Descent in p-adic Hodge theory and applications

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Descente en théorie de Hodge p-adique et applications

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THESE DE DOCTORAT

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Titre : Descente en théorie de Hodge p-adique et applications
Mots clés : théorie de Hodge p-adique, descente, topos de Faltings, opérateur de Sen, presque-cohérence.
Résumé :

Nous développons et étudions trois instances de la descente en théorie de Hodge p-adique.

Dans la première partie, nous établissons une descente cohomologique pour la théorie de Hodge p-adique de Faltings. L'approche de Faltings de la théorie de Hodge p-adique peut être schématiquement divisée en deux étapes principales : premièrement, une réduction locale du calcul de la cohomologie étale p-adique d'une variété lisse sur un corps local p-adique à un calcul de cohomologie galoisienne puis, l'établissement d'un lien entre ce dernier et les différentielles. Ces relations sont organisées au travers du topos annelé de Faltings dont la définition dépend du choix d'un modèle entier de la variété, et dont les bonnes propriétés dépendent de la lissité (logarithmique) de ce modèle. Notre résultat de descente cohomologique pour le faisceau structural du topos de Faltings permet d'étendre l'approche de Faltings à tout modèle entier, c'est-à-dire sans hypothèse de lissité. Un ingrédient essentiel de notre preuve est un résultat de descente d'algèbres perfectoïdes en topologie des arcs dû à Bhatt-Scholze. Comme première application de notre descente cohomologique, en utilisant une variante du théorème d'altération de de Jong pour les morphismes de schémas due à Gabber-Illusie-Temkin, nous généralisons le principal théorème de comparaison p-adique de Faltings à tout morphisme propre et de présentation finie de schémas cohérents sur une clôture intégrale absolue de $\mathbb{Z}_p$ (sans aucune hypothèse de lissité) pour des faisceaux étales de torsion (pas nécessairement localement constants constructibles). Comme deuxième application, nous prouvons une version locale de la filtration relative de Hodge-Tate comme conséquence de la version globale construite par Abbes-Gros.

Dans la deuxième partie, nous étendons la théorie de Sen aux variétés affines p-adiques admettant des cartes semi-stables. Toute représentation p-adique de dimension finie du groupe de Galois absolu d'un corps local p-adique à corps résiduel imparfait est caractérisée par ses opérateurs arithmétique et géométriques de Sen définis par Sen et Brinon. Nous généralisons leur construction au groupe fondamental d'une variété affine p-adique admettant une carte semi-stable, et donnons une formulation canonique de la théorie de Sen indépendamment du choix de la carte, ce qui est nouveau même dans le cas des corps locaux. Notre construction dépend d'un théorème de descente pour la correspondance de Simpson p-adique développée par Tsuji. Lorsque la représentation provient d'une $\mathbb{Q}_p$-représentation d'un groupe analytique p-adique quotient du groupe fondamental, nous décrivons l'action de son algèbre de Lie en termes d'opérateurs de Sen. C'est une généralisation d'un résultat de Sen et Ohkubo. Ces opérateurs de Sen peuvent être étendus continûment à certaines représentations de dimension infinie. Comme application, nous prouvons que les opérateurs géométriques de Sen annulent les vecteurs localement analytiques, généralisant un résultat de Pan.

Dans la troisième partie, pour un morphisme propre, plat et de présentation finie entre schémas à faisceaux structuraux presque-cohérents (au sens de Faltings), nous prouvons que les images directes supérieures de modules quasi-cohérents et presque-cohérents sont quasi-cohérents et presque-cohérents. Notre preuve utilise une approximation noethérienne dans le contexte de la presque-algèbre de Faltings, inspirée de la preuve de Kiehl de la pseudo-cohérence des images directes supérieures. Notre résultat nous permet d'étendre la preuve d'Abbes-Gros du principal théorème de comparaison p-adique de Faltings dans le cas relatif des morphismes log-lisses projectifs de schémas au cas des morphismes log-lisses propres, et donc aussi leur construction de la suite spectrale de Hodge-Tate relative.

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Title : Descent in p-adic Hodge theory and applications

Keywords : p-adic Hodge theory, descent, Faltings topos, Sen operator, almost coherence.

Abstract :

We develop and study three instances for descent in p-adic Hodge theory.

In the first part, we establish a cohomological descent for Faltings' p-adic Hodge theory. Faltings' approach in p-adic Hodge theory can be schematically divided into two main steps: firstly, a local reduction of the computation of the p-adic étale cohomology of a smooth variety over a p-adic local field to a Galois cohomology computation and then, the establishment of a link between the latter and differential forms. These relations are organized through Faltings ringed topos whose definition relies on the choice of an integral model of the variety, and whose good properties depend on the (logarithmic) smoothness of this model. Our cohomological descent result for the structural sheaf of Faltings topos makes it possible to extend Faltings' approach to any integral model, i.e. without any smoothness assumption. An essential ingredient of our proof is a descent result of perfectoid algebras in the arc-topology due to Bhatt-Scholze. As an application of our cohomological descent, using a variant of de Jong's alteration theorem for morphisms of schemes due to Gabber-Illusie-Temkin, we generalize Faltings' main p-adic comparison theorem to any proper and finitely presented morphism of coherent schemes over an absolute integral closure of $\mathbb{Z}_p$ (without any assumption of smoothness) for torsion étale sheaves (not necessarily finite locally constant). As a second application, we prove a local version of the relative Hodge-Tate filtration as a consequence of the global version constructed by Abbes-Gros.

In the second part, we extend Sen's theory to p-adic affine varieties with semi-stable charts. Any finite-dimensional p-adic representation of the absolute Galois group of a p-adic local field with imperfect residue field is characterized by its arithmetic and geometric Sen operators defined by Sen and Brinon. We generalize their construction to the fundamental group of a p-adic affine variety with a semi-stable chart, and give a canonical formation of Sen's theory independently of the choice of the chart, which is even new in the case of local fields. Our construction relies on a descent theorem in the p-adic Simpson correspondence developed by Tsuji. When the representation comes from a $\mathbb{Q}_p$-representation of a p-adic analytic group quotient of the fundamental group, we describe its Lie algebra action in terms of the Sen operators, which is a generalization of a result of Sen and Ohkubo. These Sen operators can be extended continuously to certain infinite-dimensional representations. As an application, we prove that the geometric Sen operators annihilate locally analytic vectors, generalizing a result of Pan.

In the third part, for a flat proper morphism of finite presentation between schemes with almost coherent structural sheaves (in the sense of Faltings), we prove that the higher direct images of quasi-coherent and almost coherent modules are quasi-coherent and almost coherent. Our proof uses a Noetherian approximation in the context of Faltings' almost algebra, inspired by Kiehl's proof of the pseudo-coherence of higher direct images. Our result allows us to extend Abbes-Gros' proof of Faltings' main p-adic comparison theorem in the relative case for projective log-smooth morphisms of schemes to proper ones, and thus also their construction of the relative Hodge-Tate spectral sequence.

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Contents

Introduction Générale

0.1. Aperçu

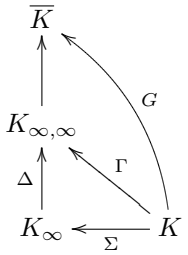
0.1.1. Le fil conducteur de cette thèse est celui de la *descente en théorie de Hodge p-adique*. Nous en développons et en étudions trois instances correspondant aux trois chapitres de cette thèse :

(1) une descente cohomologique pour le topos annelé de Faltings, ce qui nous permet de libérer l'approche de Faltings de la théorie de Hodge p-adique de l'hypothèse d'existence de bons modèles entiers ;
(2) une descente de représentations du groupe fondamental d'une variété p-adique, ce qui nous permet d'étendre canoniquement la théorie de Sen aux variétés affines p-adique admettant des cartes semi-stables ;
(3) une approximation noethérienne dans la contexte de la presque-algèbre de Faltings, ce qui nous permet de prouver la presque-cohérence d'images directes supérieures généralisant un résultat d'Abbes-Gros.

0.1.2. Commençons par la descente de représentations galoisiennes et la théorie de Sen. Soient K un corps de valuation discrète complet extension de $\mathbb{Q}_p$, $\overline{K}$ une clôture algébrique de K, $\widehat{\overline{K}}$ le complété p-adique de $\overline{K}$, G le groupe de Galois de $\overline{K}$ sur K. Lorsque le corps résiduel de K est parfait, pour toute $\widehat{\overline{K}}$-représentation W de G de dimension finie (semi-linéaire et continue), Sen **[Sen81]** lui associe un endomorphisme canonique $\widehat{\overline{K}}$-linéaire de W, appelé *l'opérateur (arithmétique) de Sen*, qui détermine la classe d'isomorphisme de la représentation W de G. Lorsque le corps résiduel de K est imparfait et admet une p-base de cardinal d, Brinon **[Bri03]** y ajoute d *opérateurs géométriques de Sen* pour déterminer la classe d'isomorphisme de la représentation W de G. Cependant, ces opérateurs (y compris l'opérateur arithmétique) dépendent du choix de la p-base, et ne sont donc pas canoniques si $d \geq 1$.

0.1.3. La construction de Sen et Brinon repose sur un théorème de descente des représentations. Plus précisément, choisissons $t_1, \ldots, t_d \in \mathcal{O}_K$ des relèvements d'une p-base du corps résiduel (que nous appellerons une système de coordonnées). On fixe également un système compatible de racines primitives p^n-ièmes de l'unité $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}}$ et un système compatible de racines p^n-ièmes $(t_{i,p^n})_{n\in\mathbb{N}}$ de t_i pour tout $1 \leq i \leq d$. Nous posons également $t_{0,p^n} = \zeta_{p^n}$ par souci de cohérence. Pour tous $n, m \in \mathbb{N} \cup \{\infty\}$, considérons l'extension de corps $K_{n,m} = K(\zeta_{p^n}, t_{1,p^m}, \ldots, t_{d,p^m})$ de K contenu dans $\overline{K}$. On pose simplement $K_{n,0} = K_n$ et on nomme les groupes de Galois comme indiqué dans le diagramme suivant

(0.1.3.1)



Alors, toute $\widehat{\overline{K}}$-représentation de dimension finie W de G se descend en une $K_{\infty,\infty}$-représentation V de Γ par un théorème de Sen et Brinon, au sens où il existe une équivalence de catégories

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma, K_{\infty,\infty}) \xrightarrow{\sim} \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}}),\ V \mapsto \widehat{\overline{K}} \otimes_{K_{\infty,\infty}} V. \tag{0.1.3.2}$$

Le groupe topologique $\Gamma = \Sigma \ltimes \Delta$ est localement isomorphe à $\mathbb{Z}_p \ltimes \mathbb{Z}_p^d$, et son algèbre de Lie $\mathrm{Lie}(\Gamma)$ sur $\mathbb{Q}_p$ est donc de dimension $1 + d$. L'action infinitésimale de $\mathrm{Lie}(\Gamma)$ sur V s'étend alors $\widehat{\overline{K}}$-linéairement en une action (non-canonique) de l'algèbre de Lie $\mathrm{Lie}(\Gamma)$ sur W, qui définit les $1 + d$ opérateurs de W construits par Brinon.

0.1.4. A priori, dès que l'on change le système de cordonnées $t_1, \dots, t_d$, la sous-représentation $V \subseteq W$ définie par descente change, ainsi que les $1+d$ opérateurs associés sur W. La première question est de savoir si on peut définir une action d'algèbre de Lie canonique sur W, qui redonnerait les opérateurs de Sen définis par Brinon une fois une base choisie. Nous y répondons positivement dans la second partie (Chapitre II) de cette thèse en considérant *l'extension de Faltings* de $\mathcal{O}_K$ définie dans [**He21**], c'est-à-dire une suite exacte canonique de $\widehat{\overline{K}}$-représentations de dimension finie de G,

$$0 \longrightarrow \widehat{\overline{K}}(1) \xrightarrow{\iota} \mathscr{E}_{\mathcal{O}_K} \xrightarrow{\jmath} \widehat{\overline{K}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow 0, \tag{0.1.4.1}$$

dans laquelle $\widehat{\overline{K}}(1)$ désigne le premier twist de Tate de $\widehat{\overline{K}}$, $\mathscr{E}_{\mathcal{O}_K} = \varprojlim_{x \mapsto px} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ est un $\widehat{\overline{K}}$-espace de dimension $1+d$ ayant une base $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\leq i\leq d}$. Cette suite se déduit de la suite fondamentale des modules des différentielles associés à la tour $\mathcal{O}_{\overline{K}}/\mathcal{O}_K/\mathbb{Z}_p$. En prenant les duaux et les twists de Tate, on obtient une suite exacte canonique

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{\overline{K}}) \xrightarrow{\jmath^*} \mathscr{E}^*_{\mathcal{O}_K}(1) \xrightarrow{\iota^*} \widehat{\overline{K}} \longrightarrow 0 \tag{0.1.4.2}$$

où $\mathscr{E}^*_{\mathcal{O}_K} = \mathrm{Hom}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}, \widehat{\overline{K}})$. Il existe une structure de $\widehat{\overline{K}}$-algèbre de Lie canonique sur $\mathscr{E}^*_{\mathcal{O}_K}(1)$ associée à la forme linéaire ι^* définie par $[f_1, f_2] = \iota^*(f_1)f_2 - \iota^*(f_2)f_1$ pour tous $f_1, f_2 \in \mathscr{E}^*_{\mathcal{O}_K}(1)$. Ce sera l'algèbre de Lie canonique remplaçant $\mathrm{Lie}(\Gamma)$. Intuitivement, le module $\mathscr{E}_{\mathcal{O}_K} = \varprojlim_{x \mapsto px} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ unifie tous les systémes de cordonnées $t_1, \dots, t_d$, et si on fixe un système, alors on obtient un isomorphisme non-canonique d'algèbres de Lie $\mathscr{E}^*_{\mathcal{O}_K}(1) \cong \mathrm{Lie}(\Gamma)$. Au Chapitre II, nous construisons une action de l'algèbre de Lie canonique $\mathscr{E}^*_{\mathcal{O}_K}(1)$ sur chaque $\widehat{\overline{K}}$-représentation de dimension finie W of G, ce qui fournit une construction canonique des opérateurs de Sen de W.

0.1.5. Nous étendons en fait la construction des opérateurs de Sen aux variétés p-adiques au Chapitre II. Tout d'abord, nous avons besoin de *l'extension de Faltings* pour les variétés p-adiques. Celle-ci a été construite par Faltings [**Fal88**] en utilisant la suite fondamentale des modules des différentielles, afin de relier la cohomologie étale et les différentielles dans sa preuve de la décomposition de Hodge-Tate. La construction de Faltings s'étend aux variétés logarithmiques. A priori, cela dépend du choix d'une carte pour la structure logarithmique. Néanmoins, nous prouvons au Chapitre II que la construction est en fait canonique en se réduisant aux extensions de Faltings canoniques sur les corps locaux, où l'un des principaux ingrédients est la descente des algèbres perfectoïdes en topologie des arcs due à Bhatt-Scholze [**BS19**]. Nous remarquons qu'Abbes-Gros [**AGT16**] ont utilisé une autre approche pour une construction canonique de l'extension de Faltings via le torseur de Higgs-Tate. Ensuite, on construit une action d'algèbre de Lie du dual de l'extension de Faltings sur les représentations du groupe fondamental d'une variété p-adique par un théorème de Tsuji [**Tsu18**] (généralisant celui de Sen et Brinon), dont la canonicité est vérifiée en se réduisant au cas des corps locaux.

0.1.6. La descente d'algèbres perfectoïdes en topologie des arcs est également l'un des ingrédients clés de la première partie (Chapitre I) de cette thèse. Le but de cette dernière est de rendre l'approche de Faltings de la théorie de Hodge p-adique « *libre des modèles entiers* » . Rappelons qu'une pierre angulaire de l'approche de Faltings est le *topos annelé de Faltings*, qui est construit pour être un pont entre la cohomologie étale p-adique et les différentielles d'une variété p-adique. Cependant, la définition du topos de Faltings dépend du choix d'un modèle entier de la variété, et les bonnes propriétés du topos de Faltings dépendent de la lissité (logarithmique) de ce modèle. Pour libérer l'approche de Faltings de l'hypothèse de lissité sur le modèle entier, nous établissons un résultat de *descente cohomologique* pour le topos annelé de Faltings en utilisant la descente d'algèbres perfectoïdes en topologie des arcs. Nous introduisons pour cela une variante du topos de Faltings pour la topologie v, satisfaisant de bonnes propriétés de descente cohomologique et qui peut être considérée comme un analogue schématique du topos v d'un espace adique. En particulier, nous établissons un résultat de descente cohomologique de ce topos au topos de Faltings. C'est un analogue de la descente cohomologique du topos v au topos pro-étale d'un espace adique établi par Scholze [**Sch17**].

Nous donnons deux applications de notre résultat de descente cohomologique. Tout d'abord, nous étendons le principal théorème de comparaison p-adique de Faltings, à la fois dans le cas absolu et dans le cas relatif, aux modèles entiers généraux sans aucune condition de lissité. Le théorème de comparaison de Faltings a été généralisé par Scholze aux variétés analytique rigides, d'abord dans le cas lisse puis dans le cas général. Notre application est un analogue de cette dernière généralisation. Même dans le cas lisse, le théorème de comparaison de Faltings et la généralisation de Scholze ne peuvent être directement déduits l'un de l'autre. Deuxièmement, nous prouvons l'existence d'une version locale de la filtration relative de Hodge-Tate comme conséquence de la version globale construite par Abbes et Gros [**AG20**] et de notre résultat de descente cohomologique. Nous voudrions

mentionner une troisième application intéressante de notre résultat à la correspondance de Simpson p-adique donnée par Xu [**Xu22**].

0.1.7. La troisième partie (Chapitre III) de cette thèse vise à généraliser un résultat d'Abbes-Gros sur la presque-cohérence des images directes supérieures par un morphisme projectif de schémas au cas d'un morphisme propre. Le résultat de presque-cohérence est l'un des ingrédients clés de la preuve d'Abbes-Gros du principal théorème de comparaison p-adique de Faltings dans le cas relatif et donc de leur construction de la suite spectrale de Hodge-Tate relative [**AG20**]. Notre généralisation nous permet d'étendre leurs arguments aux morphismes log-lisses propres. Notre preuve utilise l'approximation noethérienne pour ramener le calcul des images directes supérieures au cas noethérien. Elle est inspirée de la preuve de Kiehl de la pseudo-cohérence des images directes supérieures [**Kie72**].

0.2. Descente Cohomologique pour le Topos Annelé de Faltings

0.2.1. La preuve de Faltings de la décomposition de Hodge-Tate illustre son approche de la théorie de Hodge p-adique et le rôle de son topos annelé. Soient K un corps de valuation discrète complet extension de $\mathbb{Q}_p$ à corps résiduel parfait, $\overline{K}$ une clôture algébrique de K, $\widehat{\overline{K}}$ le complété p-adique de $\overline{K}$. Pour un K-schéma propre et lisse X, Tate a conjecturé qu'il existe une décomposition canonique $G_K = \mathrm{Gal}(\overline{K}/K)$-équivariante, appelée la *décomposition de Hodge-Tate* ([**Tat67**, Remark, page 180]),

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widehat{\overline{K}} = \bigoplus_{0\leq q\leq n} H^q(X, \Omega^{n-q}_{X/K}) \otimes_K \widehat{\overline{K}}(q-n), \tag{0.2.1.1}$$

où $\widehat{\overline{K}}(q-n)$ est le $(q-n)$-ième twist de Tate de $\widehat{\overline{K}}$. Cette conjecture a été prouvée indépendamment par Faltings [**Fal88, Fal02**], Nizioł [**Niz98, Niz08**] et Tsuji [**Tsu99, Tsu02**], et a été généralisée au cadre analytique rigide par Scholze [**Sch13a**].

0.2.2. Pour un $\mathcal{O}_K$-schéma propre et semi-stable X de fibre générique géométrique $Y = X_{\overline{K}}$, Faltings a introduit un site annelé $(\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}})$, appelé le *site annelé de Faltings*, pour servir de pont entre la cohomologie étale de Y et les différentielles sur X. Concrètement, ces liens s'établissent à travers des morphismes naturels de sites

$$Y_{\text{ét}} \xrightarrow{\psi} \mathbf{E}^{\text{ét}}_{Y\to X} \xrightarrow{\sigma} X_{\text{ét}} \tag{0.2.2.1}$$

qui vérifient les propriétes suivante :

(1) (Principal théorème de comparaison p-adique de Faltings, [**Fal02**, Thm.8, page 223], [**AG20**, 4.8.13]). Pour tout faisceau abélien localement constant constructible $\mathbb{F}$ sur $Y_{\text{ét}}$, il existe un morphisme canonique

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathcal{O}_{\overline{K}} \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{0.2.2.2}$$

qui est un presque-isomorphisme, c'est-à-dire que les groupes de cohomologie de son cône sont annulés par p^r pour tout nombre rationnel $r > 0$.

(2) (Calcul de Faltings de la cohomologie galoisienne, [**AG20**, 6.3.8]). Il existe un homomorphisme canonique de $\mathcal{O}_X \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$-modules

$$\Omega^q_{X/\mathcal{O}_K}(\log X_s) \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \longrightarrow \mathrm{R}^q\sigma_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \tag{0.2.2.3}$$

où X_s désigne la fibre spéciale du $\mathcal{O}_K$-schéma X, dont le noyau et le conoyau sont annulés par p^r pour tout nombre rationnel $r > \frac{2\dim(Y)+1}{p-1}$.

On remarquera que $\mathbb{Z}/p^n\mathbb{Z} = \psi_*(\mathbb{Z}/p^n\mathbb{Z})$. Faltings a déduit la décomposition de Hodge-Tate de la dégénérescence et du caractère scindé de la suite spectrale de Cartan-Leray associée au foncteur composé $\mathrm{R}\Gamma(X_{\text{ét}}, -) \circ \mathrm{R}\sigma_*$, nommée plus tard la *suite spectrale de Hodge-Tate* par Scholze. En utilisant le théorème d'altération de de Jong, on peut déduire la décomposition de Hodge-Tate pour un K-schéma général propre et lisse en se ramenant au cas où il admet un modèle semi-stable (cf. [**Tsu02**, A5]).

0.2.3. Afin d'énoncer notre résultat de descente cohomologique, nous commençons par la définition du site de Faltings associé à un morphisme $Y \to X$ de schémas cohérents (i.e. quasi-compacts et quasi-séparés) (cf. I.7.7). Soit $\mathbf{E}^{\text{ét}}_{Y\to X}$ la catégorie des morphismes de schémas cohérents $V \to U$ au-dessus de $Y \to X$, c'est-à-dire les diagrammes commutatifs

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{0.2.3.1}$$

tels que U soit étale au-dessus X et que V soit fini étale au-dessus $Y \times_X U$. Nous munissons $\mathbf{E}^{\text{ét}}_{Y\to X}$ de la topologie engendrée par les familles de morphismes des types suivants :

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, où M est un ensemble fini et $\coprod_{m\in M} V_m \to V$ est surjectif ;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, où N est un ensemble fini et $\coprod_{n\in N} U_n \to U$ est surjectif.

Considérons le préfaisceau $\overline{\mathscr{B}}$ sur $\mathbf{E}^{\text{ét}}_{Y\to X}$ défini par

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{0.2.3.2}$$

où U^V est la clôture intégrale de U dans V. Il s'agit en fait d'un faisceau d'anneaux, le faisceau structural de $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. I.7.6).

0.2.4. Rappelons que la descente cohomologique de la cohomologie étale le long des hyper-recouvrements propres peut être généralisée comme suit : pour un schéma cohérent S, nous munissons la catégorie des S-schémas cohérents $\mathbf{Sch}^{\text{coh}}_{/S}$ de la *topologie h* de Voevodsky, qui est engendrée par des recouvrements étales et les morphismes propres surjectifs de présentation finie. Alors, pour tout faisceau abélien de torsion $\mathcal{F}$ sur $S_{\text{ét}}$, notant $a : (\mathbf{Sch}^{\text{coh}}_{/S})_{\text{h}} \to S_{\text{ét}}$ le morphisme naturel de sites, le morphisme d'adjonction $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ est un isomorphisme.

Ce résultat reste vrai pour une topologie plus fine, la *topologie v*. Un morphisme de schémas cohérents $T \to S$ est appelé un recouvrement v si pour tout morphisme $\mathrm{Spec}(A) \to S$ où A est un anneau de valuation, il existe une extension d'anneaux de valuation $A \to B$ et un relèvement $\mathrm{Spec}(B) \to T$. En effet, un recouvrement v est une limite des recouvrements h (cf. I.3.6). Nous allons décrire la descente cohomologique pour $\overline{\mathscr{B}}$ à l'aide d'un nouveau site construit à partir de la topologie v, qui peut être considérée comme un analogue schématique du site v des espaces adiques (cf. [**Sch17**, 8.1, 14.1, 15.5]).

Définition 0.2.5 (cf. I.3.23)**.** Soit $S^\circ \to S$ une immersion ouverte de schémas cohérents telle que S soit intégralement clos dans S°. Nous définissons un site $\mathbf{I}_{S^\circ\to S}$ comme suit:

(1) La catégorie sous-jacente est la catégorie des S-schémas cohérents T intégralement clos dans $S^\circ \times_S T$.
(2) La topologie est engendrée par les familles couvrantes $\{T_i \to T\}_{i\in I}$ pour la topologie v.

Nous appelons $\mathbf{I}_{S^\circ\to S}$ le *site v des S-schémas cohérents S°-intégralement clos*, et nous appelons faisceau structural de $\mathbf{I}_{S^\circ\to S}$ le faisceau $\mathscr{O}$ sur $\mathbf{I}_{S^\circ\to S}$ associé au préfaisceau $T \mapsto \Gamma(T, \mathcal{O}_T)$.

0.2.6. Soit p un nombre premier, $\overline{\mathbb{Z}_p}$ la clôture intégrale de $\mathbb{Z}_p$ dans une clôture algébrique $\overline{\mathbb{Q}_p}$ de $\mathbb{Q}_p$. On pose $S^\circ = \mathrm{Spec}(\overline{\mathbb{Q}_p})$ et $S = \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Considérons un diagramme de schémas cohérents

$$\begin{array}{ccccc} Y & \longrightarrow & X^Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(\overline{\mathbb{Q}_p}) & \longrightarrow & \mathrm{Spec}(\overline{\mathbb{Z}_p}) & & \end{array} \tag{0.2.6.1}$$

où X^Y est la clôture intégrale de X dans Y et le carré est cartésien (nous n'imposons aucune condition sur la régularité ou la finitude de Y ou X). Le foncteur $\varepsilon^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{I}_{Y\to X^Y}$ envoyant $V \to U$ à U^V définit un morphisme naturel de sites annelés

$$\varepsilon : (\mathbf{I}_{Y\to X^Y}, \mathscr{O}) \longrightarrow (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}). \tag{0.2.6.2}$$

Notre résultat de descente cohomologique énoncé ci-dessous peut être considéré comme un analogue schématique du résultat de descente cohomologique pour le site pro-étale d'un espace adique (cf. [**Sch17**, 8.8, 14.7, 15.5]) :

Théorème 0.2.7 (Descente cohomologique pour les sites annelés de Faltings, cf. I.8.14)**.** *Pour tout faisceau abélien localement constant constructible $\mathbb{L}$ sur $\mathbf{E}^{\text{ét}}_{Y\to X}$, le morphisme canonique*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{0.2.7.1}$$

est un presque-isomorphisme.

Corollaire 0.2.8 (cf. I.8.18)**.** *Pour tout hyper-recouvrement propre $X_\bullet \to X$, si $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ désigne l'augmentation du site simplicial où $Y_\bullet = Y \times_X X_\bullet$, alors le morphisme canonique*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \tag{0.2.8.1}$$

est un presque-isomorphisme.

L'ingrédient clé de notre preuve de 0.2.7 est la descente des algèbres perfectoïdes en topologie des arcs (une topologie plus fine que la topologie v) due à Bhatt-Scholze [**BS19**, 8.10] (cf. I.5.35). L'analogue en caractéristique p de 0.2.7 est le calcul de Gabber de la cohomologie du faisceau structural en topologie h (cf. I.4). Le Théorème 0.2.7 nous permet de descendre des résultats importants pour les sites de Faltings associés aux bons modèles aux sites de Faltings associés aux modèles généraux.

0.3. Généralisations des Principaux Théorèmes de Comparaison p-adiques de Faltings

0.3.1. Nous utilisons le Théorème 0.2.7 pour généraliser le théorème de comparaison de Faltings dans le cas absolu. Soit A un anneau de valuation, extension de $\overline{\mathbb{Z}_p}$, de corps des fractions algébriquement clos. Considérons un carré cartésien de schémas cohérents

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A[\frac{1}{p}]) & \longrightarrow & \mathrm{Spec}(A) \end{array} \tag{0.3.1.1}$$

Théorème 0.3.2 (Principal théorème de comparaison p-adique de Faltings dans le cas absolu, cf. I.10.17)**.** *Supposons X propre et de présentation finie sur A. Alors, pour tout faisceau abélien localement constant constructible $\mathbb{F}$ sur $Y_{\text{ét}}$, il existe un morphisme canonique*

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{0.3.2.1}$$

qui est un presque-isomorphisme.

On remarque que le morphisme naturel $\psi : Y_{\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ induit une équivalence des catégories de faisceaux abéliens localement constants constructibles sur $Y_{\text{ét}}$ et $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. I.10.3),

$$\mathbf{LocSys}(Y_{\text{ét}}) \underset{\psi^{-1}}{\overset{\psi_*}{\rightleftarrows}} \mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X}). \tag{0.3.2.2}$$

Dans la continuité des travaux d'Abbes-Gros, le morphisme canonique (0.3.2.1) (appelé le *morphisme de comparaison de Faltings*) est construit en utilisant l'acyclicité de ψ pour $\mathbb{F}$, i.e. $\psi_*\mathbb{F} = \mathrm{R}\psi_*\mathbb{F}$ (donc $\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) = \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F})$), qui est une conséquence du résultat d'Achinger sur les schémas $K(\pi, 1)$ (cf. I.10.6 et I.10.8). Nous proposons également une nouvelle façon de construire le morphisme de comparaison de Faltings dans la catégorie dérivée des presque-modules en utilisant notre résultat de descente cohomologique 0.2.7, qui évite d'utiliser l'acyclicité de ψ. En effet, il existe des morphismes naturels de sites

$$\begin{array}{ccc} (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a} & Y_{\text{ét}} \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X} \end{array} \tag{0.3.2.3}$$

et Ψ est acyclique pour tout faisceau abélien de torsion $\mathcal{F}$ sur $Y_{\text{ét}}$, i.e. $\Psi_*(a^{-1}\mathcal{F}) = \mathrm{R}\Psi_*(a^{-1}\mathcal{F})$, qui autorise des coefficients plus généraux et dont la preuve est beaucoup plus simple que celle pour ψ (cf. I.3.27). Nous remarquons que cette nouvelle construction ne nous donnera pas un « vrai morphisme » (0.3.2.1) mais un morphisme canonique dans la catégorie dérivée des presque modules (cf. I.11.6).

0.3.3. Expliquons maintenant brièvement la stratégie pour prouver le Théorème 0.3.2 :

(1) Premièrement, nous utilisons le théorème d'altération de de Jong-Gabber-Illusie-Temkin pour les morphismes de schémas [**ILO14**, X.3] pour obtenir un morphisme surjectif, propre et de présentation finie $X' \to X$ tel que le morphisme $X' \to \mathrm{Spec}(A)$ soit la limite cofiltrée d'un système de « bons morphismes » $X'_\lambda \to T_\lambda$ de « bons modèles » sur $\mathcal{O}_{K_\lambda}$, où K_λ est une extension finie de $\mathbb{Q}_p$ (cf. I.9.11).
(2) Ensuite, on applique le théorème de comparaison de Faltings dans le cas relatif aux « bons morphismes » $X'_\lambda \to T_\lambda$ (formulé par Faltings [**Fal02**, Thm.6, page 266] et démontré par Abbes-Gros [**AG20**, 5.7.4], cf. I.10.13). Par un argument de limite, on obtient le théorème de comparaison pour X'.
(3) Enfin, en utilisant notre résultat de descente cohomologique 0.2.8, nous en déduisons le théorème de comparaison pour X.

0.3.4. Le site $\mathbf{I}_{Y\to X^Y}$ est également approprié pour globaliser le théorème de comparaison de Faltings. Considérons un carré cartésien de schémas cohérents

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{0.3.4.1}$$

où $Y \to X$ est cartésien sur $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\overline{\mathbb{Z}_p})$. En particulier, il existe un morphisme naturel de sites annelés par la fonctorialité de (0.3.2.3),

$$f_{\mathbf{I}} : (\mathbf{I}_{Y'\to X'^{Y'}}, \mathscr{O}') \longrightarrow (\mathbf{I}_{Y\to X^Y}, \mathscr{O}). \tag{0.3.4.2}$$

Théorème 0.3.5 (cf. I.11.11)**.** *Supposons $X' \to X$ propre et de présentation finie. Soient $\mathcal{F}'$ un faisceau abélien de torsion sur $Y'_{\text{ét}}$ et $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$ (cf. (0.3.2.3)). Alors, le morphisme canonique*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{0.3.5.1}$$

est un presque-isomorphisme.

On remarque que si $\mathcal{F}' = \mathbb{Z}/p^n\mathbb{Z}$ alors $\mathscr{F}' = \mathbb{Z}/p^n\mathbb{Z}$ (cf. I.3.27), et que $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ est le faiseau de cohomologie étale de Y' sur Y à coefficients $\mathcal{F}'$ en topologie v (cf. I.11.12). Très grossièrement, les objets de $\mathbf{I}_{Y\to X^Y}$ sont « localement » des spectres d'anneaux de valuation, et les « fibres » de (0.3.5.1) sont des morphismes de comparaison de Faltings (0.3.2.1) si $\mathcal{F}'$ est localement constant constructible (cf. I.11.5). Le Théorème 0.3.5 peut être considéré comme un analogue schématique du théorème de comparaison de Scholze pour la cohomologie étale p-adique pour un morphisme de variétés analytiques rigides [**Sch13b**, 3.13].

Enfin, nous généralisons le principal théorème de comparaison p-adique de Faltings dans le cas relatif en utilisant les Théorèmes 0.2.7 et 0.3.5.

Théorème 0.3.6 (Principal théorème de comparaison p-adique de Faltings dans le cas relatif, cf. I.11.13 et I.11.14)**.** *Supposons $Y' \to Y$ lisse et $X' \to X$ propre et de présentation finie. Alors, pour tout faisceau abélien localement constant constructible $\mathbb{F}'$ sur $Y'_{\text{ét}}$, il existe un morphisme canonique*

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{0.3.6.1}$$

qui est un presque-isomorphisme, et où $f_{\text{ét}} : Y'_{\text{ét}} \to Y_{\text{ét}}$ et $f_{\mathbf{E}} : \mathbf{E}^{\text{ét}}_{Y'\to X'} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ sont les morphismes naturels de sites. En particulier, il existe un morphisme canonique

$$(\psi_*\mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{0.3.6.2}$$

qui est un presque-isomorphisme, pour tout entier q.

0.4. Version Locale de la Filtration Hodge-Tate Relative

0.4.1. Une autre application de notre résultat de descente cohomologique est une généralisation de la décomposition de Hodge-Tate au cas relatif. Soit K un corps de valuation discrète complet extension de $\mathbb{Q}_p$ à corps résiduel algébriquement clos, $(f, g) : (X'^{\triangleright} \to X') \to (X^{\circ} \to X)$ un morphisme d'immersions ouvertes de schémas cohérents au-dessus de $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$. Nous supposons que les conditions suivantes sont remplies :

(1) Les schémas logarithmiques associés $(X', \mathscr{M}_{X'}), (X, \mathscr{M}_X)$ munis des structures logarithmiques compactifiantes sont adéquats (une condition technique qui est satisfaite si les immersions ouvertes $X'^{\triangleright} \to X', X^{\circ} \to X$ sont semi-stables sur $\mathcal{O}_K$, cf. I.10.11).
(2) Le morphisme de schémas logarithmiques $(X', \mathscr{M}_{X'}) \to (X, \mathscr{M}_X)$ est lisse et saturé.
(3) Le morphisme de schémas $g : X' \to X$ est propre.
(4) Le schéma $X = \mathrm{Spec}(R)$ est affine et il existe un nombre fini de diviseurs non nuls $f_1, \ldots, f_r$ de $R[1/p]$ tels que le diviseur $D = \sum_{i=1}^r \mathrm{div}(f_i)$ sur X_K ait pour support $X_K \setminus X_K^{\circ}$ et qu'à chaque hensélisation stricte de X_K les élements f_i contenus dans l'idéal maximal forment un sous-ensemble d'un système régulier de paramètres (en particulier, D est un diviseur à croisements normaux sur X_K).

Pour tout X_K°-schéma cohérent Y, on définit un Y-schéma pro-fini étale.

$$Y_{\infty} = \lim_n Y[T_1, \ldots, T_r]/(T_1^n - f_1, \ldots, T_r^n - f_r). \tag{0.4.1.1}$$

Théorème 0.4.2 (cf. I.12.7 et I.8.24)**.** *Sous les hypothèses de* 0.4.1*, soit U un schéma affine pro-étale sur X et soit V un $U^{\circ}_{\overline{K},\infty}$-schéma pro-fini étale (où $U^{\circ} = X^{\circ} \times_X U$) satisfaisant les conditions suivantes :*

(1) *La clôture intégrale de U dans V est le spectre d'une $\mathcal{O}_{\overline{K}}$-algèbre A qui est presque pré-perfectoïde au sens de* I.5.19.
(2) *Pour tous entiers $n \geq 0$ et $k \geq 0$, l'image inverse $(\mathrm{R}^n f_{\text{ét}*}\mathbb{Z}/p^k\mathbb{Z})|_{V_{\text{ét}}}$ est un faisceau constant.*

Soit $\overline{x}$ un point géométrique de V. Alors, pour tout entier $n \geq 0$, il existe une filtration décroissante finie canonique $(\mathrm{fil}^q)_{q\in\mathbb{Z}}$ sur $H^n_{\text{ét}}(X^{\flat}_{\overline{x}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \widehat{A}[1/p]$ et un isomorphisme canonique pour tout $q \in \mathbb{Z}$,

$$\mathrm{gr}^q(H^n_{\text{ét}}(X^{\flat}_{\overline{x}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \widehat{A}[\frac{1}{p}]) \cong H^q(X', \Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)}) \otimes_R \widehat{A}[\frac{1}{p}](q-n), \tag{0.4.2.1}$$

où gr^q désigne le gradué $\mathrm{fil}^q/\mathrm{fil}^{q+1}$. De plus, si U° et V sont connexes et si le corps des fractions de V est une extension galoisienne de celui de U° de groupe de Galois Γ, alors la filtration $(\mathrm{fil}^q)_{q\in\mathbb{Z}}$ et les isomorphismes (0.4.2.1) *sont Γ-équivariants.*

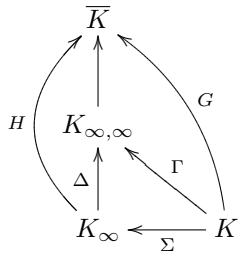
Remarque 0.4.3. Les objets $V \to U$ satisfaisant les conditions de 0.4.2 forment une famille topologiquement génératrice du site pro-étale de Faltings de $X^{\circ}_{\overline{K}} \to X$ (voir la preuve de I.8.24).

0.4.4. Cette *filtration de Hodge-Tate relative locale* découle de la filtration de Hodge-Tate relative globale construite par Abbes-Gros [**AG20**]. Leur filtration se construit via le topos de Faltings associé à $X^{\circ}_{\overline{K}} \to X$. Dans la première version de leur travail, ils ont posé la question de l'existence d'une version locale explicite. Scholze et Caraiani [**CS17**] ont construit indépendamment une filtration de Hodge-Tate relative pour des morphismes lisses et propres d'espaces adiques lisses, et Scholze a annoncé qu'il pouvait donner une version locale, répondant à la question d'Abbes-Gros. Notre construction est obtenue en appliquant notre résultat de descente cohomologique pour le topos annelé de Faltings à la filtration de Hodge-Tate relative globale d'Abbes-Gros. Dans une nouvelle version de leur manuscrit, Abbes-Gros ont donné une troisième construction de la filtration locale de Hodge-Tate dans un cadre un peu plus restrictif, en utilisant un résultat de descente cohomologique qui est un cas particulier de notre résultat.

0.5. Construction Canonique des Opérateurs de Sen sur les Corps Locaux

0.5.1. Notre stratégie pour construire les opérateurs de Sen sur les variétés p-adiques est de recoller les opérateurs de Sen définis dans le cas de corps locaux. Nous faisons d'abord un bref rappel sur la construction des opérateurs de Sen par Brinon. Soient K un corps de valuation discrète complet extension de $\mathbb{Q}_p$ dont le corps résiduel est imparfait avec une p-base de cardinal $d \geq 0$, $\overline{K}$ une clôture algébrique de K, $\widehat{\overline{K}}$ le complété p-adique de $\overline{K}$, G le groupe de Galois de $\overline{K}$ sur K. Nous choisissons $t_1, \ldots, t_d \in \mathcal{O}_K$ des relèvements d'une p-base du corps résiduel. On fixe un système compatible de racines primitives p^n-ièmes de l'unité $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}}$ et un système compatible de racines p^n-ièmes $(t_{i,p^n})_{n\in\mathbb{N}}$ de t_i pour tout $1 \leq i \leq d$. Nous posons également $t_{0,p^n} = \zeta_{p^n}$ par souci de cohérence. Pour tous $n, m \in \mathbb{N} \cup \{\infty\}$, considérons l'extension de corps $K_{n,m} = K(\zeta_{p^n}, t_{1,p^m}, \ldots, t_{d,p^m})$ sur K contenue dans $\overline{K}$. On pose simplement $K_{n,0} = K_n$ et on nomme les groupes de Galois correspondants comme indiqué dans le diagramme suivant

(0.5.1.1)



Toute $\widehat{\overline{K}}$-représentation de dimension finie W de G se descend en une $K_{\infty,\infty}$-représentation V de Γ par un théorème de Brinon (cf. II.5.17). On remarque qu'elle peut être descendue en une K_∞-représentation de Γ sur laquelle Δ agit analytiquement grâce à un théorème de Tsuji (cf. II.5.18). Ici, agir analytiquement signifie que l'action de tout élément de Δ est donnée par l'exponentielle de son action infinitésimale (cf. II.4.14). Le groupe topologique Γ est bien un groupe analytique p-adique, auquel on peut associer une algèbre de Lie $\mathrm{Lie}(\Gamma)$ sur $\mathbb{Q}_p$. L'action infinitésimale de $\mathrm{Lie}(\Gamma)$ sur V s'étend $\widehat{\overline{K}}$-linéairement en une action de l'algèbre de Lie (non-canonique) $\mathrm{Lie}(\Gamma)$ sur W, qui définit les $1 + d$ opérateurs de W par Brinon (Γ est localement isomorphe à $\mathbb{Z}_p \ltimes \mathbb{Z}_p^d$).

0.5.2. Cette action de $\mathrm{Lie}(\Gamma)$ dépend du choix de $t_1, \ldots, t_d$, ce qui empêche sa généralisation aux variétés p-adiques. La première question est de savoir si on peut définir une action d'algèbre de Lie canonique sur W, qui donne les opérateurs de Sen définis par Brinon en choisissant une base. Nous y

répondons positivement en considérant l'extension de Faltings de $\mathcal{O}_K$ définie dans **[He21]** (cf. II.5.7), c'est-à-dire une suite exacte canonique de $\widehat{\overline{K}}$-représentations de G,

$$0 \longrightarrow \widehat{\overline{K}}(1) \xrightarrow{\iota} \mathscr{E}_{\mathcal{O}_K} \xrightarrow{\jmath} \widehat{\overline{K}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow 0, \tag{0.5.2.1}$$

où $\widehat{\overline{K}}(1)$ désigne le premier twist de Tate de $\widehat{\overline{K}}$, $\mathscr{E}_{\mathcal{O}_K} = \varprojlim_{x \mapsto px} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ est un $\widehat{\overline{K}}$-espace vectoriel de dimension $1+d$ ayant une base $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\leq i\leq d}$. En prenant les duaux et les twists de Tate, on obtient une suite exacte canonique

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{\overline{K}}) \xrightarrow{\jmath^*} \mathscr{E}^*_{\mathcal{O}_K}(1) \xrightarrow{\iota^*} \widehat{\overline{K}} \longrightarrow 0 \tag{0.5.2.2}$$

où $\mathscr{E}^*_{\mathcal{O}_K} = \mathrm{Hom}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}, \widehat{\overline{K}})$. Il existe une structure de $\widehat{\overline{K}}$-algèbre de Lie canonique sur $\mathscr{E}^*_{\mathcal{O}_K}(1)$ associée à la forme linéaire ι^* défini par $[f_1, f_2] = \iota^*(f_1)f_2 - \iota^*(f_2)f_1$ pour tous $f_1, f_2 \in \mathscr{E}^*_{\mathcal{O}_K}(1)$. Ce sera l'algèbre de Lie canonique remplaçant $\mathrm{Lie}(\Gamma)$, nous servant à obtenir la définition canonique suivante des opérateurs de Sen.

Théorème 0.5.3 (cf. II.5.35, II.5.38)**.** *Soient K un corps de valuation discrète complet extension de $\mathbb{Q}_p$ dont le corps résiduel admet une p-base de cardinal fini, G son groupe de Galois absolu. Pour toute $\widehat{\overline{K}}$-représentation de dimension finie W de G, il existe un homomorphisme G-équivariant canonique d'algèbres de Lie $\widehat{\overline{K}}$-linéaires (où nous considérons l'action adjointe de G sur $\mathrm{End}_{\widehat{\overline{K}}}(W)$),*

$$\varphi_{\mathrm{Sen}}|_W : \mathscr{E}^*_{\mathcal{O}_K}(1) \longrightarrow \mathrm{End}_{\widehat{\overline{K}}}(W), \tag{0.5.3.1}$$

qui est fonctoriel en W et vérifie les propriétés suivantes :

1. *Soient $t_1, \ldots, t_e \in K$ muni de systèmes compatibles de racines p^n-ièmes $(t_{i,p^n})_{n\in\mathbb{N}} \subseteq \overline{K}$ tels que $\mathrm{d}t_1, \ldots, \mathrm{d}t_e$ soient K-linéairement indépendants dans $\widehat{\Omega}^1_{\mathcal{O}_K}[1/p]$. Considérons la tour $(K_{n,m})_{n,m\in\mathbb{N}}$ définie par ces éléments de manière analogue à* (0.5.1.1)*, prenons la même notation pour les groupes de Galois, et supposons qu'il existe une représentation V de Γ sur laquelle Δ agisse analytiquement (*II.4.14*) telle que $W = \widehat{\overline{K}} \otimes_{K_\infty} V$. Alors, Γ est naturellement localement isomorphe à $\mathbb{Z}_p \ltimes \mathbb{Z}_p^e$, et si on prend la base standard $\partial_0, \ldots, \partial_e$ de $\mathrm{Lie}(\Gamma) \cong \mathrm{Lie}(\mathbb{Z}_p \ltimes \mathbb{Z}_p^e)$, alors pour tout $f \in \mathscr{E}^*_{\mathcal{O}_K}(1)$,*

$$\varphi_{\mathrm{Sen}}|_W(f) = \sum_{i=0}^{e} f((\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1}) \otimes \varphi_{\partial_i}|_V, \tag{0.5.3.2}$$

   où $\varphi_{\partial_i}|_V$ est l'action infinitésimale de ∂_i sur V.
2. *Soient K' un corps de valuation discrète complet extension de K dont le corps résiduel admet une p-base de cardinal fini, $W' = \widehat{\overline{K'}} \otimes_{\widehat{\overline{K}}} W$. Supposons que $K' \otimes_K \widehat{\Omega}^1_{\mathcal{O}_K}[1/p] \to \widehat{\Omega}^1_{\mathcal{O}_{K'}}[1/p]$ soit injectif. Alors, il existe un diagramme commutatif naturel*

$$\begin{array}{ccc} \mathscr{E}^*_{\mathcal{O}_{K'}}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{\overline{K'}}}(W') \\ \downarrow & & \uparrow \wr \\ \widehat{\overline{K'}} \otimes_{\widehat{\overline{K}}} \mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow[\mathrm{id}_{\widehat{\overline{K'}}} \otimes \varphi_{\mathrm{Sen}}|_W]{} & \widehat{\overline{K'}} \otimes_{\widehat{\overline{K}}} \mathrm{End}_{\widehat{\overline{K}}}(W) \end{array} \tag{0.5.3.3}$$

   De plus, si K' est une extension finie de K, alors la flèche verticale gauche est un isomorphisme.

La clé de la preuve est de montrer que l'application $\varphi_{\mathrm{Sen}}|_W$ définie par la formule (0.5.3.2) ne dépend pas du choix de V ou des t_i. Pour cela, nous utilisons la variante de la correspondance de Simpson p-adique développée par Tsuji **[Tsu18]** sur $\mathcal{O}_K$ (cf. II.5.31). Un ingrédient est que l'anneau de périodes utilisé dans cette correspondance est construit comme la colimite filtrée des produits tensoriels symétriques de l'extension de Faltings (0.5.2.1) (appelée l'anneau de Hyodo, cf. II.5.13). Nous remarquons que l'hypothèse sur $K' \otimes_K \widehat{\Omega}^1_{\mathcal{O}_K}[1/p] \to \widehat{\Omega}^1_{\mathcal{O}_{K'}}[1/p]$ pour la fonctorialité est une condition technique nécessaire pour sa preuve, et nous ne savons pas comment la supprimer (cf. II.5.38).

0.6. Construction Canonique des Opérateurs de Sen sur une Variété Affine p-adique

0.6.1. Nous pouvons maintenant généraliser la construction des opérateurs de Sen aux variétés p-adiques. Soient K un corps de valuation discrète complet extension de $\mathbb{Q}_p$ à corps résiduel parfait, π une uniformisante de K. Pour simplifier, on considère un anneau A intègre, noethérien, normal et plat sur $\mathcal{O}_K$ avec $A/pA \neq 0$ tel qu'il existe un homomorphisme étale d'anneaux pour certains entiers $0 \leq r \leq c \leq d$,

$$\mathcal{O}_K[T_0, \ldots, T_r, T_{r+1}^{\pm 1}, \ldots, T_c^{\pm 1}, T_{c+1}, \ldots, T_d]/(T_0 \cdots T_r - \pi) \longrightarrow A. \tag{0.6.1.1}$$

Ainsi, $\mathrm{Spec}(A)$ est muni d'un diviseur à croisements normaux strict défini par $T_0 \cdots T_d$. Nous posons $A_{\mathrm{tr}} = A[1/T_0 \cdots T_d]$. Nous appellerons (A_{tr}, A) une paire semi-stable, et (0.6.1.1) une carte semi-stable de celle-ci. Soient $\mathcal{K}$ le corps des fractions de A, $\mathcal{K}_{\mathrm{ur}}$ l'extension maximale non ramifiée de $\mathcal{K}$ relative à (A_{tr}, A), c'est-à-dire l'union des extensions finies de corps $\mathcal{K}'$ sur $\mathcal{K}$ dans une clôture algébrique de $\mathcal{K}$ telle que la clôture intégrale A' de A dans $\mathcal{K}'$ soit finie étale sur A_{tr}. Soit $\overline{A}$ la clôture intégrale de A dans $\mathcal{K}_{\mathrm{ur}}$. Nous considérons dans le Chapitre II des paires plus générales (appelées *quasi-adéquates*) que des paires semi-stables, telles que le système inductif de sous-extensions finies de $\mathcal{K}_{\mathrm{ur}}/\mathcal{K}$ admette un sous-système cofinal constitué d'éléments $\mathcal{K}'$ tels que la paire (A'_{tr}, A') soit quasi-adéquate, où $A'_{\mathrm{tr}} = A_{\mathrm{tr}} \otimes_A A'$ (cf. II.9.5, II.11.9). Comme précédemment, on considère également l'extension de Faltings de A (cf. II.9.36), c'est-à-dire la suite exacte canonique de $\widehat{\overline{A}}[1/p]$-représentations de $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$,

$$0 \longrightarrow \widehat{\overline{A}}[\frac{1}{p}](1) \xrightarrow{\iota} \mathscr{E}_A \xrightarrow{\jmath} \widehat{\overline{A}}[\frac{1}{p}] \otimes_A \Omega^1_{(A_{\mathrm{tr}},A)} \longrightarrow 0, \tag{0.6.1.2}$$

où $\Omega^1_{(A_{\mathrm{tr}},A)}$ désigne le A-module des 1-différentielles logarithmiques de la paire (A_{tr}, A) sur $(K, \mathcal{O}_K)$, qui soit fini libre de rang d. Le $\widehat{\overline{A}}[1/p]$-module canonique $\mathscr{E}_A$ est fini libre de rang $1+d$, et il vérifie la propriété suivante (cf. II.9.36, II.9.38) : il existe un homomorphisme $\overline{A}[1/p]$-linéaire canonique $\varprojlim_{x \mapsto px} \Omega^1_{\overline{A}/A} \to \mathscr{E}_A$ tel que pour tout élément $s \in \overline{A}[1/p] \cap \overline{A}_{\mathrm{tr}}^{\times}$ avec un système compatible de racines p^n-ièmes $(s_{p^n})_{n\in\mathbb{N}} \subseteq \overline{A}[1/p]$, il existe un élément unique $\omega \in \mathscr{E}_A$ tel que l'image de $(s_{p^n}^{p^n-1} \mathrm{d}s_{p^n})_{n\in\mathbb{N}}$ soit égale à $s\omega$ (on désigne donc ω par $(\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}$). Comme précédemment, on obtient une suite exacte canonique en prenant les duaux et les twists de Tate,

$$0 \longrightarrow \mathrm{Hom}_A(\Omega^1_{(A_{\mathrm{tr}},A)}(-1), \widehat{\overline{A}}[\frac{1}{p}]) \xrightarrow{\jmath^*} \mathscr{E}_A^*(1) \xrightarrow{\iota^*} \widehat{\overline{A}}[\frac{1}{p}] \longrightarrow 0, \tag{0.6.1.3}$$

et on munit $\mathscr{E}_A^*(1)$ de la structure de $\widehat{\overline{A}}[1/p]$-algèbre de Lie canonique associée à la forme linéaire ι^*. Nous pouvons maintenant énoncer la construction canonique des opérateurs de Sen.

Théorème 0.6.2 (cf. II.11.4, II.11.7, II.11.9)**.** *Avec les notations de* 0.6.1*, pour toute $\widehat{\overline{A}}[1/p]$-représentation finie projective W d'un sous-groupe ouvert G de $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$, il existe un homomorphisme G-équivariant canonique d'algèbres de Lie $\widehat{\overline{A}}[1/p]$-linéaires (où nous considérons l'action adjointe de G sur $\mathrm{End}_{\widehat{\overline{A}}[1/p]}(W)$),*

$$\varphi_{\mathrm{Sen}}|_W : \mathscr{E}_A^*(1) \longrightarrow \mathrm{End}_{\widehat{\overline{A}}[\frac{1}{p}]}(W), \tag{0.6.2.1}$$

qui est fonctoriel en W et en G, ne dépend que de la paire (A_{tr}, A) et non du choix de la carte (0.6.1.1)*, et satisfait les propriétés suivantes :*

(1) *Soit A' la clôture intégrale de A dans une extension finie de corps $\mathcal{K}'$ sur $(\mathcal{K}_{\mathrm{ur}})^G$ contenue dans $\mathcal{K}_{\mathrm{ur}}$, et soient $t_1, \ldots, t_e \in A'[1/p] \cap A'^{\times}_{\mathrm{tr}}$ avec systèmes compatibles de racines p^n-ièmes $(t_{i,p^n})_{n\in\mathbb{N}} \subseteq \overline{A}[1/p]$ tels que $\mathrm{d}t_1, \ldots, \mathrm{d}t_e$ soient $\mathcal{K}'$-linéairement indépendants dans $\Omega^1_{\mathcal{K}'/K}$. Considérons la tour $(\mathcal{K}'_{n,m})_{n,m\in\mathbb{N}}$ définie par ces éléments de manière analogue à* (0.5.1.1) *et prenons la même notation pour les groupe de Galois, et soit $A'_{n,m}$ la clôture intégrale de A dans $\mathcal{K}'_{n,m}$, $\widetilde{A}'_{\infty} = \mathrm{colim}_n \widehat{A'_n}$. Supposons qu'il existe une $\widetilde{A}'_{\infty}[1/p]$-représentation finie projective V de Γ sur laquelle Δ agit analytiquement (*II.4.14*) telle que $W = \widehat{\overline{A}} \otimes_{\widetilde{A}'_{\infty}} V$. Alors, Γ est naturellement localement isomorphe à $\mathbb{Z}_p \ltimes \mathbb{Z}_p^e$, et si on prend la base standard $\partial_0, \ldots, \partial_e$ de $\mathrm{Lie}(\Gamma) \cong \mathrm{Lie}(\mathbb{Z}_p \ltimes \mathbb{Z}_p^e)$, alors pour tout $f \in \mathscr{E}_A^*(1)$,*

$$\varphi_{\mathrm{Sen}}|_W(f) = \sum_{i=0}^{e} f((\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1}) \otimes \varphi_{\partial_i}|_V, \tag{0.6.2.2}$$

où $\varphi_{\partial_i}|_V$ est l'action infinitésimale de ∂_i sur V.

(2) *Soient K' un corps de valuation discrète complet extension de K à corps résiduel parfait, (A'_{tr}, A') une paire semi-stable sur $\mathcal{O}_{K'}$ de corps des fractions $\mathcal{K}'$, $\overline{A} \to \overline{A'}$ un homomorphisme d'anneaux injectif au-dessus de $\mathcal{O}_K \to \mathcal{O}_{K'}$ qui induit un morphisme de paires $(A_{\mathrm{tr}}, A) \to (A'_{\mathrm{tr}}, A')$, $W' = \widehat{\overline{A'}} \otimes_{\widehat{\overline{A}}} W$. Supposons que $\mathcal{K}' \otimes_{\mathcal{K}} \Omega^1_{\mathcal{K}/K} \to \Omega^1_{\mathcal{K}'/K'}$ soit injectif. Alors, il existe un diagramme commutatif naturel*

$$\begin{array}{ccc} \mathscr{O}^*_{A'}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{\overline{A'}}[\frac{1}{p}]}(W') \\ \downarrow & & \uparrow \wr \\ \widehat{\overline{A'}} \otimes_{\widehat{\overline{A}}} \mathscr{O}^*_A(1) & \xrightarrow[\mathrm{id}_{\widehat{\overline{A'}}} \otimes \varphi_{\mathrm{Sen}}|_W]{} & \widehat{\overline{A'}} \otimes_{\widehat{\overline{A}}} \mathrm{End}_{\widehat{\overline{A}}[\frac{1}{p}]}(W) \end{array} \tag{0.6.2.3}$$

La situation décrite dans 0.5.3.(1) n'est pas très restrictive. En effet, par un théorème de descente de Tsuji [**Tsu18**, 14.2] développé pour la correspndance de Simpson p-adique pour (A_{tr}, A), la représentation W de G peut être descendue en V pour certains A'. Nous remarquons que Tsuji a prouvé le cas où $G = \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$, et nous démontrons le cas général en transférant ses arguments à une classe de paires plus générales (cf. II.10.16). La clé de la preuve de 0.6.2 est encore de vérifier que l'application $\varphi_{\mathrm{Sen}}|_W$ définie par la formule (0.6.2.2) ne dépend pas du choix de A', V ou t_i. Nous ramenons ce problème au cas des anneaux de valuation 0.5.3 en localisant en les idéaux premiers de hauteur 1 contenant p.

Définition 0.6.3. Nous appelons l'image $\Phi(W)$ de $\varphi_{\mathrm{Sen}}|_W$ le module des *opérateurs de Sen* de W. Nous noterons $\Phi^{\mathrm{geo}}(W)$ l'image de $\mathrm{Hom}_A(\Omega^1_{(A_{\mathrm{tr}},A)}(-1), \widehat{\overline{A}}[\frac{1}{p}])$ par $\varphi_{\mathrm{Sen}}|_W$ et l'appellerons le module des *opérateurs géométriques de Sen* de W. Nous appellerons l'image de $1 \in \widehat{\overline{A}}[\frac{1}{p}]$ via $\varphi_{\mathrm{Sen}}|_W$ dans $\Phi^{\mathrm{ari}}(W) = \Phi(W)/\Phi^{\mathrm{geo}}(W)$ *l'opérateur arithmétique de Sen* de W.

Le fait suivant étaie une telle définition de l'opérateur arithmétique de Sen : deux relèvements quelconques de celui-ci dans $\mathrm{End}_{\widehat{\overline{A}}[\frac{1}{p}]}(W)$ ont le même polynôme caractéristique (cf. II.11.15).

0.7. Action Infinitésimale par les Sous-groupes d'Inertie

0.7.1. Reprenons les hypothèses et notations de 0.6.1 et notons $\mathfrak{S}_p(\overline{A})$ l'ensemble des idéaux premiers de hauteur 1 de $\overline{A}$ contenant p. Pour tout $\mathfrak{q} \in \mathfrak{S}_p(\overline{A})$ d'image $\mathfrak{p} \in \mathrm{Spec}(A)$, soient $E_{\mathfrak{p}}$ le complété p-adique du corps de valuation discrète $A_{\mathfrak{p}}[1/p]$, $\overline{E}_{\mathfrak{q}}$ une clôture algègrique de $E_{\mathfrak{p}}$ avec un prolongement d'anneaux de valuation $\overline{A}_{\mathfrak{q}} \to \mathcal{O}_{\overline{E}_{\mathfrak{q}}}$. Soit $I_{\mathfrak{q}} \subseteq \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ l'image du sous-groupe d'inertie de $\mathrm{Gal}(\overline{E}_{\mathfrak{q}}/E_{\mathfrak{p}})$. Nous avons la généralisation suivante du résultat de Sen-Ohkubo, qui découle de notre stratégie de réduction ci-dessus.

Théorème 0.7.2 (cf. II.11.18, II.11.9)**.** *Soient G un sous-groupe ouvert de $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$, (V, ρ) une $\mathbb{Q}_p$-représentation de dimension finie de G, $W = \widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} V$. Alors, $\sum_{\mathfrak{q} \in \mathfrak{S}_p(\overline{A})} \mathrm{Lie}(\rho(I_{\mathfrak{q}}))$ est le plus petit sous-$\mathbb{Q}_p$-espace vectoriel S de $\mathrm{End}_{\mathbb{Q}_p}(V)$ tel que le $\widehat{\overline{A}}[1/p]$-module des opérateurs de Sen $\Phi(W)$ soit contenu dans $\widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} S$.*

Comme corollaire, on peut promouvoir les opérateurs de Sen des $\mathbb{Q}_p$-représentations en un homomorphisme universel d'algèbres de Lie.

Corollaire 0.7.3 (cf. II.11.21, II.11.23)**.** *Soit $\mathcal{G}$ un quotient d'un sous-groupe ouvert de $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ qui soit un groupe analytique p-adique. Alors, il existe un homomorphisme canonique d'algèbres de Lie $\widehat{\overline{A}}[1/p]$-linéaires $\varphi_{\mathrm{Sen}}|_{\mathcal{G}} : \mathscr{O}^*_A(1) \to \widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G})$ rendant le diagramme suivant commutatif pour toute $\mathbb{Q}_p$-représentation de dimension finie V de $\mathcal{G}$,*

$$\begin{array}{ccc} \mathscr{O}^*_A(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\overline{A}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) \\ \downarrow{\scriptstyle \varphi_{\mathrm{Sen}}|_W} & & \downarrow{\scriptstyle \mathrm{id}_{\widehat{\overline{A}}[\frac{1}{p}]} \otimes \varphi|_V} \\ \mathrm{End}_{\widehat{\overline{A}}[\frac{1}{p}]}(W) & \xleftarrow{\sim} & \widehat{\overline{A}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p}(V) \end{array} \tag{0.7.3.1}$$

où $W = \widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} V$ est l'objet associé de $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{A}}[1/p])$, et $\varphi|_V$ l'action infinitésimale de l'algèbre de Lie $\mathrm{Lie}(\mathcal{G})$ sur V.

Nous pouvons maintenant donner notre généralisation d'un résultat de Pan [**Pan22**, 3.1.2].

Théorème 0.7.4 (cf. II.12.22)**.** *Soit $\mathcal{G}$ un quotient de $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ qui soit un groupe analytique p-adique, $\mathcal{G}_H \subseteq \mathcal{G}$ l'image de $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K}_\infty)$, $\Phi_{\mathcal{G}}^{\mathrm{geo}} \subseteq \widehat{A}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}_H)$ l'image de $\mathrm{Hom}_A(\Omega^1_{(A_{\mathrm{tr}},A)}(-1), \widehat{A}[1/p])$ via $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$. Alors, l'action infinitésimale de $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ annule les vecteurs $\mathcal{G}_H$-localement analytiques de $\widehat{A}[1/p]$ (voir* II.12.18 *pour une définition précise).*

Pour sa preuve, nous devons étendre les opérateurs de Sen aux représentations de dimension infinie des fonctions analytiques sur des sous-groupes ouverts suffisamment petits de $\mathcal{G}$. C'est la raison pour laquelle nous insistons pour considérer des sous-groupes ouverts de $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ dans les théorèmes précédents, ce qui nous permet de prouver des propriétés liées aux algèbres de Lie mais nous conduisent à une classe de paires plus générales que les paires semi-stables.

Auparavant, on travaillait toujous avec des représentations à coefficients rationnels, car une extension finie A' de A n'est pas un bon modèle entier pour $A'[1/p]$. Afin d'étudier la continuité des opérateurs de Sen sur des représentations de dimension infinie, nous devons considérer les représentations à coefficients entiers comme des « réseaux » pour borner ces opérateurs. Les bonnes propriétés des opérateurs de Sen sont préservées par continuité si nous avons une bonne théorie de descente et de décomplétion pour les représentations entières sur A'. Cela n'a pas encore été bien développé car A' n'est pas un bon modèle entier. Cependant, on ne rencontre pas un tel problème si A' est un anneau de valuation (au moins pour la partie géométrique)! Nous suivons donc encore la stratégie précédente : ramener le problème au cas d'anneaux de valuation en localisant en les idéaux premiers de hauteur 1 de $\overline{A}$ contenant p ; et pour ce dernier cas, on peut appliquer les résultats de descente pour les petites représentations à coefficients entiers du groupe fondamental géométrique, développés par Faltings [**Fal05**], Abbes-Gros [**AGT16**, II.14] et Tsuji [**Tsu18**, §11, §12]. Nous prévoyons d'étudier à l'avenir si oui ou non l'image $\Phi_{\mathcal{G}}$ de $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ annule les vecteurs $\mathcal{G}$-localement analytiques de $\widehat{A}[1/p]$.

0.8. Presque-cohérence des Images Directes Supérieures

0.8.1. Un des premiers résultats importants en géométrie algébrique est le fait que la cohérence pour les modules est préservée par les images directes supérieures par un morphisme propre. Le cas noethérien est dû à Grothendieck [**EGA** III_1, 3.2.1], et le cas général est dû à Kiehl [**Kie72**, 2.9']. Le but du Chapitre III est d'étendre le corollaire suivant au cadre de la presque-algèbre, motivé par des applications en théorie de Hodge p-adique

Théorème 0.8.2 (Kiehl [**Kie72**, 2.9'], cf. [**Abb10**, 1.4.8])**.** *Soit $f : X \to S$ un morphisme de schémas vérifiant les conditions suivantes :*

(1) *f est propre et de présentation finie, et*
(2) *$\mathcal{O}_S$ est universellement cohérent.*

Alors, pour tout $\mathcal{O}_X$-module cohérent $\mathcal{M}$ et tout $q \in \mathbb{N}$, $\mathrm{R}^q f_* \mathcal{M}$ est un $\mathcal{O}_S$-module cohérent.

On dit que $\mathcal{O}_S$ est *universellement cohérent* s'il existe un recouvrement $\{S_i = \mathrm{Spec}(A_i)\}_{i\in I}$ de S par des sous-schémas affines ouverts tel que l'algèbre polynomiale $A_i[T_1, \ldots, T_n]$ soit un anneau cohérent pour tous $i \in I$ et $n \in \mathbb{N}$. En effet, une telle condition sur $\mathcal{O}_S$ implique que le $\mathcal{O}_X$-module cohérent $\mathcal{M}$ est en fait *pseudo-cohérent* relativement à S, ce qui signifie grossièrement que si on plonge X localement comme un sous-schéma fermé de $\mathbb{A}^n_{S_i}$, alors $\mathcal{M}$ admet une résolution par des modules finis libres sur $\mathbb{A}^n_{S_i}$. Le théorème 0.8.2 est un corollaire direct du résultat de Kiehl [**Kie72**, 2.9'], prouvant que le foncteur dérivé $\mathrm{R}f_*$ envoie un complexe pseudo-cohérent relatif à un complexe pseudo-cohérent.

0.8.3. La presque-algèbre a été introduite par Faltings [**Fal88, Fal02**] pour développer la théorie de Hodge p-adique. Le cadre est une paire $(R, \mathfrak{m})$ constituée d'un anneau R avec un idéal $\mathfrak{m}$ tel que $\mathfrak{m} = \mathfrak{m}^2$, et l'idée grossière est de remplacer la catégorie des R-modules par son quotient par des modules de $\mathfrak{m}$-torsion. Un « presque » analogue du Théorème 0.8.2 est nécessaire pour l'approche de Faltings de la théorie de Hodge p-adique. En effet, sous les mêmes hypothèses de 0.8.2, Abbes-Gros [**AG20**, 2.8.14] ont prouvé que $\mathrm{R}^q f_*$ transforme un $\mathcal{O}_X$-module quasi-cohérent et presque-cohérent en un $\mathcal{O}_S$-module quasi-cohérent et presque-cohérent, en se réduisant directement à 0.8.2. Ce résultat joue un rôle crucial dans la preuve du principal théorème de comparaison p-adique de Faltings dans le cas absolu (cf. [**AG20**, 4.8.13]), et donc de la décomposition de Hodge-Tate (cf. [**AG20**, 6.4.14]). Plus tard, Zavyalov [**Zav21**, 5.1.6] a étendu le même résultat de presque-cohérence aux schémas formels.

Cependant, le résultat de presque-cohérence [**AG20**, 2.8.14] n'est pas suffisant pour le principal théorème de comparaison p-adique de Faltings dans le cas relatif (donc pas non plus pour la suite spectrale de Hodge-Tate relative), car on rencontre inévitablement la situation où $\mathcal{O}_S$ est *universellement presque-cohérent* mais pas universellement cohérent. Ainsi, sous l'hypothèse

(1) f projectif, plat et de présentation finie, et
(2) $\mathcal{O}_S$ universellement presque-cohérent,

Abbes-Gros ont prouvé un résultat de presque-cohérence [**AG20**, 2.8.18] en adaptant les arguments de [**SGA 6**, III.2.2], où la condition de projectivité sur f joue un rôle crucial. C'est la raison pour laquelle le principal théorème de comparaison p-adique de Faltings dans le cas relatif (et donc la construction de la suite spectrale de Hodge-Tate relative) n'ont été prouvé que pour les morphismes log-lisses et projectifs dans [**AG20**, 5.7.4 (et 6.7.5)].

0.8.4. Au Chapitre III, nous généralisons le résultat de presque-cohérence [**AG20**, 2.8.18] aux morphismes propres, ce qui nous permet d'étendre la preuve d'Abbes-Gros du principal théorème de comparaison p-adique de Faltings dans le cas relatif aux morphismes log-lisses et propres, et donc aussi leur construction de la suite spectrale de Hodge-Tate relative (cf. III.8).

Soit R un anneau avec un idéal $\mathfrak{m}$ tel que pour tout entier $l \geq 1$, les l-ièmes puissances des éléments de $\mathfrak{m}$ engendrent $\mathfrak{m}$. La paire $(R, \mathfrak{m})$ sera notre cadre pour la presque-algèbre (cf. III.6). Le théorème principal du chapitre III est le suivant.

Théorème 0.8.5 (cf. III.7.1)**.** *Soit $f : X \to S$ un morphisme de R-schémas vérifiant les conditions suivantes :*

(1) *f est propre, plat et de présentation finie, et*
(2) *$\mathcal{O}_X$ et $\mathcal{O}_S$ sont presque-cohérents.*

Alors, pour tout $\mathcal{O}_X$-module quasi-cohérent et presque-cohérent $\mathcal{M}$ et tout $q \in \mathbb{N}$, $\mathrm{R}^q f_* \mathcal{M}$ est un $\mathcal{O}_S$-module quasi-cohérent et presque-cohérent.

Notre preuve utilise l'approximation noethérienne, et se rapproche de la preuve de Kiehl de [**Kie72**, 2.9], cf. III.7.

CHAPTER I

Cohomological Descent for Faltings' p-adic Hodge Theory and Applications

I.1. Introduction

I.1.1. Faltings and Scholze's approaches to p-adic Hodge theory share several similarities. The most recent approach, that of Scholze, generalizes Faltings' main techniques from schemes to adic spaces. Nevertheless, beyond the analogies, there is no thread connecting the two. The main difficulty stems from the difference between the nature of their keystones, namely the Faltings topos for Faltings' approach and the pro-étale topos of an adic space for Scholze's approach. Faltings' approach has the advantage of only using schemes and their classical étale topoi. But it depends on the choice of an integral model of the p-adic variety, which intervenes in the very definition of Faltings topos and whose (log-)smoothness seems necessary for the good properties of this one. On the other hand, Scholze's approach which uses adic spaces and their pro-étale topoi, does not depend on any integral model.

The initial goal of this work is to make Faltings' approach "free of integral models". For this, we establish a cohomological descent result for Faltings ringed topos. Along the way, we introduce a variant for the v-topology which satisfies good cohomological descent properties and which can be regarded as a scheme theoretic analogue of the v-topos of an adic space. In particular, we establish a cohomological descent result from this topos to Faltings topos. It is an analogue of the cohomological descent from the v-topos to the pro-étale topos of an adic space established by Scholze [**Sch17**]. We give two applications of our cohomological descent result. Firstly, we extend Faltings' main p-adic comparison theorem (which we refer to as "Faltings' comparison theorem" for short in the rest of the introduction), both in the absolute and the relative cases, to general integral models without any smoothness condition. Faltings' comparison theorem was generalized by Scholze to rigid analytic varieties, first in the smooth case and then in the general case. Our application is an analogue of this last generalization. Even in the smooth case, Faltings' comparison theorem and Scholze's generalization cannot be directly deduced from each other. Secondly, we prove a local version of the relative Hodge-Tate filtration as a consequence of the global version constructed by Abbes and Gros [**AG20**] and our cohomological descent result. We would like to mention a third interesting application of our result to the p-adic Simpson correspondence given by Xu [**Xu22**].

I.1.2. Faltings' proof of the Hodge-Tate decomposition illustrates his approach in p-adic Hodge theory and the role of his ringed topos. Let K be a complete discrete valuation field of characteristic 0 with algebraically closed residue field of characteristic $p > 0$. We fix an algebraic closure $\overline{K}$ of K and denote by $\widehat{\overline{K}}$ the p-adic completion of $\overline{K}$. For a proper smooth K-scheme X, Tate conjectured that there is a canonical $G_K = \mathrm{Gal}(\overline{K}/K)$-equivariant decomposition, now called the Hodge-Tate decomposition ([**Tat67**, Remark, page 180]),

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widehat{\overline{K}} = \bigoplus_{0\leq q\leq n} H^q(X, \Omega^{n-q}_{X/K}) \otimes_K \widehat{\overline{K}}(q-n), \tag{I.1.2.1}$$

where $\widehat{\overline{K}}(q-n)$ is the $(q-n)$-th Tate twist of $\widehat{\overline{K}}$. This conjecture was settled by Faltings [**Fal88, Fal02**] and Tsuji [**Tsu99, Tsu02**] independently, and had been generalized to rigid analytic settings by Scholze [**Sch13a**]. There is also a version for non-proper smooth varieties showed by Faltings. Let X° be an open subset of X whose complement is a normal crossings divisor D. Then, there is a canonical G_K-equivariant decomposition

$$H^n_{\text{ét}}(X^\circ_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \widehat{\overline{K}} = \bigoplus_{0\leq q\leq n} H^q(X, \Omega^{n-q}_{X/K}(\log D)) \otimes_K \widehat{\overline{K}}(q-n). \tag{I.1.2.2}$$

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I.1.3. One of the applications of our main result in this chapter is a generalization of the Hodge-Tate decomposition to the relative case. Let $(f,g):(X'^{\triangleright}\to X')\to(X^{\circ}\to X)$ be a morphism of open immersions of coherent schemes over $\mathrm{Spec}(K)\to\mathrm{Spec}(\mathcal{O}_K)$ ("coherent" stands for "quasi-compact and quasi-separated"). We assume that the following conditions hold:

(1) The associated log schemes $(X',\mathscr{M}_{X'}),(X,\mathscr{M}_X)$ endowed with compactifying log structures are adequate (a technical condition which holds if the open immersions $X'^{\triangleright}\to X', X^{\circ}\to X$ are semi-stable over $\mathcal{O}_K$, cf. I.10.11).
(2) The morphism of log schemes $(X',\mathscr{M}_{X'})\to(X,\mathscr{M}_X)$ is smooth and saturated.
(3) The morphism of schemes $g:X'\to X$ is projective.
(4) The scheme $X=\mathrm{Spec}(R)$ is affine and there exist finitely many nonzero divisors $f_1,\dots,f_r$ of $R[1/p]$ such that the divisor $D=\sum_{i=1}^r\mathrm{div}(f_i)$ on X_K has support $X_K\setminus X_K^{\circ}$ and that at each strict henselization of X_K those elements f_i contained in the maximal ideal form a subset of a regular system of parameters (in particular, D is a normal crossings divisor on X_K).

For any coherent X_K°-scheme Y, we define a pro-finite étale Y-scheme

$$Y_\infty=\lim_n Y[T_1,\dots,T_r]/(T_1^n-f_1,\dots,T_r^n-f_r). \tag{I.1.3.1}$$

Theorem I.1.4 (cf. I.12.7 and I.8.24)**.** *Under the assumptions in* I.1.3*, let U be an affine scheme pro-étale over X and let V be a pro-finite étale $U^{\circ}_{\overline{K},\infty}$-scheme (where $U^{\circ}=X^{\circ}\times_X U$) satisfying the following conditions:*

(1) *The integral closure of U in V is the spectrum of an $\mathcal{O}_{\overline{K}}$-algebra A which is almost perfectoid in the sense of* I.5.19.
(2) *For any integers $n\geq 0$ and $k\geq 0$, the pullback $(\mathrm{R}^n f_{\text{ét}*}\mathbb{Z}/p^k\mathbb{Z})|_{V_{\text{ét}}}$ is a constant sheaf.*

Let $\overline{x}$ be a geometric point of V. Then, for any integer $n\geq 0$, there is a canonical finite decreasing filtration $(\mathrm{fil}^q)_{q\in\mathbb{Z}}$ on $H^n_{\text{ét}}(X^{\triangleright}_{\overline{x}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\widehat{A}[1/p]$ and a canonical isomorphism for each $q\in\mathbb{Z}$,

$$\mathrm{gr}^q(H^n_{\text{ét}}(X^{\triangleright}_{\overline{x}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\widehat{A}[\frac{1}{p}])\cong H^q(X',\Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)})\otimes_R\widehat{A}[\frac{1}{p}](q-n), \tag{I.1.4.1}$$

where gr^q denotes the graded piece $\mathrm{fil}^q/\mathrm{fil}^{q+1}$. Moreover, if U° and V are connected and if the function field of V is a Galois extension of that of U° with Galois group Γ, then the filtration $(\mathrm{fil}^q)_{q\in\mathbb{Z}}$ and the isomorphisms (I.1.4.1) *are Γ-equivariant.*

Remark I.1.5. The objects $V\to U$ satisfying the conditions in I.1.4 form a topological generating family of the pro-étale Faltings site of $X^{\circ}_{\overline{K}}\to X$ (see the proof of I.8.24).

I.1.6. This local *relative Hodge-Tate filtration* stems from the global relative Hodge-Tate filtration constructed by Abbes-Gros [**AG20**]. Their filtration takes place on the Faltings topos associated to $X^{\circ}_{\overline{K}}\to X$. In the first version of their work, they asked for an explicit local version. Scholze and Caraiani [**CS17**] constructed independently a relative Hodge-Tate filtration for proper smooth morphisms of smooth adic spaces, and Scholze announced that he can give a local version, answering the question of Abbes-Gros. Our construction is obtained by applying our cohomological descent result for Faltings ringed topos to the global relative Hodge-Tate filtration of Abbes-Gros. In a new version of their manuscript, Abbes-Gros gave a third construction of the local Hodge-Tate filtration in a slightly more restrictive framework, using a cohomological descent result which is a special case of ours.

I.1.7. Faltings ringed topos plays a central role in the proof of the Hodge-Tate decomposition. Let $X^{\circ}\to X$ be an open immersion of coherent schemes over $\mathrm{Spec}(K)\to\mathrm{Spec}(\mathcal{O}_K)$ such that the associated log scheme $(X,\mathscr{M}_X)$ endowed with compactifying log structure is adequate. We set $Y=X^{\circ}_{\overline{K}}$. The Faltings ringed site $(\mathbf{E}^{\text{ét}}_{Y\to X},\overline{\mathscr{B}})$ was constructed by Faltings and developed by Abbes-Gros [**AGT16**, VI]. Faltings designed it as a bridge between the p-adic étale cohomology of Y and differential forms of X. Concretely, these links are established through natural morphisms of sites

$$Y_{\text{ét}}\xrightarrow{\psi}\mathbf{E}^{\text{ét}}_{Y\to X}\xrightarrow{\sigma}X_{\text{ét}} \tag{I.1.7.1}$$

which satisfy the following properties:

(1) (Faltings' comparison theorem, [**Fal02**, Thm.8, page 223], [**AG20**, 4.8.13]). Assume that X is proper over $\mathcal{O}_K$. For any finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$, there exists a canonical morphism

$$\mathrm{R}\Gamma(Y_{\text{ét}},\mathbb{F})\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathcal{O}_{\overline{K}}\longrightarrow\mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F}\otimes_{\mathbb{Z}}\overline{\mathscr{B}}), \tag{I.1.7.2}$$

which is an almost isomorphism, that is, the cohomology groups of its cone are killed by p^r for any rational number $r > 0$.

(2) (Faltings' computation of Galois cohomology, [**AG20**, 6.3.8]). There exists a canonical homomorphism of $\mathcal{O}_X \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$-modules

$$\Omega^q_{(X,\mathscr{M}_X)/(S,\mathscr{M}_S)} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \longrightarrow \mathrm{R}^q\sigma_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \tag{I.1.7.3}$$

whose kernel and cokernel are killed by p^r for any rational number $r > \frac{2\dim(Y)+1}{p-1}$.

Observing that $\mathbb{Z}/p^n\mathbb{Z} = \psi_*(\mathbb{Z}/p^n\mathbb{Z})$, Faltings deduced the Hodge-Tate decomposition from the degeneration and splitting of the Cartan-Leray spectral sequence for the composed functor $\mathrm{R}\Gamma(X_{\text{ét}}, -)\circ\mathrm{R}\sigma_*$, later named the Hodge-Tate spectral sequence by Scholze. Using de Jong's alteration theorem, one can deduce the Hodge-Tate decomposition for a general proper smooth K-scheme by reducing to the case where it admits a semi-stable model (cf. [**Tsu02**, A5]).

I.1.8. In order to state our cohomological descent result, we recall now the definition of the Faltings site associated to a morphism of coherent schemes $Y \to X$ (cf. I.7.7). Let $\mathbf{E}^{\text{ét}}_{Y\to X}$ be the category of morphisms of coherent schemes $V \to U$ over $Y \to X$, i.e. commutative diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{I.1.8.1}$$

such that U is étale over X and that V is finite étale over $Y \times_X U$. We endow $\mathbf{E}^{\text{ét}}_{Y\to X}$ with the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where M is a finite set and $\coprod_{m\in M} V_m \to V$ is surjective;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where N is a finite set and $\coprod_{n\in N} U_n \to U$ is surjective.

Consider the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ defined by

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{I.1.8.2}$$

where U^V is the integral closure of U in V. It is indeed a sheaf of rings, the structural sheaf of $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. I.7.6).

I.1.9. Recall that the cohomological descent of étale cohomology along proper hypercoverings can be generalized as follows: for a coherent S-scheme, we endow the category of coherent S-schemes $\mathbf{Sch}^{\text{coh}}_{/S}$ with Voevodsky's *h-topology* which is generated by étale coverings and proper surjective morphisms of finite presentation. Then, for any torsion abelian sheaf $\mathcal{F}$ on $S_{\text{ét}}$, denoting by $a : (\mathbf{Sch}^{\text{coh}}_{/S})_{\text{h}} \to S_{\text{ét}}$ the natural morphism of sites, the adjunction morphism $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ is an isomorphism.

This result remains true for a finer topology, the *v-topology*. A morphism of coherent schemes $T \to S$ is called a v-covering if for any morphism $\mathrm{Spec}(A) \to S$ with A a valuation ring, there exists an extension of valuation rings $A \to B$ and a lifting $\mathrm{Spec}(B) \to T$. In fact, a v-covering is a limit of h-coverings (cf. I.3.6). We will describe the cohomological descent for $\overline{\mathscr{B}}$ using a new site built from the v-topology, which can be regarded as a scheme theoretic analogue of the v-site of adic spaces (cf. [**Sch17**, 8.1, 14.1, 15.5]).

Definition I.1.10 (cf. I.3.23)**.** Let $S^\circ \to S$ be an open immersion of coherent schemes such that S is integrally closed in S°. We define a site $\mathbf{I}_{S^\circ\to S}$ as follows:

(1) The underlying category is formed by coherent S-schemes T which are integrally closed in $S^\circ \times_S T$.
(2) The topology is generated by covering families $\{T_i \to T\}_{i\in I}$ in the v-topology.

We call $\mathbf{I}_{S^\circ\to S}$ the *v-site of S°-integrally closed coherent S-schemes*, and we call the sheaf $\mathscr{O}$ on $\mathbf{I}_{S^\circ\to S}$ associated to the presheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ the structural sheaf of $\mathbf{I}_{S^\circ\to S}$.

I.1.11. Let p be a prime number, $\overline{\mathbb{Z}_p}$ the integral closure of $\mathbb{Z}_p$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We take $S^\circ = \mathrm{Spec}(\overline{\mathbb{Q}_p})$ and $S = \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Consider a diagram of coherent schemes

$$\begin{array}{ccccc} Y & \longrightarrow & X^Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(\overline{\mathbb{Q}_p}) & \longrightarrow & \mathrm{Spec}(\overline{\mathbb{Z}_p}) & & \end{array} \tag{I.1.11.1}$$

where X^Y is the integral closure of X in Y and the square is Cartesian (we don't impose any condition on the regularity or finiteness of Y or X). The functor $\varepsilon^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{I}_{Y\to X^Y}$ sending $V \to U$ to U^V defines a natural morphism of ringed sites

$$\varepsilon : (\mathbf{I}_{Y\to X^Y}, \mathscr{O}) \longrightarrow (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}). \tag{I.1.11.2}$$

Our cohomological descent results are stated as follows, which can be regarded as a scheme theoretic analogue of the cohomological descent result for the pro-étale site of an adic space (cf. [**Sch17**, 8.8, 14.7, 15.5]):

Theorem I.1.12 (Cohomological descent for Faltings ringed sites, cf. I.8.14)**.** *For any finite locally constant abelian sheaf $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{I.1.12.1}$$

is an almost isomorphism.

Corollary I.1.13 (cf. I.8.18)**.** *For any proper hypercovering $X_\bullet \to X$, if $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ denotes the augmentation of simplicial site where $Y_\bullet = Y \times_X X_\bullet$, then the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \tag{I.1.13.1}$$

is an almost isomorphism.

The key ingredient of our proof of I.1.12 is the descent of perfectoid algebras in the arc-topology (a topology finer than the v-topology) due to Bhatt-Scholze [**BS19**, 8.10] (cf. I.5.35). The analogue in characteristic p of I.1.12 is Gabber's computation of the cohomology of the structural sheaf in the h-topology (cf. I.4). Theorem I.1.12 allows us to descend important results for Faltings sites with nice models to Faltings sites associated to general models. On the other hand, its proof shows how to compute the cohomologies of Faltings ringed sites locally. Using Abhyankar's lemma, one can treat the open case which in the generic fibre is the complement of a normal crossings divisor.

Corollary I.1.14 (cf. I.8.24)**.** *Under the assumptions in* I.1.4 *and with the same notation, for any integer $n > 0$, the canonical morphism*

$$A/p^nA \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{V\to U}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \tag{I.1.14.1}$$

is an almost isomorphism.

Thus, we apply the derived functor $\mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{V\to U}, -)$ to the global relative Hodge-Tate filtration defined on the Faltings ringed site by Abbes-Gros, and then we obtain the local version I.1.4.

I.1.15. On the other hand, we use I.1.12 to generalize Faltings' comparison theorem in the absolute case. Let A be a valuation ring extension of $\overline{\mathbb{Z}_p}$ with algebraically closed fraction field. Consider a Cartesian square of coherent schemes

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A[\frac{1}{p}]) & \longrightarrow & \mathrm{Spec}(A) \end{array} \tag{I.1.15.1}$$

Theorem I.1.16 (Faltings' comparison theorem in the absolute case, cf. I.10.17)**.** *Assume that X is proper of finite presentation over A. Then, for any finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\text{ét}}$, there exists a canonical morphism*

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{I.1.16.1}$$

which is an almost isomorphism.

We remark that the natural morphism $\psi : Y_{\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ induces an equivalence of the categories of finite locally constant abelian sheaves on $Y_{\text{ét}}$ and $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. I.10.3),

$$\mathbf{LocSys}(Y_{\text{ét}}) \underset{\psi^{-1}}{\overset{\psi_*}{\rightleftarrows}} \mathbf{LocSys}(\mathbf{E}^{\text{ét}}_{Y\to X}). \tag{I.1.16.2}$$

As a continuation of the work of Abbes-Gros, the canonical morphism (I.1.16.1) (refered as *Faltings' comparison morphism*) is constructed using the acyclicity of ψ for $\mathbb{F}$, i.e. $\psi_*\mathbb{F} = \mathrm{R}\psi_*\mathbb{F}$ (so that $\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) = \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F})$), which is a consequence of Achinger's result on $K(\pi, 1)$-schemes (cf. I.10.6 and I.10.8). We also propose a new way to construct Faltings' comparison morphism in the

derived category of almost modules using our cohomological descent result I.1.12, which avoids using the acyclicity of ψ. Indeed, there are natural morphisms of sites

$$\begin{array}{ccc} (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a} & Y_{\text{ét}} \\ \Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \psi} \\ \mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X} \end{array} \tag{I.1.16.3}$$

and Ψ is acyclic for any torsion abelian sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, i.e. $\Psi_*(a^{-1}\mathcal{F}) = \mathrm{R}\Psi_*(a^{-1}\mathcal{F})$, which allows more general coefficients and whose proof is much easier than that of ψ (cf. I.3.27). We remark that this new construction won't give us a "real morphism" (I.1.16.1) but a canonical morphism in the derived category of almost modules (cf. I.11.6).

We briefly explain the strategy for proving I.1.16:

(1) Firstly, we use de Jong-Gabber-Illusie-Temkin's alteration theorem for morphisms of schemes [**ILO14**, X.3] to obtain a proper surjective morphism of finite presentation $X' \to X$ such that the morphism $X' \to \mathrm{Spec}(A)$ is the cofiltered limit of a system of "nice" morphisms $X'_\lambda \to T_\lambda$ of "nice" models over $\mathcal{O}_{K_\lambda}$, where K_λ is a finite extension of $\mathbb{Q}_p$ (cf. I.9.11).
(2) Then, we can apply Faltings' comparison theorem in the relative case to the "nice" morphisms $X'_\lambda \to T_\lambda$ (formulated by Faltings [**Fal02**, Thm.6, page 266] and proved by Abbes-Gros [**AG20**, 5.7.4], cf. I.10.13). By a limit argument, we get the comparison theorem for X'.
(3) Finally, using our cohomological descent result I.1.13, we deduce the comparison theorem for X.

I.1.17. The site $\mathbf{I}_{Y\to X^Y}$ is also appropriate to globalize Faltings' comparison theorem. Consider a Cartesian square of coherent schemes

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \Big\downarrow & & \Big\downarrow \\ Y & \longrightarrow & X \end{array} \tag{I.1.17.1}$$

where $Y \to X$ is Cartesian over $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\overline{\mathbb{Z}_p})$. In particular, there is a natural morphism of ringed sites by the functoriality of (I.1.16.3),

$$f_{\mathbf{I}} : (\mathbf{I}_{Y'\to X'^{Y'}}, \mathscr{O}') \longrightarrow (\mathbf{I}_{Y\to X^Y}, \mathscr{O}). \tag{I.1.17.2}$$

Theorem I.1.18 (cf. I.11.11)**.** *Assume that $X' \to X$ is proper of finite presentation. Let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$ and $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$* (cf. (I.1.16.3))*. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{I.1.18.1}$$

is an almost isomorphism.

We remark that if $\mathcal{F}' = \mathbb{Z}/p^n\mathbb{Z}$ then $\mathscr{F}' = \mathbb{Z}/p^n\mathbb{Z}$ (cf. I.3.27), and that $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ is the sheafification of étale cohomologies of Y' over Y with coefficient $\mathcal{F}'$ in the v-topology (cf. I.11.12). Very roughly speaking, objects of $\mathbf{I}_{Y\to X^Y}$ are "locally" the spectrums of valuation rings, and the "stalks" of (I.1.18.1) are Faltings' comparison morphisms (I.1.16.1) when $\mathcal{F}'$ is finite locally constant (cf. I.11.5). Theorem I.1.18 can be regarded as a scheme theoretical analogue of Scholze's comparison theorem for p-adic étale cohomology of a morphism of rigid analytic varieties [**Sch13b**, 3.13].

Finally, we generalize Faltings' comparison theorem in the relative case using I.1.12 and I.1.18.

Theorem I.1.19 (Faltings' comparison theorem in the relative case, cf. I.11.13 and I.11.14)**.** *Assume that $Y' \to Y$ is smooth and that $X' \to X$ is proper of finite presentation. Then, for any finite locally constant abelian sheaf $\mathbb{F}'$ on $Y'_{\text{ét}}$, there exists a canonical morphism*

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{I.1.19.1}$$

which is an almost isomorphism, and where $f_{\text{ét}} : Y'_{\text{ét}} \to Y_{\text{ét}}$ and $f_{\mathbf{E}} : \mathbf{E}^{\text{ét}}_{Y'\to X'} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ are the natural morphisms of sites. In particular, there exists a canonical morphism

$$(\psi_*\mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{I.1.19.2}$$

which is an almost isomorphism, for any integer q.

I.1.20. The paper is structured as follows. In section I.3, we establish the foundation of the site $\mathbf{I}_{S^\circ\to S}$, where proposition I.3.27 discussing the cohomological properties of $\Psi : (\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \to \mathbf{I}_{S^\circ\to S}$ is the key to our new construction of Faltings' comparison morphism (cf. I.11.6). Sections I.4 and I.5 are devoted to a detailed proof of the arc-descent for perfectoid algebras following Bhatt-Scholze [**BS19**, 8.10]. Since we use the language of schemes, the terminology "pre-perfectoid" is introduced for those algebras whose p-adic completions are perfectoid. In sections I.6 and I.7, we review the definition and some basic properties of Faltings sites and we introduce a pro-version of Faltings site to evaluate the structural sheaf on the spectrums of pre-perfectoid algebras. Then, we prove our cohomological descent result in section I.8. In section I.9, we review de Jong-Gabber-Illusie-Temkin's alteration theorem and apply it to schemes over a valuation ring of height 1. Section I.10 is devoted to proving our generalization of Faltings' comparison theorem in the absolute case. In section I.11, we give a new construction of Faltings' comparison morphism and our generalization of Faltings' comparison theorem in the relative case. Finally, we deduce from the global relative Hodge-Tate filtration an explicit local version in section I.12.

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I.2. Notation and Conventions

I.2.1. We fix a prime number p throughout this paper. For any monoid M, we denote by $\mathrm{Frob} : M \to M$ the map sending an element x to x^p and we call it the *Frobenius* of M. For a ring R, we denote by $R^\times$ the group of units of R. A ring R is called *absolutely integrally closed* if any monic polynomial $f \in R[T]$ has a root in R ([**Sta22**, 0DCK]). We remark that quotients, localizations and products of absolutely integrally closed rings are still absolutely integrally closed.

Recall that a valuation ring is a domain V such that for any element x in its fraction field, if $x \notin V$ then $x^{-1} \in V$. The family of ideals of V is totally ordered by the inclusion relation ([**Bou06**, VI.§1.2, Thm.1]). In particular, a radical ideal of V is a prime ideal. Moreover, any quotient of V by a prime ideal and any localization of V are still valuations rings ([**Sta22**, 088Y]). We remark that V is normal, and that V is absolutely integrally closed if and only if its fraction field is algebraically closed. An *extension of valuation rings* is an injective and local homomorphism of valuation rings.

I.2.2. Following [**SGA 4**$_{\mathrm{II}}$, VI.1.22], a *coherent* scheme (resp. morphism of schemes) stands for a quasi-compact and quasi-separated scheme (resp. morphism of schemes). For a coherent morphism $Y \to X$ of schemes, we denote by X^Y the integral closure of X in Y ([**Sta22**, 0BAK]). For an X-scheme Z, we say that Z is *Y-integrally closed* if $Z = Z^{Y\times_X Z}$.

I.2.3. Throughout this paper, we fix two universes $\mathbb{U}$ and $\mathbb{V}$ such that the set of natural numbers $\mathbb{N}$ is an element of $\mathbb{U}$ and that $\mathbb{U}$ is an element of $\mathbb{V}$ ([**SGA 4**$_{\mathrm{I}}$, I.0]). In most cases, we won't emphasize this set theoretical issue. Unless stated otherwise, we only consider $\mathbb{U}$-small schemes and we denote by **Sch** the category of $\mathbb{U}$-small schemes, which is a $\mathbb{V}$-small category.

I.2.4. Let C be a category. We denote by $\widehat{C}$ the category of presheaves of $\mathbb{V}$-small sets on C. If C is a $\mathbb{V}$-site ([**SGA 4**$_{\mathrm{I}}$, II.3.0.2]), we denote by $\widetilde{C}$ the topos of sheaves of $\mathbb{V}$-small sets on C. We denote by $h^C : C \to \widehat{C}$, $x \mapsto h^C_x$ the Yoneda embbeding ([**SGA 4**$_{\mathrm{I}}$, I.1.3]), and by $\widehat{C} \to \widetilde{C}$, $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{a}}$ the sheafification functor ([**SGA 4**$_{\mathrm{I}}$, II.3.4]). Unless stated otherwise, a site in this paper stands for a site where *all finite limits are representable*.

I.2.5. Let $u^+ : C \to D$ be a functor of categories. We denote by $u^{\mathrm{p}} : \widehat{D} \to \widehat{C}$ the functor that associates to a presheaf $\mathcal{G}$ of $\mathbb{V}$-small sets on D the presheaf $u^{\mathrm{p}}\mathcal{G} = \mathcal{G} \circ u^+$. If C is $\mathbb{V}$-small and D is a $\mathbb{V}$-category, then u^{p} admits a left adjoint u_{p} [**Sta22**, 00VC] and a right adjoint ${}_{\mathrm{p}}u$ [**Sta22**, 00XF] (cf. [**SGA 4**$_{\mathrm{I}}$, I.5]). So we have a sequence of adjoint functors

$$u_{\mathrm{p}},\ u^{\mathrm{p}},\ {}_{\mathrm{p}}u. \tag{I.2.5.1}$$

If moreover C and D are $\mathbb{V}$-sites, then we denote by $u_{\mathrm{s}}, u^{\mathrm{s}}, {}_{\mathrm{s}}u$ the functors of the topoi $\widetilde{C}$ and $\widetilde{D}$ of sheaves of $\mathbb{V}$-small sets induced by composing the sheafification functor with the functors $u_{\mathrm{p}}, u^{\mathrm{p}}, {}_{\mathrm{p}}u$ respectively. As we only consider finite complete sites, we say that the functor u^+ gives a *morphism of sites*, if u^+ is left exact and preserves covering families ([**SGA 4**$_{\mathrm{I}}$, IV.4.9.2]). Then, we denote by

$$u = (u^{-1}, u_*) : \widetilde{D} \to \widetilde{C} \tag{I.2.5.2}$$

the associated morphism of topoi, where $u^{-1} = u_{\mathrm{s}}$ and $u_* = u^{\mathrm{s}} = u^{\mathrm{p}}|_{\widetilde{D}}$. If moreover u is a morphism of ringed topoi, then we denote by $u^* = \mathcal{O}_{\widetilde{D}} \otimes_{u^{-1}\mathcal{O}_{\widetilde{C}}} u^{-1}$ the pullback functor of modules. We remark that the notation here, adopted by [**Sta22**], is slightly different with that in [**SGA 4**$_{\mathrm{I}}$] (cf. [**Sta22**, 0CMZ]).

I.2.6. Let C be an additive category. The *catgory of objects of C up to isogeny* (cf. [**AGT16**, III.6]) is the category $C_{\mathbb{Q}}$ with a functor $F : C \to C_{\mathbb{Q}}$ (called the localization functor) such that $\mathrm{Ob}(C_{\mathbb{Q}}) = \mathrm{Ob}(C)$ and that

$$\mathrm{Hom}_{C_{\mathbb{Q}}}(M_{\mathbb{Q}}, N_{\mathbb{Q}}) = \mathrm{Hom}_C(M, N) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{I.2.6.1}$$

where we denote by $M_{\mathbb{Q}}$ the image of an object M of C via F in $C_{\mathbb{Q}}$.

For a ringed site $(C, \mathcal{O})$, we denote by $\mathcal{O}$-$\mathbf{Mod}_{\mathbb{Q}}$ the category of $\mathcal{O}$-modules up to isogeny, whose objects are called *$\mathcal{O}_{\mathbb{Q}}$-modules*. It is an abelian category and the localization functor $\mathcal{O}$-$\mathbf{Mod} \to \mathcal{O}$-$\mathbf{Mod}_{\mathbb{Q}}$ sends injective objects to injective objects. We remark that if $\mathcal{O}$ is a $\mathbb{Q}$-module, then $\mathcal{O}$-$\mathbf{Mod} \to \mathcal{O}$-$\mathbf{Mod}_{\mathbb{Q}}$ is an equivalence. A morphism of ringed sites $u : (C, \mathcal{O}) \to (C', \mathcal{O}')$ induces a pair of adjoint functors

$$u^* : \mathcal{O}'\text{-}\mathbf{Mod}_{\mathbb{Q}} \to \mathcal{O}\text{-}\mathbf{Mod}_{\mathbb{Q}},\ u_* : \mathcal{O}\text{-}\mathbf{Mod}_{\mathbb{Q}} \to \mathcal{O}'\text{-}\mathbf{Mod}_{\mathbb{Q}}. \tag{I.2.6.2}$$

The derived functor $\mathrm{R}u_*$ commutes with the localization functor.

I.3. The v-site of Integrally Closed Schemes

Definition I.3.1. Let $X \to Y$ be a quasi-compact morphism of schemes.

(1) We say that $X \to Y$ is a *v-covering*, if for any valuation ring V and any morphism $\mathrm{Spec}(V) \to Y$, there exists an extension of valuation rings $V \to W$ (I.2.1) and a commutative diagram (cf. [**Sta22**, 0ETN])

$$\begin{array}{ccc} \mathrm{Spec}(W) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \longrightarrow & Y \end{array} \tag{I.3.1.1}$$

(2) Let π be an element of $\Gamma(Y, \mathcal{O}_Y)$. We say that $X \to Y$ is an *arc-covering* (resp. *π-complete arc-covering*), if for any valuation ring (resp. π-adically complete valuation ring) V of height ≤ 1 and any morphism $\mathrm{Spec}(V) \to Y$, there exists an extension of valuation rings (resp. π-adically complete valuation rings) $V \to W$ of height ≤ 1 and a commutative diagram (I.3.1.1) (cf. [**BM20**, 1.2], [**CS19**, 2.2.1]).

(3) We say that $X \to Y$ is an *h-covering*, if it is a v-covering and locally of finite presentation (cf. [**Sta22**, 0ETS]).

We note that an arc-covering is simply a 0-complete arc-covering.

Lemma I.3.2. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be quasi-compact morphisms of schemes, $\pi \in \Gamma(X, \mathcal{O}_X)$, $\tau \in \{h, v, \pi\text{-complete arc}\}$.*

(1) *If f is a τ-covering, then any base change of f is also a τ-covering.*
(2) *If f and g are τ-coverings, then $f \circ g$ is also a τ-covering.*
(3) *If $f \circ g$ is a τ-covering (and if f is locally of finite presentation when $\tau = h$), then f is also a τ-covering.*

Proof. It follows directly from the definitions. □

I.3.3. Let $\mathbf{Sch}^{\mathrm{coh}}$ be the category of coherent $\mathbb{U}$-small schemes, $\tau \in \{\mathrm{h}, \mathrm{v}, \mathrm{arc}\}$. We endow $\mathbf{Sch}^{\mathrm{coh}}$ with the τ-topology generated by the pretopology formed by families of morphisms $\{X_i \to X\}_{i \in I}$ with I finite such that $\coprod_{i \in I} X_i \to X$ is a τ-covering, and we denote the corresponding site by $\mathbf{Sch}^{\mathrm{coh}}_{\tau}$. It is clear that a morphism $Y \to X$ (which is locally of finite presentation if $\tau = \mathrm{h}$) is a τ-covering if and only if $\{Y \to X\}$ is a covering family in $\mathbf{Sch}^{\mathrm{coh}}_{\tau}$ by I.3.2 and [**SGA 4**$_{\mathrm{I}}$, II.1.4].

For any coherent $\mathbb{U}$-small scheme X, we endow the category $\mathbf{Sch}^{\mathrm{coh}}_{/X}$ of objects of $\mathbf{Sch}^{\mathrm{coh}}$ over X with the topology induced by the τ-topology of $\mathbf{Sch}^{\mathrm{coh}}$, i.e. the topology generated by the pretopology formed by families of X-morphisms $\{Y_i \to Y\}_{i \in I}$ with I finite such that $\coprod_{i \in I} Y_i \to Y$ is a τ-covering ([**SGA 4**$_{\mathrm{I}}$, III.5.2]). For any sheaf $\mathcal{F}$ of $\mathbb{V}$-small abelian groups on the site $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\tau}$, we denote its q-th cohomology by $H^q_{\tau}(X, \mathcal{F})$.

Lemma I.3.4. *Let $f : X \to Y$ be a quasi-compact morphism of schemes, $\pi \in \Gamma(Y, \mathcal{O}_Y)$.*

(1) *If f is proper surjective or faithfully flat, then f is a v-covering.*
(2) *If f is an h-covering and Y is affine, then there exists a proper surjective morphism $Y' \to Y$ of finite presentation and a finite affine open covering $Y' = \bigcup_{r=1}^{n} Y'_r$ such that $Y'_r \to Y$ factors through f for each r.*
(3) *If f is an h-covering and if there exists a directed inverse system $(f_\lambda : X_\lambda \to Y_\lambda)_{\lambda\in\Lambda}$ of finitely presented morphisms of coherent schemes with affine transition morphisms $\psi_{\lambda'\lambda} : X_{\lambda'} \to X_\lambda$ and $\phi_{\lambda'\lambda} : Y_{\lambda'} \to Y_\lambda$ such that $X = \lim X_\lambda$, $Y = \lim Y_\lambda$ and that f_λ is the base change of f_{λ_0} by $\phi_{\lambda\lambda_0}$ for some index $\lambda_0 \in \Lambda$ and any $\lambda \geq \lambda_0$, then there exists an index $\lambda_1 \geq \lambda_0$ such that f_λ is an h-covering for any $\lambda \geq \lambda_1$.*
(4) *If f is a v-covering, then it is a π-complete arc-covering.*
(5) *Let π' be another element of $\Gamma(Y, \mathcal{O}_Y)$ which divides π. If f is a π-complete arc-covering, then it is a π'-complete arc-covering.*
(6) *If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a π-complete arc-covering, then the morphism $\mathrm{Spec}(\widehat{B}) \to \mathrm{Spec}(\widehat{A})$ between the spectrums of their π-adic completions is also a π-complete arc-covering.*

Proof. (1), (2) are proved in [**Sta22**, 0ETK, 0ETU] respectively.

(3) To show that one can take $\lambda_1 \geq \lambda_0$ such that f_{λ_1} is an h-covering, we may assume that Y_{λ_0} is affine by replacing it by a finite affine open covering by I.3.2 and (1). Thus, applying (2) to the h-covering f and using [**EGA** IV_3, 8.8.2, 8.10.5], there exists an index $\lambda_1 \geq \lambda_0$, a proper surjective morphism $Y'_{\lambda_1} \to Y_{\lambda_1}$ and a finite affine open covering $Y'_{\lambda_1} = \bigcup_{r=1}^{n} Y'_{r\lambda_1}$ such that the morphisms $Y'_r \to Y' \to Y$ are the base changes of the morphisms $Y'_{r\lambda_1} \to Y'_{\lambda_1} \to Y_{\lambda_1}$ by the transition morphism $Y \to Y_{\lambda_1}$, and that $Y'_{r\lambda_1} \to Y_{\lambda_1}$ factors through X_{λ_1}. This shows that f_{λ_1} is an h-covering by I.3.2 and (1).

(4) With the notation in (I.3.1.1), if V is a π-adically complete valuation ring of height ≤ 1 with maximal ideal $\mathfrak{m}$, then since the family of prime ideals of W is totally ordered by the inclusion relation (I.2.1), we take the maximal prime ideal $\mathfrak{p} \subseteq W$ over $0 \subseteq V$ and the minimal prime ideal $\mathfrak{q} \subseteq W$ over $\mathfrak{m} \subseteq V$. Then, $\mathfrak{p} \subseteq \mathfrak{q}$ and $W' = (W/\mathfrak{p})_{\mathfrak{q}}$ over V is an extension of valuation rings of height ≤ 1. Since $\pi \in \mathfrak{m}$ and W' is of height ≤ 1, the π-adic completion $\widehat{W'}$ is still a valuation ring extension of V of height ≤ 1 (cf. [**Bou06**, VI.§5.3, Prop.5]), which proves (4).

(5) Since a π'-adically complete valuation ring V is also π-adically complete ([**Sta22**, 090T]), there exists a lifting $\mathrm{Spec}(W) \to X$ for any morphism $\mathrm{Spec}(V) \to Y$. After replacing W by its π'-adic completion, the conclusion follows.

(6) Let V be a π-adically complete valuation ring of height ≤ 1. Given a morphism $\widehat{A} \to V$, there exists a lifting $B \to W$ where $V \to W$ is an extension of π-adically complete valuation rings of height ≤ 1. It is clear that $B \to W$ factors through $\widehat{B}$, which proves (6). □

I.3.5. Let X be a coherent scheme, $\mathbf{Sch}^{\mathrm{fp}}_{/X}$ the full subcategory of $\mathbf{Sch}^{\mathrm{coh}}_{/X}$ formed by finitely presented X-schemes. We endow it with the topology generated by the pretopology formed by families of morphisms $\{Y_i \to Y\}_{i\in I}$ with I finite such that $\coprod_{i\in I} Y_i \to Y$ is an h-covering, and we denote the corresponding site by $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. It is clear that this topology coincides with the topologies induced from $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ and from $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}}$. The inclusion functors $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}} \xrightarrow{\xi^+} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\zeta^+} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ define morphisms of sites (I.2.5)

$$(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \xrightarrow{\zeta} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\xi} (\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}. \tag{I.3.5.1}$$

Lemma I.3.6. *Let X be a coherent scheme. Then, for any covering family $\mathfrak{U} = \{Y_i \to Y\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}}$ with I finite,*

(i) *there exists a directed inverse system $(Y_\lambda)_{\lambda\in\Lambda}$ of finitely presented X-schemes with affine transition morphisms such that $Y = \lim Y_\lambda$, and*
(ii) *for each $i \in I$, there exists a directed inverse system $(Y_{i\lambda})_{\lambda\in\Lambda}$ of finitely presented X-schemes with affine transition morphisms over the inverse system $(Y_\lambda)_{\lambda\in\Lambda}$ such that $Y_i = \lim Y_{i\lambda}$, and*
(iii) *for each $\lambda \in \Lambda$, the family $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ is a covering in $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$.*

Proof. We take a directed set A such that for each $i \in I$, we can write Y_i as a cofiltered limit of finitely presented Y-schemes $Y_i = \lim_{\alpha\in A} Y_{i\alpha}$ with affine transition morphisms ([**Sta22**, 09MV]). We see that $\coprod_{i\in I} Y_{i\alpha} \to Y$ is an h-covering for each $\alpha \in A$ by I.3.2.

We write Y as a cofiltered limit of finitely presented X-schemes $Y = \lim_{\beta\in B} Y_\beta$ with affine transition morphisms ([**Sta22**, 09MV]). By [**EGA** IV_3, 8.8.2, 8.10.5] and I.3.4.(3), for each $\alpha \in A$, there exists an index $\beta_\alpha \in B$ such that the morphism $Y_{i\alpha} \to Y$ is the base change of a finitely

presented morphism $Y_{i\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ by the transition morphism $Y \to Y_{\beta_\alpha}$ for each $i \in I$, and that $\coprod_{i\in I} Y_{i\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ is an h-covering. For each $\beta \geq \beta_\alpha$, let $Y_{i\alpha\beta}$ be the base change of $Y_{i\alpha\beta_\alpha}$ by $Y_\beta \to Y_{\beta_\alpha}$.

We define a category Λ^{op}, whose set of objects is $\{(\alpha,\beta) \in A \times B \mid \beta \geq \beta_\alpha\}$, and for any two objects $\lambda' = (\alpha',\beta')$, $\lambda = (\alpha,\beta)$, the set $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\lambda',\lambda)$ is

(i) the subset of $\prod_{i\in I} \mathrm{Hom}_{Y_{\beta'}}(Y_{i\alpha'\beta'}, Y_{i\alpha\beta'})$ formed by elements $f = (f_i)_{i\in I}$ such that for each $i \in I$, $f_i : Y_{i\alpha'\beta'} \to Y_{i\alpha\beta'}$ is affine and the base change of f_i by $Y \to Y_{\beta'}$ is the transition morphism $Y_{i\alpha'} \to Y_{i\alpha}$, if $\alpha' \geq \alpha$ and $\beta' \geq \beta$;
(ii) empty, if else.

The composition of morphisms $(g_i : Y_{i\alpha''\beta''} \to Y_{i\alpha'\beta''})_{i\in I}$ with $(f_i : Y_{i\alpha'\beta'} \to Y_{i\alpha\beta'})_{i\in I}$ in Λ^{op} is $(g_i \circ f_i' : Y_{i\alpha''\beta''} \to Y_{i\alpha\beta''})$, where f_i' is the base change of f_i by the transition morphism $Y_{\beta''} \to Y_{\beta'}$. We see that Λ^{op} is cofiltered by [**EGA** IV_3, 8.8.2]. Let Λ be the opposite category of Λ^{op}. For each $i \in I$ and $\lambda = (\alpha,\beta) \in \Lambda$, we set $Y_\lambda = Y_\beta$ and $Y_{i\lambda} = Y_{i\alpha\beta}$. It is clear that the natural functors $\Lambda \to A$ and $\Lambda \to B$ are cofinal ([**SGA** $\mathbf{4}_{\mathrm{I}}$, I.8.1.3]). After replacing Λ by a directed set ([**Sta22**, 0032]), the families $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ satisfy the required conditions. □

Lemma I.3.7. *With the notation in* I.3.5*, let $\mathcal{F}$ be a presheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, (Y_λ) a directed inverse system of finitely presented X-schemes with affine transition morphisms, $Y = \lim Y_\lambda$. Then, we have $\nu_{\mathrm{p}}\mathcal{F}(Y) = \operatorname{colim} \mathcal{F}(Y_\lambda)$, where $\nu^+ = \xi^+$ (resp. $\nu^+ = \zeta^+ \circ \xi^+$).*

Proof. Notice that the presheaf $\mathcal{F}$ is a filtered colimit of representable presheaves by [**SGA** $\mathbf{4}_{\mathrm{I}}$, I.3.4]

$$\mathcal{F} = \operatorname*{colim}_{Y' \in (\mathbf{Sch}^{\mathrm{fp}}_{/X})_{/\mathcal{F}}} h_{Y'}. \tag{I.3.7.1}$$

Thus, we may assume that $\mathcal{F}$ is representable by a finitely presented X-scheme Y' since the section functor $\Gamma(Y,-)$ commutes with colimits of presheaves ([**Sta22**, 00VB]). Then, we have

$$\nu_{\mathrm{p}} h_{Y'}(Y) = h_{\nu^+(Y')}(Y) = \mathrm{Hom}_X(Y,Y') = \operatorname{colim} \mathrm{Hom}_X(Y_\lambda, Y') = \operatorname{colim} h_{Y'}(Y_\lambda) \tag{I.3.7.2}$$

where the first equality follows from [**Sta22**, 04D2], and the third equality follows from [**EGA** IV_3, 8.14.2]. □

Proposition I.3.8. *With the notation in* I.3.5*, let $\mathcal{F}$ be an abelian sheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, (Y_λ) a directed inverse system of finitely presented X-schemes with affine transition morphisms, $Y = \lim Y_\lambda$. Let $\tau = \mathrm{h}$ and $\nu^+ = \xi^+$ (resp. $\tau = \mathrm{v}$ and $\nu^+ = \zeta^+ \circ \xi^+$). Then, for any integer q, we have*

$$H^q_\tau(Y, \nu^{-1}\mathcal{F}) = \operatorname{colim} H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y_\lambda})_{\mathrm{h}}, \mathcal{F}). \tag{I.3.8.1}$$

In particular, the canonical morphism $\mathcal{F} \longrightarrow \mathrm{R}\nu_*\nu^{-1}\mathcal{F}$ is an isomorphism.

Proof. For the second assertion, the sheaf $\mathrm{R}^q\nu_*\nu^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y \mapsto H^q_\tau(Y, \nu^{-1}\mathcal{F}) = H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y})_{\mathrm{h}}, \mathcal{F})$ by the first assertion, which is $\mathcal{F}$ if $q = 0$ and vanishes otherwise.

We claim that it suffices to show that (I.3.8.1) holds for any injective abelian sheaf $\mathcal{F} = \mathcal{I}$ on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. Indeed, if so, then we prove by induction on q that (I.3.8.1) holds in general. The case where $q \leq -1$ is trivial. We set $H^q_1(\mathcal{F}) = H^q_\tau(Y, \nu^{-1}\mathcal{F})$ and $H^q_2(\mathcal{F}) = \operatorname{colim} H^q((\mathbf{Sch}^{\mathrm{fp}}_{/Y_\lambda})_{\mathrm{h}}, \mathcal{F})$. We embed an abelian sheaf $\mathcal{F}$ to an injective abelian sheaf $\mathcal{I}$. Consider the exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ and the morphism of long exact sequences

$$\begin{array}{ccccccccc}
H^{q-1}_1(\mathcal{I}) & \longrightarrow & H^{q-1}_1(\mathcal{G}) & \longrightarrow & H^q_1(\mathcal{F}) & \longrightarrow & H^q_1(\mathcal{I}) & \longrightarrow & H^q_1(\mathcal{G}) \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5} \\
H^{q-1}_2(\mathcal{I}) & \longrightarrow & H^{q-1}_2(\mathcal{G}) & \longrightarrow & H^q_2(\mathcal{F}) & \longrightarrow & H^q_2(\mathcal{I}) & \longrightarrow & H^q_2(\mathcal{G})
\end{array} \tag{I.3.8.2}$$

If (I.3.8.1) holds for any abelian sheaf $\mathcal{F}$ for degree $q-1$, then γ_1, γ_2, γ_4 are isomorphisms and thus γ_3 is injective by the 5-lemma ([**Sta22**, 05QA]). Thus, γ_5 is also injective since $\mathcal{F}$ is an arbitrary abelian sheaf. Then, we see that γ_3 is an isomorphism, which completes the induction procedure.

For an injective abelian sheaf $\mathcal{I}$ on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$, we claim that for any covering family $\mathfrak{U} = \{(Y_i \to Y)\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_\tau$ with I finite, the augmented Čech complex associated to the presheaf $\nu_{\mathrm{p}}\mathcal{I}$

$$\nu_{\mathrm{p}}\mathcal{I}(Y) \to \prod_{i\in I} \nu_{\mathrm{p}}\mathcal{I}(Y_i) \to \prod_{i,j\in I} \nu_{\mathrm{p}}\mathcal{I}(Y_i \times_Y Y_j) \to \cdots \tag{I.3.8.3}$$

is exact. Admitting this claim, we see that $\nu_{\mathrm{p}}\mathcal{I}$ is indeed a sheaf, i.e. $\nu^{-1}\mathcal{I} = \nu_{\mathrm{p}}\mathcal{I}$, and the vanishing of higher Čech cohomologies implies that $H^q_\tau(Y, \nu^{-1}\mathcal{I}) = 0$ for $q > 0$ by I.3.6 ([**Sta22**, 03F9]), which completes the proof together with I.3.7. For the claim, we take the covering families $\mathfrak{U}_\lambda = \{Y_{i\lambda} \to Y_\lambda\}_{i\in I}$ in $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$ constructed by I.3.6. By I.3.7, the sequence (I.3.8.3) is the filtered colimit of the augmented Čech complexes

$$\mathcal{I}(Y_\lambda) \to \prod_{i\in I} \mathcal{I}(Y_{i\lambda}) \to \prod_{i,j\in I} \mathcal{I}(Y_{i\lambda} \times_{Y_\lambda} Y_{j\lambda}) \to \cdots, \tag{I.3.8.4}$$

which are exact since $\mathcal{I}$ is an injective abelian sheaf on $(\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}}$. □

Corollary I.3.9. *Let X be a coherent scheme, $\mathcal{F}$ a torsion abelian sheaf on the site $X_{\text{ét}}$ formed by coherent étale X-schemes endowed with the étale topology, $a : (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \to X_{\text{ét}}$ the morphism of sites defined by the inclusion functor. Then, the canonical morphism $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ is an isomorphism.*

Proof. Consider the morphisms of sites defined by inclusion functors

$$(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} \xrightarrow{\zeta} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} \xrightarrow{\xi} (\mathbf{Sch}^{\mathrm{fp}}_{/X})_{\mathrm{h}} \xrightarrow{\mu} X_{\text{ét}}. \tag{I.3.9.1}$$

Notice that the morphism $\mathcal{F} \to \mathrm{R}(\mu\circ\xi)_*(\mu\circ\xi)^{-1}\mathcal{F}$ is an isomorphism by [**Sta22**, 0EWG]. Hence, $\mathcal{F} \to \mathrm{R}\mu_*\mu^{-1}\mathcal{F}$ is an isomorphism by I.3.8, and thus so is $\mathcal{F} \to \mathrm{R}a_*a^{-1}\mathcal{F}$ by I.3.8. □

Corollary I.3.10. *Let $f : X \to Y$ be a proper morphism of coherent schemes, $\mathcal{F}$ a torsion abelian sheaf on $X_{\text{ét}}$. Consider the commutative diagram*

$$\begin{array}{ccc} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} & \xrightarrow{a_X} & X_{\text{ét}} \\ {\scriptstyle f_{\mathrm{v}}}\downarrow & & \downarrow{\scriptstyle f_{\text{ét}}} \\ (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a_Y} & Y_{\text{ét}} \end{array} \tag{I.3.10.1}$$

where f_{v} and $f_{\text{ét}}$ are defined by the base change by f. Then, the canonical morphism

$$a_Y^{-1}\mathrm{R}f_{\text{ét}*}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{v}*}a_X^{-1}\mathcal{F} \tag{I.3.10.2}$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{v}} & \xrightarrow{\zeta_X} & (\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{h}} & \xrightarrow{b_X} & X_{\text{ét}} \\ {\scriptstyle f_{\mathrm{v}}}\downarrow & & {\scriptstyle f_{\mathrm{h}}}\downarrow & & \downarrow{\scriptstyle f_{\text{ét}}} \\ (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{\zeta_Y} & (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{h}} & \xrightarrow{b_Y} & Y_{\text{ét}} \end{array} \tag{I.3.10.3}$$

The canonical morphism $b_Y^{-1}\mathrm{R}f_{\text{ét}*}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{h}*}b_X^{-1}\mathcal{F}$ is an isomorphism by [**Sta22**, 0EWF]. It remains to show that the canonical morphism $\zeta_Y^{-1}\mathrm{R}f_{\mathrm{h}*}b_X^{-1}\mathcal{F} \longrightarrow \mathrm{R}f_{\mathrm{v}*}a_X^{-1}\mathcal{F}$ is an isomorphism. Let Y' be a coherent Y-scheme and we set $g : X' = Y' \times_Y X \to X$. For each integer q, $\zeta_Y^{-1}\mathrm{R}^q f_{\mathrm{h}*}b_X^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y' \mapsto H^q_{\mathrm{h}}(X', b_{X'}^{-1}g_{\text{ét}}^{-1}\mathcal{F}) = H^q_{\text{ét}}(X', g_{\text{ét}}^{-1}\mathcal{F})$ by [**Sta22**, 0EWH], and $\mathrm{R}^q f_{\mathrm{v}*}a_X^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $Y' \mapsto H^q_{\mathrm{v}}(X', a_{X'}^{-1}g_{\text{ét}}^{-1}\mathcal{F}) = H^q_{\text{ét}}(X', g_{\text{ét}}^{-1}\mathcal{F})$ by I.3.9. □

Lemma I.3.11. *Let A be a product of (resp. absolutely integrally closed) valuation rings* (I.2.1).

(1) *Any finitely generated ideal of A is principal.*
(2) *Any connected component of* $\mathrm{Spec}(A)$ *with the reduced closed subscheme structure is isomorphic to the spectrum of a (resp. absolutely integrally closed) valuation ring.*

Proof. (1) is proved in [**Sta22**, 092T], and (2) follows from the proof of [**BS17**, 6.2]. □

Lemma I.3.12. *Let X be a $\mathbb{U}$-small scheme, $y \rightsquigarrow x$ a specialization of points of X. Then, there exists a $\mathbb{U}$-small family $\{f_\lambda : \mathrm{Spec}(V_\lambda) \to X\}_{\lambda\in\Lambda_{y\rightsquigarrow x}}$ of morphisms of schemes such that*

(i) *the ring V_λ is a $\mathbb{U}$-small (resp. absolutely integrally closed) valuation ring, and that*
(ii) *the morphism f_λ maps the generic point and closed point of* $\mathrm{Spec}(V_\lambda)$ *to y and x respectively, and that*

(iii) *for any morphism of schemes $f : \mathrm{Spec}(V) \to X$ where V is a (resp. absolutely integrally closed) valuation ring such that f maps the generic point and closed point of V to y and x respectively, there exists an element $\lambda \in \Lambda_{y\rightsquigarrow x}$ such that f factors through f_λ and that $V_\lambda \to V$ is an extension of valuation rings.*

Proof. Let K_y be the residue field $\kappa(y)$ of y (resp. an algebraic closure of $\kappa(y)$). Let $\mathfrak{p}_y$ be the prime ideal of the local ring $\mathcal{O}_{X,x}$ corresponding to the point y, and let $\{V_\lambda\}_{\lambda\in\Lambda_{y\rightsquigarrow x}}$ be the set of all valuation rings with fraction field K_y which contain $\mathcal{O}_{X,x}/\mathfrak{p}_y$ such that the injective homomorphism $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V_\lambda$ is local. The smallness of $\Lambda_{y\rightsquigarrow x}$ and V_λ is clear, and the inclusion $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V_\lambda$ induces a morphism $f_\lambda : \mathrm{Spec}(V_\lambda) \to X$ satisfying (ii). It remains to check (iii). The morphism f induces an injective and local homomorphism $\mathcal{O}_{X,x}/\mathfrak{p}_y \to V$. Notice that $\mathcal{O}_{X,x}/\mathfrak{p}_y \to \mathrm{Frac}(V)$ factors through K_y and that $K_y \cap V$ is a valuation ring with fraction field K_y. It is clear that $K_y \cap V \to V$ is local and injective, which shows that $K_y \cap V$ belongs to the set $\{V_\lambda\}_{\lambda\in\Lambda_{y\rightsquigarrow x}}$ constructed before. □

Lemma I.3.13. *Let $f : \mathrm{Spec}(V) \to X$ be a morphism of schemes where V is a valuation ring. We denote by a and b the closed point and generic point of $\mathrm{Spec}(V)$ respectively. If $c \in X$ is a generalization of $f(b)$, then there exists an absolutely integrally closed valuation ring W, a prime ideal $\mathfrak{p}$ of W, and a morphism $g : \mathrm{Spec}(W) \to X$ satisfying the following conditions:*

(i) *If z, y, x denote respectively the generic point, the point $\mathfrak{p}$ and the closed point of $\mathrm{Spec}(W)$, then $g(z) = c$, $g(y) = f(b)$ and $g(x) = f(a)$.*
(ii) *The induced morphism $\mathrm{Spec}(W/\mathfrak{p}) \to X$ factors through f, and the induced morphism $V \to W/\mathfrak{p}$ is an extension of valuation rings.*

Proof. According to [**EGA** II, 7.1.4], there exists an absolutely integrally closed valuation ring U and a morphism $\mathrm{Spec}(U) \to X$ which maps the generic point z and the closed point y of $\mathrm{Spec}(U)$ to c and $f(b)$ respectively. After extending U, we may assume that the morphism $y \to f(b)$ factors through b ([**EGA** II, 7.1.2]). We denote by $\kappa(y)$ the residue field of the point y. Let V' be a valuation ring extension of V with fraction field $\kappa(y)$, and let W be the preimage of V' by the surjection $U \to \kappa(y)$. Then, the maximal ideal $\mathfrak{p} = \mathrm{Ker}(U \to \kappa(y))$ of U is a prime ideal of W, and $W/\mathfrak{p} = V'$. We claim that W is an absolutely integrally closed valuation ring such that $W_\mathfrak{p} = U$. Indeed, firstly note that the fraction fields of U and W are equal as $\mathfrak{p} \subseteq W$. Let γ be an element of $\mathrm{Frac}(W) \setminus W$. If $\gamma \in U$, then $\gamma^{-1} \in W \setminus \mathfrak{p}$ by definition since $\gamma^{-1} \in U \setminus \mathfrak{p}$ and V is a valuation ring, and then $\gamma \in W_\mathfrak{p}$. If $\gamma \notin U$, then $\gamma^{-1} \in \mathfrak{p}$ since U is a valuation ring, and then $\gamma \notin W_\mathfrak{p}$. Thus, we have proved the claim, which shows that W satisfies the required conditions. □

Proposition I.3.14. *Let X be a coherent $\mathbb{U}$-small scheme, X° a quasi-compact dense open subset of X. Then, there exists a $\mathbb{U}$-small product A of absolutely integrally closed $\mathbb{U}$-small valuation rings and a v-covering $\mathrm{Spec}(A) \to X$ such that $\mathrm{Spec}(A)$ is X°-integrally closed* (I.2.2).

Proof. After replacing X by a finite affine open covering, we may assume that $X = \mathrm{Spec}(R)$. For a specialization $y \rightsquigarrow x$ of points of X, let $\{R \to V_\lambda\}_{\lambda\in\Lambda_{y\rightsquigarrow x}}$ be the $\mathbb{U}$-small set constructed in I.3.12. Let $\Lambda = \coprod_{y\in X^\circ} \Lambda_{y\rightsquigarrow x}$ where $y \rightsquigarrow x$ runs through all specializations in X such that $y \in X^\circ$. We take $A = \prod_{\lambda\in\Lambda} V_\lambda$ and $R \to A$ the natural homomorphism. As a quasi-compact open subscheme of $\mathrm{Spec}(A)$, $X^\circ \times_X \mathrm{Spec}(A)$ is the spectrum of $A[1/\pi]$ for an element $\pi = (\pi_\lambda)_{\lambda\in\Lambda} \in A$ by I.3.11.(1) ([**Sta22**, 01PH]). Notice that $\pi_\lambda \neq 0$ for any $\lambda \in \Lambda$. We see that A is integrally closed in $A[1/\pi]$. It remains to check that $\mathrm{Spec}(A) \to X$ is a v-covering. For any morphism $f : \mathrm{Spec}(V) \to X$ where V is a valuation ring, by I.3.13, there exists an absolutely integrally closed valuation ring W, a prime ideal $\mathfrak{p}$ of W and a morphism $g : \mathrm{Spec}(W) \to X$ such that g maps the generic point of W into X° and that $W/\mathfrak{p}$ is a valuation ring extension of V. By construction, there exists $\lambda \in \Lambda$ such that g factors through $\mathrm{Spec}(V_\lambda) \to X$. We see that f lifts to the composition of $\mathrm{Spec}(W/\mathfrak{p}) \to \mathrm{Spec}(V_\lambda) \to \mathrm{Spec}(A)$. □

Proposition I.3.15. *Consider a commutative diagram of schemes*

(I.3.15.1)
$$\begin{array}{ccccc} Y' & \longrightarrow & Z' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Z & \longrightarrow & X \end{array}$$

where $Z' \to Z$ and $X' \to X$ are quasi-compact. Assume that $Y' \to Y \times_X X'$ is surjective, $Y \to Z$ is dominant, $Z \to X$ is separated and $Z' \to X'$ is integral. If $X' \to X$ is a v-covering, then $Z' \to Z$ is also a v-covering.

