_t)$.
Now let $c \in \mathbb{R}$ be a constant, and let:

$$\phi = m^{-1}(\log|s| + c) \in \mathrm{PSH}(L^{\mathrm{hyb}}).$$

Then we have $\phi_t = m^{-1}(\log|s| + c\log_r|t|) \in \mathrm{PSH}(X_t, L_t)$.

Remark 1.3.4. *Let $\phi = m^{-1}(\log|s| + \frac{c}{\log r}) \in \mathrm{PSH}(L^{\mathrm{hyb}})$ as in the example above. Then ϕ is the decreasing limit of the $(\phi_j)_{j\in\mathbb{N}}$, where:*

$$\phi_j = m^{-1}(\log|s| + \frac{c_j}{\log r}),$$

where $(c_j)_{j\in\mathbb{N}}$ is a sequence of rational numbers decreasing to c. Up to replacing m by a high enough multiple depending on the denominator of c_j (and s by the according power) we may furthermore assume that $c_j \in \mathbb{Z}$, so that:

$$\phi_j = m^{-1}\log|t^{c_j}s|$$

is a pure Fubini-Study metric. Since finite maxima commute with decreasing limits, any tropical Fubini-Study metric can be written as a decreasing limit of pure Fubini-Study metrics. As a consequence, any psh metric on L^{hyb} can be written as a decreasing limit of pure *Fubini-Study metrics.*
Let us emphasize that the key point here is that the constant function $\frac{c}{\log r}$ for $c \in \mathbb{Q}$ can be written as $\log|f|$ for some non-zero $f \in A_r$, which need not hold over a general Banach ring A - it fails for instance for $A = \mathbb{Z}$.

1.3.1 Bergman metrics on the hybrid space

Let $X \longrightarrow \mathbb{D}^*$ be a degeneration of complex varieties, and L a semi-ample line bundle on X. In the sequel, it will sometimes be convenient for us to work with singular L^2-Bergman metrics in the complex world, i.e. metrics on L of the form:

$$\phi = \frac{1}{2m} \log \big(\sum_{\alpha \in A} |s_\alpha|^2 \big),$$

where $(s_\alpha)_{\alpha \in A}$ is a finite set of non-zero sections in $H^0(X, mL)$, possibly with common zeroes. It is clear that $\phi \in \mathrm{PSH}(X, L)$, and the following proposition asserts that ϕ extends naturally as a metric $\phi \in \mathrm{PSH}(L^{\mathrm{hyb}})$, replacing the square-norm with maxima at the non-archimedean limit:

Proposition 1.3.5. *Let $(s_\alpha)_{\alpha \in A}$ be finite family of global sections of mL for $m \geq 1$, and set:*

$$\phi_t = \frac{1}{2m} \log \big(\sum_{\alpha \in A} |s_{\alpha,t}|^2 \big),$$

$$\phi_0 = m^{-1} \max_{\alpha \in A} \log |s_\alpha|.$$

Then this data defines a semi-positive metric $\phi \in \mathrm{PSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$, which we call the hybrid Bergman metric associated to the family $(s_\alpha)_{\alpha \in A}$.

Proof. We may and will assume that mL is basepoint-free, up to replacing mL by dmL and the s_α's by their d-th power, for d large enough.
We thus choose a basepoint-free set $(s_\alpha)_{\alpha \in B}$ of sections of mL, where $B = A \sqcup A'$. Set:

$$\phi_{j,t} := m^{-1} \log \big(\sum_{\alpha \in B} e^{b_{\alpha,j}} |s_{\alpha,t}|^2 \big),$$

and:

$$\phi_{j,0} := m^{-1} \max_{\alpha \in B} (\log |s_\alpha| + b_{\alpha,j}),$$

where $b_{\alpha,j} = 0$ if $\alpha \in A$ and $b_{\alpha,j} = -j$ for $\alpha \in A'$. Then the fact that this defines a continuous psh metric ϕ_j on L^{hyb} follows from the following lemma, applied to the convex function $\chi(x) = \frac{1}{2} \log \big(\sum_\alpha e^{2x_\alpha + b_\alpha} \big)$. Finally, the ϕ_j's clearly decrease to ϕ by construction, so that $\phi \in \mathrm{PSH}(L^{\mathrm{hyb}})$ by prop. 1.2.20. □

Lemma 1.3.6. *Let $P \subset \mathbb{R}^N$ be the standard simplex, i.e. :*

$$P = \{(x_1, ..., x_N) \in (\mathbb{R}_{\geq 0})^N / \sum_{i=1}^N x_i = 1\},$$

and let $\chi : \mathbb{R}^N \longrightarrow \mathbb{R}_{\geq 0}$ *be a convex function such that* $\chi - \max\{x_1, ..., x_N\} = O(1)$ *(or* P*-admissible, in the sense of def. 2.1.5), such that* $\chi(x + c\mathbf{1}) = \chi(x) + c$ *for any* $c \in \mathbb{R}$, $x \in \mathbb{R}^N$. *For* $x \in \mathbb{R}^N$, *let:*

$$\chi^{\mathrm{hom}}(x) = \lim_{y \to \infty} \frac{\chi(yx)}{y},$$

which is finite since $\chi \in \mathrm{Ad}_P$.
For any set $s_1, ..., s_N$ *of sections of* mL *without common zeroes and* $\phi_i = m^{-1} \log|s_i|$, *the hybrid metric* ϕ_χ *on* L *defined by:*

$$\phi_{\chi,t} = \chi(\phi_{1,t}; ...; \phi_{N,t})$$

and:

$$\phi_{\chi,0} = \chi^{\mathrm{hom}}(\phi_{1,0}, ..., \phi_{N,0})$$

is a semi-positive continuous hybrid metric on L.

Proof. The assumption $\chi(x + c\mathbf{1}) = \chi(x) + c$ ensures that ϕ_χ is compatible with multiplication of sections by functions, hence each $\phi_{\chi,t}$ defines a continuous metric on L_t, for $t \in \bar{\mathbb{D}}_r$. We now prove that ϕ_χ is continuous on X^{hyb} using prop. 1.3.2, we assume that $m = 1$ for convenience. Given a nowhere-vanishing section s of L on a Zariski open $U \subset X$, we have:

$$\frac{\log|s(z,t)|_{\phi_{\chi,t}}}{\log_r|t|} = \frac{\log|s(z,t)| - \chi(\log|s_1(z,t)|, ..., \log|s_N(z,t)|)}{\log_r|t|},$$

while each term in the right hand side is not well-defined, the difference is - so that writing $f_i = \log|\frac{s_i}{s}|$, the condition $\chi(x + c\mathbf{1}) = \chi(x) + c$ implies that:

$$\frac{\log|s(z,t)|_{\phi_{\chi,t}}}{\log_r|t|} = \frac{\chi\big(f_1(z,t), ..., f_N(z,t)\big)}{\log_r|t|} = \frac{\max\{f_1, ..., f_N\}}{\log_r|t|} + \varepsilon(t)$$

since χ is P-admissible, where $|\varepsilon(t)| \leq \frac{C}{\log_r|t|}$ for some constant $C > 0$. As a result, since away from the zero locus of s_i the function $\frac{f_i}{\log_r|t|}$ extends continuously to U^{hyb} via $x \mapsto \log|\frac{s_i}{s}|(x)$ on U_0^{hyb}, we infer that ϕ_χ is indeed a continuous metric on $(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$. We now prove that ϕ_χ is semi-positive. Using prop. 2.1.8, we let $(\chi_j)_{j \in \mathbb{N}}$ be a sequence of P-admissible, piecewise-affine convex functions decreasing to χ, written as:

$$\chi_j = \max_{\alpha \in A_j} \big(u_\alpha + c_\alpha\big),$$

with $u_\alpha \in P$. We write $\underline{\phi} = (\phi_1, ..., \phi_N) \in \mathrm{FS}(L^{\mathrm{hyb}})^N$, and set:

$$\phi_j = \chi_j(\underline{\phi}) = \max_{\alpha \in A} \big(\langle u_\alpha, \underline{\phi}\rangle + c_\alpha \frac{\log|e|}{\log r}\big),$$

where $e \in A_r$ is $\exp(1)$ viewed a constant power series. Since the u_α's lie in the standard simplex, the $\langle u_\alpha, \underline{\phi}\rangle$ are tropical Fubini-Study metrics on L^{hyb} as convex linear combinations, so that $\phi_j \in \mathrm{FS}^\tau(L^{\mathrm{hyb}})$, and:

$$\phi_{j,t} = \max_{\alpha \in A_j} \big(\langle u_\alpha, \underline{\phi_t}\rangle + c_\alpha\big),$$

while:

$$\phi_{j,0} = \max_{\alpha \in A_j} \big(\langle u_\alpha, \underline{\phi_0} \rangle \big),$$

since $\log|e| \equiv 0$ on $X^{\mathrm{an}} = \pi_{\mathrm{hyb}}^{-1}(0)$. Moreover, the $(\phi_{j,t})_j$ decrease to ϕ_t by the choice of $(\chi_j)_j$, while $(\phi_{j,0})_j$ decreases to:

$$\phi_{\chi,0} = \chi^{\mathrm{hom}}(\underline{\phi})$$

on X^{an} since $\chi_j^{\mathrm{hom}} = \max_{\alpha \in A_j} u_\alpha$. All in all, $(\phi_j)_j$ is a decreasing sequence in $\mathrm{FS}^\tau(L^{\mathrm{hyb}})$ converging pointwise to ϕ_χ, whence $\phi_\chi \in \mathrm{PSH}(L^{\mathrm{hyb}})$. □

Example 1.3.7. *Let $X = \mathbb{CP}^N \times \mathbb{D}^*$ with the standard polarization, and let $\phi_t = \phi_{FS}$ be the standard Fubini-Study metric on X_t, i.e. :*

$$\phi_{FS} = \log(|z_0|^2 + ... + |z_N|^2),$$

where the z_i are standard homogeneous coordinates on $\mathbb{P}^n$. Let furthermore $\phi_0 = \max\{|z_0|, ..., |z_N|\}$ be the non-archimedean Fubini-Study metric on $\mathbb{P}^{n,\mathrm{an}}_K$.
Then the metric ϕ_{FS} on $\mathcal{O}(1)$ on X^{hyb} obtained by gluing the two above metrics is a continuous semi-positive metric. Note that with our definition, this metric is not *a Fubini-Study metric on $\mathcal{O}(1)$, since we would have to work with the* max *instead of the square norm on the complex fibers.*

Example 1.3.8. *Let (X, L) be a complex polarized variety, and set $V = H^0(X, mL)$, for some $m > 0$ such that mL is globally generated. We let $\mathcal{N}(V)$ be the space of Hermitian norms of V, which is a symmetric space, as fixing a reference norm yields an identification:*

$$\mathcal{N}(V) = \mathrm{GL}(V)/U(V).$$

Let $N = \dim V$, and $\mathbf{e} = (e_1, ..., e_N)$ be a basis of V. Then for each tuple $(\lambda_1, ..., \lambda_N)$ of real numbers, define the associated hermitian norm:

$$\|\sum_{i=1}^N a_i e_i\|^2 = \sum_{i=1}^N |a_i|^2 e^{-2\lambda_i}.$$

This yields an embedding:

$$\iota_{\mathbf{e}} : \mathbb{R}^N \hookrightarrow \mathcal{N}(V),$$

whose image consists precisely of the norms diagonalized by the basis $\mathbf{e}$. The image $\mathbb{A}_{\mathbf{e}}(\mathbb{R}^N) := \iota_{\mathbf{e}}(\mathbb{R}^N)$ is called the apartment associated to the basis $\mathbf{e}$. We let $I \subset \mathbb{R}$ be an interval (not necessarily bounded), and $\gamma : I \longrightarrow \mathcal{N}(V)$ a geodesic. Then there exists a basis $\mathbf{e}$ of V such that $\gamma(I) \subset \mathbb{A}_{\mathbf{e}}$, and an affine map $\alpha : I \longrightarrow \mathbb{R}^N$ such that:

$$\gamma = \iota_{\mathbf{e}} \circ \alpha.$$

More concretely, writing $\alpha(y) = (\alpha_1 y + \beta_1, ..., \alpha_n y + \beta_n)$, we have:

$$\|\sum_{i=1}^N a_i e_i\|_y^2 = \sum_{i=1}^N |a_i|^2 e^{-2\beta_i - 2\alpha_i y}.$$

Assume that $I = (y_0, +\infty)$ for some $y_0 \in \mathbb{R}$, so that γ is a geodesic ray. Then we see easily that for any non-zero $v \in V$, the limit $-\frac{\log\|v\|_y}{y}$ exists and is equal to $-\log\|v\|_{\gamma^{\mathrm{NA}}}$, where γ^{NA} is the non-archimedean norm defined by:

$$\|\sum_{i=1}^{N} a_i e_i\|_{\gamma^{\mathrm{NA}}} = \max_{i\leq N}(|a_i|_0\, e^{-\alpha_i}),$$

where $|\cdot|_0$ is the trivial norm on $\mathbb{C}$. Furthermore, two geodesic rays induce the same non-archimedean norm at infinity if and only they are parallel, i.e. $(\alpha - \alpha')$ is constant. Thus, the space $\mathcal{N}(V)^{\mathrm{NA}}$ of non-archimedean norms on V can be interpreted as the space of asymptotic directions in $\mathcal{N}(V)$.
We now explain how each geodesic ray $\gamma : [0, +\infty) \longrightarrow \mathcal{N}(V)$ gives rise to a psh hybrid metric on L, which we call the associated Bergman metric. More generally, let $W \subset V$ be a basepoint-free subspace, i.e. a subvector space such that the sections in W have no common basepoints, and let $\gamma : [0, +\infty) \longrightarrow \mathcal{N}(W)$ be a geodesic ray. By the above discussion, there exists a basis $\mathbf{e} = (s_1, ..., s_N)$ of W, and tuples $\alpha, \beta \in \mathbb{R}^N$ such that:

$$\|\sum_{i=1}^{N} a_i s_i\|^2 = \sum_{i=1}^{N} |a_i|^2 e^{-2\alpha_i y}.$$

The hybrid Bergman metric associated to γ, defined by:

$$\phi_\lambda = (2m)^{-1}(\log \sum_{i=1}^{N} |s_i|^2 e^{\frac{2\alpha_i}{\lambda}}),$$

where ϕ_λ is the pull-back to X^{hol} of $\phi_{|X_\lambda^{\mathrm{hyb}}}$ via the rescaling of the absolute value (see the proof of prop. 1.3.2), and:

$$\phi_0 = m^{-1} \max_{i\leq N}(\log|s_i| + \alpha_i),$$

is a continuous psh metric on L^{hyb}, as follows from prop. 1.3.6 applied to the convex function $\chi(x_1, ..., x_N) = (2m)^{-1}\log(\sum_{i=1} e^{2mx_i})$.
Note that ϕ_λ is the classical Bergman metric associated to the norm $\|\cdot\|_y$, while ϕ_0 is the non-archimedean Bergman metric associated to the non-archimedean norm γ^{NA}.

1.3.2 The non-archimedean limit of a psh family

Let $X \longrightarrow \mathbb{D}^*$ be a degeneration of complex varieties, and L an ample line bundle on X. We let $\phi \in \mathrm{PSH}(X, L)$ be a semipositive metric on L. Following the general heuristic of viewing non-archimedean geometry as the asymptotic limit of Kähler geometry, we will explain how under a reasonable growth condition on ϕ, the family of generic Lelong numbers of ϕ along prime vertical divisors on models of X naturally induces a non-archimedean psh metric $\phi^{\mathrm{NA}} \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$. This construction is due to [BBJ18] in the isotrivial case and [Reb21] in the general case. Let us start with a definition.

Definition 1.3.9. *[Reb21, lem. 2.3.2] The metric $\phi \in \mathrm{PSH}(X, L)$ has logarithmic growth at $t = 0$ if one of the following equivalent conditions are satisfied:*

- *there exists a normal model $(\mathscr{X}, \mathscr{L})$ such that ϕ extends as a psh metric on $\mathscr{L}$,*
- *for any normal model $(\mathscr{X}, \mathscr{L})$, there exists $a \in \mathbb{R}$ such that $\phi + a\log|t|$ extends as a psh metric on $\mathscr{L}$,*
- *there exists a normal model $(\mathscr{X}, \mathscr{L})$ and a smooth metric ϕ_0 on $\mathscr{L}$ such that $\sup_{X_t}(\phi - \phi_0) \leq C\frac{\log|t|}{\log r}$ for some constant $C > 0$.*

We will now explain how each psh ray ϕ on (X, L) with logarithmic growth induces a psh metric ϕ^{NA} on $(X^{\mathrm{an}}, L^{\mathrm{an}})$. We fix a nef model $(\mathscr{X}, \mathscr{L})$ of (X, L), so that the associated model metric $\phi_{\mathscr{L}} \in \mathrm{PSH}(L^{\mathrm{an}})$ is semipositive.
Let $v \in X^{\mathrm{an}}$ be a divisorial valuation, so that there exists a model $\mathscr{X}'$ of X and a prime divisor $E \in \mathrm{Div}_0(\mathscr{X}')$ such that $v = v_E = (-\log r) \times b_E^{-1}\,\mathrm{ord}_E$, with $b_E = \mathrm{ord}_E(t)$. We may furthermore assume that $\mathscr{X}'$ dominates $\mathscr{X}$, via a morphism $\rho : \mathscr{X}' \longrightarrow \mathscr{X}$.
By logarithmic growth, there exists $a \in \mathbb{R}$ such that the metric $\phi_a := \phi + a\log|t|$ extends as a psh metric on $(\mathscr{X}', \rho^*(\mathscr{L}))$. We choose a psh metric ϕ_E on $\mathscr{X}'$ with divisorial singularities along E, i.e. $\phi_E = \log|z_E| + O(1)$ locally, where z_E is an equation of E. We define:

$$\nu_E(\phi) = \sup\{c \in \mathbb{R}/\phi_a \leq c\phi_E + O(1)\} - a$$

the generic Lelong number of ϕ along E [BFJ08] -which is easily seen to be independent of a - and set:

$$\psi^{\mathrm{NA}}(v) := \frac{\log r}{b_E}\nu_E(\phi).$$

Theorem 1.3.10. *[BBJ18, thm. 6.2], [Reb21, thm. 3.3.1]*
The function $\psi^{\mathrm{NA}} : X^{\div} \longrightarrow \mathbb{R}$ admits a unique lower semi-continuous extension to X^{an}, and the metric ϕ^{NA} on L^{an} defined by:

$$\phi^{\mathrm{NA}} := (\phi_{\mathscr{L}} + \psi^{\mathrm{NA}}) \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}}),$$

is a semi-positive metric on L^{an}.

Example 1.3.11. *Let $(\mathscr{X}, \mathscr{L})$ be a nef model of (X, L), and $\phi \in \mathrm{PSH}(X, L)$ be bounded, and extending as a locally bounded, semi-positive metric on $(\mathscr{X}, \mathscr{L})$. Then we have $\psi_{\mathrm{NA}} \equiv 0$, so that $\phi^{\mathrm{NA}} = \phi_{\mathscr{L}}$. Hence, the non-archimedean limit of a metric extending without singularities to some model $\mathscr{L}$ is simply the associated model metric.*

Example 1.3.12. *Let:*

$$\phi = \max_{\alpha \in A}\big(\log|s_\alpha| + c_\alpha\big)$$

be a tropical Fubini-Study metric on L^{hyb}, and let $v \in X^{\div}$ be a divisorial valuation. We let $(\mathscr{X}, \mathscr{L})$ be a model of (X, L) such that $v = v_E$ for some $E \subset \mathscr{X}_0$, and the s_α's extend as holomorphic sections of $\mathscr{L}$ on $\mathscr{X}$. Then:

$$\nu_E(\phi) = (\phi_0 - \ \phi_{\mathscr{L}})(v_E),$$

where ϕ_0 is the induced FS metric on X^{an}. Indeed, extending the s_α's as holomorphic sections of $\mathscr{L}$, we have:

$$(\phi_0 - \ \phi_{\mathscr{L}})(v_E) = \max_{\alpha\in A}\big(v_E(\frac{s_\alpha}{s_{\mathscr{L}}}) + c_\alpha\big),$$

where $s_{\mathscr{L}}$ is a trivialization of $\mathscr{L}$ at the generic point of E; while:

$$\phi_t = \max_{\alpha\in A}\big(\log|s_\alpha| + c_\alpha \log_r|t|\big)$$

is easily seen to have Lelong number:

$$\nu_E(\phi) = \min_{\alpha\in A}\big(\mathrm{ord}_E(\frac{s_\alpha}{s_{\mathscr{L}}}) + \frac{c_\alpha b_E}{\log r}\big)$$

along E. We infer from this that $\psi^{\mathrm{NA}} = (\phi_0 - \phi_{\mathscr{L}})$, so that ϕ is a tropical Fubini-Study metric on L^{hyb}, then $\phi^{\mathrm{NA}} = \phi_0$ is simply its restriction to X^{an}.

We now prove that ϕ^{NA} in fact extends ϕ as a semipositive metric on the hybrid space, and that up to shrinking the disk, every *continuous* semi-positive metric on L^{hyb} arises in this way:

Theorem 1.3.13. *Let (X, L) be a polarized degeneration of complex polarized varieties over $\mathbb{D}^*$, and $\phi = (\phi_t)_{t\in\mathbb{D}^*} \in \mathrm{PSH}(X, L)$ be a semipositive metric on L, with logarithmic growth at $t = 0$. Then the metric ϕ^{hyb} on L^{hyb} defined by setting:*

$$\phi^{\mathrm{hyb}}_{|X_t^{\mathrm{hyb}}} = \frac{\log r}{\log|t|}\phi_t;$$

$$\phi_0^{\mathrm{hyb}} = \phi^{\mathrm{NA}},$$

is semi-positive, i.e. $\phi^{\mathrm{hyb}} \in \mathrm{PSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$.
Conversely, let $\phi^{\mathrm{hyb}} \in \mathrm{CPSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$ and $\varepsilon > 0$, and set:

$$\phi_t = \frac{\log|t|}{\log r}\phi_{|X_t^{\mathrm{hyb}}}$$

for $|t| < r - \varepsilon$. Then $(\phi_t)_{t\in\mathbb{D}^*_{r-\varepsilon}}$ is a psh metric on the restriction of L to $X_{|\mathbb{D}^*_{r-\varepsilon}}$, with logarithmic growth at 0, and such that $\phi^{\mathrm{NA}} = \phi_0^{\mathrm{hyb}}$.

The proof of the theorem will be provided in section 1.3.3.
Note that the continuity assumption cannot be removed: in the case where X is a point, the function η such that $\eta(0) = 1$ and $\eta \equiv 0$ on $C^{\mathrm{hyb}} \setminus 0$ is psh on the hybrid circle, but $\eta(0)$ is not the Lelong number at zero of the induced subharmonic function on the punctured disk. This says essentially that the point $0 \in C^{\mathrm{hyb}}$ is non-pluripolar, so that it is "large" in the sense of pluripotential theory: psh metrics are not uniquely determined by their restriction outside zero. It has however dense complement, so that it is negligible topologically, and thus *continuous* psh metrics are determined by their restriction outside zero.
The next proposition characterizes, given $\phi \in \mathrm{PSH}(X, L)$, the set of semi-positive metrics on L^{an} that arise as restrictions to X^{an} of hybrid metrics extending ϕ, under

a finite-energy assumption. Let $\phi \in \mathrm{PSH}(L^{\mathrm{an}})$, and assume that ϕ is a metric of finite energy. Then it follows from [Reb21] that after a suitable choice of boundary data, there exists a canonical extension $\phi^{\mathrm{hyb}} \in \mathrm{PSH}(L^{\mathrm{hyb}})$ of ϕ to the hybrid space, which is also a metric of fiberwise-finite energy. The extension is canonical in the sense that it is relatively maximal in the sense of pluripotential theory [Kli91], and the maximal extension mapping: $\mathcal{E}^1(L^{\mathrm{an}}) \longrightarrow \mathcal{E}^1(L)$ is an isometric embedding for the Darvas metric [Dar15] on $\mathcal{E}^1(L)$ and its non-archimedean analog on $\mathcal{E}^1(L^{\mathrm{an}})$; here $\mathcal{E}^1(L) \subset \mathrm{PSH}(X, L)$ denotes the set of psh metrics on L which have fiberwise-finite energy.

Proposition 1.3.14. *Let $\phi \in \mathrm{PSH}(L^{\mathrm{hyb}})$, write $\phi_0 \in \mathrm{PSH}(L^{\mathrm{an}})$ its restriction to X^{an} and $\phi^{\mathrm{NA}} \in \mathrm{PSH}(L^{\mathrm{an}})$ the non-archimedean metric associated to $\phi_{|X}$, as defined by theorem 1.3.10. Then the inequality:*

$$\phi_0 \geq \phi_{\mathrm{NA}}$$

holds on X^{an}.
Conversely, let $\phi \in \mathcal{E}^1(L)$ be a psh metric of fiberwise-finite energy. Then for any $\psi \in \mathcal{E}^1(L^{\mathrm{an}})$ such that $\psi \geq \phi^{\mathrm{NA}}$, there exists a psh extension $\tilde{\phi} \in \mathcal{E}^1(L^{\mathrm{hyb}})$ of ϕ to L^{hyb} satisfying $\tilde{\phi}_0 = \psi$.

Proof. Write $\phi = \lim_j \phi_j$ as the decreasing limit of a net in $\mathrm{FS}^\tau(L^{\mathrm{hyb}})$. It follows from example 1.3.12 that for all j, the equality $\phi_j^0 = \phi_j^{\mathrm{NA}}$ holds, hence ϕ_0 is the decreasing limit of the ϕ_j^{NA}. Since ϕ_0 and ϕ^{NA} are psh on L^{an}, they are determined by their (finite) values on divisorial points, so that it is enough to prove that if v_E is a divisorial valuation on X, and after suitably fixing a reference model metric, the inequality:

$$\lim_j(\psi_j^{\mathrm{NA}}(v_E)) \leq \psi^{\mathrm{NA}}(v_E)$$

holds, which follows from semi-continuity of Lelong numbers.
Conversely, assume that $\phi \in \mathcal{E}^1(L)$ and $\psi \in \mathcal{E}^1(L^{\mathrm{an}})$, with $\psi \geq \phi^{\mathrm{NA}}$. By [Reb21] and theorem 1.3.13, there exists $\psi^{\mathrm{hyb}} \in \mathcal{E}^1(L^{\mathrm{hyb}})$ such that $\psi_0^{\mathrm{hyb}} = \psi$. We set:

$$\eta_j = \max(\psi^{\mathrm{hyb}}, \phi^{\mathrm{hyb}} - j\log|e|),$$

then the decreasing limit of the $(\eta_j)_j$ is a psh hybrid metric, restricting to ψ on X^{an} and to ϕ on X. □

Example 1.3.15. *Let $(A, L) \longrightarrow \mathbb{D}^*$ be a polarized, maximal degeneration of abelian varieties (see def. 3.1.7). We let $\omega_t \in c_1(L_t)$ be the flat Kähler metric on L_t, then there exists a family of smooth metrics $\phi_t \in \mathrm{PSH}(X_t, L_t)$ - called the cubic metrics - such that $\omega_t = dd^c\phi_t$. Then it follows from the proof of [GO22, thm. 4.13] that $\phi \in \mathrm{PSH}(X, L)$ and has logarithmic growth at $t = 0$. Moreover, the associated non-archimedean metric ϕ^{NA} is computed explicitly in [Liu11, thm. 4.3], and [GO22, thm. 4.13] states that the induced hybrid metric ϕ^{hyb} is in fact continuous, i.e. $\phi^{\mathrm{hyb}} \in \mathrm{CPSH}(A^{\mathrm{hyb}}, L^{\mathrm{hyb}})$.*

We also prove that given a continuous hybrid metric, it is enough to test its plurisubharmonicity outside zero.

Proposition 1.3.16. *Let $\phi \in \mathrm{PSH}(X, L)$, and assume ϕ extends as a continuous metric on L^{hyb}, that we still denote ϕ. Then the extension is semi-positive, i.e. $\phi \in \mathrm{CPSH}(L^{\mathrm{hyb}})$.*

Proof. By theorem 1.3.13, it is enough to prove that $\phi_0 = \phi^{\mathrm{NA}}$. By continuity of ϕ and the following lemma, we have:

$$\phi_0(v_E) = \phi^{\mathrm{NA}}(v_E)$$

for every divisorial valuation $v_E \in X^{\div}$. Thus, after fixing a reference model metric $\phi_{\mathscr{L}}$, for every snc model $\mathscr{X}$ on which $\phi_{\mathscr{L}}$ is determined, we have $(\phi_0 - \phi_{\mathscr{L}}) \circ \rho_{\mathscr{X}} = (\phi_{\mathrm{NA}} - \phi_{\mathscr{L}}) \circ \rho_{\mathscr{X}}$ since those two continuous functions agree on the rational points of $\mathrm{Sk}(\mathscr{X})$, and the result follows from [BFJ16, prop. 7.6]. □

Lemma 1.3.17. *Let $\phi \in \mathrm{PSH}(X, L)$ with logarithmic growth and $v_E \in X^{\div}$ a divisorial valuation. Let $(\mathscr{X}, \mathscr{L})$ be an snc model of (X, L) such that v_E is determined on $\mathscr{X}$, and ϕ extends as a psh metric on $\mathscr{L}$. We fix a bounded reference metric $\phi_{\mathscr{L}}$ on $\mathscr{L}$.*
Then there exists a sequence $(z_j)_{j\in\mathbb{N}}$ in X converging to v_E in X^{hyb}, such that the sequence $(\frac{(\phi-\phi_{\mathscr{L}})(z_j)}{\log_r|t|})_j$ converges to $\psi^{\mathrm{NA}}(v_E)$.

Proof. Recall that up to a negative scaling factor, $\psi^{\mathrm{NA}}(v_E)$ is the generic Lelong number of the psh function $(\phi - \phi_{\mathscr{L}})$ along E, hence is equal to the Lelong number of $(\phi - \phi_{\mathscr{L}})$ at a very general point of E [BFJ08]. Thus, we may choose a point $z_\infty \in E$ such that z_∞ is not contained in any other irreducible component of $\mathscr{X}_0$, and such that:

$$\psi^{\mathrm{NA}}(v_E) = \frac{\log r}{b_E} \nu_{z_\infty}(\phi - \phi_{\mathscr{L}}).$$

Choose a sequence $(z_j)_j$ in X converging to z_∞ inside $\mathscr{X}$. Then by construction $(\frac{(\phi-\phi_{\mathscr{L}})(z_j)}{\log_r|t|})_j$ converges to $\psi^{\mathrm{NA}}(v_E)$, and $(z_j)_j$ converges to v_E in $\mathscr{X}^{\mathrm{hyb}}$ by the discussion following prop. 1.1.26, hence in X^{hyb}, which concludes the proof. □

The following question is taken from Favre [Fav20, question 1]:

Question 1.3.18. *Let $\phi \in \mathrm{CPSH}(L)$ be a continuous, semi-positive metric on L, and assume that ϕ^{NA} is a continuous metric on L^{an}.*
Then is it true that $\phi^{\mathrm{hyb}} \in \mathrm{CPSH}(L^{\mathrm{hyb}})$?

In view of the proof of theorem 1.3.13, this amounts to proving that assuming $\phi^{\mathrm{NA}} \in \mathrm{CPSH}$, we have an estimate of the form:

$$\phi_m - \phi \leq \varepsilon_m \big|\log|t|\big|^{-1},$$

on X where ϕ_m are the Bergman kernels regularizing ϕ, and $\varepsilon_m \xrightarrow[m\to\infty]{} 0$ is independent of t. Such a bound seems difficult to attain without a uniform estimate on the oscillation of ϕ.

1.3.3 Proof of theorem 1.3.13

This section is devoted to the proof of theorem 1.3.13. If $\rho > 0$, we will write $\mathbb{D}_\rho := \{|t| < \rho\}$ the open disk of radius ρ, and if $\mathscr{X}$ is a model of X, $\mathscr{X}_\rho := \mathscr{X}_{|\mathbb{D}_\rho}$.
We let $\phi \in \mathrm{PSH}(X, L)$ be a psh metric on L with logarithmic growth, so that after choosing an ample model $(\mathscr{X}, \mathscr{L})$ of (X, L), there exists $c \in \mathbb{R}$ such that $\phi_c = \phi + c\log|t|$ extends as a psh metric on $\mathscr{L}$ - to alleviate notation, we will still write $\phi = \phi_c$. The basic idea of the proof of theorem 1.3.10 is that if $\mathscr{I}_m := \mathscr{I}(m\phi)$ is the multiplier ideal of the psh metric $m\phi$, then the sequence of piecewise-affine functions $m^{-1}\phi_{\mathscr{I}(m\phi)}$ on X^{an} decrease pointwise to the relative potential $(\phi - \phi_{\mathscr{L}})$ on X^{an}. However for $m \gg 1$, up to a controlled error term, the sheaf $\mathcal{O}_{\mathscr{X}}(mL \otimes \mathscr{I}_m)$ is relatively globally generated on $\mathscr{X}$, so that the $m^{-1}\phi_{\mathscr{I}(m\phi)}$ are a sequence of $\phi_{\mathscr{L}}$-psh functions on X^{an}, hence $\phi \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$.
We will roughly apply the same idea here, except we will also have to regularize ϕ itself by a sequence of psh metrics with analytic singularities of the form $\mathscr{I}(m\phi)$. The procedure we apply to produce such a sequence is standard in complex pluripotential theory and goes back to the work of Demailly [Dem92], so that we merely outline the proof here and rather refer the reader to the appendix 5.4 for the technical details.
We let $\psi \in \mathrm{PSH}(\mathscr{X}, \mathscr{L})$ be a smooth metric, whose curvature form $\omega := dd^c\psi$ is a Kähler form on $\mathscr{X}$. We choose $\varepsilon \in (0, 1-r)$ and $m_0 \in \mathbb{N}$ such that for all $m \geq 0$, the sheaf $\mathcal{O}_{\mathscr{X}}((m+m_0)\mathscr{L} \otimes \mathscr{I}_m)$ is globally generated over $\mathscr{X}_{r+\varepsilon}$ (see prop. 5.4.5).
We write:

$$\psi_{m,m_0} = \big(m\phi + m_0\psi\big) \in \mathrm{PSH}(\mathscr{X}, (m+m_0)\mathscr{L})$$

and recall that we write $\mathscr{I}_m = \mathscr{I}(\psi_{m,m_0}) = \mathscr{I}(m\phi)$ for the multiplier ideal of the psh metric ψ_{m,m_0} on $(\mathscr{X}, (m+m_0)\mathscr{L})$.
We will regularize ϕ by the Bergman metrics associated to the multiplier ideal $\mathscr{I}_m$. More explicitly, set $V_{m,m_0} := H^0(\mathscr{X}_{r+\varepsilon}, (m+m_0)\mathscr{L} \otimes \mathscr{I}_m)$ and define $\mathscr{H}_{m,m_0} \subset V_{m,m_0}$ as the following Hilbert space:

$$\mathscr{H}_{m,m_0} = \{s \in V_{m,m_0} / \|s\|^2 := \int_{\mathscr{X}_{r+\varepsilon}} |s|^2_{\psi_{m,m_0}} \omega^{n+1} < \infty\}.$$

For every couple (m, m_0), we may choose a Hilbert basis $\mathscr{B}_{m,m_0} = (s_{m,m_0,l})_{l\in\mathbb{N}}$ of $\mathscr{H}_{m,m_0}$, and we now set:

$$\phi_{m,m_0} = \frac{1}{2(m+m_0)} \log(\sum_{l\in\mathbb{N}} |s_{m,m_0,l}|^2),$$

and:

$$\phi^{\mathrm{NA}}_{m,m_0} = \phi_{\mathscr{L}} + (m+m_0)^{-1}\phi_{\mathscr{I}_m}.$$

It is clear that $\phi_{m,m_0} \in \mathrm{PSH}(\mathscr{X}, \mathscr{L})$, and $\phi^{\mathrm{NA}}_{m,m_0} \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$. We claim that this defines a semi-positive hybrid metric on L:

Proposition 1.3.19. *For every $m \in \mathbb{N}_{>0}$, the hybrid metric $\phi^{\mathrm{hyb}}_{m,m_0}$ on L defined by:*

$$\phi^{\mathrm{hyb}}_{m,m_0,t} = (\phi_{m,m_0})_{|X_t}$$

and:

$$\phi^{\mathrm{hyb}}_{m,m_0,0} = \phi^{\mathrm{NA}}_m$$

is semi-positive, i.e. $\phi^{\mathrm{hyb}}_{m,m_0} \in \mathrm{PSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$.

Proof. For $q \in \mathbb{N}$, set:

$$\phi_{m,m_0,q} = \frac{1}{2(m+m_0)} \log(\sum_{l \leq q} |s_{m,m_0,l}|^2),$$

and:

$$\phi^{\mathrm{NA}}_{m,m_0,q} = (m+m_0)^{-1} \max_{l \leq q}(\log|s_{m,m_0,l}|).$$

It follows from prop. 1.3.5 that this defines a semi-positive hybrid metric $\phi^{\mathrm{hyb}}_{m,m_0,q} \in \mathrm{PSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$, we will prove that the $(\phi^{\mathrm{hyb}}_{m,m_0,q})_{q\in\mathbb{N}}$ converge uniformly to $\phi^{\mathrm{hyb}}_{m,m_0}$ on X^{hyb}.
By prop. 5.4.6, the $\phi_{m,m_0,q}$ converge uniformly to ϕ_{m,m_0} on $\mathscr{X}_{r+\varepsilon}$ as $q \to \infty$, so that it remains to prove uniform convergence over X^{an}.
We have that

$$\phi^{\mathrm{NA}}_{m,m_0,q} = \phi_{\mathscr{L}} + (m+m_0)^{-1}\phi_{\mathscr{J}_q},$$

where $\mathscr{J}_q = \mathscr{I}\big((s_{m,m_0,l})_{l\leq q}\big) \equiv \mathscr{I}_m$ over $\mathscr{X}_{r+\varepsilon}$ for $q \gg 1$ by the strong noetherian property for coherent sheaves and global generation (see the proof of prop. 5.4.6), which concludes. □

Moreover, after extracting a subsequence, the $(m+m_0)^{-1}\phi_{m,m_0}$ decrease to ϕ on $\mathscr{X}_{r+\varepsilon}$ by theorem 5.4.4 and its proof, while the $\phi^{\mathrm{NA}}_{m,m_0}$ decrease to ϕ^{NA} on X^{an} by the proof of [Reb21, thm. 3.3.1]. This proves that $\phi^{\mathrm{hyb}} \in \mathrm{PSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$.
For the converse, if $\phi \in \mathrm{CPSH}(L^{\mathrm{hyb}})$, and $s \in H^0(U, L)$ is a local trivialization, then the function on U:

$$z \mapsto \frac{\log|s(z)|_{\phi_t}}{\log|t|}$$

extends continuously to X^{hyb} by 1.3.2, hence is bounded. This proves that ϕ induces a psh metric with logarithmic growth on L, hence we can define ϕ^{NA} as above. By prop. 1.3.14, we have $\phi^{\mathrm{NA}} \leq \phi_0$, while the semi-continuity of the hybrid metric induced by $(\phi, \phi^{\mathrm{NA}})$ implies $\phi_0 = \lim_{t\to 0}\phi_t \leq \phi^{\mathrm{NA}}$, which concludes.
This can also be deduced in the following way:

Lemma 1.3.20. *The map:*

$$(\cdot)^{\mathrm{NA}} : \mathrm{CPSH}(L^{\mathrm{hyb}}) \longrightarrow \mathrm{CPSH}(L^{\mathrm{an}})$$

is well-defined and continuous with respect to the topologies of uniform convergence on both spaces.

Proof. This follows from the fact that $(\cdot)^{\mathrm{NA}}$ is order-preserving [Reb21, thm. 3.3.1], and that $(\phi + c)^{\mathrm{NA}} = \phi^{\mathrm{NA}} + c$ for $c \in \mathbb{R}$.
Indeed, writing $\phi \in \mathrm{CPSH}(L^{\mathrm{hyb}})$ as the uniform limit of a net $(\phi_j)_j$ in FS^τ, the fact that $(\cdot)^{\mathrm{NA}}$ is order-preserving implies that the $(\phi^{\mathrm{NA}}_j)_j$ converge uniformly to ϕ^{NA}, which is then continuous. □

It now follows that $(\cdot)^{\mathrm{NA}}$ and the restriction map to X^{an}:

$$(\cdot)_0 : \mathrm{CPSH}(L^{\mathrm{hyb}}) \longrightarrow \mathrm{CPSH}(L^{\mathrm{an}})$$

are both continuous, and coincide on the dense set FS^{τ} by example 1.3.12, and thus coincide on $\mathrm{PSH}(L^{\mathrm{hyb}})$. This concludes the proof of theorem 1.3.13.

1.3.4 The isotrivial case

Let (X, L) be a projective complex variety, and write $(X_0^{\mathrm{an}}, L_0^{\mathrm{an}})$ the associated Berkovich space obtained by endowing the field of complex numbers with the trivial absolute value. The latter analytic space has proven itself to be a powerful tool in Kähler geometry, and in particular has been central in the variational proof of the Yau-Tian-Donaldson conjecture by Berman-Boucksom-Jonsson [BBJ18].
Without going into the details, the proof involves the study of various convex functionals on the space $\mathrm{PSH}(X, L)$, and relating their slopes at infinity on geodesic rays with the corresponding non-archimedean functionals on the space $\mathrm{PSH}(X_0^{\mathrm{an}}, L_0^{\mathrm{an}})$.

Definition 1.3.21. *Let (X, L) be a normal projective complex variety. A family $(\phi_y)_{y>0}$ of psh metrics on L is called a psh ray if and only the $\mathbb{S}^1$-invariant metric:*

$$\Phi(x, t) := \phi_{-\log|t|}(x)$$

on $(X \times \mathbb{D}^*, p_1^* L)$ is psh.

Let ϕ_0 be a smooth, positively curved reference metric on L. If $(\phi_y)_{y>0}$ is a psh ray on X, then the function $y \mapsto \sup_X(\phi_y - \phi_0)$ is convex, so that its slope at infinity:

$$p_{\max} := \lim_{y \to \infty} \frac{\sup_X(\phi_y - \phi_0)}{y}$$

exists in $\mathbb{R} \cup \{+\infty\}$ and is independent of the choice of ϕ_0. It is immediate that $p_{\max} < +\infty$ if and only there exists $C > 0$ such that $\sup_X \phi_y \leq Cy$ (with slight abuse of notation), in which case we will say that the ray $(\phi_y)_{y \in (y_0, +\infty)}$ has *linear growth*. This is easily seen to be equivalent to the fact that the psh metric Φ on $X \times \mathbb{D}^*$ has logarithmic growth.
Following the general heuristic of viewing non-archimedean geometry as the asymptotic limit of Kähler geometry, each psh ray $(\phi_y)_y$ on (X, L) with linear growth induces a psh metric ϕ^{NA} on $(X_0^{\mathrm{an}}, L_0^{\mathrm{an}})$, defined as follows. We fix a smooth reference metric ϕ_0 on L, whose curvature form $\omega_0 = dd^c\phi_0$ is a Kähler form on X, and write $\psi_y := (\phi_y - \phi_0) \in \mathrm{PSH}(X, \omega_0)$.
By linear growth, there exists $a \in \mathbb{R}$ such that the function:

$$\Psi_a(x, t) = \psi_{-\log|t|}(x) + a \log|t|$$

on $X \times \mathbb{D}^*$ is bounded from above near $X \times \{0\}$, hence extends as a quasi-psh function on $X \times \mathbb{D}$, that we still denote by Ψ_a. Now if $v \in X_0^{\mathrm{an}}$ is a divisorial valuation on X, and $w = \gamma(v) \in (X_K)^{\mathrm{an}}$ denotes the Gauss extension of v to the base change $X_K := X \times_{\mathbb{C}} K$, one defines $w(\Psi_a)$ as the generic Lelong number of Ψ_a along the center of w, as in section 1.3.2. In other words, if ϕ is a psh ray on X and $v \in X^{\div}$, then:

$$\phi^{\mathrm{NA}}(v) = \Phi^{\mathrm{NA}}(\gamma(v)),$$

Theorem 1.3.22. *[BBJ18, thm. 6.2] The function $\psi^{\mathrm{NA}} : X^{\div} \longrightarrow \mathbb{R}$ extends continuously to X^{an}, and the metric ϕ^{NA} on L^{an} defined by:*

$$\phi^{\mathrm{NA}} = (\psi^{\mathrm{NA}} + \phi^{\mathrm{triv}}) \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}}).$$

This is essentially a special case of thm. 1.3.10, since the trivial metric ϕ^{triv} is the restriction to the Gauss section (see section 1.1.4) of the model metric $\phi_{\mathscr{L}} \in \mathrm{PSH}(X_K^{\mathrm{an}}, L_K^{\mathrm{an}})$, where $\mathscr{L} = p_1^* L$ is the trivial model, living on $X \times \mathbb{D}$.

Proposition 1.3.23. *Let $r \in (0,1)$. Then the metric ϕ^{hyb} on L^{hyb} defined by:*

$$\phi_0^{\mathrm{hyb}} = \phi^{\mathrm{NA}},$$

$$\phi^{\mathrm{hyb}}_{|X_\lambda} = \lambda \phi_{1/\lambda}$$

is semi-positive on X_0^{hyb}. Moreover, we have:

$$\Phi^{\mathrm{hyb}} = F^* \phi^{\mathrm{hyb}},$$

where $F : X_K^{\mathrm{hyb}} \longrightarrow X_0^{\mathrm{hyb}}$ is the base change map from prop. 1.1.28.

Proof. The equality $\Phi^{\mathrm{hyb}} = F^* \phi^{\mathrm{hyb}}$ is straightforward from the construction of ϕ^{hyb}. To prove that ϕ^{hyb} is psh, the argument is the same as in the proof of theorem 1.3.13, except we need the sections s_{m,m_0} from the proof of prop. 1.3.19 to be equivariant with respect to the $\mathbb{S}^1$-action on $X \times \mathbb{D}$. To achieve this, with the notation of the previous section for $\ell \in \mathbb{Z}$, we let $\mathscr{H}_{m,m_0,\ell} \subset \mathscr{H}_{m,m_0}$ be the space of ℓ-equivariant sections, i.e. sections s such that $(e^{i\theta})^* s = e^{i\ell\theta} \cdot s$. Then $\mathscr{H}_{m,m_0}$ is the completion of $\bigoplus_{\ell \in \mathbb{Z}} \mathscr{H}_{m,m_0,\ell}$, so that we may choose a Hilbert basis of $\mathscr{H}_{m,m_0}$ adapted to the weight decomposition. The rest of the proof of theorem 1.3.13 carries out without changes after replacing $\log|s_{m,m_0,\ell}|$ on X_0^{an} by $(\log|s_{m,m_0,\ell}| - \ell)$ for $s_{m,m_0,\ell} \in \mathscr{H}_{m,m_0,\ell}$, so that we omit the details. □

1.4 The Monge-Ampère operator

1.4.1 The case of a valued field

Let K be a complete valued field, X an n-dimensional projective scheme over K, and let $L_1, ..., L_n$ be $\mathbb{Q}$-line bundles on X. By Ostrowski's theorem, either $K = \mathbb{R}$ or $\mathbb{C}$ with (a power of) the usual absolute value, either K is non-archimedean.
Let us start by assuming that $K = \mathbb{C}$ and that X is smooth. It then follows from the seminal work of Bedford-Taylor [BT76] that the mixed Monge-Ampère pairing:

$$(\phi_1, ..., \phi_n) \mapsto dd^c\phi_1 \wedge ... \wedge dd^c\phi_n,$$

a priori defined when each ϕ_i is a smooth Hermitian metric on L_i, actually extends in a unique way to semi-positive, locally bounded metrics - and in particular to continuous psh metrics. The pairing was then further extended to semi-positive singular metrics by Boucksom-Eyssidieux-Guedj-Zeriahi [BEGZ10]. More precisely, there exists a class $\mathcal{E}^1(X, L) \subset \mathrm{PSH}(X, L)$ of finite-energy metrics on (X, L), such

that the above mixed Monge-Ampère pairing extends uniquely to a multilinear, measure-valued pairing on $\mathcal{E}^1(X, L_1) \times ... \times \mathcal{E}^1(X, L_n)$. Note that the space $\mathcal{E}^1(X, L)$ contains $\mathrm{CPSH}(X, L)$.
We now assume K is non-archimedean, and $\phi_i \in \mathrm{DFS}(X, L_i)$ for every $i = 1, ..., n$ are differences of Fubini-Study metrics. Then one can associate to the ϕ_i's a signed Radon measure, the mixed Monge-Ampère measure $\mathrm{MA}(\phi_1, ..., \phi_n)$, with similar properties as in the complex analytic case.

Example 1.4.1. *Assume that $K = k((t))$ is a discretely-valued field of characteristic zero. Then pure Fubini-Study metrics on L_i are the same as model metrics on L_i, so that by multilinearity we may assume that $\phi_i = \phi_{\mathscr{L}_i}$, where $(\mathscr{X}_i, \mathscr{L}_i)$ is a nef model of (X, L_i). Up to passing to a higher model, we may assume $\mathscr{X}_1 = ... = \mathscr{X}_n$. Then the Monge-Ampère measure has the following explicit description [CL06]:*

$$\mathrm{MA}(\phi_1, ..., \phi_n) = \sum_E b_E(\mathscr{L}_1 \cdot .... \cdot \mathscr{L}_n \cdot E)\delta_{v_E},$$

where the sum ranges over the irreducible components E of $\mathscr{X}_0$, $b_E = \mathrm{ord}_E(t)$ and δ_{v_E} is the Dirac mass at the associated divisorial point $v_E = b_E^{-1}\,\mathrm{ord}_E$.

In general, the mixed Monge-Ampère measure satisfies the following basic properties:

Proposition 1.4.2. *[BE21, prop. 8.3] Let $\phi_i \in \mathrm{DFS}(L_i)$ for $i = 1, ..., n$.*

- *The pairing $(\phi_1, ..., \phi_n) \mapsto \mathrm{MA}(\phi_1, ..., \phi_n)$ is symmetric and multilinear;*
- *if $\phi_i \in \mathrm{FS}(L_i)$ for all i, then $\mathrm{MA}(\phi_1, ..., \phi_n)$ is a positive Radon measure;*
- *the total mass $\int_{X^{\mathrm{an}}} \mathrm{MA}(\phi_1, ..., \phi_n) = (L_1 \cdot ... \cdot L_n)$;*
- *if $L_0 = L_1 = \mathcal{O}_X$ and $\phi_0, \phi_1 \in \mathrm{DFS}(X)$, then:*

$$\int_X \phi_0\,\mathrm{MA}(\phi_1, ..., \phi_n) = \int_X \phi_1\,\mathrm{MA}(\phi_0, \phi_2, ..., \phi_n).$$

The Monge-Ampère measure can then be extended to more general metrics:

Theorem 1.4.3. *[BE21, thm. 8.4]*
Let K be a complete valued field, X/K an n-dimensional projective scheme and L_1,...,L_n line bundles on X. Then the Monge-Ampère operator:

$$(\phi_1, ..., \phi_n) \mapsto dd^c\phi_1 \wedge ... \wedge dd^c\phi_n$$

admits a unique extension to continuous psh metrics on the L_i. The extension is furthermore continuous with respect to the topology of uniform convergence and the weak topology of Radon measures.

Remark 1.4.4. *While the class $\mathcal{E}^1(X, L)$ of finite-energy psh metrics on L is defined over any non-archimedean field K, it is unclear in general how to extend the Monge-Ampère operator on the latter. Note however that in the case where K is discretely-valued of characteristic zero, this extension was constructed in [BFJ16].*

Remark 1.4.5. *The Chambert-Loir - Ducros [CLD12] approach to pluripotential theory on Berkovich spaces makes sense of the curvature current $dd^c\phi$ of a continuous psh metric, and its wedge products, in a spirit close to the work of Bedford-Taylor in the complex case. Notably, [CLD12, thm. 6.9.3] states that the wedge product $dd^c\phi_1 \wedge ... \wedge dd^c\phi_n$ coincides with the mixed Monge-Ampère measure as described in example 1.4.1 when ϕ_1,..., ϕ_n are psh model metrics, whence the notation.*

1.4.2 The Monge-Ampère equation

Let X be a smooth n-dimensional complex manifold, and L an ample line bundle on X. Then Yau's celebrated solution to the Calabi conjecture asserts the following:

Theorem 1.4.6. *[Yau78] Let μ be a smooth volume form on X, normalized to have total mass 1. Then there exists a unique (up to an additive constant) smooth, positive definite metric ϕ on L such that:*

$$(dd^c\phi)^n = (L^n)\mu.$$

The main motivation for this theorem was the case where X is a Calabi-Yau manifold, and $\mu = i^{n^2}\Omega \wedge \bar{\Omega}$ is the square-norm of a nowhere-vanishing holomorphic n-form on X: in that case, the curvature form $\omega = dd^c\phi$ is a smooth Kähler Ricci-flat metric on X.
Throughout the years, various generalizations of the above theorem in a more singular setting have appeared in the literature: let us simply mention Kołodziej's result [Kol98], that states that under the same assumptions on (X, L), then the statement of the above theorem holds for a much wider range of probability measures on X (for instance, measures μ with L^p-density for some $p > 1$), when we don't require for the solution to be smooth - here the solution is a continuous psh metric $\phi \in \mathrm{CPSH}(X, L)$, and the equality $(dd^c\phi)^n = \mu$ is understood in the sense of [BT76].
We now let $K = k((t))$ be a discretely-valued field of equicharacteristic zero. The following result can be understood as an analog of Kołodziej's result over K:

Theorem 1.4.7. *([BFJ15, thm. A], [BGGJ+19])*
Let (X, L) be a smooth polarized variety over K. Let μ be a probability measure on X^{an}, supported on the skeleton of some snc R-model of X. Then there exists a unique (up to an additive constant) continuous, semi-positive metric ϕ on L satisying the non-archimedean Monge-Ampère equation:

$$\mathrm{MA}(\phi) = \mu.$$

The above theorem was proved when X is defined over the function field of a curve over k in [BFJ15], and then extended to varieties over non-archimedean fields of residual characteristic zero in [BGGJ+19].

