The Constrained-Routing and Spectrum Assignment Problem : Polyhedral Analysis and Algorithms

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The Constrained-Routing and Spectrum Assignment Problem: Polyhedral Analysis and Algorithms

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Abstract

In this thesis[1], we study a variant of the Routing and Spectrum Assignment problem (RSA), namely the Constrained-Routing and Spectrum Assignment (C-RSA). The C-RSA problem is a key issue when dimensioning and managing a new generation of optical networks, called spectrally flexible optical networks. The C-RSA can be stated as follows. Given an undirected, loopless, and connected graph G, an optical spectrum $\mathbb{S}$ of available contiguous frequency slots, and a multiset of traffic demands K between pairs of origins and destinations, the C-RSA consists of assigning for each traffic demand $k \in K$ a path in G between its origin and destination, and an interval of contiguous frequency slots in $\mathbb{S}$ so that some technological constraints are satisfied, and some linear objective function is optimized. First, we propose an integer linear programming formulation for the C-RSA. We identify several families of valid inequalities for the associated polytope. Some of these inequalities are obtained by using the so-called conflict graphs. Moreover, we prove that these inequalities are facet-defining for the associated polytope under some necessary and sufficient conditions. In addition, we develop separation algorithms for these inequalities. Using these results, we devise a Branch-and-Cut (B&C) algorithm for the problem, and discuss experimental results. A second part of the sis is devoted to an extended formulation for the C-RSA. A column generation algorithm is developed to solve its linear relaxation. We prove that the related pricing problem is equivalent to the so-called resource constrained shortest path problem, which is well known to be NP-hard. For this, we propose a pseudo-polynomial time based dynamic programming algorithm. Using this, we devise Branch-and-Price (B&P) and Branch-and-Cut-and-Price (B&C&P) algorithms to solve the problem. An extensive experimental study with comparisons between the different B&C, B&P, and B&C&P algorithms is also presented.

Finally, we turn our attention to the Spectrum Assignment (SA) sub-problem. This

[1]This work was supported by the French National Research Agency grant ANR-17-CE25-0006, FlexOptim Project.

has been shown to be equivalent of wavelength assignment, interval coloring, and dynamic storage allocation problems that are well known to be NP-hard. To the best of our knowledge, a polyhedral approach to the SA problem has not been considered before, even to its equivalent problems. For this, first, we propose an integer linear programming compact formulation and investigate the facial structure of the associated polytope. Moreover, we identify several classes of valid inequalities for the polytope and prove that these inequalities are facet-defining. We further discuss the related separation problems. Using this, we devise a Branch-and-Cut (B&C) algorithm for the SA problem, along with some computational results are presented.

Keywords: optical network, network design, integer programming, polyhedron, facet, separation, branch-and-cut, branch-and-price, branch-and-cut-and-price, dynamic programming.

Long Résumé

Pour faire face à une croissance continue de la demande de trafic liée à l'augmentation de la bande passante, les opérateurs de réseaux ont dû faire évoluer l'architecture de leurs réseaux. En conséquence, une nouvelle génération de réseau de transport optique flexible appelée "Spectrally Flexible Optical Networks" (SFONs) a été introduite en 2008 comme une technologie prometteuse en raison de sa flexibilité et de son efficacité par rapport à l'ancienne technologie connue sous le nom "Optical Wavelength Division Multiplexing (WDM)". Les SFONs ont suscité un intérêt intense de la part des laboratoires de recherche, ainsi que dans l'industrie.

Nous étudions dans cette thèse l'un des problèmes clés lors de dimensionnement et planification des SFONs, le problème du routage contraint et assignation spectrale, connue sous le nom " Constrained-Routing and Spectrum Assignment " (CRSA) selon la terminologie anglaise. Il se compose de deux parties: le routage contraint (sélectionner pour chaque demande en trafic un chemin optique physique qui connecte sa source avec sa destination à travers le réseau sans dépasser une longueur maximale de chemin (en km) fixée pour chaque demande en trafic), et l'assignation d'un spectre (assigner à chaque demande en trafic un seul intervalle de "slot" consécutifs (contrainte de contiguïté) au long de son chemin du routage de sorte que le même intervalle de slots consécutifs doit être utilisé sur tous les liens qui appartiennent à son chemin optique physique (contrainte de continuité), et les intervalles de slots consécutifs alloués par un ensemble de demandes dont les chemins ne sont pas des liens disjoints dans le réseau ne peuvent pas partager aucun slot sur les liens partagés (contrainte de non-chevauchement), tout en optimisant une ou plusieurs fonctions objectives linéaires. Le problème CRSA est bien connu comme un problème NP-difficile et très difficile en pratique aussi que de nombreuses études de recherche ont été menées dans ce contexte depuis sa première apparition en 2010. Certains des algorithmes de résolution proposés dans la littérature sont basés sur des formulations mathématiques utilisant la programmation linéaire (mixte) en nombres entiers qui n'ont pas pu résoudre des instances de grande taille, ainsi que des heuris-

tiques et métaheuristiques qui ne peuvent pas garantir l'optimalité de solutions. Il a été jugé approprié de proposer des nouveaux modèles mathématiques plus souples et efficaces en se basant sur la programmation linéaire en nombres entiers, de concevoir et de développer des algorithmes exacts qui pourraient offrir des améliorations prometteuses par rapport aux méthodes existantes. À notre connaissance, l'étude polyédrale n'a pas encore fait l'objet de recherches récentes pour ce problème.

Nous fournissons donc une analyse théorique approfondie et concevons des algorithmes exacts de type coupes, branchements et génération de colonnes pour résoudre le problème CRSA en considérant des réseaux de taille réaliste. Pour ce faire, notre contribution consiste à introduire un programme linéaire en nombres entiers basée sur des coupes, où le nombre de variables n'augmente que de manière polynomiale avec la taille de l'instance traitée. En outre, nous étudions la structure polyédrale du polyèdre associé, et dérivons plusieurs classes d'inégalités valides. Nous donnons quelques conditions nécessaires et suffisantes pour que certaines inégalités valides soient des facettes pour le polyèdre associé. Nous élaborons ensuite des procédures de séparation pour ces inégalités valides. Ces inégalités sont ensuite utilisées dans la relaxation linéaire afin d'obtenir des bornes duales plus serrées. En se basant sur ça, nous développons un algorithme de coupes et branchements pour le problème CRSA.

D'autre part, nous avons proposé une nouvelle formulation étendue basée sur des chemins, où les variables sont associées à tous les chemins possibles pour chaque demande en trafic induisant donc une explosion de nombre de variables qui croissent de manière exponentielle et en parallèle avec la croissance de la taille de l'instance traitée. Nous développons également un algorithme de génération de colonnes pour la résolution de sa relaxation linéaire. Les inégalités valides de la formulation de coupes restent aussi valides pour le polyèdre associé à cette formulation étendue. Nous développons ensuite un algorithme exact qui combine un algorithme de coupes et branchements avec un algorithme de génération de colonnes pour résoudre le problème CRSA.

D'autre part, vu la complexité du problème, le problème CRSA peut être décomposé en deux sous-problèmes de telle sorte que le routage contraint précède l'assignation du spectre (CR+SA). Nous analysons la structure polyédrale du sous-problème d'assignation du spectre (SA) lorsque le routage est déjà établi. Tout d'abord, nous proposons une formulation compacte pour le problème SA. Nous étudions ensuite la structure du polyèdre associé. Nous définissons quelques classes supplémentaires

d'inégalités valides et introduire quelques inégalités pour bien gérer la symétrie afin de supprimer les solutions symétriques obtenues lors de la résolution du problème. Nous donnons également quelques conditions nécessaires et suffisantes pour que certaines inégalités valides définissent des facettes pour le polyèdre. Des procédures de séparation sont ensuite proposées pour certaines de ces inégalités valides et qui seront utilisées par la suite pour obtenir des bornes plus étroites dans la relaxation linéaire. Nous élaborons ensuite un algorithme de coupes et branchements pour le sous problème SA.

A la fin de chaque étape, nous examinons plus en profondeur l'efficacité et le comportement de nos algorithmes, et augmentons leurs efficacités grâce à plusieurs améliorations basant sur des heuristiques primales et aussi quelques techniques de branchement qui pourraient offrir une promesse d'amélioration par rapport aux méthodes existantes compte tenu des réseaux de taille réaliste de SndLib, et d'autres de taille réelle. Nous menons aussi une étude comparative d'efficacité entre les différents algorithmes proposés dans cette thèse.

Mots clés : réseaux optiques flexibles, polytope, inégalité valide, facette, separation, algorithme de coupes et branchements, algorithme de génération de colonnes et branchements.

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Introduction

The global Internet Protocol (IP) traffic is expected to reach 396 exabytes per month by 2022, up from 194.4 Exabytes per month in 2020 [23]. Optical transport networks are then facing a serious challenge related to continuous growth in bandwidth capacity due to the growth of global communication services and networking: mobile internet network (e.g., 5th generation mobile network), cloud computing (e.g., data centers), Full High-definition (HD) interactive video (e.g., TV channel, social networks) [19], etc... as shown in Figure 1. To sustain the network operators face this trend of increase in bandwidth, a new generation of optical transport network architecture called Spectrally Flexible Optical Networks (SFONs) (called also FlexGrid Optical Networks) has been introduced as promising technology because of their flexibility, scalability, efficiency, reliability, and survivability [17][19] compared with the traditional FixedGrid Optical Wavelength Division Multiplexing (WDM)[92][93]. In SFONs the optical spectrum is divided into small spectral units, called frequency slots [102]. They have the same frequency of 12.5 GHz where WDM uses 50 GHz [108] as recommended by ITU-T [2]. This can be seen as an improvement in resource utilization.

The concept of slots was proposed initially by Masahiko Jinno et al. in 2008 [57], and later explored by the same authors in 2010 [113]. We refer the reader to [63] for more information about the architectures, technologies, and control of SFONs.

The Routing and Spectrum Assignment (RSA) problem plays a primary role when dimensioning and designing of SFONs which is the main task for the development of this next generation of optical networks. It consists of assigning for each traffic demand, a physical optical path, and an interval of contiguous slots (called also channels) while optimizing some linear objective(s) and satisfying the following constraints [50]:

a) *spectrum contiguity*: an interval of contiguous slots should be allocated to each demand k with a width equal to the number of slots requested by demand k;

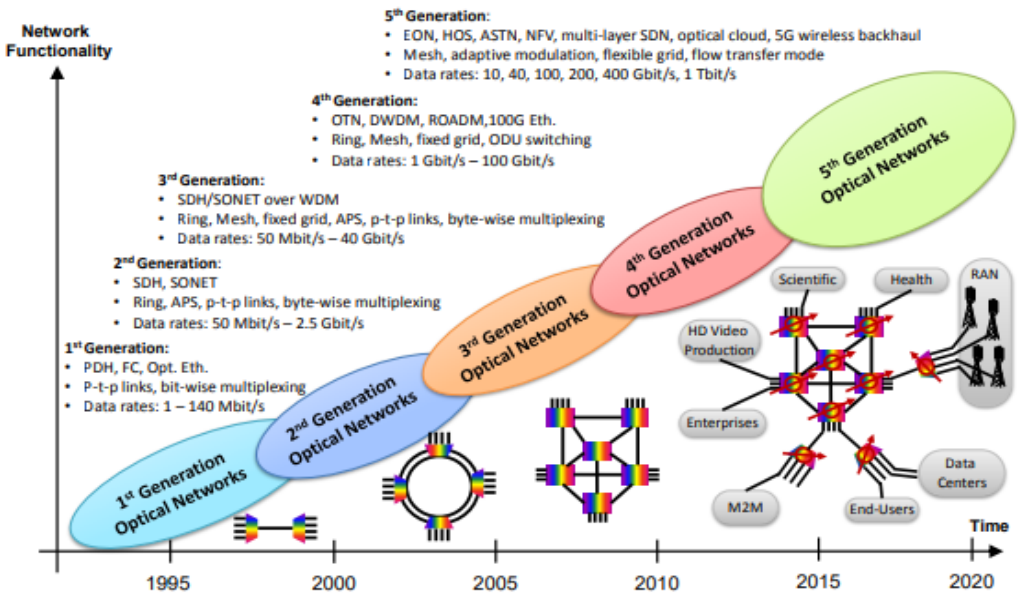


Figure 1: Historical Evolution of Optical Transport Networks.

b) *spectrum continuity*: the interval of contiguous slots allocated to each traffic demand stills the same along the chosen path;

c) *non-overlapping spectrum*: the intervals of contiguous slots of demands whose paths are not edge-disjoints in the network cannot share any slot over the shared edges.

Numerous research studies have been conducted on the RSA problem since its first appearance. The RSA is known to be NP-hard [107][109], and more complex than the historical Routing and Wavelength Assignment (RWA) problem [53]. Various (mixed) integer linear programming (ILP) formulations and algorithms have been proposed to solve it. A detailed survey of spectrum management techniques for SFONs is presented in [109] where the authors classified variants of the RSA problem into: offline RSA which has been initiated in [83], and online or dynamic RSA which has been initiated in [114] and recently developed in [78][119]. Numerous aspects are investigated in the tutorial [16]. This work focuses on the offline RSA problem. There exist two classes of ILP formulations used to solve the RSA problem, called edge-path and edge-node formulations. The ILP edge-path formulation is majorly used in the literature where variables are associated with all possible physical optical paths inducing an explosion of a number of variables and constraints which grow exponentially and in parallel with the growth of the instance size: number of

demands, the total number of slots, and topology size: number of links and nodes [50]. We observe that several papers which use the edge-path formulation as an ILP formulation to solve the RSA problem, use a set of precomputed-paths without guaranty of optimality e.g. in [22], [83], [84], [85], [112], [121], [100]. On the other hand, column generation techniques have been used by Klinkowski et al. [98], Jaumard et al. [56], and recently by Enoch [34] to solve the relaxation of the RSA taking into account all the possible paths for each traffic demand. To improve the LP bounds of the RSA relaxation, Klinkowsky et al. proposed a class of valid inequalities induced by cliques separable using a branch-and-bound algorithm [87]. On the other hand, Klinkowski et al. [88] propose a branch-and-cut-and-price method based on an edge-path formulation for the RSA problem. Recently, Fayez et al. [37], and Xuan et al. [116], proposed a decomposition approach to solve the RSA separately (i.e., R+SA) based on a recursive algorithm and an ILP edge-path formulation.
To overcome the drawbacks of the edge-path formulation usage, a compact edge-node formulation has been introduced as an alternative for it. It holds a polynomial number of variables and constraints that grow only polynomially with the size of the instance. We found just a few works in the literature that use the edge-node formulation to solve the RSA problem e.g., [13], [112], [121]. Bertero et al. [10] present a comparative study between several edge-node formulations and introduce new ILP ones.
On the other hand, and due to the NP-hardness of the C-RSA problem, several heuristics [30],[71],[102], and recently in [55], greedy algorithms [65], metaheuristics as tabu search [46], simulated annealing [88], genetic algorithms [43], [52], [53], [29], ant colony algorithms [60], and a hybrid meta-heuristic approach [97], have been used to approach large scale instances of the RSA problem. Furthermore, recent works start using artificial intelligence [96], see for example [61][62], and deep-learning [18], and machine-learning [101][120][117][48] to get more perefermonce. Selvakumar et al. give a survey [106] in which they summarise the most contributions done for the RSA problem before 2019.

In this thesis, we are interested in the resolution of a complex variant of the RSA problem, called the Constrained-Routing and Spectrum Assignment (C-RSA) problem. Here we suppose that the network should also satisfy the transmission-reach constraint for each traffic demand according to the actual service requirements. To the best of our knowledge a few related works on the RSA, take into account this additional constraint so that the length of the chosen path for each traffic demand

should not exceed a certain length (in kms). Recently, Hadhbi et al. [50][51] introduced a novel tractable ILP based on the cut formulation for the C-RSA problem with a polynomial number of variables and an exponential number of constraints that are separable in polynomial time using network flow algorithms. Computational results show that their cut formulation solves larger instances compared with those of Velasco et al. [112] and Cai et al. [13]. It has also been used as a basic formulation in the study of Colares et al. [24], and also by Chouman et al. [20][21] to show the impact of several objective functions on the optical networks state. Note that Velasco et al. [112], Cai et al. [13], and Bertero et al. [10], did not take into account the transmission-reach constraint.

However, so far the exact algorithms proposed in the literature could not solve large-scale instances. We believe that a cutting-plane-based approach could be powerful for the problem. To the best of our knowledge, such an approach has not been yet considered except the works done by Bianchetti et al. [11] for the RSA problem. For this, the main aim of this work is to investigate thoroughly theoretical properties of the C-RSA problem. To this end, we aim at providing a deep polyhedral analysis of the C-RSA problem, and based on this, devise branch-and-cut and branch-and-cut-and-price algorithms for solving large-scale instances of the problem. So we will introduce a new ILP formulation called *cut formulation* for the C-RSA problem which can be seen as an improved formulation for the one introduced by Hadhbi et al. [50][51]. We investigate the facial structure of the associated polytope. We further identify several classes of valid inequalities to obtain tighter LP bounds. Some of these inequalities are obtained by using conflict graphs related to the problem. We then devise separation procedures and give sufficient conditions under which these inequalities are facet defining. Using this, we develop a Branch-and-Cut (B&C) algorithm, along with computational results are presented using large-scale instances. On the other hand, we introduce an extended ILP formulation, called *path formulation*. A column generation algorithm is proposed to solve its linear relaxation. We further adapt the valid inequalities proposed for the *cut formulation* to obtain also tighter bounds for the *path formulation*. Based on this, we develop a Branch-and-Cut-and-Price (B&C&P) algorithm to solve the problem. Computational results are presented using this algorithm. We finally provide a comparative study between the B&C and B&C&P algorithms is presented by using two types of instances: random and realistic ones. The results show that the B&C&P algorithm is more efficient. Furthermore, we have studied the influence of the valid inequalities. The results show that some of them, in particular, clique and cover

inequalities are quite efficient. Several enhancements are further investigated and used to speed up and increase the efficiency of our approaches. They are based on a primal heuristic used to produce feasible solutions from fractional solutions given at each node of the branching tree. It allows obtaining good primal bounds and prune some uninteresting nodes of the branching tree. We have also introduced some symmetry-breaking inequalities to manage the equivalent sub-problems in the branching tree.

Several concepts are exploited throughout this dissertation. We start this dissertation by presenting the basic notions of combinatorial optimization, complexity, graph theory, and further give some notations that are used through this manuscript.

In Chapter 2, we present the C-RSA problem. We introduce an integer linear programming formulation namely cut formulation. We then carry out an investigation of the related polytope, the convex hull of all its solutions. Moreover, we describe the classes of valid inequalities and study their facial structure. In particular, we introduce symmetry-breaking inequalities.

In Chapter 3, we discuss the separation procedures for the valid inequalities and describe a Branch-and-Cut algorithm. The comparative study is presented in this chapter, it shows the impact of the additional valid inequalities using several mixed-integer linear program solvers.

In Chapter 4, we give the extended ILP formulation. We present the column generation algorithm to solve its linear relaxation, and the Branch-and-Cut-and-Price (B&C&P) algorithm, along with some computational results are presented. In this chapter, we also provide the comparative analysis of performance between the different algorithms.

Chapter 5 is devoted to the Spectrum Assignment (SA) sub-problem. First, we propose an integer linear programming compact formulation, and investigate the facial structure of the associated polytope. Fuerthremore, we describe several valid inequalities, some of them come from those that are already proposed for the C-RSA. We also give sufficient conditions under which these inequalities are facet defining. Based on these results, we develop a B&C algorithm to solve the problem. Furthermore, we describe symmetry-breaking inequalities for the SA, and provide some lower bounds. Finally, we present an extensive experimental study while showing the impact of the valid inequalities and symmetry-breaking inequalities on the effectiveness of the B&C algorithm.

Chapter 1

Preliminary Notions

In this chapter, we present some basic notions of combinatorial optimization, and polyhedra approaches.

1.1 Combinatorial Optimization

Operations research is a discipline related to computer science and applied mathematics. In this dissertation, we are interested in one of its branches, called *combinatorial optimization*. The optimization problems related to combinatorial optimization can be formulated as follows. Let $E = \{e_1, ..., e_n\}$ be a finite set, namely basic set. Suppose that each element e_i, it is associated a weight $c(e_i) \in \mathbb{R}$ with $i \in \{1, ..., n\}$. Let F denote a family of subsets of E. The problem aims to identify one subset $\mathcal{F}$ from F with the smallest or largest weight given by the sum $\sum_{e_i \in \mathcal{F}} c(e_i)$. Such a problem is called *combinatorial optimization problem* where the set F represents the set of all feasible solutions of the problem. In general, the set F contains an exponential number of solutions. For this, it's known to be very hard to solve combinatorial optimization problems by enumerating all its feasible solutions. To deal with this, various approaches have been developed to approach combinatorial optimization problems. They use different tools, complexity theory, combinatorial optimization, graph theory, linear and non-linear programming, integer programming, mixed integer programming. In the next section, we discuss some concepts from complexity theory.

1.2 Complexity Theory

Several researchers in computer science and mathematics are interested in working on the classification of problems into easy or hard problems, and further on the

algorithmic complexity whose objective is to find the most efficient algorithm. This has been initiated by Cook [25], Edmonds [33] and Karp [81].

Theory of complexity [Garey and Johnson, 1979] [41] classifies problems into two essential classes: the class P (polynomial time) class, and the class NP (Non-deterministic polynomial time). In addition, the problems of the NP class are shared into two subclasses: the class of NP-complete problems, and the class of NP-hard problems.

Before defining each class, we first give a general definition of a problem. In general, a *problem* is a question having parameters given in input such that an answer is needed for it, called *solution*. A problem is described by giving: a general description of all its parameters, and certain *constraints*. An instance of a problem is obtained by specifying the value of each input parameter of the problem. For this, one can propose an algorithm to solve the problem. An algorithm for solving a given problem is a procedure that is decomposable into a sequence of finite operations. It allows giving a solution for each instance of the problem. In general, the complexity of an algorithm depends on the *size* of the problem that reflects the number of parameters needed to describe an instance. The algorithm is said to be polynomial if the maximum number of its operations necessary to solve an instance of size n is bounded by a polynomial function f in n (i.e., $f(n)$). This means that there exists a scalar c such that the number of its operations necessary is equal to $c.f(n)$. As a result, the notation big $\mathcal{O}$ is appeared to express the complexity of an algorithm.

There exists two types of problems: optimization problems and decision problems. In optimization problems, we want to minimize (or maximize) a function while satisfying a set of constraints. On the other hand, in the a decision problem, the solution is binary like yes / no or 0/1.

An easy problem that can be solved by a polynomial algorithm with respect to its size, is called a problem of class P. A problem is NP if one can verify in polynomial time that a given solution is feasible. A problem is called NP-complete if it belongs NP, and every other problem in NP can be reduced to it in polynomial time [41].

The Satisfiability Problem (SAT) is the first problem that has been shown to be NP-complete. This was proved in 1971 by Stephen Cook [25] [42].

NP-hard problems are difficult as the NP-complete ones. If a decision problem associated with a optimization problem P is NP-complete then P is said to be NP-hard [42]. Furthermore, note that every problem of the class P is in NP ($P \subseteq NP$).

However, the converse is still open. It constitutes a well-known mathematical problem which is part of the 7 problems of the millennium prize. The question $P = NP$?

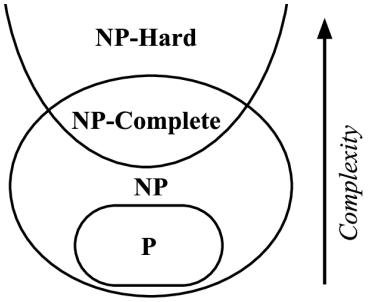


Figure 1.1: Relation between P, NP, NP-complete and NP-hard problems.

is one of the most important questions that has not yet been solved. The answer to this question by "yes" is to prove that all the problems of the NP class are in the P class. Cook has proved in [Cook, 1971] that all the problems of the NP class are reducible to the SAT problem, which means that if someone finds a polynomial algorithm for this problem, the question $P = NP$? is then solved ![42], i.e. we will be able to solve all NP-complete problems in polynomial time.

1.3 Polyhedral Approach and Branch-and-Cut Algorithm

1.3.1 Elements of the Polyhedral Theory

In this section, we will introduce some definitions and properties of polyhedraltheory. Schrijver [104], Nemhauser and Wolsey [72], Wolsey [115] and Schrijver [105] are the most useful references [118].
Let x be a vector in $\mathbb{R}^n$, with n a positive integer. x is said to be a linear combination of vectors $x_1, x_2, .., x_k \in \mathbb{R}^n$ if there exist k scaler $\lambda_1, \lambda_2.., \lambda_k$ such that $x = \sum_{\imath=1}^{k} \lambda_i x_i$. Furthermore, if $\sum_{\imath=1}^{k} \lambda_i = 1$, then x is said to be affine combination of $x_1, x_2, .., x_k$. We say that x is convex combination of $x_1, x_2, .., x_k$ if x is affine combination of $x_1, x_2, .., x_k$ and $\lambda_i \in \mathbb{R}_+$. The vectors $x_1, x_2, .., x_k$ are affinely independent if $\lambda_i = 0$ for each $i \in \{1, ..., k\}$, is the unique solution of the system

$$\sum_{\imath=1}^{k} \lambda_i x_i = 0, \sum_{\imath=1}^{k} \lambda_i = 1,$$

Given a set $S = \{x_1, ..., x_k\}$, the convex hull denoted by $conv(\mathcal{S})$, is the set of all the convex combinations of solutions of S that is

$$conv(\mathcal{S}) = \{x \in \mathbb{R}^n | \quad \sum_{i=1}^{k} \lambda_i x_i, \forall \lambda_i \geq 0 \text{ and } \sum_i \lambda_i = 1\}.$$

This definition ensures that $S \subset conv(\mathcal{S})$.

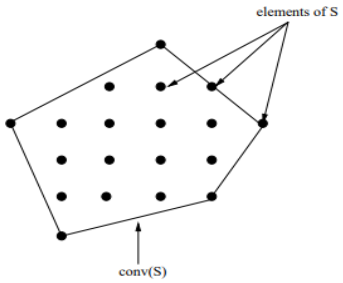


Figure 1.2: conv(S) vs S.

A *polyhedron* P is the set of solutions of a linear system $Ax \leq b$. That is $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. A bounded polyhedron is called a *polytope*.
The dimension of polyhedron P is one less than the maximum number of solution in P that are affinely independent.
An inequality $ax \leq \alpha$ is valid for a polyhedron P if and only if for every solution $\bar{x} \in P$, $a\bar{x} \leq \alpha$. It is said to be violated by a solution $\bar{x}$ if $a\bar{x} > \alpha$. A set $F \subset P$ is called face if there exists a valid inequality $ax \leq \alpha$ for the polyhedron P such that

$$F = \{x \in P, ax = \alpha\}.$$

We say that the valid inequality $ax \leq \alpha$ supports a face F if $F \neq \varnothing$.
A face F is said to be proper face if $F \neq \varnothing$ and $F \neq P$. If F is a proper face of P, and $dim(F) = dim(P) - 1$, then F is called a facet.
A face F of P is a facet if there doesn't exist any proper face F' of P containing F. If P is full-dimensional polyhedron, then $ax \leq \alpha$ defines a facet P if and only if F is a proper face and there exists a facet defining inequality$bx \leq \beta$ and a scalar$\rho \neq 0$ such that $F \subset \{x \in P | bx = \beta\}$ and $b = \rho a$. If P is not full dimensional polyhedron, then $ax \leq \alpha$ defines a facet of polyhedron P if and only if F is a proper face and there exists a facet of P induced by an inequality $bx \leq \beta$, a scalar $\rho \neq 0$ and a vector λ such that $F \subset \{x \in P | bx = \beta\}$ and $b = \rho a + \lambda A^{=}$.
A solution $x \in P$ is an extreme point of P if x is a face of P of dimension 0. Furthermore, it cannot be written as a convex combination of other points in P. Figure 1.3 illustrates the polyhedron P, valid inequality, face, facet and extreme point.

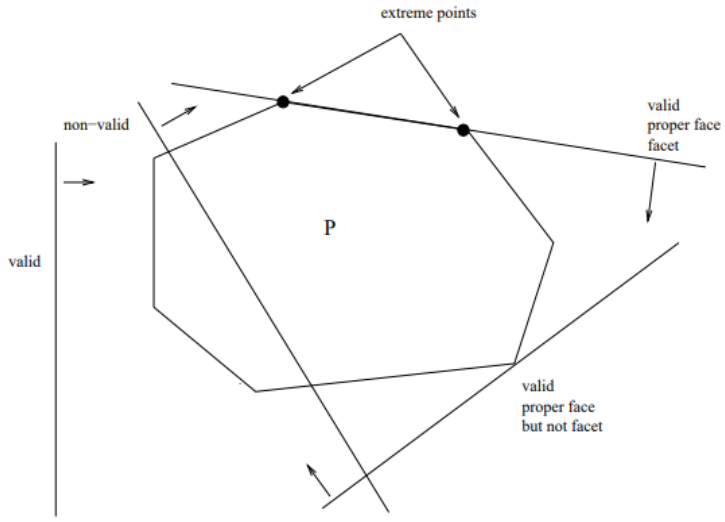


Figure 1.3: Geometric interpretation for the polyhedron P, valid inequality, face, facet and extreme point.

1.3.2 Cutting Plane Method

Let **P** be a combinatorial optimization problem and $\mathcal{S}$ the set of its feasible solutions. The problem P can be written as $\min\{cx|x \in \mathcal{S}\}$, where c denotes the weight vector associated with the variables x of the problem. Consider the convex hull $conv(\mathcal{S})$ of the feasible solutions of **P**. The problem **P** is then equivalent to the linear program $\min\{cx|x \in conv(\mathcal{S})\}$.

The polyhedral approach, introduced by Edmonds [33] consists in describing the polyhedron $conv(\mathcal{S})$ by a set of linear inequalities. This reduces the problem **P** to solving a linear program. However, a complete description of the polyhedron may contain an exponential number of linear inequalities. The optimization problem on the polyhedron $conv(\mathcal{S})$ can be solved having all its linear inequalities. However, one can have a partial characterization of the polyhedron $conv(\mathcal{S})$. This may be sufficient to solve the problem in polynomial time using the so-called cutting-plane method. This method is based on the so-called separation problem defined as follows. Let C be a class of valid inequalities for the polyhedron $conv(\mathcal{S})$. The separation problem associated with C consists in deciding whether a given solution x satisfies all inequalities of C, and if not to find an inequality of C violated by x. Grötschel, Losvàsz, and Schrijver [47] have shown that an optimization problem over C can be solved in polynomial time if and only if the separation problem associated with C can be solved in polynomial time. This may permit to solve the optimization

problem in polynomial time as a sequence of linear programs. Each program is obtained by adding new valid inequalities obtained by solving the related separation problem. For this, we start by solving a linear program containing a small set of valid inequalities. Let us denote by x the obtained optimal solution. We solve the separation problem for C. If x satisfies all the constraints of C, then x i optimal. Otherwise, at least one constraint violated by x is identified. These should be added to the current linear program. This process is repeated until an optimal solution is found.

1.3.3 Branch-and-Cut Algorithm

The cutting-plane method provides only an optimal solution for the linear relaxation of the problem. This solution may not be integer which means that it is not feasible for the original problem. In this case, we pass to the branching step which consists in dividing the problem into two Sub-problems by choosing a fractional variable x_i and setting x_i to $x_i = 1$ and $x_i = 0$. We then apply the cutting-plane method for each of the sub-problem. We continue this process until an optimal solution is obtained for the problem. This method is known as Branch-and-Cut method since it combines a branching method with a cutting plane method at each node of the tree.

1.4 Column Generation and Branch-and-Cut-and-Price Algorithms

Sometimes mathematical formulations of a problem contain a huge number of variables that can be exponential. These are known as "extended formulation". To solve such problems, we use column generation based algorithm. We begin the optimization with a restricted linear program containning a feasible basis. At each iteration, the column generation algorithm checks if there exists a missing variable having a negative reduced cost, and add it to the current restricted linear program. This is the "Pricing Problem". In fact, this consists in determining variables with negative reduced cost. This procedure is repeated until no new variable with negative reduced cost is found. The final solution is optimal for the linear relaxation of the underlaying problem. Furthermore, if it is integral, then it is optimal for the problem. If not, we create two subproblems called children by branching on some fractional variables (variable branching rule) or on some constraints using the Ryan & Foster branching rule [99] (constraint branching rule). Such an algorithm

is called a Branch-and-Price. A Branch-and-Cut-and-Price algorithm combines a Branch-and-Price algorithm with a cutting-plane procedure.

1.5 Graph Theory

In this section, we introduce some elementary definitions in graph theory that are very useful throughout the dessertation, Diestel [28], and Golumbic [45] are the most useful references on graph theory [118].
A graph is a pair $G = (V, E)$, where V is a finite set of nodes (called also vertices) linked by a set of edges (called also links) E which can be oriented or not.
A path p in graph $G = (V, E)$ from node a to node b, is a sequence of nodes such that for each pair of successive nodes v_i,v_{i+1}, there exist an edge e $(v_i, v_{i+1}) \in E$.
For any subset of nodes $X \subseteq V$ with $X \neq \emptyset$, let $\delta(X)$ denote the set of edges having one extremity in X and the other one in $\bar{X} = V \setminus X$ which is called a cut. When X is a singleton (i.e., $X = \{v\}$), we use $\delta(v)$ instead of $\delta(\{v\})$ to denote the set of edges incidents with a node $v \in V$. The cardinality of a set K is denoted by $|K|$.
A vertex coloring of G is an assignment of colors to the vertices of G so that two adjacent vertices v and v' cannot get the same color. Same rule for edges, an edge coloring of G is an assignment of colors to the edges of G so that two adjacent edges e and e' cannot get the same color. We say that graph G is t-colorable if no more than t different colors assigned in G.
G' is called a weighted graph if each node in G' is associated with weight.
An interval t-coloring of a weighted graph $G' = (V, E, w)$ is a function $c : V- > \{1, 2, ..., t\}$ such that $c(v)+w(v)-1 \leq t$. We assign an interval $[c(v), ..., c(v)+w(v)-1]$ of consecutive integers satisfying $w(v)$ of each vertex v that the intervals of colors assigned to two adjacent vertices do not overlap. If interval t-coloring is feasible for a graph G' then G' is said to be interval t-colorable [107]. The interval chromatic number of G', denoted by χ is the least integer number t such that G' has a interval t-coloring [107].

1.6 Flexible Optical Networks

The two last decades have seen a big developement in telecommunication networks with a continuous growth in demands. To face this trend of increase in bandwidth, network operators have had to make their network architectures and management evolve. Two significant changes appeared recently in the optical network architec-

ture. First the bandwidth-greedy *FixedGrid* architecture for Optical Wavelength Division Multiplexing (WDM) (called also wavelength routed networks) [92][93] based on fixed spectrum grid is being replaced by the *FlexGrid* architecture that is capable of supporting variable data rate (in Gb/s) through flexible spectrum. In this concept, the optical spectrum is divided into slots having the same frequency of 12.5 GHz (where FixedGrid networks use 50 GHz, the width of a wavelength) as recommended by ITU-T [2]. See for example Figure 1.4 which shows a fixed-grid with 4 wavelengths of 50 GHz to serve 4 demandes of two of 10 Gb/s, one of 40 Gb/s, one of 100 Gb/s. However, in the flex-grid we use just 9 slots of frequency 12.5 GHz to serve these demands.

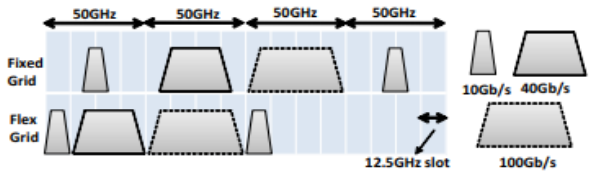


Figure 1.4: FixedGrid Vs FlexGrid.

The concept of slot was proposed initially by Masahiko Jinno et al. [57], and later explored and more developed by the same authors in 2010 [113]. In SFONs any optical path can elastically span as many contiguous slots as needed. This technology provides a more efficient use of the spectral domain than the traditional Fixed Grid WDM. Furthermore, a new generation of transponders is becoming available namely, bandwidth-variable transponders (BV-Ts) and bandwidth variable wavelength cross-connects (BV-WXCs) [113]. These can manage data rates up to 400 Gb/s which cannot be accommodated by a 50 GHz wavelength, and restores the signal which is necessary to re-amplify, re-shape and re-time the passive optical signal (which is called (3R) signal regeneration rule) when the transmission-reach of signals is limited which represents the maximum length (in kms) for the routing of each traffic demand.

The network operators have confronted several optimization problems, in particular some variants of routing and resource allocation problems that arise when designing or planning optical networks. The classical Routing and Wavelength Assignment (RWA) problem is the key issue for design FixedGrid WDM networks. In this problem, we are given an optical network and a set of demands where each demand has an origin and destination. The task is to find a path for each demand and a wavelength

such that a single 50 GHz wavelength is assigned to each demand. It was extended by Chlamtac et al. [15]. It is known to be a NP-hard problem [15]. It is equivalent to the n-graph-coloring problem where the number n of the colors corresponds to the number of wavelengths and finding the minimum number of wavelengths to route all the traffic demands is equivalent to finding the chromatic number of the conflict graph (where the demands are represented by the nodes and two nodes are linked by an edge if the path of the associated demands share an edge) when the paths are already established. It has been considered as a special case of the integer multicommodity flow (MCF) problem where some specific constraints [12] are added and should be respected. Several models and algorithms have been proposed to solve the RWA problem. However, in SFONs, RWA cannot handle the changes from wavelength to contiguous slots. As a result, a new problem is appeared to deal with this, called Routing and Spectrum Assignment (RSA) problem. It can be stated as follows. Given an optical network G and a multiset of traffic demands K, it aims at determining for each traffic demand $k \in K$ a path and an interval of contiguous slots while optimizing some linear objective(s) and satisfying the following constraints [50]:

1. *spectrum contiguity*: an interval of contiguous slots should be allocated to each demand k with width equals to the number of slots requested by demand k;

2. *spectrum continuity*: the interval of contiguous slots allocated to each traffic demand stills the same along the chosen path;

3. *non-overlapping spectrum*: the intervals of contiguous slots of demands whose paths are not edge-disjoints in the network cannot share any slot over the shared-edge.

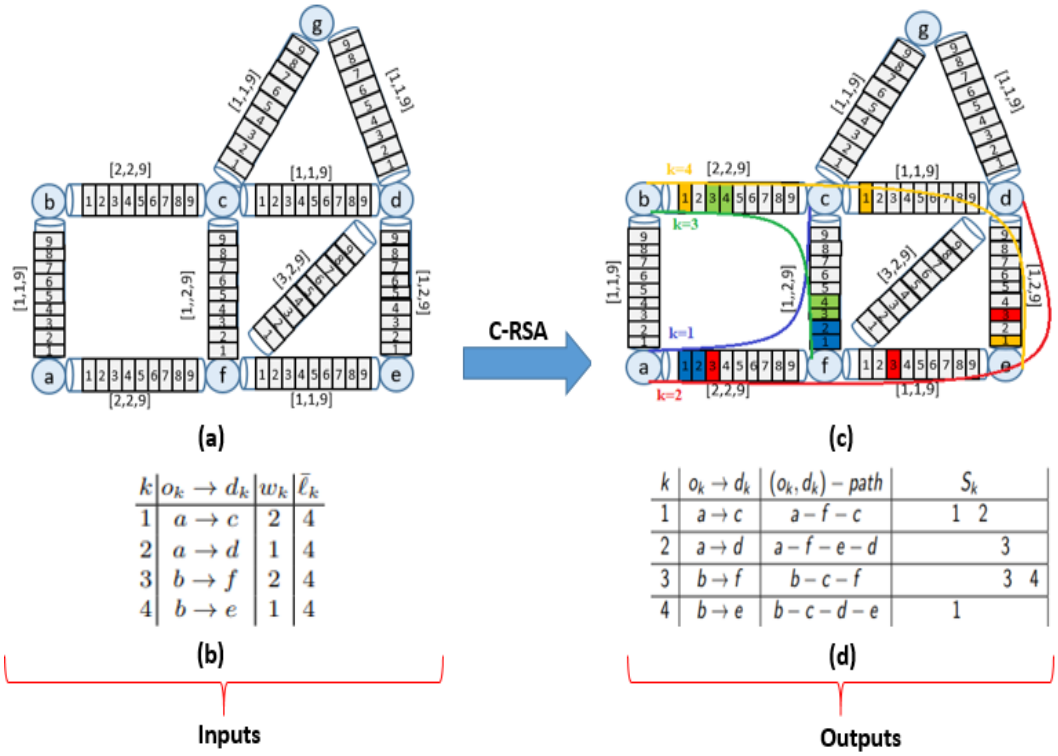
The RSA problem is harder than the RWA problem because of the continuity constraint that has not been taken into account when defining the RWA problem. In our work, we focus on a variant of the RSA problem, called Constrained-Routing and Spectrum Assignment Problem (C-RSA).

Chapter 2

Cut Formulation and Polyhedra for the C-RSA Problem

2.1 The Constrained-Routing and Spectrum Assignment Problem

The Constrained-Routing and Spectrum Assignment Problem can be stated as follows. We consider an optical spectrum of $\bar{s} \in \mathbb{Z}_+$ available contiguous frequency slots, denoted by $\mathbb{S} = \{1, \ldots, \bar{s}\}$. A SFON topology can be represented by an undirected, loopless, and connected graph $G = (V, E)$, with V is the set of vertices representing the optical nodes (data centers, users, stations,...), and E the set of links representing optical-fibers. A length $\ell_e \in \mathbb{R}_+$ (in kms), a cost $c_e \in \mathbb{R}_+$, and a set of $\bar{s}$ of contiguous frequency slots are associated with each edge e. Let K be a set of non-splittable traffic demands. Each demand $k \in K$ has an origin node $o_k \in V$, a destination node $d_k \in V \setminus \{o_k\}$, a slot-width $w_k \in \mathbb{Z}_+$, and a transmission-reach $\bar{\ell_k} \in \mathbb{R}_+$ (in kms). The C-RSA consists in determining for each demand $k \in K$, a (o_k,d_k)-path p_k in G (non-splittable demands) such that $\sum_{e \in E(p_k)} l_e \leq \bar{l}_k$ (*tranmission-reach constraint*), and an interval of contiguous frequency slots $S_k \subseteq \mathbb{S}$ of width equal to w_k (*continuity and contiguity constraint*) such that $S_k \cap S_{k'} = \emptyset$ for each pair of demands $k, k' \in K$ ($k \neq k'$) with $E(p_k) \cap E(p_{k'}) \neq \emptyset$ (*non-overlapping constraint*). It aims at minimizing the total cost of the paths used for routing the demands K (i.e., $\sum_{k \in K} \sum_{e \in E(p_k)} c_e$), where $E(p_k)$ denotes the set of edges in p_k. Fig. 2.1 provides a feasible solution for an instance of the C-RSA problem containing 4 demands routed in a graph G consisting of 7 nodes and 10 edges. Each edge e is specified by a triplet $[l_e, c_e, \bar{s}]$ with $\bar{s} = 9$.



k	$o_k \to d_k$	w_k	$\bar{\ell}_k$
1	$a \to c$	2	4
2	$a \to d$	1	4
3	$b \to f$	2	4
4	$b \to e$	1	4

k	$o_k \to d_k$	$(o_k, d_k) - path$	S_k
1	$a \to c$	$a - f - c$	1 2
2	$a \to d$	$a - f - e - d$	3
3	$b \to f$	$b - c - f$	3 4
4	$b \to e$	$b - c - d - e$	1

Figure 2.1: Set of established paths and spectrums in graph G (Fig. 2(a)) for the set of demands $\{k_1, k_2, k_3, k_4\}$ defined in Table 2(b).

2.2 Cut Formulation

Here we introduce an integer linear programming formulation for the C-RSA problem, called formulation. It can be seen as a reformulation of the one introduced by Hadhbi et al. [50]. For $k \in K$ and $e \in E$, let x_e^k be a variable which takes 1 if demand k goes through edge e and 0 if not, and for $k \in K$ and $s \in \mathbb{S}$, let z_s^k be a variable which takes 1 if slot s is the last slot allocated for the routing of demand k and 0 if not. The contiguous slots $s' \in \{s - w_k + 1, ..., s\}$ should be assigned to demand k whenever $z_s^k = 1$.

The C-RSA problem is equivalent to the following linear integer program

$$\min \sum_{k \in K} \sum_{e \in E} c_e x_e^k, \tag{2.1}$$

subject to

$$\sum_{e \in \delta(X)} x_e^k \geq 1, \forall k \in K, \forall X \subseteq V,\ |X \cap \{o_k, d_k\}| = 1, \tag{2.2}$$

$$\sum_{e \in E} l_e x_e^k \leq \bar{\ell_k}, \forall k \in K, \tag{2.3}$$

$$z_s^k = 0, \forall k \in K, \forall s \in \{1, ..., w_k - 1\}, \tag{2.4}$$

$$\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1, \forall k \in K, \tag{2.5}$$

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq 3, \forall (e, k, k', s) \in Q_{e,s}, \tag{2.6}$$

$$0 \leq x_e^k \leq 1, \forall k \in K, \forall e \in E, \tag{2.7}$$

$$0 \leq z_s^k \leq 1, \forall k \in K, \forall s \in \mathbb{S}, \tag{2.8}$$

$$x_e^k \in \{0, 1\}, \forall k \in K, \forall e \in E, \tag{2.9}$$

$$z_s^k \in \{0, 1\}, \forall k \in K, \forall s \in \mathbb{S}. \tag{2.10}$$

where $Q_{e,s}$ denotes the set of all the quadruples (e, k, k', s) where $e \in E, k \in K, k' \in K$, $k \neq k'$ and $s \in \mathbb{S}$.

Inequalities (2.2) ensure that there is an (o_k, d_k)-path between o_k and d_k for each demand k, and guarantee that all the demands should be routed. They are called cut inequalities. By optimizing the objective function (2.1), and given that the cost of all edges are positive, this ensures that there is exactly one (o_k, d_k)-path between o_k and d_k which will be selected as optimal path for demand k. Inequalities (2.3) express the length limit on the routing paths which is called "the transmission-reach constraint". Equations (2.4) express the fact that a demand k cannot use slot $s \leq w_k - 1$ as the last-slot. The slots $s \in \{1, ..., w_k - 1\}$ are called forbidden last slots for demand k. Inequalities (2.5) should normally be equalities ensuring that exactly one slot $s \in \{w_k, \ldots, \bar{s}\}$ must be assigned to demand k as last-slot. Here we relax this constraint. Inequalities (2.6) express the contiguity and non-overlapping constraints. They capture the fact that every slot s over edge e can be assigned to at most one demand $k \in K$. Inequalities (2.7)-(2.8) are the trivial inequalities, and constraints (2.9)-(2.10) are the integrality constraints.

Note that the linear relaxation of the C-RSA can be solved in polynomial time given that inequalities (2.2) can be separated in polynomial time using network flows.

2.3 Associated Polytope

Let $\mathcal{P}(G, K, \mathbb{S})$ be the polytope, convex hull of the solutions of (2.1)-(2.10).
In this section, we discuss the facial structure of the polytope $\mathcal{P}(G, K, \mathbb{S})$. First, we describe some structural properties. These will be used for determining the dimension of $\mathcal{P}(G, K, \mathbb{S})$.
For each demand k and node v, one can compute a shortest path between each of the pairs of nodes (o_k, v), (v, d_k). If the length of the $(o_k, d_k)-$paths formed by the concatenation of the shortest paths (o_k, v) and (v, d_k) is greater that $\bar{l}_k$ then node v cannot be in a path routing demand k, and we then say that v is a *forbidden node* for demand k. Let V_0^k denote the set of forbidden nodes for demand $k \in K$. Note that using Dijkstra's algorithm, one can identify in polynomial time the forbidden nodes V_0^k for each demand $k \in K$. On the other hand and regarding the edges, for each demand k and edge $e = ij$, one can compute a shortest path between each of the pairs of nodes (o_k, i), and (j, d_k), and (o_k, j), and (i, d_k). If the length of the $(o_k, d_k)-$path formed by e together with the shortest paths (o_k, i) and (j, d_k) (resp. (o_k, j) and (i, d_k)) greater that $\bar{l}_k$ then edge ij cannot be in a path routing of demand k, and we then say that ij is a *forbidden edge* for demand k due to the transmission-reach constraint. Let E_t^k denote the set of forbidden edges due to the transmission-reach constraint for demand $k \in K$. Note that using Dijkstra's algorithm, one can identify in polynomial time the forbidden edges E_t^k for each demand $k \in K$. Table 2.1 shows the set of forbidden edges E_t^k and forbidden nodes V_0^k for each demand k in K Fig. 2.1(b).

k	$o_k \Rightarrow d_k$	w_k	$\bar{\ell}_k$	V_0^k	E_t^k
1	$a \Rightarrow c$	2	4	$\{e, d, g\}$	$\{cg, dg, de, df, cd, ef\}$
2	$a \Rightarrow d$	1,00	4	$\{g\}$	$\{cg, dg, df\}$
3	$b \Rightarrow f$	2	4	$\{e, d, g\}$	$\{cg, dg, de, df, cd, ef\}$
4	$b \Rightarrow e$	1,00	4	$\{g\}$	$\{cg, dg, df\}$

Table 2.1: Sets of V_0^k and E_0^k of the example of Fig. 2.1(b).

Consider a subset of nodes W in $V \setminus V_0^k$ with $o_k \in W$ and $d_k \in V \setminus W$. Let f be an edge in a cut $\delta(W)$ such that all the edges $e \in \delta(W) \setminus \{f\}$ are forbidden for demand k. As a consequence, edge f is an *essential edge* for demand k. As the forbidden edges, the essential edges can be determined in polynomial time using network flows. Let E_1^k denote the set of essential edges of demand k, and K_e denote the subset of

demands in K having e as essential edge. Therefore,

$$x_e^k = 1, \text{ for all } k \in K \text{ and } e \in E_1^k. \tag{2.11}$$

In addition to the forbidden edges thus obtained due to the transmission-reach constraints, there may exist edges that may be forbidden because of lack of resources for demand k. This is the case when, for instance, the residual capacity of the edge in question does not allow a demand to use this edge, i.e., $w_k > \bar{s} - \sum_{k' \in K_e} w_{k'}$. Let E_c^k denote the set of forbidden edges for demand k, $k \in K$, due to the resource constraints. Let $E_0^k = E_t^k \cup E_c^k$ for $k \in K$. Hence,

$$x_e^k = 0, \text{ for all } k \in K \text{ and } e \in E_0^k. \tag{2.12}$$

As a result of the pre-processing stage, a non-compatibility between the demands may appear due to a lack of resources. For an edge e, two demands k and k' with $e \notin E_0^k \cup E_1^k \cup E_0^{k'} \cup E_1^{k'}$, are said to be *non-compatible* if the residual capacity of edge e does not allow to route the two demands k, k' together through e, i.e., $w_k + w_{k'} > \bar{s} - \sum_{k'' \in K_e} w_{k''}$. Let K_c^e denote the set of pairs of demands (k, k') in K that are non-compatible for edge e.

On the other hand, a non-compatibility between the edges for a demand may appear due to the transmission-reach constraint. Consider a demand k. Two edges $e = ij \notin E_0^k \cap E_1^k, e' = lm \notin E_0^k \cap E_1^k$ are said *non-compatible edges* if the length of all (o_k, d_k)-paths formed by $e = ij$ and $e' = lm$ together are greater than $\bar{l}_k$. Note that we are able to determine the non-compatible edges for each demand k in polynomial time using shortest-path algorithms.

2.3.1 Dimension

We first describe some properties that are useful to determine the dimension of $\mathcal{P}(G, K, \mathbb{S})$. First the following is easily seen to be true.

Proposition 2.3.1. *The follows equation system* (2.13) *is of full rank*

$$\begin{cases} x_e^k = 0, \textit{ for all } k \in K \textit{ and } e \in E_0^k, \\ x_e^k = 1, \textit{ for all } k \in K \textit{ and } e \in E_1^k, \\ z_s^k = 0, \textit{ for all } k \in K \textit{ and } s \in \{1, ..., w_k - 1\}. \end{cases} \tag{2.13}$$

The rank of system (2.13) is given by

$$r = \sum_{k \in K} (|E_0^k| + |E_1^k| + (w_k - 1)).$$

Let Q denote a matrix associated with the system (2.13) which contains r lines linear independents.

A solution of the C-RSA problem is given by two sets E_k and S_k for each demand $k \in K$ where E_k is a set of edges used for the routing of demand k, and S_k is a set of slots assigned to demand k. For the sake of presentaion, we will denote by E_k a feasible path, and by S_k the last slots assigned to demand k.
Below are some assumptions that will be considered

a) graph G contains at least one feasible path between o_k and d_k, for all $k \in K$,

b) the number of slots $\bar{s}$ is largely sufficient to route all the demands,

c) for each demand $k \in K$ and $e \in E \setminus (E_0^k \cup E_1^k)$, there exists at least a feasible route E_k between o_k and d_k such that $\sum_{e' \in E_k} \ell_{e'} + \ell_e \leq \bar{l}_k$, and for each $e' \in E_k$, the edges (e, e') are compatible edges for demand k.

Let $\mathcal{S}^i = (E^i, S^i)$ be a solution of the C-RSA problem such that $E^i = (E_1^i, E_2^i, ..., E_{|K|-1}^i, E_{|K|}^i)$ and $S^i = (S_1^i, S_2^i, ..., S_{|K|-1}^i, S_{|K|}^i)$, and let $(x^{\mathcal{S}}, z^{\mathcal{S}})$ be its incidence vector.
Note that when the routing of demands is trivial or already established, one can find a feasible spectrum assignment S^i in polynomial time using some heuristics and greedy algorithm as the well-known First-Fit algorithm [2]. This will be used throughout each proof of polyhedron dimension or facial structure of some valid inequalities such that the set of demands K is considered as an ordered set of demands, i.e., $K = \{k_1, k_2, ..., k_{|K|}\}$.

Proposition 2.3.2. *System* (2.13) *defines a minimal equation system for* $\mathcal{P}(G, K, \mathbb{S})$.

Proof. Consider an equation $\mu x + \sigma z = \lambda$ of $\mathcal{P}(G, K, \mathbb{S})$. To prove that $\mu x + \sigma z$ is a linear combination of equations system (2.13), it is sufficient to prove that there exists $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \gamma Q$.
We will show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$.
Consider a demand k in K and a slot s in $\{w_k, ..., \bar{s}\}$. Consider the solution $\mathcal{S}^0 = (E^0, S^0)$ given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^0$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^0 given by

$$I_i^0 = [\bigcap_{k_j \in D_i^0} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s - w_k\} \cup \{s + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}^0 \cap E_k^0 \neq \emptyset$ or $I_i^0 = \bigcap_{k_j \in D_i^0} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D_i^0 = \{k_j \in \{k_1, ..., k_{i-1}\} : E_{k_i}^0 \cap E_{k_j}^0 \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^0$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ if $E_{k_i}^0 \cap E_k^0 \neq \emptyset$ (we take into account the possibility of adding slot s in the set of last slots S_k^0 assigned to demand k in solution $\mathcal{S}^0$).

We let $S_{k_i}^0 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^0$ is feasible for the problem, and its incidence vector $(x^{\mathcal{S}^0}, z^{\mathcal{S}^0})$ belongs to $\mathcal{P}(G, K, \mathbb{S})$. Let $\mathcal{S}^1 = (E^1, S^1)$ be the solution obtained from $\mathcal{S}^0$ by adding slot s as last slot to demand k. Solution $\mathcal{S}^1$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^1}, z^{\mathcal{S}^1})$ belongs to $\mathcal{P}(G, K, \mathbb{S})$. Hence, solutions $\mathcal{S}^0$ and $\mathcal{S}^0$ satisfy equation $\mu x + \sigma z = \lambda$. We then obtain that

$$\mu x^{\mathcal{S}^0} + \sigma z^{\mathcal{S}^0} = \mu x^{\mathcal{S}^1} + \sigma z^{\mathcal{S}^1} = \mu x^{\mathcal{S}^0} + \sigma z^{\mathcal{S}^0} + \sigma_s^k.$$

It follows that $\sigma_s^k = 0$.
In a similar way, we can show that

$$\sigma_s^k = 0, \text{ for all } k \text{ and } s \in \{w_k, ..., \bar{s}\}$$

Next, we will show that $\mu_e^k = 0$ for all $k \in K$ and $e \in E \setminus (E_0^k \cup E_1^k)$.
Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Consider the solution $\mathcal{S}'^0 = (E'^0, S'^0)$ such that

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^0$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) we select slot $s_k = w_k$ as last slot for demand k in solution $\mathcal{S}'^0$,

c) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^0$ given by

$$I_i'^0 = [\bigcap_{k_j \in D_i'^0} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}'^0 \cap (E_k'^0 \cup \{e\}) \neq \emptyset$ or $I_i'^0 = \bigcap_{k_j \in D_i'^0} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not.

where $D_i'^0 = \{k_j \in \{k_1, ..., k_{i-1}\} \setminus \{k\} : E_{k_i}'^0 \cap E_{k_j}'^0 \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i'^0$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_k + 1, ..., s_k\} = \emptyset$ if $E_{k_i}'^0 \cap (E_k'^0 \cup \{e\}) \neq \emptyset$ (we take into account the possibility of adding edge e in the set of edges $E_k'^0$ to route demand k).

We let $S_{k_i}'^0 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^0$ is clearly feasible for the problem, , and its incidence vector $(x^{\mathcal{S}'^0}, z^{\mathcal{S}'^0})$ belongs to $\mathcal{P}(G, K, \mathbb{S})$. Let $\mathcal{S}^2 = (E^2, S^2)$ be the solution obtained from $\mathcal{S}'^0$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}'^0$ that $E_k^2 = E_k'^0 \cup \{e\}$, and remove slot s already selected for demand k as last slot in $\mathcal{S}'^0$ and replaced it by a new slot s' such that s' is the smallest slot index in $\{w_k, ..., \bar{s}\}$ such that $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each demand k' with $E_k'^0 \cap E_{k'}'^0 \neq \emptyset$. $\mathcal{S}^2$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^2}, z^{\mathcal{S}^2})$ belongs to $\mathcal{P}(G, K, \mathbb{S})$. Hence, solutions $\mathcal{S}'^0$ and $\mathcal{S}^2$ satisfy equation $\mu x + \sigma z = \lambda$. It follows that

$$\mu x^{\mathcal{S}'^0} + \sigma z^{\mathcal{S}'^0} = \mu x^{\mathcal{S}^2} + \sigma z^{\mathcal{S}^2} = \mu x^{\mathcal{S}'^0} + \mu_e^k + \sigma z^{\mathcal{S}'^0} - \sigma_s^k + \sigma_{s'}^k.$$

Since $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$, $\mu_e^k = 0$ for demand k and edge e.
In a similar way, we can show that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and } e \in E \setminus (E_0^k \cup E_1^k).$$

Therefore all the equations of the polytope $\mathcal{P}(G, K, \mathbb{S})$ are given only in terms of the variables x_e^k with $e \in E_0^k \cup E_1^k$ and z_s^k with $s \in \{1, ..., w_k\}$. We distinguish 3 blocks of lines in the matrix Q associated with the system (2.13)

a) block Q^1 corresponds to the equations $x_e^k = 0$ for all $k \in K$ and $e \in E_0^k$ such that $rang(Q^1) = \sum_{k \in K} |E_0^k|$,

b) block Q^2 corresponds to the equations $x_e^k = 1$ for all $k \in K$ and $e \in E_1^k$ such that $rang(Q^2) = \sum_{k \in K} |E_1^k|$,

c) block Q^3 corresponds to the equations $z_s^k = 0$ for all $k \in K$ and $s \in \{1, ..., w_k - 1\}$ such that $rang(Q^3) = \sum_{k \in K} w_k - 1$.

Note that the 3 blocks of the matrix Q are independents. Let $Q^k = \begin{pmatrix} Q_k^1 \\ Q_k^2 \\ Q_k^3 \end{pmatrix}$ be the submatrix of matrix Q associated to the equations (2.12) and (2.11) and involving

variables x_e^k for all $e \in E_0^k \cup E_1^k$, and variables z_s^k with $s \in \{1, ..., w_k\}$ for demand k. Note that a forbidden edge can never be an essential edge at the same time. Otherwise, the problem is infeasible. Furthermore, there is no dependency between essential edges for each demand k and also for different demands in K. Same thing for the forbidden edges. We want to show that $\mu^k = \gamma_1^k Q_k^1 + \gamma_2^k Q_k^2$ and $\sigma^k = \gamma_3^k Q_3^k$. The only solution of these two systems is given by

$$\mu_e^k = \gamma_1^{k,e}, \text{ for all } k \in K \text{ and } e \in E_0^k, \tag{2.14}$$

$$\mu_e^k = \gamma_2^{k,e}, \text{ for all } k \in K \text{ and } e \in E_1^k, \tag{2.15}$$

$$\sigma_s^k = \gamma_3^{k,s}, \text{ for all } k \in K \text{ and } s \in \{1, ..., w_k - 1\}. \tag{2.16}$$

We conclude at the end that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e} & \text{if } e \in E_0^k \\ \gamma_2^{k,e} & \text{if } e \in E_1^k \\ 0 & otherwise, \end{cases} \tag{2.17}$$

yielding

$$\mu^k = \gamma_1^k Q_k^1 + \gamma_2^k Q_k^2 \text{ for each } k \in K.$$

Moreover, for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\} \\ 0 & \text{otherwise,} \end{cases} \tag{2.18}$$

i.e., $\sigma^k = \gamma_3^k Q_k^3$.
As a consequence, $(\mu, \sigma) = \gamma Q$ as desired. □

As a consequence, we have the following result.

Theorem 2.3.1. *The dimension of $\mathcal{P}(G, K, \mathbb{S})$ is given by*

$$dim(\mathcal{P}(G, K, \mathbb{S})) = |K| * (|E| + |\mathbb{S}|) - r.$$

2.3.2 Facial Investigation

In this section, we describe facets defining inequalities for the polytope $\mathcal{P}(G, K, \mathbb{S})$ from the cut formulation (2.2)-(2.10), and the ones from the valid inequalities. First, we characterize when the basic inequalities (2.2)-(2.10) define facets.

Theorem 2.3.2. *Consider a demand $k \in K$, and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Then, inequality $x_e^k \geq 0$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$.*

Proof. Let F_e^k be the face induced by inequality $x_e^k \geq 0$, that is

$$F_e^k = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : x_e^k = 0\}.$$

Denote inequality $x_e^k \geq 0$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G,K,\mathbb{S})$ and $F = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_e^k \subseteq F$. To prove that F_e^k is a facet of $\mathcal{P}(G,K,\mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.
First, we will show that $\sigma_s^{k'} = 0$ for all $k' \in K$ and $s \in \{w_{k'}, ..., \bar{s}\}$.
Consider a slot s in $\{w_k, ..., \bar{s}\}$, and solution $\mathcal{S}^3 = (E^3, S^3)$ such that

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^3$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let E_k^3 be the set of edges involved in a shortest path between o_k and d_k which does not use edge e,

c) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^3 given by

$$I_i^3 = [\bigcap_{k_j \in D_i^3} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s - w_k\} \cup \{s + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}^3 \cap E_k^3 \neq \emptyset \text{ or } I_i^3 = \bigcap_{k_j \in D_i^3} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^3 = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}^3 \cap E_{k_j}^3 \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^3$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ if $E_{k_i}^3 \cap E_k^3 \neq \emptyset$ (we take into account the possibility of adding slot s in the set of last slots S_k^3 assigned to demand k in solution $\mathcal{S}^3$).

We let $S_{k_i}^3 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^3$ is clearly feasible for the problem, and its incidence vector $(x^{\mathcal{S}^3}, z^{\mathcal{S}^3})$ belongs to F_e^k. Now, let $\mathcal{S}^4 = (E^4, S^4)$ a solution obtained from $\mathcal{S}^3$ by adding slot s as last slot to demand k. Solution $\mathcal{S}^4$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^4}, z^{\mathcal{S}^4})$ belongs to F_e^k. Hence, solutions $\mathcal{S}^3$ and $\mathcal{S}^4$ satisfy equation $\mu x + \sigma z = \tau$. As a consequence, we have

$$\mu x^{\mathcal{S}^3} + \sigma z^{\mathcal{S}^3} = \mu x^{\mathcal{S}^4} + \sigma z^{\mathcal{S}^4} = \mu x^{\mathcal{S}^3} + \sigma z^{\mathcal{S}^3} + \sigma_s^k.$$

Hence, $\sigma_s^k = 0$.
In a similar way, we can show that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \text{ and } s' \in \{w_{k'}, ..., \bar{s}\}.$$

Next, we will show that $\mu_{e'}^{k'} = 0$ for all demand $k' \in K \setminus \{k\}$ and edge $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$, and $\mu_{e'}^{k} = 0$ for edge $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$.
Consider an edge $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$ chosen arbitrarily. Let $\mathcal{S}'^3 = (E'^3, S'^3)$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^3$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let $E_k'^3$ be the set of edges involved in a shortest path between o_k and d_k which does not use edge e',

c) we select slot $s_k = w_k$ as last slot for demand k in solution $\mathcal{S}'^3$,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^3$ given by

$$I_i'^3 = [\bigcap_{k_j \in D_i'^3} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}'^3 \cap (E_k'^3 \cup \{e'\}) \neq \emptyset$ or $I_i'^3 = \bigcap_{k_j \in D_i'^3} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D_i'^3 = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}'^3 \cap E_{k_j}'^3 \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i'^3$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_k + 1, ..., s_k\} = \emptyset$ if $E_{k_i}'^3 \cap (E_k'^3 \cup \{e'\}) \neq \emptyset$ (we take into account the possibility of adding edge e in the set of edges $E_k'^3$ to route demand k).

We let $S_{k_i}'^3 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^3$ is clearly feasible for the problem, and its incidence vector $(x^{\mathcal{S}'^3}, z^{\mathcal{S}'^3})$ belongs to F_e^k. Let $\mathcal{S}^5 = (E^5, S^5)$ be the solution obtained from $\mathcal{S}'^3$ by adding edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}'^3$ that $E_k^5 = E_k'^3 \cup \{e'\}$, and removing slot s selected for demand k in $\mathcal{S}'^3$ and replaced it by a new slot $s' \in \{w_k, ..., \mathbb{S}\}$ (i.e., $S_k^5 = (S_k'^3 \setminus \{s\}) \cup \{s'\}$ such that $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S_{k'}'^3$ with $E_k^5 \cap E_{k'}'^3 \neq \emptyset$). $\mathcal{S}^5$ is clearly feasible for the problem.

The corresponding incidence vector $(x^{\mathcal{S}^5}, z^{\mathcal{S}^5})$ belongs to F_e^k. Hence, solutions $\mathcal{S}'^3$ and $\mathcal{S}^5$ satisfy equation $\mu x + \sigma z = \tau$. As a consequence, we have

$$\mu x^{\mathcal{S}'^3} + \sigma z^{\mathcal{S}'^3} = \mu x^{\mathcal{S}^5} + \sigma z^{\mathcal{S}^5} = \mu x^{\mathcal{S}'^3} + \mu_{e'}^k + \sigma z^{\mathcal{S}'^3} - \sigma_s^k + \sigma_{s'}^k.$$

Since $\sigma_s^k = 0$, it follows that $\mu_{e'}^k = 0$.
As e' is chosen arbitrarily, we have that

$$\mu_{e'}^k = 0, \text{ for all } e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\}).$$

Using similar argument as above, we can show that

$$\mu_{e'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } e' \in E \setminus (E_0^{k'} \cup E_1^{k'}).$$

By (2.17) and (2.18), we know that

$$\begin{cases} \mu_{e'}^{k'} = \gamma_1^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} = \gamma_2^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} = \gamma_3^{k',s'} & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

Overall, we obtain that

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'} & \text{if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'} & \text{if } e' \in E_1^{k'}, \\ \rho & \text{if } k' = k \text{ and } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $e' \in E$, and

$$\sigma_{s'}^{k'} = \begin{cases} \gamma_3^{k',s'} & \text{if } s' \in \{1, ..., w_{k'} - 1\}. \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $s' \in \mathbb{S}$.
Clearly, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

Theorem 2.3.3. *Consider a demand $k \in K$, and a slot $s \in \{w_k, .., \bar{s}\}$. Then, inequality $z_s^k \geq 0$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$.*

Proof. Let F_s^k denote the face induced by inequality $z_s^k \geq 0$, that is

$$F_s^k = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : z_s^k = 0\}.$$

Denote inequality $z_s^k \geq 0$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_s^k \subseteq F$. To prove that F_s^k is a facet of $\mathcal{P}(G, K, \mathbb{S})$, it suffices to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.
First, we will show that $\mu_{e'}^{k'} = 0$ for all demand $k' \in K$ and edge $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$. Consider an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^6 = (E^6, S^6)$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^6$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{s\}$ as last slot for demand k,

c) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^6 given by

$$I_i^6 = [\bigcap_{k_j \in D_i^6} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}^6 \cap (E_k^6 \cup \{e\}) \neq \emptyset$ or $I_i^6 = \bigcap_{k_j \in D_i^6} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D_i^6 = \{k_j \in \{k_1, ..., k_{i-1}\} : E_{k_i}^6 \cap E_{k_j}^6 \neq \emptyset\}$. This verifies that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^6$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_k + 1, ..., s_k\} = \emptyset$ if $E_{k_i}^6 \cap (E_k^6 \cup \{e\}) \neq \emptyset$ (we take into account the possibility of adding edge e in the set of edges E_k^6 to route demand k).

We let $S_{k_i}^6 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^6$ is clearly feasible for the problem, and its incidence vector $(x^{\mathcal{S}^6}, z^{\mathcal{S}^6})$ belongs to F_s^k. Based on this, we consider a solution $\mathcal{S}^7 = (E^7, S^7)$ obtained from $\mathcal{S}^6$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^6$ that $E_k^7 = E_k^6 \cup \{e\}$. $\mathcal{S}^7$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^7}, z^{\mathcal{S}^7})$ belongs to F_s^k. Hence, solutions $\mathcal{S}^6$ and $\mathcal{S}^7$ satisfy equation $\mu x + \sigma z = \tau$. As a consequence, we have

$$\mu x^{\mathcal{S}^6} + \sigma z^{\mathcal{S}^6} = \mu x^{\mathcal{S}^7} + \sigma z^{\mathcal{S}^7} = \mu x^{\mathcal{S}^6} + \mu_e^k + \sigma z^{\mathcal{S}^6}.$$

As a result, $\mu_e^k = 0$ for demand k and edge e.
In a similar way, we can show that

$$\mu_e^{k'} = 0, \text{ for all } k' \in K \text{ and } e \in E \setminus (E_0^{k'} \cup E_1^{k'}).$$

Next, we will show that, $\sigma_{s'}^{k'} = 0$ for all $k' \in K \setminus \{k\}$ and $s' \in \{w_{k'}, ..., \bar{s}\}$, and $\sigma_{s'}^{k} = 0$ for all slots $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$.
Consider a slot s' in $\{w_k, ..., \bar{s}\} \setminus \{s\}$. Let $\mathcal{S}'^6 = (E'^6, S'^6)$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^6$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{s, s'\}$ as last slot for demand k,

c) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^6$ given by

$$I_i'^6 = [\bigcap_{k_j \in D_i'^6} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}'^6 \cap E_k'^6 \neq \emptyset \text{ or } I_i'^6 = \bigcap_{k_j \in D_i'^6} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i'^6 = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}'^6 \cap E_{k_j}'^6 \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., \tilde{s}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i'^6$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., \tilde{s}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E_{k_i}'^6 \cap E_k'^6 \neq \emptyset$ (we take into account the possibility of adding slot s' in the set of last slots $\tilde{s}_k'^6$ assigned to demand k in solution $\mathcal{S}'^6$).

We let $S_{k_i}'^6 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^6$ is clearly feasible for the problem, its incidence vector $(x^{\mathcal{S}'^6}, z^{\mathcal{S}'^6})$ belongs to F_s^k. Then consider the solution $\mathcal{S}^8$ obtained from $\mathcal{S}'^6$ by adding slot s' as last slot to demand k. Solution $\mathcal{S}^8$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^8}, z^{\mathcal{S}^8})$ belongs to F_s^k. Hence, solutions $\mathcal{S}'^6$ and $\mathcal{S}^8$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^6} + \sigma z^{\mathcal{S}'^6} = \mu x^{\mathcal{S}^8} + \sigma z^{\mathcal{S}^8} = \mu x^{\mathcal{S}'^6} + \sigma z^{\mathcal{S}'^6} + \sigma_{s'}^k.$$

Hence, $\sigma_{s'}^k = 0$.
In a similar way, we can show that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \text{ and } s' \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \neq s' \text{ if } k = k'.$$

By (2.17) and (2.18), we know that

$$\begin{cases} \mu_{e'}^{k'} = \gamma_1^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} = \gamma_2^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} = \gamma_3^{k',s'} & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k' \in K$ and $e \in E$

$$\mu_e^{k'} = \begin{cases} \gamma_1^{k',e} & \text{if } e \in E_0^{k'}, \\ \gamma_2^{k',e} & \text{if } e' \in E_1^{k'}, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k' \in K$ and $s' \in \mathbb{S}$

$$\sigma_{s'}^{k'} = \begin{cases} \gamma_3^{k',s'} & \text{if } s' \in \{1, ..., w_{k'} - 1\}, \\ 0 & \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \text{ and } k' \neq k, \\ 0 & \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \setminus \{s\} \text{ and } k' = k, \\ \rho & \text{if } s' = s \text{ and } k' = k. \end{cases}$$

Clearly, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

Proposition 2.3.3. *Consider a demand $k \in K$. Let (e, e') be a pair of non-compatible edges for demand k. Then, the inequality*

$$x_e^k + x_{e'}^k \leq 1, \tag{2.19}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It is trivial due to the transmission-reach constraint and given the definition of non-compatible edges for demand k. □

Based on the definition of a non-compatible demands for an edge e, we introduce the following inequality.