Proof. Notice that $Z' \to Z \times_X X'$ is still integral as $Z \to X$ is separated. After replacing $X' \to X$ by $Z \times_X X' \to Z$, we may assume that $Z = X$. Let $\mathrm{Spec}(V) \to Z$ be a morphism of

schemes where V is a valuation ring. Since $Y \to Z$ is dominant, by I.3.13, there exists a morphism $\mathrm{Spec}(W) \to Z$ where W is an absolutely integrally closed valuation ring, a prime ideal $\mathfrak{p}$ of W such that $W/\mathfrak{p}$ is a valuation ring extension of V and that the generic point ξ of $\mathrm{Spec}(W)$ is over the image of $Y \to Z$. After extending W ([**Sta22**, 00IA]), we may assume that there exists a lifting $\xi \to Y$ of $\xi \to Z$. The morphism $\mathrm{Spec}(W) \to Z = X$ admits a lifting $\mathrm{Spec}(W') \to X'$ where $W \to W'$ is an extension of valuation rings. We claim that after extending W', $\mathrm{Spec}(W') \to X'$ factors through Z'. Indeed, if ξ' denotes the generic point of $\mathrm{Spec}(W')$, as $Y' \to Y \times_X X'$ is surjective, after extending W', we may assume that there exists an X'-morphism $\xi' \to Y'$ which is over $\xi \to Y$. Since $\mathrm{Spec}(W')$ is integrally closed in ξ' and Z' is integral over X', the morphism $\mathrm{Spec}(W') \to X'$ factors through Z' ([**Sta22**, 035I]). Finally, let $\mathfrak{q} \in \mathrm{Spec}(W')$ which lies over $\mathfrak{p} \in \mathrm{Spec}(W)$, then we get a lifting $\mathrm{Spec}(W'/\mathfrak{q}) \to Z'$ of $\mathrm{Spec}(V) \to Z$, which shows that $Z' \to Z$ is a v-covering. □

I.3.16. Let $S^\circ \to S$ be an open immersion of coherent schemes such that S is S°-integrally closed (I.2.2). For any S-scheme X, we set $X^\circ = S^\circ \times_S X$. We denote by $\mathbf{I}_{S^\circ \to S}$ the category formed by coherent S-schemes which are S°-integrally closed. Note that any S°-integrally closed coherent S-scheme X is also X°-integrally closed by definition. It is clear that the category $(\mathbf{I}_{S^\circ \to S})_{/X}$ of objects of $\mathbf{I}_{S^\circ \to S}$ over X is canonically equivalent to the category $\mathbf{I}_{X^\circ \to X}$.

Lemma I.3.17 ([**Sta22**, 03GV])**.** *Let $Y \to X$ be a coherent morphism of schemes, $X' \to X$ a smooth morphism of schemes, $Y' = Y \times_X X'$. Then, we have $X'^{Y'} = X^Y \times_X X'$.*

Lemma I.3.18. *Let $(Y_\lambda \to X_\lambda)_{\lambda \in \Lambda}$ be a directed inverse system of morphisms of coherent schemes with affine transition morphisms $Y_{\lambda'} \to Y_\lambda$ and $X_{\lambda'} \to X_\lambda$ ($\lambda' \geq \lambda$). We set $Y = \lim Y_\lambda$ and $X = \lim X_\lambda$. Then, $(X_\lambda^{Y_\lambda})_{\lambda \in \Lambda}$ is a directed inverse system of coherent schemes with affine transition morphisms and we have $X^Y = \lim X_\lambda^{Y_\lambda}$.*

Proof. We fix an index $\lambda_0 \in \Lambda$. After replacing X_{λ_0} by an affine open covering, we may assume that X_{λ_0} is affine (I.3.17). We write $X_\lambda = \mathrm{Spec}(A_\lambda)$ and $B_\lambda = \Gamma(Y_\lambda, \mathcal{O}_{Y_\lambda})$ for each $\lambda \geq \lambda_0$, and we set $A = \mathrm{colim}\, A_\lambda$ and $B = \mathrm{colim}\, B_\lambda$. Then, we have $X = \mathrm{Spec}(A)$ and $B = \Gamma(Y, \mathcal{O}_Y)$ ([**Sta22**, 009F]). Let R_λ (resp. R) be the integral closure of A_λ in B_λ (resp. A in B). By definition, we have $X_\lambda^{Y_\lambda} = \mathrm{Spec}(R_\lambda)$ and $X^Y = \mathrm{Spec}(R)$. The conclusion follows from the fact that $R = \mathrm{colim}\, R_\lambda$. □

Lemma I.3.19. *Let $S^\circ \to S$ be an open immersion of coherent schemes.*

1. *If X is an S°-integrally closed coherent S-scheme, then the open subscheme X° is scheme theoretically dense in X.*
2. *If X is an S°-integrally closed coherent S-scheme and X' is a coherent smooth X-scheme, then X' is also S°-integrally closed.*
3. *If $(X_\lambda)_{\lambda \in \Lambda}$ is a directed inverse system of S°-integrally closed coherent S-scheme with affine transition morphisms, then $X = \lim_{\lambda \in \Lambda} X_\lambda$ is also S°-integrally closed.*
4. *If $Y \to X$ is a morphism of coherent schemes over $S^\circ \to S$ such that Y is integral over X°, then the integral closure X^Y is S°-integrally closed with $(X^Y)^\circ = Y$.*

Proof. (1), (2), (3) follow from [**Sta22**, 035I], I.3.17 and I.3.18 respectively. For (4), $(X^Y)^\circ = X^\circ \times_X X^Y$ is the integral closure of X° in $X^\circ \times_X Y = Y$ by I.3.17, which is Y itself. □

I.3.20. We take the notation in I.3.16. The inclusion functor

$$\Phi^+ : \mathbf{I}_{S^\circ \to S} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/S},\ X \longmapsto X, \tag{I.3.20.1}$$

admits a right adjoint

$$\sigma^+ : \mathbf{Sch}^{\mathrm{coh}}_{/S} \longrightarrow \mathbf{I}_{S^\circ \to S},\ X \longmapsto \overline{X} = X^{X^\circ}. \tag{I.3.20.2}$$

Indeed, σ^+ is well-defined by I.3.19.(4), and the adjointness follows from the functoriality of taking integral closures. We remark that $\overline{X}^\circ = X^\circ$. On the other hand, the functor

$$\Psi^+ : \mathbf{I}_{S^\circ \to S} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/S^\circ},\ X \longmapsto X^\circ, \tag{I.3.20.3}$$

admits a left adjoint

$$\alpha^+ : \mathbf{Sch}^{\mathrm{coh}}_{/S^\circ} \longrightarrow \mathbf{I}_{S^\circ \to S},\ Y \longmapsto Y. \tag{I.3.20.4}$$

Lemma I.3.21. *With the notation in* I.3.16*, let $\varphi : I \to \mathbf{I}_{S^\circ \to S}$ be a functor sending i to X_i. If $X = \lim X_i$ represents the limit of φ in the category of coherent S-schemes, then the integral closure $\overline{X} = X^{X^\circ}$ represents the limit of φ in $\mathbf{I}_{S^\circ \to S}$ with $\overline{X}^\circ = X^\circ$.*

Proof. It follows directly from the adjoint pair (Φ^+, σ^+) (I.3.20). □

It follows from I.3.21 that for a diagram $X_1 \to X_0 \leftarrow X_2$ in $\mathbf{I}_{S^\circ \to S}$, the fibred product is representable by

$$X_1 \overline{\times}_{X_0} X_2 = (X_1 \times_{X_0} X_2)^{X_1^\circ \times_{X_0^\circ} X_2^\circ}. \tag{I.3.21.1}$$

Proposition I.3.22. *With the notation in* I.3.16*, let $\mathscr{C}$ be the set of families of morphisms $\{X_i \to X\}_{i\in I}$ of $\mathbf{I}_{S^\circ \to S}$ with I finite such that $\coprod_{i\in I} X_i \to X$ is a v-covering. Then, $\mathscr{C}$ forms a pretopology of $\mathbf{I}_{S^\circ \to S}$.*

Proof. Let $\{X_i \to X\}_{i\in I}$ be an element of $\mathscr{C}$. Firstly, we check that for a morphism $X' \to X$ of $\mathbf{I}_{S^\circ \to S}$, the base change $\{X_i' \to X'\}_{i\in I}$ also lies in $\mathscr{C}$, where $Z_i = X_i \times_X X'$ and $X_i' = Z_i^{Z_i^\circ}$ by I.3.21. Applying I.3.15 to the following diagram

$$\begin{array}{ccccc} \coprod_{i\in I} Z_i^\circ & \longrightarrow & \coprod_{i\in I} X_i' & \longrightarrow & \coprod_{i\in I} Z_i \\ \downarrow & & \downarrow & & \downarrow \\ X'^\circ & \longrightarrow & X' & \longrightarrow & X' \end{array} \tag{I.3.22.1}$$

we deduce that $\coprod_{i\in I} X_i' \to X'$ is also a v-covering, which shows the stability of $\mathscr{C}$ under base change.

For each $i \in I$, let $\{X_{ij} \to X_i\}_{j\in J_i}$ be an element of $\mathscr{C}$. We need to show that the composition $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ also lies in $\mathscr{C}$. This follows immediately from the stability of v-coverings under composition. We conclude that $\mathscr{C}$ forms a pretopology. □

Definition I.3.23. With the notation in I.3.16, we endow the category $\mathbf{I}_{S^\circ \to S}$ with the topology generated by the pretopology defined in I.3.22, and we call $\mathbf{I}_{S^\circ \to S}$ the *v-site of S°-integrally closed coherent S-schemes*.

By definition, any object in $\mathbf{I}_{S^\circ \to S}$ is quasi-compact. Let $\mathscr{O}$ be the sheaf on $\mathbf{I}_{S^\circ \to S}$ associated to the presheaf $X \mapsto \Gamma(X, \mathcal{O}_X)$. We call $\mathscr{O}$ the structural sheaf of $\mathbf{I}_{S^\circ \to S}$.

Proposition I.3.24. *With the notation in* I.3.16*, let $f : X' \to X$ be a covering in $\mathbf{I}_{S^\circ \to S}$ such that f is separated and that the diagonal morphism $X'^\circ \to X'^\circ \times_{X^\circ} X'^\circ$ is surjective. Then, the morphism of representable sheaves $h^{\mathrm{a}}_{X'} \to h^{\mathrm{a}}_X$ is an isomorphism.*

Proof. We need to show that for any sheaf $\mathcal{F}$ on $\mathbf{I}_{S^\circ \to S}$, $\mathcal{F}(X) \to \mathcal{F}(X')$ is an isomorphism. Since the composition of $X'^\circ \to X'^\circ \times_{X^\circ} X'^\circ \to X' \overline{\times}_X X'$ factors through the closed immersion $X' \to X' \overline{\times}_X X'$ (as f is separated), the closed immersion $X' \to X' \overline{\times}_X X'$ is surjective (I.3.19.(1)). Consider the following sequence

$$\mathcal{F}(X) \to \mathcal{F}(X') \rightrightarrows \mathcal{F}(X' \overline{\times}_X X') \to \mathcal{F}(X'). \tag{I.3.24.1}$$

The right arrow is injective as $X' \to X' \overline{\times}_X X'$ is a v-covering. Thus, the middle two arrows are actually the same. Thus, the first arrow is an isomorphism by the sheaf property of $\mathcal{F}$. □

Proposition I.3.25. *With the notation in* I.3.16*, let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of presheaves on $\mathbf{I}_{S^\circ \to S}$. Assume that*

(i) *the morphism $\mathcal{F}_1(\mathrm{Spec}(V)) \to \mathcal{F}_2(\mathrm{Spec}(V))$ is an isomorphism for any S°-integrally closed S-scheme $\mathrm{Spec}(V)$ where V is an absolutely integrally closed valuation ring, and that*

(ii) *for any directed inverse system of S°-integrally closed affine schemes $(\mathrm{Spec}(A_\lambda))_{\lambda\in\Lambda}$ over S the natural morphism $\operatorname{colim} \mathcal{F}_i(\mathrm{Spec}(A_\lambda)) \to \mathcal{F}_i(\mathrm{Spec}(\operatorname{colim} A_\lambda))$ is an isomorphism for $i = 1, 2$ (cf.* I.3.19.(3)*).*

Then, the morphism of the associated sheaves $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is an isomorphism.

Proof. Let A be a product of absolutely integrally closed valuation rings such that $X = \mathrm{Spec}(A)$ is an S°-integrally closed S-scheme. Let $\mathrm{Spec}(V)$ be a connected component of $\mathrm{Spec}(A)$ with the reduced closed subscheme structure. Then, V is an absolutely integrally closed valuation ring by I.3.11.(2), and $\mathrm{Spec}(V)$ is also S°-integrally closed since it has nonempty intersection with the dense open subset X° of X. Notice that each connected component of an affine scheme is the intersection of some open and closed subsets ([**Sta22**, 04PP]). Moreover, since A is reduced, we have $V = \operatorname{colim} A'$, where the colimit is taken over all the open and closed subschemes $X' = \mathrm{Spec}(A')$ of X which contain $\mathrm{Spec}(V)$. By assumption, we have an isomorphism

$$\operatorname{colim} \mathcal{F}_1(X') \xrightarrow{\sim} \operatorname{colim} \mathcal{F}_2(X'). \tag{I.3.25.1}$$

For two elements $\xi_1, \xi_1' \in \mathcal{F}_1(X)$ with $\alpha_X(\xi_1) = \alpha_X(\xi_1')$ in $\mathcal{F}_2(X)$, by (I.3.25.1) and a limit argument, there exists a finite disjoin union $X = \coprod_{i=1}^r X_i'$ such that the images of ξ_1 and ξ_1' in $\mathcal{F}_1(X_i')$ are the same. Therefore, $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is injective by I.3.14. On the other hand, for an element $\xi_2 \in \mathcal{F}_2(X)$,

by (I.3.25.1) and a limit argument, there exists a finite disjoin union $X = \coprod_{i=1}^{r} X'_i$ such that there exists an element $\xi_{1,i} \in \mathcal{F}_1(X'_i)$ for each i such that the image of ξ_2 in $\mathcal{F}_2(X'_i)$ is equal to $\alpha_{X'_i}(\xi_{1,i})$. Therefore, $\mathcal{F}_1^{\mathrm{a}} \to \mathcal{F}_2^{\mathrm{a}}$ is surjective by I.3.14. □

I.3.26. We take the notation in I.3.16. Endowing $\mathbf{Sch}^{\mathrm{coh}}$ with the v-topology (I.3.3), we see that the functors σ^+ and Ψ^+ defined in I.3.20 are left exact (as they have left adjoints) and continuous by I.3.15 and I.3.22. Therefore, they define morphisms of sites (I.2.5)

$$(\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \xrightarrow{\Psi} \mathbf{I}_{S^\circ \to S} \xrightarrow{\sigma} (\mathbf{Sch}^{\mathrm{coh}}_{/S})_{\mathrm{v}}. \tag{I.3.26.1}$$

Proposition I.3.27. *With the notation in* I.3.26*, let* $a : (\mathbf{Sch}^{\mathrm{coh}}_{/S^\circ})_{\mathrm{v}} \to S^\circ_{\text{ét}}$ *be the morphism of site defined by the inclusion functor* (I.3.9)*.*

(1) *For any torsion abelian sheaf* $\mathcal{F}$ *on* $S^\circ_{\text{ét}}$*, the canonical morphism* $\Psi_*(a^{-1}\mathcal{F}) \to \mathrm{R}\Psi_*(a^{-1}\mathcal{F})$ *is an isomorphism.*

(2) *For any locally constant torsion abelian sheaf* $\mathbb{L}$ *on* $\mathbf{I}_{S^\circ \to S}$*, the canonical morphism* $\mathbb{L} \to \mathrm{R}\Psi_*\Psi^{-1}\mathbb{L}$ *is an isomorphism.*

Proof. (1) For each integer q, the sheaf $\mathrm{R}^q\Psi_*(a^{-1}\mathcal{F})$ is the sheaf associated to the presheaf $X \mapsto H^q_{\mathrm{v}}(X^\circ, a^{-1}\mathcal{F}) = H^q_{\text{ét}}(X^\circ, f_{\text{ét}}^{-1}\mathcal{F})$ by I.3.9, where $f_{\text{ét}} : X^\circ_{\text{ét}} \to S^\circ_{\text{ét}}$ is the natural morphism. If X is the spectrum of a nonzero absolutely integrally closed valuation ring V, then $X^\circ = \mathrm{Spec}(V[1/\pi])$ for a nonzero element $\pi \in V$ by I.3.11.(1) and I.3.19.(1), which is also the spectrum of an absolutely integrally closed valuation ring (I.2.1). In this case, $H^q_{\text{ét}}(X^\circ, f_{\text{ét}}^{-1}\mathcal{F}) = 0$ for $q > 0$, which proves (1) by I.3.25 and [**SGA 4**$_{\mathrm{II}}$, VII.5.8].

(2) The problem is local on $\mathbf{I}_{S^\circ \to S}$. We may assume that $\mathbb{L}$ is the constant sheaf with value L. Then, $\mathrm{R}^q\Psi_*\Psi^{-1}\mathbb{L} = 0$ for $q > 0$ by applying (1) on the constant sheaf with value L on $S^\circ_{\text{ét}}$. For $q = 0$, notice that $\mathbb{L}$ is also the sheaf associated to the presheaf $X \mapsto H^0_{\text{ét}}(X, L)$, while $\Psi_*\Psi^{-1}\mathbb{L}$ is the sheaf $X \mapsto H^0_{\text{ét}}(X^\circ, L)$ by the discussion in (1). If X is the spectrum of a nonzero absolutely integrally closed valuation ring, then so is X° and so that $H^0_{\text{ét}}(X, L) = H^0_{\text{ét}}(X^\circ, L) = L$. The conclusion follows from I.3.25 and [**SGA 4**$_{\mathrm{II}}$, VII.5.8]. □

I.4. The arc-Descent of Perfect Algebras

Definition I.4.1. For any $\mathbb{F}_p$-algebra R, we denote by R_{perf} the filtered colimit

$$R_{\mathrm{perf}} = \operatorname*{colim}_{\mathrm{Frob}} R \tag{I.4.1.1}$$

indexed by $(\mathbb{N}, \leq)$, where the transition map associated to $i \leq (i+1)$ is the Frobenius of R.

It is clear that the endo-functor of the category of $\mathbb{F}_p$-algebras, $R \mapsto R_{\mathrm{perf}}$, commutes with colimits.

I.4.2. We define a presheaf $\mathcal{O}_{\mathrm{perf}}$ on the category $\mathbf{Sch}^{\mathrm{coh}}_{\mathbb{F}_p}$ of coherent $\mathbb{U}$-small $\mathbb{F}_p$-schemes X by

$$\mathcal{O}_{\mathrm{perf}}(X) = \Gamma(X, \mathcal{O}_X)_{\mathrm{perf}}. \tag{I.4.2.1}$$

For any morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine $\mathbb{F}_p$-schemes, we consider the augmented Čech complex of the presheaf $\mathcal{O}_{\mathrm{perf}}$,

$$0 \to A_{\mathrm{perf}} \to B_{\mathrm{perf}} \to B_{\mathrm{perf}} \otimes_{A_{\mathrm{perf}}} B_{\mathrm{perf}} \to \cdots. \tag{I.4.2.2}$$

Lemma I.4.3 ([**Sta22**, 0EWT])**.** *The presheaf* $\mathcal{O}_{\mathrm{perf}}$ *is a sheaf on* $\mathbf{Sch}^{\mathrm{coh}}_{\mathbb{F}_p}$ *with respect to the fppf topology* ([**Sta22**, 021L])*. Moreover, for any coherent* $\mathbb{F}_p$*-scheme* X *and any integer* q*,*

$$H^q_{\mathrm{fppf}}(X, \mathcal{O}_{\mathrm{perf}}) = \operatorname*{colim}_{\mathrm{Frob}} H^q(X, \mathcal{O}_X). \tag{I.4.3.1}$$

Proof. Firstly, we remark that for any integer q, the functor $H^q_{\mathrm{fppf}}(X, -)$ commutes with filtered colimit of abelian sheaves on $(\mathbf{Sch}^{\mathrm{coh}}_{/X})_{\mathrm{fppf}}$ for any coherent scheme X ([**Sta22**, 0739]). Since the presheaf $\mathcal{O}$ sending X to $\Gamma(X, \mathcal{O}_X)$ on $\mathbf{Sch}^{\mathrm{coh}}_{\mathbb{F}_p}$ is an fppf-sheaf, we have $H^0_{\mathrm{fppf}}(X, \mathcal{O}^{\mathrm{a}}_{\mathrm{perf}}) = \mathrm{colim}_{\mathrm{Frob}} H^0_{\mathrm{fppf}}(X, \mathcal{O}) = \mathcal{O}_{\mathrm{perf}}(X)$. Thus, $\mathcal{O}_{\mathrm{perf}}$ is an fppf-sheaf. Moreover, $H^q_{\mathrm{fppf}}(X, \mathcal{O}_{\mathrm{perf}}) = \mathrm{colim}_{\mathrm{Frob}} H^q_{\mathrm{fppf}}(X, \mathcal{O}) = \mathrm{colim}_{\mathrm{Frob}} H^q(X, \mathcal{O}_X)$ by faithfully flat descent ([**Sta22**, 03DW]). □

Lemma I.4.4. *Let* $\tau \in \{$*fppf, h, v, arc*$\}$*. The following propositions are equivalent:*

(1) *The presheaf* $\mathcal{O}_{\mathrm{perf}}$ *on* $\mathbf{Sch}^{\mathrm{coh}}_{\mathbb{F}_p}$ *is a* τ*-sheaf and* $H^q_\tau(X, \mathcal{O}_{\mathrm{perf}}) = \mathrm{colim}_{\mathrm{Frob}} H^q(X, \mathcal{O}_X)$ *for any coherent* $\mathbb{F}_p$*-scheme* X *and any integer* q*.*

(2) *For any* τ*-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes, the augmented Čech complex* (I.4.2.2) *is exact.*

PROOF. For an affine scheme $X = \mathrm{Spec}(A)$, $H^q(X, \mathcal{O}_X)$ vanishes for $q > 0$ and $H^0(X, \mathcal{O}_X) = A$. For (1) $\Rightarrow$ (2), the exactness of (I.4.2.2) follows from the Čech-cohomology-to-cohomology spectral sequence associated to the τ-covering $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ [**Sta22**, 03AZ]. Therefore, (1) and (2) hold for $\tau = \mathrm{fppf}$ by I.4.3. Conversely, the exactness of (I.4.2.2) shows the sheaf property for any τ-covering of an affine scheme by affine schemes, which implies the fppf-sheaf $\mathcal{O}_{\mathrm{perf}}$ is a τ-sheaf (cf. [**Sta22**, 0ETM]). The vanishing of higher Čech cohomologies implies that $H^q_\tau(X, \mathcal{O}_{\mathrm{perf}}) = 0$ for any affine $\mathbb{F}_p$-scheme X and any $q > 0$ ([**Sta22**, 03F9]). Therefore, for a coherent $\mathbb{F}_p$-scheme X, $H^q_\tau(X, \mathcal{O}_{\mathrm{perf}})$ can be computed by the hyper-Čech cohomology of a hypercovering of X formed by affine open subschemes ([**Sta22**, 01GY]). In particular, we have $H^q_\tau(X, \mathcal{O}_{\mathrm{perf}}) = H^q_{\mathrm{fppf}}(X, \mathcal{O}_{\mathrm{perf}})$ for any integer q, which completes the proof by I.4.3. □

Lemma I.4.5 (Gabber)**.** *The augmented Čech complex* (I.4.2.2) *is exact for any h-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes.*

PROOF. This is a result of Gabber, cf. [**BST17**, 3.3] or [**Sta22**, 0EWU], and I.4.4. □

Lemma I.4.6 ([**BS17**, 4.1])**.** *The augmented Čech complex* (I.4.2.2) *is exact for any v-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes.*

PROOF. We write B as a filtered colimit of finitely presented A-algebras $B = \mathrm{colim}\, B_\lambda$. Then, $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ is an h-covering for each λ by I.3.2. Notice that $B_{\mathrm{perf}} = \mathrm{colim}\, B_{\lambda,\mathrm{perf}}$, then the conclusion follows from applying I.4.5 on $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ and taking colimit. □

Lemma I.4.7 ([**BS17**, 6.3])**.** *For any valuation ring V and any prime ideal $\mathfrak{p}$ of V, the sequence*

$$0 \longrightarrow V \xrightarrow{\alpha} V/\mathfrak{p} \oplus V_{\mathfrak{p}} \xrightarrow{\beta} V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \longrightarrow 0 \tag{I.4.7.1}$$

is exact, where $\alpha(a) = (a, a)$ and $\beta(a, b) = a - b$. If moreover V is a perfect $\mathbb{F}_p$-algebra, then for any perfect V-algebra R, the base change of (I.4.7.1) *by $V \to R$,*

$$0 \longrightarrow R \longrightarrow R/\mathfrak{p}R \oplus R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \longrightarrow 0 \tag{I.4.7.2}$$

is exact.

PROOF. The sequence (I.4.7.1) is exact if and only if $\mathfrak{p} = \mathfrak{p}V_{\mathfrak{p}}$. Let $a \in \mathfrak{p}$ and $s \in V \setminus \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, $s/a \notin V$, thus $a/s \in V$ as V is a valuation ring. Moreover, we must have $a/s \in \mathfrak{p}$ as $\mathfrak{p}$ is a prime ideal. This shows the equality $\mathfrak{p} = \mathfrak{p}V_{\mathfrak{p}}$.

The second assertion follows directly from the fact that $\mathrm{Tor}^A_q(B, C) = 0$ for any $q > 0$ and any diagram $B \leftarrow A \to C$ of perfect $\mathbb{F}_p$-algebras ([**BS17**, 3.16]). □

Lemma I.4.8 ([**BM20**, 4.8])**.** *The augmented Čech complex* (I.4.2.2) *is exact for any arc-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes with A a valuation ring.*

PROOF. We follow the proof of Bhatt-Mathew [**BM20**, 4.8]. Let $B = \mathrm{colim}\, B_\lambda$ be a filtered colimit of finitely presented A-algebras. Then, $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(A)$ is also an arc-covering by I.3.2. Thus, we may assume that B is a finitely presented A-algebra.

An interval $I = [\mathfrak{p}, \mathfrak{q}]$ of a valuation ring A is a pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of A. We denote by $A_I = (A/\mathfrak{p})_{\mathfrak{q}}$. The set $\mathcal{I}$ of intervals of A is partially ordered under inclusion. Let $\mathcal{P}$ be the subset consisting of intervals I such that the lemma holds for $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$. It suffices to show that $\mathcal{P} = \mathcal{I}$.

(1) If the valuation ring A_I is of height ≤ 1, we claim that $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$ is automatically a v-covering. Indeed, there is an extension of valuation rings $A_I \to V$ of height ≤ 1 which factors through $B \otimes_A A_I$. As $A_I \to V$ is faithfully flat, $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$ is a v-covering by I.3.2 and I.3.4.(1). Therefore, $I \in \mathcal{P}$ by I.4.6.

(2) For any interval $J \subseteq I$, if $I \in \mathcal{P}$ then $J \in \mathcal{P}$. Indeed, applying $\otimes_{\mathbb{F}_p}(A_J)_{\mathrm{perf}}$ to the exact sequence (I.4.2.2) for $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$, we still get an exact sequence by the Tor-independence of perfect $\mathbb{F}_p$-algebras ([**BS17**, 3.16]).

(3) If $\mathfrak{p} \subseteq A$ is not maximal, then there exists $\mathfrak{q} \supsetneq \mathfrak{p}$ with $I = [\mathfrak{p}, \mathfrak{q}] \in \mathcal{P}$. Indeed, if there is no such I with the height of A_I no more than 1, then $\mathfrak{p} = \bigcap_{\mathfrak{q} \supsetneq \mathfrak{p}} \mathfrak{q}$, and thus,

$$A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \operatorname*{colim}_{I=[\mathfrak{p},\mathfrak{q}],\mathfrak{q}\supsetneq\mathfrak{p}} A_I. \tag{I.4.8.1}$$

Since $\mathrm{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \to \mathrm{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ is an h-covering as $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a field (and we have assumed that B is of finite presentation over A), there exists an interval I in the above colimit, such that $\mathrm{Spec}(B \otimes_A A_I) \to \mathrm{Spec}(A_I)$ is also an h-covering by I.3.4.(3). Therefore, this I lies in $\mathcal{P}$ by I.4.6.

(4) If $\mathfrak{p} \subseteq A$ is nonzero, then there exists $\mathfrak{q} \subsetneq \mathfrak{p}$ with $I = [\mathfrak{q}, \mathfrak{p}] \in \mathcal{P}$. This is similar to (3).
(5) If $I, J \in \mathcal{P}$ are overlapping, then $I \cup J \in \mathcal{P}$. Indeed, by (2) and replacing A by $A_{I \cup J}$, we may assume that $I = [0, \mathfrak{p}]$, $J = [\mathfrak{p}, \mathfrak{m}]$ with $\mathfrak{m}$ the maximal ideal. In particular, $A_I = A_{\mathfrak{p}}$, $A_J = A/\mathfrak{p}$, and $A_{I \cap J} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Since for each $R = \otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$ we have the short exact sequence (I.4.7.2), we get $I \cup J \in \mathcal{P}$.

In general, by Zorn's lemma, the above five properties of $\mathcal{P}$ guarantee that $\mathcal{P} = \mathcal{I}$ (cf. [**BM20**, 4.7]). □

Lemma I.4.9 (cf. [**BM20**, 3.30])**.** *The augmented Čech complex* (I.4.2.2) *is exact for any arc-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes with* A *a product of valuation rings.*

Proof. We follow closely the proof of I.3.25. Let $\mathrm{Spec}(V)$ be a connected component of $\mathrm{Spec}(A)$ with the reduced closed subscheme structure. Then, V is a valuation ring by I.3.11.(2). By I.4.8, the augmented Čech complex

$$0 \to V_{\mathrm{perf}} \to (B \otimes_A V)_{\mathrm{perf}} \to (B \otimes_A V)_{\mathrm{perf}} \otimes_{V_{\mathrm{perf}}} (B \otimes_A V)_{\mathrm{perf}} \to \cdots \tag{I.4.9.1}$$

is exact. Notice that each connected component of an affine scheme is the intersection of some open and closed subsets ([**Sta22**, 04PP]). Moreover, since A is reduced, we have $V = \mathrm{colim}\, A'$, where the colimit is taken over all the open and closed subschemes $\mathrm{Spec}(A')$ which contain $\mathrm{Spec}(V)$.

Therefore, by a limit argument, for an element $f \in \otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$ which maps to zero in $\otimes^{n+1}_{A_{\mathrm{perf}}} B_{\mathrm{perf}}$, as $\mathrm{Spec}(A)$ is quasi-compact, we can decompose $\mathrm{Spec}(A)$ into a finite disjoint union $\coprod_{i=1}^N \mathrm{Spec}(A_i)$ such that there exists $g_i \in \otimes^{n-1}_{A_{i,\mathrm{perf}}} (B \otimes_A A_i)_{\mathrm{perf}}$ which maps to the image f_i of f in $\otimes^n_{A_{i,\mathrm{perf}}} (B \otimes_A A_i)_{\mathrm{perf}}$. Since we have

$$\otimes^n_{A_{\mathrm{perf}}} B_{\mathrm{perf}} = \prod_{i=1}^N \otimes^n_{A_{i,\mathrm{perf}}} (B \otimes_A A_i)_{\mathrm{perf}}, \tag{I.4.9.2}$$

the element $g = (g_i)_{i=1}^N$ maps to f, which shows the exactness of (I.4.2.2). □

Proposition I.4.10 ([**BS19**, 8.10])**.** *Let* $\tau \in \{$*fppf, h, v, arc*$\}$.

(1) *The presheaf* $\mathcal{O}_{\mathrm{perf}}$ *is a* τ*-sheaf over* $\mathbf{Sch}^{\mathrm{coh}}_{\mathbb{F}_p}$*, and for any coherent* $\mathbb{F}_p$*-scheme* X *and any integer* q*,*

$$H^q_\tau(X, \mathcal{O}_{\mathrm{perf}}) = \operatorname*{colim}_{\mathrm{Frob}} H^q(X, \mathcal{O}_X). \tag{I.4.10.1}$$

(2) *For any* τ*-covering* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine* $\mathbb{F}_p$*-schemes, the augmented Čech complex*

$$0 \to A_{\mathrm{perf}} \to B_{\mathrm{perf}} \to B_{\mathrm{perf}} \otimes_{A_{\mathrm{perf}}} B_{\mathrm{perf}} \to \cdots \tag{I.4.10.2}$$

is exact.

Proof. We follow closely the proof of Bhatt-Scholze [**BS19**, 8.10]. (1) and (2) are equivalent by I.4.4, and they hold for $\tau \in \{$fppf, h, v$\}$ by I.4.3, I.4.5 and I.4.6. In particular,

$$H^0_{\mathrm{v}}(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{perf}}) = A_{\mathrm{perf}} \text{ and } H^q_{\mathrm{v}}(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{perf}}) = 0,\ \forall q > 0. \tag{I.4.10.3}$$

We take a hypercovering in the v-topology $\mathrm{Spec}(A_\bullet) \to \mathrm{Spec}(A)$ such that A_n is a product of valuation rings for each degree n by I.3.14 and [**Sta22**, 094K and 0DB1]. The associated sequence

$$0 \to A_{\mathrm{perf}} \to A_{0,\mathrm{perf}} \to A_{1,\mathrm{perf}} \to \cdots \tag{I.4.10.4}$$

is exact by the hyper-Čech-cohomology-to-cohomology spectral sequence [**Sta22**, 01GY].

Consider the double complex (A_i^j) where the i-th row $A_i^\bullet$ is the base change of (I.4.10.2) by $A_{\mathrm{perf}} \to A_{i,\mathrm{perf}}$, i.e. the augmented Čech complex (I.4.2.2) associated to $\mathrm{Spec}(B \otimes_A A_i) \to \mathrm{Spec}(A_i)$ (we set $A_{-1} = A$). On the other hand, the j-th column $A_\bullet^j$ is the associated sequence (I.4.10.4) to the hypercovering $\mathrm{Spec}(A_\bullet \otimes_A (\otimes^j_A B)) \to \mathrm{Spec}(\otimes^j_A B)$, which is exact by the previous discussion. Since $A^\bullet_{-1} \to \mathrm{Tot}(A_i^j)^{j \geq 0}_{i \geq 0}$ is a quasi-isomorphism ([**Sta22**, 0133]), for the exactness of the (-1)-row $A^\bullet_{-1}$, we only need to show the exactness of the i-th row $A_i^\bullet$ for any $i \geq 0$ but this has been proved in I.4.9 thanks to our choice of the hypercovering, which completes the proof. □

I.5. Almost Pre-perfectoid Algebras

Definition I.5.1.

(1) A *pre-perfectoid field* K is a valuation field whose valuation ring $\mathcal{O}_K$ is non-discrete, of height 1 and of residue characteristic p, and such that the Frobenius map on $\mathcal{O}_K/p\mathcal{O}_K$ is surjective.
(2) A *perfectoid field* K is a pre-perfectoid field which is complete for the topology defined by its valuation (cf. [**Sch12**, 3.1]).
(3) A *pseudo-uniformizer* π of a pre-perfectoid field K is a nonzero element of the maximal ideal $\mathfrak{m}_K$ of $\mathcal{O}_K$.

A *morphism of pre-perfectoid fields* $K \to L$ is a homomorphism of fields which induces an extension of valuation rings $\mathcal{O}_K \to \mathcal{O}_L$.

Lemma I.5.2. *Let K be a pre-perfectoid field with a pseudo-uniformizer π. Then, the fraction field $\widehat{K}$ of the π-adic completion of $\mathcal{O}_K$ is a perfectoid field.*

Proof. The π-adic completion $\widehat{\mathcal{O}_K}$ of $\mathcal{O}_K$ is still a non-discrete valuation ring of height 1 with residue characteristic p (cf. [**Bou06**, VI.§5.3, Prop.5]). If $p \neq 0$, then it is canonically isomorphic to the p-adic completion of $\mathcal{O}_K$, so that there is a canonical isomorphism $\mathcal{O}_K/p\mathcal{O}_K \xrightarrow{\sim} \widehat{\mathcal{O}_K}/p\widehat{\mathcal{O}_K}$, from which we see that $\widehat{K}$ is a perfectoid field. If $p = 0$, then the Frobenius induces a surjection $\mathcal{O}_K \to \mathcal{O}_K$ if and only if $\mathcal{O}_K$ is perfect. Thus, $\widehat{\mathcal{O}_K}$ is also perfect, and we see that $\widehat{K}$ is a perfectoid field. □

I.5.3. Let K be a pre-perfectoid field. There is a unique (up to scalar) ordered group homomorphism $v_K : K^\times \to \mathbb{R}$ such that $v_K^{-1}(0) = \mathcal{O}_K^\times$, where the group structure on $\mathbb{R}$ is given by the addition. In particular, $\mathcal{O}_K \setminus 0 = v_K^{-1}(\mathbb{R}_{\geq 0})$ and $\mathfrak{m}_K \setminus 0 = v_K^{-1}(\mathbb{R}_{>0})$ (cf. [**Bou06**, VI.§4.5 Prop.7] and [**Bou07**, V.§2 Prop.1, Rem.2]). The non-discrete assumption on $\mathcal{O}_K$ implies that the image $v_K(K^\times) \subseteq \mathbb{R}$ is dense. We set $v_K(0) = +\infty$.

Lemma I.5.4 ([**Sch12**, 3.2])**.** *Let K be a pre-perfectoid field. Then, for any pseudo-uniformizer π of K, there exists $\pi_n \in \mathfrak{m}_K$ for each integer $n \geq 0$ such that $\pi_0 = \pi$ and $\pi_n = u_n \cdot \pi_{n+1}^p$ for some unit $u_n \in \mathcal{O}_K^\times$, and $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$.*

Proof. If $v_K(\pi) < v_K(p)$, since the Frobenius is surjective on $\mathcal{O}_K/p$, there exists $\pi_1 \in \mathcal{O}_K$ such that $v_K(\pi - \pi_1^p) \geq v_K(p)$. Then, $v_K(\pi) = v_K(\pi_1^p)$ and thus $\pi = u \cdot \pi_1^p$ with $u \in \mathcal{O}_K^\times$. In general, since $v_K(K^\times) \subseteq \mathbb{R}$ is dense, any pseudo-uniformizer π is a finite product of pseudo-uniformizers whose valuation values are strictly less than $v_K(p)$, from which we get a p-th root π_1 of π up to a unit. Since π_1 is also a pseudo-uniformizer, we get π_n inductively. As $v_K(\pi_n)$ tends to zero when n tends to infinity, $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$. □

I.5.5. Let K be a pre-perfectoid field. We briefly review almost algebra over $(\mathcal{O}_K, \mathfrak{m}_K)$ for which we mainly refer to [**AG20**, 2.6], [**AGT16**, V] and [**GR03**]. Remark that $\mathfrak{m}_K \otimes_{\mathcal{O}_K} \mathfrak{m}_K \cong \mathfrak{m}_K^2 = \mathfrak{m}_K$ is flat over $\mathcal{O}_K$.

An $\mathcal{O}_K$-module M is called *almost zero* if $\mathfrak{m}_K M = 0$. A morphism of $\mathcal{O}_K$-modules $M \to N$ is called an *almost isomorphism* if its kernel and cokernel are almost zero. Let $\mathscr{N}$ be the full subcategory of the category $\mathcal{O}_K$-**Mod** of $\mathcal{O}_K$-modules formed by almost zero objects. It is clear that $\mathscr{N}$ is a Serre subcategory of $\mathcal{O}_K$-**Mod** ([**Sta22**, 02MO]). Let $\mathcal{S}$ be the set of almost isomorphisms in $\mathcal{O}_K$-**Mod**. Since $\mathscr{N}$ is a Serre subcategory, $\mathcal{S}$ is a multiplicative system, and moreover the quotient abelian category $\mathcal{O}_K$-**Mod**$/\mathscr{N}$ is representable by the localized category $\mathcal{S}^{-1}\mathcal{O}_K$-**Mod** (cf. [**Sta22**, 02MS]). We denote $\mathcal{S}^{-1}\mathcal{O}_K$-**Mod** by $\mathcal{O}_K^{\mathrm{al}}$-**Mod**, whose objects are called *almost $\mathcal{O}_K$-modules* or simply $\mathcal{O}_K^{\mathrm{al}}$-*modules* (cf. [**AG20**, 2.6.2]). We denote by

$$\alpha^* : \mathcal{O}_K\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod},\ M \longmapsto M^{\mathrm{al}} \tag{I.5.5.1}$$

the localization functor. It induces an $\mathcal{O}_K$-linear structure on $\mathcal{O}_K^{\mathrm{al}}$-**Mod**. For any two $\mathcal{O}_K$-modules M and N, we have a natural $\mathcal{O}_K$-linear isomorphism ([**AG20**, 2.6.7.1])

$$\mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(M^{\mathrm{al}}, N^{\mathrm{al}}) = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K \otimes_{\mathcal{O}_K} M, N). \tag{I.5.5.2}$$

The localization functor α^* admits a right adjoint

$$\alpha_* : \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K\text{-}\mathbf{Mod},\ M \longmapsto M_* = \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(\mathcal{O}_K^{\mathrm{al}}, M), \tag{I.5.5.3}$$

and a left adjoint

$$\alpha_! : \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \longrightarrow \mathcal{O}_K\text{-}\mathbf{Mod},\ M \longmapsto M_! = \mathfrak{m}_K \otimes_{\mathcal{O}_K} M_*. \tag{I.5.5.4}$$

Moreover, the natural morphisms

$$(M_*)^{\mathrm{al}} \xrightarrow{\sim} M,\ M \xrightarrow{\sim} (M_!)^{\mathrm{al}} \tag{I.5.5.5}$$

are isomorphisms for any $\mathcal{O}_K^{\mathrm{al}}$-module M (cf. [**AG20**, 2.6.8]). In particular, for any functor $\varphi : I \to \mathcal{O}_K^{\mathrm{al}}$-**Mod** sending i to M_i, the colimit and limit of φ are representable by

$$\operatorname{colim} M_i = (\operatorname{colim} M_{i*})^{\mathrm{al}},\ \lim M_i = (\lim M_{i*})^{\mathrm{al}}. \tag{I.5.5.6}$$

The tensor product in $\mathcal{O}_K$-**Mod** induces a tensor product in $\mathcal{O}_K^{\mathrm{al}}$-**Mod** by

$$M^{\mathrm{al}} \otimes_{\mathcal{O}_K^{\mathrm{al}}} N^{\mathrm{al}} = (M \otimes_{\mathcal{O}_K} N)^{\mathrm{al}} \tag{I.5.5.7}$$

making $\mathcal{O}_K^{\mathrm{al}}$-**Mod** an abelian tensor category ([**AG20**, 2.6.4]). We denote by $\mathcal{O}_K^{\mathrm{al}}$-**Alg** the category of commutative unitary monoids in $\mathcal{O}_K^{\mathrm{al}}$-**Mod** induced by the tensor structure, whose objects are called *almost $\mathcal{O}_K$-algebras* or simply *$\mathcal{O}_K^{\mathrm{al}}$-algebras* (cf. [**AG20**, 2.6.11]). Notice that R^{al} (resp. R_*) admits a canonical algebra structure for any $\mathcal{O}_K$-algebra (resp. $\mathcal{O}_K^{\mathrm{al}}$-algebra) R. Moreover, α^* and α_* induce adjoint functors between $\mathcal{O}_K$-**Alg** and $\mathcal{O}_K^{\mathrm{al}}$-**Alg** ([**AG20**, 2.6.12]). Combining with (I.5.5.5) and (I.5.5.6), we see that for any functor $\varphi : I \to \mathcal{O}_K^{\mathrm{al}}$-**Alg** sending i to R_i, the colimit and limit of φ are representable by (cf. [**GR03**, 2.2.16])

$$\operatorname{colim} R_i = (\operatorname{colim} R_{i*})^{\mathrm{al}},\ \lim R_i = (\lim R_{i*})^{\mathrm{al}}. \tag{I.5.5.8}$$

In particular, for any diagram $B \leftarrow A \to C$ of $\mathcal{O}_K^{\mathrm{al}}$-algebras, we denote its colimit by

$$B \otimes_A C = (B_* \otimes_{A_*} C_*)^{\mathrm{al}}, \tag{I.5.5.9}$$

which is clearly compatible with the tensor products of modules. We remark that α^* commutes with arbitrary colimits (resp. limits), since it has a right adjoint α_* (resp. since the forgetful functor $\mathcal{O}_K^{\mathrm{al}}$-**Alg** $\to \mathcal{O}_K^{\mathrm{al}}$-**Mod** and the localization functor $\alpha^* : \mathcal{O}_K$-**Mod** $\to \mathcal{O}_K^{\mathrm{al}}$-**Mod** commute with arbitrary limits).

I.5.6. For an element a of $\mathcal{O}_K$, we denote by $(\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-**Mod** the full subcategory of $\mathcal{O}_K^{\mathrm{al}}$-**Mod** formed by the objects on which the morphism induced by multiplication by a is zero. Notice that for an $(\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-module M, M_* is an $\mathcal{O}_K/a\mathcal{O}_K$-module. Thus, the localization functor α^* induces an essentially surjective exact functor $(\mathcal{O}_K/a\mathcal{O}_K)$-**Mod** $\to (\mathcal{O}_K/a\mathcal{O}_K)^{\mathrm{al}}$-**Mod**, which identifies the latter with the quotient abelian category $(\mathcal{O}_K/a\mathcal{O}_K)$-**Mod**$/\mathscr{N} \cap (\mathcal{O}_K/a\mathcal{O}_K)$-**Mod**.

Let π be a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit. The Frobenius on $\mathcal{O}_K/\pi\mathcal{O}_K$ induces an isomorphism $\mathcal{O}_K/\pi_1\mathcal{O}_K \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K$. The Frobenius on $(\mathcal{O}_K/\pi)$-algebras and the localization functor α^* induce a natural transformation from the base change functor $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** $\to (\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg**, $R \mapsto (\mathcal{O}_K/\pi) \otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R$ to the identity functor.

$$\begin{array}{ccccc} (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Alg} & \longrightarrow & (\mathcal{O}_K/\pi_1)^{\mathrm{al}}\text{-}\mathbf{Alg} & \xrightarrow{\sim} & (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Alg} \\ & \searrow & \Downarrow \mathrm{Frob} & \nearrow & \\ & & \mathrm{id} & & \end{array} \tag{I.5.6.1}$$

For an $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra R, we usually identify the $(\mathcal{O}_K/\pi_1)^{\mathrm{al}}$-algebra $R/\pi_1 R$ with the $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra $(\mathcal{O}_K/\pi)\otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)}R$, and we denote by $R/\pi_1 R \to R$ the natural morphism $(\mathcal{O}_K/\pi)\otimes_{\mathrm{Frob},(\mathcal{O}_K/\pi)} R \to R$ induced by the Frobenius (cf. [**GR03**, 3.5.6]). Moreover, the natural transformations induced by Frobenius for $(\mathcal{O}_K/\pi)$-**Alg** and $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** are also compatible with the functor α_*. Indeed, it follows from the fact that for any $(\mathcal{O}_K/\pi)$-algebra R, the composition of

(I.5.6.2)

$$(\mathcal{O}_K/\pi) \otimes_{(\mathcal{O}_K/\pi)} \operatorname{Hom}(\mathfrak{m}_K, R) \longrightarrow \operatorname{Hom}(\mathfrak{m}_K, (\mathcal{O}_K/\pi) \otimes_{(\mathcal{O}_K/\pi)} R) \xrightarrow{\operatorname{Hom}(\mathfrak{m}_K, \mathrm{Frob})} \operatorname{Hom}(\mathfrak{m}_K, R)$$

is the relative Frobenius on $(R^{\mathrm{al}})_* = \operatorname{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, R)$.

I.5.7. Let C be a site. A presheaf $\mathcal{F}$ of $\mathcal{O}_K$-modules on C is called *almost zero* if $\mathcal{F}(U)$ is almost zero for any object U of C. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_K$-modules on C is called an *almost isomorphism* if $\mathcal{F}(U) \to \mathcal{G}(U)$ is an almost isomorphism for any object U of C (cf. [**AG20**, 2.6.23]). Let $\mathscr{N}$ be the full subcategory of the category $\mathcal{O}_K$-$\mathbf{Mod}_C$ of sheaves of $\mathcal{O}_K$-modules on C formed by almost zero objects. Similarly, $\mathscr{N}$ is a Serre subcategory of $\mathcal{O}_K$-$\mathbf{Mod}_C$. Let $\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ be the full subcategory of the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ formed by complexes with almost zero cohomologies. It is a strictly full saturated triangulated subcategory ([**Sta22**, 06UQ]). We also say that the objects of $\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ are *almost zero*. Let $\mathcal{S}$ be the set of arrows in $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ which induce almost isomorphisms on cohomologies. We also call the elements of $\mathcal{S}$ *almost isomorphisms*. Then, $\mathcal{S}$ is a saturated multiplicative system ([**Sta22**, 05RG]), and moreover

the quotient triangulated category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)/\mathbf{D}_{\mathscr{N}}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ is representable by the localized triangulated category $\mathcal{S}^{-1}\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C)$ ([**Sta22**, 05RI]). The natural functor

$$\mathcal{S}^{-1}\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_C) \longrightarrow \mathbf{D}(\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}_C) \tag{I.5.7.1}$$

is an equivalence by [**Sta22**, 06XM] and (I.5.5.5) (cf. [**GR03**, 2.4.9]).

Lemma I.5.8. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, M a flat $\mathcal{O}_K$-module. We fix a system of p^n-th roots $(\pi_n)_{n\geq 0}$ of π up to units* (I.5.4)*, then the map*

$$\bigcap_{n\geq 0} \pi_n^{-1}M \to (M^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, M),\ a \mapsto (x \mapsto xa) \tag{I.5.8.1}$$

where $\pi_n^{-1}M \subseteq M[1/\pi]$, is an isomorphism of $\mathcal{O}_K$-modules. Moreover, for an extension of valuation rings $\mathcal{O}_K \to R$ of height 1*, we have $R = \bigcap_{n\geq 0}\pi_n^{-1}R$ and the above isomorphism coincides with the unit map $R \to (R^{\mathrm{al}})_*$.*

Proof. Since $\mathfrak{m}_K$ is generated by $\{\pi_n\}_{n\geq 0}$, any $\mathcal{O}_K$-linear morphism $f : \mathfrak{m}_K \to M$ is determined by its values $f(\pi_n) \in M$. Notice that $(\pi/\pi_n)\cdot f(\pi_n) = f(\pi)$ and M is π-torsion free, so that f must be given by the multiplication by an element $a = f(\pi)/\pi \in M[1/\pi]$. It is clear that such a multiplication sends $\mathfrak{m}_K$ to M if and only if $a \in \bigcap_{n\geq 0}\pi_n^{-1}M$, which shows the first assertion. If $\mathcal{O}_K \to R$ is an extension of valuation rings of height 1, then we directly deduce from the valuation map $v : R[1/\pi]\setminus 0 \to \mathbb{R}$ (I.5.3) the equality $R = \bigcap_{n\geq 0}\pi_n^{-1}R$. □

Lemma I.5.9. *Let K be a pre-perfectoid field, R an $\mathcal{O}_K$-algebra, $\mathcal{O}_K \to V$ an extension of valuation rings of height* 1*. Then, the canonical map*

$$\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V) \longrightarrow \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg}}(R^{\mathrm{al}}, V^{\mathrm{al}}) \tag{I.5.9.1}$$

is bijective.

Proof. There are natural maps

$$\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V) \to \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg}}(R^{\mathrm{al}}, V^{\mathrm{al}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, (V^{\mathrm{al}})_*) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Alg}}(R, V), \tag{I.5.9.2}$$

where the middle isomorphism is given by adjunction and the last isomorphism is induced by the inverse of the unit map $V \to (V^{\mathrm{al}})_*$ by I.5.8. The composition is the identity map, which completes the proof. □

Definition I.5.10. Let K be a pre-perfectoid field. We say that an $\mathcal{O}_K^{\mathrm{al}}$-module M (resp. an $\mathcal{O}_K$-module M) is *flat* (resp. *almost flat*) if the functor $\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod} \to \mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}$ given by tensoring with M is exact (resp. M^{al} is flat).

Remark I.5.11. In general, one can define the flatness of a morphism of $\mathcal{O}_K^{\mathrm{al}}$-algebras (cf. [**GR03**, 3.1.1.(i)]). We say that a morphism of $\mathcal{O}_K$-algebras $A \to B$ is *almost flat* if $A^{\mathrm{al}} \to B^{\mathrm{al}}$ is flat.

Lemma I.5.12. *Let K be a pre-perfectoid field with a pseudo-uniformizer π. Then, an $\mathcal{O}_K^{\mathrm{al}}$-module M is flat if and only if M_* is π-torsion free. In particular, an $\mathcal{O}_K$-module N is almost flat if and only if the submodule of π-torsion elements of N is almost zero.*

Proof. First of all, for any $\mathcal{O}_K^{\mathrm{al}}$-modules L_1 and L_2, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(M \otimes_{\mathcal{O}_K^{\mathrm{al}}} L_1, L_2) = \mathrm{Hom}_{\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Mod}}(L_1, \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(M_*, L_{2*})^{\mathrm{al}}) \tag{I.5.12.1}$$

by (I.5.5.2), (I.5.5.5) and (I.5.5.7). Therefore, the functor defined by tensoring with M admits a right adjoint, and thus it is right exact. Consider the sequence

$$0 \longrightarrow \mathcal{O}_K^{\mathrm{al}} \xrightarrow{\cdot\pi} \mathcal{O}_K^{\mathrm{al}} \longrightarrow (\mathcal{O}_K/\pi\mathcal{O}_K)^{\mathrm{al}} \longrightarrow 0, \tag{I.5.12.2}$$

which is exact since the localization functor α^* is exact. If M is flat, tensoring the above sequence with M and applying α_*, we deduce that M_* is π-torsion free since α_* is left exact (as a right adjoint to α^*). Conversely, if M_* is π-torsion free, then it is flat over $\mathcal{O}_K$. For any injective morphism $L_1 \to L_2$ of $\mathcal{O}_K^{\mathrm{al}}$-modules, $L_{1*} \to L_{2*}$ is also injective, and it remains injective after tensoring with M_*. Therefore, $L_1 \to L_2$ also remains injective after tensoring with M since α^* is exact. This shows that M is flat.

The second assertion follows from the almost isomorphism $N \to (N^{\mathrm{al}})_*$ and the fact that $(N^{\mathrm{al}})_* = \mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, N)$ has no nonzero almost zero element. □

Lemma I.5.13. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, M a flat $\mathcal{O}_K^{\mathrm{al}}$-module, x an element of $\mathcal{O}_K$. Then, the canonical morphism $M_*/xM_* \to (M/xM)_*$ is injective, and for any $\epsilon \in \mathfrak{m}_K$, the image of $\varphi_\epsilon : (M/\epsilon xM)_* \to (M/xM)_*$ is M_*/xM_*. In particular, the canonical morphism*

$$\varprojlim_n M_*/\pi^n M_* \longrightarrow (\varprojlim_n M/\pi^n M)_* \tag{I.5.13.1}$$

is an isomorphism of $\mathcal{O}_K$-modules.

Proof. We follow the proof of [**Sch12**, 5.3]. Applying the left exact functor α_* to the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0, \tag{I.5.13.2}$$

we see that $M_*/xM_* \to (M/xM)_*$ is injective.

To show that the image of φ_ϵ is M_*/xM_*, it suffices to show that φ_ϵ factors through M_*/xM_*. We identify $(M/xM)_*$ with $\mathrm{Hom}_{\mathcal{O}_K\text{-}\mathbf{Mod}}(\mathfrak{m}_K, M_*/xM_*)$ by (I.5.5.5) and (I.5.5.2) so that M_*/xM_* identifies with the subset consisting of the $\mathcal{O}_K$-morphisms $\mathfrak{m}_K \to M_*/xM_*$ sending y to ya for some element $a \in M_*/xM_*$. For an $\mathcal{O}_K$-morphism $f : \mathfrak{m}_K \to M_*/\epsilon xM_*$, let b be an element of M_* which lifts $f(\epsilon)$. Notice that M_* is π-torsion free by I.5.12. With notation in I.5.8, we have $b \equiv (\epsilon/\pi_n)\cdot f(\pi_n) \mod \epsilon xM_*$ for n big enough so that the element $b/\epsilon \in M_*[1/\pi]$ lies in $\bigcap_{n\geq 0} \pi_n^{-1}M_* = M_*$. Moreover, $\pi_n \cdot (b/\epsilon) \equiv f(\pi_n) \mod xM_*$ for n big enough. As $\varphi_\epsilon(f)$ is determined by its values on π_n for n big enough, it follows that $\varphi_\epsilon(f) = a$, where a is the image of b/ϵ in M_*/xM_*.

Finally, the previous result implies that the inverse system $((M/\pi^n M)_*)_{n\geq 1}$ is Mittag-Leffler so that the "in particular" part follows immediately from the fact that α_* commutes with arbitrary limits (as a right adjoint to α^*) ([**Sta22**, 0596]). □

Definition I.5.14. Let K be a pre-perfectoid field. For any $\mathcal{O}_K$-algebra R, we define a perfect ring $R^\flat$ as the projective limit

$$R^\flat = \varprojlim_{\mathrm{Frob}} R/pR \tag{I.5.14.1}$$

indexed by $(\mathbb{N}, \leq)$, where transition map associated to $i \leq (i+1)$ is the Frobenius on R/pR. We call $R^\flat$ the *tilt* of R.

Lemma I.5.15 ([**Sch12**, 3.4])**.** *Let K be a perfectoid field with a pseudo-uniformizer π dividing p.*

(1) *The projection induces an isomorphism of multiplicative monoids*

$$\varprojlim_{\mathrm{Frob}} \mathcal{O}_K \longrightarrow \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/\pi\mathcal{O}_K. \tag{I.5.15.1}$$

In particular, the right hand side is canonically isomorphic to $(\mathcal{O}_K)^\flat$ as a ring.

(2) *We denote by*

$$\sharp : (\mathcal{O}_K)^\flat \longrightarrow \mathcal{O}_K,\ x \mapsto x^\sharp, \tag{I.5.15.2}$$

the composition of the inverse of (I.5.15.1) *and the projection onto the first component. Then $v_K \circ \sharp : (\mathcal{O}_K)^\flat \setminus 0 \to \mathbb{R}_{\geq 0}$ defines a valuation of height 1 on $(\mathcal{O}_K)^\flat$.*

(3) *The fraction field $K^\flat$ of $(\mathcal{O}_K)^\flat$ is a perfectoid field of characteristic p and the element*

$$\pi^\flat = (\cdots, \pi_1^{1/p^2}, \pi_1^{1/p}, \pi_1, 0) \in (\mathcal{O}_K)^\flat \tag{I.5.15.3}$$

is a pseudo-uniformizer of $K^\flat$, where $\pi = u \cdot \pi_1^p$ with $\pi_1 \in \mathfrak{m}_K$ and $u \in \mathcal{O}_K^\times$.

(4) *We have $\mathcal{O}_{K^\flat} = (\mathcal{O}_K)^\flat$, and there is a canonical isomorphism*

$$\mathcal{O}_{K^\flat}/\pi^\flat\mathcal{O}_{K^\flat} \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K \tag{I.5.15.4}$$

induced by (1) *and the projection onto the first component.*

I.5.16. We see that the tilt defines a functor $\mathcal{O}_K\text{-}\mathbf{Alg} \to \mathcal{O}_{K^\flat}\text{-}\mathbf{Alg}$, $R \mapsto R^\flat$, which preserves almost zero objects and almost isomorphisms. For an $\mathcal{O}_K^{\mathrm{al}}$-algebra R, we set $R^\flat = ((R_*)^\flat)^{\mathrm{al}}$ and call it the *tilt* of R, which induces a functor $\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Alg} \to \mathcal{O}_{K^\flat}^{\mathrm{al}}\text{-}\mathbf{Alg}$, $R \mapsto R^\flat$. Note that the tilt functor commutes with the localization functor α^* up to a canonical isomorphism, and commutes with the functor α_* up to a canonical almost isomorphism.

Definition I.5.17 ([**Sch12**, 5.1])**.** Let K be a perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit.

(1) A *perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra* is an $\mathcal{O}_K^{\mathrm{al}}$-algebra R such that
   (i) R is flat over $\mathcal{O}_K^{\mathrm{al}}$;

(ii) the Frobenius of $R/\pi R$ induces an isomorphism $R/\pi_1 R \to R/\pi R$ of $\mathcal{O}_K^{\mathrm{al}}$-algebras (I.5.6);
(iii) the canonical morphism $R \to \varprojlim_n R/\pi^n R$ is an isomorphism in $\mathcal{O}_K^{\mathrm{al}}$-**Alg**.

We denote by $\mathcal{O}_K^{\mathrm{al}}$-**Perf** the full subcategory of $\mathcal{O}_K^{\mathrm{al}}$-**Alg** formed by perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras.

(2) A *perfectoid* $(\mathcal{O}_K/\pi)^{\mathrm{al}}$*-algebra* is a flat $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra R such that the Frobenius map induces an isomorphism $R/\pi_1 R \xrightarrow{\sim} R$. We denote by $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Perf** the full subcategory of $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Alg** formed by perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebras.

Lemma I.5.18. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit. Then, for an $\mathcal{O}_K$-algebra R, the following conditions are equivalent:*

(1) *The almost algebra $\widehat{R}^{\mathrm{al}}$ associated to the π-adic completion $\widehat{R}$ of R is a perfectoid $\mathcal{O}_{\widehat{K}}^{\mathrm{al}}$-algebra.*
(2) *The $\mathcal{O}_{\widehat{K}}$-module $\widehat{R}$ is almost flat, and the Frobenius of $R/\pi R$ induces an almost isomorphism $R/\pi_1 R \to R/\pi R$.*

Proof. We have seen that $\widehat{K}$ is a perfectoid field in I.5.2 and π is obviously a pseudo-uniformizer of $\widehat{K}$. Since the localization functor $\alpha^* : \mathcal{O}_K$-**Alg** $\to \mathcal{O}_K^{\mathrm{al}}$-**Alg** commutes with arbitrary limits and colimits (I.5.5), we have a canonical isomorphism $\widehat{R}^{\mathrm{al}} \xrightarrow{\sim} \varprojlim_n \widehat{R}^{\mathrm{al}}/\pi^n \widehat{R}^{\mathrm{al}}$. Thus, the third condition in I.5.17.(1) holds for $\widehat{R}^{\mathrm{al}}$. Since there are canonical isomorphisms

$$R/\pi_1 R \xrightarrow{\sim} \widehat{R}/\pi_1 \widehat{R},\ R/\pi R \xrightarrow{\sim} \widehat{R}/\pi \widehat{R}, \tag{I.5.18.1}$$

the conditions (1) and (2) are clearly equivalent. □

Definition I.5.19. Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit. We say that an $\mathcal{O}_K$-algebra is *almost pre-perfectoid* if it satisfies the equivalent conditions in I.5.18.

We remark that in I.5.19, if a morphism of $\mathcal{O}_K$-algebras $R \to R'$ induces an almost isomorphism $R/\pi^n R \to R'/\pi^n R'$ for each $n \geq 1$, then the morphism of the π-adic completions $\widehat{R} \to \widehat{R'}$ is an almost isomorphism since α^* commutes with limits. In particular, R is almost pre-perfectoid if and only if R' is almost pre-perfectoid.

Lemma I.5.20. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra. If R is almost flat (resp. flat) over $\mathcal{O}_K$, then the π-adic completion $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$.*

Proof. For any integer $n > 0$, there is a canonical isomorphism

$$R/\pi^n R \xrightarrow{\sim} \widehat{R}/\pi^n \widehat{R}. \tag{I.5.20.1}$$

Let $x \in \widehat{R}$ be a π-torsion element. Since any π-torsion element of R is almost zero (resp. zero) by I.5.12, for any $\epsilon \in \mathfrak{m}_K$ (resp. $\epsilon = 1$), the image of ϵx in $\widehat{R}/\pi^n \widehat{R}$ lies in $\pi^{n-1}\widehat{R}/\pi^n \widehat{R}$. Therefore, $\epsilon x \in \bigcap_{n>0} \pi^{n-1}\widehat{R} = 0$, which amounts to say that $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$. □

Lemma I.5.21. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit, R a flat $\mathcal{O}_K$-algebra. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an injection $R/\pi_1 R \to R/\pi R$.*
(2) *For any $x \in R[1/\pi]$, if $x^p \in R$, then $x \in R$.*

Proof. We follow the proof of [**Sch12**, 5.7]. Assume first that $R/\pi_1 R \to R/\pi R$ is injective. Let $x \in R[1/\pi]$ with $x^p \in R$, k the minimal natural number such that $y = \pi_1^k x \in R$. If $k \geq 1$, then $y^p = \pi_1^{pk} x^p \in \pi R$. Therefore, $y \in \pi_1 R$ by the injectivity of the Frobenius. However, as R is π-torsion free, we have $y' = y/\pi_1 = \pi_1^{k-1} x \in R$ which contradicts the minimality of k.

Conversely, for any $x \in R$ with $x^p \in \pi R$, we have $(x/\pi_1)^p \in R$. Thus, $x/\pi_1 \in R$ by assumption, i.e. $x \in \pi_1 R$, which implies the injectivity of the Frobenius. □

Lemma I.5.22. *Let K be a pre-perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit, R an $\mathcal{O}_K$-algebra which is almost flat. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an almost injection (resp. almost isomorphism) $R/\pi_1 R \to R/\pi R$.*
(2) *The Frobenius induces an injection (resp. isomorphism) $(R^{\mathrm{al}})_*/\pi_1 (R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi (R^{\mathrm{al}})_*$.*

Proof. We follow the proof of [**Sch12**, 5.6]. Notice that the Frobenius is compatible with the functors α^* and α_* (I.5.6). (2) $\Rightarrow$ (1) follows from the almost isomorphism $R \to (R^{\mathrm{al}})_*$. The "injection" part of (1) $\Rightarrow$ (2) follows from the inclusions (I.5.13)

$$(R^{\mathrm{al}})_*/\pi_1 (R^{\mathrm{al}})_* \subseteq ((R/\pi_1 R)^{\mathrm{al}})_*,\ (R^{\mathrm{al}})_*/\pi (R^{\mathrm{al}})_* \subseteq ((R/\pi R)^{\mathrm{al}})_*. \tag{I.5.22.1}$$

For the "isomorphism" part of (1) $\Rightarrow$ (2), notice that $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$ is almost surjective. Let π_2 be a p-th root of π_1 up to a unit (I.5.4). Then, for an element x of $(R^{\mathrm{al}})_*$, there exist elements y and x' of $(R^{\mathrm{al}})_*$ such that $\pi_2^p x = y^p + \pi_2^{p^2} x'$. Thus, $x = y'^p + \pi_2^{p^2-p} x'$ where $y' = y/\pi_2 \in (R^{\mathrm{al}})_*[1/\pi]$ (as $(R^{\mathrm{al}})_*$ is flat over $\mathcal{O}_K$ by I.5.12). In fact, y' lies in $(R^{\mathrm{al}})_*$ by I.5.21 and the "injection" part of (1) $\Rightarrow$ (2). By applying this process to x', there exist elements y'' and x'' of $(R^{\mathrm{al}})_*$ such that $x' = y''^p + \pi_2^{p^2-p} x''$. In conclusion, we have $x = y'^p + \pi_2^{p^2-p}(y''^p + \pi_2^{p^2-p} x'') \equiv (y' + \pi_2^{p-1} y'')^p \mod \pi(R^{\mathrm{al}})_*$, which shows the surjectivity of $(R^{\mathrm{al}})_*/\pi_1(R^{\mathrm{al}})_* \to (R^{\mathrm{al}})_*/\pi(R^{\mathrm{al}})_*$. □

Lemma I.5.23. *Let K be a pre-perfectoid field, R an almost flat $\mathcal{O}_K$-algebra, π, π' pseudo-uniformizers dividing p with p-th roots π_1, π_1' respectively up to units. Then, the following conditions are equivalent:*

(1) *The Frobenius induces an almost injection (resp. almost surjection) $R/\pi_1 R \to R/\pi R$.*
(2) *The Frobenius induces an almost injection (resp. almost surjection) $R/\pi_1' R \to R/\pi' R$.*

In particular, the definitions I.5.17.(1) *and* I.5.19 *do not depend on the choice of the pseudo-uniformizer.*

Proof. Notice that $(R^{\mathrm{al}})_*$ is flat over $\mathcal{O}_K$ by I.5.12. The "injection" part follows from I.5.21 and I.5.22. For the "surjection" part, we assume that $R/\pi_1 R \to R/\pi R$ is almost surjective. Let $\epsilon \in \mathfrak{m}_K$. We can take a pseudo-uniformizer $\widetilde{\pi}$ of K dividing p with $\widetilde{\pi}_1^p = \widetilde{\pi}$ and $v_K(\pi)/3 < v_K(\widetilde{\pi}) < v_K(\pi)/2$. For any $x \in R$, by the almost surjectivity, we have $\epsilon x = y^p + \widetilde{\pi}^2 z$ for some $y, z \in R$. We also have $\widetilde{\pi} z = v^p + \pi w$ for some $v, w \in R$, then $\epsilon x = y^p + \widetilde{\pi} v^p + \widetilde{\pi}\pi w$. Since $y^p + \widetilde{\pi} v^p \equiv (y + \widetilde{\pi}_1 v)^p \mod pR$, $R'/\pi_1' R \to R/\pi' R$ is almost surjective for any pseudo-uniformizer π' dividing p with $v_K(\pi') < 4v_K(\pi)/3$. By induction, we see that $R'/\pi_1' R \to R/\pi' R$ is almost surjective in general. □

Proposition I.5.24. *Let K be a pre-perfectoid field of characteristic p with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra, $\widehat{R}$ the π-adic completion of R. Then, R is almost pre-perfectoid if and only if $(\widehat{R}^{\mathrm{al}})_*$ is perfect.*

Proof. Note that $\mathcal{O}_K$ is perfect by definition. If R is almost pre-perfectoid, then $\widehat{R}$ is almost flat so that $(\widehat{R}^{\mathrm{al}})_*$ is π-adically complete by taking $M = \widehat{R}^{\mathrm{al}}$ in I.5.13. Moreover, the Frobenius induces an isomorphism $(\widehat{R}^{\mathrm{al}})_*/\pi^n(\widehat{R}^{\mathrm{al}})_* \to (\widehat{R}^{\mathrm{al}})_*/\pi^{pn}(\widehat{R}^{\mathrm{al}})_*$ for any integer $n \geq 1$ by I.5.22 and I.5.23, which implies that $(\widehat{R}^{\mathrm{al}})_*$ is perfect. Conversely, assume that $(\widehat{R}^{\mathrm{al}})_*$ is perfect. For any π-torsion element $f \in (\widehat{R}^{\mathrm{al}})_*$, we have $\pi^{1/p^n} f = 0$ for any integer $n \geq 0$, which shows that $\widehat{R}$ is almost flat by I.5.12. Moreover, it is clear that the Frobenius induces an isomorphism $(\widehat{R}^{\mathrm{al}})_*/\pi(\widehat{R}^{\mathrm{al}})_* \to (\widehat{R}^{\mathrm{al}})_*/\pi^p(\widehat{R}^{\mathrm{al}})_*$, which shows that R is almost pre-perfectoid by I.5.22 and I.5.23. □

Proposition I.5.25. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost flat, R' the integral closure of R in $R[1/\pi]$. If the Frobenius induces an almost injection $R/\pi_1 R \to R/\pi R$, then $R \to R'$ is an almost isomorphism.*

Proof. Since $R \to (R^{\mathrm{al}})_*$ is an almost isomorphism, we may replace R by $(R^{\mathrm{al}})_*$ so that we may assume that $R = (R^{\mathrm{al}})_*$, $R \subseteq R[1/\pi]$ by I.5.12 and for any $x \in R[1/\pi]$ such that $x^p \in R$, then $x \in R$ by I.5.21 and I.5.22. It suffices to show that R is integrally closed in $R[1/\pi]$. Suppose that $x \in R[1/\pi]$ is integral over R. There is an integer $N > 0$ such that x^r is an R-linear combination of $1, x, \ldots, x^N$ for any $r > 0$. Therefore, there exists an integer $k > 0$ such that $\pi^k x^r \in R$ for any $r > 0$. Taking $r = p^n$, we get $x \in \bigcap_{n \geq 0} \pi_n^{-1} R = (R^{\mathrm{al}})_* = R$ by I.5.8, which completes our proof. □

Lemma I.5.26. *Let R be a ring, π a nonzero divisor of R, $\widehat{R}$ the π-adic completion of R, $\varphi : R[1/\pi] \to \widehat{R}[1/\pi]$ the canonical morphism. Then, $\varphi^{-1}(\pi^n \widehat{R}) = \pi^n R$ for any integer n.*

Proof. Remark that $\widehat{R}$ is also π-torsion free by I.5.20. For an element $x/\pi^k \in R[1/\pi]$ (where $x \in R$, $k \geq 0$) such that $\varphi(x/\pi^k) = \pi^n y$ for some $y \in \widehat{R}$. After enlarging k, we may assume that $k + n > 0$. Thus, we deduce from the canonical isomorphism $R/\pi^{k+n} R \to \widehat{R}/\pi^{k+n}\widehat{R}$ that $x \in \pi^{k+n} R$, which completes the proof. □

Lemma I.5.27. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra such that its π-adic completion $\widehat{R}$ is almost flat (resp. flat) over $\mathcal{O}_{\widehat{K}}$, $R[\pi^\infty]$ the R-submodule of elements of R killed by some power of π. Then, $(R[\pi^\infty])^\wedge$ is almost zero (resp. zero) and the canonical morphism $\widehat{R} \to (R/R[\pi^\infty])^\wedge$ is surjective and is an almost isomorphism (resp. an isomorphism).*

Proof. The exact sequence $0 \to R[\pi^\infty] \to R \to R/R[\pi^\infty] \to 0$ induces an exact sequence of the π-adic completions

$$0 \longrightarrow (R[\pi^\infty])^\wedge \longrightarrow \widehat{R} \longrightarrow (R/R[\pi^\infty])^\wedge \longrightarrow 0, \tag{I.5.27.1}$$

since $R/R[\pi^\infty]$ is flat over $\mathcal{O}_K$ ([**Sta22**, 0315]). As $\widehat{R}[\pi^\infty]$ is almost zero (resp. zero) by assumption (I.5.12), the canonical morphism $R[\pi^\infty]^{\mathrm{al}} \to \widehat{R}^{\mathrm{al}}$ (resp. $R[\pi^\infty] \to \widehat{R}$) factors through 0, and thus so is the morphism $(R[\pi^\infty])^{\wedge\mathrm{al}} \to \widehat{R}^{\mathrm{al}}$ (resp. $(R[\pi^\infty])^\wedge \to \widehat{R}$). The conclusion follows from the exactness of (I.5.27.1). □

Lemma I.5.28. *Let K be a pre-perfectoid field. Given a commutative diagram of $\mathcal{O}_K$-algebras*

$$\begin{array}{ccc} B & \xrightarrow{g} & B' \\ \uparrow & & \uparrow \\ A & \xrightarrow{f} & A' \end{array} \tag{I.5.28.1}$$

we denote by C (resp. C') the integral closure of A in B (resp. of A' in B'). Assume that f and g are almost isomorphisms. Then, the morphism $C \to C'$ is an almost isomorphism.

Proof. Since $C \to C'$ is almost injective as g is so, it remains to show the almost surjectivity. For any $\epsilon \in \mathfrak{m}_K$ and $x' \in C'$ with identity $x'^n + a'_{n-1}x'^{n-1} + \cdots + a'_1x' + a'_0 = 0$ in B' where $a'_{n-1}, \ldots, a'_0 \in A'$, there exist $a_{n-1}, \ldots, a_0 \in A$ and $x \in B$ such that $f(a_i) = \epsilon^{n-i}a'_i$ $(0 \leq i < n)$ and $g(x) = \epsilon x'$. Thus, $g(x^n + a_{n-1}x^{n-1} + \cdots a_1x + a_0) = 0$. Since g is almost injective, we see that $\epsilon x \in C$. It follows that $C \to C'$ is almost surjective. □

Proposition I.5.29. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, A an $\mathcal{O}_K$-algebra such that its π-adic completion $\widehat{A}$ is almost flat over $\mathcal{O}_{\widehat{K}}$. We denote by B (resp. B') the integral closure of A in $A[1/\pi]$ (resp. of $\widehat{A}$ in $\widehat{A}[1/\pi]$). Then, the canonical morphism of π-adic completions $\widehat{B} \to \widehat{B'}$ is an almost isomorphism of $\mathcal{O}_K$-algebras.*

Proof. We take a system of p^k-th roots $(\pi_k)_{k\geq 0}$ of π up to units (I.5.4). By I.5.27 and I.5.28, we can replace A by its image $A/A[\pi^\infty]$ in $A[1/\pi]$, so that we may assume that A is π-torsion free (and thus so is $\widehat{A}$). Let $\varphi : A[1/\pi] \to \widehat{A}[1/\pi]$ be the canonical morphism. It suffices to show that φ induces an almost isomorphism $B/\pi^nB \to B'/\pi^nB'$ for any $n > 0$.

For any element $x' \in B'$, there exists $r > 0$ such that $\pi^r x'^{p^k} \in \widehat{A}$ for any $k > 0$. We take an element $x_{ki} \in A$ such that $\varphi(x_{ki}) - \pi^r x'^{p^i} \in \pi^{rp^k}\widehat{A}$ for $i = 0, k$. Thus, $\varphi(x_{k0}^{p^k}) - \varphi(\pi^{r(p^k-1)}x_{kk}) \in \pi^{rp^k}\widehat{A}$. By I.5.26, we see that $x_{k0}^{p^k}/\pi^{r(p^k-1)} - x_{kk} \in \pi^r A$. In particular, $(x_{k0}/\pi_k^{r(p^k-1)})^{p^k} \in A$, which implies that $x_{k0}/\pi_k^{r(p^k-1)} \in B$. Notice that $\varphi(x_{k0}/\pi_k^{r(p^k-1)}) - (\pi/\pi_k^{p^k-1})^r x' \in \pi^{r(p^k-1)}\widehat{A}$. Since k is an arbitrary positive integer, we see that $B/\pi^nB \to B'/\pi^nB'$ is almost surjective.

For any element $x \in B$ such that $\varphi(x/\pi^n) \in B'$, there exists $r > 0$ such that $\pi^r\varphi(x/\pi^n)^{p^k} \in \widehat{A}$ for any $k > 0$. We take $y \in A$ such that $\pi^r\varphi(x/\pi^n)^{p^k} - \varphi(y) \in \pi\widehat{A}$, and then we see that $\pi^r(x/\pi^n)^{p^k} - y \in \pi A$ by I.5.26. In particular, $(x/\pi_k^{np^k-r})^{p^k} \in A$, which implies that $x/\pi_k^{np^k-r} \in B$. Since k is an arbitrary positive integer, we see that $B/\pi^nB \to B'/\pi^nB'$ is almost injective. □

Corollary I.5.30. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, R' the integral closure of R in $R[1/\pi]$. Then, the morphism of π-adic completions $\widehat{R} \to \widehat{R'}$ is an almost isomorphism. In particular, R' is also almost pre-perfectoid.*

Proof. We consider the following commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & R' & \longrightarrow & R[\frac{1}{\pi}] \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{R} & \longrightarrow & R'' & \longrightarrow & \widehat{R}[\frac{1}{\pi}] \end{array} \tag{I.5.30.1}$$

where R'' is the integral closure of $\widehat{R}$ in $\widehat{R}[1/\pi]$. Since $\widehat{R} \to R''$ is an almost isomorphism by I.5.25, R'' is also perfectoid. The conlusion follows from the fact that $\widehat{R'} \to \widehat{R''}$ is an almost isomorphism by I.5.29. □

Theorem I.5.31 (Tilting correspondence [**Sch12**, 5.2, 5.21])**.** *Let K be a perfectoid field, π a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit.*

1. *The functor $\mathcal{O}_K^{\mathrm{al}}$-**Perf** $\to (\mathcal{O}_K/\pi)^{\mathrm{al}}$-**Perf**, $R \mapsto R/\pi R$, is an equivalence of categories.*
2. *The functor $\mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Perf** $\to (\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-**Perf**, $R \mapsto R/\pi^\flat R$ is an equivalence of categories, and the functor $(\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-**Perf** $\to \mathcal{O}_{K^\flat}^{\mathrm{al}}$-**Perf**, $R \mapsto R^\flat$ is a quasi-inverse.*

(3) *Let R be a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra with tilt $R^\flat$. Then, R is isomorphic to $\mathcal{O}_L^{\mathrm{al}}$ for some perfectoid field L over K if and only if $R^\flat$ is isomorphic to $\mathcal{O}_{L'}^{\mathrm{al}}$ for some perfectoid field L' over $K^\flat$.*

In conclusion, we have natural equivalences

$$\mathcal{O}_K^{\mathrm{al}}\text{-}\mathbf{Perf} \xrightarrow{\sim} (\mathcal{O}_K/\pi)^{\mathrm{al}}\text{-}\mathbf{Perf} \xrightarrow{\sim} (\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}\text{-}\mathbf{Perf} \xleftarrow{\sim} \mathcal{O}_{K^\flat}^{\mathrm{al}}\text{-}\mathbf{Perf}, \tag{I.5.31.1}$$

where the middle equivalence is given by the isomorphism (I.5.15.4) $\mathcal{O}_{K^\flat}/\pi^\flat\mathcal{O}_{K^\flat} \xrightarrow{\sim} \mathcal{O}_K/\pi\mathcal{O}_K$. We remark that the natural isomorphisms of the equivalence in (2) are defined as follows: for a perfectoid $\mathcal{O}_{K^\flat}^{\mathrm{al}}$-algebra R, the natural isomorphism $R \xrightarrow{\sim} (R/\pi^\flat R)^\flat$ is induced by the homomorphism of $\mathcal{O}_{K^\flat}$-algebras $R_* \to (R_*/\pi^\flat R_*)^\flat$ sending x to $(\cdots, x^{1/p^2}, x^{1/p}, x)$ (notice that R_* is perfect by I.5.24); for a perfectoid $(\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}}$-algebra R, the natural isomorphism $R^\flat/\pi^\flat R^\flat \xrightarrow{\sim} R$ is induced by the projection on the first component $(R_*)^\flat \to R_*$ of $\mathcal{O}_{K^\flat}$-algebras (cf. [**Sch12**, 5.17]). Consequently, for a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra R, the morphism of $(\mathcal{O}_{K^\flat}/\pi^\flat)^{\mathrm{al}} = (\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebras

$$R^\flat/\pi^\flat R^\flat \longrightarrow R/\pi R \tag{I.5.31.2}$$

induced by the projection on the first component is an isomorphism.

Proposition I.5.32. *Let K be a perfectoid field with a pseudo-uniformizer π of K dividing p, $B \leftarrow A \to C$ a diagram of perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras. Then, the π-adically completed tensor product $B\widehat{\otimes}_A C$ is also perfectoid.*

Proof. We follow closely the proof of [**Sch12**, 6.18]. Firstly, we claim that $(B \otimes_A C)/\pi$ is flat over $(\mathcal{O}_K/\pi)^{\mathrm{al}}$. Since $(B \otimes_A C)/\pi = (B^\flat \otimes_{A^\flat} C^\flat)/\pi^\flat$, it suffices to show the flatness of $B^\flat \otimes_{A^\flat} C^\flat$ over $\mathcal{O}_{K^\flat}^{\mathrm{al}}$, which amounts to say that the submodule of $\pi^\flat$-torsion elements of $(B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$ is almost zero as $B^\flat \otimes_{A^\flat} C^\flat = ((B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat)^{\mathrm{al}}$. If $f \in (B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$ is a $\pi^\flat$-torsion element, then by perfectness of $(B_*)^\flat \otimes_{(A_*)^\flat} (C_*)^\flat$, we have $(\pi^\flat)^{1/p^n} f = 0$ for any $n > 0$, which proves the claim.

Thus, $(B \otimes_A C)/\pi$ is a perfectoid $(\mathcal{O}_K/\pi)^{\mathrm{al}}$-algebra. It corresponds to a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra D by I.5.31 and it induces a morphism $B\widehat{\otimes}_A C \to D$ by universal property of π-adically completed tensor product. We use dévissage to show that $(B \otimes_A C)/\pi^n \to D/\pi^n$ is an isomorphism for any integer $n > 0$. By induction,

$$\begin{array}{ccccccc} (B \otimes_A C)/\pi^n & \xrightarrow{\cdot\pi} & (B \otimes_A C)/\pi^{n+1} & \longrightarrow & (B \otimes_A C)/\pi & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow D/\pi^n & \xrightarrow{\cdot\pi} & D/\pi^{n+1} & \longrightarrow & D/\pi & \longrightarrow & 0 \end{array} \tag{I.5.32.1}$$

the vertical arrows on the left and right are isomorphisms. By snake's lemma in the abelian category $\mathcal{O}_K^{\mathrm{al}}$-**Mod** ([**Sta22**, 010H]), we know that the vertical arrow in the middle is also an isomorphism. In conclusion, $B\widehat{\otimes}_A C \to D$ is an isomorphism, which completes the proof. □

Corollary I.5.33. *Let K be a pre-perfectoid field, $B \leftarrow A \to C$ a diagram of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. Then, the tensor product $B \otimes_A C$ is also almost pre-perfectoid.*

Proof. Since α^* commutes with arbitrary limits and colimits (I.5.5), we have $(B\widehat{\otimes}_A C)^{\mathrm{al}} = \widehat{B}^{\mathrm{al}}\widehat{\otimes}_{\widehat{A}^{\mathrm{al}}}\widehat{C}^{\mathrm{al}}$, which is perfectoid by I.5.32. □

Lemma I.5.34. *Let K be a perfectoid field, $\mathcal{O}_K \to V$ an extension of valuation rings of height 1. Then, there exists an extension of perfectoid fields $K \to L$ and an extension of valuation rings $V \to \mathcal{O}_L$ over $\mathcal{O}_K$.*

Proof. Let π be a pseudo-uniformizer of K, E the fraction field of V, $\overline{E}$ an algebraic closure of E, $\overline{V}$ the integral closure of V in $\overline{E}$. Let $\mathfrak{m}$ be a maximal ideal of $\overline{V}$. It lies over the unique maximal ideal of V as $V \to \overline{V}$ is integral. Setting $W = \overline{V}_{\mathfrak{m}}$, according to [**Bou06**, VI.§8.6, Prop.6], $V \to W$ is an extension of valuation rings of height 1. Since W is integrally closed in the algebraically closed fraction field $\overline{E}$, the Frobenius is surjective on W/pW. Thus, the fraction field of W is a pre-perfectoid field over K. Passing to completion, we get an extension of perfectoid fields $K \to L$ by I.5.2. □

Theorem I.5.35 ([**BS19**, 8.10])**.** *Let K be a pre-perfectoid field with a pseudo-uniformizer π dividing p, $R \to R'$ a homomorphism of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. If $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a π-complete arc-covering, then for any integer $n \geq 1$, the augmented Čech complex*

$$0 \to R/\pi^n \to R'/\pi^n \to (R' \otimes_R R')/\pi^n \to \cdots \tag{I.5.35.1}$$

is almost exact.

PROOF. We follow Bhatt-Scholze's proof [**BS19**, 8.10]. After replacing $\mathcal{O}_K$, R, R' by their π-adic completions, we may assume that K is a perfectoid field and that R^{al} and R'^{al} are perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebras such that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a π-complete arc-covering by I.3.4.(6). Moreover, since $R \to (R^{\mathrm{al}})_*$ is an almost isomorphism, after replacing R by $(R^{\mathrm{al}})_*$ and R' by $(R^{\mathrm{al}})_* \otimes_R R'$, we may assume further that $R = (R^{\mathrm{al}})_*$. Then, the Frobenius induces an isomorphism (resp. almost isomorphism) $R/\pi_1 R \xrightarrow{\sim} R/\pi R$ (resp. $R'/\pi_1 R' \to R'/\pi R'$) by I.5.22, where π_1 is a p-th root of π up to a unit. Thus, we see that the projection on the first component induces an isomorphism (note that $R = (R^{\mathrm{al}})_*$ is π-torsion free by I.5.12)

$$R^\flat/\pi^\flat R^\flat \xrightarrow{\sim} R/\pi R \tag{I.5.35.2}$$

and an almost isomorphism (by the preceding isomorphism for $(R'^{\mathrm{al}})_*$ or by (I.5.31.2))

$$R'^\flat/\pi^\flat R'^\flat \longrightarrow R'/\pi R'. \tag{I.5.35.3}$$

In particular, $\mathrm{Spec}(R'^\flat/\pi^\flat) \to \mathrm{Spec}(R^\flat/\pi^\flat)$ is an arc-covering as $\mathrm{Spec}(R'/\pi) \to \mathrm{Spec}(R/\pi)$ is so.

On the other hand, since the localization functor α^* commutes with arbitrary limits and colimits (I.5.5), $(\widehat{\otimes}_R^k R')^{\mathrm{al}} = \widehat{\otimes}_{R^{\mathrm{al}}}^k R'^{\mathrm{al}}$ is still a perfectoid $\mathcal{O}_K^{\mathrm{al}}$-algebra by I.5.32 for any $k \geq 0$. In particular, $\widehat{\otimes}_R^k R'$ is almost flat over $\mathcal{O}_K$. Then, by dévissage, it suffices to show the almost exactness of the augmented Čech complex when $n = 1$, i.e. the almost exactness of

$$0 \to R^\flat/\pi^\flat \to R'^\flat/\pi^\flat \to (R'^\flat \otimes_{R^\flat} R'^\flat)/\pi^\flat \to \cdots. \tag{I.5.35.4}$$

We claim that the natural morphism $X = \mathrm{Spec}(R'^\flat) \coprod \mathrm{Spec}(R^\flat[1/\pi^\flat]) \to Y = \mathrm{Spec}(R^\flat)$ is an arc-covering. Firstly, we see that $X \to Y = \mathrm{Spec}(R^\flat/\pi^\flat) \bigcup \mathrm{Spec}(R^\flat[1/\pi^\flat])$ is surjective as we have shown that $\mathrm{Spec}(R'^\flat/\pi^\flat) \to \mathrm{Spec}(R^\flat/\pi^\flat)$ is an arc-covering. Therefore, we only need to consider the test map $\mathrm{Spec}(V) \to Y$ where V is a valuation ring of height 1. There are three cases:

(1) If $\pi^\flat$ is invertible in V, then we get a natural lifting $R^\flat[1/\pi^\flat] \to V$.
(2) If $\pi^\flat$ is zero in V, then $R^\flat \to V$ factors through $R^\flat/\pi^\flat \xrightarrow{\sim} R/\pi$, and thus there is a lifting $R'^\flat/\pi^\flat \to R'/\pi \to W$.
(3) Otherwise, $\mathcal{O}_{K^\flat} \to V$ is an extension of valuation rings. After replacing V by an extension (I.5.34), we may assume that $V[1/\pi^\flat]$ is a perfectoid field over $K^\flat$ with valuation ring V. By tilting correspondence I.5.31, it corresponds to a perfectoid field over K with valuation ring $V^\sharp$, together with an $\mathcal{O}_K$-morphism $R \to V^\sharp$ by I.5.9. Since $R \to R'$ gives a π-complete arc-covering, there is an extension $V^\sharp \to W$ of valuation rings of height 1 and a lifting $R' \to W$. After replacing W by an extension (I.5.34), we may assume that $W[1/\pi]$ is a perfectoid field over K with valuation ring W. By tilting correspondence I.5.31 and I.5.9, we get a lifting $R'^\flat \to W^\flat$ of $R^\flat \to V$.

Now we apply I.4.10 to the arc-covering $X \to Y$ of perfect affine $\mathbb{F}_p$-schemes. We get an exact augmented Čech complex

$$0 \to R^\flat \to R'^\flat \times R^\flat[\frac{1}{\pi^\flat}] \to (R'^\flat \times R^\flat[\frac{1}{\pi^\flat}]) \otimes_{R^\flat} (R'^\flat \times R^\flat[\frac{1}{\pi^\flat}]) \to \cdots. \tag{I.5.35.5}$$

Since each term is a perfect $\mathbb{F}_p$-algebra, the submodule of $\pi^\flat$-torsion elements is almost zero, in other words, each term is almost flat over $\mathcal{O}_{K^\flat}$. Modulo $\pi^\flat$, we get the almost exactness of (I.5.35.4), which completes the proof. □

Definition I.5.36. Let K be a pre-perfectoid field, $A \to B$ a morphism of $\mathcal{O}_K$-algebras.

(1) We say that $A \to B$ is *almost étale* if $A^{\mathrm{al}} \to B^{\mathrm{al}}$ is an étale morphism of $\mathcal{O}_K^{\mathrm{al}}$-algebras in the sense of [**GR03**, 3.1.1.(iv)].
(2) We say that $A \to B$ is *almost finite étale* if it is almost étale and if B^{al} is an almost finitely presented A^{al}-module in the sense of [**GR03**, 2.3.10] (cf. [**Sch12**, 4.13], [**AGT16**, V.7.1]).

We remark that in I.5.36 if $A \to B$ is a morphism of K-algebras, then it is almost étale (resp. almost finite étale) if and only if it is étale (resp. finite étale).

Proposition I.5.37. *Let K be a pre-perfectoid field, $\mathscr{C}$ the full subcategory of the category of $\mathcal{O}_K$-algebras formed by those $\mathcal{O}_K$-algebras which are almost pre-perfectoid.*

(1) *The subcategory $\mathscr{C}$ is stable under taking colimits and products.*
(2) *Let $A \to B$ be an almost étale morphism of $\mathcal{O}_K$-algebras. If $A \in \mathrm{Ob}(\mathscr{C})$, then $B \in \mathrm{Ob}(\mathscr{C})$.*

PROOF. Let π be a pseudo-uniformizer of K dividing p with a p-th root π_1 up to a unit.

(1) The subcategory $\mathscr{C}$ is stable under taking tensor products by I.5.33. Let $(R_\lambda)_{\lambda\in\Lambda}$ be a directed system of objects in $\mathscr{C}$ and $R = \mathrm{colim}_{\lambda\in\Lambda} R_\lambda$. It is clear that the Frobenuis induces an almost

isomorphism $R/\pi_1 R \to R/\pi R$. On the other hand, $\widehat{R}$ is the π-adic completion of $\mathrm{colim}_{\lambda\in\Lambda}\widehat{R_\lambda}$. Since the latter is almost flat over $\mathcal{O}_{\widehat{K}}$, so is $\widehat{R}$ (I.5.20). Thus, $\mathscr{C}$ is stable under taking colimits.

Let $(R_\lambda)_{\lambda\in\Lambda}$ be a set of objects in $\mathscr{C}$. Since $R/\pi R = \prod_{\lambda\in\Lambda} R_\lambda/\pi R_\lambda$, the Frobenius induces an almost isomorphism $R/\pi_1 R \to R/\pi R$. Moreover, the submodule of π-torsion elements of $\widehat{R} = \prod_{\lambda\in\Lambda}\widehat{R_\lambda}$ is almost zero, which implies that $\widehat{R}$ is almost flat over $\mathcal{O}_{\widehat{K}}$ (I.5.12). We conclude that $\mathscr{C}$ is stable under taking products.

(2) Since B is almost flat over A, it is almost flat over $\mathcal{O}_K$ and thus $\widehat{B}$ is almost flat over $\mathcal{O}_{\widehat{K}}$ (I.5.20). Since B is almost étale over A, the map $B/\pi_1 B \to B/\pi B$ induced by the Frobenius is almost isomorphic to the base change of the map $A/\pi_1 A \to A/\pi A$ by $A \to B$ ([**GR03**, 3.5.13]), which completes the proof. □

Lemma I.5.38. *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost flat and almost pre-perfectoid, R' an R-algebra which is almost finite étale. Then, the integral closure of R in R' is almost isomorphic to both R' and the integral closure of R in $R'[1/\pi]$.*

Proof. Notice that R' is also almost flat and almost pre-perfectoid by I.5.37. Since R' is almost finitely generated over R as an R-module, the elements of $\mathfrak{m}_K R'$ are integral over R (cf. [**GR03**, 2.3.10]). Thus, the integral closure of R in R' is almost isomorphic to R'. On the other hand, since R' is almost isomorphic to its integral closure in $R'[1/\pi]$ by I.5.25, the integral closure of R in R' is almost isomorphic to the integral closure of R in $R'[1/\pi]$ by I.5.28. □

I.5.39. We recall some basic definitions about affinoid algebras used in [**Sch12**] in order to prove the almost purity theorem I.5.41 by reducing to *loc.cit.* Let K be a complete valuation field of height 1. A *Tate K-algebra* is a topological K-algebra $\mathcal{R}$ whose topology is generated by the open subsets $a\mathcal{R}_0$ for a subring $\mathcal{R}_0 \subset \mathcal{R}$ and any $a \in K^\times$. We denote by $\mathcal{R}^\circ$ the subring of power-bounded elements of $\mathcal{R}$, which is thus an $\mathcal{O}_K$-algebra. An *affinoid K-algebra* is a pair $(\mathcal{R}, \mathcal{R}^+)$ consisting of a Tate K-algebra $\mathcal{R}$ and a subring $\mathcal{R}^+$ of $\mathcal{R}^\circ$ which is open and integrally closed in $\mathcal{R}$. A morphism of affinoid K-algebras $(\mathcal{R}, \mathcal{R}^+) \to (\mathcal{R}', \mathcal{R}'^+)$ is a morphism of topological K-algebras $f : \mathcal{R} \to \mathcal{R}'$ with $f(\mathcal{R}^+) \subseteq \mathcal{R}'^+$. Such a morphism is called *finite étale* in the sense of [**Sch12**, 7.1.(i)] if $\mathcal{R}'$ is finite étale over $\mathcal{R}$ endowed with the canonical topology as a finitely generated $\mathcal{R}$-module and if $\mathcal{R}'^+$ is the integral closure of $\mathcal{R}^+$ in $\mathcal{R}'$.

For a perfectoid field K and an affinoid K-algebra $(\mathcal{R}, \mathcal{R}^+)$, the inclusion $\mathcal{R}^+ \subseteq \mathcal{R}^\circ$ is an almost isomorphism. Indeed, for any $\epsilon \in \mathfrak{m}_K$ and any power-bounded element $x \in \mathcal{R}^\circ$, we have $(\epsilon x)^n \in \mathcal{R}^+$ for $n \in \mathbb{N}$ large enough as $\mathcal{R}^+$ is open. Thus, $\epsilon x \in \mathcal{R}^+$ as $\mathcal{R}^+$ is integrally closed. We remark that $(\mathcal{R}, \mathcal{R}^+)$ is perfectoid in the sense of [**Sch12**, 6.1] if and only if $\mathcal{R}^\circ$ is bounded and almost perfectoid over $\mathcal{O}_K$ ([**Sch12**, 5.5, 5.6]).

I.5.40. There is a typical example for constructing affinoid algebras from commutative algebras (cf. [**And18**, Sorite 2.3.1]). Let K be a complete valuation field of height 1 with a pseudo-uniformizer π, R a flat $\mathcal{O}_K$-algebra. The K-algebra $R[1/\pi]$ endowed with the π-adic topology defined by R is a Tate K-algebra. Let $\overline{R}$ be the integral closure of R in $R[1/\pi]$. It is clear that any element of $\overline{R}$ is power-bounded. Thus, $(R[1/\pi], \overline{R})$ is an affinoid K-algebra.

Let S be a finite $R[1/\pi]$-algebra endowed with the canonical topology. More precisely, the topology can be defined as follows: we take a finite R-algebra R' contained in S which contains a family of generators of the $R[1/\pi]$-algebra S; then the canonical topology of $S = R'[1/\pi]$ is the π-adic topology defined by R' (which is independent of the choice of R'). Let $\overline{R'}$ be the integral closure of R' in $R'[1/\pi]$, which is also the integral closure of R in $R'[1/\pi]$. We remark that $(R[1/\pi], \overline{R}) \to (R'[1/\pi], \overline{R'})$ is a finite étale morphism of affinoid K-algebras if and only if $R[1/\pi] \to R'[1/\pi]$ is finite étale.

Theorem I.5.41 (Almost purity, [**Sch12**, 7.9])**.** *Let K be a pre-perfectoid field with a pseudo-uniformizer π, R an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, R' the integral closure of R in a finite étale $R[1/\pi]$-algebra. Then, R' is almost pre-perfectoid and the π-adic completion $\widehat{R'}$ is almost finite étale over $\widehat{R}$.*

Moreover, if R is π-torsion free and if $(R, \pi R)$ is a henselian pair, then R' is almost finite étale over R.

Proof. For the first statement, by I.5.27, we can replace R by its image $R/R[\pi^\infty]$ in $R[1/\pi]$ (which does not change R'), so that we may assume that R is π-torsion free (and thus so is $\widehat{R}$). Let S (resp. S') be the integral closure of $\widehat{R}$ in $\widehat{R}[1/\pi]$ (resp. of $R' \otimes_R \widehat{R}$ in $R' \otimes_R \widehat{R}[1/\pi]$). Then, we obtain a finite étale morphism of affinoid $\widehat{K}$-algebras $(\widehat{R}[1/\pi], S) \to (R' \otimes_R \widehat{R}[1/\pi], S')$ by I.5.40.

Since $\widehat{R}$ is almost perfectoid, $\widehat{R} \to S$ is an almost isomorphism (I.5.25). Thus, S is bounded and almost perfectoid over $\mathcal{O}_{\widehat{K}}$. In other words, $(\widehat{R}[1/\pi], S)$ is a perfectoid affinoid $\widehat{K}$-algebra. Then, by

almost purity ([**Sch12**, 7.9.(iii)]), the $\mathcal{O}_{\widehat{K}}$-algebra S' is almost perfectoid (thus $S' \to \widehat{S}'$ is an almost isomorphism by definition) and almost finite étale over S.

On the other hand, the two $\mathcal{O}_{\widehat{K}}$-algebras R' and $R' \otimes_R \widehat{R}$ have the same π-adic completion $\widehat{R'}$. Thus, the π-adic completions of the integral closures of R' and $R' \otimes_R \widehat{R}$ in $R'[1/\pi]$ and $R' \otimes_R \widehat{R}[1/\pi]$ respectively are almost isomorphic to that of $\widehat{R'}$ in $\widehat{R'}[1/\pi]$ by I.5.29. In other words, $\widehat{R'} \to \widehat{S}'$ is an almost isomorphism. In conclusion, R' is almost pre-perfectoid, and $\widehat{R'}$ is almost finite étale over $\widehat{R}$.

We assume moreover that R is π-torsion free and $(R, \pi R)$ is a henselian pair. Recall that the category of almost $\mathcal{O}_K$-algebras finite étale over R^{al} (resp. over $\widehat{R}^{\mathrm{al}}$) is equivalent to that over $(R/\pi R)^{\mathrm{al}}$ via the base change functor ([**GR03**, 5.5.7.(iii)]). Hence, there exists an R-algebra R'' which is almost finite étale over R such that $(R'' \otimes_R \widehat{R})^{\mathrm{al}}$ is isomorphic to $\widehat{R'}^{\mathrm{al}}$. On the other hand, recall that the category of finite étale $R[1/\pi]$-algebras is equivalent to the category of finite étale $\widehat{R}[1/\pi]$-algebras via the base change functor ([**GR03**, 5.4.53]). Notice that $R''[1/\pi] \otimes_R \widehat{R} \cong \widehat{R'}[1/\pi]$ by the construction of R'' and that $R'[1/\pi] \otimes_R \widehat{R} \cong \widehat{R'}[1/\pi]$ by the almost isomorphisms $\widehat{R'} \to \widehat{S}' \leftarrow S'$. Hence, there is an isomorphism $R''[1/\pi] \cong R'[1/\pi]$. By I.5.38, we see that R'' is almost isomorphic to R', which completes the proof. □

I.6. Brief Review on Covanishing Fibred Sites

We give a brief review on covanishing fibred sites, which are developed by Abbes and Gros [**AGT16**, VI]. We remark that [**AGT16**, VI] does not require the sites to admit finite limits (I.2.4).

I.6.1. A *fibred site* E/C is a fibred category $\pi : E \to C$ whose fibres are sites such that for a cleavage and for every morphism $f : \beta \to \alpha$ in C, the inverse image functor $f^+ : E_\alpha \to E_\beta$ gives a morphism of sites (so that the same holds for any cleavage) (cf. [**SGA 4**$_{\mathrm{II}}$, VI.7.2]).

Let x be an object of E over $\alpha \in \mathrm{Ob}(C)$. We denote by

$$\iota_\alpha^+ : E_\alpha \to E \tag{I.6.1.1}$$

the inclusion functor of the fibre category E_α over α into the whole category E. A *vertical covering* of x is the image by ι_α^+ of a covering family $\{x_m \to x\}_{m \in M}$ in E_α. We call the topology generated by all vertical coverings the *total topology* on E (cf. [**SGA 4**$_{\mathrm{II}}$, VI.7.4.2]).

Assume further that C is a site. A *Cartesian covering* of x is a family $\{x_n \to x\}_{n \in N}$ of morphisms of E such that there exists a covering family $\{\alpha_n \to \alpha\}_{n \in N}$ in C with x_n isomorphic to the pullback of x by $\alpha_n \to \alpha$ for each n.

Definition I.6.2 ([**AGT16**, VI.5.3])**.** A *covanishing fibred site* is a fibred site E/C where C is a site. We associate to E the *covanishing topology* which is generated by all vertical coverings and Cartesian coverings. We simply call a covering family for the covanishing topology a *covanishing covering*.

Definition I.6.3. Let E/C be a covanishing fibred site. We call a composition of a Cartesian covering followed by vertical coverings a *standard covanishing covering*. More precisely, a standard covanishing covering is a family of morphisms of E

$$\{x_{nm} \to x\}_{n \in N, m \in M_n} \tag{I.6.3.1}$$

such that there is a Cartesian covering $\{x_n \to x\}_{n \in N}$ and for each $n \in N$ a vertical covering $\{x_{nm} \to x_n\}_{m \in M_n}$.

Proposition I.6.4 ([**AGT16**, VI.5.9])**.** *Let E/C be a covanishing fibred site. Assume that in each fibre any object is quasi-compact, then a family of morphisms $\{x_i \to x\}_{i \in I}$ of E is a covanishing covering if and only if it can be refined by a standard covanishing covering.*

I.6.5. Let E/C be a fibred category. Fixing a cleavage of E/C, to give a presheaf $\mathcal{F}$ on E is equivalent to give a presheaf $\mathcal{F}_\alpha$ on each fibre category E_α and transition morphisms $\mathcal{F}_\alpha \to f^{\mathrm{p}}\mathcal{F}_\beta$ for each morphism $f : \beta \to \alpha$ in C satisfying a cocycle relation (cf. [**SGA 4**$_{\mathrm{II}}$, VI.7.4.7]). Thus, we simply denote a presheaf $\mathcal{F}$ on E by

$$\mathcal{F} = (\mathcal{F}_\alpha)_{\alpha \in \mathrm{Ob}(C)}, \tag{I.6.5.1}$$

where $\mathcal{F}_\alpha = \iota_\alpha^{\mathrm{p}}\mathcal{F}$ is the restriction of $\mathcal{F}$ on the fibre category E_α. If E/C is a fibred site, then $\mathcal{F}$ is a sheaf with respect to the total topology on E if and only if $\mathcal{F}_\alpha$ is a sheaf on E_α for each α ([**SGA 4**$_{\mathrm{II}}$, VI.7.4.7]). Moreover, we have the following description of a covanishing sheaf.

Proposition I.6.6 ([**AGT16**, VI.5.10])**.** *Let E/C be a covanishing fibred site. Then, a presheaf $\mathcal{F}$ on E is a sheaf if and only if the following conditions hold:*

(v) *The presheaf $\mathcal{F}_\alpha = \iota_\alpha^{\mathrm{p}}\mathcal{F}$ on E_α is a sheaf for any $\alpha \in \mathrm{Ob}(C)$.*

(c) *For any covering family* $\{f_i : \alpha_i \to \alpha\}_{i\in I}$ *of* C*, if we set* $\alpha_{ij} = \alpha_i \times_\alpha \alpha_j$ *and* $f_{ij} : \alpha_{ij} \to \alpha$*, then the sequence of sheaves on* E_α*,*

$$\mathcal{F}_\alpha \to \prod_{i\in I} f_{i*}\mathcal{F}_{\alpha_i} \rightrightarrows \prod_{i,j\in I} f_{ij*}\mathcal{F}_{\alpha_{ij}}, \tag{I.6.6.1}$$

is exact.

I.7. Faltings Ringed Sites

I.7.1. Let $Y \to X$ be a morphism of $\mathbb{U}$-small coherent schemes, and let $\mathbf{E}_{Y\to X}$ be the category of morphisms $V \to U$ of $\mathbb{U}$-small coherent schemes over the morphism $Y \to X$, namely, the category of commutative diagrams of coherent schemes

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{I.7.1.1}$$

Given a functor $I \to \mathbf{E}_{Y\to X}$ sending i to $V_i \to U_i$, if $\lim V_i$ and $\lim U_i$ are representable in the category of coherent schemes, then $\lim(V_i \to U_i)$ is representable by $\lim V_i \to \lim U_i$. We say that a morphism $(V' \to U') \to (V \to U)$ of objects of $\mathbf{E}_{Y\to X}$ is *Cartesian* if $V' \to V \times_U U'$ is an isomorphism. It is clear that the Cartesian morphisms in $\mathbf{E}_{Y\to X}$ are stable under base change.

Consider the functor

$$\phi^+ : \mathbf{E}_{Y\to X} \longrightarrow \mathbf{Sch}^{\mathrm{coh}}_{/X},\ (V \to U) \longmapsto U. \tag{I.7.1.2}$$

The fibre category over U is canonically equivalent to the category $\mathbf{Sch}^{\mathrm{coh}}_{/U_Y}$ of coherent U_Y-schemes, where $U_Y = Y \times_X U$. The base change by $U' \to U$ gives an inverse image functor $\mathbf{Sch}^{\mathrm{coh}}_{/U_Y} \to \mathbf{Sch}^{\mathrm{coh}}_{/U'_Y}$, which endows $\mathbf{E}_{Y\to X}/\mathbf{Sch}^{\mathrm{coh}}_{/X}$ with a structure of fibred category. We define a presheaf on $\mathbf{E}_{Y\to X}$ by

$$\overline{\mathscr{B}}(V \to U) = \Gamma(U^V, \mathcal{O}_{U^V}), \tag{I.7.1.3}$$

where U^V is the integral closure of U in V.

Definition I.7.2. Let $Y \to X$ be a morphism of coherent schemes. A morphism $(V' \to U') \to (V \to U)$ in $\mathbf{E}_{Y\to X}$ is called *étale*, if $U' \to U$ is étale and $V' \to V \times_U U'$ is finite étale.

Lemma I.7.3. *Let* $Y \to X$ *be a morphism of coherent schemes,* $(V'' \to U'') \xrightarrow{g} (V' \to U') \xrightarrow{f} (V \to U)$ *morphisms in* $\mathbf{E}_{Y\to X}$*.*

(1) *If* f *is étale, then any base change of* f *is also étale.*
(2) *If* f *and* g *are étale, then* $f \circ g$ *is also étale.*
(3) *If* f *and* $f \circ g$ *are étale, then* g *is also étale.*

Proof. It follows directly from the definitions. □

I.7.4. Let $Y \to X$ be a morphism of coherent schemes. We still denote by $X_{\text{ét}}$ (resp. $X_{\text{fét}}$) the site formed by coherent étale (resp. finite étale) X-schemes endowed with the étale topology. Let $\mathbf{E}^{\text{ét}}_{Y\to X}$ be the full subcategory of $\mathbf{E}_{Y\to X}$ formed by $V \to U$ étale over the final object $Y \to X$. It is clear that $\mathbf{E}^{\text{ét}}_{Y\to X}$ is stable under finite limits in $\mathbf{E}_{Y\to X}$. Then, the functor (I.7.1.2) induces a functor

$$\phi^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \longrightarrow X_{\text{ét}},\ (V \to U) \longmapsto U, \tag{I.7.4.1}$$

which endows $\mathbf{E}^{\text{ét}}_{Y\to X}/X_{\text{ét}}$ with a structure of fibred sites, whose fibre over U is the finite étale site $U_{Y,\text{fét}}$. We endow $\mathbf{E}^{\text{ét}}_{Y\to X}$ with the associated covanishing topology, that is, the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where M is a finite set and $\coprod_{m\in M} V_m \to V$ is surjective;
(c) $\{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$, where N is a finite set and $\coprod_{n\in N} U_n \to U$ is surjective.

It is clear that any object of $\mathbf{E}^{\text{ét}}_{Y\to X}$ is quasi-compact by I.6.4. We still denote by $\overline{\mathscr{B}}$ the restriction of the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}_{Y\to X}$ to $\mathbf{E}^{\text{ét}}_{Y\to X}$ if there is no ambiguity.

Lemma I.7.5. *Let* $Y \to X$ *be a morphism of coherent schemes. Then, the presheaf on* $\mathbf{Sch}^{\mathrm{coh}}_{/Y}$ *sending* Y' *to* $\Gamma(X^{Y'}, \mathcal{O}_{X^{Y'}})$ *is a sheaf with respect to the fpqc topology* ([**Sta22**, 022A]).

PROOF. We may regard $\mathcal{O}_{X^{Y'}}$ as a quasi-coherent $\mathcal{O}_X$-algebra over X. It suffices to show that for a finite family of morphisms $\{Y_i \to Y\}_{i\in I}$ with $Y' = \coprod_{i\in I} Y_i$ faithfully flat over Y, the sequence of quasi-coherent $\mathcal{O}_X$-algebras

$$0 \longrightarrow \mathcal{O}_{X^Y} \longrightarrow \mathcal{O}_{X^{Y'}} \rightrightarrows \mathcal{O}_{X^{Y'\times_Y Y'}} \tag{I.7.5.1}$$

is exact. Thus, we may assume that $X = \mathrm{Spec}(R)$ is affine. We set $A_0 = \Gamma(Y, \mathcal{O}_Y)$, $A_1 = \Gamma(Y', \mathcal{O}_{Y'})$, $A_2 = \Gamma(Y' \times_Y Y', \mathcal{O}_{Y'\times_Y Y'})$, $R_0 = \Gamma(X^Y, \mathcal{O}_{X^Y})$, $R_1 = \Gamma(X^{Y'}, \mathcal{O}_{X^{Y'}})$, $R_2 = \Gamma(X^{Y'\times_Y Y'}, \mathcal{O}_{X^{Y'\times_Y Y'}})$. Notice that R_i is the integral closure of R in A_i for $i = 0, 1, 2$ ([**Sta22**, 035F]). Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_0 & \longrightarrow & R_1 & \rightrightarrows & R_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \rightrightarrows & A_2 \end{array} \tag{I.7.5.2}$$

We see that the vertical arrows are injective and the second row is exact by faithfully flat descent. Notice that $R_0 = A_0 \cap R_1$, since they are both the integral closure of R in A_0 as $A_0 \subseteq A_1$. Thus, the first row is also exact, which completes the proof. □

Proposition I.7.6. *Let $Y \to X$ be a morphism of coherent schemes. Then, the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$ is a sheaf.*

PROOF. It follows directly from I.6.6, whose first condition holds by I.7.5, and whose second condition holds by I.3.17 (cf. [**AGT16**, III.8.16]). □

Definition I.7.7 ([**Fal02**, page 214], [**AGT16**, VI.10.1])**.** We call $(\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}})$ the *Faltings ringed site* of the morphism of coherent schemes $Y \to X$.

It is clear that the localization $(\mathbf{E}^{\text{ét}}_{Y\to X})_{/(V\to U)}$ of $\mathbf{E}^{\text{ét}}_{Y\to X}$ at an object $V \to U$ is canonically equivalent to the Faltings ringed site $\mathbf{E}^{\text{ét}}_{V\to U}$ of the morphism $V \to U$ by I.6.4 (cf. [**AGT16**, VI.10.14]).

I.7.8. Let $Y \to X$ be a morphism of coherent schemes. Consider the natural functors

$$\psi^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \longrightarrow Y_{\text{ét}},\ (V \to U) \longmapsto V, \tag{I.7.8.1}$$

$$\beta^+ : Y_{\text{fét}} \longrightarrow \mathbf{E}^{\text{ét}}_{Y\to X},\ V \longmapsto (V \to X), \tag{I.7.8.2}$$

$$\sigma^+ : X_{\text{ét}} \longrightarrow \mathbf{E}^{\text{ét}}_{Y\to X},\ U \longmapsto (Y \times_X U \to U). \tag{I.7.8.3}$$

They are left exact and continuous (cf. [**AGT16**, VI 10.6, 10.7]). Then, we obtain a commutative diagram of sites associated functorially to the morphism $Y \to X$ by I.2.5,

$$\begin{array}{ccccc} & & Y_{\text{ét}} & & \\ & \swarrow^{\rho} & \downarrow{\scriptstyle \psi} & \searrow & \\ Y_{\text{fét}} & \xleftarrow[\beta]{} & \mathbf{E}^{\text{ét}}_{Y\to X} & \xrightarrow[\sigma]{} & X_{\text{ét}} \end{array} \tag{I.7.8.4}$$

where $\rho : Y_{\text{ét}} \to Y_{\text{fét}}$ is defined by the inclusion functor, and the unlabelled arrow $Y_{\text{ét}} \to X_{\text{ét}}$ is induced by the morphism $Y \to X$. Moreover, if $\mathcal{O}_{X_{\text{ét}}}$ denotes the structural sheaf on $X_{\text{ét}}$ sending U to $\Gamma(U, \mathcal{O}_U)$, then σ^+ actually defines a morphism of ringed sites

$$\sigma : (\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}) \longrightarrow (X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}). \tag{I.7.8.5}$$

We will study more properties of these morphisms in the remaining sections.

Lemma I.7.9. *Let X be the spectrum of an absolutely integrally closed valuation ring, Y a quasi-compact open subscheme of X. Then, for any presheaf $\mathcal{F}$ on $\mathbf{E}^{\text{ét}}_{Y\to X}$, we have $\mathcal{F}^{\mathrm{a}}(Y \to X) = \mathcal{F}(Y \to X)$. In particular, the associated topos of $\mathbf{E}^{\text{ét}}_{Y\to X}$ is local* ([**SGA 4**$_{\text{II}}$, VI.8.4.6]).

PROOF. Notice that Y is also the spectrum of an absolutely integrally closed valuation ring by I.3.11.(1) and that absolutely integrally closed valuation rings are strictly Henselian. Thus, any covering of $Y \to X$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ can be refined by the identity covering by I.6.4. We see that $\mathcal{F}^{\mathrm{a}}(Y \to X) = \mathcal{F}(Y \to X)$ for any presheaf $\mathcal{F}$. For the last assertion, it suffices to show that the section functor $\Gamma(Y \to X, -)$ commutes with colimits of sheaves. For a colimit of sheaves $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$, let $\mathcal{G}$ be the colimit of presheaves $\mathcal{G} = \mathrm{colim}\, \mathcal{F}_i$. Then, we have $\mathcal{F} = \mathcal{G}^{\mathrm{a}}$ and $\Gamma(Y \to X, \mathcal{F}) = \Gamma(Y \to X, \mathcal{G}) = \mathrm{colim}\, \Gamma(Y \to X, \mathcal{F}_i)$. □

I.7.10. Let $(Y_\lambda \to X_\lambda)_{\lambda\in\Lambda}$ be a $\mathbb{U}$-small directed inverse system of morphisms of $\mathbb{U}$-small coherent schemes with affine transition morphisms $Y_{\lambda'} \to Y_\lambda$ and $X_{\lambda'} \to X_\lambda$ ($\lambda' \geq \lambda$). We set $(Y \to X) = \lim_{\lambda\in\Lambda}(Y_\lambda \to X_\lambda)$. We regard the directed set Λ as a filtered category and regard the inverse system $(Y_\lambda \to X_\lambda)_{\lambda\in\Lambda}$ as a functor $\varphi : \Lambda^{\mathrm{op}} \to \mathbf{E}$ from the opposite category of Λ to the category of morphisms of $\mathbb{U}$-small coherent schemes. Consider the fibred category $\mathbf{E}^{\text{ét}}_\varphi \to \Lambda^{\mathrm{op}}$ defined by φ whose fibre category over λ is $\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ and whose inverse image functor $\varphi^+_{\lambda'\lambda} : \mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda} \to \mathbf{E}^{\text{ét}}_{Y'_\lambda\to X'_\lambda}$ associated to a morphism $\lambda' \to \lambda$ in Λ^{op} is given by the base change by the transition morphism $(Y_{\lambda'} \to X_{\lambda'}) \to (Y_\lambda \to X_\lambda)$ (cf. [**AGT16**, VI.11.2]). Let $\varphi^+_\lambda : \mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ be the functor defined by the base change by the transition morphism $(Y \to X) \to (Y_\lambda \to X_\lambda)$.

Recall that the filtered colimit of categories $(\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda})_{\lambda\in\Lambda}$ is representable by the category $\underrightarrow{\mathbf{E}}^{\text{ét}}_\varphi$ whose objects are those of $\mathbf{E}^{\text{ét}}_\varphi$ and whose morphisms are given by ([**SGA 4**$_{\mathrm{II}}$, VI 6.3, 6.5])

$$\mathrm{Hom}_{\underrightarrow{\mathbf{E}}^{\text{ét}}_\varphi}((V \to U),(V' \to U')) = \operatorname*{colim}_{\substack{(V''\to U'')\to(V\to U)\\ \text{Cartesian}}} \mathrm{Hom}_{\mathbf{E}^{\text{ét}}_\varphi}((V'' \to U''),(V' \to U')), \tag{I.7.10.1}$$

where the colimit is taken over the opposite category of the cofiltered category of Cartesian morphisms with target $V \to U$ of the fibred category $\mathbf{E}^{\text{ét}}_\varphi$ over Λ^{op} (distinguish with the Cartesian morphisms defined in I.7.1). We see that the functors φ^+_λ induces an equivalence of categories by [**EGA** IV$_3$, 8.8.2, 8.10.5] and [**EGA** IV$_4$, 17.7.8]

$$\underrightarrow{\mathbf{E}}^{\text{ét}}_\varphi \xrightarrow{\sim} \mathbf{E}^{\text{ét}}_{Y\to X}. \tag{I.7.10.2}$$

Recall that the cofiltered limit of sites $(\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda})_{\lambda\in\Lambda}$ is representable by $\underrightarrow{\mathbf{E}}^{\text{ét}}_\varphi$ endowed with the coarsest topology such that the natural functors $\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda} \to \underrightarrow{\mathbf{E}}^{\text{ét}}_\varphi$ are continuous ([**SGA 4**$_{\mathrm{II}}$, VI.8.2.3]).

Lemma I.7.11. *With the notation in* I.7.10*, for any covering family $\mathfrak{U} = \{f_k : (V_k \to U_k) \to (V \to U)\}_{k\in K}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ with K finite, there exists an index $\lambda_0 \in \Lambda$ and a covering family $\mathfrak{U}_{\lambda_0} = \{f_{k\lambda_0} : (V_{k\lambda_0} \to U_{k\lambda_0}) \to (V_{\lambda_0} \to U_{\lambda_0})\}_{k\in K}$ in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$ such that f_k is the base change of $f_{k\lambda_0}$ by the transition morphism $(Y \to X) \to (Y_{\lambda_0} \to X_{\lambda_0})$.*

Proof. There is a standard covanishing covering $\mathfrak{U}' = \{g_{nm} : (V'_{nm} \to U'_n) \to (V \to U)\}_{n\in N,m\in M_n}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ with N, M_n finite, which refines $\mathfrak{U}$ by I.6.4. The equivalence (I.7.10.2) implies that there exists an index $\lambda_1 \in \Lambda$ and families of morphisms $\mathfrak{U}'_{\lambda_1} = \{g_{nm\lambda_1} : (V'_{nm\lambda_1} \to U'_{n\lambda_1}) \to (V_{\lambda_1} \to U_{\lambda_1})\}_{n\in N,m\in M_n}$ (resp. $\mathfrak{U}_{\lambda_1} = \{f_{k\lambda_1} : (V_{k\lambda_1} \to U_{k\lambda_1}) \to (V_{\lambda_1} \to U_{\lambda_1})\}_{k\in K}$) in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_1}\to X_{\lambda_1}}$ such that g_{nm} (resp. f_k) is the base change of $g_{nm\lambda_1}$ (resp. $f_{k\lambda_1}$) by the transition morphism $(Y \to X) \to (Y_{\lambda_1} \to X_{\lambda_1})$ and that $\mathfrak{U}'_{\lambda_1}$ refines $\mathfrak{U}_{\lambda_1}$. For each $\lambda \geq \lambda_1$, let $g_{nm\lambda} : (V'_{nm\lambda} \to U'_{n\lambda}) \to (V_\lambda \to U_\lambda)$ (resp. $f_{k\lambda} : (V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)$) be the base change of $g_{nm\lambda_1}$ (resp. $f_{k\lambda_1}$) by the transition morphism $(Y_\lambda \to X_\lambda) \to (Y_{\lambda_1} \to X_{\lambda_1})$. Since the morphisms $\coprod_{n\in N} U'_n \to U$ and $\coprod_{m\in M_n} V'_{nm} \to V \times_U U'_n$ are surjective, there exists an index $\lambda_0 \geq \lambda_1$ such that the morphisms $\coprod_{n\in N} U'_{n\lambda_0} \to U_{\lambda_0}$ and $\coprod_{m\in M_n} V'_{nm\lambda_0} \to V_{\lambda_0} \times_{U_{\lambda_0}} U'_{n\lambda_0}$ are also surjective by [**EGA** IV$_3$, 8.10.5], i.e. $\mathfrak{U}'_{\lambda_0} = \{g_{nm\lambda_0}\}_{n\in N,m\in M_n}$ is a standard covanishing covering in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$. Thus, $\mathfrak{U}_{\lambda_0} = \{f_{k\lambda_0}\}_{k\in K}$ is a covering family in $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$. □

Proposition I.7.12 ([**AGT16**, VI.11])**.** *With the notation in* I.7.10*, $\mathbf{E}^{\text{ét}}_{Y\to X}$ represents the limit of sites $(\mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda})_{\lambda\in\Lambda}$, and $\overline{\mathscr{B}} = \mathrm{colim}_{\lambda\in\Lambda}\, \varphi_\lambda^{-1}\overline{\mathscr{B}}$.*

Proof. The first statement is proved in [**AGT16**, VI.11.3]. It also follows directly from the discussion in I.7.10 and I.7.11. For the second statement, notice that $\mathrm{colim}_{\lambda\in\Lambda}\varphi_\lambda^{-1}\overline{\mathscr{B}} = (\mathrm{colim}_{\lambda\in\Lambda}\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})^{\mathrm{a}}$ ([**Sta22**, 00WI]). It suffices to show that $\overline{\mathscr{B}}(V \to U) = \mathrm{colim}_{\lambda\in\Lambda}(\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})(V \to U)$ for each object $V \to U$ of $\mathbf{E}^{\text{ét}}_{Y\to X}$. It follows from the equivalence (I.7.10.2) that there exists an index $\lambda_0 \in \Lambda$ and an object $V_{\lambda_0} \to U_{\lambda_0}$ of $\mathbf{E}^{\text{ét}}_{Y_{\lambda_0}\to X_{\lambda_0}}$ such that $V \to U$ is the base change of $V_{\lambda_0} \to U_{\lambda_0}$ by the transition morphism. For each $\lambda \geq \lambda_0$, let $V_\lambda \to U_\lambda$ be the base change of $V_{\lambda_0} \to U_{\lambda_0}$ by the transition morphism $(Y_\lambda \to X_\lambda) \to (Y_{\lambda_0} \to X_{\lambda_0})$. Then, we have $\mathrm{colim}_{\lambda\in\Lambda}(\varphi_{\lambda,\mathrm{p}}\overline{\mathscr{B}})(V \to U) = \mathrm{colim}_{\lambda\in\Lambda}\overline{\mathscr{B}}(V_\lambda \to U_\lambda)$ by [**SGA 4**$_{\mathrm{II}}$, VI 8.5.2, 8.5.7]. The conclusion follows from $\overline{\mathscr{B}}(V \to U) = \mathrm{colim}_{\lambda\in\Lambda}\overline{\mathscr{B}}(V_\lambda \to U_\lambda)$ by I.3.18. □

Definition I.7.13. A morphism $X \to S$ of coherent schemes is called *pro-étale* (resp. *pro-finite étale*), if there is a directed inverse system of étale (resp. finite étale) S-schemes $(X_\lambda)_{\lambda\in\Lambda}$ with affine transition morphisms such that there is an isomorphism of S-schemes $X \cong \lim_{\lambda\in\Lambda} X_\lambda$. We call such an inverse system $(X_\lambda)_{\lambda\in\Lambda}$ a *pro-étale presentation* (resp. *pro-finite étale presentation*) of X over S.

Lemma I.7.14. *Let $X \xrightarrow{g} Y \xrightarrow{f} S$ be morphisms of coherent schemes.*

(1) *If f is pro-étale (resp. pro-finite étale), then f is flat (resp. flat and integral).*
(2) *Any base change of a pro-étale (resp. pro-finite étale) morphism is pro-étale (resp. pro-finite étale).*
(3) *If f and g are pro-étale (resp. pro-finite étale), then so is $f \circ g$.*
(4) *If f and $f \circ g$ are pro-étale (resp. pro-finite étale), then so is g.*
(5) *If f is pro-étale with a pro-étale presentation $Y = \lim Y_\beta$, and if g is étale (resp. finite étale), then there is an index β_0 and an étale (resp. finite étale) S-morphism $g_{\beta_0} : X_{\beta_0} \to Y_{\beta_0}$ such that g is the base change of g_{β_0} by $Y \to Y_{\beta_0}$.*
(6) *Let Z and Z' be coherent schemes pro-étale over S with pro-étale presentations $Z = \lim Z_\alpha$, $Z' = \lim Z'_\beta$, then*

$$\mathrm{Hom}_S(Z, Z') = \lim_\beta \operatorname*{colim}_\alpha \mathrm{Hom}_S(Z_\alpha, Z'_\beta). \tag{I.7.14.1}$$

Proof. (1) and (2) follow directly from the definition.

(3) We follow closely the proof of I.3.6. Let $X = \lim X_\alpha$ and $Y = \lim Y_\beta$ be pro-étale (resp. pro-finite étale) presentations over Y and over S respectively. As Y_β are coherent, for each α, there is an index β_α and an étale (resp. finite étale) Y_{β_α}-scheme $X_{\alpha\beta_\alpha}$ such that $X_\alpha \to Y$ is the base change of $X_{\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ ([**EGA** IV$_3$, 8.8.2, 8.10.5], [**EGA** IV$_4$, 17.7.8]). For each $\beta \geq \beta_\alpha$, let $X_{\alpha\beta} \to Y_\beta$ be the base change of $X_{\alpha\beta_\alpha} \to Y_{\beta_\alpha}$ by $Y_\beta \to Y_{\beta_\alpha}$. Then, we have $X = \lim_{\alpha,\beta\geq\beta_\alpha} X_{\alpha\beta}$ by [**EGA** IV$_3$, 8.8.2] (cf. I.3.6), which is pro-finite étale over S. For (5), one can take $X = X_\alpha$.

(6) We have

$$\mathrm{Hom}_S(Z, Z') = \lim_\beta \mathrm{Hom}_S(Z, Z'_\beta) = \lim_\beta \operatorname*{colim}_\alpha \mathrm{Hom}_S(Z_\alpha, Z'_\beta) \tag{I.7.14.2}$$

where the first equality follows from the universal property of limits of schemes, and the second follows from the fact that $Z'_\beta \to S$ is locally of finite presentation ([**EGA** IV$_3$, 8.14.2]). For (4), we take $Z = X$ and $Z' = Y$. Then, for each index β, we have an S-morphism $X_\alpha \to Y_\beta$ for α big enough, which is also étale (resp. finite étale). Then, $X = \lim_\alpha X_\alpha = \lim_{\alpha,\beta} X_\alpha \times_{Y_\beta} Y$ is pro-étale (resp. pro-finite étale) over Y. □

Remark I.7.15. A pro-étale (resp. pro-finite étale) morphism of $\mathbb{U}$-small coherent schemes $X \to S$ admits a $\mathbb{U}$-small pro-étale (resp. pro-finite étale) presentation. Indeed, let $X = \lim_{\lambda\in\Lambda} X_\lambda$ be a presentation of $X \to S$. We may regard Λ as a filtered category with an initial object 0. Consider the category $\mathscr{C} = {}_X\backslash X_{0,\text{ét,aff}}$ (resp. $\mathscr{C} = {}_X\backslash X_{0,\text{fét}}$) of affine (resp. finite) étale X_0-schemes which are under X. Notice that $\mathscr{C}$ is essentially $\mathbb{U}$-small and that the natual functor $\Lambda \to \mathscr{C}^{\mathrm{op}}$ is cofinal by I.7.14.(6) ([**SGA 4**$_\mathrm{I}$, I.8.1.3]). Therefore, after replacing $\mathscr{C}^{\mathrm{op}}$ by a $\mathbb{U}$-small directed set Λ', we obtain a $\mathbb{U}$-small presentation $X = \lim_{X'\in\Lambda'} X'$ ([**SGA 4**$_\mathrm{I}$, I.8.1.6]).

Definition I.7.16. For any $\mathbb{U}$-small coherent scheme X, we endow the category of $\mathbb{U}$-small coherent pro-étale (resp. pro-finite étale) X-schemes with the topology generated by the pretopology formed by families of morphisms

$$\{f_i : U_i \to U\}_{i\in I} \tag{I.7.16.1}$$

such that I is finite and that $U = \bigcup f_i(U_i)$. This defines a site $X_{\text{proét}}$ (resp. $X_{\text{profét}}$), called the *pro-étale site* (resp. *pro-finite étale site*) of X.

It is clear that the localization $X_{\text{proét}/U}$ (resp. $X_{\text{profét}/U}$) of $X_{\text{proét}}$ (resp. $X_{\text{profét}}$) at an object U is canonically equivalent to the pro-étale (resp. pro-finite étale) site $U_{\text{proét}}$ (resp. $U_{\text{profét}}$) of U. By definition, any object in $X_{\text{proét}}$ (resp. $X_{\text{profét}}$) is quasi-compact.

I.7.17. We compare our definitions of pro-étale site and pro-finite étale site with some other definitions existing in the literature. But we don't use the comparison result in this paper.