1.4.3 Family of Monge-Ampère-measures

Let now A be an integral Banach ring, and X/A a projective scheme. Since we may view X^{an} as the family of analytic spaces $\{X^{\mathrm{an}}_{\mathscr{H}_x}\}_{x\in\mathscr{M}(A)}$, which are analytic spaces over fields, one may define a Monge-Ampère operator on each fiber.

Definition 1.4.8. *Let A be an integral Banach ring, X/A an n-dimensional projective scheme, and $L_1, ..., L_n$ line bundles on X. If $(\phi_1, ..., \phi_n)$ is a tuple of continuous psh metrics on the L_i, we define the associated family of Monge-Ampère measures as follows. For $x \in \mathscr{M}(A)$, write $X_x = \pi^{-1}(x)$ which is an analytic space over $\mathscr{H}(x)$, and $\iota_x : X_x \hookrightarrow X^{\mathrm{an}}$ the inclusion. Then we set:*

$$(\mathrm{MA}(\phi_1, ..., \phi_n))_x := (\iota_x)_*\big(\mathrm{MA}((\phi_1)_{|X_x}, ..., (\phi_n)_{|X_x})\big).$$

This is a family of measures on X^{an} parametrized by $\mathscr{M}(A)$.

A natural question is to ask whether or not this family of measures is continuous, at least in the weak sense:

Question 1.4.9. *Let X be a flat, projective A-scheme, $L_1, ..., L_n$ as above, and let $\phi_i \in \mathrm{CPSH}(L_i)$ for each i. Is it true that for $\psi \in \mathcal{C}^0(X^{\mathrm{an}})$, the function:*

$$x \mapsto \int_{X^{\mathrm{an}}} \psi(\mathrm{MA}(\phi_1, ..., \phi_n))_x$$

is continuous on $\mathscr{M}(A)$?

The flatness assumption on X is necessary to ensure that the total mass of the measure:

$$(\mathrm{MA}(\phi_1, ..., \phi_n))_x(X^{\mathrm{an}}) = ((L_1)_{|X_{\mathscr{H}(x)}} \cdot ... \cdot (L_n)_{|X_{\mathscr{H}(x)}})$$

is indeed independent of $x \in \mathscr{M}(A)$.
By density of $\mathrm{DFS}(X) \subset \mathcal{C}^0(X)$ and $\mathrm{FS}(L_i) \subset \mathrm{CPSH}(L_i)$, and by using the following very general Chern-Levine-Nirenberg estimate, it is enough to prove the above statement for $\psi \in \mathrm{DFS}(X^{\mathrm{an}})$ and $\phi_i \in \mathrm{FS}(L_i)$.
While the answer to the above question seems unclear without further assumptions over the Banach ring A, we are able to provide an affirmative answer in the case of hybrid spaces in the next section. We also expect the statement to hold when A is the integer ring of a function field (for instance $A = \mathbb{Z}$), we mention for instance the work [Poi22] which discusses related questions.

Lemma 1.4.10. *[BE21, lem. 8.6] Let K be a non-archimedean field, and X/K. Let $L_0, ..., L_n$ be line bundles on X, and $\phi_i, \phi_i' \in \mathrm{FS}(L_i)$ for each i. Then:*

$$\left| \int_{X^{\mathrm{an}}} (\phi_0 - \phi_0') \,\mathrm{MA}(\phi_1, ..., \phi_n) - \int_{X^{\mathrm{an}}} (\phi_0 - \phi_0') \,\mathrm{MA}(\phi_1', ..., \phi_n') \right| \leq C \sum_{i=0}^{n} \sup_{X^{\mathrm{an}}} |\phi_i - \phi_i'|.$$

1.4.4 Hybrid metrics and admissible data

Throughout this section, we let $X \xrightarrow{\pi} \mathbb{D}^*$ be a smooth degeneration of complex manifolds, relatively polarized by an ample line bundle L; and write X^{hyb} the associated hybrid space. In this set-up, after fixing a reference Fubini-Study metric on (X, L), Favre [Fav20] defines a certain class of *model functions* $\phi_{\mathcal{F}} : X^{\mathrm{hyb}} \longrightarrow \mathbb{R}$ associated to admissible data on X; and uniform functions which are uniform limits of model functions. We will explain how those are nothing but hybrid Fubini-Study and hybrid cpsh metrics on L. In particular, the following result is a mere reformulation of [Fav20, thm. 4.2]:

Theorem 1.4.11. *[Fav20, thm. 4.2]*
Let (X,L) be as above, and $\phi \in \mathrm{CPSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$. Then the associated family of Monge-Ampère measures is continuous on X^{hyb} in the weak sense: for any $f \in \mathcal{C}^0(X^{\mathrm{hyb}})$, we have:

$$\int_{X_t} f(dd^c\phi_t)^n \xrightarrow[t\longrightarrow 0]{} \int_{X^{\mathrm{an}}} f\,\mathrm{MA}(\phi_0).$$

Recall that $\mathrm{DFS}(X^{\mathrm{hyb}})$ is dense in $\mathcal{C}^0(X^{\mathrm{hyb}})$, while $\mathrm{FS}^\tau(L^{\mathrm{hyb}})$ is dense in $\mathrm{CPSH}(L^{\mathrm{hyb}})$ for the topology of uniform convergence. As a result, by lemma 1.4.10, it is enough to prove the above convergence with $f \in \mathrm{DFS}(X^{\mathrm{hyb}})$, $\phi \in \mathrm{FS}^\tau(L^{\mathrm{hyb}})$.
As in [Fav20], we fix an snc model $\mathscr{X}$ of X such that L has an ample model $\mathscr{L}$ on $\mathscr{X}$, so that we get a relative embedding $\iota : \mathscr{X} \hookrightarrow \mathbb{CP}^N \times \mathbb{D}$ by sections of $m\mathscr{L}$ for $m \geq 1$, and write $\phi_{\mathrm{ref}} = m^{-1}\iota^*\phi_{FS}$ (here the Fubini-Study metric is taken in the usual sense, with sums of squares). By example 1.3.7 and pullback, we have $\phi_{\mathrm{ref}} \in \mathrm{CPSH}(L^{\mathrm{hyb}})$.
Then a regular admissible datum $\mathcal{F} = \{\mathscr{X}', d, D, s_1, ..., s_l\}$ consists of the following: $p : \mathscr{X}' \longrightarrow \mathscr{X}$ is an snc model dominating $\mathscr{X}$, $D \in \mathrm{Div}_0(\mathscr{X}')$ is a vertical divisor on $\mathscr{X}'$, and $(s_1, ..., s_l)$ is a tuple of sections of $p^*(\mathscr{L}^d)(D)$ without common zeroes. An regular admissible datum defines a model function $\phi_{\mathcal{F}} : X^{\mathrm{hyb}} \longrightarrow \mathbb{R}$ as follows:

$$\phi_{\mathcal{F}} = \max_{i=1,...,l} \log\|s_i\|_{\phi_{\mathrm{ref}}} = \max_{i=1,...,l}(\phi_i - \phi_{\mathrm{ref}})$$

with $\phi_i = d^{-1}\log|s_i|$.
Thus, we naturally define:

$$\psi_{\mathcal{F}} = \phi_{\mathrm{ref}} + \phi_{\mathcal{F}} = \max_{i\leq l} \phi_i,$$

which is a Fubini-Study metric on L^{hyb}, since the s_i's have no common zeroes.
Conversely, any pure Fubini-Study metric on L^{hyb}, i.e. a metric of the form:

$$\phi = d^{-1}\max_{i\leq l}\log|s_i|$$

for $d \geq 1$ and $s_1, ..., s_l \in H^0(X, dL)$ without common zeroes, defines a regular admissible datum. Indeed, we may extend $s_1, ..., s_l$ to meromorphic sections of $d\mathscr{L}$ on $\mathscr{X}$, and set:

$$\mathscr{I} =< s_1, ..., s_l >,$$

which is a vertical fractional ideal sheaf on $\mathscr{X}$. It follows from [Fav20, prop. 2.2] that if $p : \mathscr{X}' \longrightarrow \mathscr{X}$ is a log-resolution of $\mathscr{X}$, this yields an admissible datum:

$$\mathcal{F} = \{\mathscr{X}', d, D, p^*s_1, ..., p^*s_l\},$$

where $D \in \mathrm{Div}_0(\mathscr{X}')$ is such that $p^*(\mathscr{L}^d \otimes \mathscr{I}) = (p^*\mathscr{L})^d \otimes \mathcal{O}_{\mathscr{X}'}(D)$, and it follows from the previous computation that the associated model function $\phi_{\mathcal{F}}$ satisfies:

$$\phi_{\mathcal{F}} = \max_{i=1,...,l} \log\|s_i\|_{\phi_{\mathrm{ref}}} = \phi - \phi_{\mathrm{ref}},$$

with $\phi = d^{-1}\max_{i\leq l}\log|s_i|$. As a result, model functions on X^{hyb} in the sense of [Fav20] are precisely the continuous functions $\psi : X^{\mathrm{hyb}} \longrightarrow \mathbb{R}$ such that $(\phi_{\mathrm{ref}} + \psi) \in \mathrm{FS}^\tau(L^{\mathrm{hyb}})$ is a pure Fubini-Study metric, so that by remark 1.3.4, uniform functions are those such that $(\phi_{\mathrm{ref}} + \psi) \in \mathrm{CPSH}(L^{\mathrm{hyb}})$. This proves that the statement of theorem 1.4.11 is equivalent to the statement [Fav20, thm. 4.2].

Chapter 2

The toric case

In this chapter, we focus on the case where (Z, L) is a polarized toric variety over $\mathbb{C}$, and we investigate toric metrics on $(Z^{\mathrm{hyb}}, L^{\mathrm{hyb}})$. When working over $\mathbb{C}$ equipped with either the archimedean or trivial absolute value, semi-positive toric metrics on $(Z^{\mathrm{an}}, L^{\mathrm{an}})$ are well-know to correspond to a certain class of convex functions on the vector space $N_{\mathbb{R}}$ where the fan of Z lives.
The main result of this section, theorem 2.2.12, states that semi-positive metrics on $(Z^{\mathrm{hyb}}, L^{\mathrm{hyb}})$ that are *fiberwise-toric*, correspond to families parametrized by $\mathscr{M}(\mathbb{C}^{\mathrm{hyb}}) \simeq [0, 1]$ of convex functions on $N_{\mathbb{R}}$ as above, which vary in a convex way with respect to the coordinate λ on the base.

2.1 Preliminaries

Throughout this section, N denotes a lattice of rank n, $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R})$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing $M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}$. We say a function $\phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is piecewise-affine if there exists a rational polytopal subdivision $\mathcal{T}$ of $N_{\mathbb{R}}$ such that ϕ is affine on each face of the subdivision; it is said to be rational piecewise-affine if each affine piece of ϕ has rational slopes (note that we allow the constant term to be any real number). We will sometimes say ϕ is *adapted* to the subdivision $\mathcal{T}$.

2.1.1 Convex geometry

Definition 2.1.1. *A (rational) convex polytope $P \subset M_{\mathbb{R}}$ is the convex hull in $M_{\mathbb{R}}$ of finitely many points in $M_{\mathbb{Q}}$.*
The polytope is furthermore integral if we can choose the points to be in M.

If $P \subset M_{\mathbb{R}}$ is a polytope, a *face* of P is any intersection of P with a halfspace such that none of the interior points of the polytope lie on the boundary of the halfspace.

Definition 2.1.2. *Let $P \subset M_{\mathbb{R}}$ be a convex polytope. Its normal fan $\Sigma = \Sigma_P$ is the complete fan in $N_{\mathbb{R}}$ whose cones are the normal cones to the faces of P.*
More explicitly, $\Sigma = \{\sigma_F\}_{F \text{ face of } P}$, where:

$$\sigma_F = \{x \in N_{\mathbb{R}} / \forall u \in F, \langle u, x \rangle = \max_{y \in P} \langle u, y \rangle\}.$$

Definition 2.1.3. *Let $P \subset M_{\mathbb{R}}$ be a convex polytope. Its support function $\Psi_P : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is defined by the formula:*

$$\Psi_P(x) = \sup_{u \in P} \langle u, x \rangle.$$

It is a rational piecewise-linear convex function on $N_{\mathbb{R}}$, linear on each face of the normal fan Σ_P.

The fact that Ψ_P is piecewise-linear follows from the fact that for fixed x, the function $u \mapsto \langle u, x \rangle$ reaches its maximum at a vertex of P, hence the equality:

$$\Psi_P(x) = \max_{u \in V} \langle u, x \rangle,$$

where $V \subset P$ is the set of vertices of P.
Note that the data of (Σ_P, Ψ_P) recovers the polytope P uniquely.

Definition 2.1.4. *Let $\phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be a convex function. We define the Legendre transform ϕ^* of ϕ by the formula:*

$$\phi^*(u) = \sup_{x \in N_{\mathbb{R}}} \big(\langle u, x \rangle - \phi(x) \big),$$

for $u \in M_{\mathbb{R}}$, and we write its domain $P(\phi^*) := \{ \phi^* < +\infty \} \subset M_{\mathbb{R}}$.

Definition 2.1.5. *Let $P \subset M_{\mathbb{R}}$ be a convex polytope. We say that the convex function $\phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is P-admissible if and only the equality $P(\phi^*) = P$ holds. This is equivalent to the following condition:*

$$\sup_{x \in N_{\mathbb{R}}} |\phi(x) - \Psi_P(x)| < +\infty.$$

We will write $\mathrm{Ad}_P(N_{\mathbb{R}})$ or simply Ad_P for the set of P-admissible convex functions.

Example 2.1.6. *Let $J \subset P \cap M_{\mathbb{Q}}$ be a finite subset, and $\phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be the function defined by:*

$$\phi(x) = \max_{j \in J} (\langle u_j, x \rangle + c_j),$$

where the c_j's are real constants. Then $\phi \in \mathrm{Ad}_P(N_{\mathbb{R}})$ if and only if J contains the set of vertices of P.

Proposition 2.1.7. *Let $\phi \in \mathrm{Ad}_P(N_{\mathbb{R}})$. Then the Legendre transform $\phi^* : P \longrightarrow \mathbb{R}$ is convex and continuous. Moreover, the map $\phi \mapsto \phi^*$ induces a bijection from $\mathrm{Ad}_P(N_{\mathbb{R}})$ to the set of continuous convex functions on P, whose inverse map is given by $\beta \mapsto \beta^*$, where:*

$$\beta^*(x) = \sup_{u \in P} \big(\langle u, x \rangle - \beta(u) \big)$$

for $\beta : P \longrightarrow \mathbb{R}$ a continuous convex function.

Proof. If ϕ is P-admissible, its Legendre transform is upper semi-continuous on P as a supremum of continuous functions, while it follows from [How88] that the Legendre transform ϕ^* is convex on P, hence lower semi-continuous on P. This proves the first item, and the second item follows from [Roc72, thm. 12.2]. □

The following proposition states that P-admissible functions can be monotonously approximated by rational piecewise-affine functions:

Proposition 2.1.8. *Let P be a convex polytope in $M_{\mathbb{R}}$, and let $\phi \in \mathrm{Ad}_P(N_{\mathbb{R}})$ be a P-admissible convex function.*
Then there exists a decreasing sequence $(\phi_j)_{j\in\mathbb{N}}$ of rational piecewise-affine, P-admissible convex functions converging pointwise to ϕ.

Proof. Let $\beta : P \longrightarrow \mathbb{R}$ be the Legendre transform of ϕ, and let $(x_j)_{j\in\mathbb{N}}$ be a sequence in $N_{\mathbb{Q}}$ that is dense in $N_{\mathbb{R}}$. For each $j \in \mathbb{N}$, let $(r_{l,j})_{l\in\mathbb{N}}$ be a sequence of rational numbers decreasing to $\phi(x_j)$, uniformly with respect to j (uniformity can be achieved by a diagonal extraction argument). We set:

$$\beta_j(u) = \max_{l\leq j} \big(\langle u, x_l\rangle - r_{l,j}\big),$$

so that as $j \to \infty$, the function β_j is close to the maximum of the affine functions with gradients in $\{x_l\}_{l\leq j}$ that are cutting out supporting hyperplanes of the graph of β. It is clear that β_j is rational piecewise-linear, and increasing pointwise to:

$$\beta^{**}(u) = \sup_{x\in N_{\mathbb{R}}} \big(\langle u, x\rangle - \phi(x)\big),$$

which is none other than β. Continuity of β furthermore implies that the convergence is uniform, by Dini's lemma.
We now set:

$$\phi_j(x) = \beta_j^*(x) = \sup_{u\in P} \big(\langle u, x\rangle - \beta_j(u)\big).$$

It defines an decreasing sequence of P-admissible rational piecewise-affine convex functions on $N_{\mathbb{R}}$, by the following lemma. The fact that the ϕ_j's decrease (uniformly) to ϕ is now a consequence of the monotonicity of the Legendre transform. □

Lemma 2.1.9. *Let $P \subset M_{\mathbb{R}}$ a convex polytope and $\beta : P \longrightarrow \mathbb{R}$ a function that is piecewise-affine, adapted to a rational triangulation of P. Then the Legendre transform β^*, defined for $x \in N_{\mathbb{R}}$ by:*

$$\beta^*(x) = \sup_{u\in P} \big(\langle u, x\rangle - \beta(u)\big)$$

can be written as:

$$\beta^* = \max_{i\in I}(u_i + c_i),$$

where I is a finite set, $u_i \in P \cap M_{\mathbb{Q}}$ and $c_i \in \mathbb{R}$.
Moreover, $\beta^* \in \mathrm{Ad}_P(N_{\mathbb{R}})$.

Proof. Let $\mathcal{T}$ be a rational polytopal decomposition of P such that β is affine on each face of $\mathcal{T}$ (which exists as follows for instance from [BGPS14, lem. 2.5.3]), and write the set E of vertices of $\mathcal{T}$ as $E = \{u_i, i \in I\}$. Then we have:

$$\beta^*(x) = \max_{i\in I} \big(\langle u_i, x\rangle - \beta(u_i)\big),$$

as for fixed x, the function $u \mapsto \big(\langle u, x\rangle - \beta(u)\big)$ is affine on each face of $\mathcal{T}$, hence achieves its maximum at a vertex of $\mathcal{T}$.
The second item follows from example 2.1.6, since E contains the set of vertices of P by construction. □

2.1.2 Toric varieties

Recall that N is a lattice of rank n, with dual lattice M. We write $\mathbb{T} = \mathrm{Hom}(M, \mathbb{G}_m) = N \otimes \mathbb{G}_m$ the associated split algebraic torus, which is defined over $\mathbb{Z}$. We furthermore have that $M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is the character lattice of $\mathbb{T}$, so that each $m \in M$ defines a regular invertible function on $\mathbb{T}$, which we denote by χ^m.
Let $\Sigma = \{\sigma\}_{\sigma\in\Sigma}$ be a fan inside $N_\mathbb{R}$; this is a finite collection of strictly convex rational polyhedral cones inside $N_\mathbb{R}$, stable under intersection and such that each face of a cone in Σ is itself in Σ.
If k is any field, one can associate to the fan Σ a normal k-scheme of finite type as follows: for $\sigma \in \Sigma$, set $M_\sigma = M \cap \sigma^\vee$, which is a semi-group. We then define the affine toric variety associated to σ by:

$$Z_\sigma := \mathrm{Spec}\, k[M_\sigma];$$

one can then see that for $\sigma' \subset \sigma$, the ring $k[M_{\sigma'}]$ is a localization of $k[M_\sigma]$, so that the collection of $\{Z_\sigma\}_{\sigma\in\Sigma}$ can be glued to define a variety over k:

$$Z_\Sigma = \bigcup_{\sigma\in\Sigma} Z_\sigma.$$

Since the cone $\{0_{N_\mathbb{R}}\} \in \Sigma$, the variety Z_Σ contains the torus $\mathbb{T} = \mathrm{Spec}\, k[M]$ as a dense open subset. Its complement $\Delta_Z := Z \setminus \mathbb{T}$ is a reduced Weil divisor which we call the toric boundary of Z, it is furthermore an anticanonical divisor when Z is Gorenstein. We write it as the sum of its irreducible components $\Delta_Z = \sum_{l\in L} Z_l$.
The fan Σ encodes various algebro-geometric information about Z, for instance:

Proposition 2.1.10. *The following hold:*

- *the variety Z is proper over k if and only the fan Σ is complete, i.e. $\bigcup_{\sigma\in\Sigma} \sigma = N_\mathbb{R}$;*
- *the variety Z is smooth over k if and only if the fan Σ is regular, which means that each top-dimensional cone of Σ is $\mathrm{GL}(N)$-isomorphic to a standard orthant. In this case, the boundary Δ_Z is a simple normal crossing divisor.*

2.1.3 Line bundles and toric divisors

Assume now that Z is smooth over k (which implies it has simple normal crossing boundary).

Definition 2.1.11. *Let L be a line bundle on Z. A toric structure on L is the data of an isomorphism $L \simeq \mathcal{O}_Z(D)$, where D is a toric Cartier divisor; together with a nowhere-vanishing section s_D of $L_{|\mathbb{T}}$, which is a monomial on $\mathbb{T}$ under the previous isomorphism $L_\mathbb{T} \simeq \mathcal{O}_\mathbb{T}$.*
A toric line bundle on Z is a line L endowed with a toric structure.

Since each Cartier divisor on Z can be moved via the torus action to a $\mathbb{T}$-invariant Cartier divisor, which is thus supported on the boundary, every line bundle L on Z admits a toric structure.

The fan allows us to give a rather explicit description of the Picard group of Z. Indeed, components of the toric boundary provide a canonical set of generators of $\mathrm{Pic}(Z)$, and the relations between them can be described as follows.
Write $\mathrm{Div}^{\mathbb{T}}(Z) = \oplus_{l\in L}\mathbb{Z}Z_l \simeq \mathbb{Z}^L$ the abelian group of Weil divisors supported on the boundary. The canonical map $q : \mathrm{Div}^{\mathbb{T}}(Z) \longrightarrow \mathrm{Pic}(Z)$ sends a divisor to its class; the map $p : M \longrightarrow \mathrm{Div}^{\mathbb{T}}(Z)$ sends a monomial χ^m to the principal divisor $\mathrm{div}(\chi^m)$.

Lemma 2.1.12 ([Ful93, 3.4]). *The following sequence*

$$0 \longrightarrow M \xrightarrow{p} \mathrm{Div}^{\mathbb{T}}(Z) \xrightarrow{q} \mathrm{Pic}(Z) \longrightarrow 0$$

is exact.

Let us rephrase this in term of coordinates, after fixing an isomorphism $N \simeq \mathbb{Z}^r$ and denoting $L = \{u_1, \ldots, u_s\}$ the primitive generators of the 1-dimensional cones of Σ. Since by [Ful93, Lemma p.61], we have $\mathrm{ord}_{Z_l}(z^m) = \langle u_l, m\rangle$, we obtain $\mathrm{div}(\chi^m) = \sum_{l\in L}\langle u_l, m\rangle Z_l$ and hence $p(m) = (\langle u_l, m\rangle)_{l\in L}$. We deduce the following explicit description of $\mathrm{Pic}(Z)$:

Corollary 2.1.13. *Let $u_l = (u_{l,1}, \ldots, u_{l,r})$ for $l \in \{1, \ldots, s\}$. Then $\mathrm{Pic}(Z)$ is generated by the line bundles $\mathcal{O}_Z(Z_l)$, with the r relations:*

$$\mathcal{O}_Z(\sum_{l=1}^{m} u_{l,1}Z_l) = \ldots = \mathcal{O}_Z(\sum_{l=1}^{m} u_{l,r}Z_l) = 0.$$

In particular, the divisors in the r-tuple

$$\Delta = \sum_{l\in L} u_l \otimes Z_l \ \in \ N \otimes \mathrm{Div}^{\mathbb{T}}(Z) \simeq (\mathrm{Div}^{\mathbb{T}}(Z))^r$$

are principal.

We now want to describe how the fan Σ encodes the intersection theory on Z, under the assumption that Z is proper. Each 1-cycle in Z being numerically equivalent to a sum of toric strata, it is enough to study the intersection numbers $(C \cdot Z_l)$, where Z_l is a boundary component of Z and C is a 1-dimensional toric stratum, which is isomorphic to $\mathbb{P}^1$ by properness and the fact that Z has snc boundary. The stratum C is thus a rational curve with two marked points p and q, which are the intersection points of C with two components of Δ_Z, denoted here by Z_p and Z_q, with corresponding rays ρ_p and ρ_q. The curve C corresponds to an $(r-1)$-dimensional cone σ_C of Σ, while the points p and q correspond to the maximal cones generated by $< \sigma_C, \rho_p >$ and $< \sigma_C, \rho_q >$.

Lemma 2.1.14 ([Ful93, p. 99]). *The primitive generators of the rays of the fan satisfy the following relation:*

$$u_p + u_q = -\sum_{u_l\in\sigma_C} (C \cdot Z_l)u_l.$$

Observing that we have $(C \cdot Z_p) = (C \cdot Z_q) = 1$, and $(C \cdot Z_l) = 0$ for any other l, this may be rewritten in a more synthetic way:

$$\sum_{l\in L}(C \cdot Z_l)u_l = 0. \tag{2.1.15}$$

Note that this lemma holds even if Z is not proper.

2.1.4 Positivity

Throughout this section, we assume that Z is smooth, so that each cone $\sigma \in \Sigma$ can be written as:

$$\sigma = \sum_{l=1}^{\dim \sigma} \mathbb{R}_{\geq 0} v_l,$$

with $v_l \in N$ the primitive generators of the extremal rays of σ.

Definition 2.1.16. *Let $D = \sum_{l \in L} a_l Z_l$ be a toric divisor on Z. For each maximal cone $\sigma \in \Sigma$, we let:*

$$\Psi_D : \sigma \longrightarrow \mathbb{R}$$

be the unique linear map such that $\Psi_D(v_l) = a_l$ for each primitive generator u_l of an extremal ray of σ.
This defines a piecewise-linear function:

$$\Psi_D : \Sigma \longrightarrow \mathbb{R},$$

called the support function of D, it is a convex function if and only $\mathcal{O}_Z(D)$ is globally generated.

Note that our definition differs from the one in [Ful93] by a minus sign. It follows from lemma 2.1.13 that if D_1, D_2 are two toric divisors such that $\mathcal{O}_Z(D_1) \simeq \mathcal{O}_Z(D_2)$, then there exists $m \in M$ such that $D_1 - D_2 = \mathrm{div}(\chi^m)$, so that the difference $\Psi_{D_1} - \Psi_{D_2} = \langle m, \cdot \rangle$ is a linear form on $N_{\mathbb{R}}$.

Definition 2.1.17. *Let D be a toric divisor on Z. The associated convex polyhedron is defined by:*

$$P_D = \{u \in M_{\mathbb{R}} \mid \langle u, \cdot \rangle \leq \Psi_D\}.$$

Similarly to the previous remark, shifting D by a principal toric divisor amounts to translating P_D inside $M_{\mathbb{R}}$. The polytope P_D computes the space of sections of $\mathcal{O}_Z(D)$:

Proposition 2.1.18. *[Ful93, lem. p. 66] If $m \in M \cap P_D$, then the monomial χ^{-m} defines a regular section $s_m = \chi^{-m} s_D \in H^0(Z, \mathcal{O}_Z(D))$. Moreover,*

$$H^0(Z, \mathcal{O}_Z(D)) = \bigoplus_{m \in P_D \cap M} k \cdot s_m.$$

In the case where $\mathcal{O}_Z(D)$ is semi-ample (or equivalently, globally generated), the convex function Ψ_D can be written as a finite maximum of linear functions:

$$\Psi_D = \max_{j \in J} \langle m_j, \cdot \rangle,$$

so that P_D is the convex hull of the $\{m_j, j \in J\}$ and is a convex polytope, whose support function equates the support function Ψ_D of D. Hence, we will say that a convex function $\phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is D-admissible if $\phi \in \mathrm{Ad}_{P_D}(N_{\mathbb{R}})$.

Lemma 2.1.19. *Let $J \subset (P_D \cap M)$ be a finite set. Then the s_{m_j}'s, for $j \in J$, have no common zeroes if and only J contains the set of vertices of P_D, if and only $\max_{j\in J}\langle m_j, \cdot\rangle$ is D-admissible.*

Proof. The first equivalence is [Ful93, ex. p. 69], while the second one is example 2.1.6. □

Finally, we have the following vanishing theorem for nef divisors on a proper toric variety.

Proposition 2.1.20. *Let D be a nef Cartier divisor on a proper toric variety Z. Then $H^i(Z, \mathcal{O}_Z(D)) = 0$ for $i > 0$.*

Proof. The divisor D being nef is equivalent to it being globally generated, by [Mus02, Theorem 3.1]. Thus, the result follows directly from [Ful93, p. 74]. □

2.2 Toric metrics

2.2.1 Analytification of toric varieties

Let A be a Banach ring, and let $\mathbb{T}$ be a split algebraic torus over A, with character lattice $M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. The Berkovich analytification $\mathbb{T}^{\mathrm{an}}$ of $\mathbb{T}_A = \mathrm{Spec}\, A[M]$ comes with a canonical valuation map:

$$\mathrm{val} : \mathbb{T}^{\mathrm{an}} \longrightarrow N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R}),$$

$$x \mapsto (m \mapsto -\log|\chi^m(x)|).$$

In other words, $\langle m, \mathrm{val}(x)\rangle = v_x(\chi^m)$, whence the notation. Fixing coordinates $T_1, ..., T_n$ on the torus, which amounts to fixing a basis of M, the above map is given by:

$$\mathrm{val}(x) = (-\log|T_1(x)|, ..., -\log|T_n(x)|) \in \mathbb{R}^n.$$

Now let Σ be a fan inside $N_{\mathbb{R}}$, and let Z_Σ/A be the associated toric A-scheme, defined as in a case of a field, by patching together the $\mathrm{Spec}\, A[M_\sigma]$ for $\sigma \in \Sigma$. For instance, if there exists a subfield $k \subset A$, then $Z_{\Sigma,A} = Z_{\Sigma,k} \times_k A$ is the base change of the k-toric variety with fan Σ to A.
We want to define a partial compactification N_Σ of $N_{\mathbb{R}}$ so that the map val defined above extends as a continuous map:

$$\mathrm{val}_\Sigma : Z_\Sigma^{\mathrm{an}} \longrightarrow N_\Sigma.$$

To that extent, we set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, it is a semi-group for the standard addition. For $\sigma \in \Sigma$ a cone, the corresponding toric affine chart is given by $Z_\sigma = \mathrm{Spec}\, A[M_\sigma]$. It follows from the universal property of the ring $A[M_\sigma]$ that the map:

$$Z_\sigma(A) \longrightarrow \mathrm{Hom}_{\mathrm{sg}}(M_\sigma, (A, \times)),$$

$$z \mapsto (m \mapsto \chi^m(z))$$

is a bijection. It is thus natural to set:

$$N_\sigma := \mathrm{Hom}_{\mathrm{sg}}(M_\sigma, (\overline{\mathbb{R}}, +)),$$

since the semi-group homomorphism $v_A = -\log\|\cdot\| : A \longrightarrow \overline{\mathbb{R}}$ induces naturally a map:

$$\mathrm{val} : Z_\sigma(A) \longrightarrow N_\sigma,$$
$$z \mapsto (m \mapsto -\log\|\chi^m(z)\|).$$

extending the usual logarithm on the torus (the above expression being defined since χ^m is regular on Z_σ). We define the topology on N_σ as the coarsest making the evaluation maps $\mathrm{ev}_m : N_\sigma \longrightarrow \overline{\mathbb{R}}$ for $m \in M_\sigma$ continuous.

Proposition 2.2.1 ([Ful93, §1.2, prop. 2], [BGPS14, §4.1])**.** *If $\sigma' \subset \sigma$ are two cones of Σ, then there is a dense open immersion $N_{\sigma'} \hookrightarrow N_\sigma$. Thus, the $\{N_\sigma\}_{\sigma\in\Sigma}$ glue together to yield a topological space N_Σ containing $N_\mathbb{R}$, together with a continuous* val *map:*

$$\mathrm{val}_\Sigma : Z_\Sigma^{\mathrm{an}} \longrightarrow N_\Sigma$$

restricting to the above val *on each Z_σ.*
The topological space N_Σ is called the tropical toric variety associated to Σ.

Note that the above definition of N_Σ is independent of the base ring A.
Now assume that $A = (K, v_K)$ is a non-archimedean field. Then the map val admits a canonical section, defined as follows:

Definition 2.2.2. *Let $x \in N_\mathbb{R}$. The valuation $\gamma(x)$ on $K(M)$ defined on elements of $K[M]$ by the formula:*

$$v_{\gamma(x)}\Big(\sum_{m\in M} a_m\chi^m\Big) = \min_{m\in M}\big(v_K(a_m) + \langle m, x\rangle\big)$$

is called the Gauss point associated to x. This defines a continuous embedding:

$$\gamma : N_\mathbb{R} \longrightarrow \mathbb{T}^{\mathrm{an}},$$

satisfying $\mathrm{val}\circ\gamma = \mathrm{Id}_{N_\mathbb{R}}$.
Furthermore, for any fan Σ in $N_\mathbb{R}$, the map γ extends as a continuous embedding $\gamma : N_\Sigma \longrightarrow Z_\Sigma^{\mathrm{an}}$, satisfying the property:

$$\mathrm{val}_\Sigma \circ\gamma = \mathrm{Id}_{N_\Sigma}\,.$$

Thus, the tropical toric variety N_Σ can be naturally realized as a space of monomial valuations, inside Z_Σ^{an}. The fact that $v_{\gamma(x)}$ is indeed a valuation - i.e. the associated absolute value is multiplicative - as follows for instance from [BGPS14, prop. 4.2.12], or [Thu07, §2].

Definition 2.2.3. *We write:*

$$\rho_\Sigma : Z_\Sigma^{\mathrm{an}} \longrightarrow Z_\Sigma^{\mathrm{an}}$$

the composition $\rho_\Sigma = \gamma \circ \mathrm{val}_\Sigma$, and call its image the (toric) skeleton $\mathrm{Sk}(Z_\Sigma)$ of Z_Σ.

In the case $K = \mathbb{C}$ with the Euclidean absolute value, there also exists a canonical section of val, given as follows: since $\mathbb{T} = \mathrm{Hom}(M, \mathbb{C}^*)$, there is an embedding $\iota : N_{\mathbb{R}} \hookrightarrow \mathbb{T}^{\mathrm{hol}}$ given by:

$$x \longmapsto \iota(x) := \big(m \mapsto e^{-\langle m,x\rangle}\big).$$

It is clear that this map extends as an embedding $\iota : N_\sigma \hookrightarrow Z_\sigma^{\mathrm{hol}}$, using the real exponential as a section of $-\log|\cdot|$. In terms of coordinates on $(\mathbb{C}^*)^n$, the map ι is simply the section of $-\log|\cdot|$ given by the real exponential (up to a sign).
We now move to the case $A = \mathbb{C}^{\mathrm{hyb}}$, where we want to patch together the sections from above to construct a section of the continuous map:

$$\mathrm{Val} := (\mathrm{val} \times \pi) : Z_\Sigma^{\mathrm{hyb}} \longrightarrow N_{\mathbb{R}} \times \mathscr{M}(A).$$

To that purpose, we define $\iota_\lambda : N_{\mathbb{R}} \hookrightarrow \mathbb{T}^{\mathrm{hol}}$ by the formula:

$$x \longmapsto \iota_\lambda(x) := \big(m \mapsto e^{-\lambda^{-1}\langle m,x\rangle}\big),$$

and extend it to N_Σ cone by cone as above; this yields a continuous embedding $\iota_\lambda : N_\Sigma \hookrightarrow Z_\Sigma^{\mathrm{hol}}$. We then compose these rescaled embeddings with the homeomorphism:

$$p : Z^{\mathrm{hol}} \times (0,1] \xrightarrow{\sim} \pi^{-1}((0,1]),$$

to embed $N_\Sigma^{\mathrm{hyb}} := N_\Sigma \times [0,1]$ naturally inside the hybrid space:

Proposition 2.2.4. *Let $\iota_{\mathrm{hyb}} : N_\Sigma \times [0,1] \longrightarrow Z^{\mathrm{hyb}}$ be defined by the formulas:*

$$\iota_{\mathrm{hyb}}(x,\lambda) = p_\lambda(\iota_\lambda(x)) \text{ for } \lambda \neq 0,$$

$$\iota_{\mathrm{hyb}}(x,0) = \gamma(x).$$

Then ι_{hyb} is a continuous embedding, and is a section of Val.
Its image $\mathrm{Sk}^{\mathrm{hyb}}(Z_\Sigma) := \iota_{\mathrm{hyb}}(N_\Sigma \times [0,1])$ is called the hybrid toric skeleton of Z_Σ.

Proof. By definition of the topology on Z^{hyb}, in order to prove that ι_{hyb} is continuous, it is enough to prove that if $f \in \mathcal{O}(U)$ is an algebraic function on a Zariski open subset U of Z, then $(x,\lambda) \mapsto \log|f|_{\iota_{\mathrm{hyb}}(x,\lambda)}$ is continuous on $\iota_{\mathrm{hyb}}^{-1}(U^{\mathrm{hyb}})$. We may assume that $f \in \mathbb{C}[M]$ is a polynomial, which we write as $f = \sum_{j\in J} a_j \chi^{m_j}$, where the $a_j \in \mathbb{C}^*$, $m_j \in M$ and the index set J is finite.
We now have:

$$\log|f|_{\iota_{\mathrm{hyb}}(x,\lambda)} = \lambda \log\Big|\sum_{j\in J} a_j e^{-\lambda^{-1}\langle m_j,x\rangle}\Big|$$

for $\lambda \neq 0$, while:

$$\log|f|_{\iota_{\mathrm{hyb}}(x,0)} = -\min_{j\in J}\langle m_j, x\rangle.$$

It is straightforward to check that this indeed defines a continuous function on $N_{\mathbb{R}} \times [0,1]$, and similarly on $N_\sigma \times [0,1]$ for $\sigma \in \Sigma$: the function f is defined on Z_σ if and only if $\langle x, m_j\rangle \geq 0$ for $j \in J$ and $x \in N_\sigma$, so that we are merely allowing the exponentials in the above sum to vanish.
The fact that $\mathrm{Val} \circ \iota_{\mathrm{hyb}} = \mathrm{Id}_{N_\Sigma^{\mathrm{hyb}}}$ follows from the fact that $\mathrm{val}_{|Z_\lambda^{\mathrm{hyb}}} = \lambda\,\mathrm{val}_{\mathbb{C}}$ under the homeomorphism $Z_\lambda^{\mathrm{hyb}} \simeq Z^{\mathrm{hol}}$, so that ι_λ is a section of $\mathrm{val}_{|Z_\lambda^{\mathrm{hyb}}}$ for $\lambda > 0$.
This finally implies that ι_{hyb} is an embedding, the inverse map being given by the restriction of Val to $\mathrm{Sk}^{\mathrm{hyb}}(Z_\Sigma)$. □

Remark 2.2.5. *Let Z be a toric variety over $\mathbb{C}$, and Z_K its base change to $K = \mathbb{C}((t))$. Then we have a commutative diagram:*

$$\begin{array}{ccc} Z_K^{\mathrm{hyb}} & \xrightarrow{Q} & Z_0^{\mathrm{hyb}} \\ \downarrow{\scriptstyle \mathrm{Val}_K} & & \downarrow{\scriptstyle \mathrm{Val}_0} \\ N_\Sigma \times \bar{\mathbb{D}}_r & \xrightarrow{\lambda} & N_\Sigma \times [0,1], \end{array}$$

so that writing $t : [0,1] \longrightarrow \bar{\mathbb{D}}_r$ the map given by $t(\lambda) = r^{1/\lambda}$, we see that the map:

$$\iota_{\mathrm{hyb},K} := \iota_{\mathrm{hyb},0} \circ (\mathrm{Id}_{N_\Sigma} \times t)$$

defines a continuous section of Val_K.

2.2.2 Fiberwise-toric metrics

Throughout this section, we work with a toric scheme $Z = Z_\Sigma$ over a Banach ring A. The Berkovich analytification Z^{an} comes with two continuous maps: a val map:

$$\mathrm{val}_\Sigma : Z^{\mathrm{an}} \longrightarrow N_\Sigma,$$

and a structure morphism:

$$\pi : Z^{\mathrm{an}} \longrightarrow \mathscr{M}(A).$$

Writing $N_\Sigma(A) = N_\Sigma \times \mathscr{M}(A)$, this yields a continuous map:

$$\mathrm{val}_A : Z^{\mathrm{an}} \longrightarrow N_\Sigma(A)$$

defined by the formula $\mathrm{val}_A(x) = (\mathrm{val}(x), \pi(x))$. In particular, the torus over A is endowed with a map:

$$\mathrm{val}_A : \mathbb{T}_A^{\mathrm{an}} \longrightarrow N_{\mathbb{R}} \times \mathscr{M}(A).$$

Definition 2.2.6. *Let (L, s) be a toric line bundle on Z, and ϕ a metric on L. Then we say that ϕ is a fiberwise-toric metric on L if and only the function:*

$$\|s\|_\phi : \mathbb{T}_A^{\mathrm{an}} \longrightarrow \mathbb{R}$$

is such that $\|s(x)\|_\phi = \|s(y)\|_\phi$ for any two $x, y \in \mathbb{T}^{\mathrm{an}}$ such that $\mathrm{val}_A(x) = \mathrm{val}_A(y)$.

Here a toric line bundle on Z means, as in the case of a field, a line bundle L on Z, together with an isomorphism $L \simeq \mathcal{O}_Z(D)$ for a toric Cartier divisor D and a global section s that is a monomial on $\mathbb{T}_A$ under the latter isomorphism.

Example 2.2.7. *Assume that $A = K$ is a complete valued field. Then if $K = \mathbb{C}$, a metric is toric if and only if it is invariant under the action of the maximal compact torus $\mathbb{S} \subset \mathbb{T}$.*
If K is a non-archimedean discretely-valued field, then our definition of a toric metric matches the one from [BGPS14, def. 4.3.2].

The following description is a direct consequence of the definition:

Proposition 2.2.8. *Let ϕ be a continuous metric on the toric line bundle L. Then ϕ is a toric metric if and only for any $b \in \mathscr{M}(A)$, the restriction ϕ_b of ϕ to $Z^{\mathrm{an}}_{\Sigma,\mathscr{H}(b)}$ is a toric metric in the sense of [BGPS14, def. 4.3.2].*

For continuous toric metrics over a valued field, the semi-positivity of the metric can be read off in the combinatorial world:

Theorem 2.2.9. *[BGPS14, thm.4.8.1] Let Z be a proper toric variety over a complete valued field K, and $(\mathcal{O}_Z(D), s)$ a semi-ample toric line bundle on Z. We let $\iota : N_\Sigma \hookrightarrow Z^{\mathrm{an}}$ be the canonical section of* val *as given in section 2.2.1. Then the mapping:*

$$\|\cdot\|_\phi \longmapsto \big(\Phi := -\log\|s \circ \iota\|_\phi\big)$$

sets up a correspondence between continuous, semi-positive toric metrics ϕ on L and continuous, D-admissible convex functions Φ on $N_{\mathbb{R}}$.

Note that while we defined the class of D-admissible functions only when Z is smooth, the above theorem holds without that assumption.
Moreover, it follows from the proof of prop. 2.1.8 that if Φ is convex and D-admissible, then the (bounded) function $(\Phi - \Psi_D)$ extends continuously to N_Σ. More generally, continuous semi-positive toric metrics on L are in bijection with continuous convex functions on $N_{\mathbb{R}}$ satisfying the latter condition.
We expect a similar picture to hold over more general Banach rings A: a semi-positive continuous, fiberwise-toric metric should induce a family of convex functions on $N_{\mathbb{R}}$, parametrized by the Berkovich spectrum $\mathscr{M}(A)$, and which satisfy a suitable growth condition fiberwise over $\mathscr{M}(A)$. The semi-positivity of the metric in the direction of the base should be equivalent to the fact that this family of convex functions varies in a psh way on $\mathscr{M}(A)$. What this concretely means in general seems unclear at the moment; however in the case of the hybrid space attached to a complex toric variety, we will see in theorem 2.2.12 that the semi-positivity on the base translates into convexity with respect to the coordinate $\lambda \in [0, 1]$.
Note that when K is non-archimedean, this is proved only in the case where K is discretely-valued field in [BGPS14], but the general case follows roughly from the argument of the proof of theorem 2.2.12, together the following generalization of [BGPS14, cor. 4.7.2], so that we omit the details.

Proposition 2.2.10. *Let K be a complete non-archimedean field, and let $\phi \in \mathrm{FS}^\tau(Z_\Sigma, L)$ be a toric metric. Then there exists a finite set $(s_j)_{j\in J}$ of toric sections of mL such that:*

$$\phi = m^{-1} \max_{j\in J} \big(\log|s_j| + c_j\big).$$

In other words, toric Fubini-Study metrics on L are precisely the Fubini-Study metrics associated to toric sections of powers of L.

Proof. Write:

$$\phi = m^{-1} \max_{i\in I} \big(\log|s_i| + c_i\big)$$

for arbitrary sections $(s_i)_{i\in I}$ of mL, without common zeroes. Each s_i may be written in a basis of toric sections as a finite sum $s_i = \sum_j \lambda_{i,j}\chi^{m_j}$ with $\lambda_{i,j} \in K$, so that if

$x \in \mathbb{T}^{\mathrm{an}}$ we have:

$$\log|s_i(\rho(x))| = \max_j(v_K(\lambda_{i,j}) + \log|\chi^{m_j}(\rho(x))|)$$

by definition of the monomial valuation $\rho(x)$ (see def. 2.2.3). Since ϕ is toric, the equality $\phi(x) = \phi(\rho(x))$ holds for every $x \in \mathbb{T}^{\mathrm{an}}$, and we infer:

$$\phi = \max_{i\in I}\left(\max_{j\in J_i}\left(v_K(\lambda_{i,j}) + \log|\chi^{m_j}|\right) + c_i\right),$$

hence the result since the $(\chi_{m_j})_{j\in\cup_i J_i}$ have no common zeroes. □

Let K be a complete valued field. As a consequence of thm. 2.2.9, any semi-ample toric line bundle $L = \mathcal{O}_Z(D)$ admits a canonical continuous psh metric ϕ_{can}, such that the associated convex function $\Phi_{\mathrm{can}} = \Psi_D$. The canonical metric can be described explicitly as follows: let $V \subset P_D$ be the set of vertices, then we have:

$$\phi_{\mathrm{can}} = \max_{m\in P_D} \log|s_m|.$$

Note that the $(s_m)_{m\in V}$ have no common zeroes, as a semi-ample line bundle on a toric variety is globally generated [Ful93, p. 68]. This definition can thus be generalized to a smooth toric scheme over any Banach ring:

Definition 2.2.11. *Let A be a Banach ring, and (Z, L) a smooth toric scheme together with a semi-ample line bundle over A. The canonical metric on L is the tropical Fubini-Study metric defined by the formula:*

$$\phi_{\mathrm{can}} = \max_{m\in P_D} \log|s_m|,$$

where the s_m are defined as in prop. 2.1.18.

The canonical metric is in particular fiberwise-toric.

2.2.3 Hybrid toric metrics

If ϕ is a toric hybrid metric on L^{hyb}, it follows from the previous discussions that it is entirely determined by the function:

$$\Phi(w) = -\log\|s(\iota_{\mathrm{hyb}}(w))\|_\phi,$$

where $w \in N_\Sigma^{\mathrm{hyb}}$. Our next result asserts that in the case of continuous metrics, the semi-positivity of ϕ can also be read on the hybrid tropical toric variety:

Theorem 2.2.12. *Let $Z = Z_\Sigma$ be a toric variety over $\mathbb{C}$, and $L = (\mathcal{O}_Z(D), s_D)$ a semi-ample toric line bundle on Z. We write $\Psi_D : N_{\mathbb{R}} \times [0,1] \longrightarrow \mathbb{R}$ the associated support function, extended trivially in the λ-direction.*
There is a one-to-one correspondence between continuous, semi-positive, fiberwise-toric metrics ϕ on L^{hyb}, and continuous convex functions:

$$\Phi : N_{\mathbb{R}} \times [0,1] \longrightarrow \mathbb{R}$$

such that the function $(\Phi - \Psi_D)$ *extends as a continuous function to* $N_\Sigma \times [0,1]$. *The correspondence is explicitly given by:*

$$\phi \longmapsto \Big((x,\lambda) \mapsto -\log \|s_D(\iota_{\mathrm{hyb}}(x,\lambda))\|_\phi \Big).$$

Example 2.2.13. *Assume that* $\Phi(x,\lambda) \equiv \Phi(x)$ *is independent of* λ. *Then unraveling the definitions and using the isomorphism* $Z^{\mathrm{hyb}}_\lambda \simeq Z^{\mathrm{hol}}$ *obtained by rescaling the absolute value, the induced family of metrics on* Z^{hol} *is given by:*

$$\phi_\lambda(z) = \lambda^{-1}\Phi(-\lambda \log|z|)),$$

while the non-archimedean metric ϕ_0 *on* Z^{an}_0 *is described by the convex function* Φ, *in the setting of theorem 2.2.9.*
Similarly, pulling back on the trivial degeneration $X = Z \times \mathbb{D}^*$, *our theorem states in particular that the psh family of metrics on* $p_1^* L$*:*

$$\phi_t = (-\log|t|) \cdot \Phi(\frac{\log|z|}{\log|t|})$$

extends as a continuous psh metric to X^{hyb}, *with* $\phi_0 = \Phi \circ \mathrm{val}$. *Note that the* ϕ_t *are psh with respect to* (z,t) *since* Φ *can be written as the decreasing limit of a sequence of piecewise-affine functions.*

In order to prove theorem 2.2.12, we will need, given a convex function Φ satisfying the assumptions of the theorem, to produce a continuous metric ϕ on L^{hyb} whose associated convex function is Φ. The metric ϕ will be produced as a decreasing limit of hybrid Fubini-Study metrics, after translating the following relative version of prop. 2.1.8 in terms of toric metrics:

Proposition 2.2.14. *Let* $P \subset M_\mathbb{R}$ *be a convex polytope with support function* Ψ_P *and normal fan* Σ, *and let* $\Phi : N_\mathbb{R} \times [0,1] \longrightarrow \mathbb{R}$ *be a continuous convex function, such that the function* $(\Phi - \Psi_P)$ *extends continuously to* $N_\Sigma \times [0,1]$.
Then there exists a sequence $(\Phi_j)_{j\in\mathbb{N}}$ *of piecewise-affine convex functions on* $N_\mathbb{R} \times [0,1]$, *such that:*

- *for all* $j \in \mathbb{N}$, *the function* $(\Phi_j - \Psi_P)$ *extends continuously to* $N_\Sigma \times [0,1]$;
- *the sequence* $(\Phi_j)_j$ *decreases to* Φ *over* $N_\Sigma \times [0,1]$;
- *each* Φ_j *has rational slopes in the* $N_\mathbb{R}$*-direction, i.e. we may write:*

$$\Phi_j(x,\lambda) = \max_{\alpha\in A} \big(\langle u_\alpha, x\rangle + b_\alpha \lambda + c_\alpha \big),$$

where the $u_\alpha \in P \cap M_\mathbb{Q}$, *and the* a_α, $b_\alpha \in \mathbb{R}$.

