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Mean field games with free final time

Saeed Sadeghi Arjmand

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Saeed Sadeghi Arjmand. Mean field games with free final time. Optimization and Control [math.OC]. Institut Polytechnique de Paris, 2022. English. NNT : 2022IPPAX112 . tel-04107912

HAL Id: tel-04107912

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NNT : 2022IPPAX112

Thèse de doctorat



INSTITUT
POLYTECHNIQUE
DE PARIS



Mean field games with free final time

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École polytechnique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques pures

Thèse présentée et soutenue à Palaiseau, le 09 Décembre 2022, par

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Dedication

To my lovely family, especially my parents, for their endless love and support

Acknowledgements

Behind every name of a PhD holder, there are for sure some other names hidden which represent people who have been truly involved, compassionate and generous. I am not an exception, this also holds for me where behind of my name on this thesis, the trace of such people appears. Hence, I wish to express my feelings about them in terms of words, however words are cripple to carry out truly my pure feelings.

First of all, I would like to thank my advisers *Guilherme Mazanti* and *Anne-Sophie de Suzzoni* for having me as their PhD student in these three years.

I especially thank *Guilherme* for his availability, limitless patience and kindness without which this thesis could not have been accomplished and would not have seen the lights of the sunrise. I will never forget those days the week on which we were starting to do research from the morning until the end of evening, while even at the end of the day *Guilherme* was enthusiastically encouraging me to continue working on the ideas. Besides the Mathematics that I have learnt from you, you taught me to be patient and focused on solving when encountering problems either scientific or issues in the routine life. I hope that the end of my thesis will be a starting point of a long collaboration in the future.

Many thanks to *Pierre Cardaliaguet* and *Daniela Tonon* for accepting to be the referees of my thesis, and for the kind remarks that they have mentioned in the reports. I am also grateful to *Filippo Santambrogio*, *Bertrand Maury* and *Nizar Touzi* for being in my jury.

I thank all the researchers both at CMLS and L2S, where every single person took part in making a very nice condition and atmosphere for working and discussing about different aspects of Mathematics, as well as daily life, particularly *Charles Favre* and *Sébastien Boucksom*. I really wish to use this acknowledgment for showing and sending my best regards to two delightful and brilliant researchers, *Omid* and *Nina Amini*. I thank you from the bottom of my heart for your unconditional and pure help during my PhD. I would like to say that if I am standing here getting my PhD now in Mathematics, you both are really one of the reasons and I am really proud and pleased to have you for the rest of my life. A huge thanks to *Morteza Fotouhi* for encouraging me to pursue my education as a PhD student in Mathematics when I was doing my master at Sharif university of technology. You have been always a researcher with a warm and blessed heart to me. I am grateful to the presidents of Mathematics department at École Polytechnique *François Golse*, *Bertrand*

Rémy and the assistant to the department *Carole Juppín*. Also thanks to *Javier Fresan* and *Jérémie Bettinelli* to whom I have been TA.

I am so grateful for a collaboration with *Laurent Pfeiffer* who greatly influenced my last paper by exchanging some ideas during the work as an expert. It was a great pleasure to work with you, especially when you were really trying to have fun in the middle of research and make sure that our energy is at the highest level.

I would like to thank the assistance of CMLS, *Pascale Fuseau*, *Marine Amier* and *Béatrice Fixois* for their kindness in dealing with administrative procedure that I encountered as an international PhD student, as well as my advisor *Anne-Sophie* for helping and being understanding in some difficult situations caused by strikes in France and Covid-19 pandemic. Moreover, I must mention my office mates and PhD friends during these years who were making long working days more funny and full of happiness, for which I am thinking of *Nicolas Brigouleix*, *Dorian Chanfi*, *Lucas Gierczak*, *Arthur Touati*, *Etienne Bonnafoux*, *Lucas Ertzbischoff*, *Yichen Qin*, *Hernan Iriarte*, *Gabriel Ribeiro*, *Matthieu Piquerez* and *Elena Danesi*.

Many thanks to my physicist friends at CPHT for having lunch together and our discussions about the frontiers of Mathematics and Physics. I enjoyed a lot having and getting to know you during my PhD, particularly *Erik Linnér*, *Benjamin Bacq-Labreuil*, *Adrian Gomez Pueyo*, *Maria Chatzieftheriou*, *Evgeny Stepanov* and *Anustup Mukherjee*. I would like also thank some other friends from other universities such as *Ali Vakilinejad*, *Amir Hossein Zeraati*, *Mohammad Akrami*, *Iman Aghaa Sadeghi* and the others for having fun, lunch and discussions either in Paris or Tehran.

Last but not least, even more importantly, a huge thanks to my lovely and beautiful parents *Ali* and *Afsaneh* for their endless love and support without which I did not have the spirit to do such an accomplishment. To my kind brothers *Mehdi* and *Hamid* who are always available for me no matter when, where or how. You have been always symbol of intimacy and friendship to me.

Abstract

Motivé par des sujets économiques et d'ingénierie, vers 2006, les jeux à champ moyen ont été introduits par Jean-Michel Lasry et Pierre-Louis Lions, et Peter E. Caines, Minyi Huang et Roland P. Malhamé, indépendamment, et leur but était d'approximer les équilibres de Nash des jeux avec un grand nombre d'agents symétriques. L'équilibre de Nash est une situation dans laquelle aucun agent n'a intérêt à modifier unilatéralement sa stratégie : dans un équilibre de Nash, étant donné les choix actuels des autres agents, un agent donné ne peut améliorer son propre coût en changeant sa stratégie. Depuis leur introduction, les jeux à champ moyen ont été largement étudiés dans la littérature et plusieurs sujets de recherche ont été abordés, tant d'un point de vue théorique qu'appliqué. Cette thèse aborde des modèles de jeux à champ moyen avec temps final libre, et en plus avec une contribution principale pour les jeux à champ moyen avec des contraintes d'état.

Dans le premier chapitre, nous considérons plusieurs populations en interaction évoluant dans \mathbb{R}^d visant à atteindre des ensembles cibles donnés en un minimum de temps, en supposant que leur vitesse maximale est bornée en termes de la densité des agents autour de leur position. Le système de contrôle satisfait par chaque agent dépend de sa position, de la répartition des autres agents de la même population et de la répartition des agents des autres populations. Ainsi, les interactions entre agents se font par leur dynamique. Nous considérons dans ce chapitre l'existence d'équilibres lagrangiens à ce jeu à champ moyen, ce qui consiste à considérer une mesure de probabilité sur l'espace des trajectoires continues ; leur comportement asymptotique, en étudiant les taux de convergence des distributions des agents vers une distribution limite au sens de la distance de Wasserstein ; et leur caractérisation comme solutions d'un système de jeu à champ moyen, sous quelques hypothèses de régularité sur la dynamique des agents. En particulier, le système de jeu de champ moyen est établi sans s'appuyer sur les propriétés de semi-concavité de la fonction de valeur.

De manière similaire au premier chapitre, dans le deuxième chapitre, nous considérons un modèle de jeu à champ moyen inspiré de mouvement de foule où les agents visent à atteindre un ensemble fermé, appelé ensemble cible, en un temps minimal, mais en plus des phénomènes de congestion, qui affectent la vitesse de un agent, le modèle est considéré en présence de contraintes d'état : en gros, ces contraintes peuvent modéliser des murs, des colonnes, des clôtures, des haies ou d'autres types d'obstacles à la frontière du domaine que les agents ne peuvent pas franchir. Nous rappelons tout d'abord quelques résultats antérieurs sur l'existence d'équilibres pour de tels jeux et présentons les principales difficultés liées à la présence de contraintes d'état. Notre principale contribution est de montrer que les équilibres

du jeu satisfait un système d'équations aux dérivées partielles couplées, connu sous le nom de système de jeu à champ moyen, grâce aux techniques de caractérisation de contrôles optimaux développées dans le chapitre précédent, adaptées au cas de systèmes avec contraintes d'état. Ces techniques permettent non seulement de traiter des contraintes d'état mais nécessitent également très peu d'hypothèses de régularité sur la dynamique des agents.

Dans notre dernier chapitre, nous considérons un modèle de jeu à champ moyen pour le mouvement de foule dans lequel les piétons interagissent non seulement par leur position, mais aussi par leur vitesse. Plus précisément, chaque piéton est supposé minimiser un coût impliquant son temps pour atteindre un certain ensemble cible, un coût intégral individuel et un coût intégral d'interaction modélisant le fait que les agents veulent éviter les régions trop denses et préfèrent se déplacer avec les agents allant dans la même direction qu'eux, ce qui peut être vu comme une interaction de type Cucker–Smale. Le résultat principal que nous obtenons dans ce chapitre est l'existence d'équilibres pour un tel jeu, qui est basé sur une approche variationnelle qui repose sur la caractérisation des équilibres comme minimiseurs d'un coût fonctionnel global.

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Chapter 1

Introduction

1.1 General Framework

Mean Field Games (MFGs) refer to differential games with a continuum of agents assumed to be rational, indistinguishable, and influenced only by an average behavior of the others through mean field type interaction. Motivated by problems in economics and engineering on games with infinitely many agents [8, 9, 66], the theory of mean field games has been introduced around in 2006 by Jean-Michel Lasry and Pierre-Louis Lions [69–71], and Peter E. Caines, Minyi Huang, and Roland P. Malhamé [61–63], independently. In these works the main goal is to approximate *Nash equilibria* of games with a large number of symmetric agents.

Let us start by a short informal description of games with finitely many agents. Games with N players can be described by a set X of possible choices or strategies for an agent and a function $C_i: X^N \rightarrow \mathbb{R} \cup \{\pm\infty\}$, representing the cost of an agent. The goal of the i^{th} -agent, for $i \in \{1, \dots, N\}$, is to solve the minimization problem

$$(1.1) \quad \min_{x \in X} C_i(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_N).$$

A vector $\mathbf{X} = (x_1, \dots, x_N)$ solving all of the above minimization problems is called a *Nash equilibrium* of the game.

An N -player game is said to be symmetric if the permutations of agents do not change the game. Mathematically, this means that all the functions C_i , with $i \in \{1, \dots, N\}$, are equal to a certain function $C: X^N \rightarrow \mathbb{R} \cup \{\pm\infty\}$ which satisfies

$$C(x_1, \dots, x_N) = C(x_{\sigma(1)}, \dots, x_{\sigma(N)}) \quad \text{for any permutation } \sigma \text{ on } \{1, \dots, N\}.$$

In other words, one specific agent can choose its strategy without considering any order of others to react for solving its minimization criterion.

Mean field games belong to a class of differential games, i.e. games in which the choice of an agent is a trajectory submitted to a given differential equation. More precisely, the set X is usually a subset of the set $\mathcal{C}([0, T]; \Omega)$ of all continuous trajectories defined on a given

time interval $[0, T]$ and taking values in a given domain Ω , which is typically the closure of an open bounded subset of \mathbb{R}^d . This subset X is usually characterized as the set of all possible solutions of the control system

$$(1.2) \quad \dot{\gamma}(t) = f(t, \gamma(t), u(t)), \quad u(t) \in \mathcal{U}$$

where $f : [0, T] \times \Omega \times \mathcal{U} \rightarrow \mathbb{R}$ is the dynamics of the control system and u is a control input which takes values in a given set $\mathcal{U} \subseteq \mathbb{R}^m$. We always work with differential games and symmetric agents in the sequel. In several works in the literature, for simplicity, the domain Ω is considered as $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ due to being a compact manifold without boundary. However, in the sequel of thesis, we do not make such an assumption on Ω .

MFGs can be seen as limits of differential symmetric N -player games as the number of agents N tends to infinity. In an N -player game, the location of agents at time $t \in [0, T]$, can be described by $(\gamma_1(t), \gamma_2(t), \dots, \gamma_N(t)) \in \Omega^N$ and their distribution is given by $m_t = \frac{1}{N} \sum_{i=1}^N \delta_{\gamma_i(t)} \in \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ is the space of probability measures on Ω and $\delta_{\gamma(t)}$ denotes the Dirac measure at $\gamma(t) \in \Omega$. Since any probability measure can be approximated by a finite sum of Dirac measures in the weak topology of $\mathcal{P}(\Omega)$, an MFG is obtained by replacing the above measure m_t with any arbitrary time-dependent probability measure, still denoted by $m_t \in \mathcal{P}(\Omega)$ for simplicity, which not necessarily the sum of Dirac measures. The measure m_t is called the density of agents.

1.1.1 The MFG system

Equilibria of mean field games are usually characterized by a system of PDEs known as *MFG system*, made of a continuity equation coupled with a Hamilton–Jacobi–Bellman equation. We now briefly describe how to obtain such a system of PDEs from the formulation of the game in a simple setting following the presentation of [28]. For simplicity, assume that the control system (1.2), for a representative agent, is given by $\dot{\gamma}(t) = u(t)$. The minimization problem (1.1) solved by each agent is here to minimize a cost of the form

$$(1.3) \quad \int_0^T L(\gamma(t), u(t), m(t)) dt + \Phi(\gamma(T), m(T)),$$

in which the functions $L : \Omega \times \mathbb{R}^d \times \mathcal{P}(\Omega) \rightarrow \mathbb{R}$ and $\Phi : \Omega \times \mathcal{P}(\Omega) \rightarrow \mathbb{R}$ are called *Lagrangian* and *final cost*, respectively, and $T > 0$ is fixed. To study the optimization of (1.3) solved by each agent, one usually defines *the value function* associated with (1.3) by

$$(1.4) \quad \varphi(t_0, x) = \inf_{u \in L^2([0, T]; \mathcal{U})} \left\{ \int_{t_0}^T L(\gamma(t), u(t), m(t)) dt + \Phi(\gamma(T), m(T)) \mid \gamma(t_0) = x \right\}.$$

Notice that under some adequate assumptions on L , Φ and control u such as uniform bounds, convexity and superlinearity (see, e.g. [48, Chap. 3 and Chap. 10]), one deduces that the

infimum in (1.4) is attained. Let us now introduce *the Hamiltonian* $H: \Omega \times \mathbb{R}^d \times \mathcal{P}(\Omega) \rightarrow \mathbb{R}$ which is defined through the Fenchel transform of the Lagrangian L in (1.4),

$$H(x, p, m) = \sup_{u(t) \in \mathcal{U}} \left(p \cdot u - L(x, u, m(t)) \right), \quad \forall x \in \Omega, \forall p \in \mathbb{R}^d \text{ and } \forall m \in \mathcal{P}(\Omega).$$

Furthermore, by using classical techniques in optimal control, for instance those from [27, Chap. 7], we observe that the value function (1.4) satisfies the following PDE, called Hamilton–Jacobi–Bellman equation, in the sense of viscosity,

$$(1.5) \quad -\partial_t \varphi(t, x) + H(x, \nabla \varphi(t, x), m(t)) = 0, \quad \text{for } (t, x) \in (0, T) \times \Omega.$$

In addition, the optimal control u corresponding to an optimal trajectory γ is given by

$$(1.6) \quad u(t) = -\nabla_p H(\gamma(t), \nabla \varphi(t, \gamma(t)), m(t)),$$

and hence optimal trajectories solve the differential equation $\dot{\gamma}(t) = V(t, \gamma(t))$, where the velocity field V is given by

$$(1.7) \quad V(t, x) = -\nabla_p H(x, \nabla \varphi(t, x), m(t)).$$

We now wish to describe the evolution of m_t through a continuity equation, which is the other equation in an MFG system. Let us first recall the formal derivation of continuity equation describing the evolution of a density m_t subject to some velocity field $V: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ without source terms in the domain Ω . For simplicity, we assume the measure m_t admits a density, which we also denote by m_t or $m(t, \cdot)$. Considering the integral $\int_{\omega} m(t, x) dx$, which represents the total mass of agents in some subdomain $\omega \subseteq \Omega$ with smooth boundary, the changes of this quantity in ω is only due to the exchanges through the boundary of domain, i.e.,

$$(1.8) \quad \frac{d}{dt} \int_{\omega} m(t, x) dx = - \int_{\partial \omega} m(t, x) V(t, x) \cdot n(x) d\sigma,$$

where n is the unit outward normal vector to the boundary $\partial \omega$ and σ is the Hausdorff measure on the boundary $\partial \omega$. On the other hand, one knows by the divergence theorem,

$$(1.9) \quad \int_{\partial \omega} m(t, x) V(t, x) \cdot n(x) d\sigma = \int_{\omega} \operatorname{div} (m(t, x) V(t, x)) dx.$$

Hence by substituting (1.9) into (1.8), it follows

$$(1.10) \quad \int_{\omega} \left(\frac{\partial}{\partial t} m(t, x) + \operatorname{div} (m(t, x) V(t, x)) \right) dx = 0.$$

Since $\omega \subseteq \Omega$ is arbitrary in (1.10), we thus deduce that

$$(1.11) \quad \partial_t m + \operatorname{div}(mV) = 0 \quad \text{for } (t, x) \in (0, T) \times \Omega.$$

Hence, all in all, we have established the following MFG system, containing a Hamilton–Jacobi–Bellman equation in the form of (1.5) coupled with a continuity equation in the form of (1.11), with velocity field given by (1.7),

$$(1.12) \quad \begin{cases} \partial_t m(t, x) - \operatorname{div} \left(m(t, x) \nabla_p H(x, \nabla \varphi(t, x), m(t)) \right) = 0, & (t, x) \in (0, T) \times \Omega, \\ -\partial_t \varphi(t, x) + H(x, \nabla \varphi(t, x), m(t)) = 0, & (t, x) \in (0, T) \times \Omega, \\ m(0, x) = m_0, \quad \varphi(T, x) = \Phi(x, m(T)), & x \in \Omega, \end{cases}$$

where m_0 is the initial distribution of agents. Notice that $m_t \in \mathcal{P}(\Omega)$ satisfies the continuity equation in the sense of distributions. Under some suitable assumptions on H , there exists at least one solution $(m, \varphi) \in \mathcal{C}([0, T], \mathcal{P}(\Omega)) \times W^{1, \infty}([0, T] \times \Omega)$ to the MFG system (1.12) (see, e.g. [28, Theorem 3.1]), and moreover, uniqueness results are provided particularly by monotonicity assumption, for instance in the works [28, 69–71].

The description of the equilibrium of an MFG through (1.12) done above is usually known as the *Eulerian* approach. Other approaches have also been used in the literature, such as the *Lagrangian* approach and, when possible a *variational* approach described in the next two sections.

1.1.2 Lagrangian approach

In the previous section, we described the equilibrium of an MFG through a time-dependent probability measure $m_t \in \mathcal{P}(\bar{\Omega})$. An alternative approach to describe equilibria is to consider instead a probability measure Q on the space of continuous trajectories $\mathcal{C}([0, T]; \bar{\Omega})$, i.e. $Q \in \mathcal{P}(\mathcal{C}([0, T]; \bar{\Omega}))$. The relation between these two measures can be retrieved through the *evaluation map* $e_t: \mathcal{C}([0, T]; \bar{\Omega}) \rightarrow \bar{\Omega}$, defined by $e_t(\gamma) = \gamma(t)$. More precisely, if one considers the push-forward of Q by evaluation map at time t , it exactly gives us the measure m_t , i.e. $e_{t\#}Q = m_t$.

The definition of equilibrium is adapted to the Lagrangian setting by saying that such a Q is an equilibrium if Q -a.e. $\gamma \in \mathcal{C}([0, T]; \bar{\Omega})$ is optimal with respect to the optimization criterion. The Lagrangian approach is a classical technique in optimal transport, used in particular to study some problems in incompressible flows, traffic flows and branched transport problems (see, e.g. [6, 13, 18, 34, 85, 87]). Inspiring by this approach from optimal transport, there are some works about mean field games in the literature using Lagrangian setting (see, e.g. [12, 21, 29, 33, 75]).

Similarly to the Eulerian setting, we also have here a dependence of the optimal control problem on the evolution of agents itself, described through Q . For instance for a cost function of the form (1.3) the Lagrangian term becomes $L(\gamma(t), u(t), e_{t\#}Q)$ and similarly for the final cost Φ . When studying Lagrangian equilibria of MFGs, the dependence of the optimal control problem with respect to Q is usually an important topic to study.

1.1.3 Variational approach

This approach describes the equilibrium as a minimizer of some functional, created based upon the cost function. In the context of games with finitely many players, the variational approach is more commonly known as *potential*: the functional minimized by the equilibrium is called a *potential*, and games admitting such a functional are called *potential games*. As an illustration, consider a two-player game in which players want to minimize the cost functions

$$\begin{aligned} \text{player 1: } C_1(x_1, x_2) &= L(x_1) + H(x_1, x_2), \\ \text{player 2: } C_2(x_1, x_2) &= L(x_2) + H(x_1, x_2), \end{aligned}$$

where the functions L and H represent individual and interaction costs, respectively. Then, one can define the functional

$$J(x_1, x_2) = L(x_1) + L(x_2) + H(x_1, x_2),$$

which leads us to see immediately that minimizers of J are a Nash equilibria of our two-player game. In the context of mean field games, the variational approach was adapted in work [12], where the authors consider a model in which each agent is choosing a trajectory $\gamma(t)$ and optimizing the quantity

$$\int_0^T \left(\frac{|\dot{\gamma}(t)|^2}{2} + g(\gamma(t), m_t) \right) dt + \Phi(\gamma(T)),$$

where $g: \Omega \times \mathcal{P}(\Omega) \rightarrow \mathbb{R}$ represents the congestion relation between the agents. Then, by that, they have considered an overall cost functional defined by

$$\mathcal{J}(m, v) := \int_0^T \int_{\Omega} \left(\frac{1}{2} m_t |v|^2 + G(x, m_t) \right) dx dt + \int_{\Omega} \Phi dm_T,$$

where G is the anti-derivative of g with respect to its second variable. Under some appropriate assumptions on \mathcal{J} (see, e.g. [12, § 2]) they have concluded the existence of a minimizer of \mathcal{J} (see, e.g. [12, Theorem 2.2]) and shown that it satisfies their MFG system (see, e.g. [12, Theorem 2.3]). Notice that not all mean field games can be formulated in a variational framework. According to the model, sometimes constructing of such a functional \mathcal{J} is not easy. For more details and references, we address the reader also to the works [55, 69–71].

1.2 Minimal-time mean field games

Minimal-time MFGs deal with modeling and analysing situations in which the players may leave the game at different times. In Section 1.1, the minimization criterion was considered on the interval $[0, T]$ for some fixed positive $T > 0$, but in more realistic situations, T is not necessarily fixed. The players may indeed stop at different times according to the different parameters of the game. This motivates the study of MFG models with *free final time*, in

which time is a part of optimization criterion. For instance, in the work [56], the authors have considered competing producers with exhaustible resources who leave the market as soon as they exhaust their capacities. In that model, players wish to maximize their sale profit, while they must manage their exhaustible resource through a non-negative rate of production

Another realistic situation in which time can play a role in minimization criterion is crowd motion. This topic has been widely subject of many works from different perspectives, aiming mainly to studying questions on analysis, control and optimization of crowd behaviour (see, e.g. [42, 49, 73, 74, 78]). For instance, the work [73] describes crowd behaviour using a macroscopic perspective, which can be suited to the flow of people in emergency evacuation. In their model, the velocity field of pedestrians is unknown due to incompressibility constraints, which yields the actual velocity to be the projection of desired one onto the set of admissible velocities.

Most of the literature in crowd motion does not take into account the strategic choices of an agent, i.e. the fact that an agent may wish to anticipate the future behaviour of others and take such an anticipation into account when deciding their own trajectory. Game theory provides a natural framework to model such an anticipation, which motivated recent works to consider MFG models for the movement of large crowds. There are several works regarding this direction such as [12, 20, 33, 67]. These works generally consider the optimization criterion in a fixed time interval without any restriction on their speed. One may instead consider that the time interval is not fixed, but that each agent is free to chose their final time. This means that the agent leaves the game after that time, and they do not therefore have any contribution to minimize the cost afterwards.

It is also natural to consider constraints on the agent's maximal speed in terms of behaviour of the others around one specific agent. A motivation for such an assumption is the fact that the congestion which exists around the agent's position may work as a physical barrier and it thus prevents them to increase their speed by just paying some additional cost. There are some cases in which the authors have considered this kind of congestion such as Ref. [4], while still working on a finite time horizon.

One of the first studies of free final time problems is the work [75], where the authors have also used the Lagrangian approach in order to formulate the equilibrium. In that work, the agents are assumed to move in a certain open bounded subset $\Omega \subset \mathbb{R}^d$ aim at reaching in minimal time a given target set, considered to be the whole boundary of Ω , i.e. $\partial\Omega$. In other words, the agents are evolving in Ω , and their goal is to leave the domain in minimal time. In that reference, the authors define the *first exit time* after time t of a trajectory γ by

$$(1.13) \quad \tau(t, \gamma) = \inf \{s \geq 0 \mid \gamma(t + s) \in \partial\Omega\}.$$

In the optimal control problem, agents' dynamic $K: \mathcal{P}(\Omega) \times \Omega \rightarrow \mathbb{R}$ is considered with control system

$$(1.14) \quad \dot{\gamma}(t) = K(m_t, \gamma(t))u(t), \quad u(t) \in \mathcal{U}.$$

Notice that the function $K: \mathcal{P}(\Omega) \times \Omega \rightarrow \mathbb{R}$ represents the congestion around an agent and also interaction between an agent and the others through the density m_t . For instance, a suitable candidate of such a function is given by

$$K(m_t, x) = g \left(\int_{\Omega} \chi(x - y) \eta(y) \, dm_t(y) \right),$$

in which $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ is decreasing, the function $\chi: \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a smooth convolution kernel and $\eta: \mathbb{R}^d \rightarrow \mathbb{R}_+$ has $\mathcal{C}^{1,1}$ regularity. The value function is then defined as follows.

$$(1.15) \quad \varphi(t, x) = \inf_{\substack{\gamma \in \text{Adm} \\ \gamma(t) = x}} \tau(t, \gamma),$$

where Adm is the set of all admissible trajectories starting from x at time t (see, e.g. [75, Definition 3.1]). After performing the preliminary studies of the optimal control problem, the authors, under some appropriate assumptions on the agents' dynamics K , have proved the existence of a Lagrangian equilibrium after reformulating the notion of equilibrium as a fixed point of some set-valued map and proving the existence of such a fixed point by Kakutani fixed point theorem, and moreover, shown the equilibrium satisfies the continuity equation in the sense of distributions, coupled with a Hamilton–Jacobi equation on the value function (1.15).

More precisely and from a technical point of view, considering the target set as the whole boundary, having $\mathcal{C}^{1,1}$ regularity assumption on the function K and uniform exterior sphere property of Ω are required to show that the value function is Lipschitz continuous and semiconcave. Even though, the MFG system was established in this work, the optimal control could not be characterized as an explicit form except when the value function φ is differentiable. Hence, they only could obtain the MFG system by defining the *normalized gradient* (see, e.g. [75]) on their semiconcave value function.

Concerning the characterization of optimal control without having differentiability assumption on the value function φ , the work [47] contains some interesting results regarding to this direction. The authors have indeed shown that the semiconcavity of the value function yields to the existence of $\nabla\varphi(t, x)$ along the optimal trajectories, for $t \in (0, \varphi(0, x))$, and hence by using Pontryagin maximum principle, one observes the optimal control can be thus given by $-\frac{\nabla\varphi(t, x)}{|\nabla\varphi(t, x)|}$. Therefore, the MFG system can be established as follows.

$$\begin{cases} \partial_t m(t, x) - \operatorname{div} \left(m(t, x) K(m_t, x) \frac{\nabla\varphi(t, x)}{|\nabla\varphi(t, x)|} \right) = 0, & (t, x) \in \mathbb{R}_+^* \times \Omega, \\ -\partial_t \varphi(t, x) + |\nabla\varphi(t, x)| K(m_t, x) - 1 = 0, & (t, x) \in \mathbb{R}_+ \times \Omega, \\ m(0, x) = m_0, \\ \varphi(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

The initial and boundary condition, respectively, on m and φ are quite natural, since m_0 is the initial agents' distribution, and on the other hand, if there are some players who are already on the target set, then the minimal time starting from boundary is obviously zero.

1.3 Mean field games with state constraints

In this section, we consider an MFG problem in the presence of state constraints. In this type of models, the agents are constrained to remain in the domain $\Omega \subset \mathbb{R}^d$, i.e. $\gamma(t) \in \Omega$, for $t \in \mathbb{R}_+$. For instance, state constraints appear naturally in the context of crowd motion to model physical barriers for the movement of agents, such as walls, columns, fences, or holes, while they also appear naturally in other contexts, such as non-negativity and maximal capacity constraints for stocks of a product in economics.

Notice that, in the work [75] described previously in Section 1.2, even though the authors assume that agents' trajectories γ must satisfy $\gamma(t) \in \Omega$, their model is an unconstrained MFG, since, for them, the target set was taken as the whole boundary $\partial\Omega$. If, for instance, the target set was just a part of boundary, then they would have been in state constraints situation. Furthermore, notice that proving the existence of equilibrium is independent of being in a free state model or state constraints model. In other words, the existence of equilibrium can be still done by fixed point arguments, whether in the presence or absence of state constraints. For instance, in the work [21], the authors have considered an MFG system for which the constrained MFG equilibria has been established in the Lagrangian setting. Moreover, due to having some sort of monotonicity on a function in their MFG system (see, e.g. [21, Theorem 4.1]), uniqueness of equilibrium have been also shown.

One of the consequences of having state constraints on the model is the possible lack of semiconcavity of value function (see, e.g. [24, Example 4.4]). Classical strategies for studying mean field games usually rely on semiconcavity of the value function to establish further properties of the optimal control problem solved by each agent, such as characterizing optimal controls in terms of the gradient of the value function. In the works [21–23], the authors have concentrated on a model in the presence of state constraints and shown that the value function of their optimal control problem is semiconcave but not in the classical sense, i.e. with a linear modulus of semiconcavity (see, e.g. [27, Remark 1.1.2]). They have instead shown the value function is semiconcave with a fractional modulus of semiconcavity (see, e.g. [23, Theorem 3.1 and Corollary 3.2]) thanks to which the optimal control u has been characterized and continuity equation is deduced (see, e.g. [23, Theorem 4.5]). Notice that the equilibrium and Hamilton–Jacobi–Bellmann equation have been established with no major difficulties.

Another major issue of working with state constraints in optimal control problems consists in applying Pontryagin maximum principle in order to obtain further information on optimal trajectories and controls. Indeed, the necessary conditions provided by Pontryagin maximum principle involve an additional Radon measure in the equation of the costate, which comes from the presence of state constraints and is supported only on time instants in which the trajectory touches the boundary of the constrained set (see, e.g. [40, Theorem 5.2.1]). The presence of this additional term and its lack of regularity make it harder for one to deduce properties of the costate and of optimal trajectories.

In order to overcome the previous difficulty, [25] proposes an alternative approach which consists in applying Pontryagin maximum principle to a modified optimal control problem

in which the state constraint is replaced by a suitable (non-smooth) penalization. Such a penalization allows the trajectories to violate the constraints, and one of the main results in [25] states that, if the penalization parameter is small enough, and under suitable regularity assumptions on the constraints, optimal trajectories of the unconstrained penalized problem coincide with those of the original constrained problem.

Let us present in more details the difficulties discussed above by following the presentation of [25]. Consider the control system (1.16) submitted to a state constraint of the form $\gamma(t) \in \Omega$.

$$(1.16) \quad \begin{cases} \dot{\gamma}(t) = f(t, \gamma(t), u(t)) & \text{for a.e. } t \in [0, T], \\ \gamma(0) = x_0. \end{cases}$$

By applying the Pontryagin maximum principle with state constraints from [40, Theorem 5.2.1], there exist the functions p , h and a Radon measure μ such that

$$(1.17) \quad (\dot{h}(t), -\dot{p}(t), \dot{\gamma}(t)) \in \partial H(t, x(t), p(t) + \psi(t)),$$

where H is corresponding Hamiltonian and ∂H denotes the generalized gradient of H (see, e.g. [40, § 1.2]) and

$$\begin{aligned} \psi(t) &= \int_{[0,t)} \nu_{\Omega}(\gamma(s)) \, d\mu(s) \\ h(t) &= H(t, \gamma(t), p(t) + \psi(t)), \end{aligned}$$

in which $\nu_{\Omega}(\gamma(t))$ is in the normal cone to Ω at $\gamma(t)$, and the measure μ is supported on those times for which $\gamma(t) \in \partial\Omega$. Under appropriate assumptions on H and having enough smoothness on $\partial\Omega$, the authors in [25] show that (1.17) can be rewritten as

$$(1.18) \quad \begin{aligned} \dot{\gamma}(t) &= \nabla_p H(t, \gamma(t), p(t)) \\ -\dot{p}(t) &= \nabla_x H(t, \gamma(t), p(t)) - \lambda(t) \nu_{\Omega}(\gamma(t)) \mathbf{1}_{\partial\Omega}(\gamma(t)), \end{aligned}$$

where λ is a measurable function, depending only on H and $\partial\Omega$, and $\mathbf{1}_{\partial\Omega}$ is characteristic function on the boundary. The above formulation is more explicit and usually easier to work with than (1.16), since the term $\psi(t)$ is replaced by a more explicit term in the equation of \dot{p} .

More precisely, in order to obtain (1.18) from (1.16), the authors in [25] proceed by first defining the signed distance to the boundary to the boundary $d_{\partial\Omega}^{\pm} : \mathbb{R}^d \rightarrow \mathbb{R}$

$$d_{\partial\Omega}^{\pm}(x) = d_{\Omega}(x) - d_{\mathbb{R}^d \setminus \Omega}(x),$$

where $d_{\Omega}(x) = \inf \{|x-y| \mid y \in \Omega\}$. Then, under some regularity assumption on Hamiltonian H , they have deduced the following result ([25, Theorem 2.1]), stating that there exist a

Lipschitz continuous functions $p : [0, T] \rightarrow \mathbb{R}^d$, with $|p(t)| > 0$, and a bounded measurable function $\lambda(t) \geq 0$ such that

$$\begin{cases} \dot{\gamma}(t) = \nabla_p H(t, \gamma(t), p(t)) & \text{for a.e. } t \in [0, T] \\ -\dot{p}(t) = \nabla_x H(t, \gamma(t), p(t)) - \lambda(t) \nabla d_{\partial\Omega}^\pm(\gamma(t)), & \text{for a.e. } t \in [0, T], \end{cases}$$

The proof relies on applying a penalization technique by considering a penalized control system without state constraint. More precisely, the perturbed function f_ϵ and Hamiltonian H_ϵ are respectively defined by

$$\begin{aligned} f_\epsilon(t, x, u) &= f(t, x, u) \left(1 - \frac{1}{\epsilon} d_\Omega(x)\right)_+ \\ H_\epsilon(t, x, p) &= H(t, x, p) \left(1 - \frac{1}{\epsilon} d_\Omega(x)\right)_+, \end{aligned}$$

where the notation “+” refers to positive part. Thus, the unconstrained control system is given by

$$(1.19) \quad \begin{cases} \dot{\gamma}_\epsilon(t) = f_\epsilon(t, \gamma_\epsilon(t), u_\epsilon(t)) & \text{for a.e. } t \in [0, T], \\ \gamma_\epsilon(0) = x_0. \end{cases}$$

and, by applying the unconstrained Pontryagin maximum principle, one obtains the system

$$\begin{cases} \dot{\gamma}_\epsilon(t) = \nabla_p H(t, \gamma_\epsilon(t), p_\epsilon(t)) \left(1 - \frac{1}{\epsilon} d_\Omega(\gamma(t))\right)_+ & \text{for a.e. } t \in [0, T], \\ -\dot{p}_\epsilon(t) \in \nabla_x H_\epsilon(t, \gamma_\epsilon(t), p_\epsilon(t)), & \text{for a.e. } t \in [0, T], \end{cases}$$

where γ_ϵ represents the optimal trajectory for an unconstrained problem.

In order to relate the optimal trajectories of the original, constrained problem with those of the penalized problem, the authors in [25] show that both sets of optimal trajectories actually coincide if ϵ is small enough. Hence, optimal trajectories of the original problem can be characterized through Pontryagin maximum principle applied to the penalized problem with small enough ϵ , yielding the main result of [25].

1.4 Mean field games of controls

In all the MFG models discussed earlier in Sections 1.1, 1.2 and 1.3, the interaction between agents happens only through their state, i.e. the trajectory γ of an agent depends on the other agents only through m , which represents the distribution of positions of other agents. Mean field games of controls have been introduced to model situations in which an agent depends on the behavior of others not only through their state, but also through their controls. This kind of interaction appears naturally in many applications: in crowd motion, for instance, this would correspond to agents taking decisions based not only on other agents' positions,

but also on their velocities. This consist in having a mean field type interaction between agents not only thorough their distribution on the state, but also through their controls. Notice that this property can make the analysis of the model more difficult, by the fact that the Hamiltonian H also depends on the other agent's control. Let us now review some of the works in this direction such as [16, 31, 32, 54, 86].

For instance in the work [31], the authors have considered a model inspired by trade crowding which deals with taking decision to buy or to sell a large number of productions or contracts. Modeling of such an interaction between the price dynamics, which plays a role as the control, and trading process is quite difficult, since in reality theses dynamics describe the managing of assets by buying or selling the productions. Their framework indeed concerns the finite time interval $[0, T]$, since by the model their starting point was optimal liquidation problem for which T usually stands as a day or a week. After establishing the MFG system associated with their model [31, § 2, system (9)], they have proposed a general model for mean field games of controls [31, § 5].

In the work [32], the authors have studied the long time behaviour of solutions to the first-order MFG system with a control on the acceleration. They have considered an ergodic system for which the long-time average of value function converges to an ergodic constant which is represented as a minimum of Lagrangian over a suitable class of closed probability measures (see, e.g. [32, Theorem 2.2]). However, that ergodic system may make no sense in the first sight due to the mean field interaction in the Hamilton–Jacobi equation, and the authors have then investigated the solution to such a system (see, e.g. [32, Theorem 2.5]).

Some works have approached mean field games of controls through a random variable framework. This is the case, for instance, of [54] which considers mean field games with interactions in state and control through a random variable approach. More precisely, in their MFG system, the Hamiltonian depends on X and \dot{X} , where $X: [0, T] \rightarrow L^q(\Omega; \mathbb{R}^d)$ is a path in the space of random variables. Moreover, instead of considering the continuity equation in order to describe the agents' evolution, the control system $\dot{X} = -\nabla_p H(X, \nabla_p \varphi(X, t), X, \dot{X})$ was used, which can be seen as a continuity equation formulated in terms of the random variable X instead of its distribution. The authors have indeed shown the existence of a Lipschitz continuous value function $\varphi: \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}$ and a $\mathcal{C}^{1,1}$ random variable X such that their MFG system is satisfied (see, e.g. [54, Theorem 1]).

In contrast to the previous work [54], which was based on random variable framework, the work [16] refers to a deterministic optimal control problem involving nonlinear dynamics and mixed state-control constraints. In their model, the interaction between the agents' control occurs in a price variable and a congestion term, based on the collective behaviour of the agents. This price variable penalizes linearly the agent's control, which can be interpreted as a demand. As a matter of fact, this kind of frameworks are suitable to some models in which companies compete without market power such as the energy markets containing a large number of small storage devices. As an another application of these mixed state-control and final state constraints, one can look at the storage devices which are necessarily loaded at the end of the time frame.

To finish this part, let us discuss the work [86] in which the model is built upon a

Cucker–Smale inspired MFG with velocity interactions. The Cucker–Smale model describes the evolution of N particles which are usually representing a bunch of birds in movement. The state x_i of the i -th bird in the population is modelled by the system

$$x'_i = v_i, \quad v'_i = -\frac{1}{N} \sum_{j \neq i} \eta(x_i - x_j)(v_i - v_j),$$

where η is a decreasing function of distance between two agents and is used in the spirit of making a particle to align its velocity to that of the others. Inspired by that, a very simple example of a game model, should contain, in the cost function of agent i , a term of the form

$$\int_0^T \sum_j \frac{1}{2} \eta(x_i(t) - x_j(t)) |x'_i(t) - x'_j(t)|^2 dt,$$

From modeling point of view, another term should be contained in the cost which represents the kinetic energy of each player. The global cost, for each agent, has therefore the following form.

$$\int_0^T \left(\frac{\delta}{2} |\dot{x}_i(t)|^2 + \lambda \sum_j \frac{1}{2} \eta(x_i(t) - x_j(t)) |\dot{x}_i(t) - \dot{x}_j(t)|^2 \right) dt + \Phi(x_i(T)),$$

for some positive parameters δ and λ and final cost Φ . Since the authors have considered the model in the Lagrangian setting, they have thus used a probability measure Q on the space of continuous curves, for which the cost function in this continuous setting takes the form

$$\int_0^T \left(\frac{\delta}{2} |\dot{\gamma}(t)|^2 + \lambda \int_{\mathcal{C}([0,T];\bar{\Omega})} \frac{1}{2} \eta(\gamma(t) - \tilde{\gamma}(t)) |\dot{\gamma}(t) - \dot{\tilde{\gamma}}(t)|^2 dQ(\tilde{\gamma}) \right) dt + \Phi(\gamma(T)).$$

Notice that since the cost function is invariant under translation of all trajectories by a common time-dependent vector, then the compactness property will be therefore lost, and as a consequence, proving the existence of equilibrium by some fixed point arguments such as Kakutani, is not so straightforward. Hence, the authors have shown the existence of an equilibrium $Q \in \mathcal{P}(\mathcal{C}([0,T];\bar{\Omega}))$ in a variational framework, by building an appropriate overall functional $\mathcal{J}(Q)$, whose minimizers are equilibria of the game, where the existence of such a minimizer relies a lot on the lower semicontinuity of the functional \mathcal{J} .

1.5 Contributions

Having considered the previous sections, we are now ready to justify the aims and motivations of this thesis, which are essentially based on studying MFG models motivated by their potential applications in crowd motion. One of the motivations behind this thesis is to study models in which the agents do not necessarily stop their optimization criterion all at

the same time, while furthermore, each agent may belong to a different population with a different optimization criterion for each population. Secondly, rather than taking the agents into accounts in a free state model by putting some restrictive assumptions such as taking the target set as the whole boundary, like in Section 1.2, the agents may optimize their cost function subject to some certain state constraints, which is usually the case from practical point of view. Finally, another practical and realistic situation is due to having the agents' interaction not only through their density, but also relying their optimization criterion on the direction of the movement of the others. Let us now present our contribution to the literature as the different chapters of this thesis which are organized as follows. The first chapter is dedicated to *multipopulation minimal-time mean field games*, while the second and third ones are *nonsmooth mean field games with state constraints* and *a variational mean field game of controls with free final time and pairwise interaction*, respectively.

1.5.1 Multipopulation minimal-time mean field games

We here present the idea, novelties and difficulty of the problem, and then the results that we have established on it, based on the work [83], described in Chapter 2. Since by hypotheses of mean field game theory, the agents are indistinguishable and the individual behaviour of an agent does not affect the whole crowd, one may come up with considering some identical agents as a group or population and thus separating them into several parts, where each part is now considered as a new identical population for which the general movement direction interests us rather than individual movement of each agent. This, hence, motivates to generalize the previous minimal-time mean field games works in the literature. Similar to the work [75], described in Section 1.2, this part refers to the minimal time mean field games in which the agents may reach the target set at different times and time is indeed the main part of optimization criterion, which essentially yields that there is no definite time interval $[0, T]$ on which the agents leave the game. In contrast to lots of works in the literature, but again similar to [75] the agents are not able to move as fast as they want only by paying some additional cost. Therefore, their maximal speed is bounded by the density of the other agents.

Let us now go through the main novelties of this contribution. The first novelty is to consider N populations of agents who are evolving in the space \mathbb{R}^d , instead of inside of a compact subset of \mathbb{R}^d , with N target sets, one for each population, and specific dynamics K_i in the form (1.14), for $i \in \{1, \dots, N\}$. One of the difficulties of this setting is the fact that we work in a noncompact space, but this can be easily overcome thanks to the following result (see Chapter 2, Proposition 2.4.4).

- Consider the optimal control problem of reaching a nonempty closed set $\Gamma \subset \mathbb{R}^d$ in minimal time with dynamics $\dot{\gamma}(t) = k(t, \gamma(t))u(t)$, where $k: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is continuous and upper and lower bounded by positive constants. Let φ be the corresponding value function. Then there exist two nondecreasing maps with linear growth $\psi, T: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ depending only on target set and the agent's speed bounds such that, for every $R > 0$,

$t_0 \in \mathbb{R}_+$, $x_0 \in B_R$, we have $\varphi(t_0, x_0) \leq T(R)$ and, for every optimal γ starting from x_0 , we have $\gamma(t) \in B_{\psi(R)}$ for every $t \geq 0$.

In the above statement, B_R is the closed ball centred at the origin with radius R . This result says that, if an initial point of an agent starts from a ball with radius $R > 0$, for a fixed R but arbitrary, then its trajectory must remain in a ball with radius $\psi(R)$, and its exit time is at most $T(R)$. Thanks to this result, we can prove existence of a Lagrangian equilibrium Q for the multipopulation minimal-time mean field game considered here by following roughly the same fixed-point strategy as in [75], as described in Chapter 2, Section 2.5.1.

Concerning the *asymptotic behaviour* of the probability measure $m_t = e_{t\#}Q \in \mathcal{P}_p(\mathbb{R}^d)$ as $t \rightarrow +\infty$, we need first to make sense the evaluation map at $+\infty$ by proving its Borel measurability, for which we refer the reader to [83, § 5.2], then provide the following statement on each population (see Chapter 2, Theorem 2.5.8; the population index is omitted here for simplicity).

- Let Q be an equilibrium and $m_t = e_{t\#}Q$. There exists $m_\infty \in \mathcal{P}(\mathbb{R}^d)$ such that $m_t \rightarrow m_\infty$ as $t \rightarrow +\infty$.
- Let $p \in [1, +\infty)$, and assume that $m_0 \in \mathcal{P}_p(\mathbb{R}^d)$. Then for every $t \in [0, +\infty]$, we have $m_t \in \mathcal{P}_p(\mathbb{R}^d)$ and moreover $m_t \rightarrow m_\infty$ in the sense of usual Wasserstein distance \mathbf{W}_p , where $\mathcal{P}_p(\mathbb{R}^d) = \{\mu \in \mathcal{P}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |x|^p d\mu(x) < +\infty\}$.
- Assume that m_0 is compactly supported. Then, for every $t \in [0, +\infty]$, m_t is compactly supported and there exists $\tau \geq 0$ such that

$$m_t = m_\infty, \quad \forall t \geq \tau.$$

Finally, the main contribution of this part will be dedicated to relaxing the regularity assumption on the agents' dynamics. To the best of our knowledge, this is the first study of a minimal-time mean field game without having $\mathcal{C}^{1,1}$ regularity assumption on the dynamics of each agent. This assumption is indeed used in other works to establish the semiconcavity of value function of the optimal control problem, which is a key property to provide the differentiability of value function along the optimal trajectories, by which one can recognize the velocity field of the continuity equation and thus deduce that continuity equation is satisfied by equilibria.

We overcome this difficulty by taking advantage of a new technique which does not rely on $\mathcal{C}^{1,1}$ regularity assumption. To justify this technique, we first need to recall the *dynamic programming principle* (for more details, see, e.g [11, Proposition 2.1]).

- Consider the same optimal control problem as before and its value function φ . Then, for every $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \in \text{Adm}$ with $\gamma(t_0) = x_0$, we have

$$\varphi(t_0 + h, \gamma(t_0 + h)) + h \geq \varphi(t_0, x_0), \quad \text{for every } h \geq 0,$$

with equality for every $h \in [0, \tau(t_0, \gamma)]$, if γ is optimal.

Inspired by this inequality, and assuming the dynamics K to be continuous in both variables and Lipschitz continuous in its state variable, we have that, for every $u_0 \in \mathbb{S}^{d-1}$,

$$\frac{\varphi(t_0 + h, x_0 + hK(m_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \geq -1 + o(1), \quad \text{as } h \rightarrow 0^+.$$

The goal is now to conclude that an element u_0 is the optimal direction where the above ratio attains its infinitesimal lower bound -1 at the limit $h \rightarrow 0^+$, which essentially corresponds to those directions in which φ decreases with maximal rate. Hence, based on this explanation, we propose the following definition.

- We define the set $\mathcal{U}(t_0, x_0)$ of optimal directions at (t_0, x_0) as the set of all $u_0 \in \mathbb{S}^{d-1}$ for which there exists an optimal trajectory γ such that the corresponding optimal control u satisfies $u(t_0) = u_0$.
- We define the set $\mathcal{W}(t_0, x_0)$ of directions of maximal descent of φ at (t_0, x_0) as the set of all $u_0 \in \mathbb{S}^{d-1}$ such that

$$(1.20) \quad \lim_{h \rightarrow 0^+} \frac{\varphi(t_0 + h, x_0 + hK(m_0, x_0)u_0) - \varphi(t_0, x_0)}{h} = -1.$$

If $\mathcal{W}(t_0, x_0)$ contains exactly one element ω_0 , then $-\omega_0$ is called the *normalized gradient* of φ at (t_0, x_0) and denoted by $-\omega_0 = \widehat{\nabla} \varphi(t_0, x_0)$. In order to deduce the MFG system, one needs to specify the velocity field of continuity equation. To do so, we have provided the following result under some appropriate assumptions on the game (see Chapter 2, Theorem 2.4.14).

- For every $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have $\mathcal{U}(t_0, x_0) = \mathcal{W}(t_0, x_0)$.

The proof of the inclusion $\mathcal{U}(t_0, x_0) \subset \mathcal{W}(t_0, x_0)$ follows easily from the dynamic programming principle, but converse inclusion is more delicate to show. Given an element $u_0 \in \mathcal{W}(t_0, x_0)$, we prove that u_0 is indeed an optimal direction by reasoning by contradiction: if it is not, we construct a trajectory starting from (t_0, x_0) going in a different direction from u_0 and arriving at the target set strictly faster than the optimal time $\varphi(t_0, x_0)$, yielding the conclusion.

Since the set $\mathcal{U}(t_0, x_0)$ admits a unique element along the optimal trajectories, thanks to the result shown in Chapter 2, Proposition 2.4.12, then one concludes the same for the set $\mathcal{W}(t_0, x_0)$. Hence, the MFG system can be established by using the normalized gradient as the velocity field of continuity equation, which is satisfied in the sense of distribution, while the Hamilton–Jacobi–Bellman equation holds in the viscosity sense (see [83, Theorem 5.11]).

1.5.2 Nonsmooth mean field games with state constraints

We now present the main results of Chapter 3 of this thesis, which is based on [81]. As discussed in Section 1.3, the goal of this part is to study the model when the agents are restricted to remain in a certain $\Omega \subset \mathbb{R}^d$, where this constraint representing the fact that agents cannot cross walls, columns, fences, hedges, or other kind of obstacles. In this case,

even though providing the existence of equilibria is not a difficult issue, as it has already been done in [75], characterization of optimal control as the velocity field of continuity equation in an MFG system is a more delicate question.

This characterization can typically be done based on the semiconcavity of the value function, which may fail in state-constrained models, even with $\mathcal{C}^{1,1}$ regularity assumption on the agents' dynamics. Even though there are some techniques in the literature to overcome the state-constrained difficulty by establishing the fractional semiconcavity of the value function instead of classical semiconcavity (see, e.g. [21–23]), it is not clear how these techniques could be adapted to the present setting of mean field games with free final time. Hence, the idea is to take advantage of the technique which was discussed in Section 1.5.1 and the penalization technique in Section 1.3, in order to apply Pontryagin maximum principle without the difficulties coming from the Radon measure which appears in the optimality system.

In contrast to the Section 1.3 and more generally, we take the target set $\Gamma \subset \Omega$ as a closed subset of Ω and not necessarily $\partial\Omega$, and consider the penalized optimal control problem and set of optimal trajectories started at time t_0 and position x_0 , described in Section 1.3. In order to retrieve additional properties of optimal trajectories and controls, we apply Pontryagin maximum principle (as stated in [88]), to an unconstrained penalized problem constructed in a similar spirit to the penalization technique described in Section 1.3 (see Chapter 3, Propositions 3.4.3 and 3.4.5). Since the penalized dynamics is non-smooth, tools from non-smooth analysis (see Chapter 3, Lemma 3.4.4) are needed in order to obtain the desired properties.

Finally, still making use of the penalization technique, we can show, similarly to Section 1.3, that optimal trajectories of the penalized problem coincide with optimal trajectories of the original problem. Therefore, starting from a point inside of the domain, including the boundary, assures that the agent will remain in $\bar{\Omega}$, which is actually a particular case of the following result (see Chapter 3, Proposition 3.4.7).

- *There exists $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$, $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, if γ is optimal with respect to the penalized optimal control system, then we have $d_\Omega(\gamma(t)) \leq d_\Omega(x_0)$ for every $t \in \mathbb{R}_+$. In particular, if $x_0 \in \bar{\Omega}$, then $\gamma(t) \in \bar{\Omega}$ for every $t \in \mathbb{R}_+$.*

As a consequence of that, one observes (see Chapter 3, Theorem 3.4.8)

- *There exists $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$ and $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, the optimal trajectories of penalized control system are optimal with respect to the original control system and vice versa.*

Regarding the characterization of optimal control, we proceed similarly to Section 1.5.1. The main difficulty due to the state constraint is that, if an agent starts their movement from the boundary, i.e. $x_0 \in \partial\Omega$, then at the position $x_0 + hK(m_0, x_0)u_0$, he may have left the domain Ω , i.e. $x_0 + hK(m_0, x_0)u_0 \notin \Omega$, and as a matter of fact, the limit (1.20) does not make sense anymore. Hence, we modify the definition of *directions of maximal descent* as follows.

- For $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, we define the set $\mathcal{W}_\epsilon(t_0, x_0)$ of directions of maximal descent of φ_ϵ at (t_0, x_0) by

$$\mathcal{W}_\epsilon(t_0, x_0) = \left\{ u_0 \in \mathbb{S}^{d-1} \left| \lim_{h \rightarrow 0^+} \frac{\varphi_\epsilon(t_0 + h, x_0 + hK(m_0, x_0)u_0) - \varphi(t_0, x_0)}{h} = -1 \right. \right\}.$$

We can again show this set \mathcal{W}_ϵ coincides with \mathcal{U} , which leads us to deduce the MFG system in the presence of state constraint (see Chapter 3, Theorem 3.5.2). Moreover, some additional properties on value function can be concluded such as boundary conditions on $\mathbb{R}_+ \times (\partial\Omega \setminus \Gamma)$ (see Chapter 3, Theorem 3.4.11).

- The value function φ satisfies $\nabla\varphi(t, x) \cdot \mathbf{n}(x) \geq 0$ in the viscosity supersolution sense for every $(t, x) \in \mathbb{R}_+ \times (\partial\Omega \setminus \Gamma)$, where \mathbf{n} is the unit outward vector to the boundary $\partial\Omega \setminus \Gamma$.

1.5.3 A variational mean field game of controls with free final time and pairwise interaction

The results described in this section are those from Chapter 4, which is based on the work in preparation [84].