Proposition 2.3.4. *Consider an edge $e \in E$. Let (k, k') be a pair of non-compatible demands for edge e with $e \notin E_0^k \cup E_1^k \cup E_0^{k'} \cup E_1^{k'}$. Then, the inequality*

$$x_e^k + x_e^{k'} \leq 1, \tag{2.20}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of non-compatible demands for edge e. □

Theorem 2.3.4. *Consider a demand $k \in K$, and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Then, inequality $x_e^k \leq 1$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) there does not exist a demand $k' \in K \setminus \{k\}$ such that the two demands k and k' are non-compatible demands for edge e,

b) and there does not exist an edge $e' \in E \setminus (E_1^k \cup E_0^k \cup \{e\})$ such that the two edges e and e' are non-comptible edges for demand k.

Proof. **Neccessity.**
For demand k and edge $e \in E \setminus (E_0^k \cup E_1^k)$, if

a) there exists a demand $k' \in K \setminus \{k\}$ such that the two demands k and k' are non-compatible demands for edge e. Then, inequality $x_e^k \leq 1$ is dominated by inequality (2.20).

b) there exists an edge $e' \in E \setminus (E_1^k \cup E_0^k \cup \{e\})$ such that the two edges e and e' are non-comptible edges for demand k. Then, inequality $x_e^k \leq 1$ is dominated by inequality (2.19).

As a result, inequality $x_e^k \leq 1$ is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.
Sufficiency.
Let $F_e'^k$ denote the face induced by inequality $x_e^k \leq 1$, that is

$$F_e'^k = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : x_e^k = 1\}.$$

Denote inequality $x_e^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_e'^k \subseteq F$. To prove that $F_e'^k$ is a facet of $\mathcal{P}(G, K, \mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.
First, we will show that $\sigma_s^{k'} = 0$ for all $k' \in K$ and $s \in \{w_{k'}, ..., \bar{s}\}$.
Consider a slot s in $\{w_k, ..., \bar{s}\}$. Let $\mathcal{S}^9 = (E^9, S^9)$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^9$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let E_k^9 be the set of edges involved in a shortest path between o_k and d_k which uses edge e,

c) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{s\}$ as last slot for demand k in solution $\mathcal{S}^9$,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^9 given by

$$I_i^9 = [\bigcap_{k_j \in D_i^9} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s - w_k\} \cup \{s + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}^9 \cap E_k^9 \neq \emptyset \text{ or } I_i^9 = \bigcap_{k_j \in D_i^9} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not.}$$

where $D_i^9 = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}^9 \cap E_{k_j}^9 \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^9$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ (we take into account the possibility of adding slot s in the set of last slots S_k^9 assigned to demand k in solution $\mathcal{S}^9$).

We let $S_{k_i}^9 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^9$ is clearly feasible for the problem, and its incidence vector $(x^{\mathcal{S}^9}, z^{\mathcal{S}^9})$ belongs to $F_e'^k$. Then consider the solution $\mathcal{S}^{10} = (E^{10}, S^{10})$ obtained from $\mathcal{S}^9$ by adding slot s as last slot to demand k. Solution $\mathcal{S}^{10}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{10}}, z^{\mathcal{S}^{10}})$ belongs to $F_e'^k$. Hence, solutions $\mathcal{S}^9$ and $\mathcal{S}^{10}$ satisfy equation $\mu x + \sigma z = \tau$. We then obtain that

$$\mu x^{\mathcal{S}^9} + \sigma z^{\mathcal{S}^9} = \mu x^{\mathcal{S}^{10}} + \sigma z^{\mathcal{S}^{10}} = \mu x^{\mathcal{S}^9} + \sigma z^{\mathcal{S}^9} + \sigma_s^k.$$

Hence, $\sigma_s^k = 0$.
In a similar way, we can show that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \text{ and } s \in \{w_{k'}, ..., \bar{s}\}$$

Next, we will show that $\mu_{e'}^{k'} = 0$ for all demand $k' \in K \setminus \{k\}$ and $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$, and $\mu_{e'}^k = 0$ for demand k and $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$.
Consider an edge $e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\})$ chosen arbitrarily. Let $\mathcal{S}'^9 = (E'^9, S'^9)$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^9$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let $E_k'^9$ be the set of edges involved in a shortest path between o_k and d_k which uses edge e, and edge e' is compatible-edge with all the selected edges $e" \in E_k'^9$ in solution $\mathcal{S}'^9$, i.e., $\sum_{e" \in E_k'^9} l_{e"} + \ell_{e'} \leq \bar{l}_k$.

c) we select the slot $s_k = w_k$ as last slot for demand k,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^9$ given by

$$I_i'^9 = \bigcap_{k_j \in D_i'^9} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if } E_{k_i}'^9 \cap (E_k'^9 \cup \{e'\}) \neq \emptyset$$

$$\text{or } I_i'^9 = \bigcap_{k_j \in D_i'^9} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i'^9 = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}'^9 \cap E_{k_j}'^9 \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i'^9$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_k + 1, ..., s_k\} = \emptyset$ if $E_{k_i}'^9 \cap (E_k'^9 \cup \{e'\}) \neq \emptyset$ (we take into account the possibility of adding edge e' in the selected path $E_k'^9$ to route demand k in solution $\mathcal{S}'^9$).

We let $S_{k_i}'^9 = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^9$ is clearly feasible for the problem, and its incidence vector $(x^{\mathcal{S}'^9}, z^{\mathcal{S}'^9})$ belongs to $F_e'^k$. Let $\mathcal{S}^{11} = (E^{11}, S^{11})$ be the solution obtained from solution $\mathcal{S}'^9$ by adding edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}'^9$ which means that $E_k^{11} = E_k'^9 \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}'^9$ remain the same in solution $\mathcal{S}^{11}$, i.e., $S_k^{11} = S_k'^9$ for each $k \in K$, and $E_{k'}^{11} = E_{k'}'^9$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{11}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{11}}, z^{\mathcal{S}^{11}})$ belongs to $F_e'^k$. Hence, solutions $\mathcal{S}'^9$ and $\mathcal{S}^{11}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}'^9} + \sigma z^{\mathcal{S}'^9} = \mu x^{\mathcal{S}^{11}} + \sigma z^{\mathcal{S}^{11}} = \mu x^{\mathcal{S}'^9} + \mu_{e'}^k + \sigma z^{\mathcal{S}'^9}.$$

Hence, $\mu_{e'}^k = 0$.
In a similar way, we can show that

$$\mu_{e'}^k = 0, \text{ for all } e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\}).$$

Moreover, we re-do the same procedure for all $k' \in K \setminus \{k\}$ and $e' \in E \setminus (E_0^k \cup E_1^k)$. We conclude at the end that

$$\mu_{e'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } e' \in E \setminus (E_0^{k'} \cup E_1^{k'}),$$
$$\mu_{e'}^k = 0, \text{ for all } e' \in E \setminus (E_0^k \cup E_1^k \cup \{e\}).$$

We know from (2.17) and (2.18) that

$$\begin{cases} \mu_{e'}^{k'} = \gamma_1^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} = \gamma_2^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} = \gamma_3^{k',s'} & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'} & \text{if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'} & \text{if } e' \in E_1^{k'}, \\ \rho & \text{if } k' = k \text{ and } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k' \in K$ and $s \in \mathbb{S}$

$$\sigma_s^{k'} = \begin{cases} \gamma_3^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$ which ends the proof. □

Theorem 2.3.5. *Consider a demand $k \in K$, and a slot $s \in \{w_k, .., \bar{s}\}$. Then, inequality $z_s^k \leq 1$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if there does not exist a demand $k' \in K \setminus \{k\}$ with $E_1^k \cap E_1^{k'} \neq \emptyset$.*

Proof. **Neccessity.**

For a demand $k \in K$ and a slot $s \in \{w_k, .., \bar{s}\}$, if there exists a demand $k' \in K \setminus \{k\}$ with $E_1^k \cap E_1^{k'} \neq \emptyset$. Then, the inequality $z_s^k \leq 1$ is domined by the non-overlapping inequality (2.6) for each edge $e \in E_1^k \cap E_1^{k'}$. As a result, the inequality $z_s^k \leq 1$ is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_s'^k$ be the face induced by inequality $z_s^k \leq 1$, that is

$$F_s'^k = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : z_s^k = 1\}.$$

We denote inequality $z_s^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_s'^k \subseteq F$. To prove that $F_s'^k$ is a facet of $\mathcal{P}(G, K, \mathbb{S})$, we need to show there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

First, we will show that $\mu_e^{k'} = 0$ for all demand $k' \in K$ and edge $e \in E \setminus (E_0^{k'} \cup E_1^{k'})$. Consider an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^{12} = (E^{12}, S^{12})$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E^{12}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let E^{12}_k be the set of edges involved in a shortest path between o_k and d_k which does not use edge e, and edge e' is compatible-edge with all the selected edges $e" \in E^{12}_k$, i.e., $\sum_{e" \in E^{12}_k} l_{e"} + \ell_{e'} \leq \bar{l}_k$.

c) we select slot $s_k = s$ as last slot for demand k, and we let $S^{12}_k = \{s\}$,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{12}_i given by

$$I^{12}_i = [\bigcap_{k_j \in D^{12}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

if $E^{12}_{k_i} \cap (E^{12}_k \cup \{e\}) \neq \emptyset$ or $I^{12}_i = \bigcap_{k_j \in D^{12}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D^{12}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E^{12}_{k_i} \cap E^{12}_{k_j} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D^{12}_i$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_k + 1, ..., s_k\} = \emptyset$ if $E^{12}_{k_i} \cap (E^{12}_k \cup \{e\}) \neq \emptyset$ (we take into account the possibility of adding edge e in the selected path E^{12}_k to route demand k in solution $\mathcal{S}^{12}$).

We let $S^{12}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{12}$ is clearly feasible for the problem, its incidence vector $(x^{\mathcal{S}^{12}}, z^{\mathcal{S}^{12}})$ belongs to F'^k_s. Then consider the solution $\mathcal{S}^{13} = (E^{13}, S^{13})$ obtained from $\mathcal{S}^{12}$ by adding edge $e \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in solution $\mathcal{S}^{12}$ which means that $E^{13}_k = E^{12}_k \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{12}$ remain the same in solution $\mathcal{S}^{13}$, i.e., $S^{13}_k = S^{12}_k$ for each $k \in K$, and $E^{13}_{k'} = E^{12}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{13}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{13}}, z^{\mathcal{S}^{13}})$ belongs to F'^k_s. Hence, solutions $\mathcal{S}^{12}$ and $\mathcal{S}^{13}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{12}} + \sigma z^{\mathcal{S}^{12}} = \mu x^{\mathcal{S}^{13}} + \sigma z^{\mathcal{S}^{13}} = \mu x^{\mathcal{S}^{12}} + \mu^k_e + \sigma z^{\mathcal{S}^{12}}.$$

As a result, $\mu^k_e = 0$.
In a similar way, we can show that

$$\mu^{k'}_e = 0, \text{ for all } k' \in K \text{ and } e \in E \setminus (E^{k'}_0 \cup E^{k'}_1).$$

Next, we will show that, $\sigma_{s'}^{k'} = 0$ for all $k' \in K \setminus \{k\}$ and $s' \in \{w_{k'}, ..., \bar{s}\}$, and $\sigma_{s'}^{k} = 0$ for all slots $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$.
Consider a slot s' in $\{w_k, ..., \bar{s}\} \setminus \{s\}$. Let $\mathcal{S}'^{12} = (E'^{12}, S'^{12})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E'^{12}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) we select slot $s_k = s$ as last slot for demand k, and we let $S'^{12}_k = \{s\}$,

c) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{12}_i given by

$$I'^{12}_i = [\bigcap_{k_j \in D'^{12}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E'^{12}_{k_i} \cap E'^{12}_k \neq \emptyset \text{ or } I'^{12}_i = \bigcap_{k_j \in D'^{12}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not.}$$

where $D'^{12}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E'^{12}_{k_i} \cap E'^{12}_{k_j} \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D'^{12}_i$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E'^{12}_{k_i} \cap E'^{12}_k \neq \emptyset$ (we take into account the possibility of adding slot s' in the selected last slots S'^{12}_k to route demand k in solution $\mathcal{S}'^{12}$).

We let $S'^{12}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{12}$ is clearly feasible for the problem, its incidence vector $(x^{\mathcal{S}'^{12}}, z^{\mathcal{S}'^{12}})$ belongs to F'^{k}_{s}. Then we derive solution $\mathcal{S}^{14}$ from $\mathcal{S}'^{12}$ by adding slot s' as last slot to demand k in $\mathcal{S}'^{12}$. Solution $\mathcal{S}^{14}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{14}}, z^{\mathcal{S}^{14}})$ belongs to F'^{k}_{s}. Hence, solutions $\mathcal{S}'^{12}$ and $\mathcal{S}^{14}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{12}} + \sigma z^{\mathcal{S}'^{12}} = \mu x^{\mathcal{S}^{14}} + \sigma z^{\mathcal{S}^{14}} = \mu x^{\mathcal{S}'^{12}} + \sigma z^{\mathcal{S}'^{12}} + \sigma_{s'}^{k}.$$

Hence, $\sigma_{s'}^{k} = 0$.
In a similar way, we can show that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \text{ and } s' \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \neq s' \text{ if } k = k'.$$

By (2.17) and (2.18), we know that

$$\begin{cases} \mu_{e'}^{k'} = \gamma_1^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} = \gamma_2^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} = \gamma_3^{k',s'} & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

Overall, we obtain that

$$\mu_e^{k'} = \begin{cases} \gamma_1^{k',e} & \text{if } e \in E_0^{k'}, \\ \gamma_2^{k',e} & \text{if } e' \in E_1^{k'}, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $e \in E$, and

$$\sigma_{s'}^{k'} = \begin{cases} \gamma_3^{k',s'} & \text{if } s' \in \{1, ..., w_{k'} - 1\}, \\ 0 & \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \text{ and } k' \neq k, \\ 0 & \text{if } s' \in \{w_{k'}, ..., \bar{s}\} \setminus \{s\} \text{ and } k' = k, \\ \rho & \text{if } s' = s \text{ and } k' = k, \end{cases}$$

for each $k' \in K$ and $s' \in \mathbb{S}$.
As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

Theorem 2.3.6. *Consider a demand $k \in K$. Then, inequality* (2.5), $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$, *is facet defining for $\mathcal{P}(G, K, \mathbb{S})$.*

Proof. Let $F_{\mathbb{S}}^k$ be the face induced by inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$, that is

$$F_{\mathbb{S}}^k = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{s=w_k}^{\bar{s}} z_s^k = 1\}.$$

Denote inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a valid inequality that defines a facet F of $\mathcal{P}(G, K, \mathbb{S})$. Suppose that $F_{\mathbb{S}}^k \subseteq F$. To prove that $F_{\mathbb{S}}^k$ is a facet of $\mathcal{P}(G, K, \mathbb{S})$, we show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.
First, we will show that $\mu_e^{k'} = 0$ for all demand $k' \in K$ and edge $e \in E \setminus (E_0^{k'} \cup E_1^{k'})$. Consider an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^{15} = (E^{15}, S^{15})$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{15}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let E_k^{15} be the set of edges involved in a shortest path between o_k and d_k such that edge e is compatible-edge with all the selected edges $e" \in E_k^{15}$, i.e., $\sum_{e" \in E_k^{15}} l_{e"} + l_e \leq \bar{l}_k$,

c) we select slot $s_k = w_k$ as last slot for demand k, and let $S_k^{15} = \{s_k\}$,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{15} given by

$$I_i^{15} = [\bigcap_{k_j \in D_i^{15}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}^{15} \cap (E_k^{15} \cap \{e\}) \neq \emptyset$ or $I_i^{15} = \bigcap_{k_j \in D_i^{15}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D_i^{15} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}^{15} \cap E_{k_j}^{15} \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{15}$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ if $E_{k_i}^{15} \cap (E_k^{15} \cup \{e\}) \neq \emptyset$ (we take into account the possibility of adding edge e in the selected path E_k^{15} to route demand k in solution $\mathcal{S}^{15}$).

We let $S_{k_i}^{15} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{15}$ is feasible for the problem, its incidence vector $(x^{\mathcal{S}^{15}}, z^{\mathcal{S}^{15}})$ belongs to $F_{\mathbb{S}}^k$. Then we derive a solution $\mathcal{S}^{16} = (E^{15}, S^{15})$ obtained from $\mathcal{S}^{15}$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{15}$ which means that $E_k^{16} = E_k^{15} \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{15}$ remain the same in solution $\mathcal{S}^{16}$, i.e., $S_k^{16} = S_k^{15}$ for each $k \in K$, and $E_{k'}^{16} = E_{k'}^{15}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{16}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{16}}, z^{\mathcal{S}^{16}})$ belongs to $F_{\mathbb{S}}^k$. Hence, solutions $\mathcal{S}^{15}$ and $\mathcal{S}^{16}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{15}} + \sigma z^{\mathcal{S}^{15}} = \mu x^{\mathcal{S}^{16}} + \sigma z^{\mathcal{S}^{16}} = \mu x^{\mathcal{S}^{15}} + \mu_e^k + \sigma z^{\mathcal{S}^{15}}.$$

As a result, $\mu_e^k = 0$.
In a similar way, we can show that

$$\mu_e^{k'} = 0, \text{ for all } k' \in K \text{ and } e \in E \setminus (E_0^{k'} \cup E_1^{k'}).$$

Next, we will show that, $\sigma_{s'}^{k'} = 0$ for all $k' \in K \setminus \{k\}$ and $s' \in \{w_{k'}, ..., \bar{s}\}$.
Consider a demand k' in $K \setminus \{k\}$ and a slot s' in $\{w_{k'}, ..., \bar{s}\} \setminus \{s\}$. Let $\mathcal{S}'^{15} = (E'^{15}, S'^{15})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E'^{15}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) we select slot $s_k = w_k$ as last slot for demand k, and let $S'^{15}_k = \{s_k\}$,

c) we select the smallest slot index $s_{k'}$ from the set of slots $I'^{15}_{k'}$ given by

$$I'^{15}_{k'} = \{w_{k_i}, ..., s_k - w_k\} \cap \{s_k + w_{k_i}, ..., \bar{s}\} \setminus \{s'\} \text{ if } E'^{15}_{k'} \cap E'^{15}_k \neq \emptyset \text{ or } I'^{15}_{k'} = \{w_{k'}, ..., \bar{s}\} \setminus \{s'\} \text{ if not.}$$

d) for each demand $k_i \in K \setminus \{k, k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{15}_i given by

$$I'^{15}_i = [\bigcap_{k_j \in D'^{15}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E'^{15}_{k_i} \cap E'^{15}_{k'} \neq \emptyset \text{ or } I'^{15}_i = \bigcap_{k_j \in D'^{15}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D'^{15}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k, k'\} : E'^{15}_{k_i} \cap E'^{15}_{k_j} \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D'^{15}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s\} = \emptyset$ if $E'^{15}_{k_i} \cap E'^{15}_{k'} \neq \emptyset$ (we take into account the possibility of adding slot s' in the set of last slots $S'^{15}_{k'}$ to route demand k' in solution $\mathcal{S}'^{15}$).

We let $S'^{15}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{15}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}'^{15}}, z^{\mathcal{S}'^{15}})$ belongs to $F^k_{\mathbb{S}}$. Then we derive a solution $\mathcal{S}^{17}$ from $\mathcal{S}'^{15}$ by adding slot s' as last slot to demand k'. Solution $\mathcal{S}^{17}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{17}}, z^{\mathcal{S}^{17}})$ belongs to $F^k_{\mathbb{S}}$. Hence, solutions $\mathcal{S}'^{15}$ and $\mathcal{S}^{17}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{15}} + \sigma z^{\mathcal{S}'^{15}} = \mu x^{\mathcal{S}^{17}} + \sigma z^{\mathcal{S}^{17}} = \mu x^{\mathcal{S}'^{15}} + \sigma z^{\mathcal{S}'^{15}} + \sigma^{k'}_{s'}.$$

Hence, $\sigma^{k'}_{s'} = 0$.

In a similar way, we can show that

$$\sigma^{k'}_{s'} = 0, \text{ for all } k' \in K \text{ and } s' \in \{w_{k'}, ..., \bar{s}\}.$$

Let prove now that σ^k_s for demand k and slot s in $\{w_k, ..., \bar{s}\}$ are equivalent. Consider a slot $s' \in \{w_k, ..., \bar{s}\}$. Let $\tilde{\mathcal{S}}^{15} = (\tilde{E}^{15}, \tilde{S}^{15})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $\tilde{E}^{15}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) we select the smallest slot index s_k from $\{w_k, ..., \bar{s}\} \setminus \{s'\}$ as last slot for demand k,

c) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}^{15}_i$ given by

$$\tilde{I}^{15}_i = [\bigcap_{k_j \in \tilde{D}^{15}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } \tilde{E}^{15}_{k_i} \cap \tilde{E}^{15}_k \neq \emptyset \text{ or } \tilde{I}^{15}_i = \bigcap_{k_j \in \tilde{D}^{15}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $\tilde{D}^{15}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : \tilde{E}^{15}_{k_i} \cap \tilde{E}^{15}_{k_j} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in \tilde{D}^{15}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $\tilde{E}^{15}_{k_i} \cap \tilde{E}^{15}_k \neq \emptyset$ (we take into account the possibility of adding slot s' in the set of last slots $\tilde{S}^{15}_k$ to route demand k in solution $\tilde{\mathcal{S}}^{15}$).

We let $\tilde{S}^{15}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\tilde{\mathcal{S}}^{15}$ is clearly feasible for the problem, and its incidence vector $(x^{\tilde{\mathcal{S}}^{15}}, z^{\tilde{\mathcal{S}}^{15}})$ belongs to $F^k_{\tilde{S}}$. Then consider the solution $\mathcal{S}^{18}$ obtained from $\tilde{\mathcal{S}}^{15}$ by adding slot s' as last slot to demand k' in $\mathcal{S}^{18}_k$ and removing the last slot s assigned to k in $\tilde{S}^{15}_k$ (i.e., $S^{18}_k = (\tilde{S}^{15}_k \setminus \{s\}) \cup \{s'\}$ for demand k). Solution $\mathcal{S}^{18}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{18}}, z^{\mathcal{S}^{18}})$ belongs to $F^k_{\tilde{S}}$. Hence, solutions $\tilde{\mathcal{S}}^{15}$ and $\mathcal{S}^{18}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{15}} + \sigma z^{\tilde{\mathcal{S}}^{15}} = \mu x^{\mathcal{S}^{18}} + \sigma z^{\mathcal{S}^{18}} = \mu x^{\tilde{\mathcal{S}}^{15}} + \sigma z^{\tilde{\mathcal{S}}^{15}} + \sigma^k_{s'} - \sigma^k_s.$$

As a result, $\sigma^k_{s'} = \sigma^k_s$.
In a similar way, we can show that

$$\sigma^k_{s'} = \sigma^k_s, \text{ for all slots } s, s' \in \{w_k, ..., \bar{s}\}.$$

Consequently, we obtain that $\sigma^k_s = \rho$ for demand k and slot s in $\{w_k, ..., \bar{s}\}$.
We know from (2.17) and (2.18) that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k' \in K$ and $e \in E$

$$\mu_e^{k'} = \begin{cases} \gamma_1^{k',e} & \text{if } e \in E_0^{k'}, \\ \gamma_2^{k',e} & \text{if } e \in E_1^{k'}, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k' \in K$ and $s \in \mathbb{S}$

$$\sigma_s^{k'} = \begin{cases} \gamma_3^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ \rho & \text{if } k' = k \text{ and } s \in \{w_{k'}, ..., \bar{s}\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

Theorem 2.3.7. *Consider a demand k in K and a subset of node $X \subset V$, with $|X \cap \{o_k, d_k\}| = 1$ and $\delta(X) \cap E_1^k = \emptyset$. Then, inequality (2.2), $\sum_{e \in \delta(X)} x_e^k \geq 1$, is facet defining for $\mathcal{P}(G, K, \mathbb{S})$.*

Proof. Let F_X^k denote the face induced by inequality $\sum_{e \in (\delta(X) \setminus E_0^k)} x_e^k \geq 1$, that is

$$F_X^k = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{e \in (\delta(X) \setminus E_0^k)} x_e^k = 1\}.$$

Let $X = \{o_k\}$. Denote inequality $\sum_{e \in (\delta(X) \setminus E_0^k)} x_e^k \geq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_X^k \subseteq F$. To prove that F_X^k is a facet of $\mathcal{P}(G, K, \mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (with $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.
First, we will show that $\sigma_s^{k'} = 0$ for all $k' \in K$ and $s \in \{w_{k'}, ..., \bar{s}\}$.
Consider a slot s in $\{w_k, ..., \bar{s}\}$. Let $\mathcal{S}^{19} = (E^{19}, S^{19})$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{19}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let E_k^{19} be the set of edges involved in a shortest path between o_k and d_k. This guarantees that one edge e from $(\delta(X) \setminus E_0^k)$ is chosen to route demand k, i.e., $|(\delta(X) \setminus E_0^k) \cap E_k^{19}| = 1$,

c) for demand k, we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{s\}$ as last slot for demand k,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{19} given by

$$I_i^{19} = [\bigcap_{k_j \in D_i^{19}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s - w_k\} \cup \{s + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}^{19} \cap E_k^{19} \neq \emptyset \text{ or } I_i^{19} = \bigcap_{k_j \in D_i^{19}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{19} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}^{19} \cap E_{k_j}^{19} \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{19}$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ if $E_{k_i}^{19} \cap E_k^{19} \neq \emptyset$ (we take into account the possibility of adding slot s in the set of last slots S_k^{19} to route demand k in solution $\mathcal{S}^{19}$).

We let $S_{k_i}^{19} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{19}$ is clearly feasible for the problem, and its incidence vector $(x^{\mathcal{S}^{19}}, z^{\mathcal{S}^{19}})$ belongs to F_X^k. Then consider the solution $\mathcal{S}^{20} = (E^{20}, S^{20})$ obtained from $\mathcal{S}^{19}$ by adding slot s as last slot to demand k without modifying the paths assigned to the demands K in $\mathcal{S}^{19}$ (i.e., $E_k^{20} = E_1^{19}$ for each $k \in K$), and the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{19}$ remain the same in solution $\mathcal{S}^{20}$ i.e., $S_{k'}^{19} = S_{k'}^{20}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{20} = S_k^{19} \cup \{s\}$ for demand k. Solution $\mathcal{S}^{20}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{20}}, z^{\mathcal{S}^{20}})$ belongs to F_X^k. Hence, solutions $\mathcal{S}^{19}$ and $\mathcal{S}^{20}$ satisfy equation $\mu x + \sigma z = \tau$. We then obtain that

$$\mu x^{\mathcal{S}^{19}} + \sigma z^{\mathcal{S}^{19}} = \mu x^{\mathcal{S}^{20}} + \sigma z^{\mathcal{S}^{20}} = \mu x^{\mathcal{S}^{19}} + \sigma z^{\mathcal{S}^{19}} + \sigma_s^k.$$

Hence, $\sigma_s^k = 0$.
In a similar way, we can show that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \text{ and } s \in \{w_{k'}, ..., \bar{s}\}.$$

Next, we will show that $\mu_{e'}^k = 0$ for all demand $k' \in K \setminus \{k\}$ and $e' \in E \setminus (E_0^{k'} \cup E_1^{k'})$, and $\mu_{e'}^k = 0$ for demand k and $e' \in E \setminus (E_0^k \cup E_1^k \cup \delta(X))$.
Consider an edge $e' \in E \setminus (E_0^k \cup E_1^k \cup \delta(X))$ chosen arbitrarily. Let $\mathcal{S}'^{19} = (E'^{19}, S'^{19})$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^{19}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let $E_k'^{19}$ be the set of edges involved in a shortest path between o_k and d_k such that edge e' is compatible-edge with the selected edges $e" \in E_k'^{19}$, i.e., $\sum_{e" \in E_k'^{19}} l_{e"} + \ell_{e'} \leq \bar{l}_k$.

c) we select the slot $s_k = w_k$ as last slot for demand k, and let $S_k'^{19} = \{s_k\}$,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^{19}$ given by

$$I_i'^{19} = [\bigcap_{k_j \in D_i'^{19}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}'^{19} \cap (E_k'^{19} \cup \{e'\}) \neq \emptyset$ or $I_i'^{19} = \bigcap_{k_j \in D_i'^{19}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D_i'^{19} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}'^{19} \cap E_{k_j}'^{19} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i'^{19}$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E_{k_i}'^{19} \cap (E_k'^{19} \cup \{e\}) \neq \emptyset$ (we take into account the possibility of adding edge e' in the selected path $E_k'^{19}$ to route demand k in solution $\mathcal{S}'^{19}$).

We let $S_{k_i}'^{19} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{19}$ is feasible for the problem. its incidence vector $(x^{\mathcal{S}'^{19}}, z^{\mathcal{S}'^{19}})$ belongs to F_X^k. Let $\mathcal{S}^{21} = (E^{21}, S^{21})$ be the solution obtained from $\mathcal{S}'^{19}$ by adding edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}'^{19}$ which means that $E_k^{21} = E_k'^{19} \cup \{e'\}$. $\mathcal{S}^{21}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{21}}, z^{\mathcal{S}^{21}})$ belongs to F_X^k. Hence, solutions $\mathcal{S}'^{19}$ and $\mathcal{S}^{21}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}'^{19}} + \sigma z^{\mathcal{S}'^{19}} = \mu x^{\mathcal{S}^{21}} + \sigma z^{\mathcal{S}^{21}} = \mu x^{\mathcal{S}'^{19}} + \mu_{e'}^k + \sigma z^{\mathcal{S}'^{19}}.$$

Hence, $\mu_{e'}^k = 0$.

In a similar way, we can show that

$$\mu_{e'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } e' \in E \setminus (E_0^{k'} \cup E_1^{k'}),$$
$$\mu_{e'}^k = 0, \text{ for all } e' \in E \setminus (E_0^k \cup E_1^k \cup \delta(X)).$$

Next, we will prove that the μ_e^k for all edge $e \in (\delta(X) \setminus E_0^k)$ are equivalent.

Consider an edge $e' \in (\delta(X) \setminus E_0^k)$ such that $e' \notin E_k^{19}$. Let $\tilde{\mathcal{S}}^{19} = (\tilde{E}^{19}, \tilde{S}^{19})$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $\tilde{E}^{19}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let $\tilde{E}^{19}_{k}$ be the set of edges involved in a shortest path between o_k and d_k. This guarantees that one edge e from $(\delta(X) \setminus E^k_0)$ is chosen to route demand k, i.e., $|(\delta(X) \setminus E^k_0) \cap \tilde{E}^{19}_{k}| = 1$,

c) we select the slot $s_k = w_k$ as last slot for demand k , and let $\tilde{S}^{19}_{k} = \{s_k\}$,

d) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}^{19}_{i}$ given by

$$\tilde{I}^{19}_{i} = [\bigcap_{k_j \in \tilde{D}^{19}_{i}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

if $\tilde{E}^{19}_{k_i} \cap (\tilde{E}^{19}_{k} \cup \{e'\}) \neq \emptyset$ or $\tilde{I}^{19}_{i} = \bigcap_{k_j \in \tilde{D}^{19}_{i}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not.

where $\tilde{D}^{19}_{i} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : \tilde{E}^{19}_{k_i} \cap \tilde{E}^{19}_{k_j} \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in \tilde{D}^{19}_{i}$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $\tilde{E}^{19}_{k_i} \cap (\tilde{E}^{19}_{k} \cup \{e\}) \neq \emptyset$ (we take into account the possibility of using edge e' in the selected path $\tilde{E}^{19}_{k}$ to route demand k in solution $\tilde{\mathcal{S}}^{19}$).

We let $\tilde{S}^{19}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\tilde{\mathcal{S}}^{19}$ is clearly feasible for the problem, and corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{19}}, z^{\tilde{\mathcal{S}}^{19}})$ belongs to F^k_X. Then consider the solution $\mathcal{S}^{22}$ obtained from $\tilde{\mathcal{S}}^{19}$ by modifying the path assigned to demand k in $\tilde{\mathcal{S}}^{19}$ from $\tilde{E}^{19}_{k}$ to a path E^{22}_{k} passed through edge e' with $|(\delta(X) \setminus E^k_0) \cap E^{22}_{k}| = 1$, and modifying the last slots assigned to some demands $\tilde{K} \subset K$ from $\tilde{S}^{19}_{\tilde{k}}$ to $S^{22}_{\tilde{k}}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint. The paths assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{19}$ remain the same in $\mathcal{S}^{22}$ (i.e., $E^{22}_{k"} = \tilde{E}^{19}_{k"}$ for each $k" \in K \setminus \{k\}$), and also without modifying the last slots assigned to the demands $K \setminus \tilde{K}$ in $\tilde{\mathcal{S}}^{19}$, i.e., $\tilde{S}^{19}_{k} = S^{22}_{k}$ for each demand $k \in K \setminus \tilde{K}$. Solution $\mathcal{S}^{22}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{22}}, z^{\mathcal{S}^{22}})$ belongs to F^k_X. Hence, solutions $\tilde{\mathcal{S}}^{19}$ and $\mathcal{S}^{22}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{19}} + \sigma z^{\tilde{\mathcal{S}}^{19}} = \mu x^{\mathcal{S}^{22}} + \sigma z^{\mathcal{S}^{22}} = \mu x^{\tilde{\mathcal{S}}^{19}} + \sigma z^{\tilde{\mathcal{S}}^{19}} + \mu^k_{e'} - \mu^k_e + \sum_{\tilde{k} \in \tilde{K}} \sum_{s' \in S^{22}_{\tilde{k}}} \sigma^{\tilde{k}}_{s'} - \sum_{s \in \tilde{S}^{19}_{\tilde{k}}} \sigma^{\tilde{k}}_{s}$$
$$+ \sum_{e" \in E^{22}_{k} \setminus \{e'\}} \mu^k_{e"} - \sum_{e" \in \tilde{E}^{19}_{k} \setminus \{e\}} \mu^k_{e"}.$$

Since $\mu^k_{e"} = 0$ for all $e" \in E \setminus (E^k_0 \cup E^k_1 \cup \delta(X))$, and $\sigma^{k'}_s = 0$ for all $k' \in K$ and $s \in \{w_{k'}, ..., \bar{s}\}$, it follows that $\mu^k_{e'} = \mu^k_e$.
In a similar way, we can show that

$$\mu^k_e = \mu^k_{e'}, \text{for all pairs } e, e' \in (\delta(X) \setminus E^k_0).$$

Consequently, we obtain that $\mu^k_e = \rho$ for all $e \in (\delta(X) \setminus E^k_0)$.
By (2.17) and (2.18), we know that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k' \in K$ and $e \in E$

$$\mu^{k'}_e = \begin{cases} \gamma^{k',e}_1 & \text{if } e \in E^{k'}_0, \\ \gamma^{k',e}_2 & \text{if } e \in E^{k'}_1, \\ \rho & \text{if } k = k' \text{ and } e \in (\delta(X) \setminus E^k_0), \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k' \in K$ and $s \in \mathbb{S}$

$$\sigma^{k'}_s = \begin{cases} \gamma^{k',s}_3 & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

In what follows, we present several valid inequalities for $\mathcal{P}(G, K, \mathbb{S})$, and study their facial structure.

2.4 Valid Inequalities and Facets

We start this section by introducing some classes of valid inequalities that can be defined using Chvàtal-Gomory procedure.

2.4.1 Edge-Slot-Assignment Inequalities

Proposition 2.4.1. *Consider an edge $e \in E$ with $K_e \neq \emptyset$. Let s be a slot in $\mathbb{S}$. Then, the inequality*

$$\sum_{k \in K_e} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} \leq 1, \tag{2.21}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. Inequality (2.21) ensures that the set of demands K_e cannot share slot s over edge e, which means that slot s is assigned to at most one demand k from K_e over edge e. We know from the inequality (2.6) that for each pair of demands $k, k' \in K_e$ with $k \neq k'$

$$\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^{k} + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq 1,$$

given that $x_e^k = x_e^{k'} = 1$. After that, we use the Chvàtal-Gomory and recurrence procedures to prove that (2.21) is valid for $\mathcal{P}(G, K, \mathbb{S})$. For any subset of demands $\tilde{K} \subseteq K_e$, by using a recurrence procedure, we get that for all demands $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{k \in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^{k} \leq 1.$$

By adding the previous inequalities for all subset of demands $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{\substack{K' \subseteq \tilde{K} \\ |K'|=|\tilde{K}|-1}} \sum_{k \in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^{k} \leq \sum_{\substack{K' \subseteq \tilde{K} \\ |K'|=|\tilde{K}|-1}} 1.$$

Note that for each $k \in \tilde{K}$, the sum $\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^{k}$ appears $(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) = |\tilde{K}| - 1$ times in the previous sum. This implies that

$$\sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} (|\tilde{K}| - 1) z_{s'}^{k} \leq |\tilde{K}|.$$

By dividing the two sides of the previous sum by $|\tilde{K}| - 1$, we have

$$\sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^{k} \leq \left\lfloor \frac{|\tilde{K}|}{|\tilde{K}| - 1} \right\rfloor \Rightarrow \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^{k} \leq \left\lfloor \frac{|\tilde{K}|}{|\tilde{K}| - 1} \right\rfloor.$$

As a result,

$$\sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^{k} \leq 1 \text{ given that } \left\lfloor \frac{|\tilde{K}|}{|\tilde{K}| - 1} \right\rfloor = 1.$$

We conclude at the end that inequality (2.21) is valid for $\mathcal{P}(G, K, \mathbb{S})$. □

Inspiring from inequality (2.21), and based on the non-overlapping inequality (2.6), we define the following inequality.

Proposition 2.4.2. *Consider an edge $e \in E$. Let s be a slot in $\mathbb{S}$. Consider a triplet of demands $k, k', k" \in K$ with $e \notin E_0^k \cap E_0^{k'} \cap E_0^{k"}$, $(k, k') \notin K_c^e, (k, k") \notin K_c^e$, and $(k', k") \notin K_c^e$. Then, the inequality*

$$x_e^k + x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 4, \tag{2.22}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. Consider an edge $e \in E$. Let s be a slot in $\mathbb{S}$. Inequality (2.22) ensures that if the three demands $k, k', k"$ pass through edge e, they cannot share slot s.
Let show that inequality (2.22) can be seen as Chvàtal-Gomory cuts using Chvàtal-Gomory procedure. We know from (2.24) that

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq 3,$$
$$x_e^k + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 3,$$
$$x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 3.$$

By adding the three previous inequalities, we get the following inequality

$$2x_e^k + 2x_e^{k'} + 2x_e^{k"} + 2\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + 2\sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + 2\sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 9.$$

By dividing the two sides of the previous inequality by 2, we obtain that

$$x_e^k + x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq \left\lfloor \frac{9}{2} \right\rfloor.$$

As a result,

$$x_e^k + x_e^{k'} + x_e^{k"} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{s"=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s"}^{k"} \leq 4.$$

We conclude at the end that inequality (2.22) is valid for $\mathcal{P}(G, K, \mathbb{S})$. □

Inequality (2.22) can then be generalized for any subset of demand $\tilde{K} \subseteq K$ under certain conditions.

Proposition 2.4.3. *Consider an edge $e \in E$, and a slot s in $\mathbb{S}$. Let $\tilde{K}$ be a subset of demands of K with $e \notin E_0^k$ for each demand $k \in \tilde{K}$, $(k,k') \notin K_c^e$ for each pair of demands (k,k') in $\tilde{K}$, and $\sum_{k\in\tilde{K}} w_k \leq \bar{s} - \sum_{k"\in K_e\setminus\tilde{K}} w_{k"}$. Then, the inequality*

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k'\in\tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} \leq |\tilde{K}| + 1, \tag{2.23}$$

is valid for $\mathcal{P}(G,K,\mathbb{S})$.

Let $\binom{n}{k}$ denote the total number of possibilities to choose a k element in a set of n elements.

Proof. Inequality (2.23) ensures that if the demands $k \in \tilde{K}$ pass through edge e, they cannot share slot s. For this, we use the Chvàtal-Gomory and recurrence procedures to prove that (2.23) is valid for $\mathcal{P}(G,K,\mathbb{S})$. For any subset of demands $\tilde{K} \subseteq K$ with $e \notin E_0^k$ for each demand $k \in \tilde{K}$, by recurrence procedures we get that for all demands $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{k\in K'} x_e^k + \sum_{k\in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |K'| + 1.$$

By adding the previous inequalities for all subset of demands $K' \subseteq \tilde{K}$ with $|K'| = |\tilde{K}| - 1$

$$\sum_{\substack{K'\subseteq\tilde{K}\\|K'|=|\tilde{K}|-1}} \sum_{k\in K'} x_e^k + \sum_{\substack{K'\subseteq\tilde{K}\\|K'|=|\tilde{K}|-1}} \sum_{k\in K'} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq \sum_{\substack{K'\subseteq\tilde{K}\\|K'|=|\tilde{K}|-1}} (|K'| + 1).$$

Note that for each $k \in \tilde{K}$, the variable x_e^k and the sum $\sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k$ appear $(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1)$ times in the previous sum. This implies that

$$\sum_{k\in\tilde{K}} (\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} (\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) z_{s'}^k \leq \binom{|\tilde{K}|}{|\tilde{K}|-1} (|K'|+1)$$

Given that $|K'| = |\tilde{K}| - 1$, this is equivalent to say that

$$\sum_{k\in\tilde{K}} (\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} (\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) z_{s'}^k \leq \binom{|\tilde{K}|}{|\tilde{K}|-1} |\tilde{K}|$$

Moreover, and taking into account that $(\binom{|\tilde{K}|}{|\tilde{K}|-1} - 1) = |\tilde{K}| - 1$, we found that

$$\sum_{k\in\tilde{K}} (|\tilde{K}| - 1) x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} (|\tilde{K}| - 1) z_{s'}^k \leq |\tilde{K}|^2$$

By dividing the two sides of the previous sum by $|\tilde{K}|-1$, we have

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq \left\lfloor \frac{|\tilde{K}|^2}{|\tilde{K}|-1} \right\rfloor \Rightarrow \sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq \left\lfloor |\tilde{K}| \frac{|\tilde{K}|}{|\tilde{K}|-1} \right\rfloor.$$

After some simplifications, we obtain that

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}| + \left\lfloor \frac{|\tilde{K}|}{|\tilde{K}|-1} \right\rfloor.$$

As a result,

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}| + 1 \text{ given that } \left\lfloor \frac{|\tilde{K}|}{|\tilde{K}|-1} \right\rfloor = 1.$$

We conclude at the end that inequality (2.23) is valid for $\mathcal{P}(G,K,\mathbb{S})$. □

Inequality (2.23) can be strengthened as follows. For this, and using inequalities (2.21) and (2.6), we first show that inequality (2.6) can be strengthened without modifying its right-hand side as follows.

Proposition 2.4.4. *Consider an edge $e \in E$. Let s be a slot in $\mathbb{S}$. Consider a pair of demands $k, k' \in K$ with $e \notin E_0^k \cap E_0^{k'}$ and $(k,k') \notin K_c^e$. Then, the inequality*

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s'}^{k'} + \sum_{k"\in K_e\setminus\{k,k'\}} \sum_{s'=s}^{\min(s+w_{k"}-1,\bar{s})} z_{s'}^{k"} \leq 3, \tag{2.24}$$

is valid for $\mathcal{P}(G,K,\mathbb{S})$.

Proof. Consider an edge $e \in E$, and a pair of demands $k, k' \in K$. Let s be a slot in $\mathbb{S}$. Inequality (2.24) ensures that if the two demands k, k' pass through edge e, they cannot share slot s with the set of demands in $K_e \setminus \{k,k'\}$. This can be seen as a particular case for inequality (2.21) induced by subset of demands $\tilde{K} = \{k,k'\} \cup K_e$. □

Let generalize inequality (2.24) for any subset of demand $\tilde{K} \subseteq K$ under certain conditions.

Proposition 2.4.5. *Consider an edge $e \in E$, and a slot s in $\mathbb{S}$. Let $\tilde{K}$ be a subset of demands of K with $e \notin E_0^k$ for each demand $k \in \tilde{K}$, $(k,k') \notin K_c^e$ for each pair of demands (k,k') in $\tilde{K}$, and $\sum_{k\in\tilde{K}} w_k \leq \bar{s} - \sum_{k"\in K_e\setminus\tilde{K}} w_{k"}$. Then, the inequality*

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{k'\in K_e\setminus\tilde{K}} \sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq |\tilde{K}| + 1, \tag{2.25}$$

is valid for $\mathcal{P}(G,K,\mathbb{S})$.

This can be seen as a strengthened version of inequality (2.24).

Proof. Inequality (2.25) ensures that if the demands $k \in \tilde{K}$ pass through edge e, they cannot share slot s with the set of demands in $K_e \setminus \tilde{K}$. This can be seen be a particular case inequality (2.23) induced by $\tilde{K} \cup K_e$ for slot s over edge e.

□

Definition 2.4.1. *An interval $I = [s_i, s_j]$ represents an ordered set of contiguous slots situated between the two slots s_i and s_j with $j \geq i+1$ and $s_j \leq \bar{s}$ (e.g., interval $I = [1, 6]$ contains all slots situated between the slots $s_i = 1$ and $s_j = 6$).*

Theorem 2.4.1. *Consider an edge $e \in E$, and a slot $s \in \mathbb{S}$. Let $\tilde{K}$ be a subset of demands in K, and $\sum_{k \in \tilde{K}} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus \tilde{K}} w_{k'}$. Then, inequality (2.23) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if $K_e \setminus \tilde{K} = \emptyset$, and there does not exist an interval of contiguous slots $I = [s_i, s_j]$ such that*

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ *for each demand* $k \in \tilde{K}$,

b) and $s \in \{s_i + \max_{k' \in \tilde{K}} w_k - 1, ..., s_j - \max_{k \in \tilde{K}} w_k + 1\}$,

c) and $w_k + w_{k'} \geq |I| + 1$ *for each* $k, k' \in \tilde{K}$,

d) and $2w_k \geq |I| + 1$ *for each* $k \in \tilde{K}$.

Proof. **Neccessity.**

- if $K_e \setminus \tilde{K} \neq \emptyset$, then inequality (2.23) is dominated by inequality (2.25) without changing its right-hand side. As a result, inequality (2.23) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.
- if there exists an interval of contiguous slots $I = [s_i, s_j]$ such that the conditions $a) - d)$ are verfied. Then inequality (2.23) is dominated by another valid inequality which will be presented later. Hence, inequality (2.23) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F^{e,s}_{\tilde{K}}$ be the face induced by inequality (2.23), that is

$$F^{e,s}_{\tilde{K}} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} = |\tilde{K}| + 1\}.$$

Let denote by $\alpha x + \beta z \leq \lambda$ inequality $\sum_{k \in \tilde{K}} x^k_e + \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} \leq |\tilde{K}| + 1$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F^{e,s}_{\tilde{K}} \subseteq F$. To prove that $F^{e,s}_{\tilde{K}}$ is a facet of

$\mathcal{P}(G, K, \mathbb{S})$, we need to show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

Let first show that $\mu_{e'}^k = 0$ for each edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $e \neq e'$ if $k \in \tilde{K}$.

Consider a demand $k \in K$ and an edge $e' \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in \tilde{K}$. Let $\mathcal{S}^{38} = (E^{38}, S^{38})$ be the solution given by

a) for each demand $k_i \in K \setminus \tilde{K}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{38}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let E_k^{38} be the set of edges involved in a shortest path between o_k and d_k which uses edge e such that edge e' is compatible with all the selected edges e" $\in E_k^{38}$ of demand k in solution $\mathcal{S}^{38}$, i.e., $\sum_{e" \in E_k^{38}} l_{e"} + \ell_{e'} \leq \bar{l}_k$,

c) for each demand $k' \in \tilde{K} \setminus \{k\}$, we let $E_{k'}^{38}$ be the set of edges involved in a shortest path between $o_{k'}$ and $d_{k'}$ which does uses edge e,

d) for one demand $k' \in \tilde{K}$, we select the smallest slot index $s_{k'}$ in $\{w_{k'}, ..., \bar{s}\}$ as last slot such that $s \in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$,

e) for each demand $k_i \in K \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{38} given by

$$I_i^{38} = [\bigcap_{k_j \in D_i^{38}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E_{k_i}^{38} \cap (E_k^{38} \cup \{e'\}) \neq \emptyset \text{ or } I_i^{38} = \bigcap_{k_j \in D_i^{38}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not.}$$

such that $s \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ if $k_i \in \tilde{K}$, where $D_i^{38} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k'\} : E_{k_i}^{38} \cap E_{k_j}^{38} \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{38}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_k + 1, ..., s_k\} = \emptyset$ if $E_{k_i}^{38} \cap (E_k^{38} \cup \{e'\}) \neq \emptyset$ (we take into account the possibility of using edge e' in the selected path $E_k'^{38}$ to route demand k in solution $\mathcal{S}'^{38}$).

We let $S_{k_i}^{38} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{38}$ is feasible for the problem. its incidence vector $(x^{\mathcal{S}^{38}}, z^{\mathcal{S}^{38}})$ belongs to $F_{\tilde{K}}^{e,s}$. Then consider the solution $\mathcal{S}^{39} = (E^{39}, S^{39})$ obtained from $\mathcal{S}^{38}$ by adding edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{38}$ which means that

$E_k^{39} = E_k^{38} \cup \{e'\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{38}$ remain the same in solution $\mathcal{S}^{39}$, i.e., $S_k^{39} = S_k^{38}$ for each $k \in K$, and $E_{k'}^{39} = E_{k'}^{38}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{39}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{39}}, z^{\mathcal{S}^{39}})$ belongs to $F_{\tilde{K}}^{e,s}$. Hence, solutions $\mathcal{S}^{38}$ and $\mathcal{S}^{39}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{38}} + \sigma z^{\mathcal{S}^{38}} = \mu x^{\mathcal{S}^{39}} + \sigma z^{\mathcal{S}^{39}} = \mu x^{\mathcal{S}^{38}} + \mu_{e'}^k + \sigma z^{\mathcal{S}^{38}}.$$

As a result, $\mu_{e'}^k = 0$.

In a similar way, we can show that

$$\mu_{e'}^k = 0, \text{ for all } k \in K \text{ and } e' \in E \setminus (E_0^k \cup E_1^k) \text{ with } e \neq e' \text{ if } k \in \tilde{K}.$$

Let show that $\sigma_{s'}^k = 0$ for all $k \in K$ and $s' \in \{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s, ..., s + w_k - 1\}$ if $k \in \tilde{K}$. Let $\mathcal{S}'^{38} = (E'^{38}, S'^{38})$ be the solution given by

a) for each demand $k_i \in K \setminus \tilde{K}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^{38}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let $E_k'^{38}$ be the set of edges involved in a shortest path between o_k and d_k which uses edge e,

c) for each demand $k' \in \tilde{K} \setminus \{k\}$, we let $E_{k'}'^{38}$ be the set of edges involved in a shortest path between $o_{k'}$ and $d_{k'}$ which use edge e,

d) for one demand $k \in \tilde{K}$, we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\}$ as last slot such that $s \in \{s_k - w_k + 1, ..., s_k\}$,

e) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^{38}$ given by

$$I_i'^{38} = [\bigcap_{k_j \in D_i'^{38}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}'^{38} \cap E_k'^{38} \neq \emptyset \text{ or } I_i'^{38} = \bigcap_{k_j \in D_i'^{38}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not.}$$

where $D_i'^{38} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E_{k_i}'^{38} \cap E_{k_j}'^{38} \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i'^{38}$,
- $s \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ if $k_i \in \tilde{K}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E_{k_i}'^{38} \cap E_k'^{38} \neq \emptyset$ (we take into account the possibility of adding slot s' in the selected set of last slots $S_k'^{38}$ to route demand k in solution $\mathcal{S}'^{38}$).

We let $S'^{38}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{38}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}'^{38}}, z^{\mathcal{S}'^{38}})$ belongs to $F^{e,s}_{\tilde{K}}$. Then consider the solution $\mathcal{S}^{40}$ obtained from $\mathcal{S}'^{38}$ by adding slot s' as last slot to demand k. Solution $\mathcal{S}^{40}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{40}}, z^{\mathcal{S}^{40}})$ belongs to $F^{e,s}_{\tilde{K}}$. Hence, solutions $\mathcal{S}'^{38}$ and $\mathcal{S}^{40}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{38}} + \sigma z^{\mathcal{S}'^{38}} = \mu x^{\mathcal{S}^{40}} + \sigma z^{\mathcal{S}^{40}} = \mu x^{\mathcal{S}'^{38}} + \sigma z^{\mathcal{S}'^{38}} + \sigma^k_{s'}.$$

It follows that $\sigma^k_{s'} = 0$.
In a similar way, we can show that

$$\sigma^k_{s'} = 0, \text{ for all } k \in K \text{ and } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s, ..., s + w_k - 1\} \text{ if } k \in \tilde{K}.$$

Let prove that $\sigma^k_{s'}$ for all $k \in \tilde{K}$ and $s' \in \{s, ..., s+w_k-1\}$ are equivalent. Consider a demand $k' \in K$ and a slot $s' \in \{s, ..., s+w_{k'}-1\}$ with $k' \in \tilde{K}$. Let $\mathcal{S}^{41} = (E^{41}, S^{41})$ be a solution obtained from $\mathcal{S}^{38}$ by adding slot s' as last slot to demand k' with modifying the paths assigned to a subset of demands $\tilde{K} \subset K$ in $\mathcal{S}^{38}$ (i.e., $E^{41}_k = E^{38}_k$ for each $k \in K \setminus \tilde{K}$, and $E^{41}_k \neq E^{38}_k$ for each $k \in \tilde{K}$), and also the last slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{38}$ remain the same in $\mathcal{S}^{41}$, i.e., $S^{38}_{k"} = S^{41}_{k"}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S^{41}_{k'} = S^{38}_{k'} \cup \{s'\}$ for demand k', and modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s' \in S^{38}_k$ with $s' \in \{s_i+w_k+1, ..., s_j\}$ and $\tilde{s} \notin \{s_i+w_k+1, ..., s_j\}$ for demand k with $k \in \tilde{K}$ such that $S^{41}_k = (S^{38}_k \setminus \{s\}) \cup \{\tilde{s}\}$ such that $\{\tilde{s}-w_k+1, ..., \tilde{s}\} \cap \{s'-w_{k'}+1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S^{41}_{k'}$ with $E^{41}_k \cap E^{41}_{k'} \neq \emptyset$. Solution $\mathcal{S}^{41}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{41}}, z^{\mathcal{S}^{41}})$ belongs to $F^{e,s}_{\tilde{K}}$. Hence, solutions $\mathcal{S}^{38}$ and $\mathcal{S}^{41}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}^{38}} + \sigma z^{\mathcal{S}^{38}} = \mu x^{\mathcal{S}^{41}} + \sigma z^{\mathcal{S}^{41}} = \mu x^{\mathcal{S}^{38}} + \sigma z^{\mathcal{S}^{38}} + \sigma^{k'}_{s"} - \sigma^k_{s'} + \sigma^k_{\tilde{s}} - \sum_{k \in \tilde{K}} \sum_{e' \in E^{38}_k} \mu^k_{e'} + \sum_{k \in \tilde{K}} \sum_{e' \in E^{41}_k} \mu^k_{e'}.$$

Since $\sigma^k_{\tilde{s}} = 0$ for $\tilde{s} \notin \{s, ..., s + w_k - 1\}$ with $k \in \tilde{K}$, and $\mu^k_{e'} = 0$ for all $k \in K$ and $e' \in E \setminus (E^k_0 \cup E^k_1)$ with $e' \neq e$ if $k \in \tilde{K}$, it follows that $\sigma^{k'}_{s"} = \sigma^k_{s'}$.
In a similar way, we can show that

$$\sigma^k_{s'} = \sigma^{k'}_{s"}, \text{for all pairs } (k, k') \in \tilde{K}$$

with $s' \in \{s, ..., s+w_k-1\}$ and $s' \in \{s, ..., s+w_{k'}-1\}$. We re-do the same procedure for each two slots $s, s' \in \{s, ..., s+w_k-1\}$ for each demand $k \in K$ with $k \in \tilde{K}$ such that

$$\sigma^k_{s'} = \sigma^k_{s"}, \text{for all } k \in \tilde{K} \text{ and } s, s' \in \{s, ..., s+w_k-1\}.$$

We will prove that μ^k_e for all $k \in \tilde{K}$ are equivalent. Let $\mathcal{S}^{42} = (E^{42}, S^{42})$ be the solution given by

a) for each demand $k_i \in K \setminus \tilde{K}$ with $i \in \{1, ..., |K|\}$, we let $E^{42}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for one demand k from $\tilde{K}$, we let $E^{42}_{k_i}$ be the set of edges involved in a shortest path between o_k and d_k which uses edge e,

c) for each demand $k' \in \tilde{K} \setminus \{k\}$, we let $E^{42}_{k'}$ be the set of edges involved in a shortest path between $o_{k'}$ and $d_{k'}$ which does not use edge e,

d) for each demand $k \in \tilde{K}$, we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \cap \{s, ..., s+w_k-1\}$ as last slot,

e) for each demand $k_i \in K \setminus \tilde{K}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{42}_i given by

$$I^{42}_i = [\bigcap_{k_j \in D^{42}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s - w_k\} \cup \{s + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E^{42}_{k_i} \cap E^{42}_k \neq \emptyset, \text{ or } I^{42}_i = [\bigcap_{k_j \in D^{42}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \text{ if not,}$$

where $D^{42}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{K} : E^{42}_{k_i} \cap E^{42}_{k_j} \neq \emptyset\}$. This ensures that $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D^{42}_i$.
We let $S^{42}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

Obviously, $\mathcal{S}^{42}$ is feasible for the problem. Moreover, the corresponding incidence vector $(x^{\mathcal{S}^{42}}, z^{\mathcal{S}^{42}})$ belongs to $F^{e,s}_{\tilde{K}}$.
Consider now a demand k' in $\tilde{K}$ such that $e \notin E^{42}_{k'}$. We derive a feasible solution $\mathcal{S}^{43} = (E^{43}, S^{43})$ for the problem from $\mathcal{S}^{42}$ by

a) the paths assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{42}$ remain the same in $\mathcal{S}^{43}$ (i.e., $E^{43}_{k"} = E^{42}_{k"}$ for each $k" \in K \setminus \{k, k'\}$),

b) without modifying the last slots assigned to the demands K in $\mathcal{S}^{42}$, i.e., $S^{42}_k = S^{43}_k$ for each demand $k \in K$,

c) modifying the path assigned to demand k' in $\mathcal{S}^{42}$ from $E^{42}_{k'}$ to a path $E^{43}_{k'}$ passed through edge e (i.e., $e \in E^{43}_{k'}$) with $k' \in \tilde{K}$ such that $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s' \in S^{42}_{k'}$ and $s' \in S^{42}_{k}$ with $E^{42}_{k} \cap E^{43}_{k'} \neq \emptyset$,

d) modifying the path assigned to demand k in $\mathcal{S}^{42}$ with $e \in E^{42}_{k}$ and $k \in \tilde{K}$ from E^{42}_{k} to a path E^{43}_{k} without passing through edge e (i.e., $e \notin E^{43}_{k}$) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k"} + 1, ..., s'\} = \emptyset$ for each $k" \in K \setminus \{k, k'\}$ and $s' \in S^{42}_{k}$ and $s' \in S^{42}_{k"}$ with $E^{42}_{k"} \cap E^{43}_{k} \neq \emptyset$, and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $s' \in S^{42}_{k}$ and $s' \in S^{42}_{k'}$ with $E^{43}_{k"} \cap E^{43}_{k} \neq \emptyset$.

The corresponding incidence vector $(x^{\mathcal{S}^{43}}, z^{\mathcal{S}^{43}})$ belongs to $F^{e,s}_{\tilde{K}}$. Hence, solutions $\mathcal{S}^{42}$ and $\mathcal{S}^{43}$ satisfy equation $\mu x + \sigma z = \tau$. We then obtain that

$$\mu x^{\mathcal{S}^{42}} + \sigma z^{\mathcal{S}^{42}} = \mu x^{\mathcal{S}^{43}} + \sigma z^{\mathcal{S}^{43}} = \mu x^{\mathcal{S}^{42}} + \sigma z^{\mathcal{S}^{42}} + \mu^{k'}_{e} - \mu^{k}_{e} + \sum_{e" \in E^{43}_{k'} \setminus \{e\}} \mu^{k'}_{e"} - \sum_{e" \in E^{42}_{k'}} \mu^{k'}_{e"} + \sum_{e" \in E^{43}_{k}} \mu^{k}_{e"} - \sum_{e" \in E^{42}_{k} \setminus \{e\}} \mu^{k}_{e"}.$$

Since $\mu^{k}_{e"} = 0$ for all $k \in K$ and $e" \in E \setminus (E^{k}_{0} \cup E^{k}_{1})$ with $k \in \tilde{K}$, it follows that $\mu^{k'}_{e} = \mu^{k}_{e}$. In a similar way, we can show that

$$\mu^{k}_{e} = \mu^{k'}_{e}, \text{for all pairs } (k, k') \in \tilde{K}.$$

Furthermore, let prove that all $\sigma^{k}_{s'}$ and μ^{k}_{e} are equivalent for all $k \in \tilde{K}$ and $s' \in \{s, ..., s + w_k - 1\}$.

Now let us consider for each demand k' with $k' \in \tilde{K}$, a solution $\mathcal{S}^{44} = (E^{44}, S^{44})$ obtained from $\mathcal{S}^{42}$ as below

a) the paths assigned to the demands $K \setminus \{k'\}$ in $\mathcal{S}^{42}$ remain the same in $\mathcal{S}^{44}$ (i.e., $E^{44}_{k"} = E^{42}_{k"}$ for each $k" \in K \setminus \{k'\}$),

b) without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{42}$, i.e., $S^{42}_{k"} = S^{44}_{k"}$ for each demand $k" \in K \setminus \{k\}$,

c) modifying the set of last slots assigned to demand k' in $\mathcal{S}^{42}$ from $S^{42}_{k'}$ to $S^{44}_{k'}$ such that $S^{44}_{k'} \cap \{s, ..., s + w_{k'} - 1\} = \emptyset$.

Hence, there are $|\tilde{K}| - 1$ demands from $\tilde{K}$ that share slot s over edge e (i.e., all the demands in $\tilde{K} \setminus \{k'\}$), and two demands $\{k, k'\}$ from $\tilde{K}$ that use edge e in solution $\mathcal{S}^{44}$. Solution $\mathcal{S}^{44}$ is then feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{44}}, z^{\mathcal{S}^{44}})$ belongs to $F^{e,s}_{\tilde{K}}$. Hence, solutions $\mathcal{S}^{42}$ and $\mathcal{S}^{44}$ satisfy equation $\mu x + \sigma z = \tau$. We then obtain that

$$\mu x^{\mathcal{S}^{42}} + \sigma z^{\mathcal{S}^{42}} = \mu x^{\mathcal{S}^{44}} + \sigma z^{\mathcal{S}^{44}} = \mu x^{\mathcal{S}^{42}} + \sigma z^{\mathcal{S}^{42}} + \mu^{k'}_{e} - \sigma^{k'}_{s'} + \sum_{e" \in E^{44}_{k'} \setminus \{e\}} \mu^{k'}_{e"} - \sum_{e" \in E^{42}_{k'}} \mu^{k'}_{e"}.$$

Since $\mu^k_{e"} = 0$ for all $k \in K$ and $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $e \neq e"$ if $k \in \tilde{K}$, it follows that $\mu^{k'}_e = \sigma^{k'}_{s'}$.
In a similar way, we can show that

$$\mu^k_e = \sigma^k_{s'}, \text{for all } k \in \tilde{K} \text{ and } s' \in \{s, ..., s + w_k - 1\}.$$

Based on this, and given that all μ^k_e are equivalent for all $k \in \tilde{K}$, and that $\sigma^k_{s'}$ are equivalent for all $k \in \tilde{K}$ and $s' \in \{s, ..., s + w_{k'} - 1\}$, we obtain that

$$\mu^k_e = \sigma^{k'}_{s'}, \text{ for all } k, k' \in \tilde{K} \text{ and } s' \in \{s, ..., s + w_{k'} - 1\}.$$

Consequently,

$$\mu^k_e = \sigma^{k'}_{s'} = \rho, \text{ for all } k, k' \in \tilde{K} \text{ and } s' \in \{s, ..., s + w_{k'} - 1\}.$$

We know from (2.17) and (2.18) that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

Overall, we obtain that

$$\mu^{k'}_{e'} = \begin{cases} \gamma^{k',e'}_1 & \text{if } e' \in E^{k'}_0, \\ \gamma^{k',e'}_2 & \text{if } e' \in E^{k'}_1, \\ \rho & \text{if } k' \in \tilde{K} \text{ and } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $e' \in E$, and

$$\sigma^k_{s'} = \begin{cases} \gamma^{k,s'}_3 & \text{if } s' \in \{1, ..., w_k - 1\} \\ \rho & \text{if } k \in \tilde{K} \text{ and } s' \in \{s, ..., s + w_k - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

for each $k \in K$ and $s' \in \mathbb{S}$.
As a consequence, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

Theorem 2.4.2. *Consider an edge $e \in E$, and a slot $s \in \mathbb{S}$. Let $\tilde{K}$ be a subset of demands in K with $|\tilde{K}| \geq 3$, and $\sum_{k \in \tilde{K}} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus \tilde{K}} w_{k'}$. Then, inequality (2.25) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if there does not exist an interval of contiguous slots $I = [s_i, s_j]$ such that*

a) $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ *for each demand* $k \in \tilde{K}$,

b) and $s \in \{s_i + \max_{k' \in \tilde{K}} w_k - 1, ..., s_j - \max_{k \in \tilde{K}} w_k + 1\}$,

c) and $w_k + w_{k'} \geq |I| + 1$ *for each* $k, k' \in \tilde{K}$,

d) and $w_k + w_{k'} \geq |I| + 1$ *for each* $k \in \tilde{K}$ *and* $k' \in K_e \setminus \tilde{K}$,

e) and $2w_k \geq |I| + 1$ *for each* $k \in \tilde{K}$,

f) and $2w_{k'} \geq |I| + 1$ *for each* $k' \in K_e \setminus \tilde{K}$.

Proof. **Neccessity.**

Suppose that there exists an interval of contiguous slots $I = [s_i, s_j]$ such that all the conditions $a) - f)$ are verified. Then inequality (2.25) is dominated by another valid inequality which will be presented later. As a result, inequality (2.25) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let denote $F'^{e,s}_{\tilde{K}}$ the face induced by inequality (2.25), that is

$$F'^{e,s}_{\tilde{K}} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{k \in \tilde{K}} x^k_e + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} + \sum_{K_e \setminus \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1, \bar{s})} z^{k'}_{s'} = |\tilde{K}| + 1\}.$$

We denote inequality $\sum_{k \in \tilde{K}} x^k_e + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} + \sum_{K_e \setminus \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1, \bar{s})} z^{k'}_{s'} \leq |\tilde{K}| + 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F'^{e,s}_{\tilde{K}} \subseteq F$. We show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma^{k,e}_1 \in \mathbb{R}$ for all $k' \in K$ and $e \in E^{k'}_0, \gamma^{k,e}_2 \in \mathbb{R}$ for all $k' \in K$ and $e \in E^{k'}_1, \gamma^{k',s'}_3 \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. We re-do the same technique of proof already detailed to prove that inequality (2.23) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ such that the solutions $\mathcal{S}^{38} - \mathcal{S}^{44}$ still feasible for $F'^{e,s}_{\tilde{K}}$ given that their incidence vector are composed by $\sum_{k \in \tilde{K}} x^k_e + \sum_{k \in \tilde{K}} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} + \sum_{K_e \setminus \tilde{K}} \sum_{s'=s}^{\min(s+w_{k'}-1, \bar{s})} z^{k'}_{s'} \leq |\tilde{K}| + 1$. We conclude at the end that for each $k' \in K$ and $e' \in E$

$$\mu^{k'}_{e'} = \begin{cases} \gamma^{k',e'}_1 & \text{if } e' \in E^{k'}_0, \\ \gamma^{k',e'}_2 & \text{if } e' \in E^{k'}_1, \\ \rho & \text{if } k' \in \tilde{K} \text{ and } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s' \in \mathbb{S}$

$$\sigma_{s'}^{k} = \begin{cases} \gamma_3^{k,s'} & \text{if } s' \in \{1, ..., w_k - 1\}, \\ \rho & \text{if } k \in \tilde{K} \cup K_e \text{ and } s' \in \{s, ..., s + w_k - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

2.4.2 Edge-Interval-Capacity-Cover Inequalities

Let now introduce some valid inequalities which can be seen as cover inequalities using some notions of cover related to the problem.

Definition 2.4.2. *For an interval of contiguous slots $I = [s_i, s_j]$, a subset of demands $K' \subseteq K$ is said a cover for the interval $I = [s_i, s_j]$ if and only if $\sum_{k \in \tilde{K}} w_k > |I|$ and $w_k < |I|$ for each $k \in \tilde{K}$. Moreover, it is said to be a minimal cover if $\sum_{k' \in \tilde{K} \setminus \{k\}} w_{k'} \leq |I|$ for each demand $k \in \tilde{K}$.*

Based on these definitions, we introduce the following inequalities.

Proposition 2.4.6. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$. Let $K' \subseteq K_e$ be a minimal cover for interval $I = [s_i, s_j]$ over edge e. Then, the inequality*

$$\sum_{k \in K'} \sum_{s = s_i + w_k - 1}^{s_j} z_s^k \leq |K'| - 1, \tag{2.26}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|K'| - 1$ demands given that K' is a minimal cover for interval $I = [s_i, s_j]$ over edge e. Otherwise, the non-overlapping constraint is violated given that there exists at least one slot $s \in I$ such that $\sum_{k \in K'} \sum_{s'=s}^{s+w_k-1} z_s^k > 1$. □

Inequality (2.26) can be lifted using a sequential lifting procedure [5] to be facet defining and generate lifted facets for a sub-polytope of $\mathcal{P}(G, K, \mathbb{S})$.

Theorem 2.4.3. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$. Let $\tilde{K} \subseteq K_e$ be a minimal cover for interval $I = [s_i, s_j]$ over edge e. Let $K_e \setminus \tilde{K} = \{k_1, ..., k_q\}$ be arbitrarily ordred with $q = |K_e \setminus \tilde{K}|$. Consider the following sequence of knapsack*

problems defined as

$$\begin{cases} z_i = \max \sum_{j \in \tilde{K}} a_j + \sum_{j=1}^{i-1} \beta_j a_j, \\ \sum_{j \in \tilde{K}} w_j a_j + \sum_{j=1}^{i-1} w_{k_j} a_j \leq |I| - w_{k_i}, \\ a_j \in \{0,1\}, \forall j \in \tilde{K} \cup \{1, ..., i-1\}, \end{cases} \tag{2.27}$$

for all $i \in \{1, ..., q\}$ *with* $\beta_j = |\tilde{K}| - 1 - z_j$ *for all* $j \in \{1, ..., i-1\}$. *Then, the inequality*

$$\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{j=1}^{q} \sum_{s'=s_i+w_{k_j}-1}^{s_j} \beta_j z_{s'}^{k_j} \leq |\tilde{K}| - 1, \tag{2.28}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$. *Moreover, inequality* (5.13) *defines facet of*

$$\mathcal{P}(G, K, \mathbb{S}, \tilde{K}, e, E) = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{\substack{k' \in K \\ E_1^{k'} \cap E_1^k \neq \emptyset \text{ for all } k \in \tilde{K}}} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 0\}.$$

if there does not exist an interval of contiguous slots $I' = [s'_i, s'_j]$ *in* $[1, \bar{s}]$ *with* $I \subset I'$ *such that* $\tilde{K}$ *defines a minimal cover for the interval* I'.

Proof. It is trivial given that inequality (5.13) can never be dominated in $\mathcal{P}(G, K, \mathbb{S}, \tilde{K}, e, E)$ if there does not exist an interval of contiguous slots $I' = [s'_i, s'_j]$ in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I'. □

Inequality (2.26) can then be generalized over all edges $e \in E$. Moreover, it should be lifted to be facet definig for the polytope $\mathcal{P}(G, K, \mathbb{S})$ as follows.

Proposition 2.4.7. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots in* $[1, \bar{s}]$. *Let* $K' \subset K$ *be a minimal cover for interval* $I = [s_i, s_j]$ *such that* $E_1^k \cap E_1^{k'} \neq \emptyset$ *for each pair* $(k, k') \in K'$. *Then, the inequality*

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |K'| - 1, \tag{2.29}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. The interval I can cover at most $|K'| - 1$ demands given that K' is a minimal cover for interval I. □

Inequality (2.29) can then be lifted using a sequential lifting procedure [5] to generate several facets for the polytope $\mathcal{P}(G, K, \mathbb{S})$.