Proof. Let $\Phi^* : P \times \mathbb{R} \longrightarrow \mathbb{R}$ be the Legendre transform of Φ, then by lemma 2.2.15 below, the function $(\Phi^* - \max\{0, a\})$ extends continuously to $P \times [-\infty, +\infty]$, where a is the coordinate on $\mathbb{R}$.
Let $((x_j, \lambda_j))_{j\in\mathbb{N}}$ be a dense sequence of rational points in $N_\mathbb{R} \times [0,1]$, and let $(r_{j,l})_{l\in\mathbb{N}}$

be a sequence of rational numbers decreasing to $\phi(x_j, \lambda_j)$, uniformly with respect to j. We furthermore assume that $\lambda_0 = 0$ and $\lambda_1 = 1$.
We set:

$$\beta_j(u,a) = \max_{l \leq j} \big(\langle u, x_l \rangle + a\lambda_l - r_{j,l} \big)$$

for $(u,a) \in P \times \mathbb{R}$, it is a sequence of piecewise-affine convex functions increasing pointwise to Φ^*. The convergence also holds on $P \times [-\infty, +\infty]$ after substracting $\max\{0,a\}$, since for $j \geq 1$ and $|a| \gg 1$ depending on u, we have that $(\beta_j(u,a) - \max\{0,a\})$ does not depend on a. As a result, the convergence is in fact uniform, by Dini's lemma.
We now take Φ_j to be the Legendre transform of β_j, i.e.:

$$\Phi_j(x,\lambda) = \sup_{(u,a) \in P \times \mathbb{R}} \big(\langle u, x \rangle + a\lambda - \beta_j(u,a) \big),$$

the Φ_j's are of the form described in the statement by the argument from lemma 2.1.9, as $(\beta_j(u,a) - \max\{0,a\})$ does not depend on a for $|a| \gg 1$. Since the β_j increase uniformly to Φ^*, the Φ_j decrease uniformly to Φ^{**}, which is equal to Φ by [Roc72, thm. 12.2].
It remains to prove that $(\Phi_j - \Psi_P)$ extends continuously to $N_\Sigma \times [0,1]$. For $\sigma \in \Sigma$, and $m_\sigma \in M$ such that $(\Psi_P)_{|\sigma} \equiv m_\sigma$, since the convex hull $\mathrm{Conv}(u_\alpha)_\alpha = P$, we have that $\Phi_j(x,\lambda) = \langle m_\sigma, x \rangle + b_{j_0}\lambda + c_{j_0}$ for $x \in \sigma$ and $|x| \gg 1$ independent on λ, where j_0 is such that $u_{j_0} = m_\sigma$ This proves that $(\Phi_j - \Psi_P)$ extends continuously to $N_\Sigma \times [0,1]$, and decreasing convergence happens over $N_\Sigma \times [0,1]$ since it is uniform, which concludes the proof. □

Lemma 2.2.15. *Let $P \subset N_\mathbb{R}$ be a convex polytope, and let $\Phi : N_\mathbb{R} \times [0,1] \longrightarrow \mathbb{R}$ satisfying the assumptions of prop. 2.2.14. Then the Legendre transform:*

$$\Phi^* : P \times \mathbb{R} \longrightarrow \mathbb{R}$$

is continuous. Moreover, writing a the coordinate on $\mathbb{R}$, the function $(\Phi^* - \max\{0,a\})$ extends continuously to $P \times [-\infty, +\infty]$.

Proof. Let $(u,a) \in M_\mathbb{R} \times \mathbb{R}$, then:

$$\Phi^*(u,a) = \sup_{N_\mathbb{R} \times [0,1]} \big(\langle u, x \rangle + a\lambda - \Phi(x,\lambda) \big)$$

$$= \sup_{N_\mathbb{R} \times [0,1]} \big(\langle u, x \rangle + a\lambda - \Psi_P(x) + (\Psi_P(x) - \Phi(x,\lambda)) \big),$$

which is finite if and only if $u \in P$ since $(\Psi_P - \Phi)$ is continuous, hence bounded on $N_\Sigma \times [0,1]$. We also infer that:

$$\Phi^*(u,a) = \sup_{N_\Sigma \times [0,1]} \big(\langle u, x \rangle + a\lambda - \Phi(x,\lambda) \big)$$

whenever $u \in P$, since $\Phi(x,\lambda) = -\infty$ for $x \in N_\Sigma \setminus N_\mathbb{R}$. As a result, Φ^* is continuous on $P \times \mathbb{R}$ as a fiberwise-supremum over a compact set of a continuous function.

We now prove the second item, we assume $a \geq 0$ as the other case is treated in the same way. The convex function $\Psi_u : a \mapsto (\Phi^*(u,a) - a)$ on $\mathbb{R}_{\geq 0}$ can be written as:

$$\Psi_u = \sup_{\lambda \in [0,1]} \big((\lambda - 1)a + \eta_u(\lambda)\big),$$

where η is a continuous function of (u,λ). As a result, Ψ_u is a bounded convex function, decreasing on $\mathbb{R}_{\geq 0}$, and whose limit at $+\infty$ is $\sup_{[0,1]} \eta_u$, which is continuous with respect to u. Thus, the family $(\Psi(u,a))_{a \geq 0}$ of continuous convex functions on P decreases with respect to a to a continuous function, so that they converge uniformly by Dini's lemma, and $(\Phi^* - a)$ extends continuously to $P \times [0, +\infty]$. □

Proof of theorem 2.2.12. Let $\phi \in \mathrm{CPSH}(Z^{\mathrm{hyb}}, L^{\mathrm{hyb}})$ be a continuous psh, fiberwise-toric metric, and write:

$$\Phi : N_{\mathbb{R}} \times [0,1] \longrightarrow \mathbb{R},$$
$$(x,\lambda) \mapsto -\log \|s_D(\iota_{\mathrm{hyb}}(x,\lambda))\|_\phi,$$

it is a continuous function by prop. 2.2.4. Then writing $\phi = \lim_j \phi_j$ as the decreasing limit of a net in FS^τ, we have that $\Phi = \lim_j \Phi_j$ as a decreasing limit, so that to prove that Φ is convex on $N_{\mathbb{R}} \times [0,1]$, it is enough to prove that Φ_j is convex for $\phi_j \in \mathrm{FS}^\tau(L^{\mathrm{hyb}})$. Dropping the j subscript, we write:

$$\phi = m^{-1} \max_{\alpha \in A} (\log|s_\alpha| + c_\alpha),$$

with $s_\alpha \in H^0(Z, mL)$ and $c_\alpha \in \mathbb{R}$. By convexity of the maximum of finitely many convex functions, we are reduced to the case:

$$\phi = m^{-1} \log|s|,$$

where $s \in H^0(Z, mL)$ for some $m > 0$. Using the isomorphism $\mathcal{O}_{\mathbb{T}}(D) = s_D \mathcal{O}_{\mathbb{T}}$, we may write $s = f \times (s_D)^m$ on $\mathbb{T}$ for $f \in K[M]$, so that:

$$\Phi(x,\lambda) = m^{-1} \log \Big| \frac{s_D^m}{s}((\iota_{\mathrm{hyb}}(x,\lambda)) \Big| = m^{-1} \log|f(\iota_{\mathrm{hyb}}(x,\lambda))|.$$

By the proof of prop. 2.2.4 and with the same notation, we have:

$$\Phi(x,\lambda) = \lambda m^{-1} \log \Big| \sum_{j \in J} a_j e^{-\lambda^{-1}\langle m_j, x\rangle} \Big|$$

for $\lambda \neq 0$, while:

$$\Phi(x,0) = -m^{-1} \min_{j \in J} \langle m_j, x \rangle.$$

The fact that Φ is convex on $N_{\mathbb{R}} \times [0,1]$ is now elementary.
We now move back to the case where $\phi \in \mathrm{CPSH}(L^{\mathrm{hyb}})$ is an arbitrary continuous psh, fiberwise-toric metric on L^{hyb}. To prove that $(\Phi - \Psi_D)$ extends continuously to $N_\Sigma \times [0,1]$, we argue as in the proof of [BGPS14, prop. 4.3.10]. Let $\sigma \in \Sigma$ be a cone, then there exists $m_\sigma \in M$ such that the support function $\Psi_{|\sigma} \equiv m_\sigma$. Moreover, the section $s_\sigma := \chi^{-m_\sigma} s_D$ is a nowhere vanishing section of L on Z_σ. As a result, the function:

$$(x,\lambda) \mapsto -\log \|s_\sigma(\iota_{\mathrm{hyb}}(x,\lambda))\|_\phi$$

is continuous on $N_\sigma \times [0,1]$. However the above expression is easily seen to be equal to $\Phi(x,\lambda) - \langle m_\sigma, x\rangle = \Phi(x,\lambda) - \Psi_D(x)$, so that $(\Phi - \Psi_D)$ extends continuously to $N_\Sigma \times [0,1]$.
Conversely, let $\Phi : N_\mathbb{R} \times [0,1] \longrightarrow \mathbb{R}$ be a continuous convex function, such that $(\Phi - \Psi_D)$ extends continuously to $N_\Sigma \times [0,1]$. We want to produce a continuous, semi-positive hybrid metric ϕ on L such that $\Phi = -\log\|s_D \circ \iota_{\mathrm{hyb}}\|$. We start by applying prop. 2.2.14 to produce a decreasing sequence $(\Phi_j)_{j\in\mathbb{N}}$ of continuous convex functions on $N_\mathbb{R} \times [0,1]$, converging to Φ pointwise, and such that the function $(\Phi_j - \Psi_D)$ extends continuously to $N_\Sigma \times [0,1]$ for all $j \geq 0$. Additionally, we may write Φ_j as:

$$\Phi_j = d_j^{-1} \max_{\alpha\in A} \big(\langle m_\alpha, \cdot\rangle + b_\alpha\lambda + c_\alpha\big),$$

with the $m_\alpha \in M \cap P_D$. Since $\Phi_j(\cdot,\lambda)$ is D-admissible for any λ, it follows from lemma 2.1.19 that $(s_{m_\alpha}, \alpha \in A)$ is a family of toric sections of mL without common zeroes, so that:

$$\phi_j = m^{-1} \max_{\alpha\in A} \big(\log|s_{m_\alpha}| + b_\alpha \log|e| + c_\alpha\big)$$

is a Fubini-Study hybrid metric on L, inducing the convex function Φ_j on $N_\mathbb{R} \times [0,1]$ by the lemma below. Moreover, since the $(\Phi_j - \Psi_D)_j$ decrease to $(\Phi - \Psi_D)$ over $N_\mathbb{R} \times [0,1]$, the same holds over $N_\Sigma \times [0,1]$ by uniform convergence, which is easily seen to imply that the ϕ_j decrease to a continuous metric ϕ on L^{hyb}, which concludes the proof. □

Lemma 2.2.16. *Let $L = (\mathcal{O}_Z(D), s_D)$ be a toric line bundle on Z, and $m \in M \cap P_D$. We write $\phi = \log|s_m|$ the singular metric on L^{hyb} induced by prop. 2.1.18. Then for $(x,\lambda) \in N_\mathbb{R} \times [0,1]$, the equality:*

$$-\log\|s_D(\iota_{\mathrm{hyb}}(x,\lambda))\|_\phi = \langle m, x\rangle$$

holds.

Proof. By definition, we have:

$$-\log\|s_D(\iota_{\mathrm{hyb}}(x,\lambda))\|_\phi = \log\Big(\frac{|s_m|}{|s_D|}\Big)(\iota_{\mathrm{hyb}}(x,\lambda)),$$

but under the isomorphism $\mathcal{O}_\mathbb{T}(D) = s_D \cdot \mathcal{O}_\mathbb{T}$ we have $s_m/s_D = \chi^{-m}$, which concludes. □

2.2.4 Hybrid family of Haar measures

We conclude this chapter with a discussion on Haar measures on hybrid tori, elaborating on the framework from [BGPS14, §4.2], compare also [Poi22, thm. B]
Let Z be a complex projective toric variety, and $\pi : Z^{\mathrm{hyb}} \longrightarrow [0,1]$ the associated hybrid space. For each $\lambda > 0$, there exists a canonical homeomorphism $p_\lambda : Z^{\mathrm{hol}} \xrightarrow{\sim} Z_\lambda = \pi^{-1}(\lambda)$, where Z^{hol} denotes the complex analytification of Z.
For $x \in N_\mathbb{R}$, let $\mathbb{S}_x = \mathrm{val}^{-1}(x) \subset \mathbb{T}^{\mathrm{hol}}$, and $\iota_\lambda : \mathbb{T}^{\mathrm{hol}} \hookrightarrow Z^{\mathrm{hyb}}$ be the composition of the embedding $\mathbb{T}^{\mathrm{hol}} \subset Z^{\mathrm{hol}}$ with the isomorphism $Z^{\mathrm{hol}} \simeq Z_\lambda^{\mathrm{hyb}}$. The real compact torus $\mathbb{S}_x$ carries a canonical Haar probability measure $\mu_\mathbb{S}$, which we can pushforward

to the complex torus, and then to Z^{hyb} via the embedding ι_λ. We write the resulting measure as:

$$\mu_{x,\lambda} := (\iota_\lambda)_* \mu_{\mathbb{S}_x}$$

for $\lambda > 0$, and define:

$$\mu_{x,0} = \delta_{\gamma(x)}$$

the Dirac mass at the Gauss point $\gamma(x)$.
It is well-understood that the natural non-archimedean analog of the Haar measure on a real compact torus is the Dirac mass at the Gauss point of the affinoid torus [BGPS14, prop. 4.2.10], the following theorem shows that the analogy is furthermore continuous on the hybrid space:

Theorem 2.2.17. *Let Z be a complex projective toric variety, and Z^{hyb} the associated hybrid space. Then the family of measures $(\mu_{x,\lambda})_{(x,\lambda)\in N_{\mathbb{R}}\times[0,1]}$ on Z^{hyb} defined above is weakly continuous.*

Proof. We need to prove that if $f \in \mathcal{C}^0(Z^{\mathrm{hyb}})$, then the real-valued function:

$$(x,\lambda) \mapsto \int_{Z^{\mathrm{hyb}}} f(z) d\mu_{x,\lambda}(z)$$

is continuous on $N_{\mathbb{R}} \times [0,1]$. If $\lambda > 0$, then:

$$(x,\lambda) \mapsto \int_{Z^{\mathrm{hol}}} f(z,\lambda) d\mu_{\mathbb{S}_x}(z)$$

is continuous on $N_{\mathbb{R}} \times (0,1]$ by dominated convergence, since f is continuous and bounded on $Z^{\mathrm{hol}} \times (0,1]$.
We now need to examine what happens when $\lambda \longrightarrow 0$. By density of $\mathrm{DFS}(Z^{\mathrm{hyb}}) \subset \mathcal{C}^0(Z^{\mathrm{hyb}})$, it is enough to prove the statement for $f \in \mathrm{DFS}(Z^{\mathrm{hyb}})$, so that we may assume that there exists a line bundle L on Z such that $f = f_1 - f_2$, with $f_i \in \mathrm{FS}(Z^{\mathrm{hyb}}, L^{\mathrm{hyb}})$. Since every line bundle on Z is isomorphic to a toric one, we may and will assume that L is a toric line bundle. We thus write:

$$f_i = m^{-1} \max_{\alpha\in A}(\log|s_\alpha| + c_\alpha),$$

for $i = 1, 2$. Since the measures $\mu_{x,\lambda}$ are supported on $\mathbb{T}^{\mathrm{hyb}}$, on which each section $s_\alpha \in H^0(Z, mL)$ can be written as $P_\alpha \times s_D$, where $P_\alpha \in \mathbb{C}[M]$, we are reduced to the case where $L = \mathcal{O}_Z$ and $f = g_1 - g_2$, where:

$$g_i = m^{-1} \max_{\alpha\in A}(\log|P_\alpha| + c_\alpha),$$

with $P_\alpha \in \mathbb{C}[M]$. The result thus boils down to the lemma below. □

Lemma 2.2.18. *Let $(P_\alpha)_{\alpha\in A}$ be a finite family of polynomials in $\mathbb{C}[z_1, ..., z_n]$. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_{>0}$, we define the polydisk $\mathbb{D}_{\lambda,x} := \{\lambda \log|z_i| < -x_i \ \forall i \leq n\}$ and $\mathbb{S}_{\lambda,x}$ its distinguished boundary. Then there exists a locally bounded function B of x such that:*

$$\left| \int_{\mathbb{S}_{\lambda,x}} \max_{\alpha\in A}(\lambda \log|P_\alpha(\theta)| + c_\alpha) d\mu(\theta) - \max_{\alpha\in A}(-v_{\gamma(x)}(P_\alpha) + c_\alpha) \right| \leq \lambda \times B(x),$$

where the Haar measure μ on $\mathbb{S}_{\lambda,x}$ is normalized to have mass 1.

Proof. The left integral can be written as:

$$\int_{(\mathbb{S}^1)^n} \big(\max_{\alpha\in A} \log|P_\alpha(\exp(-\lambda^{-1}x + i\theta))| + c_\alpha \big) d\theta,$$

where $d\theta$ is the Haar probability measure on $(\mathbb{S}^1)^n$.
However if $P = \sum_{m\in M} a_m \chi^m$ and $\theta \in (\mathbb{S}^1)^n$, we have:

$$\lambda \log|P(\exp(-\lambda^{-1}x + i\theta)| + \min_{a_m\neq 0} \langle m, x\rangle = \lambda \log \Big| \sum_{m\in M} a_m e^{-\frac{\langle m-m_0,x\rangle}{\lambda} + i\theta} \Big|,$$

where m_0 achieves the minimum of $\langle m, x\rangle$ for $a_m \neq 0$. We infer that:

$$\lambda \log|P(\exp(-\lambda^{-1}x + i\theta)| + \min_{a_m\neq 0} \langle m, x\rangle = \lambda \log \Big| a e^{i\theta} + \sum_{m'} a_{m'} e^{-\frac{\langle m'-m_0,x\rangle}{\lambda} + i\theta} \Big|,$$

where a is a constant and $\langle m' - m_0, x\rangle > 0$ for all m'. Taking the maximum and averaging over $(\mathbb{S}^1)^n$, this concludes the proof. □

Chapter 3

The non-archimedean SYZ fibration

3.1 Generalities

We will now move on to the main motivation for the content of the previous sections: maximal degenerations of Calabi-Yau manifolds. We start with a definition:

Definition 3.1.1. *A Calabi-Yau manifold is a compact Kähler manifold X whose canonical bundle K_X is trivial, i.e. there exists a nowhere-vanishing holomorphic n-form Ω on X, where $n = \dim(X)$.*

Note that there are various definitions of Calabi-Yau manifolds across the literature, some authors require for instance that there exist no global holomorphic k-forms on X for $0 < k < n$; we will rather call such manifolds strict Calabi-Yau. Our broader definition includes for instance complex tori, as well as hyperkähler varieties.
One of the main reason Calabi-Yau have aroused great interest in Kähler geometry is the fact that, as originally conjectured by Calabi, they carry Kähler Ricci-flat metrics (in each Kähler class), according to the celebrated Yau theorem:

Theorem 3.1.2. *[Yau78]*
Let (X, L) be a polarized Calabi-Yau manifold, and $\Omega \in H^0(X, K_X)$ such that $\int_X i^{n^2}\Omega \wedge \bar{\Omega} = 1$. Then there exists a unique smooth Hermitian metric ϕ_{CY} on L whose curvature form $\omega = dd^c\phi_{CY}$ is a Kähler form, satisfying the complex Monge-Ampère equation:

$$\omega^n = (L^n)i^{n^2}\Omega \wedge \bar{\Omega}.$$

3.1.1 The classical SYZ picture

The original Strominger-Yau-Zaslow conjecture [SYZ96] appeared as an attempt to provide mathematic explanation to the following duality phenomenon, initially observed by physicists: a smooth Calabi-Yau threefold X should admit a *mirror dual* $\check{X}$, satisfying a certain amount of dual properties - the complex geometry of X being somehow encoded by the symplectic geometry of $\check{X}$. The simplest incarnation of this would be the quarter-rotation of the Hodge diamond when passing from X to its mirror dual:

$$h^{p,q}(X) = h^{n-q,p}(\check{X}).$$

More generally, one expects various correspondences between moduli spaces attached to X and $\check{X}$: the moduli space of holomorphic D-branes on X (which are complex submanifolds with additional structure) should be isomorphic to the moduli space of special Lagrangian D-branes on $\check{X}$. Thus, since X itself is the moduli space of points in X, one should be able to realize X as a certain moduli space of special Lagrangian submanifolds of $\check{X}$.

Definition 3.1.3. *Let (X, ω, Ω) be a Calabi-Yau manifold of dimension n, and B an n-dimensional (real) topological manifold.*
A special Lagrangian fibration $f : X \longrightarrow B$ is a continuous map which is a smooth torus fibration outside a codimension 2 subset $\Gamma \subset B$, and such that for all $b \in B \setminus \Gamma$, $\omega_{|f^{-1}(b)} \equiv 0$ and the imaginary part $\Im(\Omega)_{|f^{-1}(b)} \equiv 0$.

If $f : X \longrightarrow B$ is a special Lagrangian torus fibration, we write $B^{\mathrm{sm}} := B \setminus \Gamma$ its smooth locus, each smooth fiber $X_b = f^{-1}(b)$ for $x \in B^{\mathrm{sm}}$ is a smooth, special Lagrangian (real) torus inside X.
Loosely speaking, the SYZ conjecture predicts that if X, $\check{X}$ are two mirror dual Calabi-Yau manifolds, each of them should admit a special Lagrangian torus fibration onto the same half-dimensional base B, and those two fibrations should be dual to each other, in the sense that for $b \in B$, the fibers X_b and $\check{X}_b$ should be dual tori, i.e. $\check{X}_b \simeq H^1(X_b, \mathbb{R}/\mathbb{Z})$. Hence, given a Calabi-Yau manifold X and a special Lagrangian fibration $f : X \longrightarrow B$, the SYZ heuristic provides a program to reconstruct the mirror $\check{X}$, by dualizing the fibration f over B^{sm} and then extend this fibration over B.
The problem of compactifying a smooth Lagrangian torus fibration into a singular one has proven to be a difficult problem, and has led to many developments over the past years; we refer the reader to the series of recent or upcoming papers [RZ21a, RZ21b, RZ] for some progress on these questions.
The fibration induces the two following structures on B^{sm}: a $\mathbb{Z}$-affine structure $\nabla^{\mathbb{Z}}$, and a Riemannian metric g_B called the McLean metric. We start with a definition:

Definition 3.1.4. *An* integral affine structure *on a topological manifold is an atlas of charts (called $\mathbb{Z}$-affine charts) with transition functions in $GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n$.*

If B is a manifold endowed with an integral affine structure, it makes sense to speak about integral affine functions on B - those are continuous functions on B that are, locally in $\mathbb{Z}$-affine coordinates, of the form $f(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n + b$, with $a_i \in \mathbb{Z}$ and $b \in \mathbb{R}$. We denote by $\mathrm{Aff}_{\mathbb{R}^n}$ the sheaf of integral affine functions on $\mathbb{R}^n$.

Lemma 3.1.5 ([KS06, 2.1])**.** *An integral affine structure on a topological manifold M is equivalent to the datum of a subsheaf Aff_B of the sheaf of continuous functions on B such that (B, Aff_B) is locally isomorphic to $(\mathbb{R}^n, \mathrm{Aff}_{\mathbb{R}^n})$.*

Remark 3.1.6. *Given an integral affine structure on a topological manifold M, there is a* monodromy representation

$$T : \pi_1(B) \to GL_n(\mathbb{Z}) \ltimes \mathbb{R}^n$$

defined by covering a loop in M by affine charts and composing the corresponding transition functions. See [KS06, 2.2] for more details.

Let $b \in B^{\mathrm{sm}}$ and $v \in T_bB$ a tangent vector, then the symplectic form ω on X induces an isomorphism between T_bB and $T^*_xX_b$, the latter of which can be identified to $H^1(X_b, \mathbb{R})$ via evaluation of harmonic 1-forms (here we are using the fact that X_b is a real torus).
The lattice and metric on T_bB are now simply taken to be the pullbacks of the lattice $H^1(X_b, \mathbb{Z})$ and the Hodge metric:

$$g(\alpha, \beta) = \int_{X_b} \alpha \wedge *\beta,$$

the Hodge star being the one of the metric induced on X_b by ω.
One can check that there locally exists a convex function $K : B^{\mathrm{sm}} \longrightarrow \mathbb{R}$ such that in $\mathbb{Z}$-affine coordinates y_i, the McLean metric can be written as:

$$g_{ij} = \frac{\partial^2 K}{\partial y_i \partial y_j}.$$

Note that the above constructions also work for a Lagrangian torus fibration, not necessarily special.
Conversely, starting from the data of $(B, \nabla^{\mathbb{Z}}, K)$ a smooth $\mathbb{Z}$-affine manifold together with a (multi-valued) convex function, one can construct a toy model of Lagrangian fibration with base B as follows: set $X = X(B)$ to be the tangent torus bundle, that is, $X(B) = TB/\Lambda$, Λ being the local system of lattices associated to $\nabla^{\mathbb{Z}}$.
In term of local $\mathbb{Z}$-affine coordinates y_j on B, we can pick fiber coordinates $x_j = dy_j$, and the complex structure on X is defined by requiring the functions:

$$z_j = e^{2i\pi(x_j + iy_j)}$$

to be holomorphic. In particular, the map f is now given by:

$$f(z_1, ..., z_n) = \frac{1}{2\pi}(-\log|z_1|, ..., -\log|z_n|).$$

The Kähler form on X is defined in terms of its local potential:

$$\omega = i\partial\bar{\partial}(K \circ f),$$

which restricts to zero on any torus fiber - we even have a stronger statement, that the Kähler metric g_ω restricts to a flat metric to each of the torus fibers, hence the name *semi-flat metric*. Moreover, the complex Monge-Ampère measure:

$$\omega^n = \left(dd^c(K \circ f)\right)^n$$

is directly related to the real Monge-Ampère measure of K in affine coordinates, as we have:

$$\left(dd^c(K \circ f)\right)^n = \det\left(\frac{\partial^2 K}{\partial y_i \partial y_j}\right) dz \wedge d\bar{z},$$

where $dz = dz_1 \wedge ... \wedge dz_n$.
Later on, it was realized that the SYZ picture should rather occur as an asymptotic

phenomenon on the moduli space of Calabi-Yau manifolds, near the locus of the boundary where the manifolds degenerate in the worst possible way - in the terminology of mirror symmetry, this means that the manifolds approach the "large complex structure limit". Thus, we now let $X \longrightarrow \mathbb{D}^*$ be a degeneration of Calabi-Yau manifolds, polarized by a relatively ample line bundle L. We will assume that the degeneration is the worst possible, i.e. that the fibers X_t break into as many pieces as possible as $t \to 0$. The precise definition is as follows:

Definition 3.1.7. *The family X is maximally degenerate if for every dlt model $\mathscr{X} \longrightarrow \mathbb{D}$ of X, the equality:*

$$\dim \mathscr{D}(\mathscr{X}) = \dim(X_t)$$

holds.

Concretely, this means that for any dlt model $\mathscr{X}$ of X, there exists a non-empty intersection of n components of $\mathscr{X}_0$, where $n = \dim X_t$. The latter condition is equivalent to the fact that the monodromy action on $H^n(X_t, \mathbb{Q})$ is maximally unipotent, i.e. has a Jordan block of size $(n+1)$.

Example 3.1.8. *In the 2-dimensional case, projective maximally degenerate Calabi–Yau surfaces coincide with either $K3$ surfaces of Type III, or abelian varieties such that after semi-abelian reduction, the abelian part of the special fiber of the Néron model is trivial.*

We now let (X, L) be a polarized, maximal degeneration of Calabi-Yau manifolds. We fix a relative trivialization $\Omega \in H^0(X, K_{X/\mathbb{D}^*})$, and write $\omega = (\omega_t)_{t\in\mathbb{D}^*}$ the family of Kähler-Ricci flat metrics on X, normalized by the cohomological condition $\omega_t \in c_1(L_t)$.
In this setting, a strong version of the SYZ conjecture would predict the following: for any $|t| << 1$, the fiber X_t admits a special Lagrangian fibration:

$$f_t : X_t \longrightarrow B_t,$$

onto a compact base B_t of real dimension n, depending in a smooth way of t in a reasonable sense, and whose fibers are of size $(\log|t|^{-1})^{-1}$ with respect to the rescaled Calabi-Yau metric $\tilde{\omega}_t \in \frac{c_1(L_t)}{\left|\log|t|\right|}$. These rescaled Calabi-Yau metrics indeed have bounded diameters by [LT20].
In particular, the rescaled Ricci-flat Kähler manifolds $(X_t, \tilde{\omega}_t)$ should collapse in the Gromov-Hausdorff sense to the base $B := \lim_{t\to 0} B_t$, endowed with an asymptotic version of the MacLean metric alluded to above.

Conjecture 3.1.9. *(Weak SYZ conjecture) Let (X, L) be a polarized, maximal degeneration of Calabi-Yau manifolds. Then for any $\delta > 0$, there exists $\varepsilon = \varepsilon_\delta > 0$ such that for $|t| < \varepsilon$, there exists a special Lagrangian fibration:*

$$f_t : U_t \longrightarrow B,$$

where $U_t \subset X_t$ is an open subset of Calabi-Yau measure greater than $(1-\delta)$.
Moreover, the rescaled Calabi-Yau metrics $(X_t, \tilde{\omega}_t)$ converge in the Gromov-Hausdorff sense as $t \to 0$ to a compact n-dimensional base (B, g_B), such that:

- *(B, g_B) is a the metric completion of a smooth Riemannian manifold $(B^{\mathrm{sm}}, g_{B^{\mathrm{sm}}})$, whose complement $\Gamma = B \setminus B^{\mathrm{sm}}$ has Hausdorff codimension ≥ 2;*
- *the smooth locus B^{sm} is endowed with an integral affine structure, such that the metric g has a potential locally in affine coordinates: we may write $g_{ij} = \frac{\partial^2 K}{\partial y_i \partial y_j}$ in affine coordinates $(y_1, ..., y_n)$, for a smooth function K;*
- *the metric g solves the real Monge-Ampère equation $\det(g_{ij}) = 1$ on B^{sm}.*

This is known in the case of abelian varieties by the work of [Oda18], and for Fermat hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{C}}$ by [Li22].

3.1.2 The essential skeleton

In this section, we will explain how the base B of the SYZ fibration can be realized in the non-archimedean world.
Let $X \longrightarrow \mathbb{D}^*$ be a projective degeneration of smooth complex varieties, with relatively trivial canonical bundle. We use the same notation X/K for the base change to $K = \mathbb{C}((t))$.
We let $\Omega \in H^0(X, K_{X/\mathbb{D}^*})$ be a fiberwise nowhere-vanishing n-form on X; this induces a continuous family of volume forms $\nu_t := i^{n^2} \Omega_t \wedge \bar{\Omega}_t$ on the X_t's, and thus a continuous family of probability measures:

$$\mu_t := \frac{\nu_t}{\nu_t(X_t)}.$$

We are interested in determining the hybrid limit of the μ_t's, so that in light of the discussion following proposition 1.1.27, we fix an snc model $\mathscr{X} \longrightarrow \mathbb{D}$ of X, in order to understand the asymptotic behaviour of the measures $(\mathrm{Log}_{\mathscr{X}})_* \nu_t$.
Write $\mathscr{X}_0 = \sum_{i \in I} a_i D_i$, and:

$$K^{\log}_{\mathscr{X}/\mathbb{D}} := K_{\mathscr{X}/\mathbb{D}} + (\mathscr{X}_{0,\mathrm{red}} - \mathscr{X}_0) = \sum_{i \in I} b_i D_i.$$

We choose our relative holomorphic n-form so that it extends as a holomorphic section of $K^{\log}_{\mathscr{X}/\mathbb{D}}$. We will assume for convenience that $\min_{i \in I} \frac{b_i}{a_i} = 0$, see [BJ17] for the more general computation.
We let $(\mathscr{U}, z)$ be a coordinate chart, and J the set of irreducible components of $\mathscr{X}_0$ meeting $\mathscr{U}$. Then we have, on $\mathscr{U}$:

$$\Omega_t \wedge \frac{dt}{t} = f \prod_{j \in J} z_j^{b_j} \times \frac{dz_0}{z_0} \wedge ... \wedge \frac{dz_n}{z_n},$$

with f holomorphic and nowhere-vanishing. Restricting to the fiber X_t, this yields:

$$\int_{\mathscr{U} \cap X_t} \nu_t = \int |f|^2 \prod_{j \in J} |z_j|^{2b_j} \left| d \log z_1 \wedge ... \wedge d \log z_n \right|^2,$$

where the second integral is taken over a domain of the form $\sum_{j \in J} a_j \log|z_j| = \log|t|$ inside $\mathbb{C}^{n+1}$.

We thus work in logarithmic polar coordinates, and write $z_j = \exp(w_j + i\theta_j)$. This yields, after integrating over the angle directions and the coordinates not in J:

$$\int_{\mathscr{U}\cap X_t} \nu_t = \int_{\tau_t} F \prod_{j\in J} e^{b_j w_j} dw_j,$$

where $\tau_t = \{\sum_{j\in J} a_j w_j = -\log|t|\}$ and F is a non-vanishing, bounded continuous function. It then follows from the change of variable $w'_j = a_j w_j$ that as $t \to 0$, we have:

$$\int_{\mathscr{U}\cap X_t} \nu_t = (\log|t|^{-1})^{d(J)} \times O(1),$$

where $d(J) = \big|\{j \in J / b_j = 0\}\big| - 1$. This leads to the following definition:

Definition 3.1.10. *Let $\mathscr{X}$ be an snc model of the Calabi-Yau variety X/K, and write $\mathscr{X}_0 = \sum_{i\in I} a_i D_i$, $K^{\log}_{\mathscr{X}/\mathbb{D}} = \sum_{i\in I} b_i D_i$. For $i \in I$, set $\kappa_i = \frac{b_i}{a_i}$, and let $\kappa_{\min} = \min_{i\in I} \kappa_i$.*
A face $\tau \subset \mathrm{Sk}(\mathscr{X})$ is essential *if $\kappa_i = \kappa_{\min}$ for each vertex $v_i \in \tau$; we define the essential skeleton $\mathrm{Sk}(X) \subset \mathrm{Sk}(\mathscr{X})$ as the union of the essential faces of $\mathrm{Sk}(\mathscr{X})$, it is a subcomplex of dimension $d \leq n$.*

As suggested by the notation, the essential skeleton $\mathrm{Sk}(X)$ turns out to be independent of the choice of snc model $\mathscr{X}$, by [KS06, thm. 3]. The essential skeleton can also be defined more intrinsically as the locus where the weight function associated with a non-vanishing section $\Omega \in H^0(X, K_X)$, $\mathrm{wt}_\Omega : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ defined in [MN15] reaches its minimum; given an snc model $\mathscr{X}/R$ and a vertex $v_i \in \mathrm{Sk}(\mathscr{X})$, we simply have $\mathrm{wt}_\Omega(v_i) = \frac{b_i}{a_i}$ with the notation from the previous computation. We refer the reader to [MN15] and [NX16] for details.
We now have the following result, which computes the weak limit on the Calabi-Yau measures inside the hybrid space X^{hyb}. The support of the limiting measure is precisely the essential skeleton:

Theorem 3.1.11. *[BJ17, thm. B]*
Let $X \longrightarrow \mathbb{D}^*$ be a degeneration of Calabi-Yau manifolds, and $\Omega \in H^0(X, K_{X/\mathbb{D}^*})$ a relative trivialization of the canonical bundle. Writing $\nu_t = i^{n^2}\Omega_t \wedge \bar{\Omega}_t$, we have:

$$\nu_t(X_t) \sim C|t|^{\kappa}(\log|t|^{-1})^d,$$

as $t \to 0$, where $\kappa \in \mathbb{Q}$ and $d = \dim \mathrm{Sk}(X) \leq n$. Moreover, the family of rescaled measures:

$$\mu_t = \frac{\nu_t}{\nu_t(X_t)}$$

converge weakly on X^{hyb} to a Lebesgue-type measure supported on $\mathrm{Sk}(X) \subset X^{\mathrm{an}}$.

Here by Lebesgue-type measure we mean a weighted sum of Lebesgue measures on the top-dimensional faces - in fact when X has semi-stable reduction the weights are all equal.
This result provides heuristic explanation as to why the base B of the SYZ fibration should be the essential skeleton $\mathrm{Sk}(X)$: since the Calabi-Yau metric is obtained by

solving the complex Monge-Ampère equation $(\omega_t)^n = C_t\mu_t$, one would expect the solution to the non-archimedean Monge-Ampère equation $\mathrm{MA}(\phi) = \mu_0$ provided by theorem 1.4.7 to provide a non-archimedean incarnation of the family of Calabi-Yau metrics; and since μ_0 is supported on $\mathrm{Sk}(X)$ one would hope to translate this equation into a real Monge-Ampère equation on $\mathrm{Sk}(X)$, so that the base B appearing in conjecture 3.1.9 is $\mathrm{Sk}(X)$. We will come back to this in section 3.1.4.
As we shall now explain, there exists a class of R-models of X whose skeleta realize the essential skeleton. Those are the minimal models in the sense of MMP:

Definition 3.1.12. *Let X/K be a Calabi–Yau variety. A minimal model of X is a good dlt (def. 1.1.11) model $\mathscr{X}/R$, such that the logarithmic relative canonical divisor is trivial, i.e.*

$$K^{\log}_{\mathscr{X}/R} := K_{\mathscr{X}/R} + \mathscr{X}_{0,\mathrm{red}} - \mathscr{X}_0 \sim \mathcal{O}_{\mathscr{X}}.$$

Building on the general MMP machinery, the existence of such models is known when X is defined over an algebraic curve (and is expected to hold in the general case):

Theorem 3.1.13 ([NXY19, Theorem 1.13])**.** *Let X/K be a projective Calabi–Yau variety, and assume that X is defined over an algebraic curve. Then there exists a minimal model $\mathscr{X}/R$ of X. Furthermore, there exists a finite extension K'/K such that the base change $X_{K'}$ admits a minimal model with reduced special fiber.*

Such models are however not unique, but they turn out to have the same skeleton inside X^{an} by [NX16] (even though the triangulation may differ), which is in fact the essential skeleton of X:

Theorem 3.1.14. *[NX16, thm. 3.3.3] Let X/K be a Calabi–Yau variety, and let $\mathscr{X}/R$ be a minimal dlt model of X. Then the skeleton $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ is the essential skeleton of X.*

Note that if the minimal model is an snc model, then this is a straightforward consequence of definition 3.1.10.

3.1.3 Affinoid torus fibrations

From now on, we will assume that $X/\mathbb{D}^*$ is a maximal degeneration of Calabi-Yau manifolds. Then the metric spaces $(X_t, \tilde{\omega}_t)$ are conjectured to "look like" the total space of a special Lagrangian torus fibration over $\mathrm{Sk}(X)$, submersive away from a singular locus of real codimension 2; the affine structure on the base being induced by action-angle coordinates. Building on these considerations, it is reasonable to expect that the essential skeleton $\mathrm{Sk}(X)$ should be equipped with an integral affine structure, singular in real codimension ≥ 2. We will now explain how to try and construct such an affine structure in a purely non-archimedean setting, we write X^{an} the Berkovich analytification of X viewed as a variety over $K = \mathbb{C}((t))$.
As we saw in section 3.1.1, our local semi-flat model for the SYZ fibration is given

by $f : \mathbb{T} \longrightarrow N_{\mathbb{R}}$, the map f being simply $f(z_1, ..., z_n) = (-\log|z_1|, ..., -\log|z_n|)$. The non-archimedean counterpart of this map is the following:

$$\mathrm{val} : \mathbb{T}^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}.$$

This leads to the following definition:

Definition 3.1.15. *Let $\rho : X^{\mathrm{an}} \longrightarrow B$ be a continuous map to a topological space B. For any point $b \in B$, we say that ρ is an* affinoid torus fibration *at b if there exists an open neighbourhood U of b in B, such that the restriction to $\rho^{-1}(U)$ fits into a commutative diagram:*

$$\begin{array}{ccc} \rho^{-1}(U) & \xrightarrow{\simeq} & \mathrm{val}^{-1}(V) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{val}} \\ U & \xrightarrow{\sim} & V, \end{array}$$

V being an open subset of $\mathbb{R}^n$, the upper horizontal map an isomorphism of analytic spaces, the lower horizontal map a homeomorphism, and the map val *defined as in section 2.2.1.*

Note that the above definition implies that B is a topological manifold at b. In the case where $\mathscr{X}$ is a good dlt model of X, the base $B = \mathrm{Sk}(\mathscr{X})$ and ρ is the Berkovich retraction $\rho_{\mathscr{X}}$, this does not necessarily hold at every point of $\mathrm{Sk}(\mathscr{X})$. Given a continuous map $\rho : X^{\mathrm{an}} \longrightarrow B$, we denote by B^{sm} the locus of points in B where ρ is an affinoid torus fibration at; we call $B \setminus B^{\mathrm{sm}}$ the *discriminant* or singular locus of ρ. As in the archimedean case, the (possibly disconnected) topological manifold B^{sm} is endowed with an integral affine structure, which is the pull-back of $\mathrm{Aff}_{\mathbb{R}^n}$ via the charts in 3.1.15. An alternative, more intrinsic description of this structure is given in [KS06, 4.1, Theorem 1]: let $U \subset B^{\mathrm{sm}}$ be a connected open subset. Then if h is an invertible analytic function on $\rho^{-1}(U)$, its modulus $|h|$ is constant on the fibers of ρ by the maximum principle, so that it defines a continuous function on the base. We now have:

$$\mathrm{Aff}_{B^{\mathrm{sm}}}(U) = \{-\log|h| \,|\, h \in \mathcal{O}^{\times}_{X^{\mathrm{an}}}(\rho^{-1}(U))\}.$$

Example 3.1.16. *If $\mathscr{X}/R$ is a good dlt model of X with reduced special fiber, then the Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$ is an affinoid torus fibration over the interior of the maximal faces τ of $\mathrm{Sk}(\mathscr{X})$. Indeed, the retraction over $\mathring{\tau}$ only depends on the formal completion of $\mathscr{X}$ along the corresponding 0-dimensional stratum p, and we have that $\rho_{\mathscr{X}}^{-1}(\mathring{\tau})$ is the generic fiber of $\mathrm{Spf}\,\widehat{\mathcal{O}}_{\mathscr{X},p}$. By the dlt condition, the pair $(\mathscr{X}, \mathscr{X}_k)$ is snc at p, so that:*

$$\widehat{\mathcal{O}}_{\mathscr{X},p} \simeq R[[z_0, ..., z_n]]/(t - z_0...z_n).$$

The generic fiber $\mathfrak{X}_p^{\eta}$ embeds in the torus:

$$\mathbb{T}^{\mathrm{an}} = \left(\mathrm{Spec}\, K[z_0, ..., z_n]/\{t = z_0...z_n\}\right)^{\mathrm{an}},$$

and the Berkovich retraction is the restriction of the val *map over the open simplex:*

$$\mathrm{Int}(\tau) = \{x_0 + ... + x_n = 1\} \subset (\mathbb{R}_{>0})^{n+1}.$$

Definition 3.1.17. *Let X/K be a smooth projective variety, $B = \mathrm{Sk}(\mathscr{X})$ the skeleton of a dlt R-model of X. We say a continuous map:*

$$\rho : X^{\mathrm{an}} \longrightarrow B$$

is an admissible retraction if the following hold:

- *there exists a piecewise-affine locus $\Gamma \subset B$ of codimension ≥ 2 such that ρ is an affinoid torus fibration over $\mathrm{Sk}(X) \setminus \Gamma$;*
- *the retraction ρ is piecewise-affine: for any piecewise-affine function $\psi : B \longrightarrow \mathbb{R}$, the pullback $\psi \circ \rho : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ is a piecewise-affine function on X^{an} (see defn. 1.1.21).*

One way to produce a map satisfying the second item of the definition is to choose an snc model $\mathscr{X}' \longrightarrow \mathscr{X}$ dominating $\mathscr{X}$, and a piecewise-affine map $\pi : \mathrm{Sk}(\mathscr{X}') \longrightarrow \mathrm{Sk}(\mathscr{X})$. Then the retraction $\rho = \pi \circ \rho_{\mathscr{X}'}$ is piecewise-affine, it would be interesting to know whether or not all piecewise-affine maps $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ arise this way.
As suggested by the definition, affinoid torus fibrations are related to the local toric geometry of X and its models. We now describe a more global class of examples, that we will be using as building blocks later on.
Let $\mathscr{X}/R$ be a regular toric model of $\mathbb{T}$, i.e. a regular toric R-scheme such that $\mathscr{X} \times_R \mathrm{Spec}\, K = \mathbb{T}$, which we assume to have reduced special fiber. Such a model is described by a regular fan $\hat{\Sigma} \subset \hat{N}_{\mathbb{R}} := N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$, whose cones intersect $N_{\mathbb{R}} \times \{0\}$ only at the origin.
We consider the following closed subset of $\mathbb{T}^{\mathrm{an}}$:

$$\widehat{\mathscr{X}_\eta} := \{v_x \in \mathbb{T}^{\mathrm{an}} |\, v_x \text{ has a center on } \mathscr{X}\},$$

which admits a Berkovich retraction:

$$\rho_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \longrightarrow \mathrm{Sk}(\mathscr{X})$$

defined as in remark 1.1.18. In this case, the map $\rho_{\mathscr{X}}$ can be described explicitly as follows: let Σ_1 be the polyhedral complex in $N_{\mathbb{R}}$ obtained by intersecting the fan $\hat{\Sigma}$ with $N_{\mathbb{R}} \times \{1\}$. There is a natural identification between Σ_1 and $\mathcal{D}(\mathscr{X}_0)$, sending a vertex of $\mathscr{D}(\mathscr{X}_0)$ to the primitive generator of the corresponding ray of $\hat{\Sigma}$, and then extending on each face by linearity. Moreover, it follows from [GJKM19, Theorem A.4] that $\gamma(|\Sigma_1|) = \mathrm{Sk}(\mathscr{X}) \subset \mathbb{T}^{\mathrm{an}}$, where $\gamma : N_{\mathbb{R}} \longrightarrow \mathbb{T}^{\mathrm{an}}$ is the Gauss embedding from definition 2.2.2.

Proposition 3.1.18 ([NXY19, Example 3.5])**.** *The equality:*

$$\widehat{\mathscr{X}_\eta} = \mathrm{val}^{-1}(|\Sigma_1|)$$

holds, and $\rho_{\mathscr{X}} = \mathrm{val}_{|\widehat{\mathscr{X}_\eta}}$, hence is an affinoid torus fibration over the relative interior of $\mathrm{Sk}(\mathscr{X})$.

This also holds when X is a regular proper toric variety over K and $\mathscr{X}/R$ a regular proper toric model of X, by [GJKM19, Theorem A.4].

Proof. We start by proving the first equality. Let $x \in \mathbb{T}^{\mathrm{an}}$, we know from lemma 3.1.19 below that x has a center on $\mathscr{X}$ if and only if $\gamma(\mathrm{val}(x))$ has a center on $\mathscr{X}$. Thus, it is enough to prove that for $n \in N_{\mathbb{R}}$ and $y = \gamma(n)$, y has a center on $\mathscr{X}$ if and only if $n \in |\Sigma_1|$.
By [GJKM19, Lemma A.1], if $y \in \gamma(N_{\mathbb{R}})$ has a center on $\mathscr{X}$, it must be the closure of a torus orbit $Y \subset \mathscr{X}_0$. By [KKMSD73, Theorem 6], there exists a cone $\sigma \in \hat{\Sigma}$ such that the generic point of Y is contained in the associated toric affine chart $\mathscr{X}_\sigma = \operatorname{Spec} R[\check{\sigma} \cap \hat{M}]$. In particular, for any monomial z^m that is regular on $\mathscr{X}_\sigma$, we have $v_y(z^m) \geq 0$. In other words, writing $y = \gamma(n)$, we have $\langle n, m\rangle \geq 0$ for all $m \in \check{\sigma}$, so that $n \in \sigma$. Since $v_y(t) = 1$, $y \in \gamma(|\Sigma_1|)$.
By the same argument, if $n \in |\Sigma_1|$, there exists a cone σ such that $n \in \sigma$, which means that $v_{\gamma(n)}$ has positive value on each monomial $m \in \check{\sigma}$, and thus has a center on $\mathscr{X}_\sigma$ and in particular on $\mathscr{X}$.

To prove the second equality, since $\rho_{\mathscr{X}}$ is the identity on $\gamma(|\Sigma_1|) = \mathrm{Sk}(\mathscr{X})$, we merely have to prove that $\rho_{\mathscr{X}} = \rho_{\mathscr{X}} \circ \mathrm{val}$. However this follows directly from the definition of $\rho_{\mathscr{X}}$, and the fact that $c_{\mathscr{X}}(x) \in \overline{c_{\mathscr{X}}(\gamma(\mathrm{val}(x)))}$ for $x \in \widehat{\mathscr{X}_\eta}$ by lemma 3.1.19. Indeed, $\rho_{\mathscr{X}}(x)$ only depends on the values $v_x(z)$, where z is a local equation for a component of $\mathscr{X}_0$ at $c_{\mathscr{X}}(x)$. Since $\mathscr{X}$ is a toric model, these local equations can be taken to be monomials, so that the result follows from the fact that x and $\gamma(\mathrm{val}(x))$ take the same values on monomials. □

Lemma 3.1.19. *Let $x \in \mathbb{T}^{\mathrm{an}}$. Then x has a center on $\mathscr{X}$ if and only if $\gamma(\mathrm{val}(x))$ has a center on $\mathscr{X}$. Moreover, if this holds, we have $c_{\mathscr{X}}(x) \in \overline{c_{\mathscr{X}}(\gamma(\mathrm{val}(x)))}$.*

Proof. Let $\mathscr{X} \subset \bar{\mathscr{X}}$ be a toric compactification of $\mathscr{X}$, i.e. a proper toric R-scheme containing $\mathscr{X}$ as a torus-invariant open subset. By the valuative criterion of properness, any valuation on $\mathbb{T}^{\mathrm{an}}$ has a center on $\bar{\mathscr{X}}$. We write $c_{\bar{\mathscr{X}}}(x)$ for the center of $x \in \mathbb{T}^{\mathrm{an}}$. We start by proving that $c_{\bar{\mathscr{X}}}(\gamma(\mathrm{val}(x))$ is the generic point of the closure Z of the torus orbit $O(\sigma)$ in $\bar{\mathscr{X}}$ containing $c_{\bar{\mathscr{X}}}(x)$ (in particular, $c_{\bar{\mathscr{X}}}(x)$ must be contained in the toric interior of Z). We may work on the toric affine chart $\bar{\mathscr{X}}_\sigma = \operatorname{Spec} R[\check{\sigma} \cap \hat{M}]$ associated with Z. Since the valuation $\gamma(\mathrm{val}(x))$ is monomial, it is enough to prove that $\gamma(\mathrm{val}(x))(z^m) = v_x(z^m) \geq 0$ for $m \in \check{\sigma} \cap M$ and that $\gamma(\mathrm{val}(x))(z) > 0$ for z a local equation of any torus invariant divisor containing Z, to have that $c_{\bar{\mathscr{X}}}(\gamma(\mathrm{val}(x)))$ lies in Z. Since z^m is regular on $\bar{\mathscr{X}}_\sigma$, the first condition holds; the local equation z is monomial and $\gamma(\mathrm{val}(x))(z) = v_x(z) > 0$ since Z contains $c_{\bar{\mathscr{X}}}(x)$.

Now assume that v_x is centered on $\mathscr{X}$, i.e. $c_{\bar{\mathscr{X}}}(x) \in \mathscr{X}$. Since $c_{\bar{\mathscr{X}}}(x)$ is contained in the interior of the toric stratum Z and lies in $\mathscr{X}$, so does the generic point of Z, i.e. $c_{\bar{\mathscr{X}}}(\gamma(\mathrm{val}(x))) \in \mathscr{X}$. This implies that $\gamma(\mathrm{val}(x))$ has a center on $\mathscr{X}$. Conversely, if $\gamma(\mathrm{val}(x))$ has a center Z on $\mathscr{X}$, then $Z \subset \mathscr{X}$ and $c_{\bar{\mathscr{X}}}(x) \in Z$ as mentioned above; thus, $c_{\bar{\mathscr{X}}}(x) \in \mathscr{X}$, which concludes the proof. □

As our main interest lies in degenerations of Calabi-Yau manifolds, the above class of examples is too restrictive for us. We are thus naturally lead to consider degenerations which are only locally formally toric, so that we may apply remark 1.1.18 to produce Berkovich retractions which are locally affinoid torus fibrations.
Let us make this more precise. Following [NXY19], we will say that an R-scheme of finite type $\mathscr{Z}$ is toric if there exists a toric k-scheme of finite type $\mathcal{Z}$, together

with a toric morphism $t : \mathcal{Z} \longrightarrow \mathbb{A}^1_k$, such that $\mathscr{Z} \simeq \mathcal{Z} \times_{\mathbb{A}^1} R$. Writing $\widehat{N}$ for the lattice of 1-parameter subgroups of the torus of $\mathcal{Z}$, such a scheme is described by a fan $\widehat{\Sigma}$ in $\widehat{N}_{\mathbb{R}}$, together with a linear map $\mathrm{ord}(t) : |\widehat{\Sigma}| \longrightarrow \mathbb{R}_{\geq 0}$, defined by $\mathrm{ord}(t)(n) = \mathrm{ord}_0(t \circ n)$ for a 1-parameter subgroup $n : \mathbb{G}_m \to \mathcal{Z}$. Note that the map $\mathrm{ord}(t)$ recovers the function t uniquely, since it is a monomial.