One of the main drawbacks of the MFG models considered in Sections 1.5.1 and 1.5.2 is that they consider that agents only interact through their positions. We now consider situations in which agents also take into account other agents' controls in their own optimization criterion. This allows for more realistic models for crowd motion, by allowing a pedestrian's choice to depend not only on the position of others but also where they are going.

In contrast to Section 1.5.2, where obtaining the equilibria was not a major issue, here we need to establish existence of equilibria. For that purpose, we adopt a variational approach, discussed in Section 1.1.3. Notice that we are still in a Lagrangian setting to describe the model, however we aim at describing equilibria in a variational setting, instead of a fixed-point setting.

The model we consider here is based on that from [86], which was discussed in Section 1.4. The main novelty here is that we consider an optimization criterion with free final time, as discussed in Section 1.2, which brings some additional technical difficulties with respect to the approach taken in [86]. In addition, we also consider a more general individual cost for each agent, which is the integral of some running cost $\ell(t, \gamma(t), \dot{\gamma}(t))$ as well as a more general interaction cost $h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t))$, where ℓ and h satisfy standard assumptions. Therefore, the cost function, which each agent minimizes, can be represented as follows.

$$(1.21) \quad \int_0^{+\infty} \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \int_{\mathcal{C}} \int_0^{\tau(\gamma) \wedge \tau(\tilde{\gamma})} h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt dQ(\tilde{\gamma}) + \Psi(\tau(\gamma)),$$

where τ is the exist time defined as in (1.13) in Section 1.2, Ψ is an increasing final cost which penalizes the exist time, respectively, and $\mathcal{C} = \mathcal{C}(\mathbb{R}_+; \bar{\Omega})$ has been set, for simplicity.

In order to show existence of an equilibrium of the game, we first define a notion of *equilibrium* of this game by the following definition,

- Let $m_0 \in \mathcal{P}(\bar{\Omega})$. A measure $Q \in \mathcal{P}(\mathcal{C})$ is called *equilibrium of MFG*, with initial condition m_0 , if $e_{0\#}Q = m_0$ and

$$\int_{\mathcal{C}} F(\gamma, Q) dQ(\gamma) < +\infty, \quad F(\gamma, Q) = \inf_{\substack{\omega \in \mathcal{C} \\ \omega(0) = \gamma(0)}} F(\omega, Q), \quad Q\text{-a.e. } \gamma \in \mathcal{C},$$

where $F(\gamma, Q)$ is the cost function defined by (1.21).

We now aim at introducing a global cost functional \mathcal{J} whose minimizers should yield equilibria of the MFG. For that purpose, we consider the functional \mathcal{J} defined by

$$\mathcal{J}(Q) = \int_{\mathcal{C} \times \mathcal{C}} J(\gamma, \tilde{\gamma}) d(Q \otimes Q)(\gamma, \tilde{\gamma}),$$

where $J(\gamma, \tilde{\gamma})$ represents the individual and interaction costs of two arbitrary agents γ and $\tilde{\gamma}$ together with their final costs, i.e.,

$$\begin{aligned} J(\gamma, \tilde{\gamma}) = & \int_0^{+\infty} \ell(t, \gamma(t), \dot{\gamma}(t)) + \ell(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt \\ & + \int_0^{\tau(\gamma) \wedge \tau(\tilde{\gamma})} h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt + \Psi(\tau(\gamma)) + \Psi(\tau(\tilde{\gamma})). \end{aligned}$$

Based on the variational approach, explained in Section 1.1.3, there is a tonic relation between the minimizers of functional \mathcal{J} and equilibria of a game, which is not obvious and needs to be verified, as follows (see Chapter 4, Theorem 4.5.2).

- *Every minimizer of \mathcal{J} is an equilibrium of the game.*

On the other hand, we can also prove the following result (see Chapter 4, Theorem 4.5.3).

- *The functional \mathcal{J} admits a global minimizer.*

Hence as a consequence of that, we deduce the main result of Chapter 4, stated in Theorem 4.5.4, namely,

- *There exists an equilibrium $Q \in \mathcal{P}(\mathcal{C})$ for the game with initial condition m_0 .*

Chapter 2

Multipopulation minimal-time mean field games

In this chapter, we consider a mean field game model inspired by crowd motion in which several interacting populations evolving in \mathbb{R}^d aim at reaching given target sets in minimal time. The movement of each agent is described by a control system depending on their position, the distribution of other agents in the same population, and the distribution of agents on other populations. Thus, interactions between agents occur through their dynamics. We consider in this chapter the existence of Lagrangian equilibria to this mean field game, their asymptotic behavior, and their characterization as solutions of a mean field game system, under few regularity assumptions on agents' dynamics. In particular, the mean field game system is established without relying on semiconcavity properties of the value function.

2.1 Introduction

Mean field games (MFGs for short) are differential games with a continuum of agents assumed to be rational, indistinguishable, and influenced only by an averaged behavior of other agents through a mean-field type interaction. Following previous works in the economics literature on games with infinitely many agents [8, 9, 66], the theory of mean field games has been introduced in 2006 by the simultaneous works of Jean-Michel Lasry and Pierre-Louis Lions [69–71], and of Peter E. Caines, Minyi Huang, and Roland P. Malhamé [61–63], motivated by problems in economics and engineering and with the goal of approximating Nash equilibria of games with a large number of symmetric agents. Since their introduction, mean field games have been extensively studied in the literature and several research topics have been addressed, both from theoretical and applied perspectives. The main goal is typically to study equilibria of such games, which are usually characterized as solutions of a system of PDEs, called *MFG system*. We refer to [28, 30, 36, 37, 53] for more details and further references on mean field games.

In this chapter, we consider a mean field game model inspired by crowd motion in which a multi-population crowd wishes to arrive at given target sets in minimal time. Motivated by

modeling, control, and optimization objectives, the mathematical analysis of crowd motion is the subject of a very large number of works from diverse perspectives [43, 51, 58–60, 72, 77, 78, 80]. Among other points of view commonly adopted in the literature, the macroscopic modeling of crowds consists in approximating the location of the finitely many agents in the crowd by a continuous distribution, which is usually assumed to evolve according to some conservation law, and is the natural framework for a mean field game model of crowd motion.

Some previous works on mean field games, such as [1, 3, 12, 20, 33, 35, 46, 47, 67, 75, 76], have considered mean field games for, or related to, crowd motion. For instance, [67] proposes a MFG model for a two-population crowd with trajectories perturbed by additive Brownian motion and considers both their stationary distributions and their evolution on a prescribed time interval. Other works also considered multi-population MFGs, such as [1, 35], which study in particular a two-population MFG model motivated by urban settlements. The work [20] considers the fast exit of a crowd, whose agents are perturbed by additive Brownian motion, and proposes a mean field game model, which is studied numerically. Even though [33] is not originally motivated by the modeling of crowd motion, the MFG model studied in that reference presents a density constraint, preventing high concentration of agents, which is a natural assumption in some crowd motion models. We refer to [76] for second-order mean field games with density constraints. Numerical simulations for some variational mean field games related to crowd motion are presented in [12].

The present work is more closely related to [46, 47, 75], which present some particular characteristics with respect to most of the MFG literature. Firstly, contrarily to a large part of the MFG literature but similarly to [20] and some other works with motivation unrelated to crowd motion, such as [56], references [46, 47, 75] consider mean field games in which agents do not necessarily stop all at the same time, but may instead have different stopping times, which are actually the main part of the optimization criterion. Secondly, most of MFG models consider that agents are free to choose their speed, with high speeds penalized in the optimization criterion of each agent, but [46, 47, 75] assume instead that agents may move only up to a certain maximal speed, which depends on the average distribution of agents around their position. As detailed in [75], this assumption is intended to model crowd motion situations in which an agent may not be able to move faster by simply paying some additional cost, since the congestion provoked by other agents may work as a physical barrier for the agent to increase their speed. We refer to [46, 47, 75] for more details on the motivation of the model and its relation to other crowd motion models.

Similarly to [46, 47, 75], the MFG studied in this chapter assumes that agents want to minimize their time to reach a certain target set, their optimal control problem being thus with a free final time, and that their maximal speed is bounded in terms of the density of agents around their position. Several novelties are considered in the MFG from the present chapter. Firstly, we assume that the agents taking part in the game are not all identical, but are instead subdivided in N populations. Each population $i \in \{1, \dots, N\}$ may present different dynamics and different target sets. This additional assumption brings no major difficulty in the analysis of the MFG but allows for the representation of more realistic situations, such as two populations in a corridor starting at opposite sides, each one wanting

to reach the other side in minimal time. We also allow for the interaction of an agent with other agents of the same population to be different than their interaction with agents of other populations, in order to model the fact that it may be easier to move with other agents that want to reach the same target, and hence move in the same general direction, than to move in a crowd of people going on different directions.

Another novelty from the present chapter with respect to [46, 47, 75] is to consider that agents move on \mathbb{R}^d , instead of on a compact subset of \mathbb{R}^d . Lack of compactness of the state space brings additional difficulties in the analysis of the MFG, in particular since we are interested in situations in which the initial distribution of agents is not necessarily compactly supported, but these difficulties can be overcome by exploiting suitable properties of optimal trajectories. In particular, the time for an agent to reach their target set is no longer uniformly bounded, but we are able to provide sharp bounds on the convergence rate of the distribution of agents towards their limit distribution concentrated in the target set. We also remark that, contrarily to [46, 47, 75], the target sets are not assumed to be the boundary of a compact domain, but can be arbitrary nonempty closed subsets of \mathbb{R}^d .

Finally, we also relax the regularity assumptions on the dynamics of agents from [47, 75], requiring only continuity with respect to the distributions of other agents and Lipschitz continuity with respect to the space variable. In those references, similar assumptions were used to prove existence of Lagrangian equilibria, but additional regularity assumptions were required to characterize such equilibria as solutions of a MFG system. These additional assumptions were used in [47, 75] to obtain semiconcavity of the value function of the optimal control problem solved by each agent, which is a key step to obtain differentiability of the value function along optimal trajectories and hence deduce that the velocity field in the continuity equation of the MFG system is well-defined and continuous on the support of the distribution of agents. By not requiring these additional regularity assumptions, the present chapter uses instead different techniques to study the velocity field appearing in the continuity equation, based on a detailed study of some properties of optimal trajectories, which allows us to obtain the MFG system without relying on the semiconcavity of the value function. This is probably one of the main contributions of the present chapter and brings several interesting perspectives, in particular since these techniques might be adapted to other MFG models in which semiconcavity of the value function is known not to hold, such as in some MFGs with state constraints. We also refer the interested reader to [21–23] for other approaches for dealing with MFGs with state constraints.

The notion of MFG equilibrium is formulated in this chapter in a Lagrangian setting, which describes the motion of agents by a measure on the set of all possible trajectories, instead of the more classical approach consisting in describing the evolution of agents through a time-dependent measure on the space state. The Lagrangian approach is classical in optimal transport problems (see, e.g., [6, 13, 18, 34, 85, 87]) and has also recently been used in several works on mean field games [12, 21, 29, 33, 47, 75].

This chapter is organized as follows. Section 2.2 settles the main notations used in the chapter, while Section 2.3 describes the mean field game model considered here together with its associated optimal control problem solved by each agent, and presents the main

tools used in the sequel. Section 2.4 presents the important results on the optimal control problem needed for the sequel of the chapter. The main results on our MFG model are provided in Section 2.5, which proves the existence of an equilibrium, studies its asymptotic behavior at large times, and shows that the distribution of the agents and the value function of the optimal control problem solved by each agent can be characterized by the system of partial differential equations known as MFG system.

2.2 Notation and preliminary definitions

In this chapter, N and d are fixed positive integers. The set of nonnegative real numbers is denoted by \mathbb{R}_+ . We denote the usual Euclidean norm in \mathbb{R}^d by $|\cdot|$ and the unit sphere in \mathbb{R}^d by \mathbb{S}^{d-1} . Given $x \in \mathbb{R}^d$ and $R \geq 0$, we write $B(x, R)$ for the closed ball centered at x and of radius R . When $x = 0$, this ball is denoted simply by B_R . We use $\mathcal{P}(\mathbb{R}^d)$ to denote the set of all Borel probability measures on \mathbb{R}^d , which is assumed to be endowed with the topology of weak convergence of measures.

Given two sets A, B , a set-valued map $F : A \rightrightarrows B$ is a map that, to each $a \in A$, associates a (possibly empty) set $F(a) \subset B$.

Recall that, for two metric spaces X and Y endowed with their Borel σ -algebras and a Borel map $f : X \rightarrow Y$, the pushforward of a measure μ on X through f is the measure $f_{\#}\mu$ on Y defined by

$$f_{\#}\mu(B) = \mu(f^{-1}(B))$$

for every Borel subset B of Y . We extend the pushforward notation componentwise to vectors of measures: if $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N)$ with μ_i a measure on X for every $i \in \{1, \dots, N\}$, then we set $f_{\#}\boldsymbol{\mu} = (f_{\#}\mu_1, \dots, f_{\#}\mu_N)$.

We define, for $p \in [1, +\infty)$, the set

$$\mathcal{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |x|^p d\mu(x) < +\infty \right\}.$$

We endow $\mathcal{P}_p(\mathbb{R}^d)$ with the usual Wasserstein distance \mathbf{W}_p , defined by

$$(2.1) \quad \mathbf{W}_p(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\lambda(x, y) \mid \lambda \in \Pi(\mu, \nu) \right\}^{1/p},$$

where $\Pi(\mu, \nu) = \{ \lambda \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \mid \pi_{1\#}\lambda = \mu, \pi_{2\#}\lambda = \nu \}$ and $\pi_1, \pi_2 : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ denote the canonical projections onto the first and second factors of the product $\mathbb{R}^d \times \mathbb{R}^d$, respectively.

Given two metric spaces X and Y and $M > 0$, $\mathbf{C}(X; Y)$, $\text{Lip}(X; Y)$, and $\text{Lip}_M(X; Y)$ denote, respectively, the set of all continuous functions from X to Y , the set of all Lipschitz continuous functions from X to Y , and the subset of $\text{Lip}(X; Y)$ containing only those functions whose Lipschitz constant is at most M .

For $t \in \mathbb{R}_+$, we denote by $e_t : \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ the evaluation map at time t , defined by $e_t(\gamma) = \gamma(t)$ for every $\gamma \in \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$. We remark that $\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$, endowed with the topology of uniform convergence on compact sets, is a Polish space, which is complete when endowed, for instance, with the metric \mathbf{d} given by

$$(2.2) \quad \mathbf{d}(\gamma_1, \gamma_2) = \sum_{n>0} \frac{1}{2^n} \frac{\sup_{t \in [0, n]} |\gamma_1(t) - \gamma_2(t)|}{1 + \sup_{t \in [0, n]} |\gamma_1(t) - \gamma_2(t)|}$$

for $\gamma_1, \gamma_2 \in \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$. Whenever needed, we assume in the sequel that $\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$ is endowed with this metric.

2.3 The MFG model

For $i \in \{1, \dots, N\}$, let $\Gamma_i \subset \mathbb{R}^d$ be a closed nonempty set, $K_i : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)^{N-1} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, $m_0^i \in \mathcal{P}(\mathbb{R}^d)$, and denote for simplicity $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$, $\mathbf{K} = (K_1, \dots, K_N)$, and $\mathbf{m}_0 = (m_0^1, \dots, m_0^N)$. We consider in this chapter the following mean field game, denoted by $\text{MFG}(\mathbf{\Gamma}, \mathbf{K}, \mathbf{m}_0)$: N populations evolve in the space \mathbb{R}^d and, for $i \in \{1, \dots, N\}$, the distribution of the i -th population at time $t \geq 0$ is described by a probability measure $m_t^i \in \mathcal{P}(\mathbb{R}^d)$. The aim of each agent of population i is to minimize their time to reach their *target set* Γ_i and, in order to model congestion, we assume that the speed of an agent of population i at a position x in time t is bounded by $K_i(m_t^i, \hat{m}_t^i, x)$, where $\hat{m}_t^i \in \mathcal{P}(\mathbb{R}^d)^{N-1}$ describes the distribution of agents in the other populations and is defined by

$$(2.3) \quad \hat{m}_t^i = (m_t^1, \dots, m_t^{i-1}, m_t^{i+1}, \dots, m_t^N).$$

More precisely, we assume that the movement of a representative agent of population i is described by the control system

$$(2.4) \quad \dot{\gamma}(t) = K_i(m_t^i, \hat{m}_t^i, \gamma(t))u(t), \quad u(t) \in B_1,$$

where $\gamma(t) \in \mathbb{R}^d$ is the state of the agent and $u(t)$ is their control at time t , the control being constrained to remain in the closed unit ball B_1 .

In order to properly model congestion through the functions K_1, \dots, K_N , a reasonable assumption is that $K_i(\mu_i, \hat{\mu}_i, x)$ is small when the measures μ_1, \dots, μ_N are large around x , and that larger values of μ_j , $j \neq i$, are more penalized than larger values of μ_i , to model the fact that an agent moving with their own population is less penalized than if this same agent moves in the middle of another population going potentially in another direction. A possible form for each K_i is

$$K_i(\mu_i, \hat{\mu}_i, x) = g \left(\int_{\mathbb{R}^d} \chi(x-y) d\mu_i(y) + \sum_{\substack{j=1 \\ j \neq i}}^N \lambda_j \int_{\mathbb{R}^d} \chi(x-y) d\mu_j(y) \right),$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ is decreasing, $\chi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a smooth convolution kernel, and $\lambda_j > 1$ is a constant for $j \in \{1, \dots, N\} \setminus \{i\}$. Let us point out that we do not assume this specific form of K_i in the sequel but, under suitable regularity assumptions on g and χ , such a K_i satisfies assumptions (H2) and (H3) stated below as well as assumption (H8) from Section 2.5.3 (see, e.g., [75, Proposition 3.1] for a similar result).

The trajectory γ of an agent in population i depends on the distribution of agents of population i and also on that of agents of other populations, since the speed of γ should not exceed $K_i(m_t^i, \hat{m}_t^i, \gamma(t))$. On the other hand, the distributions m_t^i depend on how agents choose their trajectories. We are interested here in *equilibrium* situations, i.e., situations in which, starting from time evolutions of the distributions of agents $m^i : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}^d)$, the trajectories chosen by agents induce evolutions of the initial distribution of agents m_0^i that are precisely given by m_t^i .

To provide a more precise description of MFG($\Gamma, \mathbf{K}, \mathbf{m}_0$), we now introduce an auxiliary optimal control problem. Given $\Gamma \subset \mathbb{R}^d$ nonempty and closed and $k : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$, we consider the optimal control problem OCP(Γ, k) in which an agent evolving in \mathbb{R}^d wants to reach Γ in minimal time, their speed at position x and time t being bounded by $k(t, x)$. For this optimal control problem, k does not depend on the density of the agents and is considered as a given function. The relation between the optimal control problem OCP(Γ, k) and the mean field game MFG($\Gamma, \mathbf{K}, \mathbf{m}_0$) is that, for every population $i \in \{1, \dots, N\}$, an agent of population i solves OCP(Γ_i, k_i), where k_i is defined by $k_i(t, x) = K_i(m_t^i, \hat{m}_t^i, x)$ for $t \geq 0$ and $x \in \mathbb{R}^d$.

Definition 2.3.1. Let $\Gamma \subset \mathbb{R}^d$ be nonempty and closed and $k : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$.

- (a) A curve $\gamma \in \text{Lip}(\mathbb{R}_+; \mathbb{R}^d)$ is said to be *admissible* for OCP(Γ, k) if it satisfies $|\dot{\gamma}(t)| \leq k(t, \gamma(t))$ for almost every $t \in \mathbb{R}_+$. The set of all admissible curves is denoted by $\text{Adm}(k)$.
- (b) Let $t_0 \in \mathbb{R}_+$. The *first exit time* after t_0 of a curve $\gamma \in \text{Lip}(\mathbb{R}_+; \mathbb{R}^d)$ is the number $\tau_\Gamma(t_0, \gamma) \in [0, +\infty]$ defined by

$$\tau_\Gamma(t_0, \gamma) = \inf\{t \geq 0 \mid \gamma(t + t_0) \in \Gamma\}.$$

- (c) Let $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^d$. A curve $\gamma \in \text{Lip}(\mathbb{R}_+; \mathbb{R}^d)$ is said to be an *optimal trajectory* for (Γ, k, t_0, x_0) if $\gamma \in \text{Adm}(k)$, $\gamma(t) = x_0$ for every $t \in [0, t_0]$, $\tau_\Gamma(t_0, \gamma) < +\infty$, $\gamma(t) = \gamma(t_0 + \tau_\Gamma(t_0, \gamma)) \in \Gamma$ for every $t \in [t_0 + \tau_\Gamma(t_0, \gamma), +\infty)$, and

$$(2.5) \quad \tau_\Gamma(t_0, \gamma) = \inf_{\substack{\beta \in \text{Adm}(k) \\ \beta(t_0) = x_0}} \tau_\Gamma(t_0, \beta).$$

The set of all optimal trajectories for (Γ, k, t_0, x_0) is denoted by $\text{Opt}(\Gamma, k, t_0, x_0)$.

Note that admissible curves γ for OCP(Γ, k) are trajectories of the control system

$$(2.6) \quad \dot{\gamma}(t) = k(t, \gamma(t))u(t),$$

where the measurable function $u : \mathbb{R}_+ \rightarrow B_1$ is the control associated with γ . The control system (2.6) is *nonautonomous*, since k explicitly depends on t .

We now provide the definition of Lagrangian equilibrium (which we refer to simply as *equilibrium* in this chapter for simplicity) of $\text{MFG}(\mathbf{\Gamma}, \mathbf{K}, \mathbf{m}_0)$.

Definition 2.3.2. Let $\mathbf{m}_0 = (m_0^1, \dots, m_0^N) \in \mathcal{P}(\mathbb{R}^d)^N$, $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$, and $\mathbf{K} = (K_1, \dots, K_N)$ with $\Gamma_i \subset \mathbb{R}^d$ nonempty and closed and $K_i : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)^{N-1} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ for every $i \in \{1, \dots, N\}$. A vector of measures $\mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ is called a (*Lagrangian*) *equilibrium* for $\text{MFG}(\mathbf{\Gamma}, \mathbf{K}, \mathbf{m}_0)$ if $e_{0\#}\mathbf{Q} = \mathbf{m}_0$ and, for every $i \in \{1, \dots, N\}$, Q_i -almost every γ is optimal for $(\Gamma_i, k_i, 0, \gamma(0))$, where $k_i : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is defined for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ by $k_i(t, x) = K_i(m_t^i, \hat{m}_t^i, x)$, $m_t^i = e_{t\#}Q_i$, and \hat{m}_t^i is given by (2.3).

Let us now state the base assumptions on the data of $\text{MFG}(\mathbf{\Gamma}, \mathbf{K}, \mathbf{m}_0)$ and $\text{OCP}(\mathbf{\Gamma}, k)$ used throughout this chapter. Concerning $\text{MFG}(\mathbf{\Gamma}, \mathbf{K}, \mathbf{m}_0)$, we shall always assume the following hypotheses to be satisfied.

- (H1) For $i \in \{1, \dots, N\}$, Γ_i is a nonempty closed subset of \mathbb{R}^d .
- (H2) There exist positive constants K_{\min}, K_{\max} such that, for every $i \in \{1, \dots, N\}$, $K_i : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)^{N-1} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is continuous and $K_i(\mu, \nu, x) \in [K_{\min}, K_{\max}]$ for every $(\mu, \nu, x) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)^{N-1} \times \mathbb{R}^d$.
- (H3) The functions K_i are Lipschitz continuous with respect to their third variable, uniformly with respect to the first two variables, i.e., there exists $L > 0$ such that, for every $i \in \{1, \dots, N\}$, $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\nu \in \mathcal{P}(\mathbb{R}^d)^{N-1}$, and $x_1, x_2 \in \mathbb{R}^d$, we have

$$|K_i(\mu, \nu, x_1) - K_i(\mu, \nu, x_2)| \leq L|x_1 - x_2|.$$

As for $\text{OCP}(\mathbf{\Gamma}, k)$, we always assume the following hypotheses to be satisfied.

- (H4) The set Γ is a nonempty closed subset of \mathbb{R}^d .
- (H5) There exist positive constants K_{\min}, K_{\max} such that $k : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is continuous and $k(t, x) \in [K_{\min}, K_{\max}]$ for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.
- (H6) The function k is locally Lipschitz continuous with respect to its second variable, uniformly with respect to the first variable, i.e., for every $R > 0$, there exists $L > 0$ such that, for every $t \in \mathbb{R}_+$ and $x_1, x_2 \in B_R$, we have

$$|k(t, x_1) - k(t, x_2)| \leq L|x_1 - x_2|.$$

In the sequel of the chapter, we always use the following notation.

Notation 2.3.3. Given $m_0^1, \dots, m_0^N \in \mathcal{P}(\mathbb{R}^d)$, we denote by $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ the function defined for $R \geq 0$ by $\phi(R) = \min_{i \in \{1, \dots, N\}} m_0^i(B_R)$.

Notice that ϕ is nondecreasing and satisfies $\lim_{R \rightarrow +\infty} \phi(R) = 1$ and $m_0^i(B_R) \geq \phi(R)$ for every $i \in \{1, \dots, N\}$ and $R \geq 0$.

Remark 2.3.4. Even though this chapter considers multi-population mean field games, our techniques also apply to single-population mean field games, in which the function K_i in (2.4) is replaced by a function K depending on the distribution m_t of the single population at time t and on the position $\gamma(t)$ of an agent. We chose to consider the multi-population setting due to the fact that it is closer to applications, since, in most crowd motion situations in practice, different parts of crowd may wish to reach different target sets, such as people taking different exits in a metro station. In addition, the multi-population setting considered here does not bring much additional difficulty when compared, for instance, with single-population minimal-time mean field games treated in [47, 75].

2.4 Preliminary results on the optimal control problem

In this section, we collect the main properties of the optimal control problem $\text{OCP}(\Gamma, k)$ that will be of use in the sequel of the chapter. Note that $\text{OCP}(\Gamma, k)$ is a minimal-time optimal control problem, which is a classic subject in the optimal control literature (see, e.g., [11, 27, 40, 79]), but the assumptions (H4)–(H6) on $\text{OCP}(\Gamma, k)$ allow for less smooth Γ and k than those typically considered in the literature. Minimal-time optimal control problems have also been studied in connection with mean field games, for instance in [47, 75], the main difference with respect to the present chapter being that those references consider optimal control problems in a compact state space, whereas the state space in the present chapter is \mathbb{R}^d .

The first property of $\text{OCP}(\Gamma, k)$ that we consider is the existence of optimal trajectories, stated in the proposition below. Its proof can be carried out by standard techniques based on minimizing sequences and using the relative compactness of bounded subsets of $\text{Lip}_{K_{\max}}(\mathbb{R}_+; \mathbb{R}^d)$ in the topology of $\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$ and is omitted here for simplicity (see, e.g., [27, Theorem 8.1.4] for a similar proof in the case of a more general optimal exit time problem for an autonomous control system).

Proposition 2.4.1. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and assume that (H4) and (H5) are satisfied. Then, for every $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^d$, there exists an optimal trajectory γ for (Γ, k, t_0, x_0) .*

Another property of $\text{OCP}(\Gamma, k)$ that can be obtained by a straightforward argument is the following, which states that restrictions of optimal trajectories are still optimal.

Proposition 2.4.2. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma_0 \in \text{Opt}(\Gamma, k, t_0, x_0)$. Then, for every $t_1 \in [t_0, +\infty)$, denoting $x_1 = \gamma_0(t_1)$, the function $\gamma_1 : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ defined by $\gamma_1(t) = x_1$ for $t \leq t_1$ and $\gamma_1(t) = \gamma_0(t)$ for $t \geq t_1$ satisfies $\gamma_1 \in \text{Opt}(\Gamma, k, t_1, x_1)$.*

2.4.1 The value function

We consider in this section properties of the value function corresponding to the optimal control problem $\text{OCP}(\Gamma, k)$, whose definition is given next.

Definition 2.4.3. Let $\Gamma \subset \mathbb{R}^d$ be a nonempty closed set and $k : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$. The *value function* of the optimal control problem $\text{OCP}(\Gamma, k)$ is the function $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ defined for $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ by

$$(2.7) \quad \varphi(t_0, x_0) = \inf_{\substack{\gamma \in \text{Adm}(k) \\ \gamma(t_0) = x_0}} \tau_\Gamma(t_0, \gamma).$$

Our next preliminary result provides local bounds on the value function and on the norm of optimal trajectories.

Proposition 2.4.4. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4) and (H5) are satisfied. Then there exist two nondecreasing maps with linear growth $\psi, T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ depending only on Γ, K_{\min} , and K_{\max} such that, for every $R > 0, t_0 \in \mathbb{R}_+, x_0 \in B_R$, we have $\varphi(t_0, x_0) \leq T(R)$ and, for every $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$, we have $\gamma(t) \in B_{\psi(R)}$ for every $t \geq 0$.*

The bound $T(R)$ on the value function can be obtained, for instance, by remarking that a particular admissible trajectory is the one that moves with speed K_{\min} along the segment from x_0 to 0 and then along the segment from 0 to the closest point of Γ from 0. Since any optimal trajectory γ is K_{\max} -Lipschitz and arrives at the target set in time at most $T(R)$, one can easily bound $|\gamma(t)|$ by $|x_0| + K_{\max}T(R)$, yielding the bound on optimal trajectories.

In the next result we recall the dynamic programming principle, which can be proved by standard techniques in optimal control (see, e.g., [11, Proposition 2.1] and [27, (8.4)] for the corresponding result in the autonomous case).

Proposition 2.4.5. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4) and (H5) are satisfied. Then, for every $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \in \text{Adm}(k)$ with $\gamma(t_0) = x_0$, we have*

$$(2.8) \quad \varphi(t_0 + h, \gamma(t_0 + h)) + h \geq \varphi(t_0, x_0), \quad \text{for every } h \geq 0,$$

with equality for every $h \in [0, \tau_\Gamma(t_0, \gamma)]$ if $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$. Moreover, if γ is constant on $[0, t_0]$ and on $[t_0 + \tau_\Gamma(t_0, \gamma), +\infty)$ and if equality holds in (2.8) for every $h \in [0, \tau_\Gamma(t_0, \gamma)]$, then $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$.

Our next preliminary result on $\text{OCP}(\Gamma, k)$ deals with the Lipschitz continuity of the value function. Lipschitz continuity of the value function is a classical result in optimal exit time problems (see, e.g., [27, Theorem 8.2.5]), but most of the literature deals only with autonomous control systems, in which case the value function is a function of the space variable x only. A classical state augmentation technique of (2.6) would be sufficient to obtain

Lipschitz continuity of φ on both time and space, but this would require the assumption that k is locally Lipschitz continuous in the pair (t, x) , which is stronger than (H6). In order to highlight the fact that such an assumption is not necessary, we provide below a detailed proof of the Lipschitz continuity of φ , based on that of [27, Theorem 8.2.5] but containing some simplifications due to the particular structure of the problem at hand. We start with a preliminary result stating Lipschitz continuity of $x \mapsto \varphi(t, x)$ for fixed $t \in \mathbb{R}_+$.

Lemma 2.4.6. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4)–(H6) are satisfied. Then, for every $R > 0$, there exists $C_R > 0$ such that, for every $t_0 \in \mathbb{R}_+$ and $x_0, x_1 \in B_R$, we have*

$$|\varphi(t_0, x_0) - \varphi(t_0, x_1)| \leq C_R |x_0 - x_1|.$$

Proof. Let $T : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be as in the statement of Proposition 2.4.4, $R > 0$, $t_0 \in \mathbb{R}_+$, and $x_0, x_1 \in B_R$. Let $\gamma_0 \in \text{Opt}(\Gamma, k, t_0, x_0)$ and denote by u_0 the corresponding optimal control, i.e., $\dot{\gamma}_0(t) = k(t, \gamma_0(t))u_0(t)$ for a.e. $t \in \mathbb{R}_+$. Let $t_0^* = t_0 + \varphi(t_0, x_0)$ be the time at which γ_0 arrives at the target set Γ and $x_0^* = \gamma_0(t_0^*) \in \Gamma$ be the arrival position of γ_0 at Γ . We define $\gamma_1 : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ as follows: for $t \in [0, t_0]$, we set $\gamma_1(t) = x_1$; for $t \in [t_0, t_0^*]$, γ_1 is the unique solution of the differential equation $\dot{\gamma}_1(t) = k(t, \gamma_1(t))u_0(t)$ with initial condition $\gamma_1(t_0) = x_1$; for $t \in (t_0^*, t_1^*]$, we set $\gamma_1(t) = \left(1 - \frac{t-t_0^*}{t_1^*-t_0^*}\right) x_1^* + \frac{t-t_0^*}{t_1^*-t_0^*} x_0^*$, where $x_1^* = \gamma_1(t_0^*)$ and $t_1^* = t_0^* + \frac{|x_1^*-x_0^*|}{K_{\min}}$; and, for $t > t_1^*$, we set $\gamma_1(t) = \gamma_1(t_1^*) = x_0^* \in \Gamma$. In other words, γ_1 remains at x_1 until time t_0 , then it is defined as the solution of the control system (2.6) with control u_0 until time t_0^* , and finally γ_1 moves from its position x_1^* at time t_0^* to the final position x_0^* of γ_0 along the segment connecting these two points and with constant speed K_{\min} , remaining at x_0^* afterward. By construction, we have $\gamma_1 \in \text{Adm}(k)$ and $\tau_\Gamma(t_0, \gamma_1) \leq t_1^* - t_0 = \varphi(t_0, x_0) + \frac{|x_1^*-x_0^*|}{K_{\min}}$, and hence

$$(2.9) \quad \varphi(t_0, x_1) \leq \varphi(t_0, x_0) + \frac{|x_1^* - x_0^*|}{K_{\min}}.$$

Let us estimate $|x_1^* - x_0^*|$. Notice first that, since γ_0 and γ_1 are K_{\max} -Lipschitz, we have, for every $t \in [t_0, t_0^*]$ and $i \in \{0, 1\}$,

$$|\gamma_i(t)| \leq |x_i| + K_{\max}(t_0^* - t_0) \leq R + K_{\max}T(R).$$

Let $L > 0$ be the Lipschitz constant of k with respect to its second variable on $\mathbb{R}_+ \times B_{R+K_{\max}T(R)}$. We then have, for every $t \in [t_0, t_0^*]$,

$$\gamma_1(t) - \gamma_0(t) = x_1 - x_0 + \int_{t_0}^t [k(s, \gamma_1(s)) - k(s, \gamma_0(s))] u_0(s) ds,$$

and thus

$$|\gamma_1(t) - \gamma_0(t)| \leq |x_1 - x_0| + L \int_{t_0}^t |\gamma_1(s) - \gamma_0(s)| ds.$$

Hence, by Grönwall's inequality, we deduce that

$$|x_1^* - x_0^*| \leq e^{LT(R)} |x_1 - x_0|.$$

Combining with (2.9), we obtain that

$$\varphi(t_0, x_1) \leq \varphi(t_0, x_0) + \frac{e^{LT(R)}}{K_{\min}} |x_1 - x_0|.$$

The conclusion follows with $C_R = \frac{e^{LT(R)}}{K_{\min}}$ by exchanging the role of x_0 and x_1 in the above argument. \square

We can now deduce Lipschitz continuity of φ by using Lemma 2.4.6 and the dynamic programming principle from Proposition 2.4.5.

Proposition 2.4.7. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4)–(H6) are satisfied. Then, for every $R > 0$, there exists $M_R > 0$ such that, for every $(t_0, x_0), (t_1, x_1) \in \mathbb{R}_+ \times B_R$, we have*

$$|\varphi(t_0, x_0) - \varphi(t_1, x_1)| \leq M_R (|t_0 - t_1| + |x_0 - x_1|).$$

Proof. Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be as in the statement of Proposition 2.4.4, $R > 0$, and $(t_0, x_0), (t_1, x_1) \in \mathbb{R}_+ \times B_R$ and assume, with no loss of generality, that $t_0 < t_1$. Let $\gamma_0 \in \text{Opt}(\Gamma, k, t_0, x_0)$ and $x_0^* = \gamma_0(t_1)$. By Proposition 2.4.4, we have $|x_0^*| \leq \psi(R)$ and, by Lemma 2.4.6, we have

$$(2.10) \quad |\varphi(t_1, x_0^*) - \varphi(t_1, x_1)| \leq C_{\psi(R)} |x_0^* - x_1|,$$

where $C_{\psi(R)}$ denotes the Lipschitz constant of $x \mapsto \varphi(t, x)$ on $B_{\psi(R)}$ for all $t \geq 0$.

If $t_1 \leq t_0 + \varphi(t_0, x_0)$, then, by Proposition 2.4.5, since $\gamma_0 \in \text{Opt}(\Gamma, k, t_0, x_0)$, we have $\varphi(t_1, x_0^*) = \varphi(t_0, x_0) - (t_1 - t_0)$, and thus

$$(2.11) \quad |\varphi(t_0, x_0) - \varphi(t_1, x_1)| \leq |t_1 - t_0| + C_{\psi(R)} |x_0^* - x_1|.$$

Otherwise, we have $t_1 > t_0 + \varphi(t_0, x_0)$, in which case $x_0^* = \gamma_0(t_1) = \gamma_0(t_0 + \varphi(t_0, x_0)) \in \Gamma$ and thus $\varphi(t_1, x_0^*) = 0$. Combining this with (2.10) and the fact that $\varphi(t_0, x_0) < t_1 - t_0$, we deduce that (2.11) also holds in this case.

Since γ_0 is K_{\max} -Lipschitz, we have $|x_0 - x_0^*| \leq K_{\max} |t_1 - t_0|$. Hence, combining with (2.11), we deduce that

$$|\varphi(t_0, x_0) - \varphi(t_1, x_1)| \leq (C_{\psi(R)} K_{\max} + 1) |t_1 - t_0| + C_{\psi(R)} |x_0 - x_1|,$$

yielding the conclusion. \square

A classical consequence of the dynamic programming principle is that the value function φ satisfies a Hamilton–Jacobi equation in the viscosity sense, which is the topic of the next proposition, whose proof is omitted here since it can be obtained by adapting classical arguments (see, e.g., [11, Chapter IV, Proposition 2.3] and [27, Theorem 8.1.8]) to our non-autonomous setting.

Proposition 2.4.8. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4)–(H6) are satisfied. Consider the Hamilton–Jacobi equation*

$$(2.12) \quad -\partial_t \varphi(t, x) + k(t, x)|\nabla \varphi(t, x)| - 1 = 0.$$

Then φ is a viscosity solution of (2.12) on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \Gamma)$ and satisfies $\varphi(t, x) = 0$ for $(t, x) \in \mathbb{R}_+ \times \Gamma$.

We next provide the following property of φ , whose proof can be found in [47, Proposition 3.9 and Corollary 3.11].

Proposition 2.4.9. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4)–(H6) are satisfied. Then, for every $R > 0$, there exists $c > 0$ such that, for every $t_0, t_1 \in \mathbb{R}_+$ with $t_0 \neq t_1$ and $x \in B_R$, we have*

$$\frac{\varphi(t_1, x) - \varphi(t_0, x)}{t_1 - t_0} \geq c - 1.$$

In particular, if φ is differentiable at (t_0, x) , then $\partial_t \varphi(t_0, x) \geq c - 1$ and $|\nabla \varphi(t_0, x)| \geq \frac{c}{K_{\max}}$.

2.4.2 Characterization of optimal controls

Now that we have established elementary properties of the value function in Section 2.4.1, we turn to the problem of characterizing the optimal control $u : \mathbb{R}_+ \rightarrow B_1$ associated with an optimal trajectory $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$. Formally, by differentiating with respect to h the equality of the dynamic programming principle in Proposition 2.4.5 for optimal trajectories and using the Hamilton–Jacobi equation (2.12), one obtains that the optimal control u should satisfy $u(t) = -\frac{\nabla \varphi(t, \gamma(t))}{|\nabla \varphi(t, \gamma(t))|}$, an argument that can be made precise when φ is differentiable at $(t, \gamma(t))$ (see, e.g., [75, Corollary 4.1]).

If φ was semiconcave, one could deduce by standard arguments (see, e.g., [27, Section 7.3] and [47, Section 3.4]) that it is differentiable along optimal trajectories and hence obtain the above characterization of optimal controls. In particular (see, e.g., [27, Theorem 7.3.16]), φ can be shown to be semiconcave under the additional assumption that $k \in \mathbf{C}^{1,1}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R})$ (i.e., k is \mathbf{C}^1 and its differential is locally Lipschitz continuous). On the other hand, under our standing assumptions (H4)–(H6), neither semiconcavity nor differentiability of φ along optimal trajectories are guaranteed, and, up to the authors’ knowledge, it is an open question if these properties hold or not.

The goal of this section is to provide an alternative characterization of u when k is not necessarily more regular than locally Lipschitz continuous. This is done mainly for two reasons. Firstly, regularity assumptions on k for $\text{OCP}(\Gamma, k)$ correspond to regularity assumptions on K_i , $i \in \{1, \dots, N\}$, for $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, and hence avoiding additional regularity assumptions on k allow to obtain more general results for mean field games. Secondly, even when k is smooth, the value function φ may fail to be semiconcave in some situations, such as in the presence of state constraints (see, e.g., [24, Example 4.4]), and

semiconcavity of φ is a key step in proving its differentiability along optimal trajectories and hence in characterizing u as above. This motivates the search for techniques for characterizing optimal controls without relying on the semiconcavity of φ .

We shall need in this section the following additional assumption on k .

(H7) The function $k : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is Lipschitz continuous with respect to both variables and locally in the second variable, i.e., for every $R > 0$, there exists $L > 0$ such that, for every $(t_1, x_1), (t_2, x_2) \in \mathbb{R}_+ \times B_R$, we have

$$|k(t_1, x_1) - k(t_2, x_2)| \leq L(|t_1 - t_2| + |x_1 - x_2|).$$

The first result we present is the following, which provides additional regularity assumptions on the optimal control u . It can be obtained by applying Pontryagin Maximum Principle to $\text{OCP}(\Gamma, k)$ and using the maximization condition to deduce a relation between the optimal control u and the costate variable in Pontryagin Maximum Principle. We refer the reader to [75, Proposition 4.6 and Corollary 4.2] for the details of the proof.

Proposition 2.4.10. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and assume that (H4), (H5), and (H7) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$, and u be the optimal control corresponding to γ . Then $u \in \text{Lip}([t_0, t_0 + \varphi(t_0, x_0)]; \mathbb{S}^{d-1})$. Moreover, its Lipschitz constant is bounded by the Lipschitz constant of k on the set $[t_0, t_0 + \varphi(t_0, x_0)] \times B_R$, where $R > 0$ is such that $\gamma(t) \in B_R$ for every $t \geq 0$.*

We now introduce the two main objects that we will use to characterize optimal controls.

Definition 2.4.11. Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4), (H5), and (H7) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$.

1. We define the set $\mathcal{U}(t_0, x_0)$ of *optimal directions* at (t_0, x_0) as the set of all $u_0 \in \mathbb{S}^{d-1}$ for which there exists $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$ such that the corresponding optimal control u satisfies $u(t_0) = u_0$.
2. We define the set $\mathcal{W}(t_0, x_0)$ of *directions of maximal descent* of φ at (t_0, x_0) as the set of all $u_0 \in \mathbb{S}^{d-1}$ such that

$$(2.13) \quad \lim_{h \rightarrow 0^+} \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} = -1.$$

Thanks to Proposition 2.4.10, optimal controls are continuous and take values in \mathbb{S}^{d-1} , and in particular the pointwise value $u(t_0)$ is well-defined. Together with Proposition 2.4.1, we immediately deduce that $\mathcal{U}(t_0, x_0) \neq \emptyset$ for every $(t_0, x_0) \in \mathbb{R}_+ \times (\mathbb{R}^d \setminus \Gamma)$. On the other hand, for $(t_0, x_0) \in \mathbb{R}^d \times \Gamma$, one observes that $\mathcal{U}(t_0, x_0) = \emptyset$, since, when $x_0 \in \Gamma$, the only optimal control is the control constantly equal to 0, but, by definition, the members of $\mathcal{U}(t_0, x_0)$ must belong to the unit sphere \mathbb{S}^{d-1} .

Note also that, if $u_0 \in \mathbb{S}^{d-1}$ and $\gamma \in \text{Adm}(k)$ is the trajectory obtained by taking a constant control $u(t) = u_0$ in (2.6), then, by Proposition 2.4.5, $\varphi(t_0+h, \gamma(t_0+h)) - \varphi(t_0, x_0) \geq -h$, yielding, using also Proposition 2.4.7, that, as $h \rightarrow 0^+$,

$$\frac{\varphi(t_0+h, x_0+hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \geq -1 + o(1).$$

Hence, an element $u_0 \in \mathcal{W}(t_0, x_0)$ can be interpreted as a direction in which the above ratio attains its infinitesimal lower bound -1 at the limit $h \rightarrow 0^+$, and corresponds thus to directions in which φ decreases with maximal rate.

Before turning to the main result of this section, Theorem 2.4.14, asserting the equality between $\mathcal{U}(t_0, x_0)$ and $\mathcal{W}(t_0, x_0)$, let us first present some elementary properties of these set-valued maps. The first one is that, along an optimal trajectory γ , $\mathcal{U}(t, \gamma(t))$ is singleton, except possibly at its initial and final points. Its proof is the same as that of [75, Proposition 4.7] and is thus omitted here.

Proposition 2.4.12. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4), (H5), and (H7) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$. Then, for every $t \in (t_0, t_0 + \varphi(t_0, x_0))$, $\mathcal{U}(t, \gamma(t))$ contains exactly one element.*

Our next result shows, on the other hand, that, at the points (t_0, x_0) where φ is differentiable, $\mathcal{W}(t_0, x_0)$ contains a unique direction of maximal descent which, as one might expect, is equal to $-\frac{\nabla\varphi(t_0, x_0)}{|\nabla\varphi(t_0, x_0)|}$, as $|\nabla\varphi(t_0, x_0)| \neq 0$ is guaranteed by Proposition 2.4.9.

Proposition 2.4.13. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4), (H5), and (H7) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times (\mathbb{R}^d \setminus \Gamma)$ be such that φ is differentiable at (t_0, x_0) . Then*

$$\mathcal{W}(t_0, x_0) = \left\{ -\frac{\nabla\varphi(t_0, x_0)}{|\nabla\varphi(t_0, x_0)|} \right\}.$$

Proof. Since φ is differentiable at (t_0, x_0) and using Proposition 2.4.8, we have, for every $u_0 \in \mathbb{S}^{d-1}$,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{\varphi(t_0+h, x_0+hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \\ = \partial_t \varphi(t_0, x_0) + k(t_0, x_0) \nabla \varphi(t_0, x_0) \cdot u_0 = -1 + k(t_0, x_0) [\nabla \varphi(t_0, x_0) \cdot u_0 + |\nabla \varphi(t_0, x_0)|]. \end{aligned}$$

Hence (2.13) holds if and only if $\nabla\varphi(t_0, x_0) \cdot u_0 = -|\nabla\varphi(t_0, x_0)|$ and, since $\nabla\varphi(t_0, x_0) \neq 0$ by Proposition 2.4.9, it follows that (2.13) holds if and only if $u_0 = -\frac{\nabla\varphi(t_0, x_0)}{|\nabla\varphi(t_0, x_0)|}$, yielding the conclusion. \square

The main result of this section is the following.

Theorem 2.4.14. *Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ and assume that (H4), (H5), and (H7) hold. Then, for every $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have $\mathcal{U}(t_0, x_0) = \mathcal{W}(t_0, x_0)$.*

Proof. We first remark that, if $x_0 \in \Gamma$, then $\mathcal{U}(t_0, x_0) = \mathcal{W}(t_0, x_0) = \emptyset$, and so we are only left to consider the case $x_0 \in \mathbb{R}^d \setminus \Gamma$.

The inclusion $\mathcal{U}(t_0, x_0) \subset \mathcal{W}(t_0, x_0)$ follows from the fact that, if $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$ and u is the corresponding optimal control, then, by Proposition 2.4.5, we have, for every $h \in (0, \varphi(t_0, x_0)]$, that

$$\frac{\varphi(t_0 + h, \gamma(t_0 + h)) - \varphi(t_0, x_0)}{h} = -1$$

and, using the facts that $\gamma(t_0 + h) = x_0 + hk(t_0, x_0)u(t_0) + o(h)$ and that φ is locally Lipschitz continuous (Proposition 2.4.7), we deduce, letting $h \rightarrow 0^+$, that $u(t_0) \in \mathcal{W}(t_0, x_0)$.

Let us now show that $\mathcal{W}(t_0, x_0) \subset \mathcal{U}(t_0, x_0)$. Let $u_0 \in \mathcal{W}(t_0, x_0)$ and $h > 0$, which is implicitly always assumed to be small enough. Then, as $h \rightarrow 0^+$,

$$(2.14) \quad \varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0) = \varphi(t_0, x_0) - h + o(h).$$

Define $\gamma_0 : [t_0, t_0 + h] \rightarrow \mathbb{R}^d$ by

$$(2.15) \quad \begin{cases} \dot{\gamma}_0(t) = k(t, \gamma_0(t))u_0, \\ \gamma_0(t_0) = x_0. \end{cases}$$

Let $x_1^h = \gamma_0(t_0 + h)$ and $t_1^h = t_0 + h$. Since $\mathbb{R}^d \setminus \Gamma$ is open, one has $x_1^h \in \mathbb{R}^d \setminus \Gamma$ for $h > 0$ small enough. Let $\gamma_1^h \in \text{Opt}(\Gamma, k, t_1^h, x_1^h)$ and u_1^h be the optimal control associated with γ_1^h . Set $\bar{u}_1^h = u_1^h(t_1^h) \in \mathbb{S}^{d-1}$ and define $\bar{\gamma}_1^h : [t_1^h, t_1^h + h] \rightarrow \mathbb{R}^d$ by

$$(2.16) \quad \begin{cases} \dot{\bar{\gamma}}_1^h(t) = k(t, \bar{\gamma}_1^h(t))\bar{u}_1^h \\ \bar{\gamma}_1^h(t_1^h) = x_1^h. \end{cases}$$

Let us also set $t_2^h = t_1^h + h$, $x_2^h = \gamma_1^h(t_2^h)$ and $\bar{x}_2^h = \bar{\gamma}_1^h(t_2^h)$. We split the sequel of the proof in two cases.

Case 1. We assume in this case that $\lim_{h \rightarrow 0^+} \bar{u}_1^h = u_0$. Let $\hat{u}_1^h \in \text{Lip}(\mathbb{R}_+; \mathbb{S}^{d-1})$ be defined by $\hat{u}_1^h(t) = \bar{u}_1^h$ for $t \in [0, t_1^h]$, $\hat{u}_1^h(t) = u_1^h(t)$ for $t \in [t_1^h, t_1^h + \varphi(t_1^h, x_1^h)]$, and $\hat{u}_1^h(t) = u_1^h(t_1^h + \varphi(t_1^h, x_1^h))$ for $t \geq t_1^h + \varphi(t_1^h, x_1^h)$. Since γ_1^h and \hat{u}_1^h are Lipschitz continuous and their Lipschitz constants do not depend on h (see Proposition 2.4.10), one deduces from Arzelà–Ascoli Theorem that there exist a positive sequence $(h_n)_{n \in \mathbb{N}}$ converging to 0 as $n \rightarrow +\infty$ and elements $\gamma^* \in \text{Lip}_{K_{\max}}(\mathbb{R}_+; \mathbb{R}^d)$ and $u^* \in \text{Lip}(\mathbb{R}_+; \mathbb{S}^{d-1})$ such that $\gamma_1^{h_n} \rightarrow \gamma^*$ and $\hat{u}_1^{h_n} \rightarrow u^*$ as $n \rightarrow +\infty$, uniformly on compact time intervals. Since $\gamma_1^h \in \text{Opt}(\Gamma, k, t_1^h, x_1^h)$ for $h > 0$ and $t_1^h \rightarrow t_0$ and $x_1^h \rightarrow x_0$ as $h \rightarrow 0^+$, one can easily show, using the continuity of φ , that $\gamma^* \in \text{Opt}(\Gamma, k, t_0, x_0)$ and the restriction of u^* to $[t_0, t_0 + \varphi(t_0, x_0)]$ is its corresponding optimal control. On the other hand, we have

$$u^*(t_0) = \lim_{n \rightarrow +\infty} \hat{u}_1^{h_n}(t_1^{h_n}) = \lim_{n \rightarrow +\infty} \bar{u}_1^{h_n} = u_0,$$

which implies that $u_0 \in \mathcal{U}(t_0, x_0)$, as required.

Case 2. We now consider the case where $(\bar{u}_1^h)_{h>0}$ does not converge to u_0 as $h \rightarrow 0^+$, and we prove that this case is not possible. Let $\epsilon > 0$ and $(h_n)_{n \in \mathbb{N}}$ be a positive sequence such that $h_n \rightarrow 0$ as $n \rightarrow +\infty$ and $|\bar{u}_1^{h_n} - u_0| \geq \epsilon$ for every $n \in \mathbb{N}$. For simplicity, we set $t_1^{h_n} = t_1^n$, $x_1^{h_n} = x_1^n$, and similarly for all other variables whose upper index is h_n . In order to clarify the constructions used in this case, we illustrate them in Figure 2.1, which represents points and curves already constructed as well as those which will be defined in the sequel of the proof.

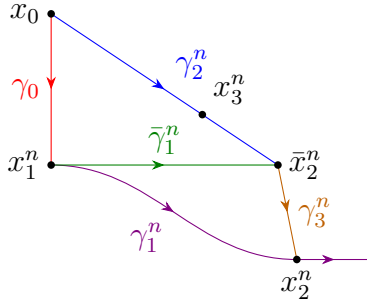


Figure 2.1: Illustration of the constructions used in the proof of Theorem 2.4.14.