Theorem 2.4.4. *Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let $\tilde{K} \subseteq K$ be a minimal cover for interval $I = [s_i, s_j]$ such that $E_1^k \cap E_1^{k'} \neq \emptyset$ for each pair $(k, k') \in \tilde{K}$. Let $K' \subseteq K \setminus \tilde{K} = \{k_1, ..., k_q\}$ such that $E_1^k \cap E_1^{k'} \neq \emptyset$ for each pair $(k, k') \in \tilde{K} \cup K'$. Consider the following sequence of knapsack problems defined as*

$$\begin{cases} z_i = \max \sum_{j \in \tilde{K}} a_j + \sum_{j=1}^{i-1} \beta_j a_j, \\ \sum_{j \in \tilde{K}} w_j a_j + \sum_{j=1}^{i-1} w_{k_j} a_j \leq |I| - w_{k_i}, \\ a_j \in \{0,1\}, \forall j \in \tilde{K} \cup \{1, ..., i-1\}, \end{cases} \tag{2.30}$$

for all $i \in \{1, ..., q\}$ with $\beta_j = |\tilde{K}| - 1 - z_j$ for all $j \in \{1, ..., i-1\}$. Then, the inequality

$$\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{j=1}^{q} \sum_{s'=s_i+w_{k_j}-1}^{s_j} \beta_j z_{s'}^{k_j} \leq |\tilde{K}| - 1, \tag{2.31}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$. Moreover, inequality (2.31) defines facet of $\mathcal{P}(G, K, \mathbb{S})$ if there does not exist an interval of contiguous slots $I' = [s'_i, s'_j]$ in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I'.

Proof. It is trivial given that inequality (2.31) can never be dominated in $\mathcal{P}(G, K, \mathbb{S})$ if there does not exist an interval of contiguous slots $I' = [s'_i, s'_j]$ in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I'. □

Inspiring from inequalities (2.26) and (2.29), we define another valid inequality induced by any subset of demands $\tilde{K}$ defining a minimal cover for any interval I as follows.

Definition 2.4.3. *Consider an inequality $\alpha x^T \leq \beta$ which is not valid for a polyhedron $\mathcal{P}(G, K, \mathbb{S})$. It is said to be* optimality cut *for $\mathcal{P}(G, K, \mathbb{S})$ if it is valid for a semi-polytope of $\mathcal{P}(G, K, \mathbb{S})$ which covers at least one optimal solution for the problem.*

Let $\mathcal{Q}(G, K, \mathbb{S}) = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{s=w_k}^{\bar{s}} z_s^k = 1, \forall k \in K\}$ be a semi-polytope of $\mathcal{P}(G, K, \mathbb{S})$. Note that each valid inequality of $\mathcal{Q}(G, K, \mathbb{S})$ which is not valid for $\mathcal{P}(G, K, \mathbb{S})$, it defines an optimality cut for $\mathcal{P}(G, K, \mathbb{S})$.

Proposition 2.4.8. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$. Let $\tilde{K}$ be a minimal cover for the interval I such that*

a) $\sum_{k\in\tilde{K}} w_k \leq \bar{s} - \sum_{k'\in K_e\setminus\tilde{K}} w_{k'}$,

b) $e \notin E_0^k$ *for each demand* $k \in \tilde{K}$,

c) $(k,k') \notin K_c^e$ *for each pair of demands* (k,k') *in* $\tilde{K}$.

Then, the inequality

$$\sum_{k\in\tilde{K}} x_e^k + \sum_{k\in\tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 2|\tilde{K}| - 1, \tag{2.32}$$

is valid for $\mathcal{Q}(G,K,\mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|\tilde{K}| - 1$ demands given that $\tilde{K}$ is a minimal cover for interval $I = [s_i, s_j]$ over edge e. It follows that if demands $\tilde{K}$ pass together through edge e (i.e., $\sum_{k\in\tilde{K}} x_e^k = |\tilde{K}|$), there are at most $|\tilde{K}| - 1$ demands that can share the interval I over edge e. We ensure that inequalities (2.32) are verified by any feasible solution having an incidence vector in $\mathcal{Q}(G,K,\mathbb{S})$. Otherwise, the non-overlapping constraint is violated such if there exists a solution $\mathbb{S}$ that violates inequality (2.32), this will certainly prove that there exists a slot $s \in I$ over edge e such that $\sum_{k\in\tilde{K}} \sum_{s'=s}^{s+w_k-1} z_{s'}^k > 1$ given that $\sum_{k\in\tilde{K}} x_e^k \leq |\tilde{K}|$ and $\sum_{k\in\tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |K|$ for any feasible solution S with incidence vector in $\mathcal{Q}(G,K,\mathbb{S})$. □

Inequality (2.32) can also be lifted using a sequential lifting procedure [5] to be facet defining and generate lifted facets for the polytope $\mathcal{Q}(G,K,\mathbb{S})$.

Theorem 2.4.5. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots in* $[1, \bar{s}]$. *Let* $\tilde{K}$ *be a minimal cover for the interval* I *such that* $\tilde{K}$ *does not define a minimal cover for an edge* e, *where* $e \notin E_0^k$ *for each demand* $k \in \tilde{K}$. *Let* $K_e \setminus \tilde{K} = \{k_1, ..., k_q\}$ *be arbitrarily ordred with* $q = |K_e \setminus \tilde{K}|$. *Consider the following sequence of knapsack problems defined as*

$$\begin{cases} z_i = \max \sum_{j\in\tilde{K}} a_j + \sum_{j=1}^{i-1} \beta_j a_j, \\ \sum_{j\in\tilde{K}} w_j a_j + \sum_{j=1}^{i-1} w_{k_j} a_j \leq |I| - w_{k_i}, \\ a_j \in \{0,1\}, \forall j \in \tilde{K} \cup \{1, ..., i-1\}, \end{cases} \tag{2.33}$$

for all $i \in \{1, ..., q\}$ *with* $\beta_j = |\tilde{K}| - 1 - z_j$ *for all* $j \in \{1, ..., i-1\}$. *Then, the inequality*

$$\sum_{k \in \tilde{K}} x_e^k + \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{j=1}^{q} \sum_{s'=s_i+w_{k_j}-1}^{s_j} \beta_j z_{s'}^{k_j} \leq 2|\tilde{K}| - 1, \tag{2.34}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. *Moreover, inequality* (2.34) *defines facet of* $\mathcal{Q}(G, K, \mathbb{S})$ *if there does not exist an interval of contiguous slots* $I' = [s'_i, s'_j]$ *in* $[1, \bar{s}]$ *with* $I \subset I'$ *such that* $\tilde{K}$ *defines a minimal cover for the interval* I'.

Proof. It is trivial given that inequality (2.34) can never be dominated in $\mathcal{Q}(G, K, \mathbb{S})$ if there does not exist an interval of contiguous slots $I' = [s'_i, s'_j]$ in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I'. □

2.4.3 Edge-Interval-Clique Inequalities

Using inequalities (2.32), and based on the set of minimal cover $\tilde{K}$ with cardinality $|\tilde{K}| = 2$, we introduce the following inequality.

Proposition 2.4.9. *Consider an edge* $e \in E$. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots. Let* $\{k, k'\}$ *be a minimal cover for the interval* I *over edge* e *such that* $e \notin E_0^k \cup E_0^{k'}$. *Then, the inequality*

$$x_e^k + x_e^{k'} + \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s=s_i+w_{k'}-1}^{s_j} z_s^{k'} \leq 3, \tag{2.35}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. *Moreover, it is valid for* $\mathcal{P}(G, K, \mathbb{S})$ *if* $2w_k > |I|$ *and* $2w_{k'} > |I|$.

Proof. Inequality (2.35) is a particular case of inequality (2.36) for a minimal cover $\tilde{K} = \{k, k'\}$. □

Using this, we introduce the following conflict graph.

Definition 2.4.4. *Consider an edge* $e \in E$. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots in* $[1, \bar{s}]$ *with* $s_i \leq s_j - 1$. *Consider the conflict graph* H_I^e *defined as follows. For each demand* $k \in K$ *with* $w_k \leq |I|$ *and* $e \notin E_0^k$, *consider a node* v_k *in* H_I^e. *Two nodes* v_k *and* $v_{k'}$ *are linked by an edge in* H_I^e *if* $w_k + w_{k'} > |I|$ *and* $(k, k') \notin K_c^e$. *This is equivalent to say that two linked nodes* v_k *and* $v_{k'}$ *means that the two demands* k, k' *define a minimal cover for the interval* I *over edge* e.

For an edge $e \in E$, the conflict graph H_I^e is a threshold graph with threshold value equals to $t = |I|$ such that for each node v_k with $e \notin E_0^k \cup E_1^k$, we associate a positive weight $\tilde{w}_{v_k} = w_k$ such that all two nodes v_k and $v_{k'}$ are linked by an edge if and only if $\tilde{w}_{v_k} + \tilde{w}_{v_{k'}} > t$ which is equivalent to the conflict graph H_I^e.

Proposition 2.4.10. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph H_I^e with $|C| \geq 3$, and $\sum_{v_k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$. Then, the inequality*

$$\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C| + 1, \tag{2.36}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, It is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in C$.

Proof. For each edge $e \in E$ and interval of contiguous slots $I \subseteq \mathbb{S}$, inequality (2.36) ensures that if the set of demands in clique C pass through edge e, they cannot share the interval $I = [s_i, s_j]$ over edge e. This means that there are at most one demand from the demands in C that can be totally covered by the interval I over edge e (i.e., all the slots assigned to the demand are in I). Inequality (2.36) can be shown as Chvàtal-Gomory cuts using Chvàtal-Gomory and recurrence procedures. For all two linked node v_k and $v_{k'}$ in H_I^e, we know from inequality (2.35)

$$x_e^k + x_e^{k'} + \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 3.$$

By adding the previous inequalities for all two linked node v_k and $v_{k'}$ in the clique set C, and by recurrence procedure we obtain that for all $K' \subseteq C$ with $|K'| = |C| - 1$

$$\sum_{v_k \in C'} x_e^k + \sum_{v_k \in C'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |K'| + 1.$$

By adding the previous inequalities for all $K' \subseteq C$ with $|K'| = |C| - 1$, we get

$$\sum_{\substack{K' \subseteq C \\ |K'|=|C|-1}} \sum_{v_k \in C'} x_e^k + \sum_{\substack{K' \subseteq C \\ |K'|=|C|-1}} \sum_{v_k \in C'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \sum_{\substack{K' \subseteq C \\ |K'|=|C|-1}} (|K'| + 1).$$

Note that for each demand k with $v_k \in C$, the variable x_e^k and the sum $\sum_{s=s_i+w_k-1}^{s_j} z_s^k$ appear $(\binom{|C|}{|C|-1} - 1)$ times in the previous sum. It follows that

$$\sum_{v_k \in C} (\binom{|C|}{|C|-1} - 1) x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} .(\binom{|C|}{|C|-1} - 1) z_s^k \leq \binom{|C|}{|C|-1} |C|.$$

Given that $(\binom{|C|}{|C|-1} - 1) = |C| - 1$, we obtain that

$$\sum_{v_k \in C} (|C| - 1) x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} (|C| - 1) z_s^k \leq |C|^2.$$

By dividing the two sides of the previous sum by $|C| - 1$, we have

$$\sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor \frac{|C|^2}{|C|-1} \right\rfloor \Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| \frac{|C|}{|C|-1} \right\rfloor$$
$$\Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| \frac{|C|-1+1}{|C|-1} \right\rfloor.$$

By doing the following simplification

$$\sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| \frac{|C|-1}{|C|-1} + \frac{|C|}{|C|-1} \right\rfloor$$
$$\Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor |C| + \frac{|C|}{|C|-1} \right\rfloor.$$

As a result,

$$\sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C| + \left\lfloor \frac{|C|}{|C|-1} \right\rfloor \Rightarrow \sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C|+1$$
$$\text{given that } \left\lfloor \frac{|C|}{|C|-1} \right\rfloor = 1.$$

We conclude at the end that inequality (2.36) is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in C$. □

Moreover, inequality (2.36) can be strengthened as follows.

Proposition 2.4.11. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph H_I^e with $|C| \geq 3$, and $\sum_{v_k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$. Let $C_e \subseteq K_e \setminus C$ be a clique in the conflict graph H_I^e such that $w_k + w_{k'} \geq |I| + 1$ for each $v_k \in C$ and $v_{k'} \in C_e$. Then, the inequality*

$$\sum_{v_k \in C} x_e^k + \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |C| + 1, \tag{2.37}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in C \cup C_e$.

Proof. For each edge $e \in E$ and interval of contiguous slots $I \subseteq \mathbb{S}$, inequality (2.37) ensures that if the set of demands in clique C pass through edge e, they cannot share the interval $I = [s_i, s_j]$ over edge e with a subset of demands in C_e. On the other hand, inequality (2.37) can be seen as a particular case of inequality (2.36) induced by a clique $C' = C \cup C_e$ given that $x_e^k = 1$ for all $v_k \in C_e$. □

Theorem 2.4.6. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph H_I^e with $|C| \geq 3$, $\sum_{k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$, and $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand k with $v_k \in C$. Then, inequality (2.36) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) there does not exist a demand $k' \in K_e \setminus C$ with $w_k + w_{k'} > |I|$ and $w_{k'} \leq |I|$,

b) and there does not exist an interval I' of contiguous slots with $I \subset I'$ such that C defines also a clique in the associated conflict graph $H_{I'}^e$.

Proof. **Neccessity.**
It is trivial given that

- if there does not exist a demand $k' \in K_e \setminus C$ with $w_k + w_{k'} > |I|$ and $w_{k'} \leq |I|$, and $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand k with $v_k \in C$. Then, inequality (2.36) can never be dominated by another inequality without changing its right-hand side. Otherwise, if there exists a demand $k' \in K_e \setminus C$ with $w_k + w_{k'} > |I|$ and $w_{k'} \leq |I|$ and $2w_{k'} > |I|$, this implies that the inequality is dominated by (2.37). Moreover, if $|\{s_i + w_k - 1, ..., s_j\}| < w_k$ for each demand k with $v_k \in C$, then inequality (2.36) is then dominated by inequality (2.23) for a set of demands $\tilde{K} = \{k \in K \text{ such that } v_k \in C\}$ and slot $s = s_i + \min_{k \in C} w_k + 1$ over edge e. Hence, inequality (2.36) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.
- if there exists an interval I' of contiguous slots with $I \subset I'$ such that C defines also a clique in the associated conflict graph $H_{I'}^e$. This implies that inequality (2.36) induced by clique C for the interval I is dominated by inequality (2.36) induced by the same clique C for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in C$. As a result, inequality (2.36) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.
Let $F_C^{H_I^e}$ denote the face induced by inequality (2.36), that is

$$F_C^{H_I^e} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k = |C| + 1\}.$$

Let denote inequality $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C|+1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G,K,\mathbb{S})$ and $F = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_C^{H_I^e} \subseteq F$. In order to prove that inequality $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |C|+1$ is facet defining for $\mathcal{P}(G,K,\mathbb{S})$, we need to show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1,...,w_{k'}-1\}$) such that $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma Q$.

We first show that $\mu_{e'}^k = 0$ for each edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $e \neq e'$ if $k \in C$.

Consider a demand $k \in K$ and an edge $e' \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $k \in C$. Let $\mathcal{S}^{53} = (E^{53}, S^{53})$ be the solution given by

a) for each demand $k_i \in K \setminus C$ with $i \in \{1,...,|K|\}$, we let $E_{k_i}^{53}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for each demand $k \in C$, we let E_k^{53} be the set of edges involved in a shortest path between o_k and d_k which uses edge e,

c) for one demand k' from C, we select the slot $s_{k'} = s_i + w_{k'} - 1$ as last slot,

d) for each demand $k_i \in C \setminus \{k'\}$ with $i \in \{1,...,|K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{53} given by

$$I_i^{53} = [\bigcap_{k_j \in D_i^{53}} \{w_{k_i},...,s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i},...,\bar{s}\}] \setminus \{s_i,...,s_j\} \text{ if } E_{k_i}^{53} \cap E_{k'}^{53} \neq \emptyset,$$

where $D_i^{53} = \{k_j \in \{k_1,...,k_{i-1}\} \cap C : E_{k_i}^{53} \cap E_{k_j}^{53} \neq \emptyset\}$.

e) for each demand $k_i \in K \setminus C$ with $i \in \{1,...,|K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{53} given by

$$I_i^{53} = [\bigcap_{k_j \in R_i^{53}} \{w_{k_i},...,s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i},...,\bar{s}\}] \cap [\{w_{k_i},...,s_k - w_k\} \cup \{s_k + w_{k_i},...,\bar{s}\}]$$

if $E_{k_i}^{53} \cap (E_k^{53} \cup \{e'\}) \neq \emptyset$ or $I_i^{53} = \bigcap_{k_j \in R_i^{53}} \{w_{k_i},...,s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i},...,\bar{s}\}$ if not,

where $R_i^{53} = \{k_j \in \{k_1,...,k_{i-1}\} \cup C$ such that $E_{k_i}^{53} \cap E_{k_j}^{53} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1,...,s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1,...,s_{k_j}\} = \emptyset$ for each $k_j \in R_i^{53}$,
- and $\{s_{k_i} - w_{k_i} + 1,...,s_{k_i}\} \cap \{s_k - w_{k_j} + 1,...,s_k\} = \emptyset$ if $E_{k_i}^{53} \cap (E_k^{53} \cup \{e'\}) \neq \emptyset$ (we take into account the possibility of using edge e' in the selected path E_k^{53} to route demand k in solution $\mathcal{S}^{53}$).

We let $S_{k_i}^{53} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1,...,|K|\}$.

$\mathcal{S}^{53}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{53}}, z^{\mathcal{S}^{53}})$ belongs to $F_C^{H_I^e}$. Then we derive a solution $\mathcal{S}^{54} = (E^{54}, S^{54})$ obtained from $\mathcal{S}^{53}$ by adding edge $e' \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{53}$ which means that $E_k^{54} = E_k^{53} \cup \{e'\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{53}$ remain the same in solution $\mathcal{S}^{54}$, i.e., $S_k^{54} = S_k^{53}$ for each $k \in K$, and $E_{k'}^{54} = E_{k'}^{53}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{54}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{54}}, z^{\mathcal{S}^{54}})$ belongs to $F_C^{H_I^e}$. Hence, solutions $\mathcal{S}^{53}$ and $\mathcal{S}^{54}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{53}} + \sigma z^{\mathcal{S}^{53}} = \mu x^{\mathcal{S}^{54}} + \sigma z^{\mathcal{S}^{54}} = \mu x^{\mathcal{S}^{53}} + \mu_{e'}^k + \sigma z^{\mathcal{S}^{53}}.$$

As a result, $\mu_{e'}^k = 0$.

In a similar way, we can show that

$$\mu_{e'}^k = 0, \text{ for all } k \in K \text{ and } e' \in E \setminus (E_0^k \cup E_1^k) \text{ with } e \neq e' \text{ if } k \in C.$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \notin C$. Let $\mathcal{S}'^{53} = (E'^{53}, S'^{53})$ be the solution given by

a) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^{53}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for each demand $k \in C$, we let $E_k'^{53}$ be the set of edges involved in a shortest path between o_k and d_k,

c) for one demand k' from C, we select the slot $s_{k'} = s_i + w_{k'} - 1$ as last slot,

d) for each demand $k_i \in C \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^{53}$ given by

$$I_i'^{53} = [\bigcap_{k_j \in D_i'^{53}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{s_i, ..., s_j\} \text{ if } E_{k_i}'^{53} \cap E_{k'}'^{53} \neq \emptyset,$$

where $D_i'^{53} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap C : E_{k_i}'^{53} \cap E_{k_j}'^{53} \neq \emptyset\}$.

e) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^{53}$ given by

$$I_i'^{53} = [\bigcap_{k_j \in R_i'^{53}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}'^{53} \cap E_k'^{53} \neq \emptyset \text{ or } I_i'^{53} = \bigcap_{k_j \in R_i'^{53}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $R_i'^{53} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C : E_{k_i}'^{53} \cap E_{k_j}'^{53} \neq \emptyset\}$.
We let $S_{k_i}'^{53} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{53}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{53}}, z^{\mathcal{S}'^{53}})$ belongs to $F_C^{H_I^e}$. Then we derive solution $\mathcal{S}^{55}$ from $\mathcal{S}'^{53}$ by adding slot s' as last slot to demand k in solution $\mathcal{S}'^{53}$, i.e., $S_k^{55} = S_k'^{53} \cup \{s'\}$. Solution $\mathcal{S}^{55}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{55}}, z^{\mathcal{S}^{55}})$ belongs to $F_C^{H_I^e}$. Hence, solutions $\mathcal{S}'^{53}$ and $\mathcal{S}^{55}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{53}} + \sigma z^{\mathcal{S}'^{53}} = \mu x^{\mathcal{S}^{55}} + \sigma z^{\mathcal{S}^{55}} = \mu x^{\mathcal{S}'^{53}} + \sigma z^{\mathcal{S}'^{53}} + \sigma_{s'}^k.$$

Hence, $\sigma_{s'}^k = 0$.
In a similar way, we can show that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in C.$$

Let prove that σ_s^k for all $v_k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalent.
Consider a demand $k' \in K$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $v_{k'} \in C$. Let $\tilde{\mathcal{S}}^{53} = (\tilde{E}^{53}, \tilde{S}^{53})$ be the solution given by

a) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we let $\tilde{E}_{k_i}^{53}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for each demand $k \in C$, we let $\tilde{E}_k^{53}$ be the set of edges involved in a shortest path between o_k and d_k which uses edge e,

c) for one demand k" from C, we select the slot $s_{k"} = s_i + w_{k"} + 1$ as last slot,

d) for each demand $k_i \in C \setminus \{k"\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}_i^{53}$ given by

$$\tilde{I}_i^{53} = [\bigcap_{k_j \in \tilde{D}_i^{53}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}] \setminus \{s_i, ..., s_j\}$$

if $\tilde{E}_{k_i}^{53} \cap \tilde{E}_{k'}^{53} \neq \emptyset$ or $\tilde{I}_i^{53} = [\bigcap_{k_j \in \tilde{D}_i^{53}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{s_i, ..., s_j\}$ if not,

where $\tilde{D}_i^{53} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap C : \tilde{E}_{k_i}^{53} \cap \tilde{E}_{k_j}^{53} \neq \emptyset\}$,

e) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}_i^{53}$ given by

$$\tilde{I}_i^{53} = [\bigcap_{k_j \in \tilde{R}_i^{53}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $\tilde{E}_{k_i}^{53} \cap \tilde{E}_{k'}^{53} \neq \emptyset$ or $\tilde{I}_i^{53} = \bigcap_{k_j \in \tilde{R}_i^{53}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $\tilde{R}_i^{53} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C : \tilde{E}_{k_i}^{53} \cap \tilde{E}_{k_j}^{53} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in \tilde{R}_i^{53}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $\tilde{E}_{k_i}^{53} \cap \tilde{E}_{k'}^{53} \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $\tilde{S}_{k'}^{53}$ to route demand k' in solution $\tilde{\mathcal{S}}^{53}$).

We let $\tilde{S}_{k_i}^{53} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\tilde{\mathcal{S}}^{53}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{53}}, z^{\tilde{\mathcal{S}}^{53}})$ belongs to $F_C^{H_I^e}$. Let $\mathcal{S}^{56} = (E^{56}, S^{56})$ be a solution obtained from $\tilde{\mathcal{S}}^{53}$ by adding slot s' as last slot to demand k' in $\tilde{\mathcal{S}}^{53}$, and modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ in $\tilde{\mathcal{S}}^{53}$ and removing the last slot $s \in \tilde{S}_k^{53}$ from $\tilde{\mathcal{S}}^{53}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for demand $k \in C$ such that $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{56}$ with $E_k^{56} \cap E_{k'}^{56} \neq \emptyset$. Solution $\mathcal{S}^{56}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{56}}, z^{\mathcal{S}^{56}})$ belongs to $F_C^{H_I^e}$. Hence, solutions $\mathcal{S}^{53}$ and $\mathcal{S}^{56}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{53}} + \sigma z^{\tilde{\mathcal{S}}^{53}} = \mu x^{\mathcal{S}^{56}} + \sigma z^{\mathcal{S}^{56}} = \mu x^{\tilde{\mathcal{S}}^{53}} + \sigma z^{\tilde{\mathcal{S}}^{53}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k.$$

Since $\sigma_{\tilde{s}}^k = 0$ for $\tilde{s} \notin \{s_i + w_k - 1, ..., s_j\}$ with $v_k \in C$, it follows that $\sigma_{s'}^{k'} = \sigma_s^k$.
In a similar way, we can show that

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_k, v_{k'}) \in C$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $v_k \in C$ such that

$$\sigma_s^k = \sigma_{s'}^k, \text{for all } v_k \in C \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\}.$$

Let prove now that μ_e^k for all $k \in K$ with $v_k \in C$ are equivalent. Consider a demand $k' \in K$ with $v_{k'}$ in C such that $e \notin E_{k'}^{57}$. For this, we derive a solution $\mathcal{S}^{"58} = (E^{"58}, S^{"58})$ from $\mathcal{S}^{53}$ by we derive a solution $\mathcal{S}^{58} = (E^{58}, S^{58})$ from $\mathcal{S}^{53}$ by

a) the paths assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{53}$ remain the same in $\mathcal{S}^{58}$ (i.e., $E_{k"}^{58} = E_{k"}^{53}$ for each $k" \in K \setminus \{k, k'\}$),

b) without modifying the last slots assigned to the demands K in $\mathcal{S}^{53}$, i.e., $S_k^{53} = S_k^{58}$ for each demand $k \in K$,

c) modifying the path assigned to demand k' in $\mathcal{S}^{53}$ from $E^{53}_{k'}$ to a path $E^{58}_{k'}$ passed through edge e (i.e., $e \in E^{58}_{k'}$) with $v_{k'} \in C$ such that $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k \in K$ and $s' \in S^{53}_{k'}$ and $s \in S^{53}_k$ with $E^{53}_k \cap E^{58}_{k'} \neq \emptyset$,

d) modifying the path assigned to demand k in $\mathcal{S}^{53}$ with $e \in E^{53}_k$ and $v_k \in C$ from E^{53}_k to a path E^{58}_k without passing through edge e (i.e., $e \notin E^{58}_k$) and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k"} + 1, ..., s'\} = \emptyset$ for each $k" \in K \setminus \{k, k'\}$ and $s \in S^{53}_k$ and $s' \in S^{53}_{k"}$ with $E^{53}_{k"} \cap E^{58}_k \neq \emptyset$, and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $s \in S^{53}_k$ and $s' \in S^{53}_{k'}$ with $E^{58}_{k"} \cap E^{58}_k \neq \emptyset$.

Solution $\mathcal{S}"^{58}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}"^{58}}, z^{\mathcal{S}"^{58}})$ belongs to $F^{He_I}_C$. Hence, solutions $\mathcal{S}^{53}$ and $\mathcal{S}"^{58}$ satisfy equation $\mu x + \sigma z = \tau$. We then obtain that

$$\mu x^{\mathcal{S}^{53}} + \sigma z^{\mathcal{S}^{53}} = \mu x^{\mathcal{S}^{58}} + \sigma z^{\mathcal{S}^{58}} = \mu x^{\mathcal{S}^{53}} + \sigma z^{\mathcal{S}^{53}} + \mu^{k'}_e - \mu^k_e + \sum_{e" \in E"^{58}_{k'} \setminus \{e\}} \mu^{k'}_{e"} - \sum_{e" \in E^{53}_{k'}} \mu^{k'}_{e"} + \sum_{e" \in E"^{58}_k} \mu^k_{e"} - \sum_{e" \in E^{53}_k \setminus \{e\}} \mu^k_{e"}.$$

Since $\mu^k_{e"} = 0$ for all $k \in K$ and $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $v_k \notin C$, it follows that $\mu^{k'}_e = \mu^k_e$.

In a similar way, we can show that

$$\mu^k_e = \mu^{k'}_e, \text{for all pairs } (v_k, v_{k'}) \in C.$$

Furthermore, let prove that all σ^k_s and μ^k_e are equivalent for all $k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$.

Now let us consider a demand $k' \in K$ with $v_{k'} \in C$, a solution $\mathcal{S}^{59} = (E^{59}, S^{59})$ obtained from $\mathcal{S}^{53}$ as below

a) the paths assigned to the demands $K \setminus \{k'\}$ in $\mathcal{S}^{53}$ remain the same in $\mathcal{S}^{59}$ (i.e., $E^{59}_{k"} = E^{53}_{k"}$ for each $k" \in K \setminus \{k'\}$),

b) without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{53}$, i.e., $S^{53}_{k"} = S^{59}_{k"}$ for each demand $k" \in K \setminus \{k\}$,

c) modifying the set of last slots assigned to demand k' in $\mathcal{S}^{53}$ from $S^{53}_{k'}$ to $S^{59}_{k'}$ such that $S^{59}_{k'} \cap \{s_i + w_{k'} - 1, ..., s_j\} = \emptyset$.

Hence, there are $|C| - 1$ demands from C that are covered by the interval I (i.e., all the demands in $C \setminus \{k'\}$), and two demands $\{k, k'\}$ from C that use edge e in solution $\mathcal{S}^{59}$. Solution $\mathcal{S}^{59}$ is then feasible for the problem. The corresponding

incidence vector $(x^{\mathcal{S}^{59}}, z^{\mathcal{S}^{59}})$ belongs to $F_C^{H_I^e}$. Hence, solutions $\tilde{\mathcal{S}}^{57}$ and $\mathcal{S}^{59}$ satisfy equation $\mu x + \sigma z = \tau$. We then obtain that

$$\mu x^{\tilde{\mathcal{S}}^{57}} + \sigma z^{\tilde{\mathcal{S}}^{57}} = \mu x^{\mathcal{S}^{59}} + \sigma z^{\mathcal{S}^{59}} = \mu x^{\tilde{\mathcal{S}}^{57}} + \sigma z^{\tilde{\mathcal{S}}^{57}} + \mu_e^{k'} - \sigma_s^{k'} + \sum_{e" \in E_{k'}^{59} \setminus \{e\}} \mu_{e"}^{k'} - \sum_{e" \in \tilde{E}_{k'}^{57}} \mu_{e"}^{k'}.$$

Since $\mu_{e"}^k = 0$ for all $k \in K$ and $e" \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e"$ if $v_k \in C$, it follows that $\mu_e^{k'} = \sigma_s^{k'}$.
In a similar way, we can show that

$$\mu_e^k = \sigma_s^k, \text{for all } v_k \in C \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}.$$

Based on this, and given that all μ_e^k are equivalent for all $v_k \in C$, and that σ_s^k are equivalent for all $v_k \in C$ and $s \in \{s_i + w_{k'} - 1, ..., s_j\}$, we obtain that

$$\mu_e^k = \sigma_s^{k'}, \text{ for all } k, k' \in C \text{ and } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

Consequently,

$$\mu_e^k = \sigma_s^{k'} = \rho, \text{ for all } k, k' \in C \text{ and } s \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

By (2.17) and (2.18), we know that

$$\begin{cases} \mu_{e'}^{k'} = \gamma_1^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} = \gamma_2^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} = \gamma_3^{k',s'} & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'} & \text{if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'} & \text{if } e' \in E_1^{k'}, \\ \rho & \text{if } k' \in C \text{ and } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\} \\ \rho & \text{if } v_k \in C \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

Theorem 2.4.7. *Consider an edge $e \in E$. Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let C be a clique in the conflict graph H_I^e with $|C| \geq 3$, $\sum_{v_k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$, and $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand k with $v_k \in C \cup C_e$. Let $C_e \subseteq K_e \setminus C$ be a clique in the conflict graph H_I^e such that $w_k + w_{k'} \geq |I| + 1$ for each $v_k \in C$ and $v_{k'} \in C_e$. Then, inequality (2.37) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) there does not exist a demand $k" \in K_e \setminus C_e$ with $w_k + w_{k"} \geq |I| + 1$ for each $v_k \in C$, and $w_{k'} + w_{k"} \geq |I| + 1$ for each $v_{k'} \in C_e$,

b) and there does not exist an interval I' of contiguous slots with $I \subset I'$ such that $C \cup C_e$ defines also a clique in the associated conflict graph $H_{I'}^e$.

Proof. **Neccessity.**

- if there exists a demand $k" \in K_e \setminus C_e$ with $w_k + w_{k"} \geq |I| + 1$ for each $v_k \in C$, and $w_{k'} + w_{k"} \geq |I| + 1$ for each $v_{k'} \in C_e$. Then, inequality (2.37) is dominated by its lifted with $C'_e = C_e \cup \{k"\}$. Moreover, if $|\{s_i + w_k - 1, ..., s_j\}| < w_k$ for each demand k with $v_k \in C \cup C_e$, then inequality (2.37) is then dominated by inequality (2.25) for a set of demands $\tilde{K} = \{k \in K \text{ such that } v_k \in C\}$ and slot $s = s_i + \min_{k \in C \cup C_e} w_k + 1$ over edge e. As a result, inequality (2.37) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.
- if there exists an interval I' of contiguous slots with $I \subset I'$ such that $C \cup C_e$ defines also a clique in the associated conflict graph $H_{I'}^e$. This implies that inequality (2.37) induced by clique $C \cup C_e$ for the interval I is dominated by inequality (2.37) induced by the same clique $C \cup C_e$ for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in C \cup C_e$. As a result, inequality (2.37) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.
Let $F_C'^{H_I^e}$ be the face induced by inequality (2.37), that is

$$F_C'^{H_I^e} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = |C| + 1\}.$$

We denote inequality $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |C|+1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_C'^{H_I^e} \subseteq F$. We use the same proof of the facial structure done for inequality (2.36) in the proof of theorem 2.4.6 to prove that inequality $\sum_{v_k \in C} x_e^k + \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{v_{k'} \in C_e} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |C|+1$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$. We first prove that $F_C'^{H_I^e}$ is a proper face based on solution $\mathcal{S}^{53}$ defined in the proof of theorem 2.4.6 which stills feasible such that

its incidence vector $(x^{\mathcal{S}^{53}}, z^{\mathcal{S}^{53}})$ belongs to $F_C'^{H_I^e}$. Furthermore, and based on the solutions $\mathcal{S}^{53}$ to $\mathcal{S}^{59}$ with corresponding incidence vector $(x^{\mathcal{S}^{53}}, z^{\mathcal{S}^{53}})$ to $(x^{\mathcal{S}^{59}}, z^{\mathcal{S}^{59}})$ belong to $F_C'^{H_I^e}$, we show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. For this, we show that

a) $\sigma_s^k = 0$ for all demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C \cup C_e$,

b) and σ_s^k are equivalent for all $v_k \in C \cup C_e$ and $s \in \{s_i + w_k - 1, ..., s_j\}$,

c) and $\mu_{e'}^k = 0$ for all demand $k \in K$ and edge $e \in E \setminus (E_0^k \cup E_1^k)$ with $e \neq e'$ if $v_k \in C$,

d) and μ_e^k are equivalent for the set of demands in C,

e) and $\sigma_s^{k'}$ and μ_e^k are equivalent for all $v_k \in C$ and $v_{k'} \in C \cup C_e$ and $s \in \{s_i + w_{k'} - 1, ..., s_j\}$.

At the end, we obtain that for each $k' \in K$ and $e' \in E$

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'} & \text{if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'} & \text{if } e' \in E_1^{k'}, \\ \rho & \text{if } k' \in C \text{ and } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\} \\ \rho & \text{if } v_k \in C \cup C_e \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

2.4.4 Interval-Clique Inequalities

We have looked at the definition of inequality (2.36), we detected that there may exist some cases that we can face which are not covered by inequality (2.36). For this, we provide the following inequality and its generalization.

Proposition 2.4.12. *Consider an interval of contiguous slots $I = [s_i, s_j]$ in $\mathbb{S}$ with $s_i \leq s_j - 1$. Let k, k' be a pair of demands in K with $E_1^k \cap E_1^{k'} \neq \emptyset$, and $w_k \leq |I|$.*

Then, the inequality

$$\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1, \tag{2.38}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$*. Moreover, it is valid for* $\mathcal{P}(G, K, \mathbb{S})$ *if* $2w_k > |I|$ *and* $2w_{k'} > |I|$*.*

Proof. It is trivial given that the interval $I = [s_i, s_j]$ cannot cover the two demands k, k' shared an essential edge with total sum of number of slots exceeds $|I|$. Furthermore, inequality (2.38) is a particular case of inequality (2.36) for $\tilde{K} = \{k, k'\}$ over each edge $e \in E_1^k \cap E_1^{k'}$. However, it will be used for a generalized inequality using the following conflict graph. □

Definition 2.4.5. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots in* $[1, \bar{s}]$ *with* $s_i \leq s_j - 1$*. Consider the conflict graph* H_I^E *defined as follows. For each demand* $k \in K$ *with* $w_k \leq |I|$*, consider a node* v_k *in* H_I^E*. Two nodes* v_k *and* $v_{k'}$ *are linked by an edge in* H_I^E *if* $w_k + w_{k'} > |I|$ *and* $E_1^k \cap E_1^{k'} \neq \emptyset$*.*

Proposition 2.4.13. *Let* $I = [s_i, s_j]$ *be an interval of contiguous slots in* $[1, \bar{s}]$ *with* $s_i \leq s_j - 1$*, and* C *be a clique in the conflict graph* H_I^E *with* $|C| \geq 3$*. Then, the inequality*

$$\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1, \tag{2.39}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$*. Moreover, it is valid for* $\mathcal{P}(G, K, \mathbb{S})$ *if* $2w_k > |I|$ *for each* $v_k \in C$*.*

Proof. It is trivial given the definition of clique set in the conflict graph H_I^E such that for all two linked node v_k and $v_{k'}$ in H_I^E, we know from inequality (2.38)

$$\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1.$$

By adding the previous inequalities for all two linked node v_k and $v_{k'}$ in the clique set C, and by recurrence procedure we obtain that for all $C' \subseteq C$ with $|C'| = |C| - 1$

$$\sum_{v_k \in C'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1.$$

By adding the previous inequalities for all $C' \subseteq C$ with $|C'| = |C| - 1$, we get

$$\sum_{\substack{C' \subseteq C \\ |C'|=|C|-1}} \sum_{v_k \in C'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \sum_{\substack{C' \subseteq C \\ |C'|=|C|-1}} 1.$$

Note that for each demand k with $v_k \in C$, the sum $\sum_{s=s_i+w_k-1}^{s_j} z_s^k$ appears $(\binom{|C|}{|C|-1} - 1) = |C| - 1$ times in the previous sum. It follows that

$$\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} (|C|-1) z_s^k \leq |C|.$$

By dividing the two sides of the previous sum by $|C| - 1$, we have so

$$\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor \frac{|C|}{|C|-1} \right\rfloor \Rightarrow \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1 \text{ given that } \left\lfloor \frac{|C|}{|C|-1} \right\rfloor = 1.$$

We conclude at the end that inequality (2.39) is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in C$. □

Theorem 2.4.8. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and C be a clique in the conflict graph H_I^E with $|C| \geq 3$, and $|\{s_i + w_k - 1, ..., s_j\}| \geq w_k$ for each demand k with $v_k \in C$. Then, inequality (2.39) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) C is a maximal clique in the conflict graph H_I^E,

b) and there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ such that $I \subset I'$ with

- *$w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,*
- *$w_k \leq |I'|$ for each $k \in C$.*

Proof. **Neccessity.**
We distinguish two cases

a) if there exists a clique C' that contains all the demands $k \in C$. Then, inequality (2.39) induced by clique C is dominated by another inequality (2.39) induced by clique C'. Hence, inequality (2.39) cannot be facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

b) if there exists an interval of contiguous slots I' in $[1, \bar{s}]$ such that $I \subset I'$ with

- $w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,
- $w_k \leq |I'|$ for each $k \in C$.

This means that inequality (2.39) induced by clique C for the interval I is dominated by inequality (2.39) induced by clique C for the interval I'. Hence, inequality (2.39) cannot be facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_C^{H_I^E}$ be the face induced by inequality (2.39), that is

$$F_C^{H_I^E} = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1\}.$$

Denote inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G,K,\mathbb{S})$ and $F = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_C^{H_I^E} \subseteq F$. In order to prove that inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1$ is facet defining for $\mathcal{P}(G,K,\mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^{60} = (E^{60}, S^{60})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{60}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for one demand k' from C, we select the slot $s_{k'} = s_i + w_{k'} - 1$,

c) for each demand $k_i \in C \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{60} given by

$$I_i^{60} = \bigcap_{k_j \in D_i^{60}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i, ..., s_j\},$$

where $D_i^{60} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap C : E_{k_i}^{60} \cap E_{k_j}^{60} \neq \emptyset\}$,

d) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{60} given by

$$I_i^{60} = [\bigcap_{k_j \in R_i^{60}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E_{k_i}^{60} \cap (E_k^{60} \cup \{e\}) \neq \emptyset \text{ or } I_i^{60} = \bigcap_{k_j \in R_i^{60}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $R_i^{60} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C$ such that $E_{k_i}^{60} \cap E_{k_j}^{60} \neq \emptyset\}$.

We let $S_{k_i}^{60} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{60}$ is feasible for the problem, and its incidence vector $(x^{\mathcal{S}^{60}}, z^{\mathcal{S}^{60}})$ belongs to $F_C^{H_I^E}$. Then we derive a solution $\mathcal{S}^{61} = (E^{61}, S^{61})$ obtained from $\mathcal{S}^{60}$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{60}$ which means that

$E_k^{61} = E_k^{60} \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{60}$ remain the same in solution $\mathcal{S}^{61}$, i.e., $S_k^{61} = S_k^{60}$ for each $k \in K$, and $E_{k'}^{61} = E_{k'}^{60}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{61}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{61}}, z^{\mathcal{S}^{61}})$ belongs to $F_C^{H_I^E}$. Hence, solutions $\mathcal{S}^{60}$ and $\mathcal{S}^{61}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{60}} + \sigma z^{\mathcal{S}^{60}} = \mu x^{\mathcal{S}^{61}} + \sigma z^{\mathcal{S}^{61}} = \mu x^{\mathcal{S}^{60}} + \mu_e^k + \sigma z^{\mathcal{S}^{60}}.$$

As a result, $\mu_e^k = 0$.

In a similar way, we can show that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and } e \in E \setminus (E_0^k \cup E_1^k).$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$.

Consider a demand $k \in K$ and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$. Let $\mathcal{S}'^{60} = (E'^{60}, S'^{60})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^{60}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for one demand k' from C, we select the slot $s_{k'} = s_i + w_k - 1$,

c) for each demand $k_i \in C \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^{60}$ given by

$$I_i'^{60} = \bigcap_{k_j \in D_i'^{60}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i, ..., s_j\},$$

where $D_i'^{60} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C : E_{k_i}'^{60} \cap E_{k_j}'^{60} \neq \emptyset\}$,

d) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^{60}$ given by

$$I_i'^{60} = [\bigcap_{k_j \in R_i'^{60}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}'^{60} \cap E_k'^{60} \neq \emptyset$ or $I_i'^{60} = \bigcap_{k_j \in R_i'^{60}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $R_i'^{60} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C$ such that $E_{k_i}^{60} \cap E_{k_j}^{60} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in R_i'^{60}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E_{k_i}'^{60} \cap E_k'^{60} \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $S_k'^{60}$ to route demand k in solution $\mathcal{S}'^{60}$).

We let $S'^{60}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{60}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{60}}, z^{\mathcal{S}'^{60}})$ belongs to $F_C^{H_I^E}$. Then consider the solution $\mathcal{S}^{62}$ obtained from $\mathcal{S}'^{60}$ by adding slot s' as last slot to demand k in $\mathcal{S}'^{60}$. Solution $\mathcal{S}^{62}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{62}}, z^{\mathcal{S}^{62}})$ belongs to $F_C^{H_I^E}$. Hence, solutions $\mathcal{S}'^{60}$ and $\mathcal{S}^{62}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{60}} + \sigma z^{\mathcal{S}'^{60}} = \mu x^{\mathcal{S}^{62}} + \sigma z^{\mathcal{S}^{62}} = \mu x^{\mathcal{S}'^{60}} + \sigma z^{\mathcal{S}'^{60}} + \sigma^k_{s'}.$$

Hence, $\sigma^k_{s'} = 0$.
In a similar way, we can show that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in C.$$

Let prove that σ^k_s for all $v_k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalent. Consider a demand $k' \in K$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $v_{k'} \in C$, and a solution $\tilde{\mathcal{S}}^{60} = (\tilde{E}^{60}, \tilde{S}^{60})$ given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $\tilde{E}^{60}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for one demand k from C, we select the slot $s_k = s_i + w_k - 1$,

c) for each demand $k_i \in C \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}^{60}_i$ given by

$$\tilde{I}^{60}_i = [\bigcap_{k_j \in \tilde{D}^{60}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}] \setminus \{s_i, ..., s_j\}$$

if $\tilde{E}^{60}_{k_i} \cap \tilde{E}^{60}_{k'} \neq \emptyset$ or $\tilde{I}^{60}_i = [\bigcap_{k_j \in \tilde{D}^{60}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{s_i, ..., s_j\}$ if not,

where $\tilde{D}^{60}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cap C : \tilde{D}^{60}_{k_i} \cap \tilde{D}^{60}_{k_j} \neq \emptyset\}$,

d) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}^{60}_i$ given by

$$\tilde{I}^{60}_i = [\bigcap_{k_j \in \tilde{R}^{60}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $\tilde{E}^{60}_{k_i} \cap \tilde{E}^{60}_{k'} \neq \emptyset$ or $\tilde{I}^{60}_i = \bigcap_{k_j \in \tilde{R}^{60}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $\tilde{R}^{60}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C$ such that $\tilde{D}^{60}_{k_i} \cap \tilde{D}^{60}_{k_j} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in \tilde{R}^{60}_i$,

- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $\tilde{E}^{60}_{k_i} \cap \tilde{E}^{60}_{k'} \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $\tilde{S}^{60}_{k'}$ to route demand k' in solution $\tilde{\mathcal{S}}^{60}$).

We let $\tilde{S}^{60}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\tilde{\mathcal{S}}^{60}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{60}}, z^{\tilde{\mathcal{S}}^{60}})$ belongs to $F_C^{H_I^E}$. Then consider the solution $\mathcal{S}^{63}$ obtained from $\tilde{\mathcal{S}}^{60}$ by adding slot s' as last slot to demand k', and modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s \in \tilde{S}^{60}_k$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for demand $k \in K$ with $v_k \in C$ such that $S^{63}_k = (\tilde{S}^{60}_k \setminus \{s\}) \cup \{\tilde{s}\}$ such that $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S^{63}_{k'}$ with $E^{63}_k \cap E^{63}_{k'} \neq \emptyset$. Solution $\mathcal{S}^{63}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{63}}, z^{\mathcal{S}^{63}})$ belongs to $F_C^{H_I^E}$. Hence, solutions $\tilde{\mathcal{S}}^{60}$ and $\mathcal{S}^{63}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{60}} + \sigma z^{\tilde{\mathcal{S}}^{60}} = \mu x^{\mathcal{S}^{63}} + \sigma z^{\mathcal{S}^{63}} = \mu x^{\tilde{\mathcal{S}}^{60}} + \sigma z^{\tilde{\mathcal{S}}^{60}} + \sigma^{k'}_{s'} - \sigma^k_s + \sigma^k_{\tilde{s}}.$$

Since $\sigma^k_{\tilde{s}} = 0$ for $\tilde{s} \notin \{s_i + w_k - 1, ..., s_j\}$ with $v_k \in C$, it follows that $\sigma^{k'}_{s'} = \sigma^k_s$.
In a similar way, we can show that

$$\sigma^k_s = \sigma^{k'}_{s'}, \text{for all pairs } (v_k, v_{k'}) \in C,$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $v_k \in C$ such that

$$\sigma^k_s = \sigma^k_{s'}, \text{for all } v_k \in C \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma^k_s = \rho$ for all $v_k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$.
By (2.17) and (2.18), we know that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu^k_e = \begin{cases} \gamma^{k,e}_1 & \text{if } e \in E^k_0, \\ \gamma^{k,e}_2 & \text{if } e \in E^k_1, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\} \\ \rho & \text{if } v_k \in C \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

Let $N(v)$ denote the set of neighbors of node v in a given graph.

Theorem 2.4.9. *Consider an interval of contiguous slots $I = [s_i, s_j]$, and a pair of demands $k, k' \in K$ with $(v_k, v_{k'})$ in G_I^E. Then, inequality (2.38) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) $N(v_k) \cap N(v_{k'}) = \emptyset$ in the conflict graph H_I^E,

b) and there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ such that $I \subset I'$ with $w_k + w_{k'} \geq |I'|$, $w_k \leq |I'|$, and $w_{k'} \leq |I'|$.

Proof. **Neccessity.**

We distinguish two cases:

a) if $N(v_k) \cap N(v_{k'}) \neq \emptyset$ in the conflict graph H_I^E, this means that there exists a clique C in the conflict graph H_I^E of cardinality equals to $|C| \geq 3$ with $k, k' \in C$. As a result, inequality (2.38) is dominated by inequality (2.39) induced by clique C. Hence, inequality (2.38) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

b) if there exists an interval of contiguous slots I' in $[1, \bar{s}]$ such that $I \subset I'$ with $w_k + w_{k'} \geq |I'|$, $w_k \leq |I'|$, and $w_{k'} \leq |I'|$. This means that inequality (2.38) induced by the two demands k, k' for the interval I is dominated by inequality (2.38) induced by the same demands for the interval I'.

Sufficiency.

We use the same proof of theorem 2.4.8 for a clique $C = \{v_k, v_{k'}\}$ in the conflict graph H_I^E. □

2.4.5 Interval-Odd-Hole Inequalities

Proposition 2.4.14. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and H be an odd-hole H in the conflict graph H_I^E with $|H| \geq 5$. Then, the inequality*

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H| - 1}{2}, \tag{2.40}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. *Moreover, it is valid for* $\mathcal{P}(G, K, \mathbb{S})$ *if* $2w_k > |I|$ *for each* $v_k \in H$.

Proof. It is trivial given the definition of odd-hole set in the conflict graph H_I^E. We strengthen the proof as belows. For each pair of nodes $(v_k, v_{k'})$ linked in H by an edge, we know that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1$. Given that H is an odd-hole which means that we have $|H|-1$ pair of nodes $(v_k, v_{k'})$ linked in H, and by doing a sum for all pairs of nodes $(v_k, v_{k'})$ linked in H, it follows that

$$\sum_{(v_k,v_{k'})\in E(H)} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |H| - 1.$$

where $E(H)$ denotes the set of edges in the sub-graph of the conflict graph H_I^E induced by H. Taking into account that each node v_k in H has two neighbors in H, this implies that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k$ appears twice in the previous inequality. As a result,

$$\sum_{(v_k,v_{k'})\in E(H)} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = \sum_{v_k\in H} 2 \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq |H| - 1.$$

By dividing the two sides of the previous sum by 2, it follows that

$$\sum_{v_k\in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2} \text{ since } |H| \text{ is an odd number.}$$

We conclude at the end that inequality (2.40) is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in H$. □

Inequality (2.40) can be strengthened without modifying its right-hand side by combining inequalities (2.39) and (2.40) as follows.

Proposition 2.4.15. *Consider an interval of contiguous slots* $I = [s_i, s_j] \subseteq \mathbb{S}$ *with* $s_i \leq s_j - 1$. *Let* H *be an odd-hole* H *in the conflict graph* H_I^E, *and* C *be a clique in the conflict graph* H_I^E *with*

a) $|H| \geq 5$,

b) and $H \cap C = \emptyset$,

c) and the nodes $(v_k, v_{k'})$ *are linked in* H_I^E *for all* $v_k \in H$ *and* $v_{k'} \in C$.

Then, the inequality

$$\sum_{v_k\in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'}\in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq \frac{|H|-1}{2}, \tag{2.41}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. *Moreover, it is valid for* $\mathcal{P}(G, K, \mathbb{S})$ *if* $2w_k > |I|$ *for each* $v_k \in C \cup H$.

Proof. It is trivial given the definition of odd-hole set and clique set in the conflict graph H_I^E such that if $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 1$ for $v_{k'} \in C$, it forces the quantity $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k$ to be equal to 0. Otherwise, we know from inequality (2.40) that the sum $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k$ is always smaller than $\frac{|H|-1}{2}$. We strengthen the proof as belows. For each pair of nodes $(v_k, v_{k'})$ linked in H by an edge, we know that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + \sum_{v_{k"} \in C} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq 1$ given that all the nodes $v_{k"} \in C$ are linked with the nodes v_k and $v_{k'}$. Given that H is an odd-hole which means that we have $|H|-1$ pair of nodes $(v_k, v_{k'})$ linked in H, and by doing a sum for all pairs of nodes $(v_k, v_{k'})$ linked in H, it follows that

$$\sum_{(v_k,v_{k'}) \in E(H)} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + \sum_{v_{k"} \in C} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H| - 1.$$

Taking into account that each node v_k in H has two neighbors in H, this implies that $\sum_{s=s_i+w_k-1}^{s_j} z_s^k$ appears twice in the previous inequality. The sum $\sum_{v_{k"} \in C} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"}$ appears $|H|-1$ times in in the previous inequality. As a result,

$$\sum_{(v_k,v_{k'}) \in E(H)} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + (|H|-1) \sum_{v_{k"} \in C} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H| - 1$$

$$\Rightarrow \sum_{v_k \in H} 2 \sum_{s=s_i+w_k-1}^{s_j} z_s^k + (|H|-1) \sum_{v_{k"} \in C} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H| - 1.$$

By dividing the two sides of the previous sum by 2, and since $|H|$ is an odd number, it follows that

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \left\lfloor \frac{|H|-1}{2} \right\rfloor \sum_{v_{k"} \in C} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2}.$$

We conclude at the end that inequality (2.41) is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in C \cup H$. □

Theorem 2.4.10. *Let H be an odd-hole in the conflict graph H_I^E with $|H| \geq 5$ and $2w_k > |I|$ for each $v_k \in H$. Then, inequality (2.40) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) for each node $v_{k'} \notin H$ in H_I^E, there exists a node $v_k \in H$ such that the induced graph $H_I^E((H \setminus \{v_k\}) \cup \{v_{k'}\})$ does not contain an odd-hole $H' = (H \setminus \{v_k\}) \cup \{v_{k'}\}$,

b) and there does not exist a node $v_{k'} \notin H$ in H_I^E such that $v_{k'}$ is linked with all nodes $v_k \in H$,

c) and there does not exist an interval I' of contiguous slots with $I \subset I'$ such that H defines also an odd-hole in the associated conflict graph $H_{I'}^E$.

Proof. **Neccessity.**

We distinguish the following cases:

a) if for a node $v_{k'} \notin H$ in H_I^E, there exists a node $v_k \in H$ such that the induced graph $H_I^E((H \setminus \{v_k\}) \cup \{v_{k'}\})$ contains an odd-hole $H' = (H \setminus \{v_k\}) \cup \{v_{k'}\}$. This implies that inequality (2.40) can be dominated by doing some lifting procedures using the following valid inequalities

$$\sum_{v_k \in H} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq \frac{|H|-1}{2}, \quad \text{and} \quad \sum_{v_{k'} \in H'} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq \frac{|H|-1}{2},$$

as follows

$$\sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + 2 \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H| - 1.$$

By adding the sum $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}$ to the previous inequality, we obtain

$$\sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + 2 \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + 2 \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H| - 1 + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'}.$$

Since $\sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1$, it follows that

$$\sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + 2 \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + 2 \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq |H|.$$

By dividing the last inequality by 2, we obtain that

$$\sum_{s'=s_i+w_k-1}^{s_j} \frac{1}{2} z_{s'}^k + \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + \sum_{v_{k"} \in H \setminus \{k,k'\}} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s"}^{k"} \leq \left\lfloor \frac{|H|}{2} \right\rfloor.$$

Given that $H' = (H \setminus \{k\}) \cup \{k'\}$ such that $|H'| = |H|$, and $|H|$ is an odd number which implies that $\left\lfloor \frac{|H|}{2} \right\rfloor = \frac{|H|-1}{2}$. As a result

$$\sum_{s'=s_i+w_k-1}^{s_j} \frac{1}{2} z_{s'}^k + \sum_{v_{k'} \in H'} \sum_{s"=s_i+w_{k'}-1}^{s_j} z_{s"}^{k'} \leq \frac{|H'|-1}{2}.$$

That which was to be demonstrated.

b) if there exists a node $v_{k'} \in H$ in H_I^E such that $v_{k'}$ is linked with all nodes $v_k \in H$. As a result, inequality (2.40) is dominated by the following inequality

$$\sum_{v_k \in H} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \frac{|H|-1}{2} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq \frac{|H|-1}{2}.$$

c) if there exists an interval I' of contiguous slots with $I \subset I'$ such that H defines also an odd-hole in the associated conflict graph $H_{I'}^E$. This implies that inequality (2.40) induced by odd-hole H for the interval I is dominated by inequality (2.40) induced by the same odd-hole H for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in H$. As a result, inequality (2.40) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

If no one of these two cases is verified, inequality (2.40) can never be dominated by another inequality without changing its right-hand side.

Sufficiency.

Let $F_H^{H_I^E}$ be the face induced by inequality (2.40), that is

$$F_H^{H_I^E} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}\}.$$

We denote inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_H^{H_I^E} \subseteq F$. In order to prove that inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$, we will show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^{64} = (E^{64}, S^{64})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{64}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) select a subset of demands $\tilde{H}$ from H with $|\tilde{H}| = \frac{|H|-1}{2}$,

c) for each demand $k_i \in \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{64} given by

$$I_i^{64} = [\bigcap_{k_j \in L_i^{64}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\}.$$

where $L_i^{64} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H} : E_{k_i}^{64} \cap E_{k_j}^{64} \neq \emptyset\}$,

d) for each demand $k_i \in H \setminus \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{64} given by

$$I_i^{64} = \bigcap_{k_j \in D_i^{64}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D_i^{64} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap H : E_{k_i}^{64} \cap E_{k_j}^{64} \neq \emptyset\}$. We let $S_{k_i}^{64} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i,

e) for each demand $k_i \in K \setminus H$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{64} given by

$$I_i^{64} = [\bigcap_{k_j \in R_i^{64}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}^{64} \cap (E_k^{64} \cup \{e\}) \neq \emptyset \text{ or } I_i^{64} = \bigcap_{k_j \in R_i^{64}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $R_i^{64} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup H$ such that $E_{k_i}^{64} \cap E_{k_j}^{64} \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in R_i^{64}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_{k_j} + 1, ..., s_k\} = \emptyset$ if $E_{k_i}^{64} \cap (E_k^{64} \cup \{e'\}) \neq \emptyset$ (we take into account the possibility of using edge e' in the selected path E_k^{64} to route demand k in solution $\mathcal{S}^{64}$).

We let $S_{k_i}^{64} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{64}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{64}}, z^{\mathcal{S}^{64}})$ belongs to $F_H^{H_I^E}$. Then we derive a solution $\mathcal{S}^{65} = (E^{65}, S^{65})$ obtained from $\mathcal{S}^{64}$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{64}$ which means that $E_k^{65} = E_k^{64} \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{64}$ remain the same in solution $\mathcal{S}^{65}$, i.e., $S_k^{65} = S_k^{64}$ for each $k \in K$, and $E_{k'}^{65} = E_{k'}^{64}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{65}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{65}}, z^{\mathcal{S}^{65}})$ belongs to $F_H^{H_I^E}$. Hence, solutions $\mathcal{S}^{64}$ and $\mathcal{S}^{65}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{64}} + \sigma z^{\mathcal{S}^{64}} = \mu x^{\mathcal{S}^{65}} + \sigma z^{\mathcal{S}^{65}} = \mu x^{\mathcal{S}^{64}} + \mu_e^k + \sigma z^{\mathcal{S}^{64}}.$$

As a result, $\mu_e^k = 0$.

In a similar way, we can show that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and } e \in E \setminus (E_0^k \cup E_1^k).$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$. Let $\mathcal{S}'^{64} = (E'^{64}, S'^{64})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E'^{64}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) select a subset of demands $\tilde{H}$ from H with $|\tilde{H}| = \frac{|H|-1}{2}$,

c) for each demand k_i from $\tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{64}_i given by

$$I'^{64}_i = [\bigcap_{k_j \in D'^{64}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D'^{64}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{H} : E'^{64}_{k_i} \cap E'^{64}_{k_j} \neq \emptyset\}$,

d) for each demand $k_i \in H \setminus \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{64}_i given by

$$I'^{64}_i = \bigcap_{k_j \in D"^{64}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D"^{64}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cap H : E"^{64}_{k_i} \cap E"^{64}_{k_j} \neq \emptyset\}$,

e) for each demand $k_i \in K \setminus H$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{64}_i given by

$$I'^{64}_i = [\bigcap_{k_j \in R'^{64}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E'^{64}_{k_i} \cap E'^{64}_k \neq \emptyset \text{ or } I'^{64}_i = \bigcap_{k_j \in R'^{64}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $R'^{64}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup H$ such that $E'^{64}_{k_i} \cap E'^{64}_{k_j} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in R'^{64}_i$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E'^{64}_{k_i} \cap E'^{64}_k \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots S'^{64}_k to route demand k in solution $\mathcal{S}'^{64}$).

We let $S'^{64}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{64}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{64}}, z^{\mathcal{S}'^{64}})$ belongs to $F_H^{H_I^E}$. Then consider the solution $\mathcal{S}^{66}$ obtained from $\mathcal{S}'^{64}$ by adding slot s' as last slot to demand k in $\mathcal{S}'^{64}$. Solution $\mathcal{S}^{66}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{66}}, z^{\mathcal{S}^{66}})$ belongs to $F_H^{H_I^E}$. Hence, solutions $\mathcal{S}'^{64}$ and $\mathcal{S}^{66}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{64}} + \sigma z^{\mathcal{S}'^{64}} = \mu x^{\mathcal{S}^{66}} + \sigma z^{\mathcal{S}^{66}} = \mu x^{\mathcal{S}'^{64}} + \sigma z^{\mathcal{S}'^{64}} + \sigma^k_{s'}.$$

Hence, $\sigma_{s'}^{k} = 0$.

In a similar way, we can show that

$$\sigma_{s}^{k} = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in H.$$

Let prove that $\sigma_{s'}^{k'}$ for all $v_{k'} \in H$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ are equivalent.

Consider a demand $k' \in K$ with $v_{k'} \in H$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. Let $\mathcal{S}^{66} = (E^{66}, S^{66})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{66}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) select a subset of demands $\tilde{H}$ from H with $|\tilde{H}| = \frac{|H|-1}{2}$,

c) for each demand k_i from $\tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{66} given by

$$I_i^{66} = [\bigcap_{k_j \in L_i^{66}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\}.$$

where $L_i^{66} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{H} : E_{k_i}^{66} \cap E_{k_j}^{66} \neq \emptyset\}$,

d) for each demand $k_i \in H \setminus \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{66} given by

$$I_i^{66} = \bigcap_{k_j \in D_i^{66}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D_i^{66} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap H : E_{k_i}^{66} \cap E_{k_j}^{66} \neq \emptyset\}$,

e) for each demand $k_i \in K \setminus H$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{66} given by

$$I_i^{66} = [\bigcap_{k_j \in R_i^{66}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E_{k_i}^{66} \cap E_{k'}^{66} \neq \emptyset \text{ or } I_i^{66} = \bigcap_{k_j \in R_i^{66}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $R_i^{66} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup H \text{ such that } E_{k_i}^{66} \cap E_{k_j}^{66} \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in R_i^{66}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E_{k_i}^{66} \cap E_{k'}^{66} \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $S_{k'}^{66}$ to route demand k' in solution $\mathcal{S}^{66}$).

We let $S_{k_i}^{66} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{66}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{66}}, z^{\mathcal{S}^{66}})$ belongs to $F_H^{H_I^E}$. Based on this, we construct a feasible solution $\mathcal{S}^{67} = (E^{67}, S^{67})$ obtained from $\mathcal{S}^{66}$ as belows

a) without changing the established paths for the demands $K \setminus \tilde{K}$ in solution $\mathcal{S}^{66}$, i.e., $E_k^{67} = E_k^{66}$ for each demand $k \in K \setminus \tilde{K}$,

b) remove the last slot $\tilde{s}$ totally covered by the interval I and which has been selected by a demand $k_i \in \{v_{k_1}, ..., v_{k_q}\}$ in solution $\mathcal{S}^{66}$ (i.e., $\tilde{s} \in S_{k_i}^{66}$ and $\tilde{s}' \in \{s_i + w_{k_i} + 1, ..., s_j\}$) such that each pair of nodes $(v_{k'}, v_{k_j})$ are not linked in odd-hole H with $j \neq i$,

c) and select a new last slot $\tilde{s}' \notin \{s_i + w_{k_i} + 1, ..., s_j\}$ for demand k_i i.e., $S_{k_i}^{67} = (S_{k_i}^{66} \setminus \{\tilde{s}\}) \cup \{\tilde{s}'\}$ such that $\{\tilde{s}' - w_{k_i} - 1, ..., \tilde{s}'\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ for each $k \in K$ and $s \in S_k^{66}$ with $E_k^{67} \cap E_{k_i}^{67} \neq \emptyset$,

d) and add slot s' to the set of last slots $S_{k'}^{66}$ assigned to demand k' in solution $\mathcal{S}^{66}$, i.e., $S_{k'}^{67} = S_{k'}^{66} \cup \{s'\}$.

Solution $\mathcal{S}^{67}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{67}}, z^{\mathcal{S}^{67}})$ belongs to $F_H^{H_I^E}$. Hence, solutions $\mathcal{S}^{66}$ and $\mathcal{S}^{67}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{S^{66}} + \sigma z^{S^{66}} = \mu x^{S^{67}} + \sigma z^{S^{67}} = \mu x^{S^{66}} + \sigma z^{S^{66}} + \sigma_{s'}^{k'} + \sigma_{\tilde{s}'}^{k_i} - \sigma_{\tilde{s}}^{k_i}.$$

Since $\sigma_s^k = 0$ for all demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k + 1, ..., s_j\}$ if $v_k \in H$, it follows that $\sigma_{\tilde{s}}^{k_i} = \sigma_{s'}^{k'}$.
In a similar way, we can show that

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_k, v_{k'}) \in H.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_k \in H$ and $s \in \{s_i + w_k - 1, ..., s_j\}$.
Overall, and using the results (2.17) and (2.18), we obtain that

$$\mu_e^k = \begin{cases} \gamma_1^{k,e} & \text{if } e \in E_0^k, \\ \gamma_2^{k,e} & \text{if } e \in E_1^k, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k \in K$ and $e \in E$, and

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\}, \\ \rho & \text{if } v_k \in H \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

for each $k \in K$ and $s \in \mathbb{S}$.
As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

Theorem 2.4.11. *Let H be an odd-hole, and C be a clique in the conflict graph H_I^E with*

a) $|H| \geq 5$,

b) and $H \cap C = \emptyset$,

c) $2w_k > |I|$ *for each* $v_k \in C \cup H$,

d) and the nodes $(v_k, v_{k'})$ *are linked in* H_I^E *for all* $v_k \in H$ *and* $v_{k'} \in C$.

Then, inequality (2.41) *is facet defining for* $\mathcal{P}(G, K, \mathbb{S})$ *if and only if*

a) for each node $v_{k"}$ *in* H_I^E *with* $v_{k"} \notin H \cup C$ *and* $C \cup \{v_{k"}\}$ *is a clique in* H_I^E, *there exists a subset of nodes* $\tilde{H} \subseteq H$ *of size* $\frac{|H|-1}{2}$ *such that* $\tilde{H} \cup \{v_{k"}\}$ *is stable in* H_I^E,

b) and there does not exist an interval I' *of contiguous slots with* $I \subset I'$ *such that* H *and* C *define also an odd-hole and its connected clique in the associated conflict graph* $H_{I'}^E$.

Proof. **Neccessity.**

a) Note that if there exists a node $v_{k"} \notin H \cup C$ in H_I^E such that $v_{k"}$ is linked with all nodes $v_k \in H$ and nodes $v_{k'} \in C$. This implies that inequality (2.41) is dominated by the following inequality

$$\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} + \frac{|H|-1}{2} \sum_{s"=s_i+w_{k"}-1}^{s_j} z_{s'}^{k"} \leq \frac{|H|-1}{2}.$$

b) if there exists an interval I' of contiguous slots with $I \subset I'$ such that H and C define also an odd-hole and its connected clique in the associated conflict graph $H_{I'}^E$. This implies that inequality (2.41) induced by odd-hole H and clique C for the interval I is dominated by inequality (2.41) induced by the same odd-hole H and clique C for the interval I' given that $\{s_i + w_k - 1, ..., s_j\} \subset I'$ for each $k \in H$.

If these cases are not verified, we ensure that inequality (2.41) can never be dominated by another inequality without modifying its right-hand side. Otherwise, inequality (2.41) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_{H,C}^{H_I^E}$ be the face induced by inequality (2.41), that is

$$F_{H,C}^{H_I^E} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \frac{|H|-1}{2} \sum_{v_{k'} \in C} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = \frac{|H|-1}{2}\}.$$

Let denote inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) :$

$\mu x + \sigma z = \tau\}$. Suppose that $F^{H_I^E}_{H,C} \subseteq F$. To prove that $F^{H_I^E}_{H,C}$ is a facet of $\mathcal{P}(G, K, \mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. For this, we show that

a) $\sigma_s^k = 0$ for all demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H \cup C$ as we did in the proof of theorem 2.4.14,

b) and $\mu_e^k = 0$ for all demand $k \in K$ and edge $e \in E \setminus (E_0^k \cup E_1^k)$ as we did in the proof of theorem 2.4.14,

c) and σ_s^k are equivalent for all $v_k \in H$ and $s \in \{s_i + w_k - 1, ..., s_j\}$ as we did in the proof of theorem 2.4.14.

Solutions $\mathcal{S}^{49} - \mathcal{S}^{69}$ still feasible for $F^{H_I^E}_{H,C}$. We should prove now that σ_s^k are equivalent for all $v_k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$.
Consider a demand $\tilde{k}'$ with $v_{\tilde{k}'} \in C$ and a slot $\tilde{s}' \in \{s_i + w_{\tilde{k}'} - 1, ..., s_j\}$. Let $\mathcal{S}^{70} = (E^{70}, S^{70})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{70}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) select a subset of demands $\tilde{H}$ from H with $|\tilde{H}| = \frac{|H|-1}{2}$,

c) for each demand $k_i \in \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{70} given by

$$I_i^{70} = [\bigcap_{k_j \in L_i^{70}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $L_i^{70} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{H} : E_{k_i}^{70} \cap E_{k_j}^{70} \neq \emptyset\}$,

d) for each demand $k_i \in H \setminus \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{70} given by

$$I_i^{70} = \bigcap_{k_j \in D_i^{70}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D_i^{70} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap H : E_{k_i}^{70} \cap E_{k_j}^{70} \neq \emptyset\}$,

e) for each demand $k_i \in K \setminus H$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{70} given by

$$I_i^{70} = [\bigcap_{k_j \in R_i^{70}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., \tilde{s}' - w_{\tilde{k}'}\} \cup \{\tilde{s}' + w_{k_i}, ..., \bar{s}\}]$$

if $E_{k_i}^{70} \cap E_{k'}^{70} \neq \emptyset$ or $I_i^{70} = \bigcap_{k_j \in R_i^{70}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $R_i^{70} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup H$ such that $E_{k_i}^{70} \cap E_{k_j}^{70} \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in R_i^{70}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{\tilde{s}' - w_{\tilde{k}'} + 1, ..., \tilde{s}'\} = \emptyset$ if $E_{k_i}^{70} \cap E_{\tilde{k}'}^{70} \neq \emptyset$ (we take into account the possibility of adding slot $\tilde{s}'$ as a last slot in the selected last slots $S_{\tilde{k}'}^{70}$ to route demand $\tilde{k}'$ in solution $\mathcal{S}^{70}$).

We let $S_{k_i}^{70} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{70}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{70}}, z^{\mathcal{S}^{70}})$ belongs to $F_{H,C}^{H_I^E}$. Then consider the solution $\mathcal{S}^{71}$ obtained from $\mathcal{S}^{70}$ as belows

a) remove all the last slots $\tilde{s}_i$ totally covered by the interval I and which has been selected by each demand $k_i \in \tilde{H}$ in solution $\mathcal{S}^{70}$ (i.e., $\tilde{s} \in S_{k_i}^{70}$ and $\tilde{s} \in \{s_i + w_{k_i} + 1, ..., s_j\}$) for each $k_i \in \tilde{H}$,

b) and select a new last slot $\tilde{s}_i' \notin \{s_i + w_{k_i} + 1, ..., s_j\}$ for each $k_i \in \tilde{H}$ i.e., $S_{k_i}^{71} = (S_{k_i}^{70} \setminus \{\tilde{s}_i\}) \cup \{\tilde{s}_i'\}$ such that $\{\tilde{s}_i' - w_{k_i} - 1, ..., \tilde{s}_i'\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ for each $k \in K$ and $s \in S_k^{70}$ with $E_k^{71} \cap E_{k_i}^{71} \neq \emptyset$ for each $k_i \in \tilde{H}$,

c) and add slot $\tilde{s}'$ to the set of last slots $S_{\tilde{k}'}^{70}$ assigned to demand $\tilde{k}'$ in solution $\mathcal{S}^{70}$, i.e., $S_{\tilde{k}'}^{71} = S_{\tilde{k}'}^{70} \cup \{\tilde{s}'\}$,

d) without changing the set of last slots assigned to the demands $K \setminus \tilde{H}$, i.e., $S_k^{71} = S_k^{70}$ for each demand $K \setminus \tilde{H}$.