Let X be a $\mathbb{U}$-small Noetherian scheme. Consider the category of pro-objects pro-$X_{\text{fét}}$ of $X_{\text{fét}}$, i.e. the category whose objects are functors $F : \mathcal{A} \to X_{\text{fét}}$ with $\mathcal{A}$ a $\mathbb{U}$-small cofiltered category and whose morphisms are given by $\mathrm{Hom}(F, G) = \lim_{\beta\in\mathcal{B}} \operatorname{colim}_{\alpha\in\mathcal{A}} \mathrm{Hom}(F(\alpha), G(\beta))$ for any $F : \mathcal{A} \to X_{\text{fét}}$ and $G : \mathcal{B} \to X_{\text{fét}}$ ([**Sch13a**, 3.2]). We may simply denote such a functor F by $(X_\alpha)_{\alpha\in A}$. Remark that $\lim_{\alpha\in\mathcal{A}} X_\alpha$ exists which is pro-finite étale over X. Consider the functor

$$\text{pro-}X_{\text{fét}} \longrightarrow X_{\text{profét}},\ (X_\alpha)_{\alpha\in A} \mapsto \lim_{\alpha\in\mathcal{A}} X_\alpha, \tag{I.7.17.1}$$

which is well-defined and fully faithful by I.7.14.(6) and essentially surjective by I.7.15. Thus, according to [**Sch13a**, 3.3] and its corrigendum [**Sch16**], Scholze's pro-finite étale site $X^{\mathrm{S}}_{\text{profét}}$ has the

underlying category $X_{\text{profét}}$ and its topology is generated by the families of morphisms

$$\{U_i \xrightarrow{f_i} U' \xrightarrow{f} U\}_{i\in I} \tag{I.7.17.2}$$

where I is finite and $\coprod_{i\in I} U_i \to U'$ is finite étale surjective, and there exists a $\mathbb{U}$-small well-ordered directed set Λ with a least index 0 and a directed inverse system of $\mathbb{U}$-small coherent pro-finite étale X-schemes $(U'_\lambda)_{\lambda\in\Lambda}$ such that $U = U'_0$, $U' = \lim_{\lambda\in\Lambda} U'_\lambda$ and that for each $\lambda \in \Lambda$ the natural morphism $U'_\lambda \to \lim_{\mu<\lambda} U'_\mu$ is finite étale surjective (cf. [**Ker16**, 5.5], I.7.14 and [**EGA** IV$_3$, 8.10.5.(vi)]). It is clear that the topology of our pro-finite étale site $X_{\text{profét}}$ is finer than that of $X^{\mathrm{S}}_{\text{profét}}$. We remark that if X is connected, then $X^{\mathrm{S}}_{\text{profét}}$ gives a site-theoretic interpretation of the continuous group cohomology of the fundamental group of X ([**Sch13a**, 3.7]). For simplicity, we don't consider $X^{\mathrm{S}}_{\text{profét}}$ in the rest of the paper, but we can replace $X_{\text{profét}}$ by $X^{\mathrm{S}}_{\text{profét}}$ for most of the statements in this paper (cf. [**Ker16**, 6]).

I.7.18. Let X be a $\mathbb{U}$-small scheme. Bhatt-Scholze's pro-étale site $X^{\mathrm{BS}}_{\text{proét}}$ has the underlying category of $\mathbb{U}$-small weakly étale X-schemes and a family of morphisms $\{f_i : Y_i \to Y\}_{i\in I}$ in $X^{\mathrm{BS}}_{\text{proét}}$ is a covering if and only if for any affine open subscheme U of Y, there exists a map $a : \{1,\dots,n\} \to I$ and affine open subschemes U_j of $Y_{a(j)}$ $(j = 1,\dots,n)$ such that $U = \bigcup_{j=1}^n f_{a(j)}(U_j)$ ([**BS15**, 4.1.1], cf. [**Sta22**, 0989]). Remark that a pro-étale morphism of coherent schemes is weakly étale by [**BS15**, 2.3.3.1]. Thus, for a coherent scheme X, $X_{\text{proét}}$ is a full subcategory of $X^{\mathrm{BS}}_{\text{proét}}$.

Lemma I.7.19. *Let X be a coherent scheme. The full subcategory $X_{\text{proét}}$ of $X^{\mathrm{BS}}_{\text{proét}}$ is a topologically generating family, and the induced topology on $X_{\text{proét}}$ coincides the topology defined in* I.7.16. *In particular, the topoi of sheaves of $\mathbb{V}$-small sets associated to the two sites are naturally equivalent.*

Proof. For a weakly étale X-scheme Y, we show that it can be covered by pro-étale X-schemes. After replacing X by a finite affine open covering and replacing Y by an affine open covering, we may assume that X and Y are affine. Then, the result follows from the fact that for any weakly étale morphism of rings $A \to B$ there exists a faithfully flat ind-étale morphism $B \to C$ such that $A \to C$ is ind-étale by [**BS15**, 2.3.4] (cf. [**BS15**, 4.1.3]). Thus, $X_{\text{proét}}$ is a topologically generating family of $X^{\mathrm{BS}}_{\text{proét}}$. A family of morphisms $\{f_i : Y_i \to Y\}_{i\in I}$ in $X_{\text{proét}}$ is a covering with respect to the induced topology if and only if for any affine open subscheme U of Y, there exists a map $a : \{1,\dots,n\} \to I$ and affine open subschemes U_j of $Y_{a(j)}$ $(j = 1,\dots,n)$ such that $U = \bigcup_{j=1}^n f_{a(j)}(U_j)$ ([**SGA 4**$_{\mathrm{I}}$, III.3.3]). Notice that Y_i and Y are coherent, thus $\{f_i\}_{i\in I}$ is a covering if and only if there exists a finite subset $I_0 \subseteq I$ such that $Y = \bigcup_{i\in I_0} f_i(Y_i)$, which shows that the induced topology on $X_{\text{proét}}$ coincides the topology defined in I.7.16. Finally, the "in particular" part follows from [**SGA 4**$_{\mathrm{I}}$, III.4.1]. □

Definition I.7.20. Let $Y \to X$ be a morphism of coherent schemes. A morphism $(V' \to U') \to (V \to U)$ in $\mathbf{E}_{Y\to X}$ is called *pro-étale* if $U' \to U$ is pro-étale and $V' \to V \times_U U'$ is pro-finite étale. A *pro-étale presentation* of such a morphism is a directed inverse system $(V_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ étale over $V \to U$ with affine transition morphisms $U_{\lambda'} \to U_\lambda$ and $V_{\lambda'} \to V_\lambda$ $(\lambda' \geq \lambda)$ such that $(V' \to U') = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$.

Lemma I.7.21. *Let $Y \to X$ be a morphism of coherent schemes, $(V'' \to U'') \xrightarrow{g} (V' \to U') \xrightarrow{f} (V \to U)$ morphisms in $\mathbf{E}_{Y\to X}$.*

1. *If f is pro-étale, then it admits a pro-étale presentation.*
2. *If f is pro-étale, then any base change of f is also pro-étale.*
3. *If f and g are pro-étale, then $f \circ g$ is also pro-étale.*
4. *If f and $f \circ g$ are pro-étale, then g is also pro-étale.*

Proof. It follows directly from I.7.14 and its arguments. □

Remark I.7.22. Similar to I.7.15, a pro-étale morphism in $\mathbf{E}_{Y\to X}$ admits a $\mathbb{U}$-small presentation.

I.7.23. Let $Y \to X$ be a morphism of coherent schemes, $\mathbf{E}^{\text{proét}}_{Y\to X}$ the full subcategory of $\mathbf{E}_{Y\to X}$ formed by objects which are pro-étale over the final object $Y \to X$. It is clear that $\mathbf{E}^{\text{proét}}_{Y\to X}$ is stable under finite limits in $\mathbf{E}_{Y\to X}$. Then, the functor (I.7.1.2) induces a functor

$$\phi^+ : \mathbf{E}^{\text{proét}}_{Y\to X} \longrightarrow X_{\text{proét}},\ (V \to U) \longmapsto U, \tag{I.7.23.1}$$

which endows $\mathbf{E}^{\text{proét}}_{Y\to X}/X_{\text{proét}}$ with a structure of fibred sites, whose fibre over U is the pro-finite étale site $U_{Y,\text{profét}}$. We endow $\mathbf{E}^{\text{proét}}_{Y\to X}$ with the associated covanishing topology, that is, the topology generated by the following types of families of morphisms

(v) $\{(V_m \to U) \to (V \to U)\}_{m\in M}$, where M is a finite set and $\coprod_{m\in M} V_m \to V$ is surjective;

(c) $\{(V\times_U U_n\to U_n)\to(V\to U)\}_{n\in N}$, where N is a finite set and $\coprod_{n\in N}U_n\to U$ is surjective.

It is clear that any object in $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ is quasi-compact by I.6.4. We still denote by $\overline{\mathscr{B}}$ the restriction of the presheaf $\overline{\mathscr{B}}$ on $\mathbf{E}_{Y\to X}$ to $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ if there is no ambiguity. We will show in I.7.32 that $\overline{\mathscr{B}}$ is a sheaf on $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$.

Definition I.7.24. We call $(\mathbf{E}^{\mathrm{proét}}_{Y\to X},\overline{\mathscr{B}})$ the *pro-étale Faltings ringed site* of the morphism of coherent schemes $Y\to X$.

It is clear that the localization $(\mathbf{E}^{\mathrm{proét}}_{Y\to X})_{/(V\to U)}$ of $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ at an object $V\to U$ is canonically equivalent to the pro-étale Faltings ringed site $\mathbf{E}^{\mathrm{proét}}_{V\to U}$ of the morphism $V\to U$ by I.6.4.

Remark I.7.25. The categories $X_{\mathrm{proét}}$, $X_{\mathrm{profét}}$ and $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ are essentially $\mathbb{V}$-small categories.

I.7.26. Let $Y\to X$ be a morphism of coherent schemes. Consider the natural functors

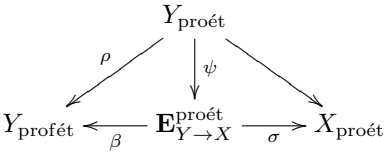
$$\psi^+:\mathbf{E}^{\mathrm{proét}}_{Y\to X}\longrightarrow Y_{\mathrm{proét}},\ (V\to U)\longmapsto V, \tag{I.7.26.1}$$

$$\beta^+:Y_{\mathrm{profét}}\longrightarrow\mathbf{E}^{\mathrm{proét}}_{Y\to X},\ V\longmapsto(V\to X), \tag{I.7.26.2}$$

$$\sigma^+:X_{\mathrm{proét}}\longrightarrow\mathbf{E}^{\mathrm{proét}}_{Y\to X},\ U\longmapsto(Y\times_X U\to U). \tag{I.7.26.3}$$

They are left exact and continuous (cf. I.7.8). Then, we obtain a commutative diagram of sites associated functorially to the morphism $Y\to X$ by I.2.5,

(I.7.26.4)



where $\rho:Y_{\mathrm{proét}}\to Y_{\mathrm{profét}}$ is defined by the inclusion functor, and the unlabelled arrow $Y_{\mathrm{proét}}\to X_{\mathrm{proét}}$ is induced by the morphism $Y\to X$. Moreover, if $\mathcal{O}_{X_{\mathrm{proét}}}$ denotes the structural sheaf on $X_{\mathrm{proét}}$ sending U to $\Gamma(U,\mathcal{O}_U)$, then σ^+ actually defines a morphism of ringed sites

$$\sigma:(\mathbf{E}^{\mathrm{proét}}_{Y\to X},\overline{\mathscr{B}})\longrightarrow(X_{\mathrm{proét}},\mathcal{O}_{X_{\mathrm{proét}}}). \tag{I.7.26.5}$$

Lemma I.7.27. *Let $Y\to X$ be a morphism of coherent schemes. Then, the inclusion functor*

$$\nu^+:\mathbf{E}^{\mathrm{ét}}_{Y\to X}\longrightarrow\mathbf{E}^{\mathrm{proét}}_{Y\to X},\ (V\to U)\longmapsto(V\to U) \tag{I.7.27.1}$$

defines a morphism of sites $\nu:\mathbf{E}^{\mathrm{proét}}_{Y\to X}\to\mathbf{E}^{\mathrm{ét}}_{Y\to X}$.

Proof. It is clear that ν^+ commutes with finite limits and sends a standard covanishing covering in $\mathbf{E}^{\mathrm{ét}}_{Y\to X}$ to a standard covanishing covering in $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ (I.6.3). Therefore, ν^+ is continuous by I.6.4 and defines a morphism of sites. □

Lemma I.7.28. *Let $Y\to X$ be a morphism of coherent schemes. Then, the topology on $\mathbf{E}^{\mathrm{ét}}_{Y\to X}$ is the topology induced from $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$.*

Proof. After I.7.27, it suffices to show that for a family of morphisms $\mathfrak{U}=\{(V_k\to U_k)\to(V\to U)\}_{k\in K}$ in $\mathbf{E}^{\mathrm{ét}}_{Y\to X}$, if $\nu^+(\mathfrak{U})$ is a covering in $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$, then $\mathfrak{U}$ is a covering in $\mathbf{E}^{\mathrm{ét}}_{Y\to X}$. We may assume that K is finite. There is a standard covanishing covering $\mathfrak{U}'=\{(V'_{nm}\to U'_n)\to(V\to U)\}_{n\in N,m\in M_n}$ in $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ with N, M_n finite, which refines $\nu^+(\mathfrak{U})$ by I.6.4. We take a directed set Ξ such that for each $n\in N$, we can take a pro-étale presentation $U'_n=\lim_{\xi\in\Xi}U'_{n\xi}$ over U, and we take a directed set Σ such that for each $n\in N$ and $m\in M_n$, we can take a pro-finite étale presentation $V'_{nm}=\lim_{\sigma\in\Sigma}V'_{nm\sigma}$ over $V\times_U U'_n$. By I.7.14 (5), for each $\sigma\in\Sigma$, there exists an index $\xi_\sigma\in\Xi$ and a finite étale morphism $V'_{nm\sigma\xi_\sigma}\to V\times_U U'_{n\xi_\sigma}$ for each n and m, whose base change by $U'_n\to U'_{n\xi_\sigma}$ is $V'_{nm\sigma}\to V\times_U U'_n$. Let $V'_{nm\sigma\xi}\to V\times_U U'_{n\xi}$ be the base change of $V'_{nm\sigma\xi_\sigma}\to V\times_U U'_{n\xi_\sigma}$ by the transition morphism $U'_{n\xi}\to U'_{n\xi_\sigma}$ for each $\xi\geq\xi_\sigma$. Since $\coprod_{m\in M_n}V'_{nm\sigma}\to V\times_U U'_n$ is surjective, after enlarging ξ_σ, we may assume that $\coprod_{m\in M_n}V'_{nm\sigma\xi}\to V\times_U U'_{n\xi}$ is also surjective for $\xi\geq\xi_\sigma$ by [**EGA** IV$_3$, 8.10.5.(vi)]. It is clear that $\coprod_{n\in N}U'_{n\xi}\to U$ is surjective for each $\xi\in\Xi$. Therefore, $\mathfrak{U}'_{\sigma\xi}=\{(V'_{nm\sigma\xi}\to U'_{n\xi})\to(V\to U)\}_{n\in N,m\in M_n}$ is a standard covanishing covering in $\mathbf{E}^{\mathrm{ét}}_{V\to U}$ for each $\sigma\in\Sigma$ and $\xi\geq\xi_\sigma$. Notice that for each $n\in N$ and $m\in M_n$, there exists $k\in K$ such that the morphism $(V'_{nm\sigma\xi}\to U'_{n\xi})\to(V\to U)$ factors through $(V_k\to U_k)$ for σ,ξ big enough by I.7.14 (6), which shows that $\mathfrak{U}$ is a covering in $\mathbf{E}^{\mathrm{ét}}_{Y\to X}$. □

Lemma I.7.29. *Let $Y \to X$ be a morphism of coherent schemes, $\mathfrak{U} = \{(V_k \to U_k) \to (V \to U)\}_{k\in K}$ a covering in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ with K finite. Then, there exist pro-étale presentations $(V \to U) = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$, $(V_k \to U_k) = \lim_{\lambda\in\Lambda}(V_{k\lambda} \to U_{k\lambda})$ over $Y \to X$ and compatible étale morphisms $(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)$ such that the family $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ is a covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$.*

Proof. We follow closely the proof of I.3.6. We take a directed set A such that for each $k \in K$ we can take a pro-étale presentation $(V_k \to U_k) = \lim_{\alpha\in A}(V_{k\alpha} \to U_{k\alpha})$ over $(V \to U)$. Then, $\mathfrak{U}_\alpha = \{(f_{k\alpha} : V_{k\alpha} \to U_{k\alpha}) \to (V \to U)\}_{k\in K}$ is a covering family in $\mathbf{E}^{\text{ét}}_{V\to U}$ for each $\alpha \in A$ by I.7.28.

Let $(V \to U) = \lim_{\beta\in B}(V_\beta \to U_\beta)$ be a pro-étale presentation over $Y \to X$. For each $\alpha \in A$, there exists an index $\beta_\alpha \in B$ and a covering family $\mathfrak{U}_{\alpha\beta_\alpha} = \{f_{k\alpha\beta_\alpha} : (V_{k\alpha\beta_\alpha} \to U_{k\alpha\beta_\alpha}) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})\}_{k\in K}$ such that $f_{k\alpha}$ is the base change of $f_{k\alpha\beta_\alpha}$ by the transition morphism $(V \to U) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})$ (I.7.11). For each $\beta \geq \beta_\alpha$, let $f_{k\alpha\beta} : (V_{k\alpha\beta} \to U_{k\alpha\beta}) \to (V_\beta \to U_\beta)$ be the base change of $f_{k\alpha\beta_\alpha}$ by the transition morphism $(V_\beta \to U_\beta) \to (V_{\beta_\alpha} \to U_{\beta_\alpha})$. We take $\Lambda = \{(\alpha, \beta) \in A \times B \mid \beta \geq \beta_\alpha\}$, $(V_\lambda \to U_\lambda) = (V_\beta \to U_\beta)$ and $(V_{k\lambda} \to U_{k\lambda}) = (V_{k\alpha\beta} \to U_{k\alpha\beta})$ for each $\lambda = (\alpha, \beta) \in \Lambda$. Then, the families $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ meet the requirements in the lemma (cf. I.3.6). □

Lemma I.7.30. *Let $Y \to X$ be a morphism of coherent schemes, $\mathcal{F}$ a presheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$, $V \to U$ an object of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ with a pro-étale presentation $(V \to U) = \lim(V_\lambda \to U_\lambda)$. Then, we have $\nu_{\mathrm{p}}\mathcal{F}(V \to U) = \operatorname{colim}\mathcal{F}(V_\lambda \to U_\lambda)$, where $\nu^+ : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ is the inclusion functor.*

Proof. Notice that the presheaf $\mathcal{F}$ is a filtered colimit of representable presheaves by [**SGA 4**$_{\mathrm{I}}$, I.3.4]

$$\mathcal{F} = \operatorname*{colim}_{(V'\to U')\in(\mathbf{E}^{\text{ét}}_{Y\to X})_{/\mathcal{F}}} h^{\text{ét}}_{V'\to U'}. \tag{I.7.30.1}$$

Thus, we may assume that $\mathcal{F}$ is representable by $V' \to U'$ since the section functor $\Gamma(V \to U, -)$ commutes with colimits of presheaves ([**Sta22**, 00VB]). Then, we have

$$\begin{aligned}\nu_{\mathrm{p}}h^{\text{ét}}_{V'\to U'}(V \to U) =& h^{\mathrm{pro\acute{e}t}}_{V'\to U'}(V \to U)\\ =& \operatorname{Hom}_{\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}}((V \to U), (V' \to U'))\\ =& \operatorname{colim}\operatorname{Hom}_{\mathbf{E}^{\text{ét}}_{Y\to X}}((V_\lambda \to U_\lambda), (V' \to U'))\\ =& \operatorname{colim} h^{\text{ét}}_{V'\to U'}(V_\lambda \to U_\lambda)\end{aligned} \tag{I.7.30.2}$$

where the first equality follows from [**Sta22**, 04D2], and the third equality follows from [**EGA** IV$_3$, 8.14.2] since U' and V' are locally of finite presentation over X and $Y \times_X U'$ respectively. □

Proposition I.7.31. *Let $Y \to X$ be a morphism of coherent schemes, $\mathcal{F}$ an abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$, $V \to U$ an object of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ with a pro-étale presentation $(V \to U) = \lim(V_\lambda \to U_\lambda)$. Then, for any integer q, we have*

$$H^q(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{V\to U}, \nu^{-1}\mathcal{F}) = \operatorname{colim} H^q(\mathbf{E}^{\text{ét}}_{V_\lambda\to U_\lambda}, \mathcal{F}), \tag{I.7.31.1}$$

where $\nu : \mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ is the morphism of sites defined by the inclusion functor (I.7.27). *In particular, the canonical morphism $\mathcal{F} \longrightarrow \mathrm{R}\nu_*\nu^{-1}\mathcal{F}$ is an isomorphism.*

Proof. We follow closely the proof of I.3.8. For the second assertion, since $\mathrm{R}^q\nu_*\nu^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $(V \to U) \mapsto H^q(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{V\to U}, \nu^{-1}\mathcal{F}) = H^q(\mathbf{E}^{\text{ét}}_{V\to U}, \mathcal{F})$ by the first assertion, which is $\mathcal{F}$ if $q = 0$ and vanishes otherwise.

For the first assertion, we may assume that $\mathcal{F} = \mathcal{I}$ is an abelian injective sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$ (cf. I.3.8). We claim that for any covering in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$, $\mathfrak{U} = \{(V_k \to U_k) \to (V \to U)\}_{k\in K}$ with K finite, the augmented Čech complex associated to the presheaf $\nu_{\mathrm{p}}\mathcal{I}$,

$$0 \to \nu_{\mathrm{p}}\mathcal{I}(V \to U) \to \prod_k \nu_{\mathrm{p}}\mathcal{I}(V_k \to U_k) \to \prod_{k,k'} \nu_{\mathrm{p}}\mathcal{I}(V_k \times_V V_{k'} \to U_k \times_U U_{k'}) \to \cdots \tag{I.7.31.2}$$

is exact. Admitting this claim, we see that $\nu_{\mathrm{p}}\mathcal{I}$ is indeed a sheaf, i.e. $\nu^{-1}\mathcal{I} = \nu_{\mathrm{p}}\mathcal{I}$, and the vanishing of higher Čech cohomologies implies that $H^q(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{V\to U}, \nu^{-1}\mathcal{I}) = 0$ for any $q > 0$, which completes the proof together with I.7.30. For the claim, let $(V \to U) = \lim_{\lambda\in\Lambda}(V_\lambda \to U_\lambda)$ and $(V_k \to U_k) = \lim_{\lambda\in\Lambda}(V_{k\lambda} \to U_{k\lambda})$ be the pro-étale presentations constructed in I.7.29. The family $\mathfrak{U}_\lambda = \{(V_{k\lambda} \to U_{k\lambda}) \to (V_\lambda \to U_\lambda)\}_{k\in K}$ is a covering in $\mathbf{E}^{\text{ét}}_{Y\to X}$. By I.7.30, the sequence (I.7.31.2) is the filtered colimit of the augmented Čech complexes

$$0 \to \mathcal{I}(V_\lambda \to U_\lambda) \to \prod_k \mathcal{I}(V_{k\lambda} \to U_{k\lambda}) \to \prod_{k,k'} \mathcal{I}(V_{k\lambda} \times_{V_\lambda} V_{k'\lambda} \to U_{k\lambda} \times_{U_\lambda} U_{k'\lambda}) \to \cdots \tag{I.7.31.3}$$

which are exact since $\mathcal{I}$ is an injective abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y\to X}$. □

Corollary I.7.32. *With the notation in* I.7.31*, the presheaf* $\overline{\mathscr{B}}$ *on* $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ *is a sheaf, and the canonical morphisms* $\nu^{-1}\overline{\mathscr{B}}\to\overline{\mathscr{B}}$ *and* $\overline{\mathscr{B}}\to \mathrm{R}\nu_*\overline{\mathscr{B}}$ *are isomorphisms. If moreover* p *is invertible on* Y*, then for each integer* $n\geq 0$*, the canonical morphisms* $\nu^{-1}(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}})\to\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}$ *and* $\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}\to \mathrm{R}\nu_*(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}})$ *are isomorphisms.*

Proof. For any pro-étale presentation $(V\to U)=\lim(V_\lambda\to U_\lambda)$, we have $\nu^{-1}\overline{\mathscr{B}}(V\to U)=\operatorname{colim}\overline{\mathscr{B}}(V_\lambda\to U_\lambda)=\overline{\mathscr{B}}(V\to U)$ by I.7.30 and I.3.18. This verifies that $\overline{\mathscr{B}}$ is a sheaf on $\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ and that $\nu^{-1}\overline{\mathscr{B}}\to\overline{\mathscr{B}}$ is an isomorphism. The second isomorphism follows from the first and I.7.31. For the last assertion, notice that the multiplication by p^n is injective on $\overline{\mathscr{B}}$, so that the conclusion follows from the exact sequence

$$0\longrightarrow\overline{\mathscr{B}}\xrightarrow{\cdot p^n}\overline{\mathscr{B}}\longrightarrow\overline{\mathscr{B}}/p^n\overline{\mathscr{B}}\longrightarrow 0. \tag{I.7.32.1}$$

□

I.7.33. We regard the ordered set $\mathbb{N}$ of natural numbers as a filtered category (there is an arrow $i\to j$ if $i\leq j$). Let E be a site. We denote by $E^{\mathbb{N}}$ the fibred site $E\times\mathbb{N}$ over $\mathbb{N}$, and we endow the category $E^{\mathbb{N}}$ with the total topology which makes it into a site without final objects (I.6.1, cf. [**AGT16**, III.7, VI.7]). Giving a presheaf $\mathcal{F}$ on $E^{\mathbb{N}}$ is equivalent to giving a directed inverse system of presheaves $(\mathcal{F}_n)_{n\geq 0}$ on E. We write $\mathcal{F}=(\mathcal{F}_n)_{n\geq 0}$. Moreover, $\mathcal{F}$ is a sheaf on $E^{\mathbb{N}}$ if and only if each $\mathcal{F}_n$ is a sheaf on E.

I.7.34. Let $Y\to X$ be a morphism of coherent schemes. We obtain a fibred site $\mathbf{E}^{\mathrm{ét},\mathbb{N}}_{Y\to X}$ (resp. $\mathbf{E}^{\mathrm{proét},\mathbb{N}}_{Y\to X}$) over $\mathbb{N}$ by I.7.33. We define a sheaf $\breve{\overline{\mathscr{B}}}$ on $\mathbf{E}^{\mathrm{ét},\mathbb{N}}_{Y\to X}$ (resp. $\mathbf{E}^{\mathrm{proét},\mathbb{N}}_{Y\to X}$) by

$$\breve{\overline{\mathscr{B}}}=(\overline{\mathscr{B}}/p^n\overline{\mathscr{B}})_{n\geq 0}. \tag{I.7.34.1}$$

The inclusion functor $\nu^+:\mathbf{E}^{\mathrm{ét}}_{Y\to X}\to\mathbf{E}^{\mathrm{proét}}_{Y\to X}$ defines a morphism of ringed fibred sites ([**SGA 4**$_{\mathrm{II}}$, VI.7.2.2])

$$\breve{\nu}:(\mathbf{E}^{\mathrm{proét},\mathbb{N}}_{Y\to X},\breve{\overline{\mathscr{B}}})\longrightarrow(\mathbf{E}^{\mathrm{ét},\mathbb{N}}_{Y\to X},\breve{\overline{\mathscr{B}}}), \tag{I.7.34.2}$$

which induces a morphism of the associated ringed topoi with respect to the total topology ([**SGA 4**$_{\mathrm{II}}$, VI.7.4.13.1], cf. [**AGT16**, III.7.18]). If moreover the prime number p is invertible on Y, then the canonical morphisms

$$\breve{\nu}^{-1}\breve{\overline{\mathscr{B}}}\longrightarrow\breve{\overline{\mathscr{B}}}\ \text{ and }\ \breve{\overline{\mathscr{B}}}\longrightarrow\mathrm{R}\breve{\nu}_*\breve{\overline{\mathscr{B}}} \tag{I.7.34.3}$$

are isomorphisms by I.7.32 ([**AGT16**, VI.7.7]). Combining with the canonical diagrams (I.7.8.4) and (I.7.26.4), we obtain a canonical commutative diagram of fibred sites over $\mathbb{N}$,

$$\begin{array}{ccccc} Y^{\mathbb{N}}_{\mathrm{proét}} & \xrightarrow{\breve{\psi}} & \mathbf{E}^{\mathrm{proét},\mathbb{N}}_{Y\to X} & \xrightarrow{\breve{\sigma}} & X^{\mathbb{N}}_{\mathrm{proét}} \\ \downarrow{\scriptstyle\breve{\nu}} & & \downarrow{\scriptstyle\breve{\nu}} & & \downarrow{\scriptstyle\breve{\nu}} \\ Y^{\mathbb{N}}_{\mathrm{ét}} & \xrightarrow{\breve{\psi}} & \mathbf{E}^{\mathrm{ét},\mathbb{N}}_{Y\to X} & \xrightarrow{\breve{\sigma}} & X^{\mathbb{N}}_{\mathrm{ét}} \end{array} \tag{I.7.34.4}$$

where we ambiguously denote by $\breve{\nu}$ the morphisms induced by the inclusion functors of the underlying categories. Moreover, if $\breve{\mathcal{O}}_{X_{\mathrm{ét}}}$ (resp. $\breve{\mathcal{O}}_{X_{\mathrm{proét}}}$) denotes the sheaf $(\mathcal{O}_{X_{\mathrm{ét}}}/p^n\mathcal{O}_{X_{\mathrm{ét}}})_{n\geq 0}$ (resp. $(\mathcal{O}_{X_{\mathrm{proét}}}/p^n\mathcal{O}_{X_{\mathrm{proét}}})_{n\geq 0}$), then the diagram (I.7.34.4) induces a commutative diagram of ringed fibred sites

$$\begin{array}{ccc} (\mathbf{E}^{\mathrm{proét},\mathbb{N}}_{Y\to X},\breve{\overline{\mathscr{B}}}) & \xrightarrow{\breve{\sigma}} & (X^{\mathbb{N}}_{\mathrm{proét}},\breve{\mathcal{O}}_{X_{\mathrm{proét}}}) \\ \downarrow{\scriptstyle\breve{\nu}} & & \downarrow{\scriptstyle\breve{\nu}} \\ (\mathbf{E}^{\mathrm{ét},\mathbb{N}}_{Y\to X},\breve{\overline{\mathscr{B}}}) & \xrightarrow{\breve{\sigma}} & (X^{\mathbb{N}}_{\mathrm{ét}},\breve{\mathcal{O}}_{X_{\mathrm{ét}}}) \end{array} \tag{I.7.34.5}$$

Definition I.7.35. We call $(\mathbf{E}^{\mathrm{ét},\mathbb{N}}_{Y\to X},\breve{\overline{\mathscr{B}}})$ (resp. $(\mathbf{E}^{\mathrm{proét},\mathbb{N}}_{Y\to X},\breve{\overline{\mathscr{B}}})$) the *p-adic Faltings ringed site* (resp. *p-adic pro-étale Faltings ringed site*) of the morphism of coherent schemes $Y\to X$. It is a ringed site without final objects, which can be regarded as a fibred site over $\mathbb{N}$.

I.8. Cohomological Descent of the Structural Sheaves

Definition I.8.1. Let K be a pre-perfectoid field of mixed characteristic $(0,p)$, $Y \to X$ a morphism of coherent schemes such that $Y \to X^Y$ is over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$, where X^Y denotes the integral closure of X in Y. We say that $Y \to X$ is *Faltings acyclic* if X is affine and if for any integer $n > 0$ the canonical morphism

$$A/p^n A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \tag{I.8.1.1}$$

is an almost isomorphism (cf. I.5.7), where A denotes the $\mathcal{O}_K$-algebra $\overline{\mathscr{B}}(Y \to X)$ (i.e. $X^Y = \mathrm{Spec}(A)$).

Remark I.8.2. In I.8.1, the canonical morphism $\mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \to \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}})$ is an isomorphism as (I.7.34.3) are isomorphisms.

Lemma I.8.3. *Let K be a pre-perfectoid field of mixed characteristic $(0,p)$, $Y \to X$ a morphism of coherent schemes such that $Y \to X^Y$ is over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$, $A = \overline{\mathscr{B}}(Y \to X)$. Assume that $Y \to X$ is Faltings acyclic. Then, we have:*

1. *For any pseudo-uniformizer π of K, the canonical morphism $A/\pi A \to \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi\overline{\mathscr{B}})$ is an almost isomorphism.*
2. *Let $\widehat{A}$ be the p-adic completion of A. Then, the canonical morphism $\widehat{A} \to \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\overline{\mathscr{B}}})$ is an almost isomorphism.*

Proof. (1) There exists an integer $n > 0$ such that $\pi' = p^n/\pi$ is a pseudo-uniformizer of K. Since A and $\overline{\mathscr{B}}$ are flat over $\mathcal{O}_K$, we have a natural morphism of exact sequences

(I.8.3.1)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/\pi' A & \longrightarrow & A/p^n A & \longrightarrow & A/\pi A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
0 & \longrightarrow & H^0(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi'\overline{\mathscr{B}}) & \longrightarrow & H^0(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) & \longrightarrow & H^0(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi\overline{\mathscr{B}}) & &
\end{array}$$

By definition, α_2 is an almost isomorphism. Thus, α_1 is almost injective. Since any pseudo-uniformizer of K is of the form $\pi' = p^n/\pi$ for some pseudo-uniformizer π of K and $n > 0$, α_3 is almost injective. By diagram chasing, we see that α_1 is an almost isomorphism (and thus so is α_3). It remains to show that $H^q(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi\overline{\mathscr{B}})$ is almost zero for $q > 0$. Since $H^q(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}})$ is almost zero. By the long exact sequence associated to the short exact sequence $0 \to \overline{\mathscr{B}}/\pi'\overline{\mathscr{B}} \to \overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \to \overline{\mathscr{B}}/\pi\overline{\mathscr{B}} \to 0$, we see that $H^1(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi'\overline{\mathscr{B}})$ is almost zero and that $H^q(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi\overline{\mathscr{B}}) \to H^{q+1}(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/\pi'\overline{\mathscr{B}})$ is an almost isomorphism. By induction, we complete the proof.

(2) Recall that for any integer $q \geq 0$ there exists a canonical exact sequence ([**AGT16**, VI.7.10])

$$0 \to \mathrm{R}^1 \lim_{n\to\infty} H^{q-1}(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \to H^q(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\overline{\mathscr{B}}}) \to \lim_{n\to\infty} H^q(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) \to 0. \tag{I.8.3.2}$$

The conclusion follows from the almost isomorphisms (I.8.1.1). □

Proposition I.8.4. *Let K be a pre-perfectoid field of mixed characteristic $(0,p)$, $Y \to X$ a morphism of coherent schemes such that $Y \to X^Y$ is over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$, $A = \overline{\mathscr{B}}(Y \to X)$. Assume that $Y \to X$ is Faltings acyclic and that $X = \mathrm{Spec}(R)$ with R being p-torsion free. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module of finite presentation with $M = \mathcal{M}(X)$ such that $M[1/p]$ is a projective $R[1/p]$-module. Then, the canonical morphism*

$$M \otimes_R \widehat{A}[\frac{1}{p}] \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\sigma}^*\breve{\mathcal{M}})[\frac{1}{p}] \tag{I.8.4.1}$$

is an isomorphism, where $\breve{\sigma} : (\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\overline{\mathscr{B}}}) \to (X^{\mathbb{N}}_{\mathrm{pro\acute{e}t}}, \breve{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}})$ is defined in I.7.34 *and $\breve{\mathcal{M}} = \mathcal{M} \otimes_{\mathcal{O}_{X_{\mathrm{pro\acute{e}t}}}} \breve{\mathcal{O}}_{X_{\mathrm{pro\acute{e}t}}} = (\mathcal{M}/p^n\mathcal{M})_{n\geq 0}$.*

Proof. Let N be the kernel of a surjective R-linear homomorphism $\phi : R^{\oplus n} \to M$. We take a splitting $R^{\oplus n}[1/p] = M[1/p] \oplus N[1/p]$. Composing with the inclusion $(M \oplus N) \subseteq (M[1/p] \oplus N[1/p])$, we get an injective map $\widetilde{\phi} : M \oplus N \to R^{\oplus n}[1/p]$. Since M and N are finitely generated, there exists an integer $k \geq 1$ such that $\varphi = p^k\widetilde{\phi} : M \oplus N \to R^{\oplus n} \subseteq R^{\oplus n}[1/p]$. It is clear that φ is injective. We claim that the cokernel of φ is killed by p^{2k}. Indeed, the composition of the maps $M \xrightarrow{\varphi} R^{\oplus n} \xrightarrow{\phi} M$ is $p^k\mathrm{id}_M$. Thus, for any $x \in R^{\oplus n}$, the element $y = p^k x - \varphi(\phi(x))$ of $R^{\oplus n}$ lies in N. Thus, $p^{2k}x = \varphi(p^k\phi(x) + y)$, which proves the claim. Then, there exists an R-linear homomorphism $\psi : R^n \to M \oplus N$ such that $\varphi \circ \psi = p^{4k}\mathrm{id}_{R^n}$ and $\psi \circ \varphi = p^{4k}\mathrm{id}_{M\oplus N}$ ([**AG20**, 2.6.3]).

Let $\mathcal{N}$ be the quasi-coherent $\mathcal{O}_X$-module associated to N. Then, for each integer q, the morphism φ induces an $\widehat{A}$-linear homomorphism

$$H^q(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\sigma}^*\breve{\mathcal{M}}) \oplus H^q(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\sigma}^*\breve{\mathcal{N}}) \longrightarrow H^q(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\mathscr{B}}^{\oplus n}), \tag{I.8.4.2}$$

whose kernel and cokernel are killed by p^{4k} by the existence of ψ. For $q \neq 0$, we have $H^q(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\mathscr{B}}^{\oplus n})[1/p] = 0$ by I.8.3.(2), thus $H^q(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\sigma}^*\breve{\mathcal{M}})[1/p] = 0$. For $q = 0$, we have $H^0(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\mathscr{B}}^{\oplus n})[1/p] = \widehat{A}^{\oplus n}[1/p]$ by I.8.3.(2). On the other hand, there is a canonical morphism

$$M \otimes_R \widehat{A} \oplus N \otimes_R \widehat{A} \to H^0(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\sigma}^*\breve{\mathcal{M}}) \oplus H^0(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\sigma}^*\breve{\mathcal{N}}) \tag{I.8.4.3}$$

whose composition with (I.8.4.2) is compatible with $\varphi \otimes_R \mathrm{id}_{\widehat{A}} : M \otimes_R \widehat{A} \oplus N \otimes_R \widehat{A} \to \widehat{A}^{\oplus n}$. Thus, $H^0(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{Y\to X}, \breve{\sigma}^*\mathcal{M})[1/p] = M \otimes_R \widehat{A}[1/p]$. □

Lemma I.8.5. *Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. Then, the functor*

$$\epsilon^+ : \mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X} \longrightarrow \mathbf{I}_{Y\to X^Y},\ (V \to U) \longmapsto U^V, \tag{I.8.5.1}$$

is well-defined, left exact and continuous. Moreover, we have $Y \times_{X^Y} U^V = V$.

Proof. Since $U' = X^Y \times_X U$ is integral over U, we have $U^V = U'^V$. Applying I.3.19.(4) to $V \to U'$ over $Y \to X^Y$, we see that the X^Y-scheme U^V is Y-integrally closed with $Y \times_{X^Y} U^V = V$, and thus the functor ϵ^+ is well-defined. Let $(V_1 \to U_1) \to (V_0 \to U_0) \leftarrow (V_2 \to U_2)$ be a diagram in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$. By I.3.21, $U_1^{V_1} \overline{\times}_{U_0^{V_0}} U_2^{V_2} = (U_1^{V_1} \times_{U_0^{V_0}} U_2^{V_2})^{V_1\times_{V_0} V_2} = (U_1 \times_{U_0} U_2)^{V_1\times_{V_0} V_2}$ which shows the left exactness of ϵ^+. For the continuity, notice that any covering in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ can be refined by a standard covanishing covering (I.6.4). For a Cartesian covering family $\mathfrak{U} = \{(V \times_U U_n \to U_n) \to (V \to U)\}_{n\in N}$ with N finite, we apply I.3.15 to the commutative diagram

$$\begin{array}{ccccc} \coprod_{n\in N} V \times_U U_n & \longrightarrow & \coprod_{n\in N} U_n^{V\times_U U_n} & \longrightarrow & \coprod_{n\in N} U_n \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U^V & \longrightarrow & U \end{array} \tag{I.8.5.2}$$

then we see that $\epsilon^+(\mathfrak{U})$ is a covering family in $\mathbf{I}_{Y\to X^Y}$. For a vertical covering family $\mathfrak{U} = \{(V_m \to U) \to (V \to U)\}_{m\in M}$ with M finite, we apply I.3.15 to the commutative diagram

$$\begin{array}{ccccc} \coprod_{m\in M} V_m & \longrightarrow & \coprod_{m\in M} U^{V_m} & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U^V & \longrightarrow & U \end{array} \tag{I.8.5.3}$$

then we see that $\epsilon^+(\mathfrak{U})$ is also a covering family in $\mathbf{I}_{Y\to X^Y}$. □

I.8.6. Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. Then, there are morphisms of sites

$$\epsilon : \mathbf{I}_{Y\to X^Y} \longrightarrow \mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}, \tag{I.8.6.1}$$

$$\varepsilon : \mathbf{I}_{Y\to X^Y} \longrightarrow \mathbf{E}^{\acute{\mathrm{e}}\mathrm{t}}_{Y\to X} \tag{I.8.6.2}$$

defined by (I.8.5.1) and the composition of (I.8.5.1) with (I.7.27.1) respectively. We temporarily denote by $\mathscr{O}^{\mathrm{pre}}$ the presheaf on $\mathbf{I}_{Y\to X^Y}$ sending W to $\Gamma(W, \mathcal{O}_W)$ (thus $\mathscr{O} = (\mathscr{O}^{\mathrm{pre}})^{\mathrm{a}}$). Notice that we have $\overline{\mathscr{B}} = \epsilon^{\mathrm{p}}\mathscr{O}^{\mathrm{pre}}$ (resp. $\overline{\mathscr{B}} = \varepsilon^{\mathrm{p}}\mathscr{O}^{\mathrm{pre}}$). The canonical morphism $\epsilon^{\mathrm{p}}\mathscr{O}^{\mathrm{pre}} \to \epsilon^{\mathrm{p}}\mathscr{O}$ (resp. $\varepsilon^{\mathrm{p}}\mathscr{O}^{\mathrm{pre}} \to \varepsilon^{\mathrm{p}}\mathscr{O}$) induces a canonical morphism $\overline{\mathscr{B}} \to \epsilon_*\mathscr{O}$ (resp. $\overline{\mathscr{B}} \to \varepsilon_*\mathscr{O}$).

I.8.7. Let K be a pre-perfectoid field (I.5.1) of mixed characteristic $(0, p)$, $\eta = \mathrm{Spec}(K)$, $S = \mathrm{Spec}(\mathcal{O}_K)$, $Y \to X$ a morphism of coherent schemes such that X^Y is an S-scheme with generic fibre $(X^Y)_\eta = Y$. In particular, X^Y is an object of $\mathbf{I}_{\eta\to S}$.

Lemma I.8.8. *For any ring R, there is an R-algebra R_∞ satisfying the following conditions:*

(i) *The scheme $\mathrm{Spec}(R_\infty[1/p])$ is pro-finite étale and faithfully flat over $\mathrm{Spec}(R[1/p])$.*
(ii) *The R-algebra R_∞ is the integral closure of R in $R_\infty[1/p]$.*
(iii) *Any unit t of R_∞ admits a p-th root $t^{1/p}$ in R_∞.*

Moreover, if p lies in the Jacobson radical $J(R)$ of R, and if there is a p^2-th root $p_2 \in R$ of p up to a unit, and we write $p_1 = p_2^p$, then we may require further that

(iv) *the Frobenius of R_∞/pR_∞ induces an isomorphism $R_\infty/p_1R_\infty \to R_\infty/pR_\infty$.*

Proof. Setting $B_0 = R[1/p]$, we construct inductively a ring B_{n+1} ind-finite étale over B_n and we denote by R_n the integral closure of R in B_n. For $n \geq 0$, we set

$$B_{n+1} = \operatorname*{colim}_{T \subseteq R_n^\times} \bigotimes_{B_n}^{t \in T} B_n[X]/(X^p - t) \tag{I.8.8.1}$$

where the colimit runs through all finite subsets T of the subset $R_n^\times$ of units of R_n and the transition maps are given by the inclusion relation of these finite subsets T. Notice that $B_n[X]/(X^p - t)$ is finite étale and faithfully flat over B_n, thus B_{n+1} is ind-finite étale and faithfully flat over B_n. Now we take $B_\infty = \operatorname{colim}_n B_n$. The integral closure R_∞ of R in B_∞ is equal to $\operatorname{colim}_n R_n$ by I.3.18. We claim that R_∞ satisfies the first three conditions. Firstly, we see inductively that $B_n = R_n[1/p]$ $(0 \leq n \leq \infty)$ by I.3.17. Thus, (i), (ii) follow immediately. For (iii), notice that we have $R_\infty^\times = \operatorname{colim}_n R_n^\times$. For an unit $t \in R_\infty^\times$, we suppose that it is the image of $t_n \in R_n^\times$. By construction, there exists an element $x_{n+1} \in R_{n+1}$ such that $x_{n+1}^p = t_n$. Thus, t admits a p-th root in R_∞.

For (iv), the injectivity follows from the fact that R_∞ is integrally closed in $R_\infty[1/p]$ (cf. I.5.21). For the surjectivity, let $a \in R_\infty$. Firstly, since R_∞ is integral over R, p also lies in the Jacobson radical $J(R_\infty)$ of R_∞. Thus, $1 + p_1 a \in R_\infty^\times$, and then by (iii) there is $b \in R_\infty$ such that $b^p = 1 + p_1 a$. We write $(b-1)^p = p_1 a'$ for some $a' \in a + p_1 R_\infty$. Thus, $1 + a' - a \in R_\infty^\times$, and then by (iii) there is $c \in R_\infty$ such that $c^p = 1 + a' - a$. On the other hand, since R_∞ is integrally closed in $R_\infty[1/p]$, we have $x = (b-1)/p_2 \in R_\infty$. Now we have $(x - c + 1)^p \equiv x^p - c^p + 1 \equiv a \pmod{pR_\infty}$, which completes the proof. □

Remark I.8.9. In I.8.8, it follows from the construction that $\operatorname{Spec}(R_\infty[1/p]) \to \operatorname{Spec}(R[1/p])$ is a covering in $\operatorname{Spec}(R[1/p])^{\mathrm{S}}_{\mathrm{proét}}$ (I.7.17).

Proposition I.8.10. *With the notation in* I.8.7*, for any object $V \to U$ in $\mathbf{E}^{\mathrm{proét}}_{Y \to X}$, there exists a covering $\{(V_i \to U_i) \to (V \to U)\}_{i \in I}$ with I finite such that for each $i \in I$, $U_i^{V_i}$ is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid* (I.5.19)*.*

Proof. After replacing U by an affine open covering, we may assume that $U = \operatorname{Spec}(A)$. Consider the category $\mathscr{C}$ of étale A-algebras B such that $A/pA \to B/pB$ is an isomorphism, and the colimit $A^{\mathrm{h}} = \operatorname{colim} B$ over $\mathscr{C}$. In fact, $\mathscr{C}$ is filtered and $(A^{\mathrm{h}}, pA^{\mathrm{h}})$ is the Henselization of the pair (A, pA) (cf. [**Sta22**, 0A02]). It is clear that $\operatorname{Spec}(A^{\mathrm{h}}) \coprod \operatorname{Spec}(A[1/p]) \to \operatorname{Spec}(A)$ is a covering in $U_{\mathrm{proét}}$. So we reduce to the situation where $p \in J(A)$ or $p \in A^\times$. The latter case is trivial, since the p-adic completion of $R = \Gamma(U^V, \mathcal{O}_{U^V})$ is zero if p is invertible in A. Therefore, we may assume that $p \in J(A)$ in the following.

Since $R = \Gamma(U^V, \mathcal{O}_{U^V})$ is integral over A, we also have $p \in J(R)$. Applying I.8.8 to the $\mathcal{O}_K$-algebra R, we obtain a covering $V_\infty = \operatorname{Spec}(R_\infty[1/p]) \to V = \operatorname{Spec}(R[1/p])$ in $V_{\mathrm{proét}}$ such that $R_\infty = \Gamma(U^{V_\infty}, \mathcal{O}_{U^{V_\infty}})$ is an $\mathcal{O}_K$-algebra which is almost pre-perfectoid by I.5.4 and I.5.20. □

Proposition I.8.11. *With the notation in* I.8.7*, if W is an object of $\mathbf{I}_{\eta \to S}$ such that W is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, then for any integer $n > 0$, the canonical morphism*

$$\Gamma(W, \mathcal{O}_W)/p^n\Gamma(W, \mathcal{O}_W) \to \mathrm{R}\Gamma(\mathbf{I}_{W_\eta \to W}, \mathscr{O}/p^n\mathscr{O}) \tag{I.8.11.1}$$

is an almost isomorphism (I.5.7)*.*

Proof. Let $\mathscr{C}$ be the full-subcategory of $\mathbf{I}_{\eta \to S}$ formed by the spectrums of $\mathcal{O}_K$-algebras which are almost pre-perfectoid. It is stable under fibred product by I.5.33, I.5.30 and I.3.21, and it forms a topologically generating family for the site $\mathbf{I}_{\eta \to S}$ by I.8.5 and I.8.10. It suffices to show that for any covering in $\mathbf{I}_{\eta \to S}$, $\mathfrak{U} = \{W_k \to W\}_{k \in K}$ consisting of objects of $\mathscr{C}$ with K finite, the augmented Čech complex associated to the presheaf $W \mapsto \Gamma(W, \mathcal{O}_W)/p^n\Gamma(W, \mathcal{O}_W)$ on $\mathbf{I}_{\eta \to S}$ (whose associated sheaf is just $\mathscr{O}/p^n\mathscr{O}$),

$$0 \to \Gamma(W, \mathcal{O}_W)/p^n \to \prod_k \Gamma(W_k, \mathcal{O}_{W_k})/p^n \to \prod_{k,k'} \Gamma(W_k \overline{\times}_W W_{k'}, \mathcal{O}_{W_k \overline{\times}_W W_{k'}})/p^n \to \cdots \tag{I.8.11.2}$$

is almost exact. Indeed, the almost exactness shows firstly that $\Gamma(W, \mathcal{O}_W)/p^n \to H^0(\mathbf{I}_{W_\eta \to W}, \mathscr{O}/p^n\mathscr{O})$ is an almost isomorphism (cf. [**Sta22**, 00W1]), so that the augmented Čech complex associated to the sheaf $\mathscr{O}/p^n\mathscr{O}$ is also almost exact. Then, the conclusion follows from the almost vanishing of the higher Čech cohomologies of $\mathscr{O}/p^n\mathscr{O}$ by [**Sta22**, 03F9].

We set $R = \Gamma(W, \mathcal{O}_W)$ and $R' = \prod_{k \in K} \Gamma(W_k, \mathcal{O}_{W_k})$. They are almost pre-perfectoid, and $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is a v-covering by definition. Thus, the almost exactness of (I.8.11.2) follows from I.5.33, I.5.30 and I.5.35. □

Theorem I.8.12. *With the notation in* I.8.7*, let* $\epsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ *be the morphism of sites defined in* I.8.6*. Then, for any integer* $n > 0$*, the canonical morphism*

$$\overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \to \mathrm{R}\epsilon_*(\mathscr{O}/p^n\mathscr{O}) \tag{I.8.12.1}$$

is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_{\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}})$ (I.5.7).

Proof. Since $\mathrm{R}^q\epsilon_*(\mathscr{O}/p^n\mathscr{O})$ is the sheaf associated to the presheaf $(V \to U) \mapsto H^q(\mathbf{I}_{V\to U^V}, \mathscr{O}/p^n\mathscr{O})$ and any object in $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ can be covered by those objects whose image under ϵ^+ are the spectrums of $\mathcal{O}_K$-algebras which are almost pre-perfectoid by I.8.10, the conclusion follows from I.8.11. □

Corollary I.8.13. *With the notation in* I.8.7*, let* $V \to U$ *be an object of* $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y\to X}$ *such that* U *is affine and that the integral closure* $U^V = \mathrm{Spec}(A)$ *is the spectrum of an* $\mathcal{O}_K$*-algebra* A *which is almost pre-perfectoid. Then,* $V \to U$ *is Faltings acyclic.*

Proof. It follows directly from I.8.12 and I.8.11. □

Corollary I.8.14. *With the notation in* I.8.7*, let* $\varepsilon : \mathbf{I}_{Y\to X^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ *be the morphism of sites defined in* I.8.6*. Then, for any finite locally constant abelian sheaf* $\mathbb{L}$ *on* $\mathbf{E}^{\text{ét}}_{Y\to X}$*, the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{I.8.14.1}$$

is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_K\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ (I.5.7).

Proof. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$, thus we may assume that $\mathbb{L}$ is the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$. Then, the conclusion follows from I.8.12 and I.7.32. □

Remark I.8.15. In I.8.14, if $\mathbb{L}$ is a bounded complex of abelian sheaves on $\mathbf{E}^{\text{ét}}_{Y\to X}$ with finite locally constant cohomology sheaves, then the canonical morphism $\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O})$ is also an almost isomorphism. Indeed, after replacing $\mathbb{L}$ by $\mathbb{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}_p$, we may assume that $\mathbb{L}$ is a complex of $\mathbb{Z}/p^n\mathbb{Z}$-modules for some integer n ([**Sta22**, 0DD7]). Then, there exists a covering family $\{(Y_i \to X_i) \to (Y \to X)\}_{i\in I}$ in $\mathbf{E}^{\text{ét}}_{Y\to X}$ such that the restriction of $\mathbb{L}$ on $\mathbf{E}^{\text{ét}}_{Y_i\to X_i}$ is represented by a bounded complex of finite locally constant $\mathbb{Z}/p^n\mathbb{Z}$-modules ([**Sta22**, 094G]). Then, the conclusion follows directly from I.8.14.

Corollary I.8.16. *With the notation in* I.8.7*, let* $Y \to X_i$ *(*$i = 1, 2$*) be a morphism of coherent schemes such that* X_i^Y *is an* S*-scheme with generic fibre* $(X_i^Y)_\eta = Y$*,* $\mathbb{L}$ *a finite locally constant abelian sheaf on* $\mathbf{E}^{\text{ét}}_{Y\to X_2}$*. If there is a morphism* $f : X_1 \to X_2$ *under* Y *such that the natural morphism* $g : X_1^Y \to X_2^Y$ *is a separated v-covering and that* $g^{-1}(Y) = Y$*, and if we denote by* $u : \mathbf{E}^{\text{ét}}_{Y\to X_1} \to \mathbf{E}^{\text{ét}}_{Y\to X_2}$ *the corresponding morphism of sites, then the natural morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}u_*(u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \tag{I.8.16.1}$$

is an almost isomorphism.

Proof. The morphism u is defined by the functor $u^+ : \mathbf{E}^{\text{ét}}_{Y\to X_2} \to \mathbf{E}^{\text{ét}}_{Y\to X_1}$ sending $(V \to U_2)$ to $(V \to U_1) = (V \to X_1 \times_{X_2} U_2)$. We set $V_0 = Y \times_{X_1} U_1 = Y \times_{X_2} U_2$. According to I.3.17, $U_1^{V_0} \to U_2^{V_0}$ is the base change of $X_1^Y \to X_2^Y$ by $U_2 \to X_2$, and thus it is a separated v-covering. Notice that V_0 is an open subscheme in both $U_1^{V_0}$ and $U_2^{V_0}$, and moreover $V_0 = V_0 \times_{U_2^{V_0}} U_1^{V_0}$. Applying I.3.15 to the commutative diagram

$$\begin{array}{ccccc} V & \longrightarrow & U_1^V & \longrightarrow & U_1^{V_0} \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & U_2^V & \longrightarrow & U_2^{V_0} \end{array} \tag{I.8.16.2}$$

it follows that $U_1^V \to U_2^V$ is also a separated v-covering. Let $\varepsilon_i : \mathbf{I}_{Y\to X_i^Y} \to \mathbf{E}^{\text{ét}}_{Y\to X_i}$ $(i = 1, 2)$ be the morphisms of sites defined in I.8.6. The sheaf $\mathrm{R}^q u_*(u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}})$ is associated to the presheaf $(V \to U_2) \mapsto H^q(\mathbf{E}^{\text{ét}}_{V\to U_1}, u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}})$. We have

$$\begin{aligned} H^q(\mathbf{E}^{\text{ét}}_{V\to U_1}, u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) &\to H^q(\mathbf{I}_{V\to U_1^V}, \varepsilon_1^{-1}u^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \\ &= H^q(\mathbf{I}_{V\to U_2^V}, \varepsilon_2^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \mathscr{O}) \leftarrow H^q(\mathbf{E}^{\text{ét}}_{V\to U_2}, \mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \end{aligned} \tag{I.8.16.3}$$

where the equality follows from the fact that the morphism of representable sheaves associated to $U_1^V \to U_2^V$ on $\mathbf{I}_{\eta\to S}$ is an isomorphism by I.3.24, and where the two arrows are almost isomorphisms by I.8.14, which completes the proof. □

I.8.17. Let Δ be the category formed by finite ordered sets $[n] = \{0, 1, \ldots, n\}$ ($n \geq 0$) with non-decreasing maps ([**Sta22**, 0164]). For a functor from its opposite category Δ^{op} to the category $\mathbf{E}$ of morphisms of coherent schemes sending $[n]$ to $Y_n \to X_n$, we simply denote it by $Y_\bullet \to X_\bullet$. Then, we obtain a fibred site $\mathbf{E}^{\text{ét}}_{Y_\bullet \to X_\bullet}$ over Δ^{op} whose fibre category over $[n]$ is $\mathbf{E}^{\text{ét}}_{Y_n \to X_n}$ and the inverse image functor $f^+ : \mathbf{E}^{\text{ét}}_{Y_n \to X_n} \to \mathbf{E}^{\text{ét}}_{Y_m \to X_m}$ associated to a morphism $f : [m] \to [n]$ in Δ^{op} is induced the base change by the morphism $(Y_m \to X_m) \to (Y_n \to X_n)$ associated to f. We endow $\mathbf{E}^{\text{ét}}_{Y_\bullet \to X_\bullet}$ with the total topology (I.6.1) and call it the *simplicial Faltings site* associated to $Y_\bullet \to X_\bullet$ ([**Sta22**, 09WE.(A)]). The sheaf $\overline{\mathscr{B}}$ on each $\mathbf{E}^{\text{ét}}_{Y_n \to X_n}$ induces a sheaf $\overline{\mathscr{B}}_\bullet = (\overline{\mathscr{B}})_{[n] \in \mathrm{Ob}(\Delta)}$ on $\mathbf{E}^{\text{ét}}_{Y_\bullet \to X_\bullet}$ with the notation in I.6.5.

For an augmentation $(Y_\bullet \to X_\bullet) \to (Y \to X)$ in $\mathbf{E}$ ([**Sta22**, 018F]), we obtain an augmentation of simplicial site $a : \mathbf{E}^{\text{ét}}_{Y_\bullet \to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y \to X}$ ([**Sta22**, 0D6Z.(A)]). We denote by $a_n : \mathbf{E}^{\text{ét}}_{Y_n \to X_n} \to \mathbf{E}^{\text{ét}}_{Y \to X}$ the natural morphism induced by $(Y_n \to X_n) \to (Y \to X)$. Notice that for any sheaf $\mathcal{F}$ on $\mathbf{E}^{\text{ét}}_{Y \to X}$, we have $a^{-1}\mathcal{F} = (a_n^{-1}\mathcal{F})_{[n] \in \mathrm{Ob}(\Delta)}$ with the notation in I.6.5 ([**Sta22**, 0D70]).

Corollary I.8.18. *With the notation in* I.8.7, *let $\mathbb{L}$ a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y \to X}$, $X_\bullet \to X$ an augmentation of simplicial coherent scheme. If we set $Y_\bullet = Y \times_X X_\bullet$ and denote by a : $\mathbf{E}^{\text{ét}}_{Y_\bullet \to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y \to X}$ the augmentation of simplicial site, assuming that $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\eta \to S}$, then the canonical morphism*

$$\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \to \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet) \tag{I.8.18.1}$$

is an almost isomorphism.

Proof. Let $b : \mathbf{I}_{Y_\bullet \to X_\bullet^{Y_\bullet}} \to \mathbf{I}_{Y \to X^Y}$ be the augmentation of simplicial site associated to the augmentation of simplicial object $X_\bullet^{Y_\bullet} \to X^Y$ of $\mathbf{I}_{\eta \to S}$ ([**Sta22**, 09X8]). The functorial morphism of sites $\varepsilon : \mathbf{I}_{Y \to X^Y} \to \mathbf{E}^{\text{ét}}_{Y \to X}$ defined in I.8.6 induces a commutative diagram of topoi ([**Sta22**, 0D99])

$$\begin{array}{ccc} \mathbf{I}^{\sim}_{Y_\bullet \to X_\bullet^{Y_\bullet}} & \xrightarrow{\varepsilon_\bullet} & \mathbf{E}^{\text{ét}\sim}_{Y_\bullet \to X_\bullet} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ \mathbf{I}^{\sim}_{Y \to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}\sim}_{Y \to X} \end{array} \tag{I.8.18.2}$$

We denote by $a_n : \mathbf{E}^{\text{ét}}_{Y_n \to X_n} \to \mathbf{E}^{\text{ét}}_{Y \to X}$ and $b_n : \mathbf{I}_{Y_n \to X_n^{Y_n}} \to \mathbf{I}_{Y \to X^Y}$ the natural morphisms of sites. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{R}a_*(a^{-1}\mathbb{L} \otimes \overline{\mathscr{B}}_\bullet) & \xleftarrow{\alpha_1} & \mathbb{L} \otimes \overline{\mathscr{B}} & \xrightarrow{\alpha_2} & \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathbb{L} \otimes \mathscr{O}) \\ {\scriptstyle \alpha_3}\downarrow & & & & \downarrow{\scriptstyle \alpha_4} \\ \mathrm{R}a_*\mathrm{R}\varepsilon_{\bullet *}\varepsilon_\bullet^{-1}(a^{-1}\mathbb{L} \otimes \overline{\mathscr{B}}_\bullet) & \xrightarrow{\alpha_5} & \mathrm{R}c_*(c^{-1}\mathbb{L} \otimes \mathscr{O}_\bullet) & = & \mathrm{R}\varepsilon_*\mathrm{R}b_*b^{-1}(\varepsilon^{-1}\mathbb{L} \otimes \mathscr{O}) \end{array} \tag{I.8.18.3}$$

where $c = a \circ \varepsilon_\bullet = \varepsilon \circ b$, and α_2 (resp. α_5) is induced by the canonical morphism $\varepsilon^{-1}\overline{\mathscr{B}} \to \mathscr{O}$ (resp. $\varepsilon_\bullet^{-1}\overline{\mathscr{B}}_\bullet \to \mathscr{O}_\bullet$), and other arrows are the canonical morphisms.

Notice that α_2 is an almost isomorphism by I.8.14, and that α_4 is an isomorphism by [**Sta22**, 0D8N] as $X_\bullet^{Y_\bullet} \to X^Y$ is a hypercovering in $\mathbf{I}_{\eta \to S}$. It remains to show that $\alpha_5 \circ \alpha_3$ is an almost isomorphism. With the notation in I.6.5, we have

$$a^{-1}\mathbb{L} \otimes \overline{\mathscr{B}}_\bullet = (a_n^{-1}\mathbb{L} \otimes \overline{\mathscr{B}})_{[n] \in \mathrm{Ob}(\Delta)} \text{ and} \tag{I.8.18.4}$$

$$c^{-1}\mathbb{L} \otimes \mathscr{O}_\bullet = (\varepsilon_n^{-1}a_n^{-1}\mathbb{L} \otimes \mathscr{O})_{[n] \in \mathrm{Ob}(\Delta)}. \tag{I.8.18.5}$$

Moreover, by [**Sta22**, 0D97] we have

$$\mathrm{R}^q\varepsilon_{\bullet *}(c^{-1}\mathbb{L} \otimes \mathscr{O}_\bullet) = (\mathrm{R}^q\varepsilon_{n*}(\varepsilon_n^{-1}a_n^{-1}\mathbb{L} \otimes \mathscr{O}))_{[n] \in \mathrm{Ob}(\Delta)} \tag{I.8.18.6}$$

for each integer q. Therefore, $a^{-1}\mathbb{L} \otimes \overline{\mathscr{B}}_\bullet \to \mathrm{R}\varepsilon_{\bullet *}(c^{-1}\mathbb{L} \otimes \mathscr{O}_\bullet)$ is an almost isomorphism by I.8.14 and so is $\alpha_5 \circ \alpha_3$. □

Lemma I.8.19. *With the notation in* I.8.7, *assume that X^Y is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid. Let $V \to U$ be an object of $\mathbf{E}^{\text{proét}}_{Y \to X}$ with U affine. Then, U^V is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid.*

PROOF. Consider the following commutative diagram:

(I.8.19.1)
$$\begin{array}{ccccc} V & \longrightarrow & U^V & & \\ \downarrow & & \downarrow & & \\ Y\times_X U & \longrightarrow & U^{Y\times_X U} & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & X^Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_K) & & \end{array}$$

Since taking integral closures commutes with étale base change and filtered colimits (I.3.17, I.3.18), all the squares in (I.8.19.1) are Cartesian (I.3.19). Notice that $U^{Y\times_X U}$ is integral over U and thus affine. Since $U^{Y\times_X U}$ is pro-étale over X^Y, it is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid by I.5.37. As V is pro-finite étale over $Y\times_X U$, by almost purity I.5.41 and I.5.37, we see that U^V is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid. □

I.8.20. Let K be a pre-perfectoid field of mixed characteristic $(0,p)$ which contains a compatible system $\{\zeta_n\}_{n\geq 1}$ of primitive roots of unity, $\eta=\mathrm{Spec}(K)$, $S=\mathrm{Spec}(\mathcal{O}_K)$, $Y\to X$ a morphism of coherent schemes such that X^Y is an S-scheme and that the induced morphism $Y\to X^Y$ is an open immersion over $\eta\to S$. Remark that the morphism $X^Y_\eta\to X$ over $\eta\to S$ is in the situation I.8.7. We assume further that there exist finitely many nonzero divisors $f_1,\dots,f_r$ of $\Gamma(X^Y_\eta,\mathcal{O}_{X^Y_\eta})$ such that the divisor $D=\sum_{i=1}^r \mathrm{div}(f_i)$ on X^Y_η has support $X^Y_\eta\setminus Y$ and that at each strict henselization of X^Y_η those elements f_i contained in the maximal ideal form a subset of a regular system of parameters (in particular, D is a normal crossings divisor on X^Y_η, and we allow D to be empty, i.e. $r=0$). We set

(I.8.20.1)
$$Y_\infty=\lim_n Y[T_1,\dots,T_r]/(T_1^n-f_1,\dots,T_r^n-f_r),$$

where the limit is taken over $\mathbb{N}$ with the division relation. We see that Y_∞ is faithfully flat and pro-finite étale over Y.

Proposition I.8.21 (Abhyankar's lemma)**.** *Under the assumptions in* I.8.20 *and with the same notation, for any finite étale Y_∞-scheme V_∞, the integral closure $X^{V_\infty}_\eta$ is finite étale over $X^{Y_\infty}_\eta$.*

PROOF. We set $Z=X^Y_\eta$. Passing to a strict henselization of Z where D is non-empty, we may assume that Z is local and regular and that $f_1,\dots,f_r$ ($r\geq 1$) are all contained in the maximal ideal. We set $Y_n=Y[T_1,\dots,T_r]/(T_1^n-f_1,\dots,T_r^n-f_r)$ and $Z_n=Z[T_1,\dots,T_r]/(T_1^n-f_1,\dots,T_r^n-f_r)$ for any integer $n>0$. Notice that Z_n is still local and regular (thus isomorphic to $X^{Y_n}_\eta$) and that $g_0=f_0^{1/n},\dots,g_r=f_r^{1/n}$ form a subset of a regular system of parameters for Z_n (see the proof of [**SGA 1**, XIII.5.1]). Using [**EGA** IV$_3$, 8.8.2, 8.10.5] and [**EGA** IV$_4$, 17.7.8], there exists an integer $n_0>0$ and a finite étale Y_{n_0}-scheme V_{n_0} such that $V_\infty=Y_\infty\times_{Y_{n_0}}V_{n_0}$. We set $V_n=Y_n\times_{Y_{n_0}}V_{n_0}$ for any $n\geq n_0$. According to [**SGA 4**$_{\mathrm{III}}$, XIII.5.2], there exists a multiple n_1 of n_0 such that $Z_{n_1}^{V_{n_1}}$ is finite étale over Z_{n_1}. As $Z_\infty=\lim Z_n$ is normal, $Z_\infty\times_{Z_{n_1}}Z_{n_1}^{V_{n_1}}$ is also normal and thus isomorphic to $Z_\infty^{V_\infty}=X_\eta^{V_\infty}$, which shows that the latter is also finite étale over $Z_\infty=X_\eta^{Y_\infty}$. □

Corollary I.8.22. *Under the assumptions in* I.8.20 *and with the same notation, the natural functor $\mathbf{E}^{\mathrm{proét}}_{X_\eta^{Y_\infty}\to X}\to\mathbf{E}^{\mathrm{proét}}_{Y_\infty\to X}$ sending $V\to U$ to $Y_\infty\times_{X_\eta^{Y_\infty}}V\to U$ induces an equivalence of ringed sites $(\mathbf{E}^{\mathrm{proét}}_{Y_\infty\to X},\overline{\mathscr{B}})\to(\mathbf{E}^{\mathrm{proét}}_{X_\eta^{Y_\infty}\to X},\overline{\mathscr{B}})$.*

PROOF. For the equivalence of categories, it suffices to show that the induced functor $u^+:\mathbf{E}^{\text{ét}}_{X_\eta^{Y_\infty}\to X}\to\mathbf{E}^{\text{ét}}_{Y_\infty\to X}$ is an equivalence by I.7.14.(6). Since u^+ is a morphism of fibred categories over $X_{\text{ét}}$, it suffices to show that for each object U of $X_{\text{ét}}$, the fibre functor $u_U^+:U^{Y_\infty\times_X U}_{\eta,\text{fét}}\to(Y_\infty\times_X U)_{\text{fét}}$ induced by u^+ is an equivalence of categories. Notice that if we replace $Y\to X$ in I.8.20 by $Y\times_X U\to U$, then $(Y\times_X U)_\infty=Y_\infty\times_X U$. Therefore, the equivalence of categories follows from applying I.8.21 to $Y\times_X U\to U$.

To show the equivalence of categories identifies their topologies, it suffices to show that it identifies the vertical coverings and Cartesian coverings in I.7.23. For a finite family $\{(V_m\to U)\to(V\to U)\}_{m\in M}$ in $\mathbf{E}^{\mathrm{proét}}_{X_\eta^{Y_\infty}\to X}$, its image in $\mathbf{E}^{\mathrm{proét}}_{Y_\infty\to X}$ is $\{(Y_\infty\times_{X_\eta^{Y_\infty}}V_m\to U)\to((Y_\infty\times_{X_\eta^{Y_\infty}}V\to U)\}_{m\in M}$. Notice that $Y_\infty\times_{X_\eta^{Y_\infty}}V$ is a dense open subset of V as V is flat over $X_\eta^{Y_\infty}$ ([**EGA** IV$_2$, 2.3.7]), and the same holds for V_m. Thus, the integral morphism $\coprod_{m\in M}V_m\to V$ is surjective if and only

if $\coprod_{m\in M} Y_\infty \times_{X_\eta^{Y_\infty}} V_m \to Y_\infty \times_{X_\eta^{Y_\infty}} V$ is surjective. On the other hand, it is tautological that the equivalence identifies the Cartesian coverings. Hence, the two sites are naturally equivalent.

The identification of the structural sheaves by the equivalence of sites follows from the fact that V is integrally closed in $Y_\infty \times_{X_\eta^{Y_\infty}} V$ for any object $V \to U$ of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{X_\eta^{Y_\infty}\to X}$ as V is pro-étale over $X_\eta^{Y_\infty}$ (I.3.19). □

Corollary I.8.23. *Under the assumptions in* I.8.20 *and with the same notation, let $V \to U$ be an object of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y_\infty\to X}$ such that U^V is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, and let $V' \to U'$ be an object of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{V\to U}$ with U' affine. Then, $U'^{V'}$ is the spectrum of an $\mathcal{O}_K$-algebra which is almost pre-perfectoid.*

Proof. It follows directly from I.8.22 and I.8.19. □

Theorem I.8.24. *Under the assumptions in* I.8.20 *and with the same notation, let $V \to U$ be an object of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y_\infty\to X}$. Then, the following statements are equivalent:*

1. *The morphism $V \to U$ is Faltings acyclic.*
2. *The scheme U is affine and $U^V = \mathrm{Spec}(A)$ is the spectrum of an $\mathcal{O}_K$-algebra A which is almost pre-perfectoid.*

Proof. (2) ⇒ (1): Let $V' \to U$ be an object of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{X_\eta^{Y_\infty}\to X}$ whose image under the equivalence in I.8.22 is isomorphic to $V \to U$. Then, $U^{V'} = \mathrm{Spec}(A)$, $V' = U_\eta^{V'}$, and $\mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{V\to U}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}}) = \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t}}_{V'\to U}, \overline{\mathscr{B}}/p^n\overline{\mathscr{B}})$. The conclusion follows from I.8.13.

(1) ⇒ (2): Firstly, notice that the objects $V' \to U'$ of $\mathbf{E}^{\mathrm{pro\acute{e}t}}_{Y_\infty\to X}$ satisfying the condition (2) form a topological generating family by I.8.22 and I.8.10. Let $p_1 \in \mathcal{O}_K$ be a p-th root of p up to a unit. Then, we see that the Frobenius induces an almost isomorphism $\overline{\mathscr{B}}/p_1\overline{\mathscr{B}} \to \overline{\mathscr{B}}/p\overline{\mathscr{B}}$ by evaluating these sheaves at the objects $V' \to U'$. The Frobenius also induces an almost isomorphism $A/p_1A \to A/pA$ by I.8.3.(1), which shows that A is almost pre-perfectoid. □

I.9. Complements on Logarithmic Geometry

We briefly recall some notions and facts of logarithmic geometry which will be used in the rest of the paper. We refer to [**Kat89, Kat94, GR04, Ogu18**] for a systematic development of logarithmic geometry, and to [**AGT16**, II.5] for a brief summary of the theory.

I.9.1. We only consider logarithmic structures in étale topology. More precisely, let X be a scheme, $X_{\text{ét}}$ the étale site of X, $\mathcal{O}_{X_{\text{ét}}}$ the structure sheaf on $X_{\text{ét}}$, $\mathcal{O}^\times_{X_{\text{ét}}}$ the subsheaf of units of $\mathcal{O}_{X_{\text{ét}}}$. A logarithmic struture on X is a homomorphism of sheaves of monoids $\alpha : \mathscr{M} \to \mathcal{O}_{X_{\text{ét}}}$ on $X_{\text{ét}}$ which induces an isomorphism $\alpha^{-1}(\mathcal{O}^\times_{X_{\text{ét}}}) \xrightarrow{\sim} \mathcal{O}^\times_{X_{\text{ét}}}$. We denote by $(X, \mathscr{M})$ the associated logarithmic scheme (cf. [**AGT16**, II.5.11]).

I.9.2. Let $(X, \mathscr{M})$ be a coherent log scheme (cf. [**AGT16**, II.5.15]). Then, there is a maximal open subscheme X^{tr} of X on which $\mathscr{M}$ is trivial, and moreover it is functorial in $(X, \mathscr{M})$ ([**Ogu18**, III.1.2.8]). Let $(X, \mathscr{M}) \to (S, \mathscr{L}) \leftarrow (Y, \mathscr{N})$ be a diagram of fine and saturated log schemes (cf. [**AGT16**, II.5.15]). Then, the fibred product is representable in the category of fine and saturated log schemes by $(Z, \mathscr{P}) = (X, \mathscr{M})\times^{\mathrm{fs}}_{(S,\mathscr{L})}(Y, \mathscr{N})$. We remark that $Z^{\mathrm{tr}} = X^{\mathrm{tr}} \times_{S^{\mathrm{tr}}} Y^{\mathrm{tr}}$, that $Z \to X \times_S Y$ is finite, and that $Z^{\mathrm{tr}} \to Z$ is Cartesian over $X^{\mathrm{tr}} \times_{S^{\mathrm{tr}}} Y^{\mathrm{tr}} \to X \times_S Y$ ([**Ogu18**, III 2.1.2, 2.1.6]). Moreover, if $X^{\mathrm{tr}} = X$, then $Z = X \times_S Y$ ([**Ogu18**, III.2.1.3]).

I.9.3. For an open immersion $j : Y \to X$, we denote by $j_{\text{ét}} : Y_{\text{ét}} \to X_{\text{ét}}$ the morphism of their étale sites defined by the base change by j. Let $\mathscr{M}_{Y\to X}$ be the preimage of $j_{\text{ét}*}\mathcal{O}^\times_{Y_{\text{ét}}}$ under the natural map $\mathcal{O}_{X_{\text{ét}}} \to j_{\text{ét}*}\mathcal{O}_{Y_{\text{ét}}}$, and we endow X with the logarithmic structure $\mathscr{M}_{Y\to X} \to \mathcal{O}_{X_{\text{ét}}}$, which is called the compactifying log structure associated to the open immersion j ([**Ogu18**, III.1.6.1]). Sometimes we write $\mathscr{M}_{Y\to X}$ as $\mathscr{M}_X$ if Y is clear in the context.

I.9.4. Let $(X, \mathscr{M})$ be a fine and saturated log scheme which is regular ([**Kat94**, 2.1], [**Niz06**, 2.3]). Then, X is locally Noetherian and normal, and X^{tr} is regular and dense in X ([**Kat94**, 4.1]). Moreover, there is a natural isomorphism $\mathscr{M} \xrightarrow{\sim} \mathscr{M}_{X^{\mathrm{tr}}\to X}$ ([**Kat94**, 11.6], [**Niz06**, 2.6]). We remark that if X is a regular scheme with a strict normal crossings divisor D, then $(X, \mathscr{M}_{X\setminus D\to X})$ is fine, saturated and regular ([**Ogu18**, III.1.11.9]).

Let $f : (X, \mathscr{M}) \to (S, \mathscr{L})$ be a smooth (resp. saturated) morphism of fine and saturated log schemes (cf. [**AGT16**, II 5.25, 5.18]). Then, f remains smooth (resp. saturated) under the base change in the category of fine and saturated log schemes ([**Ogu18**, IV.3.1.2, IV.3.1.11], resp. [**Ogu18**, III.2.5.3]). We remark that if f is smooth, then $f^{\mathrm{tr}} : X^{\mathrm{tr}} \to S^{\mathrm{tr}}$ is a smooth morphism of schemes.

If moreover $(S, \mathscr{L})$ is regular, then $(X, \mathscr{M})$ is also regular ([**Ogu18**, IV.3.5.3]). We also remark that if f is saturated, then for any fibred product in the category of fine and saturated log schemes $(Z, \mathscr{P}) = (X, \mathscr{M}) \times^{\mathrm{fs}}_{(S,\mathscr{L})} (Y, \mathscr{N})$, we have $Z = X \times_S Y$ ([**Tsu19**, II.2.13]).

I.9.5. Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, k the residue field of $\mathcal{O}_K$, π a uniformizer of $\mathcal{O}_K$. We set $\eta = \mathrm{Spec}(K)$, $S = \mathrm{Spec}(\mathcal{O}_K)$ and $s = \mathrm{Spec}(k)$. Then, $(S, \mathscr{M}_{\eta\to S})$ is fine, saturated and regular, since $\mathbb{N} \to \Gamma(S, \mathscr{M}_{\eta\to S})$ sending 1 to π forms a chart of $(S, \mathscr{M}_{\eta\to S})$ (cf. [**AGT16**, II.5.13, II.6.1]). Recall that an open immersion $Y \to X$ of quasi-compact and separated schemes over $\eta \to S$ is strictly semi-stable ([**dJ96**, 6.3]) if and only if the following conditions are satisfied ([**dJ96**, 6.4], [**EGA** IV_4, 17.5.3]):

(i) For each point x of the generic fibre X_η, there is an open neighborhood $U \subseteq X_\eta$ of x and a smooth K-morphism

$$f : U \longrightarrow \mathrm{Spec}(K[s_1, \ldots, s_m]) \tag{I.9.5.1}$$

such that f maps x to the point associated to the maximal ideal $(s_1, \ldots, s_m)$ and that $U \setminus Y$ is the inverse image of the closed subset defined by $s_1 \cdots s_m = 0$.

(ii) For each point x of the special fibre X_s, there is an open neighborhood $U \subseteq X$ of x and a smooth $\mathcal{O}_K$-morphism

$$f : U \longrightarrow \mathrm{Spec}(\mathcal{O}_K[t_1, \ldots, t_n, s_1, \ldots, s_m]/(\pi - t_1 \cdots t_n)) \tag{I.9.5.2}$$

such that f maps x to the point associated to the maximal ideal $(t_1, \ldots, t_n, s_1, \ldots, s_m)$ and that $U \setminus Y$ is the inverse image of the closed subset defined by $t_1 \cdots t_n \cdot s_1 \cdots s_m = 0$.

We call an open immersion $Y \to X$ of quasi-compact and separated schemes over $\eta \to S$ is semi-stable if for any point x of X there is an étale neighborhood U of x such that $Y \times_X U \to U$ is strictly semi-stable. In this case, $(X, \mathscr{M}_{Y\to X})$ is fine, saturated and regular which is smooth and saturated over $(S, \mathscr{M}_{\eta\to S})$, since for any point x of X there is an étale neighborhood U of x such that there exists a chart for the morphism $(U, \mathscr{M}_{Y\times_X U\to U}) \to (S, \mathscr{M}_{\eta\to S})$ subordinate to the morphism $\mathbb{N} \to \mathbb{N}^n \oplus \mathbb{N}^m$ sending 1 to $(1, \ldots, 1, 0, \ldots, 0)$ such that the induced morphism $U \to S \times_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{\mathbb{N}^n\oplus\mathbb{N}^m}$ is smooth (cf. [**Ogu18**, IV.3.1.18]).

I.9.6. Recall that a morphism of schemes $f : X \to S$ is called *generically finite* if there exists a dense open subscheme U of S such that $f^{-1}(U) \to U$ is finite. We remark that for a morphism $f : X \to S$ of finite type between Noetherian schemes which maps generic points to generic points, f is generically finite if and only if the residue field of any generic point η of X is a finite field extension of the residue field of $f(\eta)$ ([**ILO14**, II.1.1.7]).

I.9.7. Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, L an algebraically closed valuation field of height 1 extension of K with valuation ring $\mathcal{O}_L$, $\overline{K}$ the algebraic closure of K in L.

Consider the category $\mathscr{C}$ of open immersions between integral affine schemes $U \to T$ over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$ under $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ such that T is of finite type over $\mathcal{O}_K$ and that $\mathrm{Spec}(L) \to U$ is dominant. Let $\mathscr{C}_{\mathrm{car}}$ be the full subcategory of $\mathscr{C}$ formed by those objects $U \to T$ Cartesian over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$.

$$\begin{array}{ccc} \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_L) \\ \downarrow & & \downarrow \\ U = \mathrm{Spec}(B) & \longrightarrow & T = \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_K) \end{array} \tag{I.9.7.1}$$

We note that the objects of $\mathscr{C}$ are of the form $(U = \mathrm{Spec}(B) \to T = \mathrm{Spec}(A))$ where A (resp. B) is a finitely generated $\mathcal{O}_K$-subalgebra of $\mathcal{O}_L$ (resp. K-subalgebra of L) with $A \subseteq B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an open immersion.

Lemma I.9.8. *With the notation in* I.9.7*, we have:*

(1) *The category $\mathscr{C}$ is cofiltered, and the subcategory $\mathscr{C}_{\mathrm{car}}$ is initial in $\mathscr{C}$.*

(2) *The morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ represents the cofiltered limit of morphisms $U \to T$ indexed by $\mathscr{C}$ in the category of morphisms of schemes* (cf. I.7.1).

(3) *There exists a directed inverse system $(U_\lambda \to T_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathscr{C}_{\mathrm{car}}$ over a directed inverse system $(\mathrm{Spec}(K_\lambda) \to \mathrm{Spec}(\mathcal{O}_{K_\lambda}))_{\lambda\in\Lambda}$ of objects of $\mathscr{C}_{\mathrm{car}}$ such that K_λ is a finite field extension of K in L, that $\overline{K} = \bigcup_{\lambda\in\Lambda} K_\lambda$, that $U_\lambda \to T_\lambda$ is strictly semi-stable over $\mathrm{Spec}(K_\lambda) \to \mathrm{Spec}(\mathcal{O}_{K_\lambda})$* (I.9.5)*, and that $(U_\lambda \to T_\lambda)_{\lambda\in\Lambda}$ forms an initial full subcategory of $\mathscr{C}_{\mathrm{car}}$.*

PROOF. (1) For a diagram $(U_1 \to T_1) \to (U_0 \to T_0) \leftarrow (U_2 \to T_2)$ in $\mathscr{C}$, let T be the scheme theoretic image of $\mathrm{Spec}(L) \to T_1 \times_{T_0} T_2$ and let U be the intersection of $U_1 \times_{U_0} U_2$ with T. It is clear that T is of finite type over $\mathcal{O}_K$ as $\mathcal{O}_K$ is Noetherian, that U and T are integral and affine, that $\mathrm{Spec}(L) \to U$ is dominant, and that $\mathrm{Spec}(L) \to T$ factors through $\mathrm{Spec}(\mathcal{O}_L)$. Thus, $U \to T$ is an object of $\mathscr{C}$, which shows that $\mathscr{C}$ is cofiltered. For an object $(U = \mathrm{Spec}(B) \to T = \mathrm{Spec}(A))$ of $\mathscr{C}$, we write $\mathcal{O}_L$ as a filtered union of finitely generated A-subalgebras A_i. Let π be a uniformizer of K. Notice that $L = \mathcal{O}_L[1/\pi] = \mathrm{colim}\, A_i[1/\pi]$ and that $\mathrm{Hom}_{K\text{-}\mathbf{Alg}}(B, L) = \mathrm{colim}\, \mathrm{Hom}_{K\text{-}\mathbf{Alg}}(B, A_i[1/\pi])$ by [**EGA** IV$_3$, 8.14.2.2]. Thus, there exists an index i such that $\mathrm{Spec}(A_i[1/\pi]) \to \mathrm{Spec}(A_i)$ is an object of $\mathscr{C}_{\mathrm{car}}$ over $U \to T$.

(2) It follows immediately from the arguments above.

(3) Consider the category $\mathscr{D}$ of morphisms of $\mathscr{C}_{\mathrm{car}}$,

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{K'}) \end{array} \tag{I.9.8.1}$$

such that K' is a finite field extension of K. Similarly, this category is also cofiltered with limit of diagrams of schemes $(\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)) \to (\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(\mathcal{O}_{\overline{K}}))$. It suffices to show that the full subcategory of $\mathscr{D}$ formed by strictly semi-stable objects is initial. For any object $U \to T$ of $\mathscr{C}_{\mathrm{car}}$, by de Jong's alteration theorem [**dJ96**, 6.5], there exists a proper surjective and generically finite morphism $T' \to T$ of integral schemes such that $U' = U \times_T T' \to T'$ is strictly semi-stable over $\mathrm{Spec}(K') \to \mathrm{Spec}(\mathcal{O}_{K'})$ for a finite field extension $K \to K'$. Since L is algebraically closed, the dominant morphism $\mathrm{Spec}(L) \to U$ lifts to a dominant morphism $\mathrm{Spec}(L) \to U'$ (I.9.6), which further extends to a lifting $\mathrm{Spec}(\mathcal{O}_L) \to T'$ of $\mathrm{Spec}(\mathcal{O}_L) \to T$ by the valuative criterion. After replacing T' by an affine open neighborhood of the image of the closed point of $\mathrm{Spec}(\mathcal{O}_L)$, we obtain a strictly semi-stable object of $\mathscr{D}$ over $(U \to T) \to (\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K))$, which completes the proof. □

Theorem I.9.9 ([**ILO14**, X 3.5, 3.7])**.** *Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$, $(Y \to X) \to (U \to T)$ a morphism of dominant open immersions over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$ between irreducible $\mathcal{O}_K$-schemes of finite type such that $X \to T$ is proper surjective. Then, there exists a commutative diagram of dominant open immersions between irreducible $\mathcal{O}_K$-schemes of finite type*

$$\begin{array}{ccc} (Y' \to X') & \xrightarrow{(\beta^\circ, \beta)} & (Y \to X) \\ {\scriptstyle (f'^\circ, f')}\downarrow & & \downarrow{\scriptstyle (f^\circ, f)} \\ (U' \to T') & \xrightarrow[(\alpha^\circ, \alpha)]{} & (U \to T) \end{array} \tag{I.9.9.1}$$

satisfying the following conditions:

(i) *We have $Y' = \beta^{-1}(Y) \cap f'^{-1}(U')$, i.e. $Y' \to X'$ is Cartesian over $U' \times_U Y \to T' \times_T X$* (cf. I.7.1).

(ii) *The morphism $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ induced by (f'°, f') is a smooth and saturated morphism of fine, saturated and regular log schemes.*

(iii) *The morphisms α and β are proper surjective and generically finite, and f' is projective surjective.*

PROOF. We may assume that T is nonempty. Recall that $\mathrm{Spec}(\mathcal{O}_K)$ is universally $\mathbb{Q}$-resolvable ([**ILO14**, X.3.3]) by de Jong's alteration theorem [**dJ96**, 6.5]. Thus, T is also universally $\mathbb{Q}$-resolvable by [**ILO14**, X 3.5, 3.5.2] so that we can apply [**ILO14**, X.3.5] to the proper surjective morphism f and the nowhere dense closed subset $X \setminus Y$. Then, we obtain a commutative diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow[\alpha]{} & T \end{array} \tag{I.9.9.2}$$

and dense open subsets $U' \subseteq T'$, $Y' = \beta^{-1}(Y) \cap f'^{-1}(U') \subseteq X'$ such that $(X', \mathscr{M}_{Y' \to X'})$ and $(T', \mathscr{M}_{U' \to T'})$ are fine, saturated and regular, that $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ is smooth, that α, β are proper surjective and generically finite morphisms which map generic points to generic points, and that f' is projective (since f is proper, cf. [**ILO14**, X 3.1.6, 3.1.7]). Since X (resp. T)

is irreducible and X' (resp. T') is a disjoint union of normal integral schemes (I.9.4), after firstly replacing X' by an irreducible component and then replacing T' by the irreducible component under X', we may assume that X' and T' are irreducible. Then, $Y' \to U'$ is dominant (so that f' is projective surjective), since it is smooth and Y' is nonempty ([**EGA** IV$_2$, 2.3.4]). We claim that α maps U' into U. Indeed, if there exists a point $u \in U'$ with $\alpha(u) \notin U$, then $f'^{-1}(u) \cap Y' = \emptyset$. However, endowing u with the trivial log structure, the log scheme $(u, \mathcal{O}^{\times}_{u_{\text{ét}}})$ is fine, saturated and regular, and the fibred product in the category of fine and saturated log schemes

$$(u, \mathcal{O}^{\times}_{u_{\text{ét}}}) \times^{\text{fs}}_{(T', \mathscr{M}_{U' \to T'})} (X', \mathscr{M}_{Y' \to X'}) \tag{I.9.9.3}$$

is regular with underlying scheme $f'^{-1}(u)$ (I.9.4, I.9.2). Thus, $f'^{-1}(u) \cap Y'$ is dense in $f'^{-1}(u)$, which contradicts the assumption that $f'^{-1}(u) \cap Y' = \emptyset$ since f' is surjective. Thus, we obtain a diagram (I.9.9.1) satisfying all the conditions except the saturatedness of $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$.

To make $(X', \mathscr{M}_{Y' \to X'}) \to (T', \mathscr{M}_{U' \to T'})$ saturated, we apply [**ILO14**, X.3.7] to the morphism (f'°, f'). We obtain a Cartesian morphism $(\gamma^{\circ}, \gamma) : (U'' \to T'') \to (U' \to T')$ of dominant open immersions such that $(T'', \mathscr{M}_{U'' \to T''})$ is a fine, saturated and regular log scheme, that γ is a proper surjective and generically finite morphism which maps generic points of T'' to the generic point of T', and that the fibred product in the category of fine and saturated log schemes

$$(T'', \mathscr{M}_{U'' \to T''}) \times^{\text{fs}}_{(T', \mathscr{M}_{U' \to T'})} (X', \mathscr{M}_{Y' \to X'}) \tag{I.9.9.4}$$

is saturated over $(T'', \mathscr{M}_{U'' \to T''})$. The fibred product (I.9.9.4) is still smooth over $(T'', \mathscr{M}_{U'' \to T''})$, and thus it is regular (I.9.4). Let X'' be the underlying scheme of it and let $Y'' = (X'')^{\text{tr}}$. Then, the fibred product (I.9.9.4) is isomorphic to $(X'', \mathscr{M}_{Y'' \to X''})$ (I.9.4). Thus, we obtain a commutative diagram of dominant open immersions of schemes

$$\begin{array}{ccc} (Y'' \to X'') & \xrightarrow{(\delta^{\circ}, \delta)} & (Y' \to X') \\ {\scriptstyle (f''^{\circ}, f'')}\downarrow & & \downarrow{\scriptstyle (f'^{\circ}, f')} \\ (U'' \to T'') & \xrightarrow[(\gamma^{\circ}, \gamma)]{} & (U' \to T') \end{array} \tag{I.9.9.5}$$

Notice that $Y'' = U'' \times_{U'} Y'$ and $X'' \to T'' \times_{T'} X'$ is finite, and that $Y'' \to X''$ is Cartesian over $U'' \times_{U'} Y' \to T'' \times_{T'} X'$ (I.9.2). Thus, we see that $Y'' \to X''$ is Cartesian over $U'' \times_U Y \to T'' \times_T X$ and that f'' is projective. Since T' (resp. X') is irreducible and T'' (resp. X'') is a disjoint union of normal integral schemes (I.9.4), after firstly replacing T'' by an irreducible component and then replacing X'' by an irreducible component on which the restriction of δ° is dominant, we may assume that T'' and X'' are irreducible. In particular, δ is generically finite and so is $\beta \circ \delta$ (I.9.6), and again $Y'' \to U''$ is dominant so that f'' is projective surjective. □

Lemma I.9.10. *Let X be a scheme of finite type over a valuation ring A of height 1. Then, the underlying topological space of X is Noetherian.*

Proof. Let η and s be the generic point and closed point of $\mathrm{Spec}(A)$ respectively. Then, the generic fibre X_η and the special fibre X_s are both Noetherian. As a union of X_η and X_s, the underlying topological space of X is also Noetherian ([**Sta22**, 0053]). □

Proposition I.9.11. *With the notation in* I.9.7 *and* I.9.8*, let $Y \to X$ be a quasi-compact dominant open immersion over $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ such that $X \to \mathrm{Spec}(\mathcal{O}_L)$ is proper of finite presentation. Then, there exists a proper surjective $\mathcal{O}_L$-morphism of finite presentation $X' \to X$, an index $\lambda_1 \in \Lambda$, and a directed inverse system of open immersions $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$ over $(U_\lambda \to T_\lambda)_{\lambda \geq \lambda_1}$ satisfying the following conditions for each $\lambda \geq \lambda_1$:*

(i) *We have $Y' = Y \times_X X' = \lim_{\lambda \geq \lambda_1} Y'_\lambda$ and $X' = \lim_{\lambda \geq \lambda_1} X'_\lambda$.*
(ii) *The log scheme $(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda})$ is fine, saturated and regular.*
(iii) *The morphism $(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda}) \to (T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda})$ is smooth and saturated, and $X'_\lambda \to T_\lambda$ is projective.*
(iv) *If moreover $Y = \mathrm{Spec}(L) \times_{\mathrm{Spec}(\mathcal{O}_L)} X$, then we can require that $Y'_\lambda = U_\lambda \times_{T_\lambda} X'_\lambda$.*

Proof. We follow closely the proof of [**ALPT19**, 5.2.19]. Since the underlying topological space of X is Noetherian by I.9.10, each irreducible component Z of X admits a closed subscheme structure such that $Z \to X$ is of finite presentation ([**Sta22**, 01PH]). After replacing X by the disjoint union of its irreducible components, we may assume that X is irreducible. Then, the generic fibre of $X \to \mathrm{Spec}(\mathcal{O}_L)$ is also irreducible as an open subset of X. Using [**EGA** IV$_3$, 8.8.2, 8.10.5], there exists an index $\lambda_0 \in \Lambda$, a proper T_{λ_0}-scheme X_{λ_0}, and an open subscheme Y_{λ_0} of $U_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$, such that $X = \mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} X_{\lambda_0}$ and that $Y = \mathrm{Spec}(L) \times_{U_{\lambda_0}} Y_{\lambda_0}$. Let η denote the generic point of

X, η_{λ_0} the image of η under the morphism $X \to X_{\lambda_0}$, Z_{λ_0} the scheme theoretic closure of η_{λ_0} in X_{λ_0}. Notice that $\mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} Z_{\lambda_0} \to X$ is a surjective finitely presented closed immersion. After replacing X by $\mathrm{Spec}(\mathcal{O}_L) \times_{T_{\lambda_0}} Z_{\lambda_0}$ and replacing X_{λ_0} by Z_{λ_0}, we may assume that $X \to X_{\lambda_0}$ is a dominant morphism of irreducible schemes. Since T_{λ_0} is irreducible and L is algebraically closed, the generic fibre of $f : X_{\lambda_0} \to T_{\lambda_0}$ is geometrically irreducible. In particular, if ξ_{λ_0} (resp. η_{λ_0}) denotes the generic point of T_{λ_0} (resp. X_{λ_0}), then $\eta = \mathrm{Spec}(L) \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ ([**EGA** IV$_2$, 4.5.9]). In the situation of (iv), we can moreover assume that $Y_{\lambda_0} = U_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$.

By I.9.9, there exists a commutative diagram of dominant open immersions of irreducible schemes,

$$\begin{array}{ccc} (Y'_{\lambda_0} \to X'_{\lambda_0}) & \xrightarrow{(\beta^\circ,\beta)} & (Y_{\lambda_0} \to X_{\lambda_0}) \\ \downarrow{\scriptstyle (f'^\circ,f')} & & \downarrow{\scriptstyle (f^\circ,f)} \\ (U'_{\lambda_0} \to T'_{\lambda_0}) & \xrightarrow[(\alpha^\circ,\alpha)]{} & (U_{\lambda_0} \to T_{\lambda_0}) \end{array} \tag{I.9.11.1}$$

where $Y'_{\lambda_0} \to X'_{\lambda_0}$ is Cartesian over $U'_{\lambda_0} \times_{U_{\lambda_0}} Y_{\lambda_0} \to T'_{\lambda_0} \times_{T_{\lambda_0}} X_{\lambda_0}$, and where $(X'_{\lambda_0}, \mathscr{M}_{Y'_{\lambda_0} \to X'_{\lambda_0}}) \to (T'_{\lambda_0}, \mathscr{M}_{U'_{\lambda_0} \to T'_{\lambda_0}})$ is a smooth and saturated morphism of fine, saturated and regular log schemes, and where α and β are proper surjective and generically finite, and where f' is projective surjective. We take a dominant morphism $\gamma^\circ : \mathrm{Spec}(L) \to U'_{\lambda_0}$ which lifts $\mathrm{Spec}(L) \to U_{\lambda_0}$ since L is algebraically closed and α is generically finite, the morphism $\mathrm{Spec}(\mathcal{O}_L) \to T_{\lambda_0}$ lifts to $\gamma : \mathrm{Spec}(\mathcal{O}_L) \to T'_{\lambda_0}$ by the valuative criterion. We set $Y' = \mathrm{Spec}(L) \times_{U'_{\lambda_0}} Y'_{\lambda_0}$ and $X' = \mathrm{Spec}(\mathcal{O}_L) \times_{T'_{\lambda_0}} X'_{\lambda_0}$. It is clear that $Y' \to X'$ is Cartesian over $Y \to X$ by base change. Let ξ'_{λ_0} (resp. η'_{λ_0}) be the generic points of T'_{λ_0} (resp. X'_{λ_0}). Since the generic fibre of f is geometrically irreducible, $\xi'_{\lambda_0} \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ is a single point and η'_{λ_0} maps to it ([**EGA** IV$_2$, 4.5.9]). Since $\mathrm{Spec}(L) \times_{\xi_{\lambda_0}} \eta_{\lambda_0}$ is the generic point of X, we see that $X' \to X$ is proper surjective and of finite presentation. It remains to construct $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$.

After replacing T'_{λ_0} by an affine open neighborhood of the image of the closed point of $\mathrm{Spec}(\mathcal{O}_L)$, lemma I.9.8 implies that there exists an index $\lambda_1 \geq \lambda_0$ such that the transition morphism $(U_{\lambda_1} \to T_{\lambda_1}) \to (U_{\lambda_0} \to T_{\lambda_0})$ factors through $(U'_{\lambda_0} \to T'_{\lambda_0})$. For each index $\lambda \geq \lambda_1$, consider the fibred product in the category of fine and saturated log schemes

$$(X'_\lambda, \mathscr{M}_{Y'_\lambda \to X'_\lambda}) = (T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda}) \times^{\mathrm{fs}}_{(T'_{\lambda_0}, \mathscr{M}_{U'_{\lambda_0} \to T'_{\lambda_0}})} (X'_{\lambda_0}, \mathscr{M}_{Y'_{\lambda_0} \to X'_{\lambda_0}}), \tag{I.9.11.2}$$

which is a fine, saturated and regular log scheme smooth and saturated over $(T_\lambda, \mathscr{M}_{U_\lambda \to T_\lambda})$ (I.9.2, I.9.4). Moreover, we have $Y'_\lambda = U_\lambda \times_{U'_{\lambda_0}} Y'_{\lambda_0}$, $X'_\lambda = T_\lambda \times_{T'_{\lambda_0}} X'_{\lambda_0}$, and in the situation of (iv), $Y'_\lambda = U_\lambda \times_{T_\lambda} X'_\lambda$ by base change. Therefore, $(Y'_\lambda \to X'_\lambda)_{\lambda \geq \lambda_1}$ meets our requirements. □

I.10. Faltings' Main p-adic Comparison Theorem: the Absolute Case

Lemma I.10.1. *Let Y be a coherent scheme, V a finite étale Y-scheme. Then, there exists a finite étale surjective morphism $Y' \to Y$ such that $Y' \times_Y V$ is isomorphic to a finite disjoint union of Y'.*

Proof. If Y is connected, let $\overline{y}$ be a geometric point of Y, $\pi_1(Y, \overline{y})$ the fundamental group of Y with base point $\overline{y}$. Then, $Y_{\text{fét}}$ is equivalent to the category of finite $\pi_1(Y, \overline{y})$-sets so that the lemma holds ([**Sta22**, 0BND]).

In general, for any connected component Z of Y, let $(Y_\lambda)_{\lambda \in \Lambda_Z}$ be the directed inverse system of all open and closed subschemes of Y which contain Z and whose transition morphisms are inclusions. Notice that $\lim_{\lambda \in \Lambda_Z} Y_\lambda$ is a closed subscheme of Y with underlying topological space Z by [**Sta22**, 04PL] and [**EGA** IV$_3$, 8.2.9]. We endow Z with the closed subscheme structure of $\lim_{\lambda \in \Lambda_Z} Y_\lambda$. The first paragraph shows that there exists a finite étale surjective morphism $Z' \to Z$ such that $Z' \times_Y V = \coprod_{i=1}^r Z'$. Using [**EGA** IV$_3$, 8.8.2, 8.10.5] and [**EGA** IV$_4$, 17.7.8], there exists an index $\lambda_0 \in \Lambda_Z$, a finite étale surjective morphism $Y'_{\lambda_0} \to Y_{\lambda_0}$ and an isomorphism $Y'_{\lambda_0} \times_Y V = \coprod_{i=1}^r Y'_{\lambda_0}$. Notice that Y'_{λ_0} is also finite étale over Y. Since Z is an arbitrary connected component of Y, the conclusion follows from the quasi-compactness of Y. □

Lemma I.10.2. *Let Y be a coherent scheme, $\rho : Y_{\text{ét}} \to Y_{\text{fét}}$ the morphism of sites defined by the inclusion functor. Then, the functor $\rho^{-1} : \widetilde{Y_{\text{fét}}} \to \widetilde{Y_{\text{ét}}}$ of the associated topoi induces an equivalence $\rho^{-1} : \mathbf{LocSys}(Y_{\text{fét}}) \to \mathbf{LocSys}(Y_{\text{ét}})$ between the categories of finite locally constant abelian sheaves with quasi-inverse ρ_*.*

Proof. Since any finite locally constant sheaf on $Y_{\text{ét}}$ (resp. $Y_{\text{fét}}$) is representable by a finite étale Y-scheme by faithfully flat descent (cf. [**Sta22**, 03RV]), the Yoneda embeddings induce a

commutative diagram

$$\begin{array}{ccccc} \mathbf{LocSys}(Y_{\mathrm{f\acute{e}t}}) & \longrightarrow & Y_{\mathrm{f\acute{e}t}} & \xrightarrow{h^{\mathrm{f\acute{e}t}}} & \widetilde{Y_{\mathrm{f\acute{e}t}}} \\ \downarrow{\scriptstyle \rho^{-1}} & & \downarrow & & \downarrow \\ \mathbf{LocSys}(Y_{\mathrm{\acute{e}t}}) & \longrightarrow & Y_{\mathrm{\acute{e}t}} & \xrightarrow{h^{\mathrm{\acute{e}t}}} & \widetilde{Y_{\mathrm{\acute{e}t}}} \end{array} \tag{I.10.2.1}$$

where the horizontal arrows are fully faithful. In particular, ρ^{-1} is fully faithful. For a finite locally constant abelian sheaf $\mathbb{F}$ on $Y_{\mathrm{\acute{e}t}}$, let V be a finite étale Y-scheme representing $\mathbb{F}$ and let $h_V^{\mathrm{\acute{e}t}}$ (resp. $h_V^{\mathrm{f\acute{e}t}}$) be the representable sheaf of V on $Y_{\mathrm{\acute{e}t}}$ (resp. $Y_{\mathrm{f\acute{e}t}}$). We have $\mathbb{F} = h_V^{\mathrm{\acute{e}t}} = \rho^{-1} h_V^{\mathrm{f\acute{e}t}}$ ([**Sta22**, 04D3]). By I.10.1, $h_V^{\mathrm{f\acute{e}t}}$ is finite locally constant. It is clear that the adjunction morphism $h_V^{\mathrm{f\acute{e}t}} \to \rho_* h_V^{\mathrm{\acute{e}t}}$ is an isomorphism, which shows that $h_V^{\mathrm{f\acute{e}t}}$ is an abelian sheaf. Thus, ρ^{-1} is essentially surjective. Moreover, the argument also shows that ρ_* induces a functor $\rho_* : \mathbf{LocSys}(Y_{\mathrm{\acute{e}t}}) \to \mathbf{LocSys}(Y_{\mathrm{f\acute{e}t}})$ which is a quasi-inverse of ρ^{-1}. □

Proposition I.10.3. *With the notation in* I.7.8*, the functors between the categories of finite locally constant abelian sheaves*

$$\mathbf{LocSys}(Y_{\mathrm{f\acute{e}t}}) \xrightarrow{\beta^{-1}} \mathbf{LocSys}(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}) \xrightarrow{\psi^{-1}} \mathbf{LocSys}(Y_{\mathrm{\acute{e}t}}) \tag{I.10.3.1}$$

are equivalences with quasi-inverses β_* and ψ_* respectively.

Proof. Notice that for any finite locally constant abelian sheaf $\mathbb{G}$ on $Y_{\mathrm{f\acute{e}t}}$, the canonical morphism $\beta^{-1}\mathbb{G} \to \psi_*\rho^{-1}\mathbb{G}$, which is induced by the adjunction $\mathrm{id} \to \psi_*\psi^{-1}$ and by the identity $\psi^{-1}\beta^{-1} = \rho^{-1}$, is an isomorphism by I.10.2 and the proof of [**AGT16**, VI.6.3.(iii)]. For a finite locally constant abelian sheaf $\mathbb{F}$ over $Y_{\mathrm{\acute{e}t}}$, we write $\mathbb{F} = \rho^{-1}\mathbb{G}$ by I.10.2. Then, $\mathbb{F} = \psi^{-1}\beta^{-1}\mathbb{G} \xrightarrow{\sim} \psi^{-1}\psi_*\rho^{-1}\mathbb{G} = \psi^{-1}\psi_*\mathbb{F}$, whose inverse is the adjunction map $\psi^{-1}\psi_*\mathbb{F} \to \mathbb{F}$ since the composition of $\psi^{-1}(\beta^{-1}\mathbb{G}) \to \psi^{-1}(\psi_*\psi^{-1})(\beta^{-1}\mathbb{G}) = (\psi^{-1}\psi_*)\psi^{-1}(\beta^{-1}\mathbb{G}) \to \psi^{-1}(\beta^{-1}\mathbb{G})$ is the identity. On the other hand, for a finite locally constant abelian sheaf $\mathbb{L}$ over $\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}$, we claim that $\mathbb{L} \to \psi_*\psi^{-1}\mathbb{L}$ is an isomorphism. The problem is local on $\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}$. Thus, we may assume that $\mathbb{L}$ is the constant sheaf with value L where L is a finite abelian group. Let $\underline{L}$ be the constant sheaf with value L on $Y_{\mathrm{f\acute{e}t}}$. Then, $\mathbb{L} = \beta^{-1}\underline{L}$, and the isomorphism $\mathbb{L} = \beta^{-1}\underline{L} \xrightarrow{\sim} \psi_*\rho^{-1}\underline{L} = \psi_*\psi^{-1}\mathbb{L}$ coincides with the adjunction map $\mathbb{L} \to \psi_*\psi^{-1}\mathbb{L}$. Therefore, $\psi^{-1} : \mathbf{LocSys}(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}) \to \mathbf{LocSys}(Y_{\mathrm{\acute{e}t}})$ is an equivalence with quasi-inverse ψ_*. The conclusion follows from I.10.2. □

I.10.4. Let $f : (Y' \to X') \to (Y \to X)$ be a morphism of morphisms between coherent schemes over $\mathrm{Spec}(\mathbb{Q}_p) \to \mathrm{Spec}(\mathbb{Z}_p)$. The base change by f induces a commutative diagram of sites

$$\begin{array}{ccc} Y'_{\mathrm{\acute{e}t}} & \xrightarrow{\psi'} & \mathbf{E}^{\mathrm{\acute{e}t}}_{Y'\to X'} \\ {\scriptstyle f_{\mathrm{\acute{e}t}}}\downarrow & & \downarrow{\scriptstyle f_{\mathbf{E}}} \\ Y_{\mathrm{\acute{e}t}} & \xrightarrow{\psi} & \mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X} \end{array} \tag{I.10.4.1}$$

Let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\mathrm{\acute{e}t}}$. Remark that the sheaf $\overline{\mathscr{B}}$ on $\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}$ is flat over $\mathbb{Z}$. Consider the natural morphisms in the derived category $\mathbf{D}(\overline{\mathscr{B}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\mathrm{\acute{e}t}}_{Y\to X}})$,

$$(\mathrm{R}\psi_*\mathrm{R}f_{\mathrm{\acute{e}t}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \xleftarrow{\alpha_1} (\mathrm{R}f_{\mathbf{E}*}\psi'_*\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \xrightarrow{\alpha_2} \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'), \tag{I.10.4.2}$$

where α_1 is induced by the canonical morphism $\psi'_*\mathbb{F}' \to \mathrm{R}\psi'_*\mathbb{F}'$, and α_2 is the canonical morphism.

I.10.5. We keep the notation in I.10.4 and assume that X is the spectrum of an absolutely integrally closed valuation ring A and that Y is a quasi-compact open subscheme of X. Applying the functor $\mathrm{R}\Gamma(Y \to X, -)$ on (I.10.4.2), we obtain the natural morphisms in the derived category $\mathbf{D}(A\text{-}\mathbf{Mod})$ by I.7.9,

$$\mathrm{R}\Gamma(Y'_{\mathrm{\acute{e}t}}, \mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A \xleftarrow{\alpha_1} \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y'\to X'}, \psi'_*\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A \xrightarrow{\alpha_2} \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y'\to X'}, \psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'). \tag{I.10.5.1}$$

Definition I.10.6 ([**AG20**, 4.8.13, 5.7.4])**.** Under the assumptions in I.10.4 (resp. I.10.5) and with the same notation, if α_1 is an isomorphism (for instance, if the canonical morphism $\psi'_*\mathbb{F}' \to \mathrm{R}\psi'_*\mathbb{F}'$ is an isomorphism), then we call the canonical morphism

$$\alpha_2 \circ \alpha_1^{-1} : (\mathrm{R}\psi_*\mathrm{R}f_{\mathrm{\acute{e}t}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{I.10.6.1}$$

$$(\text{resp. } \alpha_2 \circ \alpha_1^{-1} : \mathrm{R}\Gamma(Y'_{\mathrm{\acute{e}t}}, \mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\mathrm{\acute{e}t}}_{Y'\to X'}, \psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')) \tag{I.10.6.2}$$

the *relative* (resp. *absolute*) *Faltings' comparison morphism* associated to $f : (Y' \to X') \to (Y \to X)$ and $\mathbb{F}'$. In this case, we say that *Faltings' comparison morphisms exist*.

Theorem I.10.7 ([**Ach17**, Cor.6.9], cf. [**AG20**, 4.4.2])**.** *Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p. We fix an algebraic closure $\overline{K}$ of K. Let X be an $\mathcal{O}_K$-scheme of finite type, $\mathbb{F}$ a finite locally constant abelian sheaf on $X_{\overline{K},\text{ét}}$, $\psi : X_{\overline{K},\text{ét}} \to \mathbf{E}^{\text{ét}}_{X_{\overline{K}}\to X}$ the morphism of sites defined in* I.7.8*. Then, the canonical morphism $\psi_*\mathbb{F} \to \mathrm{R}\psi_*\mathbb{F}$ is an isomorphism.*

Corollary I.10.8. *Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p. We fix an algebraic closure $\overline{K}$ of K. Let X be a coherent $\mathcal{O}_{\overline{K}}$-scheme, $Y = \mathrm{Spec}(\overline{K}) \times_{\mathrm{Spec}(\mathcal{O}_{\overline{K}})} X$, $\mathbb{F}$ a finite locally constant abelian sheaf on $Y_{\text{ét}}$, $\psi : Y_{\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the morphism of sites defined in* I.7.8*. Then, the canonical morphism $\psi_*\mathbb{F} \to \mathrm{R}\psi_*\mathbb{F}$ is an isomorphism.*

We emphasize that we don't need any finiteness condition of X over $\mathcal{O}_{\overline{K}}$ in I.10.8. In fact, one can replace $\mathcal{O}_{\overline{K}}$ by $\overline{\mathbb{Z}_p}$ without loss of generality, where $\overline{\mathbb{Z}_p}$ is the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$. We keep working over $\mathcal{O}_{\overline{K}}$ only for the continuation of our usage of notation.

Proof of I.10.8. We take a directed inverse system $(X_\lambda \to \mathrm{Spec}(\mathcal{O}_{K_\lambda}))_{\lambda\in\Lambda}$ of morphisms of finite type of schemes by Noetherian approximation, such that K_λ is a finite field extension of K and $\overline{K} = \bigcup_{\lambda\in\Lambda} K_\lambda$, and that the transition morphisms $X_{\lambda'} \to X_\lambda$ are affine and $X = \lim_{\lambda\in\Lambda} X_\lambda$ (cf. [**Sta22**, 09MV]). For each $\lambda \in \Lambda$, we set $Y_\lambda = \mathrm{Spec}(\overline{K}) \times_{\mathrm{Spec}(\mathcal{O}_{K_\lambda})} X_\lambda$. Notice that $Y = \lim Y_\lambda$. Then, there exists an index $\lambda_0 \in \Lambda$ and a finite locally constant abelian sheaf $\mathbb{F}_{\lambda_0}$ on $Y_{\lambda_0,\text{ét}}$ such that $\mathbb{F}$ is the pullback of $\mathbb{F}_{\lambda_0}$ by $Y_{\text{ét}} \to Y_{\lambda_0,\text{ét}}$ (cf. [**Sta22**, 09YU]). Let $\mathbb{F}_\lambda$ be the pullback of $\mathbb{F}_{\lambda_0}$ by $Y_{\lambda,\text{ét}} \to Y_{\lambda_0,\text{ét}}$ for each $\lambda \geq \lambda_0$. Notice that $\mathcal{O}_{K_\lambda}$ also satisfies the conditions in I.10.7. Let $\psi_\lambda : Y_{\lambda,\text{ét}} \to \mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ be the morphism of sites defined in I.7.8, $\varphi_\lambda : \mathbf{E}^{\text{ét}}_{Y\to X} \to \mathbf{E}^{\text{ét}}_{Y_\lambda\to X_\lambda}$ the morphism of sites defined by the transition morphism. Then, we have $\mathrm{R}^q\psi_{\lambda*}\mathbb{F}_\lambda = 0$ for each integer $q > 0$ by I.10.7, and moreover

$$\mathrm{R}^q\psi_*\mathbb{F} = \operatorname*{colim}_{\lambda\geq\lambda_0} \varphi_\lambda^{-1}\mathrm{R}^q\psi_{\lambda*}\mathbb{F}_\lambda = 0 \tag{I.10.8.1}$$

by I.7.12, [**SGA 4**$_{\text{II}}$, VII.5.6] and [**SGA 4**$_{\text{II}}$, VI.8.7.3] whose conditions are satisfied because each object in each concerned site is quasi-compact. □

Lemma I.10.9. *With the notation in* I.10.4*, let $\mathbb{F}$ be a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, the canonical morphism $f_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \psi'_* f_{\text{ét}}^{-1}\mathbb{F}$ is an isomorphism.*

Proof. The base change morphism $f_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \psi'_* f_{\text{ét}}^{-1}\mathbb{F}$ is the composition of the adjunction morphisms ([**SGA 4**$_{\text{III}}$, XVII.2.1.3])

$$f_{\mathbf{E}}^{-1}\psi_*\mathbb{F} \to \psi'_*\psi'^{-1}(f_{\mathbf{E}}^{-1}\psi_*\mathbb{F}) = \psi'_* f_{\text{ét}}^{-1}(\psi^{-1}\psi_*\mathbb{F}) \to \psi'_* f_{\text{ét}}^{-1}\mathbb{F} \tag{I.10.9.1}$$

which are both isomorphisms by I.10.3. □

I.10.10. Let K be a complete discrete valuation field of characteristic 0 with valuation ring $\mathcal{O}_K$ whose residue field k is algebraically closed (a condition required by [**AG20**, 4.1.3, 5.1.3]) of characteristic $p > 0$, $\overline{K}$ an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\eta = \mathrm{Spec}(K)$, $\overline{\eta} = \mathrm{Spec}(\overline{K})$, $S = \mathrm{Spec}(\mathcal{O}_K)$, $\overline{S} = \mathrm{Spec}(\mathcal{O}_{\overline{K}})$, $s = \mathrm{Spec}(k)$. Remark that $\overline{K}$ is a pre-perfectoid field with valuation ring $\mathcal{O}_{\overline{K}}$ so we are also in the situation I.8.7.

I.10.11. With the notation in I.10.10, let X be an S-scheme, Y an open subscheme of the generic fibre X_η. We simply denote by $\mathscr{M}_X$ the compactifying log structure $\mathscr{M}_{X_\eta\to X}$ (I.9.3). Following [**AGT16**, III.4.7], we say that $Y \to X$ is *adequate* over $\eta \to S$ if the following conditions are satisfied:

(i) X is of finite type over S;
(ii) Any point of the special fibre X_s admits an étale neighborhood U such that $U_\eta \to \eta$ is smooth and that $U_\eta \setminus Y$ is the support of a strict normal crossings divisor on U_η;
(iii) $(X, \mathscr{M}_{Y\to X})$ is a fine log scheme and the structure morphism $(X, \mathscr{M}_{Y\to X}) \to (S, \mathscr{M}_S)$ is smooth and saturated.

In this case, $(X, \mathscr{M}_{Y\to X}) \to (S, \mathscr{M}_S)$ is adequate in the sense of [**AGT16**, III.4.7]. We remark that for any adequate $(S, \mathscr{M}_S)$-log scheme $(X, \mathscr{M})$, $X^{\text{tr}} \to X$ is adequate over $\eta \to S$ and $(X, \mathscr{M}) = (X, \mathscr{M}_{X^{\text{tr}}\to X})$ (cf. I.9.4, I.9.5). Note that if $Y \to X$ is semi-stable over $\eta \to S$ then it is adequate (cf. I.9.5).

I.10.12. We recall the statement of Faltings' main p-adic comparison theorem following Abbes-Gros [**AG20**]. We take the notation and assumptions in I.10.10. Firstly, recall that for any adequate open immersion of schemes $X^\circ \to X$ over $\eta \to S$ and any finite locally constant abelian sheaf $\mathbb{F}$ on $X^\circ_{\overline{\eta},\text{ét}}$, the canonical morphism $\psi_*\mathbb{F} \to \mathrm{R}\psi_*\mathbb{F}$ is an isomorphism, where $\psi : X^\circ_{\overline{\eta},\text{ét}} \to \mathbf{E}^{\text{ét}}_{X^\circ_{\overline{\eta}} \to X}$ is the morphism of sites defined in I.7.8 ([**AG20**, 4.4.2]).

Let $(X'^{\triangleright} \to X') \to (X^\circ \to X)$ be a morphism of adequate open immersions of schemes over $\eta \to S$ such that $X' \to X$ is projective and that the induced morphism $(X', \mathscr{M}_{X'^{\triangleright} \to X'}) \to (X, \mathscr{M}_{X^\circ \to X})$ is smooth and saturated. Let $Y' = \overline{\eta} \times_\eta X'^{\triangleright}$, $Y = \overline{\eta} \times_\eta X^\circ$, $f : (Y' \to X') \to (Y \to X)$ the natural morphism, $\mathbb{F}'$ a finite locally constant abelian sheaf on $Y'_{\text{ét}}$. By the first paragraph, we have the relative Faltings' comparison morphism associated to f and $\mathbb{F}'$ (I.10.6.1),

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\mathbb{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}'). \tag{I.10.12.1}$$

Remark that under our assumption, the sheaf $\mathrm{R}^q f_{\text{ét}*}\mathbb{F}'$ on $Y_{\text{ét}}$ is finite locally constant for each integer q ([**AG20**, 2.2.25]).

Theorem I.10.13 ([**Fal02**, Thm.6, page 266], [**AG20**, 5.7.4])**.** *Under the assumptions in* I.10.12 *and with the same notation, the relative Faltings' comparison morphism associated to f and $\mathbb{F}'$ is an almost isomorphism in the derived category $\mathbf{D}(\mathcal{O}_{\overline{K}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$* (I.5.7)*, and it induces an almost isomorphism*

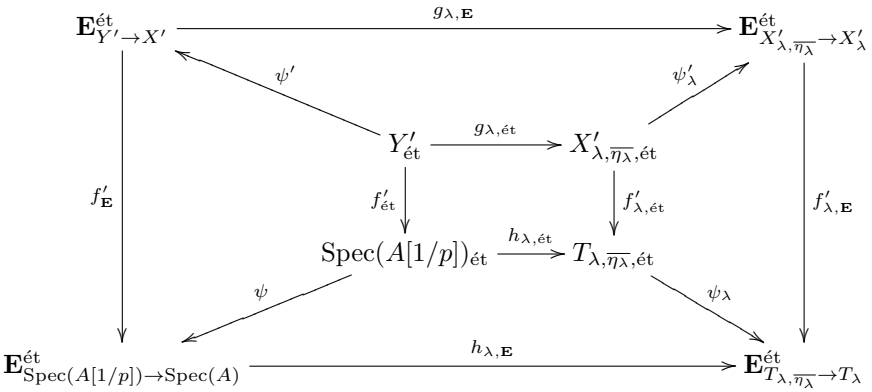
$$(\psi_*\mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{I.10.13.1}$$

of $\mathcal{O}_{\overline{K}}$-modules for each integer q.

Proposition I.10.14. *With the natation in* I.10.10*, let A be an absolutely integrally closed valuation ring of height* 1 *extension of $\mathcal{O}_{\overline{K}}$, X a proper A-scheme of finite presentation, $Y = \mathrm{Spec}(A[1/p]) \times_{\mathrm{Spec}(A)} X$, $\mathbb{F}$ a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, there exists a proper surjective morphism $X' \to X$ of finite presentation such that the relative and absolute Faltings' comparison morphisms associated to $f' : (Y' \to X') \to (\mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A))$ and $\mathbb{F}'$ (which exist by* I.10.8*) are almost isomorphisms, where $Y' = Y \times_X X'$ and $\mathbb{F}'$ is the pullback of $\mathbb{F}$ on $Y'_{\text{ét}}$.*

Proof. Since the underlying topological space of X is Noetherian by I.9.10, each irreducible component Z of X admits a closed subscheme structure such that $Z \to X$ is of finite presentation ([**Sta22**, 01PH]). After replacing X by the disjoint union of its irreducible components, we may assume that X is irreducible. If Y is empty, then we take $X' = X$ and thus the relative (resp. absolute) Faltings' comparison morphism associated to f' and $\mathbb{F}'$ is an isomorphism between zero objects. If Y is not empty, then we are in the situation of I.9.11.(iv) by taking $\mathcal{O}_L = A$. With the notation in I.9.11, we check that the morphism $X' \to X$ meets our requirements. We set $\eta_\lambda = \mathrm{Spec}(K_\lambda)$, $S_\lambda = \mathrm{Spec}(\mathcal{O}_{K_\lambda})$, $T_{\lambda,\overline{\eta_\lambda}} = \overline{\eta} \times_{\eta_\lambda} U_\lambda$, $X'_{\lambda,\overline{\eta_\lambda}} = \overline{\eta} \times_{\eta_\lambda} Y'_\lambda$, and denote by $f'_\lambda : (X'_{\lambda,\overline{\eta_\lambda}} \to X'_\lambda) \to (T_{\lambda,\overline{\eta_\lambda}} \to T_\lambda)$ the natural morphism. We obtain a commutative diagram

(I.10.14.1)



Firstly notice that the site $Y'_{\text{ét}}$ (resp. $\mathrm{Spec}(A[1/p])_{\text{ét}}$) is the limit of the sites $X'_{\lambda,\overline{\eta_\lambda},\text{ét}}$ (resp. $T_{\lambda,\overline{\eta_\lambda},\text{ét}}$) and the site $\mathbf{E}^{\text{ét}}_{Y'\to X'}$ (resp. $\mathbf{E}^{\text{ét}}_{\mathrm{Spec}(A[1/p])\to\mathrm{Spec}(A)}$) is the limit of the sites $\mathbf{E}^{\text{ét}}_{X'_{\lambda,\overline{\eta_\lambda}}\to X'_\lambda}$ (resp. $\mathbf{E}^{\text{ét}}_{T_{\lambda,\overline{\eta_\lambda}}\to T_\lambda}$) ([**SGA 4**$_{\text{II}}$, VII.5.6] and I.7.12). There exists an index $\lambda_0 \in \Lambda$ and a finite locally constant abelian sheaf $\mathbb{F}'_{\lambda_0}$ on $X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ such that $\mathbb{F}'$ is the pullback of $\mathbb{F}'_{\lambda_0}$ by $Y'_{\text{ét}} \to X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ (cf. [**Sta22**, 09YU]). Let $\mathbb{F}'_\lambda$ be the pullback of $\mathbb{F}'_{\lambda_0}$ by $X'_{\lambda,\overline{\eta_\lambda},\text{ét}} \to X'_{\lambda_0,\overline{\eta_{\lambda_0}},\text{ét}}$ for each $\lambda \geq \lambda_0$. We also have $\overline{\mathscr{B}}' = \mathrm{colim}\, g^{-1}_{\lambda,\mathbf{E}}\overline{\mathscr{B}}'$ (resp. $\overline{\mathscr{B}} = \mathrm{colim}\, h^{-1}_{\lambda,\mathbf{E}}\overline{\mathscr{B}}$) by I.7.12. According to [**SGA 4**$_{\text{II}}$, VI.8.7.3], whose conditions are satisfied

because each object in each concerned site is quasi-compact, there are canonical isomorphisms for each integer q,

$$(\mathrm{R}^q(\psi\circ f'_{\text{ét}})_*\mathbb{F}')\otimes_{\mathbb{Z}}\overline{\mathscr{B}}\xrightarrow{\sim}\operatorname{colim} h_{\lambda,\mathbf{E}}^{-1}((\mathrm{R}^q(\psi_\lambda\circ f'_{\lambda,\text{ét}})_*\mathbb{F}'_\lambda)\otimes_{\mathbb{Z}}\overline{\mathscr{B}}), \tag{I.10.14.2}$$

$$\mathrm{R}^q f'_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}')\xrightarrow{\sim}\operatorname{colim} h_{\lambda,\mathbf{E}}^{-1}\mathrm{R}^q f'_{\lambda,\mathbf{E}*}(\psi'_{\lambda*}\mathbb{F}'_\lambda\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'). \tag{I.10.14.3}$$

On the other hand, $(X'_\lambda,\mathscr{M}_{X'_\lambda})\to(T_\lambda,\mathscr{M}_{T_\lambda})$ is a smooth and saturated morphism of adequate $(S_\lambda,\mathscr{M}_{S_\lambda})$-log schemes with $X'_\lambda\to T_\lambda$ projective for each $\lambda\in\Lambda$ by construction. Thus, we are in the situation of I.10.13, which implies that the relative Faltings' comparison morphism associated to f'_λ and $\mathbb{F}'_\lambda$,

$$(\mathrm{R}^q(\psi_\lambda\circ f'_{\lambda,\text{ét}})_*\mathbb{F}'_\lambda)\otimes_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow\mathrm{R}^q f'_{\lambda,\mathbf{E}*}(\psi'_{\lambda*}\mathbb{F}'_\lambda\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \tag{I.10.14.4}$$

is an almost isomorphism for each $\lambda\geq\lambda_0$. Combining with (I.10.14.2) and (I.10.14.3), we see that the relative Faltings' comparison morphism associated to f' and $\mathbb{F}'$,

$$\mathrm{R}\psi_*(\mathrm{R}f'_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow\mathrm{R}f'_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'), \tag{I.10.14.5}$$

is an almost isomorphism (and thus so is the absolute one). □

Corollary I.10.15. *Under the assumptions in* I.10.14 *and with the same notation, there exists a proper hypercovering $X_\bullet\to X$ of coherent schemes* ([**Sta22**, 0DHI]) *such that for each degree n, the relative and absolute Faltings' comparison morphisms associated to $f_n:(Y_n\to X_n)\to(\operatorname{Spec}(A[1/p])\to\operatorname{Spec}(A))$ and $\mathbb{F}_n$ (which exist by* I.10.8*) are almost isomorphisms, where $Y_n=Y\times_X X_n$ and $\mathbb{F}_n$ is the pullback of $\mathbb{F}$ by the natural morphism $Y_{n,\text{ét}}\to Y_{\text{ét}}$. In particular, $Y_\bullet\to Y$ is a proper hypercovering and $X_\bullet^{Y_\bullet}\to X^Y$ is a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$.*

Proof. Let $\mathscr{C}$ be the category of proper A-schemes of finite presentation endowed with the pretopology formed by families of morphisms $\{f_i:X_i\to X\}_{i\in I}$ with I finite and $X=\bigcup_{i\in I}f_i(X_i)$. Consider the functor $u^+:\mathscr{C}\to\mathbf{I}_{\operatorname{Spec}(A[1/p])\to\operatorname{Spec}(A)}$ sending X to X^Y where $Y=\operatorname{Spec}(A[1/p])\times_{\operatorname{Spec}(A)}X$. It is well-defined by I.3.19.(4) and commutes with fibred products by I.3.21 and continuous by I.3.15. Lemma I.10.14 allows us to take a hypercovering $X_\bullet\to X$ in $\mathscr{C}$ meeting our requirement by [**Sta22**, 094K and 0DB1]. We see that $Y_\bullet\to Y$ is a proper hypercovering and that $X_\bullet^{Y_\bullet}\to X^Y$ is a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$ by the properties of u^+ ([**Sta22**, 0DAY]). □

Lemma I.10.16. *Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$, A a $\overline{\mathbb{Z}_p}$-algebra which is an absolutely integrally closed valuation ring, X a proper A-scheme of finite presentation, $Y=\operatorname{Spec}(A[1/p])\times_{\operatorname{Spec}(A)}X$, $\mathbb{F}$ a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Let $A'=((A/\cap_{n>0}p^nA)_{\sqrt{pA}})^\wedge$ (p-adic completion), $X'=X_{A'}$, $Y'=Y_{A'}$, $\mathbb{F}'$ the pullback of $\mathbb{F}$ on $Y'_{\text{ét}}$. Then, the following statements are equivalent:*

1. *The absolute Faltings' comparison morphism associated to $f:(Y\to X)\to(\operatorname{Spec}(A[1/p])\to\operatorname{Spec}(A))$ and $\mathbb{F}$ (which exists by* I.10.8*) is an almost isomorphism.*
2. *The absolute Faltings' comparison morphism associated to $f':(Y'\to X')\to(\operatorname{Spec}(A'[1/p])\to\operatorname{Spec}(A'))$ and $\mathbb{F}'$ (which exists by* I.10.8*) is an almost isomorphism.*

Proof. If p is zero (resp. invertible) in A, then the absolute Faltings' comparison morphisms are both isomorphisms between zero objects, since Y and Y' are empty (resp. the abelian sheaves $\mathbb{F}$ and $\mathbb{F}'$ are zero after inverting p). Thus, we may assume that p is a nonzero element of the maximal ideal of A. Notice that $\cap_{n>0}p^nA$ is the maximal prime ideal of A not containing p and that $\sqrt{pA}$ is the minimal prime ideal of A containing p (I.2.1). Thus, $(A/\cap_{n>0}p^nA)_{\sqrt{pA}}$ is an absolutely integrally closed valuation ring of height 1 extension of $\overline{\mathbb{Z}_p}$ (I.2.1) and thus so is its p-adic completion A'.