Integrating (2.15) on $[t_0, t_1^n]$, we get

$$x_1^n - x_0 = \int_{t_0}^{t_1^n} k(s, \gamma_0(s)) ds u_0,$$

and, proceeding similarly for (2.16), we get

$$\bar{x}_2^n - x_1^n = \int_{t_1^n}^{t_2^n} k(s, \bar{\gamma}_1^n(s)) ds \bar{u}_1^n.$$

Denote the integrals in the right-hand side of the above equalities by I_0^n and I_1^n , respectively. We have

$$\begin{aligned} |\bar{x}_2^n - x_0|^2 &= (I_0^n u_0 + I_1^n \bar{u}_1^n) \cdot (I_0^n u_0 + I_1^n \bar{u}_1^n) \\ &= (I_0^n)^2 + (I_1^n)^2 + 2I_0^n I_1^n u_0 \cdot \bar{u}_1^n \\ &= |x_1^n - x_0|^2 + |\bar{x}_2^n - x_1^n|^2 + 2I_0^n I_1^n u_0 \cdot \bar{u}_1^n. \end{aligned}$$

We know that $|\bar{u}_1^n - u_0| \geq \epsilon$, which leads us to observe that there exists $\alpha \in (0, 1)$ such that $u_0 \cdot \bar{u}_1^n < \alpha$ for every $n \in \mathbb{N}$. Thus

$$|\bar{x}_2^n - x_0|^2 < |x_1^n - x_0|^2 + |\bar{x}_2^n - x_1^n|^2 + 2\alpha I_0^n I_1^n.$$

Define

$$\rho := \sqrt{1 - (1 - \alpha) \frac{K_{\min}^2}{2K_{\max}^2}},$$

then obviously $\rho < 1$ and

$$\begin{aligned}
|\bar{x}_2^n - x_0|^2 &< (|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|)^2 - 2(1 - \alpha)I_0^n I_1^n \\
&= \left(1 - (1 - \alpha) \frac{2I_0^n I_1^n}{(I_0^n + I_1^n)^2}\right) (|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|)^2 \\
(2.17) \quad &\leq \rho^2 (|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|)^2,
\end{aligned}$$

where we use that $I_i^n \in [hK_{\min}, hK_{\max}]$ for $i \in \{1, 2\}$. Let $u_2^n = \frac{\bar{x}_2^n - x_0}{|\bar{x}_2^n - x_0|}$ (with the convention $u_2^n = 0$ if $\bar{x}_2^n = x_0$) and define $\gamma_2^n : [t_0, t_0 + \tau^n] \rightarrow \mathbb{R}^d$ by

$$(2.18) \quad \begin{cases} \dot{\gamma}_2^n(t) = k(t, \gamma_2^n(t))u_2^n \\ \gamma_2^n(t_0) = x_0, \end{cases}$$

where $\tau^n \geq 0$ is chosen so that $\gamma_2^n(t_0 + \tau^n) = \bar{x}_2^n$.

Claim. *As $n \rightarrow +\infty$, we have $\tau^n \leq 2\rho h_n + O(h_n^2)$.*

Proof. Note that we have nothing to prove in the case $\bar{x}_2^n = x_0$, and hence we assume $\bar{x}_2^n \neq x_0$ in the sequel. If $|\bar{x}_2^n - x_0| \leq \rho|x_1^n - x_0|$, we let $x_3^n = \bar{x}_2^n$, otherwise we choose x_3^n as the unique point in the segment (x_0, \bar{x}_2^n) such that $|x_3^n - x_0| = \rho|x_1^n - x_0|$. In both cases, we have $|x_3^n - x_0| = \bar{\rho}|x_1^n - x_0|$ for some $\bar{\rho} \leq \rho$. Let τ_1^n be the time that γ_2^n takes to reach the point x_3^n , i.e., $\gamma_2^n(t_0 + \tau_1^n) = x_3^n$. (Note that $\tau_1^n = \tau^n$ in the case $|\bar{x}_2^n - x_0| \leq \rho|x_1^n - x_0|$.) We show that $\tau_1^n \leq \rho h_n + O(h_n^2)$. To obtain that, we observe, by integrating (2.15) and (2.18) and doing a change of variables, that

$$\begin{aligned}
\int_{t_0}^{t_0 + \tau_1^n} k(s, \gamma_2^n(s)) ds &= |x_3^n - x_0| = \bar{\rho}|x_1^n - x_0| = \bar{\rho} \int_{t_0}^{t_0 + h_n} k(s, \gamma_0(s)) ds \\
&= \int_{t_0}^{t_0 + \bar{\rho}h_n} k\left(t_0 + \frac{s - t_0}{\bar{\rho}}, \gamma_0\left(t_0 + \frac{s - t_0}{\bar{\rho}}\right)\right) ds \\
&= \int_{t_0}^{t_0 + \bar{\rho}h_n} k(s, \gamma_2^n(s)) ds \\
&\quad + \int_{t_0}^{t_0 + \bar{\rho}h_n} \left[k\left(t_0 + \frac{s - t_0}{\bar{\rho}}, \gamma_0\left(t_0 + \frac{s - t_0}{\bar{\rho}}\right)\right) - k(s, \gamma_2^n(s)) \right] ds \\
(2.19) \quad &= \int_{t_0}^{t_0 + \bar{\rho}h_n} k(s, \gamma_2^n(s)) ds + O(h_n^2),
\end{aligned}$$

in which the last equality follows from the Lipschitz continuity of k and the fact that

$$\begin{aligned}
\left| \gamma_0\left(t_0 + \frac{s - t_0}{\bar{\rho}}\right) - \gamma_2^n(s) \right| &\leq \left| \gamma_0\left(t_0 + \frac{s - t_0}{\bar{\rho}}\right) - x_0 \right| + |x_0 - \gamma_2^n(s)| \\
&\leq K_{\max} \left[\frac{s - t_0}{\bar{\rho}} + (s - t_0) \right].
\end{aligned}$$

Define $F : [0, \tau^n] \rightarrow \mathbb{R}_+$ by $F(t) = \int_{t_0}^{t_0+t} k(s, \gamma_2^n(s)) ds$, then obviously F is increasing, which implies that F^{-1} is well-defined on the range of F . Since $F'(t) = k(t, \gamma_2^n(t))$, F is K_{\max} -Lipschitz continuous and, since $(F^{-1})'(t) = \frac{1}{F'(F^{-1}(t))}$, we also deduce that F^{-1} is $\frac{1}{K_{\min}}$ -Lipschitz continuous. Therefore, by (2.19), we deduce that

$$\tau_1^n = F^{-1}(F(\bar{\rho}h_n) + O(h_n^2)) = \bar{\rho}h_n + O(h_n^2) \leq \rho h_n + O(h_n^2).$$

This concludes the proof of the claim in the case $|\bar{x}_2^n - x_0| \leq \rho|x_1^n - x_0|$, since $\tau_1^n = \tau^n$ in that case.

Otherwise, we have $\bar{\rho} = \rho$ and $|x_3^n - x_0| = \rho|x_1^n - x_0|$, and thus, from (2.17), we get

$$|\bar{x}_2^n - x_0| < \rho(|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|) = |x_3^n - x_0| + \rho|\bar{x}_2^n - x_1^n|.$$

On the other hand, since x_3^n belongs to the segment (x_0, \bar{x}_2^n) , we have $|\bar{x}_2^n - x_0| = |\bar{x}_2^n - x_3^n| + |x_3^n - x_0|$, hence the inequality $|\bar{x}_2^n - x_3^n| \leq \rho|\bar{x}_2^n - x_1^n|$ holds. Suppose τ_2^n is the time the trajectory γ_2^n takes to go from x_3^n to \bar{x}_2^n , i.e., $\gamma_2^n(t_0 + \tau_1^n + \tau_2^n) = \bar{x}_2^n$, and note that $\tau^n = \tau_1^n + \tau_2^n$. As before, we compare the times between $|\bar{x}_2^n - x_3^n|$ and $|\bar{x}_2^n - x_1^n|$. Let $\beta \leq \rho$ be such that $|\bar{x}_2^n - x_3^n| = \beta|\bar{x}_2^n - x_1^n|$. Proceeding similarly to (2.19), we get

$$\begin{aligned} \int_0^{\tau_2^n} k(s + t_0 + \tau_1^n, \gamma_2^n(s + t_0 + \tau_1^n)) ds &= |\bar{x}_2^n - x_3^n| = \beta|\bar{x}_2^n - x_1^n| = \beta \int_{t_1^n}^{t_2^n} k(s, \bar{\gamma}_1^n(s)) ds \\ &= \int_0^{\beta h_n} k\left(\frac{s}{\beta} + t_0 + h_n, \bar{\gamma}_1^n\left(\frac{s}{\beta} + t_0 + h_n\right)\right) ds \\ &= \int_0^{\beta h_n} k(s + t_0 + \tau_1^n, \gamma_2^n(s + t_0 + \tau_1^n)) ds \\ &\quad + \int_0^{\beta h_n} \left[k\left(\frac{s}{\beta} + t_0 + h_n, \bar{\gamma}_1^n\left(\frac{s}{\beta} + t_0 + h_n\right)\right) - k(s + t_0 + \tau_1^n, \gamma_2^n(s + t_0 + \tau_1^n)) \right] ds \\ &= \int_0^{\beta h_n} k(s + t_0 + \tau_1^n, \gamma_2^n(s + t_0 + \tau_1^n)) ds + O(h_n^2), \end{aligned}$$

in which the last equality follows from the Lipschitz continuity of k and the facts that $\tau_1^n = O(h_n)$ and

$$\begin{aligned} &\left| \bar{\gamma}_1^n\left(\frac{s}{\beta} + t_0 + h_n\right) - \gamma_2^n(s + t_0 + \tau_1^n) \right| \\ &\leq \left| \bar{\gamma}_1^n\left(\frac{s}{\beta} + t_0 + h_n\right) - x_1^n \right| + |x_1^n - x_0| + |x_0 - \gamma_2^n(s + t_0 + \tau_1^n)| \\ &\leq K_{\max} \left[\frac{s}{\beta} + h_n + s + \tau_1^n \right]. \end{aligned}$$

Arguing similarly to above, we deduce that $\tau_2^n = \beta h_n + O(h_n^2)$. Therefore the time τ^n to reach \bar{x}_2^n from x_0 satisfies

$$\tau^n = (\rho + \beta)h_n + O(h_n^2) \leq 2\rho h_n + O(h_n^2). \quad \square$$

Let us now compare the trajectories $\bar{\gamma}_1^n$ and γ_1^n on $[t_1^n, t_2^n]$. Let $\delta_1^n(t) = \gamma_1^n(t) - \bar{\gamma}_1^n(t)$. Hence, from the ODEs satisfied by the trajectories $\bar{\gamma}_1^n$ and γ_1^n , we have

$$\begin{aligned}\delta_1^n(t) &= \int_{t_1^n}^t \left[k(s, \gamma_1^n(s))u_1^n(s) - k(s, \bar{\gamma}_1^n(s))\bar{u}_1^n \right] ds \\ &= \int_{t_1^n}^t \left[k(s, \gamma_1^n(s)) - k(s, \bar{\gamma}_1^n(s)) \right] u_1^n(s) ds + \int_{t_1^n}^t k(s, \bar{\gamma}_1^n(s))(u_1^n(s) - \bar{u}_1^n) ds.\end{aligned}$$

Since u_1^n is the optimal control, by Proposition 2.4.10, it is Lipschitz continuous. Therefore, denoting by $L > 0$ the Lipschitz constant of k on a bounded set containing the trajectories γ_1^n and $\bar{\gamma}_1^n$ for every n , we have

$$|\delta_1^n(t)| \leq L \int_{t_1^n}^t |\delta_1^n(s)| ds + K_{\max} \int_{t_1^n}^t L|s - t_1^n| ds,$$

and hence, by using Grönwall's inequality,

$$|\delta_1^n(t)| \leq LK_{\max} \frac{(t - t_1^n)^2}{2} e^{L(t-t_1^n)}.$$

In particular, if we set $t = t_1^n + h_n$, then

$$|x_2^n - \bar{x}_2^n| \leq LK_{\max} \frac{h_n^2}{2} e^{Lh_n} = O(h_n^2).$$

Let $u_3^n = \frac{x_2^n - \bar{x}_2^n}{|x_2^n - \bar{x}_2^n|}$ (with the convention $x_3^n = 0$ if $x_2^n = \bar{x}_2^n$) and γ_3^n be the solution of

$$(2.20) \quad \begin{cases} \dot{\gamma}_3^n(t) = k(t, \gamma_3^n(t))u_3^n \\ \gamma_3^n(t_0 + \tau^n) = \bar{x}_2^n. \end{cases}$$

Using the lower bound K_{\min} on k and the fact that $|x_2^n - \bar{x}_2^n| = O(h_n^2)$, one can easily deduce that the time σ^n from \bar{x}_2^n to x_2^n along γ_3^n (i.e., $\gamma_3^n(t_0 + \tau^n + \sigma^n) = x_2^n$) satisfies $\sigma^n = O(h_n^2)$.

We have thus constructed two ways to go from x_0 to x_2^n . The first one is to choose the path containing x_0 , x_1^n , and x_2^n , which corresponds to the concatenation of the trajectories γ_0 on $[t_0, t_1^n]$ and γ_1^n on $[t_1^n, t_2^n]$, and the second one is the path containing x_0 , \bar{x}_2^n , and x_2^n , which corresponds to the concatenation of the trajectories γ_2^n on $[t_0, t_0 + \tau^n]$ and γ_3^n on $[t_0 + \tau^n, t_0 + \tau^n + \sigma^n]$. Letting T_1^n and T_2^n be the times for going from x_0 to x_2^n along these two paths, respectively, we have, by construction and the claim, that $T_1^n = 2h_n$ and $T_2^n = \tau^n + \sigma^n \leq 2\rho h_n + O(h_n^2)$. Hence, since $\rho < 1$, we have, for n large enough, that $T_2^n < T_1^n$.

From (2.14), we deduce that

$$\varphi(t_0, x_0) = \varphi(t_1^n, x_1^n) + h_n + o(h_n) = \varphi(t_2^n, x_2^n) + T_1^n + o(h_n),$$

where the last equality comes from Proposition 2.4.5 and the fact that $\gamma_1^n \in \text{Opt}(\Gamma, k, t_1^n, x_1^n)$. On the other hand, since the path from x_0 to x_2^n going through \bar{x}_2^n is an admissible trajectory for k , we have, by Proposition 2.4.5, that $\varphi(t_0, x_0) \leq T_2^n + \varphi(t_0 + T_2^n, x_2^n)$. Hence

$$(2.21) \quad \varphi(t_2^n, x_2^n) + T_1^n + o(h_n) \leq T_2^n + \varphi(t_0 + T_2^n, x_2^n).$$

We also know that $t_0 + T_2^n < t_0 + T_1^n = t_2^n$ for n large enough. Therefore, by Proposition 2.4.9, there exists a constant $c > 0$ such that

$$\varphi(t_2^n, x_2^n) > \varphi(t_0 + T_2^n, x_2^n) + (c - 1)(t_2^n - t_0 - T_2^n) = \varphi(t_0 + T_2^n, x_2^n) + (c - 1)(T_1^n - T_2^n),$$

and, using (2.21), we get $(c - 1)(T_1^n - T_2^n) + T_1^n + o(h_n) \leq T_2^n$, which leads to

$$2h_n + o(h_n) = T_1^n + o(h_n) \leq T_2^n \leq 2\rho h_n + O(h_n^2).$$

Divide above inequality by h_n to observe that

$$2 + o(1) \leq 2\rho + O(h_n).$$

Finally by letting $n \rightarrow +\infty$, we conclude that $\rho \geq 1$, which is a contradiction. Therefore Case 2 will never happen and this ends the proof. \square

Motivated by Proposition 2.4.13, we introduce the following definition.

Definition 2.4.15. Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ under the assumptions (H4), (H5), and (H7) and let \mathcal{W} be as in Definition 2.4.11. If $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ is such that $\mathcal{W}(t_0, x_0)$ contains exactly one element $-\omega_0$, then ω_0 is called the *normalized gradient* of φ at (t_0, x_0) and denoted by $\omega_0 = \widehat{\nabla\varphi}(t_0, x_0)$.

As an immediate consequence of Proposition 2.4.12 and Theorem 2.4.14, we obtain the following characterization of optimal controls.

Corollary 2.4.16. Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ under the assumptions (H4), (H5), and (H7). Let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\gamma \in \text{Opt}(\Gamma, k, t_0, x_0)$, and u be the optimal control associated with γ . Then, for every $t \in (t_0, t_0 + \varphi(t_0, x_0))$, φ admits a normalized gradient at $(t, \gamma(t))$ and $u(t) = -\widehat{\nabla\varphi}(t, \gamma(t))$, i.e.,

$$(2.22) \quad \dot{\gamma}(t) = -k(t, \gamma(t))\widehat{\nabla\varphi}(t, \gamma(t)).$$

Combining Proposition 2.4.10 and Corollary 2.4.16, for every optimal trajectory γ , we obtain that $t \mapsto \widehat{\nabla\varphi}(t, \gamma(t))$ is Lipschitz continuous for t between the initial and exit times of γ . However, this provides no information on the regularity of $(t, x) \mapsto \widehat{\nabla\varphi}(t, x)$, which is the topic of our next result.

Proposition 2.4.17. Consider the optimal control problem $\text{OCP}(\Gamma, k)$ and its value function φ under the assumptions (H4), (H5), and (H7). Then $\widehat{\nabla\varphi}$ is continuous on its domain of definition.

Proof. Let \mathcal{U} be as in Definition 2.4.11 and $D \subset \mathbb{R}_+ \times (\mathbb{R}^d \setminus \Gamma)$ be the domain of definition of $\widehat{\nabla\varphi}$, i.e., $D = \{(t_0, x_0) \in \mathbb{R}_+ \times (\mathbb{R}^d \setminus \Gamma) \mid \mathcal{U}(t_0, x_0) \text{ is a singleton}\}$. Let $(t_n, x_n)_{n \in \mathbb{N}}$ be a sequence in D converging as $n \rightarrow +\infty$ to some $(t_0, x_0) \in D$ and let $\bar{u}_n = -\widehat{\nabla\varphi}(t_n, x_n)$. We want to show that $\bar{u}_n \rightarrow -\widehat{\nabla\varphi}(t_0, x_0)$ as $n \rightarrow +\infty$ and, since $(\bar{u}_n)_{n \in \mathbb{N}}$ is a sequence in the compact set \mathbb{S}^{d-1} , it suffices to show that $-\widehat{\nabla\varphi}(t_0, x_0)$ is the unique adherent point of $(\bar{u}_n)_{n \in \mathbb{N}}$. Let \bar{u}_0 be an adherent point of $(\bar{u}_n)_{n \in \mathbb{N}}$ and consider a subsequence of $(\bar{u}_n)_{n \in \mathbb{N}}$ converging to \bar{u}_0 , which we still denote by $(\bar{u}_n)_{n \in \mathbb{N}}$ for simplicity.

Since $\bar{u}_n \in \mathcal{U}(t_n, x_n)$, there exists a sequence of optimal trajectories $(\gamma_n)_{n \in \mathbb{N}}$, $\gamma_n \in \text{Opt}(\Gamma, k, t_n, x_n)$, and a corresponding sequence of optimal controls $(u_n)_{n \in \mathbb{N}}$ such that $u_n(t_n) = \bar{u}_n$. From Proposition 2.4.10 and Arzelà–Ascoli Theorem, there exist elements γ^* and u^* such that, up to extracting a subsequence, $\gamma_n \rightarrow \gamma^*$ and $u_n \rightarrow u^*$ uniformly on compact time intervals. One immediately verifies that $u^*(t_0) = \bar{u}_0$, $\gamma^*(t_0) = x_0$, and that $\gamma^* \in \text{Opt}(\Gamma, k, t_0, x_0)$ and u^* is its associated optimal control, which shows that $\bar{u}_0 \in \mathcal{U}(t_0, x_0) = \{-\widehat{\nabla\varphi}(t_0, x_0)\}$, as required. \square

2.5 Minimal-time mean field games

After having collected in Section 2.4 several preliminary results on the optimal control problem $\text{OCP}(\Gamma, k)$, we now turn to the study of the main problem considered in the chapter, the multi-population minimal-time mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$. We address existence of equilibria in Section 2.5.1, study their asymptotic behavior for large time in Section 2.5.2, and characterize equilibria as solutions of a system of PDEs in Section 2.5.3.

Recall that, according to the presentation provided in Section 2.3, equilibria of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ are described in terms of vectors of measures $\mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$. Given such a vector of measures, we shall consider the N optimal control problems $\text{OCP}(\Gamma_i, k_{\mathbf{Q},i})$, with $k_{\mathbf{Q},i}$ given by $k_{\mathbf{Q},i}(t, x) = K_i(m_t^i, \hat{m}_t^i, x)$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and where $m_t^i = e_{t\#}Q_i$ and \hat{m}_t^i is defined in (2.3). We will denote the value function of $\text{OCP}(\Gamma_i, k_{\mathbf{Q},i})$ by $\varphi_{\mathbf{Q},i}$, and we omit \mathbf{Q} from the notation of both $k_{\mathbf{Q},i}$ and $\varphi_{\mathbf{Q},i}$ when it is clear from the context. For simplicity of notation, we also write $\text{Adm}_i(\mathbf{Q})$ for $\text{Adm}(k_i)$ and $\text{Opt}_i(\Gamma, \mathbf{Q}, t_0, x_0)$ for $\text{Opt}(\Gamma_i, k_i, t_0, x_0)$.

2.5.1 Existence of equilibria

The goal of this part is to establish existence of equilibria for $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, which is done by recasting the existence of an equilibrium in terms of the existence of a fixed point of a certain set-valued map and applying a suitable fixed-point theorem. This section follows closely [75, Section 5] but, due to the facts that assumptions (H1)–(H3) are weaker than those from [75] and that we work here with mean field games in the non-compact state space \mathbb{R}^d , several proofs must be adapted in a nontrivial way to the present setting. The main result to be proved in this section is the following.

Theorem 2.5.1. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ under assumptions (H1)–(H3). Then there exists an equilibrium $\mathbf{Q} \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ for $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$.*

Let us start by showing an additional continuity property of the value function.

Lemma 2.5.2. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ under the assumptions (H1)–(H3). Then, for every $i \in \{1, \dots, N\}$, $(t, x, \mathbf{Q}) \mapsto \varphi_{\mathbf{Q},i}(t, x)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$.*

Proof. Fix $i \in \{1, \dots, N\}$ and let $(t_n, x_n, \mathbf{Q}_n)_{n \in \mathbb{N}}$ be a sequence taking values in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ converging to some (t_*, x_*, \mathbf{Q}_*) , and denote $\mathbf{Q}_n = (Q_{1,n}, \dots, Q_{N,n})$ and $\mathbf{Q}_* = (Q_{1,*}, \dots, Q_{N,*})$. For $n \in \mathbb{N}$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, define $k_n(t, x) = K(m_{n,t}^i, \hat{m}_{n,t}^i, x)$ and $k_*(t, x) = K(m_{*,t}^i, \hat{m}_{*,t}^i, x)$, where $m_{n,t}^i = e_{t\#} Q_{n,i}$, $m_{*,t}^i = e_{t\#} Q_{*,i}$, and $\hat{m}_{n,t}^i$ and $\hat{m}_{*,t}^i$ are defined as in (2.3). Note that, by continuity of $Q \mapsto e_{t\#} Q$, we have that $k_n(t, x) \rightarrow k_*(t, x)$ for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. For simplicity of notation, we write φ_n and φ_* for $\varphi_{\mathbf{Q}_n,i}$ and $\varphi_{\mathbf{Q}_*,i}$, respectively.

By Proposition 2.4.4, $(\varphi_n(t_n, x_n))_{n \in \mathbb{N}}$ is a bounded sequence and thus, to prove that it converges to $\varphi_*(t_*, x_*)$, it suffices to show that $\varphi_*(t_*, x_*)$ is the unique adherent point of $(\varphi_n(t_n, x_n))_{n \in \mathbb{N}}$. Let κ_* be an adherent point of $(\varphi_n(t_n, x_n))_{n \in \mathbb{N}}$ and consider the subsequence of $(\varphi_n(t_n, x_n))_{n \in \mathbb{N}}$ which converges to κ_* , which we still denote by $(\varphi_n(t_n, x_n))_{n \in \mathbb{N}}$ for simplicity.

For $n \in \mathbb{N}$, let $\gamma_n \in \text{Opt}_i(\Gamma, \mathbf{Q}_n, t_n, x_n)$. Since $(\gamma_n)_{n \in \mathbb{N}}$ is an equibounded and equi-Lipschitz sequence, by Arzelà–Ascoli Theorem, up to extracting a subsequence, which we still denote by $(\gamma_n)_{n \in \mathbb{N}}$, there exists $\gamma_* \in \text{Lip}_{K_{\max}}(\mathbb{R}_+; \mathbb{R}^d)$ such that $\gamma_n \rightarrow \gamma_*$ as $n \rightarrow +\infty$ (uniformly on compact time intervals). For every $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, we have $\left| \frac{\gamma_n(t_2) - \gamma_n(t_1)}{t_2 - t_1} \right| \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} k_n(s, \gamma_n(s)) ds$ and, using (H3) and letting $n \rightarrow +\infty$, we deduce that $\left| \frac{\gamma_*(t_2) - \gamma_*(t_1)}{t_2 - t_1} \right| \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} k_*(s, \gamma_*(s)) ds$, yielding that $\gamma_* \in \text{Adm}_i(\mathbf{Q}_*)$. Moreover, since $\gamma_n(t_n) = x_n$, γ_n is constant on $[0, t_n]$ and $[t_n + \varphi_n(t_n, x_n), +\infty)$, and $\gamma_n(t_n + \varphi_n(t_n, x_n)) \in \Gamma$ for every $n \in \mathbb{N}$, we easily deduce that $\gamma_*(t_*) = x_*$, γ_* is constant on $[0, t_*]$ and $[t_* + \kappa_*, +\infty)$, and $\gamma_*(t_* + \kappa_*) \in \Gamma$, yielding in particular that $\varphi_*(t_*, x_*) \leq \kappa_*$.

Let us assume, to obtain a contradiction, that $\varphi_*(t_*, x_*) < \kappa_*$. For simplicity, let $\varsigma_n = t_n + \varphi_*(t_*, x_*)$, $\varsigma_* = t_* + \varphi_*(t_*, x_*)$, $\xi_n = \gamma_n(\varsigma_n)$, and $\xi_* = \gamma_*(\varsigma_*) \in \Gamma$. Note that, since $\varphi_*(t_*, x_*) < \kappa_*$, we have $\xi_n \notin \Gamma$ for n large enough. Let $\tilde{\gamma}_n \in \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$ be defined by

$$\tilde{\gamma}_n(t) = \begin{cases} \gamma_n(t) & \text{if } 0 \leq t \leq \varsigma_n, \\ \xi_n + \frac{\xi_* - \xi_n}{|\xi_* - \xi_n|} K_{\min}(t - \varsigma_n) & \text{if } \varsigma_n \leq t \leq \varsigma_n + \frac{|\xi_* - \xi_n|}{K_{\min}}, \\ \xi_* & \text{if } t \geq \varsigma_n + \frac{|\xi_* - \xi_n|}{K_{\min}}. \end{cases}$$

Clearly, $\tilde{\gamma}_n \in \text{Adm}_i(\mathbf{Q})$ and $\tau_\Gamma(t_n, \tilde{\gamma}_n) \leq \varphi_*(t_*, x_*) + \frac{|\xi_* - \xi_n|}{K_{\min}}$. Since $\varphi_*(t_*, x_*) < \kappa_*$, we have $\tau_\Gamma(t_n, \tilde{\gamma}_n) < \frac{\varphi_*(t_*, x_*) + \kappa_*}{2} < \kappa_*$ for n large enough, implying that $\varphi_n(t_n, x_n) < \frac{\varphi_*(t_*, x_*) + \kappa_*}{2} < \kappa_*$

for n large enough and contradicting thus the fact that $\varphi_n(t_n, x_n) \rightarrow \kappa_*$ as $n \rightarrow +\infty$. Hence, one has necessarily $\varphi_n(t_n, x_n) \rightarrow \varphi_*(t_*, x_*)$ as $n \rightarrow +\infty$, as required. \square

The next result, which is an immediate consequence of Proposition 2.4.4, states an a priori property of equilibria.

Lemma 2.5.3. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, assume that (H1) and (H2) are satisfied, and let ϕ be the function from Notation 2.3.3. Then there exists a nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for every equilibrium $\mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, $t \geq 0$, $i \in \{1, \dots, N\}$, and $R > 0$, we have*

$$Q_i(\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})) \geq \phi(R).$$

In particular, denoting $m_t^i = e_{t\#}Q_i$, we have $m_t^i(B_{\psi(R)}) \geq \phi(R)$.

Lemma 2.5.3 shows that it suffices to look for equilibria of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ in the set

$$(2.23) \quad \mathfrak{Q} = \{ \mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N \mid e_{0\#}\mathbf{Q} = \mathbf{m}_0 \text{ and} \\ \forall i \in \{1, \dots, N\}, \forall R > 0, Q_i(\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})) \geq \phi(R) \},$$

where ϕ and ψ are as in the statement of Lemma 2.5.3. We next provide elementary properties of \mathfrak{Q} .

Lemma 2.5.4. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, assume that (H1) and (H2) are satisfied, and let \mathfrak{Q} be the set defined in (2.23). Then \mathfrak{Q} is nonempty, convex, and compact with respect to the topology of weak convergence of measures.*

Proof. The set \mathfrak{Q} is clearly convex and, to see that it is nonempty, define $b : \mathbb{R}^d \rightarrow \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$ as the function which associates with each $x \in \mathbb{R}^d$ the function $b(x)$ given by $b(x)(t) = x$ for every $t \in \mathbb{R}_+$. It is immediate to check that $b_{\#}\mathbf{m}_0 \in \mathfrak{Q}$, and hence \mathfrak{Q} is nonempty.

To prove that \mathfrak{Q} is compact, notice first that $\mathfrak{Q} = \mathfrak{Q}_1^N \cap \mathfrak{Q}_2$, where

$$\mathfrak{Q}_1 = \{ Q \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)) \mid \forall R > 0, Q(\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})) \geq \phi(R) \}$$

and $\mathfrak{Q}_2 = \{ \mathbf{Q} \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N \mid e_{0\#}\mathbf{Q} = \mathbf{m}_0 \}$. Since $\mathbf{Q} \mapsto e_{0\#}\mathbf{Q}$ is continuous, \mathfrak{Q}_2 is closed, and hence it suffices to show that \mathfrak{Q}_1 is compact. By Prokhorov Theorem (see, e.g., [6, Theorem 5.1.3]), it suffices to show that \mathfrak{Q}_1 is tight and closed. Tightness of \mathfrak{Q}_1 follows immediately from the facts that $\phi(R) \rightarrow 1$ as $R \rightarrow +\infty$ and that, by Arzelà–Ascoli Theorem, for every $R > 0$, $\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})$ is compact in the topology of uniform convergence on compact sets.

To see that \mathfrak{Q}_1 is closed, let $(Q_n)_{n \in \mathbb{N}}$ be a sequence in \mathfrak{Q}_1 converging to some $Q \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))$. For every $R > 0$, $\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})$ is closed and thus, by using [14, Theorem 2.1], one obtains

$$Q(\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})) \geq \limsup_{n \rightarrow \infty} Q_n(\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})) \geq \phi(R),$$

which proves that $Q \in \mathfrak{Q}_1$. Hence \mathfrak{Q}_1 is closed. \square

We now recast the definition of equilibrium of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ in terms of fixed points of a set-valued map defined on \mathfrak{Q} . Let $F : \mathfrak{Q} \rightrightarrows \mathfrak{Q}$ associate with each $\mathbf{Q} \in \mathfrak{Q}$ the subset $F(\mathbf{Q})$ of \mathfrak{Q} defined by

$$(2.24) \quad F(\mathbf{Q}) = \left\{ \tilde{\mathbf{Q}} = (\tilde{Q}_1, \dots, \tilde{Q}_N) \in \mathfrak{Q} \mid \forall i \in \{1, \dots, N\}, \tilde{Q}_i\text{-almost every } \gamma \text{ satisfies } \gamma \in \text{Opt}_i(\Gamma, \mathbf{Q}, 0, \gamma(0)) \right\}.$$

Clearly, $\mathbf{Q} \in \mathfrak{Q}$ is an equilibrium of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ if and only if it is a fixed point of F , i.e., $\mathbf{Q} \in F(\mathbf{Q})$. For every $i \in \{1, \dots, N\}$, we consider the set $\text{Opt}_i(\mathbf{Q}) \subset \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$ containing all optimal trajectories of the i -th population for \mathbf{Q} and starting at time 0, i.e.,

$$(2.25) \quad \text{Opt}_i(\mathbf{Q}) = \bigcup_{x_0 \in \mathbb{R}^d} \text{Opt}_i(\Gamma, \mathbf{Q}, 0, x_0).$$

The set $F(\mathbf{Q})$ can be rewritten in terms of $\text{Opt}_i(\mathbf{Q})$ as

$$(2.26) \quad F(\mathbf{Q}) = \left\{ \tilde{\mathbf{Q}} = (\tilde{Q}_1, \dots, \tilde{Q}_N) \in \mathfrak{Q} \mid \forall i \in \{1, \dots, N\}, \tilde{Q}_i(\text{Opt}_i(\mathbf{Q})) = 1 \right\}.$$

Lemma 2.5.5. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ under the assumptions (H1)–(H3) and let \mathfrak{Q} and Opt_i , $i \in \{1, \dots, N\}$, be defined as in (2.23) and (2.25). For every $R > 0$ and $i \in \{1, \dots, N\}$, define $\widehat{\text{Opt}}_{i,R} : \mathfrak{Q} \rightrightarrows \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$ by*

$$\widehat{\text{Opt}}_{i,R}(\mathbf{Q}) = \text{Opt}_i(\mathbf{Q}) \cap \text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)}).$$

Then $\widehat{\text{Opt}}_{i,R}$ is upper semicontinuous.

The proof of Lemma 2.5.5 is based on the continuity of the value function from Lemma 2.5.2 and follows the same lines as that of [75, Lemma 5.4], being thus omitted here.

Lemma 2.5.6. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ under the assumptions (H1)–(H3) and let \mathfrak{Q} and F be defined as in (2.23) and (2.24), respectively. Then F is upper semicontinuous and, for every $\mathbf{Q} \in \mathfrak{Q}$, $F(\mathbf{Q})$ is nonempty, convex, and compact.*

Proof. Given $\mathbf{Q} \in \mathfrak{Q}$, it follows immediately from (2.26) that $F(\mathbf{Q})$ is convex, and one can easily prove that it is nonempty and compact by adapting the arguments of [75, Lemma 5.3] (see also [47, Lemma 4.7(a)]).

Since \mathfrak{Q} is compact and F has closed values, to prove that F is upper semicontinuous it is sufficient to show that its graph is closed. Let $(\mathbf{Q}_n)_{n \in \mathbb{N}}$ be a sequence in \mathfrak{Q} with $\mathbf{Q}_n \rightarrow \mathbf{Q}$ for some $\mathbf{Q} \in \mathfrak{Q}$ and $(\tilde{\mathbf{Q}}_n)_{n \in \mathbb{N}}$ be a sequence in \mathfrak{Q} with $\tilde{\mathbf{Q}}_n \in F(\mathbf{Q}_n)$ for every $n \in \mathbb{N}$ and $\tilde{\mathbf{Q}}_n \rightarrow \tilde{\mathbf{Q}}$ for some $\tilde{\mathbf{Q}} \in \mathfrak{Q}$. We denote $\mathbf{Q}_n = (Q_{n,1}, \dots, Q_{n,N})$ and $\tilde{\mathbf{Q}}_n = (\tilde{Q}_{n,1}, \dots, \tilde{Q}_{n,N})$.

For each $n \in \mathbb{N}$, since $\tilde{\mathbf{Q}}_n \in F(\mathbf{Q}_n)$, we have $\tilde{Q}_{n,i}(\text{Opt}_i(\mathbf{Q}_n)) = 1$ for every $i \in \{1, \dots, N\}$ and, since $\tilde{\mathbf{Q}}_n \in \mathfrak{Q}$, we also have that $\tilde{Q}_{n,i}(\text{Lip}_{K_{\max}}(\mathbb{R}_+; B_{\psi(R)})) \geq \phi(R)$ for

every $i \in \{1, \dots, N\}$ and $R > 0$, where ϕ is the function from Notation 2.3.3. Hence $\tilde{Q}_{n,i}(\widehat{\text{Opt}}_{i,R}(\mathbf{Q}_n)) \geq \phi(R)$ for every $n \in \mathbb{N}$, $i \in \{1, \dots, N\}$, and $R > 0$.

For every $\epsilon \in (0, 1)$ and $i \in \{1, \dots, N\}$, let $V_\epsilon^i = \{\gamma \in \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d) \mid \mathbf{d}(\gamma, \text{Opt}_i(\mathbf{Q})) \leq \epsilon\}$, where \mathbf{d} is given by (2.2), and note that V_ϵ^i is a neighborhood of $\widehat{\text{Opt}}_{i,R}(\mathbf{Q})$ for every $R > 0$. Let $R_0 > 0$ be such that $\phi(R_0) \geq 1 - \epsilon$. Since $\widehat{\text{Opt}}_{i,R_0}$ is upper semicontinuous by Lemma 2.5.5, there exists a neighborhood W_ϵ of \mathbf{Q} in Ω such that $\widehat{\text{Opt}}_{i,R_0}(\hat{\mathbf{Q}}) \subset V_\epsilon^i$ for every $i \in \{1, \dots, N\}$ and $\hat{\mathbf{Q}} \in W_\epsilon$. From the convergence $\mathbf{Q}_n \rightarrow \mathbf{Q}$, one concludes that there exists N_ϵ such that, for every $n \geq N_\epsilon$, one has $\mathbf{Q}_n \in W_\epsilon$, and thus $\widehat{\text{Opt}}_{i,R_0}(\mathbf{Q}_n) \subset V_\epsilon^i$.

Since $\tilde{Q}_{n,i}(\widehat{\text{Opt}}_{i,R_0}(\mathbf{Q}_n)) \geq \phi(R_0) \geq 1 - \epsilon$, one obtains that $\tilde{Q}_{n,i}(V_\epsilon^i) \geq 1 - \epsilon$ for every $n \geq N_\epsilon$ and $i \in \{1, \dots, N\}$. Since $\tilde{\mathbf{Q}}_n \rightarrow \tilde{\mathbf{Q}}$ and V_ϵ^i is closed, we have $\tilde{Q}_i(V_\epsilon^i) \geq \limsup_{n \rightarrow \infty} \tilde{Q}_{n,i}(V_\epsilon^i) \geq 1 - \epsilon$. On the other hand, since $\text{Opt}_i(\mathbf{Q})$ is closed and $(V_\epsilon^i)_{\epsilon \in (0,1)}$ is a nondecreasing family of sets with $\bigcap_{\epsilon \in (0,1)} V_\epsilon^i = \text{Opt}_i(\mathbf{Q})$, we conclude that $\tilde{Q}_i(\text{Opt}_i(\mathbf{Q})) = \lim_{\epsilon \rightarrow 0} \tilde{Q}_i(V_\epsilon^i) = 1$. Hence $\tilde{\mathbf{Q}} \in F(\mathbf{Q})$, which concludes the proof that the graph of F is closed. \square

Let us now conclude the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. By Lemmas 2.5.4 and 2.5.6 and Kakutani fixed point theorem (see, e.g., [57, § 7, Theorem 8.6]), F admits a fixed point, i.e., there exists $\mathbf{Q} \in \Omega$ such that $\mathbf{Q} \in F(\mathbf{Q})$, which means \mathbf{Q} is an equilibrium for MFG($\Gamma, \mathbf{K}, \mathbf{m}_0$). \square

Remark 2.5.7. Theorem 2.5.1 asserts the existence of an equilibrium for MFG($\Gamma, \mathbf{K}, \mathbf{m}_0$), but uniqueness does not necessarily hold. An example of this fact in the single-population case is presented in [75, Remark 7.1] under the assumption $K_1 \equiv 1$, in which there is no interaction between agents.

Let us provide a heuristic example illustrating why uniqueness is not expected in the multi-population case even when agents interact. Consider the case $d = N = 2$, m_0^1 is the uniform measure on $B((-1, 0), R)$, m_0^2 is the uniform measure on $B((1, 0), R)$, $0 < R < 1$, $\Gamma_1 = B((1, 0), R)$, and $\Gamma_2 = B((-1, 0), R)$, and assume that K_1 and K_2 are such that agents are more penalized by the other population than by their own population, i.e., $K_i(\mu, \nu, x) < K_i(\nu, \mu, x)$ if ν is larger than μ in a neighbourhood of x , for $i = 1, 2$. In this case, we may expect heuristically the phenomenon of *lane formation*, in which the populations will group in separate lanes, so that each population gets to its target set while avoiding interaction with the other population (see, for instance, [43, 51] for more details on lane formation in other kinds of models for crowd motion and in experiments). If the lanes at an equilibrium are asymmetric (which is expected if our model reproduces the behaviour usually observed in experiments), then we obtain another different equilibrium with the same initial conditions by performing the symmetry transformation $(x_1, x_2) \mapsto (x_1, -x_2)$, and hence we do not expect uniqueness of equilibrium in this case.

2.5.2 Asymptotic behavior

In this part, we characterize the behavior of m_t^i as $t \rightarrow +\infty$, where $m_t^i = e_{t\#}Q_i$ and $\mathbf{Q} = (Q_1, \dots, Q_N)$ is an equilibrium of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$. Intuitively, one expects m_t^i to converge to a measure concentrated on the target set Γ_i and, in addition to proving this result in the general case, we also provide convergence rates when the initial measure m_0^i has finite p moments for some $p \in [1, +\infty)$ and prove finite-time convergence when the initial measure has bounded support.

In order to characterize the limit of m_t^i as $t \rightarrow +\infty$, let us introduce some notation. Let $\mathbf{C}_{\text{lim}}(\mathbb{R}_+; \mathbb{R}^d) = \{\gamma \in \mathbf{C}(\mathbb{R}_+; \mathbb{R}^d) \mid \lim_{t \rightarrow +\infty} \gamma(t) \text{ exists and is finite}\}$, which is a Borel subset of $\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d)$, and define $e_\infty : \mathbf{C}_{\text{lim}}(\mathbb{R}_+; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ by $e_\infty(\gamma) = \lim_{t \rightarrow +\infty} \gamma(t)$, which is a Borel-measurable function. By definition of optimal trajectories, $\text{Opt}_i(\mathbf{Q}) \subset \mathbf{C}_{\text{lim}}(\mathbb{R}_+; \mathbb{R}^d)$ for every $\mathbf{Q} \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$, and thus $e_{\infty\#}\mathbf{Q} \in \mathcal{P}(\mathbb{R}^d)^N$ is well-defined for every equilibrium \mathbf{Q} of a mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$.

We are now in position to state and prove the main result of this section.

Theorem 2.5.8. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ under assumptions (H1) and (H2). Let $\mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ be an equilibrium of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, $\mathbf{m}_t = (m_t^1, \dots, m_t^N)$ be defined by $\mathbf{m}_t = e_{t\#}\mathbf{Q}$ for $t \in [0, +\infty]$, and ψ be the function whose existence is asserted in Proposition 2.4.4.*

1. For every $i \in \{1, \dots, N\}$, we have $m_t^i \rightarrow m_\infty^i$ as $t \rightarrow +\infty$.
2. Let $p \in [1, +\infty)$, $i \in \{1, \dots, N\}$, and assume that $m_0^i \in \mathcal{P}_p(\mathbb{R}^d)$. Then, for every $t \in [0, +\infty]$, we have $m_t^i \in \mathcal{P}_p(\mathbb{R}^d)$. Moreover, there exist constants $\alpha > 0$ and $t_0 \geq 0$ such that

$$(2.27) \quad \mathbf{W}_p(m_t^i, m_\infty^i)^p \leq 2^p \int_{\mathbb{R}^d \setminus B_{\alpha(t-t_0)}} \psi(|x|)^p dm_0^i(x), \quad \forall t \geq t_0.$$

3. Let $i \in \{1, \dots, N\}$ and assume that m_0^i is compactly supported. Then, for every $t \in [0, +\infty]$, m_t^i is compactly supported and there exists $\tau \geq 0$ such that

$$m_t^i = m_\infty^i, \quad \forall t \geq \tau.$$

Remark 2.5.9. Note that, by Proposition 2.4.4, ψ has linear growth and thus, together with the assumption that $m_0^i \in \mathcal{P}_p(\mathbb{R}^d)^N$, one immediately obtains that the right-hand side of (2.27) tends to 0 as $t \rightarrow +\infty$. When more information on the distribution of m_0^i is available, the right-hand side of (2.27) allows one to obtain estimates on the convergence rate of m_t^i as $t \rightarrow +\infty$ in the Wasserstein distance.

Proof of Theorem 2.5.8. To show 1, let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be continuous and bounded and fix $i \in \{1, \dots, N\}$. We then have, using the continuity and boundedness of f and Lebesgue's

dominated convergence theorem, that

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) dm_t^i(x) &= \int_{\mathbf{C}_{\text{lim}}(\mathbb{R}_+; \mathbb{R}^d)} f(\gamma(t)) dQ_i(\gamma) \\ &\xrightarrow{t \rightarrow +\infty} \int_{\mathbf{C}_{\text{lim}}(\mathbb{R}_+; \mathbb{R}^d)} f\left(\lim_{t \rightarrow +\infty} \gamma(t)\right) dQ_i(\gamma) = \int_{\mathbb{R}^d} f(x) dm_\infty^i(x), \end{aligned}$$

yielding the required convergence.

Let us now prove 2. For $t \in [0, +\infty]$, we have, using Proposition 2.4.4, that

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^p dm_t^i(x) &= \int_{\text{Opt}_i(\mathbf{Q})} |\gamma(t)|^p dQ_i(\gamma) \\ &\leq \int_{\text{Opt}_i(\mathbf{Q})} \psi(|\gamma(0)|)^p dQ_i(\gamma) = \int_{\mathbb{R}^d} \psi(|x|)^p dm_0^i(x), \end{aligned}$$

where $\gamma(\infty)$ is defined as $\lim_{t \rightarrow +\infty} \gamma(t)$. Since ψ has linear growth, it follows that $m_t^i \in \mathcal{P}_p(\mathbb{R}^d)$ for every $t \in [0, +\infty]$.

Let T be the function whose existence is asserted in Proposition 2.4.4 and $\alpha > 0$, $t_0 \geq 0$ be such that $T(R) \leq \frac{R}{\alpha} + t_0$ for every $R > 0$. Let $t \in [t_0, +\infty)$. Note that, using the notations introduced in Section 2.2, we have $(e_t, e_\infty)_{\#} Q_i \in \Pi(m_t^i, m_\infty^i)$ and thus, by (2.1), we have

$$\mathbf{W}_p(m_t^i, m_\infty^i)^p \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d(e_t, e_\infty)_{\#} Q_i(x, y) = \int_{\text{Opt}_i(\mathbf{Q})} |e_t(\gamma) - e_\infty(\gamma)|^p dQ_i(\gamma).$$

If $\gamma \in \text{Opt}_i(\mathbf{Q})$ is such that $|\gamma(0)| \leq \alpha(t - t_0)$, then, since $T(|\gamma(0)|) \leq t$, we have, as a consequence of Proposition 2.4.4, that $\gamma(t) \in \Gamma_i$ and γ is constant on $[t, +\infty)$, yielding that $e_t(\gamma) = e_\infty(\gamma)$. Thus

$$\mathbf{W}_p(m_t^i, m_\infty^i)^p \leq \int_{\text{Opt}_i(\mathbf{Q}) \cap \{\gamma \mid |\gamma(0)| \notin B_{\alpha(t-t_0)}\}} |e_t(\gamma) - e_\infty(\gamma)|^p dQ_i(\gamma).$$

By using the fact from Proposition 2.4.4 that $e_t(\gamma) \in B_{\psi(|\gamma(0)|)}$ for all $t \in [0, +\infty]$ and $\gamma \in \text{Opt}_i(\mathbf{Q})$, one has $|e_t(\gamma)| \leq \psi(|\gamma(0)|)$ and thus

$$\begin{aligned} \mathbf{W}_p(m_t^i, m_\infty^i)^p &\leq \int_{\text{Opt}_i(\mathbf{Q}) \cap \{\gamma \mid |\gamma(0)| \notin B_{\alpha(t-t_0)}\}} 2^p \psi(|\gamma(0)|)^p dQ_i(\gamma) \\ &= 2^p \int_{\mathbb{R}^d \setminus B_{\alpha(t-t_0)}} \psi(|x|)^p dm_0^i(x), \end{aligned}$$

as required.

Finally, to prove 3, let $R_0 > 0$ be such that the support of m_0^i is included in B_{R_0} and notice that, as a consequence of Proposition 2.4.4, the support of m_t^i is included in $B_{\psi(R_0)}$ for every $t \in [0, +\infty]$. Letting T be as in the statement of Proposition 2.4.4 and $\tau = T(R_0)$, we deduce that, for every $t \geq \tau$ and $\gamma \in \text{Opt}_i(\mathbf{Q})$ with $|\gamma(0)| \leq R_0$, we have $e_t(\gamma) = e_\infty(\gamma)$, which concludes the proof since Q_i is supported in $\text{Opt}_i(\mathbf{Q}) \cap \{\gamma \mid |\gamma(0)| \leq R_0\}$. \square

2.5.3 The MFG system

As a final step in the study of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, we characterize its equilibria as solutions of a system of partial differential equations, called the *MFG system*. Given an equilibrium $\mathbf{Q} = (Q_1, \dots, Q_N)$, by Proposition 2.4.8, the value functions $\varphi_{\mathbf{Q},i}$, $i \in \{1, \dots, N\}$, corresponding to each population are already known to satisfy a Hamilton–Jacobi equation, and we are thus left to prove that the measures $m_t^i = e_{t\#}Q_i$ are also solutions of suitable partial differential equations. Since Q_i is concentrated on optimal trajectories, which satisfy (2.22) thanks to Corollary 2.4.16, one expects $t \mapsto m_t^i$ to be a solution to a continuity equation with velocity field $-\widehat{\nabla} \varphi_{\mathbf{Q},i}$.

In order for the above reasoning to be made precise, one must verify that the assumptions of Corollary 2.4.16 are satisfied. Since (H7) requires k to be locally Lipschitz continuous both in time and space, we shall make here the following stronger assumption on $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$.

(H8) There exists $p \geq 1$ such that $\mathbf{m}_0 \in \mathcal{P}_p(\mathbb{R}^d)^N$ and, for every $i \in \{1, \dots, N\}$, $K_i : \mathcal{P}_p(\mathbb{R}^d) \times \mathcal{P}_p(\mathbb{R}^d)^{N-1} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ is Lipschitz continuous with respect to all its variables (using the Wasserstein distance \mathbf{W}_p in $\mathcal{P}_p(\mathbb{R}^d)$) and locally in the last variable, i.e., for every $R > 0$, there exists $L > 0$ such that, for every $(\mu_1, \nu_1, x_1), (\mu_2, \nu_2, x_2) \in \mathcal{P}_p(\mathbb{R}^d) \times \mathcal{P}_p(\mathbb{R}^d)^{N-1} \times B_R$, we have

$$|K_i(\mu_1, \nu_1, x_1) - K_i(\mu_2, \nu_2, x_2)| \leq L(\mathbf{W}_p(\mu_1, \mu_2) + \mathbf{W}_p(\nu_1, \nu_2) + |x_1 - x_2|).$$

Remark 2.5.10. If $\mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ is such that $Q_i(\text{Lip}_c(\mathbb{R}_+; \mathbb{R}^d)) = 1$ for some $c > 0$ and every $i \in \{1, \dots, N\}$, and if $e_{t\#}Q_i \in \mathcal{P}_p(\mathbb{R}^d)$ for some $p \geq 1$ and every $t \geq 0$ and $i \in \{1, \dots, N\}$, then one immediately verifies, by considering the coupling measure $(e_t, e_s)\#Q_i \in \Pi(m_t^i, m_s^i)$ in (2.1), that $t \mapsto e_{t\#}Q_i$ is Lipschitz continuous with respect to the distance \mathbf{W}_p in $\mathcal{P}_p(\mathbb{R}^d)$. Hence, if $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ satisfies (H8) and \mathbf{Q} is an equilibrium of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, the corresponding optimal control problems $\text{OCP}(\Gamma_i, k_{\mathbf{Q},i})$, $i \in \{1, \dots, N\}$, satisfy (H7).

Theorem 2.5.11. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ under assumptions (H1), (H2), and (H8) and assume that $\mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ is an equilibrium of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$. Consider the value functions $\varphi_i = \varphi_{\mathbf{Q},i}$ and the time-dependent measures $m_i(t, \cdot) = m_t^i = e_{t\#}Q_i$ for $i \in \{1, \dots, N\}$. Then $(m^1, \dots, m^N, \varphi_1, \dots, \varphi_N)$ solves the MFG system*

$$(2.28) \quad \begin{cases} \partial_t m_i(t, x) - \text{div} \left(m_i(t, x) K_i(m_t^i, \hat{m}_t^i, x) \widehat{\nabla} \varphi_i(t, x) \right) = 0, & (t, x) \in \mathbb{R}_+^* \times (\mathbb{R}^d \setminus \Gamma_i), \\ -\partial_t \varphi_i(t, x) + |\nabla \varphi_i(t, x)| K_i(m_t^i, \hat{m}_t^i, x) - 1 = 0, & (t, x) \in \mathbb{R}_+ \times (\mathbb{R}^d \setminus \Gamma_i), \\ m_i(0, \cdot) = m_0^i, \\ \varphi_i(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma_i, \end{cases}$$

for all $i \in \{1, \dots, N\}$, where the first and second equations are satisfied, respectively, in the sense of distributions and in the viscosity sense.

Note that the Hamilton–Jacobi equations on φ_i and the corresponding boundary conditions follow immediately from Proposition 2.4.8, and the continuity equations on m_i can be established using (2.22) and the fact that, from Proposition 2.4.12, Theorem 2.4.14, Definition 2.4.15, and Proposition 2.4.17, $\widehat{\nabla}\varphi_i$ is continuous on the support of m_t^i . We refer to [75, Theorem 6.1] and [47, Theorem 4.12] for more details on the proof in the case of a single population, but we stress the fact, contrarily to those references, we establish here (2.22), and hence the continuity equations on m_i , under weaker assumptions on K_i and without relying on semiconcavity properties of φ_i . Notice also that the coupling between the different populations occur through the terms \hat{m}_t^i , which are defined in (2.3).

Theorem 2.5.11 shows that any equilibrium \mathbf{Q} of a mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ satisfies the MFG system (2.28). To prove that (2.28) actually characterizes equilibria of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$, we also need a converse statement, namely that solutions of (2.28) yield equilibria of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$. Such a converse statement has been sketched in [75, Remark 6.1] for single-population minimal-time mean field games. We now provide a more detailed argument in our present setting.

Theorem 2.5.12. *Consider the mean field game $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ under assumptions (H1), (H2), and (H8), and assume in addition that $\mathbf{m}_0 = (m_0^1, \dots, m_0^N)$ is such that m_0^i is compactly supported for every $i \in \{1, \dots, N\}$. For $i \in \{1, \dots, N\}$, let $\varphi_i : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $t \mapsto m_i(t, \cdot) \in \mathcal{P}(\mathbb{R}^d)$ be continuous functions. Assume that, for every $i \in \{1, \dots, N\}$ and $t > 0$, $\widehat{\nabla}\varphi_i(t, \cdot)$ exists and is continuous in the support of $m_i(t, \cdot)$, and that $(m_1, \dots, m_N, \varphi_1, \dots, \varphi_N)$ satisfies (2.28), where the first equation is satisfied in the sense of distributions and the second equation is satisfied in the viscosity sense. Then there exists an equilibrium $\mathbf{Q} = (Q_1, \dots, Q_N) \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))^N$ of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ such that, for every $i \in \{1, \dots, N\}$, $m_t^i = m_i(t, \cdot) = e_{t\#}Q_i$ for every $t \geq 0$ and φ_i is the value function of $\text{OCP}(\Gamma_i, k_{\mathbf{Q}, i})$.*

Proof. Let $k_i : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ be defined for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ by $k_i(t, x) = K_i(m_t^i, \hat{m}_t^i, x)$, where \hat{m}_t^i is defined as in (2.3), and consider the optimal control problem $\text{OCP}(\Gamma_i, k_i)$. Since φ_i is lower bounded by 0, satisfies the second equation of (2.28) in the viscosity sense, and also satisfies the fourth equation of (2.28), we deduce from [11, Chapter IV, Corollary 4.3] that φ_i is the value function of $\text{OCP}(\Gamma_i, k_i)$. Note that [11, Chapter IV, Corollary 4.3] is stated for autonomous control systems, but it can be applied to the non-autonomous control system $\dot{\gamma}(t) = k_i(t, \gamma(t))u(t)$ by considering the augmented state $\tilde{x}(t) = (t, \gamma(t))$. Moreover, [11, Chapter IV, Corollary 4.3] assumes that the target set has compact boundary, but we get the conclusion in our framework by reasoning locally and using Proposition 2.4.4.