Definition 3.1.20. *Let $\mathscr{X}$ be a normal R-scheme of finite type, such that $(\mathscr{X}, \mathscr{X}_0)$ is a good dlt pair, and let Y be a stratum of $\mathscr{X}_0$. We say that $\mathscr{X}$ is* toric along Y *if there exists a toric R-scheme $\mathscr{Z}$, a stratum W of $\mathscr{Z}_0$ and a formal isomorphism over R*

$$\widehat{\mathscr{X}_{/Y}} \simeq \widehat{\mathscr{Z}_{/W}}.$$

The upshot is now that if $\mathscr{X}$ is toric along Y, then the Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ is an affinoid torus fibration over $\mathrm{Star}(\tau_Y)$. Although this follows from [NXY19, Theorem 6.1] (end of the proof) and [NXY19, §3.4], we sketch an argument for reader's convenience.
As mentioned in remark 1.1.18, the retraction $\rho_{\mathscr{X}}$ over $\mathrm{Star}(\tau_Z)$ only depends on $\widehat{\mathscr{X}_{/Z}}$, so that we may assume that $\mathscr{X}$ is a toric R-scheme. The equality $\rho_{\mathscr{X}} = \mathrm{val}$ now holds over $\mathrm{Star}(\tau_Z)$ by 3.1.18, so that it follows from 3.1.15 that $\rho_{\mathscr{X}}$ is an affinoid fibration over $\mathrm{Star}(\tau_Z)$.
An obvious necessary condition for $\mathscr{X}$ to be toric along Y is for Y to be a toric variety itself; our next result (joint with E. Mazzon) states that under a positivity assumption on the conormal bundle of $Y \subset \mathscr{X}$, the converse holds:

Theorem 3.1.21. *[MPS21, thm. B] Let X/K be a smooth projective variety of dimension n, and $\mathscr{X}/R$ be a dlt model of X with reduced special fiber $\mathscr{X}_0 = \sum_\alpha D_\alpha$, such that every D_α is a Cartier divisor.*
Let $Z = D_0 \cap D_1 \cap \ldots \cap D_{n-r}$ be an r-dimensional stratum of $\mathscr{X}_k$, such that:

- *Z is a proper toric variety with toric boundary $\Delta_Z = \sum_{\alpha \,|\, Z \not\subset D_\alpha} Z \cap D_\alpha$;*
- *the conormal bundle $\nu^*_{Z/\mathscr{X}}$ is a nef vector bundle on Z;*
- *for each $\alpha \notin \{0, ..., n-r\}$, the intersection $D_\alpha \cap Z$ is either empty or connected.*

Then the formal completion $\widehat{\mathscr{X}_{/Z}}$ is isomorphic to the formal completion of the normal bundle $\mathcal{N} = \nu_{Z/\mathscr{X}}$ along the zero section. In particular, $\mathscr{X}$ is toric along Z (in the sense of definition 3.1.20).

The proof of the above theorem will be provided in section 3.2. As an immediate consequence of the discussing above, we have

Corollary 3.1.22. *The retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$ is an n-dimensional affinoid torus fibration over $\mathrm{Star}(\tau_Z)$. In particular, the integral affine structure induced by $\rho_{\mathscr{X}}$ over the interior of the maximal faces of $\mathrm{Sk}(\mathscr{X})$ extends to $\mathrm{Star}(\tau_Z)$ with no singularities.*

Example 3.1.23. *Let X/K be a smooth, maximally degenerate Calabi-Yau manifold, and $\mathscr{X}/R$ be a good minimal dlt model of X with reduced special fiber. Then [NXY19, Theorem 6.1] asserts that the Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X}) =$*

$\mathrm{Sk}(X)$ is an affinoid torus fibration away from the codimension 2 locus of the triangulation.
This statement is proved by showing that $\mathscr{X}$ is toric along the 1-dimensional strata of the special fiber - which are rational curves, hence toric - which yields on the way a complete description of such models along these strata. In this very case, the positivity assumption on the conormal bundle can in fact always be achieved via blowing-up certain zero-dimensional strata, so that thm. 5.1.1 can be seen as a generalization to higher-dimensional strata of [NXY19, prop. 5.4]. This description also provides a way to compute the singular $\mathbb{Z}$-affine structure induced on $\mathrm{Sk}(X)$, as well as its monodromy. In sections 3.3.2 and 4.2.1 we give more details and examples of such computations.

3.1.4 The Monge-Ampère comparison property

In this section, we explain the expected connection between the non-archimedean Monge-Ampère operator and the real Monge-Ampère operator on skeleta.
Let N be a lattice, $\phi : U \longrightarrow \mathbb{R}$ a convex function defined on an open subset $U \subset N_{\mathbb{R}}$, and let $d\lambda$ be the Lebesgue measure on the dual $M_{\mathbb{R}} := \mathrm{Hom}(N, \mathbb{R})$.
For $x_0 \in U$, we define the gradient image of ϕ at x_0:

$$\nabla\phi(x_0) = \{l \in M_{\mathbb{R}} / \phi(x_0) + l(x - x_0) \leq \phi(x)\ \forall x \in U\}.$$

Geometrically, this is the set of covectors cutting out affine hyperplanes in $N_{\mathbb{R}}$ that meet the graph Γ of ϕ at x_0, and are below Γ on all of U.
For instance, if ϕ has $\mathcal{C}^1$-regularity, it follows immediately from the convexity of ϕ that the gradient image of ϕ at x_0 only contains the differential $d\phi_{x_0}$ of ϕ in the usual sense.
More generally, if $E \subset U$ is a Borel set, we set $\nabla\phi(E) = \cup_{x_0 \in E} \nabla\phi(x_0)$.

Definition 3.1.24. *Let $\phi : U \longrightarrow \mathbb{R}$ be a convex function defined on an open subset $U \subset N_{\mathbb{R}}$.*
The real Monge-Ampère measure of ϕ is defined as:

$$\mathscr{M}(\phi)(E) = d\lambda(\nabla\phi(E)),$$

for any Borel set $E \subset U$.

If ϕ has $\mathcal{C}^2$-regularity, then this is nothing but the density measure $\det(\nabla^2\phi)d\lambda^\vee$. Recall that in the semi-flat setting of section 3.1.1, the Kähler Ricci-flat metric ω is given by:

$$\omega = dd^c(K \circ f)$$

for a locally defined convex function $K : B^{\mathrm{sm}} \longrightarrow \mathbb{R}$, and $f : X \longrightarrow B$ the special Lagrangian fibration. The Calabi-Yau condition was then translated into the real Monge-Ampère equation for K:

$$\det\left(\frac{\partial^2 K}{\partial y_i \partial y_j}\right) = C,$$

with $C > 0$ a real constant.
Thus, the semi-flatness condition can be expressed as the fact that the local potential for the Calabi-Yau metric is constant along the fibers of the SYZ map, which reduces the complex Monge-Ampère equation to a real Monge-Ampère equation on the base of the fibration.
We now move back to the non-archimedean picture. In this setting, the analog of the Calabi-Yau metric is the solution $\phi \in \mathrm{CPSH}(L^{\mathrm{an}})$ to the non-archimedean Monge-Ampère equation:

$$\mathrm{MA}(\phi) = \mu_0,$$

where μ_0 is the limit measure from thm. 3.1.11 - when X has semi-stable reduction, this is simply the Lebesgue measure on the essential skeleton $\mathrm{Sk}(X)$.

Definition 3.1.25. *Let X/K be a smooth projective variety over K, $\mathscr{X}/R$ a dlt model of X, and let $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ be an admissible retraction (defn. 3.1.17). We say that a continuous, semi-positive metric ϕ on L is a* semi-flat *metric on X^{an} with respect to ρ if it restricts to a toric metric over any open subset of $\mathrm{Sk}(\mathscr{X})$ over which ρ is an affinoid torus fibration.*

A bit more precisely, this means that after pulling back L^{an} to the open $\mathrm{val}^{-1}(V)$ as in defn. 3.1.15 and choosing a trivialization, the metric ϕ extends to $\mathbb{T}^{\mathrm{an}}$ as a toric metric.
When ϕ is a toric metric, it follows from [BGPS14] that the non-archimedean Monge-Ampère equation can be reduced to a real Monge-Ampère equation:

Theorem 3.1.26. *[BGPS14] Let (Z, L) be a polarized toric variety over K.*
Let ϕ be a semi-positive toric metric on L, given by a convex function which we denote by $\Phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}$.
Then the non-archimedean Monge-Ampère measure of the metric ϕ can be computed as follows:

$$\mathrm{MA}(\phi) = n! \iota_*(\mathscr{M}(\Phi)),$$

where $\iota : N_{\mathbb{R}} \hookrightarrow Z^{\mathrm{an}}$ is the embedding given by the Gauss section.

As a result, by locality of the non-archimedean Monge-Ampère operator (see [CLD12], [BFJ15, cor. 5.2]) and the argument of [Vil20, cor. 5.10], if ϕ is a semi-flat metric with respect to an admissible retraction ρ, this implies that away from the discriminant locus of ρ, the non-archimedean Monge-Ampère measure $\mathrm{MA}(\phi)$ is supported on $\mathrm{Sk}(\mathscr{X})$, and can be expressed locally as the real Monge-Ampère measure on $\mathrm{Sk}(\mathscr{X})$ of locally defined convex functions - which may be however fairly hard to compute explicitly in terms of the metric ϕ.
In general, one would hope to write $\phi = \phi_{\mathscr{L}} + \psi$ for a certain model metric $\phi_{\mathscr{L}}$ on L, and $\psi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ a continuous function, and be able to rephrase the semi-flat condition into the invariance property:

$$\psi = \psi \circ \rho$$

over $\mathrm{Sk}(\mathscr{X}) \setminus \Gamma$. We would then obtain, locally in toric charts, an equality of measures of the form:

$$\mathrm{MA}(\phi) = n! \iota_*(\mathscr{M}(\Phi_{\mathscr{L}} + \psi)),$$

where $\Phi_{\mathscr{L}}$ is a local toric potential for the model metric $\phi_{\mathscr{L}}$. While it is unclear how to do this in general because it depends on the explicit description of the toric charts for the retraction, we are able to work this out in the case of degenerations of hypersurfaces, see remark 4.3.7.
If $\rho = \rho_{\mathscr{X}}$ is the Berkovich retraction associated to an snc model $\mathscr{X}/R$ and we restrict our attention to the interior of the maximal faces of $\mathrm{Sk}(\mathscr{X})$, then things get simpler and we have the following result:

Theorem 3.1.27. *[Vil20] Let (X, L) be a smooth projective variety over K, and $(\mathscr{X}, \mathscr{L})$ be a semi-stable R-model of (X, L). Let $\phi \in \mathrm{CPSH}(X, L)$ such that:*

$$\phi = \phi_{\mathscr{L}} + \psi,$$

where $\psi \in \mathcal{C}^0(X^{\mathrm{an}})$. Let τ be a maximal face of $\mathrm{Sk}(\mathscr{X})$, and assume that the invariance property $\psi = \psi \circ \rho_{\mathscr{X}}$ holds over $\mathrm{Int}(\tau)$. Then we have the equality of measures:

$$\mathbf{1}_{\mathrm{Int}(\tau)}\,\mathrm{MA}(\phi) = n!\mathscr{M}(\psi_{|\,\mathrm{Int}(\tau)}).$$

We now let X/K be a maximal degeneration of CY manifolds, polarized by the ample line bundle L. We write $\phi \in \mathrm{PSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$ the unique solution to the NA Monge-Ampère equation:

$$\mathrm{MA}(\phi) = \mu_0$$

Definition 3.1.28. *(Weak comparison property, [Li20a, def. 3.11]) We say that ϕ satisfies the weak NAMA/real MA comparison property if there exists an snc model $\mathscr{X}/R$ of X, together with a model $\mathscr{L}$ of L on $\mathscr{X}$, such that the function $\psi : X^{\mathrm{an}} \longrightarrow \mathbb{R}$ defined by the formula:*

$$\phi = \phi_{\mathscr{L}} + \psi$$

satisfies $\psi = \psi \circ \rho_{\mathscr{X}}$ over the interior of the maximal faces of $\mathrm{Sk}(\mathscr{X})$.

The motivation for this definition resides in the following theorem, which proves that the above condition is sufficient for the weak SYZ conjecture to hold:

Theorem 3.1.29. *[Li20a, thm. 1.3] Let $X \longrightarrow \mathbb{D}^*$ be maximal degeneration of CY manifolds, polarized by an ample line bundle L. Assume the NAMA/real MA comparison property holds.*
Then for any $\delta > 0$, there exists ε_δ such that for all $|t| \leq \varepsilon_\delta$, there exists a special Lagragian torus fibration on an open subset of X_t of (normalized) Calabi-Yau measure greater than $(1 - \delta)$.

Very loosely, the idea of the proof of the above theorem is that the solution to the non-archimedean Monge-Ampère equation ϕ provides a solution to the real Monge-Ampère equation on an open region of $\mathrm{Sk}(X)$ that has full measure; and pulling back this solution by the map $\mathrm{Log}_{\mathscr{X}}$ from prop. 1.1.26 provides an ansatz for the Calabi-Yau potentials on a region of the X_t with asymptotically full measure: if $K \subset \mathrm{Int}(\tau)$ is a compact subset, then Li obtains bounds for $|\phi_{CY,t} - \phi \circ \mathrm{Log}_{\mathscr{X}}|$ over K using uniform Skoda estimates on the highly degenerate Calabi-Yau manifolds

X_t [Li20b]. The estimates on the potential are then enough to prove the existence of a special Lagrangian fibration over $\mathrm{Log}_{\mathscr{X}}^{-1}(K)$ which is a perturbation of $\mathrm{Log}_{\mathscr{X}}$, by the results of Y. Zhang [Zha17].
While the above theorem provides us with a asymptotic SYZ fibration, it does not describe completely the Gromov-Hausdorff limit of the family: we only obtain that the disjoint union of maximal faces of $\mathrm{Sk}(X)$ endowed with a real Monge-Ampère metric embeds into the Gromov-Hausdorff limit. We expect that in order to obtain Gromov-Hausdorff convergence to the metric completion of $\mathrm{Sk}(X) \setminus \Gamma$ for a piecewise-affine locus Γ of codimension 2, we need a stronger version of the comparison property in order to obtain a solution to the real Monge-Ampère equation in codimension 1:

Definition 3.1.30. *(Strong comparison property) Let X/K be a smooth projective, maximally degenerate Calabi-Yau variety, polarized by an ample line bundle L.*
Let ϕ be the unique positive metric on L satisfying the non-archimedean Monge-Ampère equation:

$$\mathrm{MA}(\phi) = \mu_0.$$

We say that ϕ satisfies the strong comparison property if there exists an admissible retraction $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ such that ϕ is a semi-flat metric with respect to ρ.

Assume that the answer to the above question is positive, and write $\Gamma \subset \mathrm{Sk}(X)$ the discriminant locus of ρ. Then $\big(\mathrm{Sk}(X) \setminus \Gamma\big) = \bigcup_{\alpha \in A} U_\alpha$ such that ρ is an affinoid torus fibration over U_α, we write $\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}$ the convex, toric potential for ϕ on U_α. Then by the previous discussion there is an equality of measures on U_α:

$$\mathbf{1}_{U_\alpha}\mu_0 = n!\mathscr{M}(\phi_\alpha),$$

where the real Monge-Ampère measure on the right-hand side is computed with respect to the integral affine structure induced by ρ. Moreover, for $\alpha \neq \beta$, the difference $\phi_\alpha - \phi_\beta$ must be affine on $U_\alpha \cap U_\beta$, so that we may define locally $g = \nabla^2\phi := \sum_{i,j} \frac{\partial^2 \phi_\alpha}{\partial y_i \partial y_j} dy_i dy_j$ where the y_i are affine coordinates on U_α.

Conjecture 3.1.31. *(Kontsevich-Soibelman conjecture) Let (X, L) be a polarized, maximal degeneration of Calabi-Yau manifolds. Then there exists a unique admissible retraction:*

$$\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X),$$

with discriminant locus $\Gamma \subset \mathrm{Sk}(X)$, such that the solution ϕ to the non-archimedean Monge-Ampère equation:

$$\mathrm{MA}(\phi) = \mu_0$$

is a semi-flat metric with respect to ρ. Additionally, the collection of local potentials $\phi = (\phi_\alpha)_\alpha$ is a smooth, strictly convex solution of the real Monge-Ampère equation on $\mathrm{Sk}(X) \setminus \Gamma$.
Moreover, the rescaled Calabi-Yau metrics $(X_t, \tilde{\omega}_t)$ converge in the Gromov-Hausdorff sense as $t \to 0$ to a compact n-dimensional base (B, g_B), such that:

- *(B, g_B) contains the smooth Riemannian manifold $(\mathrm{Sk}(X) \setminus \Gamma, \nabla^2\phi)$, whose complement is of Hausdorff codimension at least 2,*

- *B is homeomorphic to* $\mathrm{Sk}(X)$.

Building on the results from [Li22], we are able to prove the above conjecture for degenerations of Fermat hypersurfaces, see section 4.3.
Let us point out that assuming the strong comparison property, the strict convexity of the local potential ϕ can not be deduced from a local PDE argument: there are non-smooth, non-strictly convex solutions to real Monge-Ampère equations of the form $\mathscr{M}(\phi) = d\lambda$ [Moo15], so that higher regularity of the solution would have to be obtained through a 'global' argument.

3.2 Proof of theorem 3.1.21

In this section we prove theorem 3.1.21. We recall the statement and fix the notation.

Theorem 3.2.1. *Let X/K be a smooth projective variety of dimension n, and $\mathscr{X}/R$ be a dlt model of X with reduced special fiber $\mathscr{X}_0 = \sum_\alpha D_\alpha$, such that every D_α is a Cartier divisor.*
Let $Z = D_0 \cap D_1 \cap \ldots \cap D_{n-r}$ be an r-dimensional stratum of $\mathscr{X}_k$, such that:

- *Z is a proper toric variety with toric boundary $\Delta_Z = \sum_{\alpha \,|\, Z \not\subset D_\alpha} Z \cap D_\alpha$;*
- *the conormal bundle $\nu^*_{Z/\mathscr{X}}$ is a nef vector bundle on Z;*
- *for each $\alpha \notin \{0, ..., n-r\}$, the intersection $D_\alpha \cap Z$ is either empty or connected.*

Then the formal completion $\widehat{\mathscr{X}_{/Z}}$ is isomorphic to the formal completion of the normal bundle $\mathcal{N} = \nu_{Z/\mathscr{X}}$ along the zero section. In particular, $\mathscr{X}$ is toric along Z (in the sense of definition 3.1.20).

Note that our assumptions imply that Z is the smooth complete intersection of the irreducible components D_j of $\mathscr{X}_0$ containing Z, and thus has simple normal crossing boundary Δ_Z, see remark 3.2.2. Since Z is a complete intersection, the conormal bundle $\nu^*_{Z/\mathscr{X}}$ is the direct sum of the line bundles $\mathcal{O}_Z(-D_j)$. Hence, here the nefness assumption simply means that the D_j's containing Z are anti-nef divisors on Z.

3.2.1 Notation and strategy

We set $J = \{0, 1, \ldots, n-r\}$ such that $Z = \cap_{j\in J} D_j$. Since for every irreducible component D of $\mathscr{X}_0$, the intersection $D \cap Z$ is connected by assumption, this allows us to denote by D_l with $l \in L$ the components of $\mathscr{X}_0$ intersecting Z transversally along $Z_l := Z \cap D_l$, so that the toric boundary of Z is given by $\Delta_Z = \sum_{l\in L} Z_l$.

Remark 3.2.2. *The dlt assumption on $\mathscr{X}$ and the toricness of Z ensure that Z is smooth, and that (Z, Δ_Z) is an snc pair. Indeed, the singular locus of Z is a union of torus invariant subvarieties (see [CLS11, Proposition 11.1.2]), hence the generic point of a component of the singular locus is the generic point of a stratum of Δ_Z. However, (Z, Δ_Z) is a dlt pair, thus snc at the generic point of each stratum of Δ_Z.*

Remark 3.2.3. *The smoothness of Z and the assumption that the components of $\mathscr{X}_0$ are Cartier divisors imply that $\mathscr{X}$ is regular at any point of Z. Indeed, for any point $p \in Z$ and $j \in J$, let $z_j \in \mathcal{O}_{\mathscr{X},p}$ be a local equation of D_j at p. As $\mathcal{O}_{Z,p} \simeq \mathcal{O}_{\mathscr{X},p}/(z_0, \dots, z_{n-r})$ is a regular local ring of dimension r, $(z_0, \dots, z_{n-r})$ can be extended to form a regular system of parameters for $\mathcal{O}_{\mathscr{X},p}$.*

We denote by $\Sigma \subset N_{\mathbb{R}}$ the fan of Z. Its rays are given by $\mathbb{R}_{\geqslant 0} u_l$ for $l \in L$, with primitive generators u_l; the maximal cones of Σ are in bijection with the set of unordered r-tuples $\{i_1, \dots, i_r\} \in L^r$ such that $\cap_{\beta=1}^r D_{i_\beta} \cap Z \neq \varnothing$. For a maximal cone σ of Σ, we write $L_\sigma := \{l \in L \,|\, u_l \in \sigma\}$.

Lemma 3.2.4. *For any maximal cone σ of Σ, we have* $\det((u_l)_{l \in L_\sigma}) = \pm 1$.

Proof. The smoothness of Z (see remark 3.2.2) implies that the primitive generators of σ form a $\mathbb{Z}$-basis of N, which is equivalent to the condition $\det((u_l)_{l \in L_\sigma}) = \pm 1$. □

Let $\mathcal{N} := \nu_{Z/\mathscr{X}} \xrightarrow{p} Z$ be the normal bundle of Z in $\mathscr{X}$, and denote by $Z \subset \mathcal{N}$ the zero section. We write $\mathcal{O}_{\mathscr{X}}(D_j)_{|Z} = \mathcal{O}_Z(F_j)$ as a Cartier divisor F_j on Z, so that $\mathcal{N} = \oplus_{j \in J} \mathcal{O}_Z(F_j)$. Since any Cartier divisor on Z is linearly equivalent to a toric one, for any $j = 1, \dots, n-r$, there exist integers $\lambda_{j,l}$ such that

$$\mathcal{O}_Z(F_j) = \mathcal{O}_Z\big(-\sum_{l \in L} \lambda_{j,l} Z_l\big). \tag{3.2.5}$$

For $j = 0$ we set $\lambda_{0,l} := 1 - \sum_{j \in J \setminus \{0\}} \lambda_{j,l}$ and verify that

$$\mathcal{O}_Z(F_0) = \mathcal{O}_{\mathscr{X}}(D_0 - \mathscr{X}_0)_{|Z} = \mathcal{O}_Z\Big(-\sum_{j \in J\setminus\{0\}} F_j - \sum_{l \in L} Z_l\Big) = \mathcal{O}_Z(-\sum_{l \in L} \lambda_{0,l} Z_l).$$

We obtain that $\mathcal{N} = \oplus_{j \in J} \mathcal{O}_Z\big(-\sum_{l \in L} \lambda_{j,l} Z_l\big)$ and for all l in L

$$\sum_{j \in J} \lambda_{j,l} = 1. \tag{3.2.6}$$

The normal bundle $\mathcal{N}$ is a toric variety of dimension $n+1$. The corresponding fan $\hat{\Sigma}$ lies in $N_{\mathbb{R}} \times \mathbb{R}^J$ and consists of the following cones and their faces (see [CLS11, §7.3] for a reference). Let $e_0, \dots, e_{n-r}$ be the standard basis of $\mathbb{R}^J$; given a cone $\sigma \in \Sigma$, we have

$$\hat{\sigma} = \mathrm{Cone}((0, e_0), \dots, (0, e_{n-r}), (u_l, (\lambda_{j,l})) \,|\, u_l \in \sigma) \in \hat{\Sigma}.$$

In particular, we denote the rays of $\hat{\Sigma}$ by

$$v_j = (0, e_j) \text{ for } j \in J, \quad v_l = (u_l, (\lambda_{j,l})) \text{ for } l \in L.$$

Proposition 3.2.7. *For any 1-dimensional toric stratum $C \subseteq Z$*

$$\sum_{j \in J} (C \cdot D_j) v_j + \sum_{l \in L} (C \cdot D_l) v_l = 0 \text{ in } N_{\mathbb{R}} \times \mathbb{R}^J. \tag{3.2.8}$$

Proof. The relation in 3.2.8 boils down to the two following:

$$\begin{cases} \sum_{l\in L}(C\cdot D_l)u_l = 0 \\ (C\cdot D_j) + \sum_{l\in L}\lambda_{j,l}(C\cdot D_l) = 0. \end{cases}$$

The first one follows directly from 2.1.15 in the fan Σ of Z; the second comes from the construction of λ_l, and in particular from $C\cdot D_j = C\cdot F_j = -C\cdot\sum_{l\in L}\lambda_{j,l}Z_l$. □

The map

$$\mathrm{ord}(t) : N_{\mathbb{R}}\times\mathbb{R}^J \to \mathbb{R}_{\geqslant 0} \quad (u,w)\mapsto \sum_{j=0}^{n-r} w_j$$

is $\mathbb{Z}$-linear, sends all the primitive generators of the rays of $\hat{\Sigma}$ to 1 by 3.2.6, and is compatible with $\hat{\Sigma}$ and the fan of $\mathbb{A}^1_k$. Thus, it induces a toric morphism $t : \mathcal{N}\to\mathbb{A}^1_k$ whose fiber over 0 is the toric boundary of $\mathcal{N}$. The base change $\mathscr{N} := \mathcal{N}\times_{\mathbb{A}^1} R$ to R is a toric R-scheme, whose generic fiber is isomorphic to $\mathbb{G}^n_{m,K}$. The special fiber $\mathscr{N}_0$ can be written as $\mathscr{N}_0 = \sum_{i\in J\cup L} E_i$, where the combinatoric of intersections between components is exactly the same as in $\mathscr{X}_0$.

We prove theorem 3.1.21 by constructing a formal isomorphism

$$f : \widehat{\mathscr{X}_{/Z}} \xrightarrow{\simeq} \widehat{\mathscr{N}_{/Z}}.$$

More specifically, we proceed as follows. We set the notations $\mathfrak{X} = \widehat{\mathscr{X}_{/Z}}$ and $\mathfrak{N} = \widehat{\mathscr{N}_{/Z}}$.

- (sections 3.2.2,3.2.3) Let $\sigma\in\Sigma$ be a maximal cone. Denote by Z_σ and $\mathcal{N}_\sigma := \mathcal{N}_{Z_\sigma/\mathscr{X}}$ the corresponding toric affine charts in Z and $\mathcal{N}$ respectively. This induces an open formal subscheme of $\mathfrak{N}$, which we denote by $\mathfrak{N}_\sigma$. We construct a morphism
$$f_\sigma : \mathfrak{X}\setminus\left(\cup_{l\in L\setminus L_\sigma} D_l\right) =: \mathfrak{X}_\sigma \to \mathfrak{N}_\sigma,$$
in a similar manner to [NXY19]: we construct $n+1$ divisors W^σ_j and W^σ_i on $\mathscr{X}$, whose defining equations on the chart $\mathfrak{X}_\sigma$ yields the morphism f_σ. The equations are induced by sections of $\mathcal{O}_Z(W^\sigma_j)$ and $\mathcal{O}_Z(W^\sigma_i)$: these are first constructed on Z, then extended to $\mathfrak{X}$ by the nef condition on the conormal bundle $\nu^*_{Z/\mathscr{X}}$ assumed in thm. 3.1.21, which ensures the vanishing of higher cohomology groups for the tensor powers of $\nu^*_{Z/\mathscr{X}}$.

- (sections 3.2.4,3.2.5) Let σ and σ' be two maximal cones of Σ intersecting along a face of codimension one. We establish relations among the respective divisors and construct sections on $\mathfrak{X}_{\sigma'}$ from those on $\mathfrak{X}_\sigma$. This allows us to prove that the morphisms f_σ on the charts $\mathfrak{X}_\sigma$'s can be chosen so that they are compatible on the overlaps $\mathfrak{X}_\sigma\cap\mathfrak{X}_{\sigma'}$. This yields a well defined morphism f which extends the identity on Z and preserves the ideal $\mathscr{I}_Z$, so that it turns out to be an isomorphism.

3.2.2 Construction of the divisors

We set

$$\Delta := \sum_{l\in L} u_l \otimes D_l \in N \otimes \mathrm{Div}_0(\mathscr{X}) \simeq (\mathrm{Div}_0(\mathscr{X}))^r;$$

this is an r-tuple of divisors on $\mathscr{X}$. Moreover, the restriction of any of these to Z is a principal divisor by 2.1.13. Given a maximal cone σ of Σ, for any $i \in L_\sigma$, we define

$$W_i^\sigma := -\frac{\det(\Delta, (u_l)_{l\in L_\sigma\setminus\{i\}})}{\det(u_i, (u_l)_{l\in L_\sigma\setminus\{i\}})} \in \mathrm{Div}_0(\mathscr{X})$$

where the column vectors u_l are in the same order in the numerator and in the denominator, and the denominator has value ± 1 by 3.2.4.

Lemma 3.2.9. *The divisor W_i^σ has multiplicity -1 along D_i, multiplicity 0 along D_l for $l \in L_\sigma \setminus \{i\}$, and along D_j for $j \in J$. In other words, we may write:*

$$W_i^\sigma = -D_i + \sum_{l\in L\setminus L_\sigma} c_{i,l} D_l$$

for some coefficients $c_{i,l} \in \mathbb{Z}$. Moreover, the restriction of W_i^σ to Z is principal.

Proof. The statement on the multiplicities follows from the definition of W_i^σ, as

$$W_i^\sigma = -\sum_{l\in L} \frac{\det(u_l, (u_{l'})_{l'\in L_\sigma\setminus\{i\}})}{\det(u_i, (u_{l'})_{l'\in L_\sigma\setminus\{i\}})} D_l.$$

Moreover, W_i^σ is a linear combination of the divisors of the r-tuple Δ, hence its restriction to Z is principal by 2.1.13. □

For $j \in J$, we define the divisor on $\mathscr{X}$

$$\begin{aligned} W_j^\sigma &:= -D_j - \sum_{l\in L} \lambda_{j,l} D_l - \sum_{i\in L_\sigma} \lambda_{j,i} W_i^\sigma \\ &= -D_j - \sum_{l\in L\setminus L_\sigma} \lambda_{j,l} D_l - \cancel{\sum_{l\in L_\sigma} \lambda_{j,l} D_l} - \sum_{i\in L_\sigma} \lambda_{j,i}\Big(\cancel{-D_i} + \sum_{l\in L\setminus L_\sigma} c_{i,l} D_l\Big) \\ &= -D_j + \sum_{l\in L\setminus L_\sigma} d_{j,l} D_l \quad \text{with} \quad d_{j,l} = -\lambda_{j,l} - \sum_{i\in L_\sigma} \lambda_{j,i} c_{i,l}. \end{aligned} \tag{3.2.10}$$

The restriction of W_j^σ to Z is a principal divisor, as the $W_{i\,|Z}^\sigma$ are principal and $-D_{j|Z}$ is linearly equivalent to $\sum_{l\in L} \lambda_{j,l} Z_l$ by 3.2.5.

Lemma 3.2.11. *The relation $\sum_{j\in J} W_j^\sigma + \sum_{i\in L_\sigma} W_i^\sigma = -\sum_{j\in J} D_j - \sum_{l\in L} D_l$ holds.*

Proof. Write $W := \sum_{j\in J} W_j^\sigma + \sum_{i\in L_\sigma} W_i^\sigma \in \mathrm{Div}_0(\mathscr{X})$. We have

$$\begin{aligned} \text{for } j \in J \quad & \mathrm{ord}_{D_j}(W) = \mathrm{ord}_{D_j}(W_j^\sigma) = -1 \\ \text{for } i \in L_\sigma \quad & \mathrm{ord}_{D_i}(W) = \mathrm{ord}_{D_i}(W_i^\sigma) = -1 \\ \text{for } l \in L \setminus L_\sigma \quad & \mathrm{ord}_{D_l}(W) = \sum_{j\in J} d_{j,l} + \sum_{i\in L_\sigma} c_{i,l} = -\sum_{j\in J} \lambda_{j,l} + \sum_{i\in L_\sigma} c_{i,l}(1 - \sum_{j\in J} \lambda_{j,i}) = -1 \end{aligned}$$

by lemma 3.2.9 and the relations 3.2.10 and 3.2.6. □

3.2.3 Construction of the sections for a maximal cone

Let σ be a maximal cone of Σ. We denote by $\mathscr{L}_j^\sigma$ and $\mathscr{L}_i^\sigma$ the line bundles on $\mathfrak{X}$ induced respectively by $\mathcal{O}_{\mathscr{X}}(W_j^\sigma)$ for $j \in J$, and by $\mathcal{O}_{\mathscr{X}}(W_i^\sigma)$ for $i \in L_\sigma$. Since W_j^σ and W_i^σ are principal on Z, the restrictions ${\mathscr{L}_j^\sigma}_{|Z}$ and ${\mathscr{L}_i^\sigma}_{|Z}$ are trivial line bundles on Z, thus we may choose non-zero global sections s_j^σ and s_i^σ on Z.

We now lift the sections s_j^σ and s_i^σ to global sections of $\mathscr{L}_j^\sigma$ and $\mathscr{L}_i^\sigma$, which we still denote by s_j^σ and s_i^σ. Indeed, for any $n \geqslant 1$, write $(\mathscr{X}/Z)_n$ for the (non-reduced) subscheme of $\mathscr{X}$ defined by the ideal $\mathscr{I}_Z^n$. In the exact sequence

$$H^0((\mathscr{X}/Z)_n, \mathscr{L}_j^\sigma) \longrightarrow H^0((\mathscr{X}/Z)_{n-1}, \mathscr{L}_j^\sigma) \longrightarrow H^1(Z, (\nu_{Z/\mathscr{X}}^*)^{\otimes n}),$$

and in the analogous one for $\mathscr{L}_i^\sigma$, the right-hand vanishes: the conormal bundle is a direct sum of line bundles on Z which are nef by the hypothesis in thm. 3.1.21 and so are its positive tensor powers, thus their first cohomology group vanishes by lemma 2.1.20. We thus extend the sections constructed above to all of the $(\mathscr{X}/Z)_n$ by induction, which yields an extension to $\mathfrak{X} = \varprojlim_n (\mathscr{X}/Z)_n$.

Lemma 3.2.12. *Under the isomorphism $(\mathscr{L}_j^\sigma)_{|\mathscr{X}_\sigma} \simeq \mathcal{O}_{\mathscr{X}_\sigma}(-D_j)$ induced by eq. 3.2.10 (and similarly for i), the restrictions of s_j^σ and s_i^σ to $\mathfrak{X}_\sigma$ are equations for D_j and D_i, and thus*

$$w_\sigma := t \cdot \prod_{j \in J} (s_j^\sigma)^{-1} \cdot \prod_{i \in L_\sigma} (s_i^\sigma)^{-1}$$

is an invertible function on $\mathfrak{X}_\sigma$.

Proof. We show that s_i^σ is an equation for D_i on $\mathfrak{X}_\sigma$; the proof is analogous for s_j^σ.

On Z, ${W_i^\sigma}_{|Z} = \operatorname{div}(h)$ and s_i^σ is a non-zero global section, which means that

$$\begin{aligned} {\mathscr{L}_i^\sigma}_{|Z}(Z) = \mathcal{O}_Z(W_i^\sigma)(Z) = \{f \in \mathcal{K}(Z) \,|\, \operatorname{div}(f) + \operatorname{div}(h) \geqslant 0\} &\xrightarrow{\simeq} \mathcal{O}_Z(Z) = k \\ f &\mapsto fh \\ s_i^\sigma &\mapsto s_i^\sigma h = \lambda \in k^\times. \end{aligned}$$

Let $\mathcal{U}$ be an open cover of $\mathscr{X} \setminus \left(\cup_{i' \notin J \cup L_\sigma} D_{i'} \right)$ such that $D_{i|U} = \operatorname{div}(g_U)$ for any $U \in \mathcal{U}$; this is possible as D_i is a Cartier divisor. On U, ${W_i^\sigma}_{|U} = -D_{i|U} = \operatorname{div}(g_U^{-1})$ and

$$\begin{aligned} \mathscr{L}_i^\sigma(\mathfrak{X}_\sigma \cap U) &\xrightarrow{\simeq} \mathcal{O}_{\mathfrak{X}_\sigma}(\mathfrak{X}_\sigma \cap U) \\ f &\mapsto f g_U^{-1} \\ s_i^\sigma &\mapsto s_i^\sigma g_U^{-1} \in \mathcal{O}_{\mathfrak{X}_\sigma}^\times(\mathfrak{X}_\sigma \cap U), \end{aligned}$$

where $s_i^\sigma g_U^{-1}$ is a regular invertible function on $\mathfrak{X}_\sigma \cap U$, as its reduction to Z is invertible. Finally, the section s_i^σ is defined globally on $\widehat{\mathscr{X}_{/Z}}$ and on each open $\mathfrak{X}_\sigma \cap U$ gives a local equation of the divisor D_i, hence it is a equation for D_i on $\mathfrak{X}_\sigma$. □

3.2.4 Construction for two adjacent maximal cones

Let σ and σ' be two maximal cones of Σ intersecting along a face of codimension one, and let $C \subseteq Z$ be the curve associated with the cone $\sigma \cap \sigma'$. Setting $L_{\sigma\sigma'} = L_\sigma \cap L_{\sigma'}$, we may write $L_\sigma = L_{\sigma\sigma'} \cup \{i_0\}$ and $L_{\sigma'} = L_{\sigma\sigma'} \cup \{i_\infty\}$. The sets $\mathcal{B} = ((v_i)_{i\in L_{\sigma\sigma'}}, v_{i_0}, (v_j)_{j\in J})$ and $\mathcal{B}' = ((v_i)_{i\in L_{\sigma\sigma'}}, v_{i_\infty}, (v_j)_{j\in J})$ are bases of $\hat{N} = N \oplus \mathbb{Z}^J$. They induce isomorphisms $\beta, \beta' : \mathbb{Z}^r \oplus \mathbb{Z}^J \to \hat{N}$ such that the change of basis from $\mathcal{B}$ to $\mathcal{B}'$ is

$$M_{\mathcal{B}'\mathcal{B}} = \beta' \circ \beta^{-1} = \begin{array}{c} \begin{matrix} L_{\sigma\sigma'} & i_0 & J \end{matrix} \\ \begin{pmatrix} \mathrm{Id} & (-C\cdot D_i)_{i\in L_{\sigma\sigma'}} & 0 \\ 0 & -1 & 0 \\ 0 & (-C\cdot D_j)_{j\in J} & \mathrm{Id} \end{pmatrix} \end{array} \begin{array}{c} L_{\sigma\sigma'} \\ i_\infty \\ J \end{array} \quad \text{and} \quad \begin{pmatrix} v_i \\ v_{i_\infty} \\ v_j \end{pmatrix} = M_{\mathcal{B}'\mathcal{B}}^T \begin{pmatrix} v_i \\ v_{i_0} \\ v_j \end{pmatrix}.$$

Denote by $((\varepsilon_i)_{i\in L_{\sigma\sigma'}}, \varepsilon_{i_0}, (\varepsilon_j)_{j\in J})$ the basis of $\hat{M} := \mathrm{Hom}(\hat{N}, \mathbb{Z})$ dual to $\mathcal{B}$, and $((\varepsilon'_i)_{i\in L_{\sigma\sigma'}}, \varepsilon'_{i_\infty}, (\varepsilon'_j)_{j\in J})$ the basis dual to $\mathcal{B}'$. It follows that

$$\begin{pmatrix} \varepsilon'_i \\ \varepsilon'_{i_\infty} \\ \varepsilon'_j \end{pmatrix} = M_{\mathcal{B}'\mathcal{B}} \begin{pmatrix} \varepsilon_i \\ \varepsilon_{i_0} \\ \varepsilon_j \end{pmatrix}. \tag{3.2.13}$$

The isomorphisms β and β' allow us to view

$$W^\sigma := ((W_i^\sigma)_{i\in L_{\sigma\sigma'}}, W_{i_0}^\sigma, (W_j^\sigma)_{j\in J}) \in (\mathbb{Z}^r \oplus \mathbb{Z}^J) \otimes \mathrm{Div}_0(\mathscr{X}) \simeq (\mathrm{Div}_0(\mathscr{X}))^{n+1}$$

and $W^{\sigma'}$ as elements of $\hat{N} \otimes \mathrm{Div}_0(\mathscr{X})$, that we will still denote by W^σ and $W^{\sigma'}$.

Lemma 3.2.14. *Let $C \subseteq Z$ be the curve associated with the cone $\sigma \cap \sigma'$. We have*

$$\begin{cases} W_i^{\sigma'} = W_i^\sigma - (C\cdot D_i) W_{i_0}^\sigma & \text{for } i \in L_{\sigma\sigma'} \\ W_{i_\infty}^{\sigma'} = -W_{i_0}^\sigma & \\ W_j^{\sigma'} = W_j^\sigma - (C\cdot D_j) W_{i_0}^\sigma & \text{for } j \in J \end{cases}$$

In other words, the relation $W^{\sigma'} = (M_{\mathcal{B}'\mathcal{B}} \otimes \mathrm{Id}) W^\sigma$ holds.

Proof. By 2.1.15 we have $u_{i_\infty} = -u_{i_0} - \sum_{m\in L_{\sigma\sigma'}} (C\cdot D_m) u_m$, so

$$\begin{aligned} \text{for } i \in L_{\sigma\sigma'}, \quad W_i^{\sigma'} &= -\frac{\det(\Delta, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_{i_\infty})}{\det(u_i, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_{i_\infty})} \\ &= -\frac{\det(\Delta, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_{i_0})}{\det(u_i, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_{i_0})} - \sum_{m\in L_{\sigma\sigma'}} (C\cdot D_m) \frac{\det(\Delta, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_m)}{\det(u_i, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_{i_0})} \\ &= W_i^\sigma - (C\cdot D_i) \frac{\det(\Delta, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_i)}{\det(u_i, (u_l)_{l\in L_{\sigma\sigma'}\setminus\{i\}}, u_{i_0})} = W_i^\sigma - (C\cdot D_i) W_{i_0}^\sigma; \\ \text{for } i = i_\infty, \quad W_{i_\infty}^{\sigma'} &= \frac{\det(\Delta, (u_l)_{l\in L_{\sigma\sigma'}})}{\det(u_{i_\infty}, (u_l)_{l\in L_{\sigma\sigma'}})} = -\frac{\det(\Delta, (u_l)_{l\in L_{\sigma\sigma'}})}{\det(u_{i_0}, (u_l)_{l\in L_{\sigma\sigma'}})} = -W_{i_0}^\sigma. \end{aligned}$$

For $j \in J$, we have:

$$\begin{aligned}
\sum_{i\in L_{\sigma'}} \lambda_{j,i} W_i^{\sigma'} &= -\lambda_{j,i_\infty} W_{i_0}^{\sigma} + \sum_{i\in L_{\sigma\sigma'}} \lambda_{j,i}\left(W_i^{\sigma} - (C\cdot D_i) W_{i_0}^{\sigma}\right) \\
&= \Big(-\lambda_{j,i_\infty} - \sum_{i\in L_{\sigma\sigma'}} \lambda_{j,i}(C\cdot D_i) - \lambda_{j,i_0}\Big) W_{i_0}^{\sigma} + \sum_{i\in L_\sigma} \lambda_{j,i} W_i^{\sigma} \\
&= (C\cdot D_j) W_{i_0}^{\sigma} + \sum_{i\in L_\sigma} \lambda_{i,j} W_i^{\sigma} \qquad \text{by } 3.2.8 \\
W_j^{\sigma'} &= -D_j - \sum_{l\in L} \lambda_{j,l} D_l - \sum_{i\in L_{\sigma'}} \lambda_{j,i} W_i^{\sigma'} \\
&= -D_j - \sum_{l\in L} \lambda_{j,l} D_l - \sum_{i\in L_\sigma} \lambda_{j,i} W_i^{\sigma} - (C\cdot D_j) W_{i_0}^{\sigma} = W_j^{\sigma} - (C\cdot D_j) W_{i_0}^{\sigma}.
\end{aligned}$$

These relations can be summed up as $\begin{pmatrix} W_i^{\sigma'} \\ W_{i_\infty}^{\sigma'} \\ W_j^{\sigma'} \end{pmatrix} = M_{\mathcal{B}'\mathcal{B}} \begin{pmatrix} W_i^{\sigma} \\ W_{i_0}^{\sigma} \\ W_j^{\sigma} \end{pmatrix}$, i.e. $W^{\sigma'} = (M_{\mathcal{B}'\mathcal{B}} \otimes \mathrm{Id}) W^{\sigma}$. □

The inverse $(s_{i_0}^{\sigma})^{-1}$ is a section on $\mathfrak{X}$ of $\mathscr{L}_{i_\infty}^{\sigma'}$, so by lemma 3.2.14 the sections

$$\begin{cases} s_i^{\sigma'} := s_i^{\sigma} \cdot (s_{i_0}^{\sigma})^{-(C\cdot D_i)} & \text{for } i \in L_{\sigma\sigma'} \\ s_{i_\infty}^{\sigma'} := (s_{i_0}^{\sigma})^{-1} & \\ s_j^{\sigma'} := s_j^{\sigma} \cdot (s_{i_0}^{\sigma})^{-(C\cdot D_j)} & \text{for } j \in J \end{cases}$$

are sections on $\mathfrak{X}$ of the line bundles $\mathscr{L}_i^{\sigma'}$ and $\mathscr{L}_j^{\sigma'}$. By lemma 3.2.12 these give equations for D_i and D_j on the open subscheme $\mathfrak{X}_{\sigma'}$, and on $\mathfrak{X}_\sigma \cap \mathfrak{X}_{\sigma'}$ we have

$$\begin{pmatrix} s_i^{\sigma'} \\ s_{i_\infty}^{\sigma'} \\ s_j^{\sigma'} \end{pmatrix} = M_{\mathcal{B}'\mathcal{B}} \begin{pmatrix} s_i^{\sigma} \\ s_{i_0}^{\sigma'} \\ s_j^{\sigma} \end{pmatrix} \tag{3.2.15}$$

where the additive notation on the matrix corresponds to the multiplicative notation on the sections. Moreover, on $\mathfrak{X}_\sigma \cap \mathfrak{X}_{\sigma'}$ we have

$$\begin{aligned}
w_{\sigma'} &:= t \cdot \prod_{j\in J} (s_j^{\sigma'})^{-1} \cdot \prod_{i\in L_{\sigma'}} (s_i^{\sigma'})^{-1} = t \cdot \prod_{j\in J} (s_j^{\sigma})^{-1} (s_{i_0}^{\sigma})^{C\cdot D_j} \cdot \prod_{i\in L_{\sigma\sigma'}} (s_i^{\sigma})^{-1} (s_{i_0}^{\sigma})^{C\cdot D_i} \cdot (s_{i_0}^{\sigma}) \\
&= t \cdot \prod_{j\in J} (s_j^{\sigma})^{-1} \cdot \prod_{i\in L_{\sigma\sigma'}} (s_i^{\sigma})^{-1} \cdot (s_{i_0}^{\sigma})^{\sum_{j\in J} C\cdot D_j + \sum_{i\in L_{\sigma\sigma'}} C\cdot D_i + 1} = w_\sigma,
\end{aligned} \tag{3.2.16}$$

hence the invertible function w_σ on $\mathfrak{X}_\sigma$ extends to $\mathfrak{X}_\sigma \cup \mathfrak{X}_{\sigma'}$ by $w_{\sigma'}$.

3.2.5 Construction of the morphism

Let Γ be the graph with vertices the maximal cones of Σ (hence the maximal cones of $\widehat{\Sigma}$) and with an edge between σ and σ' if and only if $\sigma \cap \sigma'$ is a common face of

codimension one. Note that since Z is proper, if $\mathbb{S} \subset N_{\mathbb{R}}$ is a sphere with center the origin, then $\Sigma \cap \mathbb{S}$ is a triangulation of $\mathbb{S}$. In particular, Γ is the 1-skeleton of the dual complex of a triangulation of the sphere, and is thus connected.

Let $\sigma_0 \in \Sigma$ be a maximal cone, and $p_0 \in \Gamma$ the corresponding vertex, that we will use as a reference point. We fix a tuple of sections s^{σ_0} of W^{σ_0} as in 3.2.3.
Let $\sigma \in \Sigma$ be a maximal cone, and $p \in \Gamma$ the corresponding vertex. By connectedness of Γ, there exists a path γ from p_0 to p, hence a sequence of maximal cones $\sigma_0, \dots, \sigma_q = \sigma$ such that $\sigma_h \cap \sigma_{h+1}$ is a codimension one face of both σ_h and σ_{h+1}, for $h = 0, \dots, q-1$. The construction of section 3.2.4 allows us to construct inductively along γ a tuple of sections s^{σ_h} of W^{σ_h}.

Lemma 3.2.17. *The tuple of sections s^σ is independent on the choice of path.*

Proof. By 3.2.15, for any $h = 0, \dots, q-1$, the sections $s^{\sigma_{h+1}}$ are constructed from s^{σ_h} by multiplication by the matrix for the change of basis from $((v_i)_{i \in L_{\sigma_h}}, (v_j)_{j \in J})$ to $((v_i)_{i \in L_{\sigma_{h+1}}}, (v_j)_{j \in J})$. Thus, by composition, the sections s^σ only depends on s^{σ_0} and the change of basis from $((v_i)_{i \in L_{\sigma_0}}, (v_j)_{j \in J})$ to $((v_i)_{i \in L_\sigma}, (v_j)_{j \in J})$. □

This provides us with a tuple of sections s^σ of W^σ for each maximal cone $\sigma \in \Sigma$, and the function

$$w_\sigma = t \cdot \prod_{j \in J} (s^\sigma_j)^{-1} \cdot \prod_{i \in L_\sigma} (s^\sigma_i)^{-1} \in \mathcal{O}(\mathfrak{X}_\sigma)^\times.$$

By 3.2.16 the w_σ glue to an invertible function w on $\mathfrak{X}$; w admits a $(n+1)$-th root on Z, since it is constant, and by Hensel's lemma we obtain an invertible function w' on $\mathfrak{X}$ such that $(w')^{n+1} = w$. We use the sections s^σ and the function w' to define a morphism

$$f_\sigma : \mathfrak{X}_\sigma \longrightarrow \mathfrak{N}_\sigma$$

as follows. Denoting by $((\varepsilon_i)_{i \in L_\sigma}, (\varepsilon_j)_{j \in J})$ the dual basis to $((v_i)_{i \in L_\sigma}, (v_j)_{j \in J})$, the toric chart $\mathfrak{N}_\sigma$ has the following explicit description:

$$\mathfrak{N}_\sigma = \operatorname{Spf} R[\chi^{\varepsilon_i}, i \in L_\sigma][[\chi^{\varepsilon_j}, j \in J]]/\{t - \chi^{\sum_{i \in L_\sigma} \varepsilon_i + \sum_{j \in J} \varepsilon_j}\}.$$

Indeed, $\mathfrak{N}_\sigma$ is the formal completion along Z of $\mathcal{N}_{\hat\sigma} \times_{\mathbb{A}^1} R$, where $\mathcal{N}_{\hat\sigma} = \operatorname{Spec} k[(\hat\sigma)^\vee \cap \hat N]$; since $\operatorname{ord}(t) = \sum_{i \in L_\sigma} \varepsilon_i + \sum_{j \in J} \varepsilon_j$ on $\widehat\sigma$, the relation $t = \chi^{\sum_{i \in L_\sigma} \varepsilon_i + \sum_{j \in J} \varepsilon_j}$ holds. The map f_σ is now defined at the level of function rings by

$$\begin{aligned} f_\sigma^\# : \mathcal{O}(\mathfrak{N}_\sigma) &\longrightarrow \mathcal{O}(\mathfrak{X}_\sigma) \\ \chi^{\varepsilon_i} &\mapsto w' s^\sigma_i \qquad \text{for } i \in L_\sigma \\ \chi^{\varepsilon_j} &\mapsto w' s^\sigma_j \qquad \text{for } j \in J \end{aligned}$$

where the sections s^σ are viewed as functions on $\mathfrak{X}_\sigma$ thanks to the proof of lemma 3.2.12.

Lemma 3.2.18. *For any pair of maximal cones σ, σ' intersecting along a codimension one face, the morphisms f_σ and $f_{\sigma'}$ coincide on the overlap $\mathfrak{X}_\sigma \cap \mathfrak{X}_{\sigma'}$.*

Proof. The cones σ and σ' correspond to adjacent vertices in Γ. Thus, by lemma 3.2.17 we construct s^σ from any path joining σ_0 to σ, and $s^{\sigma'}$ from s^σ by the relation $s^{\sigma'} = M_{\mathcal{B}'\mathcal{B}}\, s^\sigma$ in 3.2.15.