We denote by $u:(Y'\to X')\to(Y\to X)$ the natural morphism. We have $\mathbb{F}'=u_{\text{ét}}^{-1}\mathbb{F}$. The natural morphisms in (I.10.5.1) induce a commutative diagram

$$\begin{array}{ccccc}
\mathrm{R}\Gamma(Y_{\text{ét}},\mathbb{F})\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xleftarrow{\alpha_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F})\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xrightarrow{\alpha_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X},\psi_*\mathbb{F}\otimes_{\mathbb{Z}}\overline{\mathscr{B}}) \\
\downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} \\
\mathrm{R}\Gamma(Y'_{\text{ét}},\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A' & \xleftarrow{\alpha'_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A' & \xrightarrow{\alpha'_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}')
\end{array} \tag{I.10.16.1}$$

where γ_1 is induced by the canonical morphism $\mathbb{F}\to\mathrm{R}u_{\text{ét}*}u_{\text{ét}}^{-1}\mathbb{F}$, and γ_2 (resp. γ_3) is induced by the composition of $\psi_*\mathbb{F}\to\mathrm{R}u_{\mathbf{E}*}u_{\mathbf{E}}^{-1}\psi_*\mathbb{F}\to\mathrm{R}u_{\mathbf{E}*}\psi'_*u_{\text{ét}}^{-1}\mathbb{F}$ (resp. and by the canonical morphism $\overline{\mathscr{B}}\to\mathrm{R}u_{\mathbf{E}*}\overline{\mathscr{B}}'$). Since α_1 and α'_1 are isomorphisms by I.10.8, it suffices to show that γ_1 and γ_3 are almost isomorphisms.

Since $A/\cap_{n>0} p^n A \to (A/\cap_{n>0} p^n A)_{\sqrt{pA}}$ is injective whose cokernel is killed by $\sqrt{pA}$ (I.4.7), the morphism $A \to A'$ induces an almost isomorphism $A/p^nA \to A'/p^nA'$ for each n. Then, for any torsion abelian group M, the natural morphism $M \otimes_{\mathbb{Z}} A \to M \otimes_{\mathbb{Z}} A'$ is an almost isomorphism. Therefore, γ_1 is an almost isomorphism by the proper base change theorem over the strictly Henselian local ring $A[1/p]$ ([**SGA 4**$_{\text{III}}$, XII 5.5, 5.4]). For γ_3, it suffices to show that the canonical morphism $\psi_*\mathbb{F} \otimes \overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes \overline{\mathscr{B}}')$ is an almost isomorphism. The problem is local on $\mathbf{E}^{\text{ét}}_{Y\to X}$, thus we may assume that $\psi_*\mathbb{F}$ is the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$ by I.10.3. Then, $\psi'_*\mathbb{F}'$ is also the constant sheaf with value $\mathbb{Z}/p^n\mathbb{Z}$ by I.10.9. Let $V \to U$ be an object of $\mathbf{E}^{\text{proét}}_{Y\to X}$ such that $U^V = \mathrm{Spec}(R)$ is the spectrum of an $\overline{\mathbb{Z}_p}$-algebra R which is almost pre-perfectoid. Since the almost isomorphisms $R/p^n \to (R\otimes_A A')/p^n$ ($n \geq 1$) induces an almost isomorphism of the p-adic completions $\widehat{R} \to R\widehat{\otimes}_A A'$, the $\overline{\mathbb{Z}_p}$-algebra $R \otimes_A A'$ is still almost pre-perfectoid (I.5.19). The pullback of $V \to U$ in $\mathbf{E}^{\text{proét}}_{Y'\to X'}$ is the object $V_{A'} \to U_{A'}$ and $U_{A'}^{V_{A'}}$ is the spectrum of the integral closure R' of $R \otimes_A A'$ in $R \otimes_A A'[1/p]$. Since $R \otimes_A A'$ is almost pre-perfectoid, R' is also almost pre-perfectoid and the morphism $(R \otimes_A A')/p^n \to R'/p^n$ is an almost isomorphism by I.5.30. Therefore, the morphism $\overline{\mathscr{B}}/p^n\overline{\mathscr{B}} \to \mathrm{R}u_{\mathbf{E}*}(\overline{\mathscr{B}}'/p^n\overline{\mathscr{B}}')$ is an almost isomorphism by I.7.32, I.8.11 and I.8.12. □

Theorem I.10.17. *Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$, A a $\overline{\mathbb{Z}_p}$-algebra which is an absolutely integrally closed valuation ring, X a proper A-scheme of finite presentation, $Y = \mathrm{Spec}(A[1/p]) \times_{\mathrm{Spec}(A)} X$, $\mathbb{F}$ a finite locally constant abelian sheaf on $Y_{\text{ét}}$. Then, the absolute Faltings' comparison morphism associated to $f : (Y \to X) \to (\mathrm{Spec}(A[1/p]) \to \mathrm{Spec}(A))$ and $\mathbb{F}$* (I.10.6.2) *(which exists by* I.10.8*),*

$$\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}), \tag{I.10.17.1}$$

is an almost isomorphism in $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod})$ (I.5.7).

Proof. Let K be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. By I.10.16, we may assume that A is a valuation ring of height 1 extension of $\mathcal{O}_{\overline{K}}$. Let $X_\bullet \to X$ be the proper hypercovering of coherent schemes constructed in I.10.15. For each degree n the canonical morphisms (I.10.6.2)

$$\mathrm{R}\Gamma(Y_{n,\text{ét}}, \mathbb{F}_n) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longleftarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_n\to X_n}, \psi_{n*}\mathbb{F}_n) \otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_n\to X_n}, \psi_{n*}\mathbb{F}_n \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \tag{I.10.17.2}$$

are an isomorphism and an almost isomorphism, where $\mathbb{F}_n$ is the pullback of $\mathbb{F}$ by the natural morphism $Y_{n,\text{ét}} \to Y_{\text{ét}}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{R}\Gamma(Y_{\text{ét}}, \mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xleftarrow{\alpha_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F}) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xrightarrow{\alpha_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y\to X}, \psi_*\mathbb{F} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{R}\Gamma(Y_{\bullet,\text{ét}}, \mathbb{F}_\bullet) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xleftarrow{\alpha_{1\bullet}} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}, \psi_{\bullet*}\mathbb{F}_\bullet) \otimes^{\mathrm{L}}_{\mathbb{Z}} A & \xrightarrow{\alpha_{2\bullet}} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet}, \psi_{\bullet*}\mathbb{F}_\bullet \otimes_{\mathbb{Z}} \overline{\mathscr{B}}_\bullet)
\end{array} \tag{I.10.17.3}$$

where $\mathbb{F}_\bullet = (\mathbb{F}_n)_{[n]\in\mathrm{Ob}(\Delta)}$ with the notation in I.6.5. By the functorial spectral sequence of simplicial sites ([**Sta22**, 09WJ]), we deduce from (I.10.17.2) that $\alpha_{1\bullet}$ is an isomorphism and $\alpha_{2\bullet}$ is an almost isomorphism. Since α_1 is an isomorphism by I.10.8, it remains to show that the left vertical arrow is an isomorphism and the right vertical arrow is an almost isomorphism.

We denote by $a : \mathbf{E}^{\text{ét}}_{Y_\bullet\to X_\bullet} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the augmentation of simplicial site and by $a_n : \mathbf{E}^{\text{ét}}_{Y_n\to X_n} \to \mathbf{E}^{\text{ét}}_{Y\to X}$ the natural morphism of sites. Notice that $a^{-1}\psi_*\mathbb{F} = (a_n^{-1}\psi_*\mathbb{F})_{[n]\in\mathrm{Ob}(\Delta)} = (\psi_{n*}\mathbb{F}_n)_{[n]\in\mathrm{Ob}(\Delta)} = \psi_{\bullet*}\mathbb{F}_\bullet$ by I.10.9 ([**Sta22**, 0D70]). Since $X_\bullet^{Y_\bullet} \to X^Y$ forms a hypercovering in $\mathbf{I}_{\overline{\eta}\to\overline{S}}$, the right vertical arrow is an almost isomorphism by I.10.3 and I.8.18. Finally, the left vertical arrow is an isomorphism by the cohomological descent for étale cohomology [**Sta22**, 0DHL]. □

I.11. Faltings' Main p-adic Comparison Theorem: the Relative Case for More General Coefficients

I.11.1. Let $Y \to X$ be a morphism of coherent schemes such that $Y \to X^Y$ is an open immersion. We obtain from I.3.26, I.8.6 and I.7.8 a commutative diagram of sites

$$\begin{array}{ccccc}
(\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a} & Y_{\text{ét}} & & \\
\downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\psi} & \searrow^{\rho} & \\
\mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X} & \xrightarrow{\beta} & Y_{\text{fét}}
\end{array} \tag{I.11.1.1}$$

where $a : (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} \to Y_{\text{ét}}$ and $\rho : Y_{\text{ét}} \to Y_{\text{fét}}$ are defined by the inclusion functors.

Lemma I.11.2. *With the notation in* I.11.1*, for any finite locally constant abelian sheaf* $\mathbb{F}$ *on* $Y_{\text{ét}}$*, the canonical morphism* $\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_* a^{-1}\mathbb{F}$ *is an isomorphism.*

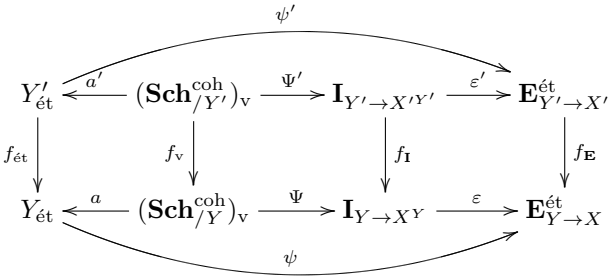
Proof. The base change morphism $\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_* a^{-1}\mathbb{F}$ is the composition of the adjunction morphisms ([**SGA 4**$_{\text{III}}$, XVII.2.1.3])

$$\varepsilon^{-1}\psi_*\mathbb{F} \to \Psi_*\Psi^{-1}(\varepsilon^{-1}\psi_*\mathbb{F}) = \Psi_* a^{-1}(\psi^{-1}\psi_*\mathbb{F}) \to \Psi_* a^{-1}\mathbb{F} \tag{I.11.2.1}$$

which are both isomorphisms by I.3.27.(2) and I.10.3. □

I.11.3. We fix an algebraic closure $\overline{\mathbb{Q}_p}$ of the p-adic number field $\mathbb{Q}_p$ and we denote by $\overline{\mathbb{Z}_p}$ the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. We set $\eta = \mathrm{Spec}(\mathbb{Q}_p)$, $\overline{\eta} = \mathrm{Spec}(\overline{\mathbb{Q}_p})$, $S = \mathrm{Spec}(\mathbb{Z}_p)$, $\overline{S} = \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Remark that $\overline{\mathbb{Q}_p}$ is a pre-perfectoid field with valuation ring $\overline{\mathbb{Z}_p}$ so we are also in the situation I.8.7. Let $f : (Y' \to X') \to (Y \to X)$ be a Cartesian morphism of morphisms of coherent schemes with a Cartesian morphism $(Y \to X^Y) \to (\overline{\eta} \to \overline{S})$ (then, $Y' \to X'^{Y'}$ is Cartesian over $\overline{\eta} \to \overline{S}$ by I.3.19.(4)). Thus, X^Y and $X'^{Y'}$ are objects of $\mathbf{I}_{\overline{\eta}\to\overline{S}}$. Consider the following commutative diagram of sites associated to f.

(I.11.3.1)



I.11.4. Following I.11.3, let $g : (\widetilde{Y} \to \widetilde{X}) \to (Y \to X)$ be a morphisms of coherent schemes such that $\widetilde{Y} \to \widetilde{X}^{\widetilde{Y}}$ is also Cartesian over $\overline{\eta} \to \overline{S}$. We denote by $g' : (\widetilde{Y'} \to \widetilde{X'}) \to (Y' \to X')$ the base change of g by f, and denote by $\widetilde{f} : (\widetilde{Y'} \to \widetilde{X'}) \to (\widetilde{Y} \to \widetilde{X})$ the natural morphism which is Cartesian by base change. Thus, $\widetilde{X}^{\widetilde{Y}}$ and $\widetilde{X'}^{\widetilde{Y'}}$ are also objects of $\mathbf{I}_{\overline{\eta}\to\overline{S}}$. We write the diagram (I.11.3.1) associated $\widetilde{f}$ equipping all labels with tildes.

Lemma I.11.5. *With the notation in* I.11.3 *and* I.11.4*, let* $\mathbb{F}'$ *be a finite locally constant abelian sheaf on* $Y'_{\text{ét}}$ *and we set* $\mathscr{F}' = \Psi'_* a'^{-1}\mathbb{F}'$*. Let* $\widetilde{X}$ *be an object of* $\mathbf{I}_{Y\to X^Y}$*,* $\widetilde{Y} = \overline{\eta} \times_{\overline{S}} \widetilde{X}$*,* $\widetilde{\mathbb{F}'} = g'^{-1}_{\text{ét}}\mathbb{F}'$*, q an integer.*

1. *The sheaf* $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ *on* $\mathbf{I}_{Y\to X^Y}$ *is canonically isomorphic to the sheaf associated to the presheaf* $\widetilde{X} \mapsto H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathbb{F}'})$*.*
2. *The sheaf* $\mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ *on* $\mathbf{I}_{Y\to X^Y}$ *is canonically almost isomorphic to the sheaf associated to the presheaf* $\widetilde{X} \mapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}}, \widetilde{\psi'}_*\widetilde{\mathbb{F}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')$*.*
3. *The canonical morphism* $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} \to (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ *is compatible with the canonical morphisms* $H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathbb{F}'}) \otimes_{\mathbb{Z}} R \xleftarrow{\alpha_1} H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}}, \widetilde{\psi'}_*\widetilde{\mathbb{F}'}) \otimes_{\mathbb{Z}} R \xrightarrow{\alpha_2} H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}}, \widetilde{\psi'}_*\widetilde{\mathbb{F}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}')$*, where* $R = \overline{\mathscr{B}}(\widetilde{Y} \to \widetilde{X})$ (cf.I.10.5.1)*.*

Proof. Let $\widetilde{\mathscr{F}'}$ be the restriction of $\mathscr{F}'$ on $\mathbf{I}_{\widetilde{Y'}\to\widetilde{X'}^{\widetilde{Y'}}}$. We have $\widetilde{\mathscr{F}'} = \widetilde{\Psi'}_*\widetilde{a'}^{-1}\widetilde{\mathbb{F}'}$. We set $\widetilde{\mathbb{L}'} = \widetilde{\psi'}_*\widetilde{\mathbb{F}'}$ which is a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}}$ by I.10.3. Notice that the canonical morphisms $\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'} \to \widetilde{\mathbb{F}'}$ and $\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \to \widetilde{\mathscr{F}'}$ are isomorphisms by I.10.3 and I.11.2 respectively.

(1) It follows from the canonical isomorphisms

$$H^q(I_{\widetilde{Y'}\to\widetilde{X'}^{\widetilde{Y'}}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) \xrightarrow{\gamma_1} H^q_{\text{v}}(\widetilde{Y'}, \widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}) = H^q_{\text{v}}(\widetilde{Y'}, \widetilde{a'}^{-1}\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}) \xleftarrow{\gamma_2} H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}), \tag{I.11.5.1}$$

where γ_1 is induced by the canonical isomorphism $\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \xrightarrow{\sim} \mathrm{R}\widetilde{\Psi'}_*\widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}$ (I.3.27.(2)), and γ_2 is induced by the canonical isomorphism $\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'} \to \mathrm{R}\widetilde{a'}_*\widetilde{a'}^{-1}\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'}$ (I.3.9).

(2) It follows from the canonical almost isomorphism

$$\gamma_3 : H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}}, \widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \longrightarrow H^q(I_{\widetilde{Y'}\to\widetilde{X'}^{\widetilde{Y'}}}, \widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}') \tag{I.11.5.2}$$

which is induced by the canonical almost isomorphism $\widetilde{\mathbb{L}'} \otimes_{\mathbb{Z}} \overline{\mathscr{B}}' \to \mathrm{R}\widetilde{\varepsilon'}_*(\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'} \otimes \mathscr{O}')$ (I.8.14).

(3) Consider the following diagram

(I.11.5.3)

$$\begin{array}{ccccc}
H^q_{\text{ét}}(\widetilde{Y'},\widetilde{\psi'}^{-1}\widetilde{\mathbb{L}'})\otimes R & \xleftarrow{\alpha_1} & H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}},\widetilde{\mathbb{L}'})\otimes R & \xrightarrow{\alpha_2} & H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}},\widetilde{\mathbb{L}'}\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \\
{\scriptstyle\gamma_2\otimes\mathrm{id}_R}\downarrow\wr & & \downarrow & & \downarrow{\scriptstyle\gamma_3} \\
H^q_{\text{v}}(\widetilde{Y'},\widetilde{\Psi'}^{-1}\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'})\otimes R & \xleftarrow[\gamma_1\otimes\mathrm{id}_R]{\sim} & H^q(I_{\widetilde{Y'}\to\widetilde{X'}^{\widetilde{Y'}}},\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'})\otimes R & \longrightarrow & H^q(I_{\widetilde{Y'}\to\widetilde{X'}^{\widetilde{Y'}}},\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}\otimes\mathscr{O}')
\end{array}$$

where the unlabelled vertical arrow is induced by the canonical morphism $\widetilde{\mathbb{L}'}\to \mathrm{R}\widetilde{\varepsilon'}_*\widetilde{\varepsilon'}^{-1}\widetilde{\mathbb{L}'}$, and the unlabelled horizontal arrow is the canonical morphism which induces $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}')\otimes_{\mathbb{Z}}\mathscr{O}\to \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}')$ on $\mathbf{I}_{Y\to X^Y}$ by sheafification. It is clear that the diagram (I.11.5.3) is commutative, which completes the proof. □

I.11.6. We remark that I.11.5 gives a new definition of the relative (resp. absolute) Faltings' comparison morphism without using I.10.8. Following I.11.3, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}'=\Psi'_*a'^{-1}\mathbb{F}'$. We set $\mathbb{L}'=\psi'_*\mathbb{F}'$, which is a finite locally constant abelian sheaf on $\mathbf{E}^{\text{ét}}_{Y'\to X'}$ by I.10.3. Remark that the canonical morphisms $\psi'^{-1}\mathbb{L}'\to\mathbb{F}'$ and $\varepsilon'^{-1}\mathbb{L}'\to\mathscr{F}'$ are isomorphisms by I.10.3 and I.11.2 respectively. We also remark that $\overline{\mathscr{B}}$, $\mathscr{O}$ are flat over $\mathbb{Z}$. The canonical morphisms in the derived category $\mathbf{D}(\overline{\mathscr{B}}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ (cf. I.10.4.2),

$$(\mathrm{R}\psi_*\mathrm{R}f_{\text{ét}*}\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\xleftarrow{\alpha_1}(\mathrm{R}f_{\mathbf{E}*}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\xrightarrow{\alpha_2}\mathrm{R}f_{\mathbf{E}*}(\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'),\tag{I.11.6.1}$$

fit into the following commutative diagram

(I.11.6.2)

$$\begin{array}{ccccc}
\mathrm{R}\psi_*(\mathrm{R}f_{\text{ét}*}\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xleftarrow{\alpha_1} & (\mathrm{R}f_{\mathbf{E}*}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xrightarrow{\alpha_2} & \mathrm{R}f_{\mathbf{E}*}(\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \\
{\scriptstyle\alpha_3}\downarrow\wr & & \downarrow & & \downarrow{\scriptstyle\alpha_4} \\
\mathrm{R}\psi_*(\mathrm{R}a_*\mathrm{R}f_{\text{v}*}\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xleftarrow[\alpha_5]{\sim} & \mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}} & \xrightarrow[\alpha_6]{} & \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}')
\end{array}$$

(1) The morphism α_3 is induced by the canonical isomorphism $\psi'^{-1}\mathbb{L}'\to \mathrm{R}a'_*a'^{-1}(\psi'^{-1}\mathbb{L}')$ by I.3.9, and thus α_3 is an isomorphism.
(2) The morphism α_5 is induced by the canonical isomorphism $\varepsilon'^{-1}\mathbb{L}'\to \mathrm{R}\Psi'_*\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}'$ by I.3.27, and thus α_5 is an isomorphism.
(3) The unlabelled arrow is induced by the canonical morphism $\mathbb{L}\to \mathrm{R}\varepsilon'_*\varepsilon'^{-1}\mathbb{L}$.
(4) The morphism α_4 is induced by the canonical almost isomorphism $\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'\to \mathrm{R}\varepsilon'_*(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}')$ by I.8.14, and thus α_4 is an almost isomorphism.
(5) The morphism α_6 is the composition of

$$\mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O})\tag{I.11.6.3}$$

$$\text{with } \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O})\longrightarrow \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}').\tag{I.11.6.4}$$

In conclusion, the arrows α_3, α_5, α_6 and α_4 induce an arrow

$$\alpha_4^{-1}\circ\alpha_6\circ\alpha_5^{-1}\circ\alpha_3:\mathrm{R}\psi_*(\mathrm{R}f_{\text{ét}*}\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\overline{\mathscr{B}}\longrightarrow \mathrm{R}f_{\mathbf{E}*}(\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}')\tag{I.11.6.5}$$

in the derived category of almost $\overline{\mathbb{Z}_p}$-modules on $\mathbf{E}^{\text{ét}}_{Y\to X}$ (I.5.7). Remark that we don't assume that α_1 is an isomorphism here. We also call (I.11.6.5) the *relative Faltings' comparison morphism*. Indeed, if α_1 is an isomorphism, then the relative Faltings' comparison morphism (I.10.6.1) induces (I.11.6.5) in $\mathbf{D}(\overline{\mathbb{Z}_p}^{\text{al}}\text{-}\mathbf{Mod})$ due to the commutativity of the diagram (I.11.6.2).

If X is the spectrum of an absolutely integrally closed valuation ring A and if $Y=\overline{\eta}\times_{\overline{S}}X$, then applying the functor $\mathrm{R}\Gamma(Y\to X,-)$ on (I.11.6.2) we obtain the natural morphisms in the derived category $\mathbf{D}(A\text{-}\mathbf{Mod})$ by I.7.9,

(I.11.6.6)

$$\begin{array}{ccccc}
\mathrm{R}\Gamma(Y'_{\text{ét}},\psi'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xleftarrow{\alpha_1} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xrightarrow{\alpha_2} & \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\mathbb{L}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \\
{\scriptstyle\alpha_3}\downarrow\wr & & \downarrow & & \downarrow{\scriptstyle\alpha_4} \\
\mathrm{R}\Gamma((\mathbf{Sch}^{\text{coh}}_{/Y'})_{\text{v}},\Psi'^{-1}\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xleftarrow[\alpha_5]{\sim} & \mathrm{R}\Gamma(\mathbf{I}_{Y'\to X'^{Y'}},\varepsilon'^{-1}\mathbb{L}')\otimes^{\mathrm{L}}_{\mathbb{Z}}A & \xrightarrow[\alpha_6]{} & \mathrm{R}\Gamma(\mathbf{I}_{Y'\to X'^{Y'}},\varepsilon'^{-1}\mathbb{L}'\otimes_{\mathbb{Z}}\mathscr{O}')
\end{array}$$

The arrows α_3, α_5, α_6 and α_4 induce an arrow

$$\alpha_4^{-1}\circ\alpha_6\circ\alpha_5^{-1}\circ\alpha_3 : \mathrm{R}\Gamma(Y'_{\text{ét}},\mathbb{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{ét}}_{Y'\to X'},\psi'_*\mathbb{F}'\otimes_{\mathbb{Z}}\overline{\mathscr{B}}') \tag{I.11.6.7}$$

in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}^{\mathrm{al}}\text{-}\mathbf{Mod})$ of almost $\overline{\mathbb{Z}_p}$-modules (I.5.7). We also call (I.11.6.7) the *absolute Faltings' comparison morphism.*

Lemma I.11.7. *With the notation in* I.11.3*, let $\mathbb{F}'$ be a finite locally constant abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_*a'^{-1}\mathbb{F}'$. Assume that $X'\to X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}\longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}') \tag{I.11.7.1}$$

is an almost isomorphism.

Proof. Following I.11.5, consider the following presheaves on $\mathbf{I}_{Y\to X^Y}$ for each integer q:

$$\mathcal{H}^q_1 : \widetilde{X}\longmapsto H^q_{\text{ét}}(\widetilde{Y'},\widetilde{\mathbb{F}'})\otimes_{\mathbb{Z}}\overline{\mathscr{B}}(\widetilde{Y}\to\widetilde{X}), \tag{I.11.7.2}$$

$$\mathcal{H}^q_2 : \widetilde{X}\longmapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}},\widetilde{\psi'}_*\widetilde{\mathbb{F}'})\otimes_{\mathbb{Z}}\overline{\mathscr{B}}(\widetilde{Y}\to\widetilde{X}), \tag{I.11.7.3}$$

$$\mathcal{H}^q_3 : \widetilde{X}\longmapsto H^q(\mathbf{E}^{\text{ét}}_{\widetilde{Y'}\to\widetilde{X'}},\widetilde{\psi'}_*\widetilde{\mathbb{F}'}\otimes_{\mathbb{Z}}\overline{\mathscr{B}}'), \tag{I.11.7.4}$$

They satisfy the limit-preserving condition I.3.25.(ii) by I.7.12, [**SGA 4**$_{\text{II}}$, VII.5.6] and [**SGA 4**$_{\text{II}}$, VI 8.5.9, 8.7.3]. Moreover, if $\widetilde{X}=\mathrm{Spec}(A)$ where A is an absolutely integrally closed valuation ring with p nonzero in A, then the canonical morphisms

$$\mathcal{H}^q_1(\mathrm{Spec}(A))\leftarrow\mathcal{H}^q_2(\mathrm{Spec}(A))\to\mathcal{H}^q_3(\mathrm{Spec}(A)) \tag{I.11.7.5}$$

are an isomorphism and an almost isomorphism by I.10.17. Thus, the canonical morphisms $\mathcal{H}^q_1\leftarrow\mathcal{H}^q_2\to\mathcal{H}^q_3$ induce an isomorphism and an almost isomorphism of their sheafifications by I.3.25. The conclusion follows from I.11.5. □

Lemma I.11.8. *Let $Y\to X$ be an open immersion of coherent schemes, $Y'\to Y$ a finite morphism of finite presentation. Then, there exists a finite morphism $X'\to X$ of finite presentation whose base change by $Y\to X$ is $Y'\to Y$.*

Proof. Firstly, assume that X is Noetherian. We have $Y'=Y\times_X X^Y$ by I.3.19.(4). We write $X^Y=\mathrm{Spec}_X(\mathcal{A})$ where $\mathcal{A}$ is an integral quasi-coherent $\mathcal{O}_X$-algebra on X, and we write $\mathcal{A}$ as a filtered colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras $\mathcal{A}=\mathrm{colim}\,\mathcal{A}_\alpha$ ([**Sta22**, 0817]). Let $\mathcal{B}_\alpha$ be the restriction of $\mathcal{A}_\alpha$ to Y. Then, $\mathcal{B}=\mathrm{colim}\,\mathcal{B}_\alpha$ is a filtered colimit of finite quasi-coherent $\mathcal{O}_Y$-algebras with injective transition morphisms. Since $Y'=\mathrm{Spec}_Y(\mathcal{B})$ is finite over Y, there exists an index α_0 such that $Y'=\mathrm{Spec}_Y(\mathcal{B}_{\alpha_0})$. Therefore, $X'=\mathrm{Spec}_X(\mathcal{A}_{\alpha_0})$ meets our requirements.

In general, we write X as a cofiltered limit of coherent schemes of finite type over $\mathbb{Z}$ with affine transition morphisms $X=\lim_{\lambda\in\Lambda}X_\lambda$ ([**Sta22**, 01ZA]). Since $Y\to X$ is an open immersion of finite presentation, using [**EGA** IV$_3$, 8.8.2, 8.10.5] there exists an index $\lambda_0\in\Lambda$, an open immersion $Y_{\lambda_0}\to X_{\lambda_0}$ and a finite morphism $Y'_{\lambda_0}\to Y_{\lambda_0}$ such that the base change of the morphisms $Y'_{\lambda_0}\to Y_{\lambda_0}\to X_{\lambda_0}$ by $X\to X_{\lambda_0}$ are the morphisms $Y'\to Y\to X$. By the first paragraph, there exists a finite morphism $X'_{\lambda_0}\to X_{\lambda_0}$ of finite presentation such that $Y'_{\lambda_0}=Y_{\lambda_0}\times_{X_{\lambda_0}}X'_{\lambda_0}$. We see that the base change $X'\to X$ of $X'_{\lambda_0}\to X_{\lambda_0}$ by $X\to X_{\lambda_0}$ meets our requirements. □

Lemma I.11.9. *With the notation in* I.11.3*, let $g:Y''\to Y'$ be a finite morphism of finite presentation, $\mathbb{F}''$ a finite locally constant abelian sheaf on $Y''_{\text{ét}}$ and we set $\mathscr{F}'=\Psi'_*a'^{-1}(g_{\text{ét}*}\mathbb{F}'')$. Assume that $X'\to X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}')\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathscr{O}\longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}'\otimes_{\mathbb{Z}}\mathscr{O}') \tag{I.11.9.1}$$

is an almost isomorphism.

Proof. There exists a Cartesian morphism $g:(Y''\to X'')\to(Y'\to X^Y\times_X X')$ of open immersions of coherent schemes such that $X''\to X^Y\times_X X'$ is finite and of finite presentation by I.11.8. Consider the diagram (I.11.3.1) associated to g:

$$\begin{array}{ccccc} Y''_{\text{ét}} & \xleftarrow{a''} & (\mathbf{Sch}^{\mathrm{coh}}_{/Y''})_{\mathrm{v}} & \xrightarrow{\Psi''} & \mathbf{I}_{Y''\to X''Y''} \\ \downarrow{\scriptstyle g_{\text{ét}}} & & \downarrow{\scriptstyle g_{\mathrm{v}}} & & \downarrow{\scriptstyle g_{\mathbf{I}}} \\ Y'_{\text{ét}} & \xleftarrow{a'} & (\mathbf{Sch}^{\mathrm{coh}}_{/Y'})_{\mathrm{v}} & \xrightarrow{\Psi'} & \mathbf{I}_{Y'\to X'Y'} \end{array} \tag{I.11.9.2}$$

We set $\mathscr{G}'' = \Psi''_* a''^{-1}\mathbb{F}''$. The base change morphism $a'^{-1}g_{\text{ét}*} \to g_{\text{v}*}a''^{-1}$ induces a canonical isomorphism $\mathscr{F}' \xrightarrow{\sim} g_{\mathbf{I}*}\mathscr{G}''$ by I.3.10. Moreover, the canonical morphism $g_{\mathbf{I}*}\mathscr{G}'' \to \mathrm{R}g_{\mathbf{I}*}\mathscr{G}''$ is an isomorphism by I.11.5.(1) and I.3.25, since $g : Y'' \to Y'$ is finite ([**SGA 4**$_{\text{II}}$, VIII.5.6]). By applying I.11.7 to g and $\mathbb{F}''$, the canonical morphism

$$(\mathrm{R}g_{\mathbf{I}*}\mathscr{G}'') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}' \longrightarrow \mathrm{R}g_{\mathbf{I}*}(\mathscr{G}'' \otimes_{\mathbb{Z}} \mathscr{O}'') \tag{I.11.9.3}$$

is an almost isomorphism. Let h be the composition of $(Y'' \to X'') \to (Y' \to X^Y \times_X X') \to (Y \to X^Y)$. Note that $X'' \to X^Y$ is also proper of finite presentation. By applying I.11.7 to h and $\mathbb{F}''$, the canonical morphism

$$(\mathrm{R}h_{\mathbf{I}*}\mathscr{G}'') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}h_{\mathbf{I}*}(\mathscr{G}'' \otimes_{\mathbb{Z}} \mathscr{O}'') \tag{I.11.9.4}$$

is an almost isomorphism. It is clear that $h_{\mathbf{I}} = f_{\mathbf{I}} \circ g_{\mathbf{I}}$. The conclusion follows from the canonical isomorphism $\mathscr{F}' \to \mathrm{R}g_{\mathbf{I}*}\mathscr{G}''$ and the canonical almost isomorphisms (I.11.9.3) and (I.11.9.4). □

Lemma I.11.10. *With the notation in* I.11.3*, let $\mathcal{F}'$ be a constructible abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$. Assume that $X' \to X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{I.11.10.1}$$

is an almost isomorphism.

Proof. We prove by induction on an integer q that the canonical morphism $(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} \to \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}')$ is an almost isomorphism. It holds trivially for each $q \leq -1$. Notice that there exists a finite morphism $g : Y'' \to Y'$ of finite presentation, a finite locally constant abelian sheaf $\mathbb{F}''$ on $Y''_{\text{ét}}$ and an injective morphism $\mathcal{F}' \to g_{\text{ét}*}\mathbb{F}''$ by [**Sta22**, 09Z7] (cf. [**SGA 4**$_{\text{III}}$, IX.2.14]). Let $\mathcal{G}'$ be the quotient of $\mathcal{F}' \to g_{\text{ét}*}\mathbb{F}''$, which is also a constructible abelian sheaf on $Y'_{\text{ét}}$ since $g_{\text{ét}*}\mathbb{F}''$ is so ([**Sta22**, 095R, 03RZ]). The exact sequence $0 \to \mathcal{F}' \to g_{\text{ét}*}\mathbb{F}'' \to \mathcal{G}' \to 0$ induces an exact sequence by I.3.27.(1),

$$0 \longrightarrow \Psi'_* a'^{-1}\mathcal{F}' \longrightarrow \Psi'_* a'^{-1}(g_{\text{ét}*}\mathbb{F}'') \longrightarrow \Psi'_* a'^{-1}\mathcal{G}' \longrightarrow 0. \tag{I.11.10.2}$$

We set $\mathscr{H}' = \Psi'_* a'^{-1}(g_{\text{ét}*}\mathbb{F}'')$ and $\mathscr{G}' = \Psi'_* a'^{-1}\mathcal{G}'$. Then, we obtain a morphism of long exact sequences

(I.11.10.3)

$$\begin{array}{ccccccccc}
(\mathrm{R}^{q-1}f_{\mathbf{I}*}\mathscr{H}') \otimes \mathscr{O} & \longrightarrow & (\mathrm{R}^{q-1}f_{\mathbf{I}*}\mathscr{G}') \otimes \mathscr{O} & \longrightarrow & (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes \mathscr{O} & \longrightarrow & (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{H}') \otimes \mathscr{O} & \longrightarrow & (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{G}') \otimes \mathscr{O} \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_5} \\
\mathrm{R}^{q-1}f_{\mathbf{I}*}(\mathscr{H}' \otimes \mathscr{O}') & \longrightarrow & \mathrm{R}^{q-1}f_{\mathbf{I}*}(\mathscr{G}' \otimes \mathscr{O}') & \longrightarrow & \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes \mathscr{O}') & \longrightarrow & \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{H}' \otimes \mathscr{O}') & \longrightarrow & \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{G}' \otimes \mathscr{O}')
\end{array}$$

Notice that γ_1 and γ_2 are almost isomorphisms by induction, and that γ_4 is an almost isomorphism by I.11.9. Thus, applying the 5-lemma ([**Sta22**, 05QA]) in the abelian category of almost $\overline{\mathbb{Z}_p}$-modules over $\mathbf{I}_{Y \to X^Y}$, we see that γ_3 is almost injective. Since $\mathcal{F}'$ is an arbitrary constructible abelian sheaf, the morphism γ_5 is also almost injective. Thus, γ_3 is an almost isomorphism. □

Theorem I.11.11. *With the notation in* I.11.3*, let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$ and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$. Assume that $X' \to X$ is proper of finite presentation. Then, the canonical morphism*

$$(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O} \longrightarrow \mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{I.11.11.1}$$

is an almost isomorphism in the derived category $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod}_{\mathbf{I}_{Y \to X^Y}})$ (I.5.7).

Proof. We write $\mathcal{F}'$ as a filtered colimit of constructible abelian sheaves $\mathcal{F}' = \operatorname{colim}_{\lambda \in \Lambda} \mathcal{F}'_\lambda$ ([**Sta22**, 03SA], cf. [**SGA 4**$_{\text{III}}$, IX.2.7.2]). We set $\mathscr{F}'_\lambda = \Psi'_* a'^{-1}\mathcal{F}_\lambda$. We have $\mathscr{F}' = \operatorname{colim}_{\lambda \in \Lambda} \mathscr{F}'_\lambda$ by [**SGA 4**$_{\text{II}}$, VI.5.1] whose conditions are satisfied since each object in each concerned site is quasi-compact. Moreover, for each integer q, we have

$$(\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}') \otimes_{\mathbb{Z}} \mathscr{O} = \operatorname*{colim}_{\lambda \in \Lambda} (\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'_\lambda) \otimes_{\mathbb{Z}} \mathscr{O}, \tag{I.11.11.2}$$

$$\mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') = \operatorname*{colim}_{\lambda \in \Lambda} \mathrm{R}^q f_{\mathbf{I}*}(\mathscr{F}'_\lambda \otimes_{\mathbb{Z}} \mathscr{O}'). \tag{I.11.11.3}$$

The conclusion follows from I.11.10. □

Lemma I.11.12. *With the notation in* I.11.3 *and* I.11.4*, let $\mathcal{F}'$ be a torsion abelian sheaf on $Y'_{\text{ét}}$, $\mathcal{H} = \mathrm{R}f_{\text{ét}*}\mathcal{F}'$, and we set $\mathscr{F}' = \Psi'_* a'^{-1}\mathcal{F}'$, $\mathscr{H} = \mathrm{R}\Psi_* a^{-1}\mathcal{H}$. Let $\widetilde{X}$ be an object of $\mathbf{I}_{Y \to X^Y}$, $\widetilde{Y} = \overline{\eta} \times_{\overline{S}} \widetilde{X}$, $\widetilde{\mathcal{F}'} = g'^{-1}_{\text{ét}}\mathcal{F}'$.*

(1) *The sheaf* $\mathrm{R}^q f_{\mathbf{I}*}\mathscr{F}'$ *is canonically isomorphic to the presheaf* $\widetilde{X} \mapsto H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathcal{F}'})$ *for each integer* q.
(2) *If* $Y' \to Y$ *is proper, then there exists a canonical isomorphism* $\mathscr{H} \xrightarrow{\sim} \mathrm{R}f_{\mathbf{I}*}\mathscr{F}'$.

Proof. Note that the canonical morphism $\mathscr{F}' \to \mathrm{R}\Psi'_* a'^{-1}\mathcal{F}'$ is an isomorphism by I.3.27.(1). Thus, $\mathrm{R}f_{\mathbf{I}*}\mathscr{F}' = \mathrm{R}(\Psi \circ f_{\mathrm{v}})_* a'^{-1}\mathcal{F}'$, whose q-th cohomology is the sheaf associated to the presheaf $\widetilde{X} \mapsto H^q_{\mathrm{v}}(\widetilde{Y'}, \widetilde{a'}^{-1}\widetilde{\mathcal{F}'}) = H^q_{\text{ét}}(\widetilde{Y'}, \widetilde{\mathcal{F}'})$ by I.3.9, and thus (1) follows. If $Y' \to Y$ is proper, then the base change morphism $a^{-1}\mathrm{R}f_{\text{ét}*} \to \mathrm{R}f_{\mathrm{v}*}a'^{-1}$ induces an isomorphism $a^{-1}\mathcal{H} \xrightarrow{\sim} \mathrm{R}f_{\mathrm{v}*}a'^{-1}\mathcal{F}'$ by I.3.10, and thus (2) follows. □

Theorem I.11.13. *With the notation in* I.11.3*, let* $\mathbb{F}'$ *be a finite locally constant abelian sheaf on* $Y'_{\text{ét}}$. *Assume that*

(i) *the morphism* $X' \to X$ *is proper of finite presentation, and that*
(ii) *the sheaf* $\mathrm{R}^q f_{\text{ét}*}\mathbb{F}'$ *is finite locally constant for each integer* q *and nonzero for finitely many* q*, and that*
(iii) *we have* $\mathrm{R}^q\psi_*\mathbb{H} = 0$ *(resp.* $\mathrm{R}^q\psi'_*\mathbb{H} = 0$*) for any finite locally constant abelian sheaf* $\mathbb{H}$ *on* $Y_{\text{ét}}$ *(resp.* $Y'_{\text{ét}}$*) and any integer* $q > 0$.

Then, the relative Faltings' comparison morphism associated to f *and* $\mathbb{F}'$ (I.10.6.1) *(which exists by* (iii)*) is an almost isomorphism in the derived category* $\mathbf{D}(\overline{\mathbb{Z}_p}\text{-}\mathbf{Mod}_{\mathbf{E}^{\text{ét}}_{Y\to X}})$ (I.5.7)*, and it induces an almost isomorphism*

$$(\psi_*\mathrm{R}^q f_{\text{ét}*}\mathbb{F}') \otimes_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}^q f_{\mathbf{E}*}(\psi'_*\mathbb{F}' \otimes_{\mathbb{Z}} \overline{\mathscr{B}}') \tag{I.11.13.1}$$

of $\overline{\mathbb{Z}_p}$*-modules for each integer* q.

Proof. We follow the discussion of I.11.6 and set $\mathscr{F}' = \Psi'_* a'^{-1}\mathbb{F}'$. The canonical morphism (I.11.6.4)

$$\mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \longrightarrow \mathrm{R}\varepsilon_*\mathrm{R}f_{\mathbf{I}*}(\mathscr{F}' \otimes_{\mathbb{Z}} \mathscr{O}') \tag{I.11.13.2}$$

is an almost isomorphism by I.11.7. It remains to show that the canonical morphism (I.11.6.3)

$$\mathrm{R}\varepsilon_*(\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*((\mathrm{R}f_{\mathbf{I}*}\mathscr{F}') \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \tag{I.11.13.3}$$

is also an almost isomorphism. With the notation in I.11.12 by taking $\mathcal{F}' = \mathbb{F}'$, the complex $\mathcal{H}$ is a bounded complex whose cohomologies $H^q(\mathcal{H})$ are finite locally constant abelian sheaves by condition (ii). Consider the commutative diagram (I.11.1.1),

$$\begin{array}{ccc} (\mathbf{Sch}^{\mathrm{coh}}_{/Y})_{\mathrm{v}} & \xrightarrow{a} & Y_{\text{ét}} \\ \Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \psi} \\ \mathbf{I}_{Y\to X^Y} & \xrightarrow{\varepsilon} & \mathbf{E}^{\text{ét}}_{Y\to X} \end{array} \tag{I.11.13.4}$$

We set $\mathcal{L} = \mathrm{R}\psi_*\mathcal{H}$. Then, $H^q(\mathcal{L}) = \psi_* H^q(\mathcal{H})$ by Cartan-Leray spectral sequence and condition (iii). Hence, $\mathcal{L}$ is a bounded complex of abelian sheaves whose cohomologies are finite locally constant by I.10.3 so that the canonical morphism

$$\mathcal{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathcal{L} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathscr{O}) \tag{I.11.13.5}$$

is an almost isomorphism by I.8.15.

On the other hand, $H^q(\mathscr{H}) = \Psi_* a^{-1} H^q(\mathcal{H})$ by Cartan-Leray spectral sequence and I.3.27.(1). Thus, the base change morphism $\varepsilon^{-1}\mathrm{R}\psi_* \to \mathrm{R}\Psi_* a^{-1}$ induces an isomorphism $\varepsilon^{-1}\mathcal{L} \xrightarrow{\sim} \mathscr{H}$ by I.11.2. Moreover, the canonical morphism $\mathcal{L} \to \mathrm{R}\varepsilon_*\varepsilon^{-1}\mathcal{L} = \mathrm{R}\varepsilon_*\mathscr{H} = \mathrm{R}\psi_*\mathrm{R}a_* a^{-1}\mathcal{H}$ is an isomorphism by I.3.9. Thus, the canonical morphism

$$(\mathrm{R}\varepsilon_*\varepsilon^{-1}\mathcal{L}) \otimes^{\mathrm{L}}_{\mathbb{Z}} \overline{\mathscr{B}} \longrightarrow \mathrm{R}\varepsilon_*(\varepsilon^{-1}\mathcal{L} \otimes_{\mathbb{Z}} \mathscr{O}) \tag{I.11.13.6}$$

is an almost isomorphism by (I.11.13.5). In conclusion, (I.11.13.3) is an almost isomorphism by (I.11.13.6) and by the canonical isomorphisms $\varepsilon^{-1}\mathcal{L} \xrightarrow{\sim} \mathscr{H} \xrightarrow{\sim} \mathrm{R}f_{\mathbf{I}*}\mathscr{F}'$. □

Remark I.11.14. We give two concrete situations where the conditions in I.11.13 are satisfied:

(1) Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\mathbb{Z}_p$ in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, $X' \to X$ a proper and finitely presented morphism of coherent $\overline{\mathbb{Z}_p}$-schemes, $Y' \to Y$ the base change of $X' \to X$ by $\mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\overline{\mathbb{Z}_p})$. Assume that $Y' \to Y$ is smooth. Then, the condition (ii) is guaranteed by [**SGA 4**$_{\mathrm{III}}$, XVI.2.2 and XVII.5.2.8.1], and the condition (iii) is guaranteed by I.10.8.

(2) Let $\mathcal{O}_K$ be a strictly Henselian discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p, $\overline{K}$ an algebraic closure of K, $X' \to X$ a proper morphism of $\mathcal{O}_K$-schemes of finite type, $Y' \to Y$ the base change of $X' \to X$ by $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(\mathcal{O}_K)$. Assume that $Y' \to Y$ is smooth. Then, the condition (ii) is guaranteed by [**SGA 4**$_{\mathrm{III}}$, XVI.2.2 and XVII.5.2.8.1], and the condition (iii) is guaranteed by I.10.7.

I.12. A Local Version of the Relative Hodge-Tate Filtration

I.12.1. Let K be a complete discrete valuation field of characteristic 0 with valuation ring $\mathcal{O}_K$ whose residue field is algebraically closed (a condition required by [**AG20**, 4.1.3, 5.1.3]) of characteristic $p > 0$, $\overline{K}$ an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\eta = \mathrm{Spec}(K)$, $\overline{\eta} = \mathrm{Spec}(\overline{K})$, $S = \mathrm{Spec}(\mathcal{O}_K)$, $\overline{S} = \mathrm{Spec}(\mathcal{O}_{\overline{K}})$. Let $(f, g) : (X'^{\triangleright} \to X') \to (X^{\circ} \to X)$ be a morphism of adequate open immersions of schemes (I.10.11) over $\eta \to S$ such that $g : X' \to X$ is proper and that the induced morphism $(X', \mathscr{M}_{X'^{\triangleright} \to X'}) \to (X, \mathscr{M}_{X^{\circ} \to X})$ is smooth and saturated. We are in fact in the situation I.10.12 but with a slightly different notation.

Consider the morphisms of sites defined in I.7.34:

$$X'^{\triangleright,\mathbb{N}}_{\overline{\eta},\text{ét}} \xrightarrow{\breve{f}_{\overline{\eta},\text{ét}}} X^{\circ,\mathbb{N}}_{\overline{\eta},\text{ét}} \xrightarrow{\breve{\psi}} \mathbf{E}^{\text{ét},\mathbb{N}}_{X^{\circ}_{\overline{\eta}} \to X} \xrightarrow{\breve{\sigma}} X^{\mathbb{N}}_{\text{ét}}. \tag{I.12.1.1}$$

We consider $\breve{\sigma}$ as a morphism of ringed sites $\breve{\sigma} : (\mathbf{E}^{\text{ét},\mathbb{N}}_{X^{\circ}_{\overline{\eta}} \to X}, \breve{\overline{\mathscr{B}}}) \to (X^{\mathbb{N}}_{\text{ét}}, \breve{\mathcal{O}}_{X_{\text{ét}}})$, and $\breve{f}_{\overline{\eta},\text{ét}}$ as a morphism of ringed sites $\breve{f}_{\overline{\eta},\text{ét}} : (X'^{\triangleright,\mathbb{N}}_{\overline{\eta},\text{ét}}, \breve{\mathbb{Z}}_p) \to (X^{\circ,\mathbb{N}}_{\overline{\eta},\text{ét}}, \breve{\mathbb{Z}}_p)$, where $\breve{\mathbb{Z}}_p = (\mathbb{Z}/p^n\mathbb{Z})_{n\geq 0}$.

Theorem I.12.2 ([**AG20**, 6.7.5, 6.7.10, 6.7.13], III.8)**.** *Under the assumptions in* I.12.1 *and with the same notation, for each integer $n \geq 0$, there is a canonical G_K-equivariant finite decreasing filtration $(\mathrm{Fil}^q)_{q\in\mathbb{Z}}$ on the $\breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$-module $\breve{\psi}_*(\mathrm{R}^n \breve{f}_{\overline{\eta},\text{ét}*}(\breve{\mathbb{Z}}_p)) \otimes_{\mathbb{Z}_p} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}}$ (cf.* I.2.6*) and a canonical G_K-equivariant isomorphism for each $q \in \mathbb{Z}$,*

$$\mathrm{Gr}^q(\breve{\psi}_*(\mathrm{R}^n \breve{f}_{\overline{\eta},\text{ét}*}(\breve{\mathbb{Z}}_p)) \otimes_{\breve{\mathbb{Z}}_p} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}}) \cong \breve{\sigma}^*(\mathrm{R}^q g_*(\Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)}) \otimes_{\mathcal{O}_{X_{\text{ét}}}} \breve{\mathcal{O}}_{X_{\text{ét}}}(q-n))_{\mathbb{Q}}, \tag{I.12.2.1}$$

where Gr^q *denotes the graded piece* $\mathrm{Fil}^q/\mathrm{Fil}^{q+1}$.

We call this filtration the *relative Hodge-Tate filtration* of the morphism $(f, g) : (X'^{\triangleright} \to X') \to (X^{\circ} \to X)$.

Remark I.12.3. We keep the notation and assumptions in I.12.1 and I.12.2.

(1) If we set $\mathrm{R}^n \breve{f}_{\overline{\eta},\text{ét}*}(\breve{\mathbb{Z}}_p) = \breve{\mathbb{L}}^{(n)} = (\mathbb{L}^{(n)}_k)_{k\geq 0}$ where $\mathbb{L}^{(n)}_k = \mathrm{R}^n f_{\overline{\eta},\text{ét}*}(\mathbb{Z}/p^k\mathbb{Z})$, then $\mathbb{L}^{(n)}_k$ is a finite locally constant abelian sheaf on $X^{\circ}_{\overline{\eta},\text{ét}}$ ([**AG20**, 2.2.25]), and the inverse system $(\mathbb{L}^{(n)}_k)_{k\geq 0}$ is Artin-Rees p-adic ([**Fu15**, 10.1.18.(iii)]).

(2) The $\mathcal{O}_X$-module $\mathcal{M}^{q,n-q} = \mathrm{R}^q g_*(\Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)})$ is coherent and its restriction to X_η is locally free ([**IKN05**, 7.2], cf. the proof of [**AG20**, 6.7.13]).

I.12.4. Under the assumptions in I.12.1, I.12.2, I.12.3 and with the same notation, assume further that $X = \mathrm{Spec}(R)$ is affine. We remark that R is p-torsion free ([**AGT16**, II.6.3.(ii)]). Let $V \to U$ be an object of $\mathbf{E}^{\text{proét}}_{X^{\circ}_{\overline{\eta}} \to X}$ satisfying the following conditions:

(1) The morphism $V \to U$ is Faltings acyclic (cf. I.8.1, I.8.24).

(2) For any integers $n \geq 0$ and $k \geq 0$, the pullback $\mathbb{L}^{(n)}_k|_{V_{\text{ét}}}$ is constant with value $H^n_{\text{ét},k}$.

We denote by A the $\mathcal{O}_K$-algebra $\overline{\mathscr{B}}(V \to U)$ (i.e. $U^V = \mathrm{Spec}(A)$), and we set $H^n_{\text{ét}} = \lim_{k\to\infty} H^n_{\text{ét},k}$.

Remark I.12.5. Let $\overline{x}$ be a geometric point of V. Then, there is a natural isomorphism $H^n_{\text{ét},k} \cong H^n_{\text{ét}}(X^{\triangleright}_{\overline{x}}, \mathbb{Z}/p^k\mathbb{Z})$ ([**AG20**, 2.2.25]). We remark that $H^n_{\text{ét}}(X^{\triangleright}_{\overline{x}}, \mathbb{Z}_p) = \lim_{k\to\infty} H^n_{\text{ét}}(X^{\triangleright}_{\overline{x}}, \mathbb{Z}/p^k\mathbb{Z})$ is a finitely generated $\mathbb{Z}_p$-module (thus so is $H^n_{\text{ét}}$), and that the morphism of inverse systems of abelian groups

$$(H^n_{\text{ét}}(X^{\triangleright}_{\overline{x}}, \mathbb{Z}_p)/p^k H^n_{\text{ét}}(X^{\triangleright}_{\overline{x}}, \mathbb{Z}_p))_{k\geq 0} \longrightarrow (H^n_{\text{ét},k})_{k\geq 0} \tag{I.12.5.1}$$

is an Artin-Rees isomorphism, by which we mean its kernel and cokernel are Artin-Rees zero (cf. I.12.3.(1) and [**Fu15**, 10.1.4]).

I.12.6. Following I.12.4, notice that $U^{\circ} = X^{\circ} \times_X U$ is affine and geometrically normal over K and that V is affine and normal, since X° is smooth over K. We assume further that the following conditions hold:

(3) The scheme V is integral and lies over a connected component $U^{\circ}_{\star}$ of U°.

(4) The function field $\mathcal{L}$ of V is a Galois extension of the function field $\mathcal{K}$ of $U^\circ_\star$ with Galois group Γ.

Let $U^\circ_{\overline{\eta},\star}$ be the connected component of $U^\circ_{\overline{\eta}}$ over which V lies. Its function field is the composite $\overline{K}\mathcal{K}$ of $\overline{K}$ and $\mathcal{K}$ in $\mathcal{L}$, which is Galois over $\mathcal{K}$ whose Galois group identifies with the closed subgroup G_L of G_K where L is the algebraic closure of K in $\mathcal{K}$. We denote by Δ the Galois group of $\mathcal{L}$ over $\overline{K}\mathcal{K}$. It is clear that $G_L = \Gamma/\Delta$.

(I.12.6.1)
$$\begin{array}{ccc} V & & \\ \downarrow{\scriptstyle \Delta} & & \\ U^\circ_{\overline{\eta},\star} & \longrightarrow & \mathrm{Spec}(\overline{K}) \\ \downarrow{\scriptstyle G_L} & & \downarrow{\scriptstyle G_K} \\ U^\circ_\star & \longrightarrow & \mathrm{Spec}(K) \end{array} \qquad (\Gamma \text{ acts on } V)$$

Since V is the integral closure of $U^\circ_\star$ in $\mathcal{L}$, the canonical homomorphism of groups

(I.12.6.2)
$$\mathrm{Aut}_{U^\circ_\star}(V)^{\mathrm{op}} \longrightarrow \mathrm{Aut}_{\mathcal{K}}(\mathcal{L}) = \Gamma$$

is an isomorphism. In particular, Γ acts naturally on V on the right. For $\gamma \in \Gamma$ with image $u \in G_K$, we denote by $f_\gamma : V \to V$ the right action of γ on V, and for any K-scheme Z, we denote by $f_u : Z_{\overline{\eta}} \to Z_{\overline{\eta}}$ the base change of the automorphism of $\overline{\eta}$ induced by u. There is a commutative diagram

(I.12.6.3)
$$\begin{array}{ccc} V & \xrightarrow{f_\gamma} & V \\ \downarrow & & \downarrow \\ U^\circ_{\overline{\eta}} & \xrightarrow{f_u} & U^\circ_{\overline{\eta}} \end{array}$$

The natural isomorphism (induced by the base change)

(I.12.6.4)
$$f^{-1}_{\gamma,\text{ét}}(\mathbb{L}^{(n)}_k|_{V_{\text{ét}}}) \xrightarrow{\sim} \mathbb{L}^{(n)}_k|_{V_{\text{ét}}}$$

defines a natural action of Γ on $H^n_{\text{ét},k}$, and thus an action on $H^n_{\text{ét}} = \lim_k H^n_{\text{ét},k}$. On the other hand, Γ also acts naturally on A as $\mathrm{Spec}(A)$ is the integral closure of U in $\mathcal{L}$, and thus acts on the Tate twist $A(i)$ via the map $\Gamma \to G_K$.

Theorem I.12.7. *Under the assumptions in* I.12.4 *and with the same notation, for each integer $n \geq 0$, there is a canonical finite decreasing filtration $(\mathrm{fil}^q)_{q\in\mathbb{Z}}$ on $H^n_{\text{ét}} \otimes_{\mathbb{Z}_p} \widehat{A}[1/p]$ and a canonical isomorphism for each $q \in \mathbb{Z}$,*

(I.12.7.1)
$$\mathrm{gr}^q(H^n_{\text{ét}} \otimes_{\mathbb{Z}_p} \widehat{A}[\frac{1}{p}]) \cong H^q(X', \Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)}) \otimes_R \widehat{A}[\frac{1}{p}](q-n),$$

where gr^q denotes the graded piece $\mathrm{fil}^q/\mathrm{fil}^{q+1}$. Moreover, under the assumptions in I.12.6 *and with the same notation, the filtration $(\mathrm{fil}^q)_{q\in\mathbb{Z}}$ and the isomorphisms* (I.12.7.1) *are Γ-equivariant.*

Proof. We set $Y = X^\circ_{\overline{\eta}}$. We start from the filtration of I.12.2. Consider the natural morphism of ringed sites (I.7.34.2) $\breve{\nu} : (\mathbf{E}^{\text{proét},\mathbb{N}}_{Y\to X}, \breve{\overline{\mathscr{B}}}) \to (\mathbf{E}^{\text{ét},\mathbb{N}}_{Y\to X}, \breve{\overline{\mathscr{B}}})$. We obtain a filtration $\breve{\nu}^*\mathrm{Fil}^q$ on $\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \breve{\overline{\mathscr{B}}}_{\mathbb{Q}})$ with graded pieces $\breve{\nu}^*\mathrm{Gr}^q = \breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n)_{\mathbb{Q}}$ (as $\breve{\nu}^* = \breve{\nu}^{-1}$ is exact, cf. (I.7.34.3)). We apply the derived functor $\mathrm{R}\Gamma(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, -)$ to these modules.

Consider the commutative diagram

(I.12.7.2)
$$\begin{array}{ccccc} V_{\text{ét}} & \xrightarrow{\psi} & \mathbf{E}^{\text{ét}}_{V\to U} & \xleftarrow{\nu} & \mathbf{E}^{\text{proét}}_{V\to U} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ Y_{\text{ét}} & \xrightarrow{\psi} & \mathbf{E}^{\text{ét}}_{Y\to X} & \xleftarrow{\nu} & \mathbf{E}^{\text{proét}}_{Y\to X} \end{array}$$

Since the canonical morphism $j^{-1}\psi_*\mathbb{L}^{(n)}_k \to \psi_* j^{-1}\mathbb{L}^{(n)}_k$ is an isomorphism by I.10.9, $j^{-1}\psi_*\mathbb{L}^{(n)}_k$ is constant with value $H^n_{\text{ét},k}$ by I.10.3. Thus, the restriction of $\breve{\nu}^{-1}\breve{\psi}_*\breve{\mathbb{L}}^{(n)}$ to $\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}$ is the inverse system of constant sheaves $(H^n_{\text{ét},k})_{k\geq 0}$. Therefore, the canonical morphism

(I.12.7.3)
$$H^n_{\text{ét},k} \otimes_{\mathbb{Z}_p} A \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{proét}}_{V\to U}, \nu^*(\psi_*\mathbb{L}^{(n)}_k \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}}))$$

is an almost isomorphism, since $V \to U$ is Faltings acyclic and $H^n_{\text{ét},k}$ is a finite abelian group. Notice that for any integer $q \geq 0$ there exists a canonical exact sequence ([**AGT16**, VI.7.10])

(I.12.7.4)
$$0 \to \mathrm{R}^1 \lim_{k\to\infty} H^{q-1}(\mathbf{E}^{\text{proét}}_{V\to U}, \nu^*(\psi_*\mathbb{L}_k^{(n)} \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}})) \to H^q(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, \breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}})) \to \lim_{k\to\infty} H^q(\mathbf{E}^{\text{proét}}_{V\to U}, \nu^*(\psi_*\mathbb{L}_k^{(n)} \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}})) \to 0.$$

Since the inverse system $(H^n_{\text{ét},k} \otimes_{\mathbb{Z}_p} A)_{k\geq 0}$ is Artin-Rees p-adic, $\mathrm{R}^1 \lim_{k\to\infty} H^0(\mathbf{E}^{\text{proét}}_{V\to U}, \nu^*(\psi_*\mathbb{L}_k^{(n)} \otimes_{\mathbb{Z}_p} \overline{\mathscr{B}}))$ is almost zero by the almost isomorphism (I.12.7.3). Moreover, we deduce that $\lim_{k\to\infty}(H^n_{\text{ét},k} \otimes_{\mathbb{Z}_p} A) \to H^0(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, \breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}}))$ is an almost isomorphism and that $H^q(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, \breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}}))$ is almost zero for $q > 0$. Since $H^n_{\text{ét}}$ is a finitely generated $\mathbb{Z}_p$-module (I.12.5), we have $\lim_{k\to\infty}(H^n_{\text{ét},k} \otimes_{\mathbb{Z}_p} A) = H^n_{\text{ét}} \otimes_{\mathbb{Z}_p} \widehat{A}$. Therefore, the canonical morphism

$$H^n_{\text{ét}} \otimes_{\mathbb{Z}_p} \widehat{A} \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, \breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}})) \tag{I.12.7.5}$$

is an almost isomorphism. Inverting p, we obtain a canonical isomorphism

$$H^n_{\text{ét}} \otimes_{\mathbb{Z}_p} \widehat{A}[\frac{1}{p}] \xrightarrow{\sim} \mathrm{R}\Gamma(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, \breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}}))[\frac{1}{p}]. \tag{I.12.7.6}$$

By taking $H^0(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, \breve{\nu}^*\mathrm{Fil}^q)$, the filtration $\breve{\nu}^*\mathrm{Fil}^q$ on $\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}}_{\mathbb{Q}})$ induces a canonical filtration fil^q on $H^n_{\text{ét}}(X^{\prime\triangleright}_{\overline{x}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \widehat{A}[1/p]$ (cf. I.2.6).

On the other hand, recall that $\mathcal{M}^{q,n-q}$ is the coherent $\mathcal{O}_X$-module associated to the coherent R-module $M^{q,n-q} = H^q(X', \Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)})$ and that $M^{q,n-q}[1/p]$ is a projective $R[1/p]$-module (I.12.3.(2)). By I.8.4, we see that the canonical morphism

$$H^q(X', \Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)}) \otimes_R \widehat{A}[\frac{1}{p}] \longrightarrow \mathrm{R}\Gamma(\mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U}, \breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q})[\frac{1}{p}] \tag{I.12.7.7}$$

is an isomorphism. Thus, the canonical isomorphisms for the graded pieces $\breve{\nu}^*\mathrm{Gr}^q = \breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n)_{\mathbb{Q}}$ induce canonical isomorphisms (I.12.7.1). This completes the proof of the first statement.

For the Γ-equivariance, let $\gamma \in \Gamma$ with image $u \in G_K$. We obtain from I.12.6 a commutative diagram

$$\begin{array}{ccc} (V \to U) & \xrightarrow{(f_\gamma, \mathrm{id}_U)} & (V \to U) \\ \downarrow & & \downarrow \\ (Y \to X) & \xrightarrow{(f_u, \mathrm{id}_X)} & (Y \to X) \end{array} \tag{I.12.7.8}$$

where the vertical arrows are the same pro-étale morphism. It induces a commutative diagram of fibred ringed sites over $\mathbb{N}$,

$$\begin{array}{ccc} \mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U} & \xrightarrow{f_\gamma} & \mathbf{E}^{\text{proét},\mathbb{N}}_{V\to U} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbf{E}^{\text{proét},\mathbb{N}}_{Y\to X} & \xrightarrow{f_u} & \mathbf{E}^{\text{proét},\mathbb{N}}_{Y\to X} \end{array} \tag{I.12.7.9}$$

where the vertical morphisms induce the same localization morphism of the associated topoi ([**AGT16**, III.7.9]). Recall that the G_K-actions on the $\overline{\breve{\mathscr{B}}}$-modules $\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}})$, $\breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n)$ on $\mathbf{E}^{\text{proét},\mathbb{N}}_{Y\to X}$ define isomorphisms

$$\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}}) \xrightarrow{\sim} f_u^{-1}(\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)} \otimes_{\breve{\mathbb{Z}}_p} \overline{\breve{\mathscr{B}}})), \tag{I.12.7.10}$$

$$\breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n) \xrightarrow{\sim} f_u^{-1}(\breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n)). \tag{I.12.7.11}$$

By I.12.2, up to isogenies, they are respectively compatible with the relative Hodge-Tate filtration $\breve{\nu}^*\mathrm{Fil}^q$ and compatible with the canonical isomorphisms for the graded pieces $\breve{\nu}^*\mathrm{Gr}^q$. Passing to

localization by j^{-1}, we obtain from the isomorphisms (I.12.7.10) and (I.12.7.11) the isomorphisms (I.12.7.12)

$$j^{-1}(\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)}\otimes_{\breve{\mathbb{Z}}_p}\overline{\breve{\mathscr{B}}}))\xrightarrow{\sim} j^{-1}f_u^{-1}(\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)}\otimes_{\breve{\mathbb{Z}}_p}\overline{\breve{\mathscr{B}}}))\xrightarrow{\sim} f_\gamma^{-1}j^{-1}(\breve{\nu}^*(\breve{\psi}_*\breve{\mathbb{L}}^{(n)}\otimes_{\breve{\mathbb{Z}}_p}\overline{\breve{\mathscr{B}}})),$$

(I.12.7.13)

$$j^{-1}(\breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n))\xrightarrow{\sim} j^{-1}f_u^{-1}(\breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n))\xrightarrow{\sim} f_\gamma^{-1}j^{-1}(\breve{\nu}^*\breve{\sigma}^*\breve{\mathcal{M}}^{q,n-q}(q-n)).$$

Applying the derived functor $\mathrm{R}\Gamma(\mathbf{E}^{\mathrm{pro\acute{e}t},\mathbb{N}}_{V\to U},-)[1/p]$ and combining with the canonical isomorphisms (I.12.7.6) and (I.12.7.7), we obtain automorphisms of $H^n_{\text{ét}}\otimes_{\mathbb{Z}_p}\widehat{A}[1/p]$ and of $H^q(X',\Omega^{n-q}_{(X',\mathscr{M}_{X'})/(X,\mathscr{M}_X)})\otimes_R \widehat{A}[1/p](q-n)$, which are exactly the semi-linear actions of $\gamma\in\Gamma$ on these $\widehat{A}[1/p]$-modules defined in I.12.6 by going through the definitions. Thus, we see that the actions of Γ are compatible with the relative Hodge-Tate filtration and the canonical isomorphisms for the graded pieces, which completes the proof. □

Remark I.12.8. The arguments for I.12.7 does not make use of the assumption that the residue field of K is algebraically closed.

Remark I.12.9. In I.12.6, we take U to be an étale neighborhood of a point of the special fibre of X which is affine and admits an adequate chart in the sense of [**AGT16**, III.4.4] (cf. [**AGT16**, III.4.7]), and take V to be the inverse limit of the normalized universal cover (V_i) of $U^\circ_\star$ at $\overline{x}$ (cf. [**SGA 1**, V.7], [**AGT16**, VI.9.8]). We set $U^{V_i}=\mathrm{Spec}(R_i)$ and $\overline{R}=\mathrm{colim}(R_i)$. Then, $\Gamma=\pi_1(U^\circ_\star,\overline{x})$ and $A=\overline{R}$. We obtain from the adequate chart finitely many nonzero divisors $f_1,\dots,f_r$ of $\Gamma(U_\eta,\mathcal{O}_{U_\eta})$ such that the divisor $D=\sum_{i=1}^r\mathrm{div}(f_i)$ has support $U_\eta\setminus U^\circ$ and that at each strict henselization of U_η those elements f_i contained in the maximal ideal form a subset of a regular system of parameters (cf. [**AG20**, 4.2.2.(ii)]). Then, A is almost pre-perfectoid and admits compatible n-th power roots of f_i ([**AGT16**, II.9.10]). Hence, $V\to U$ is Faltings acyclic by I.8.24, and thus Theorem I.12.7 holds in this setting, which gives a local version of the relative Hodge-Tate filtration answering the question of Abbes-Gros raised in the first version of [**AG20**] (cf. [**AG20**, 1.2.3]).

CHAPTER II

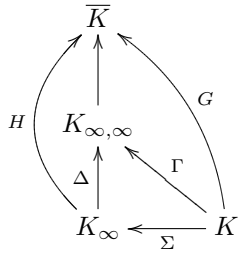
Sen Operators and Lie Algebras arising from Galois Representations over p-adic Varieties

II.1. Introduction

II.1.1. Let K be a complete discrete valuation field extension of $\mathbb{Q}_p$, $\overline{K}$ an algebraic closure of K, $\widehat{\overline{K}}$ the p-adic completion of $\overline{K}$, G the Galois group of $\overline{K}$ over K. When the residue field of K is perfect, for any finite-dimensional (continuous semi-linear) $\widehat{\overline{K}}$-representation W of G, Sen [**Sen81**] associates a canonical $\widehat{\overline{K}}$-linear endomorphism on W, called the Sen operator, which determines the isomorphism class of G-representations on W. Moreover, if W is the base change of a $\mathbb{Q}_p$-representation V of G, Sen [**Sen81**, Theorem 11] relates the infinitesimal action of the inertia subgroup of G on V to the Sen operator on W. When the residue field of K is imperfect with a p-basis of cardinality $d \geq 1$, Brinon [**Bri03**] defines $1+d$ (non-canonical) operators on W, which also determine the isomorphism class of G-representations on W. Moreover, if W is the base change of a $\mathbb{Q}_p$-representation V of G, Ohkubo [**Ohk14**] relates the space generated by these $1+d$ operators to the infinitesimal action of the inertia subgroup of G on V as Sen did for $d=0$. In this chapter, we construct Sen operators for representations of the fundamental group of a p-adic affine variety with semi-stable chart. We show that the module of Sen operators is canonically defined, independent of the choice of the chart. Indeed, we associate to each representation a canonical Lie algebra action which gives all the Sen operators. Moreover, when the representation comes from $\mathbb{Q}_p$, we relate the Sen operators to the infinitesimal action of the inertia subgroups at height-1 primes, generalizing the results of Sen-Ohkubo. As an application, we prove that the geometric Sen operators annihilate locally analytic vectors, generalizing a result of Pan.

II.1.2. In fact, our strategy for constructing the Sen operators in the relative situation is to glue the Sen operators defined in the case of valuation fields. Hence, we firstly take a brief review on Brinon's construction of Sen operators. We take $t_1, \ldots, t_d \in \mathcal{O}_K$ whose images in the residue field form a p-basis. We fix a compatible system of primitive p^n-th roots of unity $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}}$ and a compatible system of p^n-th roots $(t_{i,p^n})_{n\in\mathbb{N}}$ of t_i for $1 \leq i \leq d$. We also put $t_{0,p^n} = \zeta_{p^n}$ for consistency. For any $n, m \in \mathbb{N} \cup \{\infty\}$, consider the field extension $K_{n,m} = K(\zeta_{p^n}, t_{1,p^m}, \ldots, t_{d,p^m})$ of K contained in $\overline{K}$. We simply set $K_{n,0} = K_n$ and we name some Galois groups as indicated in the following diagram

(II.1.2.1)



Any finite-dimensional $\widehat{\overline{K}}$-representation W of G descends to a $K_{\infty,\infty}$-representation V of Γ by a theorem of Brinon (cf. II.5.17). We remark that it can be descended further to a K_∞-representation of Γ on which Δ acts analytically by a theorem of Tsuji (cf. II.5.18). Here, acting analytically means that the action of any element of Δ is given by the exponential of its infinitesimal action (cf. II.4.14). The topological group Γ is indeed a p-adic analytic group, to which one can associate a Lie algebra $\mathrm{Lie}(\Gamma)$ over $\mathbb{Q}_p$. Then, the infinitesimal action of $\mathrm{Lie}(\Gamma)$ on V extends $\widehat{\overline{K}}$-linearly to a (non-canonical) Lie algebra action of $\mathrm{Lie}(\Gamma)$ on W, which defines $1+d$ operators of W by Brinon as Γ is locally isomorphic to $\mathbb{Z}_p \ltimes \mathbb{Z}_p^d$.

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This action of $\mathrm{Lie}(\Gamma)$ depends on the choice of $t_1,\dots,t_d$, which prevents the generalization to relative situation. The first question is whether we can define a canonical Lie algebra action on W, which gives the Sen operators defined by Brinon by choosing a basis. We answer it positively by considering the Faltings extension of $\mathcal{O}_K$ defined in **[He21]** (cf. II.5.7), that is, a canonical exact sequence of $\widehat{\overline{K}}$-representations of G,

$$0\longrightarrow\widehat{\overline{K}}(1)\xrightarrow{\iota}\mathscr{E}_{\mathcal{O}_K}\xrightarrow{\jmath}\widehat{\overline{K}}\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}\longrightarrow 0, \tag{II.1.2.2}$$

where $\widehat{\overline{K}}(1)$ denotes the first Tate twist of $\widehat{\overline{K}}$, $\mathscr{E}_{\mathcal{O}_K}=\varprojlim_{x\mapsto px}\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is a $(1+d)$-dimensional $\widehat{\overline{K}}$-space with a basis $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\le i\le d}$. Taking duals and Tate twists, we obtain a canonical exact sequence

$$0\longrightarrow\mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1),\widehat{\overline{K}})\xrightarrow{\jmath^*}\mathscr{E}^*_{\mathcal{O}_K}(1)\xrightarrow{\iota^*}\widehat{\overline{K}}\longrightarrow 0 \tag{II.1.2.3}$$

where $\mathscr{E}^*_{\mathcal{O}_K}=\mathrm{Hom}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K},\widehat{\overline{K}})$. There is a canonical $\widehat{\overline{K}}$-linear Lie algebra structure on $\mathscr{E}^*_{\mathcal{O}_K}(1)$ associated to the linear form ι^* defined by $[f_1,f_2]=\iota^*(f_1)f_2-\iota^*(f_2)f_1$ for any $f_1,f_2\in\mathscr{E}^*_{\mathcal{O}_K}(1)$. This will be the canonical Lie algebra replacing $\mathrm{Lie}(\Gamma)$, so that we obtain the following canonical definition of Sen operators.

Theorem II.1.3 (cf. II.5.35, II.5.38)**.** *Let K be a complete discrete valuation field extension of $\mathbb{Q}_p$ whose residue field admits a finite p-basis, G its absolute Galois group. For any finite-dimensional $\widehat{\overline{K}}$-representation W of G, there is a canonical G-equivariant homomorphism of $\widehat{\overline{K}}$-linear Lie algebras (where we put adjoint action of G on $\mathrm{End}_{\widehat{\overline{K}}}(W)$),*

$$\varphi_{\mathrm{Sen}}|_W:\mathscr{E}^*_{\mathcal{O}_K}(1)\longrightarrow\mathrm{End}_{\widehat{\overline{K}}}(W), \tag{II.1.3.1}$$

which is functorial in W and satisfies the following properties:

(1) *Let $t_1,\dots,t_e\in K$ with compatible systems of p-power roots $(t_{i,p^n})_{n\in\mathbb{N}}\subseteq\overline{K}$ such that $\mathrm{d}t_1,\dots,\mathrm{d}t_e$ are K-linearly independent in $\widehat{\Omega}^1_{\mathcal{O}_K}[1/p]$. Consider the tower $(K_{n,m})_{n,m\in\mathbb{N}}$ defined by these elements analogously to* (II.1.2.1) *and take the same notation for Galois groups, and assume that there is a K_∞-representation V of Γ on which Δ acts analytically (*II.4.14*) such that $W=\widehat{\overline{K}}\otimes_{K_\infty}V$. Then, Γ is naturally locally isomorphic to $\mathbb{Z}_p\ltimes\mathbb{Z}_p^e$, and if we take the standard basis $\partial_0,\dots,\partial_e$ of $\mathrm{Lie}(\Gamma)\cong\mathrm{Lie}(\mathbb{Z}_p\ltimes\mathbb{Z}_p^e)$, then for any $f\in\mathscr{E}^*_{\mathcal{O}_K}(1)$,*

$$\varphi_{\mathrm{Sen}}|_W(f)=\sum_{i=0}^{e}f((\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\otimes\zeta^{-1})\otimes\varphi_{\partial_i}|_V, \tag{II.1.3.2}$$

where $\varphi_{\partial_i}|_V$ is the infinitesimal action of ∂_i on V.

(2) *Let K' be a complete discrete valuation field extension of K whose residue field admits a finite p-basis, $W'=\widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}W$. Assume that $K'\otimes_K\widehat{\Omega}^1_{\mathcal{O}_K}[1/p]\to\widehat{\Omega}^1_{\mathcal{O}_{K'}}[1/p]$ is injective. Then, there is a natural commutative diagram*

$$\begin{array}{ccc}\mathscr{E}^*_{\mathcal{O}_{K'}}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{\overline{K'}}}(W')\\ \downarrow & & \uparrow\wr\\ \widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}\mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow[\mathrm{id}_{\widehat{\overline{K'}}}\otimes\varphi_{\mathrm{Sen}}|_W]{} & \widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}\mathrm{End}_{\widehat{\overline{K}}}(W)\end{array} \tag{II.1.3.3}$$

Moreover, if K' is a finite extension of K, then the left vertical arrow is an isomorphism.

The key of its proof is to show that the map $\varphi_{\mathrm{Sen}}|_W$ defined by the formula (II.1.3.2) does not depend on the choice of V and t_i. For this, we use the variant of p-adic Simpson correspondence developed by Tsuji **[Tsu18]** over $\mathcal{O}_K$ (cf. II.5.31). One clue is that the period ring used in this correspondence is constructed as the filtered colimit of symmetric tensor products of the Faltings extension (II.1.2.2) (called the Hyodo ring, cf. II.5.13). We remark that the assumption on $K'\otimes_K\widehat{\Omega}^1_{\mathcal{O}_K}[1/p]\to\widehat{\Omega}^1_{\mathcal{O}_{K'}}[1/p]$ for the functoriality is a technical condition for its proof, and we don't know how to remove this (cf. II.5.38).

II.1.4. Now we can generalize the construction of Sen operators in the relative situation. Let K be a complete discrete valuation field extension of $\mathbb{Q}_p$ with perfect residue field, π a uniformizer of K.

For simplicity, we consider a Noetherian normal domain A flat over $\mathcal{O}_K$ with $A/pA \neq 0$ such that there exists an étale ring homomorphism for some integers $0 \leq r \leq c \leq d$,

$$\mathcal{O}_K[T_0, \ldots, T_r, T_{r+1}^{\pm 1}, \ldots, T_c^{\pm 1}, T_{c+1}, \ldots, T_d]/(T_0 \cdots T_r - \pi) \longrightarrow A. \tag{II.1.4.1}$$

Thus, $\mathrm{Spec}(A)$ is endowed with a strictly normal crossings divisor defined by $T_0 \cdots T_d$. We set $A_{\mathrm{tr}} = A[1/T_0 \cdots T_d]$. We call (A_{tr}, A) a semi-stable pair, and call (II.1.4.1) a semi-stable chart of it. Let $\mathcal{K}$ be the fraction field of A, $\mathcal{K}_{\mathrm{ur}}$ the maximal unramified extension of $\mathcal{K}$ with respect to (A_{tr}, A), i.e. the union of finite field extensions $\mathcal{K}'$ of $\mathcal{K}$ in an algebraic closure of $\mathcal{K}$ such that the integral closure A' of A in $\mathcal{K}'$ is finite étale over A_{tr}, and let $\overline{A}$ be the integral closure of A in $\mathcal{K}_{\mathrm{ur}}$. We remark that this chapter considers more general pairs (called *quasi-adequate*) than semi-stable pairs, so that the directed system of finite subextensions of $\mathcal{K}_{\mathrm{ur}}/\mathcal{K}$ admits a cofinal subsystem consisting of elements $\mathcal{K}'$ such that the pair (A'_{tr}, A') is quasi-adequate, where $A'_{\mathrm{tr}} = A_{\mathrm{tr}} \otimes_A A'$ (cf. II.9.5, II.11.9). As before, we also consider the Faltings extension of A (cf. II.9.36), that is, a canonical exact sequence of $\widehat{\overline{A}}[1/p]$-representations of $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$,

$$0 \longrightarrow \widehat{\overline{A}}[\frac{1}{p}](1) \xrightarrow{\iota} \mathscr{E}_A \xrightarrow{\jmath} \widehat{\overline{A}}[\frac{1}{p}] \otimes_A \Omega^1_{(A_{\mathrm{tr}},A)} \longrightarrow 0, \tag{II.1.4.2}$$

where $\Omega^1_{(A_{\mathrm{tr}},A)}$ denotes the A-module of logarithmic 1-differentials of the pair (A_{tr}, A) over $(K, \mathcal{O}_K)$, which is finite free of rank d. The canonical $\widehat{\overline{A}}[1/p]$-module $\mathscr{E}_A$ is finite free of rank $1+d$, which satisfies the following property (cf. II.9.36, II.9.38): there is a canonical $\overline{A}[1/p]$-linear map $\varprojlim_{x \mapsto px} \Omega^1_{\overline{A}/A} \to \mathscr{E}_A$ such that for any element $s \in \overline{A}[1/p] \cap \overline{A}_{\mathrm{tr}}^{\times}$ with a compatible system of p-power roots $(s_{p^n})_{n\in\mathbb{N}} \subseteq \overline{A}[1/p]$, there is a unique element $\omega \in \mathscr{E}_A$ such that the image of $(s_{p^n}^{p^n-1} \mathrm{d}s_{p^n})_{n\in\mathbb{N}}$ is equal to $s\omega$ (we thus denote ω by $(\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}$). As before, we obtain a canonical exact sequence by taking duals and Tate twists,

$$0 \longrightarrow \mathrm{Hom}_A(\Omega^1_{(A_{\mathrm{tr}},A)}(-1), \widehat{\overline{A}}[\frac{1}{p}]) \xrightarrow{\jmath^*} \mathscr{E}_A^*(1) \xrightarrow{\iota^*} \widehat{\overline{A}}[\frac{1}{p}] \longrightarrow 0, \tag{II.1.4.3}$$

and we endow $\mathscr{E}_A^*(1)$ with the canonical $\widehat{\overline{A}}[1/p]$-linear Lie algebra structure associated to the linear form ι^*. Now we can state the construction of Sen operators in the relative situation.

Theorem II.1.5 (cf. II.11.4, II.11.7, II.11.9)**.** *With the notation in* II.1.4*, for any finite projective $\widehat{\overline{A}}[1/p]$-representation W of an open subgroup G of $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$, there is a canonical G-equivariant homomorphism of $\widehat{\overline{A}}[1/p]$-linear Lie algebras (where we put adjoint action of G on $\mathrm{End}_{\widehat{\overline{A}}[1/p]}(W)$),*

$$\varphi_{\mathrm{Sen}}|_W : \mathscr{E}_A^*(1) \longrightarrow \mathrm{End}_{\widehat{\overline{A}}[\frac{1}{p}]}(W), \tag{II.1.5.1}$$

which is functorial in W and G, depends only on the pair (A_{tr}, A) not on the choice of the chart (II.1.4.1)*, and satisfies the following properties:*

(1) *Let A' be the integral closure of A in a finite field extension $\mathcal{K}'$ of $(\mathcal{K}_{\mathrm{ur}})^G$ contained in $\mathcal{K}_{\mathrm{ur}}$, and let $t_1, \ldots, t_e \in A'[1/p] \cap A'^{\times}_{\mathrm{tr}}$ with compatible systems of p-power roots $(t_{i,p^n})_{n\in\mathbb{N}} \subseteq \overline{A}[1/p]$ such that $\mathrm{d}t_1, \ldots, \mathrm{d}t_e$ are $\mathcal{K}'$-linearly independent in $\Omega^1_{\mathcal{K}'/K}$. Consider the tower $(\mathcal{K}'_{n,m})_{n,m\in\mathbb{N}}$ defined by these elements analogously to* (II.1.2.1) *and take the same notation for Galois groups, and let $A'_{n,m}$ be the integral closure of A in $\mathcal{K}'_{n,m}$, $\widetilde{A}'_{\infty} = \mathrm{colim}_n \widehat{A'_n}$. Assume that there is a finite projective $\widetilde{A}'_{\infty}[1/p]$-representation V of Γ on which Δ acts analytically (*II.4.14*) such that $W = \widehat{\overline{A}} \otimes_{\widetilde{A}'_{\infty}} V$. Then, Γ is naturally locally isomorphic to $\mathbb{Z}_p \ltimes \mathbb{Z}_p^e$, and if we take the standard basis $\partial_0, \ldots, \partial_e$ of $\mathrm{Lie}(\Gamma) \cong \mathrm{Lie}(\mathbb{Z}_p \ltimes \mathbb{Z}_p^e)$, then for any $f \in \mathscr{E}_A^*(1)$,*

$$\varphi_{\mathrm{Sen}}|_W(f) = \sum_{i=0}^{e} f((\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1}) \otimes \varphi_{\partial_i}|_V, \tag{II.1.5.2}$$

where $\varphi_{\partial_i}|_V$ is the infinitesimal action of ∂_i on V.

(2) *Let K' be a complete discrete valuation field extension of K with perfect residue field, (A'_{tr}, A') a semi-stable pair over $\mathcal{O}_{K'}$ with fraction field $\mathcal{K}'$, $\overline{A} \to \overline{A'}$ an injective ring homomorphism over $\mathcal{O}_K \to \mathcal{O}_{K'}$ which induces an inclusion $(A_{\mathrm{tr}}, A) \to (A'_{\mathrm{tr}}, A')$, $W' = \widehat{\overline{A'}} \otimes_{\widehat{\overline{A}}} W$.*

Assume that $\mathcal{K}' \otimes_{\mathcal{K}} \Omega^1_{\mathcal{K}/K} \to \Omega^1_{\mathcal{K}'/K'}$ is injective. Then, there is a natural commutative diagram

$$\begin{array}{ccc} \mathscr{O}^*_{A'}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{A'}[\frac{1}{p}]}(W') \\ \downarrow & & \uparrow \wr \\ \widehat{A'} \otimes_{\widehat{A}} \mathscr{O}^*_A(1) & \xrightarrow[\mathrm{id}_{\widehat{A'}} \otimes \varphi_{\mathrm{Sen}}|_W]{} & \widehat{A'} \otimes_{\widehat{A}} \mathrm{End}_{\widehat{A}[\frac{1}{p}]}(W) \end{array} \tag{II.1.5.3}$$

The situation described in II.1.3.(1) is not special. Indeed, by a descent theorem of Tsuji [**Tsu18**, 14.2] when he developed the p-adic Simpson correspondence for (A_{tr}, A), the representation W of G can be descended to V for some A'. We remark that Tsuji proved the case when $G = \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$, and we prove the general case by transferring his arguments to a more general class of pairs (cf. II.10.16). The key to the proof of II.1.5 is still checking that the map $\varphi_{\mathrm{Sen}}|_W$ defined by the formula (II.1.5.2) does not depend on the choice of A', V and t_i. We reduce this problem to the case of valuation rings II.1.3 by localizing at height-1 prime ideals of $\overline{A}$ containing p.

Definition II.1.6. We call the image $\Phi(W)$ of $\varphi_{\mathrm{Sen}}|_W$ the module of *Sen operators* of W. We call the image $\Phi^{\mathrm{geo}}(W)$ of $\mathrm{Hom}_A(\Omega^1_{(A_{\mathrm{tr}},A)}(-1), \widehat{\overline{A}}[\frac{1}{p}])$ under $\varphi_{\mathrm{Sen}}|_W$ the module of *geometric Sen operators* of W. And we call the image of $1 \in \widehat{\overline{A}}[\frac{1}{p}]$ under $\varphi_{\mathrm{Sen}}|_W$ in $\Phi^{\mathrm{ari}}(W) = \Phi(W)/\Phi^{\mathrm{geo}}(W)$ the *arithmetic Sen operator* of W.

The following evidence supports such a definition of arithmetic Sen operator: any two lifts of it in $\mathrm{End}_{\widehat{A}[\frac{1}{p}]}(W)$ have the same characteristic polynomial (cf. II.11.15).

II.1.7. We denote by $\mathfrak{S}_p(\overline{A})$ the set of height-1 prime ideals of $\overline{A}$ containing p. For any $\mathfrak{q} \in \mathfrak{S}_p(\overline{A})$ with image $\mathfrak{p} \in \mathrm{Spec}(A)$, let $E_{\mathfrak{p}}$ be the p-adic completion of the discrete valuation field $A_{\mathfrak{p}}[1/p]$, $\overline{E}_{\mathfrak{q}}$ an algebraic closure of $E_{\mathfrak{p}}$ with an embedding of valuation rings $\overline{A}_{\mathfrak{q}} \to \mathcal{O}_{\overline{E}_{\mathfrak{q}}}$. Let $I_{\mathfrak{q}} \subseteq \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ be the image of the inertia subgroup of $\mathrm{Gal}(\overline{E}_{\mathfrak{q}}/E_{\mathfrak{p}})$. We have the following generalization of Sen-Ohkubo's result, which follows from the same reduction strategy as above.

Theorem II.1.8 (cf. II.11.18, II.11.9)**.** *Let G be an open subgroup of $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$, (V, ρ) a finite-dimensional $\mathbb{Q}_p$-representation of G, $W = \widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} V$. Then, $\sum_{\mathfrak{q} \in \mathfrak{S}_p(\overline{A})} \mathrm{Lie}(\rho(I_{\mathfrak{q}}))$ is the smallest $\mathbb{Q}_p$-subspace S of $\mathrm{End}_{\mathbb{Q}_p}(V)$ such that the $\widehat{\overline{A}}[1/p]$-module of Sen operators $\Phi(W)$ is contained in $\widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} S$.*

As a corollary, one can lift the Sen operators of $\mathbb{Q}_p$-representations to a universal Lie algebra homomorphism.

Corollary II.1.9 (cf. II.11.21, II.11.23)**.** *Let $\mathcal{G}$ be a quotient of an open subgroup of $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ which is a p-adic analytic group. Then, there exists a canonical homomorphism of $\widehat{\overline{A}}[1/p]$-linear Lie algebras $\varphi_{\mathrm{Sen}}|_{\mathcal{G}} : \mathscr{O}^*_A(1) \to \widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G})$ making the following diagram commutative for any finite-dimensional $\mathbb{Q}_p$-representation V of $\mathcal{G}$,*

$$\begin{array}{ccc} \mathscr{O}^*_A(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\overline{A}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) \\ {\scriptstyle \varphi_{\mathrm{Sen}}|_W}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\widehat{\overline{A}}[\frac{1}{p}]} \otimes \varphi|_V} \\ \mathrm{End}_{\widehat{\overline{A}}[\frac{1}{p}]}(W) & \xleftarrow{\sim} & \widehat{\overline{A}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p}(V) \end{array} \tag{II.1.9.1}$$

where $W = \widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} V$ is the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{A}}[1/p])$, and $\varphi|_V$ is the infinitesimal Lie algebra action of $\mathrm{Lie}(\mathcal{G})$ on V.

Now we can give our generalization of Pan's result [**Pan22**, 3.1.2].

Theorem II.1.10 (cf. II.12.22)**.** *Let $\mathcal{G}$ be a quotient of $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ which is a p-adic analytic group, $\mathcal{G}_H \subseteq \mathcal{G}$ the image of $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K}_\infty)$, $\Phi^{\mathrm{geo}}_{\mathcal{G}} \subseteq \widehat{\overline{A}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}_H)$ the image of $\mathrm{Hom}_A(\Omega^1_{(A_{\mathrm{tr}},A)}(-1), \widehat{\overline{A}}[1/p])$ under $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$. Then, the infinitesimal action of $\Phi^{\mathrm{geo}}_{\mathcal{G}}$ annihilates the $\mathcal{G}_H$-locally analytic vectors in $\widehat{\overline{A}}[1/p]$ (see* II.12.18 *for a precise definition).*

For its proof, we need to extend Sen operators on the infinite-dimensional representations of analytic functions on sufficiently small open subgroups of $\mathcal{G}$. This is the reason why we insist to consider open subgroups of $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ in the previous theorems, which enables us to prove properties related to Lie algebras but leads us to a general class of pairs more than semi-stable pairs.

Previously, we always work with representations with rational coefficients, since a finite extension A' of A is not a nice integral model for $A'[1/p]$. But in order to investigate the continuity of Sen operators on infinite-dimensional representations, we need to consider representations with integral coefficients as "lattices" to bound these operators. Nice properties of the Sen operators are preserved by continuation if we have good descent and decompletion theory for integral representations over A'. But it has not been well developed yet as A' is not a nice integral model. However, we don't encounter such a problem if A' is a valuation ring (at least for the geometric part)! So we still follow the previous strategy: reduce the problem to the case of valuation rings by localizing at height-1 prime ideals of $\overline{A}$ containing p; and for the latter case, we can apply the descent results for small representations with integral coefficients of the geometric fundamental group, developed by Faltings [**Fal05**], Abbes-Gros [**AGT16**, II.14] and Tsuji [**Tsu18**, §11, §12]. We plan to investigate in the future whether or not the image $\Phi_{\mathcal{G}}$ of $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ annihilates the $\mathcal{G}$-locally analytic vectors in $\widehat{\overline{A}}[1/p]$.

II.1.11. The article is structured as follows. In section II.3, we briefly review the theory of p-adic analytic groups from a purely algebraic view following [**DdSMS99**]. In section II.4, we study the tower $(K_{n,m})_{n,m\in\mathbb{N}}$ (II.1.2.1) and the infinitesimal actions of representations arising from this tower in a general setting. Then, we revisit Brinon's generalization of Sen's theory in section II.5 using the p-adic Simpson correspondence developed by Tsuji, and give our canonical definition of Sen operators. For the generalization in the relative situation, we firstly introduce the main objects, quasi-adequate algebras, in section II.9. They share nice properties with semi-stable pairs up to a p-power torsion by some preparation lemmas in section II.7. Especially, we can also define Faltings extension for such general algebra. A priori, the construction of Faltings extension in the relative situation is not canonical. We show the canonicity of Faltings extension by reducing to the case of valuation rings, cf. II.9.36. To glue the Sen operators defined over valuation rings, we need a "global model" on a quasi-adequate algebra, that is, a descent of representation of the fundamental group. This is a generalization of Tsuji's result and done in the section II.10. We construct the Sen operators in section II.11, and discuss their relation with Lie algebras. Finally, we extend Sen operators to infinite-dimensional representations in section II.6 and the end of section II.11, and give an application on locally analytic vectors in the last section II.12.

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II.2. Notation and Conventions

II.2.1. Let $d \in \mathbb{N}$ be a natural number. We endow the set $(\mathbb{N}\cup\{\infty\})^d$ with the partial order defined by $\underline{m} = (m_1,\dots,m_d) \leq \underline{m'} = (m'_1,\dots,m'_d)$ if $m_i \leq m'_i$ for any $1 \leq i \leq d$. We put $|\underline{m}| = m_1 + \cdots + m_d \in \mathbb{N}\cup\{\infty\}$. For any $r \in \mathbb{N}\cup\{\infty\}$, we set $\underline{r} = (r,\dots,r) \in (\mathbb{N}\cup\{\infty\})^d$ and $\underline{r}_i = (0,\dots,r,\dots,0) \in (\mathbb{N}\cup\{\infty\})^d$ where r appears at the i-th component.

We endow the set $\mathbb{N}^d_{>0}$ with the partial order defined by $\underline{N}|\underline{N'}$ if N_i divides N'_i for any $1 \leq i \leq d$, where $\underline{N} = (N_1,\dots,N_d)$ and $\underline{N'} = (N'_1,\dots,N'_d)$, and we put $\underline{N'}/\underline{N} = (N'_1N_1^{-1},\dots,N'_dN_d^{-1}) \in \mathbb{N}^d_{>0}$.

II.2.2. All rings considered in this article are unitary and commutative. We fix a prime number p. For a ring R, we denote by Ω^1_R the R-module $\Omega^1_{R/\mathbb{Z}}$ of 1-differentials of R over $\mathbb{Z}$, and by $\widehat{\Omega}^1_R$ its p-adic completion. For an abelian group M, we set

$$T_p(M) = \varprojlim_{x\mapsto px} M[p^n] = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p]/\mathbb{Z}, M), \tag{II.2.2.1}$$

$$V_p(M) = \varprojlim_{x\mapsto px} M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], M). \tag{II.2.2.2}$$

We remark that $T_p(M)$ is a p-adically complete $\mathbb{Z}_p$-module ([**Jan88**, 4.4]), and that if $M = M[p^\infty]$ (i.e., M is p-primary torsion) then $V_p(M) = T_p(M)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we set $\mathbb{Z}_p(1) = T_p(\overline{\mathbb{Q}}_p^\times)$ which is a free $\mathbb{Z}_p$-module of rank 1 and any compatible system of primitive p^n-th roots of unity $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}}$ in $\overline{\mathbb{Q}}_p$ (i.e. $\zeta^p_{p^{n+1}} = \zeta_{p^n}$, $\zeta_1 = 1$, $\zeta_p \neq 1$) gives a basis of it. We

endow $\mathbb{Z}_p(1)$ with the natural continuous action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. For any $\mathbb{Z}_p$-module M and $r \in \mathbb{Z}$, we set $M(r) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes r}$, the r-th Tate twist of M.

II.2.3. Let A be a topological ring, M a finitely generated A-module. For any A-linear surjective homomorphism $A^n \to M$ with $n \in \mathbb{N}$, if we endow A^n with the product topology, then the quotient topology on M does not depend on the choice of the surjection. We call this topology on M the *canonical topology* ([**Tsu18**, page 820]). It is clear that any homomorphism of finitely generated A-modules is continuous with respect to the canonical topology.

If the topology on A is linear, then the canonical topology on M is also linear. For another finitely generated A-module N, the canonical topology on $M \otimes_A N$ coincides with the tensor product topology of the canonical topologies on M and N. Moreover, let $A \to A'$ be a continuous homomorphism of linearly topologized rings. Then, the canonical topology on $A' \otimes_A M$ as a finitely generated A'-module coincides with the tensor product topology of the topology on A' and the canonical topology on M.

II.2.4. Let G be a topological group, A a topological ring endowed with a continuous action by G. An *A-representation (W, ρ) of G* is a topological A-module W endowed with a continuous semi-linear action ρ of G. A morphism $W \to W'$ of A-representations of G is a continuous A-linear homomorphism compatible with the action of G. We denote by $\mathbf{Rep}_{\mathrm{cont}}(G, A)$ the category of A-representations of G. Let A_0 be a G-stable subring of A. The *(G, A_0)-finite part* of an A-representation W of G is the sum of all G-stable finitely generated A_0-submodules of W.

We say that an A-representation W of G is *finite projective* if W is a finite projective A-module endowed with the canonical topology. We denote by $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, A)$ the full subcategory of $\mathbf{Rep}_{\mathrm{cont}}(G, A)$ consisting of finite projective A-representations of G.

Assume that the topology on A is linear. For any two A-representations W, W' of G with linear topologies, the diagonal action of G on $W \otimes_A W'$ is continuous with respect to the tensor product topology. If moreover W and W' are finite projective, then so is $W \otimes_A W'$ by II.2.3. This makes $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, A)$ into an additive tensor category. Moreover, let A' be a linearly topologized ring endowed with a continuous action of a topological group G', $G' \to G$ a continuous group homomorphism, $A \to A'$ a continuous ring homomorphism compatible with the actions of G and G'. Then, the tensor product defines a natural functor

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, A) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G', A'),\ W \mapsto A' \otimes_A W. \tag{II.2.4.1}$$

II.3. Brief Review on p-adic Analytic Groups

The theory of p-adic analytic groups (which are often referred to as "p-adic Lie groups") was developed by Lazard [**Laz65**]. We mainly follow [**DdSMS99**] to give a brief review.

Definition II.3.1 ([**DdSMS99**, Theorem 4.5])**.** A pro-p group G is called *uniform* if G is topologically finitely generated, torsion free and $G/\overline{G^p}$ (resp. $G/\overline{G^4}$) is abelian if p is odd (resp. $p = 2$), where $\overline{G^n}$ denotes the closed subgroup of G generated by n-th powers for $n \in \mathbb{N}$.

In fact, the subset of p^n-th powers in a uniform pro-p group G forms a uniform and open characteristic subgroup G^{p^n} of G; these open subgroups $\{G^{p^n}\}_{n\in\mathbb{N}}$ form a fundamental system of neighbourhoods of $1 \in G$; and the map $G \to G^{p^n}$ sending x to x^{p^n} is a homeomorphism of topological spaces ([**DdSMS99**, Theorems 3.6, 4.10]).

Definition II.3.2 ([**DdSMS99**, Section 9.4])**.** A Lie algebra L over $\mathbb{Z}_p$ is called *powerful* if L is a finite free $\mathbb{Z}_p$-module and $[L, L] \subseteq pL$ (resp. $[L, L] \subseteq 4L$) if p is odd (resp. $p = 2$).

For a powerful Lie algebra L over $\mathbb{Z}_p$ and any $n \in \mathbb{N}$, it is clear that the Lie sub-algebra $p^n L$ is also powerful.

II.3.3. We associate to a uniform pro-p group G a powerful Lie algebra $L_G = (G, +_G, [\ ,\ ]_G)$ over $\mathbb{Z}_p$ as follows:

(1) The underlying set of L_G is that of G.
(2) The additive structure on L_G is given by ([**DdSMS99**, Definition 4.12])

$$x +_G y = \lim_{n\to\infty} (x^{p^n} y^{p^n})^{p^{-n}},\ \forall x, y \in G, \tag{II.3.3.1}$$

where taking p^n-th root is well-defined as the map $G \to G^{p^n}$ sending x to x^{p^n} is a homeomorphism.

(3) The Lie bracket on L_G is given by ([**DdSMS99**, Definition 4.29])

$$[x, y]_G = \lim_{n\to\infty} (x^{-p^n} y^{-p^n} x^{p^n} y^{p^n})^{p^{-2n}},\ \forall x, y \in G. \tag{II.3.3.2}$$

The Lie algebra L_G over $\mathbb{Z}_p$ is well-defined and powerful, and a minimal topological generating set $(g_1, \ldots, g_d)$ of G forms a $\mathbb{Z}_p$-linear basis of L_G ([**DdSMS99**, Theorems 4.17, 4.30 and Exercise 4.2.(ii)]). We denote by $\log : G \to L_G$ and $\exp : L_G \to G$ the identity maps. Then, the map

$$\mathbb{Z}_p^d \longrightarrow G,\ (a_1, \ldots, a_d) \mapsto \exp(a_1 \log(g_1) +_G \cdots +_G a_d \log(g_d)) \tag{II.3.3.3}$$

is a homeomorphism of topological spaces such that the image of $(p^n\mathbb{Z}_p)^d$ is G^{p^n}. This map is called the *system of coordinates of the first kind*. Moreover, for $x, y \in G$ such that $xy = yx$ (resp. for $u, v \in L_G$ such that $[u, v] = 0$), we have $\log(xy) = \log(x) +_G \log(y)$ (resp. $\exp(u +_G v) = \exp(u)\exp(v)$). In particular, for $a \in \mathbb{Z}_p$, we have $\log(x^a) = a\log(x)$ for any $x \in G$, and we have $\exp(au) = \exp(u)^a$ for any $u \in L_G$. On the other hand, the map

$$\mathbb{Z}_p^d \longrightarrow G,\ (b_1, \ldots, b_d) \mapsto g_1^{b_1} \cdots g_d^{b_d}, \tag{II.3.3.4}$$

is also a homeomorphism of topological spaces such that the image of $(p^n\mathbb{Z}_p)^d$ is G^{p^n} ([**DdSMS99**, Theorems 4.9, 4.10]). This map is called the *system of coordinates of the second kind*.

II.3.4. We associate to a powerful Lie algebra L over $\mathbb{Z}_p$ a uniform pro-p group $(L, *)$ as follows: we endow L with a group structure given by the Baker-Campbell-Hausdorff formula ([**DdSMS99**, Section 9.4])

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots. \tag{II.3.4.1}$$

The group $(L, *)$ is well-defined and uniform pro-p, and a $\mathbb{Z}_p$-linear basis of L forms a minimal topological generating set of $(L, *)$ ([**DdSMS99**, Theorem 9.8]).

Theorem II.3.5 ([**DdSMS99**, Theorem 9.10])**.** *The assignments $G \mapsto L_G$ and $L \mapsto (L, *)$ defined in* II.3.3 *and* II.3.4 *are mutually inverse isomorphisms between the category of uniform pro-p groups and the category of powerful Lie algebras over $\mathbb{Z}_p$.*

Example II.3.6. The subgroup $G = \mathrm{id} + p^\epsilon \mathrm{M}_d(\mathbb{Z}_p)$ of the general linear group $\mathrm{GL}_d(\mathbb{Z}_p)$ of degree d over $\mathbb{Z}_p$ is a uniform pro-p group, where $\epsilon = 1$ if p is odd, and $\epsilon = 2$ if $p = 2$. We have $G^{p^n} = \mathrm{id} + p^{n+\epsilon}\mathrm{M}_d(\mathbb{Z}_p)$ for any $n \in \mathbb{N}$. In fact, the matrix exponential and logarithm,

$$\exp : p^\epsilon \mathrm{M}_d(\mathbb{Z}_p) \longrightarrow \mathrm{id} + p^\epsilon \mathrm{M}_d(\mathbb{Z}_p),\ X \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} X^n, \tag{II.3.6.1}$$

$$\log : \mathrm{id} + p^\epsilon \mathrm{M}_d(\mathbb{Z}_p) \longrightarrow p^\epsilon \mathrm{M}_d(\mathbb{Z}_p),\ 1 + X \mapsto \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} X^n, \tag{II.3.6.2}$$

are mutually inverse homeomorphisms, which identify $\mathrm{id} + p^{n+\epsilon}\mathrm{M}_d(\mathbb{Z}_p)$ with $p^{n+\epsilon}\mathrm{M}_d(\mathbb{Z}_p)$. Moreover, they induce an isomorphism of $\mathbb{Z}_p$-linear Lie algebras $L_G \xrightarrow{\sim} p^\epsilon \mathrm{M}_d(\mathbb{Z}_p)$, where $p^\epsilon \mathrm{M}_d(\mathbb{Z}_p)$ is endowed with the usual matrix Lie algebra structure. We can extend the matrix logarithm to $\log : \mathrm{GL}_d(\mathbb{Z}_p) \to \mathrm{M}_d(\mathbb{Q}_p)$ by setting $\log(X) = \log(X^r)/r$ for some $r \in \mathbb{N}$ such that $X^r \in \mathrm{id} + p^\epsilon \mathrm{M}_d(\mathbb{Z}_p)$. Especially, for $d = 1$, we can take $r = p(p-1)$ so that

$$\log : \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p. \tag{II.3.6.3}$$

Lemma II.3.7 ([**DdSMS99**, Proposition 4.31])**.** *Let G be a uniform pro-p group, N a closed normal subgroup of G such that G/N is a uniform pro-p group. Then, N is also a uniform pro-p group, and the following natural sequence of powerful Lie algebras over $\mathbb{Z}_p$ is exact*

$$0 \longrightarrow L_N \longrightarrow L_G \longrightarrow L_{G/N} \longrightarrow 0. \tag{II.3.7.1}$$

Definition II.3.8 ([**DdSMS99**, Theorems 8.32, 9.4])**.** A *p-adic analytic group* is a topological group which contains a uniform pro-p open subgroup. A morphism between p-adic analytic groups is a continuous group homomorphism.

Theorem II.3.9 ([**DdSMS99**, Theorem 7.19])**.** *Any compact p-adic analytic group is isomorphic to a closed subgroup of $\mathrm{GL}_d(\mathbb{Z}_p)$ for some $d \in \mathbb{N}$.*

Lemma II.3.10 ([**DdSMS99**, proof of Theorem 4.8])**.** *Let G be a p-adic analytic group, N a closed normal subgroup of G. Then, there exists an open subgroup G_0 of G such that G_0, $N_0 = N \cap G_0$ and G_0/N_0 are all uniform pro-p groups.*

Theorem II.3.11 ([**DdSMS99**, Theorems 9.6, 9.7])**.** *Let G be a separated topological group, N a closed normal subgroup of G, H a closed subgroup of G.*

1. *If G is a p-adic analytic group, then so is H and G/N.*
2. *If N and G/N are p-adic analytic groups, then so is G.*

II.3.12. The uniform pro-p open subgroups H of a p-adic analytic group G form a fundamental system of open neighbourhoods of $1 \in G$. Moreover, given such an H, $\{H^{p^n}\}_{n\in\mathbb{N}}$ is initial in this system. Thus, for a uniform pro-p open subgroup H' of H, the corresponding Lie algebra $L_{H'}$ over $\mathbb{Z}_p$ is a $\mathbb{Z}_p$-submodule of L_H with finite index. In particular, the natural morphism

$$L_{H'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow L_H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \tag{II.3.12.1}$$

is an isomorphism of Lie algebras over $\mathbb{Q}_p$.

Definition II.3.13 ([**DdSMS99**, Section 9.5])**.** Let G be a p-adic analytic group. The filtered colimit of Lie algebras over $\mathbb{Q}_p$,

$$\mathrm{Lie}(G) = \operatorname*{colim}_H L_H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \tag{II.3.13.1}$$

where H is a uniform pro-p open subgroup of G with the corresponding Lie algebra L_H over $\mathbb{Z}_p$, is called the *Lie algebra* of G over $\mathbb{Q}_p$. We denote by $\dim(G)$ the dimension of $\mathrm{Lie}(G)$ over $\mathbb{Q}_p$ and call it the *dimension* of G.

Moreover, if G is compact, then there is a canonical continuous map, called the *logarithm map* of G,

$$\log_G : G \longrightarrow \mathrm{Lie}(G) \tag{II.3.13.2}$$

sending g to $\log_H(g^r)/r$, where r is the index of a uniform pro-p open subgroup H of G, $\log_H : H \to L_H$ is defined in II.3.3, and this definition does not depend on the choice of H.

Lemma II.3.14. *Let G be a p-adic analytic group, N a closed normal subgroup of G. Then, there is a canonical exact sequence of $\mathbb{Q}_p$-linear Lie algebras*

$$0 \longrightarrow \mathrm{Lie}(N) \longrightarrow \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G/N) \longrightarrow 0. \tag{II.3.14.1}$$

Proof. It follows from II.3.10 and II.3.7. □

II.3.15. Let G be a p-adic analytic group. For any $g \in G$, the conjugation on G sending x to gxg^{-1} is continuous, and thus induces an automorphism Ad_g of the Lie algebra $\mathrm{Lie}(G)$. The map

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(\mathrm{Lie}(G)),\ g \mapsto \mathrm{Ad}_g, \tag{II.3.15.1}$$

is a continuous group homomorphism, which makes $\mathrm{Lie}(G)$ into a finite projective $\mathbb{Q}_p$-representation of G, which we call the *adjoint representation* of G (cf. [**DdSMS99**, Exercise 9.11]). This construction is functorial in G. We remark that for $G = \mathrm{GL}_d(\mathbb{Z}_p)$, the adjoint action is given by $\mathrm{Ad}_X(Y) = XYX^{-1}$ for any $X \in \mathrm{GL}_d(\mathbb{Z}_p)$ and $Y \in \mathrm{Lie}(G) = \mathrm{M}_d(\mathbb{Q}_p)$.

II.4. Infinitesimal Actions of Representations arising from Kummer Towers

Definition II.4.1. Let A be a ring, π an element of A. We denote by $\mathfrak{S}_\pi(A)$ the set of prime ideals $\mathfrak{p}$ of height 1 containing π.

We remark that for a Noetherian normal domain A with a non-zero element π, the set $\mathfrak{S}_\pi(A)$ coincides naturally with the finite set of generic points of $\mathrm{Spec}(A/\pi A)$, and $A_\mathfrak{p}$ is a discrete valuation ring for any $\mathfrak{p} \in \mathfrak{S}_\pi(A)$.

Lemma II.4.2. *Let $A \to B$ be an injective and integral homomorphism of domains with A normal, π an element of A. Then, the inverse image of $\mathfrak{S}_\pi(A)$ via the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is $\mathfrak{S}_\pi(B)$, and the induced map $\mathfrak{S}_\pi(B) \to \mathfrak{S}_\pi(A)$ is surjective.*

Proof. Firstly, we note that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. For any $\mathfrak{q} \in \mathrm{Spec}(B)$ with image $\mathfrak{p} \in \mathrm{Spec}(A)$, if $\mathfrak{p}$ is of height 1, then so is $\mathfrak{q}$, since any point of $\mathrm{Spec}(B)$ is closed in its fibre over $\mathrm{Spec}(A)$; conversely, if $\mathfrak{q}$ is of height 1, then so is $\mathfrak{p}$, since $A \to B$ satisfies going down property ([**Sta22**, 00H8]). This shows that the inverse image of $\mathfrak{S}_\pi(A)$ via the surjection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is $\mathfrak{S}_\pi(B)$. □

Proposition II.4.3. *Let $A \to B$ be an injective and integral homomorphism of normal domains with A Noetherian, π a nonzero element of A. We assume that B is the union of a directed system $(B_\lambda)_{\lambda\in\Lambda}$ of Noetherian normal A-subalgebras.*

1. *We have $\mathfrak{S}_\pi(B) = \lim_{\lambda\in\Lambda^{\mathrm{op}}} \mathfrak{S}_\pi(B_\lambda)$, and for each $\mathfrak{q} \in \mathfrak{S}_\pi(B)$, if we denote by $\mathfrak{q}_\lambda \in \mathfrak{S}_\pi(B_\lambda)$ its image, then $(B_{\lambda,\mathfrak{q}_\lambda})_{\lambda\in\Lambda}$ is a directed system of discrete valuation rings with faithfully flat transition maps, whose colimit is $B_\mathfrak{q}$, a valuation ring of height 1.*

(2) *For any integer $n > 0$, the natural map*

$$B/\pi^n B \longrightarrow \prod_{\mathfrak{q}\in\mathfrak{S}_\pi(B)} B_{\mathfrak{q}}/\pi^n B_{\mathfrak{q}} \tag{II.4.3.1}$$

is injective, which thus induces an injective map of π-adic completions

$$\widehat{B} \longrightarrow \prod_{\mathfrak{q}\in\mathfrak{S}_\pi(B)} \widehat{B_{\mathfrak{q}}}. \tag{II.4.3.2}$$

Proof. (1) Since $\mathrm{Spec}(B) = \lim_{\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Spec}(B_\lambda)$ by [**EGA** IV$_3$, 8.2.10], we have $\mathfrak{S}_\pi(B) = \lim_{\lambda\in\Lambda^{\mathrm{op}}} \mathfrak{S}_\pi(B_\lambda)$ by II.4.2. Since B_λ is a Noetherian normal domain, its localization $B_{\lambda,\mathfrak{q}_\lambda}$ is a discrete valuation ring, and the transition map $B_{\lambda,\mathfrak{q}_\lambda} \to B_{\lambda',\mathfrak{q}_{\lambda'}}$ for $\lambda \leq \lambda'$ is local and injective, thus an extension of discrete valuation rings, which completes the proof.

(2) For each λ, the map $B_\lambda/\pi^n B_\lambda \to \prod_{\mathfrak{q}_\lambda\in\mathfrak{S}_\pi(B_\lambda)} B_{\lambda,\mathfrak{q}_\lambda}/\pi^n B_{\lambda,\mathfrak{q}_\lambda}$ is injective by [**Sta22**, 031T, 0311]. If we denote by $f_\lambda : \mathfrak{S}_\pi(B) \to \mathfrak{S}_\pi(B_\lambda)$ the natural surjection, then

$$B_{\lambda,\mathfrak{q}_\lambda}/\pi^n B_{\lambda,\mathfrak{q}_\lambda} \longrightarrow \prod_{\mathfrak{q}\in f_\lambda^{-1}(\mathfrak{q}_\lambda)} B_{\mathfrak{q}}/\pi^n B_{\mathfrak{q}} \tag{II.4.3.3}$$

is injective as $B_{\lambda,\mathfrak{q}_\lambda} \to B_{\mathfrak{q}}$ is faithfully flat. Thus, the composition of the two previous maps, $B_\lambda/\pi^n B_\lambda \to \prod_{\mathfrak{q}\in\mathfrak{S}_\pi(B)} B_{\mathfrak{q}}/\pi^n B_{\mathfrak{q}}$, is injective. The conclusion follows from taking filtered colimit on $\lambda \in \Lambda$. □

Remark II.4.4. Let $A \to B$ be an injective and integral homomorphism of normal domains with A Noetherian. We remark that if the fraction field of B is a finite separable extension of that of A, then B is finite over A ([**Sta22**, 032L]). Thus, the assumption of II.4.3 is satisfied if the fraction field of B is a separable extension of that of A.

Definition II.4.5. A *tower of normal domains* is a directed system $(A_\lambda)_{\lambda\in\Lambda}$ of normal domains with injective and integral transition morphisms. We denote by A_∞ the colimit of $(A_\lambda)_{\lambda\in\Lambda}$.

We remark that if $(\mathcal{K}_\lambda)_{\lambda\in\Lambda}$ is the tower of the fraction fields of a tower of normal domains $(A_\lambda)_{\lambda\in\Lambda}$, then in fact A_λ is the integral closure of A_{λ_0} in $\mathcal{K}_\lambda$ for $\lambda, \lambda_0 \in \Lambda$ with $\lambda \geq \lambda_0$. Moreover, for any element $\pi \in A_{\lambda_0}$, we obtain an inverse system of sets $(\mathfrak{S}_\pi(A_\lambda))_{\lambda\in\Lambda^{\mathrm{op}}_{\geq\lambda_0}}$ with surjective transition maps by II.4.2, and we have $\mathfrak{S}_\pi(A_\infty) = \lim_{\lambda\in\Lambda^{\mathrm{op}}_{\geq\lambda_0}} \mathfrak{S}_\pi(A_\lambda)$.

Lemma II.4.6. *Let A be a ring, π an element of A, $M \to N$ an injective homomorphism of π-torsion free A-modules. Assume that $M = M[1/\pi] \cap N \subseteq N[1/p]$. Then, for any integer $n > 0$, the homomorphism*

$$M/\pi^n M \longrightarrow N/\pi^n N \tag{II.4.6.1}$$

is injective. In particular, the homomorphism of the π-adic completions (endowed with the π-adic topology) $\widehat{M} \to \widehat{N}$ is a closed embedding.

Proof. Firstly, we show that $\pi^n M = M \cap \pi^n N$ (i.e. $M/\pi^n M \to N/\pi^n N$ is injective). For $x \in M \cap \pi^n N$, $\pi^{-n}x \in M[1/\pi]$ lies in N. Hence, $\pi^{-n}x \in M[1/\pi] \cap N = M$, which proves the assertion. Then, we see that $\widehat{M} \to \widehat{N}$ is injective and that the π-adic topology on $\widehat{M}$ coincides with the topology induced from the π-adic topology of $\widehat{N}$. Since $\widehat{M}$ is complete and $\widehat{N}$ is separated, $\widehat{M}$ is closed in $\widehat{N}$ ([**Bou71**, II.16, Proposition 8]). Thus, $\widehat{M}$ identifies with a closed topological subgroup of $\widehat{N}$. □

Definition II.4.7. Let K be a valuation field of height 1 with a non-zero element π in its maximal ideal, $(A_\lambda)_{\lambda\in\Lambda}$ a tower of normal domains flat over $\mathcal{O}_K$. The π-adic completions form a directed system $(\widehat{A}_\lambda)_{\lambda\in\Lambda}$ of flat $\mathcal{O}_K$-algebras, whose transition maps are closed embeddings with respect to the π-adic topology by II.4.6. We set

$$\widetilde{A}_\infty = \operatorname*{colim}_{\lambda\in\Lambda} \widehat{A}_\lambda. \tag{II.4.7.1}$$

As $\widehat{A}_\lambda/\pi^n\widehat{A}_\lambda \to \widetilde{A}_\infty/\pi^n\widetilde{A}_\infty$ is also injective for any $n \in \mathbb{N}$ by II.4.6, we see that $\widehat{A}_\lambda \to \widetilde{A}_\infty$ is also a closed embedding with respect to the π-adic topology and that $\widetilde{A}_\infty$ is π-adically separated. Thus, we always regard $\widetilde{A}_\infty$ as a topological $\mathcal{O}_K$-subalgebra of the π-adic completion $\widehat{A}_\infty$ of A_∞.

Definition II.4.8. Let M be a separated topological $\mathbb{Q}_p$-module endowed with a continuous action of a pro-p group G. For any $g \in G$ and $x \in M$, if the limit

$$\lim_{\mathbb{Z}_p \setminus \{0\} \ni a \to 0} a^{-1}(g^a - 1)(x) \tag{II.4.8.1}$$

exists in M, then we denote it by $\varphi_g(x)$ and call the assignment $x \mapsto \varphi_g(x)$ the *infinitesimal action* of g on x.

The following lemma follows directly from the definition.

Lemma II.4.9. *Let M be a separated topological $\mathbb{Q}_p$-module endowed with a continuous action of a pro-p group G.*

1. *Assume that the infinitesimal action of an element $g \in G$ exists for any $x \in M$. Then, $\varphi_g : x \mapsto \varphi_g(x)$ is a $\mathbb{Q}_p$-linear endomorphism of M, and we have $\varphi_{g^a} = a\varphi_g$ for any $a \in \mathbb{Z}_p$. Moreover, for any $g' \in G$, the infinitesimal action of $g'gg'^{-1}$ also exists for any $x \in M$, and we have $\varphi_{g'gg'^{-1}} = g' \circ \varphi_g \circ g'^{-1}$.*
2. *If the infinitesimal action of two elements $g, g' \in G$ with $gg' = g'g$ exists for any $x \in M$, then $g' \circ \varphi_g = \varphi_g \circ g'$, $\varphi_g \circ \varphi_{g'} = \varphi_{g'} \circ \varphi_g$, and $\varphi_{gg'} = \varphi_g + \varphi_{g'}$.*
3. *Let $G' \to G$ be a continuous homomorphism of pro-p groups, M' a separated topological $\mathbb{Q}_p$-module endowed with a continuous action of a pro-p group G', $f : M \to M'$ a continuous $\mathbb{Q}_p$-homomorphism compatible with the actions of G and G', $x \in M$, $g' \in G'$. Assume that the infinitesimal actions of g' and its image $g \in G$ exist for $f(x) \in M'$ and x respectively. Then, $f(\varphi_g(x)) = \varphi_{g'}(f(x))$.*

Proposition II.4.10 ([**Tsu18**, 5.3])**.** *Let A be a topological $\mathbb{Q}_p$-algebra endowed with a continuous action of a topological group G. Assume that G contains a pro-p open subgroup G_0 of finite index and that there exists a tower $(A_\lambda)_{\lambda\in\Lambda}$ of normal domains flat over $\mathbb{Z}_p$ such that there is an isomorphism of topological rings $A \cong \widetilde{A}_\infty[1/p]$ and that for any $\lambda \in \Lambda$ the subalgebra $\widehat{A}_\lambda[1/p]$ is G_0-stable and invariant by an open subgroup of G_0 (via the isomorphism $A \cong \widetilde{A}_\infty[1/p]$). Then, for any $g \in G$ and any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, A)$, there exists a unique A-linear endomorphism $\varphi_g|_W$ of W satisfying the following conditions (we simply write $\varphi_g|_W$ by φ_g if there is no ambiguity):*

1. *For any $g \in G_0$, φ_g is the infinitesimal action* (II.4.8.1) *of g on W.*
2. *For any $g \in G$ and $n \in \mathbb{N}$, we have $\varphi_{g^n} = n\varphi_g$.*
3. *For any $g \in G_0$ and $x \in W$, there exists $m_x \in \mathbb{N}$ such that for any $a \in p^{m_x}\mathbb{Z}_p$,*

$$g^a(x) = \exp(a\varphi_g)(x) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\underbrace{\varphi_g \circ \cdots \circ \varphi_g}_{k \text{ copies}})(x). \tag{II.4.10.1}$$

In particular, $\varphi_g|_W$ does not depend on the choice of G_0 or the tower $(A_\lambda)_{\lambda\in\Lambda}$. Thus, we still call it the infinitesimal action *of $g \in G$ on W.*

Proof. Firstly, assume that $G = G_0$. Since $\widetilde{A}_\infty$ is a p-adically separated flat $\mathbb{Z}_p$-algebra, there is a canonical norm on $\widetilde{A}_\infty[1/p]$ which induces its p-adic topology (cf. II.6.3). Thus, we are in the situation of [**Tsu18**, 5.3], and the conclusion follows from it. In general, for any $g \in G$, we set $\varphi_g = r^{-1}\varphi_{g^r}$ where r is the index of G_0 in G and φ_{g^r} is the infinitesimal action of $g^r \in G_0$ on finite projective A-representation W of G_0 (defined by restricting the G-action of W). One can check easily by II.4.9 that this φ_g satisfies all the required properties. □

Remark II.4.11. Let A' be a topological $\mathbb{Q}_p$-algebra endowed with a continuous action of a topological group G' satisfying the assumptions in II.4.10. Assume that there is a morphism of topological groups $G' \to G$ and a morphism of topological rings $A \to A'$ which is compatible with the actions of G and G'. For any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, A)$, the base change $W' = A' \otimes_A W$ is naturally an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G', A')$ by II.2.4. Then, for any $g' \in G'$ with image $g \in G$, we deduce from (II.4.8.1) and II.4.10 that

$$\varphi_{g'}|_{W'} = \mathrm{id}_{A'} \otimes \varphi_g|_W. \tag{II.4.11.1}$$

Lemma II.4.12. *Under the assumptions in* II.4.10*, assume further that G acts trivially on A. Then, there is a pro-p open subgroup G_1 of G such that the map*

$$\phi : \mathbb{Z}_p \times G_1 \times W \longrightarrow W, \tag{II.4.12.1}$$

sending $(0, g, x)$ to $\varphi_g(x)$ and sending (a, g, x) to $a^{-1}(g^a x - x)$ for $a \neq 0$, is continuous.

PROOF. Since G acts trivially on A by assumption, for any pro-p open subgroup G_1 of G, ϕ induces a map

$$\widetilde{\phi} : \mathbb{Z}_p \times G_1 \longrightarrow \mathrm{End}_A(W), \tag{II.4.12.2}$$

sending $(0, g)$ to φ_g and sending (a, g) to $a^{-1}(g^a - 1)$ for $a \neq 0$. We fix an A-linear surjection $A^r \to W$. As W is finite projective over A, we get an A-linear surjection $\psi : \mathrm{M}_r(A) \to \mathrm{End}_A(W)$ whose quotient topology on $\mathrm{End}_A(W)$ coincides with its canonical topology (II.2.3). As the matrix multiplication $\mathrm{M}_r(A) \times A^r \to A^r$ is continuous, the natural A-linear homomorphism $\mathrm{End}_A(W) \times W \to W$ is also continuous (II.2.3). Thus, it suffices to show that $\widetilde{\phi}$ is continuous for some G_1. Moreover, by the definition (II.4.8.1) of φ_g, $\widetilde{\phi}$ is uniformly continuous if and only if its restriction on $(\mathbb{Z}_p \setminus \{0\}) \times G_1$ is uniformly continuous ([**Bou71**, II.20, Théorème 2]). We claim that the latter holds for some G_1.

We note that the topology on A is defined by the p-adic topology of $\widetilde{A}_\infty$. As G acts continuously and A-linearly on W by assumption, the induced map $G \to \mathrm{End}_A(W)$ is continuous. For any $k \geq 2$, we take pro-p open subgroups G_k of G whose image in $\mathrm{End}_A(W)$ lies in $\psi(\mathrm{id} + p^k \mathrm{M}_r(\widetilde{A}_\infty))$. For $g_1, g_2 \in G_2$ with $g_1 g_2^{-1} \in G_k$, let $P_1, P_2 \in \mathrm{id} + p^2 \mathrm{M}_r(\widehat{A}_\lambda)$ be some liftings of the images of $g_1, g_2 \in G_2$ respectively with $P_1 P_2^{-1} \in \mathrm{id} + p^k \mathrm{M}_r(\widehat{A}_\lambda)$ for some $\lambda \in \Lambda$. In particular, $P_1 - P_2 \in p^k \mathrm{M}_r(\widehat{A}_\lambda)$, and

$$\log(P_1) - \log(P_2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} ((P_2 - \mathrm{id} + (P_1 - P_2))^n - (P_2 - \mathrm{id})^n) \in p^k \mathrm{M}_r(\widehat{A}_\lambda) \tag{II.4.12.3}$$

as $p^{2(n-1)} \in n! \cdot \mathbb{Z}_p$ for any $n \geq 1$. Similarly, for any $a_1, a_2 \in \mathbb{Z}_p \setminus \{0\}$ with $a_1 - a_2 \in p^k \mathbb{Z}_p$, we have

$$a_1^{-1}(P_1^{a_1} - \mathrm{id}) - a_1^{-1}(P_2^{a_1} - \mathrm{id}) = \sum_{n=1}^{\infty} a_1^{n-1} \frac{\log(P_1)^n - \log(P_2)^n}{n!} \in p^k \mathrm{M}_r(\widehat{A}_\lambda), \tag{II.4.12.4}$$

$$a_1^{-1}(P_2^{a_1} - \mathrm{id}) - a_2^{-1}(P_2^{a_2} - \mathrm{id}) = \sum_{n=2}^{\infty} (a_1^{n-1} - a_2^{n-1}) \frac{\log(P_2)^n}{n!} \in p^k \mathrm{M}_r(\widehat{A}_\lambda). \tag{II.4.12.5}$$

Thus, $\widetilde{\phi}(a_1, g_1) - \widetilde{\phi}(a_2, g_2)$ belongs to $\psi(p^k \mathrm{M}_r(\widetilde{A}_\infty))$, which implies that $\widetilde{\phi}$ is uniformly continuous by taking $G_1 = G_2$. □

Corollary II.4.13. *Under the assumptions in* II.4.10*, assume further that G is a compact p-adic analytic group. Let* $\mathrm{Lie}(G)$ *be the Lie algebra of G over $\mathbb{Q}_p$. Then, there is a unique morphism of Lie algebras over $\mathbb{Q}_p$,*

$$\varphi : \mathrm{Lie}(G) \longrightarrow \mathrm{End}_A(W), \tag{II.4.13.1}$$

such that its composition with the logarithm map of G (II.3.13.2) $\log_G : G \to \mathrm{Lie}(G)$ *is the map $\varphi : G \to \mathrm{End}_A(W)$ sending g to the infinitesimal action φ_g of $g \in G$ on W.*

We call φ the canonical Lie algebra action induced by the infinitesimal action of G on W, or simply the infinitesimal Lie algebra action*.*

PROOF. Recall that $\mathrm{Lie}(G) = \mathrm{colim}_{G_0} L_{G_0} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ where the colimit is taken over the system of uniform pro-p open subgroups G_0 of G (see II.3.13), and that the $\mathbb{Z}_p$-linear Lie algebra $L_{G_0} = (G_0, +_{G_0}, [\ ,\ ]_{G_0})$ is defined in II.3.3. As $\log : G_0 \to L_{G_0}$ is a homeomorphism, the uniqueness is obvious. It remains to check that the map $L_{G_0} \to \mathrm{End}_A(W)$ sending g to φ_g is compatible with addition and Lie bracket.