For $i \in \{1, \dots, N\}$, since m_i satisfies the continuity equation in (2.28) in the sense of distributions and the corresponding velocity field is bounded by K_{\max} , it follows from the Superposition Principle for continuity equations (see [5, Theorem 3.2]) that there exists $Q_i \in \mathcal{P}(\mathbf{C}(\mathbb{R}_+; \mathbb{R}^d))$ such that $m_t^i = e_{t\#}Q_i$ for every $t \geq 0$. Let $\mathbf{Q} = (Q_1, \dots, Q_N)$. Note that, since m_0^i is compactly supported and the velocity field in the continuity equation is bounded, m_t^i is also compactly supported, and thus $\mathbf{Q} \in \Omega$. We will show that \mathbf{Q} is an equilibrium of $\text{MFG}(\Gamma, \mathbf{K}, \mathbf{m}_0)$ by showing that $\mathbf{Q} \in F(\mathbf{Q})$, i.e., that Q_i is supported on $\text{Opt}_i(\mathbf{Q})$.

To see that, notice that, from the proof of [5, Theorem 3.2], it also follows that Q_i is concentrated on the solutions of $\dot{\gamma}(t) = -k_i(t, \gamma(t)) \widehat{\nabla} \varphi_i(t, \gamma(t))$, which are clearly admissible trajectories for $\text{OCP}(\Gamma_i, k_i)$ since $|\widehat{\nabla} \varphi_i(t, \gamma(t))| = 1$. We prove that such trajectories are optimal by showing that they satisfy the equality in the dynamic programming principle (2.8). Let γ be such a trajectory and notice that it is K_{\max} Lipschitz continuous. From the definition of normalized gradient, we have

$$\lim_{h \rightarrow 0^+} \frac{\varphi_i(t+h, \gamma(t) - hk_i(t, \gamma(t)) \widehat{\nabla} \varphi_i(t, \gamma(t))) - \varphi_i(t, \gamma(t))}{h} = -1.$$

Using the facts that $\gamma(t+h) = \gamma(t) + h\dot{\gamma}(t) + o(h)$ and that φ_i is Lipschitz continuous, we deduce that

$$\lim_{h \rightarrow 0^+} \frac{\varphi_i(t+h, \gamma(t+h)) - \varphi_i(t, \gamma(t))}{h} = -1.$$

Since $t \mapsto \varphi_i(t, \gamma(t))$ is Lipschitz continuous, and hence differentiable almost everywhere, we deduce that $\frac{d}{dt} \varphi_i(t, \gamma(t)) = -1$ a.e., and thus, integrating the above expression from t to $t+h$, we get that $\varphi_i(t+h, \gamma(t+h)) - \varphi_i(t, \gamma(t)) = -h$, and therefore, by Proposition 2.4.5, γ is optimal for $\text{OCP}(\Gamma_i, k_i)$. Hence Q_i is concentrated on $\text{Opt}_i(\mathbf{Q})$, concluding the proof that \mathbf{Q} is an equilibrium. \square

Chapter 3

Nonsmooth mean field games with state constraints

In this chapter, we consider a mean field game model inspired by crowd motion where agents aim to reach a closed set, called target set, in minimal time. Congestion phenomena are modeled through a constraint on the velocity of an agent that depends on the average density of agents around their position. The model is considered in the presence of state constraints: roughly speaking, these constraints may model walls, columns, fences, hedges, or other kinds of obstacles at the boundary of the domain which agents cannot cross. After providing a more detailed description of the model, the chapter recalls some previous results on the existence of equilibria for such games and presents the main difficulties that arise due to the presence of state constraints. Our main contribution is to show that equilibria of the game satisfy a system of coupled partial differential equations, known mean field game system, thanks to recent techniques to characterize optimal controls in the presence of state constraints. These techniques not only allow to deal with state constraints but also require very few regularity assumptions on the dynamics of the agents.

3.1 Introduction

Mean field games (MFGs for short) were first introduced around 2006 by Larsy and Lions [69–71] and independently by Caines, Huang, and Malhamé [61–63], motivated by problems in economics and engineering and based on some previous works on games with infinitely many players, such as those from [8, 9, 66]. MFGs are differential games with a continuum of players, assumed to be rational, indistinguishable, individually negligible, and influenced only by some “average” behavior of other players through a “mean-field type” interaction. Since their introduction, MFGs have been studied both in connection with several applications and from a theoretical point of view, in which the main goals are typically proving the existence of equilibria, characterizing such equilibria as solutions to a system of partial differential equations, called *MFG system*, or studying the connections between MFGs and games with a large (but finite) number of symmetric players. We refer to [2, 28, 30, 36, 37, 53] for

more details and further references on mean field games. In this chapter, we use the words “players” and “agents” interchangeably to refer to people taking part in the game.

Most works on mean field games consider either first- or second-order mean field games. First-order MFGs, also known as deterministic mean field games, usually assume that players’ dynamics are described by a deterministic control system, and their equilibria are characterized by a system of first-order partial differential equations, whereas second-order MFGs, also known as probabilistic mean field games, consider that players’ dynamics are determined by a stochastic control system, typically with an additive Brownian motion modeling a random drift, and their equilibria are typically described by a system of second-order partial differential equations.

This chapter considers a class of first-order mean field games inspired by crowd motion in which agents are assumed to remain inside of a specific domain while their goal is to arrive at a given target set in minimal time. In order to model congestion, the speed of each agent is constrained by a function depending on the position of the agent and on the distribution of all agents.

Crowd motion has been the subject of a very large number of works in the literature from different points of view, motivated not only by understanding but also by controlling and optimizing the crowd behavior (see, e.g., [42, 51, 58–60, 64, 65, 72, 77, 78, 80]). The natural framework for a mean field game modeling crowd motion is to adopt a *macroscopic* modeling of crowd, i.e., to describe the crowd at a given time t as infinitely many agents represented by a measure m_t on the space of possible positions, which evolves according to some conservation law, typically a continuity equation of the form $\partial_t m + \operatorname{div}(mV) = 0$, where V is the velocity field followed by the agents. While most macroscopic crowd motion models consider a given velocity field V constructed from modeling assumptions, the mean field game approach consists instead in considering that each agent will choose their trajectory by solving some optimal control problem, and the velocity field V is a consequence of the optimal choices of the agents.

Up to the authors’ knowledge, the first work to be fully dedicated to a mean field game model for crowd motion is [67], which proposes an MFG model for a two-population crowd with trajectories perturbed by additive Brownian motion and considers both their stationary distributions and their evolution on a prescribed time interval. Other works have later proposed MFG models for crowd motion taking into account different characteristics, such as [20], which considers the fast exit of a crowd and proposes a mean field game model which is studied numerically; [33], which is not originally motivated by the modeling of crowd motion but considers a MFG model with a density constraint, which is a natural assumption in some crowd motion models; [12], which presents numerical simulations for some variational mean field games related to crowd motion; or also [41], which provides a generalized MFG model for pedestrians with limited predictive abilities.

The present chapter considers the MFG model inspired by crowd motion introduced in [75] (described in details in Section 3.2 below), which contains two major differences with respect to previous MFGs for crowd motions and also to most MFG models in general. Firstly, the model from [75] assumes that each agent solves an optimal control problem with

free final time, and actually that the optimization criterion of each agent is to minimize their arrival time at a certain target set, after which they quit the game. This is in contrast with most of the MFG literature, which usually considers either optimization criteria with a given and known finite final time or infinite-horizon optimization criteria. Secondly, congestion is modeled in [75] by imposing a maximal speed for each agent which depends on their position and the distribution of other agents, while, in most of other MFG works, agents are allowed to choose their speed without constraints, but high speeds and congestion are instead penalized in the cost function. The motivation of [75] to impose a constraint on the speed of agents is to model high-congestion situations in which an agent may be unable to move faster since other agents in front of them may work as a physical barrier which cannot be crossed by simply paying a higher “cost”.

The main results of [75] are the existence of equilibria of the proposed MFG model and the characterization of equilibria through an MFG system. Even though existence of equilibria is proved in [75] under rather general assumptions, their characterization through an MFG system is only shown in the case where the target set of the agents is the whole boundary of the compact domain in which they evolve, which avoids the presence of state constraints in the minimal-time optimal control problem solved by each agent. This is a very restrictive assumption for a MFG model for crowd motion, since crowds often evolve in domains with boundaries which cannot be crossed by the agents, such as walls, columns, fences, hedges, or other obstacles. However, from a technical point of view, the major difficulty in analyzing optimal control problems with state constraints is that their value functions may fail to be semiconcave (see, e.g., [24, Example 4.4]), the latter property being important in the characterization of optimal controls (see, e.g., [27]), which is a key step in obtaining the MFG system in [75]. Other works, such as [46, 47, 83], have further explored the model from [75] and related models, but none of those works consider the case of MFGs with state constraints.

Studying optimal control problems and mean field games with state constraints turns out to be a challenging problem due to the possible lack of semiconcavity of the value function. The series of papers [21–23] represent an important step in the study of MFGs with state constraints: in those works, the authors prove that, for the optimal control problem they consider, even though the corresponding value function may fail to be semiconcave in the classical sense, i.e., with a *linear* modulus of semiconcavity, the value function is still semiconcave with a *fractional* modulus of semiconcavity, which is sufficient to obtain additional properties of optimal trajectories allowing to characterize optimal controls. However, it is not clear how to adapt that strategy to minimal-time optimal control problems. In order to circumvent this difficulty, this chapter proves the required additional properties of optimal trajectories without relying on the semiconcavity of the value function. This has also the additional advantage of requiring fewer regularity properties on the dynamics of the control system satisfied by each agent.

The main contribution of the present chapter is thus to characterize optimal controls and deduce the MFG system for the model of [75] under state constraints, which is an important step towards the study of MFG models for crowd motion with more realistic

assumptions. A first step in the characterization of optimal controls is to apply Pontryagin Maximum Principle in order to deduce additional information on optimal trajectories. To do so, we adapt the penalization technique used in [25]: we transform the minimal-time optimal control problem with state constraints into a penalized optimal control problem without state constraints and show that, similarly to [25], if the penalization parameter is small enough, optimal trajectories of the penalized problem coincide with optimal trajectories of the original problem (see Theorem 3.4.8), which allows us to deduce properties of the optimal trajectories of the original optimal control problem by applying Pontryagin Maximum Principle to the penalized problem (see Corollary 3.4.9).

With those properties of optimal trajectories, we then proceed to the study of optimal controls by following the arguments used in the recent paper [83]: our main results are Theorem 3.4.11, which provides a boundary condition for the value function of the optimal control problem solved by each agent; Theorem 3.4.23, which characterizes the optimal control at a given point as the direction of greatest decrease of the value function by adapting the arguments of [83, Theorem 4.14] to the case with state constraints; and Theorem 3.5.2, which shows that equilibria of our MFG model satisfy the MFG system.

The present chapter is an extended version of [82], which announced most of the results from this paper and provided some details of the strategy of the proofs of the main results. With respect to that reference, this chapter works under weaker assumptions on the dynamics of the control system (compare (H4)–(H7) below with [82, (H2)–(H3)]), which renders some proofs more technically involved and require different proof strategies at some points, such as in Lemma 3.3.2 and for most of the results of Section 3.4.1. This chapter also provides a more detailed introduction to the subject, details proofs that had been omitted or only sketched in [82] (and in particular the results in Section 3.4.3), and provides additional remarks and comments on the results.

The chapter is organized as follows. Some notations and standard definitions are provided in Section 3.2, together with the precise description of the optimal control problem and the mean field game model considered in this chapter, and the list of hypotheses used here. Section 3.3 presents preliminary results on the optimal control problem and the mean field game, most of which are either easy to prove or already present in the literature. The major new contributions in Section 3.3 are the proof of Lipschitz continuity of the value function of the optimal control problem under the weaker assumptions (H4) and (H5) as well as the alternative characterization of equilibria from Proposition 3.3.7, which also holds for other mean field game models (see, e.g., [86, Lemma 3.4]).

The main results of this chapter are provided in Sections 3.4 and 3.5. Section 3.4.1 uses the strategy of [25] to study an optimal control problem with state constraints through a penalized optimal control problem without state constraints, adapting the techniques of that reference to our setting in order to show that optimal trajectories of the original problem coincide with that of the penalized problem if the penalization parameter is small enough (Theorem 3.4.8). Section 3.4.2 provides a boundary condition for the value function of the optimal control problem in the part of the boundary which is not in the target set. We then provide, in Section 3.4.3, the characterization of optimal controls as directions of

maximal descent of the value function, and we use this characterization in Section 3.4.4 to introduce the notion of normalized gradient and provide its main properties. The main result concerning the mean field game model of interest is the fact that its equilibria satisfy a system of PDEs, shown in Section 3.5.

3.2 Notations and definitions

3.2.1 General notations

In this chapter, \mathbb{N} denotes the set of positive integers, d is a fixed positive integer, the sets of nonnegative and positive real numbers are denoted respectively by \mathbb{R}_+ and \mathbb{R}_+^* , \mathbb{R}^d is endowed with the usual Euclidean norm $|\cdot|$, and the unit sphere is denoted by \mathbb{S}^{d-1} . For $A \subset \mathbb{R}^d$, \bar{A} denotes its closure, ∂A denotes its boundary, and $\text{co} A$ denotes its convex hull. For any $x \in \mathbb{R}^d$ and $r \geq 0$, $B(x, r)$ (resp., $\bar{B}(x, r)$) denotes¹ the open (resp., closed) ball centered at x and with radius r . We denote this ball simply by B_r (resp., \bar{B}_r) if $x = 0$ and by B (resp., \bar{B}) if $x = 0$ and $r = 1$, i.e., B (resp., \bar{B}) is the open (resp., closed) unit ball centered at the origin.

For two sets A and B , a set-valued map from A to B is a map F that associates, with each $x \in A$, a (possibly empty) set $F(x) \subset B$. We use the notation $F : A \rightrightarrows B$ to indicate that F is a set-valued map from A to B .

Given two metric spaces X and Y and a constant $M > 0$, $\mathbf{C}(X; Y)$, $\text{Lip}(X; Y)$, and $\text{Lip}_M(X; Y)$ denote, respectively, the set of all continuous functions from X to Y , the set of all Lipschitz continuous functions from X to Y , and the subset of $\text{Lip}(X; Y)$ containing only those functions whose Lipschitz constant is at most M . We will often use these spaces to represent time-dependent functions defined for nonnegative times, in which case $X = \mathbb{R}_+$ and, for simplicity, we omit X from the previous notations, writing simply $\mathbf{C}(Y)$, $\text{Lip}(Y)$, and $\text{Lip}_M(Y)$, respectively. When $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^m$ for some positive integers k and m , we also consider the set $\mathbf{C}^1(X; Y)$ of continuously differentiable functions, the set $\mathbf{C}^{1,1}(X; Y)$ of differentiable functions with a Lipschitz continuous differential, and the set $\mathbf{C}_c^\infty(X; Y)$ of infinitely differentiable functions with compact support on X . If $f : X \rightarrow Y$ and $Z \subset X$, we write $f \in \mathbf{C}(Z; Y)$ to denote that the restriction of f to Z is continuous (even though f itself may fail to be continuous in the boundary of Z), with similar notations for the other functional spaces defined above.

For compact $A \subset \mathbb{R}^d$, the space $\mathbf{C}(A)$ is assumed to be endowed with the topology of uniform convergence on compact sets, with respect to which $\mathbf{C}(A)$ is a Polish space (see, e.g., [17, Chapter X]). For $t \in \mathbb{R}_+$, we denote by $e_t : \mathbf{C}(A) \rightarrow A$ the evaluation map at time t , defined by $e_t(\gamma) = \gamma(t)$ for every $\gamma \in \mathbf{C}(A)$.

¹In accordance with the previous notation, $\bar{B}(x, r)$ is the closure of $B(x, r)$ whenever $r > 0$. However, our notation is slightly ambiguous for $r = 0$: $B(x, 0) = \emptyset$, but $\bar{B}(x, 0) = \{x\}$ is not the closure of $B(x, 0)$. All along the chapter, we use the convention that, when referring to balls, the notation $\bar{B}(x, r)$ is the closed ball centered at x and with radius r . The same convention applies to the notation \bar{B}_r , and we remark that \bar{B} is unambiguous.

Recall that, for two metric spaces X and Y endowed with their Borel σ -algebras and a Borel-measurable map $f : X \rightarrow Y$, the pushforward of a measure μ on X through f is the measure $f_{\#}\mu$ on Y defined by $f_{\#}\mu(B) = \mu(f^{-1}(B))$ for every Borel subset B of Y . Given a Polish space X , the set of all Borel probability measures on X is denoted by $\mathcal{P}(X)$ and is endowed with the topology of weak convergence of measures (which is also sometimes referred to as narrow convergence). The support of a measure $\mu \in \mathcal{P}(X)$ is denoted by $\text{spt}(\mu)$, and is defined as the set of all points $x \in X$ such that $\mu(N_x) > 0$ for every open neighborhood N_x of x . When X is endowed with a complete metric \mathbf{d} with respect to which X is bounded, we endow $\mathcal{P}(X)$ with the Wasserstein distance \mathbf{W}_1 , defined for $\mu, \nu \in \mathcal{P}(X)$ by

$$\mathbf{W}_1(\mu, \nu) = \inf \left\{ \int_{X \times X} \mathbf{d}(x, y) \, d\lambda(x, y) \mid \lambda \in \Pi(\mu, \nu) \right\},$$

where $\Pi(\mu, \nu) = \{\lambda \in \mathcal{P}(X \times X) \mid \pi_{1\#}\lambda = \mu, \pi_{2\#}\lambda = \nu\}$ and $\pi_1, \pi_2 : X \times X \rightarrow X$ denote the canonical projections on to the first and second factors of the product $X \times X$, respectively. Recall (see, e.g., [6, Chapter 7] and [85, Chapter 5]) that \mathbf{W}_1 is compatible with the topology of weak convergence in $\mathcal{P}(X)$ and that it admits the dual formulation

$$\mathbf{W}_1(\mu, \nu) = \sup \left\{ \int_X \Phi(x) \, d(\mu - \nu) \mid \Phi \in \text{Lip}_1(X; \mathbb{R}) \right\}.$$

We shall also need in this chapter the notion of signed distance to the boundary of a set, whose definition we now provide, together with the classical definition of distance from a point to a set.

Definition 3.2.1. Let $A \subset \mathbb{R}^d$. If $A \neq \emptyset$, we denote by d_A the Euclidean distance to A , defined by $d_A(x) = \inf_{y \in A} |x - y|$. If $A \neq \emptyset$ and $A \neq \mathbb{R}^d$, the signed distance to the boundary of A , $d_{\partial A}^{\pm} : \mathbb{R}^d \rightarrow \mathbb{R}$, is defined by

$$d_{\partial A}^{\pm}(x) = d_A(x) - d_{\mathbb{R}^d \setminus A}(x).$$

Finally, we will need, in Section 3.4, the notion of normal cones of nonsmooth sets. We recall here the definitions provided in [88, Section 4.2].

Definition 3.2.2. Let k be a positive integer, $C \subset \mathbb{R}^k$ be a nonempty closed set, and $x \in C$.

1. The *proximal normal cone* to C at x is the set $N_C^{\text{P}}(x)$ defined by

$$N_C^{\text{P}}(x) = \{p \in \mathbb{R}^k \mid \exists M > 0 \text{ such that } p \cdot (y - x) \leq M|y - x|^2 \text{ for every } y \in C\}.$$

2. The *limiting normal cone* to C at x is the set $N_C(x)$ of vectors $p \in \mathbb{R}^k$ for which there exist a sequence $(x_n)_{n \in \mathbb{N}}$ in C and a sequence $(p_n)_{n \in \mathbb{N}}$ in \mathbb{R}^k such that $x_n \rightarrow x$ and $p_n \rightarrow p$ as $n \rightarrow +\infty$ and $p_n \in N_C^{\text{P}}(x_n)$ for every $n \in \mathbb{N}$.

3.2.2 The minimal-time optimal control problem

Before introducing the minimal-time mean field game of interest to this chapter, let us first consider an auxiliary minimal-time optimal control problem, which is a classical kind of problem in control theory. Let $\Omega \subset \mathbb{R}^d$ be a nonempty open bounded set and $\Gamma \subset \bar{\Omega}$ be a nonempty closed set. We consider the control system

$$(3.1) \quad \begin{cases} \dot{\gamma}(t) = k(t, \gamma(t))u(t), & u(t) \in \bar{B}, \\ \gamma(0) = x, \end{cases}$$

where $x \in \bar{\Omega}$, $k : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$, and, for $t \geq 0$, $\gamma(t) \in \bar{\Omega}$ is the state and $u(t) \in \bar{B}$ is the control. The function k describes the dynamics of the system and, due to the constraint that $u(t) \in \bar{B}$, $k(t, x)$ can be interpreted as the maximal speed at which an agent at position x at time t can move. The fact of requiring that $\gamma(t) \in \bar{\Omega}$ for every time $t \geq 0$ is a constraint we impose in the state of (3.1).

We are interested in this chapter in the optimal control problem consisting in minimizing the time a trajectory of (3.1) takes to reach the set Γ , called the *target set*. This optimal control problem, to which we refer in the sequel as $\text{OCP}(k)$ making explicit its dependence on the dynamics k , is made more precise in the following definition.

Definition 3.2.3. Let $\Omega \subset \mathbb{R}^d$ be a nonempty open bounded set, $\Gamma \subset \bar{\Omega}$ be a nonempty closed set, and $k : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$.

- (a) A curve $\gamma \in \text{Lip}(\bar{\Omega})$ is said to be *admissible* for $\text{OCP}(k)$ if there exists a measurable function $u : \mathbb{R}_+ \rightarrow \bar{B}$, called the *control* associated with γ , such that the first equation of (3.1) is satisfied for almost every $t \in \mathbb{R}_+$. The set of all admissible curves for $\text{OCP}(k)$ is denoted by $\text{Adm}(k)$.
- (b) Let $t_0 \in \mathbb{R}_+$. The *first exit time* after t_0 of a curve $\gamma \in \text{Lip}(\bar{\Omega})$ is the number $\tau(t_0, \gamma)$ defined by

$$\tau(t_0, \gamma) = \inf\{t \geq 0 \mid \gamma(t + t_0) \in \Gamma\},$$

with the convention that $\tau(t_0, \gamma) = +\infty$ if $\gamma(t + t_0) \notin \Gamma$ for every $t \geq 0$.

- (c) Let $t_0 \in \mathbb{R}_+$ and $x_0 \in \bar{\Omega}$. A curve $\gamma \in \text{Lip}(\bar{\Omega})$ is said to be an *optimal trajectory* for $\text{OCP}(k)$ from (t_0, x_0) if $\gamma \in \text{Adm}(k)$, $\gamma(t) = x_0$ for every $t \in [0, t_0]$,

$$(3.2) \quad \tau(t_0, \gamma) = \inf_{\substack{\beta \in \text{Adm}(k) \\ \beta(t_0) = x_0}} \tau(t_0, \beta),$$

and $\gamma(t) = \gamma(t_0 + \tau(t_0, \gamma))$ for every $t \in [t_0 + \tau(t_0, \gamma), +\infty)$. In this case, the associated control u of γ is called an *optimal control* associated with γ . The set of all optimal trajectories for $\text{OCP}(k)$ from (t_0, x_0) is denoted by $\text{Opt}(k, t_0, x_0)$.

(d) The *value function* of the optimal control problem $\text{OCP}(k)$ is the function $\varphi : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$ defined for $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$ by

$$(3.3) \quad \varphi(t_0, x_0) = \inf_{\substack{\gamma \in \text{Adm}(k) \\ \gamma(t_0) = x_0}} \tau(t_0, \gamma).$$

We highlight the fact that, for a curve γ to be admissible, it must not only satisfy the first equation of (3.1) for some control u and almost every time $t \geq 0$, but it must also remain inside $\bar{\Omega}$ for all times.

Remark 3.2.4. Since we are considering in Definition 3.2.3(c) optimal trajectories γ starting at time t_0 from x_0 and minimizing the time to reach Γ , these trajectories could have been defined only in the interval $[t_0, t_0 + \tau(t_0, \gamma)]$. However, in order to simplify the notations and deal in this chapter only with trajectories and controls defined on \mathbb{R}_+ , we extend an optimal trajectory to \mathbb{R}_+ by requiring it to be constant on $[0, t_0]$ and on $[t_0 + \tau(t_0, \gamma), +\infty)$, as done in Definition 3.2.3(c).

3.2.3 The minimal-time mean field game and its equilibria

As in Section 3.2.2, we fix a nonempty open bounded set $\Omega \subset \mathbb{R}^d$ and a nonempty closed set $\Gamma \subset \bar{\Omega}$. We consider a minimal-time mean field game in which a population of agents evolves on $\bar{\Omega}$ and the goal of each agent is to reach the target set Γ in minimal time. The population is described by a time-dependent probability measure $m_t \in \mathcal{P}(\bar{\Omega})$ for $t \geq 0$, and m_0 is assumed to be known. The trajectory γ of an agent starting its movement at a position $x \in \bar{\Omega}$ is assumed to satisfy the control system

$$(3.4) \quad \begin{cases} \dot{\gamma}(t) = K(m_t, \gamma(t))u(t), & u(t) \in \bar{B}, \\ \gamma(0) = x, \end{cases}$$

where $K : \mathcal{P}(\bar{\Omega}) \times \bar{\Omega} \rightarrow \mathbb{R}_+$ and $u : \mathbb{R}_+ \rightarrow \bar{B}$ is the control of the agent. Note that (3.4) corresponds to (3.1) with k defined by $k(t, x) = K(m_t, x)$ for $t \geq 0$ and $x \in \bar{\Omega}$. Each agent is assumed to choose their control in order to solve the optimal control problem $\text{OCP}(k)$ with k defined from K as before. This mean field game is denoted in the sequel by $\text{MFG}(K)$.

The function K models interactions between agents and states that the maximal speed at which an agent at position x at time t can move depends on the position x itself and on the distribution of all agents at time t , m_t . This function can be used to model congestion phenomena in crowd motion by choosing $K(m_t, x)$ to be small when m_t is “large” around x , which means that it is harder for agents to move on more crowded regions. For instance, K can be chosen as

$$(3.5) \quad K(\mu, x) = g \left(\int_{\bar{\Omega}} \chi(x - y) \eta(y) \, d\mu(y) \right),$$

where $\chi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a convolution kernel, $\eta : \bar{\Omega} \rightarrow \mathbb{R}_+$ is a weight function in the space $\bar{\Omega}$, and $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a decreasing function. The function χ may represent, for instance,

the region around an agent at which they look in order to evaluate local congestion, while η may be a function that is larger in regions difficult to move, such as regions with obstacles or other kinds of difficult terrain, and which can also be used to discount people who already reached the target set, as it was done in [47, 75]. Note, however, that we do *not* assume this specific form for K in this chapter.

Note that, since K depends on m_t for all $t \geq 0$, the optimal trajectories taken by the agents depend on m_t . On the other hand, m_t itself describes the evolution of the agents, and hence is determined by their choices of trajectories. We are interested in this chapter in *equilibrium* situations, in which, roughly speaking, starting from time evolution of the distribution of agents $m : \mathbb{R}_+ \rightarrow \mathcal{P}(\bar{\Omega})$, the optimal trajectories chosen by agents induce an evolution of the initial distribution of agents m_0 that is precisely given by $t \mapsto m_t$.

We provide a more precise notion of equilibrium in what is known as the Lagrangian framework, in which, instead of describing the evolution of agents as a time-dependent measure $m : \mathbb{R}_+ \rightarrow \mathcal{P}(\bar{\Omega})$, we rely instead on a measure Q on the set of all possible trajectories $\mathcal{P}(\mathbf{C}(\bar{\Omega}))$. Note that, given a measure $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$, one can obtain the associated time-dependent measure m by setting $m_t = e_{t\#}Q$ for $t \geq 0$. The Lagrangian approach is a classical approach in optimal transport problems (see, e.g., [6, 85]) which has been used to define equilibria of first-order mean field games in some recent works, such as [12, 21, 26, 29, 33, 47, 50, 75, 83].

Definition 3.2.5. Let $\Omega \subset \mathbb{R}^d$ be a nonempty open bounded set, $\Gamma \subset \bar{\Omega}$ be a nonempty closed set, $K : \mathcal{P}(\bar{\Omega}) \times \bar{\Omega} \rightarrow \mathbb{R}_+$, and $m_0 \in \mathcal{P}(\bar{\Omega})$. A measure $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$ is called a *Lagrangian equilibrium* of MFG(K) with initial condition m_0 if $e_{0\#}Q = m_0$ and Q -almost every $\gamma \in \mathbf{C}(\bar{\Omega})$ satisfies $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$, where $k_Q : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$ is defined for $t \geq 0$ and $x \in \bar{\Omega}$ by $k_Q(t, x) = K(e_{t\#}Q, x)$.

In the sequel of the chapter, we refer to Lagrangian equilibria simply as equilibria.

Remark 3.2.6. Another classical way to describe the evolution of agents in a mean field game, which dates back to [62] and has been used and developed in several other references such as [38, 54, 68], is to fix a probability space $(\mathbf{Z}, \mathcal{Z}, \mathbb{P})$ and describe the motion of agents through a time-dependent random variable in \mathbf{Z} , i.e., through a function X defined on \mathbb{R}_+ and such that, for every $t \geq 0$, $X(t) : \mathbf{Z} \rightarrow \bar{\Omega}$ is measurable. One can then retrieve the time-dependent measure $m_t \in \mathcal{P}(\bar{\Omega})$ as the law of $X(t)$, i.e., $m_t = X(t)\#\mathbb{P}$, but X contains more information than m , since X also carries information on the correlation of the distributions of agents at different times, for instance. This formulation using random variables has the additional advantage of being also adapted to study of other kinds of mean field games, such as second-order mean field games or mean field games of controls.

In addition, if² $\mathbb{R}_+ \ni t \mapsto X(t, z) \in \bar{\Omega}$ is continuous for almost every $z \in \mathbf{Z}$ and the function $\Xi : \mathbf{Z} \rightarrow \mathbf{C}(\bar{\Omega})$ defined by setting $\Xi(z) = X(\cdot, z)$ for a.e. $z \in \mathbf{Z}$ is measurable, then the probability measure $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$ describing the evolution of agents from a Lagrangian perspective can be retrieved as $Q = \Xi\#\mathbb{P}$.

²Here, $X(t, z)$ is a simplified notation for $X(t)(z)$.

Conversely, given a measure $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$ describing the evolution of agents, one can consider the probability space $(\mathbf{C}(\bar{\Omega}), \mathcal{B}, Q)$, where \mathcal{B} is the Borel σ -algebra of $\mathbf{C}(\bar{\Omega})$, and in this case the evolution of agents can be described by the time-dependent random variable X defined by $X(t) = e_t$ for $t \geq 0$.

3.2.4 Hypotheses

Along the chapter, we will need several assumptions on Ω , Γ , K , and k , which we collect in this subsection. We start with the following assumptions on the sets Ω and Γ .

(H1) The set $\Omega \subset \mathbb{R}^d$ is nonempty, open, bounded, and connected.

(H2) The set $\Gamma \subset \bar{\Omega}$ is nonempty and closed.

(H3) The boundary $\partial\Omega$ is a compact $\mathbf{C}^{1,1}$ submanifold of \mathbb{R}^d of dimension $d - 1$.

Whenever we assume that (H3) holds, we use $\mathbf{n}(x)$ to denote the outward unit normal vector to $\bar{\Omega}$ at the point $x \in \partial\Omega$.

When studying the optimal control problem $\text{OCP}(k)$, we shall need the following assumptions on the function k .

(H4) The function $k : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$ is continuous and there exist positive constants K_{\min} and K_{\max} such that $k(t, x) \in [K_{\min}, K_{\max}]$ for every $(t, x) \in \mathbb{R}_+ \times \bar{\Omega}$.

(H5) The function k is Lipschitz continuous with respect to its second variable, uniformly with respect to the first variable, i.e., there exists $L > 0$ such that, for every $t \in \mathbb{R}_+$ and $x_1, x_2 \in \bar{\Omega}$, we have

$$|k(t, x_1) - k(t, x_2)| \leq L|x_1 - x_2|.$$

The counterpart of Hypotheses (H4) and (H5) concerning the function K from the mean field game $\text{MFG}(K)$ are the following.

(H6) The function $K : \mathcal{P}(\bar{\Omega}) \times \bar{\Omega} \rightarrow \mathbb{R}_+$ is continuous and there exist positive constants K_{\min} and K_{\max} such that $K(\mu, x) \in [K_{\min}, K_{\max}]$ for every $(\mu, x) \in \mathcal{P}(\bar{\Omega}) \times \bar{\Omega}$.

(H7) The function K is Lipschitz continuous with respect to its second variable, uniformly with respect to the first variable, i.e., there exists $L > 0$ such that, for every $\mu \in \mathcal{P}(\bar{\Omega})$ and $x_1, x_2 \in \bar{\Omega}$, we have

$$|K(\mu, x_1) - K(\mu, x_2)| \leq L|x_1 - x_2|.$$

Note that (H6) and (H7) are satisfied in the particular case where K is chosen as in (3.5) if g is Lipschitz continuous, upper bounded, and lower bounded by a positive constant, χ is Lipschitz continuous, and η is continuous.

3.3 Preliminary results

This section presents results on $\text{OCP}(k)$ and $\text{MFG}(K)$ that will be useful in the sequel, most of which are either present in other references or easy to prove using classical techniques. It turns out that the results we will present in this section require fewer regularity assumptions on Ω than (H1) and (H3); namely, we replace here (H3) by the following assumption.

(H3') There exists $D > 0$ such that, for every $x, y \in \bar{\Omega}$, there exists a curve γ included in $\bar{\Omega}$ connecting x to y and of length at most $D|x - y|$.

Hypothesis (H3') means that the geodesic distance in $\bar{\Omega}$ is equivalent to the usual Euclidean distance, and it holds in particular when (H1) and (H3) are satisfied.

The first result we present concerns three elementary properties of $\text{OCP}(k)$: existence of optimal trajectories, boundedness of the value function, and the dynamic programming principle. The proof is omitted, since all properties are either easy to prove or classical. Indeed, existence of optimal trajectories can be proved easily using compactness of a minimizing sequence (see [27, Theorem 8.1.4] for a proof in the autonomous case, i.e., when $k : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$ does not depend on its first variable). Boundedness of the value function follows from the fact that, using (H3'), one can easily construct, for every point $x \in \bar{\Omega}$, an admissible trajectory connecting it to a point in Γ with constant speed K_{\min} and that arrives in Γ in time at most $\frac{Dd_\Gamma(x)}{K_{\min}}$, and the continuous function d_Γ is bounded in the compact set $\bar{\Omega}$. Finally, the proof of the dynamic programming principle is classical: the autonomous case can be found, for instance, in [11, Proposition 2.1] and [27, (8.4)], and the corresponding proofs can be easily adapted to our nonautonomous setting.

Proposition 3.3.1. *Consider $\text{OCP}(k)$ under hypotheses (H1), (H2), (H3'), (H4), and (H5).*

1. *For every $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, there exists an optimal trajectory γ for $\text{OCP}(k)$ from (t_0, x_0) .*
2. *There exists $T > 0$ such that, for every $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, the value function φ satisfies $\varphi(t_0, x_0) \leq T$.*
3. *For every $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$ and $\gamma \in \text{Adm}(k)$ such that $\gamma(t_0) = x_0$, we have, for every $h \geq 0$,*

$$(3.6) \quad \varphi(t_0 + h, \gamma(t_0 + h)) + h \geq \varphi(t_0, x_0),$$

with equality if $\gamma \in \text{Opt}(k, t_0, x_0)$ and $h \in [0, \tau(t_0, \gamma)]$. Conversely, if $\gamma \in \text{Adm}(k)$ satisfies $\gamma(t_0) = x_0$, γ is constant on $[0, t_0]$ and on $[t_0 + \tau(t_0, \gamma), +\infty)$, and equality holds in (3.6) for every $h \in [0, \tau(t_0, \gamma)]$, then $\gamma \in \text{Opt}(k, t_0, x_0)$.

Our next result deals with Lipschitz continuity of the value function φ . This kind of result is classical for optimal control problems with free final time (see, e.g., [27, Proposition 8.2.5] for a proof in the autonomous case), and a complete proof for the nonautonomous

optimal control problem $\text{OCP}(k)$ was given in [75, Propositions 4.2 and 4.3]. However, that reference uses the stronger assumption that $k \in \text{Lip}(\mathbb{R}_+ \times \bar{\Omega}; \mathbb{R}_+)$. When the optimal control problem does not have state constraints, this assumption can be relaxed to (H5) by first showing Lipschitz continuity of φ with respect to x , which can be done by adapting the classical proof of [27, Proposition 8.2.5], and then using the dynamic programming principle to deduce Lipschitz continuity also with respect to t . This strategy was described in [47, Proposition 3.8] and carried out in details in [83, Lemma 4.7 and Proposition 4.8], however those proofs rely on the absence of state constraints and cannot be easily generalized to optimal control problems with state constraints. For that reason, we present here a new proof, which is inspired by that of [75, Propositions 4.2 and 4.3] but uses a technique introduced in the proof of [47, Proposition 3.9] in order to replace the assumption $k \in \text{Lip}(\mathbb{R}_+ \times \bar{\Omega}; \mathbb{R}_+)$ by the weaker assumption (H5). As a first step, we prove Lipschitz continuity of φ in space for fixed time.

Lemma 3.3.2. *Consider the optimal control problem $\text{OCP}(k)$ and its value function φ and assume that (H1), (H2), (H3'), (H4), and (H5) are satisfied. Then there exists $C > 0$ such that, for every $t_0 \in \mathbb{R}_+$ and $x_0, x_1 \in \bar{\Omega}$, we have*

$$(3.7) \quad |\varphi(t_0, x_0) - \varphi(t_0, x_1)| \leq C|x_0 - x_1|.$$

Proof. Let $D > 0$ be as in (H3'), $L > 0$ be as in (H5), and $T > 0$ be as in the statement of Proposition 3.3.12. It suffices to show that there exists $C > 0$ such that, for every $t_0 \in \mathbb{R}_+$ and $x_0, x_1 \in \bar{\Omega}$, we have

$$(3.8) \quad \varphi(t_0, x_1) - \varphi(t_0, x_0) \leq C|x_0 - x_1|,$$

since, in this case, (3.7) can be deduced by exchanging the role of x_0 and x_1 .

Let $t_0 \in \mathbb{R}_+$ and $x_0, x_1 \in \bar{\Omega}$. Let $\gamma_0 \in \text{Opt}(k, t_0, x_0)$ and denote by u_0 the corresponding optimal control, i.e., $\dot{\gamma}_0(t) = k(t, \gamma_0(t))u_0(t)$ for a.e. $t \in \mathbb{R}_+$. Let $t_0^* = t_0 + \varphi(t_0, x_0)$ be the time at which γ_0 arrives at the target set Γ .

Let us first describe informally the idea of the proof. We will construct an admissible trajectory $\gamma_1 \in \text{Lip}(\bar{\Omega})$ which remains constant at x_1 in the time interval $[0, t_0]$, then moves from x_1 to x_0 in a time interval $[t_0, t_1]$, and then follows the same path of γ_0 , but with a change in the time scale since it starts from x_0 at time $t_1 > t_0$. This trajectory will then arrive at $x_0^* \in \Gamma$, and we will prove that its arrival time at Γ satisfies

$$(3.9) \quad \tau(t_0, \gamma_1) \leq \varphi(t_0, x_0) + C|x_0 - x_1|,$$

which yields the conclusion since $\varphi(t_0, x_1) \leq \tau(t_0, \gamma_1)$. The difficult part of the proof is to perform a suitable change in time scale guaranteeing both that γ_1 is admissible and that its arrival time at Γ satisfies the above inequality.

Applying (H3') and renormalizing the speed of the curve whose existence is asserted in that hypothesis, we obtain the existence of $t_1 \geq t_0$ and a Lipschitz continuous curve $\sigma : [t_0, t_1] \rightarrow \bar{\Omega}$ such that $\sigma(t_0) = x_1$, $\sigma(t_1) = x_0$, $|\dot{\sigma}(t)| = K_{\min}$ for almost every $t \in [t_0, t_1]$, and $t_1 - t_0 \leq \frac{D|x_1 - x_0|}{K_{\min}}$.

Let us now define $\phi : [t_1, +\infty) \rightarrow [t_0, +\infty)$ as a solution of the problem

$$\begin{cases} \dot{\phi}(t) = \frac{k(t, \gamma_0(\phi(t)))}{k(\phi(t), \gamma_0(\phi(t)))} & \text{for } t \geq t_1, \\ \phi(t_1) = t_0. \end{cases}$$

Note that, since $(t, s) \mapsto \frac{k(t, \gamma_0(s))}{k(s, \gamma_0(s))}$ is continuous (but not necessarily Lipschitz continuous in its second argument), a solution ϕ to the above problem exists and is of class \mathbf{C}^1 (but it may not be unique). Moreover, $\dot{\phi}(t) \in \left[\frac{K_{\min}}{K_{\max}}, \frac{K_{\max}}{K_{\min}} \right]$ for every $t \in [t_1, +\infty)$, which implies that $\phi : [t_1, +\infty) \rightarrow [t_0, +\infty)$ is increasing and surjective, and hence invertible, and both ϕ and ϕ^{-1} are Lipschitz continuous, with Lipschitz constant $\frac{K_{\max}}{K_{\min}}$. We define $\sigma_1 : [t_1, +\infty) \rightarrow \bar{\Omega}$ by $\sigma_1(t) = \gamma_0(\phi(t))$, which, by construction, satisfies $\sigma_1(t_1) = x_0$, $\sigma_1 \in \text{Lip}(\bar{\Omega})$, and $\dot{\sigma}_1(t) = k(t, \sigma_1(t))u_1(t)$ for a.e. $t \in [t_1, +\infty)$, where u_1 is defined for $t \in [t_1, +\infty)$ by $u_1(t) = u_0(\phi(t))$. In particular, since ϕ^{-1} is Lipschitz continuous, u_1 is measurable. We define $t_1^* = \phi^{-1}(t_0^*)$ and remark that $\sigma_1(t_1^*) = \gamma_0(t_0^*) = x_0^* \in \Gamma$.

Finally, we define $\gamma_1 \in \text{Lip}(\bar{\Omega})$ by $\gamma_1(t) = x_1$ for $t \in [0, t_0]$, $\gamma_1(t) = \sigma(t)$ for $t \in [t_0, t_1]$, and $\gamma_1(t) = \sigma_1(t)$ for $t \in [t_1, +\infty)$. By construction, we have $\gamma_1 \in \text{Adm}(k)$ and $\tau(t_0, \gamma_1) \leq t_1^* - t_0$. We are thus only left to show (3.9).

Note that, for every $t \in [t_1, +\infty)$, we have $k(\phi(t), \gamma_0(\phi(t)))\dot{\phi}(t) = k(t, \gamma_0(\phi(t)))$ and thus, by integrating this identity and performing a change of variables, we deduce that, for every $t \geq t_1$,

$$(3.10) \quad \int_{t_0}^{\phi(t)} k(s, \gamma_0(s)) \, ds = \int_{t_1}^t k(s, \gamma_0(\phi(s))) \, ds.$$

Let $G : [t_0, +\infty) \rightarrow [0, +\infty)$ be defined for $\theta \geq t_0$ by

$$G(\theta) = \int_{t_0}^{\theta} k(s, \gamma_0(s)) \, ds.$$

Then G is differentiable, with $\dot{G}(\theta) = k(\theta, \gamma_0(\theta)) \in [K_{\min}, K_{\max}]$ for every $\theta \geq t_0$. In particular, G is K_{\max} -Lipschitz continuous, invertible, and its inverse is $\frac{1}{K_{\min}}$ -Lipschitz continuous. Moreover, using (3.10), we have, for every $t \geq t_1$, that

$$G(\phi(t)) = \int_{t_1}^t k(s, \gamma_0(\phi(s))) \, ds, \quad G(t) = \int_{t_0}^t k(s, \gamma_0(s)) \, ds.$$

Hence, for every $t \geq t_1$, we have

$$\begin{aligned} |\phi(t) - t| &= \left| G^{-1} \left(\int_{t_1}^t k(s, \gamma_0(\phi(s))) \, ds \right) - G^{-1} \left(\int_{t_0}^t k(s, \gamma_0(s)) \, ds \right) \right| \\ &\leq \frac{1}{K_{\min}} \left| \int_{t_1}^t k(s, \gamma_0(\phi(s))) \, ds - \int_{t_0}^t k(s, \gamma_0(s)) \, ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{K_{\min}} \int_{t_1}^t |k(s, \gamma_0(\phi(s))) - k(s, \gamma_0(s))| ds + \frac{1}{K_{\min}} \int_{t_0}^{t_1} k(s, \gamma_0(s)) ds \\
&\leq \frac{LK_{\max}}{K_{\min}} \int_{t_1}^t |\phi(s) - s| ds + \frac{K_{\max}}{K_{\min}} (t_1 - t_0).
\end{aligned}$$

Thus, by Grönwall's inequality, we have, for every $t \geq t_1$,

$$|\phi(t) - t| \leq (t_1 - t_0) \frac{K_{\max}}{K_{\min}} e^{\frac{LK_{\max}}{K_{\min}}(t-t_1)},$$

which yields, for $t = t_1^*$, that

$$|t_1^* - t_0^*| \leq (t_1 - t_0) \frac{K_{\max}}{K_{\min}} e^{\frac{LK_{\max}}{K_{\min}}(t_1^* - t_1)}.$$

Note that $0 \leq t_1^* - t_1 = \phi^{-1}(t_0^*) - \phi^{-1}(t_0) \leq \frac{K_{\max}}{K_{\min}}(t_0^* - t_0) = \frac{K_{\max}}{K_{\min}}\varphi(t_0, x_0) \leq \frac{TK_{\max}}{K_{\min}}$. Moreover, $t_1^* = t_0 + \tau(t_0, \gamma_1)$ and $t_0^* = t_0 + \varphi(t_0, x_0)$, showing that $|t_1^* - t_0^*| = |\tau(t_0, \gamma_1) - \varphi(t_0, x_0)|$. Hence, we deduce that

$$\tau(t_0, \gamma_1) - \varphi(t_0, x_0) \leq (t_1 - t_0) \frac{K_{\max}}{K_{\min}} \exp\left(\frac{TLK_{\max}^2}{K_{\min}^2}\right).$$

Recalling that $0 \leq t_1 - t_0 \leq \frac{D|x_1 - x_0|}{K_{\min}}$, we finally obtain (3.9) with $C = \frac{DK_{\max}}{K_{\min}^2} \exp\left(\frac{TLK_{\max}^2}{K_{\min}^2}\right)$. \square

Now, exactly as in [83, Proposition 4.8], we can deduce Lipschitz continuity of φ by using Lemma 3.3.2 and the dynamic programming principle from Proposition 3.3.13. We provide a proof of this fact here for sake of completeness.

Proposition 3.3.3. *Consider the optimal control problem $\text{OCP}(k)$ and its value function φ and assume that (H1), (H2), (H3'), (H4), and (H5) are satisfied. Then there exists $M > 0$ such that, for every $(t_0, x_0), (t_1, x_1) \in \mathbb{R}_+ \times \bar{\Omega}$, we have*

$$|\varphi(t_0, x_0) - \varphi(t_1, x_1)| \leq M (|t_0 - t_1| + |x_0 - x_1|).$$

Proof. We denote by $C > 0$ the Lipschitz constant from Lemma 3.3.2. Let $(t_0, x_0), (t_1, x_1) \in \mathbb{R}_+ \times \bar{\Omega}$ and assume, with no loss of generality, that $t_0 < t_1$. Fix $\gamma_0 \in \text{Opt}(k, t_0, x_0)$ and set $x_0^* = \gamma_0(t_1)$. By Lemma 3.3.2, we have

$$(3.11) \quad |\varphi(t_1, x_0^*) - \varphi(t_1, x_1)| \leq C|x_0^* - x_1|.$$

If $t_1 \leq t_0 + \varphi(t_0, x_0)$, then, by Proposition 3.3.13, since $\gamma_0 \in \text{Opt}(k, t_0, x_0)$, we have $\varphi(t_1, x_0^*) = \varphi(t_0, x_0) - (t_1 - t_0)$, and thus

$$(3.12) \quad |\varphi(t_0, x_0) - \varphi(t_1, x_1)| \leq |t_1 - t_0| + C|x_0^* - x_1|.$$

Otherwise, we have $t_1 > t_0 + \varphi(t_0, x_0)$, which shows that $x_0^* = \gamma_0(t_1) = \gamma_0(t_0 + \varphi(t_0, x_0)) \in \Gamma$, and thus $\varphi(t_1, x_0^*) = 0$. Combining this with (3.11) and the fact that $\varphi(t_0, x_0) < t_1 - t_0$, we deduce that (3.12) also holds in this case.

Since γ_0 is K_{\max} -Lipschitz, we have $|x_0^* - x_0| = |\gamma_0(t_1) - \gamma_0(t_0)| \leq K_{\max}|t_1 - t_0|$. Hence, combining with (3.12), we deduce that

$$|\varphi(t_0, x_0) - \varphi(t_1, x_1)| \leq (CK_{\max} + 1)|t_1 - t_0| + C|x_0 - x_1|,$$

yielding the conclusion. \square

The next preliminary result we present is the fact that φ satisfies a Hamilton–Jacobi equation. This kind of result is classical and can be obtained by adapting classical proofs used in the autonomous case, such as those of [11, Chapter IV, Proposition 2.3] and [27, Theorem 8.1.8]. The statement provided here can also be found in [47, Proposition 3.5] and [75, Theorem 4.1].

Proposition 3.3.4. *Consider OCP(k) under hypotheses (H1), (H2), (H3'), (H4), and (H5). The value function φ of OCP(k) satisfies the Hamilton–Jacobi equation*

$$(3.13) \quad -\partial_t \varphi(t, x) + |\nabla \varphi(t, x)|k(t, x) - 1 = 0$$

in the following sense: φ is a viscosity subsolution of (3.13) in $\mathbb{R}_+ \times (\Omega \setminus \Gamma)$ and a viscosity supersolution of (3.13) in $\mathbb{R}_+ \times (\bar{\Omega} \setminus \Gamma)$. Moreover, φ satisfies $\varphi(t, x) = 0$ for every $(t, x) \in \mathbb{R}_+ \times \Gamma$.

We next provide in Proposition 3.3.5 two properties of φ , the first one providing a lower bound on the rate of change of φ over time at a fixed position, and the second one characterizing the optimal control at points at which φ and the optimal trajectory are differentiable. The first result was shown in [75, Proposition 4.4], once again under the stronger assumption that $k \in \text{Lip}(\mathbb{R}_+ \times \bar{\Omega}; \mathbb{R}_+)$, but its proof was later refined in [47, Proposition 3.9] in order to use only assumptions (H4) and (H5). Even though [47] considers only minimal-time mean field games without state constraints, the proof of [47, Proposition 3.9] remains unchanged when state constraints are present. The second result of Proposition 3.3.5 follows as a consequence of the first one and Proposition 3.3.4, as detailed, for instance, in [75, Corollary 4.1].

Proposition 3.3.5. *Consider OCP(k) under hypotheses (H1), (H2), (H3'), (H4), and (H5), and let φ be the value function of OCP(k).*

1. *There exists $c > 0$ such that, for every $x \in \bar{\Omega}$ and $t_0, t_1 \in \mathbb{R}_+$ with $t_0 \neq t_1$, we have*

$$\frac{\varphi(t_1, x) - \varphi(t_0, x)}{t_1 - t_0} \geq c - 1.$$

In particular, if φ is differentiable at (t_0, x) , then $\partial_t \varphi(t_0, x) \geq c - 1$ and $|\nabla \varphi(t_0, x)| \geq c$.

2. For every $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, if $\gamma \in \text{Opt}(k, t_0, x_0)$, $t \in [t_0, t_0 + \varphi(t_0, x_0))$, and φ is differentiable at $(t, \gamma(t))$, then $|\nabla\varphi(t, \gamma(t))| \neq 0$ and

$$\dot{\gamma}(t) = -k(t, \gamma(t)) \frac{\nabla\varphi(t, \gamma(t))}{|\nabla\varphi(t, \gamma(t))|}.$$

We now turn to preliminary results concerning the mean field game $\text{MFG}(K)$. The main result on $\text{MFG}(K)$ we present here is the following, asserting existence of equilibria.

Proposition 3.3.6. *Consider the mean field game $\text{MFG}(K)$ under hypotheses (H1), (H2), (H3'), (H6), and (H7). Then, for every $m_0 \in \mathcal{P}(\bar{\Omega})$, there exists an equilibrium $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$ for $\text{MFG}(K)$ with initial condition m_0 .*

A result similar to Proposition 3.3.6 was shown in [75, Theorem 5.1], but requiring the stronger assumption that $K \in \text{Lip}(\mathcal{P}(\bar{\Omega}) \times \bar{\Omega}; \mathbb{R}_+)$ instead of (H7). Proofs using only the weaker assumption (H7) were provided in [47, Theorem 4.4] and [83, Theorem 5.1]. Both references consider only mean field games without state constraints, but the only point in those proofs where the absence of state constraints is used is to show Lipschitz continuity of value functions of optimal control problems with functions k satisfying (H4) and (H5). Since Proposition 3.3.3 above proves this fact also in the presence of state constraints, the proof of Proposition 3.3.6 can be carried out exactly as in [47, Theorem 4.4] and [83, Theorem 5.1], and it is thus omitted here.

A natural question regarding the definition of an equilibrium of a mean field game is whether we can replace “ Q -almost every $\gamma \in \mathbf{C}(\bar{\Omega})$ satisfies $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$ ” by “every $\gamma \in \text{spt}(Q)$ satisfies $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$ ” in Definition 3.2.5. The last result of this section provides an affirmative answer to that question.