The functions χ^ε transform into $\chi^{\varepsilon'}$ via the change of dual bases, which is given by $\varepsilon' = M_{\mathcal{B}'\mathcal{B}}\,\varepsilon$ in 3.2.13. Comparing the two formulas, it follows that $f_\sigma = f_{\sigma'}$ on $\mathfrak{X}_\sigma \cap \mathfrak{X}_{\sigma'}$. □

Proposition 3.2.19. *The morphism of formal R-schemes $f : \widehat{\mathscr{X}_{/Z}} \longrightarrow \widehat{\mathscr{N}_{/Z}}$ obtained by gluing the morphisms f_σ is an isomorphism.*

Proof. We follow the argument in [NXY19, Proposition 5.4].
Since the source and the target have same dimension and are integral, it is enough to check that f is a closed immersion.
If $\mathscr{J}$ is the largest ideal of definition of $\widehat{\mathscr{N}_{/Z}}$, i.e. the defining ideal of $Z \subset \mathscr{N}$, then $f^*\mathscr{J} = \mathscr{I}_Z$ is the largest ideal of definition of $\widehat{\mathscr{X}_{/Z}}$. Indeed, since Z is cut out inside $\mathscr{X}$ by the D_j for $j \in J$, the ideal $\mathscr{I}_Z$ is locally generated by the s_j for $j \in J$; the same reasoning shows that $\mathscr{J}$ is locally generated by the χ^{ε_j}. The equality $f^*\mathscr{J} = \mathscr{I}_Z$ now follows directly from the local definition of f.
We use [Gro61, 4.8.10] and the fact that f induces an isomorphism on the reductions to infer that f is a closed immersion, and thus an isomorphism by equality of dimensions. □

This concludes the proof of theorem 3.1.21: $\mathscr{X}$ is toric along Z.

3.3 Integral affine structures

Let X/K be a smooth n-dimensional maximally degenerate Calabi-Yau variety. In this section we compute the transition functions between the charts of the integral affine structure on $\mathrm{Sk}(X)$ associated with a minimal model of X, or obtained by combining several minimal models. This relies on and generalizes the construction in [NXY19].
We then focus on certain degenerations of quartic $K3$ surfaces, and later of quintic 3-folds: we apply theorem 3.1.21 to reconstruct integral affine structures on the essential skeleton, and provide explicit formulas for the monodromy transformations around the singularities.

3.3.1 Integral affine structure near a toric vertex

Let X/K be a smooth variey, and let $\mathscr{X}/R$ be a dlt model of X with reduced special fiber. We assume that D is a toric irreducible component of $\mathscr{X}_0$, whose toric boundary is the intersection of D with other irreducible components of $\mathscr{X}_0$.
We let Σ_D be the fan of D inside $N_\mathbb{R}$, and let:

$$\phi_{v_D} : \mathrm{Star}(v_D) \longrightarrow N_\mathbb{R}$$

be the unique embedding sending v_D to the origin, each vertex $v = v_{D_l}$ to the primitive generator e_l of the ray in Σ_D corresponding to the boundary component $Z_l = D_l \cap D$, and that is affine on each cell of $\mathrm{Star}(v_D)$.

Definition 3.3.1. *The* fan structure *on* $\mathrm{Star}(v_D)$ *is the only integral affine structure on* $\mathrm{Star}(v_D)$ *such that the map* ϕ_{v_D} *is* $\mathbb{Z}$*-affine.*

Under the assumptions of theorem 3.1.21, we proved that $\mathscr{X}$ is toric along D, and that $\rho_{\mathscr{X}}$ is an affinoid torus fibration over $\mathrm{Star}(v_D)$. We claim that the above construction provides an explicit description of the $\mathbb{Z}$-affine structure induced by $\rho_{\mathscr{X}}$ on $\mathrm{Star}(v_D)$ - note that it only depends on D and not on how D sits inside $\mathscr{X}$.

Corollary 3.3.2. *In the setting of theorem 3.1.21, let* $Z = D$ *be an irreducible component of* $\mathscr{X}_0$*. Then the integral affine structure induced by* $\rho_{\mathscr{X}}$ *over the interior of* $\mathrm{Star}(v_D)$ *is the fan structure defined above.*

Proof. By the proof of theorem 3.1.21 and prop. 3.1.18 we have the following diagram:

$$\begin{array}{ccc} \mathfrak{X}_D^{\eta} & \xrightarrow{\simeq} & \mathfrak{N}_D^{\eta} \\ \downarrow{\scriptstyle \rho_{\mathscr{X}}} & & \downarrow{\scriptstyle \mathrm{val}} \\ \mathrm{Star}(v_D) & \xrightarrow[\varphi]{\simeq} & \mathrm{Int}(\Sigma_1) \end{array}$$

where the upper arrow is an isomorphism of analytic spaces, and the lower one a homeomorphism. Here $\mathfrak{X}_D^{\eta}$ and $\mathfrak{N}_D^{\eta}$ are the generic fibers (in the sense of Berkovich) of the formal completions $\widehat{\mathscr{X}_{/D}}$ and $\widehat{\mathscr{N}_{/D}}$ respectively, and $\mathrm{Int}(\Sigma_1)$ denotes the interior of the polyhedral complex Σ_1 obtained by intersecting the fan $\hat{\Sigma} \subset N_{\mathbb{R}} \times \mathbb{R}$ of the normal bundle of D in $\mathscr{X}$ with $N_{\mathbb{R}} \times \{1\}$. In particular, $\mathrm{Int}(\Sigma_1)$ is embedded in $\Sigma_D \simeq N_{\mathbb{R}} \simeq \mathbb{R}^n$, the polyhedral decomposition of $\mathrm{Star}(v_D)$ is the same as Σ_D, and the vertex v_D corresponds to the origin. By definition, the integral affine structure on $\mathrm{Star}(v_D)$ is the pullback via φ of the integral affine structure on Σ_D, and this concludes the proof. □

3.3.2 Integral affine structure induced by a model

Assume that X/K is a smooth, maximally degenerate Calabi-Yau variety, and let $\mathscr{X}/R$ be a minimal model of X; we assume that the special fiber $\mathscr{X}_0 = \sum_{i \in I} D_i$ is reduced. We consider a one-dimensional stratum $C = D_1 \cap \ldots \cap D_n$ of $\mathscr{X}_0$, which is therefore a smooth rational curve, and is such that $(\mathscr{X}, \mathscr{X}_0)$ is an snc pair in a formal neighbourhood of C by [NXY19, Corollary 4.6]. Since (C, Δ_C) is log Calabi–Yau, we may write its boundary as $\Delta_C = p_0 + p_\infty$, where $p_0 = C \cap D_0$ and $p_\infty = C \cap D_\infty$ for two irreducible components D_0, D_∞ of $\mathscr{X}_0$ meeting C transversally.

Following [NXY19], we write $b_i = -(C \cdot D_i)$ for $i = 1, \ldots, n$; from $C \cdot \mathscr{X}_0 = 0$ we infer $\sum_{i=1}^n b_i = 2$. The $\mathrm{Star}(\tau_C)$ consists on the union of two maximal faces corresponding to the zero-dimensional strata p_0, p_∞, meeting along τ_C. The goal of this section is to describe the integral affine structure on $\mathrm{Star}(\tau_C)$ in terms of the intersection numbers b_i's, with no assumption on their positivity.

Proposition 3.3.3. *Let* $\rho_{\mathscr{X}}$ *be the retraction associated with the model* $\mathscr{X}$*, and endow* $\mathrm{Sk}(X)$ *with the* $\mathbb{Z}$*-affine structure induced by* $\rho_{\mathscr{X}}$ *away from the codimension*

2 faces of $\mathrm{Sk}(\mathscr{X})$*. Then* $\mathrm{Star}(\tau_C)$ *is* $\mathbb{Z}$*-affine isomorphic to the union of the simplices* $< v_0, v_1, ..., v_n >$ *and* $< v_1, ..., v_n, v_\infty >$ *in* $\mathbb{R}^n$ *where*

$$v_0 = (1, 0, ..., 0),\ v_1 = (0, 1, ..., 0), ...,\ v_n = 0 \text{ and } v_\infty = (-1, b_1, ..., b_{n-1}).$$

Proof. We write $b = \min_{i \leq n} b_i$; we assume b to be negative or zero by the condition $b_1 + ... + b_n = 2$, as the case $n = 2$ and $b_1 = b_2 = 1$ is already treated in the proof of [NXY19, prop. 5.4].

The blow-up $\mathscr{X}_1$ of the point p_∞ in $\mathscr{X}$ yields a new irreducible component $D_{\infty,1}$ (we denote the strict transforms by the same letters for notational simplicity) with multiplicity $N_{\infty,1} = n + 1$, the point $p_{\infty,1} = C \cap D_{\infty,1}$ and the intersection numbers $b_{i,1} := -(C \cdot D_i)_{\mathscr{X}_1} = b_i + 1$. If we repeat the process s times, we obtain the models $\mathscr{X}_s$, the exceptional divisors $D_{\infty,s}$ with multiplicity $N_{\infty,s} = ns + 1$, the points $p_{\infty,s} = C \cap D_{\infty,s}$ and the intersection numbers $b_{i,s} := -(C \cdot D_i)_{\mathscr{X}_s} = b_i + s$.

For $s = 1 - b$, we have $\min_{i \leq n}\{b_{i,1-b}\} > 0$, and by [NXY19] the integral affine structure induced by $\mathscr{X}_{1-b}$ on $\mathrm{Star}(\tau_C)$ is given by $v_0, \ldots, v_n$ and

$$v_{\infty,1-b} = \frac{1}{n(1-b)+1}(-1, b_1 + 1 - b, \ldots, b_{n-1} + 1 - b). \tag{3.3.4}$$

The sequence of blow-ups $\mathscr{X}_{s+1} \to \mathscr{X}_s$ induces (weighted) barycentric subdivisions of the faces $\tau_{p_{\infty,s}}$ with vertices such that

$$N_{\infty,s+1} v_{\infty,s+1} = N_{\infty,s} v_{\infty,s} + \sum_{i=1}^{n} v_i. \tag{3.3.5}$$

Combining 3.3.4 and 3.3.5, at each step we obtain that

$$v_{\infty,s} = \frac{1}{ns+1}(-1, b_1 + s, \ldots, b_{n-1} + s),$$

and in particular $v_\infty = (-1, b_1, \ldots, b_{n-1})$. The proposition follows from the following lemma. □

Lemma 3.3.6. *Let* $B = \tau_1 \cup \tau_2$ *be the union of two* n*-dimensional simplices along a face of codimension one. Assume we are given a* $\mathbb{Z}$*-affine structure on* B*, compatible with those on the* τ_i*'s. Suppose there exists a sequence of (weighted) star subdivisions of* τ_1 *such that* $B' := \mathrm{Star}(\tau_1 \cap \tau_2)$ *(with respect to this subdivision) can be embedded in* $\mathbb{R}^n$ *compatibly with the* $\mathbb{Z}$*-affine structure. Then this embedding extends to* B*, and the* $\mathbb{Z}$*-affine structure on* B *is uniquely recovered by this embedding.*

Proof. The assumptions yield two charts for the $\mathbb{Z}$-affine structure on B: the $\mathbb{Z}$-affine subsets B' and τ_1. These two charts are glued along $B' \cap \tau_1$ which is a simplex and thus has no non-trivial $\mathbb{Z}$-automorphisms preserving the vertices, hence the affine structure on B is uniquely determined. By induction on the number of star subdivisions, there exists a unique subset $\tilde{B} \subset \mathbb{R}^n$ such that $B' \subset \mathbb{R}^n$ can be obtained as the result of the same star subdivisions of $\tilde{B}$, and uniqueness of the affine structure ensures a $\mathbb{Z}$-affine isomorphism $B \simeq \tilde{B}$. □

Remark 3.3.7. *Consider an irreducible component D_i of $\mathscr{X}_k$ and write $\Delta_{D_i} = \sum_{j\neq i} D_j \cap D_i$. By adjunction, the pair (D_i, Δ_{D_i}) is log Calabi–Yau, i.e. D_i is a smooth projective variety over k and Δ_{D_i} is a divisor such that $K_{D_i} + \Delta_{D_i}$ is trivial. By [EM21, Theorem 6.14] there exists a Lagrangian torus fibration*

$$\phi : \mathcal{U} \to B \subseteq \mathrm{Star}(v_{D_i}) \setminus W$$

where $\mathcal{U}$ is a symplectic tubular neighborhood of the 1-dimensional strata of Δ_{D_i}, B is a retract of $\mathrm{Star}(v_{D_i})\setminus W$, and W is the union of cells of codimension $\geqslant 2$ in $\mathrm{Sk}(\mathscr{X})$. The fibration ϕ is constructed gluing toric moment maps defined in the neighborhood of each stratum curve of Δ_{D_i}. Evans and Mauri compare the monodromy T_ϕ induced by ϕ on B to the monodromy $T_{\rho_{\mathscr{X}}}$ induced by the affinoid torus fibration

$$\rho_{\mathscr{X}} : \rho_{\mathscr{X}}^{-1}(\mathrm{Star}(v_{D_i}) \setminus W) \to \mathrm{Star}(v_{D_i}) \setminus W$$

and conclude that they are dual. This means that given a loop $\gamma \in \pi_1(B) \simeq \pi_1(\mathrm{Star}(v_{D_i}) \setminus W)$, we have $T_{\rho_{\mathscr{X}}}(\gamma) = (T_\phi(\gamma)^{-1})^T$. Thus the affine structure constructed in [NXY19] has a symplectic topological analog. The duality is due to the fact that the image of the moment maps lives in $M_\mathbb{R}$, while the image of the tropicalization map val *is in $N_\mathbb{R}$.*

Case of K3 surfaces

Let X/K be a maximally degenerate $K3$ surface and let $\mathscr{X}/R$ be a minimal model of X with reduced special fiber $\mathscr{X}_0 = \sum_{i\in I} D_i$. The dual complex $\mathcal{D}(\mathscr{X}_0)$ is well-known to be a triangulated 2-sphere, whose vertices correspond to the irreducible components of $\mathscr{X}_0$.
We focus our attention to such a vertex v_D, and hence to the corresponding irreducible component D of $\mathscr{X}_0$, which has boundary $\Delta_D := \sum_{i=1}^r (D_i \cap D) = \sum_{i=1}^r C_i$. Since the simple normal crossing curve $\Delta_D \in |-K_D|$ is an anticanonical curve by adjunction, it follows from general surface theory that Δ_D is a cycle of rational curves $(C_i)_{i\leq r}$, whose geometry is encoded by the $b_i = -(C_i \cdot D) = -(C_i^2)_D - 2$. We label the curves so that for $i \leq r$, $C_i \cap C_{i+1} \neq \varnothing$, with convention $C_{r+1} = C_1$.

One can associate to the pair (D, Δ_D) a pseudo-fan, which is a singular affine structure on $\mathbb{R}^2$, singular at most at 0. The singularity at 0 is a way to measure the defect of (D, Δ_D) of being toric: the affine structure affine extends smoothly at 0 if and only (D, Δ_D) is a toric pair [Eng18, Proposition 3.9].
The construction, as explained in [GHK15, §1.2], is the following. For each node $p_i = C_i \cap C_{i+1}$, consider a cone $\sigma_i := \mathbb{R}_{\geq 0} v_i + \mathbb{R}_{\geq 0} v_{i+1} \subset \mathbb{R}^2$, (v_i, v_{i+1}) being a basis of the lattice $\mathbb{Z}^2$. The cones σ_i and σ_{i+1} are then glued to each other along $\mathbb{R}_{\geq 0} v_{i+1}$, and the affine structure is extended through the edge by pretending that the pair (D, Δ_D) is toric. If the pair was toric, the σ_i's would be the maximal cones of its fan, and the relation

$$v_{i+2} + v_i = b_{i+1} v_{i+1}$$

would hold by Eq. 2.1.15, so that the chart $\psi_i : \sigma_i \cup \sigma_{i+1}$ that defines the $\mathbb{Z}$-affine structure satisfies $\psi_i(0) = 0$, $\psi_i(v_i) = (1, 0)$, $\psi_i(v_{i+1}) = (0, 1)$ and $\psi_i(v_{i+2}) = (-1, b_{i+1})$, and is extended by dilatation. The unions of the σ_i's, glued along the

successive edges is homeomorphic to $\mathbb{R}^2$, and we obtain this way a $\mathbb{Z}$-affine structure away from the origin, extending to 0 if and only the pair is toric.

It follows from prop. 3.3.3 that the singular $\mathbb{Z}$-affine structure induced by the Berkovich retraction $\rho_{\mathscr{X}}$ coincides with the one described above. We now determine the monodromy around the singularities.

Corollary 3.3.8. *Let D be an irreducible component of $\mathscr{X}_0$, with boundary $\Delta_D = \sum_{i=1}^r C_i$. Writing $b_i = -(C_i^2)_D$, the monodromy $T_{\rho_{\mathscr{X}}}$ of the $\mathbb{Z}$-affine structure induced by $\rho_{\mathscr{X}}$ around v_D is given by*

$$T_{\rho_{\mathscr{X}}} = \begin{pmatrix} b_r & 1 \\ -1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} b_2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to the basis (v_{D_r}, v_{D_1}) and origin v_D.

Proof. By prop. 3.3.3 the integral affine structure on $\mathrm{Star}(\tau_{C_i})$ identifies the set of vertices $(v_{D_{i-1}}, v_{D_i}, v_D, v_{D_{i+1}})$ with:

$$(v_0 = (1,0), v_1 = (0,1), v_2 = (0,0), v_\infty = (-1, b_i)),$$

while on $\mathrm{Star}(\tau_{C_{i+1}})$ identifies $(v_{D_i}, v_{D_i+1}, v_D, v_{D_{i+2}})$ with=

$$(v_0 = (1,0), v_1 = (0,1), v_2 = (0,0), v_\infty = (-1, b_{i+1})).$$

It follows that the transition map from the chart $\mathrm{Star}(\tau_{C_i})$ to $\mathrm{Star}(\tau_{C_{i+1}})$ of the integral affine structure on $\mathrm{Star}(\tau_{C_i}) \cap \mathrm{Star}(\tau_{C_{i+1}})$ is given by the matrix $\begin{pmatrix} b_i & 1 \\ -1 & 0 \end{pmatrix}$. Thus, the composition of such matrices gives the monodromy around v_D, along a loop oriented as the path connecting $v_{D_1}, v_{D_2}, \ldots, v_{D_r}, v_{D_1}$. □

Remark 3.3.9. *It is well-known (see for instance [GHK15]) that $T_{\rho_{\mathscr{X}}} = \mathrm{Id}$ if and only the pair $(D, \Delta_D = \sum_{i=1}^r C_i)$ is toric, or if and only if the charge Q vanishes, where*

$$Q = \chi_{top}(D \setminus \Delta_D) = 12 + \sum_{i=1}^{r} (b_i - 3).$$

Example 3.3.10. *If S is a K3 surface of Type III, and $\mathscr{X}/R$ a minimal model of S, then the map $\rho_{\mathscr{X}}$ is an affinoid torus fibration away from the vertices of $\mathcal{D}(\mathscr{X}_k)$. The induced $\mathbb{Z}$-affine structure with isolated singularities on the 2-sphere matches the one constructed in [GHK15, 1.2] and [Eng18, Proposition 3.10], and has no singularity at a vertex v_D if and only if the corresponding component D of $\mathscr{X}_k$ is toric.*

3.3.3 Integral affine structure induced by combining several models

In this short section, we explain how to compute the monodromy of the integral affine structure induced by a class of admissible retractions that are not necessarily associated to a model of X (and rather to several of them). These results can be used for instance to compute the monodromy of the retraction we will construct in thm. 4.1.8 for degenerations of hypersurfaces. We start with a general definition.

Definition 3.3.11. *Let τ be a simplex of dimension m and consider the first barycentric subdivision τ' of τ. For each vertex v of τ, we denote the star of v in τ' by $\widetilde{\mathrm{Star}}(v)$ and define Γ_{m-1} to be the polyhedral complex of dimension $m-1$ given by*

$$\Gamma_{m-1} := \tau \setminus \bigcup_{v \in \tau} \widetilde{\mathrm{Star}}(v) \subset \tau.$$

For instance, if $m = 2$, Γ_1 is the union of the three line segments joining the barycenter of the triangle to the barycenters of the edges.

We return to the setting of section 3.3 - that is, let X/K be a smooth n-dimensional maximally degenerate Calabi–Yau variety. Assume we are given two minimal models $\mathscr{X}$, $\mathscr{X}'$ of X such that $\mathcal{D}(\mathscr{X}_0) = \mathcal{D}(\mathscr{X}'_0)$, so that $\mathrm{Sk}(\mathscr{X}) = \mathrm{Sk}(\mathscr{X}') = \mathrm{Sk}(X)$, not only as subsets but also with the same cellular decomposition. We fix an ordered labelling $(1, \dots, s)$ of the vertices of $\mathrm{Sk}(X)$, equivalently of the irreducible components of the special fiber of $\mathscr{X}$ (resp. $\mathscr{X}'$).
Fix a codimension 1 face τ of $\mathcal{D}(\mathscr{X}_0) = \mathcal{D}(\mathscr{X}'_0)$, with vertices $v_{i_1}, \dots, v_{i_n}$. We write C (resp. C') for the corresponding strata curves of $\mathscr{X}$ (resp. $\mathscr{X}'$), and D_{i_l} (resp. D'_{i_l}), $l = 1, \dots, n$ the corresponding components. We then have $C = D_{i_1} \cap \ldots \cap D_{i_n}$, and similarly for C'. We write $b_{i_l} = -(C \cdot D_{i_l})_{\mathscr{X}}$ the intersection number computed inside $\mathscr{X}$, and similarly, $b'_{i_l} = -(C' \cdot D'_{i_l})_{\mathscr{X}'}$ for $l = 1, \dots, n$. The $(n-1)$-dimensional face τ is contained in two maximal faces τ_{p_0} and τ_{p_∞} of $\mathcal{D}(\mathscr{X}_0)$, since the boundary of C in $(\mathscr{X}, \mathscr{X}_0)$ consists of two strata points $p_0 = C \cap D_{i_0}$ and $p_\infty = C \cap D_{i_\infty}$; we assume $i_0 < i_\infty$. We set $\mathcal{S} := \tau_{p_0} \cup \tau_{p_\infty} \subset \mathrm{Sk}(X)$, and $\Gamma := \Gamma_{n-2} \subset \tau_C$ as in definition 3.3.11.
Given two vertices $v_{i_m}, v_{i_{m'}}$ of τ_C, with corresponding components D_{i_m} and $D_{i_{m'}}$ of $\mathscr{X}_0$ containing C, we assume $i_m < i_{m'}$ and construct a loop γ as follows:

- γ is contained in $\mathcal{S} \setminus \Gamma$;
- γ goes around the segment joining the barycenter of τ_C with the barycenter of the edge between v_{i_m} and $v_{i_{m'}}$;
- γ has an orientation induced by the fixed ordered labelling on the vertices of $\mathrm{Sk}(X)$ in the following way: in $\mathcal{S} \setminus \Gamma$, γ is homotopy equivalent to the closed path given by the edges which connect in order

$$v_{D_{i_0}}, v_{D_{i_m}}, v_{D_{i_\infty}}, v_{D_{i_{m'}}}.$$

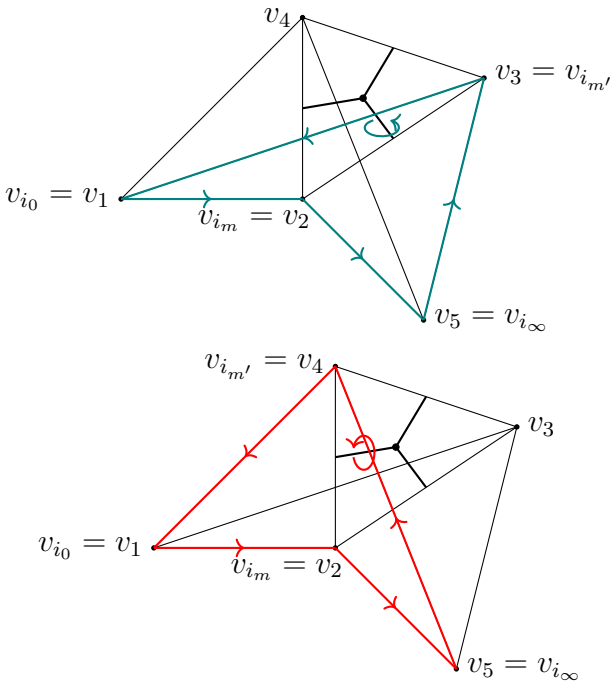


Figure 3.1: Two examples of loops γ in the case $n = 3$ and $C = D_2 \cap D_3 \cap D_4$

Suppose we are given a retraction $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ such that

$$\rho = \begin{cases} \rho_{\mathscr{X}} & \text{over } U_{i_m} := \mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup \widetilde{\mathrm{Star}}(v_{i_m}) \\ \rho_{\mathscr{X}'} & \text{over } U_{i_{m'}} := \mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup \widetilde{\mathrm{Star}}(v_{i_{m'}}). \end{cases}$$

Proposition 3.3.12. *The monodromy along the loop γ, of the $\mathbb{Z}$-affine structure induced by ρ on $U_{i_m} \cup U_{i_{m'}}$ is*

$$T_\rho(\gamma) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ b_{i_1} - b'_{i_1} & 1 & 0 & \dots & 0 \\ b_{i_2} - b'_{i_2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ b_{i_{n-1}} - b'_{i_{n-1}} & 0 & 0 & \dots & 1 \end{pmatrix} \tag{3.3.13}$$

with respect to the basis $(v_{D_{i_0}}, v_{D_{i_1}}, \dots, v_{D_{i_{n-1}}})$ and origin $v_{D_{i_n}}$.

Proof. We need to compute the parallel transport of the vectors

$$(u_0, \dots, u_{n-1}) := (v_{D_{i_0}} - v_{D_{i_n}}, v_{D_{i_1}} - v_{D_{i_n}}, \dots, v_{D_{i_{n-1}}} - v_{D_{i_n}})$$

along the loop γ. By prop. 3.3.3 the $\mathbb{Z}$-affine structure on $\mathrm{Star}(\tau_C)$ induced by $\rho_{\mathscr{X}}$ is described by the chart which has the following vertices:

$v_0 = (1, 0, \dots, 0)$, $v_1 = (0, 1, \dots, 0)$, $\dots$, $v_n = 0$ and $v_\infty = (-1, b_{i_1}, \dots, b_{i_{n-1}})$;

while the $\mathbb{Z}$-affine structure induced by $\rho_{\mathscr{X}'}$ is given by:

$v'_0 = (1, 0, \dots, 0)$, $v'_1 = (0, 1, \dots, 0)$, $\dots$, $v'_n = 0$ and $v'_\infty = (-1, b'_{i_1}, \dots, b'_{i_{n-1}})$.

Moreover, the vectors u_l correspond to the vectors v_l (resp. v'_l) in the chart for $\rho_{\mathscr{X}}$ (resp. $\rho_{\mathscr{X}'}$). We now have $v_0 = -v_\infty + \sum_{l=1}^{n-1} b_{i_l} v_l$, so that the vectors we are transporting are written on τ_{p_∞}

$$(-v_\infty + \sum_{l=1}^{n-1} b_{i_l} v_l \, , v_1, \dots, v_{n-1})$$

in the chart for $\rho_{\mathscr{X}}$. These are thus mapped to the tuple $(-v'_\infty + \sum_{l=1}^{n-1} b_{i_l} v'_l, v'_1, \dots, v'_{n-1})$ by the chart for $\rho_{\mathscr{X}'}$. We now transport back across τ_C in the chart for $\rho_{\mathscr{X}'}$, to get the tuple of vectors

$$(-v'_0 - \sum_{l=1}^{n-1} b'_{i_l} v'_l + \sum_{l=1}^{n-1} b_{i_l} v'_l \, , v'_1, \dots, v'_{n-1})$$

according to the relation $-v'_\infty = -v'_0 - \sum_{l=1}^{n-1} b'_{i_l} v'_l$. We now see that after parallel transport the vectors $(u_0, \dots, u_{n-1})$ have changed to

$$(u_0 + \sum_{l=1}^{n-1} (b_{i_l} - b'_{i_l}) u_l \, , u_1, \dots, u_{n-1}),$$

hence the formula 3.3.13 for the monodromy matrix. □

Case of K3 surfaces

We focus on the case of a maximally degenerate $K3$ surface X/K. We have $C = D_{i_m} \cap D_{i_{m'}}$, $\Gamma = \{a\}$ is a point in the interior of τ_C and γ is a loop around a, oriented as the path joining in order $v_{D_{i_0}}, v_{D_{i_m}}, v_{D_{i_\infty}}, v_{D_{i_{m'}}}$. We assume we have a retraction $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ such that

$$\rho = \begin{cases} \rho_{\mathscr{X}} & \text{over } \mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup [v_{i_m}, a) \\ \rho_{\mathscr{X}'} & \text{over } \mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup [v_{i_{m'}}, a) \end{cases}$$

where $[v_{i_\cdot}, a)$ is the part of the edge τ_C joining the vertex to a, but not including a. Then prop. 3.3.12 may be rewritten as follows; in particular the obtained singularites are of Kodaira type I_m for some $m \in \mathbb{N}$:

Corollary 3.3.14. *The monodromy along the loop γ, of the $\mathbb{Z}$-affine structure induced by ρ on $\mathrm{Star}(\tau_C) \setminus \{a\}$, is*

$$T_\rho(\gamma) = \begin{pmatrix} 1 & 0 \\ b_{i_m} - b'_{i_m} & 1 \end{pmatrix} \tag{3.3.15}$$

with respect to the basis $(v_{D_{i_0}}, v_{D_{i_m}})$ and origin $v_{D_{i_{m'}}}$.

Chapter 4

Degenerations of Calabi-Yau hypersurfaces

In this chapter, we will be focusing on degenerations of Calabi-Yau manifolds obtained in the following way:

$$X = \{z_0...z_{n+1} + tF_{n+2} = 0\} \subset \mathbb{P}^{n+1} \times \mathbb{D}^*,$$

where F_{n+2} is a general homogeneous polynomial of degree $(n+2)$. In this setting, the closure $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{D}$ of X inside the projective space provides us with a somewhat canonical model of our degeneration, whose special fiber is the toric boundary of $\mathbb{P}^{n+1}$. In particular, the assumptions of theorem 3.1.21 hold for every component of $\mathscr{X}_0$, but as we shall see, while the model $\mathscr{X}$ is indeed a minimal dlt model of X - and in particular $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$ - it is not *good*, i.e. the irreducible components of $\mathscr{X}_0$ are not $\mathbb{Q}$-Cartier, so that it is not possible to apply def. 1.1.17 to produce a Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$.
Nevertheless, if v_i is a vertex of $\mathrm{Sk}(\mathscr{X})$, the toric nature of the corresponding component D_i allows us to define in a natural way an integral affine structure with no singularities on $\mathrm{Star}(v_i)$. These integral affine charts cover $\mathrm{Sk}(X)$ but are however not compatible, so that the local affine structures do not glue to an integral affine structure on the whole $\mathrm{Sk}(X)$. However, after a somewhat *ad hoc* choice of codimension 2 discriminant locus $\Gamma \subset \mathrm{Sk}(X)$, we obtain a singular affine structure on $\mathrm{Sk}(X) \setminus \Gamma$, which recovers the fan of D_i near each vertex v_i; this integral affine structure was initially defined in [GS06].
The main result of this chapter, theorem 4.1.8, is the construction of a retraction $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$, which is an affinoid torus fibration away from the codimension 2 locus Γ, and which recovers the above-mentioned singular affine structure on $\mathrm{Sk}(X)$. This builds on the construction by Yamamoto [Yam21] of tropical contractions, from tropicalizations of Calabi-Yau complete intersections in toric varieties to their essential skeleton, and generalizes the construction from [MPS21, thm. B] to any dimension.
While the retraction constructed above depends on a certain choice - namely, an interior point of each face of $\mathrm{Sk}(\mathscr{X})$ - in the case of Fermat hypersurfaces, where:

$$F(z) = z_0^{n+2} + ... + z_{n+1}^{n+2},$$

the $\mathfrak{S}_{n+2}$-symmetry of the problem suggests that choosing the barycenter of each face is a natural way to make the retraction ρ we construct canonical. It indeed turns out that using the results from [Li22] on the convergence of the Calabi-Yau metrics, we are able to describe explicitly the solution to the non-archimedean Monge-Ampère in this setting, and prove that it satisfies the comparison property with respect to the retraction ρ.
This chapter is organized as follows: in section 4.1, we set the scene and describe the construction of the tropical contraction from [Yam21], before studying some small resolutions of the model $\mathscr{X}$; and conclude with a proof of thm. 4.1.8. In section 4.2, we provide an alternative construction of the retraction ρ in dimension 2 and 3, following [MPS21]; we also provide a refinement of the construction in dimension 2. Finally, in section 4.3, we focus on the Fermat family of hypersurfaces, where we apply the results of [Li22] to describe the solution of the non-archimedean Monge-Ampère equation in this case, and prove that the comparison property holds for the retraction constructed in thm. 4.1.8. Along the way, we prove that the local $\mathcal{C}^\infty$-limit obtained for the Fermat family in [Li22] is unique and does not depend on a choice of subsequence.

4.1 The SYZ fibration for hypersurfaces

4.1.1 Degenerations of Calabi-Yau hypersurfaces

We are interested in degenerations of Calabi-Yau manifolds obtained in the following way:

$$X = \{z_0 ... z_{n+1} + tF_{n+2} = 0\} \subset \mathbb{P}^{n+1} \times \mathbb{D}^*,$$

where $F_{n+2} \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}}(n+2))$ is a general section; we also denote by X the base change to $K = \mathbb{C}((t))$. The fibers X_t are Calabi-Yau manifolds by the adjunction formula, and even simply-connected, strict Calabi-Yau whenever $n > 1$ (that is, $h^i(X_t, \mathcal{O}_{X_t}) = 0$ for $0 < i < n$) by the Lefschetz theorem on hyperplane sections. We will see that in this case, the essential skeleton $\mathrm{Sk}(X)$ can be realized explicitly using tropical geometry, and there is a natural way of endowing it with an integral affine structure, which is singular in codimension 2. The construction we outline here is a special case of [Gro05], but since the hypersurface case is far less technical we will outline the details.
We start by identifying the essential skeleton $\mathrm{Sk}(X)$ with the Berkovich skeleton of the model $\mathscr{X}$:

Proposition 4.1.1. *Let $F \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}}(n+2))$ be a general section. Then the hypersurface:*

$$\mathscr{X} = \{z_0 ... z_{n+1} + tF = 0\} \subset \mathbb{P}^{n+1}_R$$

is a minimal dlt model of the smooth, maximally degenerate Calabi-Yau manifold $X = \mathscr{X}_K$; in particular $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$.

More precisely, the statement of the above proposition holds whenever F is not identically zero on any toric stratum Z of $\mathbb{P}^{n+1}$, and the intersection $Z \cap \{F = 0\}$ is smooth; such F are called *admissible* in [HJMM22].

Remark 4.1.2. *As shown by the proof, the model $\mathscr{X}$ will not be good for general choice of F - the irreducible components of $\mathscr{X}_0$ are not $\mathbb{Q}$-Cartier at the singular points of $\mathscr{X}$ as soon as $n \geq 2$. Thus, there does not exist a Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ in the sense of def. 1.1.17, which is consistent with the statement of theorem 3.1.21: if such a retraction existed it would be an affinoid torus fibration over the whole $\mathrm{Sk}(X) \simeq \mathbb{S}^n$, yielding an integral affine structure with no singularities on an n-sphere, which is impossible.*

Proof. It follows from an elementary computation that for general F, the singular locus of the total space $\mathscr{X}$ is given by:

$$\mathrm{Sing}(\mathscr{X}) = \bigcup_{i \neq j} \{z_i = z_j = t = F = 0\} \subset \mathscr{X}_0,$$

which is of codimension 3 in the total space $\mathscr{X}$. The points where the singularity is the most severe are the intersection of the strata curves of $\mathscr{X}_0$ with the locus $\{F = 0\}$, since F is general this locus does not meet the zero-dimensional strata of $\mathscr{X}_0$. Thus, we may focus on the curve $\{z_1 = ... = z_n = 0\} \subset \mathscr{X}_0$, where the singularity is étale-locally of the form:

$$\mathscr{U} = \{z_1....z_n + tw = 0\} \subset \mathbb{A}^{n+1}_R.$$

so that by [CLS11, prop. 11.4.24], the pair $(\mathscr{U}, \mathscr{U}_0)$ is log canonical. We also infer that at the generic point of a stratum, $\mathscr{X}$ is smooth while the components of $\mathscr{X}_0$ meet transversally, which proves that $\mathscr{X}$ is a dlt model of X.
Moreover, $\mathscr{X}_0$ is reduced and $K_{\mathscr{X}/R} = \mathcal{O}_{\mathscr{X}}$ by adjunction, which concludes. □

From now on, we will assume that F is general, so the statement of the above lemma holds. We view $Z_\Sigma = \mathbb{P}^{n+1}$ as a smooth toric Fano variety of dimension $(n+1)$, with simple normal crossing boundary $\Delta = \sum_{l=0}^{n+1} D_l$ the sum of coordinate hyperplanes. We additionally let $L = \mathcal{O}_{\mathbb{P}}(n+2)$ be the anticanonical polarization on $\mathbb{P}^{n+1}$, and we write $P \subset M_{\mathbb{R}}$ the associated polytope, as in def. 2.1.17. If $(v_1, ..., v_{n+1})$ is a basis of N, we set $v_0 := -(v_1 + ... + v_{n+1})$ and the v_l's for $l = 0, ..., (n+1)$ are the primitive generators of the rays of Σ.
Writing $P^* \subset N_{\mathbb{R}}$ the polar dual of P, that is:

$$P^* = \{x \in N_{\mathbb{R}} / \langle u, x \rangle \leq 1 \ \forall u \in P\},$$

we see that here $P^* = \mathrm{Conv}(v_0, ..., v_n)$.
One of the reasons such degenerations provide an interesting class of examples is that their essential skeleton can be realized in the toric world, as we will now explain. The dual complex of the toric boundary $\Delta \subset \mathbb{CP}^{n+1}$ can naturally be realized inside $N_{\mathbb{R}}$: if D_l is a coordinate hyperplane, it corresponds to $e_l \in N$ the primitive integral generator of the corresponding ray of Σ. Since $\mathscr{X}_0 = \Delta$, we see that we have a canonical isomorphism of simplicial complexes between the dual complex $\mathscr{D}(\mathscr{X}_0)$ and the boundary ∂P^*, sending a vertex $v_{D_l} \in \mathscr{D}(\mathscr{X}_0)$ to $v_l \in N$. Realizing $\mathscr{D}(\mathscr{X}_0) \simeq \mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ inside the Berkovich analytification, we have the more precise statement:

Proposition 4.1.3. *Let $X \subset \mathbb{P}_K^{n+1}$ as above. Then the composition:*

$$X^{\mathrm{an}} \subset \mathbb{P}^{n+1,\mathrm{an}} \xrightarrow{\mathrm{val}_\Sigma} N_\Sigma$$

induces by restriction a isomorphism of simplicial complexes from $\mathrm{Sk}(\mathscr{X})$ *to* ∂P^*.

Proof. It is enough to prove that if $D_{l_1}, ..., D_{l_{n+1}}$ are $(n+1)$ components of $\Delta = \mathscr{X}_0$ meeting at a point p, then val maps the corresponding maximal face τ_p of $\mathrm{Sk}(\mathscr{X})$ homeomorphically to:

$$\tau_\sigma = \mathrm{Conv}(v_{l_1}, ..., v_{l_{n+1}}) \subset N_{\mathbb{R}},$$

where $\sigma = \sum_{i=1}^{n+1} \mathbb{R}_{\geq 0} v_{l_i}$. Thus, we let $v \in \tau_p$ be a quasi-monomial valuation, which we see as a semi-valuation on $\mathbb{P}^{n+1}$. Using prop. 1.1.14, we identify v with the monomial weight $w = (w_1, ..., w_{n+1})$, where $w_i = v(f_i)$ for any local equation f_i of D_{l_i} at p. We have $\sum_{i=1}^{n+1} w_i = 1$, by the normalization $v_w(t) = 1$ and the equation for the hypersurface. We let l_0 be the index such that $p \notin D_{l_0}$, and take as a local equation:

$$f_i = \frac{z_{l_i}}{z_{l_0}},$$

where $[z_0 : ... : z_{n+1}]$ are standard homogeneous coordinates on $\mathbb{P}_K^{n+1}$ - this is possible as F does not vanish at p, since it is general. Then by definition of the val map, we have:

$$\langle \mathrm{val}(v), m \rangle = v(\chi^m)$$

for any $m \in M$. We let $(m_1, ..., m_{n+1})$ be the basis of M dual to $(v_{l_1}, ..., v_{l_n})$, so that χ^{m_i} is an equation for D_{l_i} on the toric affine chart $\mathbb{P}_\sigma$. We now have $\chi^{m_i} = f_i$, so that writing $m = \sum_{i=1}^{n+1} \lambda_i m_i$, we infer from prop. 1.1.14:

$$v(\chi^m) = \sum_{i=1}^{n+1} w_i \lambda_i,$$

or equivalently $\mathrm{val}(v) = \sum_{i=1}^{n+1} w_i v_{l_i}$, which concludes the proof. □

We are thus naturally led to study the image inside N_Σ of X^{an} by the valuation map val_Σ. This is well-known to be the (extended) tropicalization $\overline{\mathrm{Trop}}(X)$ of the hypersurface, defined as follows. Write the generic section F as a sum of monomials:

$$F(z) = \sum_{m \in P_{\mathbb{Z}}} c_m z^m$$

and let $L : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be the piecewise-affine function obtained by replacing, in the equation $(tF + z_0 ... z_{n+1})$ for X, the products by sums and the sums by minima. More explicitly, L is defined by the formula:

$$L(x) = \min\{0, \min_{c_m \neq 0}(1 - \langle m, x \rangle)\},$$

and $\mathrm{Trop}(X) \subset N_{\mathbb{R}}$ the locus where the minimum is achieved by at least two different terms. From now on, we will make the following combinatorial assumption:

Condition 4.1.4. *Whenever $m \in P_{\mathbb{Z}}$ is a vertex, we have $c_m \neq 0$ - equivalently, the Newton polytope of the section F is equal to the whole P.*

This immediately implies the equality:

$$L(x) = \min\{0, \min_{m \in V(P)} (1 - \langle m, x \rangle)\}.$$

Proposition 4.1.5. *The closure $\overline{\mathrm{Trop}}(X) \subset N_\Sigma$ is the image of X^{an} via the map:*

$$\mathrm{val}_\Sigma : Z_\Sigma^{\mathrm{an}} \longrightarrow N_\Sigma.$$

Moreover, the complement of $\mathrm{Trop}(X)$ in $N_{\mathbb{R}}$ admits one bounded connected component, which is the interior of P^*.

Proof. The first item follows from the fundamental theorem of tropical geometry [MS15, thm. 3.1.1], while the second one follows from the fact that the polytope P^* is the locus where $L = 0$. □

We will now construct a piecewise-affine subset $\Gamma \subset \mathrm{Sk}(X)$ of codimension 2, and an integral affine structure on $\mathrm{Sk}(X) \setminus \Gamma$, compatible with the integral affine structure on the interior of the maximal faces of $\mathrm{Sk}(\mathscr{X})$, and compatible with the fan structure (definition 3.3.1) in the neighbourhood of each vertex. The construction is non-canonical, as it depends on the following choice: for each face τ of $\mathrm{Sk}(\mathscr{X})$ of dimension > 0, we choose a point $a_\tau \in \mathrm{Int}(\tau)$. In the sequel, we will call the collection $\underline{a} = (a_\tau)_\tau$ a *choice of branch cuts*. If τ is a vertex of $\mathrm{Sk}(\mathscr{X})$, we will write $a_\tau = \tau$.
We let $\widetilde{\mathscr{P}}$ be the subdivision of $\mathrm{Sk}(\mathscr{X})$ whose faces are of the form:

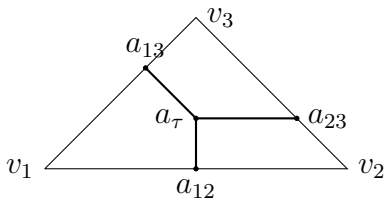
$$\tau' = \mathrm{Conv}(a_{\tau_0}, ..., a_{\tau_l}),$$

whenever $\tau_0 \subset ... \subset \tau_l$ is an ascending chain of faces of $\mathrm{Sk}(\mathscr{X})$.
The discriminant locus in now given as:

$$\Gamma := \bigcup \mathrm{Conv}(a_{\tau_0}, ..., a_{\tau_l}),$$

where the union is taken over ascending chains of faces $\tau_0 \subset ... \subset \tau_l$ of $\mathrm{Sk}(\mathscr{X})$, with $\dim \tau_0 \geq 1$ and $\dim \tau_l \leq (n-1)$. It is clear from the definition that it defines a codimension 2 piecewise-affine subset of $\mathrm{Sk}(X)$, which generalizes definition 3.3.11. We will sometimes write $\Gamma = \Gamma_{\underline{a}}$ whenever we wish to emphasize the dependency on the choice of branch cuts $\underline{a}$.

Example 4.1.6. *Assume $n = 2$, then we are choosing a point $a_e \in \mathrm{Int}(e)$ in each edge of $\mathrm{Sk}(\mathscr{X})$, in addition to a point $a_\tau \in \mathrm{Int}(\tau)$ in each maximal face τ of $\mathrm{Sk}(\mathscr{X})$. The discriminant locus Γ consists of the set $\{a_e\}_e$, which is finite.*
Assume now that $n = 3$, then we are choosing an interior point in each d-cell of $\mathrm{Sk}(\mathscr{X})$ for $d = 1, 2, 3$. The discriminant locus Γ is a trivalent graph, contained in the union of the 2-cells of $\mathrm{Sk}(\mathscr{X})$, and its intersection with the 2-face $\tau = \mathrm{Conv}(v_1, v_2, v_3)$ is depicted below:



Let $v \in \mathrm{Sk}(\mathscr{X})$ be a vertex, and Σ_v be the fan of D_v in $\mathbb{R}^n$. As in definition 3.3.1, we let:

$$\phi_v : \mathrm{Star}(v) \longrightarrow \mathbb{R}^n$$

be the unique piecewise-affine homeomorphism sending v to $0 \in \mathbb{R}^n$, and each vertex $v_l \in \mathrm{Star}(v)$ to the primitive generator of the corresponding ray in Σ_v.
Moreover, if $\tau \subset \mathrm{Sk}(\mathscr{X})$ is a maximal face, we let $\phi_\tau : \tau \longrightarrow \mathbb{R}^n$ be an integral affine homeomorphism onto a standard simplex.
As mentioned above, the affine charts ϕ_v do not glue to define an integral affine structure on $\mathrm{Sk}(\mathscr{X})$, i.e. the transition functions between them are not integral affine. To remedy this, we simply shrink the stars of the vertices using the subdivision $\widetilde{\mathscr{P}}$ so that they don't intersect anymore. More precisely, for a vertex v of the skeleton, let $\widetilde{\mathrm{Star}}(v)$ be the star of v in the polyhedral complex $\widetilde{\mathscr{P}}$. Then the singular affine structure on $\mathrm{Sk}(\mathscr{X})$ is defined as follows:

Definition 4.1.7. *The collection of charts* $\left((\mathrm{Int}(\widetilde{\mathrm{Star}}(v)), \phi_v)_v \cup (\mathrm{Int}(\tau), \phi_\tau)_\tau\right)$ *is our induced integral affine structure on* $\mathrm{Sk}(X) \setminus \Gamma$.

The main theorem of this section is the following:

Theorem 4.1.8. *Let* $\underline{a} = (a_\tau)_\tau$ *be a choice of branch cuts in* $\mathrm{Sk}(\mathscr{X})$, *and* $\delta_{\underline{a}} : \overline{\mathrm{Trop}}(X) \longrightarrow \mathrm{Sk}(X)$ *the associated tropical contraction [Yam21, thm. 5.1]. Then the composition:*

$$\rho_{\underline{a}} := \delta_{\underline{a}} \circ \mathrm{val}_\Sigma : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$$

is an affinoid torus fibration over $\mathrm{Sk}(X) \setminus \Gamma$. *Moreover, the induced integral affine structure on* $\mathrm{Sk}(X) \setminus \Gamma$ *coincides with the one from definition 4.1.7.*

In the cases $n = 2, 3$, the retraction $\rho_{\underline{a}}$ was constructed in [MPS21] using purely non-archimedean techniques; this alternative construction will be explained in section 4.2.
Let $v = v_i \in \mathrm{Sk}(\mathscr{X})$ be a vertex, and let $U_i \subset \mathrm{Sk}(\mathscr{X})$ be the open star of v_i with respect to the simplicial decomposition $\widetilde{\mathscr{P}}$. Then we need to prove that the map $\rho = \rho_{\underline{a}}$ is an affinoid torus fibration over each U_i, as well as over the interior of each maximal face $\tau \subset \mathrm{Sk}(\mathscr{X})$. The proof will proceed in two steps:

- (section 4.1.2) we mainly review the construction of the tropical contraction, following [Yam21]. The contraction is in fact constructed face by face, and we will describe it in particular near a vertex and over a maximal face. The latter, in combination with prop. 4.1.3, will readily imply that ρ is an affinoid torus fibration over a maximal face of $\mathrm{Sk}(\mathscr{X})$.

- (section 4.1.3) we then focus on a vertex $v_i \in \mathrm{Sk}(\mathscr{X})$, and construct a small log resolution $\mathscr{X}_i \longrightarrow \mathscr{X}$, which is an isomorphism over the corresponding component D_i. In particular, we may apply theorem 3.1.21 to conclude that the associated retraction $\rho_{\mathscr{X}_i}$ is an affinoid torus fibration over $\mathrm{Star}(v_i)$, and we conclude by proving that the equality $\rho = \rho_{\mathscr{X}_i}$ holds over U_i.

4.1.2 The tropical contraction

We let $X \subset \mathbb{P}^{n+1}_K$ be a maximal degeneration of Calabi-Yau hypersurfaces as above. The goal of this section is to describe in a fairly simple way the construction of a certain tropical contraction $\delta : \overline{\mathrm{Trop}}(X) \longrightarrow \mathrm{Sk}(X)$, due to Yamamoto [Yam21]. Here tropical contraction means that the map δ is locally modeled on projections of the form $N_\Sigma \longrightarrow N_{\Sigma'}$, where Σ' is the fan of a (closed) orbit in Z_Σ.

Theorem 4.1.9. *[Yam21, thm. 1.2] For any choice of branch cuts $\underline{a} = (a_\tau)_\tau$, there exists a tropical contraction:*

$$\delta_{\underline{a}} : \overline{\mathrm{Trop}}(X) \longrightarrow \mathrm{Sk}(X),$$

where we view $\mathrm{Sk}(X) \subset \overline{\mathrm{Trop}}(X)$ via the embedding of prop. 4.1.3. The tropical contraction preserves the integral affine structures, in the sense that:

$$\delta_* \, \mathrm{Aff}_{\overline{\mathrm{Trop}}(X)} = j_*(\mathrm{Aff}_{\mathrm{Sk}(X)\setminus\Gamma}),$$

where Aff *denotes the sheaf of integral affine functions on $\overline{\mathrm{Trop}}(X)$ (see [Yam21, def. 2.10]) and $\mathrm{Sk}(X) \setminus \Gamma$ respectively, and $j : \mathrm{Sk}(X) \setminus \Gamma \hookrightarrow \mathrm{Sk}(X)$ the inclusion.*

Let us point out that original construction is far more general, as it applies to a large range of Calabi-Yau complete intersections inside toric varieties. However the codimension one case is substantially simpler, so that we will outline the details of the construction in this case.
We fix once and for all a choice $\underline{a}$ of branch cuts, and consider the associated subdivision $\widetilde{\mathscr{P}}$ of $\mathrm{Sk}(\mathscr{X})$. Our first step will be to describe $\overline{\mathrm{Trop}(X)}$ more explicitly. Recall that the open subset $\mathrm{Trop}(X) \subset \overline{\mathrm{Trop}(X)}$ is given as the corner locus of the function:

$$L(x) = \min\{0, \min_{m \in V(P)}(1 - \langle m, x\rangle)\},$$

hence the following natural stratification. If $x \in \mathrm{Trop}(X)$, we let $J(x) \subset I := \{0\} \cup V(P)$ the subset of indices that realize the minimum; note that $|J(x)| \geq 2$ since $x \in \mathrm{Trop}(X)$. Given a subset $J \subset I$, the associated stratum $T_J \subset \mathrm{Trop}(X)$ is the set of $x \in \mathrm{Trop}(X)$ such that $J(x) \supseteq J$. We furthermore write $\overline{T}_J \subset N_\Sigma$ its closure, which is contained in $\overline{\mathrm{Trop}}(X)$.

Definition 4.1.10. *Let $\tau \subset \partial P^*$ be a face, and write $\tau = \mathrm{Conv}(e_{i_1}, ..., e_{i_r})$. The associated cone $\sigma(\tau) \in \Sigma$ is given by:*

$$\sigma(\tau) = \mathrm{Cone}(e_{i_1}, ..., e_{i_r}),$$

and its closure $\bar{\sigma}(\tau) \subset N_\Sigma$. We additionally set $\tilde{\tau} := \tau + \bar{\sigma}(\tau) \subset N_\Sigma$.