As G_0 is topologically finitely generated, there exists $\lambda \in \Lambda$ and a finite projective $\widehat{A}_\lambda[1/p]$-representation W_λ of G_0 such that $W = A \otimes_{\widehat{A}_\lambda[1/p]} W_\lambda$ ([**Tsu18**, 5.2.(1)]). By II.4.11, it suffices to check that the map $L_{G_0} \to \mathrm{End}_{\widehat{A}_\lambda[1/p]}(W_\lambda)$ sending g to φ_g is compatible with addition and Lie bracket. We take a uniform pro-p open subgroup G_1 of G_0 such that $\widehat{A}_\lambda[1/p]$ is G_1-invariant by the assumptions in II.4.10. After replacing G_1 by $G_1^{p^n}$ for some $n \in \mathbb{N}$, we may assume by II.4.12 that the map

$$\phi_\lambda : \mathbb{Z}_p \times G_1 \times W_\lambda \longrightarrow W_\lambda, \tag{II.4.13.2}$$

sending $(0, g, x)$ to $\varphi_g(x)$ and sending (a, g, x) to $a^{-1}(g^a x - x)$ for $a \neq 0$, is continuous. As the map $L_{G_0} \to \mathrm{End}_{\widehat{A}_\lambda[1/p]}(W_\lambda)$ is compatible with multiplication by an integer by II.4.10.(2), it suffices to check that its restriction on L_{G_1} is compatible with addition and Lie bracket. For any $g, g' \in G_1$ and $x \in W_\lambda$, applying the continuity of (II.4.13.2) to the convergent sequence $\{(p^n, (g^{p^n} g'^{p^n})^{p^{-n}}, x)\}_{n \in \mathbb{N}} \subseteq \mathbb{Z}_p \times G_1 \times W_\lambda$ with limit $(0, g +_{G_1} g', x)$ by (II.3.3.1), we get

$$\varphi_{g +_{G_1} g'}(x) = \lim_{n \to \infty} p^{-n}(g^{p^n} g'^{p^n} x - x). \tag{II.4.13.3}$$

On the other hand,

$$(\varphi_g+\varphi_{g'})(x)=\lim_{n\to\infty}p^{-n}(g^{p^n}x+g'^{p^n}x-2x). \tag{II.4.13.4}$$

Thus, $\varphi_{g+_{G_1}g'}(x)-(\varphi_g+\varphi_{g'})(x)=\lim_{n\to\infty}p^{-n}(g^{p^n}-1)(g'^{p^n}x-x)=0$, since the action $G_1\times W_\lambda\to W_\lambda$ is continuous and $\lim_{n\to\infty}p^{-n}(g'^{p^n}x-x)=\varphi_{g'}(x)$. Similarly, applying the continuity of (II.4.13.2) to the convergent sequence $\{(p^{2n},(g^{-p^n}g'^{-p^n}g^{p^n}g'^{p^n})^{p^{-2n}},x)\}_{n\in\mathbb{N}}\subseteq\mathbb{Z}_p\times G_1\times W_\lambda$ with limit $(0,[g,g']_{G_1},x)$ by (II.3.3.2), we get

$$\varphi_{[g,g']_{G_1}}(x)=\lim_{n\to\infty}p^{-2n}(g^{-p^n}g'^{-p^n}g^{p^n}g'^{p^n}x-x). \tag{II.4.13.5}$$

On the other hand, applying the continuity of (II.4.13.2) to the convergent sequence $\{(p^n,g,p^{-n}(g'^{p^n}x-x))\}_{n\in\mathbb{N}}\subseteq\mathbb{Z}_p\times G_1\times W_\lambda$ with limit $(0,g,\varphi_{g'}(x))$, we get

$$(\varphi_g\circ\varphi_{g'}-\varphi_{g'}\circ\varphi_g)(x)=\lim_{n\to\infty}p^{-2n}(g^{p^n}g'^{p^n}x-g'^{p^n}g^{p^n}x). \tag{II.4.13.6}$$

Thus, $\varphi_{[g,g']_{G_1}}(x)-(\varphi_g\circ\varphi_{g'}-\varphi_{g'}\circ\varphi_g)(x)=\lim_{n\to\infty}p^{-2n}(g^{-p^n}g'^{-p^n}-1)(g^{p^n}g'^{p^n}x-g'^{p^n}g^{p^n}x)=0$ by the continuity of the action $G_1\times W_\lambda\to W_\lambda$ and (II.4.13.6). □

Definition II.4.14 (cf. [**Tsu18**, 14.1])**.** Under the assumptions in II.4.10, let H be a subgroup of G. For an object W in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,A)$, we say that W is *H-analytic* if $\exp(\varphi_h)(x)=\sum_{k=0}^{\infty}\varphi_h^k(x)/k!$ converges to $h(x)$ for any $x\in W$ and any $h\in H$. We denote by $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},H\text{-an}}(G,A)$ the full subcategory of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,A)$ formed by H-analytic objects.

II.4.15. Let K be a field of characteristic not equal to p, $\overline{K}$ an algebraic closure of K, $(\zeta_{p^n})_{n\in\mathbb{N}}$ a compatible system of primitive p-power roots of unity in $\overline{K}$, $G=\mathrm{Gal}(\overline{K}/K)$. For any $n\in\mathbb{N}\cup\{\infty\}$, we define a Galois extension of K in $\overline{K}$ by

$$K_n=K(\zeta_{p^k}\mid k\in\mathbb{N}_{\leq n}). \tag{II.4.15.1}$$

Consider the cyclotomic character

$$\chi:G\longrightarrow\mathbb{Z}_p^\times \tag{II.4.15.2}$$

which is defined by $\sigma(\zeta_{p^n})=\zeta_{p^n}^{\chi(\sigma)}$ for any $\sigma\in G$ and $n\in\mathbb{N}$. It factors through an injection $\Sigma=\mathrm{Gal}(K_\infty/K)\hookrightarrow\mathbb{Z}_p^\times$, and does not depend on the choice of the system $(\zeta_{p^n})_{n\in\mathbb{N}}$. In particular, Σ is either finite cyclic or isomorphic to the direct product of a finite cyclic group with $\mathbb{Z}_p$.

We fix $d\in\mathbb{N}$. Let $t_1,\dots,t_d$ be elements of K with compatible systems of p-power roots $(t_{i,p^n})_{n\in\mathbb{N}}$ in $\overline{K}$ (where $1\leq i\leq d$). For consistency, sometimes we also denote ζ_{p^n} by t_{0,p^n}. For any $n\in\mathbb{N}\cup\{\infty\}$ and any $\underline{m}=(m_1,\dots,m_d)\in(\mathbb{N}\cup\{\infty\})^d$, we define an extension of K in $\overline{K}$ by

$$K_{n,\underline{m}}=K_n(t_{1,p^{k_1}},\cdots,t_{d,p^{k_d}}\mid k_i\in\mathbb{N}_{\leq m_i},\ 1\leq i\leq d). \tag{II.4.15.3}$$

It is a Galois extension of K if $n\geq\max\{m_1,\dots,m_d\}$. Consider the continuous map

$$\xi=(\xi_1,\dots,\xi_d):G\longrightarrow\mathbb{Z}_p^d \tag{II.4.15.4}$$

defined by $\tau(t_{i,p^n})=\zeta_{p^n}^{\xi_i(\tau)}t_{i,p^n}$ for any $\tau\in G$, $n\in\mathbb{N}$ and $1\leq i\leq d$. Notice that for any $\sigma,\tau\in G$,

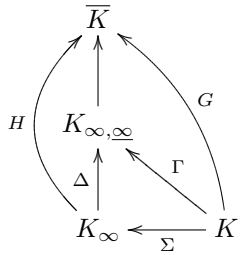
$$\xi(\sigma\tau)=\xi(\sigma)+\chi(\sigma)\xi(\tau) \tag{II.4.15.5}$$

Thus, ξ is a continuous 1-cocycle. It becomes a group homomorphism when restricted to $H=\mathrm{Gal}(\overline{K}/K_\infty)$, which factors through an injection $\Delta=\mathrm{Gal}(K_{\infty,\underline{\infty}}/K_\infty)\hookrightarrow\mathbb{Z}_p^d$. In particular, Δ is isomorphic to $\mathbb{Z}_p^r$ for some $0\leq r\leq d$. For any $\sigma\in\Gamma=\mathrm{Gal}(K_{\infty,\underline{\infty}}/K)$ and $\tau\in\Delta$, we have

$$\sigma\tau\sigma^{-1}=\tau^{\chi(\sigma)}, \tag{II.4.15.6}$$

by the definition of χ. We have named some Galois groups as indicated in the following diagram:

(II.4.15.7)



We remark that Γ is a compact p-adic analytic group as an extension of Σ by Δ, and there is a natural exact sequence of Lie algebras over $\mathbb{Q}_p$ by II.3.14,

$$0 \longrightarrow \mathrm{Lie}(\Delta) \longrightarrow \mathrm{Lie}(\Gamma) \longrightarrow \mathrm{Lie}(\Sigma) \longrightarrow 0. \tag{II.4.15.8}$$

Notice that the group homomorphism $\log\circ\chi : \Gamma \to \mathbb{Z}_p$ induces a homomorphism of $\mathbb{Q}_p$-linear Lie algebras $\log\circ\chi : \mathrm{Lie}(\Gamma) \to \mathrm{Lie}(\mathbb{Z}_p) = \mathbb{Q}_p$ which factors through $\mathrm{Lie}(\Sigma)$, where $\log : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ is the p-adic logarithm map (II.3.6.3). We deduce from (II.4.15.6) and (II.3.3.2) that for any $x \in \mathrm{Lie}(\Gamma)$ and $y \in \mathrm{Lie}(\Delta)$,

$$[x, y] = \log(\chi(x)) \cdot y. \tag{II.4.15.9}$$

II.4.16. Let A be a Noetherian normal domain flat over $\mathbb{Z}_p$ with fraction field K, $t_i \in K$ with a compatible system of p-power roots $(t_{i,p^n})_{n\in\mathbb{N}}$ in $\overline{K}$ for any $1 \leq i \leq d$. With the notation in II.4.15, let $A_{n,\underline{m}}$ be the integral closure of A in $K_{n,\underline{m}}$ for any $n \in \mathbb{N}\cup\{\infty\}$ and $\underline{m} \in (\mathbb{N}\cup\{\infty\})^d$. We remark that $A_{n,\underline{m}}$ is a Noetherian normal domain finite over A if $n, \underline{m}$ are finite by II.4.4. Endowing $\mathbb{N}^{1+d}$ with the product order (cf. II.2.1), we call the tower of Noetherian normal domains $(A_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ the *Kummer tower of A* defined by $\zeta_{p^n}, t_{1,p^n}, \ldots, t_{d,p^n}$.

Notice that Γ acts continuously on $\widetilde{A}_{\infty,\underline{\infty}}[1/p]$ (defined in II.4.7) which satisfies the assumptions in II.4.13. Then, for any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma, \widetilde{A}_{\infty,\underline{\infty}}[1/p])$, there is a canonical morphism of Lie algebras over $\mathbb{Q}_p$ induced by the infinitesimal action of Γ on W,

$$\varphi : \mathrm{Lie}(\Gamma) \longrightarrow \mathrm{End}_{\widetilde{A}_{\infty,\underline{\infty}}[1/p]}(W). \tag{II.4.16.1}$$

Lemma II.4.17 (cf. [**Bri03**, Propositions 5, 7])**.** *With the notation in* II.4.16*, for any $\sigma \in \Gamma$, $\tau \in \Delta$ and any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma, \widetilde{A}_{\infty,\underline{\infty}}[1/p])$, we have*

$$\sigma \circ \varphi_\tau \circ \sigma^{-1} = \chi(\sigma) \cdot \varphi_\tau, \tag{II.4.17.1}$$

$$\varphi_\sigma \circ \varphi_\tau - \varphi_\tau \circ \varphi_\sigma = \log(\chi(\sigma)) \cdot \varphi_\tau, \tag{II.4.17.2}$$

as $\widetilde{A}_{\infty,\underline{\infty}}[1/p]$-linear endomorphisms on W.

Proof. As $\sigma\tau\sigma^{-1} = \tau^{\chi(\sigma)}$ (II.4.15.6), we have $\sigma \circ \varphi_\tau \circ \sigma^{-1} = \varphi_{\sigma\tau\sigma^{-1}} = \varphi_{\tau^{\chi(\sigma)}} = \chi(\sigma) \cdot \varphi_\tau$ (cf. II.4.9). As $[\log_\Gamma(\sigma), \log_\Gamma(\tau)] = \log(\chi(\sigma)) \cdot \log_\Gamma(\tau)$ by (II.4.15.9), we have $\varphi_\sigma \circ \varphi_\tau - \varphi_\tau \circ \varphi_\sigma = \varphi_{[\log_\Gamma(\sigma), \log_\Gamma(\tau)]} = \log(\chi(\sigma)) \cdot \varphi_\tau$. □

Proposition II.4.18 (cf. [**Bri03**, Proposition 5], [**Tsu18**, 14.17])**.** *With the notation in* II.4.16*, assume that K_∞ is an infinite extension of K and that $\mathfrak{S}_p(A_{\infty,\underline{\infty}})$ is finite. Then, the infinitesimal action φ_τ of any element $\tau \in \Delta$ on any object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma, \widetilde{A}_{\infty,\underline{\infty}}[1/p])$ is nilpotent.*

Proof. We follow the proof of [**Tsu18**, 14.17]. Notice that Σ identifies to an open subgroup of $\mathbb{Z}_p^\times$ via the cyclotomic character (II.4.15.2). Thus, there exists $\sigma \in \Gamma$ such that $\log(\chi(\sigma)) \neq 0$. For any $\mathfrak{q} \in \mathfrak{S}_p(A_{\infty,\underline{\infty}})$, the localization $A_{\infty,\underline{\infty},\mathfrak{q}}$ is a valuation ring of height 1 by II.4.3.(1), and we denote by $L_\mathfrak{q}$ the fraction field of the p-adic completion of $A_{\infty,\underline{\infty},\mathfrak{q}}$. Then, the natural map

$$\widehat{A}_{\infty,\underline{\infty}}[1/p] \longrightarrow \prod_{\mathfrak{q}\in\mathfrak{S}_p(A_{\infty,\underline{\infty}})} L_\mathfrak{q} \tag{II.4.18.1}$$

is injective by II.4.3.(2).

For an object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma, \widetilde{A}_{\infty,\underline{\infty}}[1/p])$ and $g \in \Gamma$, we denote by $W_\mathfrak{q}$ (resp. $\varphi_{g,\mathfrak{q}}$) the scalar extension of W (resp. φ_g) to $L_\mathfrak{q}$. As $\mathfrak{S}_p(A_{\infty,\underline{\infty}})$ is finite, it suffices to show that $\varphi_{\tau,\mathfrak{q}}$ is nilpotent by the injection (II.4.18.1) and II.4.11. Let $P_\mathfrak{q}(T) \in L_\mathfrak{q}[T]$ be the characteristic polynomial of $\varphi_{\sigma,\mathfrak{q}}$. We take an integer $N > 0$ large enough such that $P_\mathfrak{q}(T - N\log(\chi(\sigma)))$ is prime to $P_\mathfrak{q}(T)$. Thus, the endomorphism $P_\mathfrak{q}(\varphi_{\sigma,\mathfrak{q}} - N\log(\chi(\sigma)))$ on $W_\mathfrak{q}$ is an automorphism. By (II.4.17.2), we have

$$P_\mathfrak{q}(\varphi_{\sigma,\mathfrak{q}} - N\log(\chi(\sigma))) \circ (\varphi_{\tau,\mathfrak{q}})^N = (\varphi_{\tau,\mathfrak{q}})^N \circ P_\mathfrak{q}(\varphi_{\sigma,\mathfrak{q}}) = 0. \tag{II.4.18.2}$$

Hence, we have $(\varphi_{\tau,\mathfrak{q}})^N = 0$. □

Proposition II.4.19. *With the notation in* II.4.16*, consider the following statements:*

(1) *We have $A/pA \neq 0$ (thus $\mathfrak{S}_p(A)$ is non-empty), and for any $\mathfrak{p} \in \mathfrak{S}_p(A)$, if we denote by $\widehat{K}^\mathfrak{p}$ the completion of K with respect to the discrete valuation ring $A_\mathfrak{p}$ and consider the Kummer tower $(\widehat{K}^\mathfrak{p}_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ of $\widehat{K}^\mathfrak{p}$ defined by $\zeta_{p^n}, t_{1,p^n}, \ldots, t_{d,p^n}$, then the image of the continuous homomorphism* (II.4.15.4) $\mathrm{Gal}(\widehat{K}^\mathfrak{p}_{\infty,\underline{\infty}}/\widehat{K}^\mathfrak{p}_\infty) \to \mathbb{Z}_p^d$ *is open.*

(2) *There exists* $n_0 \in \mathbb{N}$ *such that for any* $n \in \mathbb{N}_{\geq n_0}$, *the cyclotomic character* (II.4.15.2) $\chi : G \to \mathbb{Z}_p^\times$ *and the p-adic logarithm map* (II.3.6.3) $\log : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ *induce an isomorphism*

$$\log \circ \chi : \mathrm{Gal}(K_{\infty,\underline{\infty}}/K_{n,\underline{\infty}}) \xrightarrow{\sim} p^n\mathbb{Z}_p; \tag{II.4.19.1}$$

and there exists $\underline{m_0} \in \mathbb{N}^d$ *such that for any* $\underline{m} \in (\mathbb{N}^d)_{\geq \underline{m_0}}$ *the continuous* 1*-cocycle* (II.4.15.4) $\xi : G \to \mathbb{Z}_p^d$ *induces an isomorphism*

$$\xi : \mathrm{Gal}(K_{\infty,\underline{\infty}}/K_{\infty,\underline{m}}) \xrightarrow{\sim} p^{m_1}\mathbb{Z}_p \times \cdots \times p^{m_d}\mathbb{Z}_p \tag{II.4.19.2}$$

where $\underline{m} = (m_1, \ldots, m_d)$.

(3) *There exists* $(n_0, \underline{m_0}) \in \mathbb{N}^{1+d}$ *such that for any* $(n, \underline{m}) \in (\mathbb{N}^{1+d})_{\geq (n_0,\underline{m_0})}$, *the natural map* $\mathfrak{S}_p(A_{n,\underline{m}}) \to \mathfrak{S}_p(A_{n_0,\underline{m_0}})$ *is a bijection.*

(4) *The cardinality of* $\mathfrak{S}_p(A_{n,\underline{m}})$, *when* $(n, \underline{m})$ *varies in* $\mathbb{N}^{1+d}$, *is bounded.*

Then, (1) *implies* (2), (3) *and* (4)*; and* (3) *is equivalent to* (4).

Proof. Notice that for elements $(n', \underline{m'}) \geq (n, \underline{m})$ in $\mathbb{N}^{1+d}$, the natural map $\mathfrak{S}_p(A_{n',\underline{m'}}) \to \mathfrak{S}_p(A_{n,\underline{m}})$ is surjective by II.4.2. Since $\mathbb{N}^{1+d}$ is directed, we see that (3) and (4) are equivalent.

Now we assume (1). We take an integer $m_0 \in \mathbb{N}$ such that $(p^{m_0}\mathbb{Z}_p)^d$ lies in the image of the injective homomorphism $\xi : \mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}}_{\infty}) \to \mathbb{Z}_p^d$ (II.4.15.4). We identify $(p^{m_0}\mathbb{Z}_p)^d$ with an open normal subgroup of $\mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}}_{\infty})$. We claim that the invariant subextension of $\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}$ by $p^{m_1}\mathbb{Z}_p \times \cdots \times p^{m_d}\mathbb{Z}_p \subseteq (p^{m_0}\mathbb{Z}_p)^d$ is $\widehat{K}^{\mathfrak{p}}_{\infty,\underline{m}}$ for any $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d_{\geq m_0}$. Indeed, the invariant subextension contains $\widehat{K}^{\mathfrak{p}}_{\infty,\underline{m}}$ by the definition of ξ. On the other hand, $\mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}}_{\infty,\underline{m}})$ identifies with a closed subgroup of $p^{m_1}\mathbb{Z}_p \times \cdots \times p^{m_d}\mathbb{Z}_p$ via ξ. Thus, the claim follows from the Galois theory. In particular, ξ induces a natural isomorphism

$$\xi : \mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}}_{\infty,\underline{m}}) \xrightarrow{\sim} p^{m_1}\mathbb{Z}_p \times \cdots \times p^{m_d}\mathbb{Z}_p. \tag{II.4.19.3}$$

We claim that $\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}$ is an infinite extension of $\widehat{K}^{\mathfrak{p}}_{0,\underline{\infty}}$. Otherwise, $\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}$ is an extension of a finite extension of $\widehat{K}^{\mathfrak{p}}$ by adding $t_{1,p^n}, \ldots, t_{d,p^n}$, so that the dimension of the p-adic analytic group $\mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}})$ is no more than d by (II.4.15.4). On the other hand, $\widehat{K}^{\mathfrak{p}}$ is a complete discrete valuation field, while the valuation on $\widehat{K}^{\mathfrak{p}}_{\infty}$ is non-discrete of height 1. Thus, $\mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty}/\widehat{K}^{\mathfrak{p}})$ is an open subgroup of $\mathbb{Z}_p^\times$, which implies that the dimension of the p-adic analytic group $\mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}})$ is $1+d$ under the assumption (1) by (II.4.15.8). We get a contradiction, which proves the claim. Thus, the image of the cyclotomic character

$$\chi : \mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}}_{0,\underline{\infty}}) \longrightarrow \mathbb{Z}_p^\times \tag{II.4.19.4}$$

is open. We take $n_0 \in \mathbb{N}_{\geq 2}$ such that $1 + p^{n_0}\mathbb{Z}_p$ lies in the image of (II.4.19.4). Similarly as above, the invariant subextension of $\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}$ by $1 + p^n\mathbb{Z}_p \subseteq 1 + p^{n_0}\mathbb{Z}_p$ is $\widehat{K}^{\mathfrak{p}}_{n,\underline{\infty}}$ by the definition of χ, for any $n \in \mathbb{N}_{\geq n_0}$. In particular, χ and log induce an isomorphism

$$\log \circ \chi : \mathrm{Gal}(\widehat{K}^{\mathfrak{p}}_{\infty,\underline{\infty}}/\widehat{K}^{\mathfrak{p}}_{n,\underline{\infty}}) \xrightarrow{\sim} p^n\mathbb{Z}_p. \tag{II.4.19.5}$$

For any $(n, \underline{m}) \in (\mathbb{N}^{1+d})_{\geq (n_0,\underline{m_0})}$, we have

$$[\widehat{K}^{\mathfrak{p}}_{\infty,\underline{m}} : \widehat{K}^{\mathfrak{p}}_{\infty,\underline{m_0}}] \leq [\widehat{K}^{\mathfrak{p}}_{n,\underline{m}} : \widehat{K}^{\mathfrak{p}}_{n,\underline{m_0}}] \leq [K_{n,\underline{m}} : K_{n,\underline{m_0}}] \leq p^{\sum_{i=1}^d (m_i - m_0)}. \tag{II.4.19.6}$$

By (II.4.19.3), we see that the inequalities in (II.4.19.6) are equalities, which implies (II.4.19.2). In particular, each fibre of $\mathfrak{S}_p(A_{n,\underline{m}}) \to \mathfrak{S}_p(A_{n,\underline{m_0}})$ consists of a single element (cf. [**Bou06**, VI.§8.5, Cor.3]). Similarly,

$$[\widehat{K}^{\mathfrak{p}}_{n,\underline{\infty}} : \widehat{K}^{\mathfrak{p}}_{n_0,\underline{\infty}}] \leq [\widehat{K}^{\mathfrak{p}}_{n,\underline{m_0}} : \widehat{K}^{\mathfrak{p}}_{n_0,\underline{m_0}}] \leq [K_{n,\underline{m_0}} : K_{n_0,\underline{m_0}}] \leq p^{n-n_0}. \tag{II.4.19.7}$$

By (II.4.19.5), we see that the inequalities in (II.4.19.7) are equalities, which implies (II.4.19.1). In particular, each fibre of $\mathfrak{S}_p(A_{n,\underline{m_0}}) \to \mathfrak{S}_p(A_{n_0,\underline{m_0}})$ consists of a single element. Therefore, we obtain (3). □

We will give in II.5.22 and II.11.2 some differential criteria for checking the condition II.4.19.(1) for a Kummer tower.

II.4.20. With the notation in II.4.16, for further computation (e.g. II.5.28), we introduce a standard basis of $\mathrm{Lie}(\Gamma)$ under the assumption that the Kummer tower $(A_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ satisfies the condition II.4.19.(2). We name some Galois groups as indicated in the following diagram for any $n\in\mathbb{N}$:

(II.4.20.1)
$$\begin{array}{ccc} K_{\infty,\underline{\infty}} & \xleftarrow{\Sigma_{n,\underline{\infty}}} & K_{n,\underline{\infty}} \\ {\scriptstyle\Delta}\uparrow & \nwarrow^{\Gamma_n} & \uparrow \\ K_\infty & \xleftarrow[\Sigma_n]{} & K_n \end{array}$$

By the assumption II.4.19.(2), there is an isomorphism for some $n_0\in\mathbb{N}$,

(II.4.20.2)
$$\log\circ\chi : \Sigma_{n_0,\underline{\infty}} \xrightarrow{\sim} \Sigma_{n_0} \xrightarrow{\sim} p^{n_0}\mathbb{Z}_p,$$

and ξ identifies Δ with an open subgroup of $\mathbb{Z}_p^d$. The isomorphism $\Sigma_{n_0,\underline{\infty}} \xrightarrow{\sim} \Sigma_{n_0}$ identifies Γ_{n_0} with the semi-direct product $\Sigma_{n_0}\ltimes\Delta$ defined by $\sigma\in\Sigma_{n_0}$ acting on $\tau\in\Delta$ by $\tau\mapsto\tau^{\chi(\sigma)}$ (cf. (II.4.15.6)). Moreover, there is an open embedding of topological groups

(II.4.20.3)
$$(\log\circ\chi,\xi) : \Gamma_{n_0} = \Sigma_{n_0}\ltimes\Delta \longrightarrow \mathbb{Z}_p\ltimes\mathbb{Z}_p^d$$

where $\mathbb{Z}_p\ltimes\mathbb{Z}_p^d$ is the semi-direct product of $\mathbb{Z}_p$ acting on $\mathbb{Z}_p^d$ by multiplication. It induces an isomorphism of $\mathbb{Q}_p$-linear Lie algebras

(II.4.20.4)
$$\mathrm{Lie}(\Gamma) \xrightarrow{\sim} \mathrm{Lie}(\mathbb{Z}_p\ltimes\mathbb{Z}_p^d).$$

Let $\partial_i\in\mathrm{Lie}(\Gamma)$ be the image of $(0,\dots,1,\dots,0)\in\mathbb{Z}_p\ltimes\mathbb{Z}_p^d$ (where 1 appears at the i-th component) via the logarithm map of $\mathbb{Z}_p\ltimes\mathbb{Z}_p^d$ and (II.4.20.4). We deduce from (II.4.15.9) that for any $1\le i,j\le d$,

(II.4.20.5)
$$[\partial_0,\partial_i]=\partial_i \quad\text{and}\quad [\partial_i,\partial_j]=0,$$

and we deduce from (II.4.15.8) that $\partial_0,\partial_1,\dots,\partial_d$ form a $\mathbb{Q}_p$-basis of $\mathrm{Lie}(\Gamma)$, which we call the *standard basis*. Moreover, if we extend $\xi_1,\dots,\xi_d$ to $\mathbb{Q}_p$-linear forms on $\mathrm{Lie}(\Delta)$, then we see that they form a dual basis of $\partial_1,\dots,\partial_d$.

Consider an object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma,\widetilde{A}_{\infty,\underline{\infty}}[1/p])$ and the canonical Lie algebra action (II.4.16.1) $\varphi:\mathrm{Lie}(\Gamma)\to\mathrm{End}_{\widetilde{A}_{\infty,\underline{\infty}}[1/p]}(W)$ induced by the infinitesimal action of Γ on W. For any $g\in G$, we set

(II.4.20.6)
$$\varphi_g^\chi = \log(\chi(g))\varphi_{\partial_0},$$

which defines a continuous group homomorphism $\varphi^\chi : G\to\mathrm{End}_{\widetilde{A}_{\infty,\underline{\infty}}[1/p]}(W)$ factoring through Γ with $\varphi^\chi|_{\Sigma_{0,\underline{\infty}}}=\varphi|_{\Sigma_{0,\underline{\infty}}}$ and $\varphi^\chi|_\Delta=0$. We also set

(II.4.20.7)
$$\varphi_g^\xi = \sum_{i=1}^d \xi_i(g)\varphi_{\partial_i},$$

which defines a continuous 1-cocycle $\varphi^\xi : G\to\mathrm{End}_{\widetilde{A}_{\infty,\underline{\infty}}[1/p]}(W)$ factoring through Γ with $\varphi^\xi|_\Delta=\varphi|_\Delta$ and $\varphi^\xi|_{\Sigma_{0,\underline{\infty}}}=0$.

Lemma II.4.21. *Under the assumptions in* II.4.20*, for any $g\in\Gamma$ with $\log(\chi(g))\neq 0$, we have*

(II.4.21.1)
$$\log_\Gamma(g) = \log(\chi(g))\partial_0 + \frac{\log(\chi(g))}{\chi(g)-1}\sum_{i=1}^d \xi_i(g)\partial_i \in \mathrm{Lie}(\Gamma).$$

In particular, for any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma,\widetilde{A}_{\infty,\underline{\infty}}[1/p])$, we have

(II.4.21.2)
$$\varphi_g = \varphi_g^\chi + \frac{\log(\chi(g))}{\chi(g)-1}\varphi_g^\xi$$

as $\widetilde{A}_{\infty,\underline{\infty}}[1/p]$-linear endomorphisms of W.

Proof. Since for any $r\ge 1$ we have $\log(\chi(g^r))=r\log(\chi(g))$ and $\xi(g^r)/(\chi(g^r)-1)=\xi(g)/(\chi(g)-1)$ by (II.4.15.5), it suffices to prove (II.4.21.1) for g^r. Thus, we may assume that $g\in\Gamma_{n_0}$ and let $g=\tau\sigma$ be the unique decomposition for some $\tau\in\Delta$ and $\sigma\in\Sigma_{n_0,\underline{\infty}}$. Since $\log(\chi(g))=\log(\chi(\sigma))$ and $\xi(g)=\xi(\tau)$, we have $\log(\chi(g))\partial_0=\log_\Gamma(\sigma)$ and $\sum_{i=1}^d\xi_i(g)\partial_i=\log_\Gamma(\tau)$. It remains to check that in $\mathrm{Lie}(\Gamma)$, we have

(II.4.21.3)
$$\log_\Gamma(g) = \log_\Gamma(\sigma) + \frac{\log(\chi(g))}{\chi(g)-1}\log_\Gamma(\tau).$$

By iteratively using the identity $\sigma\tau\sigma^{-1}=\tau^{\chi(\sigma)}$, we get $g^{p^n}=\tau^{\frac{\chi(\sigma)^{p^n}-1}{\chi(\sigma)-1}}\sigma^{p^n}$. After enlarging n_0, we may assume that Γ_{n_0} is contained in a uniform pro-p open subgroup Γ' of Γ. Thus, by (II.3.3.1),

$$g+_{\Gamma'}\sigma^{-1}=\lim_{n\to\infty}(g^{p^n}\sigma^{-p^n})^{p^{-n}}=\lim_{n\to\infty}\tau^{\frac{\chi(\sigma)^{p^n}-1}{p^n(\chi(\sigma)-1)}}=\tau^{\frac{\log(\chi(g))}{\chi(g)-1}}, \tag{II.4.21.4}$$

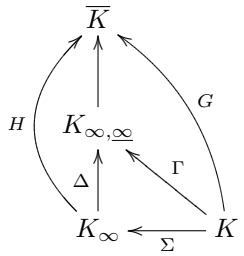
which completes the proof. □

II.5. Revisiting Brinon's Generalization of Sen's Theory after Tsuji

In this section, we revisit Brinon's generalization [**Bri03**] of Sen's theory following Tsuji [**Tsu18**, §15]. More precisely, we establish a p-adic Simpson correspondence over a complete discrete valuation field of mixed characteristic (cf. II.5.31). Then, we give a canonical definition of Sen operators, which does not depend on choosing a p-basis of the residue field (and its p-power roots) (cf. II.5.37).

II.5.1. We use the following notation in this section. Let K be a complete discrete valuation field of characteristic 0 whose residue field κ is of characteristic $p>0$ such that $[\kappa:\kappa^p]=p^d<\infty$ (i.e. κ admits a finite p-basis, cf. [**EGA** IV_1, 21.1.9]). We fix an algebraic closure $\overline{K}$ of K, and denote by $\widehat{\overline{K}}$ its p-adic completion. Let $t_1,\dots,t_d$ be d elements of $\mathcal{O}_K^\times$ with compatible systems of p-power roots $(t_{1,p^n})_{n\in\mathbb{N}},\cdots,(t_{d,p^n})_{n\in\mathbb{N}}$ in $\mathcal{O}_{\overline{K}}^\times$ such that the images of $t_1,\dots,t_d$ in κ form a p-basis. We consider the Kummer tower $(\mathcal{O}_{K_{n,\underline{m}}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ of $\mathcal{O}_K$ defined by $\zeta_{p^n},t_{1,p^n},\dots,t_{d,p^n}$ (II.4.16). We take again the notation in II.4.15.

(II.5.1.1)



Lemma II.5.2. *There exists a complete discrete valuation subfield K' of K with $\mathcal{O}_{K'}/p\mathcal{O}_{K'}=\kappa$ such that K is a totally ramified finite extension of K' and that $t_1,\dots,t_d\in\mathcal{O}_{K'}$.*

Proof. Let π be a uniformizer of K. By Cohen structure theorem [**EGA** IV_1, 19.8.8], there exists a complete discrete valuation ring R extension of $\mathbb{Z}_p$ with a local injective homomorphism $f:R\to\mathcal{O}_K$ which induces an isomorphism $f_1:R/pR\xrightarrow{\sim}\mathcal{O}_K/\pi\mathcal{O}_K=\kappa$. We take $s_1,\dots,s_d\in R$ lifting the images of $t_1,\dots,t_d\in\mathcal{O}_K$ in κ respectively. We claim that it suffices to find a series of homomorphisms $(f_n:R/p^nR\to\mathcal{O}_K/\pi^n\mathcal{O}_K)_{n\geq 2}$ such that $f_n(s_i)=t_i$ and f_n lifts f_{n-1}. Indeed, this series defines a homomorphism $f_\infty:R\to\mathcal{O}_K$ by taking limit on n, which sends s_i to t_i and identifies the residue fields. Thus, f_∞ is finite ([**Sta22**, 031D]) and thus a totally ramified extension of discrete valuation rings. The claim follows.

We construct $(f_n)_{n\geq 2}$ inductively. Suppose that we have constructed f_{n-1}. We fix a lifting $\widetilde{f}_{n-1}:R/p^nR\to\mathcal{O}_K/\pi^n\mathcal{O}_K$ of f_{n-1}, and consider the commutative diagram

(II.5.2.1)

$$\begin{array}{ccc}\mathcal{O}_K/\pi^n\mathcal{O}_K & \longrightarrow & \mathcal{O}_K/\pi^{n-1}\mathcal{O}_K\\ \uparrow & \overset{\widetilde{f}_{n-1}}{\nearrow} & \uparrow{\scriptstyle f_{n-1}}\\ \mathbb{Z}/p^n\mathbb{Z} & \longrightarrow & R/p^nR\end{array}$$

There is a map

$$\mathrm{Hom}_R(\Omega^1_{(R/p^nR)/(\mathbb{Z}/p^n\mathbb{Z})},\pi^{n-1}\mathcal{O}_K/\pi^n\mathcal{O}_K)\longrightarrow\mathrm{Hom}_{\mathbb{Z}\text{-Alg}}(R/p^nR,\mathcal{O}_K/\pi^n\mathcal{O}_K) \tag{II.5.2.2}$$

sending D to $\widetilde{f}_{n-1}+D\circ\mathrm{d}_{R/p^nR}$. Recall that $\Omega^1_{(R/p^nR)/(\mathbb{Z}/p^n\mathbb{Z})}$ is a finite free R/p^n-module with basis $\mathrm{d}s_1,\dots,\mathrm{d}s_d$ ([**He21**, 3.2]). We can take D sending $\mathrm{d}s_i$ to $t_i-\widetilde{f}_{n-1}(s_i)$, as $t_i=f_{n-1}(s_i)$ by the induction hypothesis. Taking $f_n=\widetilde{f}_{n-1}+D\circ\mathrm{d}_{R/p^nR}$, we see that $t_i=f_n(s_i)$ and f_n lifts f_{n-1}, which completes the induction. □

Lemma II.5.3. *Let K' be a subfield of K as in* II.5.2, *$(\mathcal{O}_{K'_{n,\underline{m}}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ the Kummer tower of $\mathcal{O}_{K'}$ defined by $\zeta_{p^n},t_{1,p^n},\dots,t_{d,p^n}$ (*II.4.16*).*

(1) *The extension K'_n over K' is totally ramified, and $\mathcal{O}_{K'_n}=\mathcal{O}_{K'}[X_0]/(\frac{X_0^{p^n}-1}{X_0^{p^{n-1}}-1})=\mathcal{O}_{K'}[\zeta_{p^n}]$.*

(2) *The extension* $K'_{n,\underline{m}}$ *over* K'_n *is weakly unramified, and* $\mathcal{O}_{K'_{n,\underline{m}}} = \mathcal{O}_{K'_n}[X_1, \dots, X_d]/(X_1^{p^{m_1}} - t_1, \dots, X_d^{p^{m_d}} - t_d) = \mathcal{O}_{K'}[\zeta_{p^n}, t_{1,p^{m_1}}, \dots, t_{d,p^{m_d}}]$, *where* $\underline{m} = (m_1, \dots, m_d)$.
(3) *The cyclotomic character* (II.4.15.2) $\chi : \mathrm{Gal}(K'_{\infty,\underline{\infty}}/K'_{0,\underline{\infty}}) \to \mathbb{Z}_p^\times$ *is an isomorphism.*
(4) *The continuous homomorphism* (II.4.15.4) $\xi : \mathrm{Gal}(K'_{\infty,\underline{\infty}}/K'_\infty) \to \mathbb{Z}_p^d$ *is an isomorphism.*

Proof. The first two assertions follow from the fact that p is a uniformizer of the complete discrete valuation field K' and that $t_1, \dots, t_d$ form a p-basis of its residue field (cf. [**Ser79**, I.§6]). One can deduce easily the last two assertions from the first two by the arguments of II.4.19. □

Proposition II.5.4. *The Kummer tower* $(\mathcal{O}_{K_{n,\underline{m}}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ *satisfies the condition* II.4.19.(1).

Proof. We take K' as in II.5.2. Since K is a finite extension of K', the Galois group Δ identifies with an open subgroup of $\mathrm{Gal}(K'_{\infty,\underline{\infty}}/K'_\infty)$. The conclusion follows from II.5.3.(4). □

Remark II.5.5. Let $n_0 \in \mathbb{N}_{\geq 2}$ such that $K \cap K'_{\infty,\underline{\infty}} \subseteq K'_{n_0,\underline{n_0}}$. Then, for any $(n, \underline{m}) \in \mathbb{N}^{1+d}_{\geq n_0}$, the natural map $\mathrm{Gal}(K_{\infty,\underline{\infty}}/K_{n,\underline{m}}) \to \mathrm{Gal}(K'_{\infty,\underline{\infty}}/K'_{n,\underline{m}})$ is an isomorphism by Galois theory. In particular, the conclusion of II.4.19.(2) for $(\mathcal{O}_{K_{n,\underline{m}}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ holds for any $(n, \underline{m}) \in \mathbb{N}^{1+d}_{\geq n_0}$ by II.5.3.

II.5.6. Recall that the $\mathcal{O}_K$-module $\widehat{\Omega}^1_{\mathcal{O}_K}$ (defined in II.2.2) is finitely generated whose free part has rank d, and that $\widehat{\Omega}^1_{\mathcal{O}_K}[1/p]$ admits a K-basis $\mathrm{d}\log(t_1), \dots, \mathrm{d}\log(t_d)$ (cf. [**He21**, 3.3]). For simplicity, we set (cf. II.2.2)

$$\mathscr{E}_{\mathcal{O}_K} = V_p(\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}) = \varprojlim_{x\mapsto px} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}. \tag{II.5.6.1}$$

It is a $\widehat{\overline{K}}$-module as $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$ is p-primary torsion ([**He21**, 4.2]), and endowed with the natural action of G. For any $(s_{p^n})_{n\in\mathbb{N}} \in V_p(\overline{K}^\times)$, we take $k \in \mathbb{N}$ sufficiently large such that $p^k s_1, p^k s_1^{-1} \in \mathcal{O}_{\overline{K}}$ (thus $p^k s_{p^n}^{\pm 1} \in \mathcal{O}_{\overline{K}}$). The element $p^{-2k}(p^k s_{p^n}^{-1} \mathrm{d}(p^k s_{p^n}))_{n\in\mathbb{N}} \in \mathscr{E}_{\mathcal{O}_K}$ does not depend on the choice of k, which we denote by $(\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}$. Similarly, we define $\mathrm{d}\log(s) \in \widehat{\Omega}^1_{\mathcal{O}_K}[1/p]$ for any $s \in K^\times$.

Theorem II.5.7 ([**He21**, 4.4])**.** *There is a canonical* G*-equivariant exact sequence of* $\widehat{\overline{K}}$*-modules, called the* Faltings extension *of* $\mathcal{O}_K$*,*

$$0 \longrightarrow \widehat{\overline{K}}(1) \xrightarrow{\iota} \mathscr{E}_{\mathcal{O}_K} \xrightarrow{\jmath} \widehat{\overline{K}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow 0, \tag{II.5.7.1}$$

satisfying the following properties:

(1) *We have* $\iota((\zeta_{p^n})_{n\in\mathbb{N}}) = (\mathrm{d}\log(\zeta_{p^n}))_{n\in\mathbb{N}}$.
(2) *For any* $s \in K^\times$ *and any compatible system of* p*-power roots* $(s_{p^n})_{n\in\mathbb{N}}$ *of* s *in* $\overline{K}$*,* $\jmath(\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}) = \mathrm{d}\log(s)$.
(3) *The* $\widehat{\overline{K}}$*-linear surjection* $\jmath$ *admits a section sending* $\mathrm{d}\log(t_i)$ *to* $(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}$ *for any* $1 \leq i \leq d$.

In particular, $\mathscr{E}_{\mathcal{O}_K}$ *is a finite free* $\widehat{\overline{K}}$*-module with basis* $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\leq i\leq d}$*, where* $t_{0,p^n} = \zeta_{p^n}$*, on which* G *acts continuously with respect to the canonical topology (where* $\widehat{\overline{K}}$ *is endowed with the* p*-adic topology defined by its valuation ring).*

Proof. The sequence (II.5.7.1) is constructed in [**He21**, 4.4] and (1), (3) are proved there. Notice that (2) follows from the constructing process [**He21**, (4.4.5)] (see also II.9.32 for a detailed proof). For the "in particular" part, it remains to check the continuity of the G-action. We set $\alpha_i = (\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}} \in \mathscr{E}_{\mathcal{O}_K}$ for any $0 \leq i \leq d$. For any $g \in G$ and $1 \leq i \leq d$, we have

$$g(\alpha_0) = \chi(g)\alpha_0 \quad \text{and} \quad g(\alpha_i) = \xi_i(g)\alpha_0 + \alpha_i, \tag{II.5.7.2}$$

where $\chi : G \to \mathbb{Z}_p^\times$ is the cyclotomic character (II.4.15.2) and $\xi = (\xi_1, \dots, \xi_d) : G \to \mathbb{Z}_p^d$ is the continuous 1-cocycle (II.4.15.4). The elements $\alpha_0, \dots, \alpha_d$ generate a finite free $\mathcal{O}_{\widehat{\overline{K}}}$-submodule $\mathscr{E}^+_{\mathcal{O}_K}$ of $\mathscr{E}_{\mathcal{O}_K}$ which is G-stable. For any $r \in \mathbb{N}$, each element of $\mathscr{E}_{\mathcal{O}_K}/p^r\mathscr{E}^+_{\mathcal{O}_K} = \oplus_{i=0}^d(\overline{K}/p^r\mathcal{O}_{\overline{K}})\alpha_i$ is fixed by an open subgroup of G, which implies that the map $G \times (\mathscr{E}_{\mathcal{O}_K}/p^r\mathscr{E}^+_{\mathcal{O}_K}) \to \mathscr{E}_{\mathcal{O}_K}/p^r\mathscr{E}^+_{\mathcal{O}_K}$ (given by the action of G) is continuous with respect to the discrete topology on $\mathscr{E}_{\mathcal{O}_K}/p^r\mathscr{E}^+_{\mathcal{O}_K}$. Taking inverse limit on $r \in \mathbb{N}$, we see that $G \times \mathscr{E}_{\mathcal{O}_K} \to \mathscr{E}_{\mathcal{O}_K}$ is continuous with respect to the limit topology on $\mathscr{E}_{\mathcal{O}_K}$, which indeed coincides with the canonical topology (II.2.3). □

Remark II.5.8. The Faltings extension (II.5.7.1) is functorial in the following sense: let K' be a complete discrete valuation field extension of K whose residue field admits a finite p-basis, $\overline{K} \to \overline{K'}$

a compatible embedding of the algebraic closures of K and K'. It defines a natural map $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \to \Omega^1_{\mathcal{O}_{\overline{K'}}/\mathcal{O}_{K'}}$ by pullback and thus a natural morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\overline{K}}(1) & \xrightarrow{\iota} & \mathscr{E}_{\mathcal{O}_K} & \xrightarrow{\jmath} & \widehat{\overline{K}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{\overline{K'}}(1) & \xrightarrow{\iota} & \mathscr{E}_{\mathcal{O}_{K'}} & \xrightarrow{\jmath} & \widehat{\overline{K'}} \otimes_{\mathcal{O}_{K'}} \widehat{\Omega}^1_{\mathcal{O}_{K'}} & \longrightarrow & 0 \end{array} \tag{II.5.8.1}$$

Moreover, if K' is a finite extension of K, then $K' \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \to K' \otimes_{\mathcal{O}_{K'}} \widehat{\Omega}^1_{\mathcal{O}_{K'}}$ is an isomorphism (cf. the proof of [**He21**, 3.3]). Thus, the vertical maps in (II.5.8.1) are isomorphisms.

Corollary II.5.9. *The connecting map of the Faltings extension* (II.5.7.1) *induces a canonical* $\widehat{K_\infty}$*-linear isomorphism*

$$\widehat{K_\infty} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \xrightarrow{\sim} H^1(H, \widehat{\overline{K}}(1)), \tag{II.5.9.1}$$

sending $\mathrm{d}\log(t_i)$ *to* $\xi_i \otimes \zeta$*, where* H^1 *denotes the continuous group cohomology,* $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}} \in \mathbb{Z}_p(1)$ *and* $\xi = (\xi_1, \dots, \xi_d) : H \to \mathbb{Z}_p^d$ *is the continuous* 1*-cocycle* (II.4.15.4).

Proof. We remark that the Faltings extension (II.5.7.1) is an exact sequence of finite projective $\widehat{\overline{K}}$-representations of G, which admits a continuous splitting (not G-equivariant), so that we obtain a long exact sequence of continuous group cohomologies (cf. [**Tat76**, §2]). The corollary follows from Hyodo's computation of $H^1(H, \widehat{\overline{K}}(1))$ (cf. [**Hyo86**, 2-1, 5-1]). We will give a detailed proof in II.13.2. □

Remark II.5.10. A similar result for $H^1(G, \widehat{\overline{K}}(1))$ is given in [**He21**, 4.5], relying on Hyodo's computation.

II.5.11. We set

$$\mathscr{C}_{\mathrm{HT}} = \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)), \tag{II.5.11.1}$$

where Sym^n is taking the homogeneous part of degree n of the symmetric algebra, and the transition map $\mathrm{Sym}^n \to \mathrm{Sym}^{n+1}$ is defined by sending $[x_1 \otimes \cdots \otimes x_n]$ to $[1 \otimes x_1 \otimes \cdots \otimes x_n]$ (where 1 denotes the image of $1 \in \widehat{\overline{K}}$ via (II.5.7.1)). It is a $\widehat{\overline{K}}$-module endowed with the natural action of G. There is a natural G-equivariant exact sequence of $\widehat{\overline{K}}$-modules induced by (II.5.7.1),

$$0 \to \mathrm{Sym}^{n-1}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) \to \mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) \to \widehat{\overline{K}} \otimes_{\mathcal{O}_K} (\mathrm{Sym}^n_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K})(-n) \to 0. \tag{II.5.11.2}$$

The $\widehat{\overline{K}}$-module $\mathscr{C}_{\mathrm{HT}}$ admits a natural $\widehat{\overline{K}}$-algebra structure induced by the multiplication morphisms $\mathrm{Sym}^n \otimes \mathrm{Sym}^m \to \mathrm{Sym}^{n+m}$.

Corollary II.5.12. *We set* $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}} \in \mathbb{Z}_p(1)$ *and denote by* $\zeta^{-1} \in \mathbb{Z}_p(-1) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1), \mathbb{Z}_p)$ *the dual basis of* ζ*.*

(1) *There is an isomorphism of* $\widehat{\overline{K}}$*-algebras,*

$$\widehat{\overline{K}}[T_1, \dots, T_d] \xrightarrow{\sim} \mathscr{C}_{\mathrm{HT}}, \tag{II.5.12.1}$$

sending the variable T_i *to* $(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1}$ *for any* $1 \le i \le d$*.*

(2) *If we endow* $\widehat{\overline{K}}[T_1, \dots, T_d]$ *with the semi-linear* G*-action by transport of structure via* (II.5.12.1)*, then for any* $g \in G$ *and* $1 \le i \le d$*, we have*

$$g(T_i) = \chi(g)^{-1}(\xi_i(g) + T_i), \tag{II.5.12.2}$$

where $\chi : G \to \mathbb{Z}_p^\times$ *is the cyclotomic character* (II.4.15.2) *and* $\xi = (\xi_1, \dots, \xi_d) : G \to \mathbb{Z}_p^d$ *is the continuous* 1*-cocycle* (II.4.15.4)*. In particular,* G *acts continuously on* $\mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))$ *with respect to the canonical topology for any* $n \in \mathbb{N}$*.*

(3) *The canonical map* $\jmath$ *in* (II.5.7.1) *induces a canonical isomorphism of* $\mathscr{C}_{\mathrm{HT}}$*-modules,*

$$\Omega^1_{\mathscr{C}_{\mathrm{HT}}/\widehat{\overline{K}}} \xrightarrow{\sim} \mathscr{C}_{\mathrm{HT}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1), \tag{II.5.12.3}$$

and the universal differential map $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}} : \mathscr{C}_{\mathrm{HT}} \to \mathscr{C}_{\mathrm{HT}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$ *sends* T_i *to* $\mathrm{d}\log(t_i) \otimes \zeta^{-1}$ *for any* $1 \le i \le d$*.*

PROOF. It follows directly from II.5.7 and its arguments. □

Definition II.5.13 (cf. [**Hyo89**, §1], [**AGT16**, II.15], [**Tsu18**, §15]). The $\widehat{\overline{K}}$-algebra $\mathscr{C}_{\mathrm{HT}}$ constructed in (II.5.11.1) is called the *Hyodo ring* of $\mathcal{O}_K$.

Corollary II.5.14. *We have*

$$\operatorname*{colim}_{n\in\mathbb{N}} H^q(H, \operatorname{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))) = \begin{cases} \widehat{K_\infty} & \text{if } q=0, \\ 0 & \text{otherwise,} \end{cases} \tag{II.5.14.1}$$

where H^q denotes the continuous group cohomology, and $\operatorname{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))$ is endowed with the canonical topology as a finite-dimensional $\widehat{\overline{K}}$-module. In particular, $(\mathscr{C}_{\mathrm{HT}})^H = \widehat{K_\infty}$.

PROOF. It follows from the argument of [**Hyo89**, (1.2.2)], which relies on the cohomological property II.5.9 of the Faltings extension. We will give a detailed proof in II.13.5. □

Remark II.5.15. One can also obtain a similar result for $H^q(G, \operatorname{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)))$ by the argument of [**Hyo89**, (1.2.1)].

II.5.16. Taking a Tate twist of the dual of the Faltings extension (II.5.7.1) of $\mathcal{O}_K$, we obtain a canonical exact sequence of finite projective $\widehat{\overline{K}}$-representations of G,

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{\overline{K}}) \xrightarrow{\jmath^*} \mathscr{E}^*_{\mathcal{O}_K}(1) \xrightarrow{\iota^*} \widehat{\overline{K}} \longrightarrow 0 \tag{II.5.16.1}$$

where $\mathscr{E}^*_{\mathcal{O}_K} = \operatorname{Hom}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}, \widehat{\overline{K}})$. There is a canonical G-equivariant $\widehat{\overline{K}}$-linear Lie algebra structure on $\mathscr{E}^*_{\mathcal{O}_K}(1)$ associated to the linear form ι^*, defined by the Lie bracket for any $f_1, f_2 \in \mathscr{E}^*_{\mathcal{O}_K}(1)$,

$$[f_1, f_2] = \iota^*(f_1)f_2 - \iota^*(f_2)f_1. \tag{II.5.16.2}$$

Thus, $\operatorname{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{\overline{K}})$ is a Lie ideal of $\mathscr{E}^*_{\mathcal{O}_K}(1)$, and $\widehat{\overline{K}}$ is the quotient, and the induced Lie algebra structures on them are trivial. Any $\widehat{\overline{K}}$-linear splitting of (II.5.16.1) identifies $\mathscr{E}^*_{\mathcal{O}_K}(1)$ with the semi-direct product of Lie algebras of $\widehat{\overline{K}}$ acting on $\operatorname{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{\overline{K}})$ by multiplication. Let $\{T_i = (\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1}\}_{0\le i\le d}$ (where $t_{0,p^n} = \zeta_{p^n}$) denote the basis of $\mathscr{E}_{\mathcal{O}_K}(-1)$, and let $\{T_i^*\}_{0\le i\le d}$ be the dual basis of $\mathscr{E}^*_{\mathcal{O}_K}(1)$. Then, we see that the Lie bracket on $\mathscr{E}^*_{\mathcal{O}_K}(1)$ is determined by

$$[T_0^*, T_i^*] = T_i^* \quad \text{and} \quad [T_i^*, T_j^*] = 0, \tag{II.5.16.3}$$

for any $1 \le i, j \le d$. Indeed, this dual basis induces an isomorphism of $\widehat{\overline{K}}$-linear Lie algebras

$$\widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathbb{Z}_p \ltimes \mathbb{Z}_p^d) \xrightarrow{\sim} \mathscr{E}^*_{\mathcal{O}_K}(1),\ 1 \otimes \partial_i \mapsto T_i^*, \tag{II.5.16.4}$$

where $\{\partial_i\}_{0\le i\le d}$ is the standard basis of $\mathrm{Lie}(\mathbb{Z}_p \ltimes \mathbb{Z}_p^d)$ (cf. II.4.20).

Theorem II.5.17 ([**Bri03**, Théorème 1, 2], [**Ohk11**, §9]). *The functor*

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma, K_{\infty,\underline{\infty}}) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}}),\ V \mapsto \widehat{\overline{K}} \otimes_{K_{\infty,\underline{\infty}}} V \tag{II.5.17.1}$$

is an equivalence of categories.

Proposition II.5.18 ([**Tsu18**, 14.16]). *The functor (cf.* II.4.14*)*

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma, K_\infty) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Gamma, K_{\infty,\underline{\infty}}),\ V \mapsto K_{\infty,\underline{\infty}} \otimes_{K_\infty} V \tag{II.5.18.1}$$

is an equivalence.

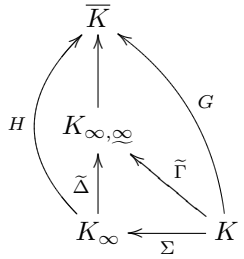
PROOF. It follows from the proof of [**Tsu18**, 14.16]. We note that the lemma [**Tsu18**, 14.17] used in the proof holds by II.4.18, and the lemma [**Tsu18**, 14.18] holds since any finite field extension of K is still a complete discrete valuation field. □

Lemma II.5.19 ([**Tsu18**, 15.3.(2)]). *For any object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Sigma, K_\infty)$, the (Σ, K_∞)-finite part of $\widehat{K_\infty} \otimes_{K_\infty} V$ (see* II.2.4*) is V.*

PROOF. It follows from the argument of [**Tsu18**, 15.3.(2)] (cf. II.10.10). □

II.5.20. We shall give an explicit way in II.5.28 to construct Higgs bundles from representations, which generalizes [**Tsu18**, 15.1.(4)]. Firstly, we introduce another Kummer tower more general than the one considered in II.5.1. We fix $e \in \mathbb{N}$. Let $\widetilde{t}_1, \dots, \widetilde{t}_e$ be elements of K with compatible systems of p-power roots $(\widetilde{t}_{1,p^n})_{n\in\mathbb{N}}, \dots, (\widetilde{t}_{e,p^n})_{n\in\mathbb{N}}$ in $\overline{K}$. Consider the Kummer tower $(\mathcal{O}_{K_{n,\underline{l}}})_{(n,\underline{l})\in\mathbb{N}^{1+e}}$ of $\mathcal{O}_K$ defined by $\zeta_{p^n}, \widetilde{t}_{1,p^n}, \dots, \widetilde{t}_{e,p^n}$. We take the notation in II.4.15 for this Kummer tower by adding tildes.

(II.5.20.1)



We have the continuous 1-cocycle

$$\widetilde{\xi} = (\widetilde{\xi}_1, \dots, \widetilde{\xi}_e) : G \longrightarrow \mathbb{Z}_p^e, \tag{II.5.20.2}$$

describing the action of G on $\widetilde{t}_{1,p^n}, \dots, \widetilde{t}_{e,p^n}$, cf. (II.4.15.4). We define $1+e$ elements in $\mathscr{E}_{\mathcal{O}_K}(-1) \subseteq \mathscr{C}_{\mathrm{HT}}$ by

$$\widetilde{T}_0 = 1,\ \widetilde{T}_1 = (\mathrm{d}\log(\widetilde{t}_{1,p^n}))_n \otimes \zeta^{-1},\ \cdots,\ \widetilde{T}_e = (\mathrm{d}\log(\widetilde{t}_{e,p^n}))_n \otimes \zeta^{-1}. \tag{II.5.20.3}$$

Similarly to II.5.12.(2), for any $g \in G$ and $1 \leq i \leq e$, we have

$$g(\widetilde{T}_i) = \chi(g)^{-1}(\widetilde{\xi}_i(g) + \widetilde{T}_i). \tag{II.5.20.4}$$

We remark that $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_i) = \mathrm{d}\log(\widetilde{t}_i) \otimes \zeta^{-1} \in K \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$ by II.5.7.(2).

Lemma II.5.21. *With the notation in* II.5.20*, we write*

$$(\widetilde{T}_1, \dots, \widetilde{T}_e) = (T_1, \dots, T_d)A + B \tag{II.5.21.1}$$

as elements of $\mathscr{E}_{\mathcal{O}_K}(-1) \subseteq \mathscr{C}_{\mathrm{HT}}$*, where* $A = (a_{ij}) \in \mathrm{M}_{d\times e}(\widehat{\overline{K}}), B = (b_j) \in \mathrm{M}_{1\times e}(\widehat{\overline{K}})$*. Then,* $A \in \mathrm{M}_{d\times e}(K)$*, and we have*

$$(\widetilde{\xi}_1, \dots, \widetilde{\xi}_e) = (\xi_1, \dots, \xi_d)A \tag{II.5.21.2}$$

as vectors with value in the continuous group cohomology group $H^1(H, \widehat{\overline{K}})$*. In particular, we have*

$$\dim(\widetilde{\Delta}) \geq \mathrm{rank}(A). \tag{II.5.21.3}$$

Proof. Notice that $(\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_1), \dots, \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_e)) = (\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_1), \dots, \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_d))A$ and that $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_1), \dots, \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_d)$ form a basis of $K \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$. Thus, $A \in \mathrm{M}_{d\times e}(K)$ as $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_i) = \mathrm{d}\log(\widetilde{t}_i) \otimes \zeta^{-1} \in K \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$. We act on (II.5.21.1) by $g \in G$, then by (II.5.20.4),

$$(\widetilde{\xi}_1(g), \dots, \widetilde{\xi}_e(g)) = (\xi_1(g), \dots, \xi_d(g))A + \chi(g)g(B) - B. \tag{II.5.21.4}$$

Thus, (II.5.21.2) follows from the fact that $\chi(H) = 1$ and the map $H \to \widehat{\overline{K}}$ sending g to $g(b_j) - b_j$ is a 1-coboundry. In particular, the image of the composition of the natural maps

$$\mathrm{Hom}(\widetilde{\Delta}, \mathbb{Q}_p) \to \mathrm{Hom}(H, \mathbb{Q}_p) \to H^1(H, \widehat{\overline{K}}) \tag{II.5.21.5}$$

contains $(\xi_1, \dots, \xi_d)A$. Since $\xi_1, \dots, \xi_d$ form a $\widehat{K_\infty}$-basis of $H^1(H, \widehat{\overline{K}})$ by (II.5.9.1), we see that $\dim(\widetilde{\Delta}) \geq \mathrm{rank}(A)$. □

Proposition II.5.22. *With the notation in* II.5.20*, the following conditions are equivalent:*

1. *The group* $\widetilde{\Delta}$ *has dimension* e*, i.e. the Kummer tower* $(\mathcal{O}_{K_{n,\underline{l}}})_{(n,\underline{l})\in\mathbb{N}^{1+e}}$ *satisfies the condition* II.4.19.(1)*.*
2. *The* $1+e$ *elements* $\widetilde{T}_0, \widetilde{T}_1, \cdots, \widetilde{T}_e$ *of* $\mathscr{E}_{\mathcal{O}_K}(-1) \subseteq \mathscr{C}_{\mathrm{HT}}$ *are linearly independent over* $\widehat{\overline{K}}$*.*
3. *The* e *elements* $\mathrm{d}\widetilde{t}_1, \dots, \mathrm{d}\widetilde{t}_e$ *of* $\widehat{\Omega}^1_{\mathcal{O}_K}[1/p]$ *are linearly independent over* K*.*

PROOF. It is clear that (2) and (3) are equivalent by the splitting of the Faltings extension defined in II.5.7.(3). We see that (3) implies (1) by (II.5.21.3). It remains to check (1) $\Rightarrow$ (2).

We set $\alpha_i = \widetilde{T}_i \otimes \zeta \in \mathscr{E}_{\mathcal{O}_K}$ $(0 \leq i \leq e)$. Let $l \in \mathbb{N}$ be the smallest integer such that $\alpha_{i_1}, \ldots, \alpha_{i_l}$ are linearly dependent for some $0 \leq i_1 < \cdots < i_l \leq e$. Assume that $l \geq 1$. We write $x_{i_1}\alpha_{i_1} + \cdots + x_{i_l}\alpha_{i_l} = 0$ for some $x_{i_1}, \ldots, x_{i_l} \in \widehat{\overline{K}} \setminus \{0\}$. By (II.5.20.4), for any $g \in G$ and $1 \leq i \leq e$, we have

$$g(\alpha_0) = \chi(g)\alpha_0 \quad \text{and} \quad g(\alpha_i) = \widetilde{\xi}_i(g)\alpha_0 + \alpha_i. \tag{II.5.22.1}$$

We consider the cases where $i_1 = 0$ and $i_1 > 0$ separately.

If $i_1 = 0$, then we may assume that $(i_1, i_2, \ldots, i_l) = (0, 1, \ldots, l-1)$. Since $\alpha_0 \neq 0$, we have $l > 1$ and we may assume that $x_{l-1} = 1$. Thus,

(II.5.22.2)

$$\begin{aligned} 0 &= \sum_{i=0}^{l-1} x_i\alpha_i - g\Big(\sum_{i=0}^{l-1} x_i\alpha_i\Big) \\ &= \left(x_0 - \chi(g)g(x_0) - \sum_{i=1}^{l-1} \widetilde{\xi}_i(g)g(x_i)\right)\alpha_0 + (x_1 - g(x_1))\alpha_1 + \cdots + (x_{l-2} - g(x_{l-2}))\alpha_{l-2}. \end{aligned}$$

By the minimality of l, we have $x_0 = \chi(g)g(x_0) + \sum_{i=1}^{l-1} \widetilde{\xi}_i(g)g(x_i)$, $x_1 = g(x_1)$, $\ldots$, $x_{l-2} = g(x_{l-2})$. Thus, $x_0 = \chi(g)g(x_0) + \sum_{i=1}^{l-1} \widetilde{\xi}_i(g)x_i$, which is a contradiction as we can take $(\chi(g), \widetilde{\xi}_1(g), \ldots, \widetilde{\xi}_e(g))$ running through an open subgroup of $0 \times \mathbb{Z}_p^e$ by varying g in $\widetilde{\Delta}$.

If $i_1 > 0$, then we may assume that $(i_1, i_2, \ldots, i_l) = (1, 2, \ldots, l)$ and that $x_l = 1$. Thus,

$$\begin{aligned} 0 &= \sum_{i=1}^{l} x_i\alpha_i - g\Big(\sum_{i=1}^{l} x_i\alpha_i\Big) \\ &= \left(-\sum_{i=1}^{l-1} \widetilde{\xi}_i(g)g(x_i)\right)\alpha_0 + (x_1 - g(x_1))\alpha_1 + \cdots + (x_{l-1} - g(x_{l-1}))\alpha_{l-1}. \end{aligned} \tag{II.5.22.3}$$

We get a contradiction in a similar way. □

II.5.23. Following II.5.20, we assume that the equivalent conditions in II.5.22 hold, and we take the notation in II.4.20 by adding tildes. Recall that for any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\widetilde{\Gamma}, K_{\infty,\infty})$, there is a canonical Lie algebra action induced by the infinitesimal action of $\widetilde{\Gamma}$ on W defined in (II.4.16.1),

$$\varphi : \mathrm{Lie}(\widetilde{\Gamma}) \to \mathrm{End}_{K_{\infty,\infty}}(W). \tag{II.5.23.1}$$

Let $\widetilde{\partial}_0 \in \mathrm{Lie}(\widetilde{\Sigma}_{0,\infty})$ and $\widetilde{\partial}_1, \ldots, \widetilde{\partial}_e \in \mathrm{Lie}(\widetilde{\Delta})$ be the standard basis defined in II.4.20, and we put for any $g \in G$,

$$\varphi_g^{\widetilde{\chi}} = \log(\chi(g))\varphi_{\widetilde{\partial}_0}, \quad \text{and} \quad \varphi_g^{\widetilde{\xi}} = \sum_{i=1}^{d} \widetilde{\xi}_i(g)\varphi_{\widetilde{\partial}_i}. \tag{II.5.23.2}$$

Lemma II.5.24. *Under the assumption in* II.5.23 *and with the same notation, the map*

$$\psi : \mathscr{E}^*_{\mathcal{O}_K}(1) = \mathrm{Hom}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1), \widehat{\overline{K}}) \longrightarrow \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\widetilde{\Gamma}), \tag{II.5.24.1}$$

sending f to $\sum_{i=0}^{e} f(\widetilde{T}_i) \otimes \widetilde{\partial}_i$, is surjective and induces a morphism of exact sequences of $\widehat{\overline{K}}$-linear Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{\overline{K}}) & \xrightarrow{\jmath^*} & \mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow{\iota^*} & \widehat{\overline{K}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \wr} & & \\ 0 & \longrightarrow & \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\widetilde{\Delta}) & \longrightarrow & \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\widetilde{\Gamma}) & \longrightarrow & \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\Sigma) & \longrightarrow & 0 \end{array} \tag{II.5.24.2}$$

where the first row is (II.5.16.1).

PROOF. The surjectivity of ψ follows from II.5.22.(2). It remains to check ψ is compatible with Lie brackets. With the notation in II.5.21, we have

$$\psi(T_0^*) = \widetilde{\partial}_0 + \sum_{j=1}^{e} b_j\widetilde{\partial}_j, \quad \text{and} \quad \psi(T_i^*) = \sum_{j=1}^{e} a_{ij}\widetilde{\partial}_j, \ \forall 1 \leq i \leq d, \tag{II.5.24.3}$$

where $(T_0^*, \dots, T_d^*)$ is the dual basis of $(T_0, \dots, T_d)$ defined in II.5.16. Thus, $[\psi(T_0^*), \psi(T_i^*)] = \psi(T_i^*)$ and $[\psi(T_i^*), \psi(T_j^*)] = 0$ for any $1 \leq i, j \leq d$, which completes the proof. □

Remark II.5.25. The first vertical map in (II.5.24.2) can be defined without taking bases. Consider the canonical maps

$$\mathrm{Hom}(\widetilde{\Delta}, \mathbb{Q}_p) \to \mathrm{Hom}(H, \mathbb{Q}_p) \to H^1(H, \widehat{\overline{K}}) \xrightarrow{\sim} \widehat{K_\infty} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1) \tag{II.5.25.1}$$

where the first arrow is induced by the surjection $H \to \widetilde{\Delta}$, and the last arrow is induced by the connecting map of the Faltings extension of $\mathcal{O}_K$ (II.5.9.1) which sends ξ_i to $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_i)$ for any $1 \leq i \leq d$. Thus, the composition of (II.5.25.1) sends $\widetilde{\xi}_i$ to $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_i)$ for any $1 \leq i \leq e$ by II.5.21. Since $\{\widetilde{\xi}_i\}_{1 \leq i \leq e}$ is the dual basis of the basis $\{\widetilde{\partial}_i\}_{1 \leq i \leq e} \subseteq \mathrm{Lie}(\widetilde{\Delta})$ by construction, the composition of (II.5.25.1) induces a natural injective K-linear map

$$\mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{Lie}(\widetilde{\Delta}), K) \longrightarrow K \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1) \tag{II.5.25.2}$$

which sends f to $\sum_{i=1}^{e} f(\widetilde{\partial}_i) \otimes \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_i)$. Its dual is a natural surjective K-linear map

$$\mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), K) \longrightarrow K \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\widetilde{\Delta}) \tag{II.5.25.3}$$

which sends f to $\sum_{i=1}^{e} f(\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_i)) \otimes \widetilde{\partial}_i$, and induces the first vertical map in (II.5.24.2) by extending scalars.

II.5.26. Let $(C, \mathcal{O})$ be a ringed site, $\mathcal{M}$ an $\mathcal{O}$-module. A *Higgs field* on an $\mathcal{O}$-module $\mathcal{F}$ *with coefficients in* $\mathcal{M}$ is an $\mathcal{O}$-linear morphism $\theta : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{M}$ such that $\theta^{(1)} \circ \theta = 0$, where $\theta^{(1)}$ is the $\mathcal{O}$-linear morphism $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{M} \to \mathcal{F} \otimes_{\mathcal{O}} \wedge^2 \mathcal{M}$ defined by $\theta^{(1)}(x \otimes \omega) = \theta(x) \wedge \omega$ for any local sections x of $\mathcal{F}$ and ω of $\mathcal{M}$.

If $\mathcal{M}$ is a finite free $\mathcal{O}$-module with basis $\omega_1, \dots, \omega_d$, then to give a Higgs field θ on $\mathcal{F}$ is equivalent to give d endomorphisms θ_i $(1 \leq i \leq d)$ of the $\mathcal{O}$-module $\mathcal{F}$ which commute with each other. For any local section x of $\mathcal{M}$, we have

$$\theta(x) = \theta_1(x) \otimes \omega_1 + \cdots \theta_d(x) \otimes \omega_d. \tag{II.5.26.1}$$

We call the d-tuple $(\theta_1, \dots, \theta_d)$ the *coordinates of the Higgs field* θ with respect to the $\mathcal{O}$-basis $\omega_1, \dots, \omega_d$ of $\mathcal{M}$.

Assume that $\mathcal{M}$ is a finite projective $\mathcal{O}$-module. We say that a Higgs field θ on an $\mathcal{O}$-module $\mathcal{F}$ is *nilpotent* if there is a finite decreasing filtration by $\mathcal{O}$-submodules $\mathcal{F} = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^n = 0$ such that $\theta(\mathcal{F}^i) \subseteq \mathcal{F}^{i+1} \otimes_{\mathcal{O}} \mathcal{M}$ for any $0 \leq i < n$.

One checks easily by II.5.12 that the universal differential map

$$\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}} : \mathscr{C}_{\mathrm{HT}} \to \mathscr{C}_{\mathrm{HT}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1) \tag{II.5.26.2}$$

is a G-equivariant $\widehat{\overline{K}}$-linear Higgs field on the $\widehat{\overline{K}}$-module $\mathscr{C}_{\mathrm{HT}}$ with coefficients in $\widehat{\overline{K}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$.

Definition II.5.27 (cf. [**Tsu18**, page 872])**.** We define a category $\mathbf{HB}_{\mathrm{nilp}}(\Sigma, K_\infty, \widehat{\Omega}^1_{\mathcal{O}_K}(-1))$ as follows:

(1) An object (M, ρ, θ) is a finite projective K_∞-representation (M, ρ) of Σ (cf. II.2.4) endowed with a K_∞-linear nilpotent Higgs field $\theta : M \to M \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$ (with coefficients in $K_\infty \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$) which is Σ-equivariant (i.e. $\theta \circ \rho(\sigma) = (\rho(\sigma) \otimes \chi^{-1}(\sigma)) \circ \theta$ for any $\sigma \in \Sigma$).
(2) A morphism $(M, \rho, \theta) \to (M', \rho', \theta')$ is a Σ-equivariant K_∞-linear morphism $f : M \to M'$ which is compatible with the Higgs fields (i.e. $\theta' \circ f = (f \otimes 1) \circ \theta$).

It is an additive tensor category, where the tensor product is given by $(M, \rho, \theta) \otimes (M', \rho', \theta') = (M \otimes_{K_\infty} M', \rho \otimes \rho', \theta \otimes 1 + 1 \otimes \theta')$.

Proposition II.5.28 (cf. [**Tsu18**, page 873])**.** *Under the assumption in* II.5.23 *and with the same notation, let* (V, ρ) *be an object of* $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}, \widetilde{\Delta}\text{-an}}(\widetilde{\Gamma}, K_\infty)$.

(1) *For any* $g \in \widetilde{\Gamma}$, *we set*

$$\overline{\rho}(g) = \exp(-\varphi_g^{\widetilde{\xi}})\rho(g). \tag{II.5.28.1}$$

Then, $\overline{\rho}|_{\widetilde{\Delta}} = 1$ *and* $\overline{\rho}|_{\widetilde{\Sigma}_{0,\infty}} = \rho|_{\widetilde{\Sigma}_{0,\infty}}$. *Moreover,* $(V, \overline{\rho})$ *is an object of* $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Sigma, K_\infty)$.

(2) *The K_∞-linear homomorphism*

$$\theta_V : V \longrightarrow V \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1),\ v \mapsto -\sum_{i=1}^{e} \varphi_{\widetilde{\partial}_i}(v) \otimes \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_i) \tag{II.5.28.2}$$

is a nilpotent Higgs field which is Σ-equivariant via $\overline{\rho}$.

Therefore, there is a functor

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\widetilde{\Delta}\text{-an}}(\widetilde{\Gamma}, K_\infty) \longrightarrow \mathbf{HB}_{\mathrm{nilp}}(\Sigma, K_\infty, \widehat{\Omega}^1_{\mathcal{O}_K}(-1)),\ (V,\rho) \mapsto (V, \overline{\rho}, \theta_V), \tag{II.5.28.3}$$

which relies on the choice of $\widetilde{t}_1, \ldots, \widetilde{t}_e$.

Proof. (1) Notice that the infinitesimal Lie algebra action of $\mathrm{Lie}(\widetilde{\Delta})$ on V (II.4.13),

$$\mathrm{Lie}(\widetilde{\Delta}) \longrightarrow \mathrm{End}_{K_\infty}(V), \tag{II.5.28.4}$$

is nilpotent by II.4.11 and II.4.18 (whose assumptions are satisfied as K is a complete discrete valuation field). Thus, $\exp(-\varphi^{\widetilde{\xi}}_g)$ and $\overline{\rho}(g)$ are well-defined endomorphisms of V. Since (V,ρ) is $\widetilde{\Delta}$-analytic (II.4.14), if $g \in \widetilde{\Delta}$ then $\rho(g) = \exp(\varphi_g) = \exp(\varphi^{\widetilde{\xi}}_g)$ as $\varphi^{\widetilde{\xi}}|_{\widetilde{\Delta}} = \varphi|_{\widetilde{\Delta}}$. Thus, $\overline{\rho}(g) = \mathrm{id}_V$. It is clear that $\overline{\rho}$ is continuous and K_∞-semi-linear. It remains to show that $\overline{\rho}(g_1g_2) = \overline{\rho}(g_1)\overline{\rho}(g_2)$ for any $g_1, g_2 \in \widetilde{\Gamma}$ (so that $\overline{\rho}$ is a K_∞-representation of $\widetilde{\Gamma}/\widetilde{\Delta} = \Sigma$).

$$\begin{aligned}
\overline{\rho}(g_1g_2) &= \exp(-\sum_{i=1}^{e} \widetilde{\xi}_i(g_1g_2)\varphi_{\widetilde{\partial}_i})\rho(g_1g_2) \quad \text{(by (II.5.23.2))} \\
&= \exp(-\sum_{i=1}^{e} (\widetilde{\xi}_i(g_1) + \chi(g_1)\widetilde{\xi}_i(g_2))\varphi_{\widetilde{\partial}_i})\rho(g_1g_2) \quad \text{(by (II.4.15.5))} \\
&= \exp(-\sum_{i=1}^{e} \widetilde{\xi}_i(g_1)\varphi_{\widetilde{\partial}_i})\rho(g_1)\exp(-\sum_{i=1}^{e} \widetilde{\xi}_i(g_2)\varphi_{\widetilde{\partial}_i})\rho(g_2) \quad \text{(by (II.4.17.1))} \\
&= \overline{\rho}(g_1)\overline{\rho}(g_2).
\end{aligned} \tag{II.5.28.5}$$

(2) Since the endomorphisms $\varphi_{\widetilde{\partial}_i}$ on V are nilpotent and commute with each other, θ_V is a nilpotent Higgs field. It remains to check the Σ-equivariance of θ_V. For $v \in V$ and $g \in \widetilde{\Gamma}$, we have

$$\begin{aligned}
\overline{\rho}(g)(\theta_V(v)) &= \exp(-\sum_{i=1}^{e} \widetilde{\xi}_i(g)\varphi_{\widetilde{\partial}_i})\rho(g)(-\sum_{j=1}^{e} \varphi_{\widetilde{\partial}_j}(v) \otimes \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_j)) \quad \text{(by (II.5.23.2))} \\
&= \exp(-\sum_{i=1}^{e} \widetilde{\xi}_i(g)\varphi_{\widetilde{\partial}_i})(-\sum_{j=1}^{e} \chi(g)\varphi_{\widetilde{\partial}_j}(\rho(g)(v)) \otimes \chi(g)^{-1}\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_j)) \quad \text{(by (II.4.17.1))} \\
&= -\sum_{j=1}^{e} \varphi_{\widetilde{\partial}_j}(\exp(-\sum_{i=1}^{e} \widetilde{\xi}_i(g)\varphi_{\widetilde{\partial}_i})\rho(g)(v)) \otimes \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_j) \quad (\text{as } \varphi_{\widetilde{\partial}_i}\varphi_{\widetilde{\partial}_j} = \varphi_{\widetilde{\partial}_j}\varphi_{\widetilde{\partial}_i}) \\
&= \theta_V(\overline{\rho}(g)(v)).
\end{aligned} \tag{II.5.28.6}$$

□

Remark II.5.29. (1) The minus sign in the definition of θ_V is designed for II.5.31.(4) (cf. II.5.33).

(2) There is another definition for the Higgs field θ_V (II.5.28.2). Consider the canonical maps

$$V \longrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{Lie}(\widetilde{\Delta}), V) \longrightarrow V \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1), \tag{II.5.29.1}$$

where the first map is induced by the infinitesimal Lie algebra algebra action of $\mathrm{Lie}(\widetilde{\Delta})$ on V which sends $v \in V$ to $(\widetilde{\partial} \mapsto \varphi_{\widetilde{\partial}}(v))$, and the second map is induced by extending scalars from the canonical map (II.5.25.2) which sends f to $\sum_{i=1}^{e} f(\widetilde{\partial}_i) \otimes \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_i)$. It is clear that the composition is $-\theta_V$.

II.5.30. Let $M = (M, \rho, \theta)$ be an object of $\mathbf{HB}_{\mathrm{nilp}}(\Sigma, K_\infty, \widehat{\Omega}^1_{\mathcal{O}_K}(-1))$. On the $\mathscr{C}_{\mathrm{HT}}$-module $\mathscr{C}_{\mathrm{HT}} \otimes_{K_\infty} M$, we define a semi-linear action of G by the diagonal action $g \otimes \overline{g}$ for any $g \in G$ with image $\overline{g} \in \Sigma$, and a G-equivariant $\widehat{\overline{K}}$-linear Higgs field $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}} \otimes 1 + 1 \otimes \theta$ with value in $\widehat{\overline{K}} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$. In particular, its $\widehat{\overline{K}}$-submodule

$$\mathbb{V}(M) = (\mathscr{C}_{\mathrm{HT}} \otimes_{K_\infty} M)^{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}} \otimes 1 + 1 \otimes \theta = 0} \tag{II.5.30.1}$$

is endowed with the induced semi-linear action of G.

Let $W=(W,\rho)$ be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$. On the $\mathscr{C}_{\mathrm{HT}}$-module $\mathscr{C}_{\mathrm{HT}}\otimes_{\widehat{\overline{K}}}W$, we define a semi-linear action of G by the diagonal action $g\otimes g$ for any $g\in G$, and a G-equivariant $\widehat{\overline{K}}$-linear Higgs field $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1$. In particular, its K_∞-submodule

$$\mathbb{D}(W)=((\mathscr{C}_{\mathrm{HT}}\otimes_{\widehat{\overline{K}}}W)^H)^{(\Sigma,K_\infty)\text{-fini}} \tag{II.5.30.2}$$

where $(-)^{(\Sigma,K_\infty)\text{-fini}}$ is taking the (Σ,K_∞)-finite part (cf. II.2.4), is endowed with the induced semi-linear action of Σ and the induced Σ-equivariant K_∞-linear Higgs field with value in $K_\infty\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}(-1)$ (cf. II.5.14 and II.5.19).

We remark that the definitions of $\mathbb{V}$ and $\mathbb{D}$ do not depend on the choice of $t_1,\dots,t_d$.

Theorem II.5.31 (cf. [**Tsu18**, 15.1])**.** *We keep the notation in* II.5.1.

(1) *For any object M of $\mathbf{HB}_{\mathrm{nilp}}(\Sigma,K_\infty,\widehat{\Omega}^1_{\mathcal{O}_K}(-1))$, $\mathbb{V}(M)$ is a finite-dimensional $\widehat{\overline{K}}$-module on which G acts continuously with respect to the canonical topology (thus $\mathbb{V}(M)$ is an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$). Moreover, the canonical $\mathscr{C}_{\mathrm{HT}}$-linear morphism (which is G-equivariant and compatible with Higgs fields by definition)*

$$\mathscr{C}_{\mathrm{HT}}\otimes_{\widehat{\overline{K}}}\mathbb{V}(M)\longrightarrow\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M \tag{II.5.31.1}$$

is an isomorphism.

(2) *For any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$, $\mathbb{D}(W)$ is a finite-dimensional $\widehat{\overline{K}}$-module on which Σ acts continuously with respect to the canonical topology (thus $\mathbb{D}(W)$ is an object of $\mathbf{HB}_{\mathrm{nilp}}(\Sigma,K_\infty,\widehat{\Omega}^1_{\mathcal{O}_K}(-1))$). Moreover, the canonical $\mathscr{C}_{\mathrm{HT}}$-linear morphism (which is G-equivariant and compatible with Higgs fields by definition)*

$$\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}\mathbb{D}(W)\longrightarrow\mathscr{C}_{\mathrm{HT}}\otimes_{\widehat{\overline{K}}}W \tag{II.5.31.2}$$

is an isomorphism.

(3) *The functors*

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})\underset{\mathbb{V}}{\overset{\mathbb{D}}{\rightleftarrows}}\mathbf{HB}_{\mathrm{nilp}}(\Sigma,K_\infty,\widehat{\Omega}^1_{\mathcal{O}_K}(-1)) \tag{II.5.31.3}$$

are equivalences of additive tensor categories, quasi-inverse to each other.

(4) *Under the assumption in* II.5.23 *and with the same notation, let (V,ρ) be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\widetilde{\Delta}\text{-an}}(\widetilde{\Gamma},K_\infty)$, $M=(V,\overline{\rho},\theta_V)$ the object of $\mathbf{HB}_{\mathrm{nilp}}(\Sigma,K_\infty,\widehat{\Omega}^1_{\mathcal{O}_K}(-1))$ defined by the functor* (II.5.28.3)*. Then, there exists a natural G-equivariant $\widehat{\overline{K}}$-linear isomorphism*

$$\mathbb{V}(M)\xrightarrow{\sim}\widehat{\overline{K}}\otimes_{K_\infty}V \tag{II.5.31.4}$$

which depends on the choice of $\widetilde{t}_1,\dots,\widetilde{t}_e$.

Proof. We follow the proof of [**Tsu18**, 15.1]. We identify $\mathscr{C}_{\mathrm{HT}}=\widehat{\overline{K}}[T_1,\dots,T_d]$ by (II.5.12.1).

(1) Let $\theta:M\to M\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}(-1)$ denote the K_∞-linear Higgs field of M. We write

$$\theta(x)=\sum_{i=1}^d\theta_i(x)\otimes\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_i),\ \forall x\in M, \tag{II.5.31.5}$$

$$\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(f)=\sum_{i=1}^d\frac{\partial f}{\partial T_i}\otimes\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_i),\ \forall f\in\mathscr{C}_{\mathrm{HT}}, \tag{II.5.31.6}$$

where θ_i (resp. $\frac{\partial}{\partial T_i}$) are K_∞-linear (resp. $\widehat{\overline{K}}$-linear) endomorphisms of M (resp. $\mathscr{C}_{\mathrm{HT}}$), which commute with each other. Since θ_i are nilpotent by definition, we can define a $\mathscr{C}_{\mathrm{HT}}$-linear isomorphism

$$\iota=\exp(-\sum_{i=1}^d T_i\otimes\theta_i):\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M\xrightarrow{\sim}\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M, \tag{II.5.31.7}$$

whose inverse is given by $\iota^{-1}=\exp(\sum_{i=1}^{d}T_i\otimes\theta_i)$. We claim that the following diagram is commutative.

$$\begin{array}{ccc}\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M & \xrightarrow{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1} & \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}(-1)\\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota\otimes 1}\\ \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M & \xrightarrow{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1+1\otimes\theta} & \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}(-1)\end{array}\tag{II.5.31.8}$$

Indeed, we have

$$\begin{aligned}&(\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1)\circ\exp(\sum_{i=1}^{d}T_i\otimes\theta_i)\\ =&\exp(\sum_{i=1}^{d}T_i\otimes\theta_i)\circ(\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1)+\sum_{j=1}^{d}\left(\exp(\sum_{i=1}^{d}T_i\otimes\theta_i)\circ(1\otimes\theta_j)\right)\otimes\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_j)\\ =&(\exp(\sum_{i=1}^{d}T_i\otimes\theta_i)\otimes 1)\circ(\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1+1\otimes\theta).\end{aligned}\tag{II.5.31.9}$$

Thus, the restriction of ι induces a $\widehat{\overline{K}}$-linear isomorphism

$$\iota_0:\widehat{\overline{K}}\otimes_{K_\infty}M=(\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M)^{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1=0}\xrightarrow{\sim}\mathbb{V}(M)=(\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M)^{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1+1\otimes\theta=0},\tag{II.5.31.10}$$

from which we see that $\mathbb{V}(M)$ is finite-dimensional over $\widehat{\overline{K}}$ and thus contained in the finite-dimensional $\widehat{\overline{K}}$-submodule $\mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))\otimes_{K_\infty}M$ of $\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M$ for some integer $n>0$. Since $\mathbb{V}(M)$ is a direct summand of $\mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))\otimes_{K_\infty}M$, the topology on $\mathbb{V}(M)$ induced from $\mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))\otimes_{K_\infty}M$ coincides with the canonical topology as a finite-dimensional $\widehat{\overline{K}}$-module. Since G acts continuously on $\mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))\otimes_{K_\infty}M$ by II.5.12.(2), it acts also continuously on $\mathbb{V}(M)$ with respect to the canonical topology, which means that $\mathbb{V}(M)$ is an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$. Finally, notice that the composition of the $\mathscr{C}_{\mathrm{HT}}$-linear maps

$$\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M\xrightarrow{\mathrm{id}_{\mathscr{C}_{\mathrm{HT}}}\otimes\iota_0}\mathscr{C}_{\mathrm{HT}}\otimes_{\widehat{\overline{K}}}\mathbb{V}(M)\xrightarrow{\text{(II.5.31.1)}}\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M\tag{II.5.31.11}$$

is the isomorphism ι. Thus, (II.5.31.1) is an isomorphism, which completes the proof of (1).

(4) Since the K_∞-endomorphisms $\varphi_{\widetilde{\partial}_i}$ $(1\leq i\leq e)$ on V are nilpotent and commute with each other, we can define a $\mathscr{C}_{\mathrm{HT}}$-linear isomorphism

$$\jmath=\exp(-\sum_{i=1}^{e}\widetilde{T}_i\otimes\varphi_{\widetilde{\partial}_i}):\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M\xrightarrow{\sim}\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}V,\tag{II.5.31.12}$$

whose inverse is given by $\jmath^{-1}=\exp(\sum_{i=1}^{e}\widetilde{T}_i\otimes\varphi_{\widetilde{\partial}_i})$. We claim that $\jmath$ is G-equivariant. Indeed, for any $g\in G$ and $x\in M$, we have

(II.5.31.13)

$$\begin{aligned}g(\jmath(1\otimes x))&=\exp(-\sum_{i=1}^{e}\chi(g)^{-1}(\widetilde{\xi}_i(g)+\widetilde{T}_i)\otimes\chi(g)\varphi_{\widetilde{\partial}_i})(1\otimes\rho(g)(x))\quad\text{(by (II.5.20.4), (II.4.17.1))}\\ &=\exp(-\sum_{i=1}^{e}\widetilde{T}_i\otimes\varphi_{\widetilde{\partial}_i})\exp(-\sum_{i=1}^{e}1\otimes\widetilde{\xi}_i(g)\varphi_{\widetilde{\partial}_i})(1\otimes\rho(g)(x))\\ &=\jmath(1\otimes\overline{\rho}(g)(x))\quad\text{(by (II.5.23.2)).}\end{aligned}$$

On the other hand, by the same argument as (II.5.31.9), we see that $\jmath$ is compatible with Higgs fields, i.e. the following diagram is commutative.

$$\begin{array}{ccc}\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M & \xrightarrow{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1+1\otimes\theta_V} & \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}(-1)\\ \downarrow{\scriptstyle\jmath} & & \downarrow{\scriptstyle\jmath\otimes 1}\\ \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}V & \xrightarrow{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1} & \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}V\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}(-1)\end{array}\tag{II.5.31.14}$$

Thus, the restriction of $\jmath$ induces a G-equivariant $\widehat{\overline{K}}$-linear isomorphism

$$\mathbb{V}(M)=(\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M)^{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1+1\otimes\theta_V=0}\xrightarrow{\sim}\widehat{\overline{K}}\otimes_{K_\infty}V=(\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}V)^{\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}\otimes 1=0}. \tag{II.5.31.15}$$

(2) We apply (4) to the case where $e=d$ and $(\widetilde{t}_1,\dots,\widetilde{t}_e)=(t_1,\dots,t_d)$ (with the same p-power roots). By II.5.17, II.5.18 and (4), we see that the functor $\mathbb{V}$ is essentially surjective. Thus, we may assume that $W=\mathbb{V}(M)$ for some object M of $\mathbf{HB}_{\mathrm{nilp}}(\Sigma,K_\infty,\widehat{\Omega}^1_{\mathcal{O}_K}(-1))$. Taking the H-invariant part of the isomorphism (II.5.31.1), by II.5.14 we get a canonical isomorphism

$$(\mathscr{C}_{\mathrm{HT}}\otimes_{\widehat{\overline{K}}}W)^H\xrightarrow{\sim}\widehat{K_\infty}\otimes_{K_\infty}M. \tag{II.5.31.16}$$

Taking the (Σ,K_∞)-finite part, we get by II.5.19 a canonical Σ-equivariant K_∞-linear isomorphism $\mathbb{D}(W)\xrightarrow{\sim}M$ compatible with Higgs fields, which completes the proof of (2).

(3) The proof of (2) shows that the canonical morphism $\mathbb{D}\circ\mathbb{V}\to\mathrm{id}$ is an isomorphism. Taking the Higgs field zero part of the isomorphism (II.5.31.2), we see that $\mathbb{V}\circ\mathbb{D}\to\mathrm{id}$ is an isomorphism. Using the isomorphisms (II.5.31.1) and (II.5.31.2), we see that $\mathbb{V}$ and $\mathbb{D}$ are compatible with tensor products. This completes the proof. □

Remark II.5.32. Let L be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic p. If $\mathcal{O}_K$ is the p-adic completion of the localization of an adequate $\mathcal{O}_L$-algebra A at some $\mathfrak{p}\in\mathfrak{S}_p(A)$ (cf. II.9.5), then II.5.31 (except (4)) is a special case of [**Tsu18**, 15.2].

Remark II.5.33. Our construction of (II.5.28.2) has a sign difference with Tsuji's. The essential reason is that the G-action defined by Tsuji on the Hyodo ring $\mathscr{C}_{\mathrm{HT}}=\widehat{\overline{K}}[T_1,\dots,T_d]$ is given by $g(T_i)=\chi(g)^{-1}(-\xi_i(g)+T_i)$ (cf. [**Tsu18**, page 872] and II.14.6).

Lemma II.5.34. *Under the assumption in* II.5.23 *and with the same notation, let* (V,ρ) *(resp.* $(\widetilde{V},\rho')$*) be an object of* $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma,K_\infty)$ *(resp.* $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\widetilde{\Delta}\text{-an}}(\widetilde{\Gamma},K_\infty)$*). Consider the* K_∞*-linear endomorphisms* $\{\varphi_{\partial_i}|_V\}_{0\le i\le d}$ *on* V *(resp.* $\{\varphi_{\widetilde{\partial}_i}\}_{0\le i\le d}$ *on* $\widetilde{V}$*) defined by the infinitesimal Lie algebra action of* $\mathrm{Lie}(\Gamma)$ *(resp.* $\mathrm{Lie}(\widetilde{\Gamma})$*). We write*

$$(\widetilde{T}_1,\dots,\widetilde{T}_e)=(T_1,\dots,T_d)A+B \tag{II.5.34.1}$$

as elements of $\mathscr{E}_{\mathcal{O}_K}(-1)\subseteq\mathscr{C}_{\mathrm{HT}}$*, where* $A=(a_{ij})\in\mathrm{M}_{d\times e}(K), B=(b_j)\in\mathrm{M}_{1\times e}(\widehat{\overline{K}})$ *(cf.* II.5.21*). Assume that there is an isomorphism* $\beta:\widehat{\overline{K}}\otimes_{K_\infty}\widetilde{V}\xrightarrow{\sim}\widehat{\overline{K}}\otimes_{K_\infty}V$ *in* $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$*. Then, there are identities of* $\widehat{\overline{K}}$*-linear endomorphisms*

$$\beta^{-1}\circ(1\otimes\varphi_{\partial_1}|_V,\dots,1\otimes\varphi_{\partial_d}|_V)\circ\beta=(1\otimes\varphi_{\widetilde{\partial}_1}|_{\widetilde{V}},\dots,1\otimes\varphi_{\widetilde{\partial}_e}|_{\widetilde{V}})A^{\mathrm{T}}, \tag{II.5.34.2}$$

$$\beta^{-1}\circ(1\otimes\varphi_{\partial_0}|_V)\circ\beta=1\otimes\varphi_{\widetilde{\partial}_0}|_{\widetilde{V}}+(1\otimes\varphi_{\widetilde{\partial}_1}|_{\widetilde{V}},\dots,1\otimes\varphi_{\widetilde{\partial}_e}|_{\widetilde{V}})B^{\mathrm{T}}, \tag{II.5.34.3}$$

where A^{T} *and* B^{T} *are the transposes of* A *and* B*.*

Proof. Let $M=(V,\overline{\rho},\theta_V)$ (resp. $\widetilde{M}=(\widetilde{V},\overline{\rho'},\theta_{\widetilde{V}})$) be the object of $\mathbf{HB}_{\mathrm{nilp}}(\Sigma,K_\infty,\widehat{\Omega}^1_{\mathcal{O}_K}(-1))$ defined by the functor (II.5.28.3). Consider the commutative diagram

$$\begin{array}{ccc}\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}\widetilde{M} & \xrightarrow{\tilde{\jmath}} & \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}\widetilde{V}\\ \downarrow{\scriptstyle \jmath^{-1}\circ(1\otimes\beta)\circ\tilde{\jmath}} & & \downarrow{\scriptstyle 1\otimes\beta}\\ \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M & \xrightarrow{\jmath} & \mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}V\end{array} \tag{II.5.34.4}$$

where $\jmath=\exp(-\sum_{i=1}^d T_i\otimes\varphi_{\partial_i}|_V)$ (resp. $\tilde{\jmath}=\exp(-\sum_{i=1}^e\widetilde{T}_i\otimes\varphi_{\widetilde{\partial}_i}|_{\widetilde{V}})$) is the $\mathscr{C}_{\mathrm{HT}}$-linear isomorphism defined in (II.5.31.12). Notice that

$$((\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}\widetilde{M})^H)^{(\Sigma,K_\infty)\text{-fini}}=(\widehat{K_\infty}\otimes_{K_\infty}\widetilde{M})^{(\Sigma,K_\infty)\text{-fini}}=\widetilde{M} \tag{II.5.34.5}$$

where the first equality follows from II.5.14, and the second equality follows from II.5.19. Since $1\otimes\beta$, $\jmath$ and $\tilde{\jmath}$ are G-equivariant and compatible with Higgs fields by the proof of II.5.31.(4), $\jmath^{-1}\circ(1\otimes\beta)\circ\tilde{\jmath}$ induces a Σ-equivariant K_∞-linear map compatible with Higgs fields,

$$h:\widetilde{M}=((\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}\widetilde{M})^H)^{(\Sigma,K_\infty)\text{-fini}}\longrightarrow((\mathscr{C}_{\mathrm{HT}}\otimes_{K_\infty}M)^H)^{(\Sigma,K_\infty)\text{-fini}}=M, \tag{II.5.34.6}$$

so that we actually have $\jmath^{-1}\circ(1\otimes\beta)\circ\tilde{\jmath}=1\otimes h$ (we remark that $h=\mathbb{D}(1\otimes\beta)$).

Notice that $\varphi_{\partial_0}|_V \circ \varphi_{\partial_i}^n|_V = \varphi_{\partial_i}^n|_V \circ (\varphi_{\partial_0}|_V + n)$ for any $n \in \mathbb{N}$ and $1 \leq i \leq d$ by (II.4.17.2). Thus, we have

$$\begin{aligned}(\text{II.5.34.7}) \qquad & \jmath^{-1} \circ (1 \otimes \varphi_{\partial_0}|_V) \circ \jmath \\ =& \exp(\sum_{i=1}^d T_i \otimes \varphi_{\partial_i}|_V) \circ (1 \otimes \varphi_{\partial_0}|_V) \circ \exp(-\sum_{i=1}^d T_i \otimes \varphi_{\partial_i}|_V) \\ =& \exp(\sum_{i=1}^d T_i \otimes \varphi_{\partial_i}|_V) \circ \exp(-\sum_{i=1}^d T_i \otimes \varphi_{\partial_i}|_V) \circ (1 \otimes \varphi_{\partial_0}|_V - \sum_{i=1}^d T_i \otimes \varphi_{\partial_i}|_V) \\ =& 1 \otimes \varphi_{\partial_0}|_V - \sum_{i=1}^d T_i \otimes \varphi_{\partial_i}|_V.\end{aligned}$$

Since h is compatible with Higgs fields, we have $\sum_{j=1}^e (h \circ \varphi_{\widetilde{\partial}_j}|_{\widetilde{V}}) \otimes \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_j) = \sum_{i=1}^d (\varphi_{\partial_i}|_V \circ h) \otimes \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_i)$ by the definition (II.5.28.2). Notice that $(\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_1), \ldots, \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(\widetilde{T}_e)) = (\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_1), \ldots, \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_d))A$ and that $\mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_1), \ldots, \mathrm{d}_{\mathscr{C}_{\mathrm{HT}}}(T_d)$ are linearly independent. Thus, we have

$$(\text{II.5.34.8}) \qquad (\varphi_{\partial_1}|_V \circ h, \ldots, \varphi_{\partial_d}|_V \circ h) = (h \circ \varphi_{\widetilde{\partial}_1}|_{\widetilde{V}}, \ldots, h \circ \varphi_{\widetilde{\partial}_e}|_{\widetilde{V}})A^{\mathrm{T}},$$

which implies (II.5.34.2), since $\jmath$ and $\widetilde{\jmath}$ commute with $1 \otimes \varphi_{\partial_i}|_V$ and $1 \otimes \varphi_{\widetilde{\partial}_i}|_{\widetilde{V}}$ respectively. Since $\overline{\rho'}|_{\widetilde{\Sigma}_{0,\infty}} = \rho'|_{\widetilde{\Sigma}_{0,\infty}}$ by II.5.28.(1), we have an identification

$$(\text{II.5.34.9}) \qquad \varphi_{\widetilde{\partial}_0}|_{\widetilde{V}} = \varphi_{\widetilde{\partial}_0}|_{\widetilde{M}} \in \mathrm{End}_{K_\infty}(\widetilde{V}) = \mathrm{End}_{K_\infty}(\widetilde{M}),$$

where $\varphi_{\widetilde{\partial}_0}|_{\widetilde{V}}$ is given by the infinitesimal action of $\widetilde{\partial}_0 \in \mathrm{Lie}(\widetilde{\Sigma}_{0,\infty})$ via ρ' on V, and $\varphi_{\widetilde{\partial}_0}|_{\widetilde{M}}$ is given by the infinitesimal action of $\widetilde{\partial}_0 \in \mathrm{Lie}(\widetilde{\Sigma}_{0,\infty}) = \mathrm{Lie}(\Sigma)$ via $\overline{\rho'}$ on M. Similarly, we have $\varphi_{\partial_0}|_V = \varphi_{\partial_0}|_M$. Since $h : \widetilde{M} \to M$ is Σ-equivariant, we deduce that $h \circ \varphi_{\widetilde{\partial}_0}|_{\widetilde{V}} = \varphi_{\partial_0}|_V \circ h$ by II.4.9.(3). Using these properties, we have

$$\begin{aligned}(\text{II.5.34.10}) \qquad & (1 \otimes h)^{-1} \circ (1 \otimes \varphi_{\partial_0}|_V - \sum_{i=1}^d T_i \otimes \varphi_{\partial_i}|_V) \circ (1 \otimes h) \\ =& 1 \otimes \varphi_{\widetilde{\partial}_0}|_{\widetilde{V}} - \sum_{i=1}^d T_i \otimes (h^{-1} \circ \varphi_{\partial_i}|_V \circ h) \\ =& 1 \otimes \varphi_{\widetilde{\partial}_0}|_{\widetilde{V}} - \sum_{i=1}^d \sum_{j=1}^e T_i a_{ij} \otimes \varphi_{\widetilde{\partial}_i}|_{\widetilde{V}} \\ =& 1 \otimes \varphi_{\widetilde{\partial}_0}|_{\widetilde{V}} - \sum_{j=1}^e (\widetilde{T}_j - b_j) \otimes \varphi_{\widetilde{\partial}_i}|_{\widetilde{V}}.\end{aligned}$$

By the argument of (II.5.34.7), we see that

$$(\text{II.5.34.11}) \qquad \widetilde{\jmath} \circ (1 \otimes \varphi_{\widetilde{\partial}_0}|_{\widetilde{V}} - \sum_{j=1}^e (\widetilde{T}_j - b_j) \otimes \varphi_{\widetilde{\partial}_i}|_{\widetilde{V}}) \circ \widetilde{\jmath}^{-1} = 1 \otimes \varphi_{\widetilde{\partial}_0}|_{\widetilde{V}} + \sum_{j=1}^e b_j \otimes \varphi_{\widetilde{\partial}_j}|_{\widetilde{V}}$$

which completes the proof. □

Theorem II.5.35. *Let K be a complete discrete valuation field extension of $\mathbb{Q}_p$ whose residue field admits a finite p-basis, $G = \mathrm{Gal}(\overline{K}/K)$. Then, for any object W of $\mathbf{Rep}_{\mathrm{cont}}^{\mathrm{proj}}(G, \widehat{\overline{K}})$, there is a canonical homomorphism of $\widehat{\overline{K}}$-linear Lie algebras (see* II.5.16*)*

$$(\text{II.5.35.1}) \qquad \varphi_{\mathrm{Sen}}|_W : \mathscr{E}_{\mathcal{O}_K}^*(1) \longrightarrow \mathrm{End}_{\widehat{\overline{K}}}(W),$$

which is G-equivariant with respect to the canonical action on $\mathscr{E}_{\mathcal{O}_K}^*(1)$ defined in II.5.16 *and the adjoint action on $\mathrm{End}_{\widehat{\overline{K}}}(W)$ (i.e. $g \in G$ sends an endomorphism ϕ to $g \circ \phi \circ g^{-1}$), and functorial in W, i.e. it defines a canonical functor*

$$(\text{II.5.35.2}) \qquad \varphi_{\mathrm{Sen}} : \mathbf{Rep}_{\mathrm{cont}}^{\mathrm{proj}}(G, \widehat{\overline{K}}) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}(\mathscr{E}_{\mathcal{O}_K}^*(1), \widehat{\overline{K}}),$$

from the category of finite projective (continuous semi-linear) $\widehat{\overline{K}}$-representations of the profinite group G to the category of finite projective $\widehat{\overline{K}}$-linear representations of the Lie algebra $\mathscr{E}_{\mathcal{O}_K}^*(1)$.

Moreover, under the assumption in II.5.23 *and with the same notation, assume that there is an object* $\widetilde{V}$ *of* $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\widetilde{\Delta}\text{-an}}(\widetilde{\Gamma}, K_\infty)$ *such that* $W = \widehat{\overline{K}} \otimes_{K_\infty} \widetilde{V}$. *Then, for any* $f \in \mathscr{E}^*_{\mathcal{O}_K}(1) = \mathrm{Hom}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1), \widehat{\overline{K}})$,

$$\varphi_{\mathrm{Sen}}|_W(f) = \sum_{i=0}^{e} f(\widetilde{T}_i) \otimes \varphi_{\widetilde{\partial}_i}|_{\widetilde{V}}. \tag{II.5.35.3}$$

Proof. Recall that the base change functor

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma, K_\infty) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}}),\ V \mapsto \widehat{\overline{K}} \otimes_{K_\infty} V, \tag{II.5.35.4}$$

is an equivalence by II.5.17 and II.5.18. Thus, there is an essentially unique object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma, K_\infty)$ such that $W = \widehat{\overline{K}} \otimes_{K_\infty} V$. We claim that $\varphi_{\mathrm{Sen}}|_W$ defined by the formula (II.5.35.3) does not depend on the choice of $\widetilde{V}$ and $\widetilde{t}_i$ (so that $\varphi_{\mathrm{Sen}}|_W$ is canonically defined by the essential surjectivity of (II.5.35.4), and functorial in W by the fully faithfulness of (II.5.35.4)). With the notation in II.5.34, for any $f \in \mathrm{Hom}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1), \widehat{\overline{K}})$, we have

$$\begin{aligned} &(f(\widetilde{T}_0), f(\widetilde{T}_1), \dots, f(\widetilde{T}_e)) \otimes (\varphi_{\widetilde{\partial}_0}, \varphi_{\widetilde{\partial}_1}, \dots, \varphi_{\widetilde{\partial}_e})^{\mathrm{T}} \\ =&(f(T_0), f(T_1), \dots, f(T_d)) \begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} \otimes (\varphi_{\widetilde{\partial}_0}, \varphi_{\widetilde{\partial}_1}, \dots, \varphi_{\widetilde{\partial}_e})^{\mathrm{T}} \\ =&(f(T_0), f(T_1), \dots, f(T_d)) \otimes (\varphi_{\partial_0}, \varphi_{\partial_1}, \dots, \varphi_{\partial_d})^{\mathrm{T}} \end{aligned} \tag{II.5.35.5}$$

where the last equality follows from II.5.34, which proves the claim.

Notice that the map $\varphi_{\mathrm{Sen}}|_W$ defined by (II.5.35.3) fits into the following commutative diagram

$$\begin{array}{ccc} \mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_W} & \mathrm{End}_{\widehat{\overline{K}}}(W) \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle \mathrm{id}_{\widehat{\overline{K}}} \otimes \varphi|_{\widetilde{V}}} & \\ & \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\widetilde{\Gamma}) & \end{array} \tag{II.5.35.6}$$

where ψ is the surjection (II.5.24.1), $\varphi|_{\widetilde{V}} : \mathrm{Lie}(\widetilde{\Gamma}) \to \mathrm{End}_{K_\infty}(\widetilde{V})$ is the infinitesimal Lie algebra algebra action (II.4.16.1). This shows that $\varphi_{\mathrm{Sen}}|_W$ is a morphism of $\widehat{\overline{K}}$-linear Lie algebras.

It remains to check the G-equivariance of $\varphi_{\mathrm{Sen}}|_W$. For any $g \in G$, we have

$$g \circ \varphi_{\widetilde{\partial}_i} \circ g^{-1} = \chi(g)\varphi_{\widetilde{\partial}_i},\ \forall 1 \le i \le e, \tag{II.5.35.7}$$

$$g \circ \varphi_{\widetilde{\partial}_0} \circ g^{-1} = \varphi_{\widetilde{\partial}_0} - \sum_{i=1}^{e} \widetilde{\xi}_i(g)\varphi_{\widetilde{\partial}_i}, \tag{II.5.35.8}$$

where the first follows from (II.4.17.1), and the second follows from II.4.21 and the identity $\widetilde{\xi}(g\sigma_0 g^{-1}) = \widetilde{\xi}(g)(1 - \chi(\sigma_0))$ for any $\sigma_0 \in \widetilde{\Sigma}_{0,\infty}$ by (II.4.15.5). Therefore,

(II.5.35.9)

$$\begin{aligned} \varphi_{\mathrm{Sen}}|_W(g \cdot f) =& \sum_{i=0}^{e} g(f(g^{-1}\widetilde{T}_i)) \otimes \varphi_{\widetilde{\partial}_i} \\ =&(g \otimes 1)\left(f(\widetilde{T}_0) \otimes \varphi_{\widetilde{\partial}_0} + \sum_{i=1}^{e} \chi(g)(f(\widetilde{T}_i) + \widetilde{\xi}_i(g^{-1})f(\widetilde{T}_0)) \otimes \varphi_{\widetilde{\partial}_i} \right) \quad \text{(by (II.5.20.4))} \\ =&(g \otimes 1)\left(f(\widetilde{T}_0) \otimes (\varphi_{\widetilde{\partial}_0} - \sum_{i=1}^{e} \widetilde{\xi}_i(g)\varphi_{\widetilde{\partial}_i}) + \chi(g) \sum_{i=1}^{e} f(\widetilde{T}_i) \otimes \varphi_{\widetilde{\partial}_i} \right) \quad \text{(by (II.4.15.5))} \\ =& \sum_{i=0}^{e} g(f(\widetilde{T}_i)) \otimes (g \circ \varphi_{\widetilde{\partial}_i} \circ g^{-1}) = g \circ \varphi_{\mathrm{Sen}}|_W(f) \circ g^{-1} \end{aligned}$$

which shows the G-equivariance. □

Remark II.5.36. The same argument also shows that the $\widehat{\overline{K}}$-linear map

$$W \longrightarrow W \otimes_{\widehat{\overline{K}}} \mathscr{E}_{\mathcal{O}_K}(-1) \tag{II.5.36.1}$$

sending x to $\sum_{i=0}^{e}(\mathrm{id}_{\widehat{K}}\otimes\varphi_{\widetilde{\partial}_i}|_{\widetilde{V}})(x)\otimes\widetilde{T}_i$, is G-equivariant and does not depend on the choice of $\widetilde{V}$ or $\widetilde{t}_i$. It naturally induces the map $\varphi_{\mathrm{Sen}}|_W$ (II.5.35.1). We note that it is not a Higgs field.

Definition II.5.37. Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$. We denote by $\Phi(W)$ the image of $\varphi_{\mathrm{Sen}}|_W$, and by $\Phi^{\mathrm{geo}}(W)$ the image of $\mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1),\widehat{\overline{K}})$ under $\varphi_{\mathrm{Sen}}|_W$. We call an element of $\Phi(W)\subseteq\mathrm{End}_{\widehat{\overline{K}}}(W)$ a *Sen operator* of W. We call an element of $\Phi^{\mathrm{geo}}(W)\subseteq\mathrm{End}_{\widehat{\overline{K}}}(W)$ a *geometric Sen operator* of W. And we call the image of $1\in\widehat{\overline{K}}$ in $\Phi^{\mathrm{ari}}(W)=\Phi(W)/\Phi^{\mathrm{geo}}(W)$ the *arithmetic Sen operator* of W.

Namely, we defined a canonical morphism of exact sequences of $\widehat{\overline{K}}$-linear Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1),\widehat{\overline{K}}) & \xrightarrow{\jmath^*} & \mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow{\iota^*} & \widehat{\overline{K}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Phi^{\mathrm{geo}}(W) & \longrightarrow & \Phi(W) & \longrightarrow & \Phi^{\mathrm{ari}}(W) & \longrightarrow & 0 \end{array} \tag{II.5.37.1}$$

which factors through (II.5.24.2) under the assumption of (II.5.35.3).

Proposition II.5.38. *Let K' be a complete discrete valuation field extension of K whose residue field admits a finite p-basis, $\overline{K'}$ an algebraic closure of K' containing $\overline{K}$, $G'=\mathrm{Gal}(\overline{K'}/K')$, W an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$, $W'=\widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}W$ the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G',\widehat{\overline{K'}})$. Assume that $K'\otimes_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K}\to K'\otimes_{\mathcal{O}_{K'}}\widehat{\Omega}^1_{\mathcal{O}_{K'}}$ is injective. Then, there is a natural commutative diagram*

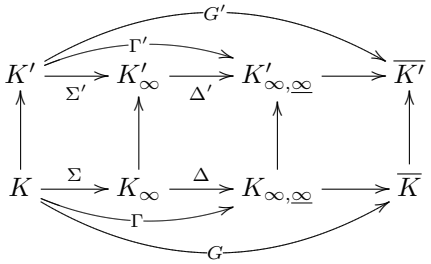
$$\begin{array}{ccc} \mathscr{E}^*_{\mathcal{O}_{K'}}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{\overline{K'}}}(W') \\ \downarrow & & \uparrow\wr \\ \widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}\mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow{\mathrm{id}_{\widehat{\overline{K'}}}\otimes\varphi_{\mathrm{Sen}}|_W} & \widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}\mathrm{End}_{\widehat{\overline{K}}}(W) \end{array} \tag{II.5.38.1}$$

where φ_{Sen} are the canonical Lie algebra actions defined in II.5.35*, the left vertical arrow is the surjection induced by taking dual of the natural injection $\widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}\mathscr{E}_{\mathcal{O}_K}(-1)\to\mathscr{E}_{\mathcal{O}_{K'}}(-1)$ (cf.* II.5.8*), and the right vertical arrow is the canonical isomorphism. In particular, the inverse of the right vertical arrow induces a natural isomorphism*

$$\Phi(W')\xrightarrow{\sim}\widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}\Phi(W) \tag{II.5.38.2}$$

which is compatible with geometric and arithmetic Sen operators.

Proof. Let $t'_{1,p^n},\dots,t'_{d,p^n}\in\overline{K'}$ be the images of $t_{1,p^n},\dots,t_{d,p^n}\in\overline{K}$. Then, there is a commutative diagram



(II.5.38.3)

Since $\mathrm{d}\log(t'_1),\dots,\mathrm{d}\log(t'_d)$ are K'-linearly independent in $K'\otimes_{\mathcal{O}_{K'}}\widehat{\Omega}^1_{\mathcal{O}_{K'}}$ by assumption, Δ' is also of dimension d by II.5.22. In particular, we have a natural isomorphism $\mathrm{Lie}(\Gamma')\xrightarrow{\sim}\mathrm{Lie}(\Gamma)$ which identifies their standard bases $\{\partial'_i\}_{1\le i\le d}$ and $\{\partial_i\}_{1\le i\le d}$ defined in II.4.20. Let V be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma,K_\infty)$ such that $W=\widehat{\overline{K}}\otimes_{K_\infty}V$. Then, the object $V'=K'_\infty\otimes_{K_\infty}V$ of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta'\text{-an}}(\Gamma',K'_\infty)$ satisfies that $W'=\widehat{\overline{K'}}\otimes_{K'_\infty}V'$. By II.4.11, the natural identification $\mathrm{End}_{\widehat{\overline{K'}}}(W')=\widehat{\overline{K'}}\otimes_{\widehat{\overline{K}}}\mathrm{End}_{\widehat{\overline{K}}}(W)$ identifies $\mathrm{id}_{\widehat{\overline{K'}}}\otimes\varphi_{\partial'_i}|_{V'}$ with $\mathrm{id}_{\widehat{\overline{K'}}}\otimes\varphi_{\partial_i}|V$. This shows that the diagram (II.5.38.1) is commutative which induces an isomorphism (II.5.38.2). □

Lemma II.5.39. *Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$.*

(1) *Geometric Sen operators of W are nilpotent and commute with each other.*

(2) *If $\phi \in \Phi(W)$ is a lifting of the arithmetic Sen operator and $\theta \in \Phi^{\mathrm{geo}}(W)$ is a geometric Sen operator, then $[\phi, \theta] = \theta$.*
(3) *Moreover, we have $\phi(W^G) = 0$ and $\theta(W^H) = 0$.*

PROOF. (1) follows from II.5.28.(2) and the definition of geometric Sen operator (II.5.35.3). (2) follows from the Lie algebra structure of $\mathscr{E}^*_{\mathcal{O}_K}(1)$ defined in II.5.16. (3) follows from the definition of Sen operators, and we will give a detailed proof in II.11.12. □

Lemma II.5.40. *Any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}})$ admits a canonical and functorial finite ascending filtration $\{\mathrm{F}_n\}_{n\in\mathbb{N}}$ stable under the Lie algebra action $\varphi_{\mathrm{Sen}}|_W$ (II.5.35.1) such that any geometric Sen operator sends $\mathrm{F}_{n+1}W$ into F_nW. In particular, the arithmetic Sen operator of W acts naturally on the graded object $\mathrm{Gr}^{\mathrm{F}}W = \oplus_{n\in\mathbb{N}}\mathrm{Gr}^{\mathrm{F}}_n W$, where $\mathrm{Gr}^{\mathrm{F}}_n W = \mathrm{F}_nW/\mathrm{F}_{n-1}W$.*

PROOF. We set $\mathrm{F}_0W = 0$, and

$$\mathrm{F}_1W = \bigcap_{\theta\in\Phi^{\mathrm{geo}}(W)} \mathrm{Ker}(\theta). \tag{II.5.40.1}$$

By II.5.39, one checks easily that F_1W is stable by any Sen operator of W. If $W \neq 0$, then F_1W is non-zero, as θ is nilpotent. It is also functorial in W by II.5.35. Then, for any $n \in \mathbb{N}_{>0}$, F_nW is defined inductively by $\mathrm{Gr}^{\mathrm{F}}_n W = \mathrm{F}_1(W/\mathrm{F}_{n-1}W) = \bigcap_{\theta\in\Phi^{\mathrm{geo}}(W)} \mathrm{Ker}(\theta|_{W/\mathrm{F}_{n-1}W})$. □

Proposition II.5.41. *Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}})$. Any lifting of the arithmetic Sen operator $\phi \in \Phi(W)$ of W has the same characteristic polynomial, whose coefficients are in K.*

PROOF. By II.5.40, there exists a $\widehat{\overline{K}}$-basis of W with respect to which the matrix of ϕ is upper triangular and the matrix of any geometric operator θ of W is strictly upper triangular. We see that ϕ and $\phi + \theta$ have the same characteristic polynomial. The coefficients of this polynomial lie in K by [**Bri03**, Proposition 5.(a)]. □

Lemma II.5.42. *Let F be a field, V, W two F-linear spaces, $\Phi \subseteq W \otimes_F V$ a subset, $\mathfrak{F}_\Phi = \{f \in \mathrm{Hom}_F(V, F) \mid f_W(\Phi) = 0\}$, where $f_W = \mathrm{id}_W \otimes f \in \mathrm{Hom}_F(W \otimes_F V, W)$. Then, $V_\Phi = \bigcap_{f\in\mathfrak{F}_\Phi} \mathrm{Ker}(f)$ is the smallest F-linear subspace V' of V such that $\Phi \subseteq W \otimes_F V'$. Moreover, an F-linear subspace V' of V is equal to V_Φ if and only if $\mathfrak{F}_\Phi$ is equal to $\mathfrak{F}_{V'} = \{f \in \mathrm{Hom}_F(V, F) \mid f(V') = 0\}$.*

PROOF. Firstly, we claim that $\Phi \subseteq W \otimes_F V_\Phi$. Consider the exact sequence

$$0 \longrightarrow V_\Phi \longrightarrow V \xrightarrow{(f)_{f\in\mathfrak{F}_\Phi}} \prod_{f\in\mathfrak{F}_\Phi} F. \tag{II.5.42.1}$$

Since W is flat over F, we have an exact sequence

$$0 \longrightarrow W \otimes_F V_\Phi \longrightarrow W \otimes_F V \xrightarrow{(\mathrm{id}_W\otimes f)_{f\in\mathfrak{F}_\Phi}} W \otimes_F \left(\prod_{f\in\mathfrak{F}_\Phi} F\right). \tag{II.5.42.2}$$

Since $W \otimes_F (\prod_{f\in\mathfrak{F}_\Phi} F) \subseteq \prod_{f\in\mathfrak{F}_\Phi} W$, the subset $\Phi \subseteq W \otimes_F V$ is mapped to zero in (II.5.42.2), which proves the claim.

Secondly, for any F-linear subspace V' of V, it is clear that $V' \subseteq \bigcap_{f\in\mathfrak{F}_{V'}} \mathrm{Ker}(f)$. This is actually an equality, since for any element $v \in V \setminus V'$ there exists $f \in \mathfrak{F}_{V'}$ such that $f(v) \neq 0$.

Assume that $\Phi \subseteq W \otimes_F V'$. Then, $\mathfrak{F}_{V'} \subseteq \mathfrak{F}_\Phi$ so that $V_\Phi = \bigcap_{f\in\mathfrak{F}_\Phi} \mathrm{Ker}(f) \subseteq \bigcap_{f\in\mathfrak{F}_{V'}} \mathrm{Ker}(f) = V'$. It shows that V_Φ is the smallest F-linear subspace V' of V such that $\Phi \subseteq W \otimes_F V'$. In particular, we have $\mathfrak{F}_{V_\Phi} \subseteq \mathfrak{F}_\Phi$. On the other hand, the definition of V_Φ implies that $\mathfrak{F}_\Phi \subseteq \mathfrak{F}_{V_\Phi}$. Thus, $\mathfrak{F}_{V_\Phi} = \mathfrak{F}_\Phi$ and the final assertion follows. □

Theorem II.5.43 ([**Sen81**, Theorem 11], [**Ohk14**, 3.1])**.** *Let I be the inertia subgroup of G, (V, ρ) an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \mathbb{Q}_p)$, $W = \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} V$ the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}})$. Then, $\mathrm{Lie}(\rho(I))$ is the smallest $\mathbb{Q}_p$-subspace S of $\mathrm{End}_{\mathbb{Q}_p}(V)$ such that the space of Sen operators $\Phi(W)$ is contained in $\widehat{\overline{K}} \otimes_{\mathbb{Q}_p} S$.*

Remark II.5.44. We don't know whether or not $\mathrm{Lie}(\rho(I \cap H))$ is the smallest $\mathbb{Q}_p$-subspace S of $\mathrm{End}_{\mathbb{Q}_p}(V)$ such that the space of geometric Sen operators $\Phi^{\mathrm{geo}}(W)$ is contained in $\widehat{\overline{K}} \otimes_{\mathbb{Q}_p} S$, where $H = \mathrm{Gal}(\overline{K}/K_\infty)$. Recall that Sen-Ohkubo's proof of II.5.43 relies on Sen's ramification theorem on a Galois extension of a complete discrete valuation field whose Galois group is a p-adic analytic group ([**Sen73**, Lemma 3], [**Ohk14**, 1.3]). Thus, it seems that we couldn't apply their techniques directly to this question. Nevertheless, we have the following weaker result.

Corollary II.5.45. *With the notation in* II.5.43*, the space of geometric Sen operators* $\Phi^{\mathrm{geo}}(W)$ *is contained in* $\widehat{K} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\rho(I \cap H))$.

Proof. Note that $\Phi^{\mathrm{geo}}(W) = [\Phi(W), \Phi(W)]$ by the Lie algebra structure on $\mathscr{E}^*_{\mathcal{O}_K}(1)$ (cf. II.5.16), and that $[\mathrm{Lie}(\rho(I)), \mathrm{Lie}(\rho(I))] \subseteq \mathrm{Lie}(\rho(I \cap H))$ as $I/(I \cap H) \subseteq G/H = \Sigma$ is abelian. The conclusion follows directly from the fact that $\Phi(W) \subseteq \widehat{K} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\rho(I))$ by II.5.43. □

II.6. Extending Sen operators to Infinite-Dimensional Representations

We extend Sen operators to certain infinite-dimensional representations and their completions over a complete discrete valuation ring. The goal is to show that geometric Sen operators still annihilate the Galois invariant part of the representation.

II.6.1. By saying that a topological abelian group M is "complete", we always mean that it is separated and every Cauchy net of M admits a limit point ([**GR04**, 8.2.6]). The forgetful functor from the category of complete topological abelian groups to the category of topological abelian groups admits a left adjoint, called the *completion* and denoted by $M \mapsto \widehat{M}$. The canonical map $M \to \widehat{M}$ has dense image and induces the topology on M from that of $\widehat{M}$ ([**GR04**, 8.2.8]). The adjoint property implies that for any continuous group homomorphism of topological abelian groups $f : M \to N$, there is a unique continuous group homomorphism of the completions $\widehat{f} : \widehat{M} \to \widehat{N}$ making the following diagram commutative

(II.6.1.1)
$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ \widehat{M} & \xrightarrow{\widehat{f}} & \widehat{N} \end{array}$$

where the vertical arrows are the canonical maps. We call $\widehat{f}$ the *continuation* (or *completion*) of f.

II.6.2. We briefly review the definition of normed modules mainly following [**Tsu18**, §5], and we also refer to [**BGR84**] for a systematic development. A (non-Archimedean) *norm* on an abelian group M is a map $|\ | : M \to \mathbb{R}_{\geq 0}$ such that $|x| = 0$ if and only if $x = 0$, and that $|x - y| \leq \max\{|x|, |y|\}$ for any $x, y \in M$. For any $r \in \mathbb{R}_{>0}$, we denote the closed ball of radius r by

(II.6.2.1)
$$M^{\leq r} = \{x \in M \mid |x| \leq r\}.$$

The metric topology makes M into a separated topological abelian group, where $\{M^{\leq r}\}_{r \in \mathbb{R}_{>0}}$ forms a fundamental system of closed neighbourhoods of 0. The norm map extends uniquely over the completion $\widehat{M}$ of M, and M naturally identifies with a dense normed subgroup of $\widehat{M}$.

A *normed ring* R is a ring endowed with a norm such that $|xy| \leq |x||y|$ for any $x, y \in R$. We remark that there is a natural normed ring structure on $\widehat{R}$ induced by that of R. Given a normed ring R, a *normed R-module* is an R-module M endowed with a norm such that $|ax| \leq |a||x|$ for any $a \in R$ and $x \in M$. Moreover, we call R a *Banach ring* if it is complete, and we call M an *R-Banach module* if M is complete.

II.6.3. For any valuation field K of height 1 extension of $\mathbb{Q}_p$, we fix a valuation map $v_K : K^\times \to \mathbb{Q}$ normalized by $v_K(p) = 1$, and we endow K with the norm $|\ |_K$ defined by $|x|_K = p^{-v_K(x)}$ for any $x \in K^\times$. Let V be a normed K-module. Since $p^n V^{\leq 1} = V^{\leq p^{-n}}$ for any $n \in \mathbb{N}$, $V^{\leq 1}$ is a p-adically separated flat $\mathcal{O}_K$-module, and the induced metric topology on $V^{\leq 1}$ coincides with its p-adic topology. We have $V = V^{\leq 1}[1/p]$.

Conversely, given a p-adically separated flat $\mathcal{O}_K$-module M, we can define a norm on $M[1/p]$ by setting $|x| = p^{-v_M(x)}$ for any $x \in M[1/p] \setminus \{0\}$ where $v_M(x)$ is the biggest integer such that $x \in p^{v_M(x)}M$. The metric topology defined by this norm on $M[1/p]$ coincides with its p-adic topology defined by M, and makes $M[1/p]$ into a normed K-module with $M[1/p]^{\leq 1} = M$. We remark that V (resp. $M[1/p]$) is complete if and only if $V^{\leq 1}$ (resp. M) is p-adically complete.

Lemma II.6.4. *Let K be a complete valuation field of height 1 extension of $\mathbb{Q}_p$, V a normed K-module. Then, the induced topology on any finite-dimensional K-subspace V_0 of V coincides with its canonical topology (cf.* II.2.3*).*

Proof. The norm on V induces a norm on V_0, whose associated metric topology defines the induced topology on V_0. On the other hand, any K-linear isomorphism $V_0 \cong K^n$ defines a norm on V_0 which induces the canonical topology on V_0. The conclusion follows from the fact that any two norms on a finite-dimensional K-space are equivalent ([**BGR84**, 2.3.3.5]). □

Proposition II.6.5. *With the notation in* II.5.1*, let W be a normed $\widehat{\overline{K}}$-module endowed with a continuous semi-linear action of G. Assume that the $(G,\widehat{\overline{K}})$-finite part of W is equal to W itself (cf.* II.2.4*). Then, there exists a unique $\widehat{\overline{K}}$-linear Lie algebra action of $\mathscr{E}^*_{\mathcal{O}_K}(1)$ on W,*

$$\varphi_{\mathrm{Sen}}|_W : \mathscr{E}^*_{\mathcal{O}_K}(1) \longrightarrow \mathrm{End}_{\widehat{\overline{K}}}(W), \tag{II.6.5.1}$$

such that for any G-equivariant continuous $\widehat{\overline{K}}$-linear homomorphism $W_0 \to W$ from an object W_0 of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$, $\varphi_{\mathrm{Sen}}|_W$ is compatible with the Lie algebra action $\varphi_{\mathrm{Sen}}|_{W_0}$ of $\mathscr{E}^*_{\mathcal{O}_K}(1)$ defined in II.5.35*.*

Proof. The maps $W_0 \to W$ form a category $\mathcal{I}$. Indeed, it is the localization of the category $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$ at the presheaf given by the restriction of the presheaf $\mathrm{Hom}(-,W)$ on $\mathbf{Rep}_{\mathrm{cont}}(G,\widehat{\overline{K}})$ represented by W (cf. [**SGA 4**$_{\mathrm{I}}$, I.3.4.0]). We claim that $\mathcal{I}$ is filtered. Indeed, for any object $W_0 \to W$ of $\mathcal{I}$, its image $W_1 \subseteq W$ is G-stable. Since the topology on W_1 induced from W coincides with the canonical topology as a finite-dimensional $\widehat{\overline{K}}$-space by II.6.4, we see that W_1 is a finite projective $\widehat{\overline{K}}$-representation of G and a subrepresentation of W and that $W_0 \to W_1$ is a morphism in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{K}})$. As direct sums exist in $\mathcal{I}$, one checks easily that $\mathcal{I}$ is filtered. By assumption and the previous argument, $W = \mathrm{colim}_{\mathcal{I}} W_0$ as $\widehat{\overline{K}}$-modules. Since the Lie algebra $\mathscr{E}^*_{\mathcal{O}_K}(1)$ acts functorially on each W_0, it defines a unique action on W compatible with that on each W_0. □

Remark II.6.6. Let W be a $\widehat{\overline{K}}$-Banach space endowed with a continuous semi-linear action of G such that the $(G,\widehat{\overline{K}})$-finite part W^{f} of W is dense in W. If we endow W^{f} with the induced topology, then its completion coincides with W ([**GR04**, 8.2.8.(iii)]). By II.6.5, we obtain a canonical Lie algebra action $\varphi_{\mathrm{Sen}}|_{W^{\mathrm{f}}}$ on W^{f}. If the operators on W^{f} defined by $\varphi_{\mathrm{Sen}}|_{W^{\mathrm{f}}}$ are continuous, then we can extend this action uniquely to a Lie algebra action $\varphi_{\mathrm{Sen}}|_W$ on W by continuation (cf. II.6.1).

However, in this work we haven't found a simple condition to guarantee the continuity of the Sen operators on W^{f}. Instead, we consider two types of dense subrepresentations of W^{f} and discuss the continuity of Sen operators on them. Roughly speaking, the first type (considered in the rest of this section) is the union of representations with "small lattices", which is ad hoc but suitable for doing descent and decompletion (so that nice properties are preserved after continuation, cf. II.6.19). The other type (considered in the end of section II.11) is the union of representations defined over $\mathbb{Q}_p$, which is more canonical but we need to reduce to the first type for proving properties (cf. II.11.26).

II.6.7. Let K be a valuation field of height 1 extension of $\mathbb{Q}_p$ with a valuation map $v_K : K^\times \to \mathbb{Q}$ normalized by $v_K(p) = 1$, A a p-adically complete flat $\mathcal{O}_K$-algebra, M a p-torsion free p-adically complete A-module. Consider an A-linear endomorphism ϕ on M such that $\phi(M) \subseteq \alpha M$ for some element $\alpha \in \mathfrak{m}_K$ with $v_K(\alpha) > \frac{1}{p-1}$. As $v_K(n!) \leq \frac{n-1}{p-1}$ for any $n \in \mathbb{N}_{>0}$, the series for any $x \in M$,

$$\exp(\phi)(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\phi^n(x), \tag{II.6.7.1}$$

$$\log(1+\phi)(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\phi^n(x) \tag{II.6.7.2}$$

are well-defined and converge in M with respect to the p-adic topology. They define two A-linear endomorphisms $\exp(\phi) \in \mathrm{id} + \alpha\mathrm{End}_A(M)$ and $\log(1+\phi) \in \alpha\mathrm{End}_A(M)$ of M such that $\exp(\log(1+\phi)) = 1+\phi$ and $\log(\exp(\phi)) = \phi$. Thus, we deduce easily that for any $n \in \mathbb{N}$, $p^{-n}((1+\phi)^{p^n} - 1) = p^{-n}(\exp(p^n \log(1+\phi)) - 1) \in \alpha\mathrm{End}_A(M)$, and that

$$\log(1+\phi)(x) = \lim_{n\to\infty} p^{-n}((1+\phi)^{p^n} - 1)(x), \tag{II.6.7.3}$$

namely, $\log(1+\phi)$ is the infinitesimal action of $1+\phi$ on M (cf. II.4.8).

Definition II.6.8 (cf. [**AGT16**, II.13.1, II.13.2])**.** Let K be a valuation field of height 1 extension of $\mathbb{Q}_p$, $\mathfrak{a}$ an ideal of $\mathcal{O}_K$, A a p-adically complete flat $\mathcal{O}_K$-algebra endowed with a continuous action of a topological group G by homomorphisms of $\mathcal{O}_K$-algebras.

(1) For any object M of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,A)$, we say that M is $\mathfrak{a}$-*small* if M is a finite free A-module admitting a basis consisting of elements that are G-invariant modulo αM for some $\alpha \in \mathfrak{a}$. We denote by $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(G,A)$ the full subcategory of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,A)$ consisting of $\mathfrak{a}$-small objects.