Proposition 3.3.7. *Consider the mean field game $\text{MFG}(K)$ under hypotheses (H1), (H2), (H3'), (H6), and (H7), and let $m_0 \in \mathcal{P}(\bar{\Omega})$. A measure $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$ is an equilibrium of $\text{MFG}(K)$ with initial condition m_0 if and only if $e_{0\#}Q = m_0$ and every $\gamma \in \text{spt}(Q)$ satisfies $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$, where $k_Q : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$ is defined for $t \geq 0$ and $x \in \bar{\Omega}$ by $k_Q(t, x) = K(e_{t\#}Q, x)$.*

Proof. Notice first that $t \mapsto e_{t\#}Q$ is Lipschitz continuous with Lipschitz constant K_{\max} , since, for $t_0, t_1 \in \mathbb{R}_+$, we have

$$\begin{aligned} \mathbf{W}_1(e_{t_0\#}Q, e_{t_1\#}Q) &= \sup_{\Phi \in \text{Lip}_1(\bar{\Omega}; \mathbb{R})} \int_{\bar{\Omega}} \Phi(x) \, d(e_{t_0\#}Q - e_{t_1\#}Q)(x) \\ &= \sup_{\Phi \in \text{Lip}_1(\bar{\Omega}; \mathbb{R})} \int_{\mathbf{C}(\bar{\Omega})} [\Phi(\gamma(t_0)) - \Phi(\gamma(t_1))] \, dQ(\gamma) \leq K_{\max}|t_0 - t_1|. \end{aligned}$$

In particular, k_Q satisfies (H4) and (H5).

The result is proved if we show that, for every $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$, $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$ for every $\gamma \in \text{spt}(Q)$ if and only if $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$ for Q -almost every $\gamma \in \mathbf{C}(\bar{\Omega})$. Since

$\text{spt}(Q)$ is a set of full Q measure, we clearly have $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$ for Q -almost every $\gamma \in \mathbf{C}(\bar{\Omega})$ if $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$ for every $\gamma \in \text{spt}(Q)$.

To prove the converse implication, assume that $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$ for Q -almost every $\gamma \in \mathbf{C}(\bar{\Omega})$. Fix $\gamma \in \text{spt}(Q)$. By definition of support, for every open neighborhood V of γ in $\mathbf{C}(\bar{\Omega})$, we have $Q(V) > 0$, and thus, by assumption, there exists $\gamma_V \in V$ such that $\gamma_V \in \text{Opt}(k_Q, 0, \gamma_V(0))$. In particular, there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ such that $\gamma_n \in \text{Opt}(k_Q, 0, \gamma_n(0))$ for every $n \in \mathbb{N}$ and $\gamma_n \rightarrow \gamma$ in the topology of $\mathbf{C}(\bar{\Omega})$ (i.e., uniformly on compact subsets of \mathbb{R}_+) as $n \rightarrow +\infty$.

Since $\gamma_n \in \text{Opt}(k_Q, 0, \gamma_n(0))$, we have in particular $\gamma_n \in \text{Adm}(k_Q)$, which implies that, for every $t_0, t_1 \in \mathbb{R}_+$, we have

$$|\gamma_n(t_1) - \gamma_n(t_0)| \leq \int_{t_0}^{t_1} k_Q(s, \gamma_n(s)) \, ds.$$

Letting $n \rightarrow +\infty$, we deduce that

$$|\gamma(t_1) - \gamma(t_0)| \leq \int_{t_0}^{t_1} k_Q(s, \gamma(s)) \, ds,$$

which shows in particular that γ is K_{\max} -Lipschitz continuous. Moreover, dividing by $|t_1 - t_0|$ and letting $t_1 \rightarrow t_0$, we deduce that $|\dot{\gamma}(t)| \leq k_Q(t, \gamma(t))$ for almost every $t \in \mathbb{R}_+$, showing that $\gamma \in \text{Adm}(k_Q)$.

To prove optimality of γ , notice that $\gamma_n(\varphi_Q(0, \gamma_n(0))) \in \Gamma$, where φ_Q denotes the value function of $\text{OCP}(k_Q)$. Since Γ is closed and, by Proposition 3.3.3, φ_Q is continuous, we deduce by letting $n \rightarrow +\infty$ that $\gamma(\varphi_Q(0, \gamma(0))) \in \Gamma$, showing that $\tau(0, \gamma) \leq \varphi_Q(0, \gamma(0))$. On the other hand, since $\gamma \in \text{Adm}(k_Q)$, we have $\varphi_Q(0, \gamma(0)) \leq \tau(0, \gamma)$ by the definition (3.3) of φ_Q , yielding that $\tau(0, \gamma) = \varphi_Q(0, \gamma(0))$. Thus $\gamma \in \text{Opt}(k_Q, 0, \gamma(0))$, as required. \square

3.4 Further properties of the optimal control problem

We now provide some further results on the optimal control problem $\text{OCP}(k)$. In the references [47, 75, 83], additional properties of the value function and of optimal trajectories, such as $\mathbf{C}^{1,1}$ regularity of optimal trajectories or differentiability of the value function along optimal trajectories, were obtained only for optimal control problems without state constraints. The technique used in those references is to apply the (unconstrained) Pontryagin Maximum Principle to obtain further properties of optimal trajectories that can then be used to deduce additional results on $\text{OCP}(k)$. The main difficulty in generalizing this technique is that, even though versions of Pontryagin Maximum Principle taking into account control systems with state constraints such as (3.1) exist in the literature (see, e.g., [40, Chapter 5]), it is a difficult problem to get nice additional properties of $\text{OCP}(k)$ from their conclusions.

The strategy we follow in this section relies instead on a penalization technique adapted from [25], which consists in considering an optimal control problem with no state constraints but such that the maximal speed of each agent decays fast to 0 as the agent moves away from

$\bar{\Omega}$. It is then possible to show that, if the penalization parameter is small enough, optimal trajectories of the penalized problem never leave $\bar{\Omega}$ and thus they coincide with optimal trajectories of the original problem with state constraints. As consequences, we provide a boundary condition for the Hamilton–Jacobi equation (3.13) and a characterization of optimal controls, the latter being a key ingredient for showing, in Section 3.5, that equilibria of $\text{MFG}(K)$ satisfy an MFG system.

3.4.1 Penalized optimal control problem

We present in this section the penalized optimal control problem we consider in order to study optimal trajectories of $\text{OCP}(k)$. Before turning to the core of this section, let us first recall, in the next proposition, some classical consequences of (H3) on the signed distance to $\partial\Omega$, whose proofs can be found, for instance, in [45, Theorems 5.1 and 5.7].

Proposition 3.4.1. *Let $\Omega \subset \mathbb{R}^d$ be a set satisfying (H3).*

1. *The signed distance $d_{\partial\Omega}^{\pm}$ is Lipschitz continuous on \mathbb{R}^d , with Lipschitz constant equal to 1.*
2. *There exists a neighborhood W of $\partial\Omega$ such that the signed distance to $\partial\Omega$ satisfies $d_{\partial\Omega}^{\pm} \in \mathbf{C}^{1,1}(W; \mathbb{R})$. Moreover, $|\nabla d_{\partial\Omega}^{\pm}(x)| = 1$ for every $x \in W$.*
3. *For every $x \in \partial\Omega$, $\nabla d_{\partial\Omega}^{\pm}(x)$ is the outward unit normal to Ω at x .*

Let us now introduced the penalized optimal control problem. We fix Ω , Γ , and k satisfying assumptions (H1)–(H3), (H4), and (H5), and we extend k to a continuous function defined on $\mathbb{R}_+ \times \mathbb{R}^d$ and taking values in $[K_{\min}, K_{\max}]$ such that, for every $t \in \mathbb{R}_+$, $x \mapsto k(t, x)$ is L -Lipschitz continuous on \mathbb{R}^d , with L independent of t . Such an extension can be constructed, for instance, by proceeding as in [10].

For $\epsilon > 0$, we define $k_{\epsilon} : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ by

$$(3.14) \quad k_{\epsilon}(t, x) = k(t, x) \left(1 - \frac{1}{\epsilon} d_{\Omega}(x) \right)_+,$$

where the notation a_+ is defined by $a_+ = \max(0, a)$ for $a \in \mathbb{R}$. In particular, k_{ϵ} is continuous, nonnegative, upper bounded by K_{\max} , $x \mapsto k_{\epsilon}(t, x)$ is Lipschitz continuous for every $t \in \mathbb{R}_+$ with a Lipschitz constant $L_{\epsilon} = L + \frac{K_{\max}}{\epsilon}$ independent of t , $k_{\epsilon}(t, x) = k(t, x)$ for every $(t, x) \in \mathbb{R}_+ \times \bar{\Omega}$, and $k_{\epsilon}(t, x) = 0$ if $d_{\Omega}(x) \geq \epsilon$.

We consider the penalized control system

$$(3.15) \quad \dot{\gamma}(t) = k_{\epsilon}(t, \gamma(t))u(t), \quad t \geq 0,$$

where $\gamma(t) \in \mathbb{R}^d$ is the state (which is no longer constrained to remain in $\bar{\Omega}$) and $u(t) \in \bar{B}$ is the control. We denote by $\text{OCP}_{\epsilon}(k_{\epsilon})$ the minimal-time optimal control problem of finding, for any $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, a trajectory $\gamma \in \text{Lip}(\mathbb{R}^d)$ with $\gamma(t_0) = x_0$ and a measurable

control $u : \mathbb{R}_+ \rightarrow \bar{B}$ satisfying (3.15) and minimizing the time at which γ reaches the target set Γ for the first time after t_0 . Similarly to Definition 3.2.3, we denote by $\text{Adm}_\epsilon(k_\epsilon)$ the set of admissible curves for (3.15), i.e., the set of $\gamma \in \text{Lip}(\mathbb{R}^d)$ such that (3.15) is satisfied for some measurable $u : \mathbb{R}_+ \rightarrow \bar{B}$. The definition of the first exit time $\tau(t_0, \gamma)$ from Definition 3.2.3(b) is extended to curves $\gamma \in \text{Lip}(\mathbb{R}^d)$, and we define optimal trajectories for $\text{OCP}_\epsilon(k_\epsilon)$ as in Definition 3.2.3(c), but requiring in addition that $\tau(t_0, \gamma) < +\infty$ for a trajectory γ to be considered optimal from $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$. The set of optimal trajectories for $\text{OCP}_\epsilon(k_\epsilon)$ from $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ is denoted by $\text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$. The value function $\varphi_\epsilon : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ of $\text{OCP}_\epsilon(k_\epsilon)$ is defined similarly to Definition 3.2.3(d).

Standard arguments allow to show that $\text{OCP}_\epsilon(k_\epsilon)$ also satisfies a dynamic programming principle: if $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \in \text{Adm}_\epsilon(k_\epsilon)$ is such that $\gamma(t_0) = x_0$, then, for every $h \geq 0$, we have

$$(3.16) \quad \varphi_\epsilon(t_0 + h, \gamma(t_0 + h)) + h \geq \varphi_\epsilon(t_0, x_0),$$

with equality if $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$ and $h \in [0, \varphi_\epsilon(t_0, x_0)]$. However, contrarily to $\text{OCP}(k)$, we may have nonexistence of optimal trajectories for $\text{OCP}_\epsilon(k_\epsilon)$. More precisely, $\text{Opt}_\epsilon(k_\epsilon, t_0, x_0) = \emptyset$ if $d_\Omega(x_0) \geq \epsilon$, but one can show by standard arguments that this set is nonempty as soon as $d_\Omega(x_0) < \epsilon$. In addition, $\varphi_\epsilon(t, x) < +\infty$ if and only if $d_\Omega(x) < \epsilon$.

The first result we show for $\text{OCP}_\epsilon(k_\epsilon)$ is the following property of optimal trajectories.

Proposition 3.4.2. *Consider the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ under assumptions (H1)–(H3), (H4), and (H5). Let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$. Then $d_\Omega(\gamma(t)) < \epsilon$ for every $t \in \mathbb{R}_+$.*

Proof. Denote by $u : \mathbb{R}_+ \rightarrow \bar{B}$ an optimal control associated with γ . Assume, to obtain a contradiction, that there exists $t_1 \in \mathbb{R}_+$ such that $d_\Omega(\gamma(t_1)) \geq \epsilon$. In particular, $k_\epsilon(t, \gamma(t_1)) = 0$ for every $t \in \mathbb{R}_+$, and thus the function $\tilde{\gamma} : \mathbb{R}_+ \rightarrow \mathbb{R}^d$, defined by $\tilde{\gamma}(t) = \gamma(t_1)$ satisfies $\dot{\tilde{\gamma}}(t) = k_\epsilon(t, \tilde{\gamma}(t))u(t)$ for every $t \in \mathbb{R}_+$. Since the initial value problem consisting of this equation and the initial condition $\tilde{\gamma}(t_1) = \gamma(t_1)$ admits a unique solution, then necessarily $\tilde{\gamma} = \gamma$. However, since $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$, we have $\tau(t_0, \gamma) < +\infty$ and $\gamma(t_0 + \tau(t_0, \gamma)) \in \Gamma \subset \bar{\Omega}$, which contradicts the fact that $d_\Omega(\gamma(t_0 + \tau(t_0, \gamma))) = d_\Omega(\tilde{\gamma}(t_0 + \tau(t_0, \gamma))) = d_\Omega(\gamma(t_1)) \geq \epsilon$. \square

We now apply Pontryagin Maximum Principle to optimal trajectories of the penalized problem $\text{OCP}_\epsilon(k_\epsilon)$ starting from a point in $\mathbb{R}^d \setminus \Gamma$.

Proposition 3.4.3. *Consider the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ under assumptions (H1)–(H3), (H4), and (H5), and with k_ϵ defined from k as in (3.14). Let $(t_0, x_0) \in \mathbb{R}_+ \times (\mathbb{R}^d \setminus \Gamma)$, $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$, $T = \varphi_\epsilon(t_0, x_0)$, and $u : \mathbb{R}_+ \rightarrow \bar{B}$ be an optimal control associated with γ . Then there exist $\lambda \geq 0$ and an absolutely continuous function $p : [t_0, t_0 + T] \rightarrow \mathbb{R}^d$ such that the following assertions hold.*

1. $\lambda + \max_{t \in [t_0, t_0 + T]} |p(t)| > 0$.

2. For almost every $t \in [t_0, t_0 + T]$, we have

$$(3.17) \quad \dot{p}(t) \in \text{co} \{ \zeta \in \mathbb{R}^d \mid (\zeta, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t)) \},$$

where, for $t \geq 0$, the set $G_\epsilon(t)$ is defined by

$$(3.18) \quad G_\epsilon(t) = \{ (x, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid \exists u \in \bar{B} \text{ such that } k_\epsilon(t, x)u = v \}.$$

3. $-p(t_0 + T) \in N_\Gamma(\gamma(t_0 + T))$.

4. One has

$$u(t) = \frac{p(t)}{|p(t)|},$$

almost everywhere on $\{t \in [t_0, t_0 + T] \mid p(t) \neq 0\}$.

5. $\lambda = k_\epsilon(t_0 + T, \gamma(t_0 + T))|p(t_0 + T)|$.

Before turning to the proof of Proposition 3.4.3, we remark that, by taking $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$, we implicitly assume that $\text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$ is not empty, and thus $d_\Omega(x_0) < \epsilon$.

Proof. We apply [88, Theorem 8.4.1] to $\text{OCP}_\epsilon(k_\epsilon)$ and the optimal trajectory γ over the interval $[t_0, t_0 + T]$. Notice, first, that, since $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$, we have in particular $T < +\infty$ and, since $x_0 \in \mathbb{R}^d \setminus \Gamma$, we have $T > 0$. We denote by L_ϵ the Lipschitz constant of $x \mapsto k_\epsilon(t, x)$, and we recall that this constant is independent of $t \in \mathbb{R}_+$.

We first introduce some notation in accordance to the statement of [88, Theorem 8.4.1]. Let $g : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be defined for $(t_1, x_1, t_2, x_2) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ by $g(t_1, x_1, t_2, x_2) = t_2 - t_1$, define $C = \{(t_1, x_1, t_2, x_2) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mid t_1 = t_0, x_1 = x_0, t_2 \geq t_1, x_2 \in \Gamma\}$, and let $F_\epsilon : \mathbb{R} \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be the set-valued map defined for $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ by $F_\epsilon(t, x) = \{k_\epsilon(\max(t, 0), x)u \mid u \in \bar{B}\}$. Notice that the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ starting from the given $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$ can be rephrased as: minimize $g(t_0, \tilde{\gamma}(t_0), t_0 + T, \tilde{\gamma}(t_0 + T))$ over intervals $[t_0, t_0 + T]$ and absolutely continuous functions $\tilde{\gamma} : [t_0, t_0 + T] \rightarrow \mathbb{R}^d$ such that $\dot{\tilde{\gamma}}(t) \in F_\epsilon(t, \tilde{\gamma}(t))$ for a.e. $t \in [t_0, t_0 + T]$ and $(t_0, \tilde{\gamma}(t_0), t_0 + T, \tilde{\gamma}(t_0 + T)) \in C$. Remark also that the set $G_\epsilon(t)$ from 2 is the graph of $\mathbb{R}^d \ni x \mapsto F_\epsilon(t, x) \subset \mathbb{R}^d$.

Let us verify the assumptions of [88, Theorem 8.4.1]. First, the constant $\delta > 0$ from [88, Theorem 8.4.1] can be chosen arbitrarily. Since g is of class \mathbf{C}^∞ and Γ is closed, assumption (H1) from [88, Theorem 8.4.1] is verified. The fact that k_ϵ is continuous also allows one to easily verify assumption (H2) from [88, Theorem 8.4.1]. Using the fact that k_ϵ is L_ϵ -Lipschitz continuous in its second variable uniformly with respect to the first variable, assumption (H3) from [88, Theorem 8.4.1] is satisfied with $\beta = 0$ and $k_{F_\epsilon}(t) = L_\epsilon$ for every $t \in [t_0, t_0 + T]$. Since t_0 is fixed, assumption (H4) from [88, Theorem 8.4.1] is not needed. Finally, using once again that k_ϵ is L_ϵ -Lipschitz continuous in its second variable uniformly with respect to the first variable, we deduce that (H5) from [88, Theorem 8.4.1] holds with $c_1 = K_{\max}$, $k_1 = L_\epsilon$, and with an arbitrary $\delta_1 > 0$. Thus, all assumptions of [88, Theorem 8.4.1] are verified.

The statement of [88, Theorem 8.4.1] now asserts the existence of an absolutely continuous $p : [t_0, t_0 + T] \rightarrow \mathbb{R}^d$ and real numbers $\lambda \geq 0$, ξ , and η , satisfying (i)–(iv) and (vi) from the statement of [88, Theorem 8.4.1]. Items (i) and (ii) are exactly 1 and 2 above.

Assertion [88, Theorem 8.4.1(iii)] states that

$$(3.19) \quad (-\xi, p(t_0), \eta, -p(t_0 + T)) \in \lambda \partial g(t_0, x_0, t_0 + T, \gamma(t_0 + T)) + N_C(t_0, x_0, t_0 + T, \gamma(t_0 + T)),$$

where ∂g denotes the limiting subdifferential of g (see [88, Definition 4.3.1] for its definition). Since g is smooth, we have $\partial g(t_0, x_0, t_0 + T, \gamma(t_0 + T)) = \{\nabla g(t_0, x_0, t_0 + T, \gamma(t_0 + T))\} = \{(-1, 0, 1, 0)\}$.

Let us now compute the limiting normal cone of C at $(t_0, x_0, t_0 + T, \gamma(t_0 + T))$. For that purpose, we first compute its proximal normal cone at every $(t_1, x_1, t_2, x_2) \in C$ with $t_2 > t_1$. Let $(s_1, p_1, s_2, p_2) \in N_C^P(t_1, x_1, t_2, x_2)$. By Definition 3.2.21, there exists $M > 0$ such that

$$\begin{aligned} (s_1, p_1, s_2, p_2) \cdot (t'_1 - t_1, x'_1 - x_1, t'_2 - t_2, x'_2 - x_2) \\ \leq M [|t'_1 - t_1|^2 + |x'_1 - x_1|^2 + |t'_2 - t_2|^2 + |x'_2 - x_2|^2] \end{aligned}$$

for every $(t'_1, x'_1, t'_2, x'_2) \in C$. By definition of C , we have $t_1 = t'_1 = t_0$, $x_1 = x'_1 = x_0$, and thus

$$s_2(t'_2 - t_2) + p_2 \cdot (x'_2 - x_2) \leq M [|t'_2 - t_2|^2 + |x'_2 - x_2|^2].$$

Taking $t'_2 = t_2$, we deduce that $p_2 \cdot (x'_2 - x_2) \leq M|x'_2 - x_2|^2$ for every $x'_2 \in \Gamma$, and thus $p_2 \in N_\Gamma^P(x_2)$. Taking $x'_2 = x_2$, we deduce that $s_2(t'_2 - t_2) \leq M|t'_2 - t_2|^2$ for every $t'_2 \geq t'_1 = t_0$. In particular, for every $\rho > 0$, taking $t'_2 = t_2 + \rho$, we have $s_2\rho \leq M\rho^2$, which yields, since $\rho > 0$ is arbitrary, that $s_2 \leq 0$. On the other hand, taking $t'_2 = t_2 - \rho$ for $\rho \in (0, t_2 - t_1]$, we have $-s_2\rho \leq M\rho^2$, which yields, since $\rho \in (0, t_2 - t_1]$ is arbitrary and $t_2 > t_1$, that $-s_2 \leq 0$, and thus $s_2 = 0$. We have thus shown that³

$$N_C^P(t_1, x_1, t_2, x_2) \subset \mathbb{R} \times \mathbb{R}^d \times \{0\} \times N_\Gamma^P(x_2),$$

It now follows, using Definition 3.2.22, that⁴

$$N_C(t_0, x_0, t_0 + T, \gamma(t_0 + T)) \subset \mathbb{R} \times \mathbb{R}^d \times \{0\} \times N_\Gamma(\gamma(t_0 + T)).$$

In particular, (3.19) imposes no constraints on ξ and $p(t_0)$, and asserts that $\eta = \lambda$ and $-p(t_0 + T) \in N_\Gamma(\gamma(t_0 + T))$, showing 3.

To show 4, notice that [88, Theorem 8.4.1(iv)] states that, for almost every $t \in [t_0, t_0 + T]$, we have $p(t) \cdot \dot{\gamma}(t) \geq p(t) \cdot v$ for every $v \in F_\epsilon(t, \gamma(t))$, which means that, for almost every $t \in [t_0, t_0 + T]$, we have $k_\epsilon(t, \gamma(t))p(t) \cdot u(t) \geq k_\epsilon(t, \gamma(t))p(t) \cdot w$ for every $w \in \bar{B}$. By Proposition 3.4.2, we have that $d_\Omega(\gamma(t)) < \epsilon$ for every $t \in [t_0, t_0 + T]$, yielding from (3.14)

³The converse inclusion can be shown by straightforward arguments, but it is not necessary for our proof. Notice also that this inclusion is false in the case $t_2 = t_1$, but this case is not needed in our proof.

⁴Once again, we also have the converse inclusion, but it is not necessary for our proof.

that $k_\epsilon(t, \gamma(t)) > 0$. Hence, for almost every $t \in [t_0, t_0 + T]$, we have $p(t) \cdot u(t) \geq p(t) \cdot w$ for every $w \in \bar{B}$, and this is equivalent to 4.

Finally, define $H_\epsilon : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ for $(t, x, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ by $H_\epsilon(t, x, p) = \sup_{v \in F_\epsilon(t, x)} p \cdot v$, and notice that $H_\epsilon(t, x, p) = k_\epsilon(t, x)|p|$. In particular, remarking that H_ϵ is continuous, [88, Theorem 8.4.1(vi)] then asserts that $\eta = k_\epsilon(t_0 + T, \gamma(t_0 + T))|p(t_0 + T)|$, and thus 5 holds since $\eta = \lambda$. \square

The absolutely continuous function p from Proposition 3.4.32 is known as the *costate* associated with the optimal trajectory γ . Even though Proposition 3.4.32 provides the differential inclusion (3.17) for the costate p , this inclusion is in general hard to manipulate. We provide, in the next lemma, the main consequence of this differential inclusion that we will use in the sequel. Recall that, for $r > 0$, \bar{B}_r denotes the closed ball centered at the origin and with radius r .

Lemma 3.4.4. *Consider the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ under assumptions (H1)–(H3), (H4), and (H5), and with k_ϵ defined from k as in (3.14). Let (t_0, x_0) , γ , T , and p be as in the statement of Proposition 3.4.3 and $L_\epsilon > 0$ be the Lipschitz constant of $x \mapsto k_\epsilon(t, x)$, which is independent of t . Then, for almost every $t \in [t_0, t_0 + T]$, we have*

$$\text{co} \{ \zeta \in \mathbb{R}^d \mid (\zeta, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t)) \} \subset \bar{B}_{L_\epsilon |p(t)|}.$$

In particular, for almost every $t \in [t_0, t_0 + T]$, we have $|\dot{p}(t)| \leq L_\epsilon |p(t)|$.

Proof. Let G_ϵ be defined as in the statement of Proposition 3.4.32. Since $\bar{B}_{L_\epsilon |p(t)|}$ is convex, the result is proved if we show that, for a.e. $t \in [t_0, t_0 + T]$ and every $\zeta \in \mathbb{R}^d$, if $(\zeta, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t))$, then $|\zeta| \leq L_\epsilon |p(t)|$. Note that this inequality is trivial if $\zeta = 0$, and thus we assume from now on that $\zeta \neq 0$.

Fix $t \in [t_0, t_0 + T]$ at which p and γ are differentiable and (3.17) holds. Let $\zeta \in \mathbb{R}^d$ be such that $(\zeta, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t))$. By definition of the limiting normal cone of $G_\epsilon(t)$, there exist sequences $(x_n, v_n)_{n \in \mathbb{N}}$ in $G_\epsilon(t)$ and $(\zeta_n, p_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^d \times \mathbb{R}^d$ such that $(x_n, v_n) \rightarrow (\gamma(t), \dot{\gamma}(t))$ and $(\zeta_n, p_n) \rightarrow (\zeta, p(t))$ as $n \rightarrow +\infty$ and $(\zeta_n, p_n) \in N_{G_\epsilon(t)}^P(x_n, v_n)$ for every $n \in \mathbb{N}$. Since $\zeta \neq 0$, we assume, with no loss of generality, that $\zeta_n \neq 0$ for every $n \in \mathbb{N}$.

Take $n \in \mathbb{N}$. Since $(x_n, v_n) \in G_\epsilon(t)$, it follows from (3.18) that there exists $u_n \in \bar{B}$ such that $v_n = k_\epsilon(t, x_n)u_n$. Since $(\zeta_n, p_n) \in N_{G_\epsilon(t)}^P(x_n, v_n)$, there exists $M_n > 0$ such that

$$(3.20) \quad (\zeta_n, p_n) \cdot (y - x_n, k_\epsilon(t, y)u - k_\epsilon(t, x_n)u_n) \leq M_n [|y - x_n|^2 + |k_\epsilon(t, y)u - k_\epsilon(t, x_n)u_n|^2]$$

for every $y \in \mathbb{R}^d$ and $u \in \bar{B}$. Let $\alpha_n = ((n+1)(1+L_\epsilon^2)M_n)^{-1} > 0$ and take $y = x_n + \alpha_n \zeta_n$ and $u = u_n$ in (3.20). Then

$$\begin{aligned} \alpha_n |\zeta_n|^2 + [k_\epsilon(t, x_n + \alpha_n \zeta_n) - k_\epsilon(t, x_n)] p_n \cdot u_n \\ \leq M_n [\alpha_n^2 |\zeta_n|^2 + |k_\epsilon(t, x_n + \alpha_n \zeta_n) - k_\epsilon(t, x_n)|^2]. \end{aligned}$$

Hence

$$\alpha_n |\zeta_n|^2 \leq M_n \alpha_n^2 |\zeta_n|^2 + M_n L_\epsilon^2 \alpha_n^2 |\zeta_n|^2 + \alpha_n L_\epsilon |\zeta_n| |p_n|.$$

Since $\alpha_n > 0$ and $|\zeta_n| > 0$, we deduce that

$$(1 - M_n \alpha_n - M_n L_\epsilon^2 \alpha_n) |\zeta_n| \leq L_\epsilon |p_n|.$$

By definition of α_n , we have $M_n \alpha_n + M_n L_\epsilon^2 \alpha_n = \frac{1}{n+1} < 1$, and thus

$$|\zeta_n| \leq \frac{L_\epsilon}{1 - \frac{1}{n+1}} |p_n|.$$

The conclusion follows by letting $n \rightarrow +\infty$. \square

Using Lemma 3.4.4, we can prove additional regularity properties of optimal trajectories and optimal controls for $\text{OCP}_\epsilon(k_\epsilon)$.

Proposition 3.4.5. *Consider the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ under assumptions (H1)–(H3), (H4), and (H5), and with k_ϵ defined from k as in (3.14). Let (t_0, x_0) , γ , T , u , and p be as in the statement of Proposition 3.4.3 and $L_\epsilon > 0$ be the Lipschitz constant of k_ϵ with respect to its second variable, which is independent of its first variable. For $t \in [t_0, t_0 + T]$, define $\Xi_\epsilon(t)$ by*

$$(3.21) \quad \Xi_\epsilon(t) = \text{co} \{ \xi \in \mathbb{R}^d \mid (\xi |p(t)|, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t)) \},$$

where $G_\epsilon(t)$ is given by (3.18). Then, up to redefining u in a set of Lebesgue measure zero, the following assertions hold.

1. For every $t \in [t_0, t_0 + T]$, we have $|p(t)| \neq 0$ and $u(t) = \frac{p(t)}{|p(t)|}$.
2. For almost every $t \in [t_0, t_0 + T]$ and every $\xi \in \Xi_\epsilon(t)$, we have $|\xi| \leq L_\epsilon$.
3. We have $\gamma \in \mathbf{C}^1([t_0, t_0 + T]; \mathbb{R}^d)$, $u \in \text{Lip}_{L_\epsilon}([t_0, t_0 + T]; \mathbb{S}^{d-1})$, and (γ, u) satisfies, for almost every $t \in [t_0, t_0 + T]$, the system

$$(3.22) \quad \begin{cases} \dot{\gamma}(t) = k_\epsilon(t, \gamma(t))u(t) \\ \dot{u}(t) \in \text{Pr}_{u(t)}^\perp \Xi_\epsilon(t) \end{cases}$$

where, for $x \in \mathbb{S}^{d-1}$ and $A \subset \mathbb{R}^d$, $\text{Pr}_x^\perp A$ denotes the projection of the vectors of A onto the tangent space of \mathbb{S}^{d-1} at x , defined by $\text{Pr}_x^\perp A = \{a - (x \cdot a)x \mid a \in A\}$.

Proof. Let us first prove 1. By Lemma 3.4.4, we have that $|\dot{p}(t)| \leq |p(t)|L_\epsilon$, implying that, for every $t_1, t_2 \in [t_0, t_0 + T]$, we have

$$|p(t_2)| \leq |p(t_1)| + L_\epsilon \int_{\min\{t_1, t_2\}}^{\max\{t_1, t_2\}} |p(s)| \, ds.$$

Hence, by Grönwall's inequality, for every $t_1, t_2 \in [t_0, t_0 + T]$, we have

$$|p(t_2)| \leq |p_\epsilon(t_1)|e^{L_\epsilon|t_2-t_1|}.$$

If there exists t_1 such that $p(t_1) = 0$, then $p(t) = 0$ for every $t \in [t_0, t_0 + T]$. Letting $\lambda \geq 0$ be as in the statement of Proposition 3.4.3, we have, by Proposition 3.4.35, that $\lambda = 0$, which is a contradiction according to Proposition 3.4.31. Therefore $|p(t)| \neq 0$ for every $t \in [t_0, t_0 + T]$. We then deduce from Proposition 3.4.34 that $u(t) = \frac{p(t)}{|p(t)|}$ for almost every $t \in [t_0, t_0 + T]$, and the conclusion for every $t \in [t_0, t_0 + T]$ holds by redefining u in a set of measure zero. Hence 1 is proved.

Note that, for every $t \in [t_0, t_0 + T]$, we have $\xi \in \Xi_\epsilon(t)$ if and only if

$$\frac{\xi}{|p(t)|} \in \text{co} \{ \zeta \in \mathbb{R}^d \mid (\zeta, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t)) \}.$$

In particular, 2 follows as a consequence of 1 and Lemma 3.4.4. Notice also that, by Proposition 3.4.32, we have $\dot{p}(t) \in |p(t)|\Xi_\epsilon(t)$ for a.e. $t \in [t_0, t_0 + T]$.

Let us now prove 3. Note that, by 1, $u(t) \in \mathbb{S}^{d-1}$ for every $t \in [t_0, t_0 + T]$ and, since p is absolutely continuous, u is also absolutely continuous. Let $\xi : [t_0, t_0 + T] \rightarrow \mathbb{R}^d$ be a measurable function with $\xi(t) \in \Xi_\epsilon(t)$ and $\dot{p}(t) = |p(t)|\xi(t)$ for a.e. $t \in [t_0, t_0 + T]$. We compute, for a.e. $t \in [t_0, t_0 + T]$,

$$\dot{u}(t) = \frac{\dot{p}(t)|p(t)| - \frac{p(t)\dot{p}(t)}{|p(t)|}p(t)}{|p(t)|^2} = \xi(t) - (u(t) \cdot \xi(t))u(t),$$

which yields the differential inclusion for u in system (3.22). By 2, we deduce that $|\dot{u}(t)| \leq L_\epsilon$ for a.e. $t \in [t_0, t_0 + T]$, yielding that u is L_ϵ -Lipschitz continuous on $[t_0, t_0 + T]$. The first equation in (3.22) is simply (3.15), and its right-hand side is continuous on $[t_0, t_0 + T]$, showing that $\gamma \in \mathbf{C}^1([t_0, t_0 + T]; \mathbb{R}^d)$. \square

We will need in the sequel the following technical lemma.

Lemma 3.4.6. *Consider the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ under assumptions (H1)–(H3), (H4), and (H5), and with k_ϵ defined from k as in (3.14). Let (t_0, x_0) , γ , T , and p be as in the statement of Proposition 3.4.3 and $L > 0$ be the Lipschitz constant of $\mathbb{R}^d \ni x \mapsto k(t, x) \in [K_{\min}, K_{\max}]$, which is independent of t . For $t \in [t_0, t_0 + T]$, define $\Xi_\epsilon(t)$ as in the statement of Proposition 3.4.5. Finally, let W be as in the statement of Proposition 3.4.12.*

For almost every $t \in [t_0, t_0 + T]$, if $\gamma(t) \in W$ and $\gamma(t) \notin \bar{\Omega}$, then, for every $\xi \in \Xi_\epsilon(t)$, we have

$$(3.23) \quad \left| \xi \cdot \nabla d_{\partial\Omega}^\pm(\gamma(t)) - \frac{1}{\epsilon}k(t, \gamma(t)) \right| \leq \left(1 - \frac{1}{\epsilon}d_\Omega(\gamma(t)) \right) L.$$

Proof. Fix $t \in [t_0, t_0 + T]$ at which γ is differentiable and such that $\gamma(t) \in W$ and $\gamma(t) \notin \bar{\Omega}$. Since the set of $\xi \in \mathbb{R}^d$ such that (3.23) holds is convex, it suffices to show that, for every

$\xi \in \mathbb{R}^d$, if $(\xi|p(t)|, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t))$, then (3.23) holds, where $G_\epsilon(t)$ is given by (3.18). Fix $\xi \in \mathbb{R}^d$ such that $(\xi|p(t)|, p(t)) \in N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t))$. Let u be an optimal control associated with γ and recall that, by Proposition 3.4.5, u is Lipschitz continuous.

By definition of $N_{G_\epsilon(t)}(\gamma(t), \dot{\gamma}(t))$, there exist sequences $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d , $(u_n)_{n \in \mathbb{N}}$ in \bar{B} , $(\zeta_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d , and $(p_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that, as $n \rightarrow +\infty$, we have $x_n \rightarrow \gamma(t)$, $k_\epsilon(t, x_n)u_n \rightarrow k_\epsilon(t, \gamma(t))u(t)$, $\zeta_n \rightarrow \xi|p(t)|$, and $p_n \rightarrow p(t)$, and in addition $(\zeta_n, p_n) \in N_{G_\epsilon(t)}^P(x_n, k_\epsilon(t, x_n)u_n)$ for every $n \in \mathbb{N}$. By Proposition 3.4.51, we have $|p(t)| \neq 0$, and we then define $\xi_n = \frac{\zeta_n}{|p(t)|}$ for $n \in \mathbb{N}$. Then, as $n \rightarrow +\infty$, we have $\xi_n \rightarrow \xi$ and, since k_ϵ is continuous and⁵ $k_\epsilon(t, \gamma(t)) > 0$, we also have $u_n \rightarrow u(t)$.

For every $n \in \mathbb{N}$, since $(\zeta_n, p_n) \in N_{G_\epsilon(t)}^P(x_n, k_\epsilon(t, x_n)u_n)$, there exists $M_n > 0$ such that

$$(3.24) \quad (\xi_n|p(t)|, p_n) \cdot (y - x_n, k_\epsilon(t, y)w - k_\epsilon(t, x_n)u_n) \leq M_n [|y - x_n|^2 + |k_\epsilon(t, y)w - k_\epsilon(t, x_n)u_n|^2]$$

for every $y \in \mathbb{R}^d$ and $w \in \bar{B}$. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that $\alpha_n \rightarrow 0$ and $M_n \alpha_n \rightarrow 0$ as $n \rightarrow +\infty$. Since $\gamma(t) \in W$, $\gamma(t) \notin \bar{\Omega}$, and $d_\Omega(\gamma(t)) < \epsilon$, for n large enough, we have $x_n \in W$, $x_n \notin \bar{\Omega}$, and $d_\Omega(x_n) < \epsilon$. In particular, by Proposition 3.4.12, $\nabla d_{\partial\Omega}^\pm(x_n)$ is well-defined and $|\nabla d_{\partial\Omega}^\pm(x_n)| = 1$. For $n \in \mathbb{N}$, apply (3.24) with $y = x_n + \alpha_n \nabla d_{\partial\Omega}^\pm(x_n)$ and $w = u_n$. Then

$$\alpha_n |p(t)| \xi_n \cdot \nabla d_{\partial\Omega}^\pm(x_n) + p_n \cdot u_n [k_\epsilon(t, x_n + \alpha_n \nabla d_{\partial\Omega}^\pm(x_n)) - k_\epsilon(t, x_n)] \leq M_n \alpha_n^2 (1 + L_\epsilon^2),$$

where L_ϵ is the Lipschitz constant of $\mathbb{R}^d \ni x \mapsto k_\epsilon(t, x) \in \mathbb{R}_+$, which is independent of t . Dividing by α_n , we get

$$(3.25) \quad |p(t)| \xi_n \cdot \nabla d_{\partial\Omega}^\pm(x_n) + p_n \cdot u_n \frac{k_\epsilon(t, x_n + \alpha_n \nabla d_{\partial\Omega}^\pm(x_n)) - k_\epsilon(t, x_n)}{\alpha_n} \leq M_n \alpha_n (1 + L_\epsilon^2).$$

Since $\nabla d_{\partial\Omega}^\pm$ is continuous in W , we have, as $n \rightarrow +\infty$, that $\xi_n \cdot \nabla d_{\partial\Omega}^\pm(x_n) \rightarrow \xi \cdot \nabla d_{\partial\Omega}^\pm(\gamma(t))$, $p_n \cdot u_n \rightarrow p(t) \cdot u(t) = |p(t)|$, and, by construction, $M_n \alpha_n \rightarrow 0$.

Let us denote for simplicity $z_n = x_n + \alpha_n \nabla d_{\partial\Omega}^\pm(x_n)$. Since $\alpha_n \rightarrow 0$ as $n \rightarrow +\infty$ and $\gamma(t) \in W$, $\gamma(t) \notin \bar{\Omega}$, and $d_\Omega(\gamma(t)) < \epsilon$, we deduce that, for n large enough, $x_n \in W$, $z_n \in W$, $x_n \notin \bar{\Omega}$, $z_n \notin \bar{\Omega}$, $d_\Omega(x_n) < \epsilon$, and $d_\Omega(z_n) < \epsilon$ for n large enough. Using those facts and (3.14), we compute

$$\begin{aligned} \frac{1}{\alpha_n} [k_\epsilon(t, z_n) - k_\epsilon(t, x_n)] &= \frac{1}{\alpha_n} \left[k(t, z_n) \left(1 - \frac{1}{\epsilon} d_{\partial\Omega}^\pm(z_n) \right) - k(t, x_n) \left(1 - \frac{1}{\epsilon} d_{\partial\Omega}^\pm(x_n) \right) \right] \\ &= \frac{1}{\epsilon} k(t, z_n) \frac{d_{\partial\Omega}^\pm(x_n) - d_{\partial\Omega}^\pm(z_n)}{\alpha_n} + \beta_n \left(1 - \frac{1}{\epsilon} d_{\partial\Omega}^\pm(x_n) \right), \end{aligned}$$

where

$$\beta_n = \frac{k(t, z_n) - k(t, x_n)}{\alpha_n}.$$

⁵Recall that, by Proposition 3.4.2, we have $d_\Omega(\gamma(t)) < \epsilon$.

Note that, since k is L -Lipschitz continuous in its second argument, we have $|\beta_n| \leq L$ for n large enough. In particular, up to extracting a subsequence, there exists $\beta_* \in [-L, L]$ such that $\beta_n \rightarrow \beta_*$ as $n \rightarrow +\infty$. Moreover, since $d_{\partial\Omega}^\pm \in \mathbf{C}^{1,1}(W; \mathbb{R})$, we have $\frac{d_{\partial\Omega}^\pm(x_n) - d_{\partial\Omega}^\pm(z_n)}{\alpha_n} \rightarrow -|\nabla d_{\partial\Omega}^\pm(\gamma(t))|^2 = -1$ as $n \rightarrow +\infty$. Hence, letting $n \rightarrow +\infty$ in (3.25) and dividing by $|p(t)|$, we deduce that

$$(3.26) \quad \xi \cdot \nabla d_{\partial\Omega}^\pm(\gamma(t)) - \frac{1}{\epsilon} k(t, \gamma(t)) + \beta_* \left(1 - \frac{1}{\epsilon} d_{\partial\Omega}^\pm(\gamma(t)) \right) \leq 0.$$

Performing the same computations as above from (3.24) but now taking $y = x_n - \alpha_n \nabla d_{\partial\Omega}^\pm(x_n)$ and $w = u_n$, we get that

$$-\xi \cdot \nabla d_{\partial\Omega}^\pm(\gamma(t)) + \frac{1}{\epsilon} k(t, \gamma(t)) + \tilde{\beta}_* \left(1 - \frac{1}{\epsilon} d_{\partial\Omega}^\pm(\gamma(t)) \right) \leq 0$$

for some $\tilde{\beta}_* \in [-L, L]$. Together with (3.26), this yields (3.23), as required. \square

Proposition 3.4.2 stated that, for every $\epsilon > 0$, optimal trajectories for $\text{OCP}_\epsilon(k_\epsilon)$ always remain in an ϵ neighborhood of $\bar{\Omega}$. Using the previous results, we can now be more precise and show that, if $\epsilon > 0$ is small enough, optimal trajectories for $\text{OCP}_\epsilon(k_\epsilon)$ starting at $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$ never leave $\bar{\Omega}$.

Proposition 3.4.7. *Consider the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ under assumptions (H1)–(H3), (H4), and (H5) and k_ϵ defined from k as in (3.14). There exists $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$, $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, and $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$, we have $d_\Omega(\gamma(t)) \leq d_\Omega(x_0)$ for every $t \in \mathbb{R}_+$. In particular, if $x_0 \in \bar{\Omega}$, then $\gamma(t) \in \bar{\Omega}$ for every $t \in \mathbb{R}_+$.*

Proof. Take $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$. Notice first that, if $x_0 \in \Gamma$, then $\text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$ is equal to the singleton containing only the trajectory that remains in x_0 for all times. We thus assume, for the rest of the proof, that $x_0 \in \bar{\Omega} \setminus \Gamma$.

Let W be as in the statement of Proposition 3.4.12 and take $\epsilon_* > 0$ such that $\{x \in \mathbb{R}^d \mid d_{\partial\Omega}(x) < \epsilon_*\} \subset W$. Let C be a Lipschitz constant of $\nabla d_{\partial\Omega}^\pm$ on W and L, K_{\max} , and K_{\min} be as in (H4) and (H5). Let $\epsilon_0 = \min \left\{ \epsilon_*, \frac{K_{\min}}{L + CK_{\max}} \right\}$ and take $\epsilon \in (0, \epsilon_0)$.

Fix $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$ and let $T = \varphi_\epsilon(t_0, x_0)$. Recall that, by Proposition 3.4.2, $d_\Omega(\gamma(t)) < \epsilon$ for every $t \in \mathbb{R}_+$. Let $p : [t_0, t_0 + T] \rightarrow \mathbb{R}^d$ be the costate associated with γ , whose existence is asserted in Proposition 3.4.3. By Propositions 3.4.32 and 3.4.51, there exists a measurable function $\xi : [t_0, t_0 + T] \rightarrow \mathbb{R}^d$ such that $\xi(t) \in \Xi_\epsilon(t)$ and $\dot{p}(t) = |p(t)|\xi(t)$ for a.e. $t \in [t_0, t_0 + T]$, where $\Xi_\epsilon(t)$ is defined in (3.21).

By definition of optimal trajectories, we have that $\gamma(t) = x_0$ for every $t \in [0, t_0]$ and $\gamma(t) = \gamma(t_0 + T) \in \Gamma \subset \bar{\Omega}$ for every $t \in [t_0 + T, +\infty)$, and hence $d_\Omega(\gamma(t)) \leq d_\Omega(x_0)$ for every $t \in [0, t_0] \cup [t_0 + T, +\infty)$. Assume, to obtain a contradiction, that there exist $a, b \in [t_0, t_0 + T]$ such that $a < b$, $d_\Omega(\gamma(t)) > d_\Omega(x_0)$ for $t \in (a, b)$ and $d_\Omega(\gamma(t)) = d_\Omega(x_0)$ for $t \in \{a, b\}$. In particular, $d_\Omega(\gamma(t)) > 0$ for $t \in (a, b)$, implying that $\gamma(t) \notin \bar{\Omega}$ for $t \in (a, b)$ and $\gamma(t) \notin \Omega$ for $t \in [a, b]$.

Recall that, by Proposition 3.4.2, $d_\Omega(\gamma(t)) < \epsilon$ for every $t \in \mathbb{R}_+$, and thus, in particular, $\gamma(t) \in W$ for every $t \in [a, b]$. Hence, by Propositions 3.4.1 and 3.4.53, the map $t \mapsto d_{\partial\Omega}^\pm(\gamma(t))$ is differentiable on $(t_0, t_0 + T)$, $d_{\partial\Omega}^\pm(\gamma(t)) > d_\Omega(x_0)$ for $t \in (a, b)$, and $d_{\partial\Omega}^\pm(\gamma(t)) = d_\Omega(x_0)$ for $t \in \{a, b\}$. Thus, its derivative is nonnegative at a and nonpositive at b , i.e.,

$$\dot{\gamma}(a) \cdot \nabla d_{\partial\Omega}^\pm(\gamma(a)) \geq 0 \quad \text{and} \quad \dot{\gamma}(b) \cdot \nabla d_{\partial\Omega}^\pm(\gamma(b)) \leq 0.$$

Since, by Propositions 3.4.2 and 3.4.5, we have $\dot{\gamma}(t) = k_\epsilon(t, \gamma(t)) \frac{p(t)}{|p(t)|}$ and $\frac{k_\epsilon(t, \gamma(t))}{|p(t)|} > 0$ for every $t \in [t_0, t_0 + T]$, we deduce that

$$(3.27) \quad p(a) \cdot \nabla d_{\partial\Omega}^\pm(\gamma(a)) \geq 0 \quad \text{and} \quad p(b) \cdot \nabla d_{\partial\Omega}^\pm(\gamma(b)) \leq 0.$$

Consider now the map $\alpha : t \mapsto p(t) \cdot \nabla d_{\partial\Omega}^\pm(\gamma(t))$. Since $\gamma(t) \in W$ for every $t \in [a, b]$ and $d_{\partial\Omega}^\pm \in \mathbf{C}^{1,1}(W; \mathbb{R})$, α is absolutely continuous on $[a, b]$ and, for a.e. $t \in (a, b)$, we have, using Lemma 3.4.6 and the fact that $\nabla d_{\partial\Omega}^\pm \circ \gamma$ is CK_{\max} -Lipschitz continuous in $[a, b]$, that

$$\begin{aligned} \dot{\alpha}(t) &= \dot{p}(t) \cdot \nabla d_{\partial\Omega}^\pm(\gamma(t)) + p(t) \cdot \frac{d[\nabla d_{\partial\Omega}^\pm \circ \gamma]}{dt}(t) \\ &= |p(t)| \xi(t) \cdot \nabla d_{\partial\Omega}^\pm(\gamma(t)) + p(t) \cdot \frac{d[\nabla d_{\partial\Omega}^\pm \circ \gamma]}{dt}(t) \\ &\geq \frac{1}{\epsilon} |p(t)| k(t, \gamma(t)) - \left(1 - \frac{1}{\epsilon} d_\Omega(\gamma(t))\right) L |p(t)| - CK_{\max} |p(t)| \\ &\geq |p(t)| \left(\frac{K_{\min}}{\epsilon} - L - CK_{\max} \right) > 0, \end{aligned}$$

which contradicts (3.27). □

We are now in position to show the main result of this section.

Theorem 3.4.8. *Consider the optimal control problem $\text{OCP}(k)$ under assumptions (H1)–(H3), (H4), and (H5), as well as the problem $\text{OCP}_\epsilon(k_\epsilon)$ with k_ϵ defined from k as in (3.14). There exists $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$ and $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, we have $\text{Opt}_\epsilon(k_\epsilon, t_0, x_0) = \text{Opt}(k, t_0, x_0)$ and $\varphi_\epsilon(t_0, x_0) = \varphi(t_0, x_0)$.*

Proof. Let $\epsilon_0 > 0$ be as in the statement of Proposition 3.4.7, $\epsilon \in (0, \epsilon_0)$, and $(t_0, x_0) \in \mathbb{R} \times \bar{\Omega}$. Let $\gamma_\epsilon \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$ and $\gamma \in \text{Opt}(k, t_0, x_0)$, and denote $T_\epsilon = \varphi_\epsilon(t_0, x_0)$ and $T = \varphi(t_0, x_0)$. Clearly, $\gamma \in \text{Adm}_\epsilon(k_\epsilon)$, and thus $T_\epsilon \leq T$. On the other hand, thanks to Proposition 3.4.7, $\gamma_\epsilon \in \text{Adm}(k)$, which leads us to $T \leq T_\epsilon$. Hence $T = T_\epsilon$, which shows that we have both $\gamma_\epsilon \in \text{Opt}(k, t_0, x_0)$ and $\gamma \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$, as required. □

Thanks to Theorem 3.4.8, all properties established in this section for optimal trajectories of $\text{OCP}_\epsilon(k_\epsilon)$ also hold for optimal trajectories of $\text{OCP}(k)$. The next statement, whose proof is an immediate consequence of Proposition 3.4.53 and Theorem 3.4.8, gathers the properties of optimal trajectories of $\text{OCP}(k)$ that will be used in the sequel.

Corollary 3.4.9. *Consider the optimal control problem $\text{OCP}(k)$ under assumptions (H1)–(H3), (H4), and (H5). Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, $\gamma \in \text{Opt}(k, t_0, x_0)$, $T = \varphi(t_0, x_0)$, and $u : \mathbb{R}_+ \rightarrow \bar{B}$ be an optimal control associated with γ . Then, up to redefining u in a set of Lebesgue measure zero, we have $\gamma \in \mathbf{C}^1([t_0, t_0 + T]; \bar{\Omega})$ and $u \in \text{Lip}([t_0, t_0 + T]; \mathbb{S}^{d-1})$.*

Before concluding this section, we also provide the following result on the local Lipschitz continuity of φ_ϵ , which can be obtained as a consequence of Proposition 3.4.7. For its statement, we introduce, for $\epsilon > 0$, the set

$$\Omega_\epsilon = \{x \in \mathbb{R}^d \mid d_\Omega(x) < \epsilon\}.$$

Proposition 3.4.10. *Consider the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ and its value function φ_ϵ under assumptions (H1)–(H3), (H4), and (H5) and k_ϵ defined from k as in (3.14). Then there exists $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$, the value function $\varphi_\epsilon : \mathbb{R}_+ \times \Omega_\epsilon \rightarrow \mathbb{R}_+$ of $\text{OCP}_\epsilon(k_\epsilon)$ is locally Lipschitz continuous. More precisely, for every $\eta \in [0, \epsilon)$, φ_ϵ is Lipschitz continuous on $\mathbb{R}_+ \times \bar{\Omega}_\eta$.*

Proof. Let $\epsilon_0 > 0$ be as in the statement of Proposition 3.4.7 and fix $\epsilon \in (0, \epsilon_0)$ and $\eta \in [0, \epsilon)$. We consider the optimal control problem $\widehat{\text{OCP}}$ defined as follows: given $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}_\eta$, minimize $\tau(t_0, \gamma)$ over all Lipschitz continuous curves $\gamma : \mathbb{R}_+ \rightarrow \bar{\Omega}_\eta$ satisfying (3.15) for some measurable function $u : \mathbb{R}_+ \rightarrow \bar{B}$ and $\gamma(t_0) = x_0$. We denote by $\widehat{\varphi} : \mathbb{R}_+ \times \bar{\Omega}_\eta \rightarrow \mathbb{R}_+$ the value function of $\widehat{\text{OCP}}$ and by $\widehat{\text{Opt}}(k_\epsilon, t_0, x_0)$ the set of optimal trajectories for (t_0, x_0) , with the convention that $\gamma \in \widehat{\text{Opt}}(k_\epsilon, t_0, x_0)$ must satisfy $\gamma(t) = x_0$ for $t \in [0, t_0]$ and $\gamma(t) = \gamma(t_0 + \widehat{\varphi}(t_0, x_0))$ for $t \in [t_0 + \widehat{\varphi}(t_0, x_0), +\infty)$.

Note that $\widehat{\text{OCP}}$ is a minimal-time optimal control problem with dynamics defined by k_ϵ and with the state constraint $\gamma(t) \in \bar{\Omega}_\eta$ for every $t \geq 0$. In particular, $\widehat{\text{OCP}}$ coincides with the optimal control problem $\text{OCP}(k)$ if one replaces Ω by Ω_η and k by the restriction of k_ϵ to $\mathbb{R}_+ \times \bar{\Omega}_\eta$, and assumptions (H1)–(H3), (H4), and (H5) are satisfied for $\widehat{\text{OCP}}$, with K_{\min} in (H4) replaced by $\widehat{K}_{\min} = (1 - \frac{\eta}{\epsilon}) K_{\min}$ and L in (H5) replaced by $\widehat{L} = L_\epsilon = L + \frac{K_{\max}}{\epsilon}$. Hence, all properties of Section 3.3 apply to $\widehat{\text{OCP}}$ and thus, by Proposition 3.3.3, $\widehat{\varphi}$ is Lipschitz continuous on $\mathbb{R}_+ \times \bar{\Omega}_\eta$.