Solution $\mathcal{S}^{71}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{71}}, z^{\mathcal{S}^{71}})$ belongs to $F_{H,C}^{H_I^E}$. Hence, solutions $\mathcal{S}^{70}$ and $\mathcal{S}^{71}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{S^{70}} + \sigma z^{S^{70}} = \mu x^{S^{71}} + \sigma z^{S^{71}} = \mu x^{S^{70}} + \sigma z^{S^{70}} + \sigma_{\tilde{k}'}^{\tilde{k}'} + \sum_{k_i \in \tilde{H}} \sigma_{\tilde{s}_i'}^{k_i} - \sum_{k_i \in \tilde{H}} \sigma_{\tilde{s}_i}^{k_i}.$$

Since $\sigma_s^k = 0$ for all demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k + 1, ..., s_j\}$ if $v_k \in H \cup C$, it follows that $\sum_{k_i \in \tilde{H}} \sigma_{\tilde{s}_i}^{k_i} = \sigma_{\tilde{s}'}^{\tilde{k}'}$ for $v_{\tilde{k}'} \in C$.
In a similar way, we can show that

$$\sigma_{s'}^{k'} = \rho \frac{|H| - 1}{2}, \text{for all } v_{k'} \in C \text{ and } s' \in \{s_i + w_{k'} + 1, ..., s_j\}.$$

As a result,

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all } (v_k, v_{k'}) \in C \text{ and } s \in \{s_i + w_k + 1, ..., s_j\} \text{ and } s' \in \{s_i + w_{k'} + 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma_{s'}^{k'} = \rho^{\frac{|H|-1}{2}}$ for all $v_{k'} \in C$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$.

By (2.17) and (2.18), we know that

$$\begin{cases} \mu_{e'}^{k'} = \gamma_1^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} = \gamma_2^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} = \gamma_3^{k',s'} & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e} & \text{if } e \in E_0^k, \\ \gamma_2^{k,e} & \text{if } e \in E_1^k, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\}, \\ \rho & \text{if } v_k \in H \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ \rho^{\frac{|H|-1}{2}} & \text{if } v_k \in C \text{ and } s \in \{s_i + w_k - 1, ..., s_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

2.4.6 Edge-Slot-Assignment-Clique Inequalities

Here, we introduce another conflict graph totally different compared with the conflict graphs presented previously.

Definition 2.4.6. *Let H_S^e be a conflict graph defined as follows. For each slot $s \in \{w_k, ..., \bar{s}\}$ and demand $k \in K$ with $e \notin E_0^k$, consider a node $v_{k,s}$ in H_S^e. Two nodes $v_{k,s}$ and $v_{k',s'}$ are linked by an edge in H_S^e if*

a) $k = k'$,

b) or $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ if $k \neq k'$ and $(k, k') \notin K_c^e$.

Based on this definition, we introduce the following inequalities.

Proposition 2.4.16. *Consider an edge $e \in E$. Let C be a clique in the conflict graph H_S^e with $|C| \geq 3$, and $\sum_{k \in C} w_k \leq \bar{s} - \sum_{k' \in K_e \setminus C} w_{k'}$. Then, the inequality*

$$\sum_{v_{k,s} \in C} (x_e^k + z_s^k) \leq |C| + 1, \tag{2.42}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$.

Proof. It is trivial given the definition of a clique set in the conflict graph H_S^e such that for each two linked nodes $v_{k,s}$ and $v_{k',s'}$ in H_S^e, we have

$$x_e^k + x_e^{k'} + z_s^k + z_{s'}^{k'} \leq 3.$$

This can be generalized for a triplet of linked nodes $v_{k,s}$ and $v_{k',s'}$ and $v_{k'',s''}$ with $w_k + w_{k'} + w_{k"} \leq \bar{s} - \sum_{\tilde{k} \in K_e \setminus \{k,k',k"\}} w_{\tilde{k}}$, such that for each linked nodes $(v_{k,s}, v_{k',s'})$ and $(v_{k,s}, v_{k'',s''})$ and $(v_{k',s'}, v_{k'',s''})$, we have

$$\begin{aligned} x_e^k + x_e^{k'} + z_s^k + z_{s'}^{k'} &\leq 3, \\ x_e^k + x_e^{k"} + z_s^k + z_{s"}^{k"} &\leq 3, \\ x_e^{k'} + x_e^{k"} + z_{s'}^{k'} + z_{s"}^{k"} &\leq 3. \end{aligned}$$

By adding the three previous inequalities, we get the following inequality using the chvatal gomory procedure

$$\begin{aligned} 2x_e^k + 2x_e^{k'} + 2x_e^{k"} + 2z_s^k + 2z_{s'}^{k'} + 2z_{s"}^{k"} &\leq 9 \\ \Rightarrow x_e^k + x_e^{k'} + x_e^{k"} + z_s^k + z_{s'}^{k'} + z_{s"}^{k"} \leq 4 \text{ given that } \left\lfloor \frac{9}{2} \right\rfloor &= 4. \end{aligned}$$

This can be generalized for each clique C with $|C| \geq 4$ while showing that inequality (2.42) can be seen as Chvàtal-Gomory cuts. Using the Chvàtal-Gomory and recurrence procedures, we obtain that

$$\sum_{v_{k,s} \in C'} x_e^k + z_s^k \leq |C'| + 1,$$

for all $C' \subset C$ with $|C'| = |C| - 1$ and $|C'| \geq 3$.
By adding the previous inequalities for all $C' \subset C$ with $|C'| = |C| - 1$, and doing then some simplification, we get at the end that

$$\sum_{v_{k,s} \in C} x_e^k + z_s^k \leq \left\lfloor |C| + \frac{|C|}{|C|-1} \right\rfloor \Rightarrow \sum_{v_{k,s} \in C} x_e^k + z_s^k \leq |C| + 1,$$

given that $\left\lfloor \frac{|C|}{|C|-1} \right\rfloor = 1$. We conclude at the end that inequality (2.42) is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$. □

Theorem 2.4.12. *Consider an edge $e \in E$, and a clique C in the conflict graph H_S^e with $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$. Then, inequality (2.42) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if C is a maximal clique in the conflict graph H_S^e, and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with*

a) $[\min_{v_{k,s}\in C}(s-w_k+1), \max_{v_{k,s}\in C} s] \subset I$,

b) and $w_k + w_{k'} \geq |I|+1$ *for each* $(v_k, v_{k'}) \in C$,

c) and $2w_k \geq |I|+1$ *and* $w_k \leq |I|$ *for each* $v_k \in C$.

Proof. **Neccessity.**

If C is not maximal clique in the conflict graph H_S^e, this means that inequality (2.42) can be dominated by another inequality associated with a clique C' such that $C \subset C'$ without changing its right-hand side. Moreover, if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

a) $[\min_{v_{k,s}\in C}(s-w_k+1), \max_{v_{k,s}\in C} s] \subset I$,

b) and $w_k + w_{k'} \geq |I|+1$ for each $(v_k, v_{k'}) \in C$,

c) and $2w_k \geq |I|+1$ and $w_k \leq |I|$ for each $v_k \in C$.

Then, inequality (2.42) is dominated by inequality (2.36). As a result, inequality (2.42) cannot be facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_C^{H_S^e}$ be the face induced by inequality (2.42), that is

$$F_C^{H_S^e} = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \sum_{v_{k,s}\in C} x_e^k + z_s^k = 1\}.$$

Let denote inequality $\sum_{v_{k,s}\in C} x_e^k + z_s^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_C^{H_S^e} \subseteq F$. In order to prove that inequality $\sum_{v_{k,s}\in C} x_e^k + z_s^k \leq 1$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e'} \in \mathbb{R}$ for all $k' \in K$ and $e' \in E_0^{k'}, \gamma_2^{k,e'} \in \mathbb{R}$ for all $k' \in K$ and $e' \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'}-1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. In a similar way with the proof of theorem 2.4.1, we obtain that

$$\mu_{e'}^{k'} = \begin{cases} \gamma_1^{k',e'} & \text{if } e' \in E_0^{k'}, \\ \gamma_2^{k',e'} & \text{if } e' \in E_1^{k'}, \\ \rho & \text{if } k' \in K(C) \text{ and } e' = e, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $e' \in E$, and

$$\sigma_{s'}^k = \begin{cases} \gamma_3^{k,s'} & \text{if } s' \in \{1, ..., w_k - 1\} \\ \rho & \text{if } v_{k,s} \in C, \\ 0 & \text{otherwise.} \end{cases}$$

for each $k \in K$ and $s' \in \mathbb{S}$, where $K(C) = \{k \in K : \exists s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \in C\}$. As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

2.4.7 Slot-Assignment-Clique Inequalities

On the other hand, we detected that there may exist some cases that are not covered by inequality (2.42) and (2.25) previously introduced. For this, we provide the following definition of a conflict graph and its associated inequality.

Definition 2.4.7. *Let H_S^E be a conflict graph defined as follows. For all slot $s \in \{w_k, ..., \bar{s}\}$ and demand $k \in K$, consider a node $v_{k,s}$ in H_S^E. Two nodes $v_{k,s}$ and $v_{k',s'}$ are linked by an edge in H_S^E if and only if*

- $k = k'$,
- *or $E_1^k \cap E_1^{k'} \neq \emptyset$ and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ if $k \neq k'$.*

Proposition 2.4.17. *Let C be a clique in the conflict graph H_S^E with $|C| \geq 3$. Then, the inequality*

$$\sum_{v_{k,s} \in C} z_s^k \leq 1, \tag{2.43}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$.

Proof. It is trivial given the definition of a clique set in the conflict graph H_S^E such that for each two linked nodes $v_{k,s}$ and $v_{k',s'}$ in H_S^E, we know from the inequality (2.6) that

$$z_s^k + z_{s'}^{k'} \leq 1,$$

given that $x_e^k = x_e^{k'} = 1$ for all $e \in E_1^k \cap E_1^{k'}$ and $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$.
By adding the previous inequalities for all two nodes $v_{k,s}$ and $v_{k',s'}$ in C, and by recurrence procedure we obtain that for all $C' \subseteq C$ with $|C'| = |C| - 1$

$$\sum_{v_{k,s} \in C'} z_s^k \leq 1.$$

By adding the previous inequalities for all $C' \subseteq C$ with $|C'| = |C| - 1$, we get

$$\sum_{\substack{C' \subseteq C \\ |C'| = |C| - 1}} \sum_{v_{k,s} \in C'} z_s^k \leq \sum_{\substack{C' \subseteq C \\ |C'| = |C| - 1}} 1.$$

Note that for each demand k and slot s with $v_{k,s} \in C$, the variable z_s^k appears $(\binom{|C|}{|C|-1} - 1) = |C| - 1$ times in the previous sum. It follows that

$$\sum_{v_{k,s} \in C} (|C| - 1) z_s^k \leq |C|.$$

By dividing the two sides of the previous sum by $|C| - 1$, we have so

$$\sum_{v_{k,s} \in C} z_s^k \quad \leq \quad \left\lfloor \frac{|C|}{|C| - 1} \right\rfloor \quad \Rightarrow \quad \sum_{v_{k,s} \in C} z_s^k \quad \leq \quad 1 \text{ given that } \left\lfloor \frac{|C|}{|C| - 1} \right\rfloor \quad = \quad 1.$$

We conclude at the end that inequality (2.43) is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$. □

Theorem 2.4.13. *Consider a clique C in the conflict graph H_S^E with $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$. Then, inequality (2.43) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if C is a maximal clique in the conflict graph H_S^E, and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with*

a) $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'}) \in C$,

c) and $2w_k \geq |I| + 1$ *and* $w_k \leq |I|$ *for each* $v_k \in C$.

Proof. **Neccessity.**

If C is not maximal clique in the conflict graph H_S^E, this means that inequality (2.43) can be dominated by another inequality associated with a clique C' such that $C \subset C'$ without changing its right-hand side. Moreover, if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

a) $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,

b) and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'}) \in C$,

c) and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in C$.

Then, inequality (2.43) is dominated by inequality (2.39). As a result, inequality (2.43) cannot be facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_C^{H_S^E}$ be the face induced by inequality (2.43), that is

$$F_C^{H_S^E} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_{k,s} \in C} z_s^k = 1\}.$$

Let denote inequality $\sum_{v_{k,s}\in C} z_s^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_C^{H_S^E} \subseteq F$. In order to prove that inequality $\sum_{v_{k,s}\in C} z_s^k \leq 1$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'}-1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta)+\gamma Q$. We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^{72} = (E^{72}, S^{72})$ be the solution given by

1. for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{72}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},
2. for demand k, we let E_k^{72} be the set of edges involved in a shortest path between o_k and d_k such that edge e is compatible with all the selected edges $e \in E_k^{72}$, i.e., $\sum_{e'\in E_k^{72}} \ell_{e'} + \ell_e \leq \bar{l}_k$,
3. select one pair of demand k' and slot s' from clique C (i.e., $v_{k',s'} \in C$), and use slot $s_{k'} = s'$ as last slot,
4. for each demand $k_i \in K \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{72} given by

$$I_i^{72} = [\bigcap_{k_j\in D_i^{72}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}^{72} \cap (E_k^{72} \cup \{e\}) \neq \emptyset \text{ or } I_i^{72} = \bigcap_{k_j\in D_i^{72}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{72} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k'\} : E_{k_i}^{72} \cap E_{k_j}^{72} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{72}$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_{k_j} + 1, ..., s_k\} = \emptyset$ if $E_{k_i}^{72} \cap E_k^{72} \neq \emptyset$ (we take into account the possibility of using edge e in the selected path E_k^{72} to route demand k in solution $\mathcal{S}^{72}$).

We let $S_{k_i}^{72} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{72}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{72}}, z^{\mathcal{S}^{72}})$ belongs to $F_C^{H_S^E}$. Then we derive a solution $\mathcal{S}^{73} = (E^{73}, S^{73})$ obtained from $\mathcal{S}^{72}$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{72}$ which means that $E_k^{73} = E_k^{72} \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{72}$ remain the same in solution $\mathcal{S}^{73}$, i.e., $S_k^{73} = S_k^{72}$ for each $k \in K$, and $E_{k'}^{73} = E_{k'}^{72}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{73}$ is clearly

feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{73}}, z^{\mathcal{S}^{73}})$ belongs to $F_C^{H_S^E}$. Hence, solutions $\mathcal{S}^{72}$ and $\mathcal{S}^{73}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{72}} + \sigma z^{\mathcal{S}^{72}} = \mu x^{\mathcal{S}^{73}} + \sigma z^{\mathcal{S}^{73}} = \mu x^{\mathcal{S}^{72}} + \mu_e^k + \sigma z^{\mathcal{S}^{72}}.$$

As a result, $\mu_e^k = 0$.

In a similar way, we can show that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and } e \in E \setminus (E_0^k \cup E_1^k).$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin C$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin C$. Let $\mathcal{S}'^{72} = (E'^{72}, S'^{72})$ be the solution given by

1. for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E'^{72}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},
2. select one pair of demand k' and slot s' from clique C (i.e., $v_{k',s'} \in C$), and use slot $s_{k'} = s'$ as last slot,
3. for each demand $k_i \in K \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{72}_i given by

$$I'^{72}_i = [\bigcap_{k_j \in D'^{72}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E'^{72}_{k_i} \cap E'^{72}_k \neq \emptyset \text{ or } I'^{72}_i = \bigcap_{k_j \in D'^{72}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D'^{72}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k'\} : E'^{72}_{k_i} \cap E'^{72}_{k_j} \neq \emptyset\}$. This satisfies that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D'^{72}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E'^{72}_{k_i} \cap E'^{72}_k \neq \emptyset$ (we take into account the possibility of adding slot s' in the selected set of last slots S'^{72}_k to route demand k in solution $\mathcal{S}'^{72}$).

We let $S'^{72}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{72}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{72}}, z^{\mathcal{S}'^{72}})$ belongs to $F_C^{H_S^E}$. Then consider the solution $\mathcal{S}^{74}$ obtained from $\mathcal{S}'^{72}$ by adding slot s' as last slot to demand k in $\mathcal{S}'^{72}$. Solution $\mathcal{S}^{74}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{74}}, z^{\mathcal{S}^{74}})$ belongs to $F_C^{H_S^E}$. Hence, solutions $\mathcal{S}'^{72}$ and $\mathcal{S}^{74}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{72}} + \sigma z^{\mathcal{S}'^{72}} = \mu x^{\mathcal{S}^{74}} + \sigma z^{\mathcal{S}^{74}} = \mu x^{\mathcal{S}'^{72}} + \sigma z^{\mathcal{S}'^{72}} + \sigma_{s'}^k.$$

Hence, $\sigma^k_{s'} = 0$.

In a similar way, we can show that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin C.$$

Let prove that σ^k_s for all $v_{k,s} \in C$ are equivalent.

Consider a node $v_{k',s'}$ in C such that $s' \notin S^{72}_{k'}$. Let $\tilde{\mathcal{S}}^{72} = (\tilde{E}^{72}, \tilde{S}^{72})$ be the solution given by

1. for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $\tilde{E}^{72}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},
2. select a pair of demand k and slot s from clique C (i.e., $v_{k,s} \in C$) such that slot $s_k = s$ will be used as last slot for demand k,
3. for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}^{72}_i$ given by

$$\tilde{I}^{72}_i = [\bigcap_{k_j \in \tilde{D}^{72}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } \tilde{E}^{72}_{k_i} \cap \tilde{E}^{72}_{k'} \neq \emptyset \text{ or } \tilde{I}^{72}_i = \bigcap_{k_j \in \tilde{D}^{72}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $\tilde{D}^{72}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : \tilde{E}^{72}_{k_i} \cap \tilde{E}^{72}_{k_j} \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in \tilde{D}^{72}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $\tilde{E}^{72}_{k_i} \cap \tilde{E}^{72}_{k'} \neq \emptyset$ (we take into account the possibility of adding slot s' in the selected set of last slots $\tilde{S}^{72}_{k'}$ to route demand k' in solution $\tilde{\mathcal{S}}^{72}$).

We let $\tilde{S}^{72}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

$\tilde{\mathcal{S}}^{72}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\tilde{\mathcal{S}}^{72}}, z^{\tilde{\mathcal{S}}^{72}})$ belongs to $F_C^{H^E_S}$. Then consider the solution $\mathcal{S}^{75}$ obtained from $\tilde{\mathcal{S}}^{72}$ by adding slot s' as last slot to demand k' in $\tilde{\mathcal{S}}^{72}$, and modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s \in \tilde{S}^{72}_k$ with $v_{k,s} \in C$ and $v_{k,\tilde{s}} \notin C$ such that $S^{75}_k = (\tilde{S}^{72}_k \setminus \{s\}) \cup \{\tilde{s}\}$ and $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S^{75}_{k'}$ with $E^{75}_k \cap E^{75}_{k'} \neq \emptyset$. Solution $\mathcal{S}^{75}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{75}}, z^{\mathcal{S}^{75}})$ belongs to $F_C^{H^E_S}$. Hence, solutions $\tilde{\mathcal{S}}^{72}$ and $\mathcal{S}^{75}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\tilde{\mathcal{S}}^{72}} + \sigma z^{\tilde{\mathcal{S}}^{72}} = \mu x^{\mathcal{S}^{75}} + \sigma z^{\mathcal{S}^{75}} = \mu x^{\tilde{\mathcal{S}}^{72}} + \sigma z^{\tilde{\mathcal{S}}^{72}} + \sigma^{k'}_{s'} - \sigma^k_s + \sigma^k_{\tilde{s}}.$$

Since $\sigma^k_{\tilde{s}} = 0$ for $v_{k,\tilde{s}} \notin C$, and $\mu^k_e = 0$ for all $k \in K$ and $e \in E \setminus (E^k_0 \cup E^k_1)$, it follows that $\sigma^{k'}_{s'} = \sigma^k_s$.
In a similar way, we can show that

$$\sigma^k_s = \sigma^{k'}_{s'}, \text{for all pairs } (v_{k,s}, v_{k',s'}) \in C.$$

Consequently, we obtain that $\sigma^k_s = \rho$ for all pairs $v_{k,s} \in C$.
We know from (2.17) and (2.18) that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

As a result, we obtain that for each $k \in K$ and $e \in E$

$$\mu^k_e = \begin{cases} \gamma^{k,e}_1 & \text{if } e \in E^k_0 \\ \gamma^{k,e}_2 & \text{if } e \in E^k_1 \\ 0 & otherwise \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma^k_s = \begin{cases} \gamma^{k,s}_3 & \text{if } s \in \{1, ..., w_k - 1\} \\ \rho & \text{if } v_{k,s} \in C, \\ 0 & \text{if } v_{k,s} \notin C. \end{cases}$$

As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. □

2.4.8 Slot-Assignment-Odd-Hole Inequalities

One can strengthen inequality (2.43) by introducing the following inequalities based on the so-called odd-hole inequalities.

Proposition 2.4.18. *Let H be an odd-hole in the conflict graph H^E_S with $|H| \geq 5$. Then, the inequality*

$$\sum_{v_{k,s} \in H} z^k_s \leq \frac{|H| - 1}{2}, \tag{2.44}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ that are linked in H.

Proof. It is trivial given the definition of the odd-hole in the conflict graph H_S^E such that for each pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H by an edge, we know that $z_s^k + z_{s'}^{k'} \leq 1$. Given that H is an odd-hole which means that we have $|H|-1$ pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H, and by doing a sum over all pairs of nodes $(v_{k,s}, v_{k',s'})$ linked in H, it follows that

$$\sum_{(v_{k,s},v_{k',s'})\in E(H)} z_s^k + z_{s'}^{k'} \leq |H| - 1.$$

Taking into account that each node v_k in H has two neighbors in H, this implies that z_s^k appears twice in the previous inequality. As a result,

$$\sum_{(v_{k,s},v_{k',s'})\in E(H)} z_s^k + z_{s'}^{k'} = \sum_{v_{k,s}\in H} 2z_s^k \implies \sum_{v_{k,s}\in H} 2z_s^k \leq |H| - 1.$$

As a result,

$$\sum_{v_{k,s}\in H} z_s^k \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2} \text{ since } |H| \text{ is an odd number.}$$

We conclude at the end that inequality (2.44) is valid for $\mathcal{P}(G, K, \mathbb{S})$. □

Note that inequality (2.44) can be strengthened without modifying its right-hand side by combining inequality (2.44) and (2.43).

Proposition 2.4.19. *Let H be an odd-hole, and C be a clique in the conflict graph H_S^E with*

a) $|H| \geq 5$,

b) and $H \cap C = \emptyset$,

c) and the nodes $(v_{k,s}, v_{k',s'})$ *are linked in* H_S^E *for all* $v_{k,s} \in H$ *and* $v_{k',s'} \in C$.

Then, the inequality

$$\sum_{v_{k,s}\in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'}\in C} z_{s'}^{k'} \leq \frac{|H|-1}{2}, \tag{2.45}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$ and pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H.

Proof. It is trivial given the definition of the odd-hole and clique in H_S^E such that if $\sum_{v_{k',s'}\in C} z_{s'}^{k'} = 1$ for a $v_{k',s'} \in C \in C$ which implies that the quantity $\sum_{v_{k,s}\in H} z_s^k$ is forced to be equal to 0. Otherwise, we know from inequality (2.44) that the sum

$\sum_{v_{k,s}\in H} z_s^k$ is always smaller than $\frac{|H|-1}{2}$. We strengthen the proof as belows. For each pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H by an edge, we know that $z_s^k + z_{s'}^{k'} + \sum_{v_{k",s"}\in C} z_{s"}^{k"} \leq 1$ given that all the nodes $v_{k",s"} \in C$ are linked with the nodes $v_{k,s}$ and $v_{k',s'}$. Given that H is an odd-hole which means that we have $|H|-1$ pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H, and by doing a sum for all pairs of nodes $(v_{k,s}, v_{k',s'})$ linked in H, it follows that

$$\sum_{(v_{k,s},v_{k',s'})\in E(H)} z_s^k + z_{s'}^{k'} + \sum_{v_{k",s"}\in C} z_{s"}^{k"} \leq |H| - 1.$$

Taking into account that each node $v_{k,s}$ has two neighbors in H, this implies that z_s^k appears twice in the previous inequality. The sum $\sum_{v_{k",s"}\in C} z_{s"}^{k"}$ appears $|H|-1$ times in in the previous inequality. As a result,

$$\sum_{(v_{k,s},v_{k',s'})\in E(H)} z_s^k + z_{s'}^{k'} + (|H|-1) \sum_{v_{k",s"}\in C} z_{s"}^{k"} \leq |H| - 1$$
$$\Rightarrow \sum_{v_{k,s}\in H} 2z_s^k + (|H|-1) \sum_{v_{k",s"}\in C} z_{s"}^{k"} \leq |H| - 1.$$

By dividing the two sides of the previous sum by 2, and since $|H|$ is an odd number, it follows that

$$\sum_{v_{k,s}\in H} z_s^k + \left\lfloor \frac{|H|-1}{2} \right\rfloor \sum_{v_{k",s"}\in C} z_{s"}^{k"} \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2}.$$

We conclude at the end that inequality (2.45) is valid for $\mathcal{P}(G, K, \mathbb{S})$. □

Theorem 2.4.14. *Let H be an odd-hole in the conflict graph H_S^E with $|H| \geq 5$, and $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H. Then, inequality (2.44) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) for each node $v_{k',s'} \notin H$ in H_S^E, there exists a node $v_{k,s} \in H$ such that the induced graph $H_S^E((H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\})$ does not contain an odd-hole,

b) and there does not exist a node $v_{k',s'} \notin H$ in H_S^E such that $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$,

c) and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

- $[\min_{v_{k,s}\in H}(s - w_k + 1), \max_{v_{k,s}\in H}] \subset I$,
- *and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,*
- *and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$.*

Proof. **Neccessity.**

We distinguish the following cases:

a) if for a node $v_{k',s'} \notin H$ in H_S^E, there exists a node $v_{k,s} \in H$ such that the induced graph $H_S^E(H \setminus \{v_{k,s}\} \cup \{v_{k',s'}\})$ contains an odd-hole $H' = (H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\}$. This implies that inequality (2.44) can be dominated using some technics of lifting based on the following two inequalities $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$, and $\sum_{v_{k',s'} \in H'} z_{s'}^{k'} \leq \frac{|H'|-1}{2}$.

b) if there exists a node $v_{k',s'} \notin H$ in H_S^E such that $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$. This implies that inequality (2.44) can be dominated by the following valid inequality

$$\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} z_{s'}^{k'} \leq \frac{|H|-1}{2}.$$

c) if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

- $[\min_{v_{k,s} \in H}(s - w_k + 1), \max_{v_{k,s} \in H}] \subset I$,
- and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,
- and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$.

This implies that inequality (2.44) is dominated by inequality (2.40).

If no one of these cases is verified, inequality (2.44) can never be dominated by another inequality without changing its right-hand side. Otherwise, inequality (2.44) cannot be facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_H^{H_S^E}$ be the face induced by inequality (2.44), that is

$$F_H^{H_S^E} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}\}.$$

Denote inequality $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_H^{H_S^E} \subseteq F$. In order to prove that inequality $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$, we will show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$. Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^{76} = (E^{76}, S^{76})$ be the solution given by

1. for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E^{76}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},
2. for demand k, we let E^{76}_k be the set of edges involved in a shortest path between o_k and d_k such that edge e is compatible with all the selected edges $e \in E^{76}_k$,
3. select a subset of nodes $\tilde{H}^{76}$ from H with $|\tilde{H}^{76}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k,s}, v_{k',s'}) \in \tilde{H}^{76}$ are not linked in the conflict graph H^E_S,
4. for each pair of demand k and slot s with $v_{k,s} \in \tilde{H}^{76}$, we select slot $s_k = s$ as last slot for demand k,
5. for each demand $k_i \in K \setminus \tilde{H}^{76}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{76}_i given by

$$I^{76}_i = [\bigcap_{k_j \in D^{76}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E^{76}_{k_i} \cap (E^{76}_k \cup \{e\}) \neq \emptyset \text{ or } I^{76}_i = \bigcap_{k_j \in D^{76}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D^{76}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H}^{76} : E^{76}_{k_i} \cap E^{76}_{k_j} \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D^{76}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_{k_j} + 1, ..., s_k\} = \emptyset$ if $E^{76}_{k_i} \cap E^{76}_k \neq \emptyset$ (we take into account the possibility of using edge e in the selected path E^{76}_k to route demand k in solution $\mathcal{S}^{76}$).

We let $S^{76}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{76}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{76}}, z^{\mathcal{S}^{76}})$ belongs to $F^{H^E_S}_H$. Then we derive a solution $\mathcal{S}'^{77} = (E'^{77}, S'^{77})$ obtained from $\mathcal{S}^{76}$ by adding edge $e \in E \setminus (E^k_0 \cup E^k_1)$ for the routing of demand k in solution $\mathcal{S}^{76}$ which means that $E'^{77}_k = E^{76}_k \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{76}$ remain the same in solution $\mathcal{S}'^{77}$, i.e., $S'^{77}_k = S^{76}_k$ for each $k \in K$, and $E'^{77}_{k'} = E^{76}_{k'}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}'^{77}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}'^{77}}, z^{\mathcal{S}'^{77}})$ belongs to $F^{H^E_S}_H$. Hence, solutions $\mathcal{S}^{76}$ and $\mathcal{S}'^{77}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{76}} + \sigma z^{\mathcal{S}^{76}} = \mu x^{\mathcal{S}'^{77}} + \sigma z^{\mathcal{S}'^{77}} = \mu x^{\mathcal{S}^{76}} + \mu^k_e + \sigma z^{\mathcal{S}^{76}}.$$

As a result, $\mu^k_e = 0$.

In a similar way, we can show that

$$\mu^k_e = 0, \text{ for all } k \in K \text{ and } e \in E \setminus (E^k_0 \cup E^k_1).$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin H$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin H$. Let $\mathcal{S}'^{76} = (E'^{76}, S'^{76})$ be the solution given by

1. for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E'^{76}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},
2. select a subset of nodes $\tilde{H}'^{76}$ from H with $|\tilde{H}'^{76}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k,s}, v_{k',s'}) \in \tilde{H}'^{76}$ are not linked in the conflict graph H_S^E,
3. for each pair of demand k and slot s with $v_{k,s} \in \tilde{H}'^{76}$, we select slot $s_k = s$ as last slot for demand k,
4. for each demand $k_i \in K \setminus \tilde{H}'^{76}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{76}_i given by

$$I'^{76}_i = [\bigcap_{k_j \in D'^{76}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E'^{76}_{k_i} \cap E'^{76}_k \neq \emptyset \text{ or } I'^{76}_i = \bigcap_{k_j \in D'^{76}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D'^{76}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H}'^{76} : E'^{76}_{k_i} \cap E'^{76}_{k_j} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D'^{76}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E'^{76}_{k_i} \cap E'^{76}_k \neq \emptyset$ (we take into account the possibility of adding slot s' in the selected set of last slots S'^{76}_k to route demand k in solution $\mathcal{S}'^{76}$).

We let $S'^{76}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{76}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{76}}, z^{\mathcal{S}'^{76}})$ belongs to $F_H^{H_S^E}$. Then consider the solution $\mathcal{S}^{78}$ obtained from $\mathcal{S}'^{76}$ by adding slot s' as last slot to demand k in $\mathcal{S}'^{76}$. Solution $\mathcal{S}^{78}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{78}}, z^{\mathcal{S}^{78}})$ belongs to $F_H^{H_S^E}$. Hence, solutions $\mathcal{S}'^{76}$ and $\mathcal{S}^{78}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{76}} + \sigma z^{\mathcal{S}'^{76}} = \mu x^{\mathcal{S}^{78}} + \sigma z^{\mathcal{S}^{78}} = \mu x^{\mathcal{S}'^{76}} + \sigma z^{\mathcal{S}'^{76}} + \sigma^k_{s'}.$$

Hence, $\sigma^k_{s'} = 0$.

In a similar way, we can show that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin H.$$

Let prove that σ^k_s for all $v_{k,s} \in H$ are equivalent. Consider a node $v_{k',s'}$ in H. we consider the solution $\mathcal{S}^{80} = (E^{80}, S^{80})$ defined as follows

1. for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E^{80}_{k_i}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},
2. select a subset of nodes $\tilde{H}^{80}$ from H with $|\tilde{H}^{80}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k,s}, v_{k",s"}) \in \tilde{H}^{80}$ are not linked in the conflict graph H^E_S, and each $v_{k,s} \in \tilde{H}^{80}$ is not linked with node $v_{k',s'}$ in H^E_S,
3. for each pair of demand k and slot s with $v_{k,s} \in \tilde{H}^{80}$, we select slot $s_k = s$ as last slot for demand k,
4. for each demand $k_i \in K \setminus \tilde{H}^{80}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{80}_i given by

$$I^{80}_i = [\bigcap_{k_j \in D^{80}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E^{80}_{k_i} \cap E^{80}_k \neq \emptyset \text{ or } I^{80}_i = \bigcap_{k_j \in D^{80}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not.}$$

where $D^{80}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H}^{80} : E^{80}_{k_i} \cap E^{80}_{k_j} \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D^{80}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E^{80}_{k_i} \cap E^{80}_{k'} \neq \emptyset$ (we take into account the possibility of adding slot s' in the selected set of last slots $S^{80}_{k'}$ to route demand k' in solution $\mathcal{S}^{80}$).

We let $S^{80}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{80}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{80}}, z^{\mathcal{S}^{80}})$ belongs to $F^{H^E_S}_H$. After that, we derive the solution $\mathcal{S}'^{80} = (E'^{80}, S'^{80})$ from $\mathcal{S}^{80}$ by

a) and adding slot s' as last slot to demand k', i.e., $S'^{80}_{k'} = S^{80}_{k'} \cup \{s'\}$ for demand k',

b) and modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s \in S^{80}_k$ with $v_{k,s} \in H$ and $v_{k,\tilde{s}} \notin H$ such that $S'^{80}_k = (S^{80}_k \setminus \{s\}) \cup \{\tilde{s}\}$ for demand k such that $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S'^{80}_{k'}$ with $E'^{80}_k \cap E'^{80}_{k'} \neq \emptyset$.

The corresponding incidence vector $(x^{\mathcal{S}'^{80}}, z^{\mathcal{S}'^{80}})$ belongs to $F^{H^E_S}_H$. Hence, solutions $\mathcal{S}^{80}$ and $\mathcal{S}'^{80}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}^{80}} + \sigma z^{\mathcal{S}^{80}} = \mu x^{\mathcal{S}'^{80}} + \sigma z^{\mathcal{S}'^{80}} = \mu x^{\mathcal{S}^{80}} + \sigma z^{\mathcal{S}^{80}} + \sigma^{k'}_{s'} - \sigma^k_s + \sigma^k_{\tilde{s}}.$$

It follows that $\sigma^{k'}_{s'} = \sigma^k_s$ for demand k' and a slot $s' \in \{w_{k'}, ..., \bar{s}\}$ with $v_{k',s'} \in H$ given that $\sigma^k_{\tilde{s}} = 0$ for $v_{k,\tilde{s}} \notin H$.

By (2.17) and (2.18), we know that

$$\begin{cases} \mu_{e'}^{k'} = \gamma_1^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_0^{k'}, \\ \mu_{e'}^{k'} = \gamma_2^{k',e'} & \text{for all } k' \in K \text{ and } e' \in E_1^{k'}, \\ \sigma_{s'}^{k'} = \gamma_3^{k',s'} & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e} & \text{if } e \in E_0^k \\ \gamma_2^{k,e} & \text{if } e \in E_1^k \\ 0 & otherwise, \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\} \\ \rho & \text{if } v_{k,s} \in H, \\ 0 & \text{if } v_{k,s} \notin H. \end{cases}$$

As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

Theorem 2.4.15. *Let H be an odd-hole, and C be a clique in the conflict graph H_S^E with*

a) $|H| \geq 5$,

b) and $H \cap C = \emptyset$,

c) and the nodes $(v_{k,s}, v_{k',s'})$ *are linked in* H_S^E *for all* $v_{k,s} \in H$ *and* $v_{k',s'} \in C$,

d) $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ *for each* $(v_{k,s}, v_{k',s'}) \in C$ *and pair of nodes* $(v_{k,s}, v_{k',s'})$ *linked in* H.

Then, inequality (2.45) *is facet defining for* $\mathcal{P}(G, K, \mathbb{S})$ *if and only if*

a) for each node $v_{k",s"}$ *in* H_S^E *with* $v_{k",s"} \notin H \cup C$ *and* $C \cup \{v_{k",s"}\}$ *is a clique in* H_S^E, *there exists a subset of nodes* $\tilde{H} \subseteq H$ *of size* $\frac{|H|-1}{2}$ *such that* $\tilde{H} \cup \{v_{k",s"}\}$ *is stable in* H_S^E,

b) and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ *with*

- $[\min_{v_{k,s} \in H \cup C}(s - w_k + 1), \max_{v_{k,s} \in H \cup C}] \subset I$,
- *and* $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'})$ *linked in* H,

- *and* $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'})$ *linked in* C,
- *and* $w_k + w_{k'} \geq |I| + 1$ *for each* $v_k \in H$ *and* $v_{k'} \in C$,
- *and* $2w_k \geq |I| + 1$ *and* $w_k \leq |I|$ *for each* $v_k \in H$,
- *and* $2w_{k'} \geq |I| + 1$ *and* $w_{k'} \leq |I|$ *for each* $v_{k'} \in C$.

Proof. **Neccessity.**

We distinguish the following cases:

a) if there exists a node $v_{k",s"} \notin H \cup C$ in H_S^E such that $v_{k",s"}$ is linked with all nodes $v_{k,s} \in H$ and also with all nodes $v_{k',s'} \in C$. This implies that inequality (2.45) can be dominated by the following valid inequality

$$\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} + \frac{|H|-1}{2} z_{s"}^{k"} \leq \frac{|H|-1}{2}.$$

b) if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ satisfying the conditions of b), this implies that inequality (2.45) is dominated by inequality (2.41).

If no one of these cases is verified, inequality (2.41) can never be dominated by another inequality without changing its right-hand side. Otherwise, inequality (2.45) cannot be facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_{H,C}^{H_S^E}$ denote the face induced by inequality (2.45), that is

$$F_{H,C}^{H_S^E} = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} = \frac{|H|-1}{2}\}.$$

Let denote inequality $\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} \sum_{v_{k',s'} \in C} z_{s'}^{k'} \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_{H,C}^{H_S^E} \subseteq F$. To prove that $F_{H,C}^{H_S^E}$ is a facet of $\mathcal{P}(G, K, \mathbb{S})$, it suffices to show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. For this, we need to show that

a) $\sigma_s^k = 0$ for all demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin H \cup C$ as done in the proof of theorem 2.4.14,

b) and $\mu_e^k = 0$ for all demand $k \in K$ and edge $e \in E \setminus (E_0^k \cup E_1^k)$ as done in the proof of theorem 2.4.14,

c) and σ_s^k are equivalent for all $v_{k,s} \in H$ as done in the proof of theorem 2.4.14,

given that the solutions $\mathcal{S}^{65}-\mathcal{S}^{80}$ still feasible such that their corresponding incidence vectors belong to $F_{H,C}^{H_S^E}$. In what follows, we prove that $\sigma_{s'}^{k'}$ are equivalent for all $v_{k',s'} \in C$. For this, we consider a node $v_{s'}^{k'} \in C$, and a solution $\mathcal{S}^{82} = (E^{82}, S^{82})$ given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{82}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) select a subset of nodes $\tilde{H}^{82}$ from H with $|\tilde{H}^{82}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k,s}, v_{k",s"}) \in \tilde{H}^{82}$ are not linked in the conflict graph H_S^E,

c) for each pair of demand k and slot s with $v_{k,s} \in \tilde{H}^{82}$, we select slot $s_k = s$ as last slot for demand k,

d) for each demand $k_i \in K \setminus \tilde{H}^{82}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{82} given by

$$I_i^{82} = [\bigcap_{k_j \in D_i^{82}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}^{82} \cap E_k^{82} \neq \emptyset \text{ or } I_i^{82} = \bigcap_{k_j \in D_i^{82}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{82} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H}^{82} : E_{k_i}^{82} \cap E_{k_j}^{82} \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{82}$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E_{k_i}^{82} \cap E_{k'}^{82} \neq \emptyset$ (we take into account the possibility of adding slot s' in the selected set of last slots $S_{k'}^{82}$ to route demand k' in solution $\mathcal{S}^{82}$).

We let $S_{k_i}^{82} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{82}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{82}}, z^{\mathcal{S}^{82}})$ belongs to $F_{H,C}^{G_S^E}$. Then we derive solution $\mathcal{S}^{83}$ obtained from $\mathcal{S}^{82}$ by

a) adding slot s' as last slot to demand k', i.e., $S_{k'}^{83} = S_{k'}^{82} \cup \{s'\}$ with $v_{k',s'} \in C$,

b) and modifying the last slots assigned to each demand $k \in \{\tilde{k} \in K \text{ with } v_{\tilde{k},s} \in \tilde{H}^{82}\}$ by adding a new last slot $\tilde{s}_k$ and removing the last slot $s_k \in S_k^{82}$ with $v_{k,s_k} \in H$ and $v_{k,\tilde{s}_k} \notin H \cup C$ such that $S_k^{83} = (S_k^{82} \setminus \{s_k\}) \cup \{\tilde{s}_k\}$ for each demand $k \in \{\tilde{k} \in K \text{ with } v_{\tilde{k},s} \in \tilde{H}^{82}\}$ such that $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{83}$ with $E_k^{83} \cap E_{k'}^{83} \neq \emptyset$.

Solution $\mathcal{S}^{83}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{83}}, z^{\mathcal{S}^{83}})$ belongs to $F_{H,C}^{H_S^E}$. Hence, solutions $\mathcal{S}^{82}$ and $\mathcal{S}^{83}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}^{82}} + \sigma z^{\mathcal{S}^{82}} = \mu x^{\mathcal{S}^{83}} + \sigma z^{\mathcal{S}^{83}} = \mu x^{\mathcal{S}^{82}} + \sigma z^{\mathcal{S}^{82}} + \sigma_{s'}^{k'} - \sum_{(k,s_k)\in\tilde{H}^{82}} \sigma_{s_k}^k + \sum_{k\in K_H} \sigma_{\tilde{s}_k}^k.$$

where $K_H = \{\tilde{k} \in K \text{ with } v_{\tilde{k},s} \in \tilde{H}^{82}\}$. Since $\sigma_{\tilde{s}_k}^k = 0$ for $v_{k,\tilde{s}_k} \notin H \cup C$, it follows that $\sigma_{s'}^{k'} = \sum_{(k,s_k)\in\tilde{H}^{82}} \sigma_{s_k}^k$.
As a result, $\sigma_{s'}^{k'} = \rho\frac{|H|-1}{2}$ given that σ_s^k are equivalent for all $v_{k,s} \in H$.
Given that the pair $v_{k',s'}$ is chosen arbitrarily in clique C, we re-do the same procedure for all $v_{k',s'} \in C$. Consequently, we obtain that $\sigma_{s'}^{k'} = \rho\frac{|H|-1}{2}$ for all $v_{k',s'} \in C$.
Overall, and using the results (2.17) and (2.18), we obtain that

$$\mu_e^k = \begin{cases} \gamma_1^{k,e} & \text{if } e \in E_0^k, \\ \gamma_2^{k,e} & \text{if } e \in E_1^k, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k \in K$ and $e \in E$, and

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\}, \\ \rho & \text{if } v_{k,s} \in H, \\ \rho\frac{|H|-1}{2} & \text{if } v_{k,s} \in C, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k \in K$ and $s \in \mathbb{S}$.
As a consequence, we obtain that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

Let us now introduce some valid inequalities that are related to the routing sub-problem issus from the *transmission-reach* constraint.

2.4.9 Incompatibility-Clique Inequalities

Based on inequalities (2.19) and (2.20), we introduce the following conflict graph.

Definition 2.4.8. *Let H_E^K be a conflict graph defined as follows. For each demand k and edge $e \notin E_0^k \cup E_1^k$, consider a node v_e^k in H_E^K. Two nodes v_e^k and $v_{e'}^{k'}$ are linked by an edge in H_E^K*

a) if $k = k'$: e and e' are non compatible edges for demand k.

b) if $k \neq k'$: k and k' are non compatible demands for edge e.

Proposition 2.4.20. *Let C be a clique in H_E^K. Then, the inequality*

$$\sum_{v_e^k \in C} x_e^k \leq 1, \tag{2.46}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of a clique set in the conflict graph H_E^K. We know from inequalities (2.19) or (2.20) that for all pairs of nodes $(v_e^k, v_{e'}^{k'})$ in a clique C in H_E^K

$$x_e^k + x_{e'}^{k'} \leq 1,$$

By adding the previous inequalities for all two nodes $v_{k,e}$ and $v_{k',e'}$ in C, and by recurrence procedure we obtain that for all $C' \subseteq C$ with $|C'| = |C| - 1$

$$\sum_{v_{k,e} \in C'} x_e^k \leq 1.$$

By adding the previous inequalities for all $C' \subseteq C$ with $|C'| = |C| - 1$, we get

$$\sum_{\substack{C' \subseteq C \\ |C'| = |C|-1}} \sum_{v_{k,e} \in C'} x_e^k \leq \sum_{\substack{C' \subseteq C \\ |C'| = |C|-1}} 1.$$

Note that for each demand k and edge e with $v_{k,e} \in C$, the variable x_e^k appears $(\binom{|C|}{|C|-1} - 1) = |C| - 1$ times in the previous sum. It follows that

$$\sum_{v_{k,e} \in C} (|C| - 1) x_e^k \leq |C|.$$

By dividing the two sides of the previous sum by $|C| - 1$, we have so

$$\sum_{v_{k,e} \in C} x_e^k \leq \left\lfloor \frac{|C|}{|C| - 1} \right\rfloor \Rightarrow \sum_{v_{k,e} \in C} x_e^k \leq 1 \text{ given that } \left\lfloor \frac{|C|}{|C| - 1} \right\rfloor = 1.$$

This ends the proof. □

Theorem 2.4.16. *Consider a clique C in the conflict graph H_E^K. Then, inequality (2.46) is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if C is a maximal clique in the conflict graph H_E^K.*

Proof. It is trivial given that inequality (2.46) can never be dominated by another inequality without changing its right-hand side.
Let $F_C^{H_E^K}$ denote the face induced by inequality (2.46), that is

$$F_C^{H_E^K} = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \sum_{v_{k,e} \in C} x_e^k = 1\}.$$

Let denote inequality $\sum_{v_{k,e} \in C} x_e^k \leq 1$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G,K,\mathbb{S})$ and $F = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_C^{H_E^K} \subseteq F$. In order to prove that inequality $\sum_{v_{k,e} \in C} x_e^k \leq 1$ is facet defining for $\mathcal{P}(G,K,\mathbb{S})$, we show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1,...,w_{k'}-1\}$) such that $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma Q$.
We first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $v_{k,e} \notin C$.
Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k,e} \notin C$. Let $\mathcal{S}^{84} = (E^{84}, S^{84})$ be the solution given by

a) select one pair of demand k' and edge e' from clique C (i.e., $v_{k',e'} \in C$), we let $E_{k'}^{84}$ be the set of edges involved in a shortest path between $o_{k'}$ and $d_{k'}$ which uses edge e',

b) for each pair of demand k" and edge e" with $v_{k",e"} \in C \setminus \{v_{k,e}\}$, we let $E_{k"}^{84}$ be the set of edges involved in a shortest path between $o_{k"}$ and $d_{k"}$ which uses edge e" which does not use edge e",

c) for each demand $k_i \in K \setminus C$ with $i \in \{1,...,|K|\} \setminus \{k\}$, we let $E_{k_i}^{84}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

d) for demand k, we select the slot $s_k = w_k$ as last slot,

e) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1,...,|K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{84} given by

$$I_i^{84} = [\bigcap_{k_j \in D_i^{84}} \{w_{k_i},...,s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i},...,\bar{s}\}] \cap [\{w_{k_i},...,s_k - w_k\} \cup \{s_k + w_{k_i},...,\bar{s}\}]$$
$$\text{if } E_{k_i}^{84} \cap (E_k^{84} \cup \{e\}) \neq \emptyset \text{ or } I_i^{84} = \bigcap_{k_j \in D_i^{84}} \{w_{k_i},...,s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i},...,\bar{s}\} \text{ if not,}$$

where $D_i^{84} = \{k_j \in \{k_1,...,k_{i-1}\} \cup \{k\} : E_{k_i}^{84} \cap E_{k_j}^{84} \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1,...,s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1,...,s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{84}$,
- $\{s_{k_i} - w_{k_i} + 1,...,s_{k_i}\} \cap \{s_k - w_k + 1,...,s_k\} = \emptyset$ if $E_{k_i}^{84} \cap (E_k^{84} \cup \{e\}) \neq \emptyset$ (we take into account the possibility of using edge e in the selected path E_k^{84} to route demand k in solution $\mathcal{S}^{84}$).

We let $S_{k_i}^{84} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{84}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{84}}, z^{\mathcal{S}^{84}})$ belongs to $F_C^{H_E^K}$. Then we derive a solution $\mathcal{S}^{85} = (E^{85}, S^{85})$ obtained from $\mathcal{S}^{84}$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{84}$ which means that $E_k^{85} = E_k^{84} \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{84}$ remain the same in solution $\mathcal{S}^{85}$, i.e., $S_k^{85} = S_k^{84}$ for each $k \in K$, and $E_{k'}^{85} = E_{k'}^{84}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{85}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{85}}, z^{\mathcal{S}^{85}})$ belongs to $F_C^{H_E^K}$. Hence, solutions $\mathcal{S}^{84}$ and $\mathcal{S}^{85}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{84}} + \sigma z^{\mathcal{S}^{84}} = \mu x^{\mathcal{S}^{85}} + \sigma z^{\mathcal{S}^{85}} = \mu x^{\mathcal{S}^{84}} + \mu_e^k + \sigma z^{\mathcal{S}^{84}}.$$

As a result, $\mu_e^k = 0$.

In a similar way, we can show that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and } e \in E \setminus (E_0^k \cup E_1^k) \text{ with } v_{k,e} \notin C.$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$. Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$, and a solution $\mathcal{S}'^{84} = (E'^{84}, S'^{84})$ such that

a) select one pair of demand k' and edge e' from clique C (i.e., $v_{k',e'} \in C$), we let $E_{k'}'^{84}$ be the set of edges involved in a shortest path between $o_{k'}$ and $d_{k'}$ which uses edge e',

b) for each pair of demand k" and edge e" with $v_{k",e"} \in C \setminus \{v_{k,e}\}$, we let $E_{k"}'^{84}$ be the set of edges involved in a shortest path between $o_{k"}$ and $d_{k"}$ which uses edge e" which does not use edge e",

c) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\} \setminus \{k\}$, we let $E_{k_i}'^{84}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

d) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $I_i'^{84}$ given by

$$I_i'^{84} = [\bigcap_{k_j \in D_i'^{84}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E_{k_i}'^{84} \cap E_k'^{84} \neq \emptyset \text{ or } I_i'^{84} = \bigcap_{k_j \in D_i'^{84}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not.}$$

where $D_i'^{84} = \{k_j \in \{k_1, ..., k_{i-1}\} : E_{k_i}'^{84} \cap E_{k_j}'^{84} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i'^{84}$,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E'^{84}_{k_i} \cap E'^{84}_{k} \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the set of last slots S'^{84}_k to route demand k in solution $\mathcal{S}'^{84}$).

  We let $S'^{84}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

  $\mathcal{S}'^{84}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{84}}, z^{\mathcal{S}'^{84}})$ belongs to $F_C^{H_E^K}$. Then consider the solution $\mathcal{S}^{86}$ obtained from $\mathcal{S}'^{84}$ by adding slot s' as last slot to demand k in $\mathcal{S}'^{84}$. Solution $\mathcal{S}^{86}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{86}}, z^{\mathcal{S}^{86}})$ belongs to $F_C^{H_E^K}$. Hence, solutions $\mathcal{S}'^{84}$ and $\mathcal{S}^{86}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{84}} + \sigma z^{\mathcal{S}'^{84}} = \mu x^{\mathcal{S}^{86}} + \sigma z^{\mathcal{S}^{86}} = \mu x^{\mathcal{S}'^{84}} + \sigma z^{\mathcal{S}'^{84}} + \sigma^k_{s'}.$$

  Hence, $\sigma^k_{s'} = 0$.
  In a similar way, we can show that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin C.$$

  Let prove that μ^k_e for all $v_{k,e}$ are equivalent. Consider a node $v_{k',e'}$ in C such that $e' \notin E^{84}_{k'}$. For this, we derive solution $\mathcal{S}^{87}$ from $\mathcal{S}^{84}$ by

a) modifying the path assigned to demand k' in $\mathcal{S}^{84}$ from $E^{84}_{k'}$ to a path $E^{87}_{k'}$ passed through edge e' with $v_{k',e'} \in C$,

b) modifying the path assigned to demand k in $\mathcal{S}^{84}$ with $e \in E^{84}_k$ and $v_{k,e} \in C$ from E^{84}_k to a path E^{87}_k without passing through any edge $e" \in E \setminus (E^k_0 \cup E^k_1)$ such that $v_{k',e'}$ and $v_{k,e"}$ linked in C,

c) modifying the last slots assigned to some demands $\tilde{K} \subset K$ from $S^{84}_{\tilde{k}}$ to $S^{87}_{\tilde{k}}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint.

  The paths assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{84}$ remain the same in $\mathcal{S}^{87}$ (i.e., $E^{87}_{k"} = E^{84}_{k"}$ for each $k" \in K \setminus \{k, k'\}$), and also without modifying the last slots assigned to the demands $K \setminus \tilde{K}$ in $\mathcal{S}^{84}$, i.e., $S^{84}_k = S^{87}_k$ for each demand $k \in K \setminus \tilde{K}$. Solution $\mathcal{S}^{87}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{87}}, z^{\mathcal{S}^{87}})$ belongs to $F_C^{H_E^K}$. Hence, solutions $\mathcal{S}^{84}$ and $\mathcal{S}^{87}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}^{84}} + \sigma z^{\mathcal{S}^{84}} = \mu x^{\mathcal{S}^{87}} + \sigma z^{\mathcal{S}^{87}} = \mu x^{\mathcal{S}^{84}} + \sigma z^{\mathcal{S}^{84}} + \mu^{k'}_{e'} - \mu^k_e + \sum_{\tilde{k} \in \tilde{K}} \sum_{s' \in S^{87}_{\tilde{k}}} \sigma^{\tilde{k}}_{s'} - \sum_{s \in S^{84}_{\tilde{k}}} \sigma^{\tilde{k}}_s$$
$$+ \sum_{e" \in E^{87}_{k'} \setminus \{e'\}} \mu^{k'}_{e"} - \sum_{e" \in E^{84}_{k'}} \mu^{k'}_{e"} + \sum_{e" \in E^{87}_k} \mu^k_{e"} - \sum_{e" \in E^{84}_k \setminus \{e\}} \mu^k_{e"}.$$

Since $\mu^k_{e"} = 0$ for all $k \in K$ and $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $v_{k,e"} \notin C$, and $\sigma^k_s = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$, it follows that $\mu^{k'}_{e'} = \mu^k_e$.
Given that the pair $(v_{k,e}, v_{k',e'})$ are chosen arbitrarily in clique C, we re-do the same procedure for all pairs $(v_{k,e}, v_{k',e'})$ such that we find

$$\mu^k_e = \mu^{k'}_{e'}, \text{for all pairs } (v_{k,e}, v_{k',e'}) \in C.$$

Consequently, we obtain that $\mu^k_e = \rho$ for all $v_{k,e} \in C$.
By (2.17) and (2.18), we know that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We conclude that for each $k \in K$ and $e \in E$

$$\mu^k_e = \begin{cases} \gamma^{k,e}_1 & \text{if } e \in E^k_0, \\ \gamma^{k,e}_2 & \text{if } e \in E^k_1, \\ \rho & \text{if } v_{k,e} \in C, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma^k_s = \begin{cases} \gamma^{k,s}_3 & \text{if } s \in \{1, ..., w_k - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

2.4.10 Incompatibility-Odd-Hole Inequalities

Proposition 2.4.21. *Let H be an odd-hole in the conflict graph H^K_E with $|H| \geq 5$. Then, the inequality*

$$\sum_{v^k_e \in H} x^k_e \leq \frac{|H| - 1}{2}, \tag{2.47}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of the odd-hole in the conflict graph H^K_E. We strengthen the proof as belows. For each pair of nodes $(v^k_e, v^{k'}_{e'})$ linked in H by an edge, we know that $x^k_e + x^{k'}_{e'} \leq 1$. Given that H is an odd-hole which means that we

have $|H|-1$ pair of nodes $(v_e^k, v_{e'}^{k'})$ linked in H, and by doing a sum for all pairs of nodes $(v_e^k, v_{e'}^{k'})$ linked in H, it follows that

$$\sum_{(v_e^k, v_{e'}^{k'}) \in E(H)} x_e^k + x_{e'}^{k'} \leq |H| - 1.$$

Taking into account that each node v_e^k in H has two neighbors in H, this implies that x_e^k appears twice in the previous inequality. As a result,

$$\sum_{(v_e^k, v_{e'}^{k'}) \in E(H)} x_e^k + x_{e'}^{k'} = \sum_{v_e^k \in H} 2x_e^k \implies \sum_{v_e^k \in H} 2x_e^k \leq |H| - 1$$

$$\implies \sum_{v_e^k \in H} x_e^k \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2} \text{ since } |H| \text{ is an odd number.}$$

We conclude at the end that inequality (2.47) is valid for $\mathcal{P}(G, K, \mathbb{S})$. □

Inequality (2.47) can be strengthened without modifying its right-hand side by combining inequality (2.47) and (2.46) as follows.

Proposition 2.4.22. *Let H be an odd-hole in the conflict graph H_E^K, and C be a clique in the conflict graph H_E^K with*

a) $|H| \geq 5$,

b) and $H \cap C = \emptyset$,

c) and the nodes $(v_e^k, v_{e'}^{k'})$ *are linked in* H_E^K *for all* $v_e^k \in H$ *and* $v_{e'}^{k'} \in C$.

Then, the inequality

$$\sum_{v_e^k \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{e'}^{k'} \in C} x_{e'}^{k'} \leq \frac{|H|-1}{2}, \tag{2.48}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It is trivial given the definition of the odd-hole and clique in H_E^K such that if $\sum_{v_{e'}^{k'} \in C} x_{e'}^{k'} = 1$ for a $v_{e'}^{k'} \in C$, which implies that the quantity $\sum_{v_e^k \in H} x_e^k$ is forced to be equal to 0. Otherwise, we know from inequality (2.47) that the sum $\sum_{v_e^k \in H} x_e^k$ should be smaller than $\frac{|H|-1}{2}$. We strengthen the proof as belows. For each pair of nodes $(v_{k,e}, v_{k',e'})$ linked in H by an edge, we know that $x_e^k + x_{e'}^{k'} + \sum_{v_{k",e"} \in C} x_{e"}^{k"} \leq 1$ given that all the nodes $v_{k",e"} \in C$ are linked with the nodes $v_{k,e}$ and $v_{k',e'}$. Given that H is an odd-hole which means that we have $|H|-1$ pair of nodes $(v_{k,e}, v_{k',e'})$

linked in H, and by doing a sum for all pairs of nodes $(v_{k,e}, v_{k',e'})$ linked in H, it follows that

$$\sum_{(v_{k,e},v_{k',e'})\in E(H)} x_e^k + x_{e'}^{k'} + \sum_{v_{k",e"}\in C} x_{e"}^{k"} \leq |H| - 1.$$

Taking into account that each node $v_{k,e}$ has two neighbors in H, this implies that x_e^k appears twice in the previous inequality. The sum $\sum_{v_{k",e"}\in C} x_{e"}^{k"}$ appears $|H|-1$ times in in the previous inequality. As a result,

$$\sum_{(v_{k,e},v_{k',e'})\in E(H)} x_e^k + x_{e'}^{k'} + (|H|-1) \sum_{v_{k",e"}\in C} x_{e"}^{k"} \leq |H| - 1$$
$$\Rightarrow \sum_{v_{k,e}\in H} 2x_e^k + (|H|-1) \sum_{v_{k",e"}\in C} x_{e"}^{k"} \leq |H| - 1.$$

By dividing the two sides of the previous sum by 2, and since $|H|$ is an odd number, it follows that

$$\sum_{v_{k,e}\in H} x_e^k + \left\lfloor \frac{|H|-1}{2} \right\rfloor \sum_{v_{k",e"}\in C} x_{e"}^{k"} \leq \left\lfloor \frac{|H|-1}{2} \right\rfloor = \frac{|H|-1}{2}.$$

We conclude at the end that inequality (2.48) is valid for $\mathcal{P}(G, K, \mathbb{S})$. □

Theorem 2.4.17. *Let H be an odd-hole in the conflict graph H_E^K with $|H| \geq 5$. Then, inequality* (2.47) *is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if*

a) for each $v_{k',e'} \notin H$, there exists a node $v_{k,e} \in H$ such that the induced graph $H_E^K(H \setminus \{v_{k,e}\} \cup \{v_{k',e'}\})$ does not contain an odd-hole $H' = H \setminus \{v_{k,e}\} \cup \{v_{k',e'}\}$,

b) and there does not exist a node $v_{k',e'} \notin H$ in H_E^K such that $v_{k',e'}$ is linked with all nodes $v_{k,e} \in H$.

Proof. **Neccessity.**

We distinguish the following cases:

a) if for a node $v_{k',e'} \notin H$ in H_E^K, there exists a node $v_{k,e} \in H$ such that the induced graph $H_E^K(H \setminus \{v_{k,e}\} \cup \{v_{k',e'}\})$ contains an odd-hole $H' = (H \setminus \{v_{k,e}\}) \cup \{v_{k',e'}\}$. This implies that inequality (2.47) can be dominated using some technics of lifting based on the following two inequalities $\sum_{v_{k,e}\in H} x_e^k \leq \frac{|H|-1}{2}$, and $\sum_{v_{k',e'}\in H'} x_{e'}^{k'} \leq \frac{|H'|-1}{2}$.

b) if there exists a node $v_{k',e'} \notin H$ in H_E^K such that $v_{k',e'}$ is linked with all nodes $v_{k,e} \in H$. This implies that inequality (2.47) can be dominated by the following valid inequality

$$\sum_{v_{k,e}\in H} x_e^k + \frac{|H|-1}{2} x_{e'}^{k'} \leq \frac{|H|-1}{2}.$$

If no one of these cases is verified, inequality (2.47) can never be dominated by another inequality without changing its right-hand side. Otherwise, inequality (2.47) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.

Let $F_H^{H_E^K}$ denote the face induced by inequality (2.47), that is

$$F_H^{H_E^K} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_{k,e} \in H} x_e^k = \frac{|H| - 1}{2}\}.$$

Denote inequality $\sum_{v_{k,e} \in H} x_e^k \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G, K, \mathbb{S})$ and $F = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_H^{H_E^K} \subseteq F$. In order to prove that inequality $\sum_{v_{k,e} \in H} x_e^k = \frac{|H|-1}{2}$ is facet defining for $\mathcal{P}(G, K, \mathbb{S})$, we show that there exist $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. Let first show that $\mu_e^k = 0$ for each edge $e \in E \setminus (E_0^k \cup E_1^k)$ for each demand $k \in K$ with $v_{k,e} \notin H$.

Consider a demand $k \in K$ and an edge $e \in E \setminus (E_0^k \cup E_1^k)$. Let $\mathcal{S}^{88} = (E^{88}, S^{88})$ be the solution given by

a) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}^{88}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) for demand k, we let E_k^{88} be the set of edges involved in a shortest path between o_k and d_k such that edge e is compatible with all the selected edges $e \in E_k^{88}$, i.e., $\sum_{e' \in E_k^{88}} \ell_{e'} + \ell_e \leq \bar{l}_k$,

c) select a subset of nodes $\tilde{H}^{88}$ from H with $|\tilde{H}^{88}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k',e'}, v_{k",e"}) \in \tilde{H}^{88}$ are not linked in the conflict graph H_E^K,

d) for each pair of demand k' and edge e' with $v_{k',e'} \in \tilde{H}^{88}$, we consider a new set of edges $E_{k'}^{88}$ involved in a shortest path between $o_{k'}$ and $d_{k'}$ if edge e' is not compatible with all the selected edges $e" \in E_{k'}^{88}$, or we add edge e' in $E_{k'}^{88}$ if not, i.e., $E_{k'}^{88} = E_{k'}^{88} \cup \{e'\}$,

e) for each demand k' and edge e' with $v_{k',e'} \in H \setminus \tilde{H}^{88}$, we modify the set of edges $E_{k'}^{88}$ if $E_{k'}^{88}$ contains some edges e' that are non compatible with the selected edges $E_{k"}^{88}$ with $v_{k",e"} \in \tilde{H}^{88}$. This can be done by selecting a new set of edges $E_{k'}^{88}$ which contains all edges involved in a shortest path between $o_{k'}$ and $d_{k'}$ such that edge e' is compatible with each edge $e"$ and demand $k"$ with $v_{k",e"} \in \tilde{H}^{88}$,

f) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{88} given by

$$I_i^{88} = [\bigcap_{k_j \in D_i^{88}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s_k - w_k\} \cup \{s_k + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E_{k_i}^{88} \cap (E_k^{88} \cup \{e\}) \neq \emptyset \text{ or } I_i^{88} = \bigcap_{k_j \in D_i^{88}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{88} = \{k_j \in \{k_1, ..., k_{i-1}\} : E_{k_i}^{88} \cap E_{k_j}^{88} \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{88}$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_k - w_k + 1, ..., s_k\} = \emptyset$ if $E_{k_i}^{88} \cap (E_k^{88} \cup \{e\}) \neq \emptyset$ (we take into account the possibility of using edge e in the selected path E_k^{88} to route demand k in solution $\mathcal{S}^{88}$).

We let $S_{k_i}^{88} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}^{88}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}^{88}}, z^{\mathcal{S}^{88}})$ belongs to $F_H^{H_E^K}$. Then we derive a solution $\mathcal{S}^{89} = (E^{89}, S^{89})$ obtained from $\mathcal{S}^{88}$ by adding edge $e \in E \setminus (E_0^k \cup E_1^k)$ for the routing of demand k in solution $\mathcal{S}^{88}$ which means that $E_k^{89} = E_k^{88} \cup \{e\}$. The last slots assigned to the demands K, and paths assigned the set of demands $K \setminus \{k\}$ in $\mathcal{S}^{88}$ remain the same in solution $\mathcal{S}^{89}$, i.e., $S_k^{89} = S_k^{88}$ for each $k \in K$, and $E_{k'}^{89} = E_{k'}^{88}$ for each $k' \in K \setminus \{k\}$. $\mathcal{S}^{89}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{89}}, z^{\mathcal{S}^{89}})$ belongs to $F_H^{H_E^K}$. Hence, solutions $\mathcal{S}^{88}$ and $\mathcal{S}^{89}$ satisfy equation $\mu x + \sigma z = \tau$. It follows that

$$\mu x^{\mathcal{S}^{88}} + \sigma z^{\mathcal{S}^{88}} = \mu x^{\mathcal{S}^{89}} + \sigma z^{\mathcal{S}^{89}} = \mu x^{\mathcal{S}^{88}} + \mu_e^k + \sigma z^{\mathcal{S}^{88}}.$$

As a result, $\mu_e^k = 0$.

In a similar way, we can show that

$$\mu_e^k = 0, \text{ for all } k \in K \text{ and } e \in E \setminus (E_0^k \cup E_1^k) \text{ with } v_{k,e} \notin H.$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$. Let $\mathcal{S}'^{88} = (E'^{88}, S'^{88})$ be the solution given by

a) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we let $E_{k_i}'^{88}$ be the set of edges involved in a shortest path between o_{k_i} and d_{k_i},

b) select a subset of nodes $\tilde{H}'^{88}$ from H with $|\tilde{H}'^{88}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k',e'}, v_{k",e"}) \in \tilde{H}'^{88}$ are not linked in the conflict graph H_E^K,

c) for each pair of demand k' and edge e' with $v_{k',e'} \in \tilde{H}'^{88}$, we consider a new set of edges $E'^{88}_{k'}$ involved in a shortest path between $o_{k'}$ and $d_{k'}$ if edge e' is not compatible with all the selected edges $e" \in E'^{88}_{k'}$, or we add edge e' in $E'^{88}_{k'}$ if not, i.e., $E'^{88}_{k'} = E'^{88}_{k'} \cup \{e'\}$,

d) for each demand k' and edge e' with $v_{k',e'} \in H \setminus \tilde{H}'^{88}$, we modify the set of edges $E'^{88}_{k'}$ if $E'^{88}_{k'}$ contains some edges e' that are non compatible with the selected edges $E'^{88}_{k"}$ with $v_{k",e"} \in \tilde{H}'^{88}$. This can be done by selecting a new set of edges $E'^{88}_{k'}$ which contains all edges involved in a shortest path between $o_{k'}$ and $d_{k'}$ such that edge e' is compatible with each edge $e"$ and demand $k"$ with $v_{k",e"} \in \tilde{H}'^{88}$,

e) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I'^{88}_i given by

$$I'^{88}_i = [\bigcap_{k_j \in D'^{88}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cap \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E'^{88}_{k_i} \cap E'^{88}_k \neq \emptyset \text{ or } I'^{88}_i = \bigcap_{k_j \in D'^{88}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D'^{88}_i = \{k_j \in \{k_1, ..., k_{i-1}\} : E'^{88}_{k_i} \cap E'^{88}_{k_j} \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D'^{88}_i$,
- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E'^{88}_{k_i} \cap E'^{88}_k \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the set of last slots S'^{88}_k to route demand k in solution $\mathcal{S}'^{88}$).

We let $S'^{88}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

$\mathcal{S}'^{88}$ is feasible for the problem. Hence, the corresponding incidence vector $(x^{\mathcal{S}'^{88}}, z^{\mathcal{S}'^{88}})$ belongs to $F_H^{H_E^K}$. After that, we derive solution $\mathcal{S}^{90}$ obtained from $\mathcal{S}'^{88}$ by adding slot s' as last slot to demand k in $\mathcal{S}'^{88}$. Solution $\mathcal{S}^{90}$ is clearly feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{90}}, z^{\mathcal{S}^{90}})$ belongs to $F_H^{H_E^K}$. Hence, solutions $\mathcal{S}'^{88}$ and $\mathcal{S}^{90}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}'^{88}} + \sigma z^{\mathcal{S}'^{88}} = \mu x^{\mathcal{S}^{90}} + \sigma z^{\mathcal{S}^{90}} = \mu x^{\mathcal{S}'^{88}} + \sigma z^{\mathcal{S}'^{88}} + \sigma^k_{s'}.$$

Hence, $\sigma^k_{s'} = 0$.
In a similar way, we can show that

$$\sigma^k_s = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin H.$$

Let prove now that μ^k_e for all $v_{k,e}$ are equivalent. Consider a node $v_{k',e'}$ in H such that $e' \notin E^{88}_{k'}$. For this, we derive solution $\mathcal{S}^{91}$ from $\mathcal{S}^{88}$ by

a) modifying the path assigned to demand k' in $\mathcal{S}^{88}$ from $E^{88}_{k'}$ to a path $E^{91}_{k'}$ passed through edge e' with $v_{k',e'} \in H$,

b) and selecting a pair of demand-edge (k, e) from the set of pairs of demand-edge in H_{88} such that $v_{k',e'}$ is not linked with any node $v_{k",e"}$ in $H_{88} \setminus \{v_{k,e}\}$,

c) modifying the path assigned to demand k in $\mathcal{S}^{88}$ with $e \in E^{88}_k$ and $v_{k,e} \in H$ from E^{88}_k to a path E^{91}_k without passing through any edge $e" \in E \setminus (E^k_0 \cup E^k_1)$ such that $v_{k',e'}$ and $v_{k,e"}$ linked in H,

d) modifying the last slots assigned to some demands $\tilde{K} \subset K$ from $S^{88}_{\tilde{k}}$ to $S^{91}_{\tilde{k}}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint.