(2) For any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, A[1/p])$, we say that W is $\mathfrak{a}$-*small* if there exists a G-stable A-submodule W^+ of W generated by finitely many elements that are G-invariant modulo αW^+ for some $\alpha \in \mathfrak{a}$. We denote by $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(G, A[1/p])$ the full subcategory of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, A[1/p])$ consisting of $\mathfrak{a}$-small objects.

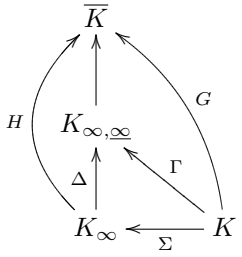
Remark II.6.9. In II.6.8.(1), the G-action on $M/\alpha M$ may not be trivial, since this action is A-semi-linear not A-linear. In II.6.8.(2), if W is $\mathfrak{a}$-small, then the canonical topology on $W = W^+[1/p]$ is induced by the p-adic topology of W^+ (cf. II.2.3). In particular, there is a natural faithful functor

$$\mathbf{Rep}^{\mathfrak{a}\text{-s}}(G, A) \longrightarrow \mathbf{Rep}^{\mathfrak{a}\text{-s}}(G, A[1/p]),\ M \mapsto M[1/p]. \tag{II.6.9.1}$$

We remark that even if $A = \mathcal{O}_K$ (so that W^+ is a finite free $\mathcal{O}_K$-module), W^+ may not admit a basis consisting of elements that are G-invariant modulo αW^+, i.e. W^+ may not be $\mathfrak{a}$-small.

II.6.10. In the rest of this section, we take again the assumptions and notation in II.5.1.

(II.6.10.1)



Consider the following assumption on K:

(∗) Let κ be the residue field of K, $\cap_{n\geq 0}\kappa^{p^n}$ the maximal perfect subfield of κ, K_{can} the algebraic closure in K of the fraction field of the Witt ring $W(\cap_{n\geq 0}\kappa^{p^n})$. Then, $K_{\mathrm{can}} \to K$ is a weakly unramified extension of complete discrete valuation fields, i.e. a uniformizer of K_{can} is still a uniformizer of K.

This assumption is considered by Hyodo [**Hyo86**, (0-5)] when computing the cohomology $H^q(G, \widehat{\overline{K}})$. We remark that there exists a finite Galois extension of K which satisfies this assumption by Epp's theorem on eliminating ramification [**Epp73**, 1.9, 2.0.(1)]. The assumption implies that for any finite field extension K'_{can} of K_{can}, we have $\mathcal{O}_{K'} = \mathcal{O}_{K'_{\mathrm{can}}} \otimes_{\mathcal{O}_{K_{\mathrm{can}}}} \mathcal{O}_K$ where $K' = K'_{\mathrm{can}}K$ ([**Sta22**, 09E7, 09EQ]). In particular, the residue field of K_n is separable over that of K for any $n \in \mathbb{N}$, and thus $t_1, \ldots, t_d$ still form a p-basis of the residue field of K_n. Hence, there is an isomorphism of $\mathcal{O}_{K_n}$-algebras for any $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$,

$$\mathcal{O}_{K_n}[T_1, \ldots, T_d]/(T_1^{p^{m_1}} - t_1, \ldots, T_d^{p^{m_d}} - t_d) \xrightarrow{\sim} \mathcal{O}_{K_{n,\underline{m}}}, \tag{II.6.10.2}$$

sending T_i to t_{i,p^m} ([**Hyo86**, 1-2]). In particular, the continuous homomorphism (II.4.15.4) $\xi : \Delta \to \mathbb{Z}_p^d$ is an isomorphism, which also implies that $K_{\infty,\underline{\infty}} = K_\infty \otimes_K K_{0,\underline{\infty}}$.

Theorem II.6.11 (Faltings, [**Fal05**], cf. [**AGT16**, II.14.4])**.** *Under the assumption* (∗) *in* II.6.10 *on K, let $\mathfrak{a}$ be the ideal of $\mathcal{O}_{K_\infty}$ consisting of elements α with normalized valuation $v_{K_\infty}(\alpha) > \frac{2}{p-1}$. Then, the functor*

$$\mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta, \mathcal{O}_{\widehat{K_\infty}}) \longrightarrow \mathbf{Rep}^{\mathfrak{a}\text{-s}}(H, \mathcal{O}_{\widehat{\overline{K}}}) \tag{II.6.11.1}$$

is an equivalence of categories.

Proof. It follows from the same arguments of [**AGT16**, II.14.4]. □

Theorem II.6.12 ([**Tsu18**, 11.2, 12.4])**.** *Under the assumption* (∗) *in* II.6.10 *on K, let $\mathfrak{a}$ be the ideal of $\mathcal{O}_{K_\infty}$ consisting of elements α with normalized valuation $v_{K_\infty}(\alpha) > \frac{2}{p-1}$. Then, the functor*

$$\mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta, \widehat{K_\infty}) \longrightarrow \mathbf{Rep}^{\mathfrak{a}\text{-s}}(H, \widehat{\overline{K}}) \tag{II.6.12.1}$$

is an equivalence of categories.

Proof. It follows from the same arguments of [**Tsu18**, 11.2, 12.4]. □

Lemma II.6.13. *Let V be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma, K_\infty)$. Then, the associated $\widehat{K_\infty}$-representation $\widehat{V} = \widehat{K_\infty} \otimes_{K_\infty} V$ of Δ is $\mathfrak{a}$-small for any nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_{K_\infty}$.*

PROOF. Since Δ is commutative and by II.4.11 and II.4.18 the infinitesimal Lie algebra action $\varphi : \mathrm{Lie}(\Delta) \to \mathrm{End}_{K_\infty}(V)$ is nilpotent, we can take a basis $v_1, \dots, v_n$ of V as in the argument of II.5.41 such that $\varphi_\tau(v_i) \in \sum_{j>i} K_\infty v_j$ for any $\tau \in \Delta$ and $1 \le i \le n$. Since V is Δ-analytic, we have $\tau(v_i) - v_i = \exp(\varphi_\tau)(v_i) - v_i \in \sum_{j>i} K_\infty v_j$. For any $k_0 \in \mathbb{N}$, after replacing $v_1, \dots, v_n$ by $p^{-k_1}v_1, \dots, p^{-k_n}v_n$ for some integers $k_n \gg \cdots \gg k_1 > k_0$, we may assume that $\tau(v_i) - v_i \in \sum_{j>i} p^{k_0}\mathcal{O}_{K_\infty} v_j$. Thus, the finite free $\mathcal{O}_{K_\infty}$-submodule V^+ of V generated by $v_1, \dots, v_n$ is Δ-stable and v_i is Δ-invariant modulo p^{k_0}. In particular, $\widehat{V}$ is $p^{k_0}\mathcal{O}_{K_\infty}$-small. □

II.6.14. Under the assumption $(*)$ in II.6.10 on K, let $\mathfrak{a}$ be the ideal of $\mathcal{O}_{K_\infty}$ consisting of elements α with normalized valuation $v_{K_\infty}(\alpha) > \frac{2}{p-1}$. There is a canonical commutative diagram

$$\begin{array}{ccccc} \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}}) & \longrightarrow & \mathbf{Rep}^{\mathfrak{a}\text{-s}}(H, \widehat{\overline{K}}) & \longleftarrow & \mathbf{Rep}^{\mathfrak{a}\text{-s}}(H, \mathcal{O}_{\widehat{\overline{K}}}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma, K_\infty) & \longrightarrow & \mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta, \widehat{K_\infty}) & \longleftarrow & \mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta, \mathcal{O}_{\widehat{K_\infty}}) \end{array} \tag{II.6.14.1}$$

where the vertical arrows are equivalences of categories by II.5.17, II.5.18, II.6.12 and II.6.11, and where the horizontal arrows of the left square are induced by these equivalences and II.6.13. It allows us to calculate geometric Sen operators using "small lattices".

Lemma II.6.15. *Under the assumption $(*)$ in* II.6.10 *on K, let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{K}})$ such that there exists an object W^+ of $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(H, \mathcal{O}_{\widehat{\overline{K}}})$ with $W = W^+[1/p]$ in $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(H, \widehat{\overline{K}})$, where $\mathfrak{a}$ is the ideal of $\mathcal{O}_{K_\infty}$ consisting of elements α with normalized valuation $v_{K_\infty}(\alpha) > \frac{2}{p-1}$. Then, there is a commutative diagram*

$$\begin{array}{ccc} \mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_W} & \mathrm{End}_{\widehat{\overline{K}}}(W) \\ {\scriptstyle\psi}\downarrow\wr & & \uparrow{\scriptstyle \mathrm{id}_{\widehat{\overline{K}}}\otimes\log(-)|_{V^+}} \\ \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\Gamma) & \xleftarrow[1\otimes\log_\Delta]{} & \Delta \end{array} \tag{II.6.15.1}$$

where $\varphi_{\mathrm{Sen}}|_W$ is the canonical Lie algebra action defined in II.5.35, *ψ is the isomorphism* (II.5.24.1), *V^+ is the essentially unique object of $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta, \mathcal{O}_{\widehat{K_\infty}})$ such that $W^+ = \mathcal{O}_{\widehat{\overline{K}}} \otimes_{\mathcal{O}_{\widehat{K_\infty}}} V^+$, and $\log(\tau)|_{V^+}$ is the $\mathcal{O}_{\widehat{K_\infty}}$-linear endomorphism on V^+ defined in* II.6.7 *for $\tau \in \Delta$.*

PROOF. Let V be the essentially unique object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta\text{-an}}(\Gamma, K_\infty)$ such that $W = \widehat{\overline{K}} \otimes_{K_\infty} V$. Then, we conclude by the equivalence (II.6.12.1) that $\widehat{V} = V^+[1/p]$ in $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta, \widehat{K_\infty})$, where $\widehat{V} = \widehat{K_\infty} \otimes_{K_\infty} V$.

$$\begin{array}{ccccc} W & \longmapsto & W = W^+[\frac{1}{p}] & \longmapsfrom & W^+ \\ \uparrow & & \uparrow & & \uparrow \\ V & \longmapsto & \widehat{V} = V^+[\frac{1}{p}] & \longmapsfrom & V^+ \end{array} \tag{II.6.15.2}$$

Consider the infinitesimal Lie algebra action $\varphi|_V : \mathrm{Lie}(\Gamma) \to \mathrm{End}_{K_\infty}(V)$.

$$\begin{array}{ccc} \mathscr{E}^*_{\mathcal{O}_K}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_W} & \mathrm{End}_{\widehat{\overline{K}}}(W) \\ {\scriptstyle\psi}\downarrow\wr & \nearrow_{\mathrm{id}_{\widehat{\overline{K}}}\otimes\varphi|_V} & \uparrow{\scriptstyle \mathrm{id}_{\widehat{\overline{K}}}\otimes\log(-)|_{V^+}} \\ \widehat{\overline{K}} \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\Gamma) & \xleftarrow[1\otimes\log_\Delta]{} & \Delta \end{array} \tag{II.6.15.3}$$

The upper triangle commutes by II.5.35. It remains to check that the lower triangle commutes, i.e. $\mathrm{id}_{\widehat{K_\infty}} \otimes \varphi_\tau|_V = \mathrm{id}_{\widehat{K_\infty}} \otimes \log(\tau)|_{V^+}$ as $\widehat{K_\infty}$-linear endomorphisms of $\widehat{V} = V^+[1/p]$ for any $\tau \in \Delta$. For any $x \in V$ whose image y in $V^+[1/p]$ lies in V^+,

$$\begin{aligned} (\mathrm{id}_{\widehat{K_\infty}} \otimes \varphi_\tau|_V)(1 \otimes x) =& 1 \otimes \lim_{n\to\infty} p^{-n}(\tau^{p^n} - 1)(x) \quad \text{(by (II.4.8.1))} \\ =& 1 \otimes \lim_{n\to\infty} p^{-n}(\tau^{p^n} - 1)(y) \\ =& (\mathrm{id}_{\widehat{K_\infty}} \otimes \log(\tau)|_{V^+})(1 \otimes y) \quad \text{(by (II.6.7.3))}, \end{aligned} \tag{II.6.15.4}$$

which completes the proof. □

On the other hand, one can control the Galois invariant part of the completion of a filtered colimit of "small lattices".

Lemma II.6.16 (cf. [**AGT16**, II.8.23]). *Under the assumption* (∗) *in* II.6.10 *on* K, *let* $(W_\lambda^+)_{\lambda\in\Lambda}$ *be a directed system of objects in* $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(H, \mathcal{O}_{\widehat{K}})$, *where* $\mathfrak{a}$ *is the ideal of* $\mathcal{O}_{K_\infty}$ *consisting of elements* α *with normalized valuation* $v_{K_\infty}(\alpha) > \frac{2}{p-1}$. *Let* $(V_\lambda^+)_{\lambda\in\Lambda}$ *be the essentially unique directed system of objects in* $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta, \mathcal{O}_{\widehat{K_\infty}})$ *such that* $W_\lambda^+ = \mathcal{O}_{\widehat{K}} \otimes_{\mathcal{O}_{\widehat{K_\infty}}} V_\lambda^+$ *by* II.6.11. *We put* $W_\infty^+ = \operatorname{colim}_{\lambda\in\Lambda} W_\lambda^+$ *and* $V_\infty^+ = \operatorname{colim}_{\lambda\in\Lambda} V_\lambda^+$. *Then, the natural map*

$$(\widehat{V_\infty^+}[1/p])^\Delta \longrightarrow (\widehat{W_\infty^+}[1/p])^H \tag{II.6.16.1}$$

is a bijection, where the completions are p*-adic.*

Proof. By virtue of [**AGT16**, II.8.23] (cf. (II.6.10.2)), the map $(V_\lambda^+/p^nV_\lambda^+)^\Delta \to (\mathcal{O}_{\widehat{K}} \otimes_{\mathcal{O}_{\widehat{K_\infty}}} V_\lambda^+/p^nV_\lambda^+)^H$ is almost injective whose cokernel is killed by any element $\alpha \in \mathcal{O}_{K_\infty}$ with $v_{K_\infty}(\alpha) > \frac{1}{p-1}$, and thus so is the map

$$(\widehat{V_\infty^+})^\Delta = \lim_{n\to\infty} \operatorname*{colim}_{\lambda\in\Lambda} (V_\lambda^+/p^nV_\lambda^+)^\Delta \longrightarrow \lim_{n\to\infty} \operatorname*{colim}_{\lambda\in\Lambda} (W_\lambda^+/p^nW_\lambda^+)^H = (\widehat{W_\infty^+})^H. \tag{II.6.16.2}$$

Inverting p, we get $(\widehat{V_\infty^+}[1/p])^\Delta = (\widehat{V_\infty^+})^\Delta[1/p] = (\widehat{W_\infty^+})^H[1/p] = (\widehat{W_\infty^+}[1/p])^H$. □

II.6.17. Let $(W_\lambda^+)_{\lambda\in\Lambda}$ be a directed system of objects in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \mathcal{O}_{\widehat{K}})$. We put $W_\infty^+ = \operatorname{colim}_{\lambda\in\Lambda} W_\lambda^+$ as $\mathcal{O}_{\widehat{K}}$-modules, and denote its p-adic completion by $\widehat{W_\infty^+}$. We set

$$W_\lambda = W_\lambda^+[\frac{1}{p}], \quad W_\infty = W_\infty^+[\frac{1}{p}], \quad \widehat{W_\infty} = \widehat{W_\infty^+}[\frac{1}{p}], \tag{II.6.17.1}$$

endowed with the p-adic topology defined by W_λ^+, W_∞^+ and $\widehat{W_\infty^+}$ respectively. We remark that $\widehat{W_\infty}$ is the completion of W_∞ as topological abelian group (by the canonical isomorphism $W_\infty^+/p^nW_\infty^+ \xrightarrow{\sim} \widehat{W_\infty^+}/p^n\widehat{W_\infty^+}$ for any $n \in \mathbb{N}$, cf. [**GR04**, 8.2.8.(iii)]). As $(W_\lambda)_{\lambda\in\Lambda}$ forms a directed system of objects in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{K})$, the canonical Lie algebra actions $\varphi_{\mathrm{Sen}}|_{W_\lambda}$ defined in II.5.35 induces a homomorphism of $\widehat{K}$-linear Lie algebras

$$\varphi_{\mathrm{Sen}}|_{W_\infty} : \mathscr{E}^*_{\mathcal{O}_K}(1) \longrightarrow \mathrm{End}_{\widehat{K}}(W_\infty). \tag{II.6.17.2}$$

We denote by $\Phi(W_\infty)$ its image, and by $\Phi^{\mathrm{geo}}(W_\infty)$ the image of $\mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{K})$. On the other hand, the compatible G-actions $G \times W_\lambda^+ \to W_\lambda^+$ induces an action $G \times W_\infty^+ \to W_\infty^+$ by taking colimit, and thus induces an action $G \times \widehat{W_\infty^+} \to \widehat{W_\infty^+}$ by taking p-adic completion.

Lemma II.6.18. *We keep the notation in* II.6.17.

(1) *The* G*-actions on* W_∞^+ *and* $\widehat{W_\infty^+}$ *are continuous with respect to the* p*-adic topology.*
(2) *If* W_∞^+ *is* p*-adically separated, then the Lie algebra action* $\varphi_{\mathrm{Sen}}|_{W_\infty}$ *of* $\mathscr{E}^*_{\mathcal{O}_K}(1)$ *on* W_∞ *coincides with the canonical Lie algebra action defined in* II.6.5.

Proof. (1) For any $x \in W_\infty^+$, there exists $x_\lambda \in W_\lambda^+$ for some $\lambda \in \Lambda$ whose image in W_∞^+ is x. For any $g \in G$ and $n \in \mathbb{N}$, as $G \times W_\lambda^+ \to W_\lambda^+$ is continuous, there exists an open subgroup G_0 of G such that $G_0 g \cdot x_\lambda \subseteq gx_\lambda + p^nW_\lambda^+$. Thus, $G_0 g \cdot (x + p^nW_\infty^+) \subseteq gx + p^nW_\infty^+$, which shows that $G \times W_\infty^+ \to W_\infty^+$ and $G \times (W_\infty^+/p^nW_\infty^+) \to W_\infty^+/p^nW_\infty^+$ are continuous. Taking limit on n, we see that $G \times \widehat{W_\infty^+} \to \widehat{W_\infty^+}$ is also continuous.

(2) If W_∞^+ is p-adically separated, then it defines a norm on W_∞ as in II.6.3 so that we can apply II.6.5 to W_∞. The conclusion follows directly from II.6.5 and the definition of (II.6.17.2). □

Theorem II.6.19. *With the notation in* II.6.17, *assume that* W_λ^+ *is* $p^3\mathbb{Z}_p$*-small (cf.* II.6.8*) for each* $\lambda \in \Lambda$. *Then, any element of* $\Phi^{\mathrm{geo}}(W_\infty)$ *acts continuously on* W_∞, *and it induces a* $\widehat{K}$*-linear homomorphism by continuation,*

$$\varphi^{\mathrm{geo}}_{\mathrm{Sen}}|_{\widehat{W_\infty}} : \mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1), \widehat{K}) \longrightarrow \mathrm{End}_{\widehat{K}}(\widehat{W_\infty}), \tag{II.6.19.1}$$

whose image $\Phi^{\mathrm{geo}}(\widehat{W_\infty})$ *acts trivially on* $(\widehat{W_\infty})^H$.

PROOF. We may replace K by a finite extension and restrict the actions of G to an open subgroup, as this does not change the action (II.6.17.2) $\varphi_{\mathrm{Sen}}|_{W_\infty}$ by II.5.38. Thus, we may assume that K satisfies the assumption $(*)$ in II.6.10 by Epp's theorem on eliminating ramification [**Epp73**, 1.9, 2.0.(1)]. By II.6.11, there is an essentially unique directed system $(V_\lambda^+)_{\lambda\in\Lambda}$ of objects in $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(\Delta,\mathcal{O}_{\widehat{K_\infty}})$ with $W_\lambda^+=\mathcal{O}_{\widehat{K}}\otimes_{\mathcal{O}_{\widehat{K_\infty}}}V_\lambda^+$ in $\mathbf{Rep}^{\mathfrak{a}\text{-s}}(H,\mathcal{O}_{\widehat{K}})$. We obtain $\mathcal{O}_{\widehat{K_\infty}}$-linear group actions $\Delta\times V_\infty^+\to V_\infty^+$ and $\Delta\times\widehat{V_\infty^+}\to\widehat{V_\infty^+}$ by taking colimit and p-adic completion.

For each $\tau\in\Delta$, the compatible endomorphisms $\log(\tau)|_{V_\lambda^+}$ define an $\mathcal{O}_{\widehat{K_\infty}}$-endomorphism $\log(\tau)|_{V_\infty^+}$ on V_∞^+ (cf. II.6.7). As $\mathrm{id}_{\widehat{K}}\otimes\log(\tau)|_{V_\infty^+}$ sends W_∞^+ to itself, we see that any element of $\Phi^{\mathrm{geo}}(W_\infty)$ acts continuously on W_∞ by II.6.15. Thus, the Lie algebra action $\varphi_{\mathrm{Sen}}|_{W_\infty}$ induces a canonical $\widehat{K}$-linear homomorphism (II.6.19.1) by continuation.

Notice that Δ acts trivially on $\widehat{V_\infty^+}/\alpha\widehat{V_\infty^+}$ for any $\alpha\in\mathcal{O}_{K_\infty}$ with $\frac{1}{p-1}<v_{K_\infty}(\alpha)\le\frac{2}{p-1}$. Thus, $\log(\tau)$ is a well-defined $\mathcal{O}_{\widehat{K_\infty}}$-endomorphism on $\widehat{V_\infty^+}$ by II.6.7 for any $\tau\in\Delta$. Since $\log(\tau)|_{\widehat{V_\infty^+}}$ is compatible with $\log(\tau)|_{V_\infty^+}$ by the formula (II.6.7.3), the uniqueness of continuation implies that the following diagram is commutative as the continuation of (II.6.15.1),

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{O}_K}(\widehat{\Omega}^1_{\mathcal{O}_K}(-1),\widehat{K}) & \xrightarrow{\varphi^{\mathrm{geo}}_{\mathrm{Sen}}|_{\widehat{W_\infty}}} & \mathrm{End}_{\widehat{K}}(\widehat{W_\infty})\\ \downarrow{\scriptstyle\psi}\wr & & \uparrow\\ \widehat{K}\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\Delta) & \xleftarrow[1\otimes\log_\Delta]{} & \Delta\end{array}\tag{II.6.19.2}$$

where the right vertical arrow is induced by the map $\Delta\to\mathrm{End}_{\mathcal{O}_{\widehat{K}}}(\widehat{W_\infty^+})$ sending τ to the p-adic completion of the endomorphism $\mathrm{id}_{\mathcal{O}_{\widehat{K}}}\otimes\log(\tau)|_{\widehat{V_\infty^+}}$ of $\mathcal{O}_{\widehat{K}}\otimes_{\mathcal{O}_{\widehat{K_\infty}}}\widehat{V_\infty^+}$. Since the endomorphism $\log(\tau)|_{\widehat{V_\infty^+}}$ is the infinitesimal action of τ by (II.6.7.3), it acts trivially on $(\widehat{V_\infty^+})^\Delta$. Therefore, $\Phi^{\mathrm{geo}}(\widehat{W_\infty})$ acts trivially on $(\widehat{V_\infty})^\Delta=(\widehat{W_\infty})^H$ by II.6.16 and (II.6.19.2). □

Remark II.6.20. Even if any element of $\Phi(W_\infty)$ acts continuously on W_∞ (which holds in many cases, cf. II.11.25), we don't know whether the induced Lie algebra action $\varphi_{\mathrm{Sen}}|_{\widehat{W_\infty}}$ by continuation is compatible with the canonical Lie algebra action $\varphi_{\mathrm{Sen}}|_{(\widehat{W_\infty})^{\mathrm{f}}}$ on the (G,K)-finite part $(\widehat{W_\infty})^{\mathrm{f}}$ (endowed with the topology induced from $\widehat{W_\infty}$) defined in II.6.5, since we don't know the continuity of the latter (cf. II.6.6). Thus, we couldn't conclude easily that $\Phi(\widehat{W_\infty})$ annihilates $(\widehat{W_\infty})^G$. To see whether it is true or not, we need to study descent and decompletion of $\mathcal{O}_{\widehat{K}}$-representations of G and also compare the Galois invariant part as in II.6.16. We plan to investigate this in the future.

II.7. Some Boundedness Conditions on a Ring Map

Definition II.7.1. Let A be a ring, π an element of A.

(1) We say that an A-module M is π-zero if it is killed by π. We say that a morphism of A-modules $f:M\to N$ is a *π-isomorphism* if its kernel and cokernel are π-zero.
(2) We say that a chain complex of A-modules $M_\bullet$ is *π-exact* if the homology group $H_n(M_\bullet)$ is π-zero for any $n\in\mathbb{Z}$. We say that a morphism of chain complexes of A-modules $f:M_\bullet\to N_\bullet$ is a *π-quasi-isomorphism* if it induces a π-isomorphism on the homology groups $H_n(M_\bullet)\to H_n(N_\bullet)$ for any $n\in\mathbb{Z}$.

Lemma II.7.2 ([**AG20**, 2.6.3])**.** *Let A be a ring, π an element of A, $f:M\to N$ a morphism of A-modules.*

(1) *If there exists an A-linear homomorphism $g:N\to M$ such that $g\circ f=\pi\mathrm{id}_M$ and $f\circ g=\pi\mathrm{id}_N$, then f is a π-isomorphism.*
(2) *If f is a π-isomorphism, then there is a unique A-linear homomorphism $g:N\to M$ sending $y\in N$ to $\pi x\in M$ where $x\in f^{-1}(\pi y)$. In particular, $g\circ f=\pi^2\mathrm{id}_M$ and $f\circ g=\pi^2\mathrm{id}_N$.*

PROOF. (1) is clear. For (2), for any $y\in N$, $f^{-1}(\pi y)$ is not empty as $\pi\mathrm{Coker}(f)=0$. The element $\pi x\in M$ does not depend on the choice of $x\in f^{-1}(\pi y)$ as $\pi\mathrm{Ker}(f)=0$. Thus, the map $g:N\to M$ is well-defined. It is clearly unique and satisfy the relations $g\circ f=\pi^2\mathrm{id}_M$ and $f\circ g=\pi^2\mathrm{id}_N$. □

Remark II.7.3. Let A be a ring, π an element of A.

(1) Let $f_\bullet : M_\bullet \to N_\bullet$ be a morphism of chain complexes of A-modules such that f_n is a π-isomorphism for any $n \in \mathbb{Z}$. Then, there is a morphism of chain complexes of A-modules $g_\bullet : N_\bullet \to M_\bullet$ defined by II.7.2.(2) such that $g_\bullet \circ f_\bullet = \pi^2 \mathrm{id}_{M_\bullet}$ and $f_\bullet \circ g_\bullet = \pi^2 \mathrm{id}_{N_\bullet}$. We see that $f_\bullet$ is a π^2-quasi-isomorphism. Moreover, for any A-linear endofunctor F of the category of A-modules, $F(f_\bullet) : F(M_\bullet) \to F(N_\bullet)$ is a π^2-quasi-isomorphism.
(2) Let $(f_n)_{n\in\mathbb{N}} : (M_n)_{n\in\mathbb{N}} \to (N_n)_{n\in\mathbb{N}}$ be a morphism of inverse systems of A-modules such that f_n is a π-isomorphism for any $n \in \mathbb{N}$. Then, there is a morphism of inverse systems of A-modules $(g_n)_{n\in\mathbb{N}} : (N_n)_{n\in\mathbb{N}} \to (M_n)_{n\in\mathbb{N}}$ defined by II.7.2.(2) such that $g_n \circ f_n = \pi^2 \mathrm{id}_{M_n}$ and $f_n \circ g_n = \pi^2 \mathrm{id}_{N_n}$. We see that the induced morphism $\mathrm{R}^q \lim(M_n)_{n\in\mathbb{N}} \to \mathrm{R}^q \lim(N_n)_{n\in\mathbb{N}}$ is a π^2-isomorphism for any $q \in \mathbb{Z}$.

Lemma II.7.4. *Let A be a ring, π an element of A, $B \to C$ a π-isomorphism of A-algebras. Then, the canonical morphism $C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A}$ is a π^{17}-isomorphism.*

Proof. Let I (resp. J) be the kernel of the multiplication map $B \otimes_A B \to B$ (resp. $C \otimes_A C \to C$). Recall the $\Omega^1_{B/A}$ (resp. $\Omega^1_{C/A}$) is canonically isomorphic to I/I^2 (resp. J/J^2). Consider the morphism of exact sequences of A-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & B \otimes_A B & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & C \otimes_A C & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{II.7.4.1}$$

Since $B \to C$ is a π-isomorphism, $B \otimes_A B \to C \otimes_A C$ is a π^4-isomorphism by II.7.2. By the snake lemma, we see that $I \to J$ is a π^5-isomorphism, and thus $I^2 \to J^2$ is a π^{10}-isomorphism. The snake lemma shows that $I/I^2 \to J/J^2$ is a π^{15}-isomorphism. On the other hand, $I/I^2 \to C \otimes_B I/I^2$ is a π^2-isomorphism by II.7.2. We conclude that $C \otimes_B I/I^2 \to J/J^2$ is a π^{17}-isomorphism. □

Proposition II.7.5. *Let A be a ring, A' and B two A-algebras, $B' = A' \otimes_A B$. Then, the cone of the canonical morphism*

$$\tau_{\leq 1}(B' \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) \longrightarrow \tau_{\leq 1}\mathrm{L}_{B'/A'} \tag{II.7.5.1}$$

is concentrated in homological degree 2, where $\tau_{\leq 1}$ is the canonical truncation of chain complexes ([**Sta22**, 0118]*), and $\mathrm{L}_{B/A}$ denotes the cotangent complex of B over A. Moreover, if $\mathrm{Tor}^A_1(A', B)$ is π-zero for some $\pi \in A$, then* (II.7.5.1) *is a π-quasi-isomorphism.*

Proof. We take a surjective homomorphism from a polynomial A-algebra P to B, and denote its kernel by I. Recall that $\tau_{\leq 1}\mathrm{L}_{B/A}$ is quasi-isomorphic to the complex $I/I^2 \to B \otimes_P \Omega^1_{P/A}$ ([**Sta22**, 08RB]). Thus, in the derived category, we have

$$\tau_{\leq 1}(B' \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) = \tau_{\leq 1}(B' \otimes^{\mathrm{L}}_B (\tau_{\leq 1}\mathrm{L}_{B/A})) = (B' \otimes_B I/I^2 \to B' \otimes_P \Omega^1_{P/A}), \tag{II.7.5.2}$$

where the first equality follows from the distinguished triangle $\tau_{\geq 2}\mathrm{L}_{B/A} \to \mathrm{L}_{B/A} \to \tau_{\leq 1}\mathrm{L}_{B/A} \to$, and the second equality can be deduced from replacing I/I^2 by a flat resolution (note that $B \otimes_P \Omega^1_{P/A}$ is a free B-module).

We set $P' = A' \otimes_A P$ and $I' = \mathrm{Ker}(P' \to B')$. Then, $\tau_{\leq 1}\mathrm{L}_{B'/A'}$ is quasi-isomorphic to the complex $I'/I'^2 \to B' \otimes_{P'} \Omega^1_{P'/A'}$. Applying the functor $P' \otimes_P -$ to the exact sequence $0 \to I \to P \to B \to 0$, we obtain an exact sequence

$$\mathrm{Tor}^P_1(P', B) \longrightarrow P' \otimes_P I \longrightarrow I' \longrightarrow 0. \tag{II.7.5.3}$$

Applying the functor $B' \otimes_{P'} -$, we get an exact sequence

$$B' \otimes_{P'} \mathrm{Tor}^P_1(P', B) \longrightarrow B' \otimes_B I/I^2 \longrightarrow I'/I'^2 \longrightarrow 0. \tag{II.7.5.4}$$

Let N be the image of the first arrow. Then, by (II.7.5.2), the cone of (II.7.5.1) is quasi-isomorphic to the complex $N[-2]$. Since $\mathrm{Tor}^P_1(P', B) = \mathrm{Tor}^A_1(A', B)$ as P is flat over A, we see that N is π-zero if $\pi \mathrm{Tor}^A_1(A', B) = 0$. □

Corollary II.7.6. *Let A be a ring, A' and B two A-algebras, $B' = A' \otimes_A B$. Assume that $\tau_{\leq 1}\mathrm{L}_{B/A}$ is π-exact for some $\pi \in A$. Then, $\tau_{\leq 1}\mathrm{L}_{B'/A'}$ is π^2-exact.*

Proof. Consider the convergent spectral sequence [**EGA** III$_2$, 6.3.2.2]

$$E^2_{i,j} = \mathrm{Tor}^B_i(B', H_j(\mathrm{L}_{B/A})) \Rightarrow \mathrm{Tor}^B_{i+j}(B', \mathrm{L}_{B/A}), \tag{II.7.6.1}$$

with $\mathrm{d}^2_{i,j} : E^2_{i,j} \to E^2_{i-2,j+1}$. Then, there is an exact sequence

$$\mathrm{Tor}_0^B(B', H_1(\mathrm{L}_{B/A})) \longrightarrow H_1(B' \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) \longrightarrow \mathrm{Tor}_1^B(B', H_0(\mathrm{L}_{B/A})) \longrightarrow 0. \tag{II.7.6.2}$$

Since $H_1(\mathrm{L}_{B/A})$ and $H_0(\mathrm{L}_{B/A})$ are killed by π as $\tau_{\leq 1}\mathrm{L}_{B/A}$ is π-exact, we see that $\pi^2 H_1(B' \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) = 0$. Notice that $H_1(B' \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) \to H_1(\mathrm{L}_{B'/A'})$ is surjective by II.7.5. We see that $\pi^2 H_1(\mathrm{L}_{B'/A'}) = 0$ and $\pi H_0(\mathrm{L}_{B'/A'}) = \pi(B' \otimes_B H_0(\mathrm{L}_{B/A})) = 0$. □

Lemma II.7.7. *Let A be a ring, π an element of A, $B \to C$ a homomorphism of A-algebras. Assume that $\tau_{\leq 1}\mathrm{L}_{C/B}$ and $\tau_{\leq 1}\mathrm{L}_{B/A}$ are both π-exact. Then, $\tau_{\leq 1}\mathrm{L}_{C/A}$ is π^3-exact.*

Proof. We see that $\pi^2 H_1(C \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) = 0$ by (II.7.6.2), and that $\pi H_0(C \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) = \pi(C \otimes_B H_0(\mathrm{L}_{B/A})) = 0$. By the fundamental distinguished triangle of cotangent complexes ([**Ill71**, II.2.1.5.6]), we obtain an exact sequence

(II.7.7.1)
$$H_1(C \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) \to H_1(\mathrm{L}_{C/A}) \to H_1(\mathrm{L}_{C/B}) \to H_0(C \otimes^{\mathrm{L}}_B \mathrm{L}_{B/A}) \to H_0(\mathrm{L}_{C/A}) \to H_0(\mathrm{L}_{C/B}) \to 0,$$

which shows that $H_1(\mathrm{L}_{C/A})$ and $H_0(\mathrm{L}_{C/A})$ are both killed by π^3. □

Proposition II.7.8. *Let A be a ring, π an element of A, B an A-algebra such that $A \to B$ is a π-isomorphism. Then, $\tau_{\leq 1}\mathrm{L}_{B/A}$ is π^{102}-exact.*

Proof. Let C be the image of A in B. We take a surjective homomorphism from a polynomial C-algebra P to B, and denote its kernel by I. Let Q be the preimage of C via the surjection $P \to B$. It is a C-subalgebra of P such that $\pi P \subseteq Q$ and that I is an ideal of Q. We remark that $C = Q/I$ and $B = P/I = C \otimes_Q P$. Consider the canonical exact sequence

$$H_1(\mathrm{L}_{P/C}) \longrightarrow H_1(\mathrm{L}_{P/Q}) \longrightarrow P \otimes_Q \Omega^1_{Q/C} \longrightarrow \Omega^1_{P/C} \longrightarrow \Omega^1_{P/Q} \longrightarrow 0. \tag{II.7.8.1}$$

Since $H_1(\mathrm{L}_{P/C}) = 0$ and $P \otimes_Q \Omega^1_{Q/C} \to \Omega^1_{P/C}$ is π^{17}-injective by II.7.4, $H_1(\mathrm{L}_{P/Q})$ is killed by π^{17}. It is clear that $\pi H_0(\mathrm{L}_{P/Q}) = \pi\Omega^1_{P/Q} = 0$ as $\pi P \subseteq Q$. It follows from II.7.6 that $\tau_{\leq 1}\mathrm{L}_{B/C}$ is π^{34}-exact as $B = C \otimes_Q P$. On the other hand, let J be the kernel of $A \to C$ which is killed by π. Then, $H_1(\mathrm{L}_{C/A}) = J/J^2$ and $H_0(\mathrm{L}_{C/A}) = 0$ ([**Ill71**, III.1.2.8.1]). It follows from II.7.7 that $\tau_{\leq 1}\mathrm{L}_{B/A}$ is π^{102}-exact. □

II.7.9. Let $A \to B$ be an injective homomorphism of normal domains flat over $\mathbb{Z}_p$. We fix an algebraic closure $\overline{\mathcal{L}}$ of the fraction field $\mathcal{L}$ of B, and let $\overline{\mathcal{K}}$ be the algebraic closure of the fraction field $\mathcal{K}$ of A in $\overline{\mathcal{L}}$. Consider an algebraic extension $\mathcal{K}'$ of $\mathcal{K}$ in $\overline{\mathcal{K}}$ and the integral closure A' of A in $\mathcal{K}'$. Let B' be the integral closure of B in the composite $\mathcal{L}' = \mathcal{L}\mathcal{K}' \subseteq \overline{\mathcal{L}}$.

(II.7.9.1)
$$\begin{array}{ccccc} \mathcal{L}' & \longleftarrow & B' & \longleftarrow & A' \longrightarrow \mathcal{K}' \\ \uparrow & & \uparrow & & \uparrow \quad \uparrow \\ \mathcal{L} & \longleftarrow & B & \longleftarrow & A \longrightarrow \mathcal{K} \end{array}$$

Let $\mathscr{P}_A$ be the family of algebraic extensions $\mathcal{K}'$ of $\mathcal{K}$ in $\overline{\mathcal{K}}$ such that

(1) there exists a valuation ring $\mathcal{O}_{K'}$ extension of $\mathbb{Z}_p$ contained in A' such that its fraction field K' is a pre-perfectoid field in the sense of I.5.1, and
(2) the $\mathcal{O}_{K'}$-algebra A' is almost pre-perfectoid in the sense of I.5.19.

In particular, $\overline{\mathcal{K}} \in \mathscr{P}_A$.

Definition II.7.10. With the notation in II.7.9, for any algebraic extension $\mathcal{K}'$ of $\mathcal{K}$ in $\overline{\mathcal{K}}$, we say that the map $A \to B$ is *bounded at* $\mathcal{K}'$ if there exists $k \in \mathbb{N}$ such that

$$p^k \mathrm{Coker}(B \otimes_A A' \to B') = 0. \tag{II.7.10.1}$$

For any $\mathcal{K}' \in \mathscr{P}_A$, we say that $A \to B$ is *pre-perfectoid at* $\mathcal{K}'$ if $\mathcal{L}' = \mathcal{L}\mathcal{K}' \in \mathscr{P}_B$ and if $A \to B$ is bounded at $\mathcal{K}'$. We say that $A \to B$ is *pre-perfectoid* if it is pre-perfectoid at any $\mathcal{K}' \in \mathscr{P}_A$.

Lemma II.7.11. *We keep the notation in* II.7.9.

(1) *If $A \to B$ is étale, then it is pre-perfectoid.*
(2) *If $A \to B$ is pre-perfectoid, then for any algebraic extension $\mathcal{K}'$ of $\mathcal{K}$ in $\overline{\mathcal{K}}$, the map $A' \to B'$ is also pre-perfectoid.*
(3) *If $B \to C$ is another injective homomorphism of normal domains flat over $\mathbb{Z}_p$ and if $A \to B$, $B \to C$ are pre-perfectoid, then so is $A \to C$.*

PROOF. (1) Note that $B \otimes_A A'$ is a finite product of normal domains as it is étale over the normal domain A' ([**AGT16**, III.3.3]). Thus, B' identifies with one of the components. If $\mathcal{K}' \in \mathscr{P}_A$, then $B \otimes_A A'$ is almost pre-perfectoid by I.5.37 and so is B' (i.e. $\mathcal{L}' \in \mathscr{P}_B$). This shows that $A \to B$ is pre-perfectoid.

(2) We only need to unwind the definition. For any algebraic extension $\mathcal{K}''$ of $\mathcal{K}'$ in $\overline{\mathcal{K}}$, let A'' be the integral closure of A in $\mathcal{K}''$, B'' the integral closure of B in $\mathcal{L}'' = \mathcal{L}\mathcal{K}'' \subseteq \overline{\mathcal{L}}$. Assume that $\mathcal{K}'' \in \mathscr{P}_{A'}$. By definition, we also have $\mathcal{K}'' \in \mathscr{P}_A$, and thus $\mathcal{L}'' \in \mathscr{P}_B$ so that $\mathcal{L}'' \in \mathscr{P}_{B'}$. Since $A \to B$ is bounded at $\mathcal{K}''$, there exists $k \in \mathbb{N}$ such that $p^k\mathrm{Coker}(B \otimes_A A'' \to B'') = 0$. Thus, $p^k\mathrm{Coker}(B' \otimes_{A'} A'' \to B'') = 0$, which means that $A' \to B'$ is bounded at $\mathcal{K}''$. This completes the proof.

(3) It also follows directly from unwinding the definition. □

Theorem II.7.12 (Almost purity, [**Sch12**, 7.9])**.** *Let K be a pre-perfectoid field with a non-zero element π in its maximal ideal, R a flat $\mathcal{O}_K$-algebra which is almost pre-perfectoid, R' the integral closure of R in a finite étale $R[1/\pi]$-algebra. Then, R' is almost pre-perfectoid and almost finite étale over R.*

PROOF. Let $(S, \pi S)$ be the henselization of the pair $(R, \pi R)$. Then, $S' = S \otimes_R R'$ is the integral closure of S in a finite étale $S[1/\pi]$-algebra. By I.5.41, we see that R' is almost pre-perfectoid and that S' is almost finite étale over S. Notice that $R \to R[1/\pi] \times S$ is faithfully flat. By almost faithfully flat descent [**AGT16**, V.8.10], we see that R' is almost finite étale over R. □

Corollary II.7.13. *Let A be a normal domain flat over $\mathbb{Z}_p$, B the integral closure of A in a domain finite étale over $A[1/p]$. Then, the map $A \to B$ is pre-perfectoid.*

PROOF. With the notation in II.7.9, for any $\mathcal{K}' \in \mathscr{P}_A$, B' is almost pre-perfectoid and almost finite étale over A' by almost purity II.7.12. As $B[1/p]$ is finite étale over $A[1/p]$, $B \otimes_A A'[1/p]$ is the integral closure of $A'[1/p]$ in $\mathcal{L} \otimes_{\mathcal{K}} \mathcal{K}'$, which is a finite product of normal domains and one of its component identifies with $B'[1/p]$. In particular, $B \otimes_A A'[1/p] \to B'[1/p]$ is surjective. We see that there exists $k \in \mathbb{N}$ such that $p^k\mathrm{Coker}(B \otimes_A A' \to B') = 0$, since the $B \otimes_A A'$-module B' is almost finitely generated. It follows from the definition that $A \to B$ is pre-perfectoid. □

II.7.14. Let A be a ring, I an ideal of A, a an element of I. The affine blowup algebra $A[I/a]$ is the A-subalgebra of $A[1/a]$ generated by the subset $\{x/a\}_{x \in I}$ ([**Sta22**, 052P]). As the ideal $IA[I/a]$ is generated by a, there is a unique morphism $\mathrm{Spec}(A[I/a]) \to X$ over $\mathrm{Spec}(A)$, where X is the blowup of $\mathrm{Spec}(A)$ in I. Moreover, if I is generated by a subset S, then $\{\mathrm{Spec}(A[I/a]) \to X\}_{a \in S}$ forms a Zariski open covering ([**Sta22**, 0804]).

Lemma II.7.15. *Let A be a ring, π an element of A, I an ideal of A containing a power of π, a an element of I, $\widehat{A}$ the π-adic completion of A, $I' = I\widehat{A}$, a' the image of a in $\widehat{A}$. Then, the natural morphism of affine blowup algebras $A[I/a] \to \widehat{A}[I'/a']$ induces an isomorphism of their π-adic completions*

$$(A[I/a])^\wedge \xrightarrow{\sim} (\widehat{A}[I'/a'])^\wedge. \tag{II.7.15.1}$$

PROOF. We denote by $\varphi : A \to \widehat{A}$ and $\phi : A[1/a] \to \widehat{A}[1/a]$ the natural morphisms. We need to show that for each integer $n > 0$ the natural morphism

$$A[I/a]/\pi^n A[I/a] \longrightarrow \widehat{A}[I'/a']/\pi^n \widehat{A}[I'/a'] \tag{II.7.15.2}$$

is an isomorphism. We claim that $\widehat{I} \to \widehat{I'}$ is an isomorphism. Indeed, since A/I is killed by a power of π, we get from the short exact sequence $0 \to I \to A \to A/I \to 0$ a short exact sequence $0 \to \widehat{I} \to \widehat{A} \to A/I \to 0$ by π-adic completion ([**Sta22**, 0BNG]). Similarly, we get from the short exact sequence $0 \to I' \to \widehat{A} \to \widehat{A}/I' \to 0$ a short exact sequence $0 \to \widehat{I'} \to \widehat{A} \to \widehat{A}/I' \to 0$. Moreover, as $\pi^n A \subseteq I$ for n large enough, we deduce from the canonical isomorphism $A/\pi^n A \xrightarrow{\sim} \widehat{A}/\pi^n \widehat{A}$ that $A/I \to \widehat{A}/I'$ is an isomorphism. Combining with the previous short exact sequences, we see that $\widehat{I} \to \widehat{I'}$ is an isomorphism.

For the surjectivity of (II.7.15.2), recall that the A-algebra $A[I/a]$ is generated by the elements $\{x/a\}_{x \in I}$. For $x' \in I'$, as $I/\pi^n I \to I'/\pi^n I'$ is surjective, there exists $x \in I$ and $y \in I'$ such that $x' = \varphi(x) + \pi^n y \in I'$. Thus, $x'/a' \equiv \phi(x/a) \mod \pi^n \widehat{A}[I'/a']$, which shows that (II.7.15.2) is surjective.

For the injectivity of (II.7.15.2), as any element of $A[I/a]$ is of the form x/a^k for some $x \in I$ and $k > 0$, we suppose that $\phi(x/a^k) = \pi^n(x'/a'^{k'})$ for some $x' \in I'$ and $k' > 0$. Thus, there exists $N > 0$ such that $a'^N(a'^k \pi^n x' - a'^{k'}\varphi(x)) = 0$ in I'. In particular, $\varphi(a^{N+k'}x) \in \pi^n I'$. Since $I/\pi^n I \to I'/\pi^n I'$

is injective, there exists $y \in I$ such that $a^{N+k'}x = \pi^n y$. Thus, $x/a^k = \pi^n(y/a^{N+k+k'}) \in \pi^n A[I/a]$, which shows that (II.7.15.2) is injective. □

Lemma II.7.16. *Let A be a ring, π an element of A, B an A-subalgebra of $A[1/\pi]$. Assume that the morphism of π-adic completions $\widehat{A} \to \widehat{B}$ is a π^n-isomorphism for some $n \in \mathbb{N}$. Then, $\pi^n\mathrm{Coker}(A \to B) = 0$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f'} & \widehat{B} \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g'} \\ A & \xrightarrow{f} & \widehat{A} \end{array} \tag{II.7.16.1}$$

For $b \in B \subseteq A[1/\pi]$, we write $\pi^n b = g(x)/\pi^m$ in $A[1/\pi]$, where $x \in A$ and $m \in \mathbb{N}$. We have $\pi^{m+n}f'(b) = g'(f(x))$. On the other hand, by the assumption $\pi^n\mathrm{Coker}(g') = 0$, there exists $y \in \widehat{A}$ such that $\pi^n f'(b) = g'(y)$. Thus, $g'(f(x) - \pi^m y) = 0$, which implies that $\pi^n(f(x) - \pi^m y) = 0$ by the assumption $\pi^n\mathrm{Ker}(g') = 0$. By the isomorphism $A/\pi^{m+n}A \xrightarrow{\sim} \widehat{A}/\pi^{m+n}\widehat{A}$, we have $\pi^n x \in \pi^{m+n}A$. Therefore, $\pi^n b = g(\pi^n x)/\pi^{m+n} \in \mathrm{Im}(g)$. □

Theorem II.7.17 ([**Sch12**, 6.3])**.** *Let K be a pre-perfectoid field with a non-zero element π of its maximal ideal, A an $\mathcal{O}_K$-algebra which is almost pre-perfectoid, I a finitely generated ideal of A containing a power of π, a an element of I.*

1. *The integral closure B of the affine blowup algebra $A[I/a]$ in $A[I/a][1/\pi] = A[1/\pi a]$ is almost pre-perfectoid.*
2. *There exists $n \in \mathbb{N}$ such that $\pi^n\mathrm{Coker}(A[I/a] \to B) = 0$.*

Proof. As $\widehat{A}$ is almost flat over $\mathcal{O}_{\widehat{K}}$ by definition, $\widehat{A} \to (A/A[\pi^\infty])^\wedge$ is surjective and is an almost isomorphism (I.5.27). Thus, after replacing A by $A/A[\pi^\infty] \subseteq A[1/\pi]$ and I by its image (which does not change B and $\mathrm{Coker}(A[I/a] \to B)$), we may assume that A is flat over $\mathcal{O}_K$. Let B' (resp. B'') be the integral closure of $\widehat{A}[I'/a']$ (resp. $(A[I/a])^\wedge$) in $\widehat{A}[I'/a'][1/\pi]$ (resp. $(A[I/a])^\wedge[1/\pi]$), where the completions are π-adic, $I' = I\widehat{A}$ and a' is the image of a in $\widehat{A}$. By II.7.15, we have $(A[I/a])^\wedge = (\widehat{A}[I'/a'])^\wedge$. Thus, there exists a canonical morphism $B' \to B''$ and a commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & B' & \longrightarrow & B'' \\ \uparrow & & \uparrow & & \uparrow \\ A[\frac{I}{a}] & \longrightarrow & \widehat{A}[\frac{I'}{a'}] & \longrightarrow & (A[\frac{I}{a}])^\wedge \end{array} \tag{II.7.17.1}$$

Since the three $\mathcal{O}_K$-algebras in the second row are flat (I.5.20) and have the same π-adic completion flat over $\mathcal{O}_{\widehat{K}}$, the π-adic completions of the three $\mathcal{O}_K$-algebras in the first row are almost isomorphic by I.5.29.

(1) By definition, the $\mathcal{O}_{\widehat{K}}$-algebra $\widehat{A}$ is almost perfectoid. We endow $\widehat{A}[1/\pi]$ with the π-adic topology defined by $\widehat{A}$ so that it becomes a Tate $\widehat{K}$-algebra in the sense of [**Sch12**, 2.6]. If $\widehat{A}^+$ denotes the integral closure of $\widehat{A}$ in $\widehat{A}[1/\pi]$ (which is almost isomorphic to $\widehat{A}$), then $(\widehat{A}[1/\pi], \widehat{A}^+)$ forms a perfectoid affinoid $\widehat{K}$-algebra in the sense of [**Sch12**, 6.1]. Similarly, we endow $\widehat{A}[I'/a'][1/\pi]$ with the π-adic topology defined by $\widehat{A}[I'/a']$ so that it becomes a Tate $\widehat{K}$-algebra. Then, $(\widehat{A}[I'/a'][1/\pi], B')$ is an affinoid $\widehat{K}$-algebra. Its completion is $((A[I/a])^\wedge[1/\pi], B'')$, which is the completed affinoid $\widehat{K}$-algebra associated to the rational subset of $\mathrm{Spa}(\widehat{A}[1/\pi], \widehat{A}^+)$ defined by I and a by the definition [**Sch12**, 2.13]). By virtue of [**Sch12**, 6.3.(ii)], $((A[I/a])^\wedge[1/\pi], B'')$ is a perfectoid affinoid $\widehat{K}$-algebra. Thus, the $\mathcal{O}_{\widehat{K}}$-algebra B'' is almost perfectoid and bounded in $(A[I/a])^\wedge[1/\pi]$ with respect to the π-adic topology defined by $(A[I/a])^\wedge$. In particular, $B'' \to \widehat{B''}$ is an almost isomorphism ([**Sch12**, 5.5]). Thus, B is almost pre-perfectoid, since $\widehat{B} \to \widehat{B''}$ is an almost isomorphism by the discussion in the beginning.

(2) Since $(A[I/a])^\wedge \to B''$ is injective as $(A[I/a])^\wedge$ is flat over $\mathcal{O}_K$, the map $(A[I/a])^\wedge \to \widehat{B}$ is almost injective by the almost isomorphisms $\widehat{B} \to \widehat{B''} \leftarrow B''$. Since B'' is bounded in $(A[I/a])^\wedge[1/\pi]$ with respect to the π-adic topology defined by $(A[I/a])^\wedge$, there exists $n \in \mathbb{N}$ such that π^n annihilates the kernel and cokernel of $(A[I/a])^\wedge \to \widehat{B}$. The conclusion follows from II.7.16. □

Corollary II.7.18. *Let A be a normal domain flat over $\mathbb{Z}_p$, I a finitely generated ideal of A containing a power of p, a an element of I, B the integral closure of the affine blowup algebra $A[I/a]$ in $A[I/a][1/p] = A[1/pa]$. Then, the map $A \to B$ is pre-perfectoid.*

Proof. With the notation in II.7.9, for any $\mathcal{K}' \in \mathscr{P}_A$, notice that the affine blowup algebra $A'[I/a]$ is the image of $A[I/a] \otimes_A A' \to A'[1/a]$. Thus, B' is the integral closure of $A'[I/a]$ in $A'[I/a][1/p] = A'[1/pa]$. Thus, it is almost pre-perfectoid and $p^k\mathrm{Coker}(A'[I/a] \to B') = 0$ for some $k \in \mathbb{N}$ by II.7.17, which completes the proof. □

II.8. Brief Review on Adequate Charts of Logarithmic Schemes

The main geometric object of this article, quasi-adequate algebras, stems from logarithmic geometry. In this section, we firstly review basic notions of logarithmic geometry. We refer to [**Kat89, Kat94, GR04, Ogu18**] for a systematic development of logarithmic geometry, and to [**AGT16**, II.5] and I.9 for a brief summary of the theory. Then, we review adequate charts of a logarithmic scheme and the induced coverings following Tsuji [**Tsu18**, §4].

II.8.1. All monoids considered in this article are unitary and commutative, and we denote the monoid structures additively. The category of monoids admits arbitrary colimits (cf. [**Ogu18**, I.1.1]), and we denote the colimit of a diagram $M_1 \leftarrow M_0 \to M_2$ by $M_1 \oplus_{M_0} M_2$. The forgetful functor from the category of groups (resp. rings) to the category of monoids (resp. with respect to the multiplicative structure) admits a left adjoint sending M to M^{gp} (resp. $\mathbb{Z}[M]$). For any monoid M, we denote by $\exp_M : M \to \mathbb{Z}[M]$ the canonical homomorphism of monoids. The forgetful functor from the category of finitely generated monoids to the category of fs (i.e. fine and saturated) monoids admits a left adjoint sending M to the saturation M^{fs} of its image in M^{gp} ([**Ogu18**, I.1.3.5, I.2.2.5]).

II.8.2. A log scheme X is a pair $(\underline{X}, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ consisting of a scheme $\underline{X}$ and a homomorphism from a sheaf of monoids to the structural sheaf on the étale site of $\underline{X}$ (equivalent to the strictly étale site of X, see below) which induces an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$, where $\mathcal{O}_X = \mathcal{O}_{\underline{X}_{\text{ét}}}$ and $\mathcal{O}_X^\times$ is the subsheaf of units. A morphism of log schemes $Y \to X$ is a pair $(f, f^\flat)$ consisting of a morphism of the underlying schemes $f : \underline{Y} \to \underline{X}$ and a homomorphism of sheaves of monoids $f^\flat : f^{-1}(\mathcal{M}_X) \to \mathcal{M}_Y$ compatible with the natural homomorphism $f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$ via α_Y and $f^{-1}(\alpha_X)$. A morphism of log schemes $Y \to X$ is *strict* if the log structure of Y is the inverse image of that of X ([**AGT16**, II.5.11]). For an open immersion of schemes $j : U \to X$, let $\mathcal{M}_{U \to X}$ be the preimage of $j_{\text{ét}*}\mathcal{O}_{U_{\text{ét}}}^\times$ via the natural map $\mathcal{O}_{X_{\text{ét}}} \to j_{\text{ét}*}\mathcal{O}_{U_{\text{ét}}}$. Then, the log structure $\alpha_{U \to X} : \mathcal{M}_{U \to X} \to \mathcal{O}_{X_{\text{ét}}}$ on X is called the *compactifying log structure* associated to the open immersion j ([**Ogu18**, III.1.6.1]).

II.8.3. A log ring is a homomorphism $M \to A$ from a monoid M to the multiplicative monoid of a ring A. We denote by $\mathrm{Spec}(M \to A)$ the log scheme with underlying scheme $\mathrm{Spec}(A)$ endowed with the log structure associated to the pre-log structure $M \to \mathcal{O}_{\mathrm{Spec}(A)_{\text{ét}}}$ induced by $M \to A$, and we set $\mathbb{A}_M = \mathrm{Spec}(\exp_M : M \to \mathbb{Z}[M])$ ([**Ogu18**, III.1.2.3]). A *chart* of a log scheme X is a homomorphism $M \to \Gamma(X, \mathcal{M}_X)$ from a monoid M to the monoid of global sections of $\mathcal{M}_X$ such that the induced morphism of log schemes $X \to \mathbb{A}_M$ is strict ([**AGT16**, II.5.13]). We say that a log scheme X is *coherent* (resp. *fs*) if strictly étale locally on X it admits a chart from a finitely generated (resp. fs) monoid M ([**AGT16**, II.5.15]).

II.8.4. The inclusion functor from the category of schemes to the category of coherent log schemes (by endowing with trivial log structures) admits a left adjoint sending X to its underlying scheme $\underline{X}$, and admits a right adjoint sending X to the maximal open subscheme X^{tr} of $\underline{X}$ on which the log structure is trivial ([**Ogu18**, III.1.2.8]). The inclusion functor from the category of fs log schemes to the category of coherent log schemes admits a right adjoint $X \mapsto X^{\mathrm{fs}}$, and we remark that the canonical morphism of underlying schemes $\underline{X}^{\mathrm{fs}} \to \underline{X}$ is finite with $(X^{\mathrm{fs}})^{\mathrm{tr}} = X^{\mathrm{tr}} \times_{\underline{X}} \underline{X}^{\mathrm{fs}} = X^{\mathrm{tr}}$ ([**Ogu18**, III.2.1.5]). The category of log schemes admits finite limits, which commute with taking underlying schemes and preserve coherence ([**Ogu18**, III.2.1.2]). By the universal property of the functor $X \mapsto X^{\mathrm{fs}}$, the category of fs log schemes also admits finite limits ([**Ogu18**, III.2.1.6]).

II.8.5. Let X be a regular fs log scheme ([**Kat94**, 2.1], [**Niz06**, 2.3]). Its underlying scheme $\underline{X}$ is locally Noetherian and normal, and X^{tr} is regular and dense in X ([**Kat94**, 4.1]). Moreover, the log structure on X is the compactifying log structure associated to the open immersion $X^{\mathrm{tr}} \to \underline{X}$ ([**Kat94**, 11.6], [**Niz06**, 2.6]). A typical example is that given a regular scheme X with a strict normal crossings divisor D, then $(X, \alpha_{X \setminus D \to X})$ is a regular fs log scheme whose open subset of triviality of log structure is $X \setminus D$ ([**Ogu18**, III.1.11.9]).

II.8.6. To any morphism of log schemes $Y \to X$, one can associate the $\mathcal{O}_Y$-module of log differentials $\Omega^1_{Y/X}$ with natural maps $\mathrm{d} : \mathcal{O}_Y \to \Omega^1_{Y/X}$ and $\mathrm{d}\log : \mathcal{M}_Y \to \Omega^1_{Y/X}$ ([**AGT16**, II.5.21]). If $Y \to X$ is strict, then $\Omega^1_{Y/X} = \Omega^1_{\underline{Y}/\underline{X}}$. If X and Y are coherent, then $\Omega^1_{Y/X}$ is quasi-coherent. If $Y \to X$ is a smooth morphism of coherent log schemes, then $\Omega^1_{Y/X}$ is locally finite free ([**Ogu18**, IV.3.2.1]). For any morphism of log rings $(\alpha : M \to A) \to (\beta : N \to B)$, if we denote by $X = \mathrm{Spec}(M \to A)$ and $Y = \mathrm{Spec}(N \to B)$, then $\Omega^1_{Y/X}$ is the quasi-coherent $\mathcal{O}_Y$-module associated to the B-module

$$\Omega^1_{(N\to B)/(M\to A)} = \frac{\Omega^1_{B/A} \oplus (B \otimes_{\mathbb{Z}} (N^{\mathrm{gp}}/M^{\mathrm{gp}}))}{F} \tag{II.8.6.1}$$

where F is the B-submodule of $\Omega^1_{B/A} \oplus (B \otimes_{\mathbb{Z}} (N^{\mathrm{gp}}/M^{\mathrm{gp}}))$ generated by the elements $(\mathrm{d}(\beta(x)), -\beta(x) \otimes x)$ for any $x \in N$ ([**Ogu18**, IV.1.2.6]). We remark that for any Cartesian diagram in the category of (resp. fs) log schemes

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array} \tag{II.8.6.2}$$

the canonical morphism $g'^{*}\Omega^1_{Y/X} \to \Omega^1_{Y'/X'}$ is an isomorphism ([**Ogu18**, IV.1.2.15]).

II.8.7. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$, S the log scheme with underlying scheme $\mathrm{Spec}(\mathcal{O}_K)$ endowed with the compactifying log structure associated to the open immersion $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$ (in particular, S is a regular fs log scheme, cf. II.8.5), $f : X \to S$ a morphism of fs log schemes. We remark that if f is smooth, then X is also regular ([**Ogu18**, IV.3.5.3]), and thus the log structure on X is the compactifying log structure associated to $X^{\mathrm{tr}} \to \underline{X}$, cf. II.8.5.

Definition II.8.8. With the notation in II.8.7, an *adequate chart* of f is a triple of homomorphisms of monoids $(\alpha : \mathbb{N} \to \Gamma(S, \mathcal{M}_S),\ \beta : P \to \Gamma(X, \mathcal{M}_X),\ \gamma : \mathbb{N} \to P)$ satisfying the following conditions:

(1) The following diagram is commutative

$$\begin{array}{ccc} \Gamma(X, \mathcal{M}_X) & \xleftarrow{\beta} & P \\ {\scriptstyle f^{\flat}}\uparrow & & \uparrow{\scriptstyle \gamma} \\ \Gamma(S, \mathcal{M}_S) & \xleftarrow{\alpha} & \mathbb{N} \end{array} \tag{II.8.8.1}$$

(2) The element $\alpha(1) \in \Gamma(S, \mathcal{M}_S) = \mathcal{O}_K \setminus \{0\}$ is a uniformizer of $\mathcal{O}_K$ (in particular, $S \to \mathbb{A}_{\mathbb{N}}$ is strict).

(3) The homomorphism β induces a strict and étale morphism $X \to S \times_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_P$ (in particular, $X \to \mathbb{A}_P$ is also strict).

(4) The monoid P is fs, and if we denote by $\gamma_\eta : \mathbb{Z} \to P_\eta = \mathbb{Z} \oplus_{\mathbb{N}} P$ the pushout of γ by the inclusion $\mathbb{N} \to \mathbb{Z}$, then there exists an isomorphism for some $c, d \in \mathbb{N}$ with $c \leq d$,

$$P_\eta \cong \mathbb{Z} \oplus \mathbb{Z}^c \oplus \mathbb{N}^{d-c} \tag{II.8.8.2}$$

identifying γ_η with the inclusion of $\mathbb{Z}$ into the first component of right hand side.

Remark II.8.9. In II.8.8, the morphism of fs log schemes $S \times_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_P \to S$ is smooth ([**AGT16**, II.5.25]), and thus so is $X \to S$. If we set $A = \mathcal{O}_K \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[P]$ and $A_{\mathrm{tr}} = \mathcal{O}_K \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[P^{\mathrm{gp}}]$, then the underlying scheme of $S \times_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_P$ is $\mathrm{Spec}(A)$, and $\mathrm{Spec}(A_{\mathrm{tr}})$ is the maximal open subscheme on which the log structure is trivial (cf. [**Ogu18**, III.1.2.10]). As $X \to S \times_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_P$ is strictly étale, $X^{\mathrm{tr}} = \mathrm{Spec}(A_{\mathrm{tr}}) \times_{\mathrm{Spec}(A)} \underline{X}$ and the log structure on X is the compactifying log structure associated to the open immersion $X^{\mathrm{tr}} \to \underline{X}$ (cf. II.8.7). Moreover, (II.8.8.2) induces an isomorphism (cf. II.8.1)

$$A[\frac{1}{p}] = K \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[\mathbb{Z}] \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[P] = K \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[P_\eta] \cong K[\mathbb{Z}^c \oplus \mathbb{N}^{d-c}], \tag{II.8.9.1}$$

and A is a Noetherian normal domain (cf. II.8.5, II.8.7).

Remark II.8.10. Our definition of adequate charts is slightly different from Abbes-Gros' definition [**AGT16**, III.4.4], where they require moreover that γ is saturated (cf. [**AG20**, 4.2.2]). Nevertheless, our adequate charts describe log smooth schemes over S by the following proposition due to Abbes-Gros and Tsuji.

Proposition II.8.11 ([**Tsu18**, 3.14, 3.16], cf. [**AGT16**, III.4.6]). *With the notation in* II.8.7*, the following conditions are equivalent:*

(1) *The morphism of fs log schemes $f : X \to S$ is smooth and the underlying generic fibre $\underline{X}_K$ is regular.*
(2) *Every geometric point of $\underline{X}$ admits a strictly étale neighbourhood U in X such that $U \to S$ factors through $S' = (\mathrm{Spec}(\mathcal{O}_{K'}), \alpha_{\mathrm{Spec}(K')\to\mathrm{Spec}(\mathcal{O}_{K'})})$ for some tamely ramified finite extension K' of K and that the induced morphism of fs log schemes $U \to S'$ admits an adequate chart.*

Proof. (2) $\Rightarrow$ (1): By the conditions in II.8.8, $\underline{U}_{K'}$ is étale over $\mathrm{Spec}(K'[\mathbb{Z}^c \oplus \mathbb{N}^{d-c}])$, thus regular. Since $U \to S'$ is smooth by II.8.9, so is $U \to S$ as S' is étale over S.

(1) $\Rightarrow$ (2): For a geometric point of the generic fibre $\underline{X}_K$, the conclusion follows directly from [**Tsu18**, 3.14] (where we take $S' = S$). For a geometric point of the special fibre of $\underline{X}$, the conclusion follows directly from [**Tsu18**, 3.16]. □

II.8.12. We follow Tsuji [**Tsu18**, §4] to construct coverings of adequate log schemes from an adequate chart. Let $\gamma : \mathbb{N} \to P$ be an injective homomorphism of fs monoids such that there exists an isomorphism for some $c, d \in \mathbb{N}$ with $c \le d$,

$$P_\eta \cong \mathbb{Z} \oplus \mathbb{Z}^c \oplus \mathbb{N}^{d-c} \tag{II.8.12.1}$$

identifying $\gamma_\eta : \mathbb{Z} \to P_\eta = \mathbb{Z} \oplus_{\mathbb{N}} P$ with the inclusion of $\mathbb{Z}$ into the first component of right hand side as in II.8.8.(4). We identify P^{gp} with $\mathbb{Z}^{1+d}$ and $\mathbb{N}^{\mathrm{gp}}$ with the first component of $\mathbb{Z}^{1+d}$. For any $e \in \mathbb{N}_{>0}$ and $\underline{r} = (r_1, \dots, r_d) \in \mathbb{N}^d_{>0}$, we define a submonoid of $\mathbb{Q}^{1+d}$ by

$$P_{e,\underline{r}} = \{x \in e^{-1}\mathbb{Z} \times r_1^{-1}\mathbb{Z} \times \cdots \times r_d^{-1}\mathbb{Z} \mid \exists k \in \mathbb{N}_{>0} \text{ s.t. } kx \in P\}. \tag{II.8.12.2}$$

It is an fs monoid ([**Tsu18**, 3.2]), and if we denote by $P_{e,\underline{r},\eta}$ the pushout $e^{-1}\mathbb{Z} \oplus_{e^{-1}\mathbb{N}} P_{e,\underline{r}}$, then there is an isomorphism

$$P_{e,\underline{r},\eta} \cong e^{-1}\mathbb{Z} \oplus r_1^{-1}\mathbb{Z} \oplus \cdots \oplus r_c^{-1}\mathbb{Z} \oplus r_{c+1}^{-1}\mathbb{N} \oplus \cdots \oplus r_d^{-1}\mathbb{N} \tag{II.8.12.3}$$

induced by (II.8.12.1) (cf. [**Tsu18**, page 810, equation (2)]).

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$, L a finite field extension of K, S (resp. S^L) the log scheme with underlying scheme $\mathrm{Spec}(\mathcal{O}_K)$ (resp. $\mathrm{Spec}(\mathcal{O}_L)$) endowed with the compactifying log structure defined by the closed point. We fix a homomorphism of monoids $\alpha : \mathbb{N} \to \mathcal{O}_K \setminus \{0\}$ sending 1 to a uniformizer π of K. For any $\underline{r} \in \mathbb{N}^d_{>0}$, consider the fibred product in the category of fs log schemes

$$X^L_{\underline{r}} = S^L \times^{\mathrm{fs}}_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_{P_{1,\underline{r}}}, \tag{II.8.12.4}$$

where the map $S^L \to \mathbb{A}_{\mathbb{N}}$ is induced by α and the inclusion $\mathcal{O}_K \to \mathcal{O}_L$. We omit the index L or $\underline{r}$ if $L = K$ or $\underline{r} = \underline{1}$ respectively. Let $\beta : P_{1,\underline{r}} \to \Gamma(X^L_{\underline{r}}, \mathcal{M}_{X^L_{\underline{r}}})$ be the induced homomorphism of monoids, and for any $1 \le i \le d$, we denote by T_{i,r_i} the image of $r_i^{-1} \cdot \underline{1}_i = (0, \dots, r_i^{-1}, \dots, 0) \in P^{\mathrm{gp}}_{1,\underline{r}}$ in $\Gamma(X^L_{\underline{r}}, \mathcal{M}^{\mathrm{gp}}_{X^L_{\underline{r}}})$.

$$\begin{array}{ccc} \Gamma(X^L_{\underline{r}}, \mathcal{M}_{X^L_{\underline{r}}}) & \xleftarrow{\beta} & P_{1,\underline{r}} \\ \uparrow & & \uparrow{\scriptstyle \gamma} \\ \Gamma(S^L, \mathcal{M}_{S^L}) & \xleftarrow{\alpha} & \mathbb{N} \end{array} \tag{II.8.12.5}$$

In section II.10, we will produce a Kummer tower from $X^L_{\underline{r}} \to X$ by varying L and $\underline{r}$. We need an adequate chart of $X^L_{\underline{r}}$ over S^L.

Lemma II.8.13 ([**Tsu18**, page 812, equation (6)]). *With the notation in* II.8.12*, the morphism of fs log schemes $X^L_{\underline{r}} \to S^L$ admits an adequate chart*

$$(\alpha' : e^{-1}\mathbb{N} \to \Gamma(S^L, \mathcal{M}_{S^L}),\ \beta' : P_{e,\underline{r}} \to \Gamma(X^L_{\underline{r}}, \mathcal{M}_{X^L_{\underline{r}}}),\ \gamma' : e^{-1}\mathbb{N} \to P_{e,\underline{r}}) \tag{II.8.13.1}$$

where e is the ramification index of L/K, α' is a homomorphism of monoids sending e^{-1} to a uniformizer π' of L, $\beta'(k_0/e, k_1/r_1, \dots, k_d/r_d) = \pi'^{k_0} \cdot T_{1,r_1}^{k_1} \cdots T_{d,r_d}^{k_d}$, and γ' is induced by the inclusion of the first component of (II.8.12.3)*. Moreover, the canonical morphism of log schemes induced by this chart*

$$X^L_{\underline{r}} \longrightarrow S^L \times_{\mathbb{A}_{e^{-1}\mathbb{N}}} \mathbb{A}_{P_{e,\underline{r}}} \tag{II.8.13.2}$$

is an isomorphism.

PROOF. For the convenience of the readers, we briefly recall Tsuji's proof. Consider the commutative diagram of fs monoids

$$\begin{array}{ccccc} P_{e,\underline{r}} & \xrightarrow{(\iota,0)} & P_{e,\underline{r}}\oplus\mathbb{Z} & \xleftarrow{(\iota,s)} & P_{1,\underline{r}} \\ \uparrow{\scriptstyle \gamma'} & & \uparrow{\scriptstyle (\gamma',\mathrm{id}_{\mathbb{Z}})} & & \uparrow{\scriptstyle \gamma} \\ e^{-1}\mathbb{N} & \xrightarrow{(\iota,0)} & e^{-1}\mathbb{N}\oplus\mathbb{Z} & \xleftarrow{(\iota,\iota)} & \mathbb{N} \end{array} \tag{II.8.13.3}$$

where we use ι to denote the inclusions, $s : P_{1,\underline{r}} \to \mathbb{Z}$ is the projection to the first component, and γ' is induced by the inclusion of the first component of (II.8.12.3). The squares in (II.8.13.3) are cocartesian in the category of fs monoids (using [**Tsu18**, 4.2] for the right). Notice that the map $\alpha : \mathbb{N} \to \Gamma(S^L, \mathcal{M}_{S^L})$ is the composition of

$$\mathbb{N} \xrightarrow{(\iota,\iota)} e^{-1}\mathbb{N}\oplus\mathbb{Z} \xrightarrow{(\alpha',s')} \Gamma(S^L, \mathcal{M}_{S^L}), \tag{II.8.13.4}$$

where α' is a homomorphism of monoids sending e^{-1} to a uniformizer π' of L, and $s' : \mathbb{Z} \to \Gamma(S^L, \mathcal{M}_{S^L}) = \mathcal{O}_L \setminus \{0\}$ sends 1 to the unit π/π'^e. Thus, the cocartesian square in (II.8.13.3) on the right induces the following commutative diagram of monoids by the definition $X^L_{\underline{r}} = S^L \times^{\mathrm{fs}}_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_{P_{1,\underline{r}}}$ (II.8.12.4).

$$\begin{array}{ccc} \Gamma(X^L_{\underline{r}}, \mathcal{M}_{X^L_{\underline{r}}}) & \xleftarrow{(\beta',s')} & P_{e,\underline{r}}\oplus\mathbb{Z} \\ \uparrow & & \uparrow{\scriptstyle (\gamma',\mathrm{id}_{\mathbb{Z}})} \\ \Gamma(S^L, \mathcal{M}_{S^L}) & \xleftarrow{(\alpha',s')} & e^{-1}\mathbb{N}\oplus\mathbb{Z} \end{array} \tag{II.8.13.5}$$

It is clear that the induced map β' sends an element $(k_0/e, k_1/r_1, \ldots, k_d/r_d) \in P_{e,\underline{r}}$ to $\pi'^{k_0}\cdot T_{1,r_1}^{k_1}\cdots T_{d,r_d}^{k_d}$. Since the morphism (II.8.13.2) is induced by the composition of (II.8.13.5) with the cocartesian square in (II.8.13.3) on the left, it is an isomorphism by the definition $X^L_{\underline{r}} = S^L \times^{\mathrm{fs}}_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_{P_{1,\underline{r}}}$ (II.8.12.4). □

Proposition II.8.14 ([**Tsu18**, 4.3, 4.5])**.** *With the notation in* II.8.12*, there exists $k_0 \in \mathbb{N}$ such that for any finite field extensions $L \subseteq L'$ of K and any elements $\underline{r}|\underline{r}'$ of $\mathbb{N}^d_{>0}$, we have*

$$p^{k_0} A^{L'}_{\underline{r}'} \subseteq \bigoplus_{\underline{k}\in I} \mathcal{O}_{L'} \otimes_{\mathcal{O}_L} A^L_{\underline{r}} \cdot \prod_{i=1}^{d} T_{i,r'_i}^{k_i} \subseteq p^{-k_0} A^{L'}_{\underline{r}'} \tag{II.8.14.1}$$

where $A^L_{\underline{r}} = \Gamma(X^L_{\underline{r}}, \mathcal{O}_{X^L_{\underline{r}}})$, and $I = \{(k_1, \ldots, k_d) \in \mathbb{N}^d \mid 0 \le k_i < r'_i/r_i,\ 1 \le i \le d\}$.

PROOF. Firstly, we note that

$$A^L_{\underline{r}}[\frac{1}{p}] = L[r_1^{-1}\mathbb{Z} \oplus \cdots \oplus r_c^{-1}\mathbb{Z} \oplus r_{c+1}^{-1}\mathbb{N} \oplus \cdots \oplus r_d^{-1}\mathbb{N}] = L[T_{1,r_1}^{\pm 1}, \ldots, T_{c,r_c}^{\pm 1}, T_{c+1,r_{c+1}}, \ldots, T_{d,r_d}] \tag{II.8.14.2}$$

by definition (II.8.12.4) (cf. [**Tsu18**, page 812, equation (5)]). Thus, $\bigoplus_{\underline{k}\in I} \mathcal{O}_{L'} \otimes_{\mathcal{O}_L} A^L_{\underline{r}} \cdot \prod_{i=1}^d T_{i,r'_i}^{k_i}$ is a finite free $\mathcal{O}_{L'} \otimes_{\mathcal{O}_L} A^L_{\underline{r}}$-subalgebra of $A^{L'}_{\underline{r}'}[1/p]$. By [**Tsu18**, 4.5.(2)], we have

$$pA^{L'}_{\underline{r}} \subseteq \mathcal{O}_{L'} \otimes_{\mathcal{O}_L} A^L_{\underline{r}} \subseteq A^{L'}_{\underline{r}}. \tag{II.8.14.3}$$

By [**Tsu18**, 4.3] and the isomorphism (II.8.13.2), there exists $k_0 \in \mathbb{N}_{>0}$ independent of $L, L', \underline{r}, \underline{r}'$ such that (cf. [**Tsu18**, 4.9])

$$p^{k_0-1} A^{L'}_{\underline{r}'} \subseteq \bigoplus_{\underline{k}\in I} A^{L'}_{\underline{r}} \cdot \prod_{i=1}^{d} T_{i,r'_i}^{k_i} \subseteq p^{1-k_0} A^{L'}_{\underline{r}'}. \tag{II.8.14.4}$$

The conclusion follows from combining (II.8.14.3) and (II.8.14.4). □

II.9. Quasi-adequate Algebras and Faltings Extension

In this section, we fix a complete discrete valuation field K of characteristic 0 with perfect residue field of characteristic $p > 0$, an algebraic closure $\overline{K}$ of K, and a compatible system of primitive n-th roots of unity $(\zeta_n)_{n\in\mathbb{N}}$ in $\overline{K}$. Sometimes we denote ζ_n by $t_{0,n}$.

II.9.1. Let A be a Noetherian normal domain flat over $\mathbb{Z}_p$, A_{tr} a localization of A with respect to a nonzero element of pA, $\mathcal{K}$ the fraction field of A, $\overline{\mathcal{K}}$ an algebraic closure of $\mathcal{K}$. The *maximal unramified extension* $\mathcal{K}_{\mathrm{ur}}$ of $\mathcal{K}$ with respect to (A_{tr}, A) is the union of all finite extensions $\mathcal{K}'$ of $\mathcal{K}$ contained in $\overline{\mathcal{K}}$ such that the integral closure of A_{tr} in $\mathcal{K}'$ is étale over A_{tr}. It is a Galois extension over $\mathcal{K}$, whose Galois group $\mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$ is denoted by G_A. We call the integral closure $\overline{A}$ of A in $\mathcal{K}_{\mathrm{ur}}$ the *maximal unramified extension* of A with respect to A_{tr}. It is G_A-stable under the natural action of G_A on $\mathcal{K}_{\mathrm{ur}}$. We remark that the integral closure of A in any finite field extension $\mathcal{K}'$ of $\mathcal{K}$ is a Noetherian normal domain finite over A by II.4.4.

Definition II.9.2. A $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$*-triple* is a triple $(A_{\mathrm{tr}}, A, \overline{A})$ consisting of a Noetherian normal domain A flat over $\mathcal{O}_K$ with $A/pA \neq 0$, a localization A_{tr} of A with respect to a nonzero element of pA, and the maximal unramified extension $\overline{A}$ of A with respect to A_{tr} contained in an algebraic closure $\overline{\mathcal{K}}$ of the fraction field $\mathcal{K}$ of A containing $\overline{K}$.

A morphism of $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triples $(A_{\mathrm{tr}}, A, \overline{A}) \to (A'_{\mathrm{tr}}, A', \overline{A'})$ is a homomorphism of $\mathcal{O}_{\overline{K}}$-algebras $f : \overline{A} \to \overline{A'}$ such that $f(A) \subseteq A'$ and $f(A_{\mathrm{tr}}) \subseteq A'_{\mathrm{tr}}$. If f is injective, then it induces an extension of the fraction fields $\mathcal{K}_{\mathrm{ur}} \to \mathcal{K}'_{\mathrm{ur}}$ and thus a natural homomorphism of Galois groups $G_{A'} \to G_A$.

We will use this definition to describe the functoriality of the Faltings extensions (cf. II.9.38).

II.9.3. Let $(A_{\mathrm{tr}}, A, \overline{A})$ be a $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triple. We denote by $\mathfrak{E}(A)$ the set of morphisms of $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triples $(A_{\mathrm{tr}}, A, \overline{A}) \to (E, \mathcal{O}_E, \mathcal{O}_{\overline{E}})$, where E is a complete discrete valuation field extension of K whose residue field admits a finite p-basis. There is a natural right action of the Galois group G_A on $\mathfrak{E}(A)$ defined by sending $v \in \mathfrak{E}(A)$ to $v \circ g \in \mathfrak{E}(A)$, where $g \in G_A$ is regarded as a ring automorphism of $\overline{A}$ (which also induces a ring automorphism of $\mathcal{O}_{\overline{K}}$).

We fix an injection

$$\mathfrak{S}_p(\overline{A}) \longrightarrow \mathfrak{E}(A),\ \mathfrak{q} \mapsto ((A_{\mathrm{tr}}, A, \overline{A}) \to (E_{\mathfrak{p}}, \mathcal{O}_{E_{\mathfrak{p}}}, \mathcal{O}_{\overline{E}_{\mathfrak{q}}})), \tag{II.9.3.1}$$

where $\mathfrak{p} \in \mathfrak{S}_p(A)$ is the image of $\mathfrak{q}$, $\mathcal{O}_{E_{\mathfrak{p}}}$ is the p-adic completion of the localization $A_{\mathfrak{p}}$, $\overline{E}_{\mathfrak{q}}$ is an algebraic closure of $E_{\mathfrak{p}}$, and $\overline{A} \to \mathcal{O}_{\overline{E}_{\mathfrak{q}}}$ is the injection induced by an extension of valuation rings $\overline{A}_{\mathfrak{q}} \to \mathcal{O}_{\overline{E}_{\mathfrak{q}}}$ over $A_{\mathfrak{p}} \to \mathcal{O}_{E_{\mathfrak{p}}}$ (cf. II.4.3). In particular, there is a natural homomorphism of Galois groups $\mathrm{Gal}(\overline{E}_{\mathfrak{q}}/E_{\mathfrak{p}}) \to \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K}) = G_A$ by II.9.2.

Lemma II.9.4. *Let $(A_{\mathrm{tr}}, A, \overline{A})$ be a $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triple. The natural map of p-adic completions $f : \widehat{\overline{A}} \to \prod_{\mathfrak{E}(A)} \mathcal{O}_{\widehat{\overline{E}}}$ is G_A-equivariant and injective.*

Proof. We note that the natural G_A-action on $\prod_{\mathfrak{E}(A)} \mathcal{O}_{\widehat{\overline{E}}}$ is given by

$$g(x_v)_{v \in \mathfrak{E}(A)} = (x_{v \circ g})_{v \in \mathfrak{E}(A)} \tag{II.9.4.1}$$

for any $g \in G_A$. Thus, for any $x \in \widehat{\overline{A}}$, the v-component of $g(f(x))$ is the $(v \circ g)$-component of $f(x)$, which is equal to the image of x under the composition of $\widehat{\overline{A}} \xrightarrow{g} \widehat{\overline{A}} \xrightarrow{f_v} \mathcal{O}_{\widehat{\overline{E}}}$, i.e. the v-component of $f(gx)$. This shows the G_A-equivariance of f. With the notation in II.9.3, the natural maps

$$\widehat{\overline{A}} \longrightarrow \prod_{\mathfrak{q} \in \mathfrak{S}_p(\overline{A})} (\overline{A}_{\mathfrak{q}})^{\wedge} \longrightarrow \prod_{\mathfrak{q} \in \mathfrak{S}_p(\overline{A})} \mathcal{O}_{\widehat{\overline{E}}_{\mathfrak{q}}} \tag{II.9.4.2}$$

are injective by II.4.3. Their composition is also the composition of the natural maps

$$\widehat{\overline{A}} \longrightarrow \prod_{\mathfrak{E}(A)} \mathcal{O}_{\widehat{\overline{E}}} \longrightarrow \prod_{\mathfrak{q} \in \mathfrak{S}_p(\overline{A})} \mathcal{O}_{\widehat{\overline{E}}_{\mathfrak{q}}} \tag{II.9.4.3}$$

where the second map is induced by (II.9.3.1), which completes the proof. □

Definition II.9.5. A $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triple $(B_{\mathrm{tr}}, B, \overline{B})$ is called *quasi-adequate* if there exists a commutative diagram of monoids

$$\begin{array}{ccc} B & \xleftarrow{\beta} & P \\ \uparrow & & \uparrow{\scriptstyle \gamma} \\ \mathcal{O}_K & \xleftarrow{\alpha} & \mathbb{N} \end{array} \tag{II.9.5.1}$$

satisfying the following conditions:

(1) The element $\alpha(1)$ is a uniformizer of $\mathcal{O}_K$.

(2) The monoid P is fs, and if we denote by $\gamma_\eta : \mathbb{Z} \to P_\eta = \mathbb{Z} \oplus_{\mathbb{N}} P$ the pushout of γ by the inclusion $\mathbb{N} \to \mathbb{Z}$, then there exists an isomorphism for some $c, d \in \mathbb{N}$ with $c \leq d$,

$$P_\eta \cong \mathbb{Z} \oplus \mathbb{Z}^c \oplus \mathbb{N}^{d-c} \tag{II.9.5.2}$$

identifying γ_η with the inclusion of $\mathbb{Z}$ into the first component of the right hand side.

(3) The homomorphism β induces an injective ring homomorphism of finite type $A = \mathcal{O}_K \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[P] \to B$ which is pre-perfectoid in the sense of II.7.10 such that $B \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{gp}}] = B_{\mathrm{tr}}$ and $A[1/p] \to B[1/p]$ is étale.

We usually denote $(B_{\mathrm{tr}}, B, \overline{B})$ by B, and call it a *quasi-adequate $\mathcal{O}_K$-algebra* for simplicity. The triple $(\alpha : \mathbb{N} \to \mathcal{O}_K,\ \beta : P \to B,\ \gamma : \mathbb{N} \to P)$ is called a *quasi-adequate chart* of B. If we fix an isomorphism (II.9.5.2), then we call the images $t_1, \ldots, t_d \in B[1/p]$ of the standard basis of $\mathbb{Z}^c \oplus \mathbb{N}^{d-c}$ a *system of coordinates* of the chart. We call d the *relative dimension* of B over $\mathcal{O}_K$ (i.e. the Krull dimension of B_{tr}). If $A \to B$ is étale, then we say that B is an *adequate $\mathcal{O}_K$-algebra* and call (α, β, γ) an *adequate chart*.

Remark II.9.6. In II.9.5, the condition $B/pB \neq 0$ imposes that $A/pA \neq 0$. As A is a Noetherian normal domain flat over $\mathcal{O}_K$ by II.8.9, if we set $A_{\mathrm{tr}} = \mathcal{O}_K \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{gp}}]$ and denote by $\overline{A}$ the maximal unramified extension of A contained in $\overline{B}$, then $(A_{\mathrm{tr}}, A, \overline{A})$ is an adequate $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triple. The inclusion $\overline{A} \subseteq \overline{B}$ induces an injective morphism of quasi-adequate $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triples $(A_{\mathrm{tr}}, A, \overline{A}) \to (B_{\mathrm{tr}}, B, \overline{B})$. If $A \to B$ is étale (so that B is adequate) and if we endow $\mathrm{Spec}(B)$ with the compactifying log structure associated to the open immersion $\mathrm{Spec}(B_{\mathrm{tr}}) \to \mathrm{Spec}(B)$, then it becomes a log scheme over $S = (\mathrm{Spec}(\mathcal{O}_K), \alpha_{\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)})$ with an adequate chart in the sense of II.8.8 induced by (α, β, γ) (cf. II.8.9).

Remark II.9.7. Let B' be a B-algebra which is a Noetherian normal domain flat over $\mathcal{O}_K$ with $B \to B'$ injective and $B'/pB' \neq 0$. We set $B'_{\mathrm{tr}} = B_{\mathrm{tr}} \otimes_B B'$ and take a maximal unramified extension $\overline{B'}$ of (B'_{tr}, B') containing $\overline{B}$. Then, B' is a quasi-adequate $\mathcal{O}_K$-algebra with the same chart of B if $B[1/p] \to B'[1/p]$ is étale and if the ring homomorphism $B \to B'$ is of finite type and pre-perfectoid. This is satisfied in each of the following cases:

(1) The ring homomorphism $B \to B'$ is étale.
(2) The B-algebra B' is the integral closure of B in a finite étale $B[1/p]$-algebra (cf. II.7.13).
(3) The B-algebra B' is the normalization of an affine blowup algebra $B[I/a]$, where I is a finitely generated ideal of B containing a power of p, and a is a non-zero element of I (cf. II.7.18).

Lemma II.9.8. *Let B be a quasi-adequate $\mathcal{O}_K$-algebra.*

(1) *A system of coordinates $t_1, \ldots, t_d \in B[1/p]$ defines a strict normal crossings divisor on the regular scheme $\mathrm{Spec}(B[1/p])$, i.e. in the localization of $B[1/p]$ at any point, those elements t_i contained in the maximal ideal form a subset of a regular system of parameters.*
(2) *Let $(\alpha : \mathbb{N} \to \mathcal{O}_K,\ \beta : P \to B,\ \gamma : \mathbb{N} \to P)$ be a quasi-adequate chart of B, $Y = \mathrm{Spec}(\beta : P \to B)$. Then, the generic fibre Y_K is regular whose log structure is the compactifying log structure associated to the open immersion $\mathrm{Spec}(B_{\mathrm{tr}}) \to \mathrm{Spec}(B[1/p])$.*

Proof. (1) follows from the fact that the system of coordinates $t_1, \ldots, t_d \in B[1/p]$ identifies $B[1/p]$ with an étale $K[\mathbb{Z}^c \oplus \mathbb{N}^{d-c}]$-algebra (cf. II.8.9). (2) follows from the observation that $Y_K = \mathrm{Spec}(\mathbb{Z}^c \oplus \mathbb{N}^{d-c} \to B[1/p])$ (cf. II.8.5). □

Lemma II.9.9. *Let B be a quasi-adequate $\mathcal{O}_K$-algebra and we fix a chart of B as in* II.9.5. *Then, there exists $k \in \mathbb{N}$ such that the canonical truncation of the cotangent complex $\tau_{\leq 1} \mathrm{L}_{B/A}$ is p^k-exact in the sense of* II.7.1.

Proof. Since $A \to B$ is a ring homomorphism of finite type between Noetherian rings, the homology groups of $\mathrm{L}_{B/A}$ are finitely generated B-modules ([**Ill71**, II.2.3.7]). The conclusion follows from the fact that for any integer n, $H_n(\mathrm{L}_{B/A})[1/p] = H_n(\mathrm{L}_{B[1/p]/A[1/p]}) = 0$ as $B[1/p]$ is étale over $A[1/p]$. □

Lemma II.9.10. *There exists $k \in \mathbb{N}$ such that for any $\mathcal{O}_K$-algebra R, the canonical map of p-adic completions $\widehat{\Omega}^1_R \to (\Omega^1_{R/\mathcal{O}_K})^\wedge$ is a p^k-isomorphism.*

Proof. Since K is a complete discrete valuation field extension of $\mathbb{Q}_p$ with perfect residue field, $\widehat{\Omega}^1_{\mathcal{O}_K}$ is killed by p^k for some $k \in \mathbb{N}$ ([**He21**, 3.3]). For any $r \in \mathbb{N}$, we see that $\Omega^1_R/p^r \to \Omega^1_{R/\mathcal{O}_K}/p^r$ is surjective whose kernel N_r is killed by p^k. Taking limit over $r \in \mathbb{N}$, since $\lim N_r$ and $\mathrm{R}^1 \lim N_r$ are also killed by p^k, the conclusion follows. □

Proposition II.9.11. *Let B be a quasi-adequate $\mathcal{O}_K$-algebra, S a multiplicative subset of B. Then, $\widehat{\Omega}^1_{S^{-1}B}[1/p]$ is a finite free $(S^{-1}B)^\wedge[1/p]$-module, where the completions are p-adic. Moreover, it admits a basis $\mathrm{d}t_1, \dots, \mathrm{d}t_d$ for any system of coordinates $t_1, \dots, t_d \in B[1/p]$.*

Proof. We note that $\widehat{\Omega}^1_{S^{-1}B}[1/p] = (\Omega^1_{S^{-1}B/\mathcal{O}_K})^\wedge[1/p]$ by II.9.10. We take a quasi-adequate chart of B as in II.9.5. By the fundamental distinguished triangle of cotangent complexes ([**Ill71**, II.2.1.5.6]), we obtain an exact sequence,

$$H_1(\mathrm{L}_{S^{-1}B/A}) \to S^{-1}B \otimes_A \Omega^1_{A/\mathcal{O}_K} \xrightarrow{\alpha} \Omega^1_{S^{-1}B/\mathcal{O}_K} \to \Omega^1_{S^{-1}B/A} \to 0. \tag{II.9.11.1}$$

By II.9.9, α is a p^k-isomorphism for some $k \in \mathbb{N}$. Taking p-adic completion, we obtain an isomorphism $(S^{-1}B \otimes_A \Omega^1_{A/\mathcal{O}_K})^\wedge[1/p] \xrightarrow{\sim} (\Omega^1_{S^{-1}B/\mathcal{O}_K})^\wedge[1/p]$ by II.7.3.(1). As A is of finite type over $\mathcal{O}_K$, $\Omega^1_{A/\mathcal{O}_K}$ is a finitely generated A-module. Since $S^{-1}B$ is Noetherian, we have $(S^{-1}B \otimes_A \Omega^1_{A/\mathcal{O}_K})^\wedge[1/p] = (S^{-1}B)^\wedge \otimes_A \Omega^1_{A/\mathcal{O}_K}[1/p]$. The conclusion follows from the fact that $A[1/p] = K[\mathbb{Z}^c \oplus \mathbb{N}^{d-c}]$ (cf. II.8.9). □

Corollary II.9.12. *Let B be a quasi-adequate $\mathcal{O}_K$-algebra with relative dimension d. Then, for any $\mathfrak{p} \in \mathfrak{S}_p(B)$ (cf.* II.4.1*), we have $[\kappa(\mathfrak{p}) : \kappa(\mathfrak{p})^p] = p^d$, where $\kappa(\mathfrak{p})$ is the residue field of B at $\mathfrak{p}$.*

Proof. Notice that $B_\mathfrak{p}$ is a discrete valuation ring extension of $\mathbb{Z}_p$. Thus, the rank of the $\widehat{B_\mathfrak{p}}[1/p]$-module $\widehat{\Omega}^1_{B_\mathfrak{p}}[1/p]$ is equal to $\log_p[\kappa(\mathfrak{p}), \kappa(\mathfrak{p})^p]$, where the completions are p-adic ([**He21**, 3.3]). The conclusion follows from II.9.11. □

II.9.13. Let B be a quasi-adequate $\mathcal{O}_K$-algebra, $\mathcal{L}$ its fraction field, $\mathcal{L}_{\mathrm{ur}}$ the fraction field of $\overline{B}$, $G = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L})$. As in II.8.12, we construct coverings of B by quasi-adequate algebras. We fix B in the rest of this section, as well as the following notation.

We fix a quasi-adequate chart $(\alpha : \mathbb{N} \to \mathcal{O}_K,\ \beta : P \to B,\ \gamma : \mathbb{N} \to P)$ of B, $(A_{\mathrm{tr}}, A, \overline{A})$ the associated adequate $\mathcal{O}_K$-algebra defined in II.9.6, and a system of coordinates $t_1, \dots, t_d \in A[1/p]$. Let $\mathcal{K}$ (resp. $\mathcal{K}_{\mathrm{ur}}$) be the fraction field of A (resp. $\overline{A}$). For $1 \le i \le d$, we fix a compatible system of k-th roots $(t_{i,k})_{k\in\mathbb{N}}$ of t_i in $\overline{A}[1/p]$. For any field extension E'/E, let $\mathscr{F}_{E'/E}$ (resp. $\mathscr{F}^{\mathrm{fini}}_{E'/E}$) be the set of algebraic (resp. finite) field extensions of E contained in E', and we endow it with the partial order defined by the inclusion relation. For any $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/K}$ and any $\underline{r} = (r_1, \dots, r_d) \in \mathbb{N}^d_{>0}$, we set

$$\mathcal{K}^L_{\underline{r}} = L\mathcal{K}(t_{i,r_i} \mid 1 \le i \le d) \quad \text{and} \quad \mathcal{L}^L_{\underline{r}} = \mathcal{L}\mathcal{K}^L_{\underline{r}} \tag{II.9.13.1}$$

where the composites of fields are taken in $\mathcal{L}_{\mathrm{ur}}$ (which contains $\mathcal{K}_{\mathrm{ur}}$). It is clear that $\mathcal{K}^L_{\underline{r}}$ (resp. $\mathcal{L}^L_{\underline{r}}$) forms a system of fields over the directed partially ordered set $\mathscr{F}^{\mathrm{fini}}_{\overline{K}/K} \times \mathbb{N}^d_{>0}$ (cf. II.2.1). Let $A^L_{\underline{r}}$ (resp. $B^L_{\underline{r}}$) be the integral closure of A (resp. B) in $\mathcal{K}^L_{\underline{r}}$ (resp. $\mathcal{L}^L_{\underline{r}}$). We note that there is an isomorphism of L-algebras

$$L[T_1^{\pm 1}, \dots, T_c^{\pm 1}, T_{c+1}, \dots, T_d] \xrightarrow{\sim} A^L_{\underline{r}}[1/p] \tag{II.9.13.2}$$

sending T_i to t_{i,r_i} as $A[1/p] = K[\mathbb{Z}^c \oplus \mathbb{N}^{d-c}]$ (cf. II.8.9).

Let S^L (resp. $X^L_{\underline{r}}$) be the log scheme with underlying scheme $\mathrm{Spec}(\mathcal{O}_L)$ (resp. $\mathrm{Spec}(A^L_{\underline{r}})$) endowed with the compactifying log structure associated to the open immersion $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_L)$ (resp. $\mathrm{Spec}(A^L_{\underline{r},\mathrm{tr}}) \to \mathrm{Spec}(A^L_{\underline{r}})$ where $A^L_{\underline{r},\mathrm{tr}} = A_{\mathrm{tr}} \otimes_A A^L_{\underline{r}}$). The following lemma II.9.14 guarantees the consistency of this notation with the notation in II.8.12. Let $Y^L_{\underline{r}}$ be the log scheme with underlying scheme $\mathrm{Spec}(B^L_{\underline{r}})$ whose log structure is the inverse image of that of $X^L_{\underline{r}}$ via the map $\mathrm{Spec}(B^L_{\underline{r}}) \to \mathrm{Spec}(A^L_{\underline{r}})$ (i.e. $Y^L_{\underline{r}} \to X^L_{\underline{r}}$ is strict). We extend the notation above to any $(L, \underline{r}) \in \mathscr{F}_{\overline{K}/K} \times (\mathbb{N} \cup \{\infty\})^d_{>0}$ by taking filtered colimits.

Lemma II.9.14. *With the notation in* II.8.12 *for the chart $(\alpha : \mathbb{N} \to \mathcal{O}_K,\ \beta : P \to B,\ \gamma : \mathbb{N} \to P)$ of B, for any $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/K}$ and $\underline{r} = (r_1, \dots, r_d) \in \mathbb{N}^d_{>0}$, the homomorphism of monoids $P_{1,\underline{r}} \to A^L_{\underline{r}}$ sending $(k_0, k_1/r_1, \dots, k_d/r_d)$ to $\alpha(1)^{k_0} \cdot t_{1,r_1}^{k_1} \cdots t_{d,r_d}^{k_d}$ induces an isomorphism between $\mathrm{Spec}(A^L_{\underline{r}})$ with the underlying scheme of $S^L \times^{\mathrm{fs}}_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{P_{1,\underline{r}}}$. In particular, $X^L_{\underline{r}} = S^L \times^{\mathrm{fs}}_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{P_{1,\underline{r}}}$, where the left hand side is defined in* II.9.13.

Proof. Let $\mathrm{Spec}(A')$ denote the underlying scheme of $S^L \times^{\mathrm{fs}}_{\mathbb{A}_\mathbb{N}} \mathbb{A}_{P_{1,\underline{r}}}$. By II.8.13 and II.8.9, A' is a Noetherian normal domain finite over A such that $A'[1/p] = L[r_1^{-1}\mathbb{N} \oplus \cdots r_c^{-1}\mathbb{N} \oplus r_{c+1}^{-1}\mathbb{Z} \oplus \cdots \oplus r_d^{-1}\mathbb{Z}]$. Thus, $A'[1/p] = A^L_{\underline{r}}[1/p]$ by (II.9.13.2), and we obtain $A' = A^L_{\underline{r}}$. The "in particular" part follows from the fact that the log structures on both sides are the compactifying log structure associated to the open immersion $\mathrm{Spec}(A^L_{\underline{r},\mathrm{tr}}) \to \mathrm{Spec}(A^L_{\underline{r}})$ by definition, II.8.13 and II.8.9. □

Proposition II.9.15. *Let $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/K}$ and $\underline{r} = (r_1, \dots, r_d) \in \mathbb{N}^d_{>0}$.*

(1) *The $(L, \mathcal{O}_L, \mathcal{O}_{\overline{K}})$-algebra $(B^L_{\underline{r},\mathrm{tr}}, B^L_{\underline{r}}, \overline{B})$ is quasi-adequate with a chart (with the notation in* II.8.12*)*

$$(\alpha^L : e_L^{-1}\mathbb{N} \to \mathcal{O}_L,\ \beta^L_{\underline{r}} : P_{e_L,\underline{r}} \to B^L_{\underline{r}},\ \gamma^L_{\underline{r}} : e_L^{-1}\mathbb{N} \to P_{e_L,\underline{r}}) \tag{II.9.15.1}$$

where e_L is the ramification index of L/K, α^L is a homomorphism of monoids sending e_L^{-1} to a uniformizer π_L of L, $\beta^L_{\underline{r}}(k_0/e_L, k_1/r_1, \dots, k_d/r_d) = \pi_L^{k_0} \cdot t_{1,r_1}^{k_1} \cdots t_{d,r_d}^{k_d}$, and $\gamma^L_{\underline{r}}$ is induced by the inclusion of the first component of (II.8.12.3)*. Moreover, $A^L_{\underline{r}} = \mathcal{O}_L \otimes_{\mathbb{Z}[e_L^{-1}\mathbb{N}]} \mathbb{Z}[P_{e_L,\underline{r}}]$.*

(2) *The scheme $\mathrm{Spec}(B^L_{\underline{r}}[1/p])$ is an open and closed subscheme of $\mathrm{Spec}(B \otimes_A A^L_{\underline{r}}[1/p])$. The two schemes are equal if and only if $\mathcal{L}^L_{\underline{r}} = \mathcal{L} \otimes_{\mathcal{K}} \mathcal{K}^L_{\underline{r}}$ (i.e. $[\mathcal{L}^L_{\underline{r}} : \mathcal{L}] = [\mathcal{K}^L_{\underline{r}} : \mathcal{K}]$).*

(3) *For any $L' \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/L}$ and any element $\underline{r}' \in \mathbb{N}^d_{>0} \cdot \underline{r}$, the morphism of generic fibres $Y^{L'}_{\underline{r}',K} \to Y^L_{\underline{r},K}$ is étale.*

Proof. As $X^L_{\underline{r}} = S^L \times^{\mathrm{fs}}_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_{P_{1,\underline{r}}}$ by II.9.14, we see that $X^{L'}_{\underline{r}',K} \to X^L_{\underline{r},K}$ is étale. In particular, $Y_K \times^{\mathrm{fs}}_X X^L_{\underline{r}}$ is étale over Y_K with underlying scheme $\mathrm{Spec}(B \otimes_A A^L_{\underline{r}}[1/p])$ (as $Y \to X$ is strict). Since Y_K is regular by II.9.8.(2), so is $Y_K \times^{\mathrm{fs}}_X X^L_{\underline{r}}$. We see that $\mathrm{Spec}(B \otimes_A A^L_{\underline{r}}[1/p])$ is the disjoint union of finitely many normal integral schemes whose set of generic points identifies with $\mathrm{Spec}(\mathcal{L} \otimes_{\mathcal{K}} \mathcal{K}^L_{\underline{r}})$ (cf. II.8.5), and thus by definition $\mathrm{Spec}(B^L_{\underline{r}}[1/p])$ is one of these components corresponding to the generic point $\mathrm{Spec}(\mathcal{L}^L_{\underline{r}})$, so that we obtain (2). This implies that $Y^L_{\underline{r},K} \to Y_K \times^{\mathrm{fs}}_X X^L_{\underline{r}}$ is an open and closed immersion. Thus, (3) follows as $Y_K \to X_K$ is strict and étale.

For (1), we take $\alpha^L, \beta^L_{\underline{r}}, \gamma^L_{\underline{r}}$ as in the statement. Note that $A^L_{\underline{r}} = \mathcal{O}_L \otimes_{\mathbb{Z}[e_L^{-1}\mathbb{N}]} \mathbb{Z}[P_{e_L,\underline{r}}]$ by II.9.14 and II.8.13, and moreover it defines an adequate $\mathcal{O}_L$-algebra. It remains to check that the ring homomorphism $A^L_{\underline{r}} \to B^L_{\underline{r}}$ satisfies the conditions in the definition II.9.5 of quasi-adequate algebras. We have seen above that $A^L_{\underline{r}}[1/p] \to B^L_{\underline{r}}[1/p]$ is étale. As $B^L_{\underline{r}}$ is finite over B by II.4.4, it is of finite type over $A^L_{\underline{r}}$. Finally, it follows from II.7.11.(2) that $A \to B$ being pre-perfectoid implies that so is $A^L_{\underline{r}} \to B^L_{\underline{r}}$. □

Lemma II.9.16. *Let $F \in \mathscr{F}_{\overline{K}/K}$ be a pre-perfectoid field. Then, the $\mathcal{O}_F$-algebra $B^F_{\underline{\infty}}$ is almost pre-perfectoid.*

Proof. Recall that $B^F_{\underline{\infty}}$ is the filtered colimit of $B^L_{\underline{r}}$ over $(L, \underline{r}) \in \mathscr{F}^{\mathrm{fini}}_{F/K} \times \mathbb{N}^d_{>0}$ (cf. II.9.13). We claim that the Frobenius on $A^F_{\underline{\infty}}/pA^F_{\underline{\infty}}$ is surjective. Recall that for any $L \in \mathscr{F}^{\mathrm{fini}}_{F/K}$ and $\underline{r} \in \mathbb{N}^d_{>0}$, there is an isomorphism $f^L_{\underline{r}} : \mathcal{O}_L \otimes_{\mathbb{Z}[e_L^{-1}\mathbb{N}]} \mathbb{Z}[P_{e_L,\underline{r}}] \xrightarrow{\sim} A^L_{\underline{r}}$ sending $1 \otimes \exp(k_0/e_L, k_1/r_1, \dots, k_d/r_d)$ to $\pi_L^{k_0} \cdot t_{1,r_1}^{k_1} \cdots t_{d,r_d}^{k_d}$ by II.9.14 and II.8.13, where π_L is a uniformizer of L and $x = (k_0/e_L, k_1/r_1, \dots, k_d/r_d) \in P_{e_L,\underline{r}}$. Since F is a non-discrete valuation field such that the Frobenius map on $\mathcal{O}_F/p\mathcal{O}_F$ is surjective, there exists $L' \in \mathscr{F}^{\mathrm{fini}}_{F/L}$ such that $e_{L'}/e_L$ is divisible by p (I.5.4). We take $\underline{r}' = p\underline{r} \in \mathbb{N}^d_{>0}$. By definition, we have $x' = (k_0/pe_L, k_1/r'_1, \dots, k_d/r'_d) \in P_{e_{L'},\underline{r}'}$ as $px' = x$. Thus, there is a unit $u \in \mathcal{O}^\times_{L'}$ such that $f^L_{\underline{r}}(x) = u \cdot f^{L'}_{\underline{r}'}(x')^p$. Since the Frobenius map on $\mathcal{O}_F/p\mathcal{O}_F$ is surjective, u admits a p-th root in $\mathcal{O}_F/p\mathcal{O}_F$, and thus $f^L_{\underline{r}}(x)$ admits a p-th root in $A^F_{\underline{\infty}}/pA^F_{\underline{\infty}}$, which proves the claim.

Since $A^F_{\underline{\infty}}$ is normal and flat over $\mathcal{O}_F$, the Frobenius map induces an injection $\phi : A^F_{\underline{\infty}}/p_1A^F_{\underline{\infty}} \to A^F_{\underline{\infty}}/pA^F_{\underline{\infty}}$ (I.5.21), where p_1 is a p-th root of p up to a unit. By the claim above, ϕ is an isomorphism, which means that the $\mathcal{O}_F$-algebra $A^F_{\underline{\infty}}$ is almost pre-perfectoid (I.5.19). The conclusion follows from the fact that $A \to B$ is pre-perfectoid. □

Lemma II.9.17. *Let $L \subseteq L'$ be elements of $\mathscr{F}^{\mathrm{fini}}_{\overline{K}/K}$, $\underline{r}|\underline{r}'$ elements of $\mathbb{N}^d_{>0}$. Assume that $\mathcal{L}^{L'}_{\underline{r}'} = \mathcal{L}^L_{\underline{r}} \otimes_{\mathcal{K}^L_{\underline{r}}} \mathcal{K}^{L'}_{\underline{r}'}$ (i.e. $[\mathcal{L}^{L'}_{\underline{r}'} : \mathcal{L}^L_{\underline{r}}] = [\mathcal{K}^{L'}_{\underline{r}'} : \mathcal{K}^L_{\underline{r}}]$). Then, the finite free $B^L_{\underline{r}}$-module*

$$D^{L'/L}_{\underline{r}'/\underline{r}} = \bigoplus_{\underline{k} \in I} \mathcal{O}_{L'} \otimes_{\mathcal{O}_L} B^L_{\underline{r}} \cdot \prod_{i=1}^d t_{i,r'_i}^{k_i}, \tag{II.9.17.1}$$

identifies naturally with a finite free $B^L_{\underline{r}}$-subalgebra of $B^{L'}_{\underline{r}'}[1/p]$, where $I = \{(k_1, \dots, k_d) \in \mathbb{N}^d \mid 0 \le k_i < r'_i/r_i,\ 1 \le i \le d\}$. Moreover, for any finite field extension L'' of L' and any element $\underline{r}'' \in \mathbb{N}^d_{>0}$ divisible by $\underline{r}'$ such that $\mathcal{L}^{L''}_{\underline{r}''} = \mathcal{L}^L_{\underline{r}} \otimes_{\mathcal{K}^L_{\underline{r}}} \mathcal{K}^{L''}_{\underline{r}''}$, we have

$$D^{L'/L}_{\underline{r}'/\underline{r}} = B^{L'}_{\underline{r}'}[1/p] \cap D^{L''/L}_{\underline{r}''/\underline{r}}. \tag{II.9.17.2}$$

PROOF. We put $C^{L'/L}_{\underline{r}'/\underline{r}} = \bigoplus_{\underline{k}\in I} \mathcal{O}_{L'} \otimes_{\mathcal{O}_L} A^L_{\underline{r}} \cdot \prod_{i=1}^d t_{i,r'_i}^{k_i}$. Thus, $D^{L'/L}_{\underline{r}'/\underline{r}} = B^L_{\underline{r}} \otimes_{A^L_{\underline{r}}} C^{L'/L}_{\underline{r}'/\underline{r}}$. By assumption, we have $B^{L'}_{\underline{r}'}[1/p] = B^L_{\underline{r}} \otimes_{A^L_{\underline{r}}} A^{L'}_{\underline{r}'}[1/p]$ (cf. II.9.15.(2)), which is equal to $D^{L'/L}_{\underline{r}'/\underline{r}}[1/p]$ as $A^{L'}_{\underline{r}'}[1/p] = C^{L'/L}_{\underline{r}'/\underline{r}}[1/p]$ (cf. II.8.14). This proves the first assertion. For the "moreover" part, notice that $\mathcal{O}_{L''} \otimes_{\mathcal{O}_{L'}} D^{L'/L}_{\underline{r}'/\underline{r}}$ is a direct summand of $D^{L''/L}_{\underline{r}''/\underline{r}}$, both regarded as $\mathcal{O}_{L''} \otimes_{\mathcal{O}_L} B^L_{\underline{r}}$-submodules of $B^{L''}_{\underline{r}''}[1/p]$. Therefore,

$$\mathcal{O}_{L''} \otimes_{\mathcal{O}_{L'}} D^{L'/L}_{\underline{r}'/\underline{r}} = (L'' \otimes_{\mathcal{O}_{L'}} D^{L'/L}_{\underline{r}'/\underline{r}}) \cap D^{L''/L}_{\underline{r}''/\underline{r}} \supseteq B^{L'}_{\underline{r}'}[1/p] \cap D^{L''/L}_{\underline{r}''/\underline{r}}. \tag{II.9.17.3}$$

It remains to check that $B^{L'}_{\underline{r}'}[1/p] \cap (\mathcal{O}_{L''} \otimes_{\mathcal{O}_{L'}} D^{L'/L}_{\underline{r}'/\underline{r}}) = D^{L'/L}_{\underline{r}'/\underline{r}}$. As $B^{L'}_{\underline{r}'}[1/p] = D^{L'/L}_{\underline{r}'/\underline{r}}[1/p]$, restricting to the coefficient of each $\prod_{i=1}^d t_{i,r'_i}^{k_i}$ by (II.9.17.1), we reduce to show that $(L' \otimes_{\mathcal{O}_L} B^L_{\underline{r}}) \cap (\mathcal{O}_{L''} \otimes_{\mathcal{O}_L} B^L_{\underline{r}}) = \mathcal{O}_{L'} \otimes_{\mathcal{O}_L} B^L_{\underline{r}}$ as submodules of $L'' \otimes_{\mathcal{O}_L} B^L_{\underline{r}}$. This follows from the identity $L' \cap \mathcal{O}_{L''} = \mathcal{O}_{L'}$ and the flatness of $B^L_{\underline{r}}$ over $\mathcal{O}_L$. □

Proposition II.9.18. *Let $F \in \mathscr{F}_{\overline{K}/K}$ be a pre-perfectoid field. Then, there exists $k_0 \in \mathbb{N}$ such that for any $L \in \mathscr{F}^{\mathrm{fini}}_{F/K}$ and $\underline{r} \in \mathbb{N}^d_{>0}$, if $\mathcal{L}^F_{\underline{\infty}} = \mathcal{L}^L_{\underline{r}} \otimes_{\mathcal{K}^L_{\underline{r}}} \mathcal{K}^F_{\underline{\infty}}$, then the natural map*

$$B^L_{\underline{r}} \otimes_{A^L_{\underline{r}}} A^{L'}_{\underline{r}'} \longrightarrow B^{L'}_{\underline{r}'} \tag{II.9.18.1}$$

is a p^{k_0}-isomorphism for any $L' \in \mathscr{F}^{\mathrm{fini}}_{F/L}$ and $\underline{r}' \in \mathbb{N}^d_{>0} \cdot \underline{r}$.

PROOF. The assumption $\mathcal{L}^F_{\underline{\infty}} = \mathcal{L}^L_{\underline{r}} \otimes_{\mathcal{K}^L_{\underline{r}}} \mathcal{K}^F_{\underline{\infty}}$ implies that $\mathcal{L}^{L'}_{\underline{r}'} = \mathcal{L}^L_{\underline{r}} \otimes_{\mathcal{K}^L_{\underline{r}}} \mathcal{K}^{L'}_{\underline{r}'}$ (i.e. $[\mathcal{L}^{L'}_{\underline{r}'} : \mathcal{L}^L_{\underline{r}}] = [\mathcal{K}^{L'}_{\underline{r}'} : \mathcal{K}^L_{\underline{r}}]$) for any $L' \in \mathscr{F}^{\mathrm{fini}}_{F/L}$ and $\underline{r}' \in \mathbb{N}^d_{>0} \cdot \underline{r}$. We take again the notation in II.9.17 and its proof, and we put $D^{F/L}_{\underline{\infty}/\underline{r}} = \mathrm{colim}_{L',\underline{r}'} D^{L'/L}_{\underline{r}'/\underline{r}}$ as $B^L_{\underline{r}}$-subalgebra of $B^F_{\underline{\infty}}[1/p]$. By II.9.17, we have

$$D^{L'/L}_{\underline{r}'/\underline{r}} = B^{L'}_{\underline{r}'}[1/p] \cap D^{F/L}_{\underline{\infty}/\underline{r}}. \tag{II.9.18.2}$$

On the other hand, since $A \to B$ is pre-perfectoid and $\mathcal{K}^F_{\underline{\infty}} \in \mathscr{P}_A$ (cf. II.7.10) by II.9.16, there exists $k_0 \in \mathbb{N}$ such that

$$p^{k_0}\mathrm{Coker}(B \otimes_A A^F_{\underline{\infty}} \to B^F_{\underline{\infty}}) = 0. \tag{II.9.18.3}$$

After enlarging k_0 (independently of L, L') by II.9.14 and II.8.14, we may also assume that

$$p^{k_0} A^{L'}_{\underline{r}'} \subseteq C^{L'/L}_{\underline{r}'/\underline{r}} \subseteq p^{-k_0} A^{L'}_{\underline{r}'}. \tag{II.9.18.4}$$

Applying the functor $B^L_{\underline{r}} \otimes_{A^L_{\underline{r}}} -$, we see that the natural maps

$$B^L_{\underline{r}} \otimes_{A^L_{\underline{r}}} A^{L'}_{\underline{r}'} \longleftarrow B^L_{\underline{r}} \otimes_{A^L_{\underline{r}}} p^{k_0} A^{L'}_{\underline{r}'} \longrightarrow D^{L'/L}_{\underline{r}'/\underline{r}} \tag{II.9.18.5}$$

are p^{4k_0}-isomorphisms by II.7.3.(1). As $D^{L'/L}_{\underline{r}'/\underline{r}} \subseteq B^{L'}_{\underline{r}'}[1/p]$, we deduce that the kernel of (II.9.18.1) is p^{k_0}-zero after replacing k_0 by a constant multiple. On the other hand, as k_0 is independent of L, L', after replacing it by a constant multiple, we deduce from (II.9.18.3) and (II.9.18.5) that $p^{k_0} B^F_{\underline{\infty}} \subseteq D^{F/L}_{\underline{\infty}/\underline{r}}$. Thus, $p^{k_0} B^{L'}_{\underline{r}'} \subseteq B^{L'}_{\underline{r}'}[1/p] \cap D^{F/L}_{\underline{\infty}/\underline{r}} = D^{L'/L}_{\underline{r}'/\underline{r}}$ by (II.9.18.2), which implies by (II.9.18.5) that the cokernel of (II.9.18.1) is p^{k_0}-zero after replacing k_0 by a constant multiple. □

Corollary II.9.19. *Let $F \in \mathscr{F}_{\overline{K}/K}$ be a pre-perfectoid field, $L \in \mathscr{F}^{\mathrm{fini}}_{F/K}$, $\underline{r} \in \mathbb{N}^d_{>0}$. Assume that $\mathcal{L}^F_{\underline{\infty}} = \mathcal{L}^L_{\underline{r}} \otimes_{\mathcal{K}^L_{\underline{r}}} \mathcal{K}^F_{\underline{\infty}}$. Then, there exists $k_0 \in \mathbb{N}$ such that the truncated cotangent complex $\tau_{\leq 1}\mathrm{L}_{B^{L'}_{\underline{r}'}/A^{L'}_{\underline{r}'}}$ is p^{k_0}-exact for any $L' \in \mathscr{F}^{\mathrm{fini}}_{F/L}$ and $\underline{r}' \in \mathbb{N}^d_{>0} \cdot \underline{r}$.*

PROOF. We set $C = B^L_{\underline{r}} \otimes_{A^L_{\underline{r}}} A^{L'}_{\underline{r}'}$.

$$\begin{array}{ccccc} B^{L'}_{\underline{r}'} & \longleftarrow & C & \longleftarrow & A^{L'}_{\underline{r}'} \\ & & \uparrow & & \uparrow \\ & & B^L_{\underline{r}} & \longleftarrow & A^L_{\underline{r}} \end{array} \tag{II.9.19.1}$$

We take $k_1 \in \mathbb{N}$ such that $\tau_{\leq 1}\mathrm{L}_{B^L_{\underline{r}}/A^L_{\underline{r}}}$ is p^{k_1}-exact by II.9.9 and II.9.15.(1). Thus, $\tau_{\leq 1}\mathrm{L}_{C/A^{L'}_{\underline{r}'}}$ is p^{2k_1}-exact by II.7.6. Since $C \to B^{L'}_{\underline{r}'}$ is a p^{k_0}-isomorphism by II.9.18, $\tau_{\leq 1}\mathrm{L}_{B^{L'}_{\underline{r}'}/C}$ is p^{102k_0}-exact by II.7.8. After replacing k_0 by $3\max\{2k_1, 102k_0\}$, we see that $\tau_{\leq 1}\mathrm{L}_{B^{L'}_{\underline{r}'}/A^{L'}_{\underline{r}'}}$ is p^{k_0}-exact for any $L' \in \mathscr{F}^{\mathrm{fini}}_{F/L}$ and $\underline{r}' \in \mathbb{N}^d_{>0} \cdot \underline{r}$ by II.7.7. □

Corollary II.9.20. *Under the assumptions in* II.9.19 *and with the same notation, let* $L' \in \mathscr{F}^{\mathrm{fini}}_{F/L}$ *and* $\underline{r}' \in \mathbb{N}^d_{>0} \cdot \underline{r}$.

(1) *The canonical morphism*

$$B^{L'}_{\underline{r}'} \otimes_{A^{L'}_{\underline{r}'}} \Omega^1_{X^{L'}_{\underline{r}'}/S^{L'}} \longrightarrow \Omega^1_{Y^{L'}_{\underline{r}'}/S^{L'}} \tag{II.9.20.1}$$

is a p^{k_0}*-isomorphism.*

(2) *Let* $L'' \in \mathscr{F}^{\mathrm{fini}}_{F/L'}$ *and* $\underline{r}'' \in \mathbb{N}^d_{>0} \cdot \underline{r}'$. *Then, the canonical morphism*

$$B^{L''}_{\underline{r}''} \otimes_{A^{L''}_{\underline{r}''}} \Omega^1_{X^{L''}_{\underline{r}''}/X^{L'}_{\underline{r}'}} \longrightarrow \Omega^1_{Y^{L''}_{\underline{r}''}/Y^{L'}_{\underline{r}'}} \tag{II.9.20.2}$$

is a p^{2k_0}*-isomorphism.*

Proof. (1) The fundamental distinguished triangle of logarithmic cotangent complexes defined by Gabber ([**Ols05**, 8.29]) gives a canonical exact sequence

$$H_1(\mathrm{L}_{Y^{L'}_{\underline{r}'}/X^{L'}_{\underline{r}'}}) \longrightarrow B^{L'}_{\underline{r}'} \otimes_{A^{L'}_{\underline{r}'}} \Omega^1_{X^{L'}_{\underline{r}'}/S^{L'}} \xrightarrow{\alpha} \Omega^1_{Y^{L'}_{\underline{r}'}/S^{L'}} \longrightarrow \Omega^1_{Y^{L'}_{\underline{r}'}/X^{L'}_{\underline{r}'}} \longrightarrow 0. \tag{II.9.20.3}$$

Since $Y^{L'}_{\underline{r}'} \to X^{L'}_{\underline{r}'}$ is strict, its logarithmic cotangent complex is quasi-isomorphic to the cotangent complex $\mathrm{L}_{B^{L'}_{\underline{r}'}/A^{L'}_{\underline{r}'}}$ ([**Ols05**, 8.22]). Thus, α is a p^{k_0}-isomorphism by II.9.19.

(2) The proof is similar to that of (1). There are canonical exact sequences

$$B^{L''}_{\underline{r}''} \otimes_{B^{L'}_{\underline{r}'}} \Omega^1_{Y^{L'}_{\underline{r}'}/X^{L'}_{\underline{r}'}} \longrightarrow \Omega^1_{Y^{L''}_{\underline{r}''}/X^{L'}_{\underline{r}'}} \xrightarrow{\alpha} \Omega^1_{Y^{L''}_{\underline{r}''}/Y^{L'}_{\underline{r}'}} \longrightarrow 0, \tag{II.9.20.4}$$

$$H_1(\mathrm{L}_{Y^{L''}_{\underline{r}''}/X^{L''}_{\underline{r}''}}) \longrightarrow B^{L''}_{\underline{r}''} \otimes_{A^{L''}_{\underline{r}''}} \Omega^1_{X^{L''}_{\underline{r}''}/X^{L'}_{\underline{r}'}} \xrightarrow{\beta} \Omega^1_{Y^{L''}_{\underline{r}''}/X^{L'}_{\underline{r}'}} \longrightarrow \Omega^1_{Y^{L''}_{\underline{r}''}/X^{L''}_{\underline{r}''}} \longrightarrow 0. \tag{II.9.20.5}$$

By II.9.19, we see that α and β are p^{k_0}-isomorphisms, and thus $\alpha \circ \beta$ is a p^{2k_0}-isomorphism. □

II.9.21. We start to compute some modules of log differentials and construct the Faltings extension of B. Since $\mathcal{L}$ is a finite extension of $\mathcal{K}$, there exists $L_0 \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/K}$ and $r_0 \in \mathbb{N}_{>0}$ such that $\mathcal{L} \cap \mathcal{K}^{\overline{K}}_{\underline{\infty}} \subseteq \mathcal{K}^{L_0}_{\underline{r_0}}$. As $\mathcal{K}^{\overline{K}}_{\underline{\infty}}$ is a Galois extension of $\mathcal{K}$, we see by Galois theory that for any $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/L_0}$ and $r \in \mathbb{N}_{>0} \cdot r_0$,

$$\mathcal{L}^L_{\underline{r}} = \mathcal{L}^{L_0}_{\underline{r_0}} \otimes_{\mathcal{K}^{L_0}_{\underline{r_0}}} \mathcal{K}^L_{\underline{r}} \quad (\text{i.e. } [\mathcal{L}^L_{\underline{r}} : \mathcal{L}^{L_0}_{\underline{r_0}}] = [\mathcal{K}^L_{\underline{r}} : \mathcal{K}^{L_0}_{\underline{r_0}}]). \tag{II.9.21.1}$$

Recall that the log structure that we put on $Y^{\overline{K}}_{\underline{\infty}}$ is the colimit of the inverse images of log structures on $Y^L_{\underline{r}}$ over $(L, \underline{r}) \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/K} \times \mathbb{N}^d_{>0}$ (cf. [**GR04**, 12.2.10]). In the rest of this section, we denote by $\overline{Y}$ the log scheme with underlying scheme $\mathrm{Spec}(\overline{B})$ whose log structure is the inverse image of that of $Y^{\overline{K}}_{\underline{\infty}}$ via the map $\mathrm{Spec}(\overline{B}) \to \mathrm{Spec}(B^{\overline{K}}_{\underline{\infty}})$. We denote by $\mathbf{e}_0, \ldots, \mathbf{e}_d$ the standard basis of $\mathbb{Z}^{1+d}$. Recall that for any $1 \leq i \leq d$ and $r \in \mathbb{N}_{>0}$, the element $t_{i,r} \in \overline{B}[1/p]$ is the image of $r^{-1}\mathbf{e}_i \in P_{1,r,\eta} = \mathbb{Z} \oplus (r^{-1}\mathbb{Z})^c \oplus (r^{-1}\mathbb{N})^{d-c}$ via the chart (II.9.15.1). For any morphism of log schemes $\overline{Y} \to Z$ over S, we denote by $\mathrm{d}\log(t_{i,r}) \in \Gamma(\mathrm{Spec}(\overline{B}), \Omega^1_{\overline{Y}/Z})$ the image of the global section $r^{-1}\mathbf{e}_i$ via the canonical map $\mathrm{d}\log : \mathcal{M}^{\mathrm{gp}}_{\overline{Y}} \to \Omega^1_{\overline{Y}/Z}$ (cf. II.8.6).

Lemma II.9.22 (Abhyankar's lemma)**.** *Let* $\mathcal{L}' \in \mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$, B' *the integral closure of* B *in* $\mathcal{L}'$. *Then, there exists* $r \in \mathbb{N}_{>0}$ *such that* $B'_{\underline{r}}[1/p]$ *is finite étale over* $B_{\underline{r}}[1/p]$, *where* $B'_{\underline{r}}$ *is the integral closure of* B' *in* $\mathcal{L}'_{\underline{r}} = \mathcal{L}'\mathcal{L}_{\underline{r}}$. *In particular, if we set* $B'_{\underline{r},\mathrm{tr}} = B_{\mathrm{tr}} \otimes_B B'_{\underline{r}}$, *then* $(B'_{\underline{r},\mathrm{tr}}, B'_{\underline{r}}, \overline{B})$ *is a quasi-adequate* $\mathcal{O}_K$*-algebra with a chart induced by the chart* (II.9.15.1) *of* $(B_{\underline{r},\mathrm{tr}}, B_{\underline{r}}, \overline{B})$.

Proof. Recall that $t_1, \ldots, t_d$ defines a strict normal crossings divisor on the regular scheme $\mathrm{Spec}(B[1/p])$ by II.9.8.(1), and that for any $r \in \mathbb{N}_{>0}$, we have $A_{\underline{r}}[1/p] = A[1/p][T_1, \ldots, T_d]/(T_1^r - t_1, \ldots, T_d^r - t_d)$ by (II.9.13.2). By Abhyankar's lemma [**SGA 1**, XIII.5.2], there exists $r \in \mathbb{N}_{>0}$ such that the integral closure C of $B \otimes_A A_{\underline{r}}[1/p]$ in $B'_{\mathrm{tr}} \otimes_A A_{\underline{r}}[1/p]$ is finite étale over $B \otimes_A A_{\underline{r}}[1/p]$. Notice that $\mathrm{Spec}(B[1/p])$ is an open and closed subscheme of $\mathrm{Spec}(B \otimes_A A_{\underline{r}}[1/p])$ by II.9.15.(2). By the same argument, we also see that $\mathrm{Spec}(B'_{\underline{r}}[1/p])$ is an open and closed subscheme of $\mathrm{Spec}(C)$. Thus, $B'_{\underline{r}}[1/p]$ is finite étale over $B_{\underline{r}}[1/p]$. The "in particular" part follows from II.9.7. □

Proposition II.9.23. *Let* $F \in \mathscr{F}_{\overline{K}/K}$ *be a pre-perfectoid field (e.g.* $F = \overline{K}$*). Then, for any* $\mathcal{L}' \in \mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}^F_{\underline{\infty}}}$, *the integral closure* B' *of* B *in* $\mathcal{L}'$ *is almost finite étale over* $B^F_{\underline{\infty}}$. *In particular, the cotangent complex* $\mathrm{L}_{\overline{B}/B^F_{\underline{\infty}}}$ *is almost zero.*

PROOF. By Abhyankar's lemma II.9.22 and a limit argument (cf. I.8.21), $B'[1/p]$ is finite étale over $B^F_{\underline{\infty}}[1/p]$. Since $B^F_{\underline{\infty}}$ is almost pre-perfectoid (II.9.16), B' is almost finite étale over $B^F_{\underline{\infty}}$ by almost purity II.7.12. Thus, the cotangent complex $\mathrm{L}_{B'/B^F_{\underline{\infty}}}$ is almost zero ([**GR03**, 2.5.37]), and so is $\mathrm{L}_{\overline{B}/B^F_{\underline{\infty}}}$ by taking filtered colimit. □

Lemma II.9.24. *Let E'/E be an extension of discrete valuation fields, Z (resp. Z') the log scheme with underlying scheme* $\mathrm{Spec}(\mathcal{O}_E)$ *(resp.* $\mathrm{Spec}(\mathcal{O}_{E'})$*) with log structure defined by the closed point. Then, the kernel of $\Omega^1_{\mathcal{O}_{E'}/\mathcal{O}_E} \to \Omega^1_{Z'/Z}$ is killed by a uniformizer of E, and its cokernel is killed by a uniformizer of E'.*

PROOF. Let π' (resp. π) be a uniformizer of E' (resp. E). We write $\pi = u\pi'^e$ for some $u \in \mathcal{O}_{E'}^\times$ and $e \in \mathbb{N}$. Then, $Z' \to Z$ admits a chart $(\alpha : \mathbb{N} \to \mathcal{O}_E,\ \beta : e^{-1}\mathbb{N} \oplus \mathbb{Z} \to \mathcal{O}_{E'},\ \gamma : \mathbb{N} \to e^{-1}\mathbb{N} \oplus \mathbb{Z})$ where $\alpha(1) = \pi$, $\beta(e^{-1}, 0) = \pi'$, $\beta(0, 1) = u$ and $\gamma(1) = (1, 1)$. By II.8.6, there is an $\mathcal{O}_{E'}$-linear surjection

$$\Omega^1_{\mathcal{O}_{E'}/\mathcal{O}_E} \oplus \mathcal{O}_{E'} \otimes (e^{-1}\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} \longrightarrow \Omega^1_{Z'/Z} \tag{II.9.24.1}$$

whose kernel M is generated by $(\mathrm{d}u, -u \otimes (0, 1))$ and $(\mathrm{d}\pi', -\pi' \otimes (e^{-1}, 0))$. Thus, the cokernel of $\Omega^1_{\mathcal{O}_{E'}/\mathcal{O}_E} \to \Omega^1_{Z'/Z}$ is killed by π'. Let $\omega \in \Omega^1_{\mathcal{O}_{E'}/\mathcal{O}_E} \cap M$. We have $\omega = a\mathrm{d}u + b\mathrm{d}\pi'$ for some $a, b \in \mathcal{O}_{E'}$ such that $b\pi' = eau$. Since $0 = \mathrm{d}\pi = \pi'^e\mathrm{d}u + eu\pi'^{e-1}\mathrm{d}\pi'$, we see that $\pi\omega = au(\pi'^e\mathrm{d}u + eu\pi'^{e-1}\mathrm{d}\pi') = 0$. □

Proposition II.9.25 ([**Fon82**, Théorème 1']). *The $\mathcal{O}_{\overline{K}}$-linear homomorphism*

$$\overline{K}/\mathcal{O}_{\overline{K}} \longrightarrow \Omega^1_{S^{\overline{K}}/S}, \tag{II.9.25.1}$$

sending p^{-n} to $\mathrm{d}\log(\zeta_{p^n})$ for any $n \in \mathbb{N}$, is a p^k-isomorphism for some $k \in \mathbb{N}$.