In order to conclude, it suffices to show that φ_ϵ and $\widehat{\varphi}$ coincide on this set. For this purpose, we proceed as in the proof of Theorem 3.4.8. Let $(t_0, x_0) \in \mathbb{R} \times \bar{\Omega}_\eta$. Let $\gamma_\epsilon \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$ and $\widehat{\gamma} \in \widehat{\text{Opt}}(k_\epsilon, t_0, x_0)$, and denote $T_\epsilon = \varphi_\epsilon(t_0, x_0)$ and $\widehat{T} = \widehat{\varphi}(t_0, x_0)$. Clearly, $\widehat{\gamma} \in \text{Adm}_\epsilon(k_\epsilon)$, and thus $T_\epsilon \leq \widehat{T}$. On the other hand, thanks to Proposition 3.4.7, we have $d_\Omega(\gamma_\epsilon(t)) \leq d_\Omega(x_0) \leq \eta$ for every $t \geq 0$, showing that γ_ϵ takes values in $\bar{\Omega}_\eta$. Hence γ_ϵ is an admissible trajectory for the optimal control problem $\widehat{\text{OCP}}$, showing that $\widehat{T} \leq T_\epsilon$. Hence $\widehat{T} = T_\epsilon$, as required. \square

3.4.2 Boundary condition of the value function

Proposition 3.3.4 in Section 3.3 asserts that φ is a viscosity solution of the Hamilton–Jacobi equation (3.13) and also provides its boundary condition on $\mathbb{R}_+ \times \Gamma$. The only information

provided in Proposition 3.3.4 on φ on the part of the boundary $\mathbb{R}_+ \times (\partial\Omega \setminus \Gamma)$ is that φ is a viscosity supersolution there. The main result of this section, Theorem 3.4.11, provides additional information on φ on $\mathbb{R}_+ \times (\partial\Omega \setminus \Gamma)$.

Theorem 3.4.11. *Consider the optimal control problem $\text{OCP}(k)$ and its value function φ under assumptions (H1)–(H3), (H4), and (H5). Then φ satisfies $\nabla\varphi(t, x) \cdot \mathbf{n}(x) \geq 0$ in the viscosity supersolution sense for every $(t, x) \in \mathbb{R}_+ \times (\partial\Omega \setminus \Gamma)$.*

Proof. We consider in this proof that k is extended to a continuous function, still denoted by k , defined in $[-1, +\infty) \times \mathbb{R}^d$ which is Lipschitz continuous with respect to its second variable, uniformly with respect to the first one. Note that, up to considering the optimal control problem $\text{OCP}(k(\cdot - 1, \cdot))$, all of the previous results on minimal-time optimal control problems apply to k with the time domain \mathbb{R}_+ replaced by $[-1, +\infty)$

Fix $(t_0, x_0) \in \mathbb{R}_+ \times (\partial\Omega \setminus \Gamma)$ and let $\xi \in \mathbf{C}^1(V; \mathbb{R})$ be such that $\xi(t_0, x_0) = \varphi(t_0, x_0)$ and $\xi(t, x) \leq \varphi(t, x)$ for every $(t, x) \in V$, where V is a neighborhood of (t_0, x_0) in $[-1, +\infty) \times \bar{\Omega}$. Assume, to obtain a contradiction, that $\nabla\xi(t_0, x_0) \cdot \mathbf{n}(x_0) < 0$. In particular, we have $\nabla\xi(t_0, x_0) \neq 0$.

Let $\gamma \in \text{Opt}(k, t_0, x_0)$, u be an optimal control for γ , and $T = \varphi(t_0, x_0)$. Since $x_0 \notin \Gamma$, we have $T > 0$ and, by Corollary 3.4.9, we have $\gamma \in \mathbf{C}^1([t_0, t_0 + T]; \bar{\Omega})$ and $u \in \text{Lip}([t_0, t_0 + T]; \mathbb{S}^{d-1})$.

Let $\sigma : [-1, +\infty) \rightarrow \mathbb{R}^d$ denote the solution of the initial value problem

$$\begin{cases} \dot{\sigma}(t) = -k(t, \sigma(t)) \frac{\nabla\xi(t_0, x_0)}{|\nabla\xi(t_0, x_0)|}, \\ \sigma(t_0) = x_0. \end{cases}$$

Since k is continuous, by construction, $\sigma \in \mathbf{C}^1([-1, +\infty), \mathbb{R}^d)$. We claim that there exists $\epsilon \in (0, 1]$ such that $\sigma(t) \in \bar{\Omega}$ and $(t, \sigma(t)) \in V$ for every $t \in [t_0 - \epsilon, t_0]$. Indeed, let W be as in Proposition 3.4.12 and notice that, since $x_0 \in \partial\Omega$, there exists $\epsilon_0 \in (0, 1]$ such that $\sigma(t) \in W$ for every $t \in [t_0 - \epsilon_0, t_0 + \epsilon_0]$. Moreover, up to reducing ϵ_0 , we also have $(t, \sigma(t)) \in V$ for every $t \in [t_0 - \epsilon_0, t_0 + \epsilon_0]$. Consider the map $\beta : t \mapsto d_{\partial\Omega}^\pm(\sigma(t))$, which is Lipschitz continuous on $[t_0 - \epsilon_0, t_0 + \epsilon_0]$. By differentiating, we have, for a.e. $t \in [t_0 - \epsilon_0, t_0 + \epsilon_0]$,

$$(3.28) \quad \dot{\beta}(t) = \dot{\sigma}(t) \cdot \nabla d_{\partial\Omega}^\pm(\sigma(t)) = -k(t, \sigma(t)) \nabla d_{\partial\Omega}^\pm(\sigma(t)) \cdot \frac{\nabla\xi(t_0, x_0)}{|\nabla\xi(t_0, x_0)|}.$$

Recalling that $\mathbf{n}(x_0) = \nabla d_{\partial\Omega}^\pm(x_0)$ and using the continuity of $\nabla d_{\partial\Omega}$, we deduce from the fact that $\nabla\xi(t_0, x_0) \cdot \mathbf{n}(x_0) < 0$ that there exists $\epsilon \in (0, \epsilon_0]$ such that $\nabla d_{\partial\Omega}^\pm(\sigma(t)) \cdot \nabla\xi(t_0, x_0) < 0$ for every $t \in [t_0 - \epsilon, t_0 + \epsilon]$. Hence, by (3.28), we have $\dot{\beta}(t) > 0$ for a.e. $t \in [t_0 - \epsilon, t_0 + \epsilon]$. Since $\beta(t_0) = 0$, we thus have $\beta(t) \leq 0$ for $t \in [t_0 - \epsilon, t_0]$, which implies that $\sigma(t) \in \bar{\Omega}$ for every $t \in [t_0 - \epsilon, t_0]$. Since $\epsilon \in (0, \epsilon_0]$, we also have in particular that $(t, \sigma(t)) \in V$ for every $t \in [t_0 - \epsilon, t_0]$.

Define $\tilde{\gamma} : [-1, +\infty) \rightarrow \bar{\Omega}$ by

$$\tilde{\gamma}(t) = \begin{cases} \sigma(t_0 - \epsilon), & \text{if } t \in [-1, t_0 - \epsilon), \\ \sigma(t), & \text{if } t \in [t_0 - \epsilon, t_0), \\ \gamma(t), & \text{if } t \geq t_0. \end{cases}$$

Hence $\tilde{\gamma}$ is an admissible trajectory for k and thus, by Proposition 3.3.13, for every $h \in [0, \epsilon]$, we have $\varphi(t_0, x_0) \geq \varphi(t_0 - h, \tilde{\gamma}(t_0 - h)) - h$, and thus $\xi(t_0, x_0) \geq \xi(t_0 - h, \tilde{\gamma}(t_0 - h)) - h$. Notice also that, for $h \in [0, \epsilon]$, we have $\tilde{\gamma}(t_0 - h) = \sigma(t_0 - h)$, yielding that $\xi(t_0, x_0) \geq \xi(t_0 - h, \sigma(t_0 - h)) - h$. Since ξ and σ are \mathbf{C}^1 functions, for $h \in [0, \epsilon]$, we have

$$\xi(t_0 - h, \sigma(t_0 - h)) = \xi(t_0, x_0) - h\partial_t \xi(t_0, x_0) - h\nabla \xi(t_0, x_0) \cdot \dot{\sigma}(t_0) + o(h).$$

Inserting this fact in the inequality $\xi(t_0, x_0) \geq \xi(t_0 - h, \sigma(t_0 - h)) - h$ for $h \in (0, \epsilon]$, using the definition of σ , dividing by h and letting $h \rightarrow 0^+$, we deduce that

$$(3.29) \quad -\partial_t \xi(t_0, x_0) + k(t_0, x_0)|\nabla \xi(t_0, x_0)| - 1 \leq 0.$$

Since $\gamma \in \text{Opt}(k, t_0, x_0)$, we have, by Proposition 3.3.13, that $\varphi(t_0, x_0) = \varphi(t_0 + h, \gamma(t_0 + h)) + h$ for every $h \in [0, T]$, and thus $\xi(t_0, x_0) \geq \xi(t_0 + h, \gamma(t_0 + h)) + h$. We also have, for $h \in [0, T]$,

$$\xi(t_0 + h, \gamma(t_0 + h)) = \xi(t_0, x_0) + h\partial_t \xi(t_0, x_0) + h\nabla \xi(t_0, x_0) \cdot \dot{\gamma}(t_0^+) + o(h),$$

where we use the fact that $\gamma \in \mathbf{C}^1([t_0, t_0 + T]; \bar{\Omega})$ and $\dot{\gamma}(t_0^+)$ denotes the limit of $\dot{\gamma}(t)$ as $t \rightarrow t_0^+$. Using the fact that γ satisfies the first equation of (3.1), we deduce as before, dividing by h and letting $h \rightarrow 0^+$, that

$$\partial_t \xi(t_0, x_0) + k(t_0, x_0)\nabla \xi(t_0, x_0) \cdot u(t_0) + 1 \leq 0.$$

Adding with (3.29), we deduce that $\nabla \xi(t_0, x_0) \cdot u(t_0) + |\nabla \xi(t_0, x_0)| \leq 0$ and, since $u(t_0) \in \bar{B}$, this implies that $u(t_0) = -\frac{\nabla \xi(t_0, x_0)}{|\nabla \xi(t_0, x_0)|}$.

Consider now the function $\alpha : t \mapsto d_{\partial\Omega}^\pm(\gamma(t))$ defined on $[t_0, t_0 + T]$. Since $\gamma(t_0) = x_0 \in \partial\Omega$, there exists $\delta_0 \in (0, T]$ such that $\gamma(t) \in W$ for every $t \in [t_0, t_0 + \delta_0]$, and thus α is \mathbf{C}^1 on $[t_0, t_0 + \delta_0]$, with

$$\dot{\alpha}(t) = \nabla d_{\partial\Omega}^\pm(\gamma(t)) \cdot \dot{\gamma}(t) = k(t, \gamma(t))\nabla d_{\partial\Omega}^\pm(\gamma(t)) \cdot u(t).$$

In particular, $\dot{\alpha}(t_0) = -k(t_0, x_0)\mathbf{n}(x_0) \cdot \frac{\nabla \xi(t_0, x_0)}{|\nabla \xi(t_0, x_0)|} > 0$, showing that there exists $\delta \in (0, \delta_0]$ such that $\alpha(t) > \alpha(t_0) = 0$ for every $t \in (t_0, t_0 + \delta]$. From the definition of α , this means that $\gamma(t) \notin \bar{\Omega}$ for $t \in (t_0, t_0 + \delta]$, which is a contradiction since $\gamma \in \text{Opt}(k, t_0, x_0)$. This contradiction establishes that $\nabla \xi(t_0, x_0) \cdot \mathbf{n}(x_0) \geq 0$, as required. \square

Remark 3.4.12. Under the assumptions of Theorem 3.4.11, if $(t_0, x_0) \in \mathbb{R}_+ \times (\partial\Omega \setminus \Gamma)$ and φ is differentiable at (t_0, x_0) , then the inequality $\nabla \varphi(t_0, x_0) \cdot \mathbf{n}(x_0) \geq 0$ holds in the classical sense.

Proposition 3.3.4 asserts that the value function φ is a viscosity solution of the Hamilton–Jacobi equation (3.13) in $\mathbb{R}_+ \times (\Omega \setminus \Gamma)$ and thus, as a consequence of standard properties of viscosity solutions, (3.13) is satisfied in the classical sense at all points $(t, x) \in \mathbb{R}_+ \times (\Omega \setminus \Gamma)$ at which φ is differentiable. Thanks to Theorem 3.4.11, we can prove that (3.13) is actually satisfied at all points $(t, x) \in \mathbb{R}_+ \times (\bar{\Omega} \setminus \Gamma)$ at which φ is differentiable.

Proposition 3.4.13. *Consider the optimal control problem $\text{OCP}(k)$ and its value function φ under assumptions (H1)–(H3), (H4), and (H5). Let $(t_0, x_0) \in \mathbb{R}_+ \times (\bar{\Omega} \setminus \Gamma)$ be such that φ is differentiable at (t_0, x_0) . Then*

$$-\partial_t \varphi(t_0, x_0) + |\nabla \varphi(t_0, x_0)|k(t_0, x_0) - 1 = 0.$$

Proof. If $x_0 \in \Omega \setminus \Gamma$, this fact is a consequence of classical properties of viscosity solutions, and so we assume in the sequel that $x_0 \in \partial\Omega \setminus \Gamma$. By Remark 3.4.12, we have that $\nabla \varphi(t_0, x_0) \cdot \mathbf{n}(x_0) \geq 0$. Moreover, by Proposition 3.3.51, we have $\nabla \varphi(t_0, x_0) \neq 0$. Since, by Proposition 3.3.4, φ is a viscosity supersolution of (3.13) in $\mathbb{R}_+ \times (\bar{\Omega} \setminus \Gamma)$, we have in particular that

$$-\partial_t \varphi(t_0, x_0) + |\nabla \varphi(t_0, x_0)|k(t_0, x_0) - 1 \geq 0.$$

We are thus left to show the converse inequality.

Let D be the set of unit vectors pointing to the inside of $\bar{\Gamma}$ at x_0 , i.e., D is the set of $u_0 \in \mathbb{S}^{d-1}$ for which there exists $h_0 > 0$ such that $x_0 + hu_0 \in \bar{\Omega}$ for every $h \in [0, h_0]$. Notice that $\{u_0 \in \mathbb{S}^{d-1} \mid u_0 \cdot \mathbf{n}(x_0) < 0\} \subset D \subset \{u_0 \in \mathbb{S}^{d-1} \mid u_0 \cdot \mathbf{n}(x_0) \leq 0\}$ and thus, in particular, $\bar{D} = \{u_0 \in \mathbb{S}^{d-1} \mid u_0 \cdot \mathbf{n}(x_0) \leq 0\}$. Since $\nabla \varphi(t_0, x_0) \cdot \mathbf{n}(x_0) \geq 0$, we deduce that $-\frac{\nabla \varphi(t_0, x_0)}{|\nabla \varphi(t_0, x_0)|} \in \bar{D}$.

Let $u \in D$ and $\sigma \in \mathbf{C}^1(\mathbb{R}_+; \mathbb{R}^d)$ be the unique solution of

$$(3.30) \quad \begin{cases} \dot{\sigma}(t) = k(t, \sigma(t))u \\ \sigma(t_0) = x_0. \end{cases}$$

Since $u \in D$, there exists $h_0 > 0$ such that $\sigma(t) \in \bar{\Omega}$ for every $t \in [t_0, t_0 + h_0]$. Let $\gamma \in \text{Adm}(k)$ be defined by $\gamma(t) = x_0$ for $t \in [0, t_0]$, $\gamma(t) = \sigma(t)$ for $t \in [t_0, t_0 + h_0]$, and $\gamma(t) = \sigma(t_0 + h_0)$ for $t \geq t_0 + h_0$. By Proposition 3.3.13, we have, for every $h \in [0, h_0]$,

$$(3.31) \quad \varphi(t_0, x_0) \leq h + \varphi(t_0 + h, \sigma(t_0 + h)).$$

On the other hand, notice that $\sigma(t_0 + h) = x_0 + hk(t_0, x_0)u + o(h)$ as $h \rightarrow 0$. Since φ is differentiable at (t_0, x_0) and Lipschitz continuous, we deduce that, as $h \rightarrow 0^+$,

$$\varphi(t_0 + h, \sigma(t_0 + h)) - \varphi(t_0, x_0) = h\partial_t \varphi(t_0, x_0) + hk(t_0, x_0)\nabla \varphi(t_0, x_0) \cdot u + o(h).$$

Dividing the above expression by h , using (3.31) and letting $h \rightarrow 0^+$, we get

$$-1 \leq \partial_t \varphi(t_0, x_0) + k(t_0, x_0)\nabla \varphi(t_0, x_0) \cdot u.$$

Since this holds for every $u \in D$ and the right-hand side of the above inequality is continuous in u , we get that

$$\partial_t \varphi(t_0, x_0) + k(t_0, x_0) \inf_{u \in \bar{D}} \nabla \varphi(t_0, x_0) \cdot u + 1 \geq 0.$$

Since $-\frac{\nabla \varphi(t_0, x_0)}{|\nabla \varphi(t_0, x_0)|} \in \bar{D}$, the above infimum is attained at $u = -\frac{\nabla \varphi(t_0, x_0)}{|\nabla \varphi(t_0, x_0)|}$, yielding that

$$-\partial_t \varphi(t_0, x_0) + |\nabla \varphi(t_0, x_0)| k(t_0, x_0) - 1 \leq 0,$$

as required. \square

Remark 3.4.14. At the light of Proposition 3.4.13, a natural question is whether φ is also a viscosity subsolution of (3.13) in $\mathbb{R}_+ \times (\bar{\Omega} \setminus \Gamma)$, since this would imply the result of Proposition 3.4.13. However, this may fail to be the case, as illustrated by the following example. Consider the case $d = 1$, $\Omega = (0, 1)$, $\Gamma = \{0\}$, and $k : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}_+$ given by $k(t, x) = 1$ for every $(t, x) \in \mathbb{R}_+ \times [0, 1]$. One easily computes that $\varphi(t, x) = x$ for every $(t, x) \in \mathbb{R}_+ \times [0, 1]$. For every $\alpha \in (-\infty, 1]$, the function $\xi : \mathbb{R}_+ \times [0, 1]$ defined for $(t, x) \in \mathbb{R}_+ \times [0, 1]$ by $\xi(t, x) = 1 + \alpha(x - 1)$ satisfies $\xi(t, 1) = \varphi(t, 1) = 1$ and $\xi(t, x) \geq \varphi(t, x)$ for every $(t, x) \in \mathbb{R}_+ \times [0, 1]$. However, for $\alpha < -1$ and $t \in \mathbb{R}_+$, we have

$$-\partial_t \xi(t, 1) + |\partial_x \xi(t, 1)| k(t, 1) - 1 = |\alpha| - 1 > 0,$$

showing that φ cannot be a viscosity subsolution of (3.13) at $(t, 1)$.

3.4.3 Characterization of optimal controls

We now turn to the problem of characterizing the optimal control $u : \mathbb{R}_+ \rightarrow \bar{B}$ associated with an optimal trajectory $\gamma \in \text{Opt}(k, t_0, x_0)$. By Proposition 3.3.52, the optimal control u can be written as $u(t) = -\frac{\nabla \varphi(t, \gamma(t))}{|\nabla \varphi(t, \gamma(t))|}$ for every $t \in [t_0, t_0 + \varphi(t_0, x_0))$ such that φ is differentiable at $(t, \gamma(t))$. For optimal control problems without state constraints, one can typically prove that φ is a semiconcave function (see, e.g., [27, Theorem 8.2.7]) and use this fact and properties of superdifferentials of semiconcave functions to deduce that φ is indeed differentiable along optimal trajectories, except possibly at their starting and ending times (see, e.g., [27, Theorem 8.4.6]).

Since value functions of optimal control problems with state constraints may fail to be semiconcave, we propose in this section an alternative way to characterize the optimal control. In particular, we are also able to provide a characterization of the optimal control under weaker assumptions: semiconcavity results usually require the dynamics of the system to be $\mathbf{C}^{1,1}$ with respect to the state, as in [27, Theorem 8.2.7], but the Lipschitz continuity assumption (H5) is actually sufficient for our strategy. The approach we follow here is an adaptation to the case with state constraints of the characterization of optimal controls from [83]. We start by introducing the set of optimal directions at a point.

Definition 3.4.15. Consider the optimal control problem $\text{OCP}(k)$ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$. We define the set $\mathcal{U}(t_0, x_0)$ of *optimal*

directions at (t_0, x_0) as the set of $u_0 \in \mathbb{S}^{d-1}$ for which there exists $\gamma \in \text{Opt}(k, t_0, x_0)$ such that the corresponding optimal control $u \in \text{Lip}([t_0, t_0 + \varphi(t_0, x_0)], \mathbb{S}^{d-1})$ associated with γ satisfies $u(t_0) = u_0$.

Notice that, by Proposition 3.3.11 and Corollary 3.4.9, the set $\mathcal{U}(t_0, x_0)$ is non-empty whenever $x_0 \in \bar{\Omega} \setminus \Gamma$. We now show that $\mathcal{U}(t_0, x_0)$ is singleton along optimal trajectories, except possibly at their initial and final points. This result was already presented in [75, Proposition 4.11] for optimal control problems without state constraints and with the stronger assumption that $k \in \text{Lip}(\mathbb{R}_+ \times \bar{\Omega}; \mathbb{R}_+)$, but the proof presented in that reference also applies to our setting. We provide the proof here for completeness and also since it is an easy and interesting consequence of Corollary 3.4.9.

Proposition 3.4.16. *Consider the optimal control problem $\text{OCP}(k)$ and its value function φ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$ and $\gamma \in \text{Opt}(k, t_0, x_0)$. Then, for every $t \in (t_0, t_0 + \varphi(t_0, x_0))$, the set $\mathcal{U}(t, \gamma(t))$ contains exactly one element.*

Proof. Let $u : \mathbb{R}_+ \rightarrow \bar{B}$ be the optimal control associated with γ , $t_1 \in (t_0, t_0 + \varphi(t_0, x_0))$, and $x_1 = \gamma(t_1)$. Since $x_1 \in \bar{\Omega} \setminus \Gamma$, we have $\mathcal{U}(t_1, x_1) \neq \emptyset$. Moreover, by Proposition 3.3.13, we have $\varphi(t_1, x_1) + t_1 - t_0 = \varphi(t_0, x_0)$ and, noticing that $\tau(t_1, \gamma) = t_0 + \varphi(t_0, x_0) - t_1 = \varphi(t_1, x_1)$, we deduce that the restriction of γ to $[t_1, t_0 + \varphi(t_0, x_0)]$ (extended by constant values to \mathbb{R}_+) is an optimal trajectory for $\text{OCP}(k)$ from (t_1, x_1) , and thus $u(t_1) \in \mathcal{U}(t_1, x_1)$.

We conclude the proof by showing that, for every $u_1 \in \mathcal{U}(t_1, x_1)$, we have $u_1 = u(t_1)$. For that purpose, take $u_1 \in \mathcal{U}(t_1, x_1)$. By definition of $\mathcal{U}(t_1, x_1)$, there exists $\tilde{\gamma} \in \text{Opt}(k, t_1, x_1)$ such that its associated control \tilde{u} satisfies $\tilde{u}(t_1) = u_1$. Consider the trajectory $\hat{\gamma} : \mathbb{R}_+ \rightarrow \bar{\Omega}$ defined by $\hat{\gamma}(t) = \gamma(t)$ for $t \leq t_1$ and $\hat{\gamma}(t) = \tilde{\gamma}(t)$ for $t \geq t_1$, and notice that its associated control \hat{u} satisfies $\hat{u}(t) = u(t)$ for $t < t_1$ and $\hat{u}(t) = \tilde{u}(t)$ for $t > t_1$. Then $\hat{\gamma} \in \text{Adm}(k)$ and, by construction, $\tau(t_0, \hat{\gamma}) = t_1 - t_0 + \tau(t_1, \tilde{\gamma}) = t_1 - t_0 + \varphi(t_1, x_1) = \varphi(t_0, x_0)$, which yields that $\hat{\gamma} \in \text{Opt}(k, t_0, x_0)$. In particular, by Corollary 3.4.9, we have $\hat{u} \in \text{Lip}([t_0, t_0 + \varphi(t_0, x_0)]; \mathbb{S}^{d-1})$, and thus $\hat{u}(t_1) = \lim_{t \rightarrow t_1^-} \hat{u}(t) = u(t_1)$ and $\hat{u}(t_1) = \lim_{t \rightarrow t_1^+} \hat{u}(t) = \tilde{u}(t_1) = u_1$, yielding that $u_1 = u(t_1)$, as required. \square

Similarly to [83], the goal of this section is to characterize $\mathcal{U}(t_0, x_0)$ as the set of directions of maximal descent of the value function φ of $\text{OCP}(k)$. For that purpose, we first introduce the notion of descent rate of φ along a given direction.

Definition 3.4.17. Consider the optimal control problem $\text{OCP}(k)$ and its value function φ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$.

1. The set of *inward pointing directions* at x_0 is the set $\text{In}(x_0)$ of vectors $u_0 \in \mathbb{S}^{d-1}$ for which there exists $h_0 > 0$ such that $x_0 + hu_0 \in \bar{\Omega}$ for every $h \in [0, h_0]$.
2. The *descent rate* of φ at (t_0, x_0) is the function $\Delta_{(t_0, x_0)} : \text{In}(x_0) \rightarrow \mathbb{R}$ defined for $u_0 \in \text{In}(x_0)$ by

$$\Delta_{(t_0, x_0)}(u_0) = \limsup_{h \rightarrow 0^+} \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h}.$$

Note that $\text{In}(x_0) = \mathbb{S}^{d-1}$ if $x_0 \in \Omega$, while, thanks to (H3), for $x_0 \in \partial\Omega$, we have

$$\{u_0 \in \mathbb{S}^{d-1} \mid u_0 \cdot \mathbf{n}(x_0) < 0\} \subset \text{In}(x_0) \subset \{u_0 \in \mathbb{S}^{d-1} \mid u_0 \cdot \mathbf{n}(x_0) \leq 0\}.$$

In general, both inclusions may be strict. As a consequence of those inclusions, we have

$$\overline{\text{In}}(x_0) = \{u_0 \in \mathbb{S}^{d-1} \mid u_0 \cdot \mathbf{n}(x_0) \leq 0\}.$$

The next proposition provides important properties of the descent rate function.

Proposition 3.4.18. *Consider the optimal control problem $\text{OCP}(k)$ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$.*

1. *For every $u_0 \in \text{In}(x_0)$, we have $\Delta_{(t_0, x_0)}(u_0) \geq -1$.*
2. *The function $\Delta_{(t_0, x_0)}$ is Lipschitz continuous on $\text{In}(x_0)$, with a Lipschitz constant depending only on K_{\max} and the Lipschitz constant of φ , and thus independent of (t_0, x_0) .*

Proof. To show 1, take $u_0 \in \text{In}(x_0)$ and let $\sigma \in \mathbf{C}^1(\mathbb{R}_+; \mathbb{R}^d)$ be the unique solution of

$$\begin{cases} \dot{\sigma}(t) = k(t, \sigma(t))u_0, \\ \sigma(t_0) = x_0. \end{cases}$$

Since $u_0 \in \text{In}(x_0)$, there exists $t_* > t_0$ such that $\sigma(t) \in \bar{\Omega}$ for every $t \in [t_0, t_*]$. Define $\gamma \in \text{Adm}(k)$ by

$$\gamma(t) = \begin{cases} x_0, & \text{if } t \in [0, t_0], \\ \sigma(t), & \text{if } t \in [t_0, t_*], \\ \sigma(t_*), & \text{if } t \geq t_*. \end{cases}$$

Applying Proposition 3.3.13 to γ , we deduce that $\varphi(t_0 + h, \gamma(t_0 + h)) + h \geq \varphi(t_0, x_0)$ for every $h \geq 0$, and thus, for $h \in (0, t_* - t_0]$, we have

$$\frac{\varphi(t_0 + h, \sigma(t_0 + h)) - \varphi(t_0, x_0)}{h} \geq -1.$$

Since $\sigma(t_0 + h) = x_0 + hk(t_0, x_0)u_0 + o(h)$ as $h \rightarrow 0^+$ and φ is Lipschitz continuous, we deduce that $\varphi(t_0 + h, \sigma(t_0 + h)) = \varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0) + o(h)$, yielding that, as $h \rightarrow 0^+$,

$$(3.32) \quad \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \geq -1 + o(1),$$

and the conclusion of 1 follows from the definition of $\Delta_{(t_0, x_0)}$.

In order to prove 2, take $u_1, u_2 \in \text{In}(x_0)$ and denote by $C > 0$ the Lipschitz constant of φ . For $h > 0$ small enough, we have

$$\begin{aligned} \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_1) - \varphi(t_0, x_0)}{h} &= \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_2) - \varphi(t_0, x_0)}{h} \\ &+ \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_1) - \varphi(t_0 + h, x_0 + hk(t_0, x_0)u_2)}{h} \\ &\leq \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_2) - \varphi(t_0, x_0)}{h} + CK_{\max}|u_1 - u_2|. \end{aligned}$$

Taking the lim sup as $h \rightarrow 0^+$, we deduce that $\Delta_{(t_0, x_0)}(u_1) \leq \Delta_{(t_0, x_0)}(u_2) + CK_{\max}|u_1 - u_2|$. Since this holds for every $u_1, u_2 \in \text{In}(x_0)$, we obtain that

$$|\Delta_{(t_0, x_0)}(u_1) - \Delta_{(t_0, x_0)}(u_2)| \leq CK_{\max}|u_1 - u_2| \quad \text{for every } u_1, u_2 \in \text{In}(x_0),$$

as required. \square

As a consequence of Proposition 3.4.182, $\Delta_{(t_0, x_0)} : \text{In}(x_0) \rightarrow \mathbb{R}$ can be extended in a unique way to a Lipschitz continuous function defined on $\overline{\text{In}}(x_0)$. In the sequel, by a slight abuse of notation, we use $\Delta_{(t_0, x_0)}$ to denote this Lipschitz continuous extension. Note that this extension still satisfies the assertions of Proposition 3.4.18.

We are now in position to provide the definition of the set of directions of maximal descent of the value function.

Definition 3.4.19. Consider the optimal control problem $\text{OCP}(k)$ and its value function φ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$. We define the set $\mathcal{W}(t_0, x_0)$ of *directions of maximal descent* of φ at (t_0, x_0) by

$$\mathcal{W}(t_0, x_0) = \Delta_{(t_0, x_0)}^{-1}(\{-1\}) = \{u_0 \in \overline{\text{In}}(x_0) \mid \Delta_{(t_0, x_0)}(u_0) = -1\}.$$

Notice that the term *maximal descent* is motivated by Proposition 3.4.181, since we are considering the elements $u_0 \in \overline{\text{In}}(x_0)$ reaching the lower bound -1 on $\Delta_{(t_0, x_0)}$.

Remark 3.4.20. If $x_0 \in \Omega$ (or, more generally, if the set $\text{In}(x_0)$ is closed), then $\mathcal{W}(t_0, x_0)$ is simply the set of $u_0 \in \mathbb{S}^{d-1}$ such that

$$(3.33) \quad \lim_{h \rightarrow 0^+} \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} = -1,$$

which is the definition of $\mathcal{W}(t_0, x_0)$ provided previously in [83, Definition 4.11] for optimal control problems without state constraints. Indeed, notice that, if $\text{In}(x_0)$ is closed, then $u_0 \in \mathcal{W}(t_0, x_0)$ if and only if

$$(3.34) \quad \limsup_{h \rightarrow 0^+} \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} = -1.$$

On the other hand, (3.32) also yields that

$$\liminf_{h \rightarrow 0^+} \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u) - \varphi(t_0, x_0)}{h} \geq -1$$

for every $u \in \text{In}(x_0)$, and hence (3.33) is equivalent to (3.34).

Our next result shows that, at the points (t_0, x_0) where φ is differentiable, $\mathcal{W}(t_0, x_0)$ contains a unique direction of maximal descent which, as one might expect, is equal to $-\frac{\nabla\varphi(t_0, x_0)}{|\nabla\varphi(t_0, x_0)|}$. This was already shown in [83, Proposition 4.13] for optimal control problems without state constraints, and that proof also carries over to the present case thanks to Proposition 3.4.13. For sake of completeness, and since this proof is quite elementary, we provide it here.

Proposition 3.4.21. *Consider the optimal control problem $\text{OCP}(k)$ and its value function φ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times (\bar{\Omega} \setminus \Gamma)$ be such that φ is differentiable at (t_0, x_0) . Then*

$$(3.35) \quad \mathcal{W}(t_0, x_0) = \left\{ -\frac{\nabla\varphi(t_0, x_0)}{|\nabla\varphi(t_0, x_0)|} \right\}.$$

Proof. Since φ is differentiable at (t_0, x_0) , we have, for every $u \in \text{In}(x_0)$,

$$(3.36) \quad \Delta_{(t_0, x_0)}(u) = \partial_t \varphi(t_0, x_0) + k(t_0, x_0) \nabla \varphi(t_0, x_0) \cdot u.$$

By continuity, the above equality also holds for every $u \in \bar{\text{In}}(x_0)$. We also have, by Proposition 3.4.13, that

$$(3.37) \quad -\partial_t \varphi(t_0, x_0) + k(t_0, x_0) |\nabla \varphi(t_0, x_0)| - 1 = 0.$$

Moreover, recall that $k(t_0, x_0) > 0$ and, by Proposition 3.3.51, $\nabla \varphi(t_0, x_0) \neq 0$. If $u_0 \in \mathcal{W}(t_0, x_0)$, then $\Delta_{(t_0, x_0)}(u_0) = -1$ and, combining with (3.36) and (3.37), we deduce that

$$k(t_0, x_0) [|\nabla \varphi(t_0, x_0)| + \nabla \varphi(t_0, x_0) \cdot u_0] = 0,$$

which yields $u_0 = -\frac{\nabla \varphi(t_0, x_0)}{|\nabla \varphi(t_0, x_0)|}$ since $u_0 \in \mathbb{S}^{d-1}$. Conversely, defining $u_0 = -\frac{\nabla \varphi(t_0, x_0)}{|\nabla \varphi(t_0, x_0)|}$, Remark 3.4.12 ensures that $u_0 \in \bar{\text{In}}(x_0)$ and it is immediate to compute, using (3.36) and (3.37), that $\Delta_{(t_0, x_0)}(u_0) = -1$, showing that $u_0 \in \mathcal{W}(t_0, x_0)$. \square

The reason why we go through the definition of descent rate $\Delta_{(t_0, x_0)}$ in order to define $\mathcal{W}(t_0, x_0)$ instead of the more direct definition provided in [83, Definition 4.11] is that, if $x_0 \in \partial\Omega$ and $u_0 \in \mathbb{S}^{d-1}$, one might have $x_0 + hk(t_0, x_0)u_0 \notin \bar{\Omega}$ for every $h > 0$ small enough. Since φ is defined only in the set $\mathbb{R}_+ \times \bar{\Omega}$, this means that the term $\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_0)$ is not well-defined for any $h > 0$ small enough, and thus the limit in the left-hand side of (3.33) does not make sense. An alternative approach, however, is to replace φ in (3.33) by the value function φ_ϵ of the penalized optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ defined in Section 3.4.1. This is the subject of our next definition.

Definition 3.4.22. Consider the optimal control problems $\text{OCP}(k)$ and $\text{OCP}_\epsilon(k_\epsilon)$ and their respective value functions φ and φ_ϵ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$. For $\epsilon > 0$, we define the set $\mathcal{W}_\epsilon(t_0, x_0)$ of *directions of maximal descent of φ_ϵ* at (t_0, x_0) by

$$\mathcal{W}_\epsilon(t_0, x_0) = \left\{ u_0 \in \mathbb{S}^{d-1} \mid \lim_{h \rightarrow 0^+} \frac{\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} = -1 \right\}.$$

Our next result concerns the relation between optimal directions (i.e., elements of $\mathcal{U}(t_0, x_0)$) and directions of maximal descent of φ (i.e., elements of $\mathcal{W}(t_0, x_0)$), and asserts that both notions actually coincide, and that they also coincide with $\mathcal{W}_\epsilon(t_0, x_0)$ for $\epsilon > 0$ small enough. The fact that $\mathcal{U}(t_0, x_0) = \mathcal{W}(t_0, x_0)$ was already established in [83, Theorem 4.14] for minimal-time optimal control problems without state constraints, and the proof we provide here follows the same lines, but passes through the optimal control problem $\text{OCP}_\epsilon(k_\epsilon)$ in order to handle the state constraints appropriately.

Theorem 3.4.23. *Consider the optimal control problems $\text{OCP}(k)$ and $\text{OCP}_\epsilon(k_\epsilon)$ and assume that (H1)–(H3), (H4), and (H5) hold. There exists $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$ and $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, we have $\mathcal{U}(t_0, x_0) = \mathcal{W}(t_0, x_0) = \mathcal{W}_\epsilon(t_0, x_0)$.*

Proof. We first remark that, if $x_0 \in \Gamma$, then $\mathcal{U}(t_0, x_0) = \mathcal{W}(t_0, x_0) = \mathcal{W}_\epsilon(t_0, x_0) = \emptyset$, and so we are only left to consider the case $x_0 \in \bar{\Omega} \setminus \Gamma$. We let $\epsilon_0 > 0$ be as in the statement of Propositions 3.4.7 and 3.4.10 and Theorem 3.4.8, and we fix $\epsilon \in (0, \epsilon_0)$. We proceed by proving that $\mathcal{U}(t_0, x_0) \subset \mathcal{W}(t_0, x_0) \subset \mathcal{W}_\epsilon(t_0, x_0) \subset \mathcal{U}(t_0, x_0)$.

Part I: Proof of the inclusion $\mathcal{U}(t_0, x_0) \subset \mathcal{W}(t_0, x_0)$. Let $u_0 \in \mathcal{U}(t_0, x_0)$ and take $\gamma \in \text{Opt}(k, t_0, x_0)$ and $u : \mathbb{R}_+ \rightarrow \bar{B}$ an optimal control associated with γ such that $u \in \text{Lip}([t_0, t_0 + \varphi(t_0, x_0)]; \mathbb{S}^{d-1})$ and $u(t_0) = u_0$. By Proposition 3.3.13, we have, for every $h \in (0, \varphi(t_0, x_0)]$, that

$$(3.38) \quad \frac{\varphi(t_0 + h, \gamma(t_0 + h)) - \varphi(t_0, x_0)}{h} = -1.$$

We claim that $u_0 \in \bar{\text{In}}(x_0)$. Indeed, this is trivial if $x_0 \in \Omega$, and, if $x_0 \in \partial\Omega$, since γ takes values in $\bar{\Omega}$ and $\dot{\gamma}(t_0) = k(t_0, x_0)u_0$, one can easily check that $u_0 \cdot \mathbf{n}(x_0) \leq 0$, implying that $u_0 \in \bar{\text{In}}(x_0)$.

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $\text{In}(x_0)$ such that $u_n \rightarrow u_0$ as $n \rightarrow +\infty$. Fix $n \in \mathbb{N}$ and notice that, as $h \rightarrow 0^+$, we have

$$\begin{aligned} \gamma(t_0 + h) &= x_0 + hk(t_0, x_0)u_0 + o(h) \\ &= x_0 + hk(t_0, x_0)u_n + hk(t_0, x_0)(u_0 - u_n) + o(h). \end{aligned}$$

Recall that φ is Lipschitz continuous (Proposition 3.3.3) and denote by $M > 0$ its Lipschitz constant. Using the fact that $x_0 + hk(t_0, x_0)u_n \in \bar{\Omega}$ for $h > 0$ small enough, we deduce that, as $h \rightarrow 0^+$,

$$|\varphi(t_0 + h, \gamma(t_0 + h)) - \varphi(t_0 + h, x_0 + hk(t_0, x_0)u_n)| \leq hMK_{\max}|u_n - u_0| + o(h)$$

Combining with (3.38) and letting $h \rightarrow 0$, we deduce that, for every $n \in \mathbb{N}$,

$$\Delta_{(t_0, x_0)}(u_n) = \limsup_{h \rightarrow 0^+} \frac{\varphi(t_0 + h, x_0 + hk(t_0, x_0)u_n) - \varphi(t_0, x_0)}{h} \leq -1 + MK_{\max}|u_n - u_0|.$$

Hence, letting $n \rightarrow +\infty$ and using Proposition 3.4.18, we deduce that $\Delta_{(t_0, x_0)}(u_0) = -1$, showing that $u_0 \in \mathcal{W}(t_0, x_0)$, as required.

Part II: Proof of the inclusion $\mathcal{W}(t_0, x_0) \subset \mathcal{W}_\epsilon(t_0, x_0)$. Let $u_0 \in \mathcal{W}(t_0, x_0)$ and consider a sequence $(u_n)_{n \in \mathbb{N}}$ in $\text{In}(x_0)$ such that $u_n \rightarrow u_0$ as $n \rightarrow +\infty$. By Proposition 3.4.10, φ_ϵ is Lipschitz continuous on $\mathbb{R}_+ \times \bar{\Omega}_{\epsilon/2}$, and we denote by $C_\epsilon > 0$ a Lipschitz constant of this map. Notice that, for $h \in \left[0, \frac{\epsilon}{2K_{\max}}\right]$, we have $d_\Omega(x_0 + hk(t_0, x_0)u_n) \leq \frac{\epsilon}{2}$ for every $n \in \mathbb{N} \cup \{0\}$.

Hence, for $h \in \left(0, \frac{\epsilon}{2K_{\max}}\right]$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \\ \leq \frac{\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_n) - \varphi(t_0, x_0)}{h} + C_\epsilon K_{\max} |u_n - u_0|. \end{aligned}$$

For $n \in \mathbb{N}$, since $u_n \in \text{In}(x_0)$, we have $x_0 + hk(t_0, x_0)u_n \in \bar{\Omega}$ for $h > 0$ small enough, and thus, by Theorem 3.4.8, we have $\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_n) = \varphi(t_0 + h, x_0 + hk(t_0, x_0)u_n)$ for h small enough. Letting $h \rightarrow 0^+$, we thus deduce that

$$\limsup_{h \rightarrow 0^+} \frac{\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \leq \Delta_{(t_0, x_0)}(u_n) + C_\epsilon K_{\max} |u_n - u_0|.$$

Taking now the limit as $n \rightarrow +\infty$, we obtain that

$$(3.39) \quad \limsup_{h \rightarrow 0^+} \frac{\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \leq -1.$$

Let $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ be defined by $\gamma(t) = x_0$ for $t \in [0, t_0]$ and as the solution of the differential equation $\dot{\gamma}(t) = k_\epsilon(t, \gamma(t))u_0$ with initial condition $\gamma(t_0) = x_0$ for $t \geq t_0$. Then $\gamma \in \text{Adm}_\epsilon(k_\epsilon)$ and, by the dynamic programming principle (3.16) for $\text{OCP}_\epsilon(k_\epsilon)$, and using also Theorem 3.4.8, we have, for $h \geq 0$,

$$\frac{\varphi_\epsilon(t_0 + h, \gamma(t_0 + h)) - \varphi(t_0, x_0)}{h} \geq -1.$$

Using that $\gamma(t_0 + h) = x_0 + hk(t_0, x_0)u_0 + o(h)$ as $h \rightarrow 0^+$, $\gamma(t_0 + h) \in \bar{\Omega}_{\epsilon/2}$ for $h \geq 0$ small enough, and that φ_ϵ is Lipschitz continuous in $\mathbb{R}_+ \times \bar{\Omega}_{\epsilon/2}$, we deduce that, as $h \rightarrow 0^+$,

$$\frac{\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} \geq -1 + o(1).$$

Together with (3.39), this shows that

$$\lim_{h \rightarrow 0^+} \frac{\varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_0) - \varphi(t_0, x_0)}{h} = -1,$$

and thus $u_0 \in \mathcal{W}_\epsilon(t_0, x_0)$, as required.

Part III: Proof of the inclusion $\mathcal{W}_\epsilon(t_0, x_0) \subset \mathcal{U}(t_0, x_0)$. Let $u_0 \in \mathcal{W}_\epsilon(t_0, x_0)$ and $h > 0$, which is implicitly always assumed to be small enough. Then, by definition of $\mathcal{W}_\epsilon(t_0, x_0)$, we have, as $h \rightarrow 0^+$,

$$(3.40) \quad \varphi_\epsilon(t_0 + h, x_0 + hk(t_0, x_0)u_0) = \varphi(t_0, x_0) - h + o(h).$$

Define $\gamma_0 : [t_0, t_0 + h] \rightarrow \mathbb{R}^d$ by

$$(3.41) \quad \begin{cases} \dot{\gamma}_0(t) = k_\epsilon(t, \gamma_0(t))u_0, \\ \gamma_0(t_0) = x_0. \end{cases}$$

Let $x_1^h = \gamma_0(t_0 + h)$ and $t_1^h = t_0 + h$. Since Γ is closed and $x_0 \notin \Gamma$, we have $x_1^h \notin \Gamma$ for $h > 0$ small enough. Let $\gamma_1^h \in \text{Opt}_\epsilon(k_\epsilon, t_1^h, x_1^h)$ and u_1^h be the optimal control associated with γ_1^h , which satisfies $u_1^h \in \text{Lip}([t_1^h, t_1^h + \varphi_\epsilon(t_1^h, x_1^h)]; \mathbb{S}^{d-1})$ by Proposition 3.4.5. Set $\bar{u}_1^h = u_1^h(t_1^h) \in \mathbb{S}^{d-1}$ and define $\bar{\gamma}_1^h : [t_1^h, t_1^h + h] \rightarrow \mathbb{R}^d$ by

$$(3.42) \quad \begin{cases} \dot{\bar{\gamma}}_1^h(t) = k_\epsilon(t, \bar{\gamma}_1^h(t))\bar{u}_1^h \\ \bar{\gamma}_1^h(t_1^h) = x_1^h. \end{cases}$$

Let us also set $t_2^h = t_1^h + h$, $x_2^h = \gamma_1^h(t_2^h)$ and $\bar{x}_2^h = \bar{\gamma}_1^h(t_2^h)$. We split the sequel of the proof in two cases.

Case 1. We assume in this case that $\lim_{h \rightarrow 0^+} \bar{u}_1^h = u_0$. Let $\hat{u}_1^h \in \text{Lip}(\mathbb{S}^{d-1})$ be defined by $\hat{u}_1^h(t) = \bar{u}_1^h$ for $t \in [0, t_1^h]$, $\hat{u}_1^h(t) = u_1^h(t)$ for $t \in [t_1^h, t_1^h + \varphi_\epsilon(t_1^h, x_1^h)]$, and $\hat{u}_1^h(t) = u_1^h(t_1^h + \varphi(t_1^h, x_1^h))$ for $t \geq t_1^h + \varphi(t_1^h, x_1^h)$. Since γ_1^h and \hat{u}_1^h are Lipschitz continuous and their Lipschitz constants do not depend on h (see Proposition 3.4.5), one deduces from Arzelà–Ascoli Theorem that there exist a positive sequence $(h_n)_{n \in \mathbb{N}}$ converging to 0 as $n \rightarrow +\infty$ and elements $\gamma^* \in \text{Lip}_{K_{\max}}(\mathbb{R}^d)$ and $u^* \in \text{Lip}(\mathbb{S}^{d-1})$ such that $\gamma_1^{h_n} \rightarrow \gamma^*$ and $\hat{u}_1^{h_n} \rightarrow u^*$ as $n \rightarrow +\infty$, uniformly on compact time intervals. Since $\gamma_1^h \in \text{Opt}_\epsilon(k_\epsilon, t_1^h, x_1^h)$ for $h > 0$ and $t_1^h \rightarrow t_0$ and $x_1^h \rightarrow x_0$ as $h \rightarrow 0^+$, one can easily show, using the continuity of φ_ϵ , that $\gamma^* \in \text{Opt}_\epsilon(k_\epsilon, t_0, x_0)$ and its corresponding optimal control coincides with u^* on $[t_0, t_0 + \varphi(t_0, x_0)]$. Moreover, since $x_0 \in \bar{\Omega}$, we have from Theorem 3.4.8 that $\text{Opt}_\epsilon(k_\epsilon, t_0, x_0) = \text{Opt}(k, t_0, x_0)$. In addition,

$$u^*(t_0) = \lim_{n \rightarrow +\infty} \hat{u}_1^{h_n}(t_1^{h_n}) = \lim_{n \rightarrow +\infty} \bar{u}_1^{h_n} = u_0,$$

which implies that $u_0 \in \mathcal{U}(t_0, x_0)$, as required.

Case 2. We now consider the case where $(\bar{u}_1^h)_{h>0}$ does not converge to u_0 as $h \rightarrow 0^+$, and we prove that this case is not possible. Let $\epsilon > 0$ and $(h_n)_{n \in \mathbb{N}}$ be a positive sequence such that $h_n \rightarrow 0$ as $n \rightarrow +\infty$ and $|\bar{u}_1^{h_n} - u_0| \geq \epsilon$ for every $n \in \mathbb{N}$. For simplicity, we set $t_1^{h_n} = t_1^n$, $x_1^{h_n} = x_1^n$, and similarly for all other variables whose upper index is h_n . In order to clarify the constructions used in this case, we illustrate them in Figure 3.1.

Notice that $|x_1^n - x_0| \leq K_{\max}h_n$, $|\gamma_1^n(t) - x_1^n| \leq K_{\max}h_n$ for every $t \in [t_1^n, t_1^n + h_n]$, and $|\bar{x}_2^n - x_1^n| \leq K_{\max}h_n$. Hence, for all $t \in [t_1^n, t_1^n + h_n]$, we have $x_1^n, \gamma_1^n(t), x_2^n, \bar{x}_2^n \in \bar{B}(x_0, 2K_{\max}h_n)$. Since $x_0 \in \bar{\Omega}$, for n large enough, this ball is included in $\bar{\Omega}_{\epsilon/2} = \{x \in \mathbb{R}^d \mid d_\Omega(x) \leq \frac{\epsilon}{2}\}$, and in particular $k_\epsilon(t, x) \geq \frac{1}{2}K_{\min}$ for every $(t, x) \in \mathbb{R}_+ \times \bar{B}(x_0, 2K_{\max}h_n)$.

Integrating (3.41) on $[t_0, t_1^n]$, we get

$$x_1^n - x_0 = \int_{t_0}^{t_1^n} k_\epsilon(s, \gamma_0(s)) ds u_0,$$

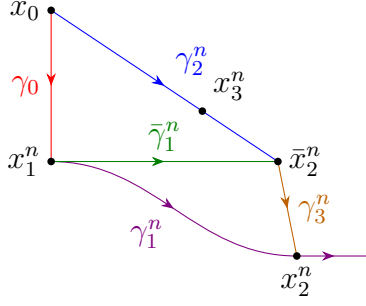


Figure 3.1: Illustration of the constructions used in the proof of Theorem 3.4.23.

and, proceeding similarly for (3.42), we get

$$\bar{x}_2^n - x_1^n = \int_{t_1^n}^{t_2^n} k_\epsilon(s, \bar{\gamma}_1^n(s)) ds \bar{u}_1^n.$$

Denote the integrals in the right-hand side of the above equalities by I_0^n and I_1^n , respectively. We have

$$\begin{aligned} |\bar{x}_2^n - x_0|^2 &= (I_0^n u_0 + I_1^n \bar{u}_1^n) \cdot (I_0^n u_0 + I_1^n \bar{u}_1^n) \\ &= (I_0^n)^2 + (I_1^n)^2 + 2I_0^n I_1^n u_0 \cdot \bar{u}_1^n \\ &= |x_1^n - x_0|^2 + |\bar{x}_2^n - x_1^n|^2 + 2I_0^n I_1^n u_0 \cdot \bar{u}_1^n. \end{aligned}$$

We know that $|\bar{u}_1^n - u_0| \geq \epsilon$, which leads us to observe that there exists $\alpha \in (0, 1)$ such that $u_0 \cdot \bar{u}_1^n < \alpha$ for every $n \in \mathbb{N}$. Thus

$$|\bar{x}_2^n - x_0|^2 < |x_1^n - x_0|^2 + |\bar{x}_2^n - x_1^n|^2 + 2\alpha I_0^n I_1^n.$$

Define

$$\rho := \sqrt{1 - (1 - \alpha) \frac{K_{\min}^2}{8K_{\max}^2}},$$

then obviously $\rho < 1$ and

$$\begin{aligned} |\bar{x}_2^n - x_0|^2 &< (|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|)^2 - 2(1 - \alpha)I_0^n I_1^n \\ &= \left(1 - (1 - \alpha) \frac{2I_0^n I_1^n}{(I_0^n + I_1^n)^2}\right) (|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|)^2 \\ (3.43) \quad &\leq \rho^2 (|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|)^2, \end{aligned}$$

where we use that $I_i^n \in [\frac{1}{2}hK_{\min}, hK_{\max}]$ for $i \in \{1, 2\}$. Let $u_2^n = \frac{\bar{x}_2^n - x_0}{|\bar{x}_2^n - x_0|}$ (with the convention $u_2^n = 0$ if $\bar{x}_2^n = x_0$) and define $\gamma_2^n : [t_0, t_0 + \tau^n] \rightarrow \mathbb{R}^d$ by

$$(3.44) \quad \begin{cases} \dot{\gamma}_2^n(t) = k_\epsilon(t, \gamma_2^n(t))u_2^n \\ \gamma_2^n(t_0) = x_0, \end{cases}$$

where $\tau^n \geq 0$ is chosen so that⁶ $\gamma_2^n(t_0 + \tau^n) = \bar{x}_2^n$.

Claim. As $n \rightarrow +\infty$, we have $\tau^n \leq 2\rho h_n + o(h_n)$.

Proof. Note that we have nothing to prove in the case $\bar{x}_2^n = x_0$, and hence we assume $\bar{x}_2^n \neq x_0$ in the sequel. If $|\bar{x}_2^n - x_0| \leq \rho|x_1^n - x_0|$, we let $x_3^n = \bar{x}_2^n$, otherwise we choose x_3^n as the unique point in the segment (x_0, \bar{x}_2^n) such that $|x_3^n - x_0| = \rho|x_1^n - x_0|$. In both cases, we have $|x_3^n - x_0| = \bar{\rho}|x_1^n - x_0|$ for some $\bar{\rho} \leq \rho$. Let τ_1^n be the time that γ_2^n takes to reach the point x_3^n , i.e., $\gamma_2^n(t_0 + \tau_1^n) = x_3^n$. (Note that $\tau_1^n = \tau^n$ in the case $|\bar{x}_2^n - x_0| \leq \rho|x_1^n - x_0|$.)