The paths assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{88}$ remain the same in $\mathcal{S}^{91}$ (i.e., $E^{91}_{k"} = E^{88}_{k"}$ for each $k" \in K \setminus \{k, k'\}$), and also without modifying the last slots assigned to the demands $K \setminus \tilde{K}$ in $\mathcal{S}^{88}$, i.e., $S^{88}_k = S^{91}_k$ for each demand $k \in K \setminus \tilde{K}$. Solution $\mathcal{S}^{91}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}^{91}}, z^{\mathcal{S}^{91}})$ belongs to $F_H^{H^K_E}$. Hence, solutions $\mathcal{S}^{88}$ and $\mathcal{S}^{91}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}^{88}} + \sigma z^{\mathcal{S}^{88}} = \mu x^{\mathcal{S}^{91}} + \sigma z^{\mathcal{S}^{91}} = \mu x^{\mathcal{S}^{88}} + \sigma z^{\mathcal{S}^{88}} + \mu^{k'}_{e'} - \mu^k_e + \sum_{\tilde{k} \in \tilde{K}} \sum_{s' \in S^{91}_{\tilde{k}}} \sigma^{\tilde{k}}_{s'} - \sum_{s \in S^{88}_{\tilde{k}}} \sigma^{\tilde{k}}_s$$
$$+ \sum_{e" \in E^{91}_{k'} \setminus \{e'\}} \mu^{k'}_{e"} - \sum_{e" \in E^{88}_{k'}} \mu^{k'}_{e"} + \sum_{e" \in E^{91}_k} \mu^k_{e"} - \sum_{e" \in E^{88}_k \setminus \{e\}} \mu^k_{e"}.$$

Since $\mu^k_{e"} = 0$ for all $k \in K$ and $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $v_{k,e"} \notin H$, and $\sigma^k_s = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$, it follows that $\mu^{k'}_{e'} = \mu^k_e$.
Given that the pair $(v_{k,e}, v_{k',e'})$ are chosen arbitrarily in odd-hole H, we re-do the same procedure for all pairs $(v_{k,e}, v_{k',e'})$ such that we find

$$\mu^k_e = \mu^{k'}_{e'}, \text{for all pairs } (v_{k,e}, v_{k',e'}) \in H.$$

Consequently, we obtain that $\mu^k_e = \rho$ for all $v_{k,e} \in H$.
We know from (2.17) and (2.18) that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

We then conclude that for each $k \in K$ and $e \in E$

$$\mu_e^k = \begin{cases} \gamma_1^{k,e} & \text{if } e \in E_0^k, \\ \gamma_2^{k,e} & \text{if } e \in E_1^k, \\ \rho & \text{if } v_{k,e} \in H, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma_s^k = \begin{cases} \gamma_3^{k,s} & \text{if } s \in \{1, ..., w_k - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

Theorem 2.4.18. *Let H be an odd-hole, and C be a clique in the conflict graph H_E^K with*

a) $|H| \geq 5$,

b) and $H \cap C = \emptyset$,

c) and the nodes $(v_{k,e}, v_{k',e'})$ are linked in H_E^K for all $v_{k,e} \in H$ and $v_{k',e'} \in C$.

Then, inequality (2.48) *is facet defining for $\mathcal{P}(G, K, \mathbb{S})$ if and only if for each node $v_{k",e"}$ in H_E^K with $v_{k",e"} \notin H \cup C$ and $C \cup \{v_{k",e"}\}$ is a clique in H_E^K, there exists a subset of nodes $\tilde{H} \subseteq H$ of size $\frac{|H|-1}{2}$ such that $\tilde{H} \cup \{v_{k",e"}\}$ is stable in H_E^K.*

Proof. **Neccessity.**
If there exists a node $v_{k",e"} \notin H \cup C$ in H_E^K such that $v_{k",e"}$ is linked with all nodes $v_{k,e} \in H$ and also with all nodes $v_{k',e'} \in C$. This implies that inequality (2.48) can be dominated by the following valid inequality

$$\sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} + \frac{|H|-1}{2} x_{e"}^{k"} \leq \frac{|H|-1}{2}.$$

As a result, inequality (2.48) is not facet defining for $\mathcal{P}(G, K, \mathbb{S})$.

Sufficiency.
Let $F_{H,C}^{H_E^K}$ be the face induced by inequality (2.48), that is

$$F_{H,C}^{H_E^K} = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{v_{k,e} \in H} x_e^k + \frac{|H|-1}{2} \sum_{v_{k',e'} \in C} x_{e'}^{k'} = \frac{|H|-1}{2}\}.$$

Let denote inequality $\sum_{v_{k,e}\in H} x_e^k + \frac{|H|-1}{2}\sum_{v_{k',e'}\in C} x_{e'}^{k'} \leq \frac{|H|-1}{2}$ by $\alpha x + \beta z \leq \lambda$. Let $\mu x + \sigma z \leq \tau$ be a facet defining inequality for $\mathcal{P}(G,K,\mathbb{S})$ and $F = \{(x,z) \in \mathcal{P}(G,K,\mathbb{S}) : \mu x + \sigma z = \tau\}$. Suppose that $F_{H,C}^{H_E^K} \subseteq F$. To prove that $F_{H,C}^{H_E^K}$ is a facet of $\mathcal{P}(G,K,\mathbb{S})$, we need to show that there exists $\rho \in \mathbb{R}$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ (such that $\gamma_1^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_0^{k'}, \gamma_2^{k,e} \in \mathbb{R}$ for all $k' \in K$ and $e \in E_1^{k'}, \gamma_3^{k',s'} \in \mathbb{R}$ for all $k' \in K$ and $s' \in \{1, ..., w_{k'} - 1\}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$. For this, we show that

a) $\sigma_s^k = 0$ for all demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ as done in the proof of theorem 2.4.17,

b) and $\mu_e^k = 0$ for all demand $k \in K$ and edge $e \in E \setminus (E_0^k \cup E_1^k)$ with $v_{k,e} \notin H \cup C$ as done in the proof of theorem 2.4.17,

c) and μ_e^k are equivalent for all $v_{k,e} \in H$ as done in the proof of theorem 2.4.17,

given that the solutions defined in the proof of theorem 2.4.17, their corresponding incidence vector belong to $F_{H,C}^{H_E^K}$.

Let prove now that $\mu_{e'}^{k'}$ are equivalent for all $v_{k',e'} \in C$.

Now let us consider a node $v_{k',e'}$ in C such that $e' \notin E_{k'}^{92}$. For this, we derive solution $\mathcal{S}'^{93}$ from $\mathcal{S}^{92}$ by

a) modifying the path assigned to demand k' in $\mathcal{S}^{92}$ from $E_{k'}^{92}$ to a path $E_{k'}'^{93}$ passed through edge e' with $v_{k',e'} \in C$,

b) and modifying the path assigned to each demand k with $v_{k,e} \in H_{92}$ in $\mathcal{S}^{92}$ with $e \in E_k^{92}$ and $v_{k,e} \in H$ from E_k^{92} to a path $E_k'^{93}$ without passing through any edge $e" \in E \setminus (E_0^k \cup E_1^k)$,

c) modifying the last slots assigned to some demands $\tilde{K} \subset K$ from $S_{\tilde{k}}^{92}$ to $S_{\tilde{k}}'^{93}$ for each $\tilde{k} \in \tilde{K}$ while satisfying non-overlapping constraint.

The paths assigned to the demands $K \setminus (K(H_{92}) \cup \{k'\})$ in $\mathcal{S}^{92}$ remain the same in $\mathcal{S}'^{93}$ (i.e., $E_{k"}'^{93} = E_{k"}^{92}$ for each $k" \in K \setminus \{k, k'\}$), and also without modifying the last slots assigned to the demands $K \setminus \tilde{K}$ in $\mathcal{S}^{92}$, i.e., $S_k^{92} = S_k'^{93}$ for each demand $k \in K \setminus \tilde{K}$. Solution $\mathcal{S}'^{93}$ is feasible for the problem. The corresponding incidence vector $(x^{\mathcal{S}'^{93}}, z^{\mathcal{S}'^{93}})$ belongs to $F_{H,C}^{H_E^K}$. Hence, solutions $\mathcal{S}^{92}$ and $\mathcal{S}'^{93}$ satisfy equation $\mu x + \sigma z = \tau$. We have so

$$\mu x^{\mathcal{S}^{92}} + \sigma z^{\mathcal{S}^{92}} = \mu x^{\mathcal{S}'^{93}} + \sigma z^{\mathcal{S}'^{93}} = \mu x^{\mathcal{S}^{92}} + \sigma z^{\mathcal{S}^{92}} + \mu_{e'}^{k'} - \sum_{v_{k,e}\in H_{92}} \mu_e^k + \sum_{\tilde{k}\in\tilde{K}} \sum_{s'\in S_{\tilde{k}}'^{93}} \sigma_{s'}^{\tilde{k}}$$

$$- \sum_{s\in S_{\tilde{k}}^{92}} \sigma_s^{\tilde{k}} + \sum_{e"\in E_{k'}'^{93}\setminus\{e'\}} \mu_{e"}^{k'} - \sum_{e"\in E_{k'}^{92}} \mu_{e"}^{k'} + \sum_{e"\in E_k'^{93}} \mu_{e"}^k - \sum_{k\in K(H_{92})} \sum_{e"\in E_k^{92}} \mu_{e"}^k.$$

Since $\mu^k_{e"} = 0$ for all $k \in K$ and $e" \in E \setminus (E^k_0 \cup E^k_1)$ with $v_{k,e"} \notin H \cup C$, and $\sigma^k_s = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$, it follows that $\mu^{k'}_{e'} = \sum_{v_{k,e} \in H_{92}} \mu^k_e$. As a result, $\mu^{k'}_{e'} = \rho\frac{|H|-1}{2}$.
Given that the pair $v_{k',e'}$ is chosen arbitrarily in clique C, we re-do the same procedure for all pairs $v_{k',e'} \in C$ such that we find

$$\mu^{k'}_{e'} = \rho\frac{|H|-1}{2}, \text{for all pairs } v_{k',e'} \in C.$$

As a result, all $\mu^{k'}_{e'} \in C$ are equivalent such that

$$\mu^{k'}_{e'} = \mu^{k"}_{e"} = \rho\frac{|H|-1}{2}, \text{for all pairs } v_{k',e'}, v_{k",e"} \in C.$$

By (2.17) and (2.18), we know that

$$\begin{cases} \mu^{k'}_{e'} = \gamma^{k',e'}_1 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_0, \\ \mu^{k'}_{e'} = \gamma^{k',e'}_2 & \text{for all } k' \in K \text{ and } e' \in E^{k'}_1, \\ \sigma^{k'}_{s'} = \gamma^{k',s'}_3 & \text{for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}. \end{cases}$$

As a result, we obtain that for each $k \in K$ and $e \in E$

$$\mu^k_e = \begin{cases} \gamma^{k,e}_1 & \text{if } e \in E^k_0, \\ \gamma^{k,e}_2 & \text{if } e \in E^k_1, \\ \rho & \text{if } v_{k,e} \in H, \\ \rho\frac{|H|-1}{2} & \text{if } v_{k,e} \in C, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k \in K$ and $s \in \mathbb{S}$

$$\sigma^k_s = \begin{cases} \gamma^{k,s}_3 & \text{if } s \in \{1, ..., w_k - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma Q$.

□

2.4.11 Tranmission-Reach-Cover Inequalities

Inequalities (2.46), (2.47) and (2.48) can be strengthened by defining a minimal cover related to the transmission-reach constraint.

Definition 2.4.9. *Consider a demand $k \in K$. A cover C for demand k related to the transmission-reach constraint is a subset of edges in $E \setminus (E^k_0 \cup E^k_1)$ such that*

$\sum_{e\in C} \ell_e > \bar{l}_k - \sum_{e'\in E_1^k} \ell_{e'}$, *and each pair of edges* $(e, e') \in C$ *are compatible edges for demand* k. *Furthermore, it is said minimal cover for demand* k *if and only if for each* $e \in C$ *we have* $\sum_{e'\in C\setminus\{e\}} \ell_{e'} \leq \bar{l}_k - \sum_{e"\in E_1^k} l_{e"}$.

Based on this, we introduce the following inequalities.

Proposition 2.4.23. *Consider a demand* $k \in K$. *Let* C *be a minimal cover related to the tranmission-reach constraint for demand* k. *Then, the inequality*

$$\sum_{e\in C} x_e^k \leq |C| - 1, \tag{2.49}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It is trivial given that C is minimal cover for demand k this means that there are at most $|C| - 1$ edges from the set of edges in C that can be used to route demand k. □

Theorem 2.4.19. *Consider a demand* k *in* K. *Let* C *be a minimal cover related to the tranmission-reach constraint for demand* k. *Then, inequality* (2.49) *is facet defining for the polytope* $\mathcal{P}(G, K, \mathbb{S}, C, k)$ *where*

$$\mathcal{P}(G, K, \mathbb{S}, C, k) = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{e'\in E\setminus(E_1^k\cup E_0^k)} x_{e'}^k = 0\}.$$

Proof. It is trivial given that inequality (2.49) can never be dominated in $\mathcal{P}(G, K, \mathbb{S}, C, k)$. □

On the other hand, one can use sequential lifting procedure [5] to sequentially lift the inequality (2.49) and generate lifted valid inequalities that are facet defining for the polytope $\mathcal{P}(G, K, \mathbb{S})$ as follows.

Theorem 2.4.20. *Let* C *be a minimal cover for a demand* $k \in K$. *Let* $E \setminus (E_1^k \cup C \cup E_0^k) = \{e_1, ..., e_q\}$ *be arbitrarily ordred with* $q = |E \setminus (E_1^k \cup C \cup E_0^k)|$. *Consider a sequence of knapsack problems defined as*

$$\begin{cases} z_i = \max \sum_{j\in C} u_j + \sum_{j=1}^{i-1} \alpha_j u_j, \\ \sum_{j\in C} l_j u_j + \sum_{j=1}^{i-1} l_j u_j \leq \bar{l}_k - \sum_{e'\in E_1^k} \ell_{e'} - l_{e_i}, \\ u_j \in \{0, 1\}, \forall j \in C \cup \{1, ..., i-1\}, \end{cases} \tag{2.50}$$

for all $i \in \{1, ..., q\}$ *with* $\alpha_j = |C| - 1 - z_j$ *for all* $j \in \{1, ..., i-1\}$. *Then, the inequality*

$$\sum_{e \in C} x_e^k + \sum_{j=1}^{q} \alpha_j x_{e_j}^k \leq |C| - 1, \tag{2.51}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$. *Moreover, it's facet defining for* $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It's trivial given that inequality (2.51) can never be dominated in $\mathcal{P}(G, K, \mathbb{S})$. □

Definition 2.4.10. *Consider a demand* $k \in K$. *Let* p *be a sub-path in* G *such that each pair of edges* $(v_{k,e}, v_{k,e'}) \in E \setminus (E_0^k \cup E_1^k)$ *are not linked by an edge in the conflict graph* $\tilde{H}_E^K$. *We say that the path* p *is infeasible for the demand* k *if it does not occur as a subpath in any feasible routing for the demand* k, *i.e., there does not exist a feasible path for demand* k *containning* p *due to the transmission-reach constraint. Moreover, it is said to be minimal infeasible if each sub-path* p' *of* p *of cardinality* $|E(p')| = |E(p)| - 1$, *can be used in a feasible routing for the demand* k.

Note that one can verify in polynomial time using Dijkstra algorithm if a sub-path p in G if it is infeasible or not for a demand $k \in K$.

Proposition 2.4.24. *Consider a demand* $k \in K$. *Let* p *be a minimal infeasible sub-path for demand* k *in* G. *Then, the inequality*

$$\sum_{e \in E(p)} x_e^k \leq |E(p)| - 1. \tag{2.52}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It is trivial given that p is minimal infeasible sub-path for demand k this means that there are at most $|E(p)| - 1$ edges from the set of edges in $E(p)$ that can be used to route demand k. □

2.4.12 Edge-Capacity-Cover Inequalities

Let provide now some inequalities related to the capacity constraint over edges.

Proposition 2.4.25. *Consider an edge* e *in* E. *Then, the inequality*

$$\sum_{k \in K \setminus K_e} w_k x_e^k \leq \bar{s} - \sum_{k' \in K_e} w_{k'}, \tag{2.53}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. The number of slots allocated in edge $e \in E$ should be less than the residual capacity of edge e which is equal to $\bar{s} - \sum_{k' \in K_e} w_{k'}$. □

Based on this, we introduce the following definitions.

Definition 2.4.11. *For an edge $e \in E$, a subset of demands $C \subseteq K$ with $e \notin E_0^k \cap E_1^k$ For each demand $k \in C$, is said a cover for edge e if $\sum_{k \in C} w_k > \bar{s} - \sum_{k' \in K_e} w_{k'}$. Moreover, it is said to be a minimal cover if $\sum_{k' \in C \setminus \{k\}} w_{k'} \leq \bar{s} - \sum_{k" \in K_e} w_{k"}$.*

Proposition 2.4.26. *Consider an edge e in E. Let C be a minimal cover in K for edge e. Then, the inequality*

$$\sum_{k \in C} x_e^k \leq |C| - 1, \tag{2.54}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. If C is minimal cover for edge $e \in E$ this means that there are at most $|C|-1$ demands from the set of demands in C that can use edge e. □

Theorem 2.4.21. *Consider an edge e in E. Let C be a minimal cover in K for edge e. Then, inequality (2.54) is facet defining for the polytope $\mathcal{P}(G, K, \mathbb{S}, C, e)$ where*

$$\mathcal{P}(G, K, \mathbb{S}, C, e) = \{(x, z) \in \mathcal{P}(G, K, \mathbb{S}) : \sum_{k' \in K \setminus (C \cup K_e)} x_e^{k'} = 0\}.$$

Proof. It is trivial given that inequality (2.54) can never be dominated in $\mathcal{P}(G, K, \mathbb{S}, C, e)$. □

One can use the sequential lifting procedure [5] to sequentially lift the inequality (2.54) and generate lifted facets for the polytope $\mathcal{P}(G, K, \mathbb{S})$ as follows.

Theorem 2.4.22. *Let C be a minimal cover for an edge $e \in E$. Let $K \setminus (K_e \cup C \cup \bar{K}_e) = \{k_1, ..., k_q\}$ be arbitrarily ordred with $q = |K \setminus (K_e \cup C \cup \bar{K}_e)|$. Consider a sequence of knapsack problems defined as*

$$\begin{cases} z_i = \max \sum_{j \in C} u_j + \sum_{j=1}^{i-1} \alpha_j u_j, \\ \sum_{j \in C} w_j u_j + \sum_{j=1}^{i-1} w_{k_j} u_j \leq \bar{s} - \sum_{k' \in K_e} w_{k'} - w_{k_i}, \\ u_j \in \{0, 1\}, \forall j \in C \cup \{1, ..., i-1\}, \end{cases} \tag{2.55}$$

for all $i \in \{1, ..., q\}$ *with* $\alpha_j = |C| - 1 - z_j$ *for all* $j \in \{1, ..., i-1\}$. *Then, the inequality*

$$\sum_{k \in C} x_e^k + \sum_{j=1}^{q} \alpha_j x_e^{k_j} \leq |C| - 1, \tag{2.56}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$. *Moreover, it's facet defining for* $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It's trivial given that inequality (2.56) can never be dominated in $\mathcal{P}(G, K, \mathbb{S})$. □

2.5 Symmetry-Breaking Inequalities

We have noticed that several symmetrical solutions may appear given that there exist several feasible solutions that have the same value of the solution (called equivalents solutions), and they can be found by doing some permutations between the slots assigned to some demands without changing the selected paths (routing) while satisfying the C-RSA constraints. There exists several methods to break the symmetry. See, for example, perturbation method proposed by Margot [66], isomorphism pruning method by Margot et al. [67][68], orbital branching method by Ostrowski et al. [75][76], orbital fixing method by Kaibel et al. [80], and symmetry-breaking constraints by Kaibel and Pfetsch [79] which is applied in our study. We aim to introduce breaking-symmetry inequalities to remove the sub-problems in the enumeration tree that are equivalent due to the equivalency of their associated solutions. For this, we derive the following inequalities.

Proposition 2.5.1. *Consider a demand* k, *slot* $s \in \{1, ..., \bar{s} - 1\}$. *Let* s' *be a slot in* $\{s, ..., \bar{s}\}$

$$\sum_{s"=s'}^{\min(s'+w_k-1, \bar{s})} z_{s"}^k - \sum_{k' \in K} \sum_{s"=s}^{\min(s+w_{k'}-1, \bar{s})} z_{s"}^{k'} \leq 0. \tag{2.57}$$

This ensures that slot s' can be assigned to demand k if and only if slot s (which precedes slot s') is already assigned to at least one demand k' in K. A similar idea was proposed by Mendez-Diaz and Zabala [70] to break the symmetry for the vertex coloring problem. Note that inequalities (5.17) are not valid for the polytope $\mathcal{P}(G, K, \mathbb{S})$ given that they cut off some feasible regions in the polytope $\mathcal{P}(G, K, \mathbb{S})$. In any case, we ensure that there exists at least one optimal solution from our original problem that remains feasible and belongs to the convex hull of non-symmetric solutions of the C-RSA problem.

2.6 Lower Bounds

In this section, we derive some lower bounds for the C-RSA. Let p_k^* denote the minimum-cost path between origin node o_k and destination node d_k for demand k with total length smaller than the transmission-reach $\bar{l}_k$. We know in advance that the optimal path chosen for each demand $k \in K$ in the optimal solution, its total cost is at least equal to the total cost of the minimum-cost path p_k^*. Based on this, we introduce the following inequalities.

Proposition 2.6.1. *Consider a demand $k \in K$. Then, the inequality*

$$\sum_{e \in E} c_e x_e^k \geq \sum_{e \in E(p_k^*)} c_e, \tag{2.58}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It's trivial given that in any feasible solution S in $\mathcal{P}(G, K, \mathbb{S})$, the total cost of the path selected to demand k is greatest than or equal to the total cost of the minimum-cost path p_k^*. □

Inequality (2.58) is then used to derive a lower bounds for the C-RSA as follows.

Proposition 2.6.2. *The inequality*

$$\sum_{k \in K} \sum_{e \in E} c_e x_e^k \geq \sum_{k \in K} \sum_{e \in E(p_k^*)} c_e, \tag{2.59}$$

is valid for $\mathcal{P}(G, K, \mathbb{S})$.

Proof. It's trivial given that the optimal value is at least equal to the sum of the total cost of minimum-cost path over all the demands in K.

□

The separation problem associated with inequality (2.59) is equivalent to solving the Resource Constrained Shortest Path (RCSP) Problem for each demand k. The RCSP is well known to be a NP-hard problem [31]. For this, we propose a pseudo-polynomial time algorithm using dynamic programming [32] to compute the minimum-cost path for each demand k while satisfying the transmission-reach constraint. For each demand $k \in K$, we associate to each node $v \in V \setminus V_0^k$ in the graph G a set of labels L^v such that each label corresponds to differents paths from th origin node o_k to the node v, and each label p is specified by a cost equals to $\sum_{e \in E(p)} c_e$, and a weight equals to $\sum_{e \in E(p)} \ell_e$. We denote by T_v the set of labels on node $v \in V$. For each demand k and slot $s \in \{w_k, ..., \bar{s}\}$, the complexity of the algorithm is bounded by $\mathcal{O}(|E \setminus E_0^k| * \bar{l}_k)$ [32]. Algorithm 1 summarizes the different steps of the dynamic programming algorithm.

Algorithm 1: Dynamic Programming Algorithm for the Computation of Lower Bounds for the C-RSA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a demand k

Result: constrained minimum-cost path p_k^* for demand k

Set $L^{o_k} = \{(0, 0)\}$ and $L^v = \emptyset$ for each node $v \in V \setminus (V_0^k \cup \{o_k\})$;
Set $T^v = \emptyset$ for each node $v \in V \setminus V_0^k$;
STOP= FALSE;
while $\cup_{v \in V}(L_v \setminus T_v) \neq \emptyset$ **do**
    Select a node $i \in V \setminus V_0^k$ and a label $p \in L^i \setminus T^i$ having the smallest value of $\sum_{e \in E(p)} c_e$;
    for *each* $e = ij \in \delta(i) \setminus E_0^k$ *such that* $\sum_{e' \in E(p)} \ell_{e'} + \ell_e \leq \bar{l}_k$ **do**
        if $j \notin V(p)$ **then**
            Set $p' = p \cup \{e\}$;
            Update the set of label $L^j = L^i \cup \{p'\}$;
        end
    end
    Set $T^i = T^i \cup \{p\}$;
end
We select one label p from the labels L^{d_k} of destination node d_k and set $p^* = p$;
return constrained minimum-cost path p_k^* for demand k;

2.7 Concluding Remarks

In this chapter, we have focused on a complex variant of the Routing and Spectrum Assignment (RSA) problem, called the Constrained-Routing and Spectrum Assignment (C-RSA). We first have proposed a new integer linear programming formulation based on the so-called cut formulation for the C-RSA. We have investigated the facial structure of the associated polytope by showing that some basic inequalities are facet-defining under certain conditions. We have further identified several families of valid inequalities to obtain tighter LP bounds. Moreover, we have studied the facial structure of these valid inequalities, and have shown that they are facet defining for the polytope under certain necessary and sufficient conditions. We have also introduced some symmetry-breaking inequalities to well manage the so-called equivalents sub-problems.

Chapter 3

Branch-and-Cut Algorithm for the C-RSA Problem

Based on theoretical results presented in chapter (2), we devise a Branch-and-Cut algorithm to solve the C-RSA problem. Our aim is to study the effectiveness of the algorithm, and assess the impact of the valid inequalities on the effectiveness of the Branch-and-Cut algorithm. First, we give an overview of the algorithm. Then, we describe the separation procedure used for each valid inequality based on exact algorithms, greedy-algorithms, and heuristics. At the end, we provide a detailed behavior study of the Branch-and-Cut algorithm.

3.1 Branch-and-Cut Algorithm

3.1.1 Description

In what follows, we describe the Branch-and-Cut algorithm. Consider an undirected, loopless, and connected graph $G = (V, E)$, which is specified by a set of nodes V, and a multiset E of links. Each link $e = ij \in E$ is associated with a length $\ell_e \in \mathbb{R}_+$ (in kms), a cost $c_e \in \mathbb{R}_+$ such that each link $e \in E$ is divided into $\bar{s} \in \mathbb{N}_+$ slots. Let $\mathbb{S} = \{1, \ldots, \bar{s}\}$ be an optical spectrum of available frequency slots with $\bar{s} \leq 320$, and K be a multiset of demands such that each demand $k \in K$ is specified by an origin node $o_k \in V$, a destination node $d_k \in V \setminus \{o_k\}$, a slot-width $w_k \in \mathbb{Z}_+$, and a transmission-reach $\bar{\ell_k} \in \mathbb{R}_+$ (in kms). We first consider a restricted linear problem denoted by LP_0 given by inequalities (2.3)-(2.5) and (2.7)-(2.10) such that the cut inequalities (2.2) and non-overlapping inequalities (2.6) are not included in LP_0.

LP_0 is so equivalent to

$$\min \sum_{k \in K} \sum_{e \in E} c_e x_e^k$$
$$\sum_{e \in E} \ell_e x_e^k \leq \bar{\ell_k}, \forall k \in K,$$
$$x_e^k = 0, \forall k \in K, \forall e \in E_0^k,$$
$$x_e^k = 1, \forall k \in K, \forall e \in E_1^k,$$
$$z_s^k = 0, \forall k \in K, \forall s \in \{1, ..., w_k - 1\},$$
$$\sum_{s=w_k}^{\bar{s}} z_s^k = 1, \forall k \in K,$$
$$0 \leq x_e^k \leq 1, \forall k \in K, \forall e \in E,$$
$$0 \leq z_s^k \leq 1, \forall k \in K, \forall s \in \mathbb{S}.$$

3.1.2 Test of Feasibility

Given an optimal solution $(\bar{x}, \bar{z})$ for the relaxation of LP_0. It is feasible for the C-RSA problem if and only if $(\bar{x}, \bar{z})$ is integral and it satisfies the cut inequalities (2.2) and non-overlapping inequalities (2.6). Usually, $(\bar{x}, \bar{z})$ does not satisfy inequalities (2.2) and (2.6). As a result, $(\bar{x}, \bar{z})$ is not feasible for the C-RSA problem. For this, we generate several valid inequalities violated by a solution $(\bar{x}, \bar{z})$ at each iteration of the Branch-and-Cut algorithm. This is known under the name of *Separation Procedure*. It consists in identifying for a given class of valid inequalities the existence of one or more inequalities of this class that are violated by the current solution. We repeat this procedure in each iteration of the algorithm until no violated inequality is identified. As a result, the final solution is optimal for the linear relaxation of the cut formulation. Furthermore, if it is integral, then it is optimal for the C-RSA problem. Otherwise, we create two subproblems called childs by branching on a fractional variable (variable branching rule) or on some constraints using the Ryan & Foster branching rule (constraint branching rule). Based on this, we devise a basic Branch-and-Cut algorithm by combining cutting-plane algorithm based on the separation of the cut inequalities (2.2) and non-overlapping inequalities (2.6), and a Branch-and-Bound algorithm.

On the other hand, to make more efficient the Branch-and-Cut algorithm, we already introduced several classes of valid inequalities used to obtain tighter LP bounds. Based on this, and at each iteration in each node of the Branch-and-Cut tree, one can identify one or more than one violated inequality by the current fractional solution for a given class of valid inequalities. Algorithm 2 summarizes the different steps of Branch-and-Cut algorithm taking into account additional valid inequalities for a

given class of valid inequalities.
For this, we study the separation problem of each class of valid inequality introduced in this dissertation as follows. Consider a fractional solution $(\bar{x}, \bar{z})$.

3.1.3 Separation of Non-Overlapping Inequalities

Let e be an edge in E and s a slot in $\mathbb{S}$. The separation problem associated with inequality (2.6) consists in identifying all pairs of demands $k, k' \in K$ such that

$$\bar{x}_e^k + \bar{x}_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} \bar{z}_{s'}^k + \sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 3.$$

For this, we propose an exact algorithm in $\mathcal{O}(|E| * \bar{s} * |K| * \log(|K|))$ which works as follows. We select each pair of demands $k, k' \in K$ with $x_e^k > 0$, $\sum_{s'=s}^{min(s+w_k-1,\bar{s})} z_{s'}^k > 0$, $\bar{x}_e^{k'} > 0$ and $\sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 0$. We then add the following inequality induced by each selected pair of demands k, k' for slot s over edge e to the current LP if it is violated

$$x_e^k + x_e^{k'} + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{s"=s}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq 3.$$

Otherwise, we conclude that such inequality does not exist for the current solution $(\bar{x}, \bar{z})$. On the other hand, given that inequalities (2.5) are taken in format of equalities when implementing the B&C algorithm (i.e., $\sum_{s=w_k}^{\bar{s}} z_s^k = 1$ for all $k \in K$). Based on this, and taking into account the non-overlapping inequalities (2.6), we propose a new non-overlapping inequality (3.1) more efficient compared to the ones of (2.6).

Proposition 3.1.1. *Consider an edge e, and a pair of demands $k, k' \in K$ with $e \notin E_0^k \cup E_0^{k'}$. Let s be a slot in $\{w_k, ..., \bar{s}\}$. Then, the inequality*

$$x_e^k + x_e^{k'} + z_s^k + \sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq 3, \tag{3.1}$$

is valid for $\mathcal{Q}(G, K, \mathbb{S})$.

The separation problem associated with inequality (3.1) consists in identifying for each edge e, demand $k \in K$, and slot $s \in \{w_k, ..., \bar{s}\}$, a demands $k' \in K$ such that

$$\bar{x}_e^k + \bar{x}_e^{k'} + \bar{z}_s^k + \sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 3.$$

Algorithm 2: Branch-and-Cut Algorithm for the C-RSA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, and a given class of valid inequality

Result: Optimal solution for the C-RSA problem

```
1  LP⟵ LP_0;
2  Stop= FALSE;
3  while STOP==FALSE do
4  |  Solve the linear program LP;
5  |  Let (x*, z*) be the optimal solution of LP;
6  |  if there exist inequalities from the cut inequalities (2.2), non-overlapping
   |   inequalities (2.6), and those of the given class that are violated by the
   |   current solution (x*, z*) then
7  |  |  Add them to LP ;
8  |  end
9  |  else
10 |  |  STOP = TRUE;
11 |  end
12 end
13 Consider the optimal solution (x*, z*) of LP ;
14 if (x*, z*) is integer for the C-RSA then
15 |  (x*, z*) is an optimal solution for the C-RSA;
16 |  End of the Branch-and-Cut algorithm ;
17 end
18 else
19 |  Create two sub-problems by branching one some fractional variables or
   |   constraints ;
20 end
21 for each sub-problem not yet solved do
22 |  go to 3 ;
23 end
24 return the best optimal solution (x*, z*) for the C-RSA problem;
```

For this, we propose an exact algorithm in $\mathcal{O}(|E| * \bar{s} * |K| * (|K| - 1))$ which works as follows. For each demand k and slot $s \in \{w_k, ..., \bar{s}\}$ over edge e with $x_e^k > 0$, $z_s^k > 0$, we select each demand $k' \in K \setminus \{k\}$ with $\bar{x}_e^{k'} > 0$ and $\sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} \bar{z}_{s"}^{k'} > 0$. We then add the following inequality to the current LP if it is violated

$$x_e^k + x_e^{k'} + z_s^k + \sum_{s"=s-w_k+1}^{\min(s+w_{k'}-1,\bar{s})} z_{s"}^{k'} \leq 3.$$

Otherwise, we conclude that there does not exist an inequality from the non-overlapping inequalities (3.1) violated by the current solution $(\bar{x}, \bar{z})$. Note that, from an efficiency point of view, inequalities (3.1) replace inequalities (2.6) in the B&C algorithm.

3.1.4 Separation of Cut Inequalities

In this section we discuss the separation problem of the cut inequalities (2.2). Its associated separation problem consists in identifying a cut inequalities (2.2) that is violated by $(\bar{x}, \bar{z})$. For each demand $k \in K$, this can be done in polynomial time [38] as shown in theorem of Ford and Fulkerson by finding a minimum cut separating the origin-node o_k and destination-node d_k. As a result, this can be done exactly [38] and very effectively in $\mathcal{O}(|V \setminus V_0^k|^2 * \sqrt{|E \setminus E_0^k|})$ for each demand k using an efficient implementation of minimum cut algorithm based on the so-called preflow push-relabel algorithm of Goldberg and Tarjan [44]. It consists in computing a maximum flow/minimum cut in G of demand k by assigning a positif weight $\bar{x}_e^k$ for each edge e in G. For this, we use a C++ library proposed by the LEMON GRAPH library [59] which calls the algorithm of Goldberg and Tarjan for the minimum cut computation. Based on this, we conclude that the separation of the cut inequalities (2.2) can be done in $\mathcal{O}(|V|^2 * \sqrt{|E|} * |K|)$ in the worst case.

3.1.5 Separation of Edge-Slot-Assignment Inequalities

Consider an edge $e \in E$ and a slot $s \in \mathbb{S}$. The separation problem associated with inequality (2.23) consists in identifying a subset of demands $\tilde{K}^* \subset K$ such that

$$\sum_{k \in \tilde{K}^*} \bar{x}_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} \bar{z}_{s'}^k > |\tilde{K}^*| + 1.$$

For this, we propose an exact algorithm in $\mathcal{O}(|K| * |E| * \bar{s})$ which works as follows. The main idea is to iteratively add each demand $k \in K$ to $\tilde{K}^*$ if and only if $x_e^k > 0$ and $\sum_{s'=s}^{min(s+w_k-1,\bar{s})} z_{s'}^k > 0$. We then add the following inequality induced by $\tilde{K}^*$

to the current LP if it is violated and satisfies some conditions about validity of inequality (2.23)

$$\sum_{k \in \tilde{K}^*} x_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq |\tilde{K}^*| + 1.$$

Otherwise, we conclude that such inequality does not exist for the current solution $(\bar{x}, \bar{z})$. Moreover, if such violated inequality is identified, it can be easily lifted by introducing inequality (2.25) induced by $\tilde{K}^*$ and a subset of demands $K_e \setminus \tilde{K}^*$ as follows

$$\sum_{k \in \tilde{K}^*} x_e^k + \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k + \sum_{k' \in K_e \setminus \tilde{K}^*} \sum_{s'=s}^{\min(s+w_{k'}-1,\bar{s})} \leq |\tilde{K}^*| + 1.$$

Remark 3.1.1. *Inequality* (3.1) *is a particular case of inequality* (2.42) *for a clique* $C = \{v_{k,s}\} \cup \{v_{k',s'} \in H_S^e$ *such that* $\{s' - w'_k + 1, ..., s'\} \cap \{s - w_k + 1, ..., s\} \neq \emptyset\}$.

3.1.6 Separation of Edge-Slot-Assignment-Clique Inequalities

Consider an edge $e \in E$. The separation algorithm for inequality (2.42) consists in identifying a maximal clique C^* in the conflict graph H_S^e such that

$$\sum_{v_{k,s} \in C^*} \bar{x}_e^k + \bar{z}_s^k > |C| + 1.$$

To do this, we use the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify a maximal clique C^* in the conflict graph H_S^e given that computing a maximal clique in such a graph is also NP-hard problem [81]. Based on this, we first assign a positive weight $\bar{z}_s^k * \bar{x}_e^k$ to each node $v_{k,s}$ in the conflict graph H_S^e. We then select a node $v_{k,s}$ in the conflict graph H_S^e having the largest weight compared with the other nodes in H_S^e, and set $C^* = \{v_{k,s}\}$. After that, we iteratively add each node $v_{k',s'}$ to the current C^* if it is linked with all the nodes $v_{k,s}$ already assigned to the current clique C^* and $\bar{z}_{s'}^{k'} > 0$ and $\bar{x}_e^{k'} > 0$. At the end, we add inequality (2.42) induced by clique C^* for edge e to the current LP if it is violated and satisfies some conditions about validity of inequality (2.42). Hence, we add the following inequality

$$\sum_{v_{k,s} \in C^*} x_e^k + z_s^k \leq |C| + 1.$$

Furthermore, it can be lifted by identifying a maximal clique N^* such that each $v_{k',s'} \in N^*$ is linked with all the nodes $v_{k,s} \in C^* \cup (N^* \setminus \{v_{k',s'}\})$ in H_S^e. For this,

we also use the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify clique N^* as follows. We first set $N^* = \{v_{k',s'}\}$ with $v_{k',s'} \notin C^*$ a node in H_S^e having the largest value of node-degree (i.e., $|\delta(v_{k',s'})|$) in H_S^e and $v_{k',s'}$ is linked with all the nodes $v_{k,s} \in C^*$ in H_S^e and $k' \in K_e$. Afterwards, we iteratively add each node $v_{k",s"} \notin C^* \cup N^*$ to the current N^* if it is linked in H_S^e with all the nodes already assigned to C^* and N^* and $k" \in K_e$. At the end, we add inequality (2.42) induced by clique $C^* \cup N^*$ to the current LP, i.e.,

$$\sum_{v_{k,s} \in C^*} (x_e^k + z_s^k) + \sum_{v_{k',s'} \in N^*} z_{s'}^{k'} \leq 1.$$

3.1.7 Separation of Edge-Interval-Clique Inequalities

Let discuss the separation problem of inequality (2.32). Consider an edge $e \in E$. We first construct a set of intervals of contiguous slots $I \in I_e$ such that each interval of contiguous slots I_e is identified by generating two slots s_i and s_j randomly in $\mathbb{S}$ with $s_j \geq s_i + 2\max_{k \in K \setminus \bar{K}_e} w_k$. Consider now an interval of contiguous slots $I = [s_i, s_j] \in I_e$ over edge e. The separation problem associated with inequality (2.32) is NP-hard [82] given that it consists in identifying a cover $\tilde{K}^*$ for the interval $I = [s_i, s_j]$ over edge e, such that

$$\sum_{k \in \tilde{K}^*} \bar{x}_e^k + \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 2|\tilde{K}^*| - 1.$$

For this, we use a greedy algorithm introduced by Nemhauser and Sigismondi [73] as follows. We first select a demand $k \in K$ having the largest number of requested slot w_k with $\bar{x}_e^k > 0$ and $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 0$, and then set $\tilde{K}^*$ to $\tilde{K}^* = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus \tilde{K}^*$ to $\tilde{K}^*$ with $\bar{x}_e^{k'} > 0$ and $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$, until a cover $\tilde{K}^*$ is obtained for the interval I over edge e with $\sum_{k \in \tilde{K}^*} w_k > |I|$. We further derive a minimal cover from the cover $\tilde{K}^*$ by deleting each demand $k \in \tilde{K}^*$ if $\sum_{k' \in \tilde{K}^* \setminus \{k\}} w_{k'} \leq |I|$. We then add inequality (2.32) induced by the minimal cover $\tilde{K}^*$ for the interval I and edge e if it is violated and satisfies some conditions about validity of inequality (2.32). The following valid inequality is then added to the current LP

$$\sum_{k \in \tilde{K}^*} x_e^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq 2|\tilde{K}^*| - 1.$$

3.1.8 Separation of Edge-Interval-Clique Inequalities

The separation problem related to inequality (2.36) is NP-hard [77][81] given that it consists in identifying a maximal clique C^* in the conflict graph H_I^e for a given

edge e and a given interval $I = [s_i, s_j]$ such that

$$\sum_{k \in C^*} \bar{x}_e^k + \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > |C^*| + 1.$$

We start our procedure of separation by constructing a set of intervals of contiguous slots $I = [s_i, s_j] \in I_e$ for a given edge $e \in E$ such that each interval of contiguous slots $I = [s_i, s_j] \in I_e$ is identified for each slot $s_i \in \mathbb{S}$ and slot s_j with $s_j \in \{s_i + \max_{k \in K \setminus \bar{K}_e} w_k, ..., \min(\bar{s}, s_i + 2\max_{k \in K \setminus \bar{K}_e} w_k)\}$. Consider now an interval of contiguous slots $I = [s_i, s_j] \in I_e$ over an edge e, and its associated conflict graph H_I^e. We then use a greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify a maximal clique in the conflict graph H_I^e as follows. We first associate a positive weight for each node v_k in H_I^e equals to $\bar{x}_e^k * \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$. We then set $C^* = \{k\}$ such that k is a demand in K having the largest number of slots w_k and weight $\bar{x}_e^k * \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$. After that, we iteratively add each demand k' having $\bar{x}_e^{k'} > 0$ and $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$ such that its corresponding node $v_{k'}$ is linked with all the nodes v_k with k already assigned to the current C^*. After that, we check if inequality (2.36) induced by the maximal clique C^* for the interval I and edge e is violated or not. If so, we add inequality (2.36) induced by the maximal clique C^* to the current LP, i.e.,

$$\sum_{k \in C^*} x_e^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq |C^*| + 1.$$

One can strengthen this additional inequality by adding inequality (2.37) induced by the maximal clique C^* and $C_e^* \subset K_e \setminus C^*$, i.e.,

$$\sum_{k \in C^*} x_e^k + \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{k' \in C_e^*} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |C^*| + 1,$$

such that

a) $w_{k'} + w_k \geq |I| + 1$ for each $k \in C^*$ and $k' \in C_e^*$,

b) $w_{k'} + w_{k"} \geq |I| + 1$ for each $k' \in C_e^*$ and $k" \in C_e^*$,

c) $w_{k'} \leq |I|$ for each $k' \in C_e^*$.

3.1.9 Separation of Interval-Clique Inequalities

Given an interval of contiguous slots $I = [s_i, s_j]$. Our separation algorithm for inequality (2.39) consists in identifying a maximal clique C^* in the conflict graph

H_I^E such that

$$\sum_{k \in C^*} \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 1.$$

As a result, its associated separation problem is NP-hard given that computing a maximal clique in a given graph is known to be a NP-hard problem [81]. For this, we also use the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify a maximal clique in the conflict graph H_I^E as follows. We first generate a set of intervals of contiguous slots denoted by I_E such that each interval of contiguous slots $I = [s_i, s_j] \in I_E$ is given for each slot $s_i \in \mathbb{S}$ and slot s_j with $s_j \in \{s_i + \max_{\substack{k \in K, \\ |E_1^k| \geq 1}} w_k, ..., \min(\bar{s}, s_i + 2 \max_{\substack{k \in K, \\ |E_1^k| \geq 1}} w_k)\}$. We then consider an interval of contiguous slots $I = [s_i, s_j] \in I_E$ and its associated conflict graph H_I^E. We associate a positive weight $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$ for each node v_k in H_I^E. We select a demand k having the largest number of slots w_k and weight $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k$, and then set $C^* = \{k\}$. After that, we iteratively add each demand k' having $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$ such that its corresponding node $v_{k'}$ is linked with all the nodes v_k with $k \in C^*$. At the end, we consider inequality (2.39) induced by the maximal clique C^* if it is violated, i.e., by adding the following inequality to the current LP

$$\sum_{k \in C^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq 1.$$

Moreover, this additional inequality can be strengthened as follows

$$\sum_{k \in C^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k + \sum_{k' \in C_e^*} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq 1,$$

where $C_E^* \subset K \setminus C^*$ such that

a) $w_{k'} + w_k \geq |I| + 1$ and $E_1^k \cap E_1^{k'} \neq \emptyset$ for each $k \in C^*$ and $k' \in C_E^*$,

b) $w_{k'} + w_{k"} \geq |I| + 1$ and $E_1^{k'} \cap E_1^{k"} \neq \emptyset$ for each $k' \in C_E^*$ and $k" \in C_E^*$,

c) $w_{k'} \leq |I|$ for each $k' \in C_E^*$.

3.1.10 Separation of Interval-Odd-Hole Inequalities

For inequality (2.40), we propose a separation algorithm that consists in identifying an odd-hole H^* in the conflict graph H_I^E for a given Interval I and a fractional solution $(\bar{x}, \bar{z})$ such that

$$\sum_{k \in H^*} \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > \frac{|H^*| - 1}{2}.$$

This can be done in polynomial time as shown by Rebennack et al. [94][95]. Based on this, we use the exact algorithm proposed by the same authors which consists of finding a minimum weighted odd-cycle in a graph. For this, we should first generate a set of intervals of contiguous slots I_E as we did before in the section 3.1.9. We then consider a conflict graph H_I^E associated with a given interval of contiguous slots $I \in I_E$. We construct an auxiliary conflict graph $H_I'^E$ which can be seen as a bipartite graph by duplicating each node v_k in H_I^E (i.e., v_k and v'_k) and two nodes are linked in $H_I'^E$ if their original nodes are linked in H_I^E. We assign to each link (v_a, v_b) in $H_I'^E$ a weight equals to $\frac{1-\sum_{s'=s_i+w_a-1}^{s_j} \bar{z}^a_{s'} - \sum_{s'=s_i+w_b-1}^{s_j} \bar{z}^b_{s'}}{2}$. We then compute for each node v_k in H_I^E, the shortest path between v_k and its copy in the auxiliary conflict graph $H_I'^E$ denoted by p_{v_k,v'_k}. After that, we check if the total sum of weight over edges belong this path is smallest than $\frac{1}{2}$,

$$\sum_{(v_a,v_b)\in E(p_{v_k,v'_k})} \frac{1-\sum_{s'=s_i+w_a-1}^{s_j} \bar{z}^a_{s'} - \sum_{s'=s_i+w_b-1}^{s_j} \bar{z}^b_{s'}}{2} < \frac{1}{2}.$$

If so, odd-hole H^* is composed by all the original nodes of nodes belong the computed shortest path p_{v_k,v'_k}, i.e., $V(p_{v_k,v'_k}) \setminus \{v'_k\}$. We then add inequality (2.40) induced by odd-hole H^* to the current LP, i.e.,

$$\sum_{k\in H^*} \sum_{s'=s_i+w_k-1}^{s_j} z^k_{s'} \leq \frac{|H^*|-1}{2}.$$

It can be lifted using the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify a maximal clique C^* in the conflict graph H_I^E such that

a) $w_{k'} + w_k \geq |I| + 1$ and $E_1^k \cap E_1^{k'} \neq \emptyset$ for each $k \in H^*$ and $k' \in C^*$,

b) $w_{k'} + w_{k"} \geq |I| + 1$ and $E_1^{k'} \cap E_1^{k"} \neq \emptyset$ for each $k' \in C^*$ and $k" \in C^*$,

c) $w_{k'} \leq |I|$ for each $k' \in C^*$.

For this, we first assign a positive weight equals to the number of slots request $w_{k'}$ by demand k' for each node $v_{k'}$ linked with all the nodes $v_k \in H^*$ in the conflict graph H_I^E. We then select the node $v_{k'}$ linked with all the nodes $v_k \in H^*$ in the conflict graph H_I^E having the largest weight, and set C^* to $\{k'\}$. After that, we iteratively add each demand $k"$ to the current clique C^* if its associated node $v_{k"}$ is linked with all the nodes $v_k \in H^*$ and nodes $v_{k'} \in C^*$. As a result, we add inequality (2.41) induced by odd-hole H^* and clique C^* to the current LP, i.e.,

$$\sum_{k\in H^*} \sum_{s'=s_i+w_k-1}^{s_j} z^k_{s'} + \frac{|H^*|-1}{2} \sum_{k'\in C^*} \sum_{s"=s_i+w_{k'}-1}^{s_j} z^{k'}_{s"} \leq \frac{|H^*|-1}{2}.$$

3.1.11 Separation of Slot-Assignment-Clique Inequalities

Now, we describe the separation algorithm for inequality (2.43). It consists in identifying a maximal clique C^* in the conflict graph H_S^E such that

$$\sum_{v_{k,s} \in C^*} \bar{z}_s^k > 1,$$

for a given fractional solution $(\bar{x}, \bar{z})$ of the current LP.
For this, we use the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify a maximal clique C^* in the conflict graph H_S^E given that computing a maximal clique in such a graph is also NP-hard problem [81]. Based on this, we first assign a positive weight $\bar{z}_s^k$ to each node $v_{k,s}$ in the conflict graph H_S^E. We then select a node $v_{k,s}$ in the conflict graph H_S^E having the largest weight compared with the other nodes in H_S^E, and set $C^* = \{v_{k,s}\}$. After that, we iteratively add each node $v_{k',s'}$ to the current C^* if it is linked with all the nodes $v_{k,s}$ already assigned to the current clique C^* and $\bar{z}_{s'}^{k'} > 0$. At the end, we add inequality (2.43) induced by clique C^* to the current LP if it is violated, i.e., we add the following inequality

$$\sum_{v_{k,s} \in C^*} z_s^k \leq 1.$$

Furthermore, it can be lifted by identifying a maximal clique N^* such that each $v_{k',s'} \in N^*$ is linked with all the nodes $v_{k,s} \in C^* \cup (N^* \setminus \{v_{k',s'}\})$ in H_S^E. For this, we also use the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify clique N^* as follows. We first set $N^* = \{v_{k',s'}\}$ with $v_{k',s'} \notin C^*$ a node in H_S^E having the largest value of node-degree (i.e., $|\delta(v_{k',s'})|$) in H_S^E and $v_{k',s'}$ is linked with all the nodes $v_{k,s} \in C^*$ in H_S^E. Afterwards, we iteratively add each node $v_{k',s'} \notin C^* \cup N^*$ to the current N^* if it is linked in H_S^E with all the nodes already assigned to C^* and N^*. At the end, we add inequality (2.43) induced by clique $C^* \cup N^*$ to the current LP, i.e.,

$$\sum_{v_{k,s} \in C^*} z_s^k + \sum_{v_{k',s'} \in N^*} z_{s'}^{k'} \leq 1.$$

3.1.12 Separation of Slot-Assignment-Odd-Hole Inequalities

The separation algorithm of inequality (2.44) can be performed by identifying an odd-hole H^* in the conflict graph H_S^E for a given fractional solution $(\bar{x}, \bar{z})$ such that

$$\sum_{v_{k,s} \in H^*} \bar{z}_s^k > \frac{|H^*| - 1}{2}.$$

This can be done in polynomial time as shown by Rebennack et al. [94][95] by finding a minimum weighted odd-cycle in the conflict graph H_S^E. For this, we first construct an auxiliary conflict graph $H_S'^E$ which can be seen also as a bipartite graph by duplicating each node $v_{k,s}$ in H_S^E (i.e., $v_{k,s}$ and $v'_{k,s}$) such that each two nodes are linked in $H_S'^E$ if their original nodes are linked in H_S^E. We assign to each link $(\tilde{v}_{k,s}, \tilde{v}_{k',s'})$ in $H_S'^E$ a weight equals to $\frac{1-\bar{z}_s^k-\bar{z}_{s'}^{k'}}{2}$. We then compute for each node $v_{k,s}$ in H_S^E, the shortest path between $v_{k,s}$ and its copy $v'_{k,s}$ in the auxiliary conflict graph $H_S'^E$ denoted by $p_{v_{k,s},v'_{k,s}}$. After that, we check if the total sum of weight over edges belonging to this path is smaller than $\frac{1}{2}$. If so, odd-hole H^* is composed by all the original nodes of nodes belong the computed shortest path $p_{v_{k,s},v'_{k,s}}$, i.e., $V(p_{v_{k,s},v'_{k,s}}) \setminus \{v'_{k,s}\}$. As a result, the following inequality (2.44) induced by odd-hole H^*

$$\sum_{v_{k,s} \in H^*} z_s^k \leq \frac{|H^*|-1}{2},$$

should be added to the current LP. Moreover, one can strengthen inequality (2.44) induced by odd-hole H^* using the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify a maximal clique C^* in the conflict graph H_S^E such that each node $v_{k',s'} \in C^*$ should have a link with all the nodes $v_{k,s} \in H^*$, and the nodes $v_{k",s"} \in C^* \setminus \{v_{k',s'}\}$ in the conflict graph H_S^E. For this, we first assign a node $v_{k',s'} \notin H^*$ to clique C^* (i.e., $C^* = \{v_{k',s'}\}$) such that $v_{k',s'}$ has the largest value of node-degree (i.e., $|\delta(v_{k',s'})|$) in H_S^E and $v_{k',s'}$ is linked with all the nodes $v_{k,s} \in H^*$ in H_S^E. After that, we iteratively add each node $v_{k',s'} \notin H^* \cup C^*$ to the current clique C^* if it is linked in H_S^E with all the nodes already assigned to odd-hole H^* and clique C^*. We then add inequality (2.45) induced by odd-hole H^* and clique C^*

$$\sum_{v_{k,s} \in H^*} z_s^k + \frac{|H^*|-1}{2} \sum_{v_{k',s'} \in C^*} z_{s'}^{k'} \leq \frac{|H^*|-1}{2},$$

3.1.13 Separation of Incompatibility-Clique Inequalities

Consider now inequality (2.46). Its associated separation algorithm consists in identifying a maximal clique C^* in the conflict graph H_E^K such that

$$\sum_{v_{k,e} \in C^*} \bar{x}_e^k > 1.$$

The separation problem related to this inequality is NP-hard given that computing a maximal clique in the conflict graph H_E^K is NP-hard problem [81]. For this, we also

use the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify a maximal clique in the conflict graph H_E^K taking into account the fractional solution $(\bar{x}, \bar{z})$ as follows. We first assign a positive weight $\bar{x}_e^k$ to each node $v_{k,e}$ in the conflict graph H_E^K. We then select a node $v_{k,e}$ in the conflict graph H_E^K having the largest weight $\bar{x}_e^k$, and set $C^* = \{v_{k,e}\}$. After that, we iteratively add each node $v_{k',e'}$ to the current C^* if it is linked with all the nodes $v_{k,e} \in C^*$ and $\bar{x}_{e'}^{k'} > 0$. At the end, the following inequality (2.46) induced by clique C^*

$$\sum_{v_{k,e} \in C^*} x_e^k \leq 1,$$

should be added to the current LP if it is violated. Furthermore, one can strengthen the additional inequality (2.46) by identifying a maximal clique N^* such that each $v_{k',e'} \in N^*$ is linked with all the nodes $v_{k,e} \in C^* \cup (N^* \setminus \{v_{k',e'}\})$ in H_E^K. For this, we also use the greedy algorithm introduced by Nemhauser and Sigismondi [73] to identify clique N^* as follows. We first set $N^* = \{v_{k',e'}\}$ with $v_{k',e'} \notin C^*$ a node in H_E^K having the largest degree $|\delta(v_{k',e'})|$ in H_E^K and should be also linked with all the nodes $v_{k,e} \in C^*$ in H_E^K. We then iteratively add each node $v_{k',e'} \notin C^* \cup N^*$ to the current N^* if it is linked in H_E^K with all the nodes already assigned to C^* and N^*. At the end, we add inequality (2.46) induced by clique $C^* \cup N^*$ to the current LP, i.e.,

$$\sum_{v_{k,e} \in C^*} x_e^k + \sum_{v_{k',e'} \in N^*} x_{e'}^{k'} \leq 1.$$

3.1.14 Separation of Incompatibility-Odd-Hole Inequalities

The separation algorithm related to inequality (2.47) can be done in polynomial time by finding a minimum weighted odd-cycle in the conflict graph H_E^K as shown by Rebennack et al. [94][95]. For this, our aims is to identify an odd-hole H^* in the conflict graph H_E^K such that

$$\sum_{v_{k,e} \in H^*} \bar{x}_e^k > \frac{|H^*| - 1}{2},$$

for a given fractional solution $(\bar{x}, \bar{z})$ of the current LP.

We start its procedure of separation by constructing an auxiliary conflict graph $H_E'^K$ by duplicating each node $v_{k,e}$ in H_E^K (i.e., $v_{k,e}$ and $v'_{k,e}$) such that each two nodes are linked in $H_E'^K$ if their original nodes are linked in H_E^K. We assign to each link $(\tilde{v}_{k,e}, \tilde{v}_{k',e'})$ in $H_E'^K$ a weight $\frac{1-\bar{x}_e^k-\bar{x}_{e'}^{k'}}{2}$. After that, we compute for each node $v_{k,e}$ in H_E^K, the shortest path between $v_{k,e}$ and its copy $v'_{k,e}$. We denote this shortest path

by $p_{v_{k,e},v'_{k,e}}$. Note that if the total sum of weight over edges belonging to this path is smaller than $\frac{1}{2}$, this means that there exists odd-hole H^* composed by all the original nodes of nodes belong the computed shortest path $p_{v_{k,e},v'_{k,s}}$, i.e., $V(p_{v_{k,e},v'_{k,s}}) \setminus \{v'_{k,s}\}$, such that its associated inequality (2.47) is violated by the current fractional solution $(\bar{x}, \bar{z})$ to the current LP. As a result, we add following inequality (2.47) induced by odd-hole H^*

$$\sum_{v_{k,e} \in H^*} x_e^k \leq \frac{|H^*| - 1}{2}.$$

Moreover, inequality (2.47) induced by odd-hole H^* can be lifted using the greedy algorithm introduced by Nemhauser and Sigismondi [73] by identifying a maximal clique C^* in the conflict graph H_E^K such that each node $v_{k',e'} \in C^*$ should have a link with all the nodes $v_{k,e} \in H^*$, and the nodes $v_{k'',e''} \in C^* \setminus \{v_{k',e'}\}$ in the conflict graph H_E^K. For this, we first assign a node $v_{k',e'} \notin H^*$ to clique C^* (i.e., $C^* = \{v_{k',e'}\}$) having the largest degree $|\delta(v_{k',e'})|$ in H_E^K, and $v_{k',e'}$ should be linked with all the nodes $v_{k,e} \in H^*$ in H_E^K. After that, we iteratively add each node $v_{k',e'} \notin H^* \cup C^*$ to the current clique C^* if it is linked in H_E^K with all the nodes already assigned to $H^* \cup C^*$. We then add inequality (2.48) induced by odd-hole H^* and clique C^*

$$\sum_{v_{k,e} \in H^*} x_e^k + \frac{|H^*| - 1}{2} \sum_{v_{k',e'} \in C^*} x_{e'}^{k'} \leq \frac{|H^*| - 1}{2}.$$

3.1.15 Separation of Transmission-Reach-Cover Inequalities

In this section, we study the separation problem of inequality (2.49). Consider a demand $k \in K$. The separation problem associated with inequality (2.49) is NP-hard [82] given that it consists in identifying a cover C^* related to the transmission-reach constraint of demand k, such that

$$\sum_{e \in c^*} \bar{x}_e^k > |C^*| - 1.$$

For this, we propose a separation algorithm based on a greedy algorithm introduced by Nemhauser and Sigismondi [73]. We first select an edge $e \in E \setminus (E_0^k \cup E_1^k)$ having the largest length ℓ_e with $\bar{x}_e^k > 0$, and set C^* to $C^* = \{e\}$. After that, we iteratively add each edge $e' \in E \setminus (E_0^k \cup E_1^k \cup C^*)$ to C^* while $\sum_{e \in C^*} \ell_e \leq \bar{l}_k$ and e' is compatible with the edges already added to the cover C^*, i.e., until a cover C^* is obtained for the demand k with $\sum_{e \in C^*} \ell_e > \bar{l}_k$. We further derive a minimal cover from the cover C^* by deleting each edge $e \in C^*$ if $\sum_{e' \in C^* \setminus \{e\}} \ell_{e'} \leq \bar{l}_k$. We then add inequality (2.49) induced by the minimal cover C^* for demand k to the current LP if it is violated,

i.e.,

$$\sum_{e \in C^*} x_e^k \leq |C^*| - 1.$$

Furthermore, inequality (2.49) induced by the minimal cover C^* can be lifted by introducing an extended coverinequality as follows

$$\sum_{e \in C^*} x_e^k + \sum_{e' \in E(C^*)} x_{e'}^k \leq |C^*| - 1,$$

where $\ell_{e'} \geq \ell_e$ for each $e \in C^*$ and $e' \in E(C^*)$ with $e' \notin E_0^k \cup E_1^k$ and e' is compatible with each edge $e \in C^*$.

3.1.16 Separation of Edge-Capacity-Cover Inequalities

Let now study the separation problem of inequality (2.54). Given an edge $e \in E$. The separation problem associated with inequality (2.54) is NP-hard [82] given that it consists in identifying a cover $\tilde{K}^*$ edge e, such that

$$\sum_{k \in \tilde{K}^*} \bar{x}_e^k > |\tilde{K}^*| - 1.$$

For this, we propose a separation algorithm based on a greedy algorithm introduced by Nemhauser and Sigismondi [73]. We first select a demand $k \in K \setminus K_e$ having largest number of requested slot w_k with $\bar{x}_e^k > 0$, and set $\tilde{K}^*$ to $\tilde{K}^* = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus (K_e \cup \tilde{K}^*)$ to $\tilde{K}^*$ while $\sum_{k \in \tilde{K}^*} w_k \leq \bar{s} - \sum_{\tilde{k} \in K_e} w_{\tilde{k}}$, i.e., until a cover $\tilde{K}^*$ is obtained for the edge e with $\sum_{k \in \tilde{K}^*} w_k > \bar{s} - \sum_{\tilde{k} \in K_e} w_{\tilde{k}}$. We further derive a minimal cover from the cover $\tilde{K}^*$ by deleting each demand $k \in \tilde{K}^*$ if $\sum_{k' \in \tilde{K}^* \setminus \{k\}} w_{k'} \leq \bar{s} - \sum_{\tilde{k} \in K_e} w_{\tilde{k}}$. We then add inequality (2.54) induced by the minimal cover $\tilde{K}^*$ for edge e to the current LP if it is violated, i.e.,

$$\sum_{k \in \tilde{K}^*} x_e^k \leq |\tilde{K}^*| - 1.$$

3.1.17 Primal Heuristic

Here, we propose a primal heuristic to boost the performance of the Branch-and-Cut algorithm. It is based on a hybrid method between a local search algorithm and a greedy-algorithm. Given an optimal fractional solution $(\bar{x}, \bar{z})$ in a certain node of the B&C tree, our primal heuristic consists in constructing an integral "feasible"

solution from $(\bar{x}, \bar{z})$. For this, we first construct several paths R_k for each demand $k \in K$ based on the fractional values $\bar{x}_e^k$ such that for each $p \in R_k$

$$\sum_{e \in \delta(X) \cap E(p)} \bar{x}_e^k \geq 1, \forall X \subset V \text{ s.t. } |X \cap \{o_k, d_k\}| = 1 \text{ and } \sum_{e \in E(p)} \ell_e \leq \bar{l_k}.$$

This can be done in polynomial time using network flow algorithms.
Afterwards, we use a local search algorithm. It consists in generating at each iteration a sequence of numeroted demands L (order) with $L = 1', 2', ..., |K|' - 1, |K|'$. Based on this sequence of demands, we use a greedy algorithm to select a path p from $R_{k'}$ and a slot $s \in \{w_{k'}, ..., \bar{s}\}$ for each demand $k' \in L$ with $\bar{z}_s^{k'} \neq 0$ and $\bar{x}_e^{k'} \neq 0$ for each $e \in E(p)$, while respecting the non-overlapping constraint with the set of demands that precede demand k' in the list L (i.e., the demands $1', 2, ..., k' - 1$). However, if there does not exist such pair of path p and slot s for demand k', we then select a path p and a slot $s \in \{w_{k'}, ..., \bar{s}\}$ for demand $k' \in L$ with $\bar{z}_s^{k'} = 0$ and $\bar{x}_e^k \neq 0$ for each $e \in E(p)$ while respecting the non-overlapping constraint with the set of demands that precede demand k' in the list L. The complexity of this algorithm is bounded by $\mathcal{O}(|K| * |\mathbb{S}| * |P| * log(|K|))$ where $|P| = \max_{k \in K} R^k$.
After that, we compute the associated total cost of the paths selected for the set of demands K in the final solution $\mathcal{S}$ given by the greedy-algorithm (i.e., $\sum_{k \in K} \sum_{e \in E_k} c_e$). Our local search algorithm generates a new sequence by doing some permutation of demands in the last sequence of demands, if the value of the solution given by greedy-algorithm is smaller than the value of the best solution found until the current iteration. Otherwise, we stop the algorithm, and we give in output the best solution found during the primal heuristic induced by the best sequence of demands having the smallest value of total cost of the selected path compared with the others generated sequences.

3.2 Computational Study

3.2.1 Implementation's Feature

We have used C++ programming language to implement the B&C algorithm under Linux using three framworks, Cplex 12.9 [26], Gurobi 9.0 [49], and "Solving Constraint Integer Programs" (Scip 7.0) [103] framework using Cplex 12.9 as LP solver. It has been tested on LIMOS high performance server with a memory size limited to 64 gb while benefiting from parallelism by activating 8 threads when using Gurobi or Scip (which is not possible when using cutting-plane based method under Cplex). We limit the CPU time to 5 hours (18000 s).

3.2.2 Description of Instances

We further present computational results using two types of graphs: real, and other realistics from SND-Lib [74] with total number of nodes $|V|$ up to 161 and total number of edges $|E|$ up to 166 as shown in Table 3.1. The demands K are randomly generated with $|K| \in \{10, 20, 30, 40, 50, 100, 150\}$, and $\bar{s}$ up to 320 slots. Note that we tested 4 instances for each triplet $(G, K, \bar{s})$ with $|K| \in \{10, 20, 30, 40, 50, 100, 150, 200, 250, 300\}$, and $\bar{s}$ up to 320 slots.

Graphs		**Number of Nodes**	**Number of Links**	**Max Node Degree**	**Min Node Degree**	**Average Node Degree**
Real Topology	**German**	17	25	5	2	2.94
	Nsfnet	14	21	4	2	3
	Spain	30	56	6	2	3.73
	Conus75	75	99	5	2	2.64
	Coronet100	100	136	5	2	2.72
Realistic Topology	**Europe**	28	41	5	2	2.92
	France	25	45	10	2	3.6
	German50	50	88	5	2	3.52
	Brain161	161	166	37	1	2.06
	Giul39	39	86	8	3	4.41
	India35	35	80	9	2	4.57
	Pioro40	40	89	5	4	4.45
	Ta65	65	108	10	1	3.32
	Zib54	54	80	10	1	2.96

Table 3.1: Characteristics of Different Topologies Used for our Experiments.

3.2.3 Computational Results

We consider 4 criteria in our computational study, average number of nodes in the B&C tree (Nb_Nd), average gap (Gap) which represents the relative error between the lower bound gotten at the end of the resolution and best upper bound, average number of violated inequalities added during the algorithm (Nbr_Cuts), and average Cpu time computation (TT).

Based on preliminary results, the cover-based inequalities (2.54) and (2.32) are shown to be efficient than the clique-based inequalities (2.43), (2.42) and (2.36). In fact, the B&C algorithm performs very well when adding the cover-based inequalities (2.54) and (2.32) under Scip and Gurobi. We notice that adding these valid inequalities allows solving to optimality some instances that are not solved to

optimality when using Cplex, Gurobi and Scip. Furthermore, they allow reducing the average gap, average number of nodes, and the average cpu time. On the other hand, we observed that the valid inequalities do not work well when using Cplex. This is due to deactivating the Cplex's cut generation such that Cplex does not work well without its proper cut generation even if the valid inequalities are shown to be efficient when using Gurobi and Scip for the instances tested. The results also show that several inequalities of the cover-based inequalities (2.54)(2.32), and clique-based inequalities (2.43), (2.42) and (2.36), are generated along the B&C algorithm. However, the number of clique-based inequalities (2.43) generated is very less compared with other inequalities. Based on these results, we conclude that the valid inequalities are very useful to obtain tighter LP bounds using Gurobi and Scip. On the other hand, the clique-based inequalities (2.46), cover-based inequalities (2.49), and the different families of odd-hole inequalities, are shown to be not efficient for the instances tested such that the number of their violated inequalities generated is very less and equal to 0 for several instances. However, they are still very interesting from a theoretical point of view. Based on this, the separation of our valid inequalities, is performed along the B&C algorithm (using Cplex, Gurobi and Scip) in the following order

a) edge-capacity-cover inequalities (2.54),

b) edge-Interval-Capacity-Cover inequalities (2.32),

c) edge-slot-assignment-clique inequalities (2.42),

d) edge-interval-clique inequalities (2.36),

e) slot-assignment-clique inequalities (2.43).

Using this, we provide a comparative study between Cplex, Gurobi and Scip. For this, we aim to evaluate the impact of the valid inequalities used within the B&C algorithm. Our first series of computational results presented in Tables 3.2, it concerns a comparaison between the results obtained for the B&C algorithm using Cplex and Scip (without or with additional valid inequalities). On the other hand, in the second series of computational results shown in Table 3.3, we present the results found for the B&C algorithm using Gurobi and Scip (without or with additional valid inequalities). In the third series shown in Table 3.4, we compare the results found by the B&C algorithm using Cplex (without or with additional valid inequalities) with those that are found when using Scip (without or with additional valid inequalities). For each instance, we run

- Cplex with benefiting of its automatic cut generation and without our additional valid inequalities (denoted by B&C_CPX in the different tables),
- Cplex using our valid inequalities and disabling its proper cut generation (denoted by Own_B&C_CPX),
- Gurobi with benefiting of its automatic cut generation and without our additional valid inequalities (denoted by B&C_GRB),
- Gurobi using our valid inequalities and disabling its proper cut generation (denoted by Own_B&C_GRB),
- Scip with benefiting of its automatic cut generation and without our additional valid inequalities (denoted by B&C_SCIP),
- Scip using our valid inequalities and disabling its proper cut generation (denoted by Own_B&C_SCIP).

To make the results and the comparison more readable, we just present some computational results using a subset of instances based on 2 real topologies: German, Nsfnet, and 2 realistic topologies: India35 and Pioro40.
We first notice that our valid inequalities allows solving several instances to optimality that are not solved to optimality when using B&C_CPX, B&C_GRB and B&C_SCIP. Furthermore, they enabled reducing the average number of nodes in the B&C tree, and also the average Cpu time for several instances. On the other hand, and when the optimality is not proven, adding valid inequalities decreases the average gap for several instances. However, there exists a few instances in which adding valid inequalities does not improve the results of B&C algorithm. We further observe that Own_B&C_SCIP is shown to be very efficient compared with Cplex and Gurobi (see for example Table 3.2 and 3.3). However, and looking at the instances that are solved to optimality by Own_B&C_GRB and Own_B&C_SCIP, we notice that we have less number of nodes and time cpu when using Own_B&C_SCIP compared with Own_B&C_GRB (see for example Table 3.3). Furthermore, Own_B&C_SCIP works much betther than SCIP, Cplex and Gurobi even when using their proper cuts such that Own_B&C_SCIP is able to solve several instances to optimality that are not solved when using B&C_CPX, B&C_GRB and B&C_SCIP. This means that we are able to beat Cplex, Gurobi and Scip using Own_B&C_SCIP. On the other hand, and considering large-scale instances with $|K| \geq 200$, we noticed that adding valid inequalities does not improve the effectiveness of the B&C algorithm such that there exist some instances that are solved to optimality using B&C_CPX and B&C_GRB

that are not solved to optimality when using Own_B&C_CPX, Own_B&C_GRB and Own_B&C_SCIP. Based on these results, we conclude that using the valid inequalities allows obtaining tighter LP bounds. They significantly improve the results yielded by the B&C_CPX, B&C_GRB, and B&C_SCIP for several instances with number of demands up to 150.

Given that Own_B&C_SCIP is able to beat Cplex, Gurobi and Scip, we turn our attention to the numerical results found when using SCIP. They are shown in the following Table 3.5.
We can see from Table 3.5 that our B&C algorithm (Own_B&C) is able to solve to optimality more instances than B&C_SCIP. Indeed, 137 instances are solved to optimality when our inequalities are used (Own_B&C) while 101 instances are solved to optimality in run B&C_SCIP. Also, when our inequalities are used, the number of nodes in the B&C tree is decreased in most cases compared to the case where they are not used. Moreover, the CPU time is, in general, smaller when our inequalities are used. Finally, when comparing the instances which are not solved to optimality, we can see that the optimality gap is smaller, for most of the instances, when our inequalities are used.

3.3 Concluding Remarks

In this chapter, we have devised a B&C algorithm, and conducted some computational experiments. Our study shows that the valid inequalities are effective for solving real and realistic instances of the problem. It could be interesting to study the impact of the symmetry breaking inequalities and the precomputed lower bounds on the performance of the Branch-and-Cut algorithm.