Let $J' \subset V(P)$ be a non-empty subset of vertices of P, this defines a face $\tau_{J'}$ of ∂P^* in the following way:

$$\tau_{J'} = \{x \in P^* / \langle m_j, x\rangle = 1 \ \forall j \in J'\},$$

this is a face of dimension $(n+1) - |J'|$. It is straightforward to check that for $|J'| \geq 2$, the region $\tilde{\tau}_{J'}$ is contained in the stratum $\overline{T}_{J'}$.
The following proposition asserts that the bounded strata of the tropicalization are the faces of ∂P^*, while the unbounded ones are obtained from non-maximal faces of ∂P^* and their associated cones:

Proposition 4.1.11. *Let $I = \{0\} \cup V(P)$, and $J \subset I$ with at least two elements. We let $J' = J \cap V(P)$.*

- *if $0 \in J$, then $\overline{T}_J = \tau_{J'} \subset \partial P^*$;*
- *if $0 \notin J$, then $\overline{T}_J = \tilde{\tau}_{J'}$.*

Note that in the latter case, since $|J| \geq 2$, the face $\tau_{J'}$ is not a maximal face of ∂P^*. The following picture depicts the tropicalization when $n = 2$: $\mathrm{Trop}(X)$ is the union of the boundary of the red polytope P^* with the portions of the linear planes passing through pairs of vertices of P^*; here we have colored $\tilde{\tau}_J$ for $J \subset \{1, 2, 3\}$.

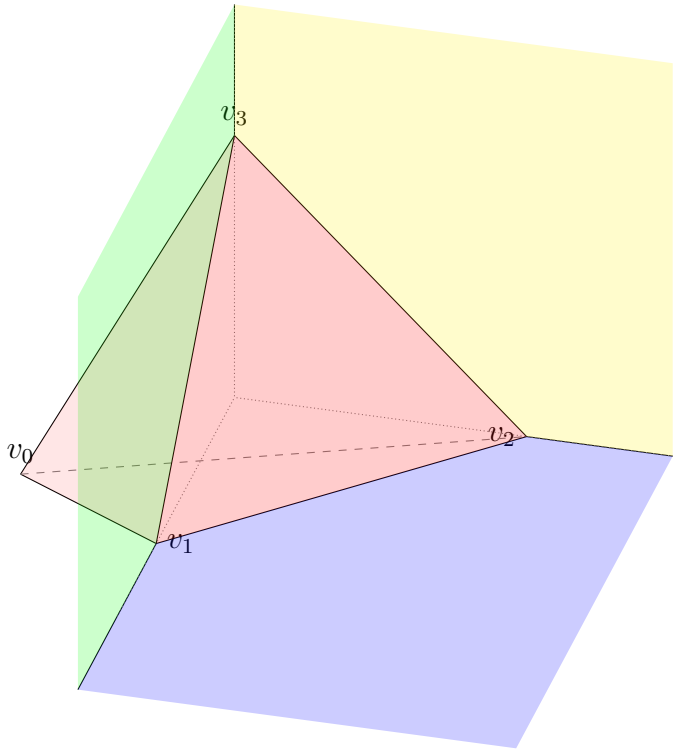


Figure 4.1: The tropicalization of a maximally degenerate K3 surface

We now move to the description of the tropical contraction. If τ is a face of ∂P^*, we set $U_\tau := \mathrm{Int}(\widetilde{\mathrm{Star}}(a_\tau))$ the open star of the point $a_\tau \in \mathrm{Int}(\tau)$ with respect to $\widetilde{\mathscr{P}}$. For $\tau' \subset \tau$ a smaller face of ∂P^*, set:

$$W_{\tau,\tau'} := U_\tau \cap U_{\tau'}$$

and $X_{\tau,\tau'} := W_{\tau,\tau'} + \overline{\sigma}(\tau')$. Then

$$X_\tau := \big(\bigcup_{\tau' \subset \tau} X_{\tau,\tau'} \big) \cap \overline{\mathrm{Trop}}(X),$$

where the union ranges over the subfaces of τ. For instance, if $\tau \subset \partial P^*$ is a maximal face, we simply have $X_\tau = U_\tau = \mathrm{Int}(\tau)$.
Then by [Yam21, lem. 5.15], the union of X_τ cover $\overline{\mathrm{Trop}}(X)$ when τ ranges over the faces of ∂P^*, so that it is enough to define local tropical contractions:

$$\delta_\tau : X_\tau \longrightarrow U_\tau,$$

which glue over the intersections $U_{\tau_1} \cap U_{\tau_2}$ by [Yam21, lem. 5.14]. Then if $x \in X_{\tau,\tau'}$, i.e. $x = w + x'$, with $w \in W_{\tau,\tau'}$ and $x' \in \overline{\sigma}(\tau')$, we set:

$$\delta_\tau(x) = w.$$

In other words, the local tropical contraction contracts the cone direction, and is locally isomorphic over $W_{\tau,\tau'}$ to the canonical projection $p : N_\Sigma \longrightarrow N_{\Sigma'}$, where Σ' is the fan in $N_\mathbb{R}/\mathrm{Span}(\sigma(\tau'))$ induced by Σ.
In the two-dimensional case, the restriction of the tropical contraction to the standard orthant is depicted as follows (we have picked the barycenters of the edges as choice of branch cuts):

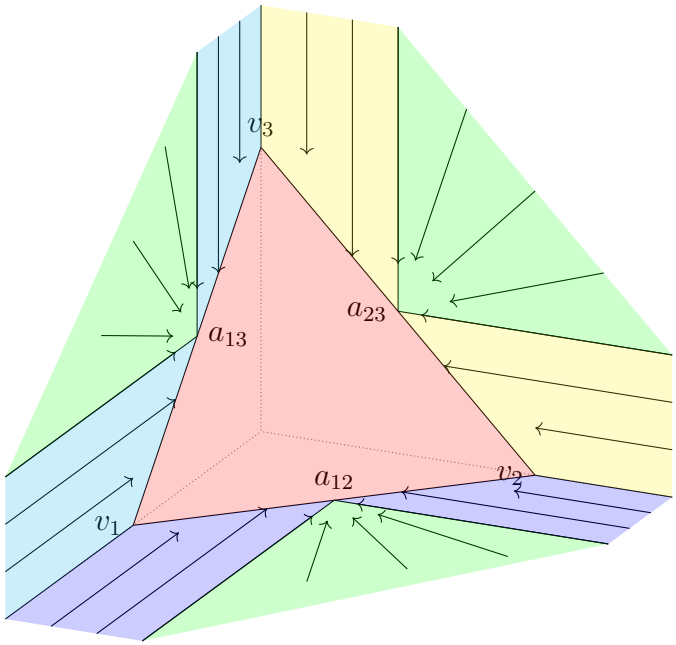


Figure 4.2: The tropical contraction in dimension 2

If $\tau = e_{12}$ is an edge, then $W_{\tau,v_i} = \mathrm{Conv}(a_{12}, v_i)$ for $i = 1, 2$, and the contractions over W_{τ,v_i} are the projection $N_\mathbb{R} \longrightarrow N_\mathbb{R}/\mathbb{R}v_i$ (the blue regions); while $W_{\tau,\tau} = a_{12}$ and $X_{\tau,\tau}$ is the lower green region, contracted to the point a_{12}.

Proposition 4.1.12. *Let τ be a maximal face of $\mathrm{Sk}(\mathscr{X})$. Then the retraction $\rho_{\underline{a}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ is an affinoid torus fibration over $\mathrm{Int}(\tau)$.*

Proof. Let τ be a maximal face of $\mathrm{Sk}(\mathscr{X}) \simeq \partial P^*$, and let $p \in \mathscr{X}_0$ be the corresponding zero-dimensional stratum. Then from the construction of δ, we have $X_\tau = U_\tau = \mathrm{Int}(\tau)$, so that $\delta^{-1}(\mathrm{Int}(\tau)) = \mathrm{Int}(\tau)$, and $\rho^{-1}((\mathrm{Int}(\tau)) = \mathrm{val}_\Sigma^{-1}((\mathrm{Int}(\tau))$. Moreover, by the proof of prop. 4.1.3, if:

$$\tau = \mathrm{Conv}(v_{l_1}, ..., v_{l_n})$$

and $\mathrm{val}_\Sigma(v) = \sum_{i=1}^{n+1} \lambda_i v_{l_i}$, then $\lambda_i = v(\frac{z_i}{z_0})$. In particular, if $\mathrm{val}_\Sigma(v) \in \mathrm{Int}(\tau)$, then $v(\frac{z_i}{z_0}) > 0$ for $i = 1, ..., (n+1)$, or in other words $z_i = 0$ at the center $c_{\mathscr{X}}(v)$. This implies that $\mathrm{val}_\Sigma(v) \in \mathrm{Int}(\tau)$, if and only $c_{\mathscr{X}}(v) = p$.
We infer that $\rho^{-1}(\mathrm{Int}(\tau)) = \mathfrak{X}_p^\eta := (\widehat{\mathscr{X}_{/p}})^\eta$, and that under the identification $\mathrm{Sk}(\mathscr{X}) = \partial P^*$, we have $\rho_{|\mathfrak{X}_p^\eta} = \rho_{\mathfrak{X}_p}$, which is indeed an affinoid torus fibration by example 3.1.16. □

In what follows, we will be mostly interested about what happens near a vertex $v_i \in \mathrm{Sk}(\mathscr{X})$, in which case the following holds:

Proposition 4.1.13. *[Yam21, lem. 5.12] Let $\tau = v_i$ be a vertex of $\mathrm{Sk}(\mathscr{X})$. Then $X_\tau = U_\tau + \overline{\mathbb{R}}_{\geq 0} v_i$, and the local tropical contraction satisfies:*

$$\delta_\tau(w + \alpha v_i) = w,$$

for $w \in U_\tau$ and $\alpha \in [0, +\infty]$.
In other words, writing $p_i : N_\Sigma \longrightarrow N_{\Sigma_i}$ the canonical projection, the following diagram commutes:

$$\begin{array}{ccccc} X_{e_i} & \xrightarrow{\delta_i} & U_i & & \subset N_{\mathbb{R}} \\ & \searrow^{p_i} & \downarrow{\scriptstyle q_i} & & \\ & & V_i & & \subset N_{\mathbb{R}}/\mathbb{R}v_i, \end{array}$$

where $q_i = (p_i)_{|U_i}$ is the homeomorphism from U_i onto its image V_i by the projection p_i.

The above proposition states that up to the homeomorphism q_i, the tropical contraction δ_i is given by the linear projection onto the quotient $N_{\mathbb{R}}/\mathbb{R}v_i$ near the vertex v_i.

4.1.3 Small resolution at a vertex

We consider a point in $\mathscr{X}^{\mathrm{sing}} \cap D_1 \cap ... \cap D_n$, the singular points in the other strata curves can be treated analogously. Etale locally around such a point, $\mathscr{X}$ is isomorphic to the (base change to R of) the toric variety $\mathscr{U} := V(z_1...z_n - wt) \subset \mathbb{A}_k^{n+2}$, where the $D_i|_{\mathscr{U}} = \{z_i = t = 0\}$ are the components of the special fiber $\mathscr{U}_0$. We still denote these by D_i; they form the toric boundary of $\mathscr{U}$ together with the :

$$D_i'|_{\mathscr{U}} = \{z_i = w = 0\}.$$

If $J \subset \{1, ..., n\}$, we let $D_J = \cap_{j \in J} D_j$. The singular locus $\mathscr{U}^{\text{sing}}$ is of codimension 3, and consists of the union over pairs $i \neq j$ of indices of the:

$$V_{ij}|_{\mathscr{U}} = \{z_i = z_j = w = t = 0\} = D_i \cap D_j \cap D'_i \cap D'_j,$$

which all intersect at the torus invariant point $p := \{z_1 = ... = z_n = w = t = 0\}$.

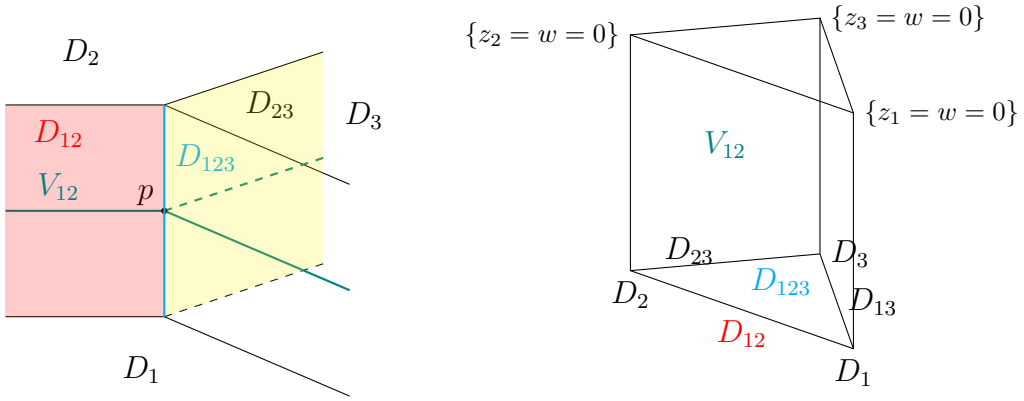


Figure 4.3: Special fiber of $\mathscr{U}$ and a *slice* of the fan of the toric variety $\mathscr{U}$

Proposition 4.1.14. *The composition of successive toric blow-ups:*

$$\mathscr{U}(n-1) = \text{Bl}_{D_{n-1}} \mathscr{U}(n-2) \longrightarrow \mathscr{U}(n-2) = ... \longrightarrow \mathscr{U}(1) = \text{Bl}_{D_1} \mathscr{U} \longrightarrow \mathscr{U},$$

where we inductively abusively denote $D_{j+1} \subset \mathscr{U}(j+1)$ the strict transform of $D_{j+1} \subset \mathscr{U}(j)$, is a small log resolution of $(\mathscr{U}, \mathscr{U}_0)$, such that the strict transform of D_n in $\mathscr{U}(n-1)$ is isomorphic to D_n.

Proof. Let $N = \mathbb{Z}^{n+2}$, with the standard basis written as $(e_1, ..., e_n, e_t, e_w)$, and $N' \subset N$ the codimension 1 sublattice:

$$N' = \{u_1 + ... + u_n = u_t + u_w\} \subset N,$$

where $(u_1, ..., u_n, u_t, u_w)$ is the dual basis in M. Then the primitive generators of the rays of the fan Σ of $\mathscr{U}$ in $N'_{\mathbb{R}}$ are the $v_i = e_i + e_t$ (corresponding to the boundary component $D_i = \{x_i = t = 0\}$) and the $v'_i = e_i + e_w$ (corresponding to the boundary component $D'_i = \{x_i = w = 0\}$), for $i = 1, ..., n$. The fan Σ has a unique maximal cone $\sigma = \sum_{i=1}^n \mathbb{R}_{\geq 0} v_i + \sum_{i=1}^n \mathbb{R}_{\geq 0} v'_i$, it is the cone over the product of an $(n-1)$-simplex and a segment.
We set $\mathscr{I}_i := \mathscr{I}_{\mathscr{U}(i-1)}(D_i)$ the ideal of the strict transform of D_i inside $\mathscr{U}(i-1)$, so that $\mathscr{U}(i)$ is the blow-up of $\mathscr{U}(i-1)$ along the toric ideal $\mathscr{I}_i$.
We claim that the blow-up $\mathscr{U}(i) \longrightarrow \mathscr{U}(i-1)$ has no exceptional divisors (i.e. divisors in $\mathscr{U}(i)$ whose image have codimension > 1), and that $\mathscr{I}_{i+1} = (x_i, t)$. We prove this by induction on $i \in \{1, ..., n-1\}$.
For $i = 1$, we are blowing-up the toric ideal $\mathscr{I} = (x_1, t)$ inside $\mathscr{U}$, this amounts to subdividing the maximal cone σ of Σ into the two subcones $\sigma \cap \{u_1 \geq u_t\}$ and $\sigma \cap \{u_1 \leq u_t\}$ - the piecewise-linear function attached to $\mathscr{I}$ is $\phi_{\mathscr{I}} = \min\{u_1, u_t\}$. It

is straightforward to check that this does not subdivide any 2-dimensional cone of Σ, so that the subdivision does not induce any new ray, and $\mathscr{U}(1)$ has no exceptional divisors. This also implies that the zero locus of $\mathscr{I}_2 = (x_2, t)$ is the strict transform $D_2 \subset \mathscr{U}(1)$.
Now assume that the claim holds for $i \geq 1$, we want to prove that $\mathscr{U}(i+1) \longrightarrow \mathscr{U}(i)$ has no exceptional divisors and that $\mathscr{I}_{i+2} = (x_{i+1}, t)$. Since we are blowing-up the ideal $\mathscr{I}_{i+1} = (x_i, t)$ inside $\mathscr{U}(i)$, this means we are subdividing the fan of $\mathscr{U}(i)$ along the hyperplane $\{u_i = u_t\}$, which is once again easily seen not to subdivide the 2-dimensional cones, hence no rays are added. By the same argument as above, this implies that inside $\mathscr{U}(i+1)$, the zero locus $V(x_{i+1}, t)$ is the strict transform of D_{i+1}.

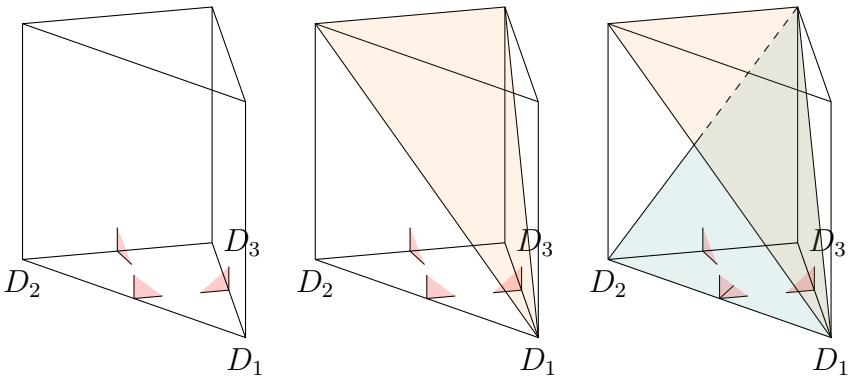


Figure 4.4: Slices of the fan of $\mathscr{U}$, $\mathscr{U}(1)$ and $\mathscr{U}(2)$

We now prove that $\mathscr{U}(n-1)$ is regular. The maximal cones of the fan $\mathscr{U}(n-1)$ are the intersection of σ with $(n-1)$ halfspaces of the form $\{\pm(u_i - u_t) \geq 0\}$; by direct computation there are n maximal cones $\sigma_1, ..., \sigma_n$, where:

$$\sigma_i = \sigma \cap \{u_i \geq u_t\} \cap \{u_{i-1} \leq u_t\},$$

with $\sigma_1 = \sigma \cap \{u_i \geq u_t\}$. This yields:

$$\sigma_i = \mathbb{R}_{\geq 0} v_1 + ... + \mathbb{R}_{\geq 0} v_i + \mathbb{R}_{\geq 0} v_i' + ... + \mathbb{R}_{\geq 0} v_n',$$

which is a regular cone as $(v_1, ..., v_i, v_i', ..., v_n')$ is a basis of N, so that the pair $(\mathscr{U}(n-1), \mathscr{U}_0(n-1))$ is snc.
Finally, we prove that the strict transform of D_n in $\mathscr{U}(n-1)$ is isomorphic to D_n. The ray $\rho_n = \mathbb{R}_{\geq 0} v_n$ viewed in the fan of $\mathscr{U}(n-1)$ is contained in only one maximal cone, namely σ_n. As a result, the fan of the strict transform of D_n is given by the maximal cone $\sigma_n / \mathbb{R} v_n$ and its faces, so that the strict transform $\tilde{D}_n$ of D_n is isomorphic to $\mathbb{A}^n \simeq D_n$. Since the restriction of the composition of blow-ups $\tilde{D}_n \longrightarrow D_n$ induces the identity on the tori, it is an isomorphism, and this concludes the proof. □

Back to the global setting, we obtain:

Corollary 4.1.15. *Let $\mathscr{X} \subset \mathbb{P}^{n+1}_R$ be as above, and $i \in \{0, ..., n+1\}$. Labelling the elements of $\{0, ..., n+1\} \setminus i$ as $\{l_1, ..., l_n\}$, the sequence of successive blow-ups:*

$$\mathscr{X}_i(n) = \mathrm{Bl}_{D_{l_n}} \mathscr{X}_i(n-1) \longrightarrow \mathscr{X}_i(n-1) = ... \longrightarrow \mathscr{X}_i(1) = \mathrm{Bl}_{D_{l_1}} \mathscr{X} \longrightarrow \mathscr{X},$$

yields an snc model $\mathscr{X}_i := \mathscr{X}_i(n)$, such that $\mathrm{Sk}(\mathscr{X}_i) = \mathrm{Sk}(\mathscr{X})$ and the strict transform $D'_i \simeq D_i$.

The upshot of this is that we obtain a Berkovich retraction $\rho_{\mathscr{X}_i} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X}_i) = \mathrm{Sk}(\mathscr{X})$, which is an affinoid torus fibration over $\mathrm{Star}(v_i)$ by theorem 3.1.21. Moreover, the integral affine structure induced on $\mathrm{Star}(v_i)$ matches the one induced by the chart ϕ_{v_i} from definition 4.1.7.
Let us write $\mathfrak{X}_i = \widehat{\mathscr{X}_{i,/D_i}}$ the formal completion. By theorem 3.1.21, writing $\mathscr{N} \longrightarrow \mathbb{A}^1_k$ the total space of normal bundle of D_i inside $\mathscr{X}_i$ and $\mathscr{N} = \mathcal{N} \times_{\mathbb{A}^1} R$, we have a formal isomorphism:

$$f : \mathfrak{X}_i \xrightarrow{\sim} \widehat{\mathscr{N}_{/D_i}},$$

where $D_i \subset \mathscr{N}$ denotes the zero section.
Thus, passing to the generic fiber, $\mathfrak{X}_i^\eta$ embeds analytically as a open subset of a torus:

$$\mathfrak{X}_i^\eta \hookrightarrow \mathbb{T}^{\mathrm{an}}_{\mathscr{N}},$$

where $\mathbb{T}_{\mathscr{N}} = \mathscr{N} \times_R K$ is the generic fiber of $\mathscr{N}$.

Lemma 4.1.16. *There is a canonical isomorphism of K-tori:*

$$\mathbb{T}_{\mathscr{N}} \simeq \mathbb{T}_{D_i} \times_k K.$$

Proof. Since $\mathcal{N}$ is trivial over $\mathbb{T}_{D_i}$, we have a natural morphism $\mathbb{T}_{D_i} \times \mathbb{G}_{m,k} \longrightarrow \mathcal{N}$, whose image is precisely the k-torus of $\mathcal{N}$. Moreover, from the description of the fan of $\mathcal{N}$ (see 3.2.8), the morphism $\mathcal{N} \longrightarrow \mathbb{A}^1_k$ is the projection onto the second factor, hence the result. □

We write Σ_i for the fan of D_i inside $N_{\mathbb{R}}/\mathbb{R}v_i$, and let $p_i : N_\Sigma \longrightarrow N_{\Sigma_i}$ be the canonical projection - on $N_{\mathbb{R}}$, this is simply the quotient map $N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}/\mathbb{R}v_i$.
The above lemma provides us with a map $\mathrm{val}_{D_i} : \mathfrak{X}_i^\eta \longrightarrow N_{\mathbb{R}}/\mathbb{R}v_i$. Note that up to identification, the map val_{D_i} is none other than the Berkovich retraction $\rho_{\mathfrak{X}_i}$ - more precisely, viewing $\mathrm{Sk}(\mathscr{X}) \subset N_{\mathbb{R}}$ via prop. 4.1.3, the relation $\mathrm{val}_{D_i} = p_i \circ \rho_{\mathfrak{X}_i}$ holds.

Proposition 4.1.17. *Let $\mathrm{val}_{\mathbb{P}} : \mathbb{P}^{n+1,\mathrm{an}} \longrightarrow N_\Sigma$. Then we have:*

$$\mathrm{val}_{D_i} = p_i \circ \mathrm{val}_{\mathbb{P}}\,.$$

Proof. We assume $i = (n+1)$ for convenience and pick standard coordinates $(\frac{z_1}{z_0}, ..., \frac{z_{n+1}}{z_0})$ on the torus of $\mathbb{P}^{n+1}$; then the projection $p_i : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is given by omitting the last coordinate. Recall the formal isomorphism:

$$f : \widehat{\mathscr{X}_{i,/D_i}} \xrightarrow{\sim} \widehat{\mathscr{N}_{/D_i}}$$

was constructed cone by cone, in the following way. We let $\sigma \in \Sigma_i$ be a maximal cone in the fan of D_i, and we assume that $\sigma = \mathrm{Cone}(e_1, ..., e_n)$ is the standard orthant.

Then the restriction f_σ of f to $\mathfrak{X}_\sigma$ was defined by the following property: if s_l is a trivializing section of $\mathcal{O}_{\mathscr{X}}(D_0 - D_l)$ on $\mathfrak{X}_\sigma$, restricting to the meromorphic function z_l/z_0 on D_i, then $s_l = f_\sigma^*(\frac{z_l}{z_0})$ (with the notation of section 3.2.2, $W_l^\sigma = (D_0 - D_l)$). Thus, if $x \in \mathfrak{X}_i^\eta$ - which means $c_{\mathscr{X}_i}(v_x) \subset D_i$, then $v_x(s_l) = v_x(\frac{z_l}{z_0})$, hence:

$$\mathrm{val}_{\mathbb{T}_{\mathscr{N}}}(x) = (v_x(\frac{z_1}{z_0}), ..., v_x(\frac{z_n}{z_0})) \in N_{\mathbb{R}}/\mathbb{R}e_{n+1}$$

while:

$$\mathrm{val}_{\mathbb{T}_{\mathbb{P}}}(x) = \big(v_x(\frac{z_1}{z_0}), ..., v_x(\frac{z_n}{z_0}), v_x(\frac{z_{n+1}}{z_0})\big) \in N_{\mathbb{R}},$$

hence the result. □

Corollary 4.1.18. *For any choice of branch cuts $\underline{a}$, the equality $\delta_{\underline{a}} \circ \mathrm{val}_{\mathbb{P}} = \rho_{\mathfrak{X}_i}$ holds on $\rho_{\mathfrak{X}_i}^{-1}(\widetilde{\mathrm{Star}}(v_i)) \subset \mathfrak{X}_i^\eta$.*

Proof. Under the identification $\mathrm{Sk}(\mathscr{X}) \subset N_\Sigma$, we have the equality $q_i \circ \rho_{\mathfrak{X}_i} = \mathrm{val}_{D_i}$, where val_{D_i} is the composition : $\mathfrak{X}_i^\eta \hookrightarrow \mathbb{T}_{D_i}^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}(D_i)$. By the above proposition, we have $q_i \circ \rho_{\mathfrak{X}_i} = p_i \circ \mathrm{val}_{\mathbb{P}}$, and by prop. 4.1.13, the equality $\delta_{\underline{a}} = (q_i)^{-1} \circ p_i$ holds over $\widetilde{\mathrm{Star}}(v_i)$, hence the result. □

Since $\rho_{\mathfrak{X}_i}$ is an affinoid torus fibration over $\mathrm{Star}(v_i)$ and in particular over $\widetilde{\mathrm{Star}}(v_i)$, this implies that $\rho_{\underline{a}} = \delta_{\underline{a}} \circ \mathrm{val}_{\mathbb{P}}$ is an affinoid torus fibration away from $\Gamma_{\underline{a}}$, and this concludes the proof of theorem 4.1.8.

4.2 An alternative construction in low dimension

In this section, we will explain the alternative construction of the map ρ from [MPS21] in dimension 2 and 3, without taking a detour into tropical geometry - in fact the $n = 2$ case originates from [KS06, § 4.2.5]. Recall that to prove that ρ was an affinoid torus fibration near a vertex v_i, we produced a model $\mathscr{X}_i \longrightarrow \mathscr{X}$ of X, that was toric along the strict transform of D_i, which produced a local model for ρ near v_i, simply given by $\rho_{\mathfrak{X}_i}$ in the notation of the previous section.
In low dimension, it is actually possible to explicitly produce an snc model $\widetilde{\mathscr{X}}/R$ of X, that simultaneously dominates all the models $\mathscr{X}_i$ - which we call the dominating model. We then produce a combinatorial retraction $\pi : \mathrm{Sk}(\widetilde{\mathscr{X}}) \longrightarrow \mathrm{Sk}(X)$, which is an explicit piecewise-affine map, and such that near a vertex v_i the map π is equal to the restriction of $\rho_{\mathfrak{X}_i}$ to $\mathrm{Sk}(\widetilde{\mathscr{X}})$. The way the map π has to be defined for this to hold is mostly dictated by the explicit pictures we are able to draw, which unfortunately has limited this approach to the low-dimensional case for the moment.
Nevertheless, one upside of this approach is that it allows us to refine the general construction; let us explain this in the $n = 2$ case. For degenerations of K3 surfaces, the discriminant locus consists of 6 singular points, located in the interior of the edges of the skeleton; however it is expected that a 'generic' singular affine structure has 24 singular point - in fact corresponding to the 24 singular points of the model $\mathscr{X} \subset \mathbb{P}^3_R$. Accordingly, for a generic K3 hypersurface, the skeleton of the dominating model contains 24 "wings" (following the terminology of [KS06]), 4 over each edge.

In the tropical world, we will see that the four wings lying over the same edge get contracted to only one 2-simplex; while the construction we will describe here allows us to contract the four wings separately, while choosing a different singular point inside the edge for each wing, and thus obtain an integral affine structure which is singular over 4 points in each edge of the skeleton.

4.2.1 Degeneration of quartic K3 surfaces

We consider $\mathscr{X} = \{z_0z_1z_2z_3 + tF_4(z_0, z_1, z_2, z_3) = 0\} \subset \mathbb{P}^3_R$, where F_4 is a generic homogeneous polynomial of degree 4. The total space $\mathscr{X}$ has 24 singular points, given by $\{z_i = 0, z_j = 0, t = 0, F_4 = 0\}$, hence 4 on each $D_i \cap D_j$; and the (étale) local model around a singular point is given by the three-dimensional ordinary double point $\mathscr{U} = \{xy = zt\} \subset \mathbb{A}^3_R$. This implies that $\mathscr{X}$ is a minimal dlt model of the K3 surface $X := \mathscr{X}_K$, but it is not good in the sense of definition 1.1.11 since the prime components of the special fiber are not $\mathbb{Q}$-Cartier.
Our goal is to construct some (to be more precise, 2^{24}) explicit good minimal models of X starting from $\mathscr{X}$, and then to study the integral affine structures on $\mathrm{Sk}(X)$ induced by the Berkovich retractions associated to these models, or by patching together several of them. To this purpose, we will apply corollaries 3.3.8 and 3.3.14.
Some good minimal models of X are obtained by performing the following small resolutions of $\mathscr{X}$. As in corollary 4.1.15, for any triple of elements i, j, k in $\{0, \dots, 3\}$ and any fixed order (i, j, k) on them, we blow-up in order the divisors D_i, D_j and D_k, and denote the resulting model by $\mathscr{X}_{ijk}$ and the morphism by

$$g_{ijk} : \mathscr{X}_{ijk} \to \mathscr{X}.$$

The exceptional locus of g_{ijk} consists of 24 smooth rational curves whose images via g_{ijk} are the singular points of $\mathscr{X}$. In particular, the strict transform of D_i is isomorphic to the blow-up of D_i along the 12 singular points in D_i; similarly for D_j at 8 points, and for D_k at the 4 remaining singular points. Instead, for $h \neq i, j, k$, D_h is isomorphic to its strict transform.
These facts follow from local computations on $\mathscr{U}$: with the notation of section 4.1.3, blowing-up $D_x := \{x = t = 0\}$ induces an exceptional curve inside the strict transform $\tilde{D}_x$, which is isomorphic to the blow-up of D_x along the origin. The claims now follow, since the singularities of $\mathscr{X}$ are isolated.
If we denote by $\tilde{D}_m$ for $m \in \{1, \dots, 4\}$ the irreducible components of the special fiber of $\mathscr{X}_{ijk}$, and by $C_{mm'} = \tilde{D}_m \cap \tilde{D}_{m'}$ the strata curves, then the intersection numbers in $\mathscr{X}_{ijk}$ are:

	$\tilde{D}_i$	$\tilde{D}_j$	$\tilde{D}_k$	$\tilde{D}_h$	$h \neq i, j, k$
C_{ij}	1	-3	1	1	
C_{ik}	1	1	-3	1	
C_{ih}	1	1	1	-3	
C_{jk}	1	1	-3	1	
C_{jh}	1	1	1	-3	
C_{kh}	1	1	1	-3	

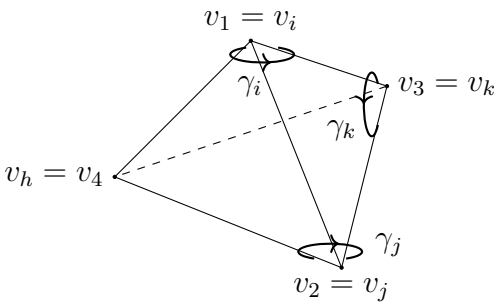
(4.2.1)

Integral affine structure induced by the model $\mathscr{X}_{ijk}$

By [NXY19] the non-archimedean SYZ fibration $\rho_{\mathscr{X}_{ijk}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X}_{ijk}) = \mathrm{Sk}(X) \simeq \mathbb{S}^2$ is an affinoid torus fibration (at least) away from the vertices of the triangulation of $\mathrm{Sk}(X)$ induced by the special fiber of $\mathscr{X}_{ijk}$, i.e. away from the v_{D_m}'s.
By theorem 3.1.21, $\rho_{\mathscr{X}_{ijk}}$ is an affinoid torus fibration over $\mathrm{Star}(\tau_{D_h})$ for $h \neq i, j, k$, as $D_h \simeq \mathbb{P}^2$ and g_{ijk} is an isomorphism on the strict transform of D_h. Moreover, by remark 3.3.9 and 4.2.1 the integral affine structure induced by $\rho_{\mathscr{X}_{ijk}}$ does not extend to v_{D_i}, v_{D_j} and v_{D_k}.

We conclude that the singular points of the affine structure on $\mathrm{Sk}(X)$ induced by $\rho_{\mathscr{X}_{ijk}}$ are precisely v_{D_i}, v_{D_j} and v_{D_k}. Corollary 3.3.8 establishes that the monodromies around these vertices are

$$T_{\rho_{\mathscr{X}_{ijk}}}(\gamma_i) = \begin{pmatrix} 21 & 8 \\ -8 & -3 \end{pmatrix}$$
in the basis (v_{D_j}, v_{D_k}) and origin v_{D_i},
$$T_{\rho_{\mathscr{X}_{ijk}}}(\gamma_j) = \begin{pmatrix} -15 & -4 \\ 4 & 1 \end{pmatrix}$$
in the basis (v_{D_k}, v_{D_h}) and origin v_{D_j},
$$T_{\rho_{\mathscr{X}_{ijk}}}(\gamma_k) = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$
in the basis (v_{D_h}, v_{D_i}) and origin v_{D_k}.



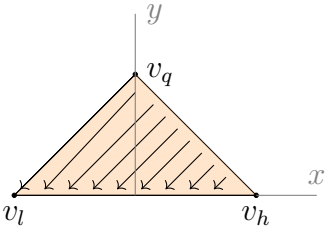
Integral affine structure induced combining more models

We recall a construction from [KS06, §4.2.5]. Consider the resolution $h : \mathscr{Z} \to \mathscr{X}$ obtained by blowing-up the 24 singular points of $\mathscr{X}$ inside $\mathbb{P}^3_R$ and then taking the closure of $\mathscr{X}$ in the blow-up. By the universal property of blowing-up (applied to $\mathscr{X}_{ijk} \longrightarrow \mathscr{X}$), this model dominates any of the models $\mathscr{X}_{ijk}$, and the special fiber is $\mathscr{Z}_0 = \sum_{i=1}^{4} D_i + \sum_{q=1}^{24} E_q$; the associated dual complex is the boundary of a tetrahedron with four additional 2-cells glued along each edge of the tetrahedron; following Kontsevich–Soibelman we call such 2-cells wings.

We parametrize each edge e of $\mathcal{D}(\mathscr{Z}_0)$ by the interval $[-1, 1]$, and each wing W_q glued to e by the 2-simplex in $\mathbb{R}^2_{(x,y)}$ bounded by e and $0 \leqslant y \leqslant 1 - |x|$.

Lemma 4.2.2. *Let W_q be a wing over the edge e_{lh}, for $l \in \{i, j, k\}$ and assume i, j, k, h all distinct. Then the retraction $\rho_{\mathscr{X}_{ijk}} : W_q \to e_{lh}$ is the contraction of W_q to the edge e_{lh} parallel to the edge e_{lq}:*

$$\rho_{\mathscr{X}_{ijk}} : W_q \to e_{lh}$$
$$(x, y) \mapsto (x - y, 0)$$



Proof. The morphism $\mathscr{Z} \to \mathscr{X}_{ijk}$ is the blow-up of the 24 exceptional curves of g_{ijk}. In particular, the exceptional divisor E_q is the preimage in $\mathscr{Z}$ of a curve contained

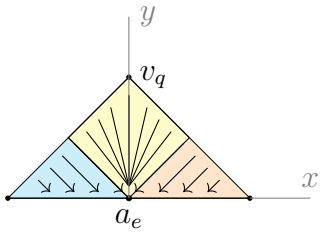
in $\tilde{D}_l$; it follows that $v_{E_q}(\tilde{z}_l) = 1$ and $v_{E_q}(\tilde{z}_h) = 0$, where $\tilde{z}_l, \tilde{z}_h$ are local equations for $\tilde{D}_l, \tilde{D}_h$ on $\mathscr{X}_{ijk}$. The Berkovich retraction $\rho_{\mathscr{X}_{ijk}}$ is linear on W_q and hence depends only on the image of v_q, which is determined by $v_{E_q}(\tilde{z}_l)$ and $v_{E_q}(\tilde{z}_h)$. Thus we conclude that $\rho_{\mathscr{X}_{ijk}}(v_q) = v_l$ and we have the result. □

Kontsevich and Soibelman define a retraction

$$\rho : X^{\mathrm{an}} \xrightarrow{\rho_{\mathscr{Z}}} \mathrm{Sk}(\mathscr{Z}) \xrightarrow{\rho'} \mathbb{S}^2 \simeq \mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$$

where $\rho_{\mathscr{Z}}$ is the Berkovich retraction onto the skeleton $\mathrm{Sk}(\mathscr{Z})$, and ρ' is a retraction of the 24 wings of $\mathrm{Sk}(\mathscr{Z})$ to the sphere given as follows. For each edge e of $\mathcal{D}(\mathscr{X}_k)$ we choose a point $a_e = (a_e, 0)$ in the interior of e, and define the retraction of W_q onto e by

$$\rho' : (x, y) \mapsto \begin{cases} (x+y, 0) & \text{if } x + y \leqslant a_e \\ (x-y, 0) & \text{if } x - y \geqslant a_e \\ (a_e, 0) & \text{otherwise.} \end{cases}$$



Picture for $a_e = 0$

We note that

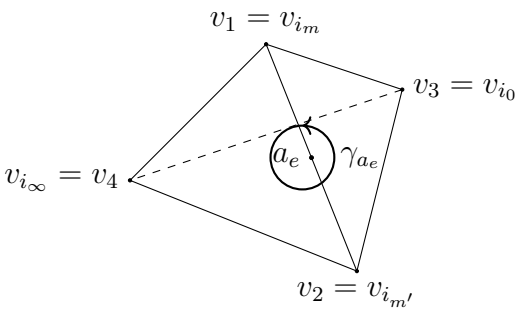
- over the interior of any 2-dimensional face $\tau \subset \mathrm{Sk}(\mathscr{X})$, ρ is equal to $\rho_{\mathscr{Z}}$, thus it is an affinoid torus fibration (see example 3.1.16).
- Around any vertex v_D, ρ is equal to $\rho_{\mathscr{X}_{ijk}}$ for any triple such that $D \neq D_i, D_j, D_k$, as follows from the previous lemma. Thus, from 4.2.1, ρ is an affinoid torus fibration around v_D, and the affine structure induced there is the fan structure induced by D, by cor. 3.3.2.
- For any edge e corresponding to $C_e = D_{i_m} \cap D_{i_{m'}}$, adopting the notation of section 3.3.3,

$$\rho = \begin{cases} \rho_{\mathscr{X}_{ijk}} & \text{for } i_m \neq i, j, k, \text{ over } \mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup [v_{i_m}, a_e) \\ \rho_{\mathscr{X}_{i'j'k'}} & \text{for } i_{m'} \neq i', j', k', \text{ over } \mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup [v_{i_{m'}}, a_e), \end{cases}$$

  and thus is an affinoid torus fibration over the union of these two open sets.

We conclude that ρ induces an integral affine structure on $\mathrm{Sk}(X)$ away from the points a_e. By corollary 3.3.14, we can compute the monodromy around the singularities. As all these computations are analogous, we exhibit the case $C_e = D_1 \cap D_2$:

$$\begin{aligned} T_\rho(\gamma_{a_e}) &= \begin{pmatrix} 1 & 0 \\ b_{1,\mathscr{X}_{234}} - b_{1,\mathscr{X}_{134}} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 3 - (-1) & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \end{aligned}$$



with respect to the basis (v_3, v_1) and origin v_2. This formula was already stated in [KS06, §4.2.5].

Lemma 4.2.3. *Let $W \subset X^{\mathrm{an}}$ be a wing over the edge $e_{ij} \subset \mathrm{Sk}(\mathscr{X})$. Then the tropicalization map:*

$$\mathrm{val}_{\mathbb{P}} : X^{\mathrm{an}} \longrightarrow N_\Sigma$$

induces an integral affine homeomorphism from the wing W to the convex hull $\mathrm{Conv}(v_i, v_j, v_i + v_j) \subset N_\Sigma$.

Note that in particular the four wings that lie over the same edge are collapsed together inside the tropicalization $\overline{\mathrm{Trop}}(X)$.

Proof. Assume for convenience that $i = 1$ and $j = 2$. We let $\mathscr{Z} \longrightarrow \mathscr{X}$ be the resolution of $\mathscr{X}$ defined above, and let $f_i = \frac{z_i}{z_0}$ be the meromorphic function on $\mathscr{Z}$ induced by the standard coordinates on $\mathbb{P}^{n+1}$. We still denote by D_i the strict transforms of the components of $\mathscr{X}_0$ inside $\mathscr{Z}$.
Let $q \in C_{12}$ be a singular point, and $E_q \subset \mathscr{Z}_0$ the corresponding exceptional divisor. Then f_1, f_2 are equations for $D_1 + E_q$, $D_2 + E_q$ respectively at the zero-dimensional stratum $p_q := E_q \cap D_1 \cap D_2$. Thus, choosing an equation f_E of E_q at p_q, and writing $w \in W$ as its monomial weights:

$$w = (w_1, w_2, w_E),$$

we have $w_i = w(f_E^{-1} f_i)$ and $w_E = w(f_E)$, while:

$$\mathrm{Trop}(w) = (w(f_1), w(f_2), 0) = (w_1 + w_E, w_2 + w_E, 0),$$

so that in coordinates Trop maps the standard simplex $w_1 + w_2 + w_E = 1$ to the convex hull $\mathrm{Conv}(v_1, v_2, v_1 + v_2) \subset \mathbb{R}^3$, hence the result. □

Proposition 4.2.4. *Let $\underline{a} = (a_e)_e$ be a choice of center of the edges. Then the retraction constructed above is the retraction $\rho_{\underline{a}} = \delta_{\underline{a}} \circ \mathrm{val}$.*

Proof. Write $\rho = \delta \circ \mathrm{val}$, and $\rho' = \pi \circ \rho_{\mathscr{Z}}$. By the proof of thm. 4.1.8, the equality $\rho = \rho'$ holds away from Γ, so that we let $x \in X^{\mathrm{an}}$ such that $\rho(x) \in \Gamma$ - or equivalently, $\rho'(x) \in \Gamma$. Then there exists a wing $W \subset \mathrm{Sk}(\mathscr{Z})$ such that $\rho_{\mathscr{Z}}(x) \in W$ - more precisely $\rho_{\mathscr{Z}}(x)$ belongs to the yellow region of the wing, as in figure 4.2.1. Hence the results follows from the lemma above and the picture 4.2. □

Dispersion of singularities

We construct a third singular integral affine structure on $\mathrm{Sk}(X)$ pushing forward the techniques developed so far. This can be viewed as a *dispersion of singularities* with respect to the integral affine structure studied in 4.2.1: on each edge we pass from one singular point around which the monodromy is $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$, to 4 singular points around each of which the monodromy is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Such singularities are called *focus-focus* and are the most standard examples of singularities for $\mathbb{Z}$-affine structures in dimension 2. Those arise for instance when considering the hyperkähler rotation of a generic

elliptic K3 surface $f : S \longrightarrow \mathbb{CP}^1$, with f an elliptic fibration: the hyperkähler rotation S^{HK} is a complex surface with same underlying topological space as S, and hence comes with a map $f^{\mathrm{HK}} : S^{\mathrm{HK}} \longrightarrow \mathbb{S}^2$, induced by f at the level of topological spaces. The map f^{HK} is no longer holomorphic, but is a symplectic torus fibration inducing a $\mathbb{Z}$-affine structure with 24 focus-focus singularities on $\mathbb{S}^2$ and acting as an SYZ fibration for S^{HK}. We refer the reader to [GW00], [OO18] for more details.

Let e be an edge of $\mathrm{Sk}(\mathscr{X})$, let $C_e = D_{e_1} \cap D_{e_2}$ be the corresponding stratum curve in $\mathscr{X}_k$. We recall that as the degree four polynomial F_4 is generic, C_e contains four singular points $p_1, \dots, p_4$ of $\mathscr{X}$, which are ordinary double points. Around each p_i, $\mathscr{X}$ is étale locally of the form $\{xy = wt\} \subset \mathbb{A}^3_R$, with x and y being local equations for D_{e_1} and D_{e_2} away from p_i. Blowing-up the singular point p_i inside the ambient space yields an exceptional divisor $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Contracting one or the other ruling of E, we obtain two distinct small resolutions of $\widehat{\mathscr{X}_{/C_e}}$ around p_i, respectively with an exceptional curve inside D_{e_1} or D_{e_2}.

For $j \in \{0, \dots, 4\}$, we denote by $\mathscr{X}_{e,j}$ the following small resolution of $\widehat{\mathscr{X}_{/C_e}}$: around p_i for $i \leqslant j$ we consider the small resolution such that the exceptional curve over p_i lies in D_{e_1}, while for $i > j$ the small resolution such that the exceptional curves lie in D_{e_2}. The gluing of these local small resolutions is done in the étale topology, so that in general the obtained models are no longer schemes but only algebraic spaces. Nevertheless, $\mathscr{X}_{e,j}$ is dominated by the scheme $\mathscr{Z}$ (defined in 4.2.1), so that we may still define a Berkovich retraction $\rho_{\mathscr{X}_{e,j}}$, as described in section 4.2.6. In particular, by prop. 4.2.11, $\rho_{\mathscr{X}_{e,j}}$ is an affinoid torus fibration over $\mathrm{Star}(\tau_{C_e})$.

We construct the following continuous retraction

$$\overline{\rho} : X^{\mathrm{an}} \xrightarrow{\rho_{\mathscr{Z}}} \mathrm{Sk}(\mathscr{Z}) \xrightarrow{\overline{\rho}'} \mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X}),$$

where $\rho_{\mathscr{Z}}$ is the Berkovich retraction onto the skeleton $\mathrm{Sk}(\mathscr{Z})$, and $\overline{\rho}'$ is a retraction of the 24 wings of $\mathrm{Sk}(\mathscr{Z})$ to the sphere given as follows. We fix four distinct, ordered, interior points $a_{e,1}, \dots, a_{e,4}$ of each edge e. Then the map $\overline{\rho}'$ on the wing W_i attached to e is defined as the map ρ' of 4.2.1, setting $a_e = a_{e,i}$, for each $i \in \{1, \dots, 4\}$.

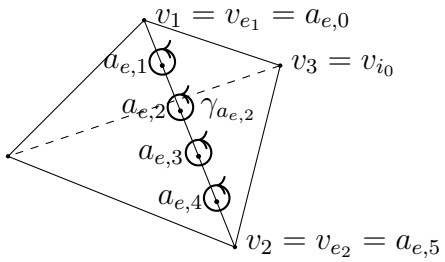
Proposition 4.2.5. *The map $\overline{\rho}$ is an affinoid torus fibration away from the 24 points $a_{e,i}$. Furthermore, the monodromy of the $\mathbb{Z}$-affine structure induced by $\overline{\rho}$, around each singular point, is $\mathrm{SL}_2(\mathbb{Z})$-conjugate to*

$$T_{\overline{\rho}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proof. Over $\mathrm{Int}(\tau)$ of any 2-dimensional face $\tau \subset \mathrm{Sk}(\mathscr{X})$, $\overline{\rho}$ is equal to $\rho_{\mathscr{Z}}$, hence is an affinoid torus fibration. Around any vertex v_D, $\overline{\rho}$ is equal to $\rho_{\mathscr{X}_{ijk}}$ for any triple such that $D \neq D_i, D_j, D_k$. It follows from 4.2.1 that $\overline{\rho}$ is an affinoid torus fibration around v_D. We denote by $p_0 + p_\infty$ the boundary of C_e, with $p_0 = C_e \cap D_{i_0}$ and $p_\infty = C_e \cap D_{i_\infty}$; we write $a_{e,0} = v_{D_{e_1}}$ and $a_{e,5} = v_{D_{e_2}}$, and denote by $(\cdot, \cdot)$ the open segment joining two points. Then, for $j \in \{0, \dots, 4\}$, $\overline{\rho}$ is equal to $\rho_{\mathscr{X}_{e,j}}$ over $\mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup (a_{e,j}, a_{e,j+1})$, thus is an affinoid torus fibration. We conclude that $\overline{\rho}$ is an affinoid torus fibration away from the points $a_{e,i}$ for $i \in \{1, \dots, 4\}$.

For a singular point $a_{e,i}$, we consider a loop γ around it and contained in $\mathrm{Int}(\tau_{p_0}) \cup \mathrm{Int}(\tau_{p_\infty}) \cup (a_{e,i-1}, a_{e,i}) \cup (a_{e,i}, a_{e,i+1})$. We apply cor. 3.3.14 to compute the monodromy along γ: the numbers $b_{e_1,\mathscr{X}_{e,i-1}}$ and $b_{e_1,\mathscr{X}_{e,i}}$ differ by 1, as the model $\mathscr{X}_{e,i}$ has an additional exceptional curves in D_{e_1} with respect to $\mathscr{X}_{e,i-1}$. Therefore, we obtain

$$\begin{aligned} T_{\overline{\rho}}(\gamma_{a_{e,i}}) &= \begin{pmatrix} 1 & 0 \\ b_{e_1,\mathscr{X}_{e,i-1}} - b_{e_1,\mathscr{X}_{e,i}} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 3-(i-1)-(3-i) & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$



with respect to the basis $(v_{D_{i_0}}, v_{D_{e_1}})$ and origin $v_{D_{e_2}}$. □

4.2.2 Study of the local resolution

We now move on to the case of a degeneration of quintic threefolds, of the form:

$$\mathscr{X} = \{z_0 ... z_4 + tF_5 = 0\} \subset \mathbb{P}^4_R,$$

where F_5 is a generic homogeneous polynomial of degree five. We will start by refining the discussion of section 4.1.3.
The singular locus $\mathscr{X}^{\mathrm{sing}}$ of the total space $\mathscr{X}$ is contained in the special fiber, and is the intersection in $\mathbb{P}^4_k$ of $\{F_5 = 0\}$ and the union of surfaces $S_{ij} = \{z_i = z_j = 0\}$ for $i \neq j$.
In particular, each D_i intersects $\mathscr{X}^{\mathrm{sing}}$ along the union of four quintic curves C_{ij} and by genericity of F_5, we may assume that C_{ij} does not intersect the torus fixed points of D_i.

$$\mathscr{X}^{\mathrm{sing}} \cap D_i = \bigcup_{\substack{j=1 \\ j \neq i}}^{5} C_{ij}$$

$$C_{ij} \subseteq D_i \cap D_j$$

$$C_{ij} \cap C_{ij'} = \{5 \text{ points}\} \text{ for } j \neq j'$$

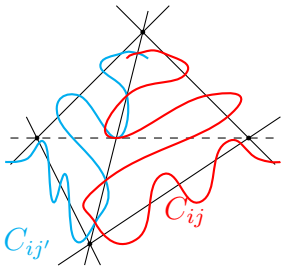


Figure 4.5: Irreducible component D_i

We consider a point in $\mathscr{X}^{\mathrm{sing}} \cap D_1 \cap D_2 \cap D_3$; the singular points in the other strata curves can be treated analogously. Etale locally around such a point, $\mathscr{X}$ is isomorphic to the toric variety $\mathscr{U} \coloneqq V(x_1x_2x_3 - wt) \subset \mathbb{A}^5_k$, where the:

$$D_i = \{x_i = t = 0\}$$

are the components of the special fiber $\{t = 0\}$ in $\mathscr{U}$. They form the toric boundary of $\mathscr{U}$ together with the:

$$D'_i = \{x_i = w = 0\}.$$

We denote the strata surfaces of the special fiber by $D_{ij} \coloneqq D_j \cap D_j$, for $i, j \in \{1,2,3\}$ and $i \neq j$, and the stratum curve by $D_{123} := \{x_1 = x_2 = x_3 = t = 0\}$. The singular locus $\mathscr{U}^{\text{sing}}$ consists of the torus invariant curves:

$$C_{ij} = \{x_i = x_j = w = t = 0\} = D_i \cap D_j \cap D'_i \cap D'_j,$$

for $\{i,j\} \subset \{1,2,3\}$. The three singular curves intersect each other at the torus invariant point $p := \{x_1 = x_2 = x_3 = w = t = 0\}$.
The toric blow-up $G_1 : \mathscr{U}_1 := \mathrm{Bl}_{D_1} \mathscr{U} \to \mathscr{U}$ along D_1 resolves the singularities along C_{12} and C_{13} except at the point p. The exceptional locus of G_1 consists of two surfaces S_{12} and S_{13} intersecting each other along a curve: these surfaces are mapped by G_1 to the respective singular curves, are contained in the strict transform of D_1, and correspond to two new edges in the slice of the fan of $\mathscr{U}_1$. The intersection of S_{12} and S_{13} corresponds to a new 2-dimensional face in the slice of the fan. With a slight abuse of notation we keep the same notation for the strict transforms in $\mathscr{U}_1$.