We first show that $\tau_1^n \leq \rho h_n + o(h_n)$. To obtain that, we observe, by integrating (3.41) and (3.44) and doing a change of variables, that

$$\begin{aligned}
(3.45) \quad \int_{t_0}^{t_0 + \tau_1^n} k_\epsilon(s, \gamma_2^n(s)) \, ds &= |x_3^n - x_0| = \bar{\rho}|x_1^n - x_0| = \bar{\rho} \int_{t_0}^{t_0 + h_n} k_\epsilon(s, \gamma_0(s)) \, ds \\
&= \int_{t_0}^{t_0 + \bar{\rho}h_n} k_\epsilon \left(t_0 + \frac{s - t_0}{\bar{\rho}}, \gamma_0 \left(t_0 + \frac{s - t_0}{\bar{\rho}} \right) \right) \, ds \\
&= \int_{t_0}^{t_0 + \bar{\rho}h_n} k_\epsilon(s, \gamma_2^n(s)) \, ds \\
&\quad + \int_{t_0}^{t_0 + \bar{\rho}h_n} \left[k_\epsilon \left(t_0 + \frac{s - t_0}{\bar{\rho}}, \gamma_0 \left(t_0 + \frac{s - t_0}{\bar{\rho}} \right) \right) - k_\epsilon(s, \gamma_2^n(s)) \right] \, ds.
\end{aligned}$$

Let us show that

$$(3.46) \quad \lim_{n \rightarrow +\infty} \frac{1}{h_n} \int_{t_0}^{t_0 + \bar{\rho}h_n} \left[k_\epsilon \left(t_0 + \frac{s - t_0}{\bar{\rho}}, \gamma_0 \left(t_0 + \frac{s - t_0}{\bar{\rho}} \right) \right) - k_\epsilon(s, \gamma_2^n(s)) \right] \, ds = 0$$

Let $\delta > 0$. Since k_ϵ is continuous in (t_0, x_0) , there exists $\eta > 0$ such that $|k_\epsilon(t, x) - k_\epsilon(t_0, x_0)| < \delta$ for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ satisfying $|t - t_0| < \eta$ and $|x - x_0| < \eta$. Since $h_n \rightarrow 0$ as $n \rightarrow +\infty$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$, we have $h_n < \eta$ and $K_{\max}h_n < \eta$. Noticing that, for every $s \in [t_0, t_0 + \bar{\rho}h_n]$, we have $\left| \frac{s - t_0}{\bar{\rho}} \right| \leq h_n < \eta$, $\left| \gamma_0 \left(t_0 + \frac{s - t_0}{\bar{\rho}} \right) - x_0 \right| \leq K_{\max}h_n < \eta$, $|s - t_0| \leq \bar{\rho}h_n < \eta$, and $|\gamma_2^n(s) - x_0| \leq \bar{\rho}K_{\max}h_n < \eta$, we deduce that, for $n \geq N$, we have, for every $s \in [t_0, t_0 + \bar{\rho}h_n]$,

$$\begin{aligned}
k_\epsilon(t_0, x_0) - \delta &< k_\epsilon \left(t_0 + \frac{s - t_0}{\bar{\rho}}, \gamma_0 \left(t_0 + \frac{s - t_0}{\bar{\rho}} \right) \right) < k_\epsilon(t_0, x_0) + \delta, \\
k_\epsilon(t_0, x_0) - \delta &< k_\epsilon(s, \gamma_2^n(s)) < k_\epsilon(t_0, x_0) + \delta.
\end{aligned}$$

Subtracting those inequalities, integrating on s in $[t_0, t_0 + \bar{\rho}h_n]$, and dividing by h_n , we deduce that

$$\frac{1}{h_n} \left| \int_{t_0}^{t_0 + \bar{\rho}h_n} \left[k_\epsilon \left(t_0 + \frac{s - t_0}{\bar{\rho}}, \gamma_0 \left(t_0 + \frac{s - t_0}{\bar{\rho}} \right) \right) - k_\epsilon(s, \gamma_2^n(s)) \right] \, ds \right| < 2\bar{\rho}\delta,$$

⁶This is possible since, for n large enough, all points in the segment from x_0 to \bar{x}_2^n lie within $\Omega_{\epsilon/2}$, on which k_ϵ is lower bounded by $\frac{1}{2}K_{\min}$.

concluding the proof of (3.46). Now, (3.45) and (3.46) imply that

$$(3.47) \quad \int_{t_0}^{t_0+\tau_1^n} k_\epsilon(s, \gamma_2^n(s)) ds = \int_{t_0}^{t_0+\bar{\rho}h_n} k_\epsilon(s, \gamma_2^n(s)) ds + o(h_n).$$

Define $F : [0, \tau^n] \rightarrow \mathbb{R}_+$ by $F(t) = \int_{t_0}^{t_0+t} k_\epsilon(s, \gamma_2^n(s)) ds$, then obviously F is continuous and increasing, which implies that F^{-1} is well-defined on the range of F . Since $\dot{F}(t) = k(t, \gamma_2^n(t))$, F is K_{\max} -Lipschitz continuous and, since $\frac{d}{dt}F^{-1}(t) = \frac{1}{F(F^{-1}(t))}$, we also deduce that F^{-1} is $\frac{2}{K_{\min}}$ -Lipschitz continuous. Therefore, by (3.47), we deduce that

$$\tau_1^n = F^{-1}(F(\bar{\rho}h_n) + o(h_n)) = \bar{\rho}h_n + o(h_n) \leq \rho h_n + o(h_n).$$

This concludes the proof of the claim in the case $|\bar{x}_2^n - x_0| \leq \rho|x_1^n - x_0|$, since $\tau_1^n = \tau^n$ in that case.

Otherwise, we have $\bar{\rho} = \rho$ and $|x_3^n - x_0| = \rho|x_1^n - x_0|$, and thus, from (3.43), we get

$$|\bar{x}_2^n - x_0| < \rho(|x_1^n - x_0| + |\bar{x}_2^n - x_1^n|) = |x_3^n - x_0| + \rho|\bar{x}_2^n - x_1^n|.$$

On the other hand, since x_3^n belongs to the segment (x_0, \bar{x}_2^n) , we have $|\bar{x}_2^n - x_0| = |\bar{x}_2^n - x_3^n| + |x_3^n - x_0|$, hence the inequality $|\bar{x}_2^n - x_3^n| \leq \rho|\bar{x}_2^n - x_1^n|$ holds. Suppose τ_2^n is the time the trajectory γ_2^n takes to go from x_3^n to \bar{x}_2^n , i.e., $\gamma_2^n(t_0 + \tau_1^n + \tau_2^n) = \bar{x}_2^n$, and note that $\tau^n = \tau_1^n + \tau_2^n$. As before, we compare the times between $|\bar{x}_2^n - x_3^n|$ and $|\bar{x}_2^n - x_1^n|$. Let $\beta \leq \rho$ be such that $|\bar{x}_2^n - x_3^n| = \beta|\bar{x}_2^n - x_1^n|$. Proceeding similarly to (3.45), we get

$$\begin{aligned} & \int_0^{\tau_2^n} k_\epsilon(s + t_0 + \tau_1^n, \gamma_2^n(s + t_0 + \tau_1^n)) ds = |\bar{x}_2^n - x_3^n| \\ &= \beta|\bar{x}_2^n - x_1^n| = \beta \int_{t_1^n}^{t_2^n} k_\epsilon(s, \bar{\gamma}_1^n(s)) ds \\ &= \int_0^{\beta h_n} k_\epsilon\left(\frac{s}{\beta} + t_0 + h_n, \bar{\gamma}_1^n\left(\frac{s}{\beta} + t_0 + h_n\right)\right) ds \\ &= \int_0^{\beta h_n} k_\epsilon(s + t_0 + \tau_1^n, \gamma_2^n(s + t_0 + \tau_1^n)) ds \\ & \quad + \int_0^{\beta h_n} \left[k_\epsilon\left(\frac{s}{\beta} + t_0 + h_n, \bar{\gamma}_1^n\left(\frac{s}{\beta} + t_0 + h_n\right)\right) - k_\epsilon(s + t_0 + \tau_1^n, \gamma_2^n(s + t_0 + \tau_1^n)) \right] ds. \end{aligned}$$

Proceeding as in the proof of (3.46), we can show that the last integral in the above expression is an $o(h_n)$ as $n \rightarrow +\infty$, and thus

$$\int_{t_0}^{t_0+\tau_2^n} k_\epsilon(s + \tau_1^n, \gamma_2^n(s + \tau_1^n)) ds = \int_{t_0}^{t_0+\beta h_n} k_\epsilon(s + \tau_1^n, \gamma_2^n(s + \tau_1^n)) ds + o(h_n).$$

Defining $F(t) = \int_{t_0}^{t_0+t} k_\epsilon(s + \tau_1^n, \gamma_2^n(s + \tau_1^n)) ds$ and arguing similarly to above, we deduce that $\tau_2^n = \beta h_n + o(h_n)$. Therefore the time τ^n to reach \bar{x}_2^n from x_0 satisfies

$$\tau^n = (\rho + \beta)h_n + o(h_n) \leq 2\rho h_n + o(h_n). \quad \square$$

Let us now compare the trajectories $\bar{\gamma}_1^n$ and γ_1^n on $[t_1^n, t_2^n]$. Let $\delta_1^n(t) = \gamma_1^n(t) - \bar{\gamma}_1^n(t)$. Hence, from the ODEs satisfied by the trajectories $\bar{\gamma}_1^n$ and γ_1^n , we have

$$\begin{aligned}\delta_1^n(t) &= \int_{t_1^n}^t \left[k_\epsilon(s, \gamma_1^n(s)) u_1^n(s) - k_\epsilon(s, \bar{\gamma}_1^n(s)) \bar{u}_1^n \right] ds \\ &= \int_{t_1^n}^t \left[k_\epsilon(s, \gamma_1^n(s)) - k_\epsilon(s, \bar{\gamma}_1^n(s)) \right] u_1^n(s) ds + \int_{t_1^n}^t k_\epsilon(s, \bar{\gamma}_1^n(s)) (u_1^n(s) - \bar{u}_1^n) ds.\end{aligned}$$

Since u_1^n is the optimal control associated with γ_1^n , by Proposition 3.4.5, it is Lipschitz continuous in $[t_1^n, t_1^n + \varphi_\epsilon(t_1^n, x_1^n)]$ and its Lipschitz constant is independent of n . Therefore, denoting by $L_\epsilon > 0$ the Lipschitz constant of k_ϵ with respect to its second variable (which is independent of the first variable) and $C > 0$ the Lipschitz constant of u_1^n , we have

$$|\delta_1^n(t)| \leq L_\epsilon \int_{t_1^n}^t |\delta_1^n(s)| ds + CK_{\max} \int_{t_1^n}^t |s - t_1^n| ds,$$

and hence, by using Grönwall's inequality,

$$|\delta_1^n(t)| \leq CK_{\max} \frac{(t - t_1^n)^2}{2} e^{L_\epsilon(t - t_1^n)}.$$

In particular, if we set $t = t_1^n + h_n$, then

$$|x_2^n - \bar{x}_2^n| \leq CK_{\max} \frac{h_n^2}{2} e^{L_\epsilon h_n} = O(h_n^2).$$

Let $u_3^n = \frac{x_2^n - \bar{x}_2^n}{|x_2^n - \bar{x}_2^n|}$ (with the convention $x_3^n = 0$ if $x_2^n = \bar{x}_2^n$) and γ_3^n be the solution of

$$(3.48) \quad \begin{cases} \dot{\gamma}_3^n(t) = k_\epsilon(t, \gamma_3^n(t)) u_3^n \\ \gamma_3^n(t_0 + \tau^n) = \bar{x}_2^n. \end{cases}$$

Using the lower bound $\frac{1}{2}K_{\min}$ on k_ϵ and the fact that $|x_2^n - \bar{x}_2^n| = O(h_n^2)$, one can easily deduce that the time σ^n from \bar{x}_2^n to x_2^n along γ_3^n (i.e., the value $\sigma^n > 0$ such that $\gamma_3^n(t_0 + \tau^n + \sigma^n) = x_2^n$) satisfies $\sigma^n = O(h_n^2)$.

We have thus constructed two ways to go from x_0 to x_2^n . The first one is to choose the path containing x_0 , x_1^n , and x_2^n , which corresponds to the concatenation of the trajectories γ_0 on $[t_0, t_1^n]$ and γ_1^n on $[t_1^n, t_2^n]$, and the second one is the path containing x_0 , \bar{x}_2^n , and x_2^n , which corresponds to the concatenation of the trajectories γ_2^n on $[t_0, t_0 + \tau^n]$ and γ_3^n on $[t_0 + \tau^n, t_0 + \tau^n + \sigma^n]$. Letting T_1^n and T_2^n be the times for going from x_0 to x_2^n along these two paths, respectively, we have, by construction and the claim, that $T_1^n = 2h_n$ and $T_2^n = \tau^n + \sigma^n \leq 2\rho h_n + o(h_n)$. Hence, since $\rho < 1$, we have, for n large enough, that $T_2^n < T_1^n$.

By definition of γ_0 , we have $x_1^n = \gamma_0(t_1^n) = x_0 + h_n k(t_0, x_0) u_0 + o(h_n)$ and, using (3.40) and the fact that φ_ϵ is Lipschitz continuous (Proposition 3.4.10), we deduce that

$$\varphi(t_0, x_0) = \varphi_\epsilon(t_1^n, x_1^n) + h_n + o(h_n).$$

Moreover, since $\gamma_1^n \in \text{Opt}_\epsilon(k_\epsilon, t_1^n, x_1^n)$, we have from (3.16) that $\varphi_\epsilon(t_1^n, x_1^n) = \varphi_\epsilon(t_2^n, x_2^n) + h_n$, yielding that

$$\varphi(t_0, x_0) = \varphi(t_2^n, x_2^n) + T_1^n + o(h_n).$$

On the other hand, since the path from x_0 to x_2^n going through \bar{x}_2^n is an admissible trajectory for k_ϵ , we have, by (3.16), that $\varphi(t_0, x_0) \leq T_2^n + \varphi_\epsilon(t_0 + T_2^n, x_2^n)$. Hence

$$(3.49) \quad \varphi_\epsilon(t_2^n, x_2^n) + T_1^n + o(h_n) \leq T_2^n + \varphi_\epsilon(t_0 + T_2^n, x_2^n).$$

We also know that $t_0 + T_2^n < t_0 + T_1^n = t_2^n$ for n large enough. Therefore⁷, by Proposition 3.3.51, there exists a constant $c > 0$ such that

$$\varphi_\epsilon(t_2^n, x_2^n) > \varphi_\epsilon(t_0 + T_2^n, x_2^n) + (c - 1)(t_2^n - t_0 - T_2^n) = \varphi_\epsilon(t_0 + T_2^n, x_2^n) + (c - 1)(T_1^n - T_2^n),$$

and, using (3.49), we get $(c - 1)(T_1^n - T_2^n) + T_1^n + o(h_n) \leq T_2^n$, which leads to

$$2h_n + o(h_n) = T_1^n + o(h_n) \leq T_2^n \leq 2\rho h_n + o(h_n).$$

Divide above inequality by h_n to observe that

$$2 + o(1) \leq 2\rho + o(1).$$

Finally by letting $n \rightarrow +\infty$, we conclude that $\rho \geq 1$, which is a contradiction. Therefore Case 2 will never happen and this ends the proof. \square

We conclude this section with a technical result showing that, for $x \in \partial\Omega \setminus \Gamma$, if all directions of $\mathcal{W}(t, x)$ point to the inside of the domain, then no optimal trajectories starting at time 0 are close to x at time t . For that purpose, we introduce

Proposition 3.4.24. *Consider the optimal control problem $\text{OCP}(k)$ and assume that (H1)–(H3), (H4), and (H5) hold. Let $x \in \partial\Omega \setminus \Gamma$, $t > 0$, and assume that there exists $w \in \mathcal{W}(t, x)$ such that $w \cdot \mathbf{n}(x) < 0$. Then there exists a neighborhood N of x in $\bar{\Omega}$ such that, for every $x_0 \in \bar{\Omega}$ and $\gamma \in \text{Opt}(k, 0, x_0)$, we have $\gamma(t) \notin N$.*

Proof. Assume, to obtain a contradiction, that there exist sequences $(x_{0,n})_{n \in \mathbb{N}}$ and $(\gamma_n)_{n \in \mathbb{N}}$ with $x_{0,n} \in \bar{\Omega}$ and $\gamma_n \in \text{Opt}(k, 0, x_{0,n})$ for every $n \in \mathbb{N}$ and such that $\gamma_n(t) \rightarrow x$ as $n \rightarrow +\infty$. Since $\bar{\Omega}$ is compact and $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence of functions which are K_{\max} -Lipschitz continuous, applying Arzelà–Ascoli Theorem, we deduce that there exist $x_0 \in \bar{\Omega}$ and $\gamma \in \text{Lip}_{K_{\max}}(\bar{\Omega})$ such that, up to extracting subsequences (which we still denote by $(x_{0,n})_{n \in \mathbb{N}}$ and $(\gamma_n)_{n \in \mathbb{N}}$ for simplicity), we have, as $n \rightarrow +\infty$, that $x_{0,n} \rightarrow x_0$ and $\gamma_n \rightarrow \gamma$ uniformly on

⁷Strictly speaking, Proposition 3.3.51 only applies to φ , and not to φ_ϵ . However, one can still get its conclusion by arguing as follows. Let $\widehat{\text{OCP}}$ be defined as in the proof of Proposition 3.4.10 with $\eta = \frac{\epsilon}{2}$ and consider its value function $\widehat{\varphi}$. By the arguments provided in that proof, $\widehat{\varphi}$ satisfies Proposition 3.3.51 and $\widehat{\varphi}$ and φ_ϵ coincide in $\Omega_{\epsilon/2}$. The conclusion then follows since all points involved here belong to $\Omega_{\epsilon/2}$ for n large enough. Note that the constant $c > 0$ depends on ϵ .

compact time intervals. By straightforward arguments based on the continuity of the value function, we deduce that $\gamma \in \text{Opt}(k, 0, x_0)$, and in addition we have $\gamma(t) = x$.

Recall that, by Corollary 3.4.9, we have $\gamma \in \mathbf{C}^1([0, T]; \bar{\Omega})$, where $T = \varphi(0, x_0)$ and φ is the value function of OCP(k). Since $t > 0$ and $x \notin \Gamma$, we have $t \in (0, T)$, and thus γ is differentiable at t . Moreover, by Proposition 3.4.16 and Theorem 3.4.23, the set $\mathcal{W}(t, x)$ contains exactly one element, which we denote by $w_0 \in \mathbb{S}^{d-1}$, and thus, by assumption, we have $w_0 \cdot \mathbf{n}(x) < 0$. By Definition 3.4.15 and Theorem 3.4.23, we deduce also that $\dot{\gamma}(t) = k(t, x)w_0$.

Let $\alpha : h \mapsto d_{\partial\Omega}^{\pm}(\gamma(t+h))$ be defined on an open neighborhood of 0 in \mathbb{R} . Then $\alpha(0) = d_{\partial\Omega}^{\pm}(x) = 0$ and $\dot{\alpha}(0) = k(t, x)\mathbf{n}(x) \cdot w_0 < 0$. In particular, there exists $h_0 \in [-t, 0)$ such that, for every $h \in [h_0, 0)$, we have $\alpha(h) > 0$, and thus $\gamma(t+h) \notin \bar{\Omega}$, which contradicts the fact that γ takes values in $\bar{\Omega}$. This contradiction yields the conclusion. \square

3.4.4 The normalized gradient

Now that Theorem 3.4.23 characterizes the set of optimal directions $\mathcal{U}(t_0, x_0)$ as the set $\mathcal{W}(t_0, x_0)$ of directions of maximal descent of φ , we provide the following definition, which is motivated by Proposition 3.4.21.

Definition 3.4.25. Consider the optimal control problem OCP(k) and its value function φ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$. If $\mathcal{W}(t_0, x_0)$ contains exactly one element, we denote this element by $-\widehat{\nabla}\varphi(t_0, x_0)$, and call it the *normalized gradient* of φ at (t_0, x_0) .

As a consequence of Proposition 3.4.16 and Theorem 3.4.23, we immediately obtain the following characterization of optimal controls.

Corollary 3.4.26. Consider the optimal control problem OCP(k) and its value function φ and assume that (H1)–(H3), (H4), and (H5) hold. Let $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, $\gamma \in \text{Opt}(k, t_0, x_0)$, and $T = \varphi(t_0, x_0)$. Then for every $t \in (t_0, t_0 + T)$, φ admits a normalized gradient at $(t, \gamma(t))$ and

$$\dot{\gamma}(t) = -k(t, \gamma(t))\widehat{\nabla}\varphi(t, \gamma(t)).$$

Combining the above result with Corollary 3.4.9, we deduce that, for every optimal trajectory γ , the map $t \mapsto \widehat{\nabla}\varphi(t, \gamma(t))$ is Lipschitz continuous as long as $\gamma(t) \in \bar{\Omega} \setminus \Gamma$ and t is larger than the initial time of γ . However, this provides no information on the regularity of $(t, x) \mapsto \widehat{\nabla}\varphi(t, x)$. We are interested in proving the continuity of this map on its domain of definition. For that purpose, we first prove the upper semi-continuity of the set-valued map \mathcal{U} (we refer to [7, Definition 1.4.1] for the definition of upper semi-continuity for set-valued maps).

Proposition 3.4.27. Consider the optimal control problem OCP(k) and assume that (H1)–(H3), (H4), and (H5) hold. Let $\mathcal{U} : \mathbb{R}_+ \times \bar{\Omega} \rightrightarrows \mathbb{S}^{d-1}$ be the set valued map introduced in Definition 3.4.15. Then \mathcal{U} is upper semi-continuous.

Proof. Since \mathbb{S}^{d-1} is a compact set, it suffices to show that \mathcal{U} has a closed graph (see, e.g., [7, Proposition 1.4.8]). Let $(t_n, x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+ \times \bar{\Omega}$ converging as $n \rightarrow +\infty$ to some $(t_0, x_0) \in \mathbb{R}_+ \times \bar{\Omega}$, $(\bar{u}_n)_{n \in \mathbb{N}}$ be a sequence such that $\bar{u}_n \in \mathcal{U}(t_n, x_n)$ for every $n \in \mathbb{N}$ and $\bar{u}_n \rightarrow \bar{u}_0$ as $n \rightarrow +\infty$ for some $\bar{u}_0 \in \mathbb{S}^{d-1}$. Since $\bar{u}_n \in \mathcal{U}(t_n, x_n)$ for every $n \in \mathbb{N}$, there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of optimal trajectories with $\gamma_n \in \text{Opt}(k, t_n, x_n)$ for every $n \in \mathbb{N}$ and a corresponding sequence of the associated optimal controls $(u_n)_{n \in \mathbb{N}}$ with $u_n \in \text{Lip}([t_n, t_n + \varphi(t_n, x_n)]; \mathbb{S}^{d-1})$ and $u_n(t_n) = \bar{u}_n$ for every $n \in \mathbb{N}$. From Corollary 3.4.9, up to modifying u_n outside of the interval $[t_n, t_n + \varphi(t_n, x_n)]$, the sequences $(\gamma_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$ are sequences of Lipschitz continuous functions with Lipschitz constants independent of n , and thus, by Arzelà–Ascoli Theorem, one finds elements $\gamma^* \in \text{Lip}(\bar{\Omega})$ and $u^* \in \text{Lip}(\mathbb{S}^{d-1})$ such that, up to a subsequence, $\gamma_n \rightarrow \gamma^*$ and $u_n \rightarrow u^*$ uniformly on compact time intervals. In particular,

$$u^*(t_0) = \lim_{n \rightarrow \infty} u_n(t_n) = \lim_{n \rightarrow \infty} \bar{u}_n = \bar{u}_0$$

and

$$\gamma^*(t_0) = \lim_{n \rightarrow \infty} \gamma_n(t_n) = \lim_{n \rightarrow \infty} x_n = x_0.$$

By using the dynamic programming principle from Proposition 3.3.13 and the Lipschitz continuity of the value function from Proposition 3.3.3, one observes that the restriction of u^* to $[t_0, t_0 + \varphi(t_0, x_0)]$ is the optimal control corresponding to the optimal trajectory γ^* . Hence $\bar{u}_0 = u^*(t_0) \in \mathcal{U}(t_0, x_0)$, concluding the proof that \mathcal{U} has a closed graph. \square

On the set of points where a set-valued map is single-valued, upper semi-continuity coincides with standard continuity of single-valued functions. As an immediate consequence of this fact, Proposition 3.4.27, and Theorem 3.4.23, we have the following result.

Corollary 3.4.28. *Consider the optimal control problem $\text{OCP}(k)$ and its value function φ and assume that (H1)–(H3), (H4), and (H5) hold. Let $\mathcal{W} : \mathbb{R}_+ \times \bar{\Omega} \rightrightarrows \mathbb{S}^{d-1}$ be the set valued map introduced in Definition 3.4.19 and $\widehat{\nabla} \varphi$ be the normalized gradient of φ . Then \mathcal{W} is upper semi-continuous and $\widehat{\nabla} \varphi$ is a continuous function on the set where it is defined.*

3.5 The MFG system

We now turn to the characterization of equilibria $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$ of a mean field game $\text{MFG}(K)$ through a system of PDEs, known as the MFG system, consisting of a continuity equation on the density of agents $t \mapsto m_t$, defined through the relation $m_t = e_{t\#} Q$, coupled with the Hamilton–Jacobi equation (3.13) on the value function φ of the optimal control problem $\text{OCP}(k)$, where k is defined from K and Q by setting $k(t, x) = K(m_t, x)$. The main difficulty in this characterization lies within the continuity equation for m_t , and more precisely on the characterization of the corresponding velocity field. Corollary 3.4.26 suggests that such velocity field should be given by the opposite of the normalized gradient of φ multiplied by the dynamics k .

Corollary 3.4.28 states the continuity of the normalized gradient $\widehat{\nabla}\varphi$ on the set of points of $\mathbb{R}_+ \times \bar{\Omega}$ where it is defined. We will now show that, for the purposes of studying the equilibria of mean field games, this set is quite “large”. More precisely, consider the mean field game $\text{MFG}(K)$ and an equilibrium $Q \in \mathcal{P}(\mathbf{C}(\bar{\Omega}))$ for this game. We will prove that, for every fixed $t > 0$, the set of points $x \in \bar{\Omega} \setminus \Gamma$ at which $\widehat{\nabla}\varphi$ does not exist has m_t measure zero, where m_t is the evaluation at time t of the equilibrium measure Q .

For that purpose, let us introduce the set

(3.50)

$$\Upsilon = \{(t, x) \in \mathbb{R}_+^* \times (\bar{\Omega} \setminus \Gamma) \mid \exists t_0 \in [0, t), \exists x_0 \in \bar{\Omega}, \exists \gamma \in \text{Opt}(k, t_0, x_0) \text{ s.t. } \gamma(t) = x\}.$$

In other words, Υ contains all points $(t, x) \in \mathbb{R}_+^* \times (\bar{\Omega} \setminus \Gamma)$ which are strictly between the starting and the final points of an optimal trajectory. In particular, it follows from Corollary 3.4.26 that $\widehat{\nabla}\varphi(t, x)$ exists for every $(t, x) \in \Upsilon$, and, by Corollary 3.4.28, $\widehat{\nabla}\varphi$ is continuous in Υ . We also introduce, for $t > 0$, the set

(3.51)

$$\Upsilon_t = \{x \in \bar{\Omega} \setminus \Gamma \mid (t, x) \in \Upsilon\}.$$

Proposition 3.5.1. *Consider the mean field game $\text{MFG}(K)$ under the assumptions (H1)–(H3), (H6), and (H7). Let Q be an equilibrium for $\text{MFG}(K)$, set $m_t = e_{t\#}Q$ for $t \geq 0$, define $k : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$ by $k(t, x) = K(m_t, x)$, consider the optimal control problem $\text{OCP}(k)$, and let Υ and Υ_t be defined as in (3.50) and (3.51), respectively. Then for every $t > 0$, we have $m_t(\bar{\Omega} \setminus (\Gamma \cup \Upsilon_t)) = 0$.*

Proof. Let $\mathbf{Opt} = \bigcup_{x_0 \in \bar{\Omega}} \text{Opt}(k, 0, x_0)$. Since Q is an equilibrium of $\text{MFG}(K)$, then $Q(\mathbf{Opt}) = 1$. For every $t > 0$, from the definition of Υ , one has that $\{\gamma \in \mathbf{Opt} \mid \gamma(t) \in \bar{\Omega} \setminus (\Gamma \cup \Upsilon_t)\} = \emptyset$, and then $m_t(\bar{\Omega} \setminus (\Gamma \cup \Upsilon_t)) = Q(\{\gamma \in \mathbf{Opt} \mid \gamma(t) \in \bar{\Omega} \setminus (\Gamma \cup \Upsilon_t)\}) = Q(\emptyset) = 0$. \square

We are now ready to provide our main result concerning the MFG system of $\text{MFG}(K)$.

Theorem 3.5.2. *Consider the mean field game $\text{MFG}(K)$ under assumptions (H1)–(H3), (H6), and (H7). Let $m_0 \in \mathcal{P}(\bar{\Omega})$, Q be an equilibrium of $\text{MFG}(K)$ with initial condition m_0 , $m_t \in \mathcal{P}(\bar{\Omega})$ be defined for $t > 0$ by $m_t = e_{t\#}Q$, $k : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}_+$ be defined from Q and K by $k(t, x) = K(e_{t\#}Q, x)$, φ be the value function of $\text{OCP}(k)$, and \mathcal{W} be the set-valued map provided in Definition 3.4.19. For $t > 0$, set*

$$\partial\Omega_t^- = \{x \in \partial\Omega \setminus \Gamma \mid \exists w \in \mathcal{W}(t, x) \text{ such that } w \cdot \mathbf{n}(x) < 0\}.$$

Then (m, φ) solves the MFG system

$$(3.52) \quad \begin{cases} \partial_t m_t(x) - \text{div} \left(m_t(x) K(m_t, x) \widehat{\nabla}\varphi(t, x) \right) = 0, & \text{in } \mathbb{R}_+^* \times (\bar{\Omega} \setminus \Gamma), \\ -\partial_t \varphi(t, x) + |\nabla\varphi(t, x)| K(m_t, x) - 1 = 0, & \text{in } \mathbb{R}_+ \times (\bar{\Omega} \setminus \Gamma), \\ m_t(x) = 0, & \text{for } t > 0 \text{ and } x \in \partial\Omega_t^-, \\ \varphi(t, x) = 0, & \text{on } \mathbb{R}_+ \times \Gamma, \\ \nabla\varphi(t, x) \cdot \mathbf{n}(x) \geq 0, & \text{on } \mathbb{R}_+ \times (\partial\Omega \setminus \Gamma), \\ m_t = m_0, & \text{in } \{0\} \times \bar{\Omega}, \end{cases}$$

where the first equation is satisfied in the sense of distributions, the second and fifth equations are satisfied in the viscosity senses of Proposition 3.3.4 and Theorem 3.4.11, respectively, and the third equation is satisfied in the following sense: for every $t > 0$ and $x \in \partial\Omega_t^-$, there exists a neighborhood N of x such that $m_t(N) = 0$.

Proof. Notice first that, since K satisfies (H6) and (H7), then k satisfies (H4) and (H5). The Hamilton–Jacobi equation on φ and its boundary conditions then follow immediately from Proposition 3.3.4 and Theorem 3.4.11. The initial condition on m_t follows from its definition.

Let us prove that m_t satisfies the continuity equation in (3.52). First, thanks to Corollary 3.4.28, $\widehat{\nabla\varphi}$ is continuous on the set Υ defined in (3.50). Let $\xi \in \mathbf{C}_c^\infty(\mathbb{R}_+^* \times (\bar{\Omega} \setminus \Gamma); \mathbb{R})$ be a test function. Take $\gamma \in \text{Opt}(k, 0, x_0)$ and let $T = \varphi(0, x_0)$. By Corollary 3.4.26, one has $\dot{\gamma}(t) = -K(m_t, \gamma(t))\widehat{\nabla\varphi}(t, \gamma(t))$ for every $t \in (0, T)$. Hence

$$(3.53) \quad \frac{d}{dt}(\xi(t, \gamma(t))) = \partial_t \xi(t, \gamma(t)) - \nabla_x \xi(t, \gamma(t)) \cdot \widehat{\nabla\varphi}(t, \gamma(t)) K(m_t, \gamma(t)).$$

Let us denote the set of all optimal trajectories by \mathbf{Opt} , i.e., $\mathbf{Opt} = \bigcup_{x_0 \in \bar{\Omega}} \text{Opt}(k, 0, x)$. Thanks to the continuity of the right-hand side of (3.53) on $\mathbb{R}_+^* \times \mathbf{Opt}$, one can integrate to observe that

$$\begin{aligned} & \int_0^\infty \int_{\mathbf{Opt}} \frac{d}{dt}(\xi(t, \gamma(t))) \, dQ(\gamma) \, dt \\ &= \int_0^\infty \int_{\mathbf{Opt}} \partial_t \xi(t, \gamma(t)) \, dQ(\gamma) \, dt \\ & \quad - \int_0^\infty \int_{\mathbf{Opt}} \nabla_x \xi(t, \gamma(t)) \cdot \widehat{\nabla\varphi}(t, \gamma(t)) K(m_t, \gamma(t)) \, dQ(\gamma) \, dt. \end{aligned}$$

Since φ is compactly supported, the left-hand side of the above equality is zero. Hence, by using the Proposition 3.5.1 and the relation between m_t and Q , one concludes that

$$\int_0^\infty \int_{\bar{\Omega}} \partial_t \xi(t, x) \, dm_t(x) \, dt = \int_0^\infty \int_{\bar{\Omega}} \nabla_x \xi(t, x) \cdot \widehat{\nabla\varphi}(t, x) K(m_t, x) \, dm_t(x) \, dt,$$

which is precisely the weak formulation of the continuity equation in (3.52).

Finally, let us prove the boundary condition on m_t . Let $t > 0$ and $x \in \partial\Omega_t^-$. Then, using the definition of m_t and Proposition 3.4.24, there exists a neighborhood N of x such that

$$m_t(N) = Q(\{\gamma \in \mathbf{Opt} \mid \gamma(t) \in N\}) = Q(\emptyset) = 0,$$

as required. \square

Chapter 4

A variational mean field game of controls with free final time and pairwise interaction

In this chapter, we consider a mean field game model for crowd motion in which pedestrians interact not only through their position, but also through their velocity. More precisely, each pedestrian is assumed to minimize a cost involving their time to reach a certain target set, an individual integral cost, and an interaction integral cost modelling the fact that agents want to avoid congestion and prefer to move together with agents going in the same direction, in which can be seen as a Cucker–Smale type interaction. The main result we obtain in this chapter is the existence of equilibria for such a game, which is based on a variational approach.

4.1 Introduction

We deal with a model that studies the collective behaviour of agents’ motion in a given domain, by taking advantage of mean field game tools. Mean field games were introduced in 2006 by by Jean-Michel Lasry and Pierre-Louis Lions [69–71], and simultaneous independent works by Peter E. Caines, Minyi Huang, and Roland P. Malhamé [61–63], based in applications in engineering and economics, and have been extensively studied in the previous years. Our aim is to propose and study here an MFG model inspired by crowd motion and using elements from a Cucker–Smale type model in order to take into account interactions between agents.

Our aim here is to provide a similar model for the movement of pedestrians, in which, similarly to [86], we consider instead that pedestrians choose their trajectories by optimizing a certain criterion, and the Cucker–Smale interaction is taken into account as an interaction term in this optimization criterion. We also replace the finite population of the Cucker–Smale model by infinitely many pedestrians, in the spirit of an MFG model. The Cucker–Smale model was introduced in [44] to study the evolution of a flock of birds, in which finitely many

birds evolve according to a prescribed law taking into account the fact that birds wish to align their velocities with others, and that this alignment effect is stronger when the birds are closer.

In our mean field game model, agents take into account not only the position but also the direction of the movements of others in their optimization criterion and, in this sense, the mean field game we consider here can be seen as an *mean field games of controls*. MFGs of controls and related models were extensively studied in the literature, for instance in [16, 31, 32, 54, 86]. The work [31] studies a model inspired by trade crowding which deals with taking decision to buy or to sell a large number of productions or contracts in which the control of others plays a role for optimization criterion. The work [54] studies a mean field game of controls in a random framework by taking the agents' trajectories in the space of random variables. In contrast to this work [54], the work [16] studies a model in a deterministic mode involving nonlinear dynamics and mixed state-control constraints through a price variable and a congestion term.

Our work is more closely related to the work [86], which considers that each agent minimizes a cost function of the form

$$(4.1) \quad \int_0^T \left(\frac{\delta}{2} |\dot{\gamma}(t)|^2 + \lambda \int_{\Gamma} \frac{1}{2} \eta(\gamma(t) - \tilde{\gamma}(t)) |\dot{\gamma}(t) - \dot{\tilde{\gamma}}(t)|^2 dQ(\tilde{\gamma}) \right) dt + \Phi(\gamma(T)),$$

where γ and $\tilde{\gamma}$ represent agents' trajectories, Q is a probability measure on the space of continuous curves Γ , Ψ is the final cost which penalizes the agent's final position and η is a decreasing interaction kernel. The authors of [86] have shown the existence of equilibria of the game, in the variational setting, by proving the existence of a minimizer of a functional cost, based upon the above cost function. Furthermore, due to having a quadratic form of individual cost and a specific interaction cost, they could manage to establish the corresponding MFG system.

One of the main novelties of this chapter is to consider the optimization criterion with free final time and not in a compact time interval $[0, T]$. Moreover, inspired by this model, another novelty of this chapter is to generalize the above cost function (4.1) by replacing a general individual cost $\ell(t, \gamma(t), \dot{\gamma}(t))$ instead of that special quadratic form, and a general interaction cost $h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t))$. The main result of this chapter is the existence of equilibria for such a model, which is done by adapting the techniques of [86] to the free final time setting. More precisely, we construct a functional \mathcal{J} on the space of probability measures on the set of continuous curves Γ and prove that, under suitable assumptions, critical points of such a function coincide with equilibria of the game. Existence of equilibria is then proved by showing existence of minimizers of the functional, which are also necessarily critical points. In addition, we also discuss two different notions of equilibria, called here "weak" and "strong", and show their equivalence under additional assumptions on ℓ .

This chapter is organized as follows. In Section 4.2, we introduced the general definitions and notations used along the chapter. Section 4.3 provides a precise description of the model and shows some useful preliminary properties, such as lower semicontinuity of the exit time function and of other functions involved in the minimization criterion of each agent. In

Sections 4.4 and 4.5, in addition to provide some definitions such as differentiability of a functional with respect to a measure and critical point in our context, we establish the relations between equilibrium, critical point and the minimizer of the functional \mathcal{J} and prove in addition the existence of a minimizer of \mathcal{J} , yielding the existence of an equilibrium for our mean field game model. Finally in Section 4.6, we address the question of the equivalence between two notions of equilibrium, weak and strong, under an additional assumption.

4.2 Notations and definitions

In this chapter, d is a fixed positive integer. The set of nonnegative real numbers is denoted by \mathbb{R}_+ . Let the space Ω be an open bounded subset of \mathbb{R}^d , Ξ be a non-empty closed subset of $\bar{\Omega}$ and consider $\mathcal{C}(\mathbb{R}_+; \bar{\Omega})$, $\mathcal{AC}(\mathbb{R}_+; \bar{\Omega})$ as the space of continuous and absolutely continuous curves from \mathbb{R}_+ to $\bar{\Omega}$, respectively, which are endowed with the topology of uniform convergence on compact time intervals. Having considered this latter topology, the space $\mathcal{C}(\mathbb{R}_+; \bar{\Omega})$ is a Polish space. For simplicity, we denote $\mathcal{C}(\mathbb{R}_+; \bar{\Omega})$ by Γ i.e., $\Gamma = \mathcal{C}(\mathbb{R}_+; \bar{\Omega})$.

We denote the space of Borel probability measures on Γ by $\mathcal{P}(\Gamma)$ with topology of weak convergence of measures. The support of $Q \in \mathcal{P}(\Gamma)$ is denoted by $\text{spt}(Q)$, and is defined as the set of all curves $\gamma \in \Gamma$ such that $Q(N_\gamma) > 0$ for every open neighborhood N_γ of γ . We denote the Dirac measure centered at a curve $\gamma_0 \in \Gamma$ by δ_{γ_0} .

For two metric spaces X and Y endowed with their Borel σ -algebras and a Borel-measurable map $f: X \rightarrow Y$, the pushforward of a measure μ on X through f is the measure $f_\# \mu$ on Y defined by $f_\# \mu(B) = \mu(f^{-1}(B))$ for every Borel subset B of Y . For $t \in \mathbb{R}_+$, we denote by $e_t: \Gamma \rightarrow \bar{\Omega}$ the evaluation map at time t , defined by $e_t(\gamma) = \gamma(t)$ for every $\gamma \in \Gamma$.

4.3 The model

We consider in this chapter the mean field game model in which a representative agent wishes to select their trajectory $\gamma \in \Gamma$ in order to minimize the cost function

$$\int_0^{+\infty} \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \int_\Gamma \int_0^{\tau(\gamma) \wedge \tau(\tilde{\gamma})} h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt dQ(\tilde{\gamma}) + \Psi(\tau(\gamma))$$

where $\tau(\gamma)$ denotes the first exit time (see Definition 4.3.1 below), the function ℓ denotes the individual cost associated to a specific trajectory, the function h deals with interaction between trajectories through their positions and controls, and Q is a probability measure on Γ denoting the distributions of the trajectories of all agents taking part in the game. We consider the infinite bound for individual cost, since in the free final time context, it may happen a trajectory never arrives at the target set and as a consequence it penalizes more and more as long as the trajectory remains in the domain, or roughly speaking engaging to the game. In contrast, the optimal trajectories reach the target set and remain constant

afterwards. More precisely, for an optimal trajectory

$$\int_0^{+\infty} \ell(t, \gamma(t), \dot{\gamma}(t)) dt = \int_0^{\tau(\gamma)} \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \int_{\tau(\gamma)}^{+\infty} \ell(t, \gamma(t), 0) dt,$$

and assuming $\ell(t, \gamma(t), 0) = 0$, we conclude

$$\int_0^{+\infty} \ell(t, \gamma(t), \dot{\gamma}(t)) dt = \int_0^{\tau(\gamma)} \ell(t, \gamma(t), \dot{\gamma}(t)) dt.$$

In order to study this game, let us introduce some notation. We define the individual cost $L(\gamma)$ of a trajectory $\gamma \in \Gamma$ by

$$(4.2) \quad L(\gamma) = \begin{cases} \int_0^{+\infty} \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \Psi(\tau(\gamma)) & \text{if } \gamma \in \mathcal{AC}(\mathbb{R}_+; \bar{\Omega}) \\ +\infty, & \text{otherwise,} \end{cases}$$

the interaction cost by

$$(4.3) \quad H(\gamma, \tilde{\gamma}) = \begin{cases} \int_0^{\tau(\gamma) \wedge \tau(\tilde{\gamma})} h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt & \text{if } \gamma \text{ and } \tilde{\gamma} \in \mathcal{AC}(\mathbb{R}_+; \bar{\Omega}) \\ +\infty, & \text{otherwise,} \end{cases}$$

and then the cost function by

$$(4.4) \quad F(\gamma, Q) = L(\gamma) + \int_{\Gamma} H(\gamma, \tilde{\gamma}) dQ(\tilde{\gamma}).$$

Notice that for the above functionals, it is possible to have $+\infty$, even if $\gamma \in \mathcal{AC}(\mathbb{R}_+; \bar{\Omega})$. In order to analyze the model, we assume the following assumptions on the functions ℓ , h , and Ψ hold.

(H1) The function $\ell: \mathbb{R}_+ \times \bar{\Omega} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is non-negative, measurable with respect to its first variable, and continuous with respect to its other variables. Moreover, it is convex in its third variable and there exist constants $\alpha > 0$ and $\theta > 1$ such that

$$\ell(t, x, p) \geq \alpha |p|^\theta, \quad \text{for all } t \in \mathbb{R}_+, x \in \bar{\Omega} \text{ and } p \in \mathbb{R}^d.$$

(H2) The function $h: \mathbb{R}_+ \times \bar{\Omega} \times \bar{\Omega} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is measurable with respect to its first variable, continuous with respect to others, jointly convex in the fourth and fifth variables, non-negative and satisfies the symmetry property, i.e.

$$h(t, x, \tilde{x}, p, \tilde{p}) = h(t, \tilde{x}, x, \tilde{p}, p), \quad \text{for all } t \in \mathbb{R}_+, x, \tilde{x} \in \bar{\Omega} \text{ and } p \in \mathbb{R}^d.$$

(H3) The function $\Psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is lower semicontinuous, non-decreasing there exist positive constants a and b such that

$$\Psi(t) \geq at - b, \quad \text{for all } t \in \mathbb{R}_+.$$

(H4) There exist constants $C > 0$ and $\beta \in (0, \theta)$, where θ is the constant from (H1) such that

$$h(t, x, \tilde{x}, p, \tilde{p}) \leq C(|p|^\beta + |\tilde{p}|^\beta), \quad \text{for all } t \in \mathbb{R}_+, x, \tilde{x} \in \bar{\Omega} \text{ and } p, \tilde{p} \in \mathbb{R}^d.$$

(H5) There exists a constant $\kappa > 0$ such that for every x_0 , there exists $\gamma \in \mathcal{AC}(\mathbb{R}_+; \bar{\Omega})$ with $\gamma(0) = x_0$ and

$$L(\gamma) \leq \kappa.$$

We now define the *first exit time* appearing in the last part of the cost function.

Definition 4.3.1. The *first exit time* is the function $\tau: \Gamma \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, defined by

$$\tau(\gamma) = \inf\{t \mid \gamma(t) \in \Xi\},$$

where by convention $\inf \emptyset = +\infty$.

Finally, we introduce the notion of *equilibrium* of the mean field game considered in this chapter.

Definition 4.3.2. Let $m_0 \in \mathcal{P}(\bar{\Omega})$. A measure $Q \in \mathcal{P}(\Gamma)$ is called an equilibrium of the MFG model, with initial condition m_0 , if $e_{0\#}Q = m_0$ and

$$\int_{\Gamma} F(\gamma, Q) dQ(\gamma) < +\infty, \quad F(\gamma, Q) = \inf_{\substack{\omega \in \Gamma \\ \omega(0) = \gamma(0)}} F(\omega, Q), \quad Q\text{-a.e. } \gamma.$$

4.3.1 Lower semicontinuity properties

Our first results prove that, under our standing assumptions (H1)–(H5), several functions we use in the sequel are lower semicontinuous with respect to their arguments.

Lemma 4.3.3. *The function τ is lower semicontinuous.*

Proof. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence converging to some γ uniformly on compact time intervals. We show

$$\liminf_{n \rightarrow \infty} \tau(\gamma_n) \geq \tau(\gamma).$$

If the left-hand side of the above inequality is infinity, then there is nothing to prove. We can then assume that $\liminf_{n \rightarrow \infty} \tau(\gamma_n) < +\infty$. Let $(\gamma_{n_k})_{k \in \mathbb{N}}$ be a subsequence such that $\lim_{k \rightarrow \infty} \tau(\gamma_{n_k}) = \liminf_{n \rightarrow \infty} \tau(\gamma_n)$. Notice that the target set Ξ is closed which implies

$$\gamma_{n_k}(\tau(\gamma_{n_k})) \longrightarrow \gamma(\liminf_{n \rightarrow \infty} \tau(\gamma_n)) \in \Xi \quad \text{as } k \rightarrow \infty.$$

Hence by the definition of τ , we observe

$$\tau(\gamma) \leq \liminf_{n \rightarrow \infty} \tau(\gamma_n). \quad \square$$

Corollary 4.3.4. *The function $\Psi \circ \tau: \Gamma \rightarrow \mathbb{R}_+$ is lower semicontinuous.*

Proof. First let $\gamma_n \rightarrow \gamma$. Since Ψ is lower semicontinuous and non-decreasing, we have

$$\liminf_{n \rightarrow \infty} \Psi(\tau(\gamma_n)) \geq \Psi(\liminf_{n \rightarrow \infty} \tau(\gamma_n)) \geq \Psi(\tau(\gamma)). \quad \square$$

Proposition 4.3.5. *Under the previous assumptions, the extended real valued function $J: \Gamma \times \Gamma \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by*

$$J(\gamma, \tilde{\gamma}) = L(\gamma) + L(\tilde{\gamma}) + H(\gamma, \tilde{\gamma})$$

is lower semicontinuous.

Proof. Let $(\gamma_n)_{n \in \mathbb{N}}$ and $(\tilde{\gamma}_n)_{n \in \mathbb{N}}$ be two sequences in Γ which converges to some γ and $\tilde{\gamma}$. The goal is to show the following inequality

$$\liminf_{n \rightarrow \infty} J(\gamma_n, \tilde{\gamma}_n) \geq J(\gamma, \tilde{\gamma}).$$

To do so, we first show

$$\liminf_{n \rightarrow \infty} L(\gamma_n) \geq L(\gamma).$$

Without loss of generality, we assume that the left hand side is strictly less than ∞ , since otherwise the inequality is trivial. By extracting a subsequence, we assume that

$$\lim_{k \rightarrow \infty} L(\gamma_{n_k}) = \liminf_{n \rightarrow \infty} L(\gamma_n).$$

By the assumption (H1) on the map ℓ in the cost function, we observe $\int_0^{+\infty} |\dot{\gamma}_n(t)|^\theta dt < C$, and on the other hand, since Ω is a bounded open subset of \mathbb{R}^d , by letting $T > 0$ be fixed, we have $\int_0^T |\dot{\gamma}_n(t)|^\theta dt < C$, for constant $C > 0$ independent from n , by which one concludes $(\gamma_n)_{n \in \mathbb{N}}$ is a bounded sequence in $W^{1,\theta}([0, T]; \bar{\Omega})$. Hence its limit $\gamma \in W^{1,\theta}([0, T]; \bar{\Omega})$ and in particular, $(\dot{\gamma}_n)_{n \in \mathbb{N}}$ converges weakly to $\dot{\gamma} \in L^\theta([0, T]; \bar{\Omega})$. Now consider the function $\int_0^T \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \Psi(\tau(\gamma))$. One observes, by applying [52, Theorem 4.5] and Corollary 4.3.4

$$\int_0^T \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \Psi(\tau(\gamma)) \leq \liminf_{n \rightarrow \infty} \int_0^T \ell(t, \gamma_n(t), \dot{\gamma}_n(t)) dt + \Psi(\tau(\gamma_n)).$$

Notice that due to the positivity of function ℓ , we have

$$\int_0^T \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \Psi(\tau(\gamma)) \leq \liminf_{n \rightarrow \infty} \int_0^{+\infty} \ell(t, \gamma_n(t), \dot{\gamma}_n(t)) dt + \Psi(\tau(\gamma_n)),$$

and since this holds for every $T > 0$, one concludes

$$(4.5) \quad L(\gamma) \leq \liminf_{n \rightarrow \infty} L(\gamma_n),$$

which means the individual cost L is indeed lower semicontinuous.