Instances			B&C_CPX			Own_B&C_CPX				B&C_SCIP			Own_B&C_SCIP			
Instances	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**
German	10	15	5534,75	**0,00**	42,37	9071,75	**0,00**	1524,25	101,76	1310,25	**0,00**	14,35	59	**0,00**	429,75	0,83
	20	45	109616,75	0,27	4382,25	68601,50	4,46	46110,75	18000	185956	0,27	3895,50	141	**0,00**	2403,50	3,89
	30	45	71995	0,38	8845,19	16945,75	18,73	53497,50	18000	401335,75	1,60	11740,04	160376,50	1,46	129867,25	8334,95
	40	45	75469	3,82	18000	363,33	60,47	41306,33	18000	315993,66	8,33	16206,36	383058,66	3,70	224642,33	16624,87
	50	55	44143	5,32	18000	470,75	61,19	52946	18000	246146,50	9,62	16675,88	251152,50	13,73	305309,75	17074,95
	100	140	1	**0,00**	50,15	33,75	60,75	20802	18000	1158,50	**0,00**	340,10	3014,25	**0,00**	17668,50	617,80
	150	210	1	**0,00**	327,54	150,50	62,40	39133,25	18000	12759	0,01	7329,06	3609	**0,00**	24782,25	3057,79
Nsfnet	10	15	72616,25	**0,00**	644,88	140046,50	**0,00**	3760,75	2640,70	13462	**0,00**	113,64	1	**0,00**	95,75	0,15
	20	20	121350,25	32,76	18000	27987,75	41,38	53257,75	18000	699646	9,51	18000	21586	**0,00**	24587	192,27
	30	30	67751	17,13	18000	6322,75	49,13	66315,50	18000	272065	40,99	18000	281569,66	3,29	340177	11048,71
	40	35	32491	21,46	18000	1687,25	59,53	62749,25	18000	225696,67	46,74	18000	119841,66	1,17	163519,33	5673,46
	50	50	21256,75	19,80	18000	2216,25	59,73	53734,50	18000	247873,25	43,09	18000	148476,50	5,91	340399,25	17405,09
	100	120	1064,50	20,28	18000	99	66,23	35031,75	18000	56598,50	57,19	18000	1	**0,00**	464,25	40,87
	150	160	145	24,53	18000	1	66,94	142594,25	18000	12663	58,50	18000	1	**0,00**	496,25	136,02
India35	10	40	968,50	**0,00**	40,87	106140	23,94	48081	18000	1907,25	**0,00**	87,60	1	**0,00**	779,75	1,80
	20	40	1	**0,00**	12,23	38307,25	41,43	33858	18000	9	**0,00**	4	7	**0,00**	2437,75	5,92
	30	40	44170,25	**0,00**	4149,40	19389,50	49,99	80759,50	18000	91798	**0,00**	7821,50	32156,75	**0,00**	10917,50	2309,66
	40	40	32029	2,80	18000	9583,50	53,19	94366,50	18000	161514	2,42	17486,08	191812	0,18	27270	17333,53
	50	80	1	**0,00**	60,98	4763	58,61	35364,75	18000	34	**0,00**	22,13	69,25	**0,00**	11105,75	112,19
	100	120	6217,50	0,71	9493,81	230	67,42	28590,25	18000	24797	0,32	9137,26	23403,75	0,44	48702,75	9494,52
	150	200	4948,25	0,28	13807,72	287,75	67,65	19196,75	18000	16809	0,21	13739,65	1026	**0,00**	19898	4101,80
Pioro40	10	40	1	**0,00**	6,23	123725,75	14,29	1705,75	18000	1	**0,00**	1,49	1	**0,00**	202,25	1,69
	20	40	1	**0,00**	15,77	52857,25	29,75	17720,25	18000	1,50	**0,00**	3,44	1	**0,00**	1255,75	4,88
	30	40	1	**0,00**	26,32	27075,50	44,66	47926,50	18000	1,50	**0,00**	5,72	6,25	**0,00**	2972,50	10,54
	40	40	32219,75	0,19	8966,91	17658,50	52,21	58468,25	18000	83597	0,20	8692,50	67151	0,12	53544,50	8711,30
	50	80	1	**0,00**	84,98	7344,75	59,79	6049,25	18000	14	**0,00**	15,93	4	**0,00**	8606	54,39
	100	80	12576	0,05	13517,75	1555,50	66,37	54491,75	18000	21281,75	0,04	9087,52	23785,75	0,04	57123,75	9916,63
	150	160	1	**0,00**	626,31	765,75	68,03	11064,75	18000	823,50	**0,00**	816,89	124,50	**0,00**	22814	1509,87

Table 3.2: Table of Comparison for the B&C Algorithm: Cplex (Without or With Additional Valid Inequalities) Vs Scip (Without or With Additional Valid Inequalities).

Instances			B&C_GRB			Own_B&C_GRB				B&C_SCIP			Own_B&C_SCIP			
Instances	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**
German	10	15	4906,25	**0,00**	981,48	3460,25	**0,00**	1866,50	149,13	1310,25	**0,00**	14,35	59	**0,00**	429,75	0,83
	20	45	20752	0,27	4889,90	28093,75	0,27	33413	4962,48	185956	0,27	3895,50	141	**0,00**	2403,50	3,89
	30	45	24321,50	0,39	9279,54	3614,50	**0,00**	19141,50	882,68	401335,75	1,60	11740,04	160376,50	1,46	129867,25	8334,95
	40	45	35451,67	5,52	18000	9920	1,82	216792,33	15453,54	315993,66	8,33	16206,36	383058,66	3,70	224642,33	16624,87
	50	55	18901,50	6,15	18000	6679	17,55	263086,50	18000	246146,50	9,62	16675,88	251152,50	13,73	305309,75	17074,95
	100	140	1	**0,00**	1634,37	1746,50	**0,00**	200920	15425,53	1158,50	**0,00**	340,10	3014,25	**0,00**	17668,50	617,80
	150	210	64,75	**0,00**	3184,56	979,75	73,06	44188,25	18000	12759	0,01	7329,06	3609	**0,00**	24782,25	3057,79
Nsfnet	10	15	15222,75	**0,00**	2087,33	4385	**0,00**	4095,25	216,83	13462	**0,00**	113,64	1	**0,00**	95,75	0,15
	20	20	51525	12,77	18000	181730	2,58	99411	18000	699646	9,51	18000	21586	**0,00**	24587	192,27
	30	30	27735	22,41	18000	19702,67	13,20	263905,66	18000	272065	40,99	18000	281569,66	3,29	340177	11048,71
	40	35	12631	34,18	18000	6239,33	24,31	307366	18000	225696,67	46,74	18000	119841,66	1,17	163519,33	5673,46
	50	50	8733,50	29,35	18000	5265,75	47,22	347095,75	18000	247873,25	43,09	18000	148476,50	5,91	340399,25	17405,09
	100	120	7790,50	29,60	18000	2253,25	41,60	326605	18000	56598,50	57,19	18000	1	**0,00**	464,25	40,87
	150	160	4255,25	33,05	18000	871	INF	80789	18000	12663	58,50	18000	1	**0,00**	496,25	136,02
India35	10	40	2139,50	**0,00**	481,60	1658,25	**0,00**	1902,50	281,53	1907,25	**0,00**	87,60	1	**0,00**	779,75	1,80
	20	40	103	**0,00**	541,13	1722,25	**0,00**	4866	572,66	9	**0,00**	4	7	**0,00**	2437,75	5,92
	30	40	19437,25	**0,00**	5132,47	1879,50	**0,00**	11196,25	914,49	91798	**0,00**	7821,50	32156,75	**0,00**	10917,50	2309,66
	40	40	28219	1,40	18000	9945	**0,00**	58588,25	7309,79	161514	2,42	17486,08	191812	0,18	27270	17333,53
	50	80	386,50	**0,00**	1221,34	2174,25	**0,00**	18163	3970,39	34	**0,00**	22,13	69,25	**0,00**	11105,75	112,19
	100	120	6036,25	0,75	10772,47	2399	95,14	30646,50	18000	24797	0,32	9137,26	23403,75	0,44	48702,75	9494,52
	150	200	4164	0,35	11292,87	726,50	98,71	7760,25	18000	16809	0,21	13739,65	1026	**0,00**	19898	4101,80
Pioro40	10	40	905,75	**0,00**	275,37	1756	**0,00**	578,50	232,46	1	**0,00**	1,49	1	**0,00**	202,25	1,69
	20	40	544,75	**0,00**	527,67	1613,75	**0,00**	2330,50	512,12	1,50	**0,00**	3,44	1	**0,00**	1255,75	4,88
	30	40	2	**0,00**	539,60	1669,25	**0,00**	5362,25	863,04	1,50	**0,00**	5,72	6,25	**0,00**	2972,50	10,54
	40	40	12237,75	0,19	9290,28	10971	0,01	41439,25	5815,90	83597	0,20	8692,50	67151	0,12	53544,50	8711,30
	50	80	10,25	**0,00**	773,69	1800	**0,00**	9409,50	2780,11	14	**0,00**	15,93	4	**0,00**	8606	54,39
	100	80	2785,50	0,54	5680,84	4749	1,76	72888,25	17854,91	21281,75	0,04	9087,52	23785,75	0,04	57123,75	9916,63
	150	160	2,25	**0,00**	4805,69	1068,25	99,51	8076	18000	823,50	**0,00**	816,89	124,50	**0,00**	22814	1509,87

Table 3.3: Table of Comparison for the B&C Algorithm: Gurobi (Without or With Additional Valid Inequalities) Vs Scip (Without or With Additional Valid Inequalities).

Instances			B&C_CPX			Own_B&C_CPX				B&C_GRB			Own_B&C_GRB			
Instances	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**
German	10	15	5534,75	**0,00**	42,37	9071,75	**0,00**	1524,25	101,76	4906,25	**0,00**	981,48	3460,25	**0,00**	1866,50	149,13
	20	45	109616,75	0,27	4382,25	68601,50	4,46	46110,75	18000	20752	0,27	4889,90	28093,75	0,27	33413	4962,48
	30	45	71995	0,38	8845,19	16945,75	18,73	53497,50	18000	24321,50	0,39	9279,54	3614,50	**0,00**	19141,50	882,68
	40	45	75469	3,82	18000	363,33	60,47	41306,33	18000	35451,67	5,52	18000	9920	1,82	216792,33	15453,54
	50	55	44143	5,32	18000	470,75	61,19	52946	18000	18901,50	6,15	18000	6679	17,55	263086,50	18000
	100	140	1	**0,00**	50,15	33,75	60,75	20802	18000	1	**0,00**	1634,37	1746,50	**0,00**	200920	15425,53
	150	210	1	**0,00**	327,54	150,50	62,40	39133,25	18000	64,75	**0,00**	3184,56	979,75	73,06	44188,25	18000
Nsfnet	10	15	72616,25	**0,00**	644,88	140046,50	**0,00**	3760,75	2640,70	15222,75	**0,00**	2087,33	4385	**0,00**	4095,25	216,83
	20	20	121350,25	32,76	18000	27987,75	41,38	53257,75	18000	51525	12,77	18000	181730	2,58	99411	18000
	30	30	67751	17,13	18000	6322,75	49,13	66315,50	18000	27735	22,41	18000	19702,67	13,20	263905,66	18000
	40	35	32491	21,46	18000	1687,25	59,53	62749,25	18000	12631	34,18	18000	6239,33	24,31	307366	18000
	50	50	21256,75	19,80	18000	2216,25	59,73	53734,50	18000	8733,50	29,35	18000	5265,75	47,22	347095,75	18000
	100	120	1064,50	20,28	18000	99	66,23	35031,75	18000	7790,50	29,60	18000	2253,25	41,60	326605	18000
	150	160	145	24,53	18000	1	66,94	142594,25	18000	4255,25	33,05	18000	871	INF	80789	18000
India35	10	40	968,50	**0,00**	40,87	106140	23,94	48081	18000	2139,50	**0,00**	481,60	1658,25	**0,00**	1902,50	281,53
	20	40	1	**0,00**	12,23	38307,25	41,43	33858	18000	103	**0,00**	541,13	1722,25	**0,00**	4866	572,66
	30	40	44170,25	**0,00**	4149,40	19389,50	49,99	80759,50	18000	19437,25	**0,00**	5132,47	1879,50	**0,00**	11196,25	914,49
	40	40	32029	2,80	18000	9583,50	53,19	94366,50	18000	28219	1,40	18000	9945	**0,00**	58588,25	7309,79
	50	80	1	**0,00**	60,98	4763	58,61	35364,75	18000	386,50	**0,00**	1221,34	2174,25	**0,00**	18163	3970,39
	100	120	6217,50	0,71	9493,81	230	67,42	28590,25	18000	6036,25	0,75	10772,47	2399	95,14	30646,50	18000
	150	200	4948,25	0,28	13807,72	287,75	67,65	19196,75	18000	4164	0,35	11292,87	726,50	98,71	7760,25	18000
Pioro40	10	40	1	**0,00**	6,23	123725,75	14,29	1705,75	18000	905,75	**0,00**	275,37	1756	**0,00**	578,50	232,46
	20	40	1	**0,00**	15,77	52857,25	29,75	17720,25	18000	544,75	**0,00**	527,67	1613,75	**0,00**	2330,50	512,12
	30	40	1	**0,00**	26,32	27075,50	44,66	47926,50	18000	2	**0,00**	539,60	1669,25	**0,00**	5362,25	863,04
	40	40	32219,75	0,19	8966,91	17658,50	52,21	58468,25	18000	12237,75	0,19	9290,28	10971	0,01	41439,25	5815,90
	50	80	1	**0,00**	84,98	7344,75	59,79	6049,25	18000	10,25	**0,00**	773,69	1800	**0,00**	9409,50	2780,11
	100	80	12576	0,05	13517,75	1555,50	66,37	54491,75	18000	2785,50	0,54	5680,84	4749	1,76	72888,25	17854,91
	150	160	1	**0,00**	626,31	765,75	68,03	11064,75	18000	2,25	**0,00**	4805,69	1068,25	99,51	8076	18000

Table 3.4: Table of Comparison for the B&C Algorithm: Cplex (Without or With Additional Valid Inequalities) Vs Gurobi (Without or With Additional Valid Inequalities).

Instances			B&C_SCIP			Own_B&C_SCIP		
Graph	$\|K\|$	$\bar{s}$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**TT**
German	10	15	1310,25	**0,00**	14,35	59	**0,00**	0,83
	20	45	185956	0,27	3895,5	141	**0,00**	3,89
	30	45	401335,75	1,60	11740,04	160376,50	1,46	8334,95
	40	45	315993,66	8,33	16206,36	383058,66	3,70	16624,87
	50	55	246146,50	9,62	16675,88	251152,50	13,73	17074,95
	100	140	1158,50	**0,00**	340,10	3014,25	**0,00**	617,80
	150	210	12759	0,01	7329,06	3609	**0,00**	3057,79
	200	260	5099,33	0,78	10095,88	3067	**0,00**	6770,75
Coronet100	10	320	1	**0,00**	10,37	1	**0,00**	462,15
	20	320	10,50	**0,00**	19,21	15	**0,00**	832,22
	30	40	66534,75	16,40	18000	11304,25	6,08	5006,31
	40	40	81051	3,96	18000	2127	**0,00**	707,54
	50	80	11385,25	0,01	4496,92	19,75	**0,00**	139,55
	100	120	12787,50	13,36	14228,34	8390,25	7,66	10920,70
	150	200	4454,50	27,12	13692,63	3165,75	29,13	15527,10
	200	280	3579,25	33,35	18000	1	38,97	18000
Nsfnet	10	15	13462	**0,00**	113,64	1	**0,00**	0,15
	20	20	699646	9,51	18000	21586	**0,00**	192,27
	30	30	272065	40,99	18000	281569,66	3,29	11048,71
	40	35	225696,67	46,74	18000	119841,66	1,17	5673,46
	50	50	247873,25	43,09	18000	148476,50	5,91	17405,09
	100	120	56598,50	57,19	18000	1	**0,00**	40,87
	150	160	12663	58,50	18000	1	**0,00**	136,02
	200	210	7726,50	54,85	18000	710	0,28	9121,79
India35	10	40	1907,25	**0,00**	87,60	1	**0,00**	1,80
	20	40	9	**0,00**	4	7	**0,00**	5,92
	30	40	91798	**0,00**	7821,5	32156,75	**0,00**	2309,66
	40	40	161514	2,42	17486,08	191812	0,18	17333,53
	50	80	34	**0,00**	22,13	69,25	**0,00**	112,19
	100	120	24797	0,32	9137,26	23403,75	0,44	9494,52
	150	200	16809	0,21	13739,65	1026	**0,00**	4101,80
	200	280	11197	0,37	13930,35	2027,75	3,69	14516,65
Pioro40	10	40	1	**0,00**	1,49	1	**0,00**	1,69
	20	40	1,50	**0,00**	3,44	1	**0,00**	4,88
	30	40	1,50	**0,00**	5,72	6,25	**0,00**	10,54
	40	40	83597	0,20	8692,5	67151	0,12	8711,30
	50	80	14	**0,00**	15,93	4	**0,00**	54,39
	100	80	21281,75	0,04	9087,52	23785,75	0,04	9916,63
	150	160	823,50	**0,00**	816,89	124,50	**0,00**	1509,87
	200	280	1503,75	**0,00**	3772,9	423,50	**0,00**	7424,98
Giul39	10	40	1	**0,00**	1,58	1	**0,00**	1,83
	20	40	1,50	**0,00**	2,92	1	**0,00**	3,71
	30	40	4	**0,00**	4,50	1	**0,00**	6,10
	40	40	4,50	**0,00**	7,17	1	**0,00**	10,15
	50	40	54420	**0,00**	4376,98	52156,75	**0,00**	4361,26
	100	40	55472,50	6,88	17781,71	54675,50	8,38	17802,83
	150	120	836	**0,00**	1050,13	11655,50	**0,00**	9411,30
	200	120	10191,25	0,24	13794,32	6518	0,01	9914,02

Table 3.5: The Impact of Valid Inequalities in the Own_B&C_SCIP Performance Using Realistic Graphs.

Chapter 4

Path Formulation and Branch-and-Cut-and-Price Algorithm for the C-RSA Problem

In this chapter, we first introduce an extended integer linear programming formulation based on the so-called *path formulation*. All the different valid inequalities presented in chapter 2, they are still valid for the path formulation. Using this, we derive a Branch-and-Cut-and-Price algorithm to solve the C-RSA problem. In this section, we describe the framework of this algorithm. First, we give an overview of the column generation algorithm. Then, we discuss the pricing problem. We further present a primal heuristic used to boost the performance of the algorithm. We give at the end some computational results and a comparative study between Branch-and-Cut and Branch-and-Cut-and-Price algorithms. We close our chapter with some concluding remarks.

4.1 Path Formulation

Let P^k denote the set of all feasible (o_k,d_k) paths in G such that for each demand $k \in K$, we have

$$\sum_{e \in E(p_k)} \ell_e \leq \bar{l}_k, \text{ for all } p_k \in P^k.$$

We consider for $k \in K$ and $p \in P^k$ and $s \in S$, a variable $y^k_{p,s}$ which takes 1 if slot s is the last slot allocated along the path p for the routing of demand k and 0 if

not, such that s represents the last slot of the interval of contiguous slots of width w_k allocated by demand $k \in K$, with $s \in S$ and $p \in P^k$. Note that all the slots $s' \in \{s - w_k + 1, ..., s\}$ should be assigned to demand k along the path p whenever $y^k_{p,s} = 1$. Let $P^k(e)$ denote set of all admissible (o_k,d_k) paths going through edge e in G for demand k.

In this case, the C-RSA is also equivalent to the following integer linear program

$$\min \sum_{k \in K} \sum_{p \in P^k} \sum_{e \in E(p)} \sum_{s=w_k}^{\bar{s}} c_e y^k_{p,s}, \tag{4.1}$$

subject to

$$\sum_{p \in P^k} \sum_{s=1}^{w_k - 1} y^k_{p,s} = 0, \forall k \in K, \tag{4.2}$$

$$\sum_{p \in P^k} \sum_{s=w_k}^{\bar{s}} y^k_{p,s} = 1, \forall k \in K, \tag{4.3}$$

$$\sum_{k \in K} \sum_{p \in P^k(e)} \sum_{s'=s}^{s+w_k-1} y^k_{p,s'} \leq 1, \forall e \in E, \forall s \in S, \tag{4.4}$$

$$y^k_{p,s} \geq 0, \forall k \in K, \forall p \in P^k, \forall s \in S, \tag{4.5}$$

$$y^k_{p,s} \in \{0, 1\}, \forall k \in K, \forall p \in P^k, \forall s \in S. \tag{4.6}$$

Inequalities (4.2) express the fact that a demand $k \in K$ cannot occupy a slot s as the last slot before her slot-width w_k. Inequalities (4.3) express the routing and spectrum constraints at the same time such that they ensure that exactly one slot $s \in \{w_k, \ldots, \bar{s}\}$ is assigned as last slot for the routing of demand k, and exactly one single path from P^k is allocated by each demand $k \in K$. Note that a slot $s \in S$ is said an allocated slot by demand k if and only if $\sum_{p \in P^k} \sum_{s'=s}^{s+w_k-1} y^k_{p,s'} = 1$ which means that s is covered by the interval of contiguous slots allocated by demand k. Inequalities (4.4) ensure that a slot s over edge e cannot be allocated to at most by one demand $k \in K$. Inequalities (4.5) are trivial inequalities, and constraints (4.6) are the integrality constraints.

To benefit from some theoretical results done in chapter 2, we introduce the two variables x^k_e and z^k_s used in the cut formulation already presented in chapter 2. As a result, all the valid inequalities for the polytope associated with the cut formulation, they are still valid for the polytope associated with the path formulation following the addition of these two variables and the two following constraints

$$x^k_e - \sum_{p \in B^k(e)} \sum_{s=w_k}^{\bar{s}} y^k_{p,s} = 0, \forall k \in K, \forall e \in E, \tag{4.7}$$

and

$$z_s^k - \sum_{p \in B^k} y_{p,s}^k = 0, \forall k \in K, \forall s \in \mathbb{S}. \tag{4.8}$$

Therefore, the C-RSA is then equivalent to the extended formulation based on the following integer linear program

$$\min \sum_{k \in K} \sum_{e \in E} c_e x_e^k, \tag{4.9}$$

$$\sum_{p \in P^k} \sum_{s=1}^{w_k - 1} y_{p,s}^k = 0, \forall k \in K, \tag{4.10}$$

$$\sum_{p \in P^k} \sum_{s=w_k}^{\bar{s}} y_{p,s}^k = 1, \forall k \in K, \tag{4.11}$$

$$x_e^k - \sum_{p \in P^k(e)} \sum_{s=w_k}^{\bar{s}} y_{p,s}^k = 0, \forall k \in K, \forall e \in E, \tag{4.12}$$

$$z_s^k - \sum_{p \in P^k} y_{p,s}^k = 0, \forall k \in K, \forall s \in \mathbb{S}, \tag{4.13}$$

$$\sum_{k \in K} \sum_{p \in P^k(e)} \sum_{s'=s}^{s+w_k-1} y_{p,s'}^k \leq 1, \forall e \in E, \forall s \in S, \tag{4.14}$$

$$y_{p,s}^k \geq 0, \forall k \in K, \forall p \in P^k, \forall s \in S, \tag{4.15}$$

$$x_e^k \geq 0, \forall k \in K, \forall e \in E, \tag{4.16}$$

$$z_s^k \geq 0, \forall k \in K, \forall s \in \mathbb{S}, \tag{4.17}$$

$$y_{p,s}^k \in \{0,1\}, \forall k \in K, \forall p \in P^k, \forall s \in S. \tag{4.18}$$

4.2 Column Generation Algorithm

As it has been mentioned previously, our path formulation contains a huge number of variables which can be exponentiel in the worst case due to the number of all feasible paths for each traffic demand. To deal with this, we use a column generation algorithm to solve its linear relaxation. For this, we begin the algorithm with a restricted linear program of our path formulation by considering a feasible subset of variables (columns). For this, we first generate a subset of feasible paths for each demand $k \in K$ denoted by $B^k \subset P^k$ such that the variables $y_{p,s}^k$ for each $k \in K$, $p \in B^k$ and $s \in \mathbb{S}$ induce a feasible basis for the restricted linear program. This means that there exists at least one feasible solution for the restricted linear program. Based on this, we derive the so-called restricted master problem (RMP)

as follows

$$\min \sum_{k \in K} \sum_{e \in E} c_e x_e^k,$$

subject to

$$\sum_{p \in B^k} \sum_{s=1}^{w_k - 1} y_{p,s}^k = 0, \forall k \in K,$$

$$\sum_{p \in B^k} \sum_{s=w_k}^{\bar{s}} y_{p,s}^k = 1, \forall k \in K,$$

$$x_e^k - \sum_{p \in B^k(e)} \sum_{s=w_k}^{\bar{s}} y_{p,s}^k = 0, \forall k \in K, \forall e \in E,$$

$$z_s^k - \sum_{p \in B^k} y_{p,s}^k = 0, \forall k \in K, \forall s \in \mathbb{S},$$

$$\sum_{k \in K} \sum_{p \in B^k(e)} \sum_{s'=s}^{s+w_k-1} y_{p,s'}^k \leq 1, \forall e \in E, \forall s \in S,$$

$$y_{p,s}^k \geq 0, \forall k \in K, \forall p \in B^k, \forall s \in S,$$

$$x_e^k \geq 0, \forall k \in K, \forall e \in E,$$

$$z_s^k \geq 0, \forall k \in K, \forall s \in \mathbb{S}.$$

At each iteration, the column generation algorithm checks if there exists a variable $y_{p,s}^k$ with $p \notin B^k$ for a demand k and slot s having a negative reduced cost using the solution of the dual problem associated with the constraints of the linear relaxation (4.1)-(4.5), and add it to B^k. This can be achieved by solving the so-called pricing problem (PP).

4.2.1 The Pricing Problem

As noted later, we consider an initial restricted master problem denoted by RMP_0 which is based on an initial subset of variables induced by a subset of feasible path $B^k \subset P^k$ for each demand $k \in K$. The pricing problem consists in finding a feasible path p for a demand k and slot s having a negative reduced cost using the optimal solution of the dual problem. For this, we consider the following dual variables

a) α associated with the equations (4.10) such that $\alpha_k \in \mathbb{R}$ for all $k \in K$,

b) β associated with the equations (4.11) such that $\beta^k \in \mathbb{R}$ for all $k \in K$,

c) μ associated with inequalities (4.14) such that $\mu_s^e \leq 0$ for all $e \in E$ and $s \in \mathbb{S}$,

d) λ associated with the equations (4.12) such that $\lambda_e^k \in \mathbb{R}$ for all $k \in K$ and $e \in E$,

e) ρ associated with the equations (4.13) such that $\rho_s^k \in \mathbb{R}$ for all $k \in K$ and $s \in \mathbb{S}$,

The dual problem of the linear relaxation (4.9)-(4.17) is equivalent to

$$\max - \sum_{k \in K} \beta^k + \sum_{e \in E} \sum_{s \in \mathbb{S}} \mu_s^e, \tag{4.19}$$

subject to

$$\beta^k - \sum_{e \in E(p)} (\lambda_e^k + \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e) - \rho_s^k \geq 0, \forall k \in K, \forall p \in P^k, \forall s \in \{w_k, ..., \bar{s}\}, \tag{4.20}$$

$$\alpha^k - \sum_{e \in E(p)} \sum_{s'=\max(1,s-w_k+1)}^{s} \mu_{s'}^e \geq 0, \forall k \in K, \forall p \in P^k, \forall s \in \{1, ..., w_k - 1\}, \tag{4.21}$$

$$c_e + \lambda_e^k \geq 0, \forall k \in K, \forall e \in E, \tag{4.22}$$

$$\alpha^k + \rho_s^k \geq 0, \forall k \in K, \forall s \in \mathbb{S}, \tag{4.23}$$

$$\mu_s^e \leq 0, \forall e \in E, \forall s \in \mathbb{S}. \tag{4.24}$$

As a result, the so-called reduced-cost $rc_s^k(p)$ related to each demand $k \in K$, path $p \in P^k$ and slot $s \in \{w_k, ..., \bar{s}\}$, is given by

$$rc_s^k = \beta^k - \rho_s^k + \min_{p \in P^k} [\sum_{e \in E(p)} -\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e], \tag{4.25}$$

Therefore, for each demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$, the pricing problem aims at finding a path p^* of P^k such that

$$rc_s^k(p^*) = \beta^k - \rho_s^k + \min_{\tilde{p} \in P^k} [\sum_{e \in E(\tilde{p})} -\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e], \tag{4.26}$$

Finding such path p^* can be seen as a separation procedure for the dual constraint (4.20) which consists in identifying a path p^* for each demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ such that

$$\beta^k - \rho_s^k + \sum_{e \in E(p^*)} (-\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e) < 0 \text{ and } \sum_{e \in E(p^*)} \ell_e \leq \bar{l}_k.$$

As a result, the pricing problem consists in solving the Resource Constrained Shortest Path (RCSP) problem. The RCSP problem is well known to be weakly NP-hard [31]. Several algorithms have been proposed in the literature to solve this problem based on dynamic programming algorithms, heuristics and some techniques related to the lagrangian decomposition. As background references we mention [14, 32, 35, 58, 64].

4.2.2 Dynamic Programming Algorithm for the Pricing Problem

In this work, we propose a pseudo-polynomial time based dynamic programming algorithm [32]. It consists in finding the minimum-cost path for each demand k and slot s while satisfying the *transmission-reach* constraint. It is based on the dynamic programming algorithm proposed by Dumitrescu et al. [32] to solve the RCSP problem. For each demand $k \in K$ and slot s, we associate to each node $v \in V$ in the graph G a set of labels L^v such that each label corresponds to differents paths from th origin node o_k to the node v, and each label p is specified by a cost equals to $\sum_{e \in E(p)}(-\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e)$, and a weight equals to $\sum_{e \in E(p)} \ell_e$. We denote by T_v the set of labels on node $v \in V$. For each demand k and slot $s \in \{w_k, ..., \bar{s}\}$, the complexity of the algorithm is bounded by $\mathcal{O}(|E \setminus E_0^k| * \bar{l}_k)$ [32].

Algorithm 3 summarizes the different steps of the dynamic programming algorithm.

4.2.3 Initial Columns

The basic subset of paths used to define the restricted master problem, they are generated using a brute-force search algorithm which creates a search tree that covers all the feasible paths P^k for each demand k. It is then used to pre-compute an initial subset B^k of feasible paths for each demand $k \in K$ taking into account the transmission-reach constraint to prune some non intersecting nodes in our search tree of this algorithm.

4.3 Branch-and-Price and Branch-and-Cut-and-Price Algorithms

Based on these features, we derive a Branch-and-Cut-and-Price algorithm for solving the C-RSA problem.

4.3.1 Description

The main purpose of this algorithm is to solve a sequence of linear programs using the column generation algorithm at each node of a Branch-and-Bound algorithm. At each iteration of the algorithm, we solve our pricing problem by identifying one or more than one new column by solving a RCSP problem for each demand k and slot $s \in \{w_k, ..., \bar{s}\}$ using the dynamic programming algorithm. We repeat this pro-

Algorithm 3: Dynamic Programming Algorithm

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, a linear program LP, a demand k and a slot $s \in \{w_k, ..., \bar{s}\}$, a set B^k of feasible paths already exists in the current LP for demand $k \in K$ and slot s, and the optimal values of the duals variables $(\alpha, \beta, \mu, \lambda, \rho)$

Result: Optimal path $p*$ for demand k and slot s

Set $L^{o_k} = \{(0, 0)\}$ and $L^v = \emptyset$ for each node $v \in V \setminus (V_0^k \cup \{o_k\})$;
Set $T^v = \emptyset$ for each node $v \in V \setminus V_0^k$;
STOP= FALSE;
while *STOP==FALSE* **do**
    if $\cup_{v \in V}(L_v \setminus T_v) = \emptyset$ **then**
        STOP= TRUE;
        Set $p^* = \emptyset$;
        We select one label p from the labels L^{d_k} of destination node d_k such that $p \notin B^k$ with $\beta^k - \rho_s^k + \sum_{e \in E(p)}(-\lambda_e^k - \sum_{s'=s-w_k+1}^{s} \mu_{s'}^e) < 0$;
        if *such label exists* **then**
            Set $p^* = p$;
        end
    end
    if $\cup_{v \in V}(L_v \setminus T_v) \neq \emptyset$ **then**
        Select a node $i \in V \setminus V_0^k$ and a label $p \in L^i \setminus T^i$ having the smallest value of $\sum_{e \in E(p)} \ell_e$;
        for *each* $e = ij \in \delta(i) \setminus E_0^k$ *such that* $\sum_{e' \in E(p)} \ell_{e'} + \ell_e \leq \bar{l}_k$ **do**
            if $j \notin V(p)$ **then**
                Set $p' = p \cup \{e\}$;
                Update the set of label $L^j = L^i \cup \{p'\}$;
            end
        end
        Set $T^i = T^i \cup \{p\}$;
    end
end
return the best optimal path p^* for demand k and slot s;

cedure in each iteration of the column generation until no new column is found (i.e., $rc_s^k \geq 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$. As a result, the final solution is optimal for the linear relaxation of the path formulation. Furthermore, if it is integral, then it is optimal for the C-RSA problem. Otherwise, we create two subproblems by branching on fractional variables (variable branching rule) or on some constraints using the Ryan & Foster branching rule [99] (constraint branching rule).
Algorithm 4 summarizes the different steps of the Branch-and-Price algorithm.
By combining the Branch-and-Price algorithm with a cutting-plane based algorithm, we devise a Branch-and-Cut-and-Price which works as follows. Consider a fractional solution $(\bar{y}, \bar{x}, \bar{z})$. At each iteration of the Branch-and-Price algorithm, and for a given class of valid inequalities, our aim is to identify the existence of one or more than one inequalities of this class that are violated by the current solution. We repeat this procedure in each iteration of the algorithm until no violated inequality is identified.
As mentioned before, the Branch-and-Cut-and-Price algorithm also uses the different classes of valid inequalities presented in chapter (2). They are performed in the order (2.54), (2.32), (2.42), (2.36), (2.43).
Algorithm 5 summarizes the different steps of the Branch-and-Cut-and-Price algorithm for a given class of valid inequalities.

4.3.2 Primal Heuristic

Here, we propose a primal heuristic based on a hybrid method between local search algorithm and a greedy-algorithm. Given a feasible fractional solution $(\bar{y}, \bar{x}, \bar{z})$, our primal heuristic consists in constructing an integral "feasible" solution from this fractional solution. For this, we propose a local search algorithm which consists in generating at each iteration a sequence of demands $L = 1', 2', ..., |K|' - 1, |K|'$. Based on this sequence of demands, our greedy algorithm selects a path p and a slot s for each demand $k' \in L$ with $y_{p,s}^{k'} \neq 0$ while respecting the non-overlapping constraint with the set of demands that precede demand k' in the list L (i.e., the demands $1', 2, ..., k' - 1$). However, if there does not exist such pair of path p and slot s for demand k', we then select a path p and a slot s for demand $k' \in L$ with $y_{p,s}^{k'} = 0$ and $s \in \{w_{k'}, ..., \bar{s}\}$ while respecting the non-overlapping constraint with the set of demands that precede demand k' in the list L. After that, we compute the associated total length of the paths selected for the set of demands K in the final solution $\mathcal{S}$ given by the greedy-algorithm. Our local search algorithm generates

Algorithm 4: Branch-And-Price Algorithm for the C-RSA

Data: C-RSA's inputs, a set B^k of precomputed feasible paths for each demand $k \in K$

Result: Optimal solution for the C-RSA problem

LP$\leftarrow$ RMP_0;
Stop= FALSE;
while *STOP==FALSE* **do**
    Solve the linear program LP;
    Let (y^*, x^*, z^*) be the optimal solution of LP;
    Consider the optimal values of the duals variables $(\alpha^*, \beta^*, \mu^*, \lambda^*, \rho^*)$;
    ADD = FALSE;
    for *each demand* $k \in K$ **do**
        for *each slot* $s \in \{w_k, ..., \bar{s}\}$ **do**
            Compute its associated reduced cost rc_s^k;
            if $rc_s^k < 0$ **then**
                Consider the optimal path p^* for demand k and slot s with $rc_s^k(p) < 0$;
                Add the new variable (column) $y_{p*,s}^k$ to the current LP;
                ADD= TRUE ;
            end
        end
    end
    if *ADD==FALSE* **then**
        STOP = TRUE;
    end
end
Consider the optimal solution y^* of LP ;
if y^* *is integer for the C-RSA* **then**
    y^* is an optimal solution for the C-RSA;
    End of the Branch-and-Price algorithm ;
end
else
    Create two sub-problems by branching one some variables or constraints ;
end
for *each sub-problem not yet solved* **do**
    go to 3 ;
end
return the best optimal solution (y^*, x^*, z^*) for the C-RSA;

Algorithm 5: Branch-and-Cut-and-Price Algorithm for the C-RSA

Data: C-RSA's inputs, and a set B^k of precomputed feasible paths for each demand $k \in K$, and a given class of valid inequality

Result: Optimal solution for the C-RSA problem

1 LP⟵ RMP_0 and set Stop= FALSE;
2 **while** *STOP==FALSE* **do**
3 Solve the linear program LP, and let (y^*, x^*, z^*) be its optimal solution;
4 Consider the optimal values of the duals variables $(\alpha^*, \beta^*, \mu^*, \lambda^*, \rho^*)$;
5 **for** *each demand* $k \in K$ **do**
6 **for** *each slot* $s \in \{w_k, ..., \bar{s}\}$ **do**
7 Compute its associated reduced cost rc_s^k;
8 **if** $rc_s^k < 0$ **then**
9 Consider the optimal path p^* for demand k and slot s with $rc_s^k(p) < 0$;
10 Add the new variable (column) $y_{p*,s}^k$ to the current LP;
11 **end**
12 **end**
13 **end**
14 **if** *there does not exist a new column to be added to the current LP* **then**
15 **if** *there exist inequalities from the given class that are violated by the current solution* y^* **then**
16 Add them to LP ;
17 **end**
18 **else**
19 STOP = TRUE;
20 **end**
21 **end**
22 **end**
23 Consider the optimal solution (y^*, x^*, z^*) of LP ;
24 **if** y^* *is integer for the C-RSA* **then**
25 y^* is an optimal solution for the C-RSA, and end of the Branch-and-Cut-and-Price algorithm ;
26 **end**
27 **else**
28 Create two sub-problems by branching one some fractional variables;
29 **end**
30 **for** *each sub-problem not yet solved* **do**
31 go to 2 ;
32 **end**
33 **return** the best optimal solution (y^*, x^*, z^*) for the C-RSA;

a new sequence by doing some permutation of demands in the last sequence of demands, if the value of the solution given by greedy-algorithm is smaller than the value of the best solution found until the current iteration. Otherwise, we stop the algorithm, and we give in output the best solution found during our primal heuristic induced by the best sequence of demands having the smallest value of total length of the selected path compared with the others generated sequences.

4.4 Computational Study

4.4.1 Implementation's Feature

The B&P and B&C&P algorithms described in the current chapter have been implemented in C++ under Linux using the "Solving Constraint Integer Programs" framework (Scip 6.0.2), and Cplex 12.9 as LP solver. These have been tested on LIMOS high-performance server with a memory size limited to 64 Gb while benefiting from parallelism by activating 8 threads, and with a CPU time limited to 5 hours (18000 s).

4.4.2 Computational Results

Throughout this section, we present the performance results of the B&C&P algorithm. Our main goal is to show the effectiveness of the valid inequalities used within the B&C&P algorithm.

Table 4.1 reports the experiment results for both the Branch-and-Price (B&P) (i.e., B&C&P without using our additional valid inequalities) and the B&C&P algorithms. Each line corresponds to the average results of 4 tested instances. Note that we deactivate the SCIP's proper cut generation for both the B&P and B&C&P algorithms given that they may change the dual problem, as well as the calculation of the reduced-cost. In order to evaluate the impact of the additional valid inequalities used within the B&C&P algorithm, we consider 5 criteria, the average number of nodes in the branching tree (Nb_Nd), the average optimality gap (Gap) which represents the relative error between the lower bound and the best upper bound obtained at the end of the resolution, the average number of generated columns (Nbr_Cols), the average number of violated inequalities added (Nbr_Cuts), and the average CPU time in seconds (TT).

The results show that the B&C&P is able to solve 187 instances to optimality while 147 instances are solved to optimality when using the B&P. Hence, our valid inequalities allow solving several instances to optimality within a reasonable amount

of CPU time. Furthermore, they enable reducing the average number of nodes in the B&C&P tree, and also the average CPU time for several instances. We also notice that several instances have been solved to optimality in the root of the B&C&P tree (i.e., Nb_Nd=1) that necessitates a large number of branching nodes when using the B&P algorithm. On the other hand, and when the optimality is not proven, adding valid inequalities decreases the gap for several instances. However, there exist a few instances very rare in which adding valid inequalities have no any impact. Moreover, some instances are still difficult to solve with both the B&P and B&C&P algorithms.

4.4.3 Comparative Study Between Branch-and-Cut and Branch-and-Cut-and-Price Algorithms

Based on the Branch-and-Cut, Branch-and-Price and Branch-and-Cut-and-Price algorithms, we present a comparison between theses algorithms using several instances with number of demands ranges in $\{10, 20, 30, 40, 50, 100, 150\}$, and $\bar{s}$ up to 320 slots. Our first series of computational results presented in Tables 4.2, 4.3, and 4.2. They concern the results obtained for the Branch-and-Cut algorithm using Cplex (without or with additional valid inequalities) compared with those of Branch-and-Price and Branch-and-Cut-and-Price using SCIP. We denote by B&C_CPX the Branch-and-Cut algorithm when using Cplex with benefiting of its automatic cut generation and without our additional valid inequalities, and by Own_B&C_CPX when using Cplex with our additional valid inequalities and disabling its proper cut generation. On the other hand, in the second series of computational results are shown in Table 4.3, we present the results found for the Branch-and-Cut algorithm using Gurobi (without or with additional valid inequalities) compared with those of Branch-and-Price and Branch-and-Cut-and-Price using SCIP. We denote by B&C_GRB when using Gurobi with benefiting of its automatic cut generation and without our additional valid inequalities, and by Own_B&C_GRB when using Gurobi with our additional valid inequalities and disabling its proper cut generation. Results obtained by the Branch-and-Cut algorithm using Scip compared with those those of Branch-and-Price and Branch-and-Cut-and-Price using SCIP, are shown in Table 4.4. Let denote by B&C_Scip the Branch-and-Cut algorithm when using Scip with benefiting of its automatic cut generation and without our additional valid inequalities, and by Own_B&C_SCIP when using our additional valid inequalities and disabling its proper cut generation.

Based on the reported results, we notice that the B&C&P algorithm seems to be very efficient compared with B&C algorithm such that it is able to provide optimal solutions for several instances, which is not the case for the B&C algorithm (without or with additional valid inequalities) within the CPU time limit (5 hours). Furthermore, several instances are solved to optimality by the B&C algorithm using Cplex, Gurobi, and Scip could also be solved to optimality within the B&C&P algorithm. The average number of explored nodes using the B&C&P algorithm is greatly reduced for several instances compared with the B&C algorithm. Moreover, the average CPU time is significantly reduced using the B&C&P algorithm compared with the B&C algorithm. On the other hand, and when using the B&P algorithm, we notice that we are able to beat Own_B&C_SCIP such that B&P is able to provide optimal solutions for several instances that are not solved to optimality by the B&C algorithm using Cplex (see Table 4.2), and Gurobi (see Table 4.3). Furthermore, we noticed that the average number of explored nodes and the average CPU time using the B&P algorithm are greatly reduced for several instances compared with the B&C algorithm using Cplex and Gurobi. However, Own_B&C_SCIP is able to beat the B&P algorithm. The results in Table 4.4 show that Own_B&C_SCIP provide optimal solutions for several instances that are not solved to optimality by the B&P algorithm. But when the optimality is verified by these two algorithms, we found that using the B&P algorithm reduces the average number of explored nodes and the average CPU time for several instances compared with Own_B&C_SCIP.

4.5 Concluding Remarks

In this chapter, we first have given an extended formulation for the problem, and solve its linear relaxation using a column generation algorithm. We have discussed the associated pricing problem. Moreover, we have investigated the polytope associated with our formulation, and introduced several classes of valid inequalities. Their separation procedures are further presented. Using this, we have devised the B&C&P algorithm. Computational experiments have convincingly shown the strength of the valid inequalities. They significantly improve the results yielded by the B&P algorithm. Hence, the B&C&P algorithm performs very well compared with the B&P algorithm. Furthermore, the B&C&P algorithm is shown to be able to beat the B&C algorithm. A computational analysis is conducted to show the effectiveness of our approach for solving large-scale instances.

Instance			B&P_SCIP				B&C&P_SCIP				
G	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**Nbr_Cuts**	**TT**
German	10	15	28	**0,00**	13,50	0,88	1	**0,00**	3,25	6,25	0,07
	20	45	39	**0,00**	0	6,31	1	**0,00**	0	7,75	0,25
	30	45	1	**0,00**	0	0,20	1	**0,00**	0	0	0,31
	40	45	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
	50	55	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
	100	140	1	**0,00**	0	9,86	2	**0,00**	0	6,25	64,73
	150	210	34	**0,00**	0	417,78	51	**0,00**	0	24,75	932,25
Nsfnet	10	15	11	**0,00**	41,50	0,37	1	**0,00**	0	0	0,02
	20	20	190,50	**0,00**	168,50	34,67	1	**0,00**	165267,25	26,75	4487,61
	30	30	3286	1,17	295,25	9032,68	1	**0,00**	74970,75	7	8926,96
	40	35	4673,50	0,47	342	18000	1	**0,00**	0	16	3,72
	50	50	2218	0,54	566	13506,61	1	**0,00**	108442,50	18,25	8932,48
	100	120	2029	2,01	1849	18000	1	**0,00**	0	0	6,36
	150	160	321,50	11,66	1847,25	18000	1	**0,00**	0	0	32,48
Coronet100	10	320	1	**0,00**	867,50	51,36	1	**0,00**	867,50	0	54,11
	20	320	1	**0,00**	663,50	31,06	1	**0,00**	663,50	0	37,09
	30	40	1504	2,94	2577,50	18000	1545	1	11460,50	861,75	18000
	40	40	2969	0,46	1196	9008,08	12	**0,00**	57205,50	152	8983,29
	50	80	771,50	0,01	1282,75	4508,32	28,50	**0,00**	690,25	14,75	80,28
	100	120	333,50	3,69	1931	6836,74	130	4,62	1919	206,25	5182,07
	150	200	124	17,25	4272	14131,79	111,50	22,90	2629,50	296	16231,28
Spain	10	15	5	**0,00**	79,75	0,58	1	**0,00**	78,75	2,50	0,19
	20	20	1645	**0,00**	871,25	3033,88	724,5	**0,00**	782,50	91	1860,90
	30	25	2422	0,77	1586,75	10943,42	3427	0,54	1896	471,75	12332,07
	40	30	1545	0,56	1205,50	9000,70	2339,5	0,61	1162,75	619	9001,09
	50	35	3117,5	0,79	3172,50	17996,72	2163	0,79	3241,25	700	18000
	100	120	977,5	0,15	4870,50	14439,73	389,5	0,08	4885,75	143,25	13897,07
	150	160	127,5	16,84	5562,75	15163,74	122,5	19,77	5539	282,25	17242,74
India35	10	40	2	**0,00**	0	0,56	1	**0,00**	0	8,50	0,28
	20	40	1	**0,00**	36	0,66	1	**0,00**	36	0	0,57
	30	40	71,50	**0,00**	109	49,93	9,50	**0,00**	34,50	43	9,51
	40	40	3975,50	0,38	2046,25	18000	2754,50	0,11	17896,50	737,50	13542,45
	50	80	1	**0,00**	69,50	3,87	1	**0,00**	69,50	24	6,37
	100	120	496	0,01	50,50	9072,59	353,50	**0,00**	98	356,25	4820,46
	150	200	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Ta65	10	40	1	**0,00**	54,50	0,58	1	**0,00**	54,50	0	0,42
	20	40	1	**0,00**	188,50	1,41	1	**0,00**	188,50	2	1,21
	30	40	1109,50	0,08	539,75	4499,45	1383,50	0,08	509,50	125,50	4500
	40	40	529	0,03	562	4517,43	1,50	**0,00**	380,25	17,75	9,95
	50	40	1591,50	0,08	1131,50	9002,35	596	0,03	991,75	337,25	4540,26
	100	80	463	0,14	2179	6683,60	362	**0,00**	2148,25	248	5566,88
	150	160	123	2,29	3821,75	5870,91	147,50	**0,00**	3743,25	232,25	5225,13
Brain161	10	40	1	**0,00**	0	0,35	1	**0,00**	0	0	0,17
	20	40	1	**0,00**	0	0,72	1	**0,00**	0	0	0,33
	30	40	8	**0,00**	0	8,51	1	**0,00**	0	1	0,62
	40	40	14	**0,00**	54,75	35,18	2,50	**0,00**	45,50	9,50	6,77
	50	40	5370	0,50	264,33	18000	10326,33	0,40	367	2087,67	18000
	100	80	1318	0,81	314	18000	2234,50	0,15	420,25	654,25	18000
	150	160	113,50	**0,00**	0	2257,49	85,50	**0,00**	0	30,25	1645,30
Zib54	10	40	1	**0,00**	27	0,40	1	**0,00**	27	0	0,27
	20	40	1	**0,00**	137,50	1,23	1	**0,00**	137,50	0	0,70
	30	40	670,5	0,09	283,50	4501,18	3	**0,00**	132,25	10	5,35
	40	40	99	**0,00**	378,75	102,17	1	**0,00**	275,75	18,75	3,15
	50	40	864,5	0,42	761,50	9063,67	840,5	0,13	649	280,75	9015,80
	100	80	1104	0,18	1600,50	16751,83	508	0,03	1787,50	343,50	8004,65
	150	160	101,5	2,28	2911,50	5938,67	125	0,26	2860,75	156,75	5550,35

Table 4.1: Influence of the Valid Inequalities: B&P Vs B&C&P.

Instances			B&C_CPX			Own_B&C_CPX				B&P_SCIP				B&C&P_SCIP				
Topology	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**Nbr_Cuts**	**TT**
German	10	15	5534,75	**0,00**	42,37	9071,75	**0,00**	1524,25	101,76	28	**0,00**	13,50	0,88	1	**0,00**	3,25	6,25	0,07
	20	45	109616,75	0,27	4382,25	68601,50	4,46	46110,75	18000	39	**0,00**	0	6,31	1	**0,00**	0	7,75	0,25
	30	45	71995	0,38	8845,19	16945,75	18,73	53497,50	18000	1	**0,00**	0	0,20	1	**0,00**	0	0	0,31
	40	45	75469	3,82	18000	363,33	60,47	41306,33	18000	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
	50	55	44143	5,32	18000	470,75	61,19	52946	18000	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
	100	140	1	**0,00**	50,15	33,75	60,75	20802	18000	1	**0,00**	0	9,86	2	**0,00**	0	6,25	64,73
	150	210	1	**0,00**	327,54	150,50	62,40	39133,25	18000	34	**0,00**	0	417,78	51	**0,00**	0	24,75	932,25
Nsfnet	10	15	72616,25	**0,00**	644,88	140046,50	**0,00**	3760,75	2640,70	11	**0,00**	41,50	0,37	1	**0,00**	0	0	0,02
	20	20	121350,25	32,76	18000	27987,75	41,38	53257,75	18000	145	**0,00**	206	7,79	1	**0,00**	0	11	0,19
	30	30	67751	17,13	18000	6322,75	49,13	66315,50	18000	4373,67	1,57	347	12041,42	1	**0,00**	99961	9,33	11902,57
	40	35	32491	21,46	18000	1687,25	59,53	62749,25	18000	4817,67	0,50	331	17990,15	1	**0,00**	0	16	3,62
	50	50	21256,75	19,80	18000	2216,25	59,73	53734,50	18000	2218	0,54	566	13506,61	1	**0,00**	108442,50	18,25	8932,48
	100	120	1064,50	20,28	18000	99	66,23	35031,75	18000	2029	2,01	1849	18000	1	**0,00**	0	0	6,36
	150	160	145	24,53	18000	1	66,94	142594,25	18000	321,50	11,66	1847,25	17996,69	1	**0,00**	0	0	32,48
India35	10	40	968,50	**0,00**	40,87	106140	23,94	48081	18000	2	**0,00**	0	0,56	1	**0,00**	0	8,50	0,28
	20	40	1	**0,00**	12,23	38307,25	41,43	33858	18000	1	**0,00**	36	0,66	1	**0,00**	36	0	0,57
	30	40	44170,25	**0,00**	4149,40	19389,50	49,99	80759,50	18000	71,50	**0,00**	109	49,93	9,50	**0,00**	34,50	43	9,51
	40	40	32029	2,80	18000	9583,50	53,19	94366,50	18000	3975,50	0,38	2046,25	17958,21	2754,50	0,11	17896,50	737,50	13542,45
	50	80	1	**0,00**	60,98	4763	58,61	35364,75	18000	1	**0,00**	69,50	3,87	1	**0,00**	69,50	24	6,37
	100	120	6217,50	0,71	9493,81	230	67,42	28590,25	18000	496	0,01	50,50	9072,59	353,50	**0,00**	98	356,25	4820,46
	150	200	4948,25	0,28	13807,72	287,75	67,65	19196,75	18000	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Pioro40	10	40	1	**0,00**	6,23	123725,75	14,29	1705,75	18000	1	**0,00**	27,25	0,26	1	**0,00**	27,25	0	0,20
	20	40	1	**0,00**	15,77	52857,25	29,75	17720,25	18000	1	**0,00**	73,75	0,52	1	**0,00**	73,75	0	0,50
	30	40	1	**0,00**	26,32	27075,50	44,66	47926,50	18000	1	**0,00**	100,50	0,81	1	**0,00**	100,50	0	0,68
	40	40	32219,75	0,19	8966,91	17658,50	52,21	58468,25	18000	101,50	**0,00**	563,25	116,31	7,50	**0,00**	261,75	19,50	9,75
	50	80	1	**0,00**	84,98	7344,75	59,79	6049,25	18000	1	**0,00**	466,25	5,27	1	**0,00**	466,25	0	4,16
	100	80	12576	0,05	13517,75	1555,50	66,37	54491,75	18000	789,50	0,01	1012,75	9062,39	663	0,01	929	282	4820,38
	150	160	1	**0,00**	626,31	765,75	68,03	11064,75	18000	1	**0,00**	1279,50	150,54	1	**0,00**	1276	0	176,69

Table 4.2: Table of Comparison Between B&C, B&P and B&C&P Algorithms: Cplex (Without or With Additional Valid Inequalities) Vs Scip (Without or With Additional Valid Inequalities).

Instances			B&C_GRB			Own_B&C_GRB				B&P_SCIP				B&C&P_SCIP				
Topology	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cols**	**Nbr_Cuts**	**TT**
German	10	15	4906,25	**0,00**	981,48	3460,25	**0,00**	1866,50	149,13	28	**0,00**	13,50	0,88	1	**0,00**	3,25	6,25	0,07
	20	45	20752	0,27	4889,90	28093,75	0,27	33413	4962,48	39	**0,00**	0	6,31	1	**0,00**	0	7,75	0,25
	30	45	24321,50	0,39	9279,54	3614,50	**0,00**	19141,50	882,68	1	**0,00**	0	0,20	1	**0,00**	0	0	0,31
	40	45	35451,67	5,52	18000	9920	1,82	216792,33	15453,54	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
	50	55	18901,50	6,15	18000	6679	17,55	263086,50	18000	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
	100	140	1	**0,00**	1634,37	1746,50	**0,00**	200920	15425,53	1	**0,00**	0	9,86	2	**0,00**	0	6,25	64,73
	150	210	64,75	**0,00**	3184,56	979,75	73,06	44188,25	18000	34	**0,00**	0	417,78	51	**0,00**	0	24,75	932,25
Nsfnet	10	15	15222,75	**0,00**	2087,33	4385	**0,00**	4095,25	216,83	11	**0,00**	41,50	0,37	1	**0,00**	0	0	0,02
	20	20	51525	12,77	18000	181730	2,58	99411	18000	145	**0,00**	206	7,79	1	**0,00**	0	11	0,19
	30	30	27735	22,41	18000	19702,67	13,20	263905,66	18000	4373,67	1,57	347	12041,42	1	**0,00**	99961	9,33	11902,57
	40	35	12631	34,18	18000	6239,33	24,31	307366	18000	4817,67	0,50	331	17990,15	1	**0,00**	0	16	3,62
	50	50	8733,50	29,35	18000	5265,75	47,22	347095,75	18000	2218	0,54	566	13506,61	1	**0,00**	108442,50	18,25	8932,48
	100	120	7790,50	29,60	18000	2253,25	41,60	326605	18000	2029	2,01	1849	18000	1	**0,00**	0	0	6,36
	150	160	4255,25	33,05	18000	871	INF	80789	18000	321,50	11,66	1847,25	17996,69	1	**0,00**	0	0	32,48
India35	10	40	2139,50	**0,00**	481,60	1658,25	**0,00**	1902,50	281,53	2	**0,00**	0	0,56	1	**0,00**	0	8,50	0,28
	20	40	103	**0,00**	541,13	1722,25	**0,00**	4866	572,66	1	**0,00**	36	0,66	1	**0,00**	36	0	0,57
	30	40	19437,25	**0,00**	5132,47	1879,50	**0,00**	11196,25	914,49	71,50	**0,00**	109	49,93	9,50	**0,00**	34,50	43	9,51
	40	40	28219	1,40	18000	9945	**0,00**	58588,25	7309,79	3975,50	0,38	2046,25	17958,21	2754,50	0,11	17896,50	737,50	13542,45
	50	80	386,50	**0,00**	1221,34	2174,25	**0,00**	18163	3970,39	1	**0,00**	69,50	3,87	1	**0,00**	69,50	24	6,37
	100	120	6036,25	0,75	10772,47	2399	95,14	30646,50	18000	496	0,01	50,50	9072,59	353,50	**0,00**	98	356,25	4820,46
	150	200	4164	0,35	11292,87	726,50	98,71	7760,25	18000	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Pioro40	10	40	905,75	**0,00**	275,37	1756	**0,00**	578,50	232,46	1	**0,00**	27,25	0,26	1	**0,00**	27,25	0	0,20
	20	40	544,75	**0,00**	527,67	1613,75	**0,00**	2330,50	512,12	1	**0,00**	73,75	0,52	1	**0,00**	73,75	0	0,50
	30	40	2	**0,00**	539,60	1669,25	**0,00**	5362,25	863,04	1	**0,00**	100,50	0,81	1	**0,00**	100,50	0	0,68
	40	40	12237,75	0,19	9290,28	10971	0,01	41439,25	5815,90	101,50	**0,00**	563,25	116,31	7,50	**0,00**	261,75	19,50	9,75
	50	80	10,25	**0,00**	773,69	1800	**0,00**	9409,50	2780,11	1	**0,00**	466,25	5,27	1	**0,00**	466,25	0	4,16
	100	80	2785,50	0,54	5680,84	4749	1,76	72888,25	17854,91	789,50	0,01	1012,75	9062,39	663	0,01	929	282	4820,38
	150	160	2,25	**0,00**	4805,69	1068,25	99,51	8076	18000	1	**0,00**	1279,50	150,54	1	**0,00**	1276	0	176,69

Table 4.3: Table of Comparison Between B&C, B&P and B&C&P Algorithms: Gurobi (Without or With Additional Valid Inequalities) Vs Scip (Without or With Additional Valid Inequalities).

Instances			B&C_SCIP			Own_B&C_SCIP				B&P_SCIP				B&C&P_SCIP				
Topology	$\|K\|$	$\|\mathbb{S}\|$	Nbr_Nd	Gap	TT	Nbr_Nd	Gap	Nbr_Cuts	TT	Nbr_Nd	Gap	Nbr_Cols	TT	Nbr_Nd	Gap	Nbr_Cols	Nbr_Cuts	TT
German	10	15	1310,25	0,00	14,35	59	0,00	429,75	0,83	28	0,00	13,50	0,88	1	0,00	3,25	6,25	0,07
	20	45	185956	0,27	3895,50	141	0,00	2403,50	3,89	39	0,00	0	6,31	1	0,00	0	7,75	0,25
	30	45	401335,75	1,60	11740,04	160376,50	1,46	129867,25	8334,95	1	0,00	0	0,20	1	0,00	0	0	0,31
	40	45	315993,66	8,33	16206,36	383058,66	3,70	224642,33	16624,87	1489,67	0,33	324,67	6000,12	1557,67	0,13	309,67	339	5998,03
	50	55	246146,50	9,62	16675,88	251152,50	13,73	305309,75	17074,95	3550,75	0,18	412,50	13506,57	1513	0,14	371	385	9020,19
	100	140	1158,50	0,00	340,10	3014,25	0,00	17668,50	617,80	1	0,00	0	9,86	2	0,00	0	6,25	64,73
	150	210	12759	0,01	7329,06	3609	0,00	24782,25	3057,79	34	0,00	0	417,78	51	0,00	0	24,75	932,25
Nsfnet	10	15	13462	0,00	113,64	1	0,00	95,75	0,15	11	0,00	41,50	0,37	1	0,00	0	0	0,02
	20	20	699646	9,51	18000	21586	0,00	24587	192,27	145	0,00	206	7,79	1	0,00	0	11	0,19
	30	30	272065	40,99	18000	281569,66	3,29	340177	11048,71	4373,67	1,57	347	12041,42	1	0,00	99961	9,33	11902,57
	40	35	225696,67	46,74	18000	119841,66	1,17	163519,33	5673,46	4817,67	0,50	331	17990,15	1	0,00	0	16	3,62
	50	50	247873,25	43,09	18000	148476,50	5,91	340399,25	17405,09	2218	0,54	566	13506,61	1	0,00	108442,50	18,25	8932,48
	100	120	56598,50	57,19	18000	1	0,00	464,25	40,87	2029	2,01	1849	18000	1	0,00	0	0	6,36
	150	160	12663	58,50	18000	1	0,00	496,25	136,02	321,50	11,66	1847,25	17996,69	1	0,00	0	0	32,48
India35	10	40	1907,25	0,00	87,60	1	0,00	779,75	1,80	2	0,00	0	0,56	1	0,00	0	8,50	0,28
	20	40	9	0,00	4	7	0,00	2437,75	5,92	1	0,00	36	0,66	1	0,00	36	0	0,57
	30	40	91798	0,00	7821,50	32156,75	0,00	10917,50	2309,66	71,50	0,00	109	49,93	9,50	0,00	34,50	43	9,51
	40	40	161514	2,42	17486,08	191812	0,18	27270	17333,53	3975,50	0,38	2046,25	17958,21	2754,50	0,11	17896,50	737,50	13542,45
	50	80	34	0,00	22,13	69,25	0,00	11105,75	112,19	1	0,00	69,50	3,87	1	0,00	69,50	24	6,37
	100	120	24797	0,32	9137,26	23403,75	0,44	48702,75	9494,52	496	0,01	50,50	9072,59	353,50	0,00	98	356,25	4820,46
	150	200	16809	0,21	13739,65	1026	0,00	19898	4101,80	292	0,01	96,50	9831,30	100,50	0,10	96,50	389,25	8155,83
Pioro40	10	40	1	0,00	1,49	1	0,00	202,25	1,69	1	0,00	27,25	0,26	1	0,00	27,25	0	0,20
	20	40	1,50	0,00	3,44	1	0,00	1255,75	4,88	1	0,00	73,75	0,52	1	0,00	73,75	0	0,50
	30	40	1,50	0,00	5,72	6,25	0,00	2972,50	10,54	1	0,00	100,50	0,81	1	0,00	100,50	0	0,68
	40	40	83597	0,20	8692,50	67151	0,12	53544,50	8711,30	101,50	0,00	563,25	116,31	7,50	0,00	261,75	19,50	9,75
	50	80	14	0,00	15,93	4	0,00	8606	54,39	1	0,00	466,25	5,27	1	0,00	466,25	0	4,16
	100	80	21281,75	0,04	9087,52	23785,75	0,04	57123,75	9916,63	789,50	0,01	1012,75	9062,39	663	0,01	929	282	4820,38
	150	160	823,50	0,00	816,89	124,50	0,00	22814	1509,87	1	0,00	1279,50	150,54	1	0,00	1276	0	176,69

Table 4.4: Table of Comparison Between B&C, B&P and B&C&P Algorithms: Scip (Without or With Additional Valid Inequalities) Vs Scip (Without or With Additional Valid Inequalities).

Chapter 5

Compact Formulation and Polyhedra for the Spectrum Assignment Sub-problem

In this chapter, we focus on the Spectrum Assignment (SA) sub-problem. First, we propose an integer linear programming compact formulation, and further investigate the facial structure of the associated polytope. Moreover, we identify several classes of valid inequalities for the polytope such that some of them come from those that are already proposed for the C-RSA. We further prove that these inequalities are facet-defining, and discuss their separation problems. Based on these results, we devise a Branch-and-Cut (B&C) algorithm for the SA problem.

5.1 The Spectrum Assignment Sub-problem

The SA problem can be stated as follows. We consider an optical spectrum of $\bar{s} \in \mathbb{Z}_+$ available contiguous frequency slots, denoted by $\mathbb{S} = \{1, \ldots, \bar{s}\}$. A spectrally flexible optical network can be represented by an undirected, loopless, and connected graph $G = (V, E)$, with V is the set of vertices representing the optical nodes (data centers, users, stations,...), and E the set of links representing the optical-fibers. A length $\ell_e \in \mathbb{R}_+$ (in kms), a cost $c_e \in \mathbb{R}_+$, and a set of $\bar{s}$ of contiguous frequency slots are associated with each edge e. Let K be a multiset of demands such that each demand k is specified by an origin node $o_k \in V$, a destination node $d_k \in V \setminus \{o_k\}$, a slot-width $w_k \in \mathbb{Z}_+$, and a routing path p_k from its origin o_k to its destination d_k through G. The SA consists of determining for each demand $k \in K$ an interval of contiguous frequency slots $S_k \subset \mathbb{S}$ of width equal to w_k (continuity and contiguity

constraints) such that $S_k \cap S_{k'} = \emptyset$ for each pair of demands $k, k' \in K$ $(k \neq k')$ with paths sharing an edge , i.e., $E(p_k) \cap E(p_{k'}) \neq \emptyset$, while optimizing the number of slots allocated in $\mathbb{S}$.
The SA is well known to be NP-hard problem [9]. It is equivalent to the problems of wavelength assignment, interval coloring, and dynamic storage allocation [9] that are well known to be NP-hard.

5.2 Compact Formulation

Here we introduce an integer linear programming compact formulation for the SA problem. For $s \in \mathbb{S}$, let u_s be a variable which takes 1 if slot s is used and 0 if not, and for $k \in K$ and $s \in \mathbb{S}$, let z_s^k be a variable which takes 1 if slot s is the last slot allocated for the routing of demand k and 0 if not. The contiguous slots $s' \in \{s - w_k + 1, ..., s\}$ should be assigned to demand k whenever $z_s^k = 1$. The SA is equivalent to the following integer linear program

$$\min \sum_{s \in \mathbb{S}} u_s, \tag{5.1}$$

subject to

$$z_s^k = 0, \quad \text{for all } k \in K \text{ and } s \in \{1, ..., w_k - 1\}, \tag{5.2}$$

$$\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1, \quad \text{for all } k \in K, \tag{5.3}$$

$$\sum_{k \in \tilde{K}_e} \sum_{s'=s}^{\min(\bar{s}, s+w_k-1)} z_s^k - u_s \leq 0, \quad \text{for all } e \in E, \text{ and } s \in \mathbb{S}, \tag{5.4}$$

$$z_s^k \geq 0, \quad \text{for all } k \in K \text{ and } s \in \mathbb{S}, \tag{5.5}$$

$$u_s \leq 1, \quad \text{for all } s \in \mathbb{S}, \tag{5.6}$$

$$z_s^k \in \{0, 1\}, \quad \text{for all } k \in K \text{ and } s \in \mathbb{S}, \tag{5.7}$$

$$u_s \in \{0, 1\}, \quad \text{for all } s \in \mathbb{S}. \tag{5.8}$$

where $\tilde{K}_e$ denotes the set of demands in K passing through edge e (i.e., $\tilde{K}_e = \{k \in K, e \in E(p_k)\}$. Equations (5.2) ensure that demand k cannot occupy a slot s as last slot before her slot-width w_k. Inequalities (5.3) ensure than more than one interval of contiguous slots can be assigned to each demand $k \in K$. It should normally be an equation form ensuring that exactly one slot $s \in \{w_k, \ldots, \bar{s}\}$ (one interval of contiguous slots) must be assigned to demand k as last-slot. Here we relax this constraint. Optimizing the spectrum-usage objective function, the equality is

guaranteed at the optimum. Inequalities (5.4) express the fact that the demands passed through the same edge e, they cannot share a slot s over edge $e \in E$ with $s \in \{1, ..., \bar{s}\}$. Inequalities (5.5)-(5.6) are the trivial inequalities, and constraints (5.7)-(5.8) are the integrality constraints.

5.3 Associated Polytope

Let $\mathcal{P}_{sa}(G, K, \mathbb{S})$ be the polytope, convex hull of the solutions for the formulation (5.1)-(5.8). Here we study the facial structure of the polytope $\mathcal{P}_{sa}(G, K, \mathbb{S})$.
A solution of the SA problem is based on the variables (u, z) is given by two sets S_k for each demand $k \in K$ and U for the spectrum-usage of $\mathbb{S}$ where

a) S_k denotes the set of index of the last slots selected for demand k such that $|S_k| \geq 1$.

b) U denotes the set of slots allocated over the spectrum $\mathbb{S}$ such that for each demand $k \in K$ and last slot $s \in S_k \Rightarrow$ each slot $s' \in \{s - w_k + 1, ..., s\}$ should be in U i.e. $s' \in U$.

We suppose that the number of slots $\bar{s}$ is largely sufficient to route all the demands, and to avoid the existence of some slots $s \in \mathbb{S}$ such that $u_s = 1$ in any feasible solution $\mathcal{S}$ of the SA problem. This means that there does not exist a slot $s \in \mathbb{S}$ such that $u_s = 1$.

5.3.1 Dimension

Let M denote the matrix associated with the equations (5.2). We ensure that the matrix M is of full rank given that the demands are independants, and the slots in $\mathbb{S}$ are independents for each demand $k \in K$. As a result, $rank(M) = \sum_{k \in K} (w_k - 1)$.
Let us denote by r' the rank of the matrix M.

Proposition 5.3.1. *The equation system* (5.2) *defines a minimal equation system for* $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Proof. To prove that $\sigma z + \mu u = \lambda$ is a linear combination of equations (5.2), it's sufficient to prove that for each demand $k \in K$, there exists for each demand $k \in K$ a $\gamma^k \in \mathbb{R}^{w_k - 1}$ such that $(\mu, \sigma) = \gamma M$. Let $u^{\mathcal{S}}$ and $z^{\mathcal{S}}$ denote the incidence vector of a solution $\mathcal{S}$ of the SA problem.
Let first show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$, and solution $\mathcal{S}^{105} = (U^{105}, S^{105})$ given by

a) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{\tilde{s}, ..., \tilde{s} + w_k - 1\}$ as last slot for demand k (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{105}),

b) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\} \setminus \{k\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{105} given by

$$I_i^{105} = [\bigcap_{k_j \in D_i^{105}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\}$$

where $D_i^{105} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \{k_1, ..., k_{i-1}\} \cup \{k\}$ with $E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{105}),

We let $S_{k_i}^{105} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

c) we let U^{105} be the set of slots used in $\mathbb{S}$ such that for each demand k and last slot $s_k \in S_k^{105}$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{105}$.

$\mathcal{S}^{105}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{105}}, z^{\mathcal{S}^{105}})$ belongs to $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Then we derive a solution $\mathcal{S}^{106} = (U^{106}, S^{106})$ obtained from $\mathcal{S}^{105}$ by adding slot $\tilde{s}$ as an used slot in U^{106} without modifying the last slots assigned to the demands K in $\mathcal{S}^{105}$ which remain the same in solution $\mathcal{S}^{106}$ i.e., $S_k^{105} = S_k^{106}$ for each demand $k \in K$. $\mathcal{S}^{106}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{106}}, z^{\mathcal{S}^{106}})$ belongs to $\mathcal{P}_{sa}(G, K, \mathbb{S})$. We then obtain that

$$\mu u^{\mathcal{S}^{105}} + \sigma z^{\mathcal{S}^{105}} = \mu u^{\mathcal{S}^{106}} + \sigma z^{\mathcal{S}^{106}} = \mu u^{\mathcal{S}^{105}} + \sigma z^{\mathcal{S}^{105}} + \mu_{\tilde{s}}.$$

Hence, $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let show now that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$.

Consider a demand k in K and a slot s in $\{w_k, ..., \bar{s}\}$. Let $\mathcal{S}^{107} = (U^{107}, S^{107})$ be the solution given by

a) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{s\}$ as last slot for demand k,

b) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{107} given by

$$I_i^{107} = [\bigcap_{k_j \in D_i^{107}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s - w_k\} \cup \{s + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_k) \neq \emptyset \text{ or } I_i^{107} = \bigcap_{k_j \in D_i^{107}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{107} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{107}$,
- and $\{s - w_k + 1, ..., s\} \cap \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} = \emptyset$ if $E(p_k) \cap E(p_{k_i}) \neq \emptyset$ (we take into account the possibility of adding slot s in the set of last slots S_k^{107} assigned to demand k in solution $\mathcal{S}^{107}$).