PROOF. Recall that there is a fractional ideal $\mathfrak{a}$ of $\overline{K}$ and an $\mathcal{O}_{\overline{K}}$-linear isomorphism

$$\overline{K}/\mathfrak{a} \xrightarrow{\sim} \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}, \tag{II.9.25.2}$$

sending p^{-n} to $\mathrm{d}\log(\zeta_{p^n})$ for any $n \in \mathbb{N}$ ([**Fon82**, Théorème 1']). The conclusion follows from the fact that $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega^1_{S^L/S}$ is a p-isomorphism for any $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/K}$ by II.9.24. □

Lemma II.9.26. *With the notation in* II.9.21*, there exists $k_0 \in \mathbb{N}$ such that for any $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/L_0}$, the $B^L_{\underline{\infty}}$-linear map*

$$B^L_{\underline{\infty}}[\frac{1}{p}]^d \longrightarrow \Omega^1_{Y^L_{\underline{\infty}}/S^L}, \tag{II.9.26.1}$$

sending $p^{-n}\mathbf{e}_i$ to $\mathrm{d}\log(t_{i,p^n})$ for any $1 \leq i \leq d$ and $n \in \mathbb{N}$, is a p^{k_0}-isomorphism.

PROOF. Since $X^L_{\underline{r}} = S^L \times^{\mathrm{fs}}_{\mathbb{A}_{\mathbb{N}}} \mathbb{A}_{P_{1,\underline{r}}}$ for any $r \in \mathbb{N}_{>0}$ by II.9.14, we have $\Omega^1_{X^L_{\underline{r}}/S^L} = A^L_{\underline{r}} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}_{1,\underline{r}}/\mathbb{N}^{\mathrm{gp}}) = A^L_{\underline{r}} \otimes_{\mathbb{Z}} (r^{-1}\mathbb{Z})^d$, which identifies $\mathrm{d}\log(t_{i,p^n})$ with $p^{-n}\mathbf{e}_i$ if $p^n | r$ (cf. II.9.21). Taking $r \to \infty$, we obtain the conclusion by II.9.20.(1). □

Lemma II.9.27. *With the notation in* II.9.21*, there exists $k \in \mathbb{N}$ such that for any $r \in \mathbb{N}_{>0} \cdot r_0$, the $B^{\overline{K}}_{\underline{r}}$-linear map*

$$B^{\overline{K}}_{\underline{r}}[\frac{1}{p}]/B^{\overline{K}}_{\underline{r}} \longrightarrow \Omega^1_{Y^{\overline{K}}_{\underline{r}}/Y^{L_0}_{\underline{r}}}, \tag{II.9.27.1}$$

sending p^{-n} to $\mathrm{d}\log(\zeta_{p^n})$ for any $n \in \mathbb{N}$, is a p^k-isomorphism.

PROOF. Since $X^L_{\underline{r}} = S^L \times^{\mathrm{fs}}_{S^{L_0}} X^{L_0}_{\underline{r}}$ for any $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/L_0}$ by II.9.14, we have $\Omega^1_{X^L_{\underline{r}}/X^{L_0}_{\underline{r}}} = A^L_{\underline{r}} \otimes_{\mathcal{O}_L} \Omega^1_{S^L/S^{L_0}}$. Taking L running through $\mathscr{F}^{\mathrm{fini}}_{\overline{K}/L_0}$, we see that there exists $k \in \mathbb{N}$ by II.9.25 such that the map $A^{\overline{K}}_{\underline{r}} \otimes_{\mathcal{O}_{\overline{K}}} \overline{K}/\mathcal{O}_{\overline{K}} \to \Omega^1_{X^{\overline{K}}_{\underline{r}}/X^{L_0}_{\underline{r}}}$ is a p^k-isomorphism for any $r \in \mathbb{N}_{>0}$. The conclusion follows from II.9.20.(2). □

Proposition II.9.28 (cf. [**AGT16**, II.7.9], [**He21**, 3.6]). *The $\overline{B}$-linear homomorphism*

$$(\overline{B}[\frac{1}{p}]/\overline{B}) \oplus \overline{B}[\frac{1}{p}]^d \longrightarrow \Omega^1_{\overline{Y}/S} \tag{II.9.28.1}$$

sending $p^{-n}\mathbf{e}_i$ to $\mathrm{d}\log(t_{i,p^n})$ for any $n \in \mathbb{N}$ and $0 \leq i \leq d$ (where $t_{0,p^n} = \zeta_{p^n}$), is a p^k-isomorphism for some $k \in \mathbb{N}$.

PROOF. With the notation in II.9.21, consider the canonical exact sequences

$$B_{\underline{\infty}}^{\overline{K}} \otimes_{B_{\underline{\infty}}^{L_0}} \Omega^1_{Y_{\underline{\infty}}^{L_0}/S^{L_0}} \xrightarrow{\alpha_1} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/S^{L_0}} \xrightarrow{\beta_1} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/Y_{\underline{\infty}}^{L_0}} \longrightarrow 0, \tag{II.9.28.2}$$

$$B_{\underline{\infty}}^{\overline{K}} \otimes_{\mathcal{O}_{\overline{K}}} \Omega^1_{S^{\overline{K}}/S^{L_0}} \xrightarrow{\alpha_2} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/S^{L_0}} \xrightarrow{\beta_2} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/S^{\overline{K}}} \longrightarrow 0. \tag{II.9.28.3}$$

By II.9.27 and II.9.25, there are p^k-isomorphisms

$$B_{\underline{\infty}}^{\overline{K}} \otimes_{\mathcal{O}_{\overline{K}}} \overline{K}/\mathcal{O}_{\overline{K}} \longrightarrow \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/Y_{\underline{\infty}}^{L_0}}, \tag{II.9.28.4}$$

$$\overline{K}/\mathcal{O}_{\overline{K}} \longrightarrow \Omega^1_{S^{\overline{K}}/S^{L_0}}. \tag{II.9.28.5}$$

On the other hand, by II.9.26, there are p^{k_0}-isomorphisms

$$B_{\underline{\infty}}^{L_0}[\frac{1}{p}]^d \longrightarrow \Omega^1_{Y_{\underline{\infty}}^{L_0}/S^{L_0}}, \tag{II.9.28.6}$$

$$B_{\underline{\infty}}^{\overline{K}}[\frac{1}{p}]^d \longrightarrow \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/S^{\overline{K}}}. \tag{II.9.28.7}$$

By considering the compositions $\beta_2 \circ \alpha_1$ and $\beta_1 \circ \alpha_2$, one checks easily that the $B_{\underline{\infty}}^{\overline{K}}$-linear homomorphism

$$(B_{\underline{\infty}}^{\overline{K}}[\frac{1}{p}]/B_{\underline{\infty}}^{\overline{K}}) \oplus B_{\underline{\infty}}^{\overline{K}}[\frac{1}{p}]^d \longrightarrow \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/S^{L_0}} \tag{II.9.28.8}$$

sending $p^{-n}\mathbf{e}_i$ to $\mathrm{d}\log(t_{i,p^n})$ for any $n \in \mathbb{N}$ and $0 \leq i \leq d$, is a p^l-isomorphism for some $l \in \mathbb{N}$. Since $S^{L_0} \to S$ is a morphism of Noetherian fs log schemes which induces an étale morphism of the generic fibres and induces a finite morphism of the underlying schemes, the homology groups of the logarithmic cotangent complex $\mathrm{L}_{S^{L_0}/S}$ defined by Gabber are p-primary torsion, finitely generated $\mathcal{O}_{L_0}$-modules ([**Ols05**, 8.30]). After enlarging l (depending only on L_0), we may assume that the map $(B_{\underline{\infty}}^{\overline{K}}[1/p]/B_{\underline{\infty}}^{\overline{K}}) \oplus B_{\underline{\infty}}^{\overline{K}}[1/p]^d \to \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/S}$ is a p^l-isomorphism. The conclusion follows from II.9.23. □

Lemma II.9.29. *With the notation in* II.9.21*, there exists $k_0 \in \mathbb{N}$ such that for any $L \in \mathscr{F}^{\mathrm{fini}}_{\overline{K}/L_0}$, the $B_{\underline{\infty}}^{L}$-linear map*

$$(B_{\underline{\infty}}^{L}[\frac{1}{p}]/r_0^{-1}B_{\underline{\infty}}^{L})^d \longrightarrow \Omega^1_{Y_{\underline{\infty}}^{L}/Y_{\underline{r_0}}^{L}}, \tag{II.9.29.1}$$

sending $p^{-n}\mathbf{e}_i$ to $\mathrm{d}\log(t_{i,p^n})$ for any $n \in \mathbb{N}$ and $1 \leq i \leq d$, is a p^{k_0}-isomorphism.

PROOF. Since $X_{\underline{r'}}^{L} = X_{\underline{r}}^{L} \times^{\mathrm{fs}}_{\mathbb{A}_{P_{1,\underline{r}}}} \mathbb{A}_{P_{1,\underline{r'}}}$ for any elements $r|r'$ in $\mathbb{N}_{>0}$ by II.9.14, we have $\Omega^1_{X_{\underline{r'}}^{L}/X_{\underline{r}}^{L}} = A_{\underline{r'}}^{L} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}_{1,\underline{r'}}/P^{\mathrm{gp}}_{1,\underline{r}}) = A_{\underline{r'}}^{L} \otimes_{\mathbb{Z}} (r'^{-1}\mathbb{Z}/r^{-1}\mathbb{Z})^d$, which identifies $\mathrm{d}\log(t_{i,p^n})$ with $p^{-n}\mathbf{e}_i$ if $p^n|r'$ (cf. II.9.21). Taking $r = r_0$ and $r' \to \infty$, we obtain the conclusion by II.9.20.(2). □

Proposition II.9.30 (cf. [**AGT16**, II.7.13], [**He21**, 4.2])**.** *The $\overline{B}$-linear homomorphism*

$$(\overline{B}[\frac{1}{p}]/\overline{B})^{1+d} \longrightarrow \Omega^1_{\overline{Y}/Y} \tag{II.9.30.1}$$

sending $p^{-n}\mathbf{e}_i$ to $\mathrm{d}\log(t_{i,p^n})$ for any $n \in \mathbb{N}$ and $0 \leq i \leq d$ (where $t_{0,p^n} = \zeta_{p^n}$), is a p^k-isomorphism for some $k \in \mathbb{N}$.

PROOF. The proof is similar to that of II.9.28. With the notation in II.9.21, consider the canonical exact sequences

$$B_{\underline{\infty}}^{\overline{K}} \otimes_{B_{\underline{\infty}}^{L_0}} \Omega^1_{Y_{\underline{\infty}}^{L_0}/Y_{\underline{r_0}}^{L_0}} \xrightarrow{\alpha_1} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/Y_{\underline{r_0}}^{L_0}} \xrightarrow{\beta_1} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/Y_{\underline{\infty}}^{L_0}} \longrightarrow 0, \tag{II.9.30.2}$$

$$B_{\underline{\infty}}^{\overline{K}} \otimes_{B_{\underline{r_0}}^{\overline{K}}} \Omega^1_{Y_{\underline{r_0}}^{\overline{K}}/Y_{\underline{r_0}}^{L_0}} \xrightarrow{\alpha_2} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/Y_{\underline{r_0}}^{L_0}} \xrightarrow{\beta_2} \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/Y_{\underline{r_0}}^{\overline{K}}} \longrightarrow 0. \tag{II.9.30.3}$$

By II.9.27, there are p^k-isomorphisms

$$B_{\underline{\infty}}^{\overline{K}} \otimes_{\mathcal{O}_{\overline{K}}} \overline{K}/\mathcal{O}_{\overline{K}} \longrightarrow \Omega^1_{Y_{\underline{\infty}}^{\overline{K}}/Y_{\underline{\infty}}^{L_0}}, \tag{II.9.30.4}$$

$$B_{\underline{r_0}}^{\overline{K}} \otimes_{\mathcal{O}_{\overline{K}}} \overline{K}/\mathcal{O}_{\overline{K}} \longrightarrow \Omega^1_{Y_{\underline{r_0}}^{\overline{K}}/Y_{\underline{r_0}}^{L_0}}. \tag{II.9.30.5}$$

On the other hand, by II.9.29, there are p^{k_0}-isomorphisms

$$(B^{L_0}_{\underline{\infty}}[\frac{1}{p}]/r_0^{-1}B^{L_0}_{\underline{\infty}})^d \longrightarrow \Omega^1_{Y^{L_0}_{\underline{\infty}}/Y^{L_0}_{\underline{r_0}}}, \tag{II.9.30.6}$$

$$(B^{\overline{K}}_{\underline{\infty}}[\frac{1}{p}]/r_0^{-1}B^{\overline{K}}_{\underline{\infty}})^d \longrightarrow \Omega^1_{Y^{\overline{K}}_{\underline{\infty}}/Y^{\overline{K}}_{\underline{r_0}}}. \tag{II.9.30.7}$$

By considering the compositions $\beta_2\circ\alpha_1$ and $\beta_1\circ\alpha_2$, one checks easily that the $B^{\overline{K}}_{\underline{\infty}}$-linear homomorphism

$$(B^{\overline{K}}_{\underline{\infty}}[\frac{1}{p}]/B^{\overline{K}}_{\underline{\infty}})^{1+d} \longrightarrow \Omega^1_{Y^{\overline{K}}_{\underline{\infty}}/Y^{L_0}_{\underline{r_0}}} \tag{II.9.30.8}$$

sending $p^{-n}\mathbf{e}_i$ to $\mathrm{d}\log(t_{i,p^n})$ for any $n\in\mathbb{N}$ and $0\leq i\leq d$, is a p^l-isomorphism for some $l\in\mathbb{N}$. Since $Y^{L_0}_{\underline{r_0}}\to Y$ is a morphism between Noetherian fs log schemes which induces an étale morphism of the generic fibres (II.9.15.(3)) and induces a finite morphism of the underlying schemes (II.4.4), the homology groups of the logarithmic cotangent complex $\mathrm{L}_{Y^{L_0}_{\underline{r_0}}/Y}$ defined by Gabber are p-primary torsion, finitely generated $B^{L_0}_{\underline{r_0}}$-modules ([**Ols05**, 8.30]). After enlarging l (depending only on L_0 and r_0), we may assume that the map $(B^{\overline{K}}_{\underline{\infty}}[\frac{1}{p}]/B^{\overline{K}}_{\underline{\infty}})^{1+d}\to\Omega^1_{Y^{\overline{K}}_{\underline{\infty}}/Y}$ is a p^l-isomorphism. The conclusion follows from II.9.23. □

II.9.31. The $\widehat{\overline{B}}[1/p]$-module $V_p(\Omega^1_{\overline{Y}/Y})=\varprojlim_{x\mapsto px}\Omega^1_{\overline{Y}/Y}$ (cf. II.2.2) is endowed with a natural action of G. For any element $(s_{p^n})_{n\in\mathbb{N}}$ of $\varprojlim_{x\mapsto x^p}\overline{B}[1/p]\cap\overline{B}^\times_{\mathrm{tr}}$, we take $l\in\mathbb{N}$ such that $p^ls_1\in\overline{B}$. Thus, $p^ls^r_{p^n}\in\overline{B}$ for any $n\in\mathbb{N}$ and $0\leq r\leq p^n$. Notice that the element $p^{-2l}((p^ls^{p^n-1}_{p^n})\mathrm{d}(p^ls_{p^n}))_{n\in\mathbb{N}}$ of $V_p(\Omega^1_{\overline{Y}/Y})$ does not depend on the choice of l. Thus, we denote this element by $(s^{p^n-1}_{p^n}\mathrm{d}s_{p^n})_{n\in\mathbb{N}}$.

Proposition II.9.32 (cf. [**AGT16**, II.7.22], [**He21**, 4.4])**.** *There is a canonical G-equivariant exact sequence of $\widehat{\overline{B}}[1/p]$-modules,*

$$0\longrightarrow\widehat{\overline{B}}[\frac{1}{p}](1)\xrightarrow{\iota}V_p(\Omega^1_{\overline{Y}/Y})\xrightarrow{\jmath}\widehat{\overline{B}}[\frac{1}{p}]\otimes_B\Omega^1_{Y/S}\longrightarrow 0, \tag{II.9.32.1}$$

satisfying the following properties:

1. *We have $\iota(1\otimes(\zeta_{p^n})_{n\in\mathbb{N}})=(\mathrm{d}\log(\zeta_{p^n}))_{n\in\mathbb{N}}$.*
2. *For any element $s\in B[1/p]\cap B^\times_{\mathrm{tr}}$ and any compatible system of p-power roots $(s_{p^n})_{n\in\mathbb{N}}$ of s in $\overline{B}[1/p]$, $\jmath((s^{p^n-1}_{p^n}\mathrm{d}s_{p^n})_{n\in\mathbb{N}})=\mathrm{d}s$ (cf.* II.9.31*).*
3. *The $\widehat{\overline{B}}[1/p]$-linear surjection $\jmath$ admits a section sending $\mathrm{d}\log(t_i)$ to $(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}$ for any $1\leq i\leq d$.*

In particular, $V_p(\Omega^1_{\overline{Y}/Y})$ is a finite free $\widehat{\overline{B}}[\frac{1}{p}]$-module with basis $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\leq i\leq d}$, where $t_{0,p^n}=\zeta_{p^n}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0\oplus\overline{B}^d & \longrightarrow & (\overline{B}[\frac{1}{p}]/\overline{B})\oplus\overline{B}[\frac{1}{p}]^d & \longrightarrow & (\overline{B}[\frac{1}{p}]/\overline{B})\oplus(\overline{B}[\frac{1}{p}]/\overline{B})^d & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \overline{B}\otimes_B\Omega^1_{Y/S} & \longrightarrow & \Omega^1_{\overline{Y}/S} & \longrightarrow & \Omega^1_{\overline{Y}/Y} & \longrightarrow & 0
\end{array} \tag{II.9.32.2}$$

where the vertical maps send $p^{-n}\mathbf{e}_i$ to $\mathrm{d}\log(t_{i,p^n})$ for any $n\in\mathbb{N}$ and $0\leq i\leq d$. There exists $k\in\mathbb{N}$ such that the vertical maps are p^k-isomorphisms by II.9.8.(2), II.9.28 and II.9.30. It is clear that the first row is exact and splits. Applying $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p/p^n\mathbb{Z}_p,-)$ to (II.9.32.2), we get a commutative diagram

(II.9.32.3)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (p^{-n}\overline{B}/\overline{B})\oplus 0 & \longrightarrow & (p^{-n}\overline{B}/\overline{B})\oplus(p^{-n}\overline{B}/\overline{B})^d & \longrightarrow & 0\oplus(\overline{B}/p^n\overline{B})^d & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(\overline{B}\otimes_B\Omega^1_{Y/S})[p^n] & \longrightarrow & (\Omega^1_{\overline{Y}/S})[p^n] & \longrightarrow & \Omega^1_{\overline{Y}/Y}[p^n] & \xrightarrow{\jmath} & (\overline{B}\otimes_B\Omega^1_{Y/S})/p^n & \longrightarrow & 0
\end{array}$$

where the connecting map $\jmath$ sends $(p^l s_{p^n}^{p^n-1})\mathrm{d}(p^l s_{p^n})$ to $\mathrm{d}(p^{2l}s)$ (with the notation in II.9.31). The vertical maps are p^{2k}-isomorphisms by II.7.3.(1), and the first row is exact and splits. Taking inverse limit on $n \in \mathbb{N}$ and inverting p, by II.7.3.(2), we get a canonical G-equivariant exact sequence, which admits a splitting (not G-equivariant),

$$0 \longrightarrow T_p(\Omega^1_{\overline{Y}/S})[\frac{1}{p}] \longrightarrow V_p(\Omega^1_{\overline{Y}/Y}) \xrightarrow{\jmath} \widehat{\overline{B}}[\frac{1}{p}] \otimes_B \Omega^1_{Y/S} \longrightarrow 0, \tag{II.9.32.4}$$

where we used the fact that $T_p(\Omega^1_{\overline{Y}/Y})[\frac{1}{p}] = V_p(\Omega^1_{\overline{Y}/Y})$ as $\Omega^1_{\overline{Y}/Y}$ is p-primary torsion, and that $\Omega^1_{Y/S}[1/p]$ is finite free over $B[1/p]$ with basis $\mathrm{d}\log(t_1), \ldots, \mathrm{d}\log(t_d)$. Notice that (II.9.32.3) also implies that $\iota : \widehat{\overline{B}}[\frac{1}{p}](1) \to T_p(\Omega^1_{\overline{Y}/S})[\frac{1}{p}]$ sending $1 \otimes (\zeta_{p^n})_{n\in\mathbb{N}}$ to $(\mathrm{d}\log(\zeta_{p^n}))_{n\in\mathbb{N}}$ is an isomorphism. The conclusion follows. □

II.9.33. A priori, the sequence (II.9.32.1) relies on the choice of the quasi-adequate chart of B. In the rest of this section, we show that it can be canonically defined, independently of the choice of a chart. Firstly, we check the compatibility of (II.9.32.1) with the Faltings extension (II.5.7.1) of a complete discrete valuation ring. Let $(B_{\mathrm{tr}}, B, \overline{B}) \to (E, \mathcal{O}_E, \mathcal{O}_{\overline{E}})$ be an element of $\mathfrak{E}(B)$ (cf. II.9.3). Let Z (resp. $\overline{Z}$) be the log scheme of underlying scheme $\mathrm{Spec}(\mathcal{O}_E)$ (resp. $\mathrm{Spec}(\mathcal{O}_{\overline{E}})$) with the compactifying log structure defined by the closed point. Notice that the map $\Omega^1_{\mathcal{O}_{\overline{E}}/\mathcal{O}_E} \to \Omega^1_{\overline{Z}/Z}$ is a p-isomorphism by II.9.24, which thus induces a natural isomorphism of $\widehat{\overline{E}}$-modules $\mathscr{E}_{\mathcal{O}_E} = V_p(\Omega^1_{\mathcal{O}_{\overline{E}}/\mathcal{O}_E}) \xrightarrow{\sim} V_p(\Omega^1_{\overline{Z}/Z})$. The map $\Omega^1_{\mathcal{O}_E/\mathcal{O}_K} \to \Omega^1_{Z/S}$ is also a p-isomorphism by II.9.24, which thus induces a natural isomorphism of E-modules $\widehat{\Omega}^1_{\mathcal{O}_E}[\frac{1}{p}] = (\Omega^1_{\mathcal{O}_E/\mathcal{O}_K})^\wedge[\frac{1}{p}] \xrightarrow{\sim} (\Omega^1_{Z/S})^\wedge[\frac{1}{p}]$. By the explicit descriptions of ι and $\jmath$, the exact sequence (II.9.32.1) fits into the following natural commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\overline{B}}[\frac{1}{p}](1) & \xrightarrow{\iota} & V_p(\Omega^1_{\overline{Y}/Y}) & \xrightarrow{\jmath} & \widehat{\overline{B}}[\frac{1}{p}] \otimes_B \Omega^1_{Y/S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{\mathfrak{E}(B)} \widehat{\overline{E}}(1) & \xrightarrow{\iota} & \prod_{\mathfrak{E}(B)} \mathscr{E}_{\mathcal{O}_E} & \xrightarrow{\jmath} & \prod_{\mathfrak{E}(B)} \widehat{\overline{E}} \otimes_{\mathcal{O}_E} \widehat{\Omega}^1_{\mathcal{O}_E} & \longrightarrow & 0
\end{array} \tag{II.9.33.1}$$

where the second sequence is the product of the Faltings extensions of $\mathcal{O}_E$ defined in II.5.7.

For any $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$ with image $\mathfrak{p} \in \mathfrak{S}_p(B)$, consider the element $(B_{\mathrm{tr}}, B, \overline{B}) \to (E_{\mathfrak{p}}, \mathcal{O}_{E_{\mathfrak{p}}}, \mathcal{O}_{\overline{E}_{\mathfrak{q}}})$ of $\mathfrak{E}(B)$ defined in II.9.3. Notice that $\{\mathrm{d}\log(t_i)\}_{1\le i\le d}$ forms an $E_{\mathfrak{p}}$-basis of $\widehat{\Omega}^1_{\mathcal{O}_{E_{\mathfrak{p}}}}[1/p]$ by II.9.11 (as $\mathcal{O}_{E_{\mathfrak{p}}}$ is the p-adic completion of the localization $B_{\mathfrak{p}}$). Thus, $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\le i\le d}$ forms an $\widehat{\overline{E}}_{\mathfrak{q}}$-basis of $\mathscr{E}_{\mathcal{O}_{E_{\mathfrak{p}}}}$. With respect to this basis, we can identify the projection of each vertical map in (II.9.33.1) to the components corresponding to $\mathfrak{S}_p(\overline{B})$ with a direct sum of the natural map $\widehat{\overline{B}}[\frac{1}{p}] \to \prod_{\mathfrak{q}\in\mathfrak{S}_p(\overline{B})} \widehat{\overline{E}}_{\mathfrak{q}}$, which is injective by II.9.4. In conclusion, the vertical maps in (II.9.33.1) are injective.

Lemma II.9.34. *For any element $s \in B[1/p] \cap B_{\mathrm{tr}}^\times$ and any compatible system of p-power roots $(s_{p^n})_{n\in\mathbb{N}}$ of s in $\overline{B}[1/p]$, there is at most one element $\omega \in V_p(\Omega^1_{\overline{Y}/Y})$ such that $s\omega = (s_{p^n}^{p^n-1}\mathrm{d}s_{p^n})_{n\in\mathbb{N}}$.*

Proof. By the discussion in II.9.33, we have $V_p(\Omega^1_{\overline{Y}/Y}) \subseteq \prod_{\mathfrak{q}\in\mathfrak{S}_p(\overline{B})} \mathscr{E}_{\mathcal{O}_{E_{\mathfrak{p}}}}$. Since s acts invertibly on the latter, the unicity of ω follows. □

Proposition II.9.35. *For any element $s \in B[1/p] \cap B_{\mathrm{tr}}^\times$ and any compatible system of p-power roots $(s_{p^n})_{n\in\mathbb{N}}$ of s in $\overline{B}[1/p]$, there is a unique element $\omega \in V_p(\Omega^1_{\overline{Y}/Y})$ such that $s\omega = (s_{p^n}^{p^n-1}\mathrm{d}s_{p^n})_{n\in\mathbb{N}}$, which we denote by $(\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}$.*

Proof. We note that for any $\mathfrak{p} \in \mathrm{Spec}(B[1/p])$, the non-units of $\{t_1, \ldots, t_d\}$ in $B_{\mathfrak{p}}$ form a subset of a regular system of parameters of the regular local ring $B_{\mathfrak{p}}$ by II.9.8.(1). Since the divisor $\mathrm{div}(s)$ on $\mathrm{Spec}(B_{\mathfrak{p}})$ defined by s is set-theoretically contained in the union of the integral effective Cartier divisors $\mathrm{div}(t_1), \ldots, \mathrm{div}(t_d)$, we can write $s = ut_1^{a_1} \cdots t_d^{a_d}$ for some $u \in B_{\mathfrak{p}}^\times$ and $a_1, \ldots, a_d \in \mathbb{N}$ ([**Sta22**, 0BCP]). Thus, we can take $f_1, \ldots, f_l \in B$ such that $\{\mathrm{Spec}(B[1/pf_i])\}_{1\le i\le l}$ forms an open covering of $\mathrm{Spec}(B[1/p])$ and that s admits an expression as before in each $B[1/pf_i]$. Consider the finitely generated ideal $I = (f_1, \ldots, f_l)$ of B. It contains a power of p as $I[1/p] = B[1/p]$. Consider the affine blowup algebra $B[I/f_i]$ (see II.7.14) and its normalization B_i. It is clear that $B_i[1/p] = B[1/pf_i]$. As the image of $B[I/f_i] \otimes_B B[I/f_j]$ in $B_i \otimes_B B_j[1/p] = B[1/pf_if_j]$ is the affine blowup algebra $B[I^2/f_if_j]$, we see that the integral closure B_{ij} of $B_i \otimes_B B_j$ in $B_i \otimes_B B_j[1/p]$ coincides with the normalization

of $B[I^2/f_if_j]$ for any $1 \leq i, j \leq l$. Thus, B_i, B_{ij} define naturally quasi-adequate $\mathcal{O}_K$-algebras with the same chart induced by B (II.9.15.1) (cf. II.9.7). For the simplicity of our arguments below, we consider the integral closures of B_i, B_{ij} in $\mathcal{L}_{ur}$ (the fraction field of $\overline{B}$), and denote them abusively by $\overline{B_i}, \overline{B_{ij}}$ (which may be smaller than the corresponding maximal unramified extensions). Therefore, $\overline{B_i}$ is the integral closure of $\overline{B}[I/f_i]$ in $\overline{B}[I/f_i][1/p] = \overline{B}[1/pf_i]$, which is almost pre-perfectoid over $\mathcal{O}_{\overline{K}}$ by II.7.17. Since $\overline{B_{ij}}$ is the integral closure of $\overline{B_i} \otimes_{\overline{B}} \overline{B_j}$ in $\overline{B_i} \otimes_{\overline{B}} \overline{B_j}[1/p] = \overline{B}[1/pf_if_j]$, it is almost pre-perfectoid and whose p-adic completion is almost isomorphic to that of $\overline{B_i} \otimes_{\overline{B}} \overline{B_j}$ by I.5.33 and I.5.30. Since $\{\mathrm{Spec}(\overline{B}[I/f_i]) \to \mathrm{Spec}(\overline{B})\}_{1\leq i\leq l}$ is the composition of a blowup with a Zariski open covering, $\{\mathrm{Spec}(\overline{B_i}) \to \mathrm{Spec}(\overline{B})\}_{1\leq i\leq l}$ is a v-covering by I.3.15. Thus, the descent of perfectoid algebras in the arc-topology (due to Bhatt-Scholze, cf. I.5.35) induces an exact augmented Čech complex

$$0 \longrightarrow \widehat{\overline{B}}[\tfrac{1}{p}] \longrightarrow \prod_{1\leq i\leq l} \widehat{\overline{B_i}}[\tfrac{1}{p}] \Longrightarrow \prod_{1\leq i,j\leq l} \widehat{\overline{B_{ij}}}[\tfrac{1}{p}]. \tag{II.9.35.1}$$

On the other hand, let Y_i (resp. $\overline{Y_i}$) be the log scheme with underlying scheme $\mathrm{Spec}(B_i)$ (resp. $\mathrm{Spec}(\overline{B_i})$) whose log structure is the inverse image of that of Y (resp. $\overline{Y}$). Although the maximal unramified extension of the quasi-adequate algebra B_i may be bigger than $\overline{B_i}$, the $\widehat{\overline{B_i}}$-module $V_p(\Omega^1_{\overline{Y_i}/Y_i})$ is finite free with basis $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\leq i\leq d}$ by the arguments of II.9.32 as $\overline{B_i}$ contains $(B_i)^{\overline{K}}_{\infty}$ and almost purity still holds for it (cf. II.9.23). Similar result also holds for B_{ij}. Thus, there is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widehat{\overline{B}}[\frac{1}{p}]^{1+d} & \longrightarrow & \prod_{1\leq i\leq l} \widehat{\overline{B_i}}[\frac{1}{p}]^{1+d} & \Longrightarrow & \prod_{1\leq i,j\leq l} \widehat{\overline{B_{ij}}}[\frac{1}{p}]^{1+d} \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
0 & \longrightarrow & V_p(\Omega^1_{\overline{Y}/Y}) & \longrightarrow & \prod_{1\leq i\leq l} V_p(\Omega^1_{\overline{Y_i}/Y_i}) & \Longrightarrow & \prod_{1\leq i,j\leq l} V_p(\Omega^1_{\overline{Y_{ij}}/Y_{ij}})
\end{array} \tag{II.9.35.2}$$

where the vertical maps are isomorphisms. Thus, the second row is also exact. Recall that in each $B_i[1/p]$, we can write $s = ut_1^{a_1}\cdots t_d^{a_d}$ for some $u \in B_i[1/p]^\times$ and $a_1, \ldots, a_d \in \mathbb{N}$. Notice that $s_{p^n}t_{1,p^n}^{-a_1}\cdots t_{d,p^n}^{-a_d} \in \mathcal{L}_{\mathrm{ur}}$. We put $u_{p^n} = s_{p^n}t_{1,p^n}^{-a_1}\cdots t_{d,p^n}^{-a_d}$, which is a p^n-th root of u and thus lies in $\overline{B_i}[1/p]$. We see that the element

$$(\mathrm{d}\log(u_{p^n}))_{n\in\mathbb{N}} + \sum_{i=1}^{d} a_i(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}} \in V_p(\Omega^1_{\overline{Y_i}/Y_i}) \tag{II.9.35.3}$$

is the unique element ω_i whose multiplication by s coincides with $(s_{p^n}^{p^n-1}\mathrm{d}s_{p^n})_{n\in\mathbb{N}}$ (cf. II.9.34). The unicity implies that ω_i and ω_j coincides in $V_p(\Omega^1_{\overline{Y_{ij}}/Y_{ij}})$. Therefore, by the exactness of the second row in (II.9.35.2), we obtain an element $\omega \in V_p(\Omega^1_{\overline{Y}/Y})$ such that $s\omega = (s_{p^n}^{p^n-1}\mathrm{d}s_{p^n})_{n\in\mathbb{N}}$, which completes the proof. □

Theorem II.9.36. *Let B be a quasi-adequate $\mathcal{O}_K$-algebra, Y_K the log scheme with underlying scheme $\mathrm{Spec}(B[1/p])$ with compactifying log structure associated to $\mathrm{Spec}(B_{\mathrm{tr}}) \to \mathrm{Spec}(B[1/p])$, $\mathscr{E}_B$ the $\widehat{\overline{B}}[1/p]$-submodule of $\prod_{\mathfrak{E}(B)} \mathscr{E}_{\mathcal{O}_E}$ (see* II.9.33*) generated by the subset*

$$\{(\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}} \mid s_1 \in B[1/p]\cap B_{\mathrm{tr}}^\times,\ s_{p^n} \in \overline{B}[1/p],\ s_{p^{n+1}}^p = s_{p^n},\ \forall n \in \mathbb{N}\}. \tag{II.9.36.1}$$

Then, $\mathscr{E}_B$ is stable under the natural $\widehat{\overline{B}}[1/p]$-semi-linear action of $G = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L})$ on $\prod_{\mathfrak{E}(B)} \mathscr{E}_{\mathcal{O}_E}$ (induced by the right G-action on $\mathfrak{E}(B)$, cf. (II.9.4.1)*), and there is a canonical G-equivariant exact sequence of $\widehat{\overline{B}}[1/p]$-modules,*

$$0 \longrightarrow \widehat{\overline{B}}[\frac{1}{p}](1) \xrightarrow{\iota} \mathscr{E}_B \xrightarrow{\jmath} \widehat{\overline{B}} \otimes_B \Omega^1_{Y_K/K} \longrightarrow 0, \tag{II.9.36.2}$$

satisfying the following properties:

1. *We have $\iota(1 \otimes (\zeta_{p^n})_{n\in\mathbb{N}}) = (\mathrm{d}\log(\zeta_{p^n}))_{n\in\mathbb{N}}$.*
2. *For any element $s \in B[1/p]\cap B_{\mathrm{tr}}^\times$ and any compatible system of p-power roots $(s_{p^n})_{n\in\mathbb{N}}$ of s in $\overline{B}[1/p]$, $\jmath((\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}) = \mathrm{d}\log(s)$.*
3. *If we take a quasi-adequate chart of B and fix compatible systems of p-power roots $(t_{i,p^n})_{n\in\mathbb{N}} \subseteq \overline{B}[1/p]$ of the coordinates $t_1, \ldots, t_d \in B[1/p]$, then the sequence* (II.9.36.2) *identifies with*

the image of the vertical maps in (II.9.33.1). *In particular, the* $\widehat{\overline{B}}[1/p]$*-linear surjection* $\jmath$ *admits a section sending* $\mathrm{d}\log(t_i)$ *to* $(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}$ *for any* $1\leq i\leq d$.

In particular, $\mathscr{E}_B$ *is a finite free* $\widehat{\overline{B}}[\frac{1}{p}]$*-module with basis* $\{(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\}_{0\leq i\leq d}$*, where* $t_{0,p^n}=\zeta_{p^n}$*, on which* G *acts continuously with respect to the canonical topology (where* $\widehat{\overline{B}}[\frac{1}{p}]$ *is endowed with the* p*-adic topology defined by* $\widehat{\overline{B}}$*).*

Proof. As the subset (II.9.36.1) is G-stable, we see that $\mathscr{E}_B$ is G-stable and thus endowed with a natural G-action. By II.9.35, $V_p(\Omega^1_{\overline{Y}/Y})$ identifies with $\mathscr{E}_B$ as a submodule of $\prod_{\mathfrak{E}(B)}\mathscr{E}_{\mathcal{O}_E}$ via (II.9.33.1). The conclusion follows directly from II.9.32 and II.9.33, where the "in particular part" follows from the same arguments of II.5.7. □

Definition II.9.37. We call the canonical sequence (II.9.36.2) the *Faltings extension* of B.

Remark II.9.38. The Faltings extension (II.9.36.2) is functorial in the following sense: let K' be a complete discrete valuation field extension of K with perfect residue field, B' a quasi-adequate $\mathcal{O}_{K'}$-algebra. Consider a commutative diagram of $(K,\mathcal{O}_K,\mathcal{O}_{\overline{K}})$-triples (see II.9.2)

$$\begin{array}{ccc}(B_{\mathrm{tr}},B,\overline{B}) & \xrightarrow{f} & (B'_{\mathrm{tr}},B',\overline{B'})\\ \uparrow & & \uparrow\\ (K,\mathcal{O}_K,\mathcal{O}_{\overline{K}}) & \longrightarrow & (K',\mathcal{O}_{K'},\mathcal{O}_{\overline{K'}})\end{array}\tag{II.9.38.1}$$

Then, f induces a natural map $\mathfrak{E}(B')\to\mathfrak{E}(B)$ sending v' to $v'\circ f$ (cf. II.9.33). It induces further a natural map

$$\prod_{\mathfrak{E}(B)}\mathscr{E}_{\mathcal{O}_E}\longrightarrow\prod_{\mathfrak{E}(B')}\mathscr{E}_{\mathcal{O}_{E'}},\ (\omega_v)_{v\in\mathfrak{E}(B)}\mapsto(\omega_{v'\circ f})_{v'\in\mathfrak{E}(B')},\tag{II.9.38.2}$$

which maps $\mathscr{E}_B$ to $\mathscr{E}_{B'}$ and is compatible with the Faltings extensions.

$$\begin{array}{ccccccccc}0&\longrightarrow&\widehat{\overline{B}}[\frac{1}{p}](1)&\xrightarrow{\iota}&\mathscr{E}_B&\xrightarrow{\jmath}&\widehat{\overline{B}}\otimes_B\Omega^1_{Y_K/K}&\longrightarrow&0\\ &&\downarrow&&\downarrow&&\downarrow&&\\ 0&\longrightarrow&\widehat{\overline{B'}}[\frac{1}{p}](1)&\xrightarrow{\iota}&\mathscr{E}_{B'}&\xrightarrow{\jmath}&\widehat{\overline{B'}}\otimes_{B'}\Omega^1_{Y'_{K'}/K'}&\longrightarrow&0\end{array}\tag{II.9.38.3}$$

Moreover, if B' is the integral closure of B in a finite extension of $\mathcal{L}$ contained in $\mathcal{L}_{\mathrm{ur}}$, then $Y'_{K'}$ is étale over Y_K by [**ILO14**, IX.2.1]. Assuming that K' is finite over K, we see that the natural map $B'\otimes_B\Omega^1_{Y_K/K}\to\Omega^1_{Y'_{K'}/K'}$ is an isomorphism, and thus the vertical maps in (II.9.38.3) are isomorphisms. In particular, $\mathscr{E}_B$ contains $(\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}$ for any element $(s_{p^n})_{n\in\mathbb{N}}$ of $\varprojlim_{x\mapsto x^p}\overline{B}[1/p]\cap\overline{B}^{\times}_{\mathrm{tr}}$.

II.9.39. As in II.5.16, taking a Tate twist of the dual of the Faltings extension (II.9.36.2) of B, we obtain a canonical exact sequence of finite projective $\widehat{\overline{B}}[1/p]$-representations of G, which splits as a sequence of $\widehat{\overline{B}}[1/p]$-modules,

$$0\longrightarrow\mathrm{Hom}_{B[\frac{1}{p}]}(\Omega^1_{Y_K/K}(-1),\widehat{\overline{B}}[\frac{1}{p}])\xrightarrow{\jmath^*}\mathscr{E}_B^*(1)\xrightarrow{\iota^*}\widehat{\overline{B}}[\frac{1}{p}]\longrightarrow 0\tag{II.9.39.1}$$

where $\mathscr{E}_B^*=\mathrm{Hom}_{\widehat{\overline{B}}[1/p]}(\mathscr{E}_B,\widehat{\overline{B}}[1/p])$. There is a canonical G-equivariant $\widehat{\overline{B}}[1/p]$-linear Lie algebra structure on $\mathscr{E}^*_{\mathcal{O}_K}(1)$ associated to the linear form ι^*,

$$[f_1,f_2]=\iota^*(f_1)f_2-\iota^*(f_2)f_1,\ \forall f_1,f_2\in\mathscr{E}_B^*(1).\tag{II.9.39.2}$$

Thus, $\mathrm{Hom}_{B[1/p]}(\Omega^1_{Y_K/K}(-1),\widehat{\overline{B}}[1/p])$ is a Lie ideal of $\mathscr{E}^*_{\mathcal{O}_K}(1)$, and $\widehat{\overline{B}}[1/p]$ is the quotient, and the induced Lie algebra structures on them are trivial. Any $\widehat{\overline{B}}[1/p]$-linear splitting of (II.9.39.1) identifies $\mathscr{E}_B^*(1)$ with the semi-direct product of Lie algebras of $\widehat{\overline{B}}[1/p]$ acting on $\mathrm{Hom}_{B[1/p]}(\Omega^1_{Y_K/K}(-1),\widehat{\overline{B}}[1/p])$ by multiplication. Let $\{T_i=(\mathrm{d}\log(t_{i,p^n}))_{n\in\mathbb{N}}\otimes\zeta^{-1}\}_{0\leq i\leq d}$ (where $t_{0,p^n}=\zeta_{p^n}$) denote the basis of $\mathscr{E}_B(-1)$, and let $\{T_i^*\}_{0\leq i\leq d}$ be the dual basis of $\mathscr{E}_B^*(1)$. Then, we see that the Lie bracket on $\mathscr{E}_B^*(1)$ is determined by

$$[T_0^*,T_i^*]=T_i^*\quad\text{and}\quad[T_i^*,T_j^*]=0,\tag{II.9.39.3}$$

for any $1 \leq i, j \leq d$. Indeed, this dual basis induces an isomorphism of $\widehat{\overline{B}}[1/p]$-linear Lie algebras

$$\widehat{\overline{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathbb{Z}_p \ltimes \mathbb{Z}_p^d) \xrightarrow{\sim} \mathscr{E}_B^*(1),\ 1 \otimes \partial_i \mapsto T_i^*, \tag{II.9.39.4}$$

where $\{\partial_i\}_{0 \leq i \leq d}$ is the standard basis of $\mathrm{Lie}(\mathbb{Z}_p \ltimes \mathbb{Z}_p^d)$ (cf. II.4.20).

II.10. Descent of Representations of Arithmetic Fundamental Groups after Tsuji

Tsuji [**Tsu18**, §14] studied the descent of representations of the arithmetic fundamental group of an adequate algebra. In this section, we show that his arguments still work for quasi-adequate algebras.

II.10.1. We construct a Kummer tower from the coverings of a quasi-adequate algebra defined in II.9.13. The following notation will be used in this section. We fix a complete discrete valuation field K of characteristic 0 with perfect residue field of characteristic $p > 0$, an algebraic closure $\overline{K}$ of K, and a compatible system of primitive n-th roots of unity $(\zeta_n)_{n \in \mathbb{N}}$ in $\overline{K}$. Let B be a quasi-adequate $\mathcal{O}_K$-algebra, $\mathcal{L}$ its fraction field, $\mathcal{L}_{\mathrm{ur}}$ the fraction field of $\overline{B}$. We fix a quasi-adequate chart of B, A the associated adequate $\mathcal{O}_K$-algebra defined in II.9.6, and a system of coordinates $t_1, \ldots, t_d \in A[1/p]$. Let $\mathcal{K}$ (resp. $\mathcal{K}_{\mathrm{ur}}$) be the fraction field of A (resp. $\overline{A}$). For any $1 \leq i \leq d$, we fix a compatible system of k-th roots $(t_{i,k})_{k \in \mathbb{N}}$ of t_i in $\overline{A}[1/p]$.

Let J be the subset of $\mathbb{N}_{>0}^d$ consisting of elements $\underline{N} = (N_1, \ldots, N_d)$ with N_i prime to p for any $1 \leq i \leq d$. We endow J with the partial order defined by the divisibility relation (cf. II.2.1). For $\underline{N} \in J$, $n \in \mathbb{N}$ and $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, we define finite field extensions of K and $\mathcal{K}$ in $\overline{K}$ and $\mathcal{K}_{\mathrm{ur}}$ respectively by

$$K_n^{(\underline{N})} = K(\zeta_{p^n N_i} \mid 1 \leq i \leq d), \qquad \mathcal{K}_{n,\underline{m}}^{(\underline{N})} = K_n^{(\underline{N})} \mathcal{K}(t_{i,p^{m_i} N_i} \mid 1 \leq i \leq d). \tag{II.10.1.1}$$

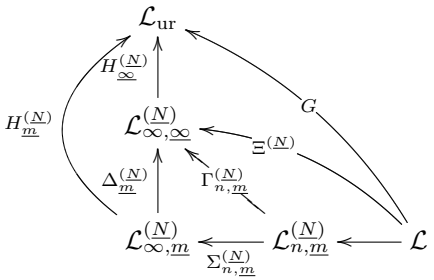
It is clear that these fields $\mathcal{K}_{n,\underline{m}}^{(\underline{N})}$ form a system of fields over the directed partially ordered set $J \times \mathbb{N} \times \mathbb{N}^d$ (cf. II.2.1). We extend this notation for one of the components of $\underline{N}, n, \underline{m}$ being ∞ by taking the filtered union, and we omit the index $\underline{N}$ or n or $\underline{m}$ if $\underline{N} = \underline{1}$ or $n = 0$ or $\underline{m} = \underline{0}$ respectively. We remark that if we take again the notation in II.9.13, then $\mathcal{K}_{n,\underline{m}}^{(\underline{N})} = \mathcal{K}_{p^{\underline{m}}\underline{N}}^{K_n^{(\underline{N})}}$. We set $\mathcal{L}_{n,\underline{m}}^{(\underline{N})} = \mathcal{L}\mathcal{K}_{n,\underline{m}}^{(\underline{N})}$ and let $A_{n,\underline{m}}^{(\underline{N})}$ (resp. $B_{n,\underline{m}}^{(\underline{N})}$) be the integral closure of A in $\mathcal{K}_{n,\underline{m}}^{(\underline{N})}$ (resp. of B in $\mathcal{L}_{n,\underline{m}}^{(\underline{N})}$).

For any $\underline{N} \in J$, the system $(B_{n,\underline{m}}^{(\underline{N})})_{(n,\underline{m}) \in \mathbb{N}^{1+d}}$ is the Kummer tower of $B^{(\underline{N})}$ defined by $\zeta_{p^n}, t_{1,p^n}, \ldots, t_{d,p^n}$ (cf. II.4.16). Following II.4.7, for any $(n, \underline{m}) \in (\mathbb{N} \cup \{\infty\})^{1+d}$, we denote by $\widehat{B}_{n,\underline{m}}^{(\underline{N})}$ the p-adic completion of $B_{n,\underline{m}}^{(\underline{N})}$, and we set

$$\widetilde{B}_{n,\underline{m}}^{(\underline{N})} = \operatorname*{colim}_{(n',\underline{m}') \in (\mathbb{N}^{1+d})_{\leq (n,\underline{m})}} \widehat{B}_{n',\underline{m}'}^{(\underline{N})}. \tag{II.10.1.2}$$

We remark that the transition maps in the colimit of (II.10.1.2) are closed embeddings with respect to the p-adic topology, and that $\widetilde{B}_{n,\underline{m}}^{(\underline{N})}$ identifies with a topological subring of $\widehat{B}_{n,\underline{m}}^{(\underline{N})}$ (both endowed with the p-adic topology) by II.4.7. We name some Galois groups as indicated in the following diagram:

(II.10.1.3)



Lemma II.10.2. *For any $\underline{N} \in J$, the Kummer tower $(B_{n,\underline{m}}^{(\underline{N})})_{(n,\underline{m}) \in \mathbb{N}^{1+d}}$ satisfies the condition* II.4.19.(1).

Proof. For any $\mathfrak{p} \in \mathfrak{S}_p(B^{(\underline{N})})$, if we denote by E the completion of the discrete valuation field $B_{\mathfrak{p}}^{(\underline{N})}[1/p]$, then we need to show that the Kummer tower $(\mathcal{O}_{E_{n,\underline{m}}})_{(n,\underline{m}) \in \mathbb{N}^{1+d}}$ (defined by $\zeta_{p^n}, t_{1,p^n}, \ldots, t_{d,p^n}$) satisfies the condition II.4.19.(1). It suffices to check that $\mathrm{d}t_1, \ldots, \mathrm{d}t_d$ form an E-basis of $\widehat{\Omega}^1_{B_{\mathfrak{p}}^{(\underline{N})}}[1/p] = \widehat{\Omega}^1_{\mathcal{O}_E}[1/p]$ by II.5.22. Since $B^{(\underline{N})}$ is a quasi-adequate $\mathcal{O}_{K^{(\underline{N})}}$-algebra with a system of coordinates $t_{1,N_1}, \ldots, t_{d,N_d}$ by II.9.15.(1), the conclusion follows from II.9.11. □

Proposition II.10.3. *There exists $n_0 \in \mathbb{N}$ such that the following statements hold for any $\underline{N} \in J$ and $(n, \underline{m}) \in \mathbb{N}_{\geq n_0}^{1+d}$:*

(1) *We have $\mathcal{L}_{\infty,\underline{\infty}}^{(\underline{N})} = \mathcal{L}_{n,\underline{m}}^{(\underline{N})} \otimes_{\mathcal{K}_{n,\underline{m}}} \mathcal{K}_{\infty,\underline{\infty}}$.*

(2) *The cyclotomic character* (II.4.15.2) *$\chi : G \to \mathbb{Z}_p^\times$ (describing the action on ζ_{p^n}) and the p-adic logarithm map* (II.3.6.3) *$\log : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ induce an isomorphism*

$$\log \circ \chi : \Sigma_{n,\underline{\infty}}^{(\underline{N})} \xrightarrow{\sim} p^n \mathbb{Z}_p. \tag{II.10.3.1}$$

(3) *The continuous 1-cocycle* (II.4.15.4) *$\xi : G \to \mathbb{Z}_p^d$ (describing the action on $t_{1,p^n}, \ldots, t_{d,p^n}$) induces an isomorphism*

$$\xi : \Delta_{\underline{m}}^{(\underline{N})} \xrightarrow{\sim} p^{m_1}\mathbb{Z}_p \times \cdots \times p^{m_d}\mathbb{Z}_p \tag{II.10.3.2}$$

where $\underline{m} = (m_1, \ldots, m_d)$.

(4) *The natural map $\mathfrak{S}_p(B_{n,\underline{m}}^{(\underline{N})}) \to \mathfrak{S}_p(B_{n_0,\underline{n_0}}^{(\underline{N})})$ is a bijection.*

Proof. Since the Kummer towers $(B_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ and $(A_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ both satisfy the condition II.4.19.(1) by II.10.2, there exists $n_0 \in \mathbb{N}$ such that the conditions II.4.19.(2, 3) hold for any $(n, \underline{m}) \in \mathbb{N}_{\geq n_0}^{1+d}$. In particular, the natural map $\mathrm{Gal}(\mathcal{L}_{\infty,\underline{\infty}}/\mathcal{L}_{n,\underline{m}}) \to \mathrm{Gal}(\mathcal{K}_{\infty,\underline{\infty}}/\mathcal{K}_{n,\underline{m}})$ is an isomorphism of pro-p groups. Thus, $\mathcal{L}_{\infty,\underline{\infty}} = \mathcal{L}_{n,\underline{m}} \otimes_{\mathcal{K}_{n,\underline{m}}} \mathcal{K}_{\infty,\underline{\infty}}$ by Galois theory. As $[\mathcal{L}_{n,\underline{m}}^{(\underline{N})} : \mathcal{L}_{n,\underline{m}}]$ is prime to p, we have $\mathcal{L}_{\infty,\underline{\infty}}^{(\underline{N})} = \mathcal{L}_{n,\underline{m}}^{(\underline{N})} \otimes_{\mathcal{L}_{n,\underline{m}}} \mathcal{L}_{\infty,\underline{\infty}}$, which implies (1). Then, we deduce easily the other statements by Galois theory. □

Lemma II.10.4. *For any $\underline{N} \in J$, the $\mathcal{O}_{K_\infty^{(\underline{N})}}$-algebra $B_{\infty,\underline{\infty}}^{(\underline{N})}$ is almost pre-perfectoid.*

Proof. As $K_\infty^{(\underline{N})}$ is a pre-perfectoid field (cf. the proof of II.13.1), the conclusion follows from the same argument of II.9.16 (where admitting roots in the prime-to-p part is unnecessary). □

Proposition II.10.5. *For any $\underline{N} \in J$, there exists $k_0 \in \mathbb{N}$ such that the natural map*

$$B_{n,\underline{m}}^{(\underline{N})} \otimes_{A_{n,\underline{m}}^{(\underline{N})}} A_{n',\underline{m'}}^{(\underline{N})} \longrightarrow B_{n',\underline{m'}}^{(\underline{N})} \tag{II.10.5.1}$$

is a p^{k_0}-isomorphism for any elements $(n', \underline{m'}) \geq (n, \underline{m})$ in $(\mathbb{N}_{\geq n_0} \cup \{\infty\})^{1+d}$, where $n_0 \in \mathbb{N}$ is defined in II.10.3.

Proof. It follows from the same argument of II.9.18 using II.10.4 instead of II.9.16. □

Lemma II.10.6 (cf. [**Tsu18**, 14.11])**.** *For any $\underline{N} \in J$, $n \in \mathbb{N}_{\geq n_0} \cup \{\infty\}$ and any elements $\underline{m'} \geq \underline{m}$ in $(\mathbb{N}_{\geq n_0} \cup \{\infty\})^d$ with $\underline{m'} \in \underline{m} + \mathbb{N}^d \subseteq (\mathbb{N} \cup \{\infty\})^d$, where $n_0 \in \mathbb{N}$ is defined in* II.10.3*, the natural homomorphism*

$$\widehat{B}_{n,\underline{m}}^{(\underline{N})} \otimes_{B_{n,\underline{m}}^{(\underline{N})}} B_{n,\underline{m'}}^{(\underline{N})}[\frac{1}{p}] \longrightarrow \widehat{B}_{n,\underline{m'}}^{(\underline{N})}[\frac{1}{p}] \tag{II.10.6.1}$$

is an isomorphism, and $B_{n,\underline{m'}}^{(\underline{N})}[1/p]$ is a finite free $B_{n,\underline{m}}^{(\underline{N})}[1/p]$-module.

Proof. Let $I = \{(k_1, \ldots, k_d) \in \mathbb{N}^d \mid 0 \leq k_i < p^{m_i' - m_i},\ 1 \leq i \leq d\}$. We deduce from the conclusion of II.8.14 (cf. II.9.14) for finite indexes that there exists $k_0 \in \mathbb{N}$ independent of $n, \underline{m}, \underline{m'}$ such that

$$p^{k_0} A_{n,\underline{m'}}^{(\underline{N})} \subseteq \bigoplus_{\underline{k}\in I} A_{n,\underline{m}}^{(\underline{N})} \cdot \prod_{i=1}^{d} t_{i,p^{m_i'}N_i}^{k_i} \subseteq p^{-k_0} A_{n,\underline{m'}}^{(\underline{N})}. \tag{II.10.6.2}$$

This shows that $A_{n,\underline{m'}}^{(\underline{N})}$ is p^{2k_0}-isomorphic to a finite free $A_{n,\underline{m}}^{(\underline{N})}$-module. The same result holds for $B_{n,\underline{m'}}^{(\underline{N})}$ over $B_{n,\underline{m}}^{(\underline{N})}$ by II.10.5, which completes the proof. □

Lemma II.10.7 (cf. [**Tsu18**, 14.10, 14.8])**.** *For any $\underline{N} \in J$, there exists $k_1 \in \mathbb{N}$ such that for any elements $(n, \underline{m}) \in (\mathbb{N}_{\geq n_0} \cup \{\infty\})^{1+d}$, where $n_0 \in \mathbb{N}$ is defined in* II.10.3*, the following statements hold:*

(1) *If $(n, \underline{m}) \in \mathbb{N}^{1+d}$, for any $r \in \mathbb{N}$, let σ be a generator of $\mathrm{Gal}(\mathcal{L}_{n+r,\underline{m}}^{(\underline{N})}/\mathcal{L}_{n,\underline{m}}^{(\underline{N})}) \cong \mathbb{Z}/p^r\mathbb{Z}$ by* (II.10.3.1)*, then we have*

$$B_{n+r,\underline{m}}^{(\underline{N})} \subseteq p^{-k_1}(B_{n,\underline{m}}^{(\underline{N})} + (\sigma - 1)(B_{n+r,\underline{m}}^{(\underline{N})})). \tag{II.10.7.1}$$

(2) *If the i-th component of $\underline{m}$ is an integer for some $1 \leq i \leq d$, for any $r \in \mathbb{N}$, let τ be a generator of* $\mathrm{Gal}(\mathcal{L}^{(\underline{N})}_{\infty,\underline{m}+\underline{r}_i}/\mathcal{L}^{(\underline{N})}_{\infty,\underline{m}}) \cong \mathbb{Z}/p^r\mathbb{Z}$ *by* (II.10.3.2)*, then we have*

$$B^{(\underline{N})}_{\infty,\underline{m}+\underline{r}_i} \subseteq p^{-k_1}(B^{(\underline{N})}_{\infty,\underline{m}} + (\tau-1)(B^{(\underline{N})}_{\infty,\underline{m}+\underline{r}_i})). \tag{II.10.7.2}$$

Proof. The conclusion holds for the case $B = A$ by [**Tsu18**, 14.8, 14.10]. It remains true in general due to II.10.5. □

Definition II.10.8 (cf. [**Tsu18**, 7.4, 7.5, 7.6])**.** We say that a tower $(R_n)_{n\in\mathbb{N}}$ of normal domains flat over $\mathcal{O}_K$ (see II.4.5) is *Tate-Sen* if it satisfies the following conditions:

(1) There exists a Noetherian normal domain R_{-1} over $\mathcal{O}_K$ contained in R_0 with $R_{-1}/pR_{-1} \neq 0$ such that R_0 is integral over R_{-1} (thus II.4.4 applies to $R_{-1} \to R_n$ for any $n \in \mathbb{N} \cup \{\infty\}$).
(2) The tower $(R_n)_{n\in\mathbb{N}}$ is a $\mathbb{Z}_p$-tower in the following sense: if $\mathcal{E}_n$ denotes the fraction field of R_n, then $\mathcal{E}_\infty$ is a Galois extension of $\mathcal{E}_0$ with Galois group isomorphic to $\mathbb{Z}_p$ and $\mathcal{E}_n$ is the $p^n\mathbb{Z}_p$-invariant part of $\mathcal{E}_\infty$ for any $n \in \mathbb{N}$.
(3) There exists $k_1 \in \mathbb{N}$ such that for any $n, r \in \mathbb{N}$ and any generator $\sigma \in \mathrm{Gal}(\mathcal{E}_{n+r}/\mathcal{E}_n)$, we have

$$R_{n+r} \subseteq p^{-k_1}(R_n + (\sigma-1)(R_{n+r})). \tag{II.10.8.1}$$

(4) For any $n \in \mathbb{N}$, $\widehat{R_n}[1/p]$ is finite over $\widehat{R_0}[1/p]$.
(5) The set $\mathfrak{S}_p(R_\infty)$ of primes ideals of R_∞ of height 1 containing p is finite.

Tsuji has established a series of decompletion results for Tate-Sen towers in [**Tsu18**, §7].

Proposition II.10.9 (cf. [**Tsu18**, 14.12])**.** *For any $\underline{N} \in J$ and $(n, \underline{m})$ in $(\mathbb{N}_{\geq n_0} \cup \{\infty\})^{1+d}$, where $n_0 \in \mathbb{N}$ is defined in* II.10.3*, the following statements hold:*

(1) *If $(n, \underline{m}) \in \mathbb{N}^{1+d}$, then the tower $(B^{(\underline{N})}_{n+r,\underline{m}})_{r\in\mathbb{N}}$ is Tate-Sen.*
(2) *If the i-th component of $\underline{m}$ is an integer for some $1 \leq i \leq d$, then the tower $(B^{(\underline{N})}_{\infty,\underline{m}+\underline{r}_i})_{r\in\mathbb{N}}$ is Tate-Sen.*

Proof. It follows from the fact that $B/pB \neq 0$, II.10.3, II.10.6, and II.10.7. □

Remark II.10.10. With the notation in II.5.1, the results of II.10.9 remain true for the Kummer tower $(\mathcal{O}_{K_{n,\underline{m}}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$. Indeed, one can check firstly for the subfield K' defined in II.5.2 with the aid of II.5.3, and then deduce the general case as we did above from A to B.

Lemma II.10.11 (cf. [**Tsu18**, 14.13])**.** *For any $\underline{N} \in J$, the following statements hold:*

(1) *The $(\Xi^{(\underline{N})}, \widehat{B}[1/p])$-finite part of $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$ is $\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$ (see* II.2.4*).*
(2) *Let V be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p])$. Then, V is the $(\Xi^{(\underline{N})}, \widehat{B}[1/p])$-finite part of $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}} V$.*

Proof. We follow the proof of [**Tsu18**, 14.13] and take $n_0 \in \mathbb{N}$ defined in II.10.3.

(1) Notice that for any $(n, \underline{m}) \in \mathbb{N}^{1+d}$, $B^{(\underline{N})}_{n,\underline{m}}$ is finite over B. Thus, $\widehat{B}^{(\underline{N})}_{n,\underline{m}}[1/p]$ is a $\Xi^{(\underline{N})}$-stable finitely generated $\widehat{B}[1/p]$-submodule of $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$ as B is Noetherian. Thus, $\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$ lies in the $(\Xi^{(\underline{N})}, \widehat{B}[1/p])$-finite part of $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$.

For the converse, let M be a $\Xi^{(\underline{N})}$-stable finitely generated $\widehat{B}[1/p]$-submodule of $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$. For any $1 \leq i \leq d$ and $\underline{m} \in \mathbb{N}^i_{\geq n_0} \times \{\infty\}^{d-i}$, applying [**Tsu18**, 7.14] to the Tate-Sen tower $(B^{(\underline{N})}_{\infty,\underline{m}+\underline{r}_i})_{r\in\mathbb{N}}$ (II.10.9.(2)), we see that the condition $M \subseteq \widehat{B}^{(\underline{N})}_{\infty,\underline{m}+\underline{\infty}_i}[\frac{1}{p}]$ implies that $M \subseteq \widehat{B}^{(\underline{N})}_{\infty,\underline{m}+\underline{r}_i}[\frac{1}{p}]$ for some $r \in \mathbb{N}$. Applying this argument in the order $i = 1, \ldots, d$, we obtain an element $\underline{m} \in \mathbb{N}^d_{\geq n_0}$ such that $M \subseteq \widehat{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p]$. Then, applying [**Tsu18**, 7.14] to the Tate-Sen tower $(B^{(\underline{N})}_{n_0+r,\underline{m}})_{r\in\mathbb{N}}$ (II.10.9.(1)), we obtain an element $n \in \mathbb{N}_{\geq n_0}$ such that $M \subseteq \widehat{B}^{(\underline{N})}_{n,\underline{m}}[1/p]$. This proves the converse part.

(2) Since $\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}$ is the colimit of $\widehat{B}^{(\underline{N})}_{n,\underline{m}}$ and $\Xi^{(\underline{N})}$ is topologically finitely generated (cf. II.10.3), there exists $(n, \underline{m}) \in \mathbb{N}^{1+d}_{\geq n_0}$ and an object V' of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widehat{B}^{(\underline{N})}_{n,\underline{m}}[1/p])$ such that $V = \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widehat{B}^{(\underline{N})}_{n,\underline{m}}} V'$ ([**Tsu18**, 5.2.(1)]). As $\widehat{B}^{(\underline{N})}_{n,\underline{m}}$ is finite over B, by (1) and [**Tsu18**, 7.3.(3)], we see that the $(\Xi^{(\underline{N})}, \widehat{B}[1/p])$-finite part of $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widehat{B}^{(\underline{N})}_{n,\underline{m}}} V'$ is $\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widehat{B}^{(\underline{N})}_{n,\underline{m}}} V'$. □

Proposition II.10.12 (cf. [**Tsu18**, 14.15]). *For any $\underline{N} \in J$, the functor*

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[\frac{1}{p}]) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[\frac{1}{p}]),\ V \mapsto \widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}} V \tag{II.10.12.1}$$

is an equivalence of categories, and a quasi-inverse is obtained by taking the $(\Xi^{(\underline{N})}, \widehat{B}[1/p])$-finite part.

Proof. We follow the proof of [**Tsu18**, 14.15]. Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p])$. For any $1 \leq i \leq d$ and $\underline{m} \in \mathbb{N}^i_{\geq n_0} \times \{\infty\}^{d-i}$, we can apply [**Tsu18**, 7.23] to the Tate-Sen tower $(B^{(\underline{N})}_{\infty,\underline{m}+\underline{r}_i})_{r\in\mathbb{N}}$ (II.10.9.(2)). Applying this argument to W in the order $i = 1, \ldots, d$, we obtain an object W' of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widehat{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p])$ for some $\underline{m} \in \mathbb{N}^d_{\geq n_0}$ such that $W = \widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widehat{B}^{(\underline{N})}_{\infty,\underline{m}}} W'$. Then, applying [**Tsu18**, 7.24] to the Tate-Sen tower $(B^{(\underline{N})}_{n_0+r,\underline{m}})_{r\in\mathbb{N}}$ (II.10.9.(1)), we obtain an object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widehat{B}^{(\underline{N})}_{n,\underline{m}}[\frac{1}{p}])$ for some $n \in \mathbb{N}_{\geq n_0}$ such that $W' = \widehat{B}^{(\underline{N})}_{\infty,\underline{m}} \otimes_{\widehat{B}^{(\underline{N})}_{n,\underline{m}}} V$. This shows that the functor (II.10.12.1) is essentially surjective. The conclusion follows from II.10.11.(2). □

Proposition II.10.13 (cf. [**Tsu18**, 14.16]). *For any $\underline{N} \in J$ and $\underline{m} \in \mathbb{N}^d_{\geq n_0}$, where $n_0 \in \mathbb{N}$ is defined in* II.10.3*, the functor (cf.* II.4.14*)*

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta^{(\underline{N})}_{\underline{m}}\text{-an}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[\frac{1}{p}]) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[\frac{1}{p}]),\ V \mapsto \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V \tag{II.10.13.1}$$

is fully faithful. Moreover, any object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p])$ lies in the essential image of the above functor for some $\underline{m} \in \mathbb{N}^d_{\geq n_0}$.

Proof. We follow the proof of [**Tsu18**, 14.16]. For any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p])$, we consider the $\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$-linear endomorphism φ_τ on W given by the infinitesimal action of $\tau \in \Delta^{(\underline{N})}$, which is nilpotent by II.4.18 (whose assumptions are satisfied by II.10.3). Then, we obtain a continuous semi-linear action $\overline{\rho}$ of $\Delta^{(\underline{N})}$ on W defined by $\overline{\rho}(\tau) = \exp(-\varphi_\tau)\rho(\tau)$ as in II.5.28.(1) (in fact, $\overline{\rho}$ can be extended to $\Gamma^{(\underline{N})}$, but we don't need this). Notice that if $W = \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$ for some object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta^{(\underline{N})}_{\underline{m}}\text{-an}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p])$, then $\overline{\rho}|_{\Delta^{(\underline{N})}_{\underline{m}}}$ acts trivially on V, which implies that

$$(W, \overline{\rho})^{\Delta^{(\underline{N})}_{\underline{m}}} = (\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}})^{\Delta^{(\underline{N})}_{\underline{m}}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V = V, \tag{II.10.13.2}$$

where the last identity follows from II.10.6. This shows that (II.10.13.1) is fully faithful.

Since $\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}$ is the filtered colimit of $\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}$, there exists $\underline{m} \in \mathbb{N}^d_{\geq n_0}$ and an object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Delta^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p])$ such that $(W, \overline{\rho}) = \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$ ([**Tsu18**, 5.2.(1)]). Moreover, since any $w \in W$ is fixed by an open subgroup of $\Delta^{(\underline{N})}$ via $\overline{\rho}$ by II.4.10, after enlarging $\underline{m}$, we may assume that $\overline{\rho}|_{\Delta^{(\underline{N})}_{\underline{m}}}$ acts trivially on V. By the discussion above, we have $(W, \overline{\rho})^{\Delta^{(\underline{N})}_{\underline{m}}} = V$. We claim that V is $\Xi^{(\underline{N})}$-stable under ρ. Indeed, for any $g \in \Xi^{(\underline{N})}$, $\tau \in \Delta^{(\underline{N})}_{\underline{m}}$ and $v \in V$, if we set $\tau' = g^{-1}\tau g \in \Delta^{(\underline{N})}_{\underline{m}}$, then

$$\rho(\tau)(\rho(g)v) = \rho(g)(\rho(\tau')v) = \rho(g)(\exp(\varphi_{\tau'})v) = \exp(\varphi_\tau)(\rho(g)v), \tag{II.10.13.3}$$

where the second equality follows from $\overline{\rho}(\tau')(v) = v$, and the last equality follows from II.4.9.(1). This shows that $\rho(g)v \in V = (W, \overline{\rho})^{\Delta^{(\underline{N})}_{\underline{m}}}$, and hence V is $\Xi^{(\underline{N})}$-stable under ρ. As $\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p] \to \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$ is a closed embedding (II.4.7), so is $V \to W$, which implies that $\Xi^{(\underline{N})}$ acts continuously on V. Moreover, V is $\Delta^{(\underline{N})}_{\underline{m}}$-analytic by definition, which completes the proof. □

Lemma II.10.14 (cf. [**Tsu18**, 14.5]). *Let $\mathcal{L}', \mathcal{L}''$ be two finite extensions of $\mathcal{L}_{\infty,\underline{\infty}}$ in $\mathcal{L}_{\mathrm{ur}}$ with $\mathcal{L}' \subseteq \mathcal{L}''$, B' and B'' the integral closures of B in $\mathcal{L}'$ and $\mathcal{L}''$ respectively. Then, the inclusion $\mathrm{Tr}_{\mathcal{L}''/\mathcal{L}'}(B'') \subseteq B'$ is almost surjective.*

Proof. Consider $\mathcal{L}^{(\infty)} = \mathrm{colim}_{\underline{N}\in J} \mathcal{L}^{(\underline{N})}$, and let $B'^{(\infty)}$ (resp. $B''^{(\infty)}$) be the integral closures of B in $\mathcal{L}'^{(\infty)} = \mathcal{L}'\mathcal{L}^{(\infty)}$ (resp. $\mathcal{L}''^{(\infty)} = \mathcal{L}''\mathcal{L}^{(\infty)}$). Notice that $B''^{(\infty)}$ is almost finite étale over $B'^{(\infty)}$ by II.9.23. In particular, $\mathrm{Tr}_{\mathcal{L}''^{(\infty)}/\mathcal{L}'^{(\infty)}}(B''^{(\infty)}) \subseteq B'^{(\infty)}$ is almost surjective ([**AGT16**, V.7.12]). Thus, for any $x \in B'$ and $\pi \in \mathfrak{m}_{K_\infty}$, there exists $\underline{N} \in J$ and $y \in B''^{(\underline{N})}$ such that $\pi x = \mathrm{Tr}_{\mathcal{L}''^{(\underline{N})}/\mathcal{L}'^{(\underline{N})}}(y)$. Notice that $l = [\mathcal{L}'^{(\underline{N})} : \mathcal{L}']$ is prime to p. We take $y' = l^{-1}\mathrm{Tr}_{\mathcal{L}''^{(\underline{N})}/\mathcal{L}''}(y) \in B''$. Thus, $\pi x = \mathrm{Tr}_{\mathcal{L}''/\mathcal{L}'}(y')$. □

Proposition II.10.15 (cf. [**Tsu18**, 14.7]). *For any $\underline{N} \in J$, the functor*

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[\frac{1}{p}]) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[\frac{1}{p}]),\ V \mapsto \widehat{\overline{B}} \otimes_{\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}} V \tag{II.10.15.1}$$

is fully faithful. Moreover, any object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$ *lies in the essential image of the above functor for some* $\underline{N} \in J$.

Proof. We follow the proof of [**Tsu18**, 14.7]. Firstly, $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}} \to (\widehat{\overline{B}})^{H^{(\underline{N})}_{\underline{\infty}}}$ is an almost isomorphism by almost Galois descent [**Tsu18**, 6.4] (whose assumptions [**Tsu18**, 6.1, 6.2] are satisfied by II.10.14). It follows immediately that the functor (II.10.15.1) is fully faithful (cf. [**Tsu18**, 6.5]). For an object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$, by almost Galois descent [**Tsu18**, 6.10.(1)], there exists an open subgroup H' of $H_{\underline{\infty}}$ such that for the integral closure B' of B in $\mathcal{L}' = \mathcal{L}^{H'}_{\mathrm{ur}}$, $W^{H'}$ is a finite projective $\widehat{B'}[1/p]$-module such that $W = \widehat{\overline{B}} \otimes_{\widehat{B'}} W^{H'}$. This remains true for any open subgroup of H' by [**Tsu18**, 6.10.(2)] so that we may assume that $\mathcal{L}'$ is Galois over $\mathcal{L}_{\infty,\underline{\infty}}$. By Abhyankar's lemma II.9.22, there exists $\underline{N} \in J$ such that $B'^{(\underline{N})}[1/p]$ is finite étale over $B^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$ (cf. I.8.21). After replacing $\mathcal{L}'$ by $\mathcal{L}'^{(\underline{N})}$, we may assume that $B'[1/p]$ is finite étale and Galois over $B^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$. Thus, $\widehat{B'}[1/p]$ is also finite étale and Galois over $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$ by [**Tsu18**, 6.15]. By Galois descent, $W^{H^{(\underline{N})}_{\underline{\infty}}}$ is a finite projective $\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p]$-representation of $\Xi^{(\underline{N})}$ such that $W = \widehat{\overline{B}} \otimes_{\widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}}} W^{H^{(\underline{N})}_{\underline{\infty}}}$. □

Theorem II.10.16 (cf. [**Tsu18**, 14.2])**.** *For any* $\underline{N} \in J$ *and* $\underline{m} \in \mathbb{N}^d_{\geq n_0}$*, where* $n_0 \in \mathbb{N}$ *is defined in* II.10.3*, the functor*

$$\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta^{(\underline{N})}_{\underline{m}}\text{-an}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[\frac{1}{p}]) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[\frac{1}{p}]),\ V \mapsto \widehat{\overline{B}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V \tag{II.10.16.1}$$

is fully faithful. Moreover, any object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$ *lies in the essential image of the above functor for some* $\underline{N} \in J$ *and* $\underline{m} \in \mathbb{N}^d_{\geq n_0}$.

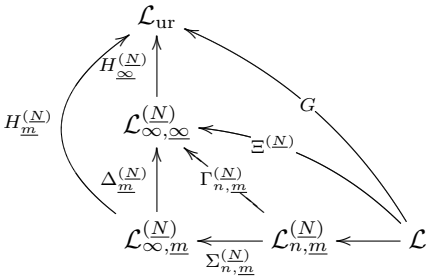
Proof. It follows from II.10.15, II.10.12 and II.10.13. □

II.11. Sen Operators over Quasi-adequate Algebras

In this section, we fix a complete discrete valuation field K of characteristic 0 with perfect residue field of characteristic $p > 0$, an algebraic closure $\overline{K}$ of K, and a quasi-adequate $\mathcal{O}_K$-algebra B of relative dimension d with fraction field $\mathcal{L}$ (see II.9.5). Let Y_K denote the log scheme with underlying scheme $\mathrm{Spec}(B[1/p])$ with compactifying log structure associated to $\mathrm{Spec}(B_{\mathrm{tr}}) \to \mathrm{Spec}(B[1/p])$.

II.11.1. In the following subsections, we introduce some notation that will be used in our construction II.11.4 of Sen operators. We fix a compatible system of primitive n-th roots of unity $(\zeta_n)_{n\in\mathbb{N}}$ contained in $\overline{K}$. As in II.5.20, we fix $e \in \mathbb{N}$ and let $t_1, \ldots, t_e$ be finitely many elements of $B[1/p] \cap B^{\times}_{\mathrm{tr}}$ with compatible systems of k-th roots $(t_{i,k})_{k\in\mathbb{N}_{>0}}$ contained in $\overline{B}[1/p]$ for any $1 \leq i \leq e$. We define the tower $(B^{(\underline{N})}_{n,\underline{m}})_{(\underline{N},n,\underline{m})\in J\times\mathbb{N}\times\mathbb{N}^e}$ and name the Galois groups as in II.10.1:

(II.11.1.1)



We remark that for any $\underline{N} \in J$, the system $(B^{(\underline{N})}_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+d}}$ is the Kummer tower of $B^{(\underline{N})}$ defined by $\zeta_{p^n}, t_{1,p^n}, \ldots, t_{e,p^n}$ (cf. II.4.16).

Proposition II.11.2 (cf. II.5.22)**.** *With the notation in* II.11.1*, the following conditions are equivalent:*

(1) *The Kummer tower* $(B_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+e}}$ *satisfies the condition* II.4.19.(1).
(2) *The* e *elements* $\mathrm{d}t_1, \ldots, \mathrm{d}t_e$ *of* $\Omega^1_{\mathcal{L}/K}$ *are linearly independent over* $\mathcal{L}$.

Proof. For any $\mathfrak{p} \in \mathfrak{S}_p(B)$, let $E_{\mathfrak{p}}$ be the completion of $\mathcal{L}$ with respect to the discrete valuation ring $B_{\mathfrak{p}}$. Recall that the $\mathcal{L}$-module $\Omega^1_{\mathcal{L}/K}$ and the $E_{\mathfrak{p}}$-module $\widehat{\Omega}^1_{B_{\mathfrak{p}}}[1/p]$ are both finite free with the same basis $\mathrm{d}t'_1, \ldots, \mathrm{d}t'_d$ given by a system of coordinates $t'_1, \ldots, t'_d \in B[1/p]$ of B by (II.8.9.1) and

II.9.11 respectively. In particular, the natural map $\Omega^1_{B_\mathfrak{p}/\mathcal{O}_K} \to \widehat{\Omega}^1_{B_\mathfrak{p}/\mathcal{O}_K}$ induces a natural isomorphism by inverting p,

$$E_\mathfrak{p} \otimes_{\mathcal{L}} \Omega^1_{\mathcal{L}/K} \xrightarrow{\sim} \widehat{\Omega}^1_{B_\mathfrak{p}}[\frac{1}{p}] = \widehat{\Omega}^1_{B_\mathfrak{p}/\mathcal{O}_K}[\frac{1}{p}]. \tag{II.11.2.1}$$

Thus, $\mathrm{d}t_1, \ldots, \mathrm{d}t_e$ are $\mathcal{L}$-linearly independent in $\Omega^1_{\mathcal{L}/K}$ if and only if they are $E_\mathfrak{p}$-linearly independent in $\widehat{\Omega}^1_{B_\mathfrak{p}}[1/p]$. The conclusion follows from II.5.22. □

II.11.3. Following II.11.1, we assume that the equivalent conditions in II.11.2 hold. Let $\partial_0 \in \mathrm{Lie}(\Sigma_{0,\underline{\infty}})$ and $\partial_1, \ldots, \partial_e \in \mathrm{Lie}(\Delta)$ be the standard bases defined in II.4.20 for the Kummer tower $(B_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+e}}$ defined by $\zeta_{p^n}, t_{1,p^n}, \ldots, t_{e,p^n}$. We remark that there are natural identifications of Lie algebras for any $\underline{N} \in J$, $n \in \mathbb{N}$ and $\underline{m} \in \mathbb{N}^e$,

$$\mathrm{Lie}(\Delta^{(\underline{N})}_{\underline{m}}) = \mathrm{Lie}(\Delta), \quad \mathrm{Lie}(\Xi^{(\underline{N})}) = \mathrm{Lie}(\Gamma), \quad \mathrm{Lie}(\Sigma^{(\underline{N})}_{n,\underline{m}}) = \mathrm{Lie}(\Sigma). \tag{II.11.3.1}$$

We define $1+e$ elements of the finite projective $\widehat{\overline{B}}[1/p]$-module $\mathscr{E}_B(-1)$ defined in II.9.36,

(II.11.3.2)
$$T_0 = (\mathrm{d}\log(\zeta_{p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1},\ T_1 = (\mathrm{d}\log(t_{1,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1},\ \ldots,\ T_e = (\mathrm{d}\log(t_{e,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1},$$

where $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}}$.

Theorem II.11.4 (cf. II.5.35)**.** *Let B be a quasi-adequate $\mathcal{O}_K$-algebra with fraction field $\mathcal{L}$, $G = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L})$. Then, for any object W of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$, there is a canonical homomorphism of $\widehat{\overline{B}}[1/p]$-linear Lie algebras (see* II.9.39*),*

$$\varphi_{\mathrm{Sen}}|_W : \mathscr{E}^*_B(1) \longrightarrow \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W), \tag{II.11.4.1}$$

which is G-equivariant with respect to the canonical action on $\mathscr{E}^*_B(1)$ defined in II.9.39 *and the adjoint action on $\mathrm{End}_{\widehat{\overline{B}}[1/p]}(W)$ (i.e. $g \in G$ sends an endomorphism ϕ to $g \circ \phi \circ g^{-1}$), and functorial in W, i.e. it defines a canonical functor*

$$\varphi_{\mathrm{Sen}} : \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[\frac{1}{p}]) \longrightarrow \mathbf{Rep}^{\mathrm{proj}}(\mathscr{E}^*_B(1), \widehat{\overline{B}}[\frac{1}{p}]), \tag{II.11.4.2}$$

from the category of finite projective (continuous semi-linear) $\widehat{\overline{B}}[1/p]$-representations of the profinite group G to the category of finite projective $\widehat{\overline{B}}[1/p]$-linear representations of the Lie algebra $\mathscr{E}^*_B(1)$.

Moreover, under the assumption in II.11.3 *and with the same notation, assume that there exists an object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta^{(\underline{N})}_{\underline{m}}\text{-an}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p])$ for some $\underline{N} \in J$ and $\underline{m} \in \mathbb{N}^d$ with $W = \widehat{\overline{B}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$. Then, for any $f \in \mathscr{E}^*_B(1) = \mathrm{Hom}_{\widehat{\overline{B}}[1/p]}(\mathscr{E}_B(-1), \widehat{\overline{B}}[1/p])$,*

$$\varphi_{\mathrm{Sen}}|_W(f) = \sum_{i=0}^{e} f(T_i) \otimes \varphi_{\partial_i}|_V, \tag{II.11.4.3}$$

where $\varphi_{\partial_i}|_V \in \mathrm{End}_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p]}(V)$ is the infinitesimal action of $\partial_i \in \mathrm{Lie}(\Xi^{(\underline{N})})$ on V defined in II.4.13.

Proof. For any $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$ with image $\mathfrak{p} \in \mathfrak{S}_p(B)$, consider the element $(B_{\mathrm{tr}}, B, \overline{B}) \to (E_\mathfrak{p}, \mathcal{O}_{E_\mathfrak{p}}, \mathcal{O}_{\overline{E}_\mathfrak{q}})$ of $\mathfrak{E}(B)$ defined in II.9.3. Consider the diagram (cf. II.9.38)

$$\begin{array}{ccccc}
\mathscr{E}^*_B(1) & \longrightarrow & \prod_\mathfrak{q} \widehat{\overline{E}}_\mathfrak{q} \otimes_{\widehat{\overline{B}}} \mathscr{E}^*_B(1) & \xleftarrow{\sim} & \prod_\mathfrak{q} \mathscr{E}^*_{\mathcal{O}_{E_\mathfrak{p}}}(1) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (\alpha_\mathfrak{q})_\mathfrak{q}} & & \downarrow{\scriptstyle \varphi_{\mathrm{Sen}}} \\
\mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W) & \longrightarrow & \prod_\mathfrak{q} \widehat{\overline{E}}_\mathfrak{q} \otimes_{\widehat{\overline{B}}} \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W) & \xleftarrow{\sim} & \prod_\mathfrak{q} \mathrm{End}_{\widehat{\overline{E}}_\mathfrak{q}}(\widehat{\overline{E}}_\mathfrak{q} \otimes_{\widehat{\overline{B}}} W)
\end{array} \tag{II.11.4.4}$$

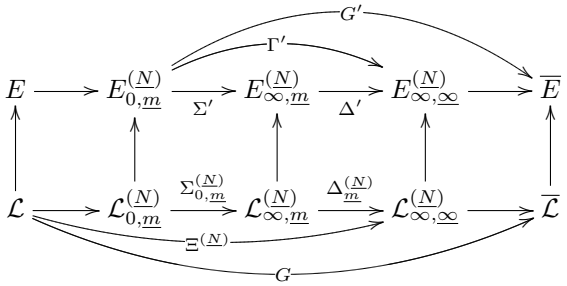
where the product is taken over $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$, and the right vertical arrow is the canonical Lie algebra action defined in II.5.35. Notice that the horizontal arrows in the right square are isomorphisms, since W is finite projective and both $\mathscr{E}_B$ and $\mathscr{E}_{\mathcal{O}_{E_\mathfrak{q}}}$ admit the same basis induced by a system of coordinates of B (cf. II.9.33). On the other hand, the horizontal arrows in the left square are injective by II.4.3. Therefore, there is at most one map α making this diagram commutative together with its base change $\alpha_\mathfrak{q}$.

We claim that taking α to be the map defined by (II.11.4.3) (under the corresponding assumptions) makes the diagram commute. Indeed, for any $f \in \mathscr{E}_B^*(1)$, if $f_{\mathfrak{q}} \in \mathscr{E}^*_{\mathcal{O}_{E_{\mathfrak{p}}}}(1)$ denotes its base change, then we need to check that

$$\varphi_{\mathrm{Sen}}|_{\widehat{E}_{\mathfrak{q}} \otimes_{\widehat{B}} W}(f_{\mathfrak{q}}) = \sum_{i=0}^{e} f_{\mathfrak{q}}(T_{i,\mathfrak{q}}) \otimes \varphi_{\partial_i}|_V, \tag{II.11.4.5}$$

where $T_{i,\mathfrak{q}} \in \mathscr{E}_{\mathcal{O}_{E_{\mathfrak{p}}}}(-1)$ is the image of the element $T_i \in \mathscr{E}_B(-1)$ defined in (II.11.3.2). For simplicity, we omit the subscripts $\mathfrak{p}$ and $\mathfrak{q}$. With the notation in II.11.1, there is a commutative diagram of fields defined by $t_{i,k}$ similarly as in II.10.1,

(II.11.4.6)



where we put the notation for corresponding Galois groups on the arrows. For $\underline{N}$ and $\underline{m}$ in the statement, we set $E' = E_{0,\underline{m}}^{(\underline{N})}$ and consider the Kummer tower $(\mathcal{O}_{E'_{n,\underline{m}'}})_{(n,\underline{m}')\in\mathbb{N}^{1+e}}$ defined by $\zeta_{p^n}, t_{1,p^n}, \dots, t_{e,p^n}$ (thus $E'_\infty = E_{\infty,\underline{m}}^{(\underline{N})}$ and $E'_{\infty,\underline{\infty}} = E_{\infty,\underline{\infty}}^{(\underline{N})}$), which satisfies the condition II.4.19.(1) by II.11.2 (as E' is a finite extension of E). Then, $W' = \widehat{\overline{E}} \otimes_{\widehat{\overline{B}}} W$ is an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G', \widehat{\overline{E}})$ and $V' = E'_\infty \otimes_{\widetilde{B}_{\infty,\underline{m}}^{(\underline{N})}} V$ is an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta'\text{-an}}(\Gamma', E'_\infty)$ such that $W' = \widehat{\overline{E}} \otimes_{E'_\infty} V'$. By II.11.2 and (II.11.3.1), we have natural identifications of Lie algebras

$$\mathrm{Lie}(\Delta') = \mathrm{Lie}(\Delta), \quad \mathrm{Lie}(\Gamma') = \mathrm{Lie}(\Gamma), \quad \mathrm{Lie}(\Sigma') = \mathrm{Lie}(\Sigma). \tag{II.11.4.7}$$

Moreover, the images of $\partial_0, \dots, \partial_e$ in $\mathrm{Lie}(\Gamma')$ are the standard basis $\partial'_0, \dots, \partial'_e$ defined in II.4.20 for the Kummer tower $(\mathcal{O}_{E'_{n,\underline{m}'}})_{(n,\underline{m}')\in\mathbb{N}^{1+e}}$ defined by $\zeta_{p^n}, t_{1,p^n}, \dots, t_{e,p^n}$. Therefore, applying the "moreover" part of II.5.35 to E' (whose assumptions are satisfied by II.11.2), for any $f' \in \mathscr{E}^*_{\mathcal{O}_{E'}}(1)$ we have

$$\varphi_{\mathrm{Sen}}|_{W'}(f') = \sum_{i=0}^{e} f'(T'_i) \otimes \varphi_{\partial'_i}|_{V'}, \tag{II.11.4.8}$$

where $T'_i \in \mathscr{E}_{\mathcal{O}_{E'}}(-1)$ is the image of $T_i \in \mathscr{E}_B(-1)$. Notice that $\varphi_{\partial'_i}|_{V'} = \mathrm{id}_{E'_\infty} \otimes \varphi_{\partial_i}|_V$ by II.4.11. The claim follows from the following canonical commutative diagram given by II.5.38 (as E' is a finite extension of E)

(II.11.4.9)

$$\begin{array}{ccc} \mathscr{E}^*_{\mathcal{O}_{E'}}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{\overline{E}}}(W') \\ \big\downarrow \wr & & \| \\ \mathscr{E}^*_{\mathcal{O}_E}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\widehat{\overline{E}} \otimes_{\widehat{\overline{B}}} W}} & \mathrm{End}_{\widehat{\overline{E}}}(\widehat{\overline{E}} \otimes_{\widehat{\overline{B}}} W) \end{array}$$

Finally, the uniqueness of α implies that (II.11.4.3) does not depend on choice of V or t_i, and the descent II.10.16 guarantees its existence and functoriality. Its G-equivariance follows from the same argument of that of (II.5.35.1). □

Remark II.11.5. The same argument also shows that the $\widehat{\overline{B}}[1/p]$-linear map

$$W \longrightarrow W \otimes_{\widehat{\overline{B}}} \mathscr{E}_B(-1) \tag{II.11.5.1}$$

sending x to $\sum_{i=0}^{e} (\mathrm{id}_{\widehat{\overline{B}}} \otimes \varphi_{\partial_i}|_V)(x) \otimes T_i$, is G-equivariant and does not depend on the choice of V or t_i. It naturally induces the map $\varphi_{\mathrm{Sen}}|_W$ (II.11.4.1). We note that it is not a Higgs field.

Definition II.11.6. Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$. We denote by $\Phi(W)$ the image of $\varphi_{\mathrm{Sen}}|_W$, and by $\Phi^{\mathrm{geo}}(W)$ the image of $\mathrm{Hom}_{B[1/p]}(\Omega^1_{Y_K/K}(-1), \widehat{\overline{B}}[1/p])$ under $\varphi_{\mathrm{Sen}}|_W$. We call an element of $\Phi(W) \subseteq \mathrm{End}_{\widehat{\overline{B}}[1/p]}(W)$ a *Sen operator* of W. We call an element of $\Phi^{\mathrm{geo}}(W) \subseteq \mathrm{End}_{\widehat{\overline{B}}[1/p]}(W)$ a *geometric Sen operator* of W. And we call the image of $1 \in \widehat{\overline{B}}[1/p]$ in $\Phi^{\mathrm{ari}}(W) = \Phi(W)/\Phi^{\mathrm{geo}}(W)$ the *arithmetic Sen operator* of W.

(II.11.6.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{B[\frac{1}{p}]}(\Omega^1_{Y_K/K}(-1), \widehat{\overline{B}}[\frac{1}{p}]) & \xrightarrow{\jmath^*} & \mathscr{E}^*_B(1) & \xrightarrow{\iota^*} & \widehat{\overline{B}}[\frac{1}{p}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Phi^{\mathrm{geo}}(W) & \longrightarrow & \Phi(W) & \longrightarrow & \Phi^{\mathrm{ari}}(W) & \longrightarrow & 0 \end{array}$$

Proposition II.11.7 (cf. II.5.38)**.** *Let K' be a complete discrete valuation field extension of K with perfect residue field, B' a quasi-adequate $\mathcal{O}_{K'}$-algebra with fraction field $\mathcal{L}'$. Consider a commutative diagram of $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triples (see* II.9.2*)*

(II.11.7.1)
$$\begin{array}{ccc} (B_{\mathrm{tr}}, B, \overline{B}) & \longrightarrow & (B'_{\mathrm{tr}}, B', \overline{B'}) \\ \uparrow & & \uparrow \\ (K, \mathcal{O}_K, \mathcal{O}_{\overline{K}}) & \longrightarrow & (K', \mathcal{O}_{K'}, \mathcal{O}_{\overline{K'}}) \end{array}$$

with $\overline{B} \to \overline{B'}$ injective. Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$, $W' = \widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} W$ the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G', \widehat{\overline{B'}}[1/p])$, where $G' = \mathrm{Gal}(\mathcal{L}'_{\mathrm{ur}}/\mathcal{L}')$. Assume that $\mathcal{L}' \otimes_{\mathcal{L}} \Omega^1_{\mathcal{L}/K} \to \Omega^1_{\mathcal{L}'/K'}$ is injective. Then, there is a natural commutative diagram

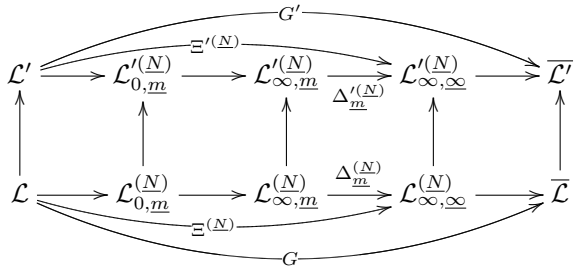
(II.11.7.2)
$$\begin{array}{ccc} \mathscr{E}^*_{B'}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{\overline{B'}}[\frac{1}{p}]}(W') \\ \downarrow & & \uparrow \wr \\ \widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} \mathscr{E}^*_B(1) & \xrightarrow{\mathrm{id}_{\widehat{\overline{B'}}} \otimes \varphi_{\mathrm{Sen}}|_W} & \widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W) \end{array}$$

where φ_{Sen} are the canonical Lie algebra actions defined in II.11.4*, the left vertical arrow is induced by taking dual of the natural map $\widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} \mathscr{E}_B(-1) \to \mathscr{E}_{B'}(-1)$ (cf.* II.9.38*), and the right vertical arrow is the canonical isomorphism. Moreover, if we denote by $\widehat{\overline{B'}}\Phi(W)$ the image of $\widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} \Phi(W)$ in $\widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W)$, then the inverse of the right vertical arrow induces a natural isomorphism*

(II.11.7.3)
$$\Phi(W') \xrightarrow{\sim} \widehat{\overline{B'}}\Phi(W),$$

which is compatible with geometric and arithmetic Sen operators.

Proof. We follow the same argument of II.5.38 using II.11.2 instead of II.5.22. We may assume that we are in the situation of the "moreover" part of II.11.4 by the descent theorem II.10.16. Let $t'_{i,k} \in \overline{B'}[1/p]$ be the image of $t_{i,k} \in \overline{B}[1/p]$. With the notation in II.11.1, there is a commutative diagram of fields

(II.11.7.4)


Since $\mathrm{d}t'_1, \ldots, \mathrm{d}t'_e$ are $\mathcal{L}'$-linearly independent in $\Omega^1_{\mathcal{L}'/K'}$ by assumption, the Kummer tower $(B'_{n,\underline{m}})_{(n,\underline{m})\in\mathbb{N}^{1+e}}$ defined by $\zeta_{p^n}, t'_{1,p^n}, \ldots, t'_{e,p^n}$ also satisfies the condition II.4.19.(1) by II.11.2. By the discussion in II.11.3, we have a natural isomorphism $\mathrm{Lie}(\Xi'^{(\underline{N})}) \xrightarrow{\sim} \mathrm{Lie}(\Xi^{(\underline{N})})$ which identifies their standard bases $\{\partial'_i\}_{0\le i\le e}$ and $\{\partial_i\}_{0\le i\le e}$. Moreover, $V' = \widetilde{B'}^{(\underline{N})}_{\infty,\underline{m}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$ is an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta'^{(\underline{N})}_{\underline{m}}\text{-an}}(\Xi'^{(\underline{N})}, \widetilde{B'}^{(\underline{N})}_{\infty,\underline{m}}[1/p])$ with $W' = \widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} W = \widehat{\overline{B'}} \otimes_{\widetilde{B'}^{(\underline{N})}_{\infty,\underline{m}}} V'$. By II.4.11, the natural identification $\mathrm{End}_{\widehat{\overline{B'}}}(W') = \widehat{\overline{B'}} \otimes_{\widehat{\overline{B}}} \mathrm{End}_{\widehat{\overline{B}}[1/p]}(W)$ identifies $\mathrm{id}_{\widehat{\overline{B'}}} \otimes \varphi_{\partial'_i}|_{V'}$ with $\mathrm{id}_{\widehat{\overline{B'}}} \otimes \varphi_{\partial_i}|_V$. This shows that the diagram (II.11.7.2) is commutative which induces an isomorphism (II.11.7.3). □

Remark II.11.8. One can also replace B' by a complete discrete valuation ring extension $\mathcal{O}_E$ of $\mathcal{O}_K$ whose residue field admits a finite p-basis. Assuming that $E \otimes_{\mathcal{L}} \Omega^1_{\mathcal{L}'/K} \to \widehat{\Omega}^1_{\mathcal{O}_E}[1/p]$ is injective, the result of II.11.7 still holds for $\mathcal{O}_E$.

Corollary II.11.9. *Let $\mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ be the directed system of finite field extensions $\mathcal{L}'$ of $\mathcal{L}$ in $\mathcal{L}_{\mathrm{ur}}$, $\mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ the subsystem consists of $\mathcal{L}'$ such that the integral closure B' of B in $\mathcal{L}'$ is a quasi-adequate $\mathcal{O}_K$-algebra, where we put $B'_{\mathrm{tr}} = B_{\mathrm{tr}} \otimes_B B'$ and $\overline{B'} = \overline{B}$.*

(1) *The subsystem $\mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ is cofinal in $\mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$.*

(2) *Let $\mathcal{L}' \in \mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ with Galois group $G' = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}') \subseteq G$, W an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$ and W' the object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G', \widehat{\overline{B}}[1/p])$ defined by restricting the G-action on W. Then, there is a natural commutative diagram*

$$\begin{array}{ccc} \mathscr{E}^*_{B'}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W'}} & \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W') \\ \big\downarrow \wr & & \| \\ \mathscr{E}^*_{B}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{W}} & \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W) \end{array} \tag{II.11.9.1}$$

Proof. (1) follows from Abhyankar's lemma II.9.22, and (2) follows from II.9.38 and II.11.7 (as $\mathcal{L}'$ is étale over $\mathcal{L}$). □

Remark II.11.10. Let $\mathcal{L}' \in \mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ with Galois group $G' = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}') \subseteq G$, $\mathcal{L}'' \in \mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ containing $\mathcal{L}'$ with Galois group $G'' = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}'') \subseteq G'$, W' an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G', \widehat{\overline{B}}[1/p])$, W'' the object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G'', \widehat{\overline{B}}[1/p])$ defined by restricting the G'-action on W'. Then, we define a Lie algebra action $\varphi_{\mathrm{Sen}}|_{W'}$ of $\mathscr{E}^*_B(1)$ on W' by assigning it to be $\varphi_{\mathrm{Sen}}|_{W''}$. This definition of $\varphi_{\mathrm{Sen}}|_{W'}$ does not depend on the choice of $\mathcal{L}''$ by II.11.9. One can check that $\varphi_{\mathrm{Sen}}|_{W'}$ is G'-equivariant by the same arguments in II.5.35.

Lemma II.11.11. *With the notation in* II.10.1*, let V be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont},\Delta^{(\underline{N})}_{\underline{m}}\text{-an}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}[1/p])$ for some $\underline{N} \in J$ and $\underline{m} \in \mathbb{N}^d$, $W = \widehat{\overline{B}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$ the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$.*

(1) *Let $V' = \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$ be the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Xi^{(\underline{N})}, \widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}}[1/p])$. Then, $W^G = V'^G$. In particular, $(\widehat{\overline{B}}[1/p])^G = \widehat{B}[1/p]$.*

(2) *Let $C = \mathrm{colim}_{\underline{m'} \in \mathbb{N}^d} \widehat{B}^{(\underline{N})}_{\infty,\underline{m'}}$, $V'' = C \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$ the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\Delta^{(\underline{N})}, C[1/p])$. Then, $W^H = V''^H$.*

Proof. (1) Recall that by the proof of II.10.15 we have $W^{H^{(\underline{N})}_{\infty}} = \widehat{B}^{(\underline{N})}_{\infty,\underline{\infty}} \otimes_{\widetilde{B}^{(\underline{N})}_{\infty,\underline{m}}} V$, and by II.10.11.(2) its $(\Xi^{(\underline{N})}, \widehat{B}[1/p])$-finite part is V'. Thus, we have $W^G \subseteq V'^G$ (and hence $W^G = V'^G$ as $V' \subseteq W$). In particular, we have $(\widehat{\overline{B}})^G \subseteq (\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}})^G$. Since $\widehat{B}^{(\underline{N})}_{n,\underline{m}} = \widehat{B} \otimes_B B^{(\underline{N})}_{n,\underline{m}}$ (as $B^{(\underline{N})}_{n,\underline{m}}$ is finite over the Noetherian ring B), we have $(\widetilde{B}^{(\underline{N})}_{\infty,\underline{\infty}})^G = \mathrm{colim}(\widehat{B}^{(\underline{N})}_{n,\underline{m}})^G = \mathrm{colim}\, \widehat{B} \otimes_B (B^{(\underline{N})}_{n,\underline{m}})^G = \widehat{B}$ (as $\widehat{B}$ is flat over B).

(2) The arguments of II.10.11 also show that $(\Delta^{(\underline{N})}, \widehat{B}^{(\underline{N})}_{\infty}[1/p])$-finite part of $W^{H^{(\underline{N})}_{\infty}}$ is V''. Thus, we have $W^H \subseteq V''^H$ (and hence $W^H = V''^H$ as $V'' \subseteq W$). □

Proposition II.11.12. *Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$. Then, any Sen operator of W vanishes on W^G, and any geometric Sen operator of W vanishes on W^H.*

Proof. By the descent theorem II.10.16, we may put ourselves in the situation of II.11.11. By II.4.10, the infinitesimal Lie algebra action of $\mathrm{Lie}(\Xi^{(\underline{N})})$ (resp. $\mathrm{Lie}(\Delta^{(\underline{N})})$) on V' (resp. V'') is well-defined and vanishes on $V'^{\Xi^{(\underline{N})}}$ (resp. $V''^{\Delta^{(\underline{N})}}$), and thus vanishes on W^G (resp. W^H) by II.11.11. On the other hand, the infinitesimal action of $\mathrm{Lie}(\Xi^{(\underline{N})})$ (resp. $\mathrm{Lie}(\Delta^{(\underline{N})})$) on V' (resp. V'') is the base change of that on V by II.4.11. Thus, these infinitesimal actions induce the (resp. geometric) Sen operators on W by extending scalars to $\widehat{\overline{B}}[1/p]$, which completes the proof. □

II.11.13. Recall that for any finite projective module M over a ring R, the trace map $\mathrm{Tr} : \mathrm{End}_R(M) \to R$ is the composition of the second map with the inverse of the first map:

$$\mathrm{End}_R(M) \xleftarrow{\sim} M \otimes_R \mathrm{Hom}_R(M, R) \longrightarrow R, \tag{II.11.13.1}$$

where for any $x \in M$ and $f \in \mathrm{Hom}_R(M, R)$, the element $x \otimes f$ is mapped to xf by the first arrow and to $f(x)$ by the second arrow. For any R-linear endomorphism ϕ of M, its determinant is defined as follows ([**Gol61**, §1]): we write $R^n = M \oplus M'$ for some $n \in \mathbb{N}$ and define $\det(\phi)$ to be the determinant of the endomorphism $\phi \oplus \mathrm{id}_{M'}$ on R^n, which does not depend on the choice of M' and is compatible with base change. Then, the characteristic polynomial $\det(T - \phi) \in R[T]$ of ϕ is defined as the determinant of the $R[T]$-linear endomorphism $T \otimes \mathrm{id}_M - \mathrm{id}_{R[T]} \otimes \phi$ of $R[T] \otimes_R M$ ([**Gol61**, §2]). Similarly, one can define the reverse characteristic polynomial $\det(1 - T\phi) \in R[T]$ as the determinant of the $R[T]$-linear endomorphism $\mathrm{id}_{R[T]} \otimes \mathrm{id}_M - T \otimes \phi$ of $R[T] \otimes_R M$. One can check by taking localizations of R that the reverse characteristic polynomial is also given by the formula (which is indeed a finite sum)

$$\det(1 - T\phi) = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}(\wedge^k \phi) T^k, \tag{II.11.13.2}$$

where $\wedge^k \phi : \wedge^k M \to \wedge^k M$ is the R-linear endomorphism of the k-th exterior power of M induced by ϕ. By the multiplicativity of determinants, we have $\det(T - \phi) = \det(T) \det(1 - T^{-1}\phi)$ in $R[T^{\pm 1}]$. We remark that the coefficients of the characteristic polynomial $\det(T)$ of the zero endomorphism are mutually orthogonal idempotents of R, and $\det(T)$ may not be a monomial if $\mathrm{Spec}(R)$ is not connected ([**Gol61**, 2.2, 2.3]).

Lemma II.11.14. *Let R be a ring endowed with an action of a group G, M a finite projective R-module endowed with a semi-linear action of G, ϕ an R-linear endomorphism of M. Then, for any $g \in G$, $\det(1 - T(g \circ \phi \circ g^{-1})) = g(\det(1 - T\phi))$.*

Proof. It is clear from the definitions that $\mathrm{Tr}(g \circ \phi \circ g^{-1}) = g(\mathrm{Tr}(\phi))$ and that $\wedge^k(g \circ \phi \circ g^{-1}) = g \circ (\wedge^k \phi) \circ g^{-1}$. The conclusion follows from (II.11.13.2). □

Proposition II.11.15. *Let W be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$. Then, any lifting $\phi \in \Phi(W)$ of the arithmetic Sen operator of W has the same (resp. reverse) characteristic polynomial. Moreover, the coefficients of the reverse characteristic polynomial lie in $\widehat{B}[1/p]$.*

Proof. The first assertion follows from the diagram (II.11.4.4) and II.5.41. For the second, notice that for any $g \in G$, $g \circ \phi \circ g^{-1}$ is also a lifting of the arithmetic Sen operator by the G-equivariance of $\varphi_{\mathrm{Sen}}|_W$. Thus, ϕ and $g \circ \phi \circ g^{-1}$ have the same (resp. reverse) characteristic polynomials. By II.11.14, we see that the coefficients of the reverse characteristic polynomial $\det(1 - T\phi)$ lie in $\widehat{\overline{B}}[1/p]^G = \widehat{B}[1/p]$ by II.11.11.(1). □

Remark II.11.16.
(1) If B is adequate, then the operator "φ_{Sen}" on the associated Higgs bundle defined by Tsuji ([**Tsu18**, page 876]) induces a lifting of the arithmetic Sen operator of W by extending scalars to $\widehat{\overline{B}}[1/p]$ (cf. [**Tsu18**, 15.1.(4)]). In particular, the characteristic polynomial of ϕ coincides with that of "φ_{Sen}". Thus, in general, we call the roots of the characteristic polynomial of ϕ the *Hodge-Tate weights* of W.
(2) If W is defined over $\widehat{B}[1/p]$, i.e. there exists an object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{B}[1/p])$ such that $W = \widehat{\overline{B}} \otimes_{\widehat{B}} V$, then the characteristic polynomial $\det(T)$ of the zero endomorphism of W has coefficients in $\widehat{B}[1/p]$. The identity $\det(T - \phi) = \det(T) \det(1 - T^{-1}\phi)$ in $\widehat{\overline{B}}[1/p][T^{\pm 1}]$ implies that the coefficients of the characteristic polynomial of ϕ also lie in $\widehat{B}[1/p]$.

II.11.17. Let $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$ with image $\mathfrak{p} \in \mathfrak{S}_p(B)$, consider the element $(B_{\mathrm{tr}}, B, \overline{B}) \to (E_{\mathfrak{p}}, \mathcal{O}_{E_{\mathfrak{p}}}, \mathcal{O}_{\overline{E}_{\mathfrak{q}}})$ of $\mathfrak{E}(B)$ defined in II.9.3. Let $I_{\mathfrak{q}} \subseteq G = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L})$ be the image of the inertia subgroup of the absolute Galois group of $E_{\mathfrak{p}}$. It is a closed subgroup of G, which we call the *inertia subgroup* of G at $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$.

Theorem II.11.18 (cf. II.5.43)**.** *Let (V, ρ) be an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \mathbb{Q}_p)$, $W = \widehat{\overline{B}}[1/p] \otimes_{\mathbb{Q}_p} V$ the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$. Then, $\sum_{\mathfrak{q} \in \mathfrak{S}_p(\overline{B})} \mathrm{Lie}(\rho(I_{\mathfrak{q}}))$ is the smallest $\mathbb{Q}_p$-subspace S of $\mathrm{End}_{\mathbb{Q}_p}(V)$ such that the $\widehat{\overline{B}}[1/p]$-module of Sen operators $\Phi(W)$ is contained in the submodule $\widehat{\overline{B}}[1/p] \otimes_{\mathbb{Q}_p} S$ of $\mathrm{End}_{\widehat{\overline{B}}[1/p]}(W) = \widehat{\overline{B}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p}(V)$.*

Proof. By II.5.42, it suffices to show that for any $\mathbb{Q}_p$-linear form f on $\mathrm{End}_{\mathbb{Q}_p}(V)$, we have $f(\mathrm{Lie}(\rho(I_{\mathfrak{q}}))) = 0$ for any $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$ if and only if $f'(\Phi(W)) = 0$, where f' is the $\widehat{\overline{B}}[1/p]$-linear form on $\mathrm{End}_{\widehat{\overline{B}}[1/p]}(W)$ defined by extending scalars from f. We set $W_{\mathfrak{q}} = \widehat{\overline{E}}_{\mathfrak{q}} \otimes_{\widehat{\overline{B}}} W = \widehat{\overline{E}}_{\mathfrak{q}} \otimes_{\mathbb{Q}_p} V$ and let $f'_{\mathfrak{q}}$

be the $\widehat{\overline{E}}_{\mathfrak{q}}$-linear form on $\mathrm{End}_{\widehat{\overline{E}}_{\mathfrak{q}}}(W_{\mathfrak{q}})$ defined by extending scalars from f. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_{\widehat{\overline{B}}[1/p]}(W) & \xrightarrow{(\iota_{\mathfrak{q}})_{\mathfrak{q}}} & \prod_{\mathfrak{q}\in\mathfrak{S}_p(\overline{B})} \mathrm{End}_{\widehat{\overline{E}}_{\mathfrak{q}}}(W_{\mathfrak{q}}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle (f'_{\mathfrak{q}})_{\mathfrak{q}}} \\ \widehat{\overline{B}}[1/p] & \longrightarrow & \prod_{\mathfrak{q}\in\mathfrak{S}_p(\overline{B})} \widehat{\overline{E}}_{\mathfrak{q}} \end{array} \tag{II.11.18.1}$$

where $\iota_{\mathfrak{q}}$ is defined by extending scalars. Notice that the horizontal arrows are injective by II.4.3. Thus, $f'(\Phi(W))=0$ if and only if $f'_{\mathfrak{q}}(\iota_{\mathfrak{q}}(\Phi(W)))=0$ for any $\mathfrak{q}\in\mathfrak{S}_p(\overline{B})$. Notice that $f(\mathrm{Lie}(\rho(I_{\mathfrak{q}})))=0$ if and only if $f'_{\mathfrak{q}}(\Phi(W_{\mathfrak{q}}))=0$ by II.5.43, and that $\Phi(W_{\mathfrak{q}})$ is generated by $\iota_{\mathfrak{q}}(\Phi(W))$ by (II.11.7.3) and II.11.8 (or directly from (II.11.4.4)). This completes the proof. □

Corollary II.11.19. *With the notation in* II.11.18*, the $\widehat{\overline{B}}[1/p]$-module of Sen operators $\Phi(W)$ is zero if and only if $\rho(I_{\mathfrak{q}})$ is finite for any $\mathfrak{q}\in\mathfrak{S}_p(\overline{B})$.*

Corollary II.11.20. *With the notation in* II.11.18*, the $\widehat{\overline{B}}[1/p]$-module of Sen operators $\Phi(W)$ is contained in $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\rho(G))$, and the $\widehat{\overline{B}}[1/p]$-module of geometric Sen operators $\Phi^{\mathrm{geo}}(W)$ is contained in $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\rho(H))$, where $H=\mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}_\infty)$.*

Proof. The first assertion is a direct result of II.11.18. Note that $\Phi^{\mathrm{geo}}(W)=[\Phi(W),\Phi(W)]$ by the Lie algebra structure on $\mathscr{E}_B^*(1)$ (cf. II.9.39), and that $[\mathrm{Lie}(\rho(G)),\mathrm{Lie}(\rho(G))]\subseteq\mathrm{Lie}(\rho(H))$ as $G/H=\Sigma$ is abelian. Thus, the second assertion follows from the first. □

Theorem II.11.21 (cf. [**Sen81**, Theorem 12])**.** *Let $\mathcal{G}$ be a quotient of G which is a p-adic analytic group. Then, there exists a unique homomorphism of $\widehat{\overline{B}}[1/p]$-linear Lie algebras $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}:\mathscr{E}_B^*(1)\to\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$ making the following diagram commutative for any object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G},\mathbb{Q}_p)$,*

$$\begin{array}{ccc} \mathscr{E}_B^*(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\overline{B}}[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}) \\ {\scriptstyle \varphi_{\mathrm{Sen}}|_W}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\widehat{\overline{B}}[\frac{1}{p}]}\otimes\varphi|_V} \\ \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W) & \xleftarrow{\sim} & \widehat{\overline{B}}[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{End}_{\mathbb{Q}_p}(V) \end{array} \tag{II.11.21.1}$$

where $W=\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}V$ is the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G,\widehat{\overline{B}}[1/p])$, $\varphi_{\mathrm{Sen}}|_W$ is the canonical Lie algebra action defined in II.11.4*, and $\varphi|_V$ is the infinitesimal Lie algebra action of $\mathrm{Lie}(\mathcal{G})$ on V (cf.* II.4.13*).*

Proof. Firstly, as $\mathcal{G}$ is a compact p-adic analytic group, it admits a faithful finite projective $\mathbb{Q}_p$-representation V by II.3.9. The faithfulness implies that the map $\varphi|_V:\mathrm{Lie}(\mathcal{G})\to\mathrm{End}_{\mathbb{Q}_p}(V)$ is injective (cf. II.4.10.(3)). Thus, the uniqueness of $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ is clear.

Consider an injective morphism $V\to V'$ of faithful finite projective $\mathbb{Q}_p$-representations of $\mathcal{G}$. Note that $W=\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}V$ is still a subrepresentation of $W'=\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}V'$. We claim that the natural surjection $\Phi(W')\to\Phi(W)$ (of the images of $\varphi_{\mathrm{Sen}}|_{W'}$ and $\varphi_{\mathrm{Sen}}|_W$) defined by restriction is also injective. Indeed, we regard $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$ as a subset of $\mathrm{End}_{\widehat{\overline{B}}[1/p]}(W)$ via $\mathrm{id}_{\widehat{\overline{B}}[1/p]}\otimes\varphi|_V$. Thus, the restriction from V' to V induces the identity map on $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$. On the other hand, $\Phi(W)$ (resp. $\Phi(W')$) is contained in $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$ by II.11.18, which shows that the surjective map $\Phi(W')\to\Phi(W)$ induced by the restriction is injective.

Therefore, we take a faithful finite projective $\mathbb{Q}_p$-representation V of $\mathcal{G}$, and we define $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ to be the composition of

$$\mathscr{E}_B^*(1)\xrightarrow{\varphi_{\mathrm{Sen}}|_W}\Phi(W)\subseteq\mathrm{Im}(\mathrm{id}_{\widehat{\overline{B}}[\frac{1}{p}]}\otimes\varphi|_V)=\widehat{\overline{B}}[\frac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}). \tag{II.11.21.2}$$

As any two faithful representations V and V' are both contained in $V\oplus V'$, we deduce easily from the above discussions that this definition of $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ does not depend on the choice of V. It follows immediately that the diagram (II.11.21.1) is commutative for faithful representations.

In general, we take a faithful finite projective $\mathbb{Q}_p$-representation V' of $\mathcal{G}$. Then, for any object V of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G},\mathbb{Q}_p)$, $V\oplus V'$ is also a faithful finite projective $\mathbb{Q}_p$-representation of $\mathcal{G}$. The previous result shows that there is a canonical commutative diagram

$$\begin{array}{ccc} \mathscr{O}_B^*(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\overline{B}}[\frac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}) \\ \downarrow{\scriptstyle \varphi_{\mathrm{Sen}}|_{W\oplus W'}} & & \downarrow{\scriptstyle \mathrm{id}_{\widehat{\overline{B}}[\frac{1}{p}]}\otimes\varphi|_{V\oplus V'}} \\ \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W\oplus W') & \xleftarrow{\sim} & \widehat{\overline{B}}[\frac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{End}_{\mathbb{Q}_p}(V\oplus V') \end{array} \tag{II.11.21.3}$$

where $W=\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}V$ and $W'=\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}V'$. Notice that the image of $\varphi_{\mathrm{Sen}}|_{W\oplus W'}=\varphi_{\mathrm{Sen}}|_W\oplus\varphi_{\mathrm{Sen}}|_{W'}$ lies in $\mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W)\oplus\mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W')$ (cf. II.11.4), and that the image of $\varphi|_{V\oplus V'}=\varphi|_V\oplus\varphi|_{V'}$ lies in $\mathrm{End}_{\mathbb{Q}_p}(V)\oplus\mathrm{End}_{\mathbb{Q}_p}(V')$. By looking at the first component, the commutativity of (II.11.21.3) implies that of (II.11.21.1). □

Definition II.11.22. Let $\mathcal{G}$ be a quotient of G which is a p-adic analytic group. We denote by $\Phi_{\mathcal{G}}$ the image of $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ (II.11.21), and by $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ the image of $\mathrm{Hom}_{B[1/p]}(\Omega^1_{Y_K/K}(-1),\widehat{\overline{B}}[1/p])$ under $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$. We call an element of $\Phi_{\mathcal{G}}\subseteq\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$ a *Sen operator* of $\mathcal{G}$. We call an element of $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ a *geometric Sen operator* of $\mathcal{G}$. And we call the image of $1\in\widehat{\overline{B}}[1/p]$ in $\Phi_{\mathcal{G}}^{\mathrm{ari}}=\Phi_{\mathcal{G}}/\Phi_{\mathcal{G}}^{\mathrm{geo}}$ the *arithmetic Sen operator* of $\mathcal{G}$.

Corollary II.11.23. *Let $\mathcal{G}$ be a quotient of G which is a p-adic analytic group.*

1. *The canonical morphism $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}:\mathscr{O}_B^*(1)\to\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$ is G-equivariant with respect to the canonical action on $\mathscr{O}_B^*(1)$, $\widehat{\overline{B}}[1/p]$ and the adjoint action on $\mathrm{Lie}(\mathcal{G})$ defined in* II.3.15.
2. *For any $\mathfrak{q}\in\mathfrak{S}_p(\overline{B})$, let $\mathcal{G}_{I_{\mathfrak{q}}}\subseteq\mathcal{G}$ be the image of the inertia subgroup $I_{\mathfrak{q}}\subseteq G$ at $\mathfrak{q}$ (see* II.11.17*). Then, $\sum_{\mathfrak{q}\in\mathfrak{S}_p(\overline{B})}\mathrm{Lie}(\mathcal{G}_{I_{\mathfrak{q}}})$ is the smallest $\mathbb{Q}_p$-subspace S of $\mathrm{Lie}(\mathcal{G})$ such that $\Phi_{\mathcal{G}}$ is contained in $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}S$.*
3. *Let $\mathcal{G}_H\subseteq\mathcal{G}$ be the image of $H=\mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}_\infty)\subseteq G$. Then, the Lie algebra $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ of geometric Sen operators of $\mathcal{G}$ is contained in $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}_H)$.*
4. *Let K' be a complete discrete valuation field extension of K with perfect residue field, B' a quasi-adequate $\mathcal{O}_{K'}$-algebra with fraction field $\mathcal{L}'$ and Galois group $G'=\mathrm{Gal}(\mathcal{L}'_{\mathrm{ur}}/\mathcal{L}')$. Consider a commutative diagram of $(K,\mathcal{O}_K,\mathcal{O}_{\overline{K}})$-triples (see* II.9.2*)*

$$\begin{array}{ccc} (B_{\mathrm{tr}},B,\overline{B}) & \longrightarrow & (B'_{\mathrm{tr}},B',\overline{B'}) \\ \uparrow & & \uparrow \\ (K,\mathcal{O}_K,\mathcal{O}_{\overline{K}}) & \longrightarrow & (K',\mathcal{O}_{K'},\mathcal{O}_{\overline{K'}}) \end{array} \tag{II.11.23.1}$$

with $\overline{B}\to\overline{B'}$ injective, and let $\mathcal{G}'$ be the image of the composition of $G'\to G\to\mathcal{G}$. Then, there is a natural commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{B'}^*(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}'}} & \widehat{\overline{B'}}[\frac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}') \\ \downarrow & & \downarrow \\ \widehat{\overline{B'}}\otimes_{\widehat{\overline{B}}}\mathscr{O}_B^*(1) & \xrightarrow{\mathrm{id}_{\widehat{\overline{B'}}}\otimes\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\overline{B'}}[\frac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}) \end{array} \tag{II.11.23.2}$$

Moreover, if we denote by $\widehat{\overline{B'}}\Phi_{\mathcal{G}}$ the image of $\widehat{\overline{B'}}\otimes_{\widehat{\overline{B}}}\Phi_{\mathcal{G}}$ in $\widehat{\overline{B'}}[\frac{1}{p}]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$, then the right vertical arrow induces a natural isomorphism

$$\Phi_{\mathcal{G}'}\xrightarrow{\sim}\widehat{\overline{B'}}\Phi_{\mathcal{G}} \tag{II.11.23.3}$$

which is compatible with geometric and arithmetic Sen operators.

(5) *Let $\mathcal{L}'$ be an element of the directed system $\mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ defined in* II.11.9, $G' = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}')$, $\mathcal{G}' \subseteq \mathcal{G}$ *the image of* $G' \subseteq G$. *Then, there is a natural commutative diagram*

$$\begin{array}{ccc} \mathscr{O}^*_{B'}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}'}} & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}') \\ \downarrow \wr & & \downarrow \wr \\ \mathscr{O}^*_{B}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) \end{array} \tag{II.11.23.4}$$

In particular, $\Phi_{\mathcal{G}'} = \Phi_{\mathcal{G}}$.

(6) *Let $\mathcal{G}'$ be a quotient of $\mathcal{G}$. Then, there is a natural commutative diagram*

$$\begin{array}{ccc} \mathscr{O}^*_{B}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) \\ \| & & \downarrow \\ \mathscr{O}^*_{B}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}'}} & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}') \end{array} \tag{II.11.23.5}$$

In particular, it induces a surjection $\Phi_{\mathcal{G}} \to \Phi_{\mathcal{G}'}$.

Proof. For (1), the $\mathcal{G}$-equivariance of $\varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ follows from that of the other three arrows in the diagram (II.11.21.1) (cf. II.11.4, II.3.15). (2) holds by applying II.11.18 to a faithful finite projective $\mathbb{Q}_p$-representation V of $\mathcal{G}$. (3) follows from the arguments of II.11.20.

For (4), as $\mathcal{G}'$ is a closed subgroup of $\mathcal{G}$, a faithful finite projective $\mathbb{Q}_p$-representation V of $\mathcal{G}$ defines a faithful finite projective $\mathbb{Q}_p$-representation V' of $\mathcal{G}'$ by restricting the action. Combining II.11.7 with II.11.21, we obtain a natural diagram

$$\begin{array}{ccccc} \mathscr{O}^*_{B'}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}'}} & \widehat{\widehat{B'}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}') & \xrightarrow{\mathrm{id}_{\widehat{\widehat{B'}}[\frac{1}{p}]} \otimes \varphi|_{V'}} & \widehat{\widehat{B'}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p}(V') \\ \downarrow & & \downarrow & & \downarrow \wr \\ \widehat{\widehat{B'}} \otimes_{\widehat{\widehat{B}}} \mathscr{O}^*_{B}(1) & \xrightarrow{\mathrm{id}_{\widehat{\widehat{B'}}} \otimes \varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\widehat{B'}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) & \xrightarrow{\mathrm{id}_{\widehat{\widehat{B'}}[\frac{1}{p}]} \otimes \varphi|_{V}} & \widehat{\widehat{B'}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p}(V) \end{array} \tag{II.11.23.6}$$

Notice that the left square is commutative, since the right square and the big rectangle are commutative and the horizontal arrows in the right square are injective. Moreover, (II.11.7.3) implies that the image $\Phi_{\mathcal{G}'}$ of $\varphi_{\mathrm{Sen}}|_{\mathcal{G}'}$ coincides with the image $\widehat{\widehat{B'}}\Phi_{\mathcal{G}}$ of $\mathrm{id}_{\widehat{\widehat{B'}}} \otimes \varphi_{\mathrm{Sen}}|_{\mathcal{G}}$ via the middle vertical arrow (which is injective). This completes the proof of (4), and (5) is a special case of (4).

For (6), a faithful finite projective $\mathbb{Q}_p$-representation V of $\mathcal{G}'$ can be regarded as an object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \mathbb{Q}_p)$. Thus, we obtain a natural diagram

$$\begin{array}{ccccc} \mathscr{O}^*_{B}(1) & \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) & \twoheadrightarrow & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}') \\ \downarrow{\scriptstyle \varphi_{\mathrm{Sen}}|_{W}} & & \downarrow{\scriptstyle \mathrm{id}_{\widehat{\widehat{B}}[\frac{1}{p}]} \otimes \varphi|_{V}} & & \downarrow{\scriptstyle \mathrm{id}_{\widehat{\widehat{B}}[\frac{1}{p}]} \otimes \varphi|_{V}} \\ \mathrm{End}_{\widehat{\widehat{B}}[\frac{1}{p}]}(W) & \xleftarrow{\sim} & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p}(V) & = & \widehat{\widehat{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p}(V) \end{array} \tag{II.11.23.7}$$

Notice that the left square is commutative by II.11.21 and the right square is obviously commutative. Since the right vertical arrow is injective, the composition of the two horizontal arrows on the top is the unique map making the big rectangle commutative, which thus coincides with $\varphi_{\mathrm{Sen}}|_{\mathcal{G}'}$ by II.11.21. □

II.11.24. Let $\mathcal{G}$ be a quotient of G which is a p-adic analytic group. The universal Lie algebra homomorphism $\varphi_{\mathrm{Sen}}|_{\mathcal{G}} : \mathscr{O}^*_B(1) \to \widehat{\widehat{B}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G})$ allows us to canonically extend Sen operators to certain infinite-dimensional representations as follows.

Let V be a $\mathbb{Q}_p$-Banach space endowed with a $\mathbb{Q}_p$-linear action of $\mathcal{G}$ such that $\mathcal{G}$ preserves the norm of V and induces a trivial action on $V^{\leq 1}/p^2V^{\leq 1}$, where $V^{\leq 1}$ is the closed ball of radius 1 in V. Such a smallness condition implies that the infinitesimal action of $\mathcal{G}$ on $V^{\leq 1}$ is well-defined and given by the following formula for any $g \in \mathcal{G}$ and $x \in V^{\leq 1}$ (cf. II.6.7)

$$\log(g)|_{V^{\leq 1}}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (g-1)^n(x). \tag{II.11.24.1}$$

It defines a Lie algebra homomorphism

$$\varphi|_V : \mathrm{Lie}(\mathcal{G}) \longrightarrow \mathrm{End}_{\mathbb{Q}_p}(V), \tag{II.11.24.2}$$

sending $\log_{\mathcal{G}}(g)$ to $\mathrm{id}_{\mathbb{Q}_p} \otimes \log(g)|_{V^{\leq 1}}$ for any $g \in \mathcal{G}$ (cf. II.12.11, II.12.15).

We put $W^+ = (\widehat{\overline{B}} \otimes_{\mathbb{Z}_p} V^{\leq 1})^\wedge$ and $W = W^+[1/p]$. They are naturally endowed with continuous group actions of G with respect to the p-adic topology defined by W^+, cf. II.6.18.(1). Notice that the $\widehat{\overline{B}}$-linear endomorphism $\mathrm{id}_{\widehat{\overline{B}}} \otimes \log(g)|_{V^{\leq 1}}$ on $\widehat{\overline{B}} \otimes_{\mathbb{Z}_p} V^{\leq 1}$ extends to an endomorphism on W^+ by taking p-adic completion. Thus, we also obtain a Lie algebra homomorphism after inverting p,

$$\widehat{\overline{B}}[\frac{1}{p}] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) \longrightarrow \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W). \tag{II.11.24.3}$$

We define $\varphi_{\mathrm{Sen}}|_W : \mathscr{O}_B^*(1) \to \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W)$ to be the composition of

$$\mathscr{O}_B^*(1) \xrightarrow{\varphi_{\mathrm{Sen}}|_{\mathcal{G}}} \widehat{\overline{B}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) \xrightarrow{\text{(II.11.24.3)}} \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W). \tag{II.11.24.4}$$

It is a G-equivariant homomorphism of $\widehat{\overline{B}}[1/p]$-linear Lie algebras. We denote by $\Phi(W)$ its image, and by $\Phi^{\mathrm{geo}}(W)$ the image of $\mathrm{Hom}_{B[1/p]}(\Omega^1_{Y_K/K}(-1), \widehat{\overline{B}}[1/p])$. It follows from the construction that any element of $\Phi(W)$ acts continuously on W with respect to the p-adic topology defined by W^+.