By doing the same procedure as before, we can show $\liminf_{n \rightarrow \infty} L(\tilde{\gamma}_n) \geq L(\tilde{\gamma})$, and we thus left to deal with interaction cost. More precisely, we show

$$\liminf_{n \rightarrow \infty} H(\gamma_n, \tilde{\gamma}_n) \geq H(\gamma, \tilde{\gamma}).$$

Without loss of generality, we assume that the left hand side is strictly less than ∞ , since otherwise the inequality is trivial. By extracting a subsequence, we assume that

$$\lim_{k \rightarrow \infty} H(\gamma_{n_k}, \tilde{\gamma}_{n_k}) = \liminf_{n \rightarrow \infty} H(\gamma_n, \tilde{\gamma}_n).$$

Knowing that $(\gamma_n)_{n \in \mathbb{N}}$ and $(\tilde{\gamma}_n)_{n \in \mathbb{N}}$ are two bounded sequences in $W^{1,\theta}([0, T]; \bar{\Omega})$ and hence their limits γ and $\tilde{\gamma}$ are in $W^{1,\theta}([0, T]; \bar{\Omega})$ together with weakly converging of $\dot{\gamma}_n$ and $\dot{\tilde{\gamma}}_n$ to $\dot{\gamma}$ and $\dot{\tilde{\gamma}}$ in the space $L^\theta([0, T]; \bar{\Omega})$, respectively, we proceed as follows. For simplicity, we first rewrite the function as follows.

$$H(\gamma, \tilde{\gamma}) = \int_0^\sigma h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt,$$

where $\sigma = \tau(\gamma) \wedge \tau(\tilde{\gamma})$, and $\sigma_n = \tau(\gamma_n) \wedge \tau(\tilde{\gamma}_n)$. In the sequel, we consider two cases to treat for σ , whether is finite or infinity, i.e. $\sigma < \infty$ or $\sigma = +\infty$. By Lemma 4.3.3, one knows $\liminf_{n \rightarrow \infty} \sigma_n \geq \sigma$, by which if $\sigma = +\infty$, then σ_n is necessary unbounded. As a matter of fact, being unbounded insists on considering $\sigma \wedge T$ and $\sigma_n \wedge T$, as the upper bounds of integrals, by which the domain of integrals can be interpreted as the closed time interval $[0, T]$, since they converge to $+\infty$, and moreover by assumption (H2) and from [52, Theorem 4.5], we have

$$\begin{aligned} \int_0^T h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt &\leq \liminf_{n \rightarrow \infty} \int_0^T h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \\ &\leq \liminf_{n \rightarrow \infty} \int_0^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt. \end{aligned}$$

Since for large enough $n \in \mathbb{N}$, we have $\sigma_n \geq T$. On the other hand, since it holds for every $T > 0$ and σ is infinity, then one concludes

$$H(\gamma, \tilde{\gamma}) \leq \liminf_{n \rightarrow \infty} H(\gamma_n, \tilde{\gamma}_n).$$

Now assume σ and σ_n are bounded. We show the Liminf of the following subtraction is

non-negative.

$$\begin{aligned}
\tilde{H}^{(n)} &:= \int_0^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt - \int_0^\sigma h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt \\
&= \int_0^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt - \underbrace{\int_0^\sigma h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt}_{\tilde{H}_1^{(n)}} \\
&\quad + \underbrace{\int_0^\sigma h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt}_{\tilde{H}_2^{(n)}} - \int_0^\sigma h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt
\end{aligned}$$

Without loss of generality, up to extracting subsequences, we can assume $\lim_{n \rightarrow \infty} \tilde{H}^{(n)} = \liminf_{n \rightarrow \infty} \tilde{H}^{(n)}$ and $\lim_{n \rightarrow \infty} \sigma_n = \liminf_{n \rightarrow \infty} \sigma_n$. One can easily apply [52, Theorem 4.5] on $\tilde{H}_2^{(n)}$ to see

$$(4.6) \quad \liminf_{n \rightarrow \infty} \int_0^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \geq \int_0^\sigma h(t, \gamma(t), \tilde{\gamma}(t), \dot{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt.$$

For $\tilde{H}_1^{(n)}$, we proceed as follows. By Lemma 4.3.3, $\lim_{n \rightarrow \infty} \sigma_n \geq \sigma$. Let us first assume $\lim_{n \rightarrow \infty} \sigma_n > \sigma$. Therefore, there exists $N \in \mathbb{N}$ such that $\sigma_n > \sigma$, for $n \geq N$. Since h is a positive function, we thus have

$$\begin{aligned}
\int_0^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt - \int_0^\sigma h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \\
= \int_\sigma^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \geq 0.
\end{aligned}$$

We now assume $\lim_{n \rightarrow \infty} \sigma_n = \sigma$. By Hypotheses (H4), we have

$$\begin{aligned}
\left| \int_\sigma^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \right| &\leq C \left(\left| \int_\sigma^{\sigma_n} |\dot{\gamma}_n(t)|^\beta dt \right| + \left| \int_\sigma^{\sigma_n} |\dot{\tilde{\gamma}}_n(t)|^\beta dt \right| \right) \\
&\leq C \left(|\sigma_n - \sigma|^{\frac{1}{\xi}} \left| \left(\int_\sigma^{\sigma_n} |\dot{\gamma}_n(t)|^{\beta \xi'} dt \right)^{\frac{1}{\xi'}} \right| + \right. \\
&\quad \left. |\sigma_n - \sigma|^{\frac{1}{\xi}} \left| \left(\int_\sigma^{\sigma_n} |\dot{\tilde{\gamma}}_n(t)|^{\beta \xi'} dt \right)^{\frac{1}{\xi'}} \right| \right),
\end{aligned}$$

where $\frac{1}{\xi} + \frac{1}{\xi'} = 1$. Now, by choosing $\xi' = \frac{\theta}{\beta}$, one observes $\int_\sigma^{\sigma_n} |\dot{\gamma}_n(t)|^{\beta \xi'} dt < C$ and $\int_\sigma^{\sigma_n} |\dot{\tilde{\gamma}}_n(t)|^{\beta \xi'} dt < C$. Moreover, $\xi' > 1$ leads us to see $\xi < +\infty$. Hence by taking the Lim, we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left| \int_0^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt - \int_0^\sigma h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \right| \\
\leq \lim_{n \rightarrow \infty} \left| \int_\sigma^{\sigma_n} h(t, \gamma_n(t), \tilde{\gamma}_n(t), \dot{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \right| = 0,
\end{aligned}$$

which essentially means $\lim_{n \rightarrow \infty} \tilde{H}_1^{(n)} = 0$. Hence, the desired inequality also holds for H , which leads us to conclude that the function J is lower semicontinuous. \square

4.4 Auxiliary results on the cost function

We define for every $Q, \tilde{Q} \in \mathcal{P}(\Gamma)$

$$(4.7) \quad \begin{aligned} \mathcal{L}(Q) &:= \int_{\Gamma} L(\gamma) dQ(\gamma), \\ \mathcal{H}(Q, \tilde{Q}) &:= \int_{\Gamma \times \Gamma} H(\gamma, \tilde{\gamma}) d(Q \otimes \tilde{Q})(\gamma, \tilde{\gamma}), \\ \mathcal{J}(Q) &:= \int_{\Gamma \times \Gamma} J(\gamma, \tilde{\gamma}) d(Q \otimes Q)(\gamma, \tilde{\gamma}), \end{aligned}$$

and, by convention, we denote $\mathcal{H}(Q, Q)$ simply by $\mathcal{H}(Q)$ in the sequel. Moreover, we consider the following definitions.

$$\mathcal{Q}_{m_0} = \left\{ Q \in \mathcal{P}(\Gamma) \mid e_{0\#}Q = m_0 \right\}.$$

and

$$\text{Dom}(\mathcal{J}) = \{ Q \in \mathcal{P}(\Gamma) \mid \mathcal{J}(Q) < +\infty \}.$$

Remark 4.4.1. According to the definitions of \mathcal{L} , \mathcal{H} and \mathcal{J} in (4.7), one can easily check that $\mathcal{J}(Q) = 2\mathcal{L}(Q) + \mathcal{H}(Q)$ and $\int_{\Gamma} F(\gamma, Q) dQ(\gamma) = \mathcal{L}(Q) + \mathcal{H}(Q)$, for every $Q \in \mathcal{P}(\Gamma)$. In addition, $\mathcal{J}(Q) < +\infty$ if and only if $\int_{\Gamma} F(\gamma, Q) dQ(\gamma) < +\infty$.

Remark 4.4.2. We point out that \mathcal{Q}_{m_0} is closed, thanks to the continuity of the map $Q \mapsto e_{0\#}Q$.

Lemma 4.4.3. *There exists a constant $C > 0$ such that for every $\gamma \in \Gamma$, one has*

$$H(\gamma, \tilde{\gamma}) \leq C(L(\gamma) + L(\tilde{\gamma}) + 1).$$

Proof. Under the assumptions (H4), one observes (up to increasing the constant C from one estimate to the next one)

$$\begin{aligned} H(\gamma, \tilde{\gamma}) &\leq \int_0^{\tau(\gamma) \wedge \tau(\tilde{\gamma})} C(|\dot{\gamma}(t)|^\beta + |\dot{\tilde{\gamma}}(t)|^\beta) dt \\ &\leq C \left(\int_0^{\tau(\gamma)} |\dot{\gamma}(t)|^\beta dt \right) + C \left(\int_0^{\tau(\tilde{\gamma})} |\dot{\tilde{\gamma}}(t)|^\beta dt \right), \end{aligned}$$

by applying Young's inequality on $|\dot{\gamma}(t)|^\beta$ and $|\dot{\tilde{\gamma}}(t)|^\beta$, and then integrating, together with assumption (H3)

$$\begin{aligned} H(\gamma, \tilde{\gamma}) &\leq C \left(\frac{\tau(\gamma)}{\xi} + \frac{1}{\xi'} \int_0^{\tau(\gamma)} |\dot{\gamma}(t)|^{\beta\xi'} dt \right) + C \left(\frac{\tau(\tilde{\gamma})}{\xi} + \frac{1}{\xi'} \int_0^{\tau(\tilde{\gamma})} |\dot{\tilde{\gamma}}(t)|^{\beta\xi'} dt \right) \\ &\leq C \left(\int_0^{+\infty} \frac{1}{\xi'^\alpha} \ell(t, \gamma(t), \dot{\gamma}(t)) dt + \frac{\Psi(\tau(\gamma)) + b}{\xi a} \right) \\ &\quad + C \left(\int_0^{+\infty} \frac{1}{\xi'^\alpha} \ell(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt + \frac{\Psi(\tau(\tilde{\gamma})) + b}{\xi a} \right) \leq C(L(\gamma) + L(\tilde{\gamma}) + 1), \end{aligned}$$

where $\frac{1}{\xi} + \frac{1}{\xi'} = 1$ and in particular, $\xi' = \frac{\theta}{\beta} > 1$ is chosen. \square

As a direct consequence of Lemma 4.4.3, we have the following Corollary.

Corollary 4.4.4. *There exists a constant $C > 0$ such that for every $Q \in \mathcal{P}(\Gamma)$, we have*

$$\mathcal{H}(Q, \tilde{Q}) \leq C(\mathcal{L}(Q) + \mathcal{L}(\tilde{Q}) + 1).$$

Proof. By definition of \mathcal{H} and Lemma 4.4.3, we have

$$\begin{aligned} \mathcal{H}(Q, \tilde{Q}) &= \int_{\Gamma \times \Gamma} H(\gamma, \tilde{\gamma}) d(Q \otimes \tilde{Q})(\gamma, \tilde{\gamma}) \\ &\leq C \int_{\Gamma} \int_{\Gamma} L(\gamma) dQ(\gamma) d\tilde{Q}(\tilde{\gamma}) + C \int_{\Gamma} \int_{\Gamma} L(\tilde{\gamma}) d\tilde{Q}(\tilde{\gamma}) dQ(\gamma) + C \\ &\leq C(\mathcal{L}(Q) + \mathcal{L}(\tilde{Q}) + 1). \end{aligned} \quad \square$$

Lemma 4.4.5. *Let domain of \mathcal{L} be defined by*

$$\text{Dom}(\mathcal{L}) = \{Q \in \mathcal{P}(\Gamma) \mid \mathcal{L}(Q) < +\infty\}.$$

Then, one has $\text{Dom}(\mathcal{J}) = \text{Dom}(\mathcal{L})$.

Proof. If one takes $Q \in \text{Dom}(\mathcal{J})$, then by Remark 4.4.1, $\mathcal{L}(Q) \leq \mathcal{J}(Q) < +\infty$ obviously holds, and hence, $\text{Dom}(\mathcal{J}) \subseteq \text{Dom}(\mathcal{L})$. For the reverse inclusion, take $Q \in \text{Dom}(\mathcal{L})$, for which by Corollary 4.4.4, one has $\mathcal{H}(Q) < +\infty$. Hence by Remark 4.4.1, $\mathcal{J}(Q) < +\infty$, which means $Q \in \text{Dom}(\mathcal{J})$. Therefore, $\text{Dom}(\mathcal{L}) \subseteq \text{Dom}(\mathcal{J})$, and this ends the proof. \square

Remark 4.4.6. Thanks to Corollary 4.4.4, the function $\mathcal{H}: \text{Dom}(\mathcal{L}) \times \text{Dom}(\mathcal{L}) \rightarrow \mathbb{R}_+$ can be extended by bilinearity to a unique bilinear function, still denoted by \mathcal{H} , on $\text{Span}(\text{Dom}(\mathcal{L})) \times \text{Span}(\text{Dom}(\mathcal{L}))$, i.e., $\mathcal{H}: \text{Span}(\text{Dom}(\mathcal{L})) \times \text{Span}(\text{Dom}(\mathcal{L})) \rightarrow \mathbb{R}_+$, where the Span is taken in the space of signed measures on Γ .

Lemma 4.4.7. *Under the previous assumptions, there exists a Borel measurable function $\Phi: \bar{\Omega} \rightarrow \Gamma$ such that for every x_0 , the trajectory $\gamma = \Phi(x_0)$ satisfies $\gamma(0) = x_0$ and $L(\gamma) \leq \kappa$.*

Proof. Let us first consider the set valued map

$$\mathcal{G}: \bar{\Omega} \ni x_0 \mapsto \left\{ \gamma \in \mathcal{AC}(\mathbb{R}_+; \bar{\Omega}) \mid \gamma(0) = x_0 \text{ and } L(\gamma) \leq \kappa \right\}.$$

We show the graph of above set valued map is closed. To do so, let $(x_n)_{n \in \mathbb{N}}$ be a sequence which converges to some x^* and $\gamma_n \in \mathcal{G}(x_n)$ converges to some $\gamma^* \in \Gamma$. By uniform convergence of trajectories, one easily observe $\gamma^*(0) = x^*$. On the other hand, by Proposition 4.3.5, one observes

$$L(\gamma^*) \leq \liminf_{n \rightarrow \infty} L(\gamma_n) \leq \kappa.$$

Hence $\gamma^* \in \mathcal{G}(x^*)$. Notice now by assumption (H5), $\mathcal{G}(x_0) \neq \emptyset$ for every $x_0 \in \bar{\Omega}$, and the set $\{\gamma \in \mathcal{C}(\mathbb{R}_+; \bar{\Omega}) \mid L(\gamma) \leq \kappa\}$, in which \mathcal{G} takes values is compact, since if one takes a sequence $(\gamma_n)_{n \in \mathbb{N}}$ such that $L(\gamma_n) \leq \kappa$, then $\int_0^\infty |\dot{\gamma}_n(t)|^\theta dt \leq \frac{\kappa}{\alpha}$. On the other hand, since $\Omega \subset \mathbb{R}^d$ is bounded, one observes $\int_0^T |\gamma_n(t)|^\theta dt$ is bounded, for every $T > 0$. Hence $\gamma_n \in W^{1,\theta}([0, T]; \bar{\Omega})$ is bounded which implies $\gamma_n \rightharpoonup \gamma \in W^{1,\theta}([0, T]; \bar{\Omega})$. By compact embedding of $W^{1,\theta}([0, T]; \bar{\Omega})$ in the space Γ , we deduce the compactness of the set $\{\gamma \in \mathcal{C}([0, T]; \bar{\Omega}) \mid L(\gamma) \leq \kappa\}$. Now by using [7, Proposition 1.4.8], we observe \mathcal{G} is upper semicontinuous and by [7, Proposition 1.4.4], the set $\mathcal{G}^{-1}(A) = \{x \in \bar{\Omega} \mid \mathcal{G}(x) \cap A \neq \emptyset\}$ is closed for every closed set A . Hence by [39, Proposition III.11] $\mathcal{G}^{-1}(B)$ is open for every open set B , hence \mathcal{G} is measurable.

Now by applying [7, Theorem 8.1.3], we conclude the set valued map \mathcal{G} has a Borel measurable selection, which is our desired function Φ . \square

Remark 4.4.8. Notice that it is not possible to apply Theorem 8.1.4 in Ref. [7] in order to extract directly a Borel measurable selection from closed graph because our space $\bar{\Omega}$ is not a complete measure space with Borel σ -algebra.

Corollary 4.4.9. *Under previous assumptions, there exists $Q \in \mathcal{Q}_{m_0}$ such that $\mathcal{L}(Q) < +\infty$, i.e., $\text{Dom}(\mathcal{L}) \cap \mathcal{Q}_{m_0} \neq \emptyset$.*

Proof. We set $Q = \Phi_{\#} m_0$, which is well-defined by Lemma 4.4.7. By definition of $\mathcal{L}(Q)$, we have

$$\int_{\Gamma} L(\gamma) dQ(\gamma) \leq \int_{\Gamma} \kappa dQ(\gamma) = \int_{\bar{\Omega}} \kappa dm_0(x) = \kappa. \quad \square$$

Proposition 4.4.10. *Under previous assumptions, $\text{Dom}(\mathcal{J})$ is a non-empty set.*

Proof. One observes that being non-empty is a consequence of Corollaries 4.4.4 and 4.4.9, and Remark 4.4.1. \square

4.4.1 Regularity of functional \mathcal{J}

Lemma 4.4.11. *Under previous assumptions, $\text{Dom}(\mathcal{L})$ is a convex set. In particular, $\text{Dom}(\mathcal{J})$ is convex.*

Proof. In order to show the convexity, we take $Q_0, Q \in \text{Dom}(\mathcal{L})$ and $t \in [0, 1]$. By linearity of \mathcal{L} , we have

$$\mathcal{L}(Q_0 + t(Q - Q_0)) = (1 - t)\mathcal{L}(Q_0) + t\mathcal{L}(Q) < +\infty.$$

Convexity of $\text{Dom}(\mathcal{J})$ is now straightforward by Lemma 4.4.5. \square

Definition 4.4.12. Let $Q_0 \in \text{Dom}(\mathcal{J})$. We say \mathcal{J} is differentiable at Q_0 , if there exists a Borel measurable functional $G : \Gamma \rightarrow \mathbb{R}$ such that for all $Q \in \text{Dom}(\mathcal{J})$, G is $(Q - Q_0)$ -integrable and we have

$$(4.8) \quad \lim_{t \rightarrow 0} \frac{\mathcal{J}(Q_0 + t(Q - Q_0)) - \mathcal{J}(Q_0)}{t} = \int_{\Gamma} G(\gamma) d(Q - Q_0)(\gamma) < +\infty,$$

for all $Q \in \text{Dom}(\mathcal{J})$. When such a functional exists, we denote it by $\frac{\delta \mathcal{J}}{\delta Q}(Q_0)$. We also denote the right-hand side of (4.8) by $\langle \frac{\delta \mathcal{J}}{\delta Q}(Q_0), Q - Q_0 \rangle$.

In our next result, we establish the fact that the functional $\frac{\delta \mathcal{J}}{\delta Q}(Q_0)$ is unique up to a constant. To do so, we are first required to establish the following result.

Lemma 4.4.13. *Assume that $Q_0 \in \text{Dom}(\mathcal{J})$. Let G_1 and G_2 be two Borel measurable functions such that*

$$\int_{\Gamma} G_1(\gamma) d(Q - Q_0)(\gamma) = \int_{\Gamma} G_2(\gamma) d(Q - Q_0)(\gamma), \quad \forall Q \in \text{Dom}(\mathcal{J}).$$

Then there exists $\lambda \in \mathbb{R}$ such that $G_1(\gamma) = G_2(\gamma) + \lambda$, for Q_0 -almost every $\gamma \in \Gamma$.

Proof. We first claim that for Q_0 -almost every $\gamma \in \Gamma$, $\delta_{\gamma} \in \text{Dom}(\mathcal{J})$. To do so, since $Q_0 \in \text{Dom}(\mathcal{J})$, by Lemma 4.4.5, one can easily check $L(\gamma) < +\infty$, for Q_0 -almost every $\gamma \in \Gamma$. Now, one observes

$$\mathcal{L}(\delta_{\gamma}) = \int_{\Gamma} L(\gamma) d\delta_{\gamma}(\gamma) = L(\gamma) < +\infty.$$

Therefore, by Lemma 4.4.5, we deduce $\delta_{\gamma} \in \text{Dom}(\mathcal{L}) = \text{Dom}(\mathcal{J})$. We now set $\lambda = \int_{\Gamma} (G_1 - G_2)(\gamma) dQ_0(\gamma)$ and observe that

$$\int_{\Gamma} G_1(\gamma) dQ(\gamma) = \int_{\Gamma} G_1(\gamma) dQ(\gamma) + \lambda, \quad \forall Q \in \text{Dom}(\mathcal{J}),$$

which yields us to take Q as δ_{γ} , for Q_0 -almost every $\gamma \in \Gamma$, and as a consequence to have $G_1(\gamma) = G_2(\gamma) + \lambda$. \square

Lemma 4.4.14. *Under the previous assumptions, for all $Q_0 \in \text{Dom}(\mathcal{J})$, the functional \mathcal{J} is differentiable at Q_0 , and moreover, $\frac{\delta \mathcal{J}}{\delta Q}(Q_0)(\gamma) = 2F(\gamma, Q_0)$ for Q_0 -almost every $\gamma \in \Gamma$.*

Proof. By taking advantage of Remark 4.4.1, for $Q \in \text{Dom}(\mathcal{J})$, we observe

$$\begin{aligned} \frac{\mathcal{J}(Q_0 + t(Q - Q_0)) - \mathcal{J}(Q_0)}{t} &= 2(\mathcal{L}(Q) + \mathcal{H}(Q_0, Q)) - 2(\mathcal{L}(Q_0) + \mathcal{H}(Q_0)) \\ &\quad + t\mathcal{H}(Q - Q_0), \end{aligned}$$

where $\mathcal{H}(Q - Q_0)$ is well-defined by Remark 4.4.6. Thanks to Corollary 4.4.4 and Remark 4.4.6, we conclude $t\mathcal{H}(Q - Q_0) \rightarrow 0$, as $t \rightarrow 0$. Hence, by Remark 4.4.1, we have

$$\begin{aligned} (4.9) \quad \lim_{t \rightarrow 0} \frac{\mathcal{J}(Q_0 + t(Q - Q_0)) - \mathcal{J}(Q_0)}{t} &= 2(\mathcal{L}(Q) + \mathcal{H}(Q_0, Q)) - 2(\mathcal{L}(Q_0) + \mathcal{H}(Q_0)) \\ &= \int_{\Gamma} 2F(\gamma, Q_0) dQ(\gamma) - \int_{\Gamma} 2F(\gamma, Q_0) dQ_0(\gamma) \\ &= \int_{\Gamma} 2F(\gamma, Q_0) d(Q - Q_0)(\gamma) < +\infty. \end{aligned}$$

Since the equality (4.9) holds for every $Q \in \text{Dom}(\mathcal{J})$, we deduce $\frac{\delta \mathcal{J}}{\delta Q}(Q_0)(\gamma) = 2F(\gamma, Q_0)$. \square

Definition 4.4.15. We call $Q_0 \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$ a *critical point* of \mathcal{J} in \mathcal{Q}_{m_0} , if $\langle \frac{\delta \mathcal{J}}{\delta Q}(Q_0), Q - Q_0 \rangle \geq 0$ holds, for every $Q \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$.

Proposition 4.4.16. *Under the previous assumptions, any local minimizer Q_0 of \mathcal{J} in $\text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$ is a critical point.*

Proof. We first take $Q \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$ and observe that

$$\frac{\mathcal{J}(Q_0 + t(Q - Q_0)) - \mathcal{J}(Q_0)}{t} \geq 0,$$

for small enough $t > 0$, since $Q_0 \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$ is a local minimizer and $Q_0 + t(Q - Q_0)$ weakly converges to Q_0 as $t \rightarrow 0$. Now, by taking the limit as $t \rightarrow 0$ and by Definition 4.4.12, we have $\langle \frac{\delta \mathcal{J}}{\delta Q}(Q_0), Q - Q_0 \rangle \geq 0$. Hence, Q_0 is a critical point. \square

4.5 Potential game

In the next result, we establish the equivalence interpretation between critical points and equilibria. Our proof is closely based on that of [86, Lemma 3.3].

Theorem 4.5.1. *Under previous assumptions $Q_0 \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$ is a critical point of \mathcal{J} in \mathcal{Q}_{m_0} if and only if Q_0 is an equilibrium with initial condition m_0 .*

Proof. Recall that, according to Remark 4.4.1, $Q_0 \in \text{Dom}(\mathcal{J})$ is equivalent to having $\int_{\Gamma} F(\gamma, Q_0) dQ_0(\gamma) < +\infty$. Assume first that $Q_0 \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$ is a critical point of \mathcal{J} in \mathcal{Q}_{m_0} . To obtain a contradiction, suppose that the set

$$\left\{ \gamma \in \Gamma \mid \exists \omega \in \Gamma, F(\omega, Q_0) < F(\gamma, Q_0), \omega(0) = \gamma(0) \right\}$$

is not Q_0 -negligible. Note that the above set is equal to

$$\bigcup_{\substack{(q,r) \in \mathbb{Q}_+^2 \\ 0 < q < r}} \left\{ \gamma \in \Gamma \mid F(\gamma, Q_0) > r, \{\omega \in \Gamma \mid \omega(0) = \gamma(0), F(\omega, Q_0) \leq q\} \neq \emptyset \right\},$$

and thus there exists a pair of positive rational numbers (q, r) with $q < r$ such that the set A defined by

$$(4.10) \quad A = \left\{ \gamma \in \Gamma \mid F(\gamma, Q_0) > r, \{\omega \in \Gamma \mid \omega(0) = \gamma(0), F(\omega, Q_0) \leq q\} \neq \emptyset \right\}$$

is not Q_0 -negligible.

For any $p \in \mathbb{R}$, define A^p as

$$A^p = \left\{ \gamma \in \Gamma \mid F(\gamma, Q_0) \leq p \right\}.$$

By the definition of A^p , we observe $\int_0^{+\infty} \ell(t, \gamma(t), \dot{\gamma}(t)) dt$ is finite for all $\gamma \in A^p$. By the assumption (H1), for a fixed $T > 0$ but arbitrary, one concludes $\gamma \in W^{1,\theta}([0, T]; \bar{\Omega})$ which implied the compactness of A^p , since $W^{1,\theta}([0, T]; \Omega)$ is compactly embedded in the space of continuous functions Γ .

Let us now define the multifunction \mathcal{S} as

$$\mathcal{S}: \Gamma \rightarrow 2^{A^q}, \quad \mathcal{S}(\gamma) = \left\{ \omega \in A^q \mid \omega(0) = \gamma(0) \right\}.$$

Since A^q is a compact set, one can easily verify \mathcal{S} has a closed graph, thus its domain $\text{Dom}(\mathcal{S}) = \{\gamma \in \Gamma \mid \mathcal{S}(\gamma) \neq \emptyset\}$ is closed and, by [7, Proposition 1.4.8], we deduce that \mathcal{S} is upper semicontinuous. Hence, by [7, Proposition 1.4.4], $\mathcal{S}^{-1}(A) = \{\gamma \in \Gamma \mid \mathcal{S}(\gamma) \cap A \neq \emptyset\}$ is closed for every closed set A . Now by definition of measurability and [39, Proposition III.11], we observe \mathcal{S} is measurable, and, by [7, Theorem 8.1.3], there exists a measurable function $s: \text{Dom}(\mathcal{S}) \rightarrow A^q$ satisfying

$$s(\gamma) \in \mathcal{S}(\gamma), \quad \forall \gamma \in \text{Dom}(\mathcal{S}).$$

Note that the set A defined in (4.10) is Borel measurable, since it can be written as

$$A = \left\{ \gamma \in \Gamma \mid F(\gamma, Q_0) > r \right\} \cap \text{Dom}(\mathcal{S}),$$

$\text{Dom}(\mathcal{S})$ is closed, and $\{\gamma \in \Gamma \mid F(\gamma, Q_0) > r\}$ is the complement of A^r , and hence it is open. We can thus define a measure $\tilde{Q} \in \mathcal{P}(\mathcal{C})$ by $\tilde{Q} = s_{\#}(Q_0|_A) + Q_0|_{(\Gamma \setminus A)}$, which is in \mathcal{Q}_{m_0} since one observes

$$\begin{aligned} e_{0\#}\tilde{Q} &= e_{0\#}(s_{\#}(Q_0|_A) + Q_0|_{(\Gamma \setminus A)}) = e_{0\#}(s_{\#}(Q_0|_A)) + e_{0\#}(Q_0|_{(\Gamma \setminus A)}) \\ &= (e_0 \circ s)_{\#}(Q_0|_A) + e_{0\#}(Q_0|_{(\Gamma \setminus A)}) = e_{0\#}(Q_0|_A) + e_{0\#}(Q_0|_{(\Gamma \setminus A)}) \\ &= e_{0\#}(Q_0|_A + Q_0|_{(\Gamma \setminus A)}) = e_{0\#}(Q_0) = m_0, \end{aligned}$$

where the equality $(e_0 \circ s)_{\#}(\cdot) = e_{0\#}(\cdot)$ is coming from the fact that the function s preserves the initial points. Then we have

$$\begin{aligned} \int_{\Gamma} F(\gamma, Q_0) d\tilde{Q}(\gamma) &= \int_{\Gamma} F(\gamma, Q_0) dQ_0|_{(\Gamma \setminus A)}(\gamma) + \int_{\Gamma} F(\gamma, Q_0) ds_{\#}Q_0|_A(\gamma) \\ &\leq \int_{\Gamma \setminus A} F(\gamma, Q_0) dQ_0(\gamma) + \int_{\Gamma} F(s(\gamma), Q_0) dQ_0|_A(\gamma) \\ &= \int_{\Gamma \setminus A} F(\gamma, Q_0) dQ_0(\gamma) + \int_A F(s(\gamma), Q_0) dQ_0(\gamma) \\ &\leq \int_{\Gamma \setminus A} F(\gamma, Q_0) dQ_0(\gamma) + \int_A q dQ_0(\gamma) \\ &= \int_{\Gamma \setminus A} F(\gamma, Q_0) dQ_0(\gamma) + qQ_0(A) \\ &< \int_{\Gamma \setminus A} F(\gamma, Q_0) dQ_0(\gamma) + rQ_0(A) \\ &< \int_{\Gamma \setminus A} F(\gamma, Q_0) dQ_0(\gamma) + \int_A F(\gamma, Q_0) dQ_0(\gamma) \\ &= \int_{\Gamma} F(\gamma, Q_0) dQ_0(\gamma), \end{aligned}$$

which is a contradiction.

For the reverse implication, assume that $Q_0 \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$ is an equilibrium with initial condition m_0 . We first set

$$\nu(x_0) = \inf_{\substack{\omega \in \Gamma, \\ \omega(0)=x_0}} F(\omega, Q_0),$$

from which one has $F(\gamma, Q_0) = \nu(\gamma(0))$ for Q_0 -almost every $\gamma \in \Gamma$. One thus observes

$$\int_{\Gamma} F(\gamma, Q_0) dQ_0(\gamma) = \int_{\Gamma} \nu(\gamma(0)) dQ_0(\gamma).$$

On the other hand, if one takes $Q \in \text{Dom}(\mathcal{J}) \cap \mathcal{Q}_{m_0}$, then

$$\int_{\Gamma} F(\omega, Q_0) dQ(\omega) \geq \int_{\Gamma} \nu(\omega(0)) dQ(\omega) = \int_{\bar{\Omega}} \nu(x_0) d(e_{0\#}Q)(x_0),$$

in which

$$\int_{\bar{\Omega}} \nu(x_0) d(e_{0\#}Q)(x_0) = \int_{\bar{\Omega}} \nu(x_0) dm_0(x_0) = \int_{\Gamma} \nu(\gamma(0)) dQ_0(\gamma).$$

Since $Q \in \mathcal{Q}_{m_0}$ was arbitrary, we thus conclude the proof. \square

Theorem 4.5.2. *Under the previous assumptions, any local minimizer Q_0 of \mathcal{J} in \mathcal{Q}_{m_0} is an equilibrium for the mean field game.*

Proof. This is a direct consequence of Proposition 4.4.16 and Theorem 4.5.1. \square

In order to show the existence of an equilibrium, it is sufficient to prove the existence of a global minimizer of \mathcal{J} in \mathcal{Q}_{m_0} , by Theorem 4.5.2.

Proposition 4.5.3. *Under the previous assumptions, there exists a global minimizer of functional \mathcal{J} in \mathcal{Q}_{m_0} .*

Proof. We show that there exists a minimizer to the functional \mathcal{J} restricted to \mathcal{Q}_{m_0} . We first consider the following minimizing sequence $(Q_n)_{n \in \mathbb{N}} \subset \mathcal{Q}_{m_0}$ of \mathcal{J} , i.e.,

$$\mathcal{J}(Q_1) \geq \mathcal{J}(Q_2) \geq \dots, \quad \lim_{n \rightarrow \infty} \mathcal{J}(Q_n) = \inf_{Q \in \mathcal{P}(\Gamma)} \mathcal{J}(Q) < \infty,$$

The fact that the infimum is finite comes from Corollary 4.4.9. We now claim that the collection $\{Q_n \mid n \geq 1\}$ is tight. For this purpose we define the set \mathcal{C}_M as

$$\mathcal{C}_M := \left\{ \gamma \in \mathcal{C} \mid L(\gamma) \leq M \right\}.$$

Notice that we can show the compactness of \mathcal{C}_M by taking a sequence $\gamma_n \in \mathcal{C}_M$ and using the compact embedding of $W^{1,\theta}([0, T]; \Omega)$ in Γ , for every $T > 0$. Without loss of generality, assume $\mathcal{J}(Q_1) < +\infty$, otherwise by refining the sequence with removing finitely many elements, we can choose a suitable $Q \in \mathcal{Q}_{m_0}$. By using Markov inequality, we have

$$Q_n \left(\left\{ \gamma \mid L(\gamma) > M \right\} \right) \leq \frac{1}{M} \mathcal{L}(Q_n) \leq \frac{1}{M} \mathcal{J}(Q_n),$$

then the sequence $(Q_n)_{n \in \mathbb{N}}$ is tight. Therefore, by Prokhorov theorem (see, e.g. [6, Theorem 5.1.3]), the sequence $(Q_n)_{n \in \mathbb{N}}$ has a converging subsequence $(Q_{n_i})_{i \in \mathbb{N}}$ to some probability measure $Q_\infty \in \mathcal{P}(\Gamma)$ which belongs to \mathcal{Q}_{m_0} by the closedness, mentioned in Remark 4.4.2. Moreover, since Γ is a polish space, the product measure $(Q_{n_i} \otimes Q_{n_i})_{i \in \mathbb{N}}$ converges to $Q_\infty \otimes Q_\infty$ (see e.g., [86, Lemma 3.5]). Let $J_c(\gamma, \tilde{\gamma})$ be defined by $J_c(\gamma, \tilde{\gamma}) = \min\{J(\gamma, \tilde{\gamma}), C\}$, for a constant $C > 0$. By [15, Corollary 8.2.5], we have

$$\begin{aligned} \int_{\Gamma \times \Gamma} J_c d(Q_\infty \otimes Q_\infty) &\leq \liminf_{n \rightarrow \infty} \int_{\Gamma \times \Gamma} J_c d(Q_n \otimes Q_n) \\ &\leq \liminf_{n \rightarrow \infty} \int_{\Gamma \times \Gamma} J d(Q_n \otimes Q_n) \\ &= \inf_{Q \in \mathcal{P}(\Gamma)} \mathcal{J}(Q). \end{aligned}$$

Since the above holds for every $C > 0$, by Lebesgue monotone convergence theorem, one observes

$$\int_{\Gamma \times \Gamma} J_c d(Q_\infty \otimes Q_\infty) \longrightarrow \int_{\Gamma \times \Gamma} J d(Q_\infty \otimes Q_\infty) \quad \text{as } C \rightarrow +\infty.$$

and then we can deduce $\mathcal{J}(Q_\infty) \leq \inf_{Q \in \mathcal{P}(\Gamma)} \mathcal{J}(Q)$. Hence, Q_∞ is the desired minimizer and this ends the proof. \square

As a direct consequences of Theorem 4.5.2 and Proposition 4.5.3, we have the following Theorem.

Theorem 4.5.4. *Under the previous assumptions, there exists an equilibrium probability measure Q_∞ in \mathcal{Q}_{m_0} of the mean field game.*

4.6 Strong equilibrium

In this section, we introduce a stronger notion of equilibrium, and we show the existence of such an equilibrium under some additional assumptions.

Definition 4.6.1. A measure $Q \in \mathcal{Q}_{m_0}$ is called strong equilibrium of MFG, if

$$\int_{\Gamma} F(\gamma, Q) dQ(\gamma) < +\infty, \quad F(\gamma, Q) = \inf_{\substack{\omega \in \Gamma \\ \omega(0) = \gamma(0)}} F(\omega, Q), \quad \forall \gamma \in \text{spt}(Q).$$

We introduce the next assumptions on ℓ and Ω .

(H6) Assumption (H1) holds and moreover there exists $\alpha^* > 0$ such that

$$\ell(t, x, p) \leq \alpha^* |p|^\theta \quad \forall p \in \mathbb{R}^d.$$

(H7) If one has $x_n \rightarrow x_0 \in \bar{\Omega}$, then

$$d_{geo}(x_n, x_0) \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We set

$$\mathbf{Opt}(Q) = \left\{ \gamma \in \Gamma \mid F(\gamma, Q) = \inf_{\substack{\omega \in \Gamma \\ \omega(0) = \gamma(0)}} F(\omega, Q) \right\},$$

containing all the optimal trajectories with respect to Q . In the sequel, we need to establish the following result which deals with the situations that one considers agent's starting point in the target set.

Lemma 4.6.2. *Let $x_0 \in \Xi$. Then the unique minimizer of the cost function (4.4) is a constant trajectory, i.e. $\gamma \equiv x_0$, for all $t \in \mathbb{R}_+$.*

Proof. We first assume that γ is constant and equals to x_0 . Then, we observe $\dot{\gamma}(t) = 0$, for $t \in \mathbb{R}_+$. By assumption (H6) and the fact that the function ℓ is positive, one deduces $\ell(t, \gamma(t), 0) = 0$. On the other hand, in the interaction part, the bound of the integral is 0, since the agent is already on the target set which means $\tau(\gamma) = 0$. Now if we take another trajectory $\tilde{\gamma} \in \Gamma$, then we have

$$(4.11) \quad F(\gamma, Q) = \Psi(\tau(\gamma)) < F(\tilde{\gamma}, Q),$$

since for this trajectory $\tilde{\gamma}$, one indeed has the positive individual and interaction parts. We may now show the uniqueness of such a solution. Let γ_1 and γ_2 be two optimal trajectories starting from a same initial condition, which means $F(\gamma_1, Q) = F(\gamma_2, Q)$. But this contradicts the fact that for every $\gamma \in \mathbf{Opt}(Q)$ the strict inequality 4.11 holds. Hence $\mathbf{Opt}(Q) = \{\gamma_1 \equiv \gamma_2 \equiv x_0\}$. \square

Lemma 4.6.3. *Let $Q \in \text{Dom}(\mathcal{L})$, then together with assumption (H6) and (H7), the set $\mathbf{Opt}(Q)$ is closed.*

Proof. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence which converges uniformly to γ on compact time intervals and satisfies

$$F(\gamma_n, Q) = \inf_{\substack{\omega \in \Gamma \\ \omega(0) = \gamma_n(0)}} F(\omega, Q) < \infty.$$

Let $x_0 = \gamma(0)$ and $x_n = \gamma_n(0)$. The aim is to show the optimality of γ , for which one takes $\tilde{\gamma} \in \Gamma$ with $\tilde{\gamma}(0) = x_0$ and constructs suitable trajectories $\tilde{\gamma}_n$ with $\tilde{\gamma}_n(0) = x_n$ to obtain the following inequality

$$(4.12) \quad F(\gamma, Q) \leq \liminf_{n \rightarrow \infty} F(\gamma_n, Q) \leq \liminf_{n \rightarrow \infty} F(\tilde{\gamma}_n, Q) = F(\tilde{\gamma}, Q),$$

in which the first equality can be easily deduced from Proposition 4.3.5, the second inequality holds thanks to the optimality of $\gamma_n \in \Gamma$. We are thus left to show the last equality in (4.12). To do so, consider $\tilde{\gamma} \in \Gamma$ with $\tilde{\gamma}(0) = \gamma(0) = x_0$. Let $\epsilon_n = d_{\text{geo}}(x_n, x_0)$ and $\varsigma_n : [0, \epsilon_n] \rightarrow \bar{\Omega}$ be a geodesic curve such that $\varsigma_n(0) = x_n$, $\varsigma_n(\epsilon_n) = \tilde{\gamma}(\epsilon_n)$ with constant speed $|\dot{\varsigma}_n(t)| = \frac{d_{\text{geo}}(x_n, \tilde{\gamma}(\epsilon_n))}{\epsilon_n}$, for every $t \in [0, \epsilon_n]$, where $\epsilon_n \rightarrow 0$, as $n \rightarrow \infty$, due to the assumption (H7). Notice that the existence of such a geodesic curve can be assured by assumption (H7) and [19, Proposition 2.5.19]. Let us define

$$\tilde{\gamma}_n(t) = \begin{cases} \varsigma_n(t) & t \leq \epsilon_n \\ \tilde{\gamma}(t) & t \geq \epsilon_n, \end{cases}$$

and show $\lim_{n \rightarrow \infty} F(\tilde{\gamma}_n, Q) = F(\tilde{\gamma}, Q)$, as $n \rightarrow \infty$, by which we will have established the last equality in (4.12). Since our model deals with free final time optimization, we must thus pay attention to the two following cases, i.e., whether $x_0 \in \bar{\Omega} \setminus \Xi$ or $x_0 \in \Xi$. Let us first treat the case $x_0 \in \bar{\Omega} \setminus \Xi$, in which due to the closeness of Ξ , for large enough $n \in \mathbb{N}$, we have $x_n \notin \Xi$. For the first part of the cost function (4.4), since $\tilde{\gamma}_n$ and $\tilde{\gamma}$ coincide on $[\epsilon_n, +\infty)$, we

only need to show the integration $\int_0^{\epsilon_n} \ell(t, \tilde{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt$ will tend to zero. By using (H6), we have

$$(4.13) \quad \begin{aligned} \int_0^{\epsilon_n} \ell(t, \tilde{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt &= \int_0^{\epsilon_n} \ell(t, \varsigma_n(t), \dot{\varsigma}_n(t)) dt \\ &\leq \alpha^* \int_0^{\epsilon_n} |\dot{\varsigma}_n(t)|^\theta dt \leq \alpha^* \frac{d_{geo}(x_n, \tilde{\gamma}(\epsilon_n))^\theta}{d_{geo}(x_n, x_0)^{\theta-1}}. \end{aligned}$$

We claim that $\frac{d_{geo}(x_n, \tilde{\gamma}(\epsilon_n))^\theta}{d_{geo}(x_n, x_0)^{\theta-1}} \rightarrow 0$ as $n \rightarrow \infty$. To see this, notice first

$$\begin{aligned} d_{geo}(x_n, \tilde{\gamma}(\epsilon_n))^\theta &\leq \left(d_{geo}(\tilde{\gamma}_n(0), \tilde{\gamma}(0)) + d_{geo}(\tilde{\gamma}(0), \tilde{\gamma}(\epsilon_n)) \right)^\theta \\ &\leq 2^{\theta-1} \left(d_{geo}(x_n, x_0)^\theta + \left(\int_0^{\epsilon_n} |\dot{\tilde{\gamma}}(t)| dt \right)^\theta \right) \end{aligned}$$

and secondly, by using the Hölder inequality

$$\left(\int_0^{\epsilon_n} |\dot{\tilde{\gamma}}(t)| dt \right)^\theta \leq \left(\int_0^{\epsilon_n} |\dot{\tilde{\gamma}}(t)|^\xi dt \right)^{\frac{\theta}{\xi}} \epsilon_n^{(1-\frac{1}{\xi})\theta}$$

in which by choosing $\xi = \theta$, we have

$$\left(\int_0^{\epsilon_n} |\dot{\tilde{\gamma}}(t)| dt \right)^\theta \leq \epsilon_n^{\theta-1} \int_0^{\epsilon_n} |\dot{\tilde{\gamma}}(t)|^\theta dt.$$

Hence

$$(4.14) \quad \frac{d_{geo}(x_n, \tilde{\gamma}(\epsilon_n))^\theta}{d_{geo}(x_n, x_0)^{\theta-1}} \leq 2^{\theta-1} \left(\epsilon_n + \int_0^{\epsilon_n} |\dot{\tilde{\gamma}}(t)|^\theta dt \right) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

since $\dot{\tilde{\gamma}} \in L^\theta(\mathbb{R}_+; \bar{\Omega})$ and hence by dominated convergence theorem

$$\int_0^{\epsilon_n} |\dot{\tilde{\gamma}}(t)|^\theta dt = \int_0^\infty |\dot{\tilde{\gamma}}(t)|^\theta \mathbb{1}_{(0, \epsilon_n)} dt \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore, the integration $\int_0^{\epsilon_n} \ell(t, \tilde{\gamma}_n(t), \dot{\tilde{\gamma}}_n(t)) dt \rightarrow 0$ as $\epsilon_n \rightarrow 0$. Moreover, thanks to assumption (H7) and triangle inequality, one can easily see $d_{geo}(x_n, \tilde{\gamma}(\epsilon_n)) \rightarrow 0$. Regarding the final cost, for large enough $n \in \mathbb{N}$, we show $\Psi(\tau(\tilde{\gamma}_n)) = \Psi(\tau(\tilde{\gamma}))$, for which we indeed prove $\|\varsigma_n - x_0\|_{L^\infty(0, \epsilon_n)} \rightarrow 0$, since by that and using the fact $x_0 \in \bar{\Omega} \setminus \Xi$, we can thus conclude. To do so, we proceed as follows. For every $t \in [0, \epsilon_n]$, we have

$$\begin{aligned} |\varsigma_n(t) - x_0| &= \left| \varsigma_n(0) + \int_0^t \dot{\varsigma}_n(s) ds - x_0 \right| \\ &\leq |x_n - x_0| + \int_0^t |\dot{\varsigma}_n(s)| ds \\ &\leq |x_n - x_0| + t \frac{d_{geo}(x_n, \tilde{\gamma}(\epsilon_n))}{\epsilon_n} \\ &\leq |x_n - x_0| + d_{geo}(x_n, \tilde{\gamma}(\epsilon_n)) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \text{ by (4.14)}. \end{aligned}$$

Hence, for the first part of cost function (4.4), we have shown

$$L(\tilde{\gamma}_n) \longrightarrow L(\tilde{\gamma}) \quad \text{as } n \rightarrow \infty.$$

For the second part of the cost function (4.4), we have

$$\begin{aligned} & \int_{\Gamma} \int_0^{\epsilon_n} h(t, \varsigma_n(t), \omega(t), \dot{\varsigma}_n(t), \dot{\omega}(t)) dt dQ(\omega) \\ & \leq C \left(\int_{\Gamma} \int_0^{\epsilon_n} |\dot{\varsigma}_n(t)|^{\beta} + |\dot{\omega}(t)|^{\beta} dt dQ(\omega) \right) \\ & \leq C \int_0^{\epsilon_n} |\dot{\varsigma}_n(t)|^{\beta} dt + C \int_{\Gamma} \int_0^{\epsilon_n} |\dot{\omega}(t)|^{\beta} dt dQ(\omega). \end{aligned}$$

We thus observe

$$\int_0^{\epsilon_n} |\dot{\varsigma}_n(t)|^{\beta} dt \leq \epsilon_n^{1-\frac{1}{\xi}} \left(\int_0^{\epsilon_n} |\dot{\varsigma}_n(t)|^{\beta\xi} dt \right)^{\frac{1}{\xi}} \leq \epsilon_n^{1-\frac{\beta}{\theta}} \left(\int_0^{\epsilon_n} |\dot{\varsigma}_n(t)|^{\theta} dt \right)^{\frac{\beta}{\theta}},$$

in which $\xi = \frac{\theta}{\beta}$ has been chosen, and as a consequence, by using the inequality (4.13) and the fact $1 - \frac{\beta}{\theta} > 0$, one concludes $\int_0^{\epsilon_n} |\dot{\varsigma}_n(t)|^{\beta} dt \rightarrow 0$ as $n \rightarrow \infty$. We also have

$$\begin{aligned} & \int_{\Gamma} \int_0^{\epsilon_n} |\dot{\omega}(t)|^{\beta} dt dQ(\omega) \\ & \leq \epsilon_n^{1-\frac{\beta}{\theta}} \left(\int_{\Gamma} \int_0^{\epsilon_n} |\dot{\omega}(t)|^{\theta} dt dQ(\omega) \right)^{\frac{\beta}{\theta}} \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{\Gamma} \int_0^{\tau(\tilde{\gamma}_n) \wedge \tau(\omega)} h(t, \tilde{\gamma}_n(t), \omega(t), \dot{\tilde{\gamma}}_n(t), \dot{\omega}(t)) dt dQ(\omega) \\ & \longrightarrow \int_{\Gamma} \int_0^{\tau(\tilde{\gamma}) \wedge \tau(\omega)} h(t, \tilde{\gamma}(t), \omega(t), \dot{\tilde{\gamma}}(t), \dot{\omega}(t)) dt dQ(\omega), \end{aligned}$$

as $n \rightarrow \infty$. This shows the optimality of γ .

We now consider the case $x_0 \in \Xi$. For this, we need to split the argument into two parts. First, if there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_{n_k} \in \Xi$, then we observe the optimal trajectories $\gamma_{n_k} \in \Gamma$, starting from x_{n_k} , are constant and equal to x_{n_k} , for every $t \in \mathbb{R}_+$ by Lemma 4.6.2. Hence by taking the limit, as $k \rightarrow \infty$, one concludes $\gamma(t) = x_0$ is constant for every $t \in \mathbb{R}_+$, and thus, again by Lemma 4.6.2, γ is optimal. However, it is possible to only have $x_n \in \Xi$ for finitely many elements of $(x_n)_{n \in \mathbb{N}}$, which means there exists $k \in \mathbb{N}$ such that $x_n \in \bar{\Omega} \setminus \Xi$, for all $n \geq k$. In this case, we proceed as follows. Let $\epsilon_n = d_{geo}(x_n, x_0)$ and $\sigma_n: [0, \epsilon_n] \rightarrow \bar{\Omega}$ be a geodesic curve such that $\sigma_n(0) = x_n$, $\sigma_n(\epsilon_n) = x_0$ with constant speed $|\dot{\sigma}_n(t)| = 1$, for every $t \in (0, \epsilon_n)$. One can easily extend the domain of σ_n to \mathbb{R}_+ by setting $\sigma_n(t) = x_0$, for all $t \geq \epsilon_n$, to observe the individual and interaction cost converge to zero, in the same manner as done before, and as a consequence

$$F(\sigma_n, Q) \longrightarrow \Psi(0^+), \quad \text{as } n \rightarrow \infty.$$

Since γ_n is optimal, we have $F(\gamma_n, Q) \leq F(\sigma_n, Q)$, and on the other hand, since $\tau(\gamma_n) > 0$, we have $\Psi(0^+) \leq \Psi(\tau(\gamma_n)) \leq F(\gamma_n, Q)$. Hence, $\lim_{n \rightarrow \infty} F(\gamma_n, Q) = \Psi(0^+)$, by which the assumption (H1) leads us to observe $\int_0^{+\infty} |\dot{\gamma}_n(t)|^\theta dt \rightarrow 0$. Therefore, by applying Fatou's lemma, one can deduce

$$0 \leq \int_0^{+\infty} |\dot{\gamma}(t)|^\theta dt \leq \liminf_{n \rightarrow \infty} \int_0^{+\infty} |\dot{\gamma}_n(t)|^\theta dt = 0.$$

Hence, $\int_0^{+\infty} |\dot{\gamma}(t)|^\theta dt = 0$ and as a consequence $|\dot{\gamma}(t)| = 0$, which means $\gamma \in W^{1,\theta}(\mathbb{R}_+; \bar{\Omega})$ is constant. Therefore, by Lemma 4.6.2, γ is optimal and this ends the proof. \square

The next corollary, which is an immediate consequence of Lemma 4.6.3, states the equivalence of weak and strong notion of equilibrium for the mean field game.

Theorem 4.6.4. *Under the assumptions (H6) and (H7), the weak and strong notion of equilibria are the same.*

Proof. Let us first assume that Q is a weak equilibrium by which we deduce existence of a Borel measurable set, $A \subset \Gamma$, with $Q(A) = 0$ such that every $\gamma \in A^c$ is optimal. Now take $\gamma \in \text{spt}(Q)$ and consider $\gamma_n \in A^c \subset \mathbf{Opt}(Q)$ such that $\gamma_n \rightarrow \gamma$, uniformly on compact time intervals. Since $\mathbf{Opt}(Q)$ is closed by Lemma 4.6.3, one thus deduces, $\gamma \in \text{spt}(Q)$ is optimal. On the other hand, the reverse implication is obvious, and this ends the proof. \square

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Titre : Jeux à champ moyen avec temps final libre

Mots clés : Jeux à champ moyen, Contrôle optimal, Analyse non lisse, Contraintes d'état, Équilibre lagrangien, Jeux à champ moyen variationnels

Résumé : Motivé par des sujets économiques et d'ingénierie, vers 2006, les jeux à champ moyen ont été introduits par Jean-Michel Lasry et Pierre-Louis Lions, et Peter E. Caines, Minyi Huang et Roland P. Malhamé, indépendamment. Cette thèse aborde des modèles de jeux à champ moyen avec temps final libre.

Dans le premier chapitre, nous considérons plusieurs populations en interaction évoluant dans \mathbb{R}^d visant à atteindre des ensembles cibles donnés en un minimum de temps. Le système de contrôle satisfait par chaque agent dépend de sa position, de la répartition des autres agents de la même population et de la répartition des agents des autres populations. Ainsi, les interactions entre agents se font par leur dynamique. Nous considérons dans ce chapitre l'existence d'équilibres lagrangiens à ce jeu à champ moyen, leur comportement asymptotique et leur caractérisation comme solutions d'un système de jeu à champ moyen, sous quelques hypothèses de régularité sur la dynamique des agents. En particulier, le système de jeu de champ moyen est établi sans s'appuyer sur les propriétés de semi-concavité de la fonction de valeur.

De manière similaire au premier chapitre, dans le deuxième chapitre, nous considérons un modèle de jeu à champ moyen inspiré du mouvement de foule où les agents visent à atteindre un ensemble fermé, appelé ensemble cible, en un temps minimal, mais en plus des phénomènes de congestion, qui affectent la vitesse de un agent, le modèle est considéré en présence de contraintes d'état : en gros, ces contraintes peuvent modéliser des murs, des

colonnes, des clôtures, des haies ou d'autres types d'obstacles à la frontière du domaine que les agents ne peuvent pas franchir. Nous rappelons tout d'abord quelques résultats antérieurs sur l'existence d'équilibres pour de tels jeux et présentons les principales difficultés liées à la présence de contraintes d'état. Notre principale contribution est de montrer que les équilibres du jeu satisfont un système d'équations aux dérivées partielles couplées, connu sous le nom de système de jeu à champ moyen, grâce à des techniques récentes de caractérisation des contrôles optimaux en présence de contraintes d'état. Ces techniques permettent non seulement de traiter des contraintes d'état mais nécessitent également très peu d'hypothèses de régularité sur la dynamique des agents.

Dans notre dernier chapitre, nous considérons un modèle de jeu à champ moyen pour le mouvement de foule dans lequel les piétons interagissent non seulement par leur position, mais aussi par leur vitesse. Plus précisément, chaque piéton est supposé minimiser un coût impliquant son temps pour atteindre un certain ensemble cible, un coût intégral individuel et un coût intégral d'interaction modélisant le fait que les agents veulent éviter les régions trop denses et préfèrent se déplacer avec les agents allant dans la même direction qu'eux, ce qui peut être vu comme une interaction de type Cucker–Smale. Le résultat principal que nous obtenons dans ce chapitre est l'existence d'équilibres pour un tel jeu, qui est basé sur une approche variationnelle.

Title : Mean field games with free final time

Keywords : Mean field games, Optimal control, Nonsmooth analysis, State constraints, Lagrangian equilibrium, Variational mean field games

Abstract : Motivated by economical and engineering topics, around 2006, mean field games were introduced by Jean-Michel Lasry and Pierre-Louis Lions, and Peter E. Caines, Minyi Huang and Roland P. Malhamé, independently. This thesis addresses some mean field games models with free final time.

In the first chapter, we consider several interacting populations evolving in \mathbb{R}^d aiming at reaching given target sets in minimal time. The control system satisfied by each agent depends on an agent's position, the distribution of other agents in the same population, and the distribution of agents on other populations. Thus, interactions between agents occur through their dynamics. We consider in this chapter the existence of Lagrangian equilibria to this mean field game, their asymptotic behavior, and their characterization as solutions of a mean field game system, under few regularity assumptions on agents' dynamics. In particular, the mean field game system is established without relying on semiconcavity properties of the value function.

Similarly to the first chapter, in the second chapter, we consider a mean field game model inspired by crowd motion where agents aim to reach a closed set, called target set, in minimal time, however in addition to congestion phenomena, which affects the velocity of an agent, the model is considered in the presence of

state constraints: roughly speaking, these constraints may model walls, columns, fences, hedges, or other kinds of obstacles at the boundary of the domain which agents cannot cross. We first recall some previous results on the existence of equilibria for such games and presents the main difficulties that arise due to the presence of state constraints. Our main contribution is to show that equilibria of the game satisfy a mean field game system, thanks to recent techniques to characterize optimal controls in the presence of state constraints. These techniques not only allow to deal with state constraints but also require very few regularity assumptions on the dynamics of the agents.

In our last chapter, we consider a mean field game model for crowd motion in which pedestrians interact not only through their position, but also through their velocity. More precisely, each pedestrian is assumed to minimize a cost involving their time to reach a certain target set, an individual integral cost, and an interaction integral cost modelling the fact that agents want to avoid congestion and prefer to move together with agents going in the same direction, in which can be seen as a Cucker–Smale type interaction. The main result we obtain in this chapter is the existence of equilibria for such a game, which is based on a variational approach.