We let $S_{k_i}^{107} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

c) we let U^{107} be the set of slots used in $\mathbb{S}$ such that for each demand k and last slot $s_k \in S_k^{107}$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{107}$.

$\mathcal{S}^{107}$ is clearly feasible for the problem given that it satisfies all the SA constraints of the compact formulation (5.1)-(5.8). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{107}}, z^{\mathcal{S}^{107}})$ belongs to $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Then consider the solution $\mathcal{S}^{108} = (E^{108}, S^{108})$ obtained from $\mathcal{S}^{107}$ by adding slot s as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{107}$ remain the same in solution $\mathcal{S}^{108}$ i.e., $S_{k'}^{107} = S_{k'}^{108}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{108} = S_k^{107} \cup \{s\}$ for demand k. Solution $\mathcal{S}^{108}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{108}}, z^{\mathcal{S}^{108}})$ belongs to $\mathcal{P}_{sa}(G, K, \mathbb{S})$. We then obtain that

$$\mu u^{\mathcal{S}^{107}} + \sigma z^{\mathcal{S}^{107}} = \mu u^{\mathcal{S}^{108}} + \sigma z^{\mathcal{S}^{108}} = \mu u^{\mathcal{S}^{107}} + \sigma z^{\mathcal{S}^{107}} + \sigma_s^k + \sum_{\tilde{s} \in \{s, ..., s - w_k + 1\} \setminus U^{107}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for each $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_s^k = 0$.
In a similar way, we can show that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\}.$$

Therefore, we obtain that all the equations of the polytope $\mathcal{P}_{sa}(G, K, \mathbb{S})$ are given only in terms of the variables z_s^k with $s \in \{1, ..., w_k\}$ for each demand $k \in K$. We distinguish $|K|$ blocks of lines in the matrix M associated with the system (5.2)

- block M^k corresponds to the equations $z_s^k = 0$ for all $s \in \{1, ..., w_k - 1\}$ such that $rang(M^k) = w_k - 1$.

Note that the $|K|$ blocks of the matrix M are independents. Furthermore, there is no dependency between slots such that for each demand k, the slots $s \in \{1, ..., w_k - 1\}$ are independants such that

$$\sum_{s=1}^{w_k-1} \sigma_s^k = \sum_{s=1}^{w_k-1} \gamma^{k,s} \Rightarrow \sum_{s=1}^{w_k-1} (\sigma_s^k - \gamma^{k,s}) = 0,$$

for each demand $k \in K$. The only solution of this system is $\sigma_s^k = \gamma^{k,s}$ for each $s \in \{1, ..., w_k - 1\}$ for demand k. As k is chosen arbitrarily in K, we re-do the same procedure for all $k' \in K \setminus \{k\}$. We then get that

$$\sigma_s^k = \gamma^{k,s}, \text{ for all } k \in K \text{ and } s \in \{1, ..., w_k - 1\}. \tag{5.9}$$

As a result, we have $(\mu, \sigma) = \gamma M$ which ends the proof. □

Theorem 5.3.1. *The dimension of $\mathcal{P}_{sa}(G, K, \mathbb{S})$ is given by*

$$dim(\mathcal{P}_{sa}(G, K, \mathbb{S})) \;=\; |K| * |\mathbb{S}| + |\mathbb{S}| - r' \;=\; |K| * |\mathbb{S}| + |\mathbb{S}| - \sum_{k \in K} (w_k - 1).$$

Proof. Given the rank of the matrix M which equals to r' and the results of proposition (5.3.1). □

5.3.2 Facial Investigation

Here we study the facial structure of the basic constraints of the compact formulation (5.1)-(5.8) that are facets defining for the polyhedron $\mathcal{P}_{sa}(G, K, \mathbb{S})$ under certain conditions.

Theorem 5.3.2. *Consider a demand $k \in K$ and a slot $s \in \{w_k, .., \bar{s}\}$. Then, inequality $z_s^k \geq 0$ is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.*

Proof. Let us denote $\tilde{F}_s^k$ the face induced by inequality $z_s^k \geq 0$, that is

$$\tilde{F}_s^k = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : z_s^k = 0\}.$$

We denote inequality $z_s^k \geq 0$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that defines a facet F of $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}_s^k \subset F = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. To prove that $\tilde{F}_s^k$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, it sufficient to show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.

First, let show that $\mu_{\tilde{s}} = 0$ for all $\tilde{s} \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$, and a solution $\mathcal{S}^{109} = (U^{109}, S^{109})$ given by

a) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus [\{\tilde{s}, ..., \tilde{s} + w_k - 1\} \cup \{s\}]$ as last slot for demand k (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{109}),

b) for each demand $k_i \in K$ with $i \in \{1, ..., |K|\} \setminus \{k\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{109} given by

$$I_i^{109} = [\bigcap_{k_j \in D_i^{109}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\}$$

where $D_i^{109} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \{k_1, ..., k_{i-1}\} \cup \{k\}$ with $E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{109}),

Let $S_{k_i}^{109} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

c) we let U^{109} be the set of slots used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s_{k'} \in S_k^{109}$ and $s' \in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have $s' \in U^{109}$.

$\mathcal{S}^{109}$ is clearly feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{109}}, z^{\mathcal{S}^{109}})$ belongs to $\tilde{F}_s^k$. Then consider the solution $\mathcal{S}'^{109} = (U'^{109}, S'^{109})$ obtained from $\mathcal{S}^{109}$ by adding slot $\tilde{s}$ as an used slot in U'^{109} without modifying the last slots assigned to the demands K in $\mathcal{S}^{109}$ which remain the same in solution $\mathcal{S}'^{109}$ i.e., $S_k^{109} = S_k'^{109}$ for each demand $k \in K$. Solution $\mathcal{S}'^{109}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}'^{109}}, z^{\mathcal{S}'^{109}})$ belongs to $\tilde{F}_s^k$. Solutions $\mathcal{S}^{109}$ and $\mathcal{S}'^{109}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{109}} + \sigma z^{\mathcal{S}^{109}} = \mu u^{\mathcal{S}'^{109}} + \sigma z^{\mathcal{S}'^{109}} = \mu u^{\mathcal{S}^{109}} + \sigma z^{\mathcal{S}^{109}} + \mu_{\tilde{s}}.$$

Hence, $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Next, we will show that, $\sigma_{s'}^k = 0$ for all $s' \in \{w_k, ..., \bar{s}\} \setminus \{s\}$.

Consider a slot s' in $\{w_k, ..., \bar{s}\} \setminus \{s\}$. Let $\mathcal{S}^{110} = (U^{110}, S^{110})$ be the solution given by

a) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{s, s'\}$ as last slot for demand k,

b) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{110} given by

$$I_i^{110} = [\bigcap_{k_j \in D_i^{110}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_k) \neq \emptyset \text{ or } I_i^{110} = \bigcap_{k_j \in D_i^{110}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{110} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{110}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E(p_k) \cap E(p_{k_i}) \neq \emptyset$ (we take into account the possibility of adding slot s' in the set of last slots S_k^{110} assigned to demand k in solution $\mathcal{S}^{110}$),

Let $S_{k_i}^{110} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

c) we let U^{110} be the set of slots used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s_{k'} \in S_{k'}^{10}$ and s" $\in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have s" $\in U^{110}$.

$\mathcal{S}^{110}$ is clearly feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{110}}, z^{\mathcal{S}^{110}})$ belongs to $\tilde{F}_s^k$. Then we derive a solution $\mathcal{S}^{112} = (U^{112}, S^{112})$ obtained from $\mathcal{S}^{110}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}_k^{110}$ remain the same in solution $\mathcal{S}^{112}$ i.e., $S_{k'}^{110} = S_{k'}^{112}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{112} = S_{'}^{110} \cup \{s'\}$ for demand k. Solution $\mathcal{S}^{112}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{112}}, z^{\mathcal{S}^{112}})$ belongs to $\tilde{F}_s^k$. Hence, solutions $\mathcal{S}^{110}$ and $\mathcal{S}^{112}$ satisfy equation $\mu u + \sigma z = \tau$. We the obtain that

$$\mu u^{\mathcal{S}^{110}} + \sigma z^{\mathcal{S}^{110}} = \mu u^{\mathcal{S}^{112}} + \sigma z^{\mathcal{S}^{112}} = \mu u^{\mathcal{S}^{110}} + \sigma z^{\mathcal{S}^{110}} + \sigma_{s'}^k + \sum_{\tilde{s} \in \{s' - w_k + 1, ..., s'\} \setminus U^{110}} \mu_{\tilde{s}}.$$

Given that $\mu_{\tilde{s}} = 0$ for all $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_{s'}^k = 0$.
In a similar way, we can show that

$$\sigma_{s'}^k = 0, \text{ for all slots } s' \in \{w_k, ..., \bar{s}\} \setminus \{s\},$$
$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } s' \in \{w_{k'}, ..., \bar{s}\}.$$

It follows that $\sigma_s^k = \rho$ for demand k and slot s in $\{w_k, ..., \bar{s}\}$.
By (5.9), we know that

$$\sigma_{s'}^{k'} = \gamma^{k', s'}, \text{ for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}.$$

Overall, we obtain that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and for each $k' \in K$ and $s' \in \mathbb{S}$

$$\sigma_{s'}^{k'} = \begin{cases} \gamma^{k',s'} & \text{if } s' \in \{1, ..., w_{k'} - 1\}, \\ \rho & \text{if } k' = k \text{ and } s' = s, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$. □

Theorem 5.3.3. *Consider a slot $s \in \mathbb{S}$. Then, inequality $u_s \leq 1$ is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.*

Proof. Let us denote F_s the face induced by inequality $u_s \leq 1$ given by

$$F_s = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : u_s = 1\}.$$

We denote inequality $u_s \leq 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that defines a facet F of $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Suppose that $F_s \subset F = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. To prove that F_s is facet defining for $P_{sa}(G, K, \mathbb{S})$, it sufficient to show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.
First, let show that $\mu_{s'} = 0$ for all $s' \in \mathbb{S} \setminus \{s\}$. Consider a slot $\tilde{s} \in \mathbb{S} \setminus \{s\}$, and a solution $\mathcal{S}^{113} = (U^{113}, S^{113})$ given by

a) for one demand $k' \in K$, we select the smallest slot index $s_{k'} \in [\{w_{k'}, ..., s'\} \cap \{s, ..., s + w_{k'} - 1\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k'} - 1\}$ as last slot,

b) for each demand $k_i \in K \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{113} given by

$$I_i^{113} = [\bigcap_{k_j \in D_i^{113}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\},$$

where $D_i^{113} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k'\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{113}$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{113}),

Let $S_{k_i}^{113} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

c) we let U^{113} be the of slots used in $\mathbb{S}$ such that for each demand k and last slot $s \in S_k^{113}$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{113}$.

$\mathcal{S}^{113}$ is clearly feasible for the SA problem, and its incidence vector $(u^{\mathcal{S}^{113}}, z^{\mathcal{S}^{113}})$ belongs to F_s. After that, we derive a solution $\mathcal{S}'^{113} = (U'^{113}, S'^{113})$ obtained from $\mathcal{S}^{113}$ by adding slot $\tilde{s}$ as an used slot in U'^{113} without modifying the last slots assigned to the demands K in $\mathcal{S}^{113}$ which remain the same in solution $\mathcal{S}'^{113}$ i.e., $S_k^{113} = S_k'^{113}$ for each demand $k \in K$. Solution $\mathcal{S}'^{113}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}'^{113}}, z^{\mathcal{S}'^{113}})$ belongs to F_s. Hence, solutions $\mathcal{S}^{113}$ and $\mathcal{S}'^{113}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{113}} + \sigma z^{\mathcal{S}^{113}} = \mu u^{\mathcal{S}'^{113}} + \sigma z^{\mathcal{S}'^{113}} = \mu u^{\mathcal{S}^{113}} + \sigma z^{\mathcal{S}^{113}} + \mu_{\tilde{s}}.$$

This implies that $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S} \setminus \{s\}.$$

Next, we will show that, $\sigma_{s'}^k = 0$ for all $k \in K$ and $s' \in \{w_k, ..., \bar{s}\}$.

Consider a demand $k \in K$ and a slot s' in $\{w_k, ..., \bar{s}\}$. Let $\mathcal{S}^{114} = (U^{114}, S^{114})$ be the solution given by

a) for one demand $k' \in K \setminus \{k\}$, we select the smallest slot index $s_{k'} \in \{w_{k'}, ..., s'\} \cap \{s, ..., s + w_{k'} - 1\}$ as last slot,

b) we select the slot s_k in $\{w_k, ..., \bar{s}\} \setminus \{s\} \setminus \{s'\}$ as last slot for demand k with $\{s_k - w_k + 1, ..., s_k\} \cap \{s_{k'} - w_{k'} + 1, ..., s_{k'}\} = \emptyset$ if $E(p_k) \cap E(p_{k'}) \neq \emptyset$,

c) for each demand $k_i \in K \setminus \{k, k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{114} given by

$$I_i^{114} = [\bigcap_{k_j \in D_i^{114}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ or $I_i^{114} = \bigcap_{k_j \in D_i^{114}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D_i^{114} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k, k'\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{114}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E(p_k) \cap E(p_{k_i}) \neq \emptyset$ (we take into account the possibility of adding slot s' in the set of last slots S_k^{114} assigned to demand k in solution $\mathcal{S}^{114}$),

Let $S_{k_i}^{114} = \{s_{k_i}\}$ be the set of last slots assigned to each demand k_i with $i \in \{1, ..., |K|\}$.

d) we let U^{114} be the set of slots used in $\mathbb{S}$.

$\mathcal{S}^{114}$ is clearly feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{114}}, z^{\mathcal{S}^{114}})$ belongs to F_s. Then consider the solution $\mathcal{S}^{115} = (U^{115}, S^{115})$ obtained from $\mathcal{S}^{114}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}_k^{114}$ remain the same in solution $\mathcal{S}^{115}$ i.e., $S_{k'}^{114} = S_{k'}^{115}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{115} = S_\prime^{114} \cup \{s'\}$ for demand k. Solution $\mathcal{S}^{115}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{115}}, z^{\mathcal{S}^{115}})$ belongs to F_s. Hence, solutions $\mathcal{S}^{114}$ and $\mathcal{S}^{115}$ satisfy equation $\mu u + \sigma z = \tau$. We the obtain that

$$\mu u^{\mathcal{S}^{114}} + \sigma z^{\mathcal{S}^{114}} = \mu u^{\mathcal{S}^{115}} + \sigma z^{\mathcal{S}^{115}} = \mu u^{\mathcal{S}^{114}} + \sigma z^{\mathcal{S}^{114}} + \sigma_{s'}^k + \sum_{\tilde{s} \in \{s'-w_k+1,\ldots,s'\} \setminus U^{114}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for all $\tilde{s} \in \mathbb{S} \setminus \{s\}$, it follows that $\sigma_{s'}^k = 0$.
In a similar way, we can show that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } s' \in \{w_{k'}, ..., \bar{s}\}.$$

It follows that $\mu_s = \rho$ for slot s in $\mathbb{S}$.
We know from (5.9) that

$$\sigma_{s'}^{k'} = \gamma^{k',s'}, \text{ for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}.$$

We conclude that

$$\mu_{s'} = \begin{cases} \rho & \text{if } s' = s, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $k' \in K$ and $s' \in \mathbb{S}$

$$\sigma_{s'}^{k'} = \begin{cases} \gamma^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$ as desired. □

Theorem 5.3.4. *For a demand $k \in K$, inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$ is facet defining for $P_{sa}(G, K, \mathbb{S})$.*

Proof. Let $\tilde{F}_{\mathbb{S}}^k$ be the face induced by inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$, that is

$$\tilde{F}_{\mathbb{S}}^k = \{(x, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \sum_{s=w_k}^{\bar{s}} z_s^k = 1\}.$$

We denote inequality $\sum_{s=w_k}^{\bar{s}} z_s^k \geq 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that defines a facet F of $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}_{\mathbb{S}}^k \subset F = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. To prove that $\tilde{F}_{\mathbb{S}}^k$ is facet defining for $P_{sa}(G, K, \mathbb{S})$, it sufficient to show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.

First, let show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$, and a solution $\mathcal{S}^{116} = (U^{116}, S^{116})$ given by

a) we select the smallest slot index s_k in $\{w_k, ..., \bar{s}\} \setminus \{\tilde{s}, ..., \tilde{s} + w_k - 1\}$ as last slot for demand k (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{116}),

b) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{116} given by

$$I_i^{116} = [\bigcap_{k_j \in D_i^{116}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\}$$

where $D_i^{116} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This verifies that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{116}$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{116}).

We let $S_{k_i}^{116} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

c) a set of slots U^{116} is then used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s \in S_{k'}^{116}$ and $s' \in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have $s' \in U^{116}$,

$\mathcal{S}^{116}$ is clearly feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{116}}, z^{\mathcal{S}^{116}})$ belongs to $\tilde{F}_{\mathbb{S}}^k$. Next, we derive a solution $\mathcal{S}'^{116} = (U'^{116}, S'^{116})$ obtained from $\mathcal{S}^{116}$ by adding slot $\tilde{s}$ as an used slot in U'^{116} without modifying the last slots assigned to the demands K in $\mathcal{S}^{116}$ which remain the same in solution $\mathcal{S}'^{116}$ i.e., $S_k^{116} = S_k'^{116}$ for each demand $k \in K$. Solution $\mathcal{S}'^{116}$ is feasible for the SA problem, and its incidence vector $(u^{\mathcal{S}'^{116}}, z^{\mathcal{S}'^{116}})$ belongs to $\tilde{F}_{\mathbb{S}}^k$. Hence, solutions $\mathcal{S}^{116}$ and $\mathcal{S}'^{116}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{116}} + \sigma z^{\mathcal{S}^{116}} = \mu u^{\mathcal{S}'^{116}} + \sigma z^{\mathcal{S}'^{116}} = \mu u^{\mathcal{S}^{116}} + \sigma z^{\mathcal{S}^{116}} + \mu_{\tilde{s}}.$$

Hence, $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Next, we will show that, $\sigma_{s'}^{k'} = 0$ for all $k' \in K \setminus \{k\}$ and $s' \in \{w_{k'}, ..., \bar{s}\}$.
Consider a demand k' in $K \setminus \{k\}$ and a slot s' in $\{w_{k'}, ..., \bar{s}\}$. Let $\mathcal{S}^{117} = (U^{117}, S^{117})$ be the solution given by

a) we select slot $s_k = w_k$ as last slot for demand k,

b) we select the smallest slot index $s_{k'}$ from the set of slots $I_{k'}^{117}$ given by

$$I_{k'}^{117} = \{w_{k_i}, ..., s_k - w_k\} \cap \{s_k + w_{k_i}, ..., \bar{s}\} \setminus \{s'\} \text{ if } E(p_{k'}) \cap E(p_k) \neq \emptyset$$
$$\text{or } I_{k'}^{117} = \{w_{k'}, ..., \bar{s}\} \setminus \{s'\} \text{ if not.}$$

c) for each demand $k_i \in K \setminus \{k, k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{117} given by

$$I_i^{117} = [\bigcap_{k_j \in D_i^{117}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$
$$\text{if } E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset \text{ or } I_i^{117} = \bigcap_{k_j \in D_i^{117}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{117} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k, k'\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{117}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E(p_{k'}) \cap E(p_{k_i}) \neq \emptyset$ (we take into account the possibility of adding slot s' in the set of last slots $S_{k'}^{117}$ assigned to demand k' in solution $\mathcal{S}^{117}$),

We let $S_{k_i}^{110} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) a set of slots U^{117} are then used in $\mathbb{S}$ such that for each demand k" $\in K$ and last slot $s \in S_{k"}^{117}$ and s" $\in \{s_{k"} - w_{k"} + 1, ..., s_{k"}\}$, we have s" $\in U^{117}$.

$\mathcal{S}^{117}$ is clearly feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{117}}, z^{\mathcal{S}^{117}})$ belongs to $\tilde{F}_{\mathbb{S}}^k$. Then consider the solution $\mathcal{S}^{118} = (U^{118}, S^{118})$ obtained from $\mathcal{S}^{117}$ by adding slot s' as last slot to demand k' without modifying the last slots assigned to the demands $K \setminus \{k'\}$ in $\mathcal{S}_k^{117}$ remain the same in solution $\mathcal{S}^{118}$ i.e., $S_k^{117} = S_k^{118}$ for each demand $k \in K \setminus \{k'\}$, and $S_{k'}^{118} = S_{k'}^{117} \cup \{s'\}$ for demand k'. Solution $\mathcal{S}^{118}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{118}}, z^{\mathcal{S}^{118}})$ belongs to $\tilde{F}_{\mathbb{S}}^k$. Hence, solutions $\mathcal{S}^{117}$ and $\mathcal{S}^{118}$ satisfy equation $\mu u + \sigma z = \tau$. We the obtain that

$$\mu u^{\mathcal{S}^{117}} + \sigma z^{\mathcal{S}^{117}} = \mu u^{\mathcal{S}^{118}} + \sigma z^{\mathcal{S}^{118}} = \mu u^{\mathcal{S}^{117}} + \sigma z^{\mathcal{S}^{117}} + \sigma_{s'}^{k'} + \sum_{\tilde{s} \in \{s' - w_{k'} + 1, ..., s'\} \setminus U^{117}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for all $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_{s'}^{k'} = 0$.

In a similar way, we can show that

$$\sigma_{s'}^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } s' \in \{w_{k'}, ..., \bar{s}\}.$$

Let prove now that σ_s^k for demand k and slot s in $\{w_k, ..., \bar{s}\}$ are equivalent.

Consider a slot $s' \in \{w_k, ..., \bar{s}\}$ such that $s' \notin S_k^{119}$. Let $\tilde{\mathcal{S}}^{119} = (\tilde{U}^{119}, \tilde{S}^{119})$ be the solution given by

a) we select the smallest slot index s_k from $\{w_k, ..., \bar{s}\} \setminus \{s'\}$ as last slot for demand k,

b) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots $\tilde{I}_i^{117}$ given by

$$\tilde{I}_i^{117} = [\bigcap_{k_j \in \tilde{D}_i^{117}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ or $\tilde{I}_i^{117} = \bigcap_{k_j \in \tilde{D}_i^{117}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $\tilde{D}_i^{117} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in \tilde{D}_i^{117}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E(p_{k'}) \cap E(p_{k_i}) \neq \emptyset$ (we take into account the possibility of adding slot s' in the set of last slots $\tilde{S}_{k'}^{117}$ assigned to demand k' in solution $\tilde{\mathcal{S}}^{117}$).

Let $\tilde{S}_{k_i}^{117} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

c) we let $\tilde{U}^{117}$ be the set of slots used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s_{k'} \in \tilde{S}_{k'}^{117}$ and $s" \in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have $s" \in \tilde{U}^{117}$.

$\tilde{\mathcal{S}}^{118}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\tilde{\mathcal{S}}^{118}}, z^{\tilde{\mathcal{S}}^{118}})$ belongs to $\tilde{F}_{\mathbb{S}}^k$. Based on this, we derive a solution $\mathcal{S}^{119} = (U^{119}, S^{119})$ from $\tilde{\mathcal{S}}^{118}$ by adding slot s' as last slot to demand k and removing the last slot $s \in S_k^{118}$, i.e., $S_k^{119} = (\tilde{S}_k^{118} \setminus \{s\}) \cup \{\tilde{s}\}$ for demand k such that $\{s' - w_k + 1, ..., s'\} \cap \{s" - w_{k'} + 1, ..., s"\} = \emptyset$ for each $k' \in K$ and $s" \in S_{k'}^{119}$ with $E_k^{119} \cap E_{k'}^{119} \neq \emptyset$. The last slots assigned to the demands $K \setminus \{k\}$ in $\tilde{\mathcal{S}}^{118}$ remain the same, i.e., $\tilde{S}_{k"}^{118} = S_{k"}^{119}$ for each demand $k" \in K \setminus \{k\}$. Solution $\mathcal{S}^{119}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{119}}, z^{\mathcal{S}^{119}})$ belongs to $\tilde{F}_{\mathbb{S}}^k$. Hence, solutions $\mathcal{S}^{118}$ and $\mathcal{S}^{119}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\tilde{\mathcal{S}}^{118}} + \sigma z^{\tilde{\mathcal{S}}^{118}} = \mu u^{\mathcal{S}'^{119}} + \sigma z^{\mathcal{S}'^{119}} = \mu u^{\tilde{\mathcal{S}}^{118}} + \sigma z^{\tilde{\mathcal{S}}^{118}} - \sigma_s^k + \sigma_{s'}^k - \sum_{\tilde{s} \in U^{118} \setminus U^{119}} \mu_{\tilde{s}} + \sum_{\tilde{s}' \in U^{119} \setminus U^{118}} \mu_{\tilde{s}'}.$$

Since $\mu_{\tilde{s}} = 0$ for all $\tilde{s} \in \mathbb{S}$, it follows that $\sigma^k_{s'} = \sigma^k_s$.
In a similar way, we can show that

$$\sigma^k_{s'} = \sigma^k_s, \text{ for all slots } s, s' \in \{w_k, ..., \bar{s}\}.$$

Consequently, we obtain that $\sigma^k_s = \rho$ for demand k and slot s in $\{w_k, ..., \bar{s}\}$.
By (5.9), we know that

$$\sigma^{k'}_{s'} = \gamma^{k',s'}, \text{ for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}.$$

We then conclude that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and for each $k' \in K$ and $s \in \mathbb{S}$

$$\sigma^{k'}_s = \begin{cases} \gamma^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ \rho & \text{if } k' = k \text{ and } s \in \{w_{k'}, ..., \bar{s}\}, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$ as desired. □

5.4 Valid Inequalities and Facets

In what follows, we present several valid inequalities for $\mathcal{P}_{sa}(G, K, \mathbb{S})$, and prove that they are facet-defining under certain conditions.

5.4.1 Interval-Capacity-Cover Inequalities

We start this section by introducing some classes of valid inequalities related to the knapsack constraints. Let us introduce the following conflict graph.

Definition 5.4.1. *Consider the conflict graph H_{sa} defined as follows. For each demand $k \in K$, consider a node v_k in H_{sa}. Two nodes v_k and $v_{k'}$ are linked by an edge in H_{sa} if and only if $E(p_k) \cap E(p_{k'}) \neq \emptyset$. This is equivalent to say that two linked nodes v_k and $v_{k'}$ means that the routing paths of the demands k, k' share an edge in G.*

Based on the conflict graph H_{sa}, we introduce the following inequalities.

Proposition 5.4.1. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$. Let $K' \subset K$ be a minimal cover for interval $I = [s_i, s_j]$ such that K' defines a clique in H_{sa}. Then, the inequality*

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z^k_s \leq |K'| - 1, \tag{5.10}$$

is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Proof. The interval I can cover at most $|K'|-1$ demands given that K' is a minimal cover for interval I. □

Inequality (5.10) can be strengthened by extending each minimal cover $K' \subset K$ for an interval I as follows.

Proposition 5.4.2. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$. Let $K' \subseteq K$ be a minimal cover for interval $I = [s_i, s_j]$ such that K' defines a clique in H_{sa} , and $\Xi(K')$ be a subset of demands in $K \setminus K'$ such that $\Xi(K') = \{k \in K \setminus K'$ such that $w_k \geq w_{k'}$ and $E(p_k) \cap E(p_{k'}) \neq \emptyset \quad \forall k' \in K'\}$. Then, the inequality*

$$\sum_{k \in K'} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{k' \in \Xi(K')} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} \leq |K'| - 1, \tag{5.11}$$

is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Proof. The interval $I = [s_i, s_j]$ can cover at most $|K'|-1$ demands from the demands in $K' \cup \Xi(K')$ given that K' is a minimal cover for interval $I = [s_i, s_j]$ and the definition of the set $\Xi(K')$ such that for each pair (k, k') with $k \in K'$ and $k' \in \Xi(K')$, the set $(K' \setminus \{k\}) \cup \{k'\}$ stills defining minimal cover for the interval I over edge e. Furthermore, for each quadruplet $(k, k', \tilde{k}, \tilde{k}')$ with $k, k' \in K'$ and $\tilde{k}, \tilde{k}' \in \Xi(K')$, the set $(K' \setminus \{k, k'\}) \cup \{\tilde{k}, \tilde{k}'\}$ stills defining minimal cover for the interval I given that $w_k + w_{k'} \leq w_{\tilde{k}} + w_{\tilde{k}'}$.

□

Theorem 5.4.1. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$. Let $\tilde{K} \subset K$ be a minimal cover for interval $I = [s_i, s_j]$ such that $\tilde{K}$ defines a clique in H_{sa}. Then, inequality* (5.10) *is facet defining for the polytope $\mathcal{P}_{sa}(G, K, \mathbb{S}, I)$ if and only if there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I' and a clique in H_{sa}, where*

$$\mathcal{P}_{sa}(G, K, \mathbb{S}, \tilde{K}, I) = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \sum_{\substack{k' \in K \setminus \tilde{K} \\ (v_k, v_{k'}) \in H^r \forall k \in \tilde{K}}} \sum_{s'=s_i+w_{k'}-1}^{s_j} z_{s'}^{k'} = 0\}.$$

Proof. **Necessity**
If there exists an interval of contiguous slots I' in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I'. This means that $\{s_i + w_k - 1, ..., s_j\} \subset I'$. As a result, inequality (5.10) induced by the minimal cover $\tilde{K}$ for the interval I, it is dominated by another inequality (5.10) induced by the same minimal cover $\tilde{K}$ for the interval I'. Hence, inequality (5.10) cannot be facet defining for the polytope

$\mathcal{P}_{sa}(G, K, \mathbb{S}, I)$.

Sufficiency.

Let $\tilde{F}^I_{\tilde{K}}$ denote the face induced by inequality (5.10), that is

$$\tilde{F}^I_{\tilde{K}} = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}, \tilde{K}, I) : \sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z^k_s = |\tilde{K}| - 1\}.$$

Denote inequality $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z^k_s \leq |\tilde{K}| - 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $\mathcal{P}_{sa}(G, K, \mathbb{S}, I)$. Suppose that $\tilde{F}^I_{\tilde{K}} \subset F = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}, I) : \mu u + \sigma z = \tau\}$. In order to prove that inequality $\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z^k_s \leq |\tilde{K}| - 1$ is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S}, I)$, we show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)}$ such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.

First, we show that $\mu_s = 0$ for all $s \in \mathbb{S}$.

Consider a slot $\tilde{s} \in \mathbb{S}$. Let $\mathcal{S}^{120} = (U^{120}, S^{120})$ be the solution given by

a) for one demand k' from $\tilde{K}$, we select the smallest slot index $s_{k'}$ in $[\{w_{k'}, ..., \bar{s}\} \setminus [\{s_i + w_{k'} - 1, ..., s_j\} \cup \{\tilde{s}, ..., \tilde{s} + w_{k'} - 1\}]] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{120}),

b) for each demand $k_i \in \tilde{K} \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{120}_i given by

$$I^{120}_i = [\bigcap_{k_j \in D^{120}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}][\cap\{s_i + w_{k_i} - 1, ..., s_j\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\},$$

where $D^{120}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{K} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$,

c) for each demand $k_i \in K \setminus \tilde{K}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{120}_i given by

$$I^{120}_i = [\bigcap_{k_j \in D^{120}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\}$$

where $D^{120}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D^{120}_i$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{120}).

We let $S^{120}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) let U^{120} be the set of slots used in $\mathbb{S}$ such that for each demand $k \in K$ and last slot $s_k \in S^{120}_k$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{120}$.

$\mathcal{S}^{120}$ is clearly feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{120}}, z^{\mathcal{S}^{120}})$ belongs to $\tilde{F}^I_{\tilde{K}}$. Then consider the solution $\mathcal{S}^{121} = (U^{121}, S^{121})$ obtained from $\mathcal{S}^{120}$ by adding slot $\tilde{s}$ as an used slot in U^{121} without modifying the last slots assigned to the demands K in $\mathcal{S}^{120}$ which remain the same in solution $\mathcal{S}^{121}$ i.e., $S^{120}_k = S^{121}_k$ for each demand $k \in K$. $\mathcal{S}^{121}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{121}}, z^{\mathcal{S}^{121}})$ belongs to $\tilde{F}^I_{\tilde{K}}$. Hence, solutions $\mathcal{S}^{120}$ and $\mathcal{S}^{121}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{120}} + \sigma z^{\mathcal{S}^{120}} = \mu u^{\mathcal{S}^{121}} + \sigma z^{\mathcal{S}^{121}} = \mu u^{\mathcal{S}^{120}} + \sigma z^{\mathcal{S}^{120}} + \mu_{\tilde{s}}.$$

As a result, $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let show that $\sigma^k_s = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in \tilde{K}$. Let $\mathcal{S}^{122} = (U^{122}, S^{122})$ be the solution given by

a) for one demand k' from $\tilde{K}$, we select the smallest slot index $s_{k'}$ in $\{w_{k'}, ..., \bar{s}\} \setminus \{s_i + w_{k'} - 1, ..., s_j\}$ as last slot,

b) for each demand $k_i \in \tilde{K} \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{122}_i given by

$$I^{122}_i = [\bigcap_{k_j \in D^{122}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D^{122}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{K} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$,

c) for each demand $k_i \in K \setminus \tilde{K}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{122}_i given by

$$I^{122}_i = [\bigcap_{k_j \in D^{122}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ or $I^{122}_i = \bigcap_{k_j \in D^{122}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D^{122}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{K} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D^{122}_i$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots S^{122}_k to route demand k in solution $\mathcal{S}^{122}$).

We let $S_{k_i}^{122} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) a set of slots U^{122} are then used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s_{k'} \in S_{k'}^{120}$ and $s" \in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have $s" \in U^{120}$.

$\mathcal{S}^{122}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{122}}, z^{\mathcal{S}^{122}})$ belongs to $\tilde{F}_{\tilde{K}}^{I}$. Then we derive a solution $\mathcal{S}^{123} = (U^{123}, S^{123})$ obtained from $\mathcal{S}^{122}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{122}$, i.e., $S_{k'}^{122} = S_{k'}^{123}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{123} = S_k^{122} \cup \{s'\}$ for demand k. Solution $\mathcal{S}^{123}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{123}}, z^{\mathcal{S}^{123}})$ belongs to $\tilde{F}_{\tilde{K}}^{I}$. Hence, solutions $\mathcal{S}^{122}$ and $\mathcal{S}^{123}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{122}} + \sigma z^{\mathcal{S}^{122}} = \mu u^{\mathcal{S}^{123}} + \sigma z^{\mathcal{S}^{123}} = \mu u^{\mathcal{S}^{122}} + \sigma z^{\mathcal{S}^{122}} + \sigma_{s'}^{k} + \sum_{\tilde{s} \in U^{123} \setminus U^{122}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{122} \setminus U^{123}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_{s'}^{k} = 0$.

In a similar way, we can show that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } k \notin \tilde{K}.$$

Let prove that σ_s^k for all $k \in \tilde{K}$ and $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalent.

Consider a demand $k' \in K$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $k' \in \tilde{K}$. Let $\mathcal{S}^{124} = (U^{124}, S^{124})$ be the solution given by

a) for one demand $k"$ from $\tilde{K}$, we select the smallest slot index $s_{k"}$ in $\{w_{k"}, ..., \bar{s}\} \setminus \{s_i + w_{k"} - 1, ..., s_j\}$ as last slot,

b) for each demand $k_i \in \tilde{K} \setminus \{k"\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{124} given by

$$I_i^{124} = [\bigcap_{k_j \in D_i^{124}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D_i^{124} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{K} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$,

c) for each demand $k_i \in K \setminus \tilde{K}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{124} given by

$$I_i^{124} = [\bigcap_{k_j \in D_i^{124}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_k) \neq \emptyset \text{ or } I_i^{124} = \bigcap_{k_j \in D_i^{124}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{124} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{K} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{124}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $S_{k'}^{124}$ to route demand k' in solution $\mathcal{S}^{124}$).

We let $S_{k_i}^{124} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) let U^{124} be the set of slots used in $\mathbb{S}$ such that for each demand $k" \in K$ and last slot $s_{k"} \in S_{k"}^{124}$ and $s" \in \{s_{k"} - w_{k"} + 1, ..., s_{k"}\}$, we have $s" \in U^{124}$.

$\mathcal{S}^{124}$ is clearly feasible for the problem given that it satisfies all the constraints of cut formulation (2.2)-(2.10). Hence, the corresponding incidence vector $(u^{\mathcal{S}^{124}}, z^{\mathcal{S}^{124}})$ belongs to $\tilde{F}_{\tilde{K}}^I$. Then consider the solution $\mathcal{S}^{125} = (E^{125}, S^{125})$ obtained from $\mathcal{S}^{125}$ by adding slot s' as last slot to demand k such that the last slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{125}$ remain the same in $\mathcal{S}^{125}$, i.e., $S_{k"}^{125} = S_{k"}^{125}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S_{k'}^{125} = S_{k'}^{125} \cup \{s'\}$ for demand k', and modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s \in S_k^{125}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for demand k with $k \in \tilde{K}$ such that $S_k^{125} = (S_k^{125} \setminus \{s\}) \cup \{\tilde{s}\}$ such that $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{125}$ with $E_k^{125} \cap E_{k'}^{125} \neq \emptyset$. Solution $\mathcal{S}^{125}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{125}}, z^{\mathcal{S}^{125}})$ belongs to $\tilde{F}_{\tilde{K}}^I$. Hence, solutions $\mathcal{S}^{124}$ and $\mathcal{S}^{125}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{124}} + \sigma z^{\mathcal{S}^{124}} = \mu u^{\mathcal{S}^{125}} + \sigma z^{\mathcal{S}^{125}} = \mu u^{\mathcal{S}^{124}} + \sigma z^{\mathcal{S}^{124}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{125} \setminus U^{124}} \mu_{s"} - \sum_{s" \in U^{124} \setminus U^{125}} \mu_{s"}.$$

Since $\sigma_s^k = 0$ for $s \notin \{s_i + w_k - 1, ..., s_j\}$ with $k \in \tilde{K}$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$, it follows that $\sigma_{s'}^{k'} = \sigma_s^k$.
The pair (k, k') are chosen arbitrarily in the minimal cover $\tilde{K}$, we then re-do the same procedure for all pairs (k, k') such that we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (k, k') \in \tilde{K},$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $k \in \tilde{K}$ such that

$$\sigma_s^k = \sigma_{s'}^k, \text{for all } k \in \tilde{K} \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\}.$$

By (5.9), we know that

$$\sigma_{s'}^{k'} = \gamma^{k',s'}, \text{ for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}.$$

Overall, we obtain that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and

$$\sigma_s^{k'} = \begin{cases} \gamma^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ \rho & \text{if } k' \in \tilde{K} \text{ and } s \in \{s_i + w_{k'} - 1, , ..., s_j\}, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $s \in \mathbb{S}$.
As a consequence, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$ as desired.

□

Inequality (5.10) can then be lifted using a sequential lifting procedure [5] to be facet defining and generate several facets for the polytope $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Theorem 5.4.2. *Let $I = [s_i, s_j]$ be an interval of contiguous slots. Let $\tilde{K} \subset K$ be a minimal cover for interval $I = [s_i, s_j]$ such that $\tilde{K}$ defines a clique in H_{sa}. Let $K' \subset K \setminus \tilde{K} = \{k_1, ..., k_q\}$ such that $\tilde{K} \cup \{k_1, ..., k_q\}$ defines a clique in H_{sa}. Consider the following sequence of knapsack problems defined as*

$$\begin{cases} z_i = \max \sum_{j \in \tilde{K}} a_j + \sum_{j=1}^{i-1} \beta_j a_j, \\ \sum_{j \in \tilde{K}} w_j a_j + \sum_{j=1}^{i-1} w_{k_j} a_j \leq |I| - w_{k_i}, \\ a_j \in \{0, 1\}, \forall j \in \tilde{K} \cup \{1, ..., i-1\}, \end{cases} \tag{5.12}$$

for all $i \in \{1, ..., q\}$ with $\beta_j = |\tilde{K}| - 1 - z_j$ for all $j \in \{1, ..., i-1\}$. Then, the inequality

$$\sum_{k \in \tilde{K}} \sum_{s=s_i+w_k-1}^{s_j} z_s^k + \sum_{j=1}^{q} \sum_{s'=s_i+w_{k_j}-1}^{s_j} \beta_j z_{s'}^{k_j} \leq |\tilde{K}| - 1, \tag{5.13}$$

is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Moreover, inequality (5.13) defines facet of $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if there does not exist an interval of contiguous slots $I' = [s'_i, s'_j]$ in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I'.

Proof. It is trivial given that inequality (5.13) can never be dominated in $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if there does not exist an interval of contiguous slots $I' = [s'_i, s'_j]$ in $[1, \bar{s}]$ with $I \subset I'$ such that $\tilde{K}$ defines a minimal cover for the interval I'. □

5.4.2 Interval-Clique Inequalities

Based on the definition of the conflict graph H_I^E, we define a new conflict graph adapted to the SA problem.

Definition 5.4.2. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$. Consider the conflict graph $H_I'^E$ defined as follows. For each demand $k \in K$ with $w_k \leq |I|$, consider a node v_k in $H_I'^E$. Two nodes v_k and $v_{k'}$ are linked by an edge in $H_I'^E$ if $w_k + w_{k'} > |I|$ and $E(p_k) \cap E(p_{k'}) \neq \emptyset$.*

Let $\mathcal{Q}_{sa}(G, K, \mathbb{S}) = \{(x, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \sum_{k \in K} \sum_{s=w_k}^{\bar{s}} z_s^k = 1\}$ be a semi-polytope of $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Proposition 5.4.3. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and C be a clique in the conflict graph $H_I'^E$ with $|C| \geq 3$. Then, inequality (2.39) is also valid for $\mathcal{Q}_{sa}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in C$.*

Proof. We use the same proof of proposition (2.4.13). □

Theorem 5.4.3. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and C be a clique in the conflict graph $H_I'^E$ with $|C| \geq 3$, and $2w_k > |I|$ for each $v_k \in C$. Then, inequality (2.39) is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if and only if*

a) C is a maximal clique in the conflict graph $H_I'^E$,

b) and there does not exist an interval of contiguous slots I' in $[1, \bar{s}]$ such that $I \subset I'$ with

- *$w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,*
- *$2w_k \geq |I'| + 1$ and $w_k \leq |I'|$ for each $k \in C$.*

c) and there does not exist a slot $s \in I$ such that $s \in \{s' - w_k + 1, .., s'\}$ for each $k \in C$ and $s' \in \{s_i + w_k - 1, .., s_j\}$.

Proof. **Neccessity.**
We distinguish three cases

a) if there exists a clique C' that contains all the demands $k \in C$. Then, inequality (2.39) induced by clique C is dominated by another inequality (2.39) induced by clique C'. Hence, inequality (2.39) cannot be facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

b) if there exists an interval of contiguous slots I' in $[1, \bar{s}]$ such that $I \subset I'$ with

- $w_k + w_{k'} \geq |I'|$ for each $k, k' \in C$,
- $2w_k \geq |I'| + 1$ and $w_k \leq |I'|$ for each $k \in C$.

This means that inequality (2.39) induced by clique C for the interval I is dominated by inequality (2.39) induced by clique C for the interval I'. Hence, inequality (2.39) cannot be facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

c) if there exists a slot $s \in I$ such that $s \in \{s' - w_k + 1, .., s'\}$ for each $k \in C$ and $s' \in \{s_i + w_k - 1, .., s_j\}$, this implies that inequality (2.39) is dominated by the the non-overlapping inequality (5.4). Hence, inequality (2.39) cannot be facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Sufficiency.
Let $\tilde{F}_C^{H'_I E}$ be the face induced by inequality (2.39), that is

$$\tilde{F}_C^{H'_I E} = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = 1\}.$$

We denote inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}_C^{H'_I E} \subset F = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. In order to prove that inequality $\sum_{v_k \in C} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq 1$ is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)})$ such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.
Let first show that $\mu_s = 0$ for all $s \in \mathbb{S}$.
Consider a slot $\tilde{s} \in \mathbb{S}$. Let $\mathcal{S}^{127} = (U^{127}, S^{127})$ be the solution given by

a) for one demand k' from C, we select the smallest slot index $s_{k'} = \{s_i + w_{k'} - 1, ..., s_j\} \setminus \{\tilde{s}, ..., \tilde{s} - w_{k'} - 1\}$ as last slot (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{127}),

b) for each demand $k_i \in C \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{127} given by

$$I_i^{127} = [\bigcap_{k_j \in D_i^{127}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}][\cap \{s_i + w_{k_i} - 1, ..., s_j\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\},$$

where $D_i^{127} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap C : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$,

c) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{127} given by

$$I_i^{127} = [\bigcap_{k_j \in D_i^{127}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\}$$

where $D_i^{127} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap C : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ...s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{127}$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{127}),

We let $S_{k_i}^{127} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) a set of slots U^{127} are used in $\mathbb{S}$ such that for each demand $k \in K$ and last slot $s_k \in S_k^{127}$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{127}$.

$\mathcal{S}^{127}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{127}}, z^{\mathcal{S}^{127}})$ belongs to $\tilde{F}_C^{H_I'^E}$. Then we derive a solution $\mathcal{S}^{128} = (U^{128}, S^{128})$ obtained from $\mathcal{S}^{127}$ by adding slot $\tilde{s}$ as an used slot in U^{128} without modifying the last slots assigned to the demands K in $\mathcal{S}^{127}$ which remain the same in solution $\mathcal{S}^{128}$ i.e., $S_k^{127} = S_k^{128}$ for each demand $k \in K$. Solution $\mathcal{S}^{128}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{128}}, z^{\mathcal{S}^{128}})$ belongs to $\tilde{F}_C^{H_I'^E}$. Hence, solutions $\mathcal{S}^{127}$ and $\mathcal{S}^{128}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{127}} + \sigma z^{\mathcal{S}^{127}} = \mu u^{\mathcal{S}^{128}} + \sigma z^{\mathcal{S}^{128}} = \mu u^{\mathcal{S}^{127}} + \sigma z^{\mathcal{S}^{127}} + \mu_{\tilde{s}}.$$

Hence, $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in C$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $k \in C$. be the solution given by $\mathcal{S}^{129} = (U^{129}, S^{129})$ be the solution given by

a) for one demand k' from C, we select the slot $s_{k'} = s_i + w_{k'} - 1$ as last slot,

b) for each demand $k_i \in C \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{129} given by

$$I_i^{129} = [\bigcap_{k_j \in D_i^{129}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D_i^{129} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap C : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$,

c) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{129} given by

$$I_i^{129} = [\bigcap_{k_j \in D_i^{129}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ or $I_i^{129} = \bigcap_{k_j \in D_i^{129}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $D_i^{129} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{129}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots S_k^{129} to route demand k in solution $\mathcal{S}^{129}$),

We let $S_{k_i}^{122} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) let U^{129} be the slot used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s_{k'} \in S_{k'}^{129}$ and s" $\in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have s" $\in U^{129}$.

$\mathcal{S}^{129}$ is clearly feasible for the problem, and its incidence vector $(u^{\mathcal{S}^{129}}, z^{\mathcal{S}^{129}})$ belongs to $\tilde{F}_C^{H_I'^E}$. Then consider the solution $\mathcal{S}^{130} = (U^{130}, S^{130})$ obtained from $\mathcal{S}^{129}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{129}$, i.e., $S_{k'}^{129} = S_{k'}^{130}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{130} = S_k^{129} \cup \{s'\}$ for demand k. Solution $\mathcal{S}^{130}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{130}}, z^{\mathcal{S}^{130}})$ belongs to $\tilde{F}_C^{H_I'^E}$. Hence, solutions $\mathcal{S}^{129}$ and $\mathcal{S}^{130}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{129}} + \sigma z^{\mathcal{S}^{129}} = \mu u^{\mathcal{S}^{130}} + \sigma z^{\mathcal{S}^{130}} = \mu u^{\mathcal{S}^{129}} + \sigma z^{\mathcal{S}^{129}} + \sigma_{s'}^k + \sum_{\tilde{s} \in U^{130} \setminus U^{129}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{129} \setminus U^{130}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_{s'}^k = 0$.
In a similar way, we can show that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \notin C.$$

Let prove that σ_s^k for all $v_k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$ are equivalent. Consider a demand $k' \in K$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ with $v_{k'} \in C$, and a solution $\mathcal{S}^{131} = (U^{131}, S^{131})$ given by

a) for one demand k from C, we select theslot $s_k = s_i + w_k - 1$ as last slot,

b) for each demand $k_i \in C \setminus \{k\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{131} given by

$$I_i^{131} = [\bigcap_{k_j \in D_i^{131}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}] \setminus \{s_i, ..., s_j\}$$

if $E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset$ or $I_i^{131} = [\bigcap_{k_j \in D_i^{131}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{s_i, ..., s_j\}$ if not,

where $D_i^{131} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C$ such that $D_{k_i}^{131} \cap D_{k_j}^{131} \neq \emptyset\}$,

c) for each demand $k_i \in K \setminus C$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{131} given by

$$I_i^{131} = [\bigcap_{k_j \in D_i^{131}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset \text{ or } I_i^{131} = \bigcap_{k_j \in D_i^{131}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{131} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup C \text{ such that } E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s - w_{k_j} + 1, ..., s\} = \emptyset$ for each $k_j \in D_i^{131}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $S_{k'}^{131}$ to route demand k' in solution $\mathcal{S}^{131}$).

We let $S_{k_i}^{131} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) Let U^{131} be the set of slots used in $\mathbb{S}$ such that for each demand k and last slot $s_k \in S_k^{131}$ and $s" \in \{s_k - w_k + 1, ..., s_k\}$, we have $s" \in U^{131}$.

$\mathcal{S}^{131}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{131}}, z^{\mathcal{S}^{131}})$ belongs to $\tilde{F}_C^{H'_I E}$. Then consider the solution $\mathcal{S}^{132} = (U^{132}, S^{132})$ from $\mathcal{S}^{131}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{131}$, i.e., $S_{k"}^{131} = S_{k"}^{132}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S_{k'}^{132} = S_{k'}^{131} \cup \{s'\}$ for demand k', and with modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s \in S_k^{131}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \notin \{s_i + w_k + 1, ..., s_j\}$ for demand $k \in K$ with $v_k \in C$ such that $S_k^{132} = (S_k^{131} \setminus \{s\}) \cup \{\tilde{s}\}$ such that $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{132}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$. Solution $\mathcal{S}^{132}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{132}}, z^{\mathcal{S}^{132}})$ belongs to $\tilde{F}_C^{H'_I E}$. Hence, solutions $\mathcal{S}^{131}$ and $\mathcal{S}^{132}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{131}} + \sigma z^{\mathcal{S}^{131}} = \mu u^{\mathcal{S}^{132}} + \sigma z^{\mathcal{S}^{132}} = \mu u^{\mathcal{S}^{131}} + \sigma z^{\mathcal{S}^{131}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{132} \setminus U^{131}} \mu_{s"} - \sum_{s" \in U^{131} \setminus U^{132}} \mu_{s"}.$$

Since $\sigma_{\tilde{s}}^k = 0$ for $\tilde{s} \notin \{s_i + w_k - 1, ..., s_j\}$ with $v_k \in C$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$, it follows that $\sigma_{s'}^{k'} = \sigma_s^k$.

Given that the pair $(v_k, v_{k'})$ are chosen arbitrarily in clique C, we re-do the same procedure for all pairs $(v_k, v_{k'})$ such that we find

$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all pairs } (v_k, v_{k'}) \in C,$$

with $s \in \{s_i + w_k - 1, ..., s_j\}$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. We re-do the same procedure for each two slots $s, s' \in \{s_i + w_k - 1, ..., s_j\}$ for each demand $k \in K$ with $v_k \in C$ such that

$$\sigma_s^k = \sigma_{s'}^k, \text{for all } v_k \in C \text{ and } s, s' \in \{s_i + w_k - 1, ..., s_j\},$$
$$\sigma_s^k = \sigma_{s'}^{k'}, \text{for all } v_k, v_{k'} \in C, s \in \{s_i + w_k - 1, ..., s_j\} \text{ and } s' \in \{s_i + w_{k'} - 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma_s^k = \rho$ for all $v_k \in C$ and $s \in \{s_i + w_k - 1, ..., s_j\}$. We know from (5.9) that

$$\sigma_{s'}^{k'} = \gamma^{k',s'}, \text{ for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}.$$

As a consequence, we obtain that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and

$$\sigma_s^{k'} = \begin{cases} \gamma^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ \rho & \text{if } v_{k'} \in C \text{ and } s \in \{s_i + w_{k'} - 1, , ..., s_j\}, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $s \in \mathbb{S}$. As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$ as desired.

□

5.4.3 Interval-Odd-Hole Inequalities

Proposition 5.4.4. *Let $I = [s_i, s_j]$ be an interval of contiguous slots in $[1, \bar{s}]$ with $s_i \leq s_j - 1$, and H be an odd-hole H in the conflict graph $H_I'^E$ with $|H| \geq 5$. Then, inequality* (2.40) *is valid for $\mathcal{Q}_{sa}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}(G, K, \mathbb{S})$ if $2w_k > |I|$ for each $v_k \in H$.*

Proof. We use the same proof of proposition (5.4.4). □

Theorem 5.4.4. *Let H be an odd-hole in the conflict graph $H_I'^E$ with $|H| \geq 5$, and $2w_k > |I|$ for each $v_k \in H$. Then, inequality* (2.40) *is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if and only if*

a) for each node $v_{k'} \notin H$ in $H_I'^E$, there exists a node $v_k \in H$ such that the induced graph $H_I'^E((H \setminus \{v_k\}) \cup \{v_{k'}\})$ does not contain an odd-hole $H' = (H \setminus \{v_k\}) \cup \{v_{k'}\}$,

b) and there does not exist a node $v_{k'} \notin H$ in $H_I'^E$ such that $v_{k'}$ is linked with all nodes $v_k \in H$,

c) and there does not exist an interval I' of contiguous slots with $I \subset I'$ such that H defines also an odd-hole in the associated conflict graph $H_{I'}^E$.

Proof. **Neccessity.**

We use the same proof presented in the proof of theorem (2.4.10).

Sufficiency.

Let $\tilde{F}_H^{H_I'^E}$ be the face induced by inequality (2.40), that is

$$\tilde{F}_H^{H_I'^E} = \{(u,z) \in \mathcal{P}_{sa}(G,K,\mathbb{S}) : \sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k = \frac{|H|-1}{2}\}.$$

We denote inequality $\sum_{v_k \in H} \sum_{s=s_i+w_k-1}^{s_j} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $\mathcal{P}_{sa}(G,K,\mathbb{S})$. Suppose that $\tilde{F}_H^{H_I'^E} \subset F = \{(u,z) \in \mathcal{P}_{sa}(G,K,\mathbb{S}) : \mu u + \sigma z = \tau\}$. To prove that $\tilde{F}_H^{H_I'^E}$ is a facet of $\mathcal{P}_{sa}(G,K,\mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k-1)})$ such that $(\mu,\sigma) = \rho(\alpha,\beta) + \gamma M$.

Let first show that $\mu_s = 0$ for all $s \in \mathbb{S}$.

Consider a slot $\tilde{s} \in \mathbb{S}$. Let $\mathcal{S}^{134} = (U^{134}, S^{134})$ be the solution given by

a) select a subset of demands $\tilde{H}$ from H with $|\tilde{H}| = \frac{|H|-1}{2}$,

b) for each demand k_i from $\tilde{H}$ with $i \in \{1,...,|K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{134} given by

$$I_i^{134} = [\bigcap_{k_j \in L_i^{134}} \{w_{k_i},...,s_{k_j}-w_{k_j}\} \cup \{s_{k_j}+w_{k_i},...,\bar{s}\}] \cap [\{s_i+w_{k_i}-1,...,s_j\}] \setminus \{\tilde{s},...,\tilde{s}+w_{k_i}-1\},$$

where $L_i^{134} = \{k_j \in \{k_1,...,k_{i-1}\} \cap \tilde{H} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$.

c) for each demand $k_i \in H \setminus \tilde{H}$ with $i \in \{1,...,|K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{134} given by

$$I_i^{134} = \bigcap_{k_j \in D_i^{134}} \{w_{k_i},...,s_{k_j}-w_{k_j}\} \cup \{s_{k_j}+w_{k_i},...,\bar{s}\} \setminus [\{s_i+w_{k_i}-1,...,s_j\} \cup \{\tilde{s},...,\tilde{s}+w_{k_i}-1\}],$$

where $D_i^{134} = \{k_j \in \{k_1,...,k_{i-1}\} \cap H : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. We let $S_{k_i}^{134} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i,

d) for each demand $k_i \in K \setminus H$ with $i \in \{1,...,|K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{134} given by

$$I_i^{134} = [\bigcap_{k_j \in R_i^{134}} \{w_{k_i},...,s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i},...,\bar{s}\}] \setminus \{\tilde{s},...,\tilde{s} + w_{k_i} - 1\}$$

where $R_i^{134} = \{k_j \in \{k_1,...,k_{i-1}\} \cup H$ such that $E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1,...,s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1,...,s_{k_j}\} = \emptyset$ for each $k_j \in R_i^{134}$,

- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{134}).

  We let $S^{134}_{k_i} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

e) let U^{134} be the set of slots used in $\mathbb{S}$ such that for each demand k and last slot $s_k \in S^{134}_k$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{134}$, and and $\tilde{s} \notin U^{134}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{134}).

$\mathcal{S}^{134}$ is clearly feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{134}}, z^{\mathcal{S}^{134}})$ belongs to $\tilde{F}^{H'E}_{HI}$. Then we derive a solution $\mathcal{S}^{135} = (U^{135}, S^{135})$ obtained from $\mathcal{S}^{134}$ by adding slot $\tilde{s}$ as an used slot in U^{135} without modifying the last slots assigned to the demands K in $\mathcal{S}^{134}$ which remain the same in solution $\mathcal{S}^{135}$ i.e., $S^{134}_k = S^{135}_k$ for each demand $k \in K$. Solution $\mathcal{S}^{135}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{135}}, z^{\mathcal{S}^{135}})$ belongs to $\tilde{F}^{H'E}_{HI}$. Hence, solutions $\mathcal{S}^{134}$ and $\mathcal{S}^{135}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{134}} + \sigma z^{\mathcal{S}^{134}} = \mu u^{\mathcal{S}^{135}} + \sigma z^{\mathcal{S}^{135}} = \mu u^{\mathcal{S}^{134}} + \sigma z^{\mathcal{S}^{134}} + \mu_{\tilde{s}}.$$

Hence, $\mu_{\tilde{s}} = 0$.
In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let show that $\sigma^k_s = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$.
Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $s' \notin \{s_i + w_k - 1, ..., s_j\}$ if $v_k \in H$. Let $\mathcal{S}^{136} = (U^{136}, S^{136})$ be the solution given by

a) select a subset of demands $\tilde{H}$ from H with $|\tilde{H}| = \frac{|H|-1}{2}$,

b) for each demand k_i from $\tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{136}_i given by

$$I^{136}_i = [\bigcap_{k_j \in L^{136}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{s_i + w_{k_i} - 1, ..., s_j\}],$$

where $L^{136}_i = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{H} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$.

c) for each demand $k_i \in H \setminus \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I^{136}_i given by

$$I^{136}_i = \bigcap_{k_j \in D^{136}_i} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D_i^{136} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap H : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. We let $S_{k_i}^{136} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i,

d) for each demand $k_i \in K \setminus H$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{136} given by

$$I_i^{136} = [\bigcap_{k_j \in D_i^{136}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_k) \neq \emptyset \text{ or } I_i^{136} = \bigcap_{k_j \in D_i^{136}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{136} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup H : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{136}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots S_k^{136} to route demand k in solution $\mathcal{S}^{136}$).

We let $S_{k_i}^{136} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$,

e) let U^{136} be the set of slots used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s" \in S_{k'}^{136}$ and $s" \in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have $s" \in U^{136}$.

$\mathcal{S}^{136}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{136}}, z^{\mathcal{S}^{136}})$ belongs to $\tilde{F}_H^{H'_I E}$. After that, we derive a solution $\mathcal{S}^{137} = (U^{137}, S^{137})$ from $\mathcal{S}^{136}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{136}$, i.e., $S_{k'}^{136} = S_{k'}^{137}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{137} = S_k^{136} \cup \{s'\}$ for demand k. Solution $\mathcal{S}^{137}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{137}}, z^{\mathcal{S}^{137}})$ belongs to $\tilde{F}_H^{H'_I E}$. Hence, solutions $\mathcal{S}^{136}$ and $\mathcal{S}^{137}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{136}} + \sigma z^{\mathcal{S}^{136}} = \mu u^{\mathcal{S}^{137}} + \sigma z^{\mathcal{S}^{137}} = \mu u^{\mathcal{S}^{136}} + \sigma z^{\mathcal{S}^{136}} + \sigma_{s'}^k + \sum_{\tilde{s} \in U^{137} \setminus U^{136}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{136} \setminus U^{137}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_{s'}^k = 0$.

In a similar way, we can show that

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and } s' \in \{w_k, ..., \bar{s}\} \text{ with } s' \notin \{s_i + w_k - 1, ..., s_j\} \text{ if } v_k \in H.$$

We re-do the same procedure for all demand k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } s \notin \{s_i + w_{k'} - 1, ..., s_j\} \text{ if } v_{k'} \in H.$$

Let prove that $\sigma_{s'}^{k'}$ for all $v_{k'} \in H$ and $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$ are equivalent.

Consider a demand $k' \in K$ with $v_{k'} \in H$ and a slot $s' \in \{s_i + w_{k'} - 1, ..., s_j\}$. Let $\mathcal{S}^{138} = (U^{138}, S^{138})$ be the solution given by

a) select a subset of demands $\tilde{H}$ from H with $|\tilde{H}| = \frac{|H|-1}{2}$,

b) for each demand k_i from $\tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{138} given by

$$I_i^{138} = [\bigcap_{k_j \in L_i^{138}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap \{s_i + w_{k_i} - 1, ..., s_j\}.$$

where $L_i^{138} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap \tilde{H} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$,

c) for each demand $k_i \in H \setminus \tilde{H}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{138} given by

$$I_i^{138} = \bigcap_{k_j \in D_i^{138}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \setminus \{s_i + w_{k_i} - 1, ..., s_j\},$$

where $D_i^{138} = \{k_j \in \{k_1, ..., k_{i-1}\} \cap H : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$,

d) for each demand $k_i \in K \setminus H$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{138} given by

$$I_i^{138} = [\bigcap_{k_j \in R_i^{138}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

if $E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset$ or $I_i^{138} = \bigcap_{k_j \in R_i^{138}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}$ if not,

where $R_i^{138} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup H : E_{k_i}^{138} \cap E_{k_j}^{138} \neq \emptyset\}$. Hence,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in R_i^{138}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E_{k_i}^{138} \cap E_{k'}^{138} \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $S_{k'}^{138}$ to route demand k' in solution $\mathcal{S}^{138}$).

e) a set of slots U^{138} are used in $\mathbb{S}$ such that for each demand k and last slot $s_k \in S_k^{138}$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{138}$.

$\mathcal{S}^{138}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{138}}, z^{\mathcal{S}^{138}})$ belongs to $\tilde{F}_H^{H'_I{}^E}$. Then we derive a solution $\mathcal{S}^{139}$ from $\mathcal{S}^{138}$ as belows

a) remove the last slot $\tilde{s}$ totally covered by the interval I and which has been selected by a demand $k_i \in \{v_{k_1}, ..., v_{k_q}\}$ in solution $\mathcal{S}^{139}$ (i.e., $\tilde{s} \in S_{k_i}^{139}$ and $\tilde{s}' \in \{s_i + w_{k_i} + 1, ..., s_j\}$) such that each pair of nodes $(v_{k'}, v_{k_j})$ are not linked in odd-hole H with $j \neq i$,

b) and select a new last slot $\tilde{s}' \notin \{s_i + w_{k_i} + 1, ..., s_j\}$ for demand k_i i.e., $S^{139}_{k_i} = (S^{138}_{k_i} \setminus \{\tilde{s}\}) \cup \{\tilde{s}'\}$ such that $\{\tilde{s}' - w_{k_i} - 1, ..., \tilde{s}'\} \cap \{s - w_k + 1, ..., s\} = \emptyset$ for each $k \in K$ and $s \in S^{139}_k$ with $E^{139}_k \cap E^{139}_{k_i} \neq \emptyset$,

c) and add slot s' to the set of last slots $S^{139}_{k'}$ assigned to demand k' in solution $\mathcal{S}^{139}$, i.e., $S^{139}_{k'} = S^{138}_{k'} \cup \{s'\}$.

solution $\mathcal{S}^{139}$ is clearly feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{139}}, z^{\mathcal{S}^{139}})$ belongs to $\tilde{F}^{H'^E}_H$. Hence, solutions $\mathcal{S}^{138}$ and $\mathcal{S}^{139}$ satisfy equation $\mu u + \sigma z = \tau$. We have so

$$\mu u^{S^{138}} + \sigma z^{S^{138}} = \mu u^{S^{139}} + \sigma z^{S^{139}} = \mu u^{S^{138}} + \sigma z^{S^{138}} + \sigma^{k'}_{s'} + \sigma^{k_i}_{\tilde{s}'} - \sigma^{k_i}_{\tilde{s}} + \sum_{s" \in U^{139} \setminus U^{138}} \mu_{s"} - \sum_{s" \in U^{138} \setminus U^{139}} \mu_{s"}.$$

Since $\sigma^k_s = 0$ for all demand $k \in K$ and slot $s \in \{w_k, ..., \bar{s}\}$ with $s \notin \{s_i + w_k + 1, ..., s_j\}$ if $v_k \in H$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$, it follows that $\sigma^{k_i}_{\tilde{s}} = \sigma^{k'}_{s'}$.
Given that the pair $(v_k, v_{k'})$ are chosen arbitrarily in odd-hole H, we re-do the same procedure for all pairs $(v_k, v_{k'})$ such that we find

$$\sigma^k_s = \sigma^{k'}_{s'}, \text{for all pairs } (v_k, v_{k'}) \in H, s \in \{s_i + w_k - 1, ..., s_j\} \text{ and } \{s_i + w_{k'} - 1, ..., s_j\}.$$

Consequently, we obtain that $\sigma^k_s = \rho$ for all $v_k \in H$ and $s \in \{s_i + w_k - 1, ..., s_j\}$.
Overall, and using the result (5.9), we obtain that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and

$$\sigma^{k'}_s = \begin{cases} \gamma^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ \rho & \text{if } v_{k'} \in H \text{ and } s \in \{s_i + w_{k'} - 1, , ..., s_j\}, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k' \in K$ and $s \in \mathbb{S}$. As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$ as desired.

□

5.4.4 Slot-Assignment-Clique Inequalities

On the other hand, we also noticed that there may exist some cases that are not covered by inequality (2.25). For this, we provide an adapted definition of a conflict graph H^E_S for the SA problem and its associated inequality.

Definition 5.4.3. *Let H'^E_S be a conflict graph defined as follows. For all slot $s \in \{w_k, ..., \bar{s}\}$ and demand $k \in K$, consider a node $v_{k,s}$ in H'^E_S. Two nodes $v_{k,s}$ and $v_{k',s'}$ are linked by an edge in H'^E_S if and only if*

- $k = k'$,
- *or* $E_1^k \cap E_1^{k'} \neq \emptyset$ *and* $\{s - w_k + 1, ..., s\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ *when* $k \neq k'$.

Based on the conflict graph $H_S'^E$, we introduced the following inequalities.

Proposition 5.4.5. *Let C be a clique in the conflict graph $H_S'^E$ with $|C| \geq 3$. Then, inequality (2.43) is valid for $\mathcal{Q}_{sa}(G, K, \mathbb{S})$. Moreover, it is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$.*

Proof. We use the same proof of proposition (2.4.17). □

Theorem 5.4.5. *Consider a clique C in the conflict graph $H_S'^E$ with $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each $(v_{k,s}, v_{k',s'}) \in C$. Then, inequality (2.43) is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if and only if*

1. *C is a maximal clique in the conflict graph $H_S'^E$,*
2. *and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with*
   - $[\min_{v_{k,s} \in C}(s - w_k + 1), \max_{v_{k,s} \in C} s] \subset I$,
   - *and* $w_k + w_{k'} \geq |I| + 1$ *for each* $(v_k, v_{k'}) \in C$,
   - *and* $2w_k \geq |I| + 1$ *and* $w_k \leq |I|$ *for each* $v_k \in C$.
3. *and there does not exist a slot $s' \in \mathbb{S}$ such that $s' \in \{s - w_k + 1, .., s\}$ for each $v_{k,s} \in C$.*

Proof. **Neccessity.**
If C is not maximal clique in the conflict graph $H_S'^E$, this means that inequality (2.43) can be dominated by another inequality associated with a clique C' such that $C \subset C'$ without changing its right-hand side. Moreover, if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ satisfying the conditions of the condition 2 of the theorem. Then, inequality (2.43) is dominated by inequality (2.39). As a result, inequality (2.43) cannot be facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.
On the other hand, if there exists a slot $s' \in \mathbb{S}$ such that $s' \in \{s - w_k + 1, .., s\}$ for each $v_{k,s} \in C$, then inequality (2.43) is dominated by the non-overlapping inequality (5.4). Hence, inequality (2.43) cannot be facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$. **Sufficiency.**
Let $\tilde{F}_C^{H_S'^E}$ be the face induced by inequality (2.39), that is

$$\tilde{F}_C^{H_S'^E} = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \sum_{v_{k,s} \in C} z_s^k = 1\}.$$

We denote inequality $\sum_{v_{k,s}\in C} z_s^k \leq 1$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}_C^{H'_S E} \subset F = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. In order to prove that inequality $\sum_{v_{k,s}\in C} z_s^k \leq 1$ is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$, we show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k\in K}(w_k-1)}$) such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.

First, we show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$, and a solution $\mathcal{S}^{141} = (U^{141}, S^{141})$ given by

a) select one pair of demand k' and slot s' from clique C (i.e., $v_{k',s'} \in C$), and use slot $s_{k'} = s'$ as last slot with $\tilde{s} \notin \{s' - w_{k'} + 1, ..., s'\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{141}),

b) for each demand $k_i \in K \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{141} given by

$$I_i^{141} = [\bigcap_{k_j \in D_i^{141}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\},$$

where $D_i^{141} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k'\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{141}$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{141}).

We let $S_{k_i}^{141} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

c) Let U^{141} be the set of slots used in $\mathbb{S}$ such that for each demand k and last slot $s_k \in S_k^{141}$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{141}$.

$\mathcal{S}^{141}$ is clearly feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{141}}, z^{\mathcal{S}^{141}})$ belongs to $\tilde{F}_C^{H'_S E}$. Then we derive a solution $\mathcal{S}^{142} = (U^{142}, S^{142})$ from $\mathcal{S}^{141}$ by adding slot $\tilde{s}$ as an used slot in U^{142} without modifying the last slots assigned to the demands K in $\mathcal{S}^{141}$ which remain the same in solution $\mathcal{S}^{142}$ i.e., $S_k^{141} = S_k^{142}$ for each demand $k \in K$. $\mathcal{S}^{142}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{142}}, z^{\mathcal{S}^{142}})$ belongs to $\tilde{F}_C^{H'_S E}$. Hence, solutions $\mathcal{S}^{141}$ and $\mathcal{S}^{142}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{141}} + \sigma z^{\mathcal{S}^{141}} = \mu u^{\mathcal{S}^{142}} + \sigma z^{\mathcal{S}^{142}} = \mu u^{\mathcal{S}^{141}} + \sigma z^{\mathcal{S}^{141}} + \mu_{\tilde{s}}.$$

Hence, $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin C$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin C$. Let $\mathcal{S}^{143} = (U^{143}, S^{143})$ be the solution given by

a) select one pair of demand k' and slot s' from clique C (i.e., $v_{k',s'} \in C$), and use slot $s_{k'} = s'$ as last slot with $\{s' - w_{k'} + 1, ..., s'\} \cap \{s' - w_k + 1, ..., s\} = \emptyset$ if $E(p_k) \cap E(p_{k'}) \neq \emptyset$,

b) for each demand $k_i \in K \setminus \{k'\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{143} given by

$$I_i^{143} = [\bigcap_{k_j \in D_i^{143}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_k) \neq \emptyset \text{ or } I_i^{143} = \bigcap_{k_j \in D_i^{143}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{143} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k'\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{143}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots S_k^{143} to route demand k in solution $\mathcal{S}^{143}$).

We let $S_{k_i}^{143} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

c) a set of slots U^{143} are then used in $\mathbb{S}$ such that for each demand $k' \in K$ and last slot $s" \in S_{k'}^{143}$ and $s" \in \{s_{k'} - w_{k'} + 1, ..., s_{k'}\}$, we have $s" \in U^{143}$.

$\mathcal{S}^{143}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{143}}, z^{\mathcal{S}^{143}})$ belongs to $\tilde{F}_C^{H'^E_S}$. Then we derive a solution $\mathcal{S}^{144} = (U^{144}, S^{144})$ from $\mathcal{S}^{143}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{143}$, i.e., $S_{k'}^{143} = S_{k'}^{144}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{144} = S_k^{143} \cup \{s'\}$ for demand k. Solution $\mathcal{S}^{144}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{144}}, z^{\mathcal{S}^{144}})$ belongs to $\tilde{F}_C^{H'^E_S}$. Hence, solutions $\mathcal{S}^{143}$ and $\mathcal{S}^{144}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{143}} + \sigma z^{\mathcal{S}^{143}} = \mu u^{\mathcal{S}^{144}} + \sigma z^{\mathcal{S}^{144}} = \mu u^{\mathcal{S}^{143}} + \sigma z^{\mathcal{S}^{143}} + \sigma_{s'}^k + \sum_{\tilde{s} \in U^{144} \setminus U^{143}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{143} \setminus U^{144}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_{s'}^k = 0$.