A resolution of $\mathscr{U}$ is given by the composition of G_1 and the toric blow-up $G_{12} : \mathscr{U}_{12} := \mathrm{Bl}_{D_2} \mathscr{U}_1 \to \mathscr{U}_1$ along D_2; the latter indeed resolves the singularities along C_{23}. The exceptional locus of G_{12} is a surface S_{23}, which is mapped by G_{12} to C_{23} and is contained in the strict transform of D_2. The morphism G_{12} induces a new 2-dimensional face in the slice of the fan of $\mathscr{U}_{12}$, and a new edge corresponding to the surface S_{23}. In particular, after the blow-up G_{12}, the strict transforms of the surface S_{12} and of the divisor D_3 have empty intersection.
The small resolution $\mathscr{U}_{12}$ of $\mathscr{U}$ induces an isomorphism on the strict transform of D_{13} and of D_{23}, while its restriction to the strict transform of D_{12} is the blow-up of D_{12} along a general point. These facts can be checked computing the charts of the blow-ups G_1 and G_{12}. Alternatively, they can be verified looking at the fans of the strata surfaces D_{ij} in the slice of the fans of $\mathscr{U}$, $\mathscr{U}_1$ and $\mathscr{U}_{12}$; indeed, the fan of D_{ij} is induced by the intersection of the slice with a normal plane to the edge corresponding to D_{ij}, as in the picture in the proof of prop. 4.1.14.

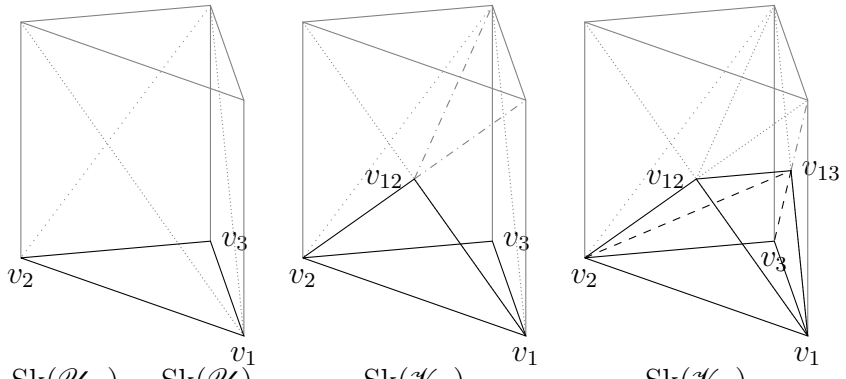
4.2.3 Local dominating model

We now introduce the local model for the dominating model $\mathscr{Z} \longrightarrow \mathscr{X}$ we will construct.
We consider the blow-up of the exceptional surfaces S_{12}, S_{13} and S_{23} one after the other:

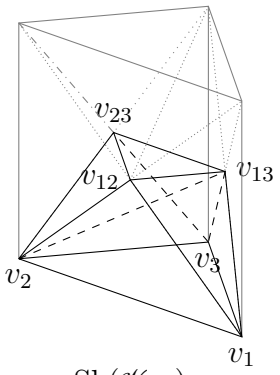
$$\begin{array}{ccccccccc} \mathscr{V}_{123} & \xrightarrow[H_{23}]{\text{blow-up of } S_{23}} & \mathscr{V}_{13} & \xrightarrow[H_{13}]{\text{blow-up of } S_{13}} & \mathscr{V}_{12} & \xrightarrow[H_{12}]{\text{blow-up of } S_{12}} & \mathscr{U}_{12} & \xrightarrow{G_{12}\circ G_1} & \mathscr{U} \\ \cup & & \cup & & \cup & & & & \downarrow \\ E_{23} & & E_{13} & & E_{12} & & & & \mathbb{A}^1_t. \end{array}$$

As these surfaces are toric strata of $\mathscr{U}_{12}$, the blow-ups are toric as well and the corresponding fans are refinements of the fan of $\mathscr{U}_{12}$. We note that

- the dual complexes of the special fibers of $\mathscr{U}_{12}, \mathscr{V}_{12}, \mathscr{V}_{13}$ and $\mathscr{V}_{123}$ are obtained from the slices of the corresponding fans by removing the vertices corresponding to D'_1, D'_2 and D'_3, as well as each face containing one of these.



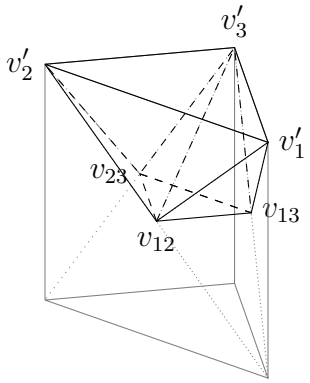
$\mathrm{Sk}(\mathscr{U}_{12}) = \mathrm{Sk}(\mathscr{U})$ $\mathrm{Sk}(\mathscr{V}_{12})$ $\mathrm{Sk}(\mathscr{V}_{13})$



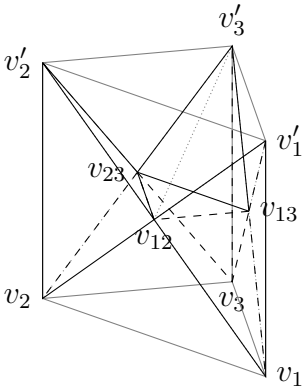
$\mathrm{Sk}(\mathscr{V}_{123})$

Then $\mathrm{Sk}(\mathscr{V}_{123})$ consists of four 3-cells: $< v_{13}, v_1, v_2, v_3 >$, $< v_{13}, v_1, v_2, v_{12} >$, $< v_{23}, v_{13}, v_2, v_3 >$ and $< v_{23}, v_{13}, v_2, v_{12} >$; it has only one edge in the interior, which is $< v_2, v_{13} >$.

- The remaining 3-dimensional simplices of the slice of the fan of $\mathscr{V}_{123}$ are

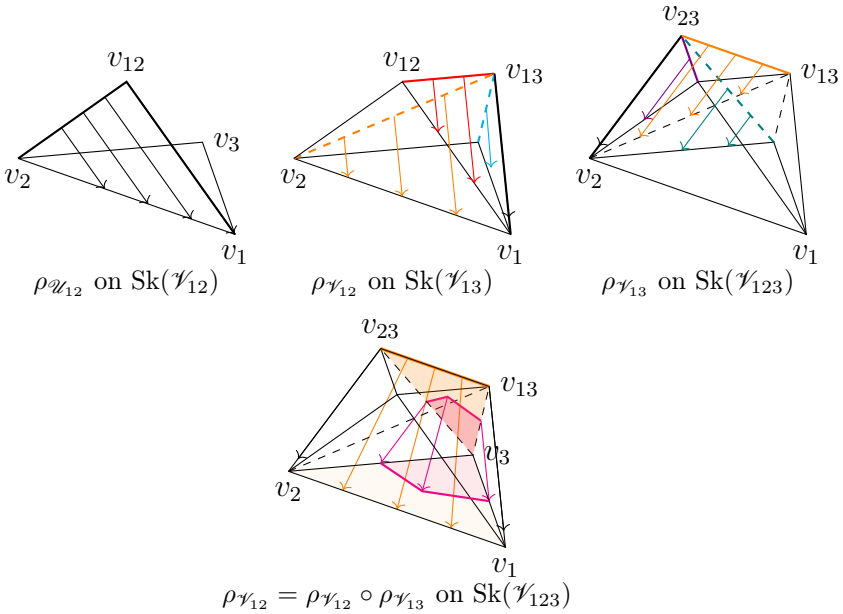


$< v_{12}, v'_1, v'_2, v'_3 >$
$< v_{13}, v'_1, v'_3, v_{12} >$
$< v_{23}, v_{12}, v'_2, v'_3 >$
$< v_{23}, v_{13}, v'_3, v_{12} >$



$< v_1, v'_1, v_{12}, v_{13} >$
$< v_2, v'_2, v_{12}, v_{23} >$
$< v_3, v'_3, v_{13}, v_{23} >$

- The Berkovich retractions associated with the models $\mathscr{U}_{12}$, $\mathscr{V}_{12}$ and $\mathscr{V}_{13}$ map v_{12} and v_{13} to v_1, and v_{23} to v_2.



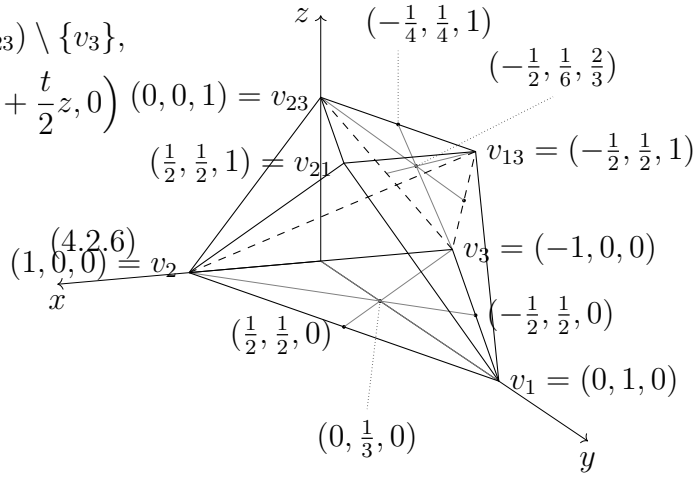
We study more in details the retraction $\rho_{\mathscr{V}_{12}}$ near the vertex v_3, as this will be relevant later in the construction of the local combinatorial retraction (see 4.2.7). We observe that $\rho_{\mathscr{V}_{12}}$ collapses the convex hull $\mathrm{Conv}(v_1, v_2, v_3, v_{13}, v_{23})$ of v_1, v_2, v_3, v_{13} and v_{23} onto the face $< v_1, v_2, v_3 >$. If we identify the skeleton $\mathrm{Sk}(\mathscr{V}_{123})$ with the polyhedron in $\mathbb{R}^3_{(x,y,z)}$ below, $\rho_{\mathscr{V}_{12}}$ on $\mathrm{Conv}(v_1, v_2, v_3, v_{13}, v_{23})$ is written explicitly as follows:

for $(x, y, z) \in \mathrm{Conv}(v_1, v_2, v_3, v_{13}, v_{23}) \setminus \{v_3\}$,

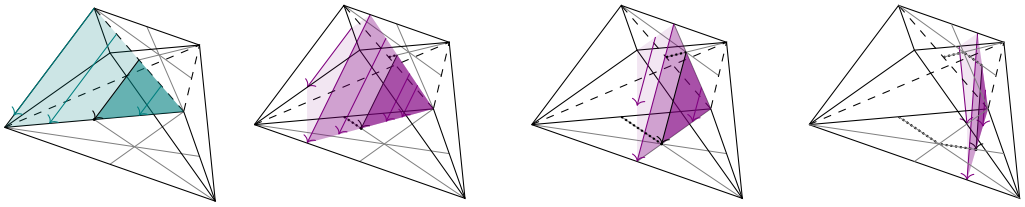
$$\rho_{\mathscr{V}_{12}}\big((x, y, z)\big) = \left(x + \left(1 - \frac{t}{2}\right) z, y + \frac{t}{2} z, 0\right)$$

where $t = \dfrac{2y}{x + y + 1}$.

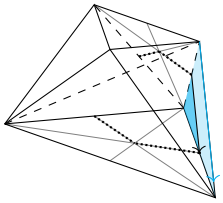
(4.2.6)



The function t on $\mathrm{Conv}(v_1, v_2, v_3)$ is the slope of the line segment joining the vertex v_3 to $tv_1 + (1 - t)v_2$ for $t \in [0, 1]$. We give a picture of the retraction $\rho_{\mathscr{V}_{12}}$ for various values of t:



$(x+z,y,0)$ $t=0$ | $(x+\frac{7}{8}z,y+\frac{1}{8}z,0)$ $t=\frac{1}{4}$ | $(x+\frac{3}{4}z,y+\frac{1}{4}z,0)$ $t=\frac{1}{2}$ | $(x+\frac{5}{8}z,y+\frac{3}{8}z,0)$ $t=\frac{3}{4}$

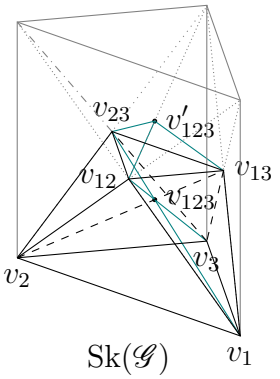


$(x+\frac{1}{2}z,y+\frac{1}{2}z,0)$ $t=1$

For purposes which will be clear in the construction of the local combinatorial retraction in section 4.2.4, we consider a further toric blow-up. Let $H_{123}:\mathscr{G}\to\mathscr{V}_{123}$ be the blow-up along the disjoint toric strata $D_2\cap E_{13}$ and $E_{12}\cap D_3'$; this yields two new components in the toric boundary, denoted by E_{123} and E_{123}'. It follows that the slice of the fan of $\mathscr{G}$ is obtained from the slice of $\mathscr{V}_{123}$ as star subdivision along the edges $<v_2,v_{13}>$ and $<v_{12},v_3'>$.

In particular, the skeleton $\mathrm{Sk}(\mathscr{G})$ is obtained from $\mathrm{Sk}(\mathscr{V}_{123})$ by

1. the star subdivision of the edge $e(v_2,v_{13})$, which turns the four 3-cells of $\mathrm{Sk}(\mathscr{V}_{123})$ into eight 3-cells;
2. adding an additional 3-cell $\tau=\mathrm{Conv}(v_{12},v_{13},v_{23},v_{123}')$, where we denote by v_{123}' the new vertex corresponding to E_{123}'.



$\mathrm{Sk}(\mathscr{G})$

The diagram below summarizes the resolutions of $\mathscr{U}$ we constructed and studied so far:

$$\begin{array}{ccccccc}\mathscr{G} & \xrightarrow[H_{123}]{\text{blow-up of } D_2\cap E_{13}, E_{12}\cap D_3'} & \mathscr{V}_{123} & \xrightarrow[H_{23}\circ H_{13}\circ H_{12}]{\text{blow-up of } S_{12},S_{13},S_{23}} & \mathscr{U}_{12} & \xrightarrow[G_{12}\circ G_1]{\text{blow-up of } D_1,D_2} & \mathscr{U}.\\ \cup & & \cup & & \cup & & \\ E_{123},E_{123}' & & E_{12},E_{13},E_{23} & & S_{12},S_{13},S_{23} & & \end{array}$$

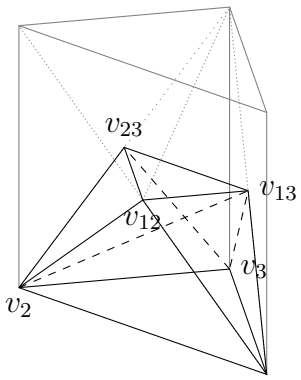
4.2.4 Local combinatorial retraction

Other resolutions of $\mathscr{U}$ can be obtained by blowing-up the divisors of the special fiber in a different order. Given any order (i_1,i_2,i_3) on $\{1,2,3\}$, we denote

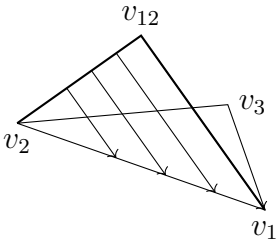
$$\begin{array}{ccccccc} \mathscr{V}_{i_1i_2i_3} & \xrightarrow{\text{blow-up of } S_{i_1i_3}, S_{i_2i_3}} & \mathscr{V}_{i_1i_2} & \xrightarrow{\text{blow-up of } S_{i_1i_2}} & \mathscr{U}_{i_1i_2} & \xrightarrow{\text{blow-up of } D_{i_1}, D_{i_2}} & \mathscr{U}\,. \\ \cup & & \cup & & \cup & & \\ E_{i_1i_3}, E_{i_2i_3} & & E_{i_1i_2} & & S_{i_1i_2}, S_{i_1i_3}, S_{i_2i_3} & & \end{array}$$

The refinement of the fan of $\mathscr{U}$ corresponding to $\mathscr{V}_{i_1i_2i_3}$ is such that the skeleton $\mathrm{Sk}(\mathscr{V}_{i_1i_2i_3}) = \mathrm{Sk}(\mathscr{V}_{123})$ as subspaces in the Berkovich space of $\mathscr{U}_K$; it is independent on the chosen order so that we simply denote this subspace by $\mathrm{Sk}(\mathscr{V})$. However, the models $\mathscr{V}_{i_1i_2i_3}$ and $\mathscr{V}_{123}$ induce in general different simplicial subdivisions and different retractions onto $< v_1, v_2, v_3 >$. For instance, the only edge in the interior of $\mathrm{Sk}(\mathscr{V}_{i_1i_2i_3})$ is $< v_{i_2}, v_{i_1i_3} >$, which indeed depends on the chosen order. Here below we illustrate the skeletons and the Berkovich retractions in a couple of examples.

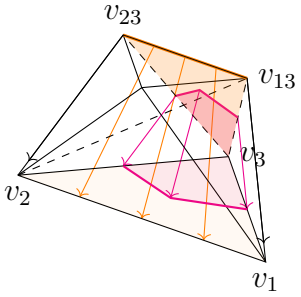
$(1,2,3)$



$\mathrm{Sk}(\mathscr{V}_{123})$

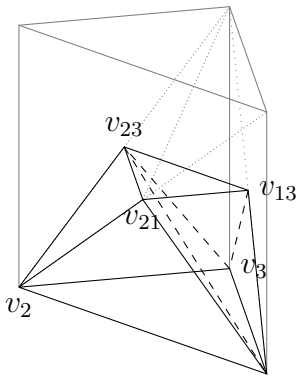


$\rho_{\mathscr{U}_{12}}$ on $\mathrm{Sk}(\mathscr{V}_{12})$

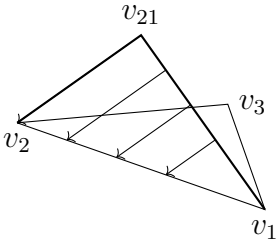


$\rho_{\mathscr{V}_{12}}$ on $\mathrm{Sk}(\mathscr{V}_{123})$

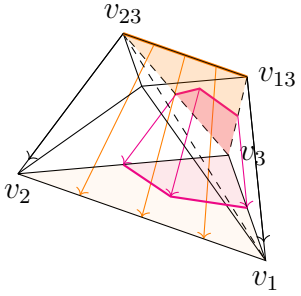
$(2,1,3)$



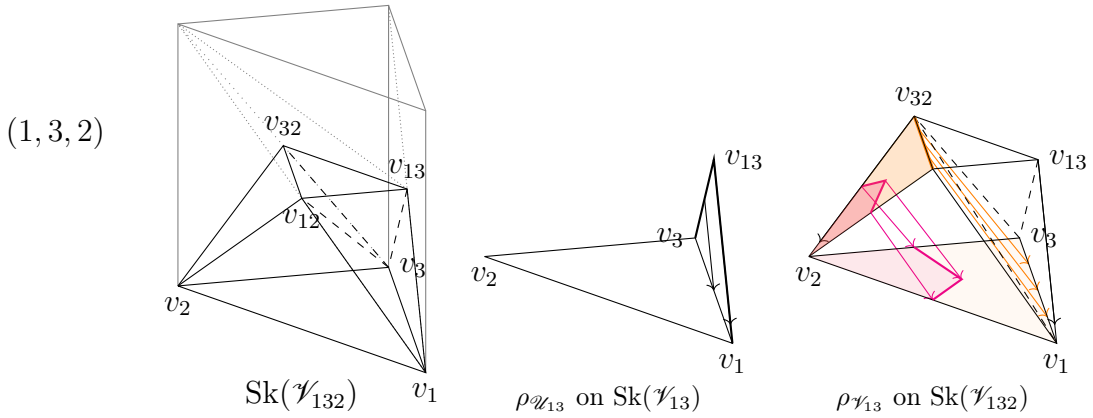
$\mathrm{Sk}(\mathscr{V}_{213})$



$\rho_{\mathscr{U}_{21}}$ on $\mathrm{Sk}(\mathscr{V}_{21})$



$\rho_{\mathscr{V}_{21}}$ on $\mathrm{Sk}(\mathscr{V}_{213})$



The blow-up of $\mathscr{V}_{i_1i_2i_3}$ along the toric strata $D_{i_2} \cap E_{i_1i_3}$ and $E_{i_1i_2} \cap D'_{i_3}$ yields a refinement of the fan which coincides with the fan of $\mathscr{G}$, constructed at the end of 4.2.3. It follows that the model $\mathscr{G}$ dominates all resolutions $\mathscr{V}_{i_1i_2i_3}$ independently on the order, hence all Berkovich retractions $\rho_{\mathscr{V}_{i_1i_2i_3}}$, $\rho_{\mathscr{V}_{i_1i_2}}$ and $\rho_{\mathscr{U}_{i_1i_2}}$ factors through $\rho_{\mathscr{G}}$.

Our goal is to construct a map π, composing the Berkovich retraction $\rho_{\mathscr{G}}$ with a collapse κ of the additional 3-cell τ and a combinatorial retraction ρ

$$\begin{array}{lccccccc}
\pi : & \mathscr{U}_K^{\mathrm{an}} & \xrightarrow{\rho_{\mathscr{G}}} & \mathrm{Sk}(\mathscr{G}) & \xrightarrow[\text{collapse}]{\kappa} & \mathrm{Sk}(\mathscr{V}) & \xrightarrow[\text{retraction}]{\rho} & \mathrm{Sk}(\mathscr{U}) \\
\| & \cup & & \cup & & \cup & & \cup \\
\text{over } \widetilde{\mathrm{Star}}(v_j) \quad \rho_{\mathscr{U}_{i_1i_2}} : \pi^{-1}(\widetilde{\mathrm{Star}}(v_j)) & & \xrightarrow{\rho_{\mathscr{G}}} & \mathrm{Sk}(\mathscr{G}) & \xrightarrow{\rho_{\mathscr{V}_{i_1i_2j}}} & \mathrm{Sk}(\mathscr{V}_{i_1i_2j}) & \xrightarrow{\rho_{\mathscr{U}_{i_1i_2}}} & \widetilde{\mathrm{Star}}(v_j)
\end{array}$$

such that, given any vertex v_j in $\mathrm{Sk}(\mathscr{U})$, the restriction of π over $\widetilde{\mathrm{Star}}(v_j)$ (the Star is taken with respect to the first barycentric subdivision, as in 3.3.11) is $\rho_{\mathscr{U}_{i_1i_2}}$ for any order (i_1, i_2, j) on $\{1, 2, 3\}$, i.e. any order where the index j is the biggest. This guarantees that around each v_j, the map π is the Berkovich retraction induced by a small resolution $\mathscr{U}_{i_1i_2}$ where the strict transform of D_j is isomorphic to D_j, so that we may apply corollary 3.3.2.

- The retraction ρ. We identify again the skeleton $\mathrm{Sk}(\mathscr{V})$ with the polyhedron in $\mathbb{R}^3$ described in 4.2.3. On the convex hull:

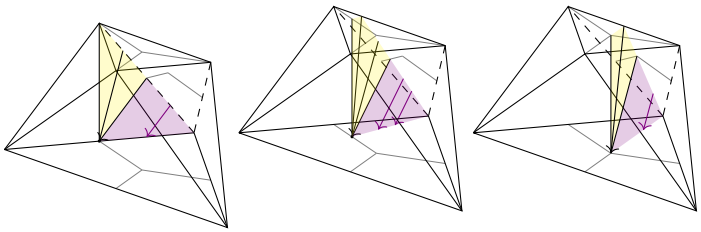
$$P = \mathrm{Conv}\left(v_{23}, (-1/4, 1/4, 1), (0, 1/3, 1), (0, 0, 0), v_3, (0, 1/3, 0)\right),$$

the retraction ρ is given as follows:

We extend

$$(x,y,z)\in P\mapsto \begin{cases}\left(x+\left(1-\frac{t}{2}\right)z, y+\frac{t}{2}z, 0\right) & \text{if } x+\left(1-\frac{t}{2}\right)z\leqslant 0\\ \left(0,\frac{y}{x+1},0\right) & \text{if } x+\left(1-\frac{t}{2}\right)z\geqslant 0\end{cases}$$
$$\text{where } t=\frac{2y}{x+y+1}. \tag{4.2.7}$$

Here is a pictorial description for certain values of t:



$t=0$ $t=\frac{1}{4}$ $t=\frac{1}{2}$

the definition of ρ to $\mathrm{Sk}(\mathscr{V})$ by symmetry along the medians of the triangles $<v_1,v_2,v_3>$ and $<v_{12},v_{13},v_{23}>$. In particular, we note that the image of $<v_{12},v_{13},v_{23}>$ is the graph in $<v_1,v_2,v_3>$ from def. 3.3.11.

- The combinatorial retraction π'. We define the collapse κ as the projection of the additional 3-cell τ of $\mathrm{Sk}(\mathscr{G})$ onto $<v_{12},v_{13},v_{23}>$ along the z-direction. We call $\pi':=\rho\circ\kappa$ the combinatorial retraction of the skeleton $\mathrm{Sk}(\mathscr{G})$ onto $\mathrm{Sk}(\mathscr{U})=<v_1,v_2,v_3>$.

- Finally, we check that $\pi'=\rho_{\mathscr{U}_{i_1i_2}}$ over $\widetilde{\mathrm{Star}}(v_j)$. As the preimage of $\widetilde{\mathrm{Star}}(v_j)$ is disjoint from $<v_{12},v_{13},v_{23},v'_{123}>$, we have to prove that $\rho=\rho_{\mathscr{U}_{i_1i_2}}$. By symmetry of ρ, it is enough to check this for v_3. Over $\mathrm{Star}(v_3)'$ we have $\rho_{\mathscr{U}_{i_1i_2}}=\rho_{\mathscr{V}_{i_1i_2}}$; there, the expression of $\rho_{\mathscr{V}_{i_1i_2}}$ determined in 4.2.6 coincides with the definition of ρ in 4.2.7, hence we conclude.

Fix an order (i,j,k,l,h) on $\{1,\dots,5\}$ and consider the small resolution $\mathscr{X}_{ijkl}$, obtained from $\mathscr{X}$ by blowing-up the divisors D_i, D_j, D_k, D_l in that order. It now follows from the local study of the singularities of $\mathscr{X}$ that this is indeed a small resolution of $\mathscr{X}$.
In $\mathscr{X}_{ijkl}$ we still denote the strict transforms of the strata of $\mathscr{X}_k$ by D_m, by $D_{mm'}=D_m\cap D_{m'}$ and by $D_{mm'm''}=D_m\cap D_{m'}\cap D_{m''}$ with $m,m',m''\in\{1,\dots,5\}$. By the study of the local model in section 4.2.2, the exceptional locus of $g_{ijkl}:\mathscr{X}_{ijkl}\to\mathscr{X}$

$$\begin{array}{ccccccccc} g_{ijkl}:\mathscr{X}_{ijkl} & \xrightarrow[G_{ijkl}]{\text{blow-up of } D_l} & \mathscr{X}_{ijk} & \xrightarrow[G_{ijk}]{\text{blow-up of } D_k} & \mathscr{X}_{ij} & \xrightarrow[G_{ij}]{\text{blow-up of } D_j} & \mathscr{X}_i & \xrightarrow[G_i]{\text{blow-up of } D_i} & \mathscr{X}\\ \cup & & \cup & & \cup & & \cup & & \\ S_{lh} & & S_{kl},S_{kh} & & S_{jk},S_{jl},S_{jh} & & S_{ij},S_{ik},S_{il},S_{ih} & & \end{array}$$

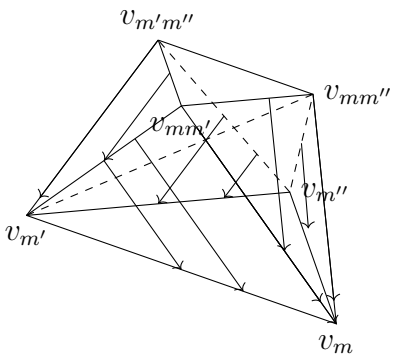
consists of ten surfaces $S_{mm'}$, with $m,m'\in\{1,\dots,5\}$ and $m<m'$ in the order (i,j,k,l,h). The surface $S_{mm'}$ is mapped via g_{ijkl} to the singular curve $C_{mm'}$, and is

contained in the strict transform of D_m. The component D_h (corresponding to the biggest index in the chosen order) is the only one isomorphic to its strict transform.

4.2.5 The global combinatorial retraction

We consider the blow-up $h_{ijkl} : \mathscr{W}_{ijkl} \to \mathscr{X}_{ijkl}$ of the surfaces $S_{mm'}$ in lexicographical order with respect to (i, j, k, l, h); we denote by $E_{mm'}$ the corresponding exceptional divisors. The skeleton $\mathrm{Sk}(\mathscr{W}_{ijkl})$ consists of the union of the skeleton $\mathrm{Sk}(X)$ with four additional 3-cells for each 2-dimensional face $< v_m, v_{m'}, v_{m''} >$ of $\mathrm{Sk}(X)$: for each ordered triple $m < m' < m''$, the union of the additional cells is isomorphic to $\mathrm{Sk}(\mathscr{V}_{123})$ from section 4.2.3, where we identify $v_1 = v_m$, $v_2 = v_{m'}$ and $v_3 = v_{m''}$. The retraction $\rho_{\mathscr{X}_{ijkl}}$ collapses the additional faces onto $< v_m, v_{m'}, v_{m''} >$ as $\rho_{\mathscr{U}_{12}} \circ \rho_{\mathscr{V}_{12}}$.

Additional 3-cells of $\mathrm{Sk}(\mathscr{W}_{ijkl})$ over $< v_m, v_{m'}, v_{m''} >$ with a pictorial description of the retraction $\rho_{\mathscr{X}_{ijkl}}$



Given another order (i', j', k', l', h') on $\{1, 2, 3, 4, 5\}$, the skeleton $\mathrm{Sk}(\mathscr{W}_{i'j'k'l'})$ coincides with $\mathrm{Sk}(\mathscr{W}_{ijkl})$ as subspace of X^{an}; we denote this simply by $\mathrm{Sk}(\mathscr{W})$. Instead, the triangulation and the retraction depend on the order. Our goal is therefore to construct a model $\mathscr{Z}$ which dominates all models $\mathscr{W}_{ijkl}$ regardless of the order, so that all retractions $\rho_{\mathscr{W}_{ijkl}}$ factors through $\rho_{\mathscr{Z}}$.

Along the lines of section 4.2.3, we define $\mathscr{Z}$ as the blow-up of $\mathscr{W}_{ijkl}$ along $D_{m'} \cap E_{mm''}$ and $E_{mm'} \cap D'_{m''}$, for all ordered triples $m < m' < m''$ in the order (i, j, k, l, h):

$$\begin{array}{ccccccc} \mathscr{Z} & & \xrightarrow[\text{for all } m<m'<m'']{\text{blow-up of } D_{m'}\cap E_{mm''}, E_{mm'}\cap D'_{m''}} & \mathscr{W}_{ijkl} & \xrightarrow[h_{ijkl}]{\text{blow-up of } S_{mm'} \text{ for all } m<m'} & \mathscr{X}_{ijkl} & \xrightarrow[g_{ijkl}]{\text{blow-up of } D_i, D_j, D_k, D_l} & \mathscr{X}. \\ \cup & & & \cup & & \cup & & \\ E_{mm'm''}, E'_{mm'm''} & & & E_{mm'} & & S_{mm'} & & \end{array}$$

We denote by $E_{mm'm''}$ and $E'_{mm'm''}$ the corresponding exceptional divisors, and deduce from the local study of these morphisms in 4.2.3 that $\mathrm{Sk}(\mathscr{Z})$ is obtained from $\mathrm{Sk}(\mathscr{W})$ by adding a new 3-cell $\tau_{mm'm''} :=< v_{mm'}, v_{mm''}, v_{m'm''}, v'_{mm'm''} >$ for each triple $m < m' < m''$.

We now define the combinatorial retraction of $\mathrm{Sk}(\mathscr{Z})$ onto $\mathrm{Sk}(X)$: given the 2-cell $< v_m, v_{m'}, v_{m''} >$, we identify $v_1 = v_m$, $v_2 = v_{m'}$ and $v_3 = v_{m''}$ and contract onto $< v_m, v_{m'}, v_{m''} >$ the additional cells of $\mathrm{Sk}(\mathscr{Z})$ over $< v_m, v_{m'}, v_{m''} >$, via the combinatorial retraction $\pi' = \rho \circ \kappa$ constructed in 4.2.4. With a slight abuse of notation, we still denote this map by π'. By construction, the composition $\pi = \pi' \circ \rho_{\mathscr{Z}}$

$$\begin{array}{ccccccc}
\pi: & X^{\mathrm{an}} & \xrightarrow{\rho_{\mathscr{X}}} & \mathrm{Sk}(\mathscr{X}) & \xrightarrow[\substack{\text{combinatorial}\\ \text{retraction}}]{\pi'} & \mathrm{Sk}(X) \\
& \| & & \cup & & \cup \\
\text{over } \widetilde{\mathrm{Star}}(v_m) & \rho_{\mathscr{X}_{ijkl}}: \pi^{-1}(\widetilde{\mathrm{Star}}(v_m)) & \xrightarrow{\rho_{\mathscr{X}}} & \mathrm{Sk}(\mathscr{X}) & \xrightarrow{\rho_{\mathscr{X}_{ijkl}}} & \widetilde{\mathrm{Star}}(v_m)
\end{array}$$

coincides with $\rho_{\mathscr{X}_{ijkl}}$ over $\mathrm{Star}(v_m)'$ for any order (i,j,k,l,m) on $\{1,2,3,4,5\}$. In other words, around each vertex v_m, the map π is the Berkovich retraction induced by a small resolution $\mathscr{X}_{ijkl}$ of $\mathscr{X}$, where the strict transform of D_m is isomorphic to D_m, thus in particular $\overset{\circ}{D}_m \subset D_m$ is a torus embedding.
By construction, around any point of $\mathrm{Sk}(X) \setminus \Gamma$, the retraction π is equal to the Berkovich retraction induced by a suitable minimal model $\mathscr{X}_{ijkl}$ of X. It follows from the results in [NXY19] that π induces an integral affine structure with singularities on $\mathrm{Sk}(X)$. By theorem 3.1.21, we obtain that this integral affine structure has no singularities outside Γ.

4.2.6 The skeleton of an analytic model

The purpose of this section is to extend the constructions of the Berkovich skeleton and retraction to the following situation: let X be a smooth family of n-dimensional projective varieties over the punctured disk $\mathbb{D}^*$, X_K the base change to $K = \mathbb{C}((t))$, and let $\mathscr{X}$ be a separated proper regular algebraic space over R, such that $\mathscr{X} \times_R K = X_K$. Writing abusively $\mathscr{X}$ the complex analytification of the latter algebraic space, this means that the morphism $\mathscr{X} \to \mathbb{D}$ is a proper holomorphic submersion, not assumed to be projective, and we will assume that $\mathscr{X}_0$ is a strict normal crossing divisor in $\mathscr{X}$ in the sense of complex analytic geometry; we still call such a space $\mathscr{X}$ an snc model of X.
Our goal is to define a skeleton $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ and a retraction $\rho_{\mathscr{X}}: X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X})$ associated with $\mathscr{X}$.

We start by recalling a construction from [BJ17, 4.2]. Let $\mathscr{X}'$ be another snc model of X that dominates $\mathscr{X}$; we write $h: \mathscr{X}' \longrightarrow \mathscr{X}$ and the special fibers $\mathscr{X}_0 = \sum_{i\in I} a_i D_i$ and $\mathscr{X}'_0 = \sum_{i\in I'} a'_i D'_i$. Then there exists an integral affine retraction

$$r_{\mathscr{X}'\mathscr{X}}: \mathcal{D}(\mathscr{X}'_0) \longrightarrow \mathcal{D}(\mathscr{X}_0),$$

as follows. Let τ' be the simplex in $\mathcal{D}(\mathscr{X}'_0)$ corresponding to a stratum $Y' \subseteq D'_0 \cap \ldots \cap D'_q$. Let Y be the minimal stratum of $\mathscr{X}_0$ such that $h(Y') \subset Y$; we denote by τ the simplex in $\mathcal{D}(\mathscr{X}_0)$ corresponding to Y and we write $D_0, \ldots, D_p$ the irreducible components of $\mathscr{X}_0$ containing Y. Then

$$h^* D_i = \sum_{j=0}^{q} a_{ij} D'_j + \sum_{h\in I'\setminus\{0,\ldots,q\}} a_{ih} D'_h$$

and we define the map $r_{\mathscr{X}'\mathscr{X}}$ on τ' by the formula:

$$\tau' \ni w = (w_0, \ldots, w_q) \mapsto r_{\mathscr{X}'\mathscr{X}}(w) = \Big(\sum_{j=0}^{q} a_{ij} w'_j\Big)_{0\leqslant i\leqslant p} \in \tau.$$

This yields a continuous integral affine map. Furthermore, we have the following transitivity property: if $\mathscr{X}'' \longrightarrow \mathscr{X}' \longrightarrow \mathscr{X}$ are three snc models of X, then $r_{\mathscr{X}''\mathscr{X}} = r_{\mathscr{X}'\mathscr{X}} \circ r_{\mathscr{X}''\mathscr{X}'}$.

Definition 4.2.8. *Let τ' be a face of $\mathcal{D}(\mathscr{X}'_0)$. We say τ' is active for $r_{\mathscr{X}'\mathscr{X}}$ if:*

- *$h : Y' \longrightarrow Y$ is a bimeromorphic morphism,*
- *the $\mathbb{Q}$-linear map inducing $r_{\mathscr{X}'\mathscr{X}} : \tau' \longrightarrow \tau$ is an isomorphism.*

We write $A_{\mathscr{X}'\mathscr{X}}$ for the union of active faces in $\mathcal{D}(\mathscr{X}'_0)$. It follows from [BJ17, Proposition 4.3] that $r_{\mathscr{X}'\mathscr{X}}$ induces a homeomorphism from $A_{\mathscr{X}'\mathscr{X}}$ onto $\mathcal{D}(\mathscr{X}_0)$.

Definition 4.2.9. *Let $\mathscr{X}$ be an snc model of X, and assume there exists a* projective *snc model $\mathscr{X}'$ and $h : \mathscr{X}' \longrightarrow \mathscr{X}$. We define the skeleton $\mathrm{Sk}(\mathscr{X}) \subset X^{\mathrm{an}}$ as the image of $A_{\mathscr{X}'\mathscr{X}}$ by the embedding $\mathcal{D}(\mathscr{X}'_0) \hookrightarrow X^{\mathrm{an}}$, and the Berkovich retraction $\rho_{\mathscr{X}} : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(\mathscr{X})$ as the composition $r_{\mathscr{X}'\mathscr{X}} \circ \rho_{\mathscr{X}'}$ (after identifying $\mathcal{D}(\mathscr{X}'_0)$ with $\mathrm{Sk}(\mathscr{X}')$).*

It follows directly from the transitivity property that this does not depend on the choice of a projective model $\mathscr{X}'$.

Lemma 4.2.10. *Let $\mathscr{X}$ be an snc model of X and assume that $\mathscr{X}$ admits a dominating projective snc model. Then for any stratum Y of $\mathscr{X}_0$, the retraction $\rho_{\mathscr{X}}$ over $\mathrm{Star}(\tau_Y)$ only depends on the formal completion $\widehat{\mathscr{X}_{/Y}}$: if $\mathscr{Y}$ is another snc model, admitting a projective resolution and such that the birational map $\mathscr{Y} \longrightarrow \mathscr{X}$ induces an isomorphism:*

$$\widehat{\mathscr{Y}_{/Y}} \xrightarrow{\sim} \widehat{\mathscr{X}_{/Y}},$$

then $\rho_{\mathscr{X}} \equiv \rho_{\mathscr{Y}}$ over $\mathrm{Star}(\tau_Y)$.

Proof. Let $\mathscr{X}'$ be a projective snc model dominating $\mathscr{X}$ and write $h : \mathscr{X}' \longrightarrow \mathscr{X}$. We denote by $D_0, \dots, D_p$ the components of $\mathscr{X}_0$ containing Y.

Let $Z' \subseteq D'_0 \cap \dots \cap D'_q$ be any stratum of $\mathscr{X}'_0$, and denote by τ' the corresponding simplex in $\mathcal{D}(\mathscr{X}'_0)$. By construction of $r_{\mathscr{X}'\mathscr{X}}$, we have $\mathrm{Int}(\tau_{Z'}) \subseteq r^{-1}_{\mathscr{X}'\mathscr{X}}(\mathrm{Star}(\tau_Y))$ if and only if $h(Z') \subseteq Y$; in this case, $h(D'_j) \cap Y \neq \varnothing$ for any $j = 0, \dots, q$. Thus, if D_α is an irreducible component of $\mathscr{X}_0$ not cutting Y, it follows that h^*D_α does not have any component along the D'_j for $j = 0, \dots, q$.
We deduce from this that for each D_i component of $\mathscr{X}_0$ containing Y and $j \in \{0, \dots, q\}$, the coefficient of D'_j in $h^*(D_i)$ is determined by h and a local equation of D_i in a formal neighbourhood of Y. This proves that $r_{\mathscr{X}'\mathscr{X}}$ over $\mathrm{Star}(\tau_Y)$ only depends on h over $\widehat{\mathscr{X}_{/Y}}$.

By remark 1.1.18, $\rho_{\mathscr{X}'}$ only depends on $\widehat{\mathscr{X}'_{/Z'}}$ above $\mathrm{Star}(\tau_{Z'})$. If moreover $h(Z') \subseteq Y$, then h induces a morphism $\widehat{\mathscr{X}'_{/Z'}} \to \widehat{\mathscr{X}_{/Y}}$, hence $\rho_{\mathscr{X}'}$ only depends on h over $\widehat{\mathscr{X}_{/Y}}$.

By the independence of $\rho_{\mathscr{X}}$ on the choice of projective model $\mathscr{X}'$ and morphism h, we conclude that $\rho_{\mathscr{X}}$ over $\mathrm{Star}(\tau_Y)$ only depends on the formal completion $\widehat{\mathscr{X}_{/Y}}$. □

Proposition 4.2.11. *Let $\mathscr{X}$ be an snc model of X and assume that $\mathscr{X}$ admits a dominating projective snc model. Let C be a one-dimensional stratum of $\mathscr{X}_0$ and assume that C is isomorphic to $\mathbb{P}^1$ and $C \setminus \mathring{C}$ consists of two points. Then $\rho_{\mathscr{X}}$ is an affinoid torus fibration over $\mathrm{Star}(\tau_C) \subset \mathrm{Sk}(\mathscr{X})$, and the induced affine structure over $\mathrm{Star}(\tau_C)$ is described as in 3.3.3.*

Proof. By lemma 4.2.10, the retraction $\rho_{\mathscr{X}}$ over $\mathrm{Star}(\tau_C)$ only depends on the formal completion $\widehat{\mathscr{X}_{/C}}$. By [Knu71, V, Theorem 2.5], since C is a scheme, the formal algebraic space $\widehat{\mathscr{X}_{/C}}$ is a formal scheme. We are therefore in the setting of [NXY19, Proposition 5.4, Theorem 6.1] and we can conclude. □

4.3 The Fermat case

4.3.1 The non-archimedean Monge-Ampère equation

Let $X \subset \mathbb{P}^{n+1} \times \mathbb{D}^*$ be the Fermat family of Calabi-Yau hypersurfaces:

$$X = \{z_0 ... z_{n+1} + t(z_0^{n+2} + ... + z_{n+1}^{n+2})\} = 0,$$

and $\mathscr{X} \subset \mathbb{P}^{n+1} \times \mathbb{D}$ its closure. We use the same notation for the respective base change to K and R. We are in the setting of section 4.1.1, as the Fermat polynomial $F = z_0^{n+2} + ... + z_{n+1}^{n+2}$ is such that prop. 4.1.1 holds. The family X is endowed with the polarization $L = \mathcal{O}_X(n+2)$, and with the family of Calabi-Yau measures:

$$\nu_t := i^{n^2} \Omega_t \wedge \bar{\Omega}_t,$$

where $(\Omega_t)_{t \in \Delta^*}$ is a holomorphic trivialization of the relative canonical bundle K_{X/Δ^*}. We set:

$$\mu_t := C_t \nu_t,$$

where $C_t > 0$ is the unique constant ensuring that $\int_{X_t} \mu_t = (L_t)^n$. By theorem 3.1.11, the family $(\mu_t)_{t \in \mathbb{D}^*}$ converges weakly on X^{hyb} to the Lebesgue measure μ_0 on $\mathrm{Sk}(X)$.
Each fiber X_t is endowed with the unique Kähler Ricci-flat metric $\omega_t \in c_1(L)$, satisfying the complex Monge-Ampère equation:

$$\omega_{CY,t}^n = \mu_t.$$

Since for all t, the Kähler form $\omega_t \in c_1(L_t)$, there exists a family of hermitian metrics $\phi_{CY,t}$ on L_t such that $\omega_{CY,t} = dd^c \phi_{CY,t}$ - note however that this family need not vary in a subharmonic way with respect to the direction of the base.
It turns out that in the non-archimedean limit, the Calabi-Yau metric (whose existence is ensured by thm. 1.4.7) is of toric nature (in an extrinsic way). Recall that $P \subset M_{\mathbb{R}}$ is the convex polytope associated to the toric boundary Δ of $\mathbb{P}^{n+1}$, so that setting $L = \mathcal{O}_{\mathbb{P}}(\Delta)$, semi-positive toric metrics on $(\mathbb{P}^{n+1,\mathrm{an}}, L^{\mathrm{an}})$ are in one-to-one correspondence with convex P-admissible function on $N_{\mathbb{R}}$, see theorem 2.2.9. We now have the following:

Theorem 4.3.1. *There exists a convex P-admissible function $u : N_{\mathbb{R}} \longrightarrow \mathbb{R}$, and induced continuous psh toric metric $\phi_u \in \mathrm{CPSH}(\mathbb{P}^{n+1,\mathrm{an}}_K, L^{\mathrm{an}})$, such that the restriction $\psi_u = (\phi_u)_{|X^{\mathrm{an}}}$ solves the non-archimedean Monge-Ampère equation:*

$$\mathrm{MA}(\psi_u) = \mu_0$$

on X^{an}.

A similar, more general statement was recently obtained independently in [HJMM22] for hypersurfaces of the form considered in section 4.1.1, building on the construction of a solution to the tropical Monge-Ampère equation on $\mathrm{Sk}(X)$.
The fact that the function u solves the real Monge-Ampère equation $\mathscr{M}(u) = \mu_0$ on $\mathrm{Sk}(X) \setminus \Gamma$ was established in [Li22] (where the branch cuts are the barycenters of the faces), however since the solution to this equation may not be unique, the convex function u was only obtained along a subsequence, and could depend on the choice thereof. However the solution to the NAMA equation is unique, so that as observed in [Li22, rem. 5.7], we get the following strenghtening of [Li22, thm. 5.1]:

Corollary 4.3.2. *The locally convex function u_∞ obtained in [Li22, thm. 5.1], as well as the generic SYZ fibration from [Li22, thm. 5.14], do not depend on the choice of a subsequence.*

We now move on to the proof of theorem 4.3.1, and start by explaining how the function u is obtained. Since in the sequel it will be more convenient to work with potentials instead of metrics, we fix once and for all the reference hybrid metric ϕ_{FS} on $\mathbb{P}^{n+1,\mathrm{hyb}}$, such that on the complex fiber X_t:

$$\phi_{FS,t} = \frac{(n+2)}{2} \log(|z_0|^2 + ... + |z_{n+1}|^2),$$

and:

$$\phi_{FS,0} = (n+2) \max(\log|z_0|, ..., \log|z_{n+1}|);$$

it is a continuous semi-positive metric on $\mathbb{P}^{n+1,\mathrm{hyb}}$ by prop. 1.3.5; we use the same notation for its restriction to X^{an}. This allows us to identify the Calabi-Yau metric with its relative potential:

$$\varphi_{CY,t} = \phi_{CY,t} - \phi_{FS,t},$$

normalized by the convention $\sup_{X_t} \varphi_{CY,t} = 0$, for all $t \in \bar{\mathbb{D}}_r$. Then Li produces, via double Legendre transform, P-admissible convex functions $u_{CY,t}$ on $N_{\mathbb{R}}$ which approximate the Calabi-Yau potential, and which satisfy suitable Lipschitz bounds. This implies that for any sequence $(t_k)_{k\in\mathbb{N}}$ going to zero inside $\mathbb{D}^*$, there exists a subsequence (which we will omit from notation) such that the admissible convex functions u_{CY,t_k} converge locally uniformly to a continuous, convex admissible function $u \in \mathrm{Ad}_P(N_{\mathbb{R}})$ - which *a priori* depends on a choice of subsequence. By theorem 2.2.12 and in particular example 2.2.13, this induces a continuous hybrid metric $\Phi_u \in \mathrm{CPSH}(\mathbb{P}^{n+1,\mathrm{hyb}}, L^{\mathrm{hyb}})$, and by restriction to X^{hyb} a continuous psh hybrid metric Ψ_u on (X, L). We write, following the notation of [Li22] (with a different sign convention), $s(t) = \frac{-1}{\log|t|}$, and $\mathrm{Log}_s(z) := \mathrm{Log}(sz)$ on the torus of $\mathbb{P}^{n+1}$, so that the restriction of Ψ_u to X_t is equal to $s^{-1}(u \circ \mathrm{Log}_s)$.

Proposition 4.3.3. *As $k \to \infty$ (up to extracting further), the sequence of complex Monge-Ampère measures $(s_k^{-n} dd^c(u \circ \mathrm{Log}_{s_k})^n)_{k\in\mathbb{N}}$ on X^{hyb} converge weakly to the Lebesgue measure μ_0 on the essential skeleton.*

Before moving to the proof of the proposition, we explain how this implies theorem 4.3.1. Writing $\psi_u = \Psi_u^{\mathrm{NA}}$ the restriction of Ψ_u to X^{an}, it follows from theorem 1.4.11 that the sequence of measures $(s_k^{-n} dd^c(u \circ \mathrm{Log}_{s_k})^n)_{k\in\mathbb{N}}$ on X^{hyb} converges weakly to $\mathrm{MA}(\psi_u)$, hence the equality. The uniqueness of u now follows from uniqueness of the solution to the non-archimedean Monge-Ampère equation.

Proof of prop. 4.3.3. By compactness of X^{hyb} and after further extraction, we may assume that the sequence of measures $\theta_k := s_k^{-n} dd^c(u \circ \mathrm{Log}_{t_k})^n$ on X^{hyb} converge weakly to a limit measure θ_∞ supported on X^{an}, of total mass L^n. Let $f \in \mathcal{C}^0(X^{\mathrm{hyb}})$, then by [Li22, lem. 5.3, proof of thm. 5.1], as $k \to \infty$ (up to extracting further) we have:

$$\Big| \int_{U_{s_k}} f\mu_{s_k} - \int_{U_{s_k}} f\theta_k \Big| \xrightarrow[k\to\infty]{} 0,$$

where U_{s_k} denotes the union of the 'toric regions' $U_w^{s_k,*}$ from [Li22, §4.5]. Moreover, we have $\mu_{s_k}(X_{s_k} \setminus U_{s_k}) \longrightarrow 0$ when $k \to \infty$, by [Li22, prop. 3.14]. Thus, if $f \geq 0$, we infer that $\int_{X^{\mathrm{an}}} f\theta_\infty \geq \int_{X^{\mathrm{an}}} f\mu_0$, hence $\theta_\infty \geq \mu_0$ as measures on X^{an}. However both measures have the same mass L^n, so that they are equal, which concludes the proof. □

Lemma 4.3.4. *[Li22, proof of thm. 5.1] Let $\mathrm{Log}_t : X_t \longrightarrow N_\Sigma$ be the restriction to X_t of $\mathrm{Log}_{\mathbb{P}}^{\mathrm{hyb}}$. Then as $k \to \infty$ (up to extracting further),*

$$(\mathrm{Log}_{t_k})_* \big(s_k^{-n} dd^c(u \circ \mathrm{Log}_{s_k})^n \big) \longrightarrow \iota_* \mu_0$$

weakly in the sense of measures on N_Σ, where $\iota : \mathrm{Sk}(X) \hookrightarrow N_\Sigma$ is the embedding from prop. 4.1.3.