In a similar way, we can show that

$$\sigma_s^k = 0, \text{ for all } k \in K \text{ and } s \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s} \notin C.$$

Let prove that σ_s^k for all $v_{k,s} \in C$ are equivalent. Consider a demand $k' \in K$ and a slot $s' \in \{w_{k'}, ..., \bar{s}\}$ with $v_{k',s'} \in C$, and a solution $\mathcal{S}^{145} = (U^{145}, S^{145})$ given by

a) select a pair of demand k and slot s from clique C (i.e., $v_{k,s} \in C$) such that slot $s_k = s$ will be used as last slot for demand k,

b) for each demand $k_i \in K \setminus \{k\}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{145} given by

$$I_i^{145} = [\bigcap_{k_j \in D_i^{145}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_{k'}\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset \text{ or } I_i^{145} = \bigcap_{k_j \in D_i^{145}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{145} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \{k\} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{145}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $S_{k'}^{145}$ to route demand k' in solution $\mathcal{S}^{145}$).

We let $S_{k_i}^{145} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

c) let U^{145} be the slots used in $\mathbb{S}$ such that for each demand $k" \in K$ and last slot $s_{k"} \in S_{k"}^{145}$ and $s" \in \{s_{k"} - w_{k"} + 1, ..., s_{k"}\}$, we have $s" \in U^{145}$.

$\mathcal{S}^{145}$ is feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{145}}, z^{\mathcal{S}^{145}})$ belongs to $\tilde{F}_C^{H'^E_S}$. After that, we derive a solution $\mathcal{S}^{146} = (E^{146}, S^{146})$ from $\mathcal{S}^{145}$ by adding slot s' as last slot to demand k' without modifying the last slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{145}$, i.e., $S_{k"}^{145} = S_{k"}^{146}$ for each demand $k" \in K \setminus \{k, k'\}$, and $S_{k'}^{146} = S_{k'}^{145} \cup \{s'\}$ for demand k', and with modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s \in S_k^{145}$ with $s \in \{s_i + w_k + 1, ..., s_j\}$ and $\tilde{s} \in \{w_k, ..., \bar{s}\}$ for demand k with $v_{k,\tilde{s}} \notin C$ such that $S_k^{146} = (S_k^{145} \setminus \{s\}) \cup \{\tilde{s}\}$. Solution $\mathcal{S}^{146}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{146}}, z^{\mathcal{S}^{146}})$ belongs to $\tilde{F}_C^{H'^E_S}$. Hence, solutions $\mathcal{S}^{145}$ and $\mathcal{S}^{146}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{145}} + \sigma z^{\mathcal{S}^{145}} = \mu u^{\mathcal{S}^{146}} + \sigma z^{\mathcal{S}^{146}} = \mu u^{\mathcal{S}^{145}} + \sigma z^{\mathcal{S}^{145}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{146} \setminus U^{145}} \mu_{s"} - \sum_{s" \in U^{145} \setminus U^{146}} \mu_{s"}.$$

Since $\sigma^k_{\tilde{s}} = 0$ for $v_{k,\tilde{s}} \notin C$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$, it follows that $\sigma^{k'}_{s'} = \sigma^k_s$.
In a similar way, we can show that

$$\sigma^k_s = \sigma^{k'}_{s'}, \text{for all pairs } (v_{k,s}, v_{k',s'}) \in C,$$

Consequently, we obtain that $\sigma^k_s = \rho$ for all $v_{k,s} \in C$.
Overall, and using the result (5.9), we obtain that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and

$$\sigma^k_s = \begin{cases} \gamma^{k,s} & \text{if } s \in \{1, ..., w_k - 1\}, \\ \rho & \text{if } v_{k,s} \in C, \\ 0 & \text{otherwise,} \end{cases}$$

for each $k \in K$ and $s \in \mathbb{S}$. As a result, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.

□

5.4.5 Slot-Assignment-Odd-Hole Inequalities

Proposition 5.4.6. *Let H be an odd-hole in the conflict graph H'^E_S with $|H| \geq 5$, and $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H. Then, inequality (2.44) is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.*

Proof. We use the same proof of proposition (2.4.14). □

Theorem 5.4.6. *Let H be an odd-hole in the conflict graph H'^E_S with $|H| \geq 5$, and $\{s - w_k + 1, ..., 1\} \cap \{s' - w_{k'} + 1, ..., s'\} \neq \emptyset$ for each pair of nodes $(v_{k,s}, v_{k',s'})$ linked in H. Then, inequality (2.44) is facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$ if and only if*

a) for each node $v_{k',s'} \notin H$ in H'^E_S, there exists a node $v_{k,s} \in H$ such that the induced graph $H'^E_S((H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\})$ does not contain an odd-hole,

b) and there does not exist a node $v_{k',s'} \notin H$ in H'^E_S such that $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$,

c) and there does not exist an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ with

- $[\min_{v_{k,s} \in H}(s - w_k + 1), \max_{v_{k,s} \in H}] \subset I$,
- *and $w_k + w_{k'} \geq |I| + 1$ for each $(v_k, v_{k'})$ linked in H,*
- *and $2w_k \geq |I| + 1$ and $w_k \leq |I|$ for each $v_k \in H$.*

Proof. **Neccessity.**
We distinguish the following cases:

a) if for a node $v_{k',s'} \notin H$ in $H_S'^E$, there exists a node $v_{k,s} \in H$ such that the induced graph $H_S'^E(H \setminus \{v_{k,s}\} \cup \{v_{k',s'}\})$ contains an odd-hole $H' = (H \setminus \{v_{k,s}\}) \cup \{v_{k',s'}\}$. This implies that inequality (2.44) can be dominated using some technics of lifting based on the following two inequalities $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$, and $\sum_{v_{k',s'} \in H'} z_{s'}^{k'} \leq \frac{|H'|-1}{2}$.

b) if there exists a node $v_{k',s'} \notin H$ in $H_S'^E$ such that $v_{k',s'}$ is linked with all nodes $v_{k,s} \in H$. This implies that inequality (2.44) can be dominated by the following valid inequality

$$\sum_{v_{k,s} \in H} z_s^k + \frac{|H|-1}{2} z_{s'}^{k'} \leq \frac{|H|-1}{2}.$$

c) if there exists an interval of contiguous slots $I = [s_i, s_j] \subset [1, \bar{s}]$ satisfying the conditions of c). Hence, inequality (2.44) is dominated by inequality (2.40).

If no one of these cases is verified, inequality (2.44) can never be dominated by another inequality without changing its right-hand side. Otherwise, inequality (2.44) cannot be facet defining for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Sufficiency.

Let $\tilde{F}_H^{H_S'^E}$ denote the face induced by inequality (2.40), that is

$$\tilde{F}_H^{H_S'^E} = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \sum_{v_{k,s} \in H} z_s^k = \frac{|H|-1}{2}\}.$$

Denote inequality $\sum_{v_{k,s} \in H} z_s^k \leq \frac{|H|-1}{2}$ by $\alpha u + \beta z \leq \lambda$. Let $\mu u + \sigma z \leq \tau$ be a valid inequality that is facet defining F of $\mathcal{P}_{sa}(G, K, \mathbb{S})$. Suppose that $\tilde{F}_H^{H_S'^E} \subset F = \{(u, z) \in \mathcal{P}_{sa}(G, K, \mathbb{S}) : \mu u + \sigma z = \tau\}$. To prove that $\tilde{F}_H^{H_S'^E}$ is a facet of $\mathcal{P}_{sa}(G, K, \mathbb{S})$, we need to show that there exist $\rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}^{\sum_{k \in K}(w_k - 1)})$ such that $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$.

We first show that $\mu_s = 0$ for all $s \in \mathbb{S}$. Consider a slot $\tilde{s} \in \mathbb{S}$, and a solution $\mathcal{S}^{148} = (U^{148}, S^{148})$ such that

a) select a subset of nodes $\tilde{H}^{148}$ from H with $|\tilde{H}^{148}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k,s}, v_{k',s'}) \in \tilde{H}^{148}$ are not linked in the conflict graph $H_S'^E$, and $\tilde{s} \notin \{s_{wk} + 1, ..., s\}$ for each $v_{k,s} \in \tilde{H}^{148}$,

b) for each pair of demand k and slot s with $v_{k,s} \in \tilde{H}^{148}$, we select slot $s_k = s$ as last slot for demand k,

c) for each demand $k_i \in K \setminus \tilde{H}^{148}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{148} given by

$$I_i^{148} = [\bigcap_{k_j \in D_i^{148}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \setminus \{\tilde{s}, ..., \tilde{s} + w_{k_i} - 1\},$$

where $D_i^{148} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H}^{148} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This guarantees that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{148}$,
- and $\tilde{s} \notin \{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\}$ (slot assignment constraint taking into account the possibility of adding slot $\tilde{s}$ in the set of used slots U^{148}).

We let $S_{k_i}^{148} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) let U^{148} be the set of slots used in $\mathbb{S}$ such that for each demand k and last slot $s_k \in S_k^{148}$ and $s' \in \{s_k - w_k + 1, ..., s_k\}$, we have $s' \in U^{148}$.

$\mathcal{S}^{148}$ is clearly feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{148}}, z^{\mathcal{S}^{148}})$ belongs to $\tilde{F}_H^{H'^E_S}$. Then consider the solution $\mathcal{S}^{149} = (U^{149}, S^{149})$ obtained from $\mathcal{S}^{148}$ by adding slot $\tilde{s}$ as an used slot in U^{149} without modifying the last slots assigned to the demands K in $\mathcal{S}^{148}$ which remain the same in solution $\mathcal{S}^{149}$ i.e., $S_k^{148} = S_k^{149}$ for each demand $k \in K$. $\mathcal{S}^{149}$ is clearly feasible for the SA problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{149}}, z^{\mathcal{S}^{149}})$ belongs to $\tilde{F}_H^{H'^E_S}$. Hence, solutions $\mathcal{S}^{148}$ and $\mathcal{S}^{149}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{148}} + \sigma z^{\mathcal{S}^{148}} = \mu u^{\mathcal{S}^{149}} + \sigma z^{\mathcal{S}^{149}} = \mu u^{\mathcal{S}^{148}} + \sigma z^{\mathcal{S}^{148}} + \mu_{\tilde{s}}.$$

It follows that $\mu_{\tilde{s}} = 0$.

In a similar way, we can show that

$$\mu_{\tilde{s}} = 0, \text{ for all slots } \tilde{s} \in \mathbb{S}.$$

Let show that $\sigma_s^k = 0$ for all $k \in K$ and $s \in \{w_k, ..., \bar{s}\}$ with $v_{k,s} \notin H$.

Consider a demand k in K and a slot s' in $\{w_k, ..., \bar{s}\}$ with $v_{k,s'} \notin H$. Let $\mathcal{S}^{150} = (U^{150}, S^{150})$ be the solution given by

a) select a subset of nodes $\tilde{H}^{150}$ from H with $|\tilde{H}^{150}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k,s}, v_{k',s'}) \in \tilde{H}^{150}$ are not linked in the conflict graph H'^E_S, ,

b) for each pair of demand k and slot s with $v_{k,s} \in \tilde{H}^{150}$, we select slot $s_k = s$ as last slot for demand k,

c) for each demand $k_i \in K \setminus \tilde{H}^{150}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{150} given by

$$I_i^{150} = [\bigcap_{k_j \in D_i^{150}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_k) \neq \emptyset \text{ or } I_i^{150} = \bigcap_{k_j \in D_i^{150}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{150} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H}^{150} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. This ensures that

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{150}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_k + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_k) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots S_k^{150} to route demand k in solution $\mathcal{S}^{150}$).

We let $S_{k_i}^{150} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) a set of slots U^{150} are used in $\mathbb{S}$ such that for each demand k" and last slot $s_{k"} \in S_{k"}^{150}$ and $s" \in \{s_{k"} - w_{k"} + 1, ..., s_{k"}\}$, we have $s" \in U^{150}$.

$\mathcal{S}^{150}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{150}}, z^{\mathcal{S}^{150}})$ belongs to $\tilde{F}_H^{H'^E_S}$. Then consider the solution $\mathcal{S}^{151} = (U^{151}, S^{151})$ obtained from $\mathcal{S}^{150}$ by adding slot s' as last slot to demand k without modifying the last slots assigned to the demands $K \setminus \{k\}$ in $\mathcal{S}^{150}$, i.e., $S_{k'}^{150} = S_{k'}^{151}$ for each demand $k' \in K \setminus \{k\}$, and $S_k^{151} = S_k^{150} \cup \{s'\}$ for demand k. Solution $\mathcal{S}^{151}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{151}}, z^{\mathcal{S}^{151}})$ belongs to $\tilde{F}_H^{H'^E_S}$. Hence, solutions $\mathcal{S}^{150}$ and $\mathcal{S}^{151}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{150}} + \sigma z^{\mathcal{S}^{150}} = \mu u^{\mathcal{S}^{151}} + \sigma z^{\mathcal{S}^{151}} = \mu u^{\mathcal{S}^{150}} + \sigma z^{\mathcal{S}^{150}} + \sigma_{s'}^k + \sum_{\tilde{s} \in U^{151} \setminus U^{150}} \mu_{\tilde{s}} - \sum_{\tilde{s} \in U^{150} \setminus U^{151}} \mu_{\tilde{s}}.$$

Since $\mu_{\tilde{s}} = 0$ for all slots $\tilde{s} \in \mathbb{S}$, it follows that $\sigma_{s'}^k = 0$.
In a similar way, we can show that

$$\sigma_{s'}^k = 0, \text{ for demand } k \text{ and } s' \in \{w_k, ..., \bar{s}\} \text{ with } v_{k,s'} \notin H.$$

We re-do the same procedure for all demand k' in $K \setminus \{k\}$ such that

$$\sigma_s^{k'} = 0, \text{ for all } k' \in K \setminus \{k\} \text{ and } s \in \{w_{k'}, ..., \bar{s}\} \text{ with } v_{k,s'} \notin H.$$

Let prove that σ_s^k for all $v_{k,s} \in H$ are equivalent.
Consider a node $v_{k',s'}$ in H. Let $\mathcal{S}^{152} = (U^{152}, S^{152})$ be the solution given by

a) select a subset of nodes $\tilde{H}^{152}$ from H with $|\tilde{H}^{152}| = \frac{|H|-1}{2}$, and each pair of nodes $(v_{k,s}, v_{k',s'}) \in \tilde{H}^{152}$ are not linked in the conflict graph $\tilde{H}_S'^E$,

b) for each pair of demand k and slot s with $v_{k,s} \in \tilde{H}^{152}$, we select slot $s_k = s$ as last slot for demand k,

c) for each demand $k_i \in K \setminus \tilde{H}^{152}$ with $i \in \{1, ..., |K|\}$, we select the smallest slot index s_{k_i} in the set of slots I_i^{152} given by

$$I_i^{152} = [\bigcap_{k_j \in D_i^{152}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\}] \cap [\{w_{k_i}, ..., s' - w_k\} \cup \{s' + w_{k_i}, ..., \bar{s}\}]$$

$$\text{if } E(p_{k_i}) \cap E(p_k) \neq \emptyset \text{ or } I_i^{152} = \bigcap_{k_j \in D_i^{152}} \{w_{k_i}, ..., s_{k_j} - w_{k_j}\} \cup \{s_{k_j} + w_{k_i}, ..., \bar{s}\} \text{ if not,}$$

where $D_i^{152} = \{k_j \in \{k_1, ..., k_{i-1}\} \cup \tilde{H}^{152} : E(p_{k_i}) \cap E(p_{k_j}) \neq \emptyset\}$. As a result,

- $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s_{k_j} - w_{k_j} + 1, ..., s_{k_j}\} = \emptyset$ for each $k_j \in D_i^{152}$,
- and $\{s_{k_i} - w_{k_i} + 1, ..., s_{k_i}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ if $E(p_{k_i}) \cap E(p_{k'}) \neq \emptyset$ (we take into account the possibility of adding slot s' as a last slot in the selected last slots $S_{k'}^{152}$ to route demand k' in solution $\mathcal{S}^{152}$).

We let $S_{k_i}^{152} = \{s_{k_i}\}$ be the set of last slots assigned to demand k_i with $i \in \{1, ..., |K|\}$.

d) let U^{152} be the set of slots used in $\mathbb{S}$ such that for each demand $k" \in K$ and last slot $s_{k"} \in S_{k"}^{152}$ and $s" \in \{s_{k"} - w_{k"} + 1, ..., s_{k"}\}$, we have $s" \in U^{152}$.

$\mathcal{S}^{152}$ is clearly feasible for the problem. Hence, the corresponding incidence vector $(u^{\mathcal{S}^{152}}, z^{\mathcal{S}^{152}})$ belongs to $\tilde{F}_H^{\tilde{H}'^E_S}$. Then consider the solution $\mathcal{S}^{153} = (U^{153}, S^{153})$ obtained from $\mathcal{S}^{152}$ such that

a) the last slots assigned to the demands $K \setminus \{k, k'\}$ in $\mathcal{S}^{152}$ remain the same in $\mathcal{S}^{153}$, i.e., $S_{k"}^{152} = S_{k"}^{153}$ for each demand $k" \in K \setminus \{k, k'\}$, where k is a demand with $v_{k,s} \in \tilde{H}^{152}$ and $s \in S_k^{152}$ such that $v_{k',s'}$ is not linked with any node $v_{k",s"} \in \tilde{H}^{152} \setminus \{v_{k,s}\}$,

b) and adding slot s' as last slot to demand k', i.e., $S_{k'}^{153} = S_{k'}^{152} \cup \{s'\}$ for demand k',

c) and modifying the last slots assigned to demand k by adding a new last slot $\tilde{s}$ and removing the last slot $s \in S_k^{152}$ with $v_{k,s} \in H$ and $v_{k,\tilde{s}} \notin H$ such that $S_k^{153} = (S_k^{152} \setminus \{s\}) \cup \{\tilde{s}\}$ for demand k such that $\{\tilde{s} - w_k + 1, ..., \tilde{s}\} \cap \{s' - w_{k'} + 1, ..., s'\} = \emptyset$ for each $k' \in K$ and $s' \in S_{k'}^{153}$ with $E(p_k) \cap E(p_{k'}) \neq \emptyset$.

Solution $\mathcal{S}^{153}$ is feasible for the SA problem. The corresponding incidence vector $(u^{\mathcal{S}^{153}}, z^{\mathcal{S}^{153}})$ belongs to $\tilde{F}_H^{H'^E_S}$. Hence, solutions $\mathcal{S}^{152}$ and $\mathcal{S}^{153}$ satisfy equation $\mu u + \sigma z = \tau$. We then obtain that

$$\mu u^{\mathcal{S}^{152}} + \sigma z^{\mathcal{S}^{152}} = \mu u^{\mathcal{S}^{153}} + \sigma z^{\mathcal{S}^{153}} = \mu u^{\mathcal{S}^{152}} + \sigma z^{\mathcal{S}^{152}} + \sigma_{s'}^{k'} - \sigma_s^k + \sigma_{\tilde{s}}^k + \sum_{s" \in U^{153} \setminus U^{152}} \mu_{s"} - \sum_{s" \in U^{152} \setminus U^{153}} \mu_{s"}.$$

Since $\sigma^k_{\tilde{s}} = 0$ for $v_{k,\tilde{s}} \notin H$, and $\mu_{s"} = 0$ for all $s" \in \mathbb{S}$, it follows that $\sigma^{k'}_{s'} = \sigma^k_s$. Consequently, we obtain that $\sigma^k_s = \rho$ for all $v_{k,s} \in H$.
By (5.9), we know that

$$\sigma^{k'}_{s'} = \gamma^{k',s'}, \text{ for all } k' \in K \text{ and } s' \in \{1, ..., w_{k'} - 1\}.$$

We then conclude that $\mu_s = 0$ for each slot $s \in \mathbb{S}$, and

$$\sigma^{k'}_s = \begin{cases} \gamma^{k',s} & \text{if } s \in \{1, ..., w_{k'} - 1\}, \\ \rho & \text{if } v_{k',s} \in H, \\ 0 & \text{otherwise,} \end{cases}$$

As a consequence, we have $(\mu, \sigma) = \rho(\alpha, \beta) + \gamma M$ as desired. □

In the next section, we will derive some symmetry breaking inequalities for the SA subproblem in which some symmetrical solutions may appeared.

5.5 Symmetry-Breaking Inequalities

In this section, we address some symmetry issues that can appear when solving the SA problem.

Proposition 5.5.1. *We ensure that for all slot* $s \in \{1, ..., \bar{s} - 1\}$

$$u_s - u_{s+1} \geq 0, \tag{5.14}$$

which means that a slot $s + 1$ *can be used if and only if slot* s *is used.*

Similar idea was proposed by Mendez-Diaz et al. [69][70] to break the symmetry for the vertex coloring problem.
To strengthen inequality (5.14), we propose the following inequalities.

Proposition 5.5.2. *Consider a slot* $s \in \{1, ..., \bar{s} - 1\}$. *Then,*

$$\sum_{k \in K} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} 2^{|K|-k} z^k_{s'} \geq \sum_{k \in K} \sum_{s'=s+1}^{\min(s+w_k, \bar{s})} 2^{|K|-k} z^k_{s'}. \tag{5.15}$$

Similar idea was proposed by Friedman [40]. However, the coefficient $2^{|K|-k}$ can provoques some numerical intractabilities for the computer machine [8]. For this, we introduce the following inequality.

Proposition 5.5.3. *We ensure that for all slot* $s \in \{1, ..., \bar{s} - 1\}$

$$\sum_{k \in K} \sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} \geq \sum_{k \in K} \sum_{s'=s+1}^{\min(s+w_k, \bar{s})} z^k_{s'}, \tag{5.16}$$

which means that the number of intervals of contiguous slots allocated which cover slot $s+1$ *(cardinality of slot-usage) cannot be greater than the number of channels allocated which cover slot* s*.*

Similar idea was proposed by Mendez-Diaz et al. [69][70] to break the symmetry for the vertex coloring problem. Our inequalities and those of Mendez-Diaz et al. [69][70] differ in their right and left hand sides.

Proposition 5.5.4. *Due to inequality* (5.14)*, we ensure that for all* $k \in K$*, and* $s^0 \in \{1, ..., \bar{s} - 1\}$ *and* $s \in \{s^0, ..., \bar{s}\}$

$$\sum_{s'=s}^{\min(s+w_k-1, \bar{s})} z^k_{s'} \leq u_{s^0}, \tag{5.17}$$

which means that for a slot $S^0 \in \{1, ..., \bar{s} - 1\}$*, a demand* k *can allocate a slot in the sub-spectrum* $\{S^0, ..., \bar{s}\}$ *if slot* S^0 *is used.*

Similar idea was proposed by Mendez-Diaz et al. [70] for the vertex coloring problem. Inequalities (5.17)} and those of Mendez-Diaz et al. [70] differ in their left hand sides.

5.6 Lower Bounds

Here we propose some lower bounds issus from the conflict graph H_{sa}. They can be seen as a valid inequalities for the polytope $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Proposition 5.6.1. *Consider an edge* $e \in E$*. Then, the inequality*

$$\sum_{s \in \mathbb{S}} u_s \geq \sum_{k \in \tilde{K}_e} w_k, \text{ for all } e \in E, \tag{5.18}$$

is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$*.*

Proof. Inequality (5.18) ensures that the number of slots used in the spectrum $\mathbb{S}$ is greater than the flow over all the edges (the flow for an edge e is equal to the number of slots that should be used over edge e). □

Inequality (5.18) can be generalized as follows using the conflict graph H_{sa}.

Proposition 5.6.2. *Let C be a clique in H_{sa}. Then, the inequality*

$$\sum_{s\in\mathbb{S}} u_s \geq \sum_{v_k\in C} w_k, \tag{5.19}$$

is valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$.

Proof. It's trivial given the definition of clique C in the conflict graph H_{sa} such that we know in advance that the demands in C share an edge in E which means that they cannot share a slot in $\mathbb{S}$. Hence, the number of allocated slots $\sum_{s\in\mathbb{S}} u_s$ is at least equal to the number of requested slots of the demands in C.

□

5.7 Upper Bounds

Let us introduce the following weighted conflict graph in which a positive integer called weight is assigned to each node.

Definition 5.7.1. *Consider the conflict graph H_w^r defined as follows. For each demand $k \in K$, consider a node v_k in H_w^r. Two nodes v_k and $v_{k'}$ are linked by an edge in H_w^r if and only if $E(p_k) \cap E(p_{k'}) \neq \emptyset$. Each node v_k is associated with a positive weight which equals to the requested number of slots w_k of demand k.*

Definition 5.7.2. *Let C be a clique in H_w^r. It's known to be the maximum weight clique in H_w^r if the total weight of the nodes in C ($\sum_{v_k\in C} w_k$) defines the maximum total weight over all cliques in H_w^r, i.e., $\sum_{v_k\in C} w_k \geq \sum_{v_{k'}\in C'} w_{k'}$ for all clique C' in H_w^r.*

Based on these definitions, we introduce the following inequality and showing that computing the upper bound for the SA is equivalent to solving the Maximum Weighted Clique Problem (MWC) which is well known to be NP-hard problem [4].

Proposition 5.7.1. *Let C be the maximum weighted clique in H_w^r. Then, the upper bound is defined as follows*

$$\sum_{s\in\mathbb{S}} u_s \leq \sum_{v_k\in C} w_k, \tag{5.20}$$

Proof. It's trivial given the definition of the maximum weighted clique C in the conflict graph H_w^r such that the maximum number of allocated slots $\sum_{s\in\mathbb{S}} u_s$ is at most equal to the number of requested slots of the demands in C. □

Inequality (5.20) is not valid for $\mathcal{P}_{sa}(G, K, \mathbb{S})$ given that there exist some feasible solutions in $\mathcal{P}_{sa}(G, K, \mathbb{S})$ which violate inequality (5.20) when for example a slot $s \in \mathbb{S}$ is used (i.e., $u_s = 1$) but there is no demand $k \in K$ which use slot s (i.e., $\sum_{k \in K} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k$). On the other hand, we ensure that all the optimization algorithms developed to solve the MWC problem can be used to compute the upper bound based on the conflict graph H_w^r.
Based on inequalities (5.19) and (5.20), we conclude that the minimum number of slots to be used by the set of demands K while satisfying the SA constraints, it's equal to the total weight of the maximum weighted clique in the conflict graph H_w^r. Based on theoretical results presented in this chapter, we devise a Branch-and-Bound (B&B) and Branch-and-Cut algorithms to solve the SA problem. Moreover, we study the effectiveness of these algorithms and assess the impact of the valid inequalities on the effectiveness of the Branch-and-Cut algorithm.

5.8 Branch-and-Cut Algorithm

5.8.1 Description

Here we describe the Branch-and-Cut algorithm. We consider the following linear problem which can be seen as a strenghtned formulation for the compact formulation (5.1)-(5.8)

$$\min \sum_{s \in \mathbb{S}} u_s, \tag{5.21}$$

$$z_s^k = 0, \quad \text{for all } k \in K \text{ and } s \in \{1, ..., w_k - 1\}, \tag{5.22}$$

$$\sum_{s=w_k}^{\bar{s}} z_s^k = 1, \quad \text{for all } k \in K, \tag{5.23}$$

$$\sum_{k \in \tilde{K}_e} \sum_{s'=s}^{\min(\bar{s}, s+w_k-1)} z_s^k - u_s \leq 0, \quad \text{for all } e \in E, \text{ and } s \in \mathbb{S}, \tag{5.24}$$

$$u_s - \sum_{k \in K} \sum_{s'=s}^{\min(s+w_k-1,\bar{s})} z_{s'}^k \leq 0, \quad \text{for all } s \in \mathbb{S}, \tag{5.25}$$

$$z_s^k \geq 0, \quad \text{for all } k \in K \text{ and } s \in \mathbb{S}, \tag{5.26}$$

$$0 \leq u_s \leq 1, \quad \text{for all } s \in \mathbb{S}, \tag{5.27}$$

$$z_s^k \in \{0, 1\}, \quad \text{for all } k \in K \text{ and } s \in \mathbb{S}, \tag{5.28}$$

$$u_s \in \{0, 1\}, \quad \text{for all } s \in \mathbb{S}. \tag{5.29}$$

Inequality (5.25) ensures that if slot s is not used by at least one demand, its associated variable u_s is forced to be equal to zero.

On the other hand, and to boost the performance of the B&B algorithm, we already introduced several classes of valid inequalities to obtain tighter LP bounds. Based on this, and at each iteration in a certain level of the B&B algorithm, one can identify one or more than one violated inequality by the current fractional solution for a given class of valid inequalities. Algorithm 6 summarizes the different steps of the Branch-and-Cut algorithm taking into account additional valid inequalities for a given class of valid inequalities.

Note that the separation procedures of the valid inequalities presented in this chapter are still the same as those presented in chapter (2) for the C-RSA. However, we need to present the separation procedure for the interval-capacity-cover inequalities (5.10) as follows. Given a fractional solution $(\bar{u}, \bar{z})$. We first consider an interval of contiguous slots $I = [s_i, s_j]$ which is identified by generating two slots s_i and s_j randomly in $\mathbb{S}$ with $s_j \geq s_i + 2\max_{k \in K} w_k$. The separation problem associated with inequality (5.10) is NP-hard [82] given that it consists in identifying a cover $\tilde{K}^*$ for the interval $I = [s_i, s_j]$, such that $\sum_{k \in \tilde{K}^*} \sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > |\tilde{K}^*| - 1$. For this, we use a greedy algorithm introduced by Nemhauser and Sigismondi [73] as follows. We first select a demand $k \in K$ having the largest number of requested slot w_k with $\sum_{s'=s_i+w_k-1}^{s_j} \bar{z}_{s'}^k > 0$, and then set $\tilde{K}^*$ to $\tilde{K}^* = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus \tilde{K}^*$ to $\tilde{K}^*$ with $\sum_{s'=s_i+w_{k'}-1}^{s_j} \bar{z}_{s'}^{k'} > 0$ and demand k' share an edge with all the demands already added $\tilde{K}^*$, until a cover $\tilde{K}^*$ is obtained for the interval I over edge e with $\sum_{k \in \tilde{K}^*} w_k > |I|$. We further derive a minimal cover from the cover $\tilde{K}^*$ by deleting each demand $k \in \tilde{K}^*$ if $\sum_{k' \in \tilde{K}^* \setminus \{k\}} w_{k'} \leq |I|$. We then add inequality (5.10) induced by the minimal cover $\tilde{K}^*$ for the interval I if it is violated, i.e., we add the following valid inequality to the current LP

$$\sum_{k \in \tilde{K}^*} \sum_{s'=s_i+w_k-1}^{s_j} z_{s'}^k \leq |\tilde{K}^*| - 1.$$

5.8.2 Primal Heuristic

Let us present now a primal heuristic useful to boost the performance of the Branch-and-Cut algorithm. It is based on a hybrid method between a local search algorithm and a greedy-algorithm. Given an optimal fractional solution $(\bar{u}, \bar{z})$ in a certain node of the B&C tree, it consists in constructing an integral solution and "feasible" if possible from this fractional solution. For this, we first use a local search algorithm to generate at each iteration a sequence of demands L numeroted with

Algorithm 6: Branch-and-Cut Algorithm for the SA

Data: An undirected, loopless, and connected graph $G = (V, E)$, a spectrum $\mathbb{S}$, a multi-set K of demands, and a given class of valid inequality

Result: Optimal solution for the SA problem

```
1  Stop= FALSE;
2  while STOP==FALSE do
3  |   Solve the linear program LP of the SA;
4  |   Let (u*, z*) be the optimal solution of LP;
5  |   if there exist inequalities from the given class that are violated by the
   |     current solution (u*, z*) then
6  |   |   Add them to LP ;
7  |   end
8  |   else
9  |   |   STOP = TRUE;
10 |   end
11 end
12 Consider the optimal solution (u*, z*) of LP ;
13 if (u*, z*) is integer for the SA then
14 |   (u*, z*) is an optimal solution for the SA;
15 |   End of the Branch-and-Cut algorithm ;
16 end
17 else
18 |   Create two sub-problems by branching one some variables or constraints
   |     ;
19 end
20 for each sub-problem not yet solved do
21 |   go to 2 ;
22 end
23 return the best optimal solution (u*, z*) for the SA;
```

$L = 1', 2', ..., |K|' - 1, |K|'$. Based on this sequence of demands, our greedy algorithm selects a slot s for each demand $k' \in L$ with $\bar{z}_s^{k'} \neq 0$, while respecting the non-overlapping constraint with the set of demands that precede demand k' in the list L (i.e., the demands $1', 2, ..., k' - 1$). However, if there does not exist such slot s for demand k', we then select a slot s for demand $k' \in L$ with $\bar{z}_s^{k'} = 0$ with $s \in \{w_{k'}, ..., \bar{s}\}$ while respecting the non-overlapping constraint with the set of demands that precede demand k' in the list L. The complexity of this algorithm can be bounded by $\mathcal{O}(|K| * |\mathbb{S}| * log(|K|))$.

Afterwards, we compute the total number of slots in $\mathbb{S}$ used by the set of demands K in the final solution $\mathcal{S}$ given by the greedy-algorithm (i.e., $\sum_{s \in \mathbb{S}} u_s$). Our local search algorithm generates a new sequence by doing some permutation of demands in the last sequence of demands, if the value of the solution given by greedy-algorithm is smaller than the value of the best solution found until the current iteration. Otherwise, we stop the algorithm, and we give in output the best solution found during our primal heuristic induced by the best sequence of demands having the smallest value of the total number of slots in $\mathbb{S}$ used compared with the others generated sequences.

5.9 Computational Study

5.9.1 Implementation's Feature

We use C++ to implement the B&B and B&C algorithms under Linux using the "Solving Constraint Integer Programs" framework (Scip 6.0.2) such that Cplex 12.9 is used as LP solver. These have also been tested on LIMOS high-performance server with a memory size limited to 64 Gb while benefiting from parallelism by activating 8 threads, and with a CPU time limited to 5 hours (18000 s). We use the same graphs presented in Table 3.1, and the same instances used in the section 3.2.2.

5.9.2 Computational Results

Preliminary results show that introducing some families of valid inequalities allows solving several instances to optimality. Moreover, they enable reducing the average number of nodes in the B&C tree, and also the average CPU time for several instances. On the other hand, the results show that the odd-hole inequalities (2.40) and (2.44) are efficient compared with those of clique-based inequalities (2.43), (2.39), and cover-based inequalities (5.10). As a result, their separation is performed along with the B&C algorithm in the following order

a) interval-odd-hole inequalities (2.40),

b) slot-assignment-odd-hole inequalities (2.44),

c) interval-clique inequalities (2.43),

d) slot-assignment-clique inequalities (2.43),

e) interval-capacity-cover inequalities (5.10).

We also consider the valid inequalities (5.19) introduced previously that are shown to be as a precomputed lower bounds for the SA problem. They can be separated as follows. For each demand $k \in K$, we use a greedy algorithm introduced by Nemhauser and Sigismondi [73] to generate a maximum clique in H_{sa} containning demand k. We first set $\tilde{C}^k$ to $\tilde{C}^k = \{k\}$. After that, we iteratively add each demand $k' \in K \setminus \tilde{C}^k$ to $\tilde{C}^k$ such that demand k' must share an edge with all the demands already generated in $\tilde{C}^k$. We further add inequality (5.19) induced by clique $\tilde{C}^k$ for demand k to the compact formulation (5.1)-(5.8)

$$\sum_{s \in \mathbb{S}} u_s \geq \sum_{k' \in \tilde{C}^k} w_{k'}.$$

Based on this, we provide a comparative study between the B&B (without additional valid inequalities) and the B&C (with additional valid inequalities) algorithms. Our objective in this study is to show the efficiency of the inequalities we have introduced for solving the SA problem. We present some computational results using several instances with a number of demand ranges in $\{10, 20, 30, 40, 50, 100, 150, 200, 250, 300\}$ and $\bar{s}$ up to 320 slots. We use two types of topologies: real, and realistic ones from SND-LIB already described in Table 3.1. We first run our B&C algorithm with SCIP in which our valid inequalities are used, and all the Scip's internal cuts are deactivated. We call this run Own_B&C_SCIP. Then, we run the B&C algorithm with SCIP, and activating all the internal cuts we had deactivated prior in run 1. We call this run B&B_SCIP. Tables 5.1 and 5.2 below report the results obtained for the two runs. For each run and each instance, we report the number of nodes in B&C tree (Nbr_Nd), the optimality gap (Gap), the number of violated inequalities added during the algorithm (Nbr_Cuts), and the total CPU Time (TT) in seconds. Finally, notice that each line of each table 5.1 and 5.2, corresponds to the average results of 4 instances.

The results show that Own_B&C_SCIP is able to solve several instances to optimality that are not solved to optimality when using the B&B_SCIP even if Scip uses its proper cuts. Furthermore, we noticed that our valid inequalties allow solving to

optimality more instances than B&B_SCIP. Also, they enable reducing the average number of nodes in the B&C tree for several instances such that there exist some cases that we are able to solve some instances in the root of the B&C tree which is not the case when using the B&B. On the other hand, and looking at the instances that are not solved to optimality (i.e., gap $> 0,00$), adding valid inequalities decreases the average gap for several instances and much more for the large instances with a number of demands $|K| \geq 150$. However, there exist a few instances very rare, for example the triplet (German, 300, 320), in which adding valid inequalities does not improve the results of the B&B algorithm. Based on these results, we ensure that using the valid inequalities allows obtaining tighter LP bounds and improve the effectiveness of the B&B algorithm such that the B&C algorithm is able to beat the B&B algorithm even if Scip use its proper cuts that are shown to be very efficient for another optimization problems studied in the literature.

5.10 Concluding Remarks

In this chapter, we have studied the Spectrum Assignment sub-problem. We have introduced an integer linear programming compact formulation, and further investigated the facial structure of the associated polyhedron. Moreover, we have derived several valid inequalities that are facet-defining under sufficient conditions. Using the polyhedral results and the separation procedures, we have devised a Branch-and-Cut (BC) algorithm to solve the problem. We have also presented experimental results. The results have shown the effectiveness of the valid inequalities such that the B&C algorithm is shown to be very performant for solving large-scale instances of the problem. It could be very interesting to study the impact of the symmetry breaking inequalities on the performance of the Branch-and-Cut algorithm.

Instances			B&B_SCIP			Own_B&C_SCIP			
Topology	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**
German	10	30	1	**0,00**	0,03	1	**0,00**	0	0,02
	20	45	1	**0,00**	0,53	1	**0,00**	11	0,66
	30	70	7	**0,00**	1,47	1	**0,00**	5	1,81
	40	90	2,5	**0,00**	1,78	5	**0,00**	5	15,89
	50	110	1	**0,00**	0,87	1	**0,00**	2,5	9,34
	100	140	1	**0,00**	12,92	1	**0,00**	6,25	90,94
	150	210	1,75	**0,00**	43,22	1	**0,00**	0,75	118,59
	200	260	1	**0,00**	176,01	9,5	**0,00**	2,5	992,34
	250	320	21	**0,00**	380,74	9	**0,00**	12	2148,45
	300	320	6	**0,00**	2584,40	1	**0,00**	1	100,62
Nsfnet	10	15	1	**0,00**	0,02	1	**0,00**	0	0,01
	20	40	2,5	**0,00**	1,83	1,5	**0,00**	0	0,53
	30	30	4	**0,00**	2,92	2,25	**0,00**	3,75	4,14
	40	70	4,5	**0,00**	2,13	28,5	**0,00**	32	16,01
	50	50	9	**0,00**	4,47	4,75	**0,00**	19,25	12,61
	100	120	14469	0,94	4552,24	5090,25	**0,00**	20,25	1565,48
	150	160	10,75	**0,00**	215,01	66	**0,00**	6,5	841,26
	200	210	37	**0,00**	986,26	23	**0,00**	2,75	2035,74
	250	285	138	1	6535,05	397,5	**0,00**	3,75	7999,81
	300	320	20,5	1,81	9932,57	27	1,02	25,5	12712,35
Spain	10	40	1	**0,00**	0,02	1	**0,00**	0,25	0,03
	20	40	1	**0,00**	0,14	1	**0,00**	4	0,08
	30	40	1	**0,00**	0,62	1	**0,00**	1	0,33
	40	40	1	**0,00**	1,27	5,75	**0,00**	17,5	8,44
	50	80	476,25	**0,00**	34,36	3,75	**0,00**	10	7,17
	100	80	169,5	0,97	4782,16	2359,5	1,09	17,75	4810,79
	150	160	106,75	0,84	10722,14	59,5	0,28	37,5	8804,63
	200	280	26	1,6	5866,57	652,75	1,20	50,5	5829,44
	250	280	1	**0,00**	3444,44	25	**0,00**	21,75	6528,09
	300	320	1,25	12,46	14696,38	15,25	5,43	127	17456,46

Table 5.1: Table of Comparison Between: B&B_SCIP Vs Own_B&C_SCIP Using Real Graphs.

Instances			B&B_SCIP			Own_B&C_SCIP			
Topology	$\|K\|$	$\|\mathbb{S}\|$	**Nbr_Nd**	**Gap**	**TT**	**Nbr_Nd**	**Gap**	**Nbr_Cuts**	**TT**
Pioro40	10	40	1	**0,00**	0,02	1	**0,00**	0	0,03
	20	40	1	**0,00**	0,53	1	**0,00**	0	0,1
	30	40	1	**0,00**	3,74	1	**0,00**	5,5	0,57
	40	40	4	**0,00**	1,32	3	**0,00**	12,5	5,84
	50	80	5	**0,00**	2,66	1,25	**0,00**	15,75	13,52
	100	80	3	**0,00**	44,31	18,5	**0,00**	77,5	2769,13
	150	160	56	1,95	9335,82	57	**0,00**	48,75	9169,93
	200	280	1	0,14	4934,59	1,25	**0,00**	28,5	3023,14
	250	280	1	**0,00**	3782,08	1	**0,00**	73,5	2580
	300	320	4,25	0,18	10548,18	3,25	0,36	96	13502,49
India35	10	80	1	**0,00**	0,04	1	**0,00**	0	0,06
	20	40	1	**0,00**	0,08	1	**0,00**	0	0,14
	30	40	2	**0,00**	3,52	1,25	**0,00**	12	6,11
	40	80	4,5	**0,00**	4,43	1	**0,00**	0	3,82
	50	160	1	**0,00**	7,64	9,25	**0,00**	7	67,06
	100	240	13,5	1,55	13278,76	10,5	0,20	64,50	10572,62
	150	400	8	4,71	18000	15	5,18	89	18000
	200	280	1	10,58	13577,39	1,25	4,11	0,75	8531,99
	250	280	1	1,45	18000	1	0,72	61	18000
	300	320	1	1,8	16858,2	3	1,97	62,25	18000
Brain161	10	40	1	**0,00**	0,08	1	**0,00**	0,50	0,17
	20	40	1	**0,00**	0,04	1	**0,00**	0	0,09
	30	40	1	**0,00**	0,36	1	**0,00**	0	0,47
	40	80	6,75	**0,00**	11,91	5,50	**0,00**	26	18,12
	50	120	9	**0,00**	25,23	3	**0,00**	16,25	25,17
	100	160	65	**0,00**	3297,48	6	**0,00**	35	1009,43
	150	320	58,5	0,26	10284,04	43,25	0,27	148,25	12232,16
	200	400	8	0,40	12172,23	1,67	0,36	45,67	18000
	250	480	1	0,86	13492,92	1,67	0,33	52	18000
	300	320	1	1,30	18000	1	0,32	11,50	18000

Table 5.2: Table of Comparison Between: B&B_SCIP Vs Own_B&C_SCIP Using Realistic Graphs.

Conclusion

In this thesis, we have studied the Constrained-Routing and Spectrum Assignment (C-RSA) problem related to the dimensioning and designing of Spectrally Flexible Optical Networks (SFONs). It's well known to be NP-hard. The main aim of this thesis was to provide a deep polyhedral investigation and design a cutting plane method for the problem and handle large-scale instances.

First, we have proposed an integer linear programming formulation namely cut formulation. We have investigated the related polytope defined by the convex hull of all its solutions. Moreover, we have identified several classes of valid inequalities for the polytope and studied their facial structure. We further have discussed their separation problems. We have also proposed a primal heuristic to obtain tighter primal bounds and enhance the resolution of the problem. These results are used to devise a Branch-and-Cut (B&C) algorithm for the C-RSA problem, along with some computational results are presented using two types of instances: random and realistic ones with $|K|$ up to 300 and $|\mathbb{S}|$ up to 320. They are composed of two types of graphs (topologies): real graphs and realistic ones from SND-LIB with $|V|$ up to 161 and $|E|$ up to $|166|$. The results have shown the significant improvement allowed by introducing the valid inequalities on obtaining tighter LP bounds and improving the effectiveness of the B&C algorithm.

In the second part of thesis, we have discussed an extended formulation based on the so-called path formulation. It can be seen as a reformulation of the cut formulation using the so-called path variables. We have developed a column generation algorithm to solve its linear relaxation. We have shown that the pricing problem is equivalent to the resource-constrained shortest path problem, which is well known to be NP-hard. For this, we have developed a pseudo-polynomial algorithm based on dynamic programming enabled solving the pricing problem in polynomial time. Using this, we have devised Branch-and-Price and Branch-and-Cut-and-Price algorithms. The results show that the Branch-and-Cut-and-Price performs very well compared with the Branch-and-Price. Hence, the significant impact and the power of the introduced

valid inequalities allowed improving the effectiveness of the B&C&P algorithm. On the other, we have presented a comparative study between the B&C, B&P, and B&C&P algorithms. The results have shown that the B&C&P algorithm is able to provide optimal solutions for several instances, which is not the case for the B&C algorithm within the CPU time limit (5 hours). Moreover, both B&C and B&P algorithms perform well. However, some instances are still difficult to solve with both B&C, B&P and B&C&P algorithms. For this, some enhancements are further investigated and integrated into our algorithms. They are based on a warm-start algorithm using some metaheuristics, and a primal heuristic using a hybrid method between a greedy algorithm and local search algorithm that is shown to be very useful to obtain good primal bounds. Moreover, we introduce some symmetry-breaking inequalities that allow avoiding the equivalents sub-problems in the different enumeration trees of B&C, B&P, and B&C&P algorithms.

Afterward, we have studied the Spectrum Assignment (SA) sub-problem when the routing is trivial or a routing path is pre-selected for each demand. First, we have presented a compact formulation for the SA problem. We have carried out an investigation of the associated polytope. Moreover, we have identified several valid inequalities for the polytope, some of them come from those that are already proposed for the C-RSA. We have proved that they are facet defining under certain necessary and sufficient conditions. They were further incorporated within a Branch-and-Cut algorithm. The results have shown the efficiency of the valid inequalities allowed enhancing the resolution of the SA problem. Hence, the Branch-and-Cut is shown to be very performant compared with the Branch-and-Bound algorithm.

Finally, it would be interesting to further investigate a combination of the different algorithms with some machine learning and reinforcement learning algorithms to well manage the B&C, B&P, and B&C&P trees and particularly for

a) the node selection [27][36],

b) variable selection and branching rule [6][36],

c) column selection [39][111],

d) cut selection [54][110],

e) and provide a deeper comparative study between the algorithms [1].

Bibliography

[1] Accorsi, L., Lodi, A., Vigo, D.: Guidelines for the Computational Testing of Machine Learning approaches to Vehicle Routing Problems. In: https://arxiv.org/abs/2109.13983, 2021, pp. 1-11.

[2] Amar, D.: Performance assessment and modeling of flexible optical networks. Thesis, Institut National des Télécommunications 2016.

[3] Balas, E.: Facets of the knapsack polytope. In: Journal of Mathematical Programming 1975, pp. 146-164.

[4] Balas, E., Chvátal, V., and Nešetřil, J: On the Maximum Weight Clique Problem. In: Mathematics of Operations Research Journal, 1987, pp. 522-535.

[5] Balas, E. and Zemel, E.: Facets of the Knapsack Polytope From Minimal Covers. In: SIAM Journal on Applied Mathematics, 1978, pp. 119-148.

[6] Balcan, M.F., Dick, T., Sandholm, T., Vitercik, E.: Learning to Branch. In: Proceedings of the 35th International Conference on Machine Learning, PMLR 80:344-353, 2018, pp. 1-10.

[7] Barry, R., and Subramaniam, S.: The max sum wavelength assignment algorithm for wdm ring networks. In: Optical Fiber Communication Conference, 1997, pp. 121–122.

[8] Bendotti, P. , Fouilhoux, P., and Rottner, C.: Sub-symmetry-breaking inequalities and application to the Unit Commitment Problem. In: The 20th Conference on Integer Programming and Combinatorial Optimization May 22–24, 2019, Ann Arbor, Michigan, USA, pp. 1-27.

[9] Bermond, J.C., Moataz, F. Z.: On spectrum assignment in elastic optical tree-networks. In: Discrete Applied Mathematics Journal 2019, pp. 40-52.

[10] Bertero, F., and Bianchetti, M., and Marenco, J.: Integer programming models for the routing and spectrum allocation problem. In: Official Journal of the Spanish Society of Statistics and Operations Research 2018, pp. 465-488.

[11] Bianchetti, M., and Marenco, J.: Valid inequalities and a branch-and-cut algorithm for the routing and spectrum allocation problem. In: Proceedings of the XI Latin and American Algorithms, Graphs and Optimization Symposium, 2021, pp. 523-531.

[12] Brun, B., and Baraketi, S.: Routing and Wavelength Assignment in Optical Networks. In: https://hal.archives-ouvertes.fr/hal-01062321, 2014, pp. 1-41.

[13] Cai, A., Shen, G., Peng, L., and Zukerman, M.: Novel Node-Arc Model and Multiiteration Heuristics for Static Routing and Spectrum Assignment in Elastic Optical Networks. In: Journal of Lightwave Technology 2013, pp. 3402-3413.

[14] Carlyle, W.M., Royset, J. O., and Wood, R. K.: Lagrangian relaxation and enumeration for solving constrained shortest-path problems. In: Networks Journal 2008, pp. 256-270.

[15] Chlamtac, I., Ganz, A., and Karmi, G.: Lightpath communications: an approach to high bandwidth optical WAN's. In: IEEE Transactions on Communications, 1992, pp. 1171-1182.

[16] Chatterjee, B.C., Ba, S., and Oki, E.: Routing and Spectrum Allocation in Elastic Optical Networks: A Tutorial. In: IEEE Communications Surveys Tutorials 2015, pp. 1776-1800.

[17] Chatterjee, B.C., Ba, S., and Oki, E.: Fragmentation Problems and Management Approaches in Elastic Optical Networks: A Survey. In: IEEE Communications Surveys Tutorials 2018, pp. 183-210.

[18] Chen, X., Guo, J., Zhu, Z., Proietti, R., Castro, A., and Yoo, S. J. B.: Deep-RMSA: A Deep-Reinforcement-Learning Routing, Modulation and Spectrum Assignment Agent for Elastic Optical Networks. In: Optical Fiber Communications Conference and Exposition (OFC) 2018, pp. 1-3.

[19] Cheng, B., Hang, C., Hu, Y., Liu, S., Yu, J., Wang, Y., and Shen, J.: Routing and Spectrum Assignment Algorithm based on Spectrum Fragment Assessment of Arriving Services. In: 28th Wireless and Optical Communications Conference (WOCC) 2019, pp. 1-4.

[20] Chouman, H., Gravey, A., Gravey, P., Hadhbi, Y., Kerivin, H., Morvan, M, and Wagler, A.: Impact of RSA Optimization Objectives on Optical Network State. In: https://hal.uca.fr/hal-03155966.

[21] Chouman, H., Luay, A., Colares, R., Gravey, A, Gravey, P., Kerivin, H., Morvan, M., Wagler, A.: Assessing the Health of Flexgrid Optical Networks. In: https://hal.archives-ouvertes.fr/hal-03123302.

[22] Christodoulopoulos, K., Tomkos, I., and Varvarigos, E.A.: Elastic Bandwidth Allocation in Flexible OFDM-Based Optical Networks. In: Lightwave Technology 2011, pp. 1354-1366.

[23] The Network Cisco's Technology News Site: Cisco Predicts More IP Traffic in the Next Five Years Than in the History of the Internet. In: https://newsroom.cisco.com.

[24] Colares, R., Kerivin, H., and Wagler, A.: An extended formulation for the Constraint Routing and Spectrum Assignment Problem in Elastic Optical Networks. In: https://hal.uca.fr/hal-03156189, 2021.

[25] Cook, S. A.: The complexity of theorem-proving procedures. In Proceedings of the third annual ACM symposium on Theory of computing, 1971. pp. 151–158.

[26] Cplex, I.I., 2020. V12. 9: User's Manual for Cplex. International Business Machines Corporation, 46(53), pp. 157.

[27] Daumé, H.H, and Eisner, H. III. J.: Learning to search in branch-and-bound algorithms. In: Advances in Neural Information Processing Systems, 2014, pp. 1-11.

[28] Diestel, R.: Graph Theory (Graduate Texts in Mathematics). In: Graduate Texts in Mathematics Springer, Heidelberg; New York, Fourth edition, 2010.

[29] Dinarte, H.A., Bruno, V.A., Daniel, A.R.C, and Raul, C. A.: Routing and spectrum assignment: A metaheuristic for hybrid ordering selection in elastic optical networks. In: Computer Networks Journal, 2020, pp. 108287.

[30] Ding, Z., Xu, Z., Zeng, X., Ma, T., and Yang, F.: Hybrid routing and spectrum assignment algorithms based on distance-adaptation combined coevolution and heuristics in elastic optical networks. In: Journal of Optical Engineering 2014, pp. 1-10.

[31] Dror, M.: Note on the Complexity of the Shortest Path Models for Column Generation in VRPTW. In: Journal of Operations Research 1994, pp. 977-978.

[32] Dumitrescu, I., and Boland, N.: Algorithms for the weight constrained shortest path problem. In: International Transactions in Operational Research, pp. 15-29.

[33] Edmonds, J.: Covers and packings in a family of sets. In: Bulletin of the American Mathematical Society, 68(5), 1962, pp. 494–499.

[34] Enoch, J.: Nested Column Generation decomposition for solving the Routing and Spectrum Allocation problem in Elastic Optical Networks. In: http://arxiv.org/abs/2001.00066, 2020.

[35] Eppstein, D.: Finding the k shortest paths. In: 35th Annual Symposium on Foundations of Computer Science, pp. 154-165.

[36] Etheve, M., Alès, Z., Bissuel, C., O. Juan, Kedad-Sidhoum, S.: On Learning Node Selection in a Branch and Bound Algorithm. In: ROADEF Conference, April, 2021, pp. 1-3.

[37] Fayez, M., Katib, I., George, N.R., Gharib, T.F., Khaleed H., and Faheem, H.M.: Recursive algorithm for selecting optimum routing tables to solve offline routing and spectrum assignment problem. In: Ain Shams Engineering Journal 2020, pp. 273-280.

[38] Ford, L. R., and Fulkerson, D. R.: Maximal flow through a network. In: Canadian Journal of Mathematics 8, pp. 399–404, 1956.

[39] Furian, N., O'Sullivan, M., Walker, C., and Çela, E.: A machine learning-based branch and price algorithm for a sampled vehicle routing problem. OR Spectrum 43, 2021, pp. 693-732.

[40] Friedman, E.J.: Fundamental Domains for Integer Programs with Symmetries. In: Combinatorial Optimization and Applications book published in Springer Berlin Heidelberg, 2007, pp. 146-153.

[41] Garey, M. R. , and Johnson, D. S.: Computers and Intractability: A Guide to the ory of Np-completeness, In: W. H. Freeman Co. publisher, volume 174, freeman New York, 1979.

[42] Gherboudj, A.: Méthodes de résolution de problémes difficiles académiques. In: PhD thesis at Constantine University, 2013.

[43] Gong, L., Zhou, X., Lu, W., and Zhu, Z.: A Two-Population Based Evolutionary Approach for Optimizing Routing, Modulation and Spectrum Assignments (RMSA) in O-OFDM Networks. In: IEEE Communications Letters 2012, pp. 1520-1523.

[44] Goldberg, A.V., and Tarjan, R. E.: A New Approach to the Maximum Flow Problem. In: Proceedings of the Eighteenth Annual Association for Computing Machinery Symposium on Theory of Computing 1986, pp. 136-146.

[45] Golumbic, M.C.: Algorithmic Graph Theory and Perfect Graphs. In: Annals of Discrete Mathematics, 29th January 2004, pp. 1-314.

[46] Goscien, R., Walkowiak, K., and Klinkowski, M.: Tabu search algorithm, Routing, Modulation and spectrum allocation, Anycast traffic, Elastic optical networks. In: Journal of Computer Networks 2015, pp. 148-165.

[47] Grötschel, M., Lovász, L., and Schrijver, A.: Geometric Algorithms and Combinatorial Optimization. In: Springer 1988.

[48] Gu, R., Yang, Z., and Ji, Y.: Machine Learning for Intelligent Optical Networks: A Comprehensive Survey. In : Journal CoRR 2020, pp. 1-42.

[49] Gurobi Optimization, LLC.: Gurobi Optimizer Reference Manual. In: https://www.gurobi.com, 2021.

[50] Hadhbi, Y., Kerivin, H., and Wagler, A.: A novel integer linear programming model for routing and spectrum assignment in optical networks. In: Federated Conference on Computer Science and Information Systems (FedCSIS) 2019, pp. 127-134.

[51] Hadhbi, Y., Kerivin, H., and Wagler, A.: Routage et Affectation Spectrale Optimaux dans des Réseaux Optiques Élastiques FlexGrid. In: Journées Polyédres et Optimisation Combinatoire (JPOC-Metz) 2019, pp. 1-4.

[52] Hai, D.H., and Hoang, K.M.: An efficient genetic algorithm approach for solving routing and spectrum assignment problem. In: Journal of Recent Advances in Signal Processing 2017.

[53] Hai, D. H., Morvan, M., and Gravey, P.: Combining heuristic and exact approaches for solving the routing and spectrum assignment problem. In: Journal of Iet Optoelectronics 2017, pp. 65-72.

[54] Huang, Z., Wang, K., Liu, F., Zhen, H.L., Zhang, W., Yuan, M., Hao, J., Yu, Y., Wang, J.: Learning to Select Cuts for Efficient Mixed-Integer Programming. In: CoRR abs/2105.13645 (2021) text to speech, 2020, pp. 1-33.

[55] He, S., Qiu, Y., and Xu, J.: Invalid-Resource-Aware Spectrum Assignment for Advanced-Reservation Traffic in Elastic Optical Network. In: Sensors 2020.

[56] Jaumard, B., and Daryalal, M.: Scalable elastic optical path networking models. In: 18th International Conference Transparent Optical Networks (ICTON) 2016, pp. 1-4.

[57] Jinno, M., Takara, H., Kozicki, B., Tsukishima, Y., Yoshimatsu, T., Kobayashi, T., Miyamoto, Y., Yonenaga, K., Takada, A., Ishida, O., and Matsuoka, S.: Demonstration of novel spectrum-efficient elastic optical path network with per-channel variable capacity of 40 Gb/s to over 400 Gb/s. In: 34th European Conference on Optical Communication 2008.

[58] Joksch, H.C.: The shortest route problem with constraints. In: Journal of Mathematical Analysis and Applications, pp. 191 - 197.

[59] https://lemon.cs.elte.hu/trac/lemon.

[60] Lezama, F., Martinez-Herrera, A.F., Castanon, G., Del-Valle-Soto, C., Sarmiento, A.M., Munoz de Cote, A. : Solving routing and spectrum allocation problems in flexgrid optical networks using pre-computing strategies. In: Journal of Photon Netw Commun 41, pp. 17-35.

[61] Liu, L., and Yin, S., and Zhang, Z., and Chu, Y., and Huang, S.: A Monte Carlo Based Routing and Spectrum Assignment Agent for Elastic Optical Networks. In: Asia Communications and Photonics Conference (ACP) 2019, pp. 1-3.

[62] Lohani, V., Sharma, A., and Singh, Y.N.: Routing, Modulation and Spectrum Assignment using an AI based Algorithm. In: 11th International Conference on Communication Systems & Networks (COMSNETS) 2019, pp. 266-271.

[63] Lopez, V., and Velasco, L.: Elastic Optical Networks: Architectures, Technologies, and Control. In: Springer Publishing Company, Incorporated 2016.

[64] Lozano, L., and Medaglia, A.L.: On an exact method for the constrained shortest path problem. In: Journal of Computers Operations Research, pp. 378-384.

[65] Mahala, N., and Thangaraj, J.: Spectrum assignment technique with first-random fit in elastic optical networks. In : 4th International Conference on Recent Advances in Information Technology (RAIT) 2018, pp. 1-4.

[66] Margot, F.: Symmetry in integer linear programming. In: 50 Years of Integer Programming 1958-2008, Springer, 2010, pp. 647–686.

[67] Margot, F.: Pruning by isomorphism in branch-and-cut. In: Mathematical Programming 2002, pp. 71–90.

[68] Margot, F.: Exploiting orbits in symmetric ilp. In: Mathematical Programming 2003, pp. 3–21.

[69] Méndez-Díaz, I., and Zabala, P.: A Polyhedral Approach for Graph Coloring[1]. In: Electronic Notes in Discrete Mathematics, 2001, pp. 178-181.

[70] Méndez-Díaz, I. and Zabala, P.: A Branch-and-Cut algorithm for graph coloring. In: Discrete Applied Mathematics Journal 2006, pp. 826-847.

[71] Mesquita, L.A.J., Assis, K., Santos, A. F., Alencar, M., and Almeida, R.C.: A Routing and Spectrum Assignment Heuristic for Elastic Optical Networks under Incremental Traffic. In: SBFoton International Optics and Photonics Conference (SBFoton IOPC) 2018, pp. 1-5.

[72] Nemhauser, G.L., and Wolsey, L.A.: Integer and Combinatorial Optimization. In: John Wiley 1988.

[73] Nemhauser, G. L., and Sigismondi, G.: A Strong Cutting Plane/Branch-and-Bound Algorithm for Node Packing. In: The Journal of the Operational Research Society 1992, pp. 443-457.

[74] Orlowski, S., Pióro, M., Tomaszewski, A. , and Wessäly, R.: SNDlib 1.0-Survivable Network Design Library. In: Proceedings of the 3rd International Network Optimization Conference (INOC 2007), Spa, Belgium, http://www.zib.de/orlowski/Paper/OrlowskiPioroTomaszewskiWessaely2007-SNDlib-INOC.pdf.gz.

[75] Ostrowski, J., Anjos, M. F., and Vannelli, A.: Symmetry in scheduling problems. In: Citeseer 2010.

[76] Ostrowski, J., Linderoth, J., Rossi, F., and Smriglio, S.: Orbital branching. In: Mathematical Programming 2011, pp. 147–178.

[77] Padberg, M.W. : On the facial structure of set packing polyhedra. In: Journal of Mathematical Programming 1973, pp. 199-215.

[78] Patel, B., Ji, H., Nayak, S., Ding, T., Pan, Y. and Aibin, M.: On Efficient Candidate Path Selection for Dynamic Routing in Elastic Optical Networks. In: 11th IEEE Annual Ubiquitous Computing 2020, pp. 889-894.

[79] Kaibel, V., and Pfetsch, M. E.: Packing and partitioning orbitopes. In: Mathematical Programming 2008, pp. 1–36.

[80] Kaibel, V., Peinhardt, M., and Pfetsch, M. E.: Orbitopal fixing. In: Discrete Optimization 2011, pp. 595–610.

[81] Karp, R. M.: Reducibility among Combinatorial Problems. In: Complexity of Computer Computations: Proceedings of a symposium on the Complexity of Computer Computations, held March 20–22, 1972, at the IBM Thomas J. Watson Research Center 1972, pp. 85-94.

[82] Klabjan, D., Nemhauser, G.L., and Tovey, C.: The complexity of cover inequality separation. In: Journal of Operations Research Letters 1998, pp. 35-40.

[83] Klinkowski, M., Pedro, J., Careglio, D., Pioro, M., Pires, J., Monteiro, P., and Sole-Pareta, J.: An overview of routing methods in optical burst switching networks. In: Optical Switching and Networking 2010, pp. 41 - 53.

[84] Klinkowski, M., and Walkowiak, K.: Routing and Spectrum Assignment in Spectrum Sliced Elastic Optical Path Network. In: IEEE Communications Letters 2011, pp. 884-886.

[85] Klinkowski, M., Careglio, D.: A routing and spectrum assignment problem in optical OFDM networks. In: European Teletraffic Seminar. "First European Teletraffic Seminar". Poznan: 2011, pp. 1-5.

[86] Klinkowski, M., and Walkowiak, K.: Offline RSA algorithms for elastic optical networks with dedicated path protection consideration. In: 2012 IV International Congress on Ultra Modern Telecommunications and Control Systems, 2012, pp. 670-676.

[87] Klinkowski, M., Pioro, M., Zotkiewicz, M., Ruiz, M., and Velasco, L.: Valid inequalities for the routing and spectrum allocation problem in elastic optical networks. In: 16th International Conference on Transparent Optical Networks (ICTON) 2014, pp. 1-5.

[88] Klinkowski, M., Pioro, M., Zotkiewicz, M., Ruiz, M., and Velasco, L.: A Simulated Annealing Heuristic for a Branch and Price-Based Routing and Spectrum Allocation Algorithm in Elastic Optical Networks. In: Intelligent Data Engineering and Automated Learning – IDEAL 2015, Springer International Publishing, pp. 290-299.

[89] Klinkowski, M., Pioro, M., Zotkiewicz, M., Walkowiak, Ruiz, M., and Velasco, L.: Spectrum allocation problem in elastic optical networks - a branch-and-price approach. In: 17th International Conference on Transparent Optical Networks (ICTON), 2015, pp. 1-5.

[90] Klinkowski, M., Zotkiewicz, M., Walkowiak, K., Pioro, M., Ruiz, M., and Velasco, L.: Solving large instances of the RSA problem in flexgrid elastic optical networks. In: Photonic Network Communications Journal, 2016, pp. 320-330.

[91] Klinkowski, M., and Walkowiak, K.: On Performance Gains of Flexible Regeneration and Modulation Conversion in Translucent Elastic Optical Networks With Superchannel Transmission. In: Journal of Lightwave Technology, 2016, pp. 5485-5495.

[92] Ramaswami, R.: Optical Networks: A Practical Perspective, 3rd Edition. In: Morgan Kaufmann Publishers Inc. 2009.

[93] Ramaswami, R., Sivarajan, K., and Sasaki, G.: Multiwavelength lightwave networks for computer communication. In: IEEE Communications Magazine 1993, pp. 78-88.

[94] Rebennack, S.: Stable Set Problem: Branch & Cut Algorithms. In: Encyclopedia of Optimization Book 2009.

[95] Rebennack, S., Reinelt, G., and Pardalos, P.M.: A tutorial on branch and cut algorithms for the maximum stable set problem. In: Journal of International Transactions in Operational Research 2012, pp. 161-199.

[96] Reihani, A.Z, Behdadfar, M., and Sebghati, M.: Artificial neural network-based adaptive modulation for elastic optical networks. In: Internet Technology Letters Journal, 11 April 2021, pp. 1-6.

[97] Ruiz, M., Pioro, M., Zotkiewicz, M., Klinkowski, M., Velasco, L.: A hybrid meta-heuristic approach for optimization of routing and spectrum assignment in Elastic Optical Network (EON). In: Journal of Enterprise Information Systems 2020, pp. 11-24.

[98] Ruiz, M., Pioro, M., Zotkiewicz, M., Klinkowski, M., Velasco, L.: Column generation algorithm for RSA problems in flexgrid optical networks. In: Photonic Network Communications 2013, pp. 53-64.

[99] Ryan, D. M. and Foster, B. A.: An integer programming approach to scheduling. In A. Wren (editor), Computer Scheduling of Public Transport Urban Passenger Vehicle and Crew Scheduling, North-Holland, Amsterdan, 1981, pp. 269-280.

[100] Salameh, B.H., Qawasmeh, R., and Al-Ajlouni, A.F.: Routing With Intelligent Spectrum Assignment in Full-Duplex Cognitive Networks Under Varying Channel Conditions. In: Journal of IEEE Communications Letters 2020, pp. 872-876.

[101] Salani, M., Rottondi, C., and Tornatore, M.: Routing and Spectrum Assignment Integrating Machine-Learning-Based QoT Estimation in Elastic Optical Networks. In: IEEE INFOCOM - IEEE Conference on Computer Communications 2019, pp. 173846.

[102] Santos, A.F.D, Assis, K., and Guimarães, M. A., and Hebraico, R.: Heuristics for Routing and Spectrum Allocation in Elastic Optical Path Networks. In: 2015, Journal Of Modern Engineering Research (IJMER), pp. 1-13.

[103] Gamrath, G., Anderson, D., Bestuzheva, K., Chen, W.K., Eifler, L., Gasse, M., Gemander, P., Gleixner, A., Gottwald, L., Halbig, K., and Hendel, G., and Hojny, C., Koch, T., Bodic, L., Maher, P. J., Matter, F., Miltenberger, M., Mühmer, E., Müller, B., Pfetsch, M.E., Schlösser, F., Serrano, F., Shinano, Y., Tawfik, C., Vigerske, S., Wegscheider, F., Weninger, D., and Witzig, J.: The Scip Optimization Suite 7.0. In: http://www.optimization-online.org/DB_HTML/2020/03/7705.html, March 2020.

[104] Schrijver, A.: Theory of Linear and Integer Programming. In: John Wiley & Sons, Chichester 1986.

[105] Schrijver, A.: Combinatorial Optimization - Polyhedra and Efficiency. In: Springer-Verlag 2003.

[106] Selvakumar, S., and Manivannan, S. S.: The Recent Contributions of Routing and Spectrum Assignment Algorithms in Elastic Optical Network (EON). In: International Journal of Innovative Technology and Exploring Engineering (IJITEE) 2020, pp. 1-11.

[107] Shirazipourazad, S., Zhou, C., Derakhshandeh, Z., and Sen, A.: On routing and spectrum allocation in spectrum-sliced optical networks. In: Proceedings IEEE INFOCOM 2013, pp. 385-389.

[108] Stern, T., Ellinas, G., and Bala, K.: Multiwavelength Optical Networks: Architectures, Design and Control (2nd ed.). In: Cambridge: Cambridge University Press, 2008.

[109] Talebi, S., Alam, F., Katib, I., Khamis, M., Salama, R., and Rouskas, G. N.: Spectrum management techniques for elastic optical networks: A survey. In: Optical Switching and Networking 2014.

[110] Tang, Y., Agrawal, S., and Faenza, Y.: Reinforcement Learning for Integer Programming: Learning to Cut. In: https://arxiv.org/abs/1906.04859, 2020, pp. 1-18.

[111] Václavík, R., Novák, A., Šůcha, P., and Hanzálek, Z.: Accelerating the Branch-and-Price Algorithm Using Machine Learning. In: European Journal of Operational Research, Volume 271, Issue 3, 2018, pp. 1055-1069.

[112] Velasco, L., Klinkowski, M., Ruiz, M., and Comellas, J.: Modeling the routing and spectrum allocation problem for flexgrid optical networks. In: Photonic Network Communications 2012, pp. 177-186.

[113] Walkowiak, K., and Aibin, M.: Elastic optical networks - a new approach for effective provisioning of cloud computing and content-oriented services. In: Przeglad Telekomunikacyjny + Wiadomosci Telekomunikacyjne 2015.

[114] Wan, X., Hua, N., and Zheng, X.: Dynamic Routing and Spectrum Assignment in Spectrum-Flexible Transparent Optical Networks. In: Journal of Optical Communications and Networking 2012, pp. 603-613.

[115] Wolsey, L.A.: Integer programming. In: Wiley-Interscience series in discrete mathematics and optimization, 1998.

[116] Xuan, H., Wang, Y., Xu, Z., Hao, S., and Wang, X.: New bi-level programming model for routing and spectrum assignment in elastic optical network. In: Opt Quant Electron 49-2017, pp. 1-16.

[117] Zhang, Y., Xin, J., and Li, X., and Huang, S.: Overview on routing and resource allocation based machine learning in optical networks. In: Journal of Optical Fiber Technology, pp. 1-21.

[118] Zhao, J.: Maximum Bounded Rooted-Tree Problem : Algorithms and Polyhedra. In: PhD thesis, June 2017, https://tel.archives-ouvertes.fr/tel-01730182.

[119] Zhou, Y., Sun, Q., and Lin, S.: Link State Aware Dynamic Routing and Spectrum Allocation Strategy in Elastic Optical Networks. In: IEEE Access 2020, pp. 45071-45083.

[120] Ziazet, J.M., and Jaumard, B.: Reinforcement Learning for Routing, Modulation And Spectrum Assignment Problem in Elastic Optical Networks. In: Partial Fulfillment of a Cooperative Masters Degree in Industrial Mathematics at AIMS-Cameroon, December 20, 2019, pp. 1-56.

[121] Zotkiewicz, M., Pioro, M., Ruiz, M., Klinkowski, M., and Velasco, L.: Optimization models for flexgrid elastic optical networks. In: 15th International Conference on Transparent Optical Networks (ICTON) 2013, pp. 1-4.