4.3.2 The comparison property

The purpose of this section is to prove the following:

Theorem 4.3.5. *Let $X \subset \mathbb{P}^{n+1} \times \mathbb{D}^*$ be the Fermat family of hypersurfaces, and set $L = \mathcal{O}_{\mathbb{P}}(n+2)$.*
We write $(\mathscr{X}, \mathscr{L})$ the dlt model of X obtained by taking the closure of X inside $\mathbb{P}^{n+1} \times \mathbb{D}^*$.
Writing the solution ϕ to the NAMA equation as:

$$\phi = \phi_{\mathscr{L}} + \psi,$$

we have that $\psi = \psi \circ \rho$ over $\mathrm{Sk}(X) \setminus \Gamma$, where $\rho : X^{\mathrm{an}} \longrightarrow \mathrm{Sk}(X)$ is the admissible retraction constructed in theorem 4.1.8, and with choice of branch cuts such that a_τ is the barycenter of τ, for each face $\tau \subset \mathrm{Sk}(\mathscr{X})$.

As we shall see, the above theorem is mostly a consequence of theorem 4.3.1 and [Li22, cor. 3.28].
Recall that the solution of the non-archimedean Monge-Ampère is explicitly given as the restriction to X^{an} of the toric metric $\phi_u \in \mathrm{CPSH}(\mathbb{P}_K^{n+1,\mathrm{an}}, L^{\mathrm{an}})$. The model metric $\phi_{\mathscr{L}}$ is also toric, and we have $\phi_{\mathscr{L}} = \phi_{u_{\mathrm{FS}}}$, where:

$$u_{\mathrm{FS}}(x) = \max_{m \in V(P)} \langle m, x \rangle$$

is the support function of the boundary of $\mathbb{P}^{n+1}$. We infer that the relative toric potential is equal to $\psi = (u - u_{\mathrm{FS}}) \circ \mathrm{val}_\Sigma$ on $\mathbb{P}^{n+1}$, hence also on X. We set $u' = (u - u_{\mathrm{FS}})$ which is a continuous function on $N_{\mathbb{R}}$, extending continuously to N_Σ.
We need to prove that if $\rho(x) \in \mathrm{Sk}(X) \setminus \Gamma$, then $\psi(x) = \psi(\rho(x))$. It is enough to treat the two following cases: $\rho(x) \in \mathrm{Int}(\tau)$ for a maximal face τ of $\mathrm{Sk}(\mathscr{X})$, or $\rho(x) \in U_i = \mathrm{Int}(\widetilde{\mathrm{Star}}(v_i))$ for a vertex v_i of the skeleton.
We start with the former. If $\rho(x) \in \mathrm{Int}(\tau)$, then $\rho(x) = \mathrm{val}_\Sigma(x)$ under the identification $\mathrm{Sk}(X) = \partial P^*$, hence $\psi(\rho(x)) = u'(\mathrm{val}_\Sigma(x)) = \psi(x)$ as ϕ is a toric metric on $\mathbb{P}^{n+1}$.
The second case relies on the following observation:

Lemma 4.3.6. *[Li22, cor. 3.28] Let $u : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ be the convex admissible function from the statement of theorem 4.3.1, and let $v_i \in \mathrm{Sk}(\mathscr{X})$ be a vertex. Then on the region $X_{v_i} = (U_i + \mathbb{R}_{\geq 0} v_i) \cap \mathrm{Trop}(X)$, the equality:*

$$u(w + \alpha v_i) = u(w) + \alpha$$

holds for any $w \in U_i$, $\alpha \in \mathbb{R}_{\geq 0}$.

This implies that if $\rho(x) \in U_i$ - which means $\mathrm{val}_\Sigma(x) \in X_{v_i}$ - then $u'(x) = u'(\delta(x))$. Indeed, writing $x = w + \alpha v_i$ with $w \in U_i$, we have:

$$u_{\mathrm{FS}}(x) = \max_{m \in V(P)} \langle m, x \rangle = u_{\mathrm{FS}}(w) + \max_{m \in V(P)} \langle m, \alpha v_i \rangle = u_{\mathrm{FS}}(w) + \alpha,$$

since v_i is a vertex of $P^* = \{\langle P, \cdot \rangle \leq 1\}$. This yields $u'(x) = u'(w) = u'(\delta(x))$ by prop. 4.1.13, and since $\psi = u' \circ \mathrm{val}_\Sigma$, we get that:

$$\psi(\rho(x)) = \psi(x)$$

whenever $\rho(x) \in U_i$. This concludes the proof of theorem 4.3.5.

Remark 4.3.7. *Let $X \subset \mathbb{P}^{n+1} \times \mathbb{D}^*$ be a maximally degenerate hypersurface as considered in theorem 4.1.8, and assume that for a certain choice $\underline{a}$ of branch cuts, the solution $\psi \in \mathcal{C}^0(X^{\mathrm{an}})$ of the non-archimedean Monge-Ampère equation:*

$$\mathrm{MA}(\phi_{\mathscr{L}} + \psi) = \mu_0$$

satisfies the comparison property $\psi = \psi \circ \rho_{\underline{a}}$ over $\mathrm{Sk}(X) \setminus \Gamma$. We write $\psi_{\mathrm{FS}} : N_{\mathbb{R}} \longrightarrow \mathbb{R}$ the toric potential for the Fubini-Study metric on the ambient $\mathbb{P}^{n+1}$ with the anti-canonical polarization, and use the same notation for its restriction to $\mathrm{Sk}(X)$.
Then the collection of local functions:

$$\psi_\tau := \psi \text{ on } \mathrm{Int}(\tau),$$

$$\psi_i := \psi + \psi_{\mathrm{FS}} - m_i \text{ on } U_i,$$

where $m_i \in M$ *satisfies* $\langle m_i, v_i \rangle = 1$*, are convex in affine coordinates on* $\mathrm{Sk}(X) \setminus \Gamma$*, and satisfy the real Monge-Ampère equation:*

$$n! \mathscr{M}(\psi) = \mu_0,$$

away from Γ*. Note that the local potential* ψ_i *equates the* u_m *from [Li22, def. 3.22]. This holds over the maximal faces of* $\mathrm{Sk}(\mathscr{X})$ *by theorem 3.1.27, so that we focus on an open subset* U_i*. It is enough to prove that under the formal isomorphism:*

$$\widehat{\mathscr{X}_{i/D_i}} \simeq \widehat{\mathscr{N}_{/D_i}}$$

provided by theorem 3.1.21, the metric $\phi = \phi_{\mathscr{L}} + \psi$ *is toric and has toric potential* ψ_i *on* U_i*. The first part follows from the assumption* $\psi = \psi \circ \rho$*, while the second one follows from the fact that the toric potential for* $(\phi_{\mathscr{L}})_{|\mathbb{T}_{D_i}}$ *is* $(\psi_{\mathrm{FS}} - m_i)$ *- recall that the torus* $\mathbb{T}_{D_i,K}$ *of* $\mathscr{N}$ *is the quotient of* $\mathbb{T}_{\mathbb{P}}$ *by the* m_i*-direction.*

4.3.3 The hybrid SYZ fibration

While the results from [Li22] do not yield a global SYZ fibration on the Fermat hypersurfaces X_t for $|t| \ll 1$, there does exist a special Lagrangian fibration on a region of X_t whose Calabi-Yau measure is arbitrarily close to 1 as $t \to 0$. We will show that this fibration converges to the retraction ρ from thm. 4.3.1 in the hybrid topology, this is similar to the results from [GO22] for finite quotients of abelian varieties.

We start by recalling the precise statement on the existence of an SYZ fibration in the generic region. By the local theory for the real Monge-Ampère equation, the locally convex u on $\mathrm{Sk}(X) \setminus \Gamma$ is smooth and strictly convex on an open subset $\mathcal{R} \subset \big(\mathrm{Sk}(X) \setminus \Gamma\big)$, whose complement has $(n-1)$-Hausdorff measure zero; in particular $\mathrm{Sk}(X) \setminus \mathcal{R}$ is connected. The estimates from [Li22] imply that the Calabi-Yau potential converges in a natural, $\mathcal{C}^\infty$-sense to u locally over $\mathcal{R}$, so that the results from [Zha17] can be used to obtain the following:

Theorem 4.3.8. *[Li22, thm. 5.13] Let* $K \subset \mathcal{R}$ *be a compact subset. There exists an open neighbourhood* $V_K \subset \mathcal{R}$ *of* K*, and writing* $U_{t,K} = \mathrm{Log}_t^{-1}(V_K)$*, a special Lagrangian fibration:*

$$f_t : U_{t,K} \longrightarrow \mathrm{Sk}(X)$$

that is a $\mathcal{C}^\infty$*-perturbation of the map* Log_t *as* $t \to 0$*. Moreover, if* $(K_n)_{n\in\mathbb{N}}$ *is a compact exhaustion of* $\mathcal{R}$*, then the* U_{t,K_n} *have Calabi-Yau measure arbitrarily close to* 1 *as* $t \to 0$*,* $n \gg 1$*.*

We claim that these generic SYZ fibrations converge in the hybrid topology to ρ:

Proposition 4.3.9. *Let* $U_{0,K} = \rho^{-1}(V_K)$ *and:*

$$U_K^{\mathrm{hyb}} = \mathrm{val}_{\mathrm{hyb}}^{-1}(V_K) = \bigsqcup_{t \in \bar{\mathbb{D}}_r} U_{t,K} \subset X^{\mathrm{hyb}}$$

for $r \ll 1$ depending on K. Then the hybrid SYZ fibration:

$$f : U_K^{\mathrm{hyb}} \longrightarrow K \subset \mathrm{Sk}(X)$$

defined by f_t on $U_{t,K}$ for $t \neq 0$ and ρ on $U_{0,K}$ is continuous.

Proof. By [Li22, §2.7, thm. 5.13], for $|t| \ll 1$, the SYZ fibration f_t over K is $\mathcal{C}^\infty$-close to the map Log_t in the scale where the torus fibers have diameters of size $O(1)$, which easily implies our statement, as Log_t converges to ρ over K in the hybrid topology. □

Chapter 5

Degenerations of canonically polarized manifolds

5.1 Statement of the main theorem

In this section, $X \xrightarrow{\pi} \mathbb{D}^*$ will denote a meromorphic degeneration of canonically polarized manifolds, and we will write $L = K_{X/\mathbb{D}^*}$ the polarization. It follows from the seminal work of Aubin and Yau ([Aub78], [Yau78]) that every fiber X_t admits a unique negatively curved Kähler-Einstein metric $\omega_t \in -c_1(X_t)$, satisfying the equation $\mathrm{Ric}(\omega_t) = -\omega_t$. The Kähler form ω_t can be written as the curvature of a smooth Hermitian metric ϕ_t on $L_t = K_{X_t}$, i.e.:

$$\omega_t = dd^c\phi_t.$$

The metric ϕ_t is unique up to addition of a constant. In this situation, the family of Hermitian metrics $(\phi_t)_{t\in\mathbb{D}^*}$ turns out to also have plurisubharmonic variation in the horizontal direction by the work of Schumacher [Sch12], and has logarithmic growth at $t = 0$ by [Sch12, thm. 3]. Thus, the family of metrics $(\phi_t)_{t\in\mathbb{D}^*}$ induces an element $\phi \in \mathrm{CPSH}(X, L)$, and it is a natural question to try and determine the non-archimedean limit ϕ^{NA} of this family, as it provides a non-archimedean analog of the Kähler-Einstein metric.
As we will explain more thoroughly in section 5.2, after a finite base change $t \mapsto t^d$ on the punctured disk, that we omit from notation, the family $X/\mathbb{D}^*$ admits a canonical model $\mathscr{X}_c/\mathbb{D}$ such that the canonical bundle $K_{\mathscr{X}_c/\mathbb{D}}$ is relatively ample; the model $\mathscr{X}_c/\mathbb{D}^*$ is furthermore unique for this property (although it is more singular than an snc - or even dlt - model). This in turn yields a canonical model metric $\phi_{K_{\mathscr{X}_c/R}} \in \mathrm{CPSH}(X^{\mathrm{an}}, K_X^{\mathrm{an}})$, which we will prove to be the non-archimedean Kähler-Einstein metric:

Theorem 5.1.1. *Let $X \xrightarrow{\pi} \mathbb{D}^*$ be a degeneration of canonically polarized manifolds, $L = K_X$, and let $\phi_{\mathrm{KE}} \in \mathrm{CPSH}(X, L)$ be the family of Kähler-Einstein metrics. We assume that the family X has semi-stable reduction over $\mathbb{D}$. Then the metric on L^{hyb} defined by:*

$$\phi_{|X} = \phi_{KE},$$

$$\phi_0 = \phi_{K_{\mathscr{X}_c/R}}$$

is continuous and semipositive, i.e. $\phi \in \mathrm{CPSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$.

If the family of Kähler-Einstein metrics ϕ were to extend as a bounded metric $\phi \in \mathrm{PSH}(\mathscr{X}_c, K_{\mathscr{X}_c/\mathbb{D}^*})$, then it would follow from example 1.3.11 that $\phi^{\mathrm{NA}} = \phi_{K_{\mathscr{X}_c/R}}$. We will see that this is however not the case, but the singularities of ϕ along the special fiber of the canonical model are mild enough for the result to still hold - they are milder than any log poles.
Using theorem 1.4.11, we have the following immediate consequence:

Corollary 5.1.2. *Let $X \longrightarrow \mathbb{D}^*$ be a meromorphic degeneration of n-dimensional canonically polarized manifolds, and let $\omega_t \in -c_1(X_t)$ be the unique Kähler-Einstein metric with negative curvature on X_t.*
The Kähler-Einstein measures $\mu_t = \omega_t^n$ converge weakly on X^{hyb} to $\mu_0 = \mathrm{MA}(\phi_{K_{\mathscr{X}_c}})$, the non-archimedean Monge-Ampère measure of the canonical model metric.
More explicitly, writing $\mathscr{X}_{c,0} = \sum_{i\in I} b_i D_i$ as the sum of its irreducible components, the measure μ_0 is a weighted sum of Dirac masses supported at the divisorial valuations v_{D_i}, given by:

$$\mu_0 = \sum_{i\in I} b_i \big((K_{\mathscr{X}_c})^n \cdot D_i\big)\delta_{v_{D_i}}.$$

This was proved in [PS22b, thm. A] in a direct way, without using thm. 1.4.11.

5.2 The canonical model

Let $X \xrightarrow{\pi} \mathbb{D}^*$ be a degeneration of smooth, canonically polarized varieties. In this case, the Minimal Model Program provides us with a *unique* canonical model $\mathscr{X}_c$ of X over the disk, at the cost of going out of the class of simple normal crossing models, and allowing some slightly worse singularities. The appropriate class of varieties for the central fiber is a higher-dimensional analog of the stable curves, the correct notion being that of *semi-log canonical models*.
If $\mathscr{X}$ is a normal model of X, saying that $\mathscr{X}_0$ is semi-log canonical (see for instance [Kol13]) is a condition on the singularities of the normalisation of $\mathscr{X}_0$, which can be seen as a mild generalization of the simple normal crossing condition; in particular we require $\mathscr{X}_0$ to be reduced and simple normal crossing in codimension 1. More precisely, the normalization morphism $\nu : \mathscr{X}_0^\nu \longrightarrow \mathscr{X}_0$ is required to yield a disjoint union $\mathscr{X}_0^\nu = \sqcup_{i\in I}(\tilde{D}_i, C_i)$ of log canonical pairs, C_i being the restriction of the *conductor* C of ν to $\tilde{D}_i$. This is a Weil divisor on $\mathscr{X}_0^\nu$, whose support is precisely the locus where the normalization ν fails to be an isomorphism, and which is simply given here by the inverse image by ν of the codimension one nodes of $\mathscr{X}_0$. It furthermore satisfies the formula : $\nu^* K_{\mathscr{X}_0} = K_{\mathscr{X}_0^\nu} + C$ (note that the canonical divisor of a semi-log canonical variety is assumed to be $\mathbb{Q}$-Cartier).
A semi-log canonical model (or *stable variety*) is now by definition a proper semi-log canonical variety, with ample canonical divisor. For instance, one-dimensional semi-log canonical models are nothing but Deligne-Mumford's stable curves.
The compactness theorem for moduli of stable varieties of higher dimension is now as follows:

Theorem 5.2.1. *([BCHM10], [KNX18]).*
Let $X \longrightarrow \mathbb{D}^*$ be an algebraic degeneration of canonically polarized manifolds.
There exists (possibly after a finite base change) a unique canonical *model $\mathscr{X}_c$ of X over the disk, satisfying the following properties:*
i) the total space $\mathscr{X}_c$ has at worst canonical singularities, while the central fiber $\mathscr{X}_{c,0}$ is reduced and has at worst semi-log canonical singularities;
ii) the relative canonical divisor $K_{\mathscr{X}_c/\mathbb{D}}$ is relatively ample.

The canonical model is constructed as follows: by the semi-stable reduction theorem [KKMSD73], there exists a finite base change $t = (t')^d$ on the punctured disk such that the family $X' = X \times_{\mathbb{D}^*_t} \mathbb{D}^*_{t'}$ admits a semi-stable model over $\mathbb{D}^*$, i.e. an snc model with reduced special fiber. Omitting the base change from notation and starting from a semi-stable model $\mathscr{X}/\mathbb{D}$ of X, we set $\mathscr{X}_c = \mathrm{Proj}_{\mathbb{D}} \bigoplus_{m\geq 0} H^0(\mathscr{X}, mK_{\mathscr{X}/\mathbb{D}})$, the main difficulty being to prove finite-generatedness of the relative canonical algebra. This is established in [BCHM10] when $X/\mathbb{D}^*$ is defined over an algebraic curve, and extended to families over the disk in [KNX18]. The uniqueness of the canonical model is now a straightforward consequence of the birational invariance of the relative canonical ring $R = \bigoplus_{m\geq 0} H^0(\mathscr{X}, mK_{\mathscr{X}/\mathbb{D}})$ - as the notation suggests, R does not depend on the choice of the model $\mathscr{X}$.

Remark 5.2.2. *If $\mathscr{X}$ is any semi-stable model of X, then the natural rational map $h : \mathscr{X} \dashrightarrow \mathscr{X}_c$ is in fact a* rational contraction *- this means that its inverse does not contract any divisors.*

5.3 Metric convergence and proof of theorem 5.1.1

The complete understanding of the Gromov-Hausdorff convergence of the fibers (X_t, g_t), is due to J. Song [Son17] (whose results were further improved recently in [SSW20]). The crucial first step, is to show that there exists on the central fiber $\mathscr{X}_{c,0} = \sum_{i\in I} D_i$ of the canonical model of X a unique Kähler-Einstein current ω_{KE}, and to derive some geometric estimates on the singularities of this current. The current ω_{KE} on the stable variety $\mathscr{X}_{c,0}$ was first constructed by Berman-Guenancia [BG14] using a variational method, while it is reconstructed in [Son17] using the techniques of [EGZ09], [Kol98], in order to obtain some stronger control on its singularities:

Theorem 5.3.1. *([Son17, thm. 1.1]).*
Let $\mathscr{X}_c \longrightarrow \mathbb{D}$ be the canonical model of X, with semi-log canonical central fiber $\mathscr{X}_{c,0}$.
There exists a unique Kähler current $\omega_{KE} \in -c_1(\mathscr{X}_{c,0})$ on $\mathscr{X}_{c,0}$, satisfying the following properties:
i) ω_{KE} is smooth and satisfies the Kähler-Einstein equation on the regular locus of $\mathscr{X}_{c,0}$;
ii) ω_{KE} has locally bounded potentials on the locus where $\mathscr{X}_{c,0}$ is log terminal;
iii) ω^n_{KE} does not charge mass on the singularities of $\mathscr{X}_{c,0}$, and $\int_{\mathscr{X}_{c,0}} \omega^n_{KE} = [K_{\mathscr{X}_{c,0}}]^n$.

Remark 5.3.2. *The fact that the above Kähler-Einstein current on $\mathscr{X}_{c,0}$ matches the one constructed in [BG14], follows from the uniqueness statement in [BG14,*

thm. A]. Moreover, the construction of [BG14] implies that $\int_{D_i} \omega_{KE}^n = (K_{\mathscr{X}_c}^n \cdot D_i)$. *Indeed, if* $\nu : \mathscr{X}_{c,0}^\nu \longrightarrow \mathscr{X}_{c,0}$ *denotes the normalization morphism, where* $\mathscr{X}_{c,0}^\nu = \sqcup_{i\in I}\tilde{D}_i$, *then the Kähler-Einstein metric* ω_{KE} *is obtained by descending the (singular) Kähler-Einstein metrics* $\omega_i \in c_1(K_{\tilde{D}_i} + C_i)$ *on the log canonical pairs* $(\tilde{D}_i, C_i)$, C_i *being the restriction of the conductor* C *of* ν *to* $\tilde{D}_i$.
Thus by construction, the mass $\int_{D_i} \omega_{KE}^n$ *equals the intersection number* $(K_{\tilde{D}_i}+C_i)^n = (\nu^* K_{\mathscr{X}_{c,0}})^n$, *the last intersection number being computed on* $\tilde{D}_i$. *Applying the projection formula, this is equal to the intersection number* $K_{\mathscr{X}_{c,0}}^n \cdot D_i = K_{\mathscr{X}_c}^n \cdot D_i$, *by adjunction and principality of* $\mathscr{X}_{c,0}$.

Let us now fix a $m > 0$ such that $mK_{\mathscr{X}_c/\mathbb{D}}$ is relatively very ample, and a relative embedding $\iota : \mathscr{X}_c \hookrightarrow \mathbb{P}^N \times \mathbb{D}$ of the canonical model inside projective space by sections of $mK_{\mathscr{X}_c/\mathbb{D}}$. We let ϕ_{FS} be the hybrid Bergman metric on $(\mathbb{P}_K^N)^{\mathrm{hyb}}$ from example 1.3.7, and still write $\phi_{\mathrm{FS}} \in \mathrm{CPSH}(X^{\mathrm{hyb}}, L^{\mathrm{hyb}})$ its pullback to X^{hyb} via the embedding ι. More explicitly, we have that

$$\phi_{\mathrm{FS},t} = m^{-1}\iota_t^* \phi_{\mathrm{FS}},$$

where ϕ_{FS} is the usual (Euclidean) Fubini-Study metric on $\mathbb{CP}^N$; while

$$\phi_{\mathrm{FS},0} = \phi_{K_{\mathscr{X}_c}/R},$$

since the model $(\mathscr{X}_c, K_{\mathscr{X}_c/R})$ is ample.
This allows us to write the Kähler-Einstein metric $\phi_{\mathrm{KE},t} = \phi_{\mathrm{FS},t} + \psi_t$, with $\psi_t \in \mathcal{C}^\infty(X_t)$. The potential ψ_t is the unique solution of the Monge-Ampère equation:

$$(\omega_{\mathrm{FS},t} + dd^c\psi_t)^n = e^{\psi_t}\omega_{\mathrm{FS},t}^n,$$

with the normalization $\int_{X_t} e^{\psi_t}\omega_{\mathrm{FS},t}^n = (K_{X_t})^n$. In order to derive uniform estimates for the family of potentials $(\psi_t)_{t\in\mathbb{D}^*}$, it is more convenient to work on a semi-stable model, as a result we perform an additional base change and consider a diagram of the form:

$$\begin{array}{ccccc} \mathscr{X} & \xrightarrow{p} & \mathscr{X}_c' & \longrightarrow & \mathscr{X}_c \\ & \searrow^{\pi'} & \downarrow & & \downarrow \\ & & \mathbb{D} & \xrightarrow{t\mapsto t^d} & \mathbb{D} \end{array}$$

where $\mathscr{X}_c'$ is the base change of the canonical model $\mathscr{X}_c$ via $t \mapsto t^d$, and $\mathscr{X}$ is a semi-stable resolution of $\mathscr{X}_c'$. We write the special fiber $\mathscr{X}_0 = \sum_{i\in I}\tilde{D}_i + \sum_{j\in J} E_j$, where $\tilde{D}_i$ is the strict transform of $D_i \subset \mathscr{X}_{c,0}' = \mathscr{X}_{c,0}$ and the E_j's are the exceptional divisors of p. For each $i \in I$, let ϕ_i be a psh metric on $\mathcal{O}_{\mathscr{X}}(\tilde{D}_i)$ with divisorial singularities along $\tilde{D}_i$, i.e. $\phi_i = \log|z_i| + O(1)$ locally, where z_i is a local equation for $\tilde{D}_i$. Similarly, we choose ψ_j with divisorial singularities along E_j, and we may assume that:

$$\sum_{i\in I}\phi_i + \sum_{j\in J}\psi_j = \log|t|.$$

In order to apply Cheeger-Colding theory to the Kähler-Einstein metrics on X_t, Song derives uniform estimates on volumes of small balls, which are obtained via comparison lemmas for volume forms. The estimate focuses on a strict transform $\tilde{D}_{i_0} \subset \mathscr{X}_0$ which we henceforth fix, and shows that the potentials ψ_t do not blow-up as we approach the interior of $\tilde{D}_{i_0}$:

Proposition 5.3.3. *([Son17, cor. 4.1, lem. 4.2])*
We have a uniform bound: $\sup_{X_t} \psi_t \leq C$.
Moreover, letting $J' \subset J$ *be the subset of exceptional divisors in* $\mathscr{X}_0$ *that meet* $\tilde{D}_{i_0}$, *the following holds: for any* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *such that for all* $t \neq 0$*:*

$$\psi_t \geq \varepsilon\big(\sum_{\substack{i \in I \\ i \neq i_0}} \phi_i + \sum_{j \in J'} \psi_j\big) - C_\varepsilon.$$

This implies smooth convergence of the Kähler-Einstein metrics ω_t *to the current* ω_{KE} *on* $\mathscr{X}_{c,0} \setminus \mathrm{Sing}(\mathscr{X}_{c,0})$ *in the following sense:*
for any point $p \in \mathscr{X}_{c,0} \setminus \mathrm{Sing}(\mathscr{X}_{c,0})$, *and any choice of neighbourhood* $\mathscr{U}$ *of* p *such that the* $U_t = \mathscr{U} \cap X_t$ *are all biholomorphic to* U_0, *and such that the* $(\omega_{\mathrm{FS},t})^n$*'s are uniformly equivalent, the pulled-back* ψ_t *converge in the* $\mathcal{C}^\infty$*-sense to* ψ_0.

Note that we have made a small abuse of notation, since the object in the right-hand side of the inequality is a metric and not a function.

Remark 5.3.4. *Even if we will not need it here, one can show that the previous theorem combined with a uniform non-collapsing condition implies pointed Gromov-Hausdorff convergence of* X_t *to a complete metric space, whose regular part (in the Cheeger-Colding sense) is precisely* $(\mathscr{X}_{c,0} \setminus \mathrm{Sing}(\mathscr{X}_{c,0}), \omega_{KE})$.
We also point out that this holds for degeneration of canonically polarized manifolds over a higher-dimensional base by the results of [SSW20], building on the semi-stable reduction theorem from [AK00], [ALT18].
The behaviour of the metrics in the region where the metric collapses is also well-understood, under the technical assumption that the canonical model is semi-stable, see [Zha15].

We are now ready to prove theorem 5.1.1. We let $\phi_0 = \phi_{K_{\mathscr{X}_{c/R}}} \in \mathrm{CPSH}(X^{\mathrm{an}}, L^{\mathrm{an}})$, and $\phi_{KE} \in \mathrm{CPSH}(X, L)$ the family of Kähler-Einstein metrics. In order to prove that the hybrid metric ϕ defined by the statement of theorem 5.1.1 is continuous and semi-positive, it is enough to prove that it defines a continuous metric on L^{hyb}, by prop. 1.3.16.
Substracting the reference metric hybrid ϕ_{FS}, whose restriction to X^{an} is the model metric $\phi_{K_{\mathscr{X}_{c/R}}}$, it is enough to prove that the potential ψ_t's converge to zero as $t \to 0$ in the hybrid topology. In other words, we need to prove that:

$$\Big|\frac{\psi_t}{\log|t|}\Big| \xrightarrow[t \to 0]{} 0,$$

which is an easy consequence of the estimates from prop. 5.3.3. This concludes the proof of theorem 5.1.1.

5.4 Degenerations of Fano manifolds

In this section, we briefly discuss the case where $X \xrightarrow{\pi} \mathbb{D}^*$ is an algebraic degeneration of Fano manifolds.
In this case, the existence of a Kähler-Einstein metric on every fiber does *not* hold in general, contrasting with the Calabi-Yau and canonically polarized cases.
There are several obstructions to the existence of a Kähler-Einstein metric on a smooth Fano manifold Z, going back at least to Lichnerowicz and Matsushima, see for instance [Fut88]. Those "classical" obstructions all come from the automorphism group of the manifold Z, which must for instance be reductive in order for a Kähler-Einstein metric to possibly exist.
Further developments in the subject in the 90's have brought people to believe that the existence of a Kähler-Einstein metric on Z is equivalent to a sophisticated form of GIT (Geometric Invariant Theory) stability for the polarized variety $(Z, -K_Z)$. The correct notion is the one of *K-stability*.
Although we will not give a precise definition of K-stability here, it is worth emphasizing that it is a purely algebro-geometric condition on $(Z, -K_Z)$, which is furthermore Zariski open on the base of any holomorphic family of Fano manifolds. It is furthermore equivalent to the condition that Z has finite automorphism group, in addition to being K-polystable, so that K-polystability should be thought as an extension of K-stability to manifolds with continuous families of automorphisms. By [CDS14], the K-polystability of $(Z, -K_Z)$ is now equivalent to the existence, of a Kähler-Einstein of positive curvature $\omega \in -c_1(K_Z)$, satisfying the equation $\mathrm{Ric}(\omega) = \omega$. This metric is furthermore unique up to $\mathrm{Aut}^0(Z)$, the connected component of the automorphism group of Z containing the identity (which is trivial if and only if Z is K-stable).
Thus, we let from now $X \longrightarrow \mathbb{D}^*$ be a degeneration of K-polystable Fano manifolds, hence Kähler-Einstein. In this case, positivity of the Ricci curvature and the classical Bonnet-Myers theorem ensure that the diameters of (X_t, g_{ω_t}) satisfy a uniform upper bound, while the Bishop-Gromov inequality imply a uniform non-collapsing condition on the (X_t, g_{ω_t}): for all small radius $r > 0$ and $p \in X_t$, we have $\mathrm{Vol}_{g_t} B(p, r) \geq cr^{2n}$, $c > 0$ being a uniform constant.
This uniform non-collapsing condition is used in [DS14] to prove that the metric spaces (X_t, g_{ω_t}) are well-behaved when $t \to 0$: the limiting object is a mildly singular Fano variety, which also admits a singular Kähler-Einstein metric in the sense of [EGZ09], and the Gromov-Hausdorff limit is unique, and homeomorphic to the limit in the sense of algebraic geometry (in the relevant Hilbert scheme).
However, the uniqueness of a model $\mathscr{X}/\mathbb{D}$ of X whose central fiber matches the Gromov-Hausdorff limit of the X_t no longer holds, as shown in the following example.

Example 5.4.1. *Consider $X = \mathbb{P}^1 \times \mathbb{D}^*$, and $\mathscr{X} = \mathbb{P}^1 \times \mathbb{D}$. We blow-up a closed point of $\mathscr{X}_0$ to obtain an snc model of X whose central fiber is a chain of two (-1)-curves, and we contract the strict transform of $\mathscr{X}_0$ in this model, which yields a new snc model $\mathscr{X}'_0$ with central fiber $\mathbb{P}^1$, which is not isomorphic to $\mathscr{X}$.*

In this case, even though the model $\mathscr{X}'$ has the 'right' central fiber, we see that the intrinsic behaviour of the metrics inside $\mathscr{X}'$ is rather badly behaved; the metric

convergence is only extrinsic - that is, after composing by suitable automorphisms. We see that the issue is due to the presence of a large automorphism group of the metrics, which makes the situation less rigid than in the negatively curved case.
Even if we assume that the X_t's have a discrete automorphism group for all $t \in \mathbb{D}^*$, it may very well happen that a model as above could be such that its irreducible central fiber $\mathscr{X}_0$ has non-trivial $\mathrm{Aut}^0(\mathscr{X}_0)$ and be non-unique, which would prevent us from determining on which one we could expect Cheeger-Gromov convergence.
However, it is proven in [BX19] that if there exists a model $\mathscr{X}/\mathbb{D}$ whose irreducible central fiber is a K-stable $\mathbb{Q}$-Fano manifold (and thus has finite automorphism group), then such a model is actually unique. We then expect that the following holds: the family of Kähler-Einstein metrics $(\phi_t)_{t\in\mathbb{D}^*}$ converge on the hybrid space to the model metric $\phi_{-K_{\mathscr{X}/\mathbb{D}}}$.
At the moment, we still lack uniform estimates on $\mathscr{X}$ on the Kähler-Einstein potentials with respect to a smooth reference family of metrics that would enable us to prove this, so that we leave this question aside for future work.

Appendix

In this appendix, we state and prove a regularization result for psh metrics in the complex analytic setting, that will be used in the proof of theorem 1.3.13. We expect that the statement below (thm. 5.4.4) is well-known to experts, as well as the techniques we use in the proof - which are largely due to Demailly [Dem92]. Nevertheless, since we could not find the precise statement required in the literature, we include a proof.
We let (Y, ω) be a Kähler manifold, and assume given a proper holomorphic map $\pi : Y \longrightarrow \Omega$, where Ω is a bounded open subset of $\mathbb{C}$. We furthermore assume that there exists a π-relatively ample line bundle L on Y such that $\omega \in c_1(L)$.
Given a semi-positive metric $\phi \in \mathrm{PSH}(Y, L)$, we want to write it as a decreasing limit of a sequence $(\phi_j)_{j\in\mathbb{N}}$ of psh metrics on L with analytic singularities along the multiplier ideal sheaf $\mathscr{I}_j := \mathscr{I}(j\phi)$. We recall the basic definitions:

Definition 5.4.2. *Let $\phi \in \mathrm{PSH}(Y, L)$ be a semi-positive metric on L. The multiplier ideal sheaf $\mathscr{I}(\phi)$ is the ideal generated by the germs of holomorphic functions f such that $|f|^2 e^{-2\phi}$ is locally integrable on Y.*

Here locally integrable means locally integrable in any coordinate chart, we also abusively view ϕ as a psh function this way.

Definition 5.4.3. *Let $\mathscr{J} \subset \mathcal{O}_Y$ be a coherent ideal sheaf on Y, and $\phi \in \mathrm{PSH}(Y, L)$. We say that ϕ has analytic singularities along $\mathscr{J}$ if ϕ can be written locally as:*

$$\phi = \log\left(|f_1|^2 + ... + |f_r|^2\right) + \chi,$$

where $(f_1, ..., f_r)$ are a set of local generators of $\mathscr{J}$ and χ is a smooth function.

Given a coherent ideal sheaf $\mathscr{J}$, one can always produce quasi-psh functions with analytic singularities along $\mathscr{J}$, using a partition of unity argument.
The rest of this appendix will be devoted to the proof of the following:

Theorem 5.4.4. *Let $\Omega \subset \mathbb{C}$ be a bounded open subset, Y a smooth Kähler manifold together with a proper holomorphic map $\pi : Y \longrightarrow \Omega$, and let L a relatively ample line bundle on Y. We let ψ be a smooth Hermitian metric on L whose curvature form $\omega = dd^c\psi$ is a Kähler metric on Y.*
Let $\phi \in \mathrm{PSH}(Y, L)$, and write $\mathscr{I}_m := \mathscr{I}(m\phi)$ the multiplier ideal of $m\phi$, for $m \in \mathbb{N}$. Then for any relatively compact, open subset $Y' \Subset Y$, there exists a sequence $(\phi_j)_{j\in\mathbb{N}} \in \mathrm{PSH}(Y', L)$ such that:

- *the ϕ_j decrease pointwise to ϕ on Y',*

- *for all $j \in \mathbb{N}$, the psh metric $2^j\phi_j$ on 2^jL has analytic singularities of the form $\mathscr{I}_{2^j}$.*

Let us fix a psh exhaustion function $\eta : \Omega \longrightarrow \mathbb{R}$, i.e. such that the sublevel sets $\Omega_c := \{\eta < c\}$ are relatively compact subsets of Ω; note that the $Y_c := \{\eta \circ \pi < c\}$ are also relatively compact in Y and weakly pseudoconvex. Since the subset $Y' \subset Y$ is relatively compact, we have $Y' \subset Y_c$ for $c \gg 1$.

Proposition 5.4.5. *For any $c \in \mathbb{R}$ such that $Y' \Subset Y_c \Subset Y$, there exists a $m_0 \gg 1$ such that for all $m \geq 1$, the sheaf $\mathcal{O}_Y((m+m_0)L \otimes \mathscr{I}_m)$ is generated by its global sections on Y_c.*

Proof. We argue as in [BBJ18, lem. 5.6], and write $n = \dim Y$.
We let H be a relatively very ample line bundle on Y, and we choose $m_0 \gg 1$ such that the line bundle $A = m_0L - K_Y - nH$ is relatively ample on Y_c.
By the relative Castelnuevo-Mumford regularity criterion (see [DEL00, lem. 1.4]), the sheaf $\mathcal{O}_Y((m+m_0)L \otimes \mathscr{I}_m)$ is π-globally generated on Y_c as soon as:

$$R^q\pi_*(\big((m+m_0)L - qH\big) \otimes \mathscr{I}_m) = 0$$

for $1 \leq q \leq n-1$, which holds by Nadel vanishing [Nad89], [BFJ16, thm. B.8]. □

We will now regularize ϕ by a sequence of psh metrics with analytic singularities of the form $\mathscr{I}(m\phi)$, up to some controlled error term. We mostly follow the argument from the proof of [GZ05, thm. 8.1]. For m_0 large enough so that prop. 5.4.5 holds, write:

$$\psi_{m,m_0} = \big(m\phi + m_0\psi\big) \in \mathrm{PSH}(Y, (m+m_0)L),$$

we have that $\mathscr{I}_m = \mathscr{I}(\psi_{m,m_0}) = \mathscr{I}(m\phi)$ is the multiplier ideal of the psh metric ψ_{m,m_0} on (Y, L).
We are naturally led to introduce the Bergman metrics associated to the multiplier ideal $\mathscr{I}_m$; for Y_c as in prop. 5.4.5, we set $V_{m,m_0} := H^0(Y_c, (m+m_0)L \otimes \mathscr{I}_m)$ and define $\mathscr{H}_{m,m_0} \subset V_{m,m_0}$ as the following Hilbert space:

$$\mathscr{H}_{m,m_0} = \{s \in V_{m,m_0} / \|s\|^2 := \int_{Y_c} |s|^2_{\psi_{m,m_0}}\omega^n < \infty\}.$$

For every m, we may choose a Hilbert basis $\mathscr{B}_{m,m_0} = (s_{m,m_0,l})_{l\in\mathbb{N}}$ of $\mathscr{H}_{m,m_0}$, and we now set:

$$\phi_{m,m_0} = \frac{1}{2(m+m_0)} \log(\sum_{l\in\mathbb{N}} |s_{m,m_0,l}|^2),$$

we have $\phi_{m,m_0} \in \mathrm{PSH}(Y_c, L)$.

Proposition 5.4.6. *For $q \in \mathbb{N}$, set:*

$$\phi_{m,m_0,q} = \frac{1}{2(m+m_0)} \log(\sum_{l\leq q} |s_{m,m_0,l}|^2).$$

Then the $\phi_{m,m_0,q}$ converge uniformly to ϕ_{m,m_0} over Y'. Moreover, ϕ_{m,m_0} has analytic singularities of the form $\mathscr{I}_m$ over Y'.

Proof. We drop the (m, m_0) subscript to alleviate notation. We choose $c' < c$ such that let $Y' \Subset Y_{c'} \Subset Y_c$, with the freedom to slightly decrease c' throughout the steps of the proof.
We set $\mathscr{J}_q = \mathscr{I}\big((s_l)_{l\leq q}\big)$, and $\mathscr{J} = \cup_{q\geq 0}\mathscr{J}_q = \mathscr{I}\big((s_l)_{l\in\mathbb{N}}\big)$. Then we have $\mathscr{J} = (m+m_0)L \otimes \mathscr{I}(m\phi)$ over Y''' by global generation, which is a coherent ideal sheaf on $Y_{c'}$ by Nadel's theorem [Nad89]. By the strong Noetherian property for coherent sheaves, the ascending chain $(\mathscr{J}_q)_{q\geq 0}$ of ideals is locally stationary, so that we have $\mathscr{J}_q \equiv \mathscr{J}$ for $q \gg 1$ on $Y_{c'}$ after shrinking.
We now set:

$$F = \sum_{l\in\mathbb{N}} |s_l|^2,$$

and:

$$F_q = \sum_{l\leq q} |s_l|^2.$$

We will prove that on Y', there exists $C_q \geq 1$ such that:

$$F_q \leq F \leq C_q F_q,$$

and that $C_q \xrightarrow[q\longrightarrow\infty]{} 1$. We mostly mimic the argument from step 2 of the proof of [DPS01, thm. 2.2.1]. Up to slightly decreasing c', we may work locally, so that we assume that the s_l are holomorphic functions. By the strong Noetherian property of coherent ideal sheaves, the sequence of ideal sheaves $\mathscr{K}_q$ on $Y_c \times Y_c$ generated by the $(s_l(z)\overline{s_l(\bar{w})})_{l\leq q}$ is locally stationary, so that it is stationary at $\mathscr{K} = \cup_{q\geq 0}\mathscr{K}_q$ on $Y_{c'} \times Y_{c'}$ for $q \gg 1$. From the bound:

$$\sum_{l\leq q} |s_l(z)\overline{s_l(\bar{w})}| \leq \Big(\big(\sum_{l\in\mathbb{N}} |s_l(z)|^2\big) \cdot \big(\sum_{l\in\mathbb{N}} |s_l(\bar{w})|^2\big) \Big)^{1/2},$$

we infer that the series $\sum_{l\leq q} s_l(z)\overline{s_l(\bar{w})}$ converges locally uniformly on $Y_{c'} \times Y_{c'}$, and thus by closedness of the space of sections of a coherent ideal sheaf, we get that the holomorphic function on $Y_{c'} \times Y_{c'}$:

$$\sum_{l\in\mathbb{N}} s_l(z)\overline{s_l(\bar{w})} \in \mathscr{K}.$$

Since $\mathscr{K} = \mathscr{K}_q$ for q large enough, we get that:

$$\sum_{l\in\mathbb{N}} s_l(z)\overline{s_l(\bar{w})} \leq C_q \sum_{l\leq q} s_l(z)\overline{s_l(\bar{w})}$$

on $Y_{c'} \times Y_{c'}$ for some $C_q > 0$, and thus:

$$F \leq C_q F_q$$

over $Y_{c'}$ for q large enough by restricting to the complex diagonal $z = \bar{w}$.
Finally, let us set $\chi_q = \phi - \phi_q$: then we proved that for any $q \gg 1$, there exists $b_q > 0$ such that $0 \leq \chi_q \leq b_q$ on $Y_{c'}$. As a result ϕ_q and ϕ have the same singularities, which are analytic singularities along $\mathscr{J}_q \equiv \mathscr{J}$ for $q \gg 1$.
Moreover, since the sum converges locally uniformly, the χ_q are continuous and decrease to zero pointwise, so that from Dini's lemma the χ_q's decrease uniformly to zero on Y', and ϕ_q converges uniformly to ϕ over Y'. □

We now want to prove that over Y', the ϕ_{m,m_0} decrease with respect to m (up to an error term) to our initial metric ϕ.
We start by proving that the ϕ_{m,m_0} converge pointwise to ϕ over Y_c. Writing $B_{m,m_0} \subset \mathscr{H}_{m,m_0}$ the unit ball with respect to the L^2-norm, we have:

$$\phi_{m,m_0} = \frac{1}{2(m+m_0)} \log\big(\sum_{l\geq 0} |s_{m,m_0,l}|^2\big) = \frac{1}{2(m+m_0)} \sup_{s\in B_{m,m_0}} \log|s|^2,$$

since for $z \in Y_c$, the quantity $\sum_{l\geq 0}|s_{m,m_0,l}(z)|^2$ is the operator norm of the evaluation map $\mathrm{ev}_z : \mathscr{H}_{m,m_0} \longrightarrow L_z^{\otimes m+m_0}$.
Covering Y_c by coordinates charts $(U_i)_{i\in I}$, we let $z \in U_i$ and $\rho > 0$ such that $B(z,\rho) \subset U_i$. For $s \in \mathscr{H}_{m,m_0}$, since $|s|^2$ is subharmonic in U_i, we have:

$$|s(z)|^2 \leq \frac{C}{\rho^{2(n+1)}} \int_{B(z,\rho)} |s|^2 \omega^{n+1} \leq C' e^{2\sup_{B(z,\rho)} \psi_{m,m_0}} \int_{Y_c} |s|^2_{\psi_{m,m_0}} \omega^{n+1},$$

so that if $s \in B_{m,m_0}$, the bound:

$$\log|s(z)|^2 \leq \sup_{B(z,\rho)} \psi_{m,m_0} + C$$

holds, hence $\phi_{m,m_0}(z) \leq \sup_{B(z,\rho)} \phi + (m+m_0)^{-1}C$.
The converse inequality follows from the Ohsawa-Takegoshi theorem [Dem15]: there exists $m_0 \gg 1$ and a universal constant $C > 0$ such that for all $m \in \mathbb{N}$ large enough and $z \in Y_c$, there exists $s \in \mathscr{H}_{m,m_0}$ such that:

$$\int_{Y_c} |s|^2 e^{-2\psi_{m,m_0}} \omega^{n+1} \leq C|s(z)|^2 e^{-2\psi_{m,m_0}(z)},$$

so that if we choose the right-hand side to be equal to one, we get $s \in B_{m,m_0}$ such that:

$$\log|s(z)| \geq \psi_{m,m_0}(z) - C,$$

hence $\phi_{m,m_0}(z) \geq \phi(z) - (m+m_0)^{-1}C$, which proves pointwise convergence on Y''.
We now prove that the ϕ_{m,m_0} are almost subadditive. We let $s \in B_{m_1+m_2,m_0} \subset \mathscr{H}_{m_1+m_2,m_0}$, and set:

$$\mathscr{H}_{m_1,m_2,m_0} = \{S \in H^0\big(Y_c \times Y_c, p_1^*((m_1 + \frac{m_0}{2})\mathscr{L})(\mathscr{I}_{m_1}) \otimes p_2^*((m_2+\frac{m_0}{2})\mathscr{L})(\mathscr{I}_{m_2})\big);$$

$$\int_{Y_c\times Y_c} |S(z_1,z_2)|^2 e^{-2\psi_{m_1+\frac{m_0}{2}}(z_1) - 2\psi_{m_2+\frac{m_0}{2}}(z_2)} (\omega_1 \otimes \omega_2)^n < \infty\},$$

where we have written $\omega_i = p_i^*\omega$.
By the Ohsawa-Takegoshi theorem, there exists $S \in \mathscr{H}_{m_1,m_2,m_0}$ with L^2-norm $\|S\| \leq C$ for a universal constant C, such that $S_{|\Delta_{Y_c}} = s$, where $\Delta_{Y_c} \subset Y_c \times Y_c$ is the diagonal. To be more precise, we let:

$$\xi : Y \times Y \longrightarrow \mathbb{R} \cup \{-\infty\}$$

be a quasi-psh function with analytic singularities along $\mathscr{I}(\Delta_Y)$, and we may assume that $\xi \leq 0$ on Y_c. Then in the notation of [Dem15, thm. 1.4] the measure $dV_{\Delta_Y,\omega}[\xi]$

is uniformly equivalent to ω^n on Δ_{Y_c}, as Δ_Y is a local complete intersection, so that our estimate follows from the aforementioned theorem with $\delta = 2$.
Since $\mathscr{H}_{m_1,m_2,m_0} = \mathscr{H}_{m_1,m_0/2}\hat{\otimes}\mathscr{H}_{m_2,m_0/2}$, the family $(s_{m_1,m_0/2,l_1} \otimes s_{m_2,m_0/2,l_2})_{(l_1,l_2)\in\mathbb{N}^2}$ form a Hilbert basis of $\mathscr{H}_{m_1,m_2,m_0}$. We may write:

$$S(z_1,z_2) = \sum_{(l_1,l_2)\in\mathbb{N}^2} c_{l_1,l_2} s_{m_1,m_0/2,l_1}(z_1) \otimes s_{m_2,m_0/2,l_2}(z_2),$$

with $\sum_{l_1,l_2}|c_{l_1,l_2}|^2 \leq C$. Thus:

$$|s(z)|^2 = |S(z,z)|^2 \leq C\big(\sum_{l_1}|s_{m_1,m_0/2,l_1}(z)|^2\big) \times \big(\sum_{l_2}|s_{m_2,m_0/2,l_2}(z)|^2\big),$$

so that:

$$\phi_{m_1+m_2,m_0} \leq \frac{C}{m_1+m_2+m_0} + \frac{(m_1+m_0/2)\phi_{m_1,m_0/2}}{m_1+m_2+m_0} + \frac{(m_2+m_0/2)\phi_{m_2,m_0/2}}{m_1+m_2+m_0}.$$

Since $\phi-\psi$ is bounded from above over Y_c, we may assume without loss of generality that $\phi-\psi \leq 0$, so that $(m+\frac{m_0}{2})^{-1}\psi_{m,m_0/2} \leq (m+m_0)^{-1}\psi_{m,m_0}$ and thus $\phi_{m_1,m_0/2} \leq \phi_{m,m_0}$. This now implies that the sequence:

$$\phi_j = \phi_{2^j-m_0,m_0} + 2^{-j-2}C$$

is decreasing to ϕ over Y_c, and has the required singularities over Y' by prop. 5.4.6.

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ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Aspects non archimédiens de la conjecture SYZ

Mots clés : Espaces de Berkovich, dégénérescences de variétés algébriques, variétés de Calabi-Yau, symétrie miroir

Résumé : Nous définissons une classe de métriques plurisousharmoniques sur l'espace hybride associé à une dégénérescence polarisée (X,L) de variétés complexes . Une telle métrique hybride induit une métrique psh sur L au sens usuel, ainsi qu'une métrique psh sur l'analytification non archimédienne de X par rapport à la valeur absolue t-adique sur C((t)). Nous démontrons que toute métrique plurisousharmonique complexe sur (X, L) admet une extension canonique à l'espace hybride. Nous étudions en particulier le cas où (Z, L) est une variété torique polarisée, où nous donnons une description combinatoire des métriques toriques hybrides continues psh sur L. Nous étudions ensuite les dégénérescences maximales de variétés de Calabi-Yau, dans le but de produire une incarnation non archimédienne de la fibration SYZ conjecturale. Pour ce faire, nous tentons dans un premier temps de comprendre la structure affine entière induite sur le squelette essentiel Sk(X) par la rétraction de Berkovich associée à un bon modèle dlt de X. Cela nous permet de construire, dans le cas des dégénérescences d'hypersurfaces, une fibration SYZ non archimédienne induisant sur Sk(X) la structure affine entière prédite par le programme de Gross-Siebert. Lorsque la famille est celle des hypersurfaces de Fermat, nous montrons en outre que la métrique de Calabi-Yau non archimédienne est invariante par cette rétraction. Enfin, nous considérons une dégénérescence de variétés canoniquement polarisées et calculons la limite non archimédienne de la famille des métriques de Kähler-Einstein dans l'espace hybride associé.

Title : Non-archimedean aspects of the SYZ conjecture

Keywords : Berkovich spaces, degenerations of algebraic varieties, Calabi-Yau manifolds, mirror symmetry

Abstract : We define a class of plurisubharmonic metrics on the hybrid space associated to a polarized degeneration (X,L) of complex manifolds over the punctured disk. Such a psh metric induces by restriction a psh metric on L in the usual sense, as well as a psh metric on the non-archimedean analytification of X with respect to the t-adic absolute value on C((t)). We prove that any complex psh metric on (X, L) admits a canonical plurisubharmonic extension to the hybrid space. We also focus on the case of a complex polarized toric variety (Z, L), where we provide a combinatorial description of continuous plurisubharmonic hybrid toric metrics on L. We then study maximal degenerations of Calabi-Yau manifolds, with the goal of constructing a non-archimedean avatar of the conjectural SYZ fibration on the complex fibers. To that extend, we study the integral affine structures induced on the skeleton Sk(X) by the Berkovich retraction associated to a good dlt model of X. This allows us to construct, in the case of degenerations of hypersurfaces, a non-archimedean SYZ fibration inducing on Sk(X) the integral affine structure predicted by the Gross-Siebert program. When the family of hypersurfaces is the Fermat one, we furthermore prove that the non-archimedean Calabi-Yau metric is invariant under this retraction. Finally, we consider degenerations of canonically polarized manifolds and compute the non-archimedean limit of the family of Kähler-Einstein metrics inside the associated hybrid space.

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