Lemma II.11.25. *With the notation in* II.11.24*, for any object V_0 of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \mathbb{Q}_p)$ and any $\mathcal{G}$-equivariant continuous $\mathbb{Q}_p$-linear homomorphism $V_0 \to V$, the Lie algebra action $\varphi_{\mathrm{Sen}}|_W$ of $\mathscr{O}_B^*(1)$ on W defined by* (II.11.24.4) *is compatible with the canonical Lie algebra action $\varphi_{\mathrm{Sen}}|_{W_0}$ of $\mathscr{O}_B^*(1)$ on W_0 defined in* II.11.4*, where $W_0 = \widehat{\overline{B}}[1/p] \otimes_{\mathbb{Q}_p} V_0$ is the associated object of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[1/p])$.*

Proof. We take a $\mathbb{Q}_p$-basis $e_1, \ldots, e_r$ of V_0 whose images under $V_0 \to V$ lie in $V^{\leq 1}$. We put $V_0^+ = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_r \subseteq V_0$. Let $\mathcal{G}_0$ be a sufficiently small open subgroup of $\mathcal{G}$ whose image under the continuous group homomorphism $\rho : \mathcal{G} \to \mathrm{GL}_r(\mathbb{Q}_p)$ (induced by the $\mathcal{G}$-action on V_0) is contained in $\mathrm{id} + p^2 \mathrm{M}_r(\mathbb{Z}_p)$. Thus, the lattice $V_0^+ = \mathbb{Z}_p^r \subseteq \mathbb{Q}_p^r = V_0$ is $\mathcal{G}_0$-stable, and $\mathcal{G}_0$ acts trivially on $V_0^+/p^2V_0^+$. Notice that the infinitesimal action of $\mathcal{G}_0$ on V_0^+ is also well-defined and given by the same formula as in (II.11.24.1). Then, the $\mathcal{G}_0$-equivariant homomorphism $V_0^+ \to V^{\leq 1}$ guarantees that the map $\widehat{\overline{B}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G}) \to \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W_0)$ induced by the infinitesimal action of $\mathcal{G}$ on V_0^+ is compatible with (II.11.24.3). Thus, the conclusion follows from II.11.21 and (II.11.24.4). □

Theorem II.11.26. *Let $\mathcal{G}$ be a quotient of G which is a p-adic analytic group, V a $\mathbb{Q}_p$-Banach space endowed with a $\mathbb{Q}_p$-linear action of $\mathcal{G}$ satisfying the following conditions:*

1. *The $(\mathcal{G}, \mathbb{Q}_p)$-finite part of V is dense in V (cf.* II.2.4*).*
2. *The $\mathcal{G}$-action preserves the norm on V, and induces a trivial action on $V^{\leq 1}/p^3V^{\leq 1}$, where $V^{\leq 1}$ is the closed ball of radius 1 in V.*

We set $W = (\widehat{\overline{B}} \otimes_{\mathbb{Z}_p} V^{\leq 1})^\wedge[1/p]$ endowed with the natural $\widehat{\overline{B}}[1/p]$-semi-linear action of G. Then, the canonical $\widehat{\overline{B}}[1/p]$-linear Lie subalgebra $\Phi^{\mathrm{geo}}_{\mathcal{G}}$ of $\widehat{\overline{B}}[1/p] \otimes_{\mathbb{Q}_p} \mathrm{Lie}(\mathcal{G})$ defined in II.11.22 *acts trivially on W^H via* (II.11.24.3)*, where $H = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}_\infty) \subseteq G$.*

Proof. The strategy is to reduce to II.6.19. We need to construct a suitable directed system of $p^3\mathbb{Z}_p$-small objects in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \widehat{\overline{B}})$ in order to reduce to the situation of II.6.19. By condition (1), we fix a directed system $(V_\lambda)_{\lambda \in \Lambda}$ of finite dimensional $\mathcal{G}$-stable $\mathbb{Q}_p$-subspaces of V such that $V_\infty = \bigcup_{\lambda \in \Lambda} V_\lambda$ is dense in V. Recall that for any $\lambda \in \Lambda \cup \{\infty\}$ and $n \in \mathbb{N}$, we have (cf. II.4.6)

$$p^n V_\lambda^{\leq 1} = V_\lambda^{\leq p^{-n}} = V_\lambda \cap p^n V^{\leq 1}. \tag{II.11.26.1}$$

In particular, $V_\lambda^{\leq 1}/p^n V_\lambda^{\leq 1} \to V^{\leq 1}/p^n V^{\leq 1}$ is injective, and $V_\infty^{\leq 1}/p^n V_\infty^{\leq 1} \to V^{\leq 1}/p^n V^{\leq 1}$ is an isomorphism as V_∞ is dense in V. Thus, $V^{\leq 1}$ identifies naturally with the p-adic completion of $V_\infty^{\leq 1}$, and the $\mathcal{G}$-action on $V_\lambda^{\leq 1}/p^3 V_\lambda^{\leq 1} \subseteq V^{\leq 1}/p^3 V^{\leq 1}$ is also trivial by condition (2). In conclusion, $(V_\lambda^{\leq 1})_{\lambda \in \Lambda}$ forms a directed system of $p^3\mathbb{Z}_p$-small objects in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \mathbb{Z}_p)$ (cf. II.6.4, II.6.8) whose transition maps are injective and injective modulo p^n for any $n \in \mathbb{N}$.

For any $\lambda \in \Lambda \cup \{\infty\}$, we put $W_\lambda^+ = \widehat{\overline{B}} \otimes_{\mathbb{Z}_p} V_\lambda^{\leq 1}$. As $\widehat{\overline{B}}$ is flat over $\mathbb{Z}_p$, $(W_\lambda^+)_{\lambda \in \Lambda}$ forms a directed system of $p^3\mathbb{Z}_p$-small objects in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \widehat{\overline{B}})$ (cf. II.6.8) whose transition maps are injective and

injective modulo p^n for any $n \in \mathbb{N}$. We put $W_\lambda = W_\lambda^+[1/p]$ and $\widehat{W_\infty} = \widehat{W_\infty^+}[1/p]$, where $\widehat{W_\infty^+}$ is the p-adic completion of W_∞^+. Remark that we have G-equivariant natural identifications

$$W^+ = (\widehat{\overline{B}} \otimes_{\mathbb{Z}_p} V^{\leq 1})^\wedge = \widehat{W_\infty^+}. \tag{II.11.26.2}$$

Thus, $W = W^+[1/p] = \widehat{W_\infty}$.

For any $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$ with image $\mathfrak{p} \in \mathfrak{S}_p(B)$, consider the element $(B_{\mathrm{tr}}, B, \overline{B}) \to (E_\mathfrak{p}, \mathcal{O}_{E_\mathfrak{p}}, \mathcal{O}_{\overline{E}_\mathfrak{q}})$ of $\mathfrak{E}(B)$ defined in II.9.3. We set $G_\mathfrak{q} = \mathrm{Gal}(\overline{E}_\mathfrak{q}/E_\mathfrak{p})$ and $H_\mathfrak{q} = \mathrm{Gal}(\overline{E}_\mathfrak{q}/E_{\mathfrak{p},\infty})$. Consider the natural commutative diagram

$$\begin{array}{ccccc} \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \mathbb{Z}_p) & \longrightarrow & \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}) & \longrightarrow & \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G_\mathfrak{q}, \mathcal{O}_{\widehat{\overline{E}}_\mathfrak{q}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \mathbb{Q}_p) & \longrightarrow & \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G, \widehat{\overline{B}}[\tfrac{1}{p}]) & \longrightarrow & \mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G_\mathfrak{q}, \widehat{\overline{E}}_\mathfrak{q}) \end{array} \tag{II.11.26.3}$$

Let $W^+_{\lambda,\mathfrak{q}} = \mathcal{O}_{\widehat{\overline{E}}_\mathfrak{q}} \otimes_{\widehat{\overline{B}}} W_\lambda^+ = \mathcal{O}_{\widehat{\overline{E}}_\mathfrak{q}} \otimes_{\mathbb{Z}_p} V_\lambda^{\leq 1}$ and $W_{\lambda,\mathfrak{q}} = W^+_{\lambda,\mathfrak{q}}[1/p]$. Since $\mathcal{O}_{\widehat{\overline{E}}_\mathfrak{q}}$ is flat over $\mathbb{Z}_p$, $(W^+_{\lambda,\mathfrak{q}})_{\lambda\in\Lambda}$ forms a directed system of $p^3\mathbb{Z}_p$-small objects in $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(G_\mathfrak{q}, \mathcal{O}_{\widehat{\overline{E}}_\mathfrak{q}})$ whose transition maps are injective and injective modulo p^n for any $n \in \mathbb{N}$. Thus, we are in the situation of II.6.19. Taking colimit of the diagram (II.11.4.4), we get a canonical commutative diagram

$$\begin{array}{ccc} \mathscr{O}^*_B(1) & \longrightarrow & \prod_\mathfrak{q} \mathscr{O}^*_{\mathcal{O}_{E_\mathfrak{p}}}(1) \\ {\scriptstyle \varphi_{\mathrm{Sen}}|_{W_\infty}}\downarrow & & \downarrow{\scriptstyle (\varphi_{\mathrm{Sen}}|_{W_{\infty,\mathfrak{q}}})_\mathfrak{q}} \\ \mathrm{End}_{\widehat{\overline{B}}[\frac{1}{p}]}(W_\infty) & \xrightarrow{(\iota_\mathfrak{q})_\mathfrak{q}} & \prod_\mathfrak{q} \mathrm{End}_{\widehat{\overline{E}}_\mathfrak{q}}(W_{\infty,\mathfrak{q}}) \end{array} \tag{II.11.26.4}$$

where the product is taken over $\mathfrak{q} \in \mathfrak{S}_p(\overline{B})$, $\iota_\mathfrak{q}$ is defined by extending scalars, and the horizontal arrows are injective. Notice that the Lie algebra action $\varphi_{\mathrm{Sen}}|_W$ defined by (II.11.24.4) is the unique continuation of $\varphi_{\mathrm{Sen}}|_{W_\infty}$ by II.11.25. The variant of II.11.25 for valuation ring case shows that $\varphi_{\mathrm{Sen}}|_{W_{\infty,\mathfrak{q}}}$ extends uniquely to $\varphi_{\mathrm{Sen}}|_{\widehat{W_{\infty,\mathfrak{q}}}}$ by continuation, whose geometric part thus coincides with that defined in II.6.19. In particular, $\iota_\mathfrak{q}(\Phi(\widehat{W_\infty})) \subset \Phi(\widehat{W_{\infty,\mathfrak{q}}})$ (resp. $\iota_\mathfrak{q}(\Phi^{\mathrm{geo}}(\widehat{W_\infty})) \subset \Phi^{\mathrm{geo}}(\widehat{W_{\infty,\mathfrak{q}}})$). Taking filtered colimit on $\lambda \in \Lambda$ and then inverse limit on $n \in \mathbb{N}$ of the natural injection $W_\lambda^+/p^n \to \prod_\mathfrak{q} W^+_{\lambda,\mathfrak{q}}/p^n$ given by II.4.3, we obtain a natural injection

$$\widehat{W_\infty} \longrightarrow \prod_\mathfrak{q} \widehat{W_{\infty,\mathfrak{q}}}, \tag{II.11.26.5}$$

which particularly sends $(\widehat{W_\infty})^G$ (resp. $(\widehat{W_\infty})^H$) into $\prod_\mathfrak{q}(\widehat{W_{\infty,\mathfrak{q}}})^{G_\mathfrak{q}}$ (resp. $\prod_\mathfrak{q}(\widehat{W_{\infty,\mathfrak{q}}})^{H_\mathfrak{q}}$). Since $\Phi^{\mathrm{geo}}(\widehat{W_{\infty,\mathfrak{q}}})$ acts trivially on $(\widehat{W_{\infty,\mathfrak{q}}})^{H_\mathfrak{q}}$ for each $\mathfrak{q}$ by II.6.19, we see that $\Phi^{\mathrm{geo}}(\widehat{W_\infty})$ acts trivially on $(\widehat{W_\infty})^H$. □

II.12. Application to Locally Analytic Vectors

This section is devoted to generalizing a result of Pan [**Pan22**, 3.1.2] to higher dimension.

II.12.1. We mainly follow [**Pan22**, 2.1] to briefly review the notion of locally analytic vectors. Let M be a finite free $\mathbb{Z}_p$-module with a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, V a $\mathbb{Q}_p$-Banach space with norm $|\ |$. We say that a map $f : M \to V$ is (strictly) *analytic* (cf. [**DdSMS99**, Definition 6.17]) if there exists $v_{\underline{m}} \in V$ for any $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ such that ($|\underline{m}| = m_1 + \cdots + m_d$)

$$\lim_{|\underline{m}|\to\infty} |v_{\underline{m}}| = 0, \tag{II.12.1.1}$$

and for any $a_1, \ldots, a_d \in \mathbb{Z}_p$, we have

$$f(a_1\mathbf{e}_1 + \cdots + a_d\mathbf{e}_d) = \sum_{\underline{m}\in\mathbb{N}^d} a_1^{m_1} \cdots a_d^{m_d} v_{\underline{m}}. \tag{II.12.1.2}$$

Notice that $f = 0$ if and only if $v_{\underline{m}}$ is zero for any $\underline{m} \in \mathbb{N}^d$. If $\mathbf{e}'_1, \ldots, \mathbf{e}'_d$ form another basis of M, then we can write $f(a_1\mathbf{e}'_1 + \cdots + a_d\mathbf{e}'_d) = \sum_{\underline{m}\in\mathbb{N}^d} a_1^{m_1} \cdots a_d^{m_d} v'_{\underline{m}}$, where $v'_{\underline{m}}$ is a $\mathbb{Z}_p$-linear combination of finitely many $v_{\underline{m}}$. In particular, we get $\sup_{\underline{m}\in\mathbb{N}^d} |v'_{\underline{m}}| \leq \sup_{\underline{m}\in\mathbb{N}^d} |v_{\underline{m}}|$, and thus they are actually equal by symmetry. In conclusion, the definition on the analyticity of f does not depend on the choice of the basis of M. We denote by $\mathscr{C}^{\mathrm{an}}(M, V)$ the $\mathbb{Q}_p$-vector space of V-valued analytic functions on

M. We note that after fixing a basis of M, taking the coefficients $v_{\underline{m}}$ identifies $\mathscr{C}^{\mathrm{an}}(M,V)$ with a subspace of $\prod_{\underline{m}\in\mathbb{N}^d} V$. We define a norm on $\mathscr{C}^{\mathrm{an}}(M,V)$ by setting

$$|f| = \sup_{\underline{m}\in\mathbb{N}^d} |v_{\underline{m}}| \in \mathbb{R}_{\geq 0}, \tag{II.12.1.3}$$

which does not depend on the choice of the basis of M and makes $\mathscr{C}^{\mathrm{an}}(M,V)$ a $\mathbb{Q}_p$-Banach space.

Lemma II.12.2. *Let M be a finite free $\mathbb{Z}_p$-module, V a $\mathbb{Q}_p$-Banach space. Then, for any $n\in\mathbb{N}$, the natural map*

$$(V^{\leq 1}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(M,\mathbb{Q}_p)^{\leq 1})/p^n \longrightarrow \mathscr{C}^{\mathrm{an}}(M,V)^{\leq 1}/p^n. \tag{II.12.2.1}$$

is an isomorphism.

Proof. After fixing a basis of M, by taking coefficients we identify $\mathscr{C}^{\mathrm{an}}(M,V)$ with

$$\{(v_{\underline{m}})\in\prod_{\underline{m}\in\mathbb{N}^d} V \mid \lim_{|\underline{m}|\to\infty}|v_{\underline{m}}|=0\}. \tag{II.12.2.2}$$

Notice that $\mathscr{C}^{\mathrm{an}}(M,V)^{\leq 1}$ identifies with

$$\{(v_{\underline{m}})\in\prod_{\underline{m}\in\mathbb{N}^d} V^{\leq 1} \mid \lim_{|\underline{m}|\to\infty}|v_{\underline{m}}|=0\} = (\oplus_{\underline{m}\in\mathbb{N}^d}V^{\leq 1})^{\wedge}, \tag{II.12.2.3}$$

where the completion is p-adic. Taking $V=\mathbb{Q}_p$, we get $\mathscr{C}^{\mathrm{an}}(M,\mathbb{Q}_p)^{\leq 1} = (\oplus_{\underline{m}\in\mathbb{N}^d}\mathbb{Z}_p)^{\wedge}$. Thus, $(V^{\leq 1}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(M,\mathbb{Q}_p)^{\leq 1})/p^n = \oplus_{\underline{m}\in\mathbb{N}^d}V^{\leq 1}/p^n = \mathscr{C}^{\mathrm{an}}(M,V)^{\leq 1}/p^n$, which completes the proof. □

Lemma II.12.3. *Let V be a $\mathbb{Q}_p$-Banach space, $g: M\to N$ a $\mathbb{Z}_p$-linear homomorphism of finite free $\mathbb{Z}_p$-modules. Then, the pullback of functions induces a map*

$$g^*: \mathscr{C}^{\mathrm{an}}(N,V)\longrightarrow\mathscr{C}^{\mathrm{an}}(M,V),\ f\mapsto f\circ g \tag{II.12.3.1}$$

which decreases the norm. Moreover, if the cokernel of g is finite, then g^* is injective; if g is injective, then g^* has dense image.

Proof. As any submodule of a finite free $\mathbb{Z}_p$-module is still finite free, we may decompose g as an injection composed with a surjection, so that we can treat the two cases separately.

Assume firstly that g is surjective. We write $M=N\oplus L$, and choose a basis $\mathbf{e}_1,\dots,\mathbf{e}_d$ for M such that $\mathbf{e}_1,\dots,\mathbf{e}_c$ form a basis of N, where $0\leq c\leq d$ are integers. We see that an analytic function $f(a_1\mathbf{e}_1+\cdots+a_c\mathbf{e}_c)=\sum_{\underline{m}\in\mathbb{N}^c}a_1^{m_1}\cdots a_c^{m_c}v_{\underline{m}}$ on N is pulled back to an analytic function $(f\circ g)(a_1\mathbf{e}_1+\cdots+a_d\mathbf{e}_d)=\sum_{\underline{m}\in\mathbb{N}^c\times\{0\}^{d-c}}a_1^{m_1}\cdots a_d^{m_d}v_{\underline{m}}$. Thus, g^* is injective and preserves norms.

Assume that g is injective. Since N/M is a direct sum of a finite free $\mathbb{Z}_p$-module with a finite $\mathbb{Z}_p$-module, by writing g as a composition of two injective maps, we can treat separately the case where g admits a retraction and the case where the cokernel of g is finite. For the first case, we can write $N=M\oplus L$. By an argument as before, we see that g^* is surjective and decreases the norm. For the second case, by expressing a basis of M as $\mathbb{Z}_p$-linear combination of that of N, we see that g^* decreases the norm as in II.12.1. Conversely, as $N[1/p]=M[1/p]$, we can express a basis of N as $\mathbb{Q}_p$-linear combination of that of M. Thus, any analytic function on N with only finitely many coefficients $v_{\underline{m}}$ non-zero is a restriction of such an analytic function on M. This shows that g^* has dense image. □

Definition II.12.4. Let $\mathcal{G}$ be a uniform pro-p group, $L_{\mathcal{G}}$ its corresponding powerful Lie algebra over $\mathbb{Z}_p$ with identity map $\exp: L_{\mathcal{G}}\to\mathcal{G}$ (see II.3.3), V a $\mathbb{Q}_p$-Banach space. We say that a V-valued function $f:\mathcal{G}\to V$ is *analytic* if $f\circ\exp: L_{\mathcal{G}}\to V$ is analytic (as $L_{\mathcal{G}}$ is a finite free $\mathbb{Z}_p$-module). We set $\mathscr{C}^{\mathrm{an}}(\mathcal{G},V)=\mathscr{C}^{\mathrm{an}}(L_{\mathcal{G}},V)$ and denote its norm by $|\ |_{\mathcal{G}}$.

Remark II.12.5. One can also use a system of coordinates of the second kind (II.3.3.4) to define analyticity of a function as in [**Pan22**, 2.1.1]. Notice that the transition map between the coordinates of the first kind and the second kind is a homeomorphism $\psi:\mathbb{Z}_p^d\to\mathbb{Z}_p^d$ such that ψ and ψ^{-1} are both analytic by [**DdSMS99**, Exercise 8.3] (or by [**Sch11**, 34.1], as $\mathcal{G}$ is p-saturable by [**DdSMS99**, Notes on page 81, Theorem 7.7, Exercise 7.10]). Thus, the two definitions of $\mathscr{C}^{\mathrm{an}}(\mathcal{G},V)$ and its norm coincide (cf. [**DdSMS99**, Theorem 6.35]).

II.12.6. Let V be a $\mathbb{Q}_p$-vector space endowed with a $\mathbb{Q}_p$-linear action of a group $\mathcal{G}$, $\mathscr{C}(\mathcal{G},V)=\prod_{\mathcal{G}}V$ the $\mathbb{Q}_p$-vector space of V-valued functions on $\mathcal{G}$. We mainly consider three $\mathcal{G}$-actions $\{1\otimes\rho_{\mathrm{L}},1\otimes$

$\rho_{\mathrm{R}}, \rho_V \otimes \rho_{\mathrm{L}}\}$ on $\mathscr{C}(\mathcal{G}, V)$, called the left translation action, right translation action and diagonal action respectively, defined as follows: for any $g, g' \in \mathcal{G}$ and $f \in \mathscr{C}(\mathcal{G}, V)$,

$$(\rho(g)f)(g') = \begin{cases} f(g^{-1}g') & \text{if } \rho = 1 \otimes \rho_{\mathrm{L}}, \\ f(g'g) & \text{if } \rho = 1 \otimes \rho_{\mathrm{R}}, \\ g(f(g^{-1}g')) & \text{if } \rho = \rho_V \otimes \rho_{\mathrm{L}}. \end{cases} \tag{II.12.6.1}$$

Lemma II.12.7 ([**Pan22**, 2.1.2])**.** *Let $\mathcal{G}$ be a uniform pro-p group, V a $\mathbb{Q}_p$-Banach space. Then, $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$ is stable under the left and right translation actions of $\mathcal{G}$ which preserve the norm $|\ |_{\mathcal{G}}$. Moreover, the left and right translation actions of $\mathcal{G}^{p^n}$ on $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)^{\leq 1}/p^n\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)^{\leq 1}$ are trivial for any $n \in \mathbb{N}$.*

Proof. For any element $g \in \mathcal{G}$, let n be the maximal integer such that $g \in \mathcal{G}^{p^n}$ and we set $g_1 = g^{p^{-n}} \in \mathcal{G} \setminus \mathcal{G}^p$ (as $\mathcal{G} \to \mathcal{G}^{p^n}$ sending x to x^{p^n} is a homeomorphism, cf. II.3.3). Then, we take $g_2, \ldots, g_d \in \mathcal{G}$ such that $(g_1, \ldots, g_d)$ forms a minimal topological generating set of $\mathcal{G}$ (i.e. it forms a $\mathbb{Z}_p$-basis of $L_{\mathcal{G}}$, cf. II.3.3). We obtain a system of coordinates of second kind (II.3.3.4),

$$\mathbb{Z}_p^d \longrightarrow \mathcal{G},\ (a_1, \ldots, a_d) \mapsto g_1^{a_1} \cdots g_d^{a_d}. \tag{II.12.7.1}$$

The left translation by $g = g_1^{p^n}$ sends the coordinate $(a_1, a_2, \ldots, a_d)$ to $(a_1 - p^n, a_2, \ldots, a_d)$. Thus, it preserves the analyticity of a V-valued function on $\mathcal{G}$ by II.12.5 as well as the norm $|\ |_{\mathcal{G}}$, and acts trivially on $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)^{\leq 1}/p^n\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)^{\leq 1}$. Thus, we obtain the conclusion for the left translation. The proof for the right translation is analogous. □

Proposition II.12.8 ([**Pan22**, 2.1.3])**.** *Let $\mathcal{G}$ be a uniform pro-p group isomorphic to a closed subgroup of $\mathrm{id} + p^2\mathrm{M}_d(\mathbb{Z}_p)$ for some $d \in \mathbb{N}$. Then, there exists a directed system of finite-dimensional $\mathbb{Q}_p$-subspaces $(V_k)_{k\geq 1}$ of $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, \mathbb{Q}_p)$ stable under the left and right translation action of $\mathcal{G}$ such that $\bigcup_{k\geq 1} V_k$ is dense in $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, \mathbb{Q}_p)$.*

Proof. We follow the proof of [**Pan22**, 2.1.3]. Consider the commutative diagram (cf. II.3.6)

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{id} + p^2\mathrm{M}_d(\mathbb{Z}_p) \\ {\scriptstyle \log_{\mathcal{G}}}\downarrow & & \uparrow{\scriptstyle \exp}\ \downarrow{\scriptstyle \log} \\ L_{\mathcal{G}} & \xrightarrow{\ \iota\ } & p^2\mathrm{M}_d(\mathbb{Z}_p) \end{array} \tag{II.12.8.1}$$

We remark that the pullback by the injection ι induces a map $\mathscr{C}^{\mathrm{an}}(p^2\mathrm{M}_d(\mathbb{Z}_p), \mathbb{Q}_p) \to \mathscr{C}^{\mathrm{an}}(L_{\mathcal{G}}, \mathbb{Q}_p)$ with dense image by II.12.3. For any $1 \leq i, j \leq d$, let $X_{ij} : \mathrm{M}_d(\mathbb{Z}_p) \to \mathbb{Z}_p$ be the map taking the value of the (i, j)-component. Let W_k (resp. V_k) be the space of $\mathbb{Q}_p$-valued functions on $\mathrm{id} + p^2\mathrm{M}_d(\mathbb{Z}_p)$ (resp. $\mathcal{G}$) of the form $P(X_{ij} | 1 \leq i, j \leq d)$ where P is a polynomial with coefficients in $\mathbb{Q}_p$ of degree $\leq k$. Since $f \circ \exp \in \mathscr{C}^{\mathrm{an}}(p^2\mathrm{M}_d(\mathbb{Z}_p), \mathbb{Q}_p)$ for any $f \in W_k$, we have $W_k \subseteq \mathscr{C}^{\mathrm{an}}(\mathrm{id} + p^2\mathrm{M}_d(\mathbb{Z}_p), \mathbb{Q}_p)$. Moreover, since $\log \circ \exp$ is the identity on $p^2\mathrm{M}_d(\mathbb{Z}_p)$, the set $\{f \circ \exp \mid f \in W_k,\ k \in \mathbb{N}\}$ is dense in $\mathscr{C}^{\mathrm{an}}(p^2\mathrm{M}_d(\mathbb{Z}_p), \mathbb{Q}_p)$. Therefore, $\bigcup_{k\in\mathbb{N}} W_k$ is dense in $\mathscr{C}^{\mathrm{an}}(\mathrm{id} + p^2\mathrm{M}_d(\mathbb{Z}_p), \mathbb{Q}_p)$. By pulling back, we see that $\bigcup_{k\in\mathbb{N}} V_k$ is dense in $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, \mathbb{Q}_p)$ by II.12.3. Moreover, it follows from the construction that each V_k is stable under the left and right translation action of $\mathcal{G}$. □

Definition II.12.9 ([**Pan22**, 2.1.4])**.** Let $\mathcal{G}$ be a uniform pro-p group, V a $\mathbb{Q}_p$-Banach space endowed with a $\mathbb{Q}_p$-linear continuous action of $\mathcal{G}$. An element $v \in V$ is called *$\mathcal{G}$-analytic* if the function $f_v : \mathcal{G} \to V$ sending g to gv is analytic. We denote by $V^{\mathcal{G}\text{-an}}$ the subset of V consisting of $\mathcal{G}$-analytic vectors.

We remark that $V^{\mathcal{G}\text{-an}}$ is stable under the action of $\mathcal{G}$ on V, since the right translation of the analytic function f_v is still analytic by II.12.7. For any continuous group homomorphism of uniform pro-p groups $\mathcal{G}' \to \mathcal{G}$, regarding V also as a $\mathbb{Q}_p$-representation of $\mathcal{G}'$, then we have $V^{\mathcal{G}\text{-an}} \subseteq V^{\mathcal{G}'\text{-an}}$ by II.12.3. The subset $V^{\mathcal{G}\text{-an}}$ is actually a $\mathbb{Q}_p$-subspace of V by the following lemma.

Lemma II.12.10 ([**Pan22**, 2.1.5])**.** *Let $\mathcal{G}$ be a uniform pro-p group, V a $\mathbb{Q}_p$-Banach space endowed with a $\mathbb{Q}_p$-linear continuous action of $\mathcal{G}$. Then, the evaluation map at $1 \in \mathcal{G}$ induces a bijection*

$$\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)^{\rho_V \otimes \rho_{\mathrm{L}} = 1} \xrightarrow{\sim} V^{\mathcal{G}\text{-}an}, \tag{II.12.10.1}$$

where $\rho_V \otimes \rho_{\mathrm{L}} : \mathscr{C}^{\mathrm{an}}(\mathcal{G}, V) \to \mathscr{C}(\mathcal{G}, V)$ is the diagonal action (see II.12.6*). Moreover, the inverse of this bijection induces a $\mathcal{G}$-equivariant inclusion*

$$V^{\mathcal{G}\text{-}an} \longrightarrow (\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V), 1 \otimes \rho_{\mathrm{R}}),\ v \mapsto (f_v : g \mapsto gv), \tag{II.12.10.2}$$

where $1 \otimes \rho_{\mathrm{R}}$ is the right translation of $\mathcal{G}$ on $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$ (see II.12.7*).*

PROOF. Notice that an element $f \in \mathscr{C}(\mathcal{G}, V)$ is fixed by $\rho_V \otimes \rho_{\mathrm{L}}$ if and only if $f(g) = g(f(1))$ for any $g \in \mathcal{G}$. Thus, in this case f is analytic if and only if $f(1) \in V$ is $\mathcal{G}$-analytic by definition, so that we have the bijection (II.12.10.1). It follows from the definition that (II.12.10.2) is $\mathcal{G}$-equivariant. □

The smallness condition will give us enough analytic vectors.

Lemma II.12.11 ([**Pan22**, 2.1.9])**.** *Let $\mathcal{G}$ be a uniform pro-p group, V a $\mathbb{Q}_p$-Banach space endowed with a $\mathbb{Q}_p$-linear action of $\mathcal{G}$. Assume that $\mathcal{G}$ preserves the norm of V and induces a trivial action on $V^{\leq 1}/p^2V^{\leq 1}$. Then, $V = V^{\mathcal{G}\text{-an}}$. In particular, for any object V_0 of $\mathbf{Rep}^{\mathrm{proj}}_{\mathrm{cont}}(\mathcal{G}, \mathbb{Q}_p)$, there exists a uniform pro-p open subgroup $\mathcal{G}_0$ of $\mathcal{G}$ such $V_0 = V_0^{\mathcal{G}_0\text{-an}}$.*

PROOF. For any $g \in \mathcal{G}$, as $(g-1)(V^{\leq 1}) \subseteq p^2V^{\leq 1}$, we have $g = \exp(\log(g))$ by II.6.7 where exp and log are defined in loc.cit. We take a minimal topological generating set $(g_1, \ldots, g_d)$ of $\mathcal{G}$ (cf. II.3.3) so that we obtain a system of coordinates of second kind (II.3.3.4),

$$\mathbb{Z}_p^d \longrightarrow \mathcal{G},\ (a_1, \ldots, a_d) \mapsto g_1^{a_1} \cdots g_d^{a_d}. \tag{II.12.11.1}$$

For any $v \in V$, we have

$$\begin{aligned} g_1^{a_1} \cdots g_d^{a_d}(v) &= \exp(a_1 \log(g_1)) \cdots \exp(a_d \log(g_d))(v) \quad \text{(cf. II.6.7)} \\ &= \left(\sum_{n=0}^{\infty} \frac{a_1^n}{n!} \log(g_1)^n\right) \cdots \left(\sum_{n=0}^{\infty} \frac{a_d^n}{n!} \log(g_d)^n\right)(v), \end{aligned} \tag{II.12.11.2}$$

which is clearly an analytic V-valued function on variables $(a_1, \ldots, a_d) \in \mathbb{Z}_p^d$ (as $\log(g_i) \in p^2\mathrm{End}_{\mathbb{Z}_p}(V^{\leq 1})$). Thus, the function $f_v : \mathcal{G} \to V$ sending g to gv is analytic by II.12.5, i.e. v is $\mathcal{G}$-analytic.

For the "in particular" part, we fix a $\mathbb{Q}_p$-basis of V_0 so that we obtain a continuous group homomorphism $\rho : \mathcal{G} \to \mathrm{GL}_d(\mathbb{Q}_p)$. Let $\mathcal{G}_0$ be a sufficiently small uniform pro-p open subgroup of $\mathcal{G}$ whose image under ρ is contained in $\mathrm{id} + p^2\mathrm{M}_d(\mathbb{Z}_p)$. Thus, the lattice $V_0^+ = \mathbb{Z}_p^d \subseteq \mathbb{Q}_p^d = V_0$ is $\mathcal{G}_0$-stable, and $\mathcal{G}_0$ acts trivially on $V_0^+/p^2V_0^+$. By the assertion we just proved above, we see that $V_0 = V_0^{\mathcal{G}_0\text{-an}}$. □

Moreover, one can get a slightly stronger result for the analytic vectors in $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$.

Lemma II.12.12. *Let $\mathcal{G}$ be a uniform pro-p group, V a $\mathbb{Q}_p$-Banach space. Endowing the $\mathbb{Q}_p$-Banach space $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$ with the left or right translation action of $\mathcal{G}$, then $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V) = \mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)^{\mathcal{G}\text{-an}}$.*

PROOF. Firstly, the action of $\mathcal{G}$ on $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$ is well-defined and continuous by II.12.7. Thus, we can talk about $\mathcal{G}$-analytic vectors in $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$ by II.12.9. We fix a $\mathbb{Z}_p$-basis of $L_{\mathcal{G}}$. For any $f \in \mathscr{C}^{\mathrm{an}}(L_{\mathcal{G}}, V)$, consider the function $\psi : L_{\mathcal{G}} \times L_{\mathcal{G}} \to V$ sending (x, y) to $f(x*y)$, where the operation $x*y$ is given by the Baker-Campbell-Hausdorff formula (II.3.4.1) (which defines the multiplication in $\mathcal{G}$). Notice that ψ is analytic by [**DdSMS99**, Lemma 9.12], i.e. there exists a unique element $v_{\underline{k},\underline{l}} \in V$ for any $\underline{k}, \underline{l} \in \mathbb{N}^d$ such that $|v_{\underline{k},\underline{l}}| \to 0$ when $|\underline{k}| + |\underline{l}| \to \infty$, and that for any $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{Z}_p^d$, we have

$$f(x * y) = \sum_{\underline{k},\underline{l} \in \mathbb{N}^d} x_1^{k_1} \cdots x_d^{k_d} y_1^{l_1} \cdots y_d^{l_d} v_{\underline{k},\underline{l}}. \tag{II.12.12.1}$$

In particular, the function $f_{\underline{k}}$ sending y to $\sum_{\underline{l} \in \mathbb{N}^d} y_1^{l_1} \cdots y_d^{l_d} v_{\underline{k},\underline{l}}$ is analytic, and we have $|f_{\underline{k}}| \to 0$ when $|\underline{k}| \to \infty$. Thus, the function on $L_{\mathcal{G}}$ sending x to ${}_xf = \sum_{\underline{k} \in \mathbb{N}^d} x_1^{k_1} \cdots x_d^{k_d} f_{\underline{k}}$ is analytic, which shows that f is a $\mathcal{G}$-analytic vector of $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$ with respect to the left translation. The proof for right translation is analogous. □

The continuity condition of the $\mathcal{G}$-action on V in definition II.12.9 is used to define infinitesimal actions as follows.

Proposition II.12.13. *Let $\mathcal{G}$ be a uniform pro-p group, V a $\mathbb{Q}_p$-Banach space endowed with a $\mathbb{Q}_p$-linear continuous action of $\mathcal{G}$. Then, for any $g \in G$, there exists a unique $\mathbb{Q}_p$-linear endomorphism φ_g of $V^{\mathcal{G}\text{-an}}$ defined by*

$$\varphi_g(v) = \lim_{\mathbb{Z}_p \setminus \{0\} \ni a \to 0} a^{-1}(g^a - 1)(v), \quad \forall v \in V^{\mathcal{G}\text{-an}}, \tag{II.12.13.1}$$

such that for any $a \in \mathbb{Z}_p$,

$$g^a(x) = \exp(a\varphi_g)(v) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\underbrace{\varphi_g \circ \cdots \circ \varphi_g}_{k \text{ copies}})(v). \tag{II.12.13.2}$$

PROOF. Let $v \in V^{\mathcal{G}\text{-an}}$ and $g \in \mathcal{G}$. As $\mathbb{Z}_p \to \mathcal{G}$ sending a to g^a is a continuous homomorphism of uniform pro-p groups, v is also $\mathbb{Z}_p$-analytic, i.e. there exists a unique element $v_k \in V$ for any $k \in \mathbb{N}$ such that $|v_k| \to 0$ when $k \to \infty$, and that for any $a \in \mathbb{Z}_p$,

$$g^a v = \sum_{k=0}^{\infty} a^k v_k. \tag{II.12.13.3}$$

We see easily that $v_0 = v$ by taking $a = 0$ and that

$$v_1 = \lim_{\mathbb{Z}_p \setminus \{0\} \ni a \to 0} a^{-1}(g^a - 1)(v). \tag{II.12.13.4}$$

Thus, $\varphi_g(v) = v_1$ is a well-defined element of V. We claim that $v_1 \in V^{\mathcal{G}\text{-an}}$ (so that φ_g is a $\mathbb{Q}_p$-linear endomorphism of $V^{\mathcal{G}\text{-an}}$) and $v_k = \varphi_g^k(v)/k!$ for any $k \geq 0$ (so that (II.12.13.2) follows). Indeed, for any $b \in \mathbb{Z}_p$, we have

$$g^{a+b} v = g^b \left(\sum_{k=0}^{\infty} a^k v_k \right) = \sum_{k=0}^{\infty} a^k g^b v_k \tag{II.12.13.5}$$

where the second equality follows from the continuity of the $\mathcal{G}$-action on V; on the other hand,

$$g^{a+b} v = \sum_{k=0}^{\infty} (a+b)^k v_k = \sum_{k=0}^{\infty} a^k \left(\sum_{l=0}^{\infty} \binom{k+l}{l} b^l v_{k+l} \right) \tag{II.12.13.6}$$

where the second equality follows from the absolute convergence condition $|v_k| \to 0$ when $k \to \infty$. Combining the two expressions for $g^{a+b}v$, we get

$$g^b v_k = \sum_{l=0}^{\infty} \binom{k+l}{l} b^l v_{k+l}, \tag{II.12.13.7}$$

which shows that $v_k \in V^{\mathcal{G}\text{-an}}$ and $v_{k+1} = \varphi_g(v_k)/(k+1)$. The claim follows by induction. □

Lemma II.12.14. *With the notation in* II.12.13*, we endow $V^{\mathcal{G}\text{-an}}$ with the norm induced from the norm $|\ |_{\mathcal{G}}$ on $\mathscr{C}^{\mathrm{an}}(\mathcal{G}, V)$ via the canonical injection* (II.12.10.2)*. Then, the map*

$$\phi : \mathbb{Z}_p \times \mathcal{G} \times V^{\mathcal{G}\text{-an}} \longrightarrow V^{\mathcal{G}\text{-an}}, \tag{II.12.14.1}$$

sending $(0, g, v)$ to $\varphi_g(v)$ and sending (a, g, v) to $a^{-1}(g^a v - v)$ for $a \neq 0$, is continuous.

PROOF. We fix a $\mathbb{Z}_p$-basis $u_1, \ldots, u_d$ of $L_{\mathcal{G}}$. By the homeomorphism $\exp : L_{\mathcal{G}} \to \mathcal{G}$, it suffices to verify the continuity of the map

$$\psi : \mathbb{Z}_p \times \mathbb{Z}_p^d \times V^{\mathcal{G}\text{-an}} \longrightarrow V^{\mathcal{G}\text{-an}} \tag{II.12.14.2}$$

sending $(a_0, a_1, \ldots, a_d, v)$ to $\phi(a_0, \exp(\sum_{i=1}^d a_i u_i), v)$. For any $v \in V^{\mathcal{G}\text{-an}}$, there exists a unique element $v_{\underline{m}} \in V$ for any $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ such that $|v_{\underline{m}}| \to 0$ when $|\underline{m}| \to \infty$, and that for any $(a_1, \ldots, a_d) \in \mathbb{Z}_p^d$,

$$\exp(\sum_{i=1}^d a_i u_i) v = \sum_{\underline{m} \in \mathbb{N}^d} a_1^{m_1} \cdots a_d^{m_d} v_{\underline{m}}. \tag{II.12.14.3}$$

Since $v_{\underline{0}} = v$ and $\exp(\sum_{i=1}^d a_i u_i)^{a_0} = \exp(\sum_{i=1}^d a_0 a_i u_i)$, we see that for $a_0 \neq 0$,

$$\psi(a_0, a_1, \ldots, a_d, v) = \sum_{\underline{m} \in \mathbb{N}^d \setminus \{\underline{0}\}} a_0^{|\underline{m}|-1} a_1^{m_1} \cdots a_d^{m_d} v_{\underline{m}}. \tag{II.12.14.4}$$

As the right hand side is continuous with respect to the variable a_0, we see that this formula remains valid for $a_0 = 0$ by the definition of φ_g (II.12.13.1). Notice that $|v|_{\mathcal{G}} = \sup_{\underline{m} \in \mathbb{N}^d} |v_{\underline{m}}|$ by definition. Thus, one deduces easily from the formula (II.12.14.4) the continuity of ψ. □

Corollary II.12.15. *With the notation in* II.12.13*, there is a canonical morphism of $\mathbb{Q}_p$-linear Lie algebras induced by the infinitesimal action of $\mathcal{G}$,*

$$\varphi : \mathrm{Lie}(\mathcal{G}) \longrightarrow \mathrm{End}_{\mathbb{Q}_p}(V^{\mathcal{G}\text{-an}}). \tag{II.12.15.1}$$

More precisely, its composition with the logarithm map $\log_{\mathcal{G}} : \mathcal{G} \to \mathrm{Lie}(\mathcal{G})$ is the map $\varphi : \mathcal{G} \to \mathrm{End}_{\mathbb{Q}_p}(V^{\mathcal{G}\text{-an}})$ sending g to the infinitesimal action φ_g (II.12.13.1) *of $g \in G$ on $V^{\mathcal{G}\text{-an}}$.*

PROOF. The proof is the same as that of II.4.13 by using II.12.14 instead of II.4.12. □

Remark II.12.16. With the notation in II.12.13, assume that $\mathcal{G}$ preserves the norm of V and induces a trivial action on $V^{\leq 1}/p^2V^{\leq 1}$. Then, $V = V^{\mathcal{G}\text{-an}}$ by II.12.11. We remark that the infinitesimal action (II.12.15.1) of $\mathcal{G}$ on V coincides with the Lie algebra action (II.11.24.2) by (II.6.7.3).

Definition II.12.17 (cf. [**Pan22**, 2.1.6])**.** Let $\mathcal{G}$ be a p-adic analytic group, V a $\mathbb{Q}_p$-Banach space endowed with a $\mathbb{Q}_p$-linear continuous action of $\mathcal{G}$. We say an element $v \in V$ is *$\mathcal{G}$-locally analytic* if it is $\mathcal{G}_0$-analytic for some uniform pro-p open subgroup $\mathcal{G}_0$ of $\mathcal{G}$. We set $V^{\mathcal{G}\text{-la}} = \bigcup_{\mathcal{G}_0} V^{\mathcal{G}_0\text{-an}}$ the $\mathbb{Q}_p$-linear subspace of V consisting of $\mathcal{G}$-locally analytic vectors.

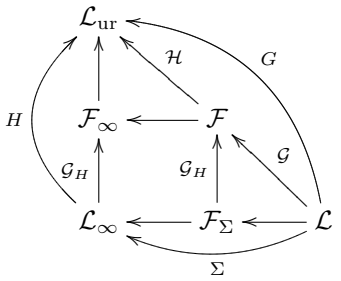
We obtain from II.12.15 a canonical morphism of Lie algebras over $\mathbb{Q}_p$,

$$\varphi : \mathrm{Lie}(\mathcal{G}) \longrightarrow \mathrm{End}_{\mathbb{Q}_p}(V^{\mathcal{G}\text{-la}}), \tag{II.12.17.1}$$

which is called the *infinitesimal Lie algebra action on locally analytic vectors*. We remark that if V is finite-dimensional, then $V = V^{\mathcal{G}\text{-la}}$ by II.12.11.

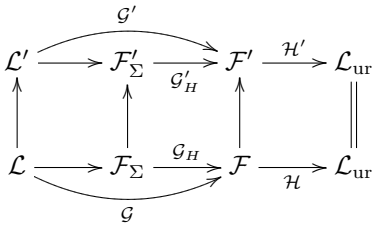
II.12.18. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$, $(B_{\mathrm{tr}}, B, \overline{B})$ a $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$-triple (see II.9.2), $\mathcal{F}$ a Galois extension of the fraction field $\mathcal{L}$ of B contained in $\mathcal{L}_{\mathrm{ur}}$ such that $\mathcal{G} = \mathrm{Gal}(\mathcal{F}/\mathcal{L})$ is a p-adic analytic group. We denote by $\mathcal{G}_H$ the image of $H = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}_\infty)$ under the surjection $G = \mathrm{Gal}(\mathcal{L}_{\mathrm{ur}}/\mathcal{L}) \to \mathcal{G}$, where $\mathcal{L}_\infty = K_\infty\mathcal{L}$ is the cyclotomic extension of $\mathcal{L}$, and we denote by $\mathcal{F}_\Sigma$ the invariant subfield of $\mathcal{F}$ by $\mathcal{G}_H$. We name some Galois groups as indicated in the following diagram

(II.12.18.1)



Indeed, we have $\mathcal{F}_\infty = K_\infty\mathcal{F} = \mathcal{L}_\infty \otimes_{\mathcal{F}_\Sigma} \mathcal{F}$ by Galois theory. Consider the directed system $\mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ of finite field extension $\mathcal{L}'$ of $\mathcal{L}$ contained in $\mathcal{L}_{\mathrm{ur}}$. For each $\mathcal{L}'$, we construct the above diagram for the Galois extension $\mathcal{F}' = \mathcal{L}'\mathcal{F}$ over $\mathcal{L}'$ contained in $\mathcal{L}_{\mathrm{ur}}$ in the same way and add prime superscript to the notation. There is a natural commutative diagram of fields

(II.12.18.2)



We remark that each Galois group with prime superscript naturally identifies with an open subgroup of the corresponding Galois group without prime superscript.

Recall that $\widehat{\overline{B}}[1/p]$ is a $\mathbb{Q}_p$-Banach algebra endowed with the canonical norm defined by $\widehat{\overline{B}}$ (see II.6.3) and the continuous action of G. Consider a $\widehat{\overline{B}}[1/p]$-Banach module W endowed with a semi-linear continuous action of G. Then, the $\mathcal{H}$-invariant part $W^{\mathcal{H}}$ is a $(\widehat{\overline{B}}[1/p])^{\mathcal{H}}$-Banach module endowed with the induced continuous action of $\mathcal{G}$. The spaces of locally analytic vectors $(W^{\mathcal{H}'})^{\mathcal{G}'\text{-la}}$ (resp. $(W^{\mathcal{H}'})^{\mathcal{G}'_H\text{-la}}$) form a directed system of $\mathbb{Q}_p$-linear subspaces of W over $\mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$. We denote its colimit by

$$W^{\mathcal{G}\text{-la}} = \operatorname*{colim}_{\mathcal{L}' \in \mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}} (W^{\mathcal{H}'})^{\mathcal{G}'\text{-la}} \tag{II.12.18.3}$$

$$(\text{resp. } W^{\mathcal{G}_H\text{-la}} = \operatorname*{colim}_{\mathcal{L}' \in \mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}} (W^{\mathcal{H}'})^{\mathcal{G}'_H\text{-la}}) \tag{II.12.18.4}$$

and we call it the subspace of *$\mathcal{G}$-locally analytic* (resp. *$\mathcal{G}_H$-locally analytic*) *vectors of W* (we take the colimit here for the flexibility of replacing $\mathcal{L}$ by a finite extension $\mathcal{L}'$ in the proof of II.12.22). It is G-stable and endowed with the infinitesimal Lie algebra action φ^{la} of $\mathrm{Lie}(\mathcal{G})$ (resp. $\mathrm{Lie}(\mathcal{G}_H)$) by

II.12.15. We extend this action $\widehat{\overline{B}}[1/p]$-linearly to

$$1\otimes\varphi^{\mathrm{la}}:(\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}))\times W^{\mathcal{G}\text{-la}}\longrightarrow W \tag{II.12.18.5}$$

$$(\text{resp. } 1\otimes\varphi^{\mathrm{la}}:(\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}_H))\times W^{\mathcal{G}_H\text{-la}}\longrightarrow W). \tag{II.12.18.6}$$

II.12.19. With the notation in II.12.18, assume that $\mathcal{G}$ is a uniform pro-p group. Then, we consider three G-actions $\{1\otimes\rho_{\mathrm{L}},1\otimes\rho_{\mathrm{R}},\rho_W\otimes\rho_{\mathrm{L}}\}$ on $\mathscr{C}^{\mathrm{an}}(\mathcal{G},W)$ defined by the same formula as in (II.12.6.1). By II.12.15 and II.12.12, there are infinitesimal Lie algebra actions $\varphi_{\mathrm{L}},\varphi_{\mathrm{R}}$ of $\mathrm{Lie}(\mathcal{G})$ on $\mathscr{C}^{\mathrm{an}}(\mathcal{G},W)$ associated to the left and right translation of $\mathcal{G}$, which commute with each other. We also extend them $\widehat{\overline{B}}[1/p]$-linearly to

$$1\otimes\varphi_{\mathrm{L}},1\otimes\varphi_{\mathrm{R}}:(\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}))\times\mathscr{C}^{\mathrm{an}}(\mathcal{G},W)\longrightarrow\mathscr{C}^{\mathrm{an}}(\mathcal{G},W). \tag{II.12.19.1}$$

Lemma II.12.20 (cf. [**Pan22**, 2.1.4, 3.3.5])**.** *With the notation in* II.12.18 *and* II.12.19*, assume that $\mathcal{G}$ is a uniform pro-p group. Then, the map*

$$(W^{\mathcal{H}})^{\mathcal{G}\text{-an}}\longrightarrow\mathscr{C}^{\mathrm{an}}(\mathcal{G},W)^{(\rho_W\otimes\rho_{\mathrm{L}})(G)=1},\ v\mapsto(f_v:g\mapsto gv) \tag{II.12.20.1}$$

is well-defined and bijective. Moreover, for any $v\in(W^{\mathcal{H}})^{\mathcal{G}\text{-an}}$ and $\phi\in\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G})$, we have

$$(1\otimes\varphi^{\mathrm{la}})(\phi,v)=-(1\otimes\varphi_{\mathrm{L}})(\phi,f_v)(1)\in W. \tag{II.12.20.2}$$

Proof. The first part follows from the same argument of II.12.10. For the "moreover" part, by assumption, we write $\phi=\sum_{i=1}^n a_i\otimes\log_{\mathcal{G}}(g_i)$ where $a_i\in\widehat{\overline{B}}[1/p]$ and $g_i\in\mathcal{G}$. Then, by the definition, we have

$$\begin{aligned}(1\otimes\varphi_{\mathrm{L}})(\phi,f_v)(1)&=\sum_{i=1}^n a_i\cdot\lim_{n\to\infty}p^{-n}(\rho_{\mathrm{L}}(g_i^{p^n})f_v-f_v)(1)\\&=\sum_{i=1}^n a_i\cdot\lim_{n\to\infty}p^{-n}(g_i^{-p^n}v-v)\\&=-\sum_{i=1}^n a_i\varphi_{g_i}(v)=-(1\otimes\varphi^{\mathrm{la}})(\phi,v),\end{aligned} \tag{II.12.20.3}$$

where the first equality follows from the fact that taking limits in $\mathscr{C}^{\mathrm{an}}(\mathcal{G},W)$ with respect to its norm commutes with evaluating at $1\in\mathcal{G}$. □

Lemma II.12.21. *With the notation in* II.12.18 *and* II.12.19*, assume that $\mathcal{G}$ is a uniform pro-p group and that $\mathcal{G}/\mathcal{G}_H$ is isomorphic to 0 or $\mathbb{Z}_p$. We take $\sigma_0\in\mathcal{G}$ whose image in $\mathcal{G}/\mathcal{G}_H$ is a topological generator, and denote by $\sigma_0^{\mathbb{Z}_p}$ the closed subgroup of $\mathcal{G}$ generated by σ_0. Then, the map*

$$(W^{\mathcal{H}})^{\mathcal{G}_H\text{-an}}\longrightarrow\{f\in\mathscr{C}^{\mathrm{an}}(\mathcal{G},W)\mid(\rho_W\otimes\rho_{\mathrm{L}})(H)f=f,\ (1\otimes\rho_{\mathrm{R}})(\sigma_0^{\mathbb{Z}_p})f=f\} \tag{II.12.21.1}$$

sending v to $(f_v:h\sigma_0^a\mapsto hv)$, where $h\in\mathcal{G}_H$ and $a\in\mathbb{Z}_p$, is well-defined and bijective. Moreover, for any $v\in(W^{\mathcal{H}})^{\mathcal{G}_H\text{-an}}$ and $\phi\in\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}_H)$, we have

$$(1\otimes\varphi^{\mathrm{la}})(\phi,v)=-(1\otimes\varphi_{\mathrm{L}})(\phi,f_v)(1)\in W. \tag{II.12.21.2}$$

Proof. If $\mathcal{G}/\mathcal{G}_H=0$, then we reduce to II.12.20. Assume that $\mathcal{G}/\mathcal{G}_H=\mathbb{Z}_p$. Firstly, we see that there is a canonical bijection by the same argument of II.12.10,

$$(W^{\mathcal{H}})^{\mathcal{G}_H\text{-an}}\xrightarrow{\sim}\mathscr{C}^{\mathrm{an}}(\mathcal{G}_H,W)^{(\rho_W\otimes\rho_{\mathrm{L}})(H)=1},\ v\mapsto(f'_v:h\mapsto hv). \tag{II.12.21.3}$$

Notice that $\mathcal{G}_H$ is a uniform pro-p group and there is an exact sequence of the powerful Lie algebras over $\mathbb{Z}_p$ by II.3.7,

$$0\longrightarrow L_{\mathcal{G}_H}\longrightarrow L_{\mathcal{G}}\longrightarrow L_{\mathcal{G}/\mathcal{G}_H}\longrightarrow 0. \tag{II.12.21.4}$$

We take a $\mathbb{Z}_p$-linear basis $h_1,\dots,h_d$ of $L_{\mathcal{G}_H}$. Then, $\sigma_0,h_1,\dots,h_d$ form a $\mathbb{Z}_p$-linear basis of $L_{\mathcal{G}}$, and thus they also form a minimal topological generating set of $\mathcal{G}$ (see II.3.4). We obtain two systems of coordinates of second kind (II.3.3.4),

$$\mathbb{Z}_p^d\longrightarrow\mathcal{G}_H,\ (a_1,\dots,a_d)\mapsto h_1^{a_1}\cdots h_d^{a_d} \tag{II.12.21.5}$$

$$\mathbb{Z}_p^{1+d}\longrightarrow\mathcal{G},\ (a_0,a_1,\dots,a_d)\mapsto h_1^{a_1}\cdots h_d^{a_d}\sigma_0^{a_0}. \tag{II.12.21.6}$$

By II.12.3 and II.12.5, the map of underlying sets $\mathcal{G}\to\mathcal{G}_H$ sending $h_1^{a_1}\cdots h_d^{a_d}\sigma_0^{a_0}$ to $h_1^{a_1}\cdots h_d^{a_d}$ induces an injective map by pullback

$$\mathscr{C}^{\mathrm{an}}(\mathcal{G}_H,W)\longrightarrow\mathscr{C}^{\mathrm{an}}(\mathcal{G},W),\ f'\mapsto(h\sigma_0^a\mapsto f'(h)), \tag{II.12.21.7}$$

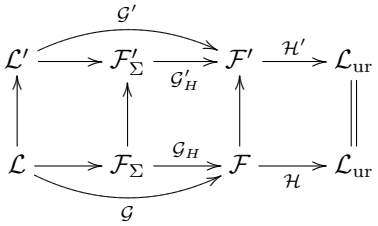
whose image is $\mathscr{C}^{\mathrm{an}}(\mathcal{G},W)^{(1\otimes\rho_{\mathrm{R}})(\sigma_0^{\mathbb{Z}_p})=1}$. This shows that (II.12.21.1) is a well-defined bijection. The "moreover" part follows from the same argument of that of II.12.20. □

Theorem II.12.22 (cf. [**Pan22**, 3.3.5])**.** *With the notation in* II.12.18*, assume that B is a quasi-adequate $\mathcal{O}_K$-algebra. Let $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ be the canonical $\widehat{\overline{B}}[1/p]$-linear Lie subalgebra of $\widehat{\overline{B}}[1/p]\otimes_{\mathbb{Q}_p}\mathrm{Lie}(\mathcal{G}_H)$ defined in* II.11.22*. Then, under the canonical infinitesimal action $1\otimes\varphi^{\mathrm{la}}$* (II.12.18.6) *by taking $W=\widehat{\overline{B}}[1/p]$, $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ annihilates $(\widehat{\overline{B}}[1/p])^{\mathcal{G}_H\text{-la}}$.*

Proof. Consider the cofinal subsystem $\mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ of $\mathscr{F}^{\mathrm{fini}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ defined in II.11.9. Recall that for any $\mathcal{L}'\in\mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$, we have natural identifications $\Phi_{\mathcal{G}}^{\mathrm{geo}}=\Phi_{\mathcal{G}'}^{\mathrm{geo}}$ by II.11.20.(5). Thus, it suffices to show that $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ annihilates $(\widehat{\overline{B}}[1/p]^{\mathcal{H}})^{\mathcal{G}_H\text{-la}}$. Moreover, after replacing $\mathcal{G}$ by a sufficiently small uniform pro-p open subgroup, it suffices to show that $\Phi_{\mathcal{G}}^{\mathrm{geo}}$ annihilates $(\widehat{\overline{B}}[1/p]^{\mathcal{H}})^{\mathcal{G}_H\text{-an}}$ (so that $\mathcal{L}$ may not lies in $\mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ from now on). Recall that there exists a sufficiently small uniform pro-p open subgroup $\mathcal{G}_0$ of $\mathcal{G}$ such that $\mathcal{G}_{0H}=\mathcal{G}_0\cap\mathcal{G}_H$ and $\mathcal{G}_0/\mathcal{G}_{0H}$ are both uniform by II.3.10. Replacing $\mathcal{G}$ by $\mathcal{G}_0$, we obtain the following conditions on $\mathcal{G}$:

(1) $\mathcal{G}$ is a uniform pro-p group isomorphic to a closed subgroup of $\mathrm{id}+p^2\mathrm{M}_d(\mathbb{Z}_p)$ for some $d\in\mathbb{N}$ (using II.3.9), and
(2) $\mathcal{G}/\mathcal{G}_H$ is isomorphic to 0 or $\mathbb{Z}_p$ (as it is a uniform pro-p subquotient of $G/H=\Sigma\subseteq\mathbb{Z}_p^\times$).

(II.12.22.1)



We take $\mathcal{L}'\in\mathscr{F}^{\mathrm{qa}}_{\mathcal{L}_{\mathrm{ur}}/\mathcal{L}}$ sufficiently large such that the image of the natural injection $\mathcal{G}'\to\mathcal{G}$ is contained in $\mathcal{G}^{p^3}$ (recall that $\mathcal{F}'=\mathcal{L}'\mathcal{F}$ by definition). Then, $\mathcal{G}'$ acts trivially on $\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1}/p^3\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1}$ via the left and right translation actions by II.12.7. Combining with II.12.8, we can apply II.11.26 to the $\mathbb{Q}_p$-Banach space $\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)$ endowed with the left translation action $1\otimes\rho_{\mathrm{L}}$ of $\mathcal{G}'$, so that $\Phi_{\mathcal{G}'}^{\mathrm{geo}}$ acts trivially on $((\widehat{\overline{B}}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1})^\wedge[1/p])^{H'}$.

Notice that there is a natural identification $(\widehat{\overline{B}}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1})^\wedge[1/p]=\mathscr{C}^{\mathrm{an}}(\mathcal{G},\widehat{\overline{B}}[1/p])$ by II.12.2, which satisfies the following properties (by firstly checking over the submodule $\widehat{\overline{B}}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1}$ and then taking p-adic completion):

(1) the action of G' on $(\widehat{\overline{B}}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1})^\wedge[1/p]$ defined in II.11.24 coincides with the diagonal action $\rho_{\widehat{\overline{B}}[1/p]}\otimes\rho_{\mathrm{L}}$ on $\mathscr{C}^{\mathrm{an}}(\mathcal{G},\widehat{\overline{B}}[1/p])$, and
(2) the action of $\Phi_{\mathcal{G}'}$ on $(\widehat{\overline{B}}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1})^\wedge[1/p]$ defined by (II.11.24.3) coincides with the infinitesimal action $1\otimes\varphi_{\mathrm{L}}$ (II.12.19.1) on $\mathscr{C}^{\mathrm{an}}(\mathcal{G},\widehat{\overline{B}}[1/p])$ induced by the left translation of $\mathcal{G}$ (cf. II.12.16).

Therefore, applying II.12.21 we see that for any $v\in((\widehat{\overline{B}}[1/p])^{\mathcal{H}})^{\mathcal{G}_H\text{-an}}$ and any $\phi\in\Phi_{\mathcal{G}}^{\mathrm{geo}}$,

$$(1\otimes\varphi^{\mathrm{la}})(\phi,v)=-(1\otimes\varphi_{\mathrm{L}})(\phi,f_v)(1)=0, \tag{II.12.22.2}$$

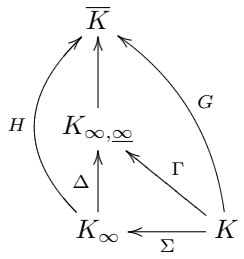
as $(f_v: h\sigma_0^a\mapsto hv)$ is an element of $\mathscr{C}^{\mathrm{an}}(\mathcal{G},\widehat{\overline{B}}[1/p])^{(\rho_{\widehat{\overline{B}}[1/p]}\otimes\rho_{\mathrm{L}})(H)=1}\subseteq((\widehat{\overline{B}}\otimes_{\mathbb{Z}_p}\mathscr{C}^{\mathrm{an}}(\mathcal{G},\mathbb{Q}_p)^{\leq 1})^\wedge[1/p])^{H'}$ killed by $\Phi_{\mathcal{G}'}^{\mathrm{geo}}=\Phi_{\mathcal{G}}^{\mathrm{geo}}$. □

Remark II.12.23. We don't know whether $\Phi_{\mathcal{G}}$ annihilates $(\widehat{\overline{B}}[1/p])^{\mathcal{G}\text{-la}}$ or not, cf. II.6.20.

II.13. Appendix: Hyodo's Computation of Galois Cohomologies

This appendix is devoted to detailed proofs of II.5.9 and II.5.14, using essentially Hyodo's arguments in [**Hyo86**] and [**Hyo89**]. We take again the notation in II.5.1.

(II.13.0.1)



Lemma II.13.1 (cf. [**Hyo86**, 2-1])**.** *Assume that K satisfies the assumption $(*)$ in* II.6.10. *Let $r, q \in \mathbb{N}$.*

(1) *The natural map $H^q(\Delta, \mathcal{O}_{K_{\infty,\underline{\infty}}}/p^r\mathcal{O}_{K_{\infty,\underline{\infty}}}) \to H^q(H, \mathcal{O}_{\overline{K}}/p^r\mathcal{O}_{\overline{K}})$ is an almost isomorphism.*

(2) *The natural map induced by the cup product $\wedge^q\mathrm{Hom}(\Delta, \mathcal{O}_{K_\infty}/p^r\mathcal{O}_{K_\infty}) \to H^q(\Delta, \mathcal{O}_{K_{\infty,\underline{\infty}}}/p^r\mathcal{O}_{K_{\infty,\underline{\infty}}})$ is injective, and admits a natural retraction with kernel killed by $\zeta_p - 1$.*

Proof. (1) If we take K' as in II.5.2, then $\mathcal{O}_{K'_{\infty,\underline{\infty}}} = \mathrm{colim}_{n\in\mathbb{N}}\mathcal{O}_{K'}[\zeta_{p^n}, t_{1,p^n}, \dots, t_{d,p^n}]$ (by II.5.3) is a non-discrete valuation ring of height 1 such that the Frobenius map on $\mathcal{O}_{K'_{\infty,\underline{\infty}}}/p\mathcal{O}_{K'_{\infty,\underline{\infty}}}$ is surjective. In other words, $K'_{\infty,\underline{\infty}}$ is a pre-perfectoid field. As K is finite over K', $\mathcal{O}_{K_{\infty,\underline{\infty}}}$ is almost étale over $\mathcal{O}_{K'_{\infty,\underline{\infty}}}$ by almost purity (II.7.12), and the conclusion follows by almost Galois descent (cf. [**AGT16**, II.6.24]).

(2) By (II.6.10.2), we can decompose $\mathcal{O}_{K_{\infty,\underline{\infty}}}$ into a direct sum of free $\mathcal{O}_{K_\infty}$-submodules of rank 1,

$$\mathcal{O}_{K_{\infty,\underline{\infty}}} = \bigoplus_{\underline{m}\in\mathbb{N}^d}\bigoplus_{\underline{k}\in J_{\leq p^{\underline{m}}}} \mathcal{O}_{K_\infty} t_{1,p^{m_1}}^{k_1}\cdots t_{d,p^{m_d}}^{k_d}, \tag{II.13.1.1}$$

where $J \subseteq \mathbb{N}_{>0}^d$ is the subset of tuples with components prime to p. It induces a natural retraction of the inclusion $\mathcal{O}_{K_\infty} \to \mathcal{O}_{K_{\infty,\underline{\infty}}}$. Notice that Δ acts on $\mathcal{O}_{K_\infty} t_{1,p^{m_1}}^{k_1}\cdots t_{d,p^{m_d}}^{k_d}$ by the multiplication by a group homomorphism from Δ to the group of roots of unity contained in $\overline{K}^\times$, which is trivial if and only if $\underline{m} = 0$. By [**AGT16**, II.8.1], we see that $\wedge^q\mathrm{Hom}(\Delta, \mathcal{O}_{K_\infty}/p^r) = H^q(\Delta, \mathcal{O}_{K_\infty}/p^r)$ and that $H^q(\Delta, \mathcal{O}_{K_\infty} t_{1,p^{m_1}}^{k_1}\cdots t_{d,p^{m_d}}^{k_d}/p^r)$ is killed by $\zeta_p - 1$ if $\underline{m} \neq 0$. □

Proposition II.13.2. *The connecting map of the Faltings extension* (II.5.7.1) *induces a canonical $\widehat{K_\infty}$-linear isomorphism*

$$\widehat{K_\infty} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \xrightarrow{\sim} H^1(H, \widehat{\overline{K}}(1)), \tag{II.13.2.1}$$

sending $\mathrm{d}\log(t_i)$ to $\xi_i \otimes \zeta$, where $\zeta = (\zeta_{p^n})_{n\in\mathbb{N}} \in \mathbb{Z}_p(1)$ and $\xi = (\xi_1, \dots, \xi_d) : H \to \mathbb{Z}_p^d$ is the continuous 1-cocycle (II.4.15.4). *Moreover, for any $q \in \mathbb{N}$, the cup product induces a natural isomorphism*

$$\widehat{K_\infty} \otimes_{\mathcal{O}_K} \wedge^q_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K} \xrightarrow{\sim} H^q(H, \widehat{\overline{K}}(q)). \tag{II.13.2.2}$$

Proof. The statement itself defines a natural map $\widehat{K_\infty} \otimes_{\mathcal{O}_K} \wedge^q_{\mathcal{O}_K}\widehat{\Omega}^1_{\mathcal{O}_K} \to H^q(H, \widehat{\overline{K}}(q))$ for any $q \in \mathbb{N}$. We only need to prove that it is an isomorphism. For $q = 0$, this follows from Ax-Sen-Tate's theorem [**Ax70**].

Assume that K satisfies the assumption $(*)$ in II.6.10. Then, the natural map

$$\wedge^q\mathrm{Hom}(\Delta, K_\infty/p^r\mathcal{O}_{K_\infty}) \longrightarrow H^q(H, \overline{K}/p^r\mathcal{O}_{\overline{K}}) \tag{II.13.2.3}$$

is a p^2-isomorphism by II.13.1 for any $q, r \in \mathbb{N}$. Consider the canonical exact sequence for any $q \in \mathbb{N}$ ([**AGT16**, II.3.10.4, II.3.10.5])

$$0 \to \mathrm{R}^1\lim_{r\in\mathbb{N}} H^{q-1}(H, \overline{K}/p^r\mathcal{O}_{\overline{K}}) \to H^q(H, \widehat{\overline{K}}) \to \lim_{r\in\mathbb{N}} H^q(H, \overline{K}/p^r\mathcal{O}_{\overline{K}}) \to 0. \tag{II.13.2.4}$$

Since the inverse system $(\wedge^{q-1}\mathrm{Hom}(\Delta, K_\infty/p^r\mathcal{O}_{K_\infty}))_{r\in\mathbb{N}}$ satisfies the Mittag-Leffler condition, we have $\mathrm{R}^1\lim_{r\in\mathbb{N}} \wedge^{q-1}\mathrm{Hom}(\Delta, K_\infty/p^r\mathcal{O}_{K_\infty}) = 0$. Thus, $\mathrm{R}^1\lim_{r\in\mathbb{N}} H^{q-1}(H, \overline{K}/p^r\mathcal{O}_{\overline{K}})$ is killed by p^4 by the p^2-isomorphism (II.13.2.3) and II.7.3.(2). Moreover, it is actually zero since multiplication by p is invertible on $H^q(H, \widehat{\overline{K}})$. Then, we get $H^q(H, \widehat{\overline{K}}) = \wedge^q\mathrm{Hom}(\Delta, \widehat{K_\infty})$ by a similar argument. Unwinding the definitions, we get the conclusion in this case.

In general, there exists a finite Galois extension K' of K which satisfies the assumption $(*)$ in II.6.10. Remark that the map $K' \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \to \widehat{\Omega}^1_{\mathcal{O}_{K'}}[1/p]$ is an isomorphism. By Ax-Sen-Tate's theorem, we have $(\widehat{K'_\infty})^{H/H'} = \widehat{K_\infty}$, where $H' = \mathrm{Gal}(\overline{K}/K'_\infty)$. Thus,

$$\widehat{K_\infty} \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K} \longrightarrow (\widehat{K'_\infty} \otimes_{\mathcal{O}_{K'}} \widehat{\Omega}^1_{\mathcal{O}_{K'}})^{H/H'} \tag{II.13.2.5}$$

is an isomorphism. In particular, the restriction map

$$\mathrm{Res} : H^q(H, \widehat{\overline{K}}(q)) \longrightarrow H^q(H', \widehat{\overline{K}}(q))^{H/H'} \tag{II.13.2.6}$$

is surjective, since the natural map $\widehat{K'_\infty} \otimes_{\mathcal{O}_{K'}} \widehat{\Omega}^1_{\mathcal{O}_{K'}} \to H^q(H', \widehat{\overline{K}}(q))$ is an isomorphism by applying the discussion above to K'. It is also injective, since there is a co-restriction map $\mathrm{Cor} : H^q(H', \widehat{\overline{K}}(q))^{H/H'} \to H^q(H, \widehat{\overline{K}}(q))$ such that $\mathrm{Cor} \circ \mathrm{Res} = [K' : K] \cdot \mathrm{id}$ (cf. [**Tat76**, §2]). Thus, we deduce the proposition for K from the special case for K'. □

II.13.3. For simplicity, we put $\widehat{\Omega}^1 = K \otimes_{\mathcal{O}_K} \widehat{\Omega}^1_{\mathcal{O}_K}(-1)$ and $\widehat{\Omega}^i = \wedge^i_K \widehat{\Omega}^1$ for any $i \in \mathbb{Z}$ (where $\widehat{\Omega}^i = 0$ if $i < 0$). Recall that there is a natural exact sequence (II.5.11.2) for any $n \in \mathbb{Z}$ (where $\mathrm{Sym}^n = 0$ if $n < 0$),

$$0 \longrightarrow \mathrm{Sym}^{n-1}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) \longrightarrow \mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) \longrightarrow \widehat{\overline{K}} \otimes_K \mathrm{Sym}^n_K \widehat{\Omega}^1 \longrightarrow 0. \tag{II.13.3.1}$$

It induces a natural exact sequence

(II.13.3.2)
$$0 \longrightarrow \widehat{\overline{K}} \otimes_K \mathrm{Sym}^{n-1}_K \widehat{\Omega}^1 \longrightarrow \mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))/\mathrm{Sym}^{n-2}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) \longrightarrow \widehat{\overline{K}} \otimes_K \mathrm{Sym}^n_K \widehat{\Omega}^1 \longrightarrow 0.$$

For any $i, j \in \mathbb{Z}$, we set

$$E_1^{i,j} = H^{i+j}(H, \widehat{\overline{K}} \otimes_K \mathrm{Sym}^{-i}_K \widehat{\Omega}^1) \tag{II.13.3.3}$$

and we denote by $\mathrm{d}_1^{i,j} : E_1^{i,j} \to E_1^{i+1,j}$ the $(i+j)$-th connecting map associated to (II.13.3.2) for $n = -i$. We remark that if we endow each $\mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))$ with the finite decreasing filtration $(\mathrm{F}^i)_{i\in\mathbb{Z}}$ given by

$$\mathrm{F}^i \mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) = \begin{cases} \mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) & \text{if } i < -n, \\ \mathrm{Sym}^{-i}_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1)) & \text{otherwise,} \end{cases} \tag{II.13.3.4}$$

then the associated spectral sequence of the group cohomology ([**Sta22**, 012M])

$${}_nE_1^{i,j} \Rightarrow H^{i+j}(H, \mathrm{Sym}^n_{\widehat{\overline{K}}}(\mathscr{E}_{\mathcal{O}_K}(-1))) \tag{II.13.3.5}$$

is convergent and is given by

$${}_nE_1^{i,j} = \begin{cases} 0 & \text{if } i < -n, \\ E_1^{i,j} & \text{otherwise,} \end{cases} \qquad {}_n\mathrm{d}_1^{i,j} = \begin{cases} 0 & \text{if } i < -n, \\ \mathrm{d}_1^{i,j} & \text{otherwise.} \end{cases} \tag{II.13.3.6}$$

Lemma II.13.4 (cf. [**AGT16**, III.5.7, III.11.11])**.** *For any $i, j \in \mathbb{Z}$, we define a K-linear map $\phi^{i,j} : \mathrm{Sym}^i_K \widehat{\Omega}^1 \otimes_K \widehat{\Omega}^j \to \mathrm{Sym}^{i-1}_K \widehat{\Omega}^1 \otimes_K \widehat{\Omega}^{j+1}$ by*

$$[x_1 \otimes \cdots \otimes x_i] \otimes \omega \mapsto \sum_{k=1}^{i} [x_1 \otimes \cdots \otimes x_{k-1} \otimes x_{k+1} \otimes \cdots \otimes x_i] \otimes (x_k \wedge \omega), \tag{II.13.4.1}$$

for any $x_1, \ldots, x_i \in \widehat{\Omega}^1$ and $\omega \in \widehat{\Omega}^j$. Then, there is a natural commutative diagram

$$\begin{array}{ccc} \widehat{K_\infty} \otimes_K (\mathrm{Sym}^{-i}_K \widehat{\Omega}^1) \otimes_K \widehat{\Omega}^{i+j} & \xrightarrow{1\otimes\phi^{-i,i+j}} & \widehat{K_\infty} \otimes_K (\mathrm{Sym}^{-i-1}_K \widehat{\Omega}^1) \otimes_K \widehat{\Omega}^{i+j+1} \\ \downarrow{\scriptstyle \alpha^{i,j}} & & \downarrow{\scriptstyle \alpha^{i+1,j}} \\ E_1^{i,j} & \xrightarrow{\mathrm{d}_1^{i,j}} & E_1^{i+1,j} \end{array} \tag{II.13.4.2}$$

where the vertical maps are the natural isomorphisms induced by (II.13.2.2).

PROOF. As in II.5.12, we set $T_k = (\mathrm{d}\log(t_{k,p^n}))_{n\in\mathbb{N}} \otimes \zeta^{-1} \in \mathscr{E}_{\mathcal{O}_K}(-1)$ for any $1 \leq k \leq d$. We remark that their images $\mathrm{d}T_k$ in $\widehat{\Omega}^1$ form a K-basis, and that $h(T_k) = \xi_k(h) + T_k$ for any $h \in H$. For any $1 \leq r_1, \dots, r_i, s_1, \dots, s_j \leq d$, we have

$$\begin{aligned}
(\mathrm{II.13.4.3}) \qquad & (\mathrm{d}_1^{-i,i+j} \circ \alpha^{-i,i+j})([\mathrm{d}T_{r_1} \otimes \cdots \otimes \mathrm{d}T_{r_i}] \otimes (\mathrm{d}T_{s_1} \wedge \cdots \wedge \mathrm{d}T_{s_j})) \\
=& \mathrm{d}_1^{-i,i+j}([\mathrm{d}T_{r_1} \otimes \cdots \otimes \mathrm{d}T_{r_i}] \otimes (\xi_{s_1} \cup \cdots \cup \xi_{s_j})) \\
=& (h \mapsto [(\xi_{r_1}(h) + T_{r_1}) \otimes \cdots \otimes (\xi_{r_i}(h) + T_{r_i}) - T_{r_1} \otimes \cdots \otimes T_{r_i}]) \cup \xi_{s_1} \cup \cdots \cup \xi_{s_j} \\
=& \sum_{k=1}^{i} [\mathrm{d}T_{r_1} \otimes \cdots \otimes \mathrm{d}T_{r_{k-1}} \otimes \mathrm{d}T_{r_{k+1}} \otimes \cdots \otimes \mathrm{d}T_{r_i}] \otimes (\xi_{r_k} \cup \xi_{s_1} \cup \cdots \cup \xi_{s_j}) \\
=& (\alpha^{-i+1,i+j} \circ (1 \otimes \phi^{i,j}))([\mathrm{d}T_{r_1} \otimes \cdots \otimes \mathrm{d}T_{r_i}] \otimes (\mathrm{d}T_{s_1} \wedge \cdots \wedge \mathrm{d}T_{s_j})).
\end{aligned}$$

□

Corollary II.13.5 (cf. [**AGT16**, III.11.12])**.** *We have*

$$(\mathrm{II.13.5.1}) \qquad \operatorname*{colim}_{n\in\mathbb{N}} H^q(H, \mathrm{Sym}^n_{\widehat{K}}(\mathscr{E}_{\mathcal{O}_K}(-1))) = \begin{cases} \widehat{K_\infty} & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $(\mathscr{C}_{\mathrm{HT}})^H = \widehat{K_\infty}$.

PROOF. For any $n \in \mathbb{N}_{>0}$, the differential map (II.13.4.1) defined in II.13.4 induces a natural sequence

$$(\mathrm{II.13.5.2}) \qquad 0 \longrightarrow \mathrm{Sym}^n_K \widehat{\Omega}^1 \xrightarrow{\phi^{n,0}} \mathrm{Sym}^{n-1}_K \widehat{\Omega}^1 \otimes_K \widehat{\Omega}^1 \xrightarrow{\phi^{n-1,1}} \cdots \xrightarrow{\phi^{2,n-2}} \widehat{\Omega}^1 \otimes_K \widehat{\Omega}^{n-1} \xrightarrow{\phi^{1,n-1}} \widehat{\Omega}^n \longrightarrow 0.$$

which is exact by [**Hyo89**, Lemma 1.2] (cf. [**AGT16**, III.5.1]). Therefore, by II.13.4, the sequence $(E_1^{\bullet,j}, \mathrm{d}_1^{\bullet,j})$ is exact for any $j \neq 0$, and nonzero only at $E_1^{0,0} = \widehat{K_\infty}$ if $j = 0$. Then, we see that for any $n \in \mathbb{N}$, the nonzero terms ${}_nE_2^{i,j}$ of the second page of the spectral sequence (II.13.3.5) appear on the positions

$$(\mathrm{II.13.5.3}) \qquad (i,j) \in \{(0,0), (-n, n+m) \mid m \in \mathbb{N}\}.$$

In particular, the spectral sequence degenerates at the second page, and we see that $H^0(H, \mathrm{Sym}^n_{\widehat{K}}(\mathscr{E}_{\mathcal{O}_K}(-1))) = \widehat{K_\infty}$ and that $H^q(H, \mathrm{Sym}^n_{\widehat{K}}(\mathscr{E}_{\mathcal{O}_K}(-1))) \to H^q(H, \mathrm{Sym}^{n+1}_{\widehat{K}}(\mathscr{E}_{\mathcal{O}_K}(-1)))$ is zero for any $q \neq 0$. □

II.14. Appendix: Faltings Extensions as Graded Pieces of De Rham Period Rings

The Faltings extension and Hyodo ring (the Hodge-Tate period ring) are also constructed as graded pieces of de Rham period ring in the literature in various p-adic geometric settings. See [**Bri08**, §5] for the good reduction case, [**Tsu11**, §2] for the semi-stable reduction case, [**Tsu18**, §15] for the adequate case, and [**Sch13a**, §6], [**Sch16**] for smooth adic spaces. This appendix is devoted to a comparison between our construction of the Faltings extension with theirs. We fix a complete discrete valuation field K of characteristic 0 with perfect residue field of characteristic $p > 0$, an algebraic closure $\overline{K}$ of K, and a compatible system of primitive n-th roots of unity $(\zeta_n)_{n\in\mathbb{N}}$ in $\overline{K}$.

Lemma II.14.1. *Let* $(A_{\mathrm{tr}}, A, \overline{A})$ *be a* $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$*-triple in the sense of* II.9.2*,* $\mathcal{K}$ *the fraction field of* A*,* $G = \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$*. Then,* $\widehat{\overline{A}}[\frac{1}{p}](1)^G = 0$*, where* (1) *denotes the first Tate twist.*

PROOF. By II.9.4, we reduce to the case where A is a complete discrete valuation ring extension of $\mathcal{O}_K$, which is proved by Tate ([**Tat67**, Theorem 2], cf. [**Hyo86**, Theorem 1]). □

Lemma II.14.2. *Let* $(A_{\mathrm{tr}}, A, \overline{A})$ *be a* $(K, \mathcal{O}_K, \mathcal{O}_{\overline{K}})$*-triple in the sense of* II.9.2*,* $\mathcal{K}$ *the fraction field of* A*,* $G = \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$*. Consider an isomorphism* $f : \mathscr{E} \to \mathscr{E}'$ *of extensions of an* $\widehat{\overline{A}}[1/p]$*-representation* W *of* G *by* $\widehat{\overline{A}}[1/p](1)$ *making the following diagram commutative.*

$$(\mathrm{II.14.2.1}) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\overline{A}}[\frac{1}{p}](1) & \xrightarrow{\iota} & \mathscr{E} & \xrightarrow{\jmath} & W & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \cdot(-1)} & & \\
0 & \longrightarrow & \widehat{\overline{A}}[\frac{1}{p}](1) & \xrightarrow{\iota'} & \mathscr{E}' & \xrightarrow{\jmath'} & W & \longrightarrow & 0
\end{array}$$

Let x *and* x' *be elements of* $\mathscr{E}$ *and* $\mathscr{E}'$ *respectively. Assume that* $\jmath(x) = -\jmath'(x') \in W^G$ *and* $g(x) - x = g(x') - x' \in \widehat{\overline{A}}[1/p](1)$ *for any* $g \in G$*. Then,* $f(x) = x'$.

PROOF. If we set $y = x' - f(x) \in \mathscr{E}'$, then $j'(y) = 0$ and $g(y) = y$ for any $g \in G$ by assumption, which implies that $y = 0$ by II.14.1. □

II.14.3. For simplicity, we only consider Tsuji's construction of de Rham period ring [**Tsu18**, §15]. The comparison with other constructions of the Faltings extension should be clear from our arguments in the following. We quickly review Tsuji's construction and state properties without proofs. Let A be an adequate $\mathcal{O}_K$-algebra of relative dimension d satisfying the following condition:

(1) We set $X = (\mathrm{Spec}(A), \alpha_{\mathrm{Spec}(A_{\mathrm{tr}})\to\mathrm{Spec}(A)})$ (cf. II.9.6). Then, the monoid $\Gamma(X, \mathcal{M}_X)/A^\times$ is finitely generated, and the identity of $\Gamma(X, \mathcal{M}_X)$ induces a chart of X (cf. [**Tsu99**, 1.3.2]).

We remark that any adequate $\mathcal{O}_K$-algebra Zariski locally satisfies this condition ([**Tsu99**, 1.3.3]). Let $\mathcal{K}$ be the fraction field of A, $G = \mathrm{Gal}(\mathcal{K}_{\mathrm{ur}}/\mathcal{K})$. Consider the tilt $\overline{A}^\flat = \varprojlim_{x\mapsto x^p} \overline{A}/p\overline{A}$ of $\overline{A}$, and we put $W_{\mathcal{O}_K}(\overline{A}^\flat) = \mathcal{O}_K \otimes_{W(\kappa)} W(\overline{A}^\flat)$, where κ is the residue field of K, and $W(-)$ is taking the ring of Witt vectors. Let $[-] : \overline{A}^\flat \to W(\overline{A}^\flat)$ denote the multiplicative lift, π a uniformizer of K with compatible system of p^n-th root $(\pi_{p^n})_{n\in\mathbb{N}}$ contained in $\mathcal{O}_{\overline{K}}$, $\xi_\pi = \pi \otimes 1 - 1 \otimes [(\pi_{p^n})_{n\in\mathbb{N}}] \in W_{\mathcal{O}_K}(\overline{A}^\flat)$. There is a canonical exact sequence

$$0 \longrightarrow W_{\mathcal{O}_K}(\overline{A}^\flat) \xrightarrow{\cdot\xi_\pi} W_{\mathcal{O}_K}(\overline{A}^\flat) \xrightarrow{\vartheta_{\mathcal{O}_K}} \widehat{\overline{A}} \longrightarrow 0 \tag{II.14.3.1}$$

where $\vartheta_{\mathcal{O}_K}$ is the homomorphism of $\mathcal{O}_K$-algebras characterized by $\vartheta_{\mathcal{O}_K}(1 \otimes [(a_n)_{n\in\mathbb{N}}]) = \lim_{n\to\infty} \widetilde{a}_n^{p^n}$ (where $\widetilde{a}_n \in \overline{A}$ is a lift of $a_n \in \overline{A}/p\overline{A}$).

Let $\overline{X}$ be the log scheme with underlying scheme $\mathrm{Spec}(\widehat{\overline{A}})$ whose log structure is associated to $\Gamma(X, \mathcal{M}_X) \to \widehat{\overline{A}}$ (different to the notation in II.9.21). Consider the fibred product of monoids

$$\begin{array}{ccc} \overline{Q} & \longrightarrow & \Gamma(X, \mathcal{M}_X) \\ \downarrow & & \downarrow \\ \varprojlim_{x\mapsto x^p} \overline{A} & \longrightarrow & \overline{A} \end{array} \tag{II.14.3.2}$$

and let $\overline{D}$ be the log scheme with underlying scheme $\mathrm{Spec}(W(\overline{A}^\flat))$ whose log structure is associated to the composition of $\overline{Q} \to \varprojlim_{x\mapsto x^p} \overline{A} \to \overline{A}^\flat \xrightarrow{[-]} W(\overline{A}^\flat)$. The condition above on the log structure of X implies that $\overline{D}$ is an fs log scheme, and the natural maps $\vartheta : W(\overline{A}^\flat) \to \widehat{\overline{A}}$ and $\overline{Q} \to \Gamma(X, \mathcal{M}_X)$ induce an exact closed immersion $i_{\overline{D}} : \overline{X} \to \overline{D}$. We put $S = (\mathrm{Spec}(\mathcal{O}_K), \alpha_{\mathrm{Spec}(K)\to\mathrm{Spec}(\mathcal{O}_K)})$ and $\overline{D}_S = S \times_{\mathrm{Spec}(W(\kappa))} \overline{D}$, where $\mathrm{Spec}(W(\kappa))$ is endowed with the trivial log structure. Consider the induced closed immersion of fs log schemes $i_{\overline{D}_S} : \overline{X} \to \overline{D}_S$ (not exact). For any $r, m \in \mathbb{N}$, let $\overline{D}^{(r)}_{S,m}$ be the r-th infinitesimal neighbourhood of the reduction mod p^m of the closed immersion $i_{\overline{D}_S}$ in the category of fine log schemes in the sense of [**Kat89**, 5.8]. Then, the natural map

$$(W_{\mathcal{O}_K}(\overline{A}^\flat)/\xi_\pi^{r+1} W_{\mathcal{O}_K}(\overline{A}^\flat))[\frac{1}{p}] \longrightarrow (\varinjlim_{m\to\infty} \Gamma(\overline{D}^{(r)}_{S,m}, \mathcal{O}_{\overline{D}^{(r)}_{S,m}}))[\frac{1}{p}] \tag{II.14.3.3}$$

is an isomorphism ([**Tsu18**, page 870, equation (33)], cf. [**Tsu11**, 2.5]). We define

$$B^+_{\mathrm{dR}}(\overline{A}) = \varprojlim_{r\to\infty} \left((\varinjlim_{m\to\infty} \Gamma(\overline{D}^{(r)}_{S,m}, \mathcal{O}_{\overline{D}^{(r)}_{S,m}}))[\frac{1}{p}] \right) \tag{II.14.3.4}$$

which is endowed with the natural action of G and a canonical G-stable decreasing filtration by ideals

$$\mathrm{Fil}^r B^+_{\mathrm{dR}}(\overline{A}) = \mathrm{Ker}\left(B^+_{\mathrm{dR}}(\overline{A}) \to (\varinjlim_{m\to\infty} \Gamma(\overline{D}^{(r-1)}_{S,m}, \mathcal{O}_{\overline{D}^{(r-1)}_{S,m}}))[\frac{1}{p}] \right) = \xi_\pi^r B^+_{\mathrm{dR}}(\overline{A}), \tag{II.14.3.5}$$

where $r \in \mathbb{N}_{>0}$ and we put $\mathrm{Fil}^r B^+_{\mathrm{dR}}(\overline{A}) = B^+_{\mathrm{dR}}(\overline{A})$ for $r \leq 0$.

We put $\overline{D}_X = X \times_{\mathrm{Spec}(W(\kappa))} \overline{D}$. Consider the induced closed immersion of fs log schemes $i_{\overline{D}_X} : \overline{X} \to \overline{D}_X$. For any $r, m \in \mathbb{N}$, let $\overline{D}^{(r)}_{X,m}$ be the r-th infinitesimal neighbourhood of the reduction mod p^m of the closed immersion $i_{\overline{D}_X}$ in the category of fine log schemes in the sense of [**Kat89**, 5.8]. We define

$$\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) = \varprojlim_{r\to\infty} \left((\varinjlim_{m\to\infty} \Gamma(\overline{D}^{(r)}_{X,m}, \mathcal{O}_{\overline{D}^{(r)}_{X,m}}))[\frac{1}{p}] \right) \tag{II.14.3.6}$$

which is endowed with the natural action of G and a canonical G-stable decreasing filtration by ideals

$$\mathrm{Fil}^r\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) = \mathrm{Ker}\left(\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) \to (\varprojlim_{m\to\infty}\Gamma(\overline{D}^{(r-1)}_{X,m}, \mathcal{O}_{\overline{D}^{(r-1)}_{X,m}}))[\frac{1}{p}]\right), \tag{II.14.3.7}$$

where $r \in \mathbb{N}_{>0}$ and we put $\mathrm{Fil}^r\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) = \mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ for $r \leq 0$. Moreover, Tsuji [**Tsu18**, page 871] defines a G-equivariant $B^+_{\mathrm{dR}}(\overline{A})$-linear integrable connection $\nabla : \mathscr{B}^+_{\mathrm{dR}}(\overline{A}) \to \mathscr{B}^+_{\mathrm{dR}}(\overline{A}) \otimes_A \Omega^1_{X/S}$ compatible with $\mathrm{d} : A \to \Omega^1_{X/S}$ and $\mathrm{d}\log : \Gamma(X, \mathcal{M}_X) \to \Omega^1_{X/S}$ satisfying that $\nabla(\mathrm{Fil}^r\mathscr{B}^+_{\mathrm{dR}}(\overline{A})) \subseteq \mathrm{Fil}^{r-1}\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) \otimes_A \Omega^1_{X/S}$. We summarize some properties of $\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ proved by Tsuji in the following.

Proposition II.14.4 ([**Tsu18**, §15])**.** *We keep the notation in* II.14.3*.*

(1) *The element $t = \log([(\zeta_{p^n})_{n\in\mathbb{N}}]) \in B^+_{\mathrm{dR}}(\overline{A})$ is a non-zero divisor such that $t^r B^+_{\mathrm{dR}}(\overline{A}) = \mathrm{Fil}^r B^+_{\mathrm{dR}}(\overline{A})$ for any $r \in \mathbb{N}$. Moreover, the map $\vartheta_{\mathcal{O}_K}$ induces a G-equivariant isomorphism*

$$t^r B^+_{\mathrm{dR}}(\overline{A})/t^{r+1}B^+_{\mathrm{dR}}(\overline{A}) \xrightarrow{\sim} \widehat{\overline{A}}[1/p](r). \tag{II.14.4.1}$$

(2) *Let $(s_{p^n})_{n\in\mathbb{N}} \in \overline{Q}$ with image $s = s_1 \in \Gamma(X, \mathcal{M}_X)$. Then, for any $r, m \in \mathbb{N}$, there is a unique element $u^{(r)}_m \in 1 + \mathrm{Ker}\left(\Gamma(\overline{D}^{(r)}_{X,m}, \mathcal{O}_{\overline{D}^{(r)}_{X,m}}) \to \Gamma(\overline{D}^{(r-1)}_{X,m}, \mathcal{O}_{\overline{D}^{(r-1)}_{X,m}})\right) \subseteq \Gamma(\overline{D}^{(r)}_{X,m}, \mathcal{O}^\times_{\overline{D}^{(r)}_{X,m}})$ such that $s = [(s_{p^n})_{n\in\mathbb{N}}] \cdot u^{(r)}_m$ in $\Gamma(\overline{D}^{(r)}_{X,m}, \mathcal{M}_{\overline{D}^{(r)}_{X,m}})$. It induces a canonical homomorphism of monoids*

$$q : \overline{Q} \longrightarrow 1 + \mathrm{Fil}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A}),\ (s_{p^n})_{n\in\mathbb{N}} \mapsto u = (u^{(r)}_m)_{r,m\in\mathbb{N}}. \tag{II.14.4.2}$$

Moreover, $\nabla(u) = u \otimes \mathrm{d}\log(s)$.

(3) *Let $\{t_i\}_{1\leq i\leq d}$ be a system of coordinates of the adequate $\mathcal{O}_K$-algebra A with compatible systems of p-power roots $(t_{i,p^n})_{n\in\mathbb{N}}$ contained in $\overline{A}[1/p]$. For sufficiently large $k \in \mathbb{N}$ such that $\pi^k t_i \in A$, the element $u_i = q((\pi_{p^n})_{n\in\mathbb{N}})^{-k} q((\pi^k_{p^n} t_{i,p^n})_{n\in\mathbb{N}}) \in 1 + \mathrm{Fil}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ does not depend on the choice of k. Moreover, $\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ is the $B^+_{\mathrm{dR}}(\overline{A})$-algebra of formal power series with d variables $u_1 - 1, \ldots, u_d - 1$, and for any $r \in \mathbb{Z}$ we have*

$$\mathrm{Fil}^r\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) = \prod_{\underline{k}\in\mathbb{N}^d} \mathrm{Fil}^{r-|\underline{k}|}B^+_{\mathrm{dR}}(\overline{A}) \cdot (u_1 - 1)^{k_1} \cdots (u_d - 1)^{k_d}. \tag{II.14.4.3}$$

Corollary II.14.5. *With the notation in* II.14.3*, there is a canonical G-equivariant exact sequence of $\widehat{\overline{A}}[1/p]$-modules,*

$$0 \longrightarrow \widehat{\overline{A}}[\frac{1}{p}](1) \xrightarrow{\iota} \mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) \xrightarrow{\jmath} \widehat{\overline{A}}[\frac{1}{p}] \otimes_A \Omega^1_{X/S} \longrightarrow 0, \tag{II.14.5.1}$$

where $\mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) = \mathrm{Fil}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})/\mathrm{Fil}^2\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$, satisfying the following properties:

(1) *The map ι is induced by taking Gr^1 on the map $B^+_{\mathrm{dR}}(\overline{A}) \to \mathscr{B}^+_{\mathrm{dR}}(\overline{A})$. In particular, we have $\iota(1 \otimes (\zeta_{p^n})_{n\in\mathbb{N}}) = \log([(\zeta_{p^n})_{n\in\mathbb{N}}]) = 1 - u_0$, where $u_0 \in 1 + \mathrm{Fil}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ is the element associated to $(\zeta_{p^n})_{n\in\mathbb{N}}$ defined in* II.14.4.(2)*.*

(2) *The map $\jmath$ is induced by taking Gr^1 on the connection $\nabla : \mathscr{B}^+_{\mathrm{dR}}(\overline{A}) \to \mathscr{B}^+_{\mathrm{dR}}(\overline{A}) \otimes_A \Omega^1_{X/S}$. In particular, for any element $s \in A[1/p] \cap A^\times_{\mathrm{tr}}$ and any compatible system of p-power roots $(s_{p^n})_{n\in\mathbb{N}}$ of s contained in $\overline{A}[1/p]$, if we denote by $u \in 1+\mathrm{Fil}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ the element associated to $(s_{p^n})_{n\in\mathbb{N}}$ defined as in* II.14.4.(2, 3)*, then we have $\jmath(u - 1) = 1 \otimes \mathrm{d}\log(s)$.*

(3) *With the notation in* II.14.4.(3)*, the $\widehat{\overline{A}}[1/p]$-linear surjection $\jmath$ admits a section sending $1 \otimes \mathrm{d}\log(t_i)$ to $u_i - 1$ for any $1 \leq i \leq d$.*

In particular, $\mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ is a finite free $\widehat{\overline{A}}[1/p]$-module with basis $\{1 - u_i\}_{0\leq i\leq d}$.

Proposition II.14.6. *With the notation in* II.14.3*, there is a unique G-equivariant $\widehat{\overline{A}}[1/p]$-linear isomorphism*

$$f : \mathscr{E}_A \xrightarrow{\sim} \mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A}), \tag{II.14.6.1}$$

where $\mathscr{E}_A$ is defined in II.9.36*, such that for any element $s \in A[1/p] \cap A^\times_{\mathrm{tr}}$ and any compatible system of p-power roots $(s_{p^n})_{n\in\mathbb{N}}$ of s contained in $\overline{A}[1/p]$, we have*

$$f((\mathrm{d}\log(s_{p^n}))_{n\in\mathbb{N}}) = 1 - u, \tag{II.14.6.2}$$

where $u \in 1 + \mathrm{Fil}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ *is the element associated to* $(s_{p^n})_{n\in\mathbb{N}}$ *defined as in* II.14.4.(2, 3). *In particular, it induces a canonical isomorphism of Faltings extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\overline{A}}[\frac{1}{p}](1) & \xrightarrow{\iota} & \mathscr{E}_A & \xrightarrow{\jmath} & \widehat{\overline{A}}[\frac{1}{p}]\otimes_A \Omega^1_{X/S} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f}\wr & & \downarrow{\scriptstyle \cdot(-1)} & & \\ 0 & \longrightarrow & \widehat{\overline{A}}[\frac{1}{p}](1) & \xrightarrow{\iota} & \mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A}) & \xrightarrow{\jmath} & \widehat{\overline{A}}[\frac{1}{p}]\otimes_A \Omega^1_{X/S} & \longrightarrow & 0 \end{array} \tag{II.14.6.3}$$

Proof. We take the notation in II.14.4.(3). Recall that $\mathscr{E}_A$ is a finite free $\widehat{\overline{A}}[1/p]$-module with basis $\{\alpha_i = \mathrm{d}\log(t_{i,p^n})_{n\in\mathbb{N}}\}_{0\le i\le d}$, where $t_{0,p^n} = \zeta_{p^n}$. Thus, the uniqueness of f is clear. For its existence, we define f to be the $\widehat{\overline{A}}[1/p]$-linear isomorphism sending $\alpha_i \in \mathscr{E}_A$ to $1 - u_i \in \mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ for any $0 \le i \le d$.

We claim that f is G-equivariant. Indeed, let $u \in 1 + \mathrm{Fil}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ be the element associated to $(s_{p^n})_{n\in\mathbb{N}}$ defined as in II.14.4.(2, 3). For any $g \in G$, there is a unique element $\xi(g) \in \mathbb{Z}_p$ such that $g(s_{p^n})_{n\in\mathbb{N}} = (\zeta_{p^n}^{\xi(g)} s_{p^n})_{n\in\mathbb{N}}$. As $s = [(s_{p^n})_{n\in\mathbb{N}}]\cdot u_m^{(r)}$, we have $u = [(\zeta_{p^n})_{n\in\mathbb{N}}]^{\xi(g)} g(u)$. Taking logarithm and using the identity $\log([(\zeta_{p^n})_{n\in\mathbb{N}}]) = 1 - u_0$ in $\mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A})$ (see II.14.5.(1)), we obtain that

$$g(1-u) = \xi(g)(1-u_0) + 1 - u \in \mathrm{Gr}^1\mathscr{B}^+_{\mathrm{dR}}(\overline{A}). \tag{II.14.6.4}$$

As f is $\widehat{\overline{A}}[1/p]$-linear and sends the basis $\alpha_0, \dots, \alpha_d$ to $1-u_0, \dots, 1-u_d$, we see that f is G-equivariant by (II.14.6.4) and (II.5.7.2).

We claim that $f(\alpha) = 1 - u$, where $\alpha = (\mathrm{d}\log(s_{p^n})_{n\in\mathbb{N}})$. Indeed, our definition of f makes the diagram (II.14.6.3) commute by II.9.36 and II.14.5. Notice that $\jmath(\alpha) = \mathrm{d}\log(s) = -\jmath(1-u) \in \Omega^1_{X/S}$ and that $g(\alpha) - \alpha = \xi(g)\alpha_0 = \xi(g)(1-u_0) = g(1-u) - (1-u) \in \widehat{\overline{A}}[1/p](1)$. Thus, we can apply II.14.2 to verify this claim.

Therefore, f satisfies all the requirements, which completes the proof. □

CHAPTER III

Almost Coherence of Higher Direct Images

III.1. Introduction

III.1.1. One of the first important results in algebraic geometry is the fact that the coherence for modules is preserved by higher direct images by a proper morphism. The Noetherian case is due to Grothendieck [**EGA** III_1, 3.2.1], and the general case is due to Kiehl [**Kie72**, 2.9']. The goal of this article is to extend the following corollary to almost algebra, motivated by applications in p-adic Hodge theory.

Theorem III.1.2 (Kiehl [**Kie72**, 2.9'], cf. [**Abb10**, 1.4.8])**.** *Let $f : X \to S$ be a morphism of schemes satisfying the following conditions:*

(1) *f is proper and of finite presentation, and*
(2) *$\mathcal{O}_S$ is universally coherent.*

Then, for any coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any $q \in \mathbb{N}$, $\mathrm{R}^q f_*\mathcal{M}$ is a coherent $\mathcal{O}_S$-module.

We say that $\mathcal{O}_S$ is *universally coherent* if there is a covering $\{S_i = \mathrm{Spec}(A_i)\}_{i\in I}$ of S by affine open subschemes such that the polynomial algebra $A_i[T_1, \ldots, T_n]$ is a coherent ring for any $i \in I$ and $n \in \mathbb{N}$. Indeed, such a condition on $\mathcal{O}_S$ implies that the coherent $\mathcal{O}_X$-module $\mathcal{M}$ is actually *pseudo-coherent* relative to S, which roughly means that if we embed X locally as a closed subscheme of $\mathbb{A}^n_{S_i}$, then $\mathcal{M}$ admits a resolution by finite free modules over $\mathbb{A}^n_{S_i}$. Theorem III.1.2 is a direct corollary of Kiehl's result [**Kie72**, 2.9'], saying that the derived pushforward $\mathrm{R}f_*$ sends a relative pseudo-coherent complex to a pseudo-coherent complex.

III.1.3. Almost algebra was introduced by Faltings [**Fal88, Fal02**] for the purpose of developing p-adic Hodge theory. The setting is a pair $(R, \mathfrak{m})$ consisting of a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$, and the rough idea is to replace the category of R-modules by its quotient by $\mathfrak{m}$-torsion modules. An "almost" analogue of Theorem III.1.2 is necessary for Faltings' approach to p-adic Hodge theory. Indeed, under the same assumptions of III.1.2, Abbes-Gros [**AG20**, 2.8.14] proved that $\mathrm{R}^q f_*$ sends a quasi-coherent and almost coherent $\mathcal{O}_X$-module to a quasi-coherent and almost coherent $\mathcal{O}_S$-module, by reducing directly to III.1.2. This result plays a crucial role in the proof of Faltings' main p-adic comparison theorem in the absolute case (cf. [**AG20**, 4.8.13]), and thus of the Hodge-Tate decomposition (cf. [**AG20**, 6.4.14]). Later, Zavyalov [**Zav21**, 5.1.6] extended the same almost coherence result to formal schemes.

However, the almost coherence result [**AG20**, 2.8.14] is not enough for Faltings' main p-adic comparison theorem in the relative case (thus neither for the relative Hodge-Tate spectral sequence), since we inevitably encounter the situation where $\mathcal{O}_S$ is *universally almost coherent* but not universally coherent. Thus, under the assumptions that

(1) f is projective, flat and of finite presentation, and that
(2) $\mathcal{O}_S$ is universally almost coherent,

Abbes-Gros proved an almost coherence result [**AG20**, 2.8.18] by adapting the arguments of [**SGA 6**, III.2.2], where the projectivity condition on f plays a crucial role. This is the reason why Faltings' main p-adic comparison theorem in the relative case (and thus the relative Hodge-Tate spectral sequence) was only proved for projective log-smooth morphisms in [**AG20**, 5.7.4 (and 6.7.5)].

III.1.4. In this article, we generalize the almost coherence result [**AG20**, 2.8.18] to proper morphisms, which allows us to extend Abbes-Gros' proof of Faltings' main p-adic comparison theorem in the relative case to proper log-smooth morphisms, and thus also their construction of the relative Hodge-Tate spectral sequence (cf. Section III.8).

Let R be a ring with an ideal $\mathfrak{m}$ such that for any integer $l \geq 1$, the l-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$. The pair $(R, \mathfrak{m})$ will be our basic setup for almost algebra (cf. Section III.6). The main theorem of this article is the following

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Theorem III.1.5 (cf. III.7.1)**.** *Let $f : X \to S$ be a morphism of R-schemes satisfying the following conditions:*

(1) *f is proper, flat and of finite presentation, and*
(2) *$\mathcal{O}_X$ and $\mathcal{O}_S$ are almost coherent.*

Then, for any quasi-coherent and almost coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any $q \in \mathbb{N}$, $\mathrm{R}^q f_* \mathcal{M}$ is a quasi-coherent and almost coherent $\mathcal{O}_S$-module.

Our proof is close to Kiehl's proof of [**Kie72**, 2.9], cf. Section III.7. We roughly explain our ideas in the following:

(1) We may assume without loss of generality that S is affine. As $\mathcal{M}$ is not of finite presentation over X in general, we couldn't descend it by Noetherian approximation. But $\mathcal{M}$ is almost coherent, for any $\pi \in \mathfrak{m}$ we can "π-resolve" $\mathcal{M}$ over a truncated Čech hypercovering $X_\bullet = (X_n)_{[n]\in\Delta_{\leq k}}$ (by affine open subschemes) of X by finite free modules $\mathcal{F}_\bullet^\bullet$ as in [**Kie72**, 2.2], where each $\mathcal{F}_n^\bullet$ is a "resolution" of $\mathcal{M}|_{X_n}$ modulo π-torsion, cf. Section III.6. By Noetherian approximation, we obtain a proper flat morphism $f_\lambda : X_\lambda \to S_\lambda$ of Noetherian schemes together with a complex of finite free modules $\mathcal{F}_{\lambda,\bullet}^\bullet$ over a truncated Čech hypercovering of X_λ descending f and $\mathcal{F}_\bullet^\bullet$.
(2) As in [**Kie72**, 1.4], the descent data of $\mathcal{M}$ over $X_\bullet$ are encoded as null homotopies of the multiplication by a certain power of π on the cone of $\alpha^*\mathcal{F}_m^\bullet \to \mathcal{F}_n^\bullet$ (where $\alpha : [m] \to [n]$ is a morphism in the truncated simplicial category $\Delta_{\leq k}$), cf. Section III.3. We can descend the latter by Noetherian approximation, from which we produce some coherent modules over X_λ, cf. Section III.5.
(3) Applying the classical coherence result for $f_\lambda : X_\lambda \to S_\lambda$, we see that the Čech complex of $\mathcal{F}_{\lambda,\bullet}^\bullet$ is "pseudo-coherent" modulo certain power of π, cf. Section III.4. The same thing holds for the Čech complex of $\mathcal{F}_\bullet^\bullet$ by base change (due to the flatness of f_λ). Since this Čech complex computes $\mathrm{R}\Gamma(X, \mathcal{M})$ up to certain degree and modulo certain power of π, the conclusion follows by varying π in $\mathfrak{m}$.

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III.2. Notation and Conventions

III.2.1. All rings considered in this article are unitary and commutative.

III.2.2. Let $\mathscr{A}$ be an abelian category. When we consider "a complex in $\mathscr{A}$", we always refer to a cochain complex in $\mathscr{A}$, and we denote it by $M^\bullet$ with differential maps $\mathrm{d}^n : M^n \to M^{n+1}$ $(n \in \mathbb{Z})$. For any $a \in \mathbb{Z}$, we denote by $\tau^{\geq a} M^\bullet$ (resp. $\sigma^{\geq a} M^\bullet$) the canonical (resp. stupid) truncation of $M^\bullet$, cf. [**Sta22**, 0118].

III.2.3. Let Δ be the category formed by finite ordered sets $[n] = \{0, 1, \ldots, n\}$ $(n \in \mathbb{N})$ with non-decreasing maps ([**Sta22**, 0164]). For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\Delta_{\leq k}$ the full subcategory of Δ formed by objects $[0], [1], \ldots, [k]$. For a category C, a contravariant functor from $\Delta_{\leq k}$ to C sending $[n]$ to X_n is called a *k-truncated simplicial object* of C, denoted by $X_\bullet$. Let $\mathcal{P}$ be a property for objects of C. We say that $X_\bullet$ has property $\mathcal{P}$ if each X_n has property $\mathcal{P}$.

III.3. Isomorphisms up to Bounded Torsion

In this section, we fix a ring R and an element π of R. Consider an abelian category $\mathscr{A}$ with a ring homomorphism $R \to \mathrm{End}(\mathrm{id}_{\mathscr{A}})$, where $\mathrm{End}(\mathrm{id}_{\mathscr{A}})$ is the ring of endomorphisms of the identity functor. Thus, π defines a functorial endomorphism on any object M of $\mathscr{A}$. We denote by $\mathbf{K}(\mathscr{A})$ the homotopy category of complexes in $\mathscr{A}$.

Definition III.3.1.
(1) We say that an object M in $\mathscr{A}$ is *π-null* if it is annihilated by π. We say that a morphism $f : M \to N$ in $\mathscr{A}$ is a *π-isomorphism* if its kernel and cokernel are π-null.
(2) We say that a complex $M^\bullet$ in $\mathscr{A}$ is *π-exact* if the cohomology group $H^n(M^\bullet)$ is π-null for any $n \in \mathbb{Z}$. We say that a morphism of complexes $f : M^\bullet \to N^\bullet$ in $\mathscr{A}$ is a *π-quasi-isomorphism* if it induces a π-isomorphism on the cohomology groups $H^n(f) : H^n(M^\bullet) \to H^n(N^\bullet)$ for any $n \in \mathbb{Z}$.

Lemma III.3.2 ([**AG20**, 2.6.3])**.** *Let $f : M \to N$ be a morphism in $\mathscr{A}$.*

(1) *If there exists a morphism* $g : N \to M$ *in* $\mathscr{A}$ *such that* $g \circ f = \pi \cdot \mathrm{id}_M$ *and* $f \circ g = \pi \cdot \mathrm{id}_N$, *then* f *is a* π*-isomorphism.*
(2) *If* f *is a* π*-isomorphism, then* $\pi \cdot \mathrm{id}_N$ *and* $\pi \cdot \mathrm{id}_M$ *uniquely factor through* $N \to \mathrm{Im}(f)$ *and* $\mathrm{Im}(f) \to M$ *respectively, whose composition* $g : N \to M$ *satisfies that* $g \circ f = \pi^2 \cdot \mathrm{id}_M$ *and* $f \circ g = \pi^2 \cdot \mathrm{id}_N$.

Lemma III.3.3. *Let* $f : M^\bullet \to N^\bullet$ *be a morphism of complexes in* $\mathscr{A}$. *Assume the following conditions:*

(1) *for any* $i > 0$, $M^i = N^i = 0$;
(2) *there is* $n \in \mathbb{N}$ *such that for any* $-n \leq i \leq 0$, M^i *is projective and* $\pi \cdot H^i(N^\bullet) = 0$.

Then, there exists a morphism $s^i : M^i \to N^{i-1}$ *for any* $-n \leq i \leq 0$, *such that*

$$\pi^{1-i} \cdot f^i = \pi \cdot s^{i+1} \circ \mathrm{d}^i + \mathrm{d}^{i-1} \circ s^i \tag{III.3.3.1}$$

as morphisms from M^i *to* N^i, *where we put* $s^1 = 0$. *In particular, the morphism of canonically truncated complexes*

$$\pi^{n+1} \cdot f : \tau^{\geq -n} M^\bullet \longrightarrow \tau^{\geq -n} N^\bullet \tag{III.3.3.2}$$

is homotopic to zero.

Proof. We construct s^i by induction. Setting $0 = s^1 = s^2 = \cdots$, we may assume that we have already constructed the homomorphisms for any degree strictly bigger than i with identities (III.3.3.1). As $\pi^{-i} \cdot f^{i+1} = \pi \cdot s^{i+2} \circ \mathrm{d}^{i+1} + \mathrm{d}^i \circ s^{i+1}$, we see that

$$\mathrm{d}^i \circ (\pi^{-i} \cdot f^i - s^{i+1} \circ \mathrm{d}^i) = \pi^{-i} \cdot f^{i+1} \circ \mathrm{d}^i - (\pi^{-i} \cdot f^{i+1} - \pi \cdot s^{i+2} \circ \mathrm{d}^{i+1}) \circ \mathrm{d}^i = 0. \tag{III.3.3.3}$$

Thus, the map $\pi^{-i} \cdot f^i - s^{i+1} \circ \mathrm{d}^i : M^i \to N^i$ factors through $\mathrm{Ker}(\mathrm{d}^i : N^i \to N^{i+1})$. The assumption $\pi \cdot H^i(N^\bullet) = 0$ implies that the map $\pi^{1-i} \cdot f^i - \pi \cdot s^{i+1} \circ \mathrm{d}^i : M^i \to N^i$ factors through $\mathrm{Im}(\mathrm{d}^{i-1} : N^{i-1} \to N^i)$. As M^i is projective, there exists a morphism $s^i : M^i \to N^{i-1}$ such that $\pi^{1-i} \cdot f^i - \pi \cdot s^{i+1} \circ \mathrm{d}^i = \mathrm{d}^{i-1} \circ s^i$, which completes the induction. In particular, for any $i \geq -n$, we have

$$\pi^{n+1} \cdot f^i = (\pi^{n+i+1} \cdot s^{i+1}) \circ \mathrm{d}^i + \mathrm{d}^{i-1} \circ (\pi^{n+i} \cdot s^i). \tag{III.3.3.4}$$

Recall that $\tau^{\geq -n} M^\bullet = (0 \to M^{-n}/\mathrm{Im}(\mathrm{d}^{-n-1}) \to M^{1-n} \to \cdots \to M^0 \to 0)$. Thus, we see that $\pi^{n+1} \cdot f : \tau^{\geq -n} M^\bullet \to \tau^{\geq -n} N^\bullet$ is homotopic to zero. □

Proposition III.3.4. *Let* $P^\bullet$ *be a complex of projective objects in* $\mathscr{A}$, $M^\bullet$ *a* π*-exact complex in* $\mathscr{A}$. *Assume that there are integers* $a \leq b$ *such that* P^i *and* M^i *vanish for any* $i \notin [a, b]$. *Then, the* R*-module* $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet)$ *is* π^{b-a+1}*-null.*

Proof. It follows directly from III.3.3. □

Corollary III.3.5. *Let* $P^\bullet$ *be a complex of projective objects in* $\mathscr{A}$, $f : M^\bullet \to N^\bullet$ *a* π*-quasi-isomorphism of complexes in* $\mathscr{A}$. *Assume that there are integers* $a \leq b$ *such that* P^i, M^i *and* N^i *vanish for any* $i \notin [a, b]$. *Then, the map* $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet)$ *induced by* f *is a* $\pi^{2(b-a+3)}$*-isomorphism of* R*-modules.*

Proof. There is an exact sequence of R-modules

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet[-1]) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet) \tag{III.3.5.1}$$

where $C^\bullet$ is the cone of f ([**Sta22**, 0149]). As $C^\bullet$ and $C^\bullet[-1]$ are π^2-exact and vanish outside $[a-1, b+1]$, the outer two R-modules are $\pi^{2(b-a+3)}$-null by III.3.4, whence we draw the conclusion. □

Lemma III.3.6. *Let* $g : P^\bullet \to N^\bullet$, $f : M^\bullet \to N^\bullet$ *be morphisms of complexes in* $\mathscr{A}$. *Assume that there are integers* $a \leq b$ *such that*

(1) $M^i = N^i = 0$ *for any* $i > b$, *and that*
(2) P^i *is projective for any* $i \in [a, b]$ *and zero for any* $i \notin [a, b]$, *and that*
(3) *the map* $H^i(f) : H^i(M^\bullet) \to H^i(N^\bullet)$ *is a* π*-isomorphism for* $i > a$ *and* π*-surjective for* $i = a$.

Then, $\pi^{2(b-a+1)} \cdot g$ *lies in the image of the map* $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet)$ *induced by* f.

PROOF. Let $C^\bullet$ be the cone of f, $\iota : N^\bullet \to C^\bullet$ the canonical morphism. Applying the homological functor $\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(P^\bullet, -)$ to the distinguished triangle $\tau^{\leq a-1}C^\bullet \to C^\bullet \to \tau^{\geq a}C^\bullet \to (\tau^{\leq a-1}C^\bullet)[1]$ in the derived category $\mathbf{D}(\mathscr{A})$, we obtain an exact sequence of R-modules ([**Sta22**, 0149, 064B])

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, \tau^{\leq a-1}C^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, \tau^{\geq a}C^\bullet). \tag{III.3.6.1}$$

The first term is zero by assumption (2), and $\tau^{\geq a}C^\bullet$ is π^2-exact by assumption (3). As $\tau^{\geq a}C^\bullet$ vanishes outside $[a, b]$ by assumption (1), the third term is $\pi^{2(b-a+1)}$-null by III.3.4. We see that $\pi^{2(b-a+1)} \cdot (\iota \circ g)$ is zero in $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet)$. Therefore, the conclusion follows from the exact sequence ([**Sta22**, 0149])

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet). \tag{III.3.6.2}$$

□

III.4. Pseudo-coherence up to Bounded Torsion

In this section, we fix integers $a \leq b$, a ring R and an element π of R. We remark that the universal bound l that shall appear in each statement of this section depends only on the difference $b - a$ but not on R or π.

Definition III.4.1. Let $M^\bullet$ be a complex of R-modules.

(1) A π-$[a, b]$-*pseudo resolution* of $M^\bullet$ is a morphism $f : P^\bullet \to M^\bullet$ of complexes of R-modules, where $P^\bullet$ is a complex of finite free R-modules such that $P^i = 0$ for any $i \notin [a, b]$, and where the map of cohomology groups $H^i(f) : H^i(P^\bullet) \to H^i(M^\bullet)$ is a π-isomorphism for $i > a$ and π-surjective for $i = a$.
(2) We say that $M^\bullet$ is π-$[a, b]$-*pseudo-coherent* if $M^i = 0$ for any $i > b$ and if it admits a π-$[a, b]$-pseudo resolution. We say that an R-module M is π-$[a, b]$-*pseudo-coherent* if the complex $M[0]$ is π-$[a, b]$-pseudo-coherent.

We follow the presentation of [**Sta22**, 064N] to establish some basic properties of this notion. The author does not know whether this notion is Zariski local on R or not (cf. [**Sta22**, 066D]). This ad hoc notion only serves for the proof of our main theorem.

Lemma III.4.2. *For any integers $a' \geq a$ and $b' \geq b$ with $a' \leq b'$, a π-$[a, b]$-pseudo-coherent complex of R-modules is also π-$[a', b']$-pseudo-coherent.*

PROOF. We only need to treat the case $a = a'$ and the case $b = b'$ separately. If $a = a'$, then it is clear that $M^i = 0$ for any $i > b'$ and a π-$[a, b]$-pseudo resolution of $M^\bullet$ is also a π-$[a, b']$-pseudo resolution. If $b = b'$, then a π-$[a, b]$-pseudo resolution $P^\bullet \to M^\bullet$ induces a π-$[a', b]$-pseudo resolution $\sigma^{\geq a'}P^\bullet \to M^\bullet$. □

Lemma III.4.3. *Let $M^\bullet$ and $N^\bullet$ be complexes of R-modules vanishing in degrees $> b$, $\alpha : M^\bullet \to N^\bullet$ a morphism inducing a π-isomorphism on cohomology groups $H^i(\alpha) : H^i(M^\bullet) \to H^i(N^\bullet)$ for any $i \geq a$.*

(1) *If $M^\bullet$ is π-$[a, b]$-pseudo-coherent, then $N^\bullet$ is π^2-$[a, b]$-pseudo-coherent.*
(2) *If $N^\bullet$ is π-$[a, b]$-pseudo-coherent, then $M^\bullet$ is π^l-$[a, b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*

PROOF. (1) We take a π-$[a, b]$-pseudo resolution $f : P^\bullet \to M^\bullet$. In particular, $H^i(f)$ is a π-isomorphism for any $i > a$ and π-surjective for $i = a$. Hence, $H^i(\alpha \circ f) = H^i(\alpha) \circ H^i(f)$ is π^2-isomorphism for any $i > a$ and π^2-surjective for $i = a$, which shows that $\alpha \circ f : P^\bullet \to N^\bullet$ is a π^2-$[a, b]$-pseudo resolution.

(2) Let $g : P^\bullet \to N^\bullet$ be a π-$[a, b]$-pseudo resolution. We obtain from III.3.6 a morphism $f : P^\bullet \to M^\bullet$ lifting $\pi^l \cdot g$ up to homotopy for $l = 2(b - a + 1)$. Thus, for any $i \in \mathbb{Z}$, we have

$$H^i(\pi^l \cdot g) = H^i(\alpha) \circ H^i(f). \tag{III.4.3.1}$$

Notice that $H^i(\pi^l \cdot g)$ is a π^{l+1}-isomorphism for $i > a$ and π^{l+1}-surjective for $i = a$, and that $H^i(\alpha)$ is a π-isomorphism for $i \geq a$. We see that $H^i(f)$ is a π^{l+2}-isomorphism for $i > a$ and π^{l+2}-surjective for $i = a$. Thus, $f : P^\bullet \to M^\bullet$ is a π^{l+2}-$[a, b]$-pseudo resolution. □

Proposition III.4.4. *Let $M^\bullet$ and $N^\bullet$ be two complexes of R-modules vanishing in degree $> b$. Assume that they are isomorphic in the derived category $\mathbf{D}(R)$. Then, if $M^\bullet$ is π-$[a, b]$-pseudo-coherent, then $N^\bullet$ is π^l-$[a, b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*

PROOF. Let $P^\bullet \to N^\bullet$ be a bounded above projective resolution with the same top degree. The assumption implies that there is a quasi-isomorphism of complexes $P^\bullet \to M^\bullet$ ([**Sta22**, 064B]). The conclusion follows from applying III.4.3 to $P^\bullet \to M^\bullet$ and $P^\bullet \to N^\bullet$. □

Lemma III.4.5. *Let $\alpha : M_1^\bullet \to M_2^\bullet$ be a morphism of π-$[a,b]$-pseudo-coherent complexes of R-modules. Given π-$[a,b]$-pseudo resolutions $f_i : P_i^\bullet \to M_i^\bullet$ $(i = 1, 2)$, there exists a morphism of complexes $\alpha' : P_1^\bullet \to P_2^\bullet$ such that $(\pi^l \cdot \alpha) \circ f_1$ is homotopic to $f_2 \circ \alpha'$ for an integer $l \geq 0$ depending only on $b - a$.*

PROOF. It follows directly from III.3.6. □

Lemma III.4.6. *Let $M_1^\bullet \xrightarrow{\alpha} M_2^\bullet \xrightarrow{\beta} M_3^\bullet \xrightarrow{\gamma} M_1^\bullet[1]$ be a distinguished triangle in the homotopy category $\mathbf{K}(R)$. Assume that $M_1^\bullet$ is π-$[a+1, b+1]$-pseudo-coherent, $M_2^\bullet$ is π-$[a,b]$-pseudo-coherent, and $M_3^i = 0$ for any $i > b$. Then, $M_3^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*

PROOF. We take a π-$[a+1, b+1]$-pseudo resolution $f_1 : P_1^\bullet \to M_1^\bullet$ and a π-$[a,b]$-pseudo resolution $f_2 : P_2^\bullet \to M_2^\bullet$. By III.4.5, there exists a morphism $\alpha' : P_1^\bullet \to P_2^\bullet$ lifting $\pi^l \cdot \alpha$ in $\mathbf{K}(R)$ for an integer $l \geq 0$ depending only on $b - a$. If we denote its cone by $P_3^\bullet$, then we have a morphism of distinguished triangles in $\mathbf{K}(R)$.

$$\begin{array}{ccccccc} P_1^\bullet & \xrightarrow{\alpha'} & P_2^\bullet & \xrightarrow{\beta'} & P_3^\bullet & \xrightarrow{\gamma'} & P_1^\bullet[1] \\ \downarrow{\scriptstyle \pi^l \cdot f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle \pi^l \cdot f_1[1]} \\ M_1^\bullet & \xrightarrow{\alpha} & M_2^\bullet & \xrightarrow{\beta} & M_3^\bullet & \xrightarrow{\gamma} & M_1^\bullet[1] \end{array} \tag{III.4.6.1}$$

Let $C_1^\bullet$ $C_2^\bullet$, $C_3^\bullet$ be the cones of $\pi^l \cdot f_1$, f_2, f_3 respectively. We obtain a distinguished triangle $C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to C_1^\bullet[1]$ in $\mathbf{K}(R)$ ([**Sta22**, 05R0]). By assumption, $\tau^{\geq (a+1)} C_1^\bullet$ is $\pi^{2(l+1)}$-exact and $\tau^{\geq a} C_2^\bullet$ is π^2-exact. Thus, we see that $\tau^{\geq a} C_3^\bullet$ is $\pi^{2(l+2)}$-exact. As $\sigma^{\geq a} P_3^\bullet$ vanishes outside $[a,b]$, $\sigma^{\geq a} P_3^\bullet \to M_3^\bullet$ is a $\pi^{2(l+2)}$-$[a,b]$-pseudo resolution. □

Proposition III.4.7. *Let $0 \longrightarrow M_1^\bullet \xrightarrow{\alpha} M_2^\bullet \xrightarrow{\beta} M_3^\bullet \longrightarrow 0$ be an exact sequence of complexes of R-modules.*

1. *Assume that $M_1^\bullet$ is π-$[a+1, b+1]$-pseudo-coherent, $M_2^\bullet$ is π-$[a,b]$-pseudo-coherent, and $M_3^i = 0$ for any $i > b$. Then, $M_3^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*
2. *Assume that $M_1^\bullet$ and $M_3^\bullet$ are π-$[a,b]$-pseudo-coherent, and $M_2^i = 0$ for any $i > b$. Then, $M_2^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*
3. *Assume that $M_2^\bullet$ is π-$[a-1, b-1]$-pseudo-coherent, $M_3^\bullet$ is π-$[a-2, b-1]$-pseudo-coherent, and $M_1^i = 0$ for any $i > b$. Then, $M_1^\bullet$ is π^l-$[a-1, b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*

PROOF. Let $C^\bullet$ be the cone of $\alpha : M_1^\bullet \to M_2^\bullet$. Then, the natural morphism $C^\bullet \to M_3^\bullet$ is a quasi-isomorphism.

(1) In this case, $C^i = 0$ for any $i > b$, and actually $C^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$ by applying III.4.6 to the distinguished triangle $M_1^\bullet \to M_2^\bullet \to C^\bullet \to M_1^\bullet[1]$ in $\mathbf{K}(R)$. Thus, $M_3^\bullet$ is π^{2l}-$[a,b]$-pseudo-coherent by III.4.3.(1).

(2) In this case, $C^i = 0$ for any $i > b$, and $C^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$ by III.4.3.(2). Thus, $M_2^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$ by applying III.4.6 to the distinguished triangle $C^\bullet[-1] \to M_1^\bullet \to M_2^\bullet \to C^\bullet$ in $\mathbf{K}(R)$.

(3) In this case, $C^i = 0$ for any $i > b - 1$, and $C^\bullet$ is π^l-$[a-2, b-1]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$ by III.4.3.(2). Thus, $M_1^\bullet$ is π^l-$[a-1, b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$ by applying III.4.6 to the distinguished triangle $M_2^\bullet[-1] \to C^\bullet[-1] \to M_1^\bullet \to M_2^\bullet$ in $\mathbf{K}(R)$. □

Corollary III.4.8. *Let $M^\bullet$ be a complex of R-modules vanishing in degrees $> b$. Assume that the cohomology group $H^i(M^\bullet)$ is π-$[a-i, b-i]$-pseudo-coherent for any $i \in [a,b]$. Then, $M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*

PROOF. We proceed by induction on $b - a$. If $a = b$, then $\tau^{\geq a} M^\bullet = H^a(M^\bullet)[-a]$ is π-$[a,b]$-pseudo-coherent by assumption. Thus, $M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending

only on $b-a$ by III.4.3.(2). In general, consider the exact sequence of complexes of R-modules

$$0 \longrightarrow \tau^{\leq(b-1)}M^\bullet \longrightarrow M^\bullet \longrightarrow (M^{b-1}/\mathrm{Ker}(\mathrm{d}^{b-1})[1-b] \to M^b[-b]) \longrightarrow 0. \tag{III.4.8.1}$$

As the natural morphism of complexes $N^\bullet = (M^{b-1}/\mathrm{Ker}(\mathrm{d}^{b-1})[1-b] \to M^b[-b]) \to H^b(M^\bullet)[-b]$ is a quasi-isomorphism, $N^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by III.4.3.(2). Notice that $\tau^{\leq(b-1)}M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a-1$ by induction. The conclusion follows from III.4.7.(2). □

Lemma III.4.9. *Let $M^\bullet$ be a π-exact complex of R-modules vanishing outside $[a,b]$, $N^\bullet$ a complex of R-modules. Then, $M^\bullet \otimes^{\mathrm{L}}_R N^\bullet$ is π^l-exact (i.e. any complex representing the derived tensor product $M^\bullet \otimes^{\mathrm{L}}_R N^\bullet$ is π^l-exact) for an integer $l \geq 0$ depending only on $b-a$.*

Proof. We proceed by induction on $b-a$. If $a=b$, then the multiplication by π on $M^\bullet = H^a(M^\bullet)[-a]$ factors through zero. Hence, $M^\bullet \otimes^{\mathrm{L}}_R N^\bullet$ is π-exact. In general, consider the distinguished triangle in the derived category $\mathbf{D}(R)$,

$$(\tau^{\leq(b-1)}M^\bullet) \otimes^{\mathrm{L}}_R N^\bullet \longrightarrow M^\bullet \otimes^{\mathrm{L}}_R N^\bullet \longrightarrow H^b(M^\bullet)[-b] \otimes^{\mathrm{L}}_R N^\bullet \longrightarrow (\tau^{\leq(b-1)}M^\bullet) \otimes^{\mathrm{L}}_R N^\bullet[1]. \tag{III.4.9.1}$$

Notice that $(\tau^{\leq(b-1)}M^\bullet) \otimes^{\mathrm{L}}_R N^\bullet$ is π^l-exact for an integer $l \geq 0$ depending only on $b-a$ by induction. By the long exact sequence of cohomology groups, we see that $M^\bullet \otimes^{\mathrm{L}}_R N^\bullet$ is π^l-exact for an integer $l \geq 0$ depending only on $b-a$. □

Proposition III.4.10. *Let $M^\bullet$ be a π-$[a,b]$-pseudo-coherent complex of R-modules, S an R-algebra. Then, $\tau^{\geq a}(S \otimes^{\mathrm{L}}_R M^\bullet)$ is represented by a π^l-$[a,b]$-pseudo-coherent complex of S-modules for an integer $l \geq 0$ depending only on $b-a$.*

Proof. We take a bounded above flat resolution $F^\bullet \to M^\bullet$ with the same top degree. By III.4.3.(2), $\sigma^{\geq a-1}F^\bullet$ is a π^l-$[a,b]$-pseudo-coherent complex of flat R-modules for an integer $l \geq 0$ depending only on $b-a$. Let $P^\bullet \to \sigma^{\geq a-1}F^\bullet$ be a π^l-$[a,b]$-pseudo resolution, and let $C^\bullet$ be its cone. Consider the distinguished triangle in $\mathbf{K}(S)$,

$$S \otimes_R P^\bullet \longrightarrow S \otimes_R \sigma^{\geq a-1}F^\bullet \longrightarrow S \otimes_R C^\bullet \longrightarrow S \otimes_R P^\bullet[1]. \tag{III.4.10.1}$$

Notice that $\tau^{\geq a}C^\bullet$ is a π^{2l}-exact complex vanishing outside $[a,b]$ and that $S \otimes_R C^\bullet \cong S \otimes^{\mathrm{L}}_R C^\bullet$ in $\mathbf{D}(S)$ by construction. After enlarging l by III.4.9, we may assume that $\tau^{\geq a}(S \otimes_R C^\bullet) = \tau^{\geq a}(S \otimes^{\mathrm{L}}_R \tau^{\geq a}C^\bullet)$ is π^l-exact. By the long exact sequence associated to (III.4.10.1), we see that $S \otimes_R P^\bullet \to S \otimes_R \sigma^{\geq a-1}F^\bullet$ is a π^{2l}-$[a,b]$-pseudo resolution of complexes of S-modules, and thus so is the composition

$$S \otimes_R P^\bullet \longrightarrow S \otimes_R \sigma^{\geq a-1}F^\bullet \longrightarrow \tau^{\geq a}(S \otimes_R \sigma^{\geq a-1}F^\bullet) = \tau^{\geq a}(S \otimes_R F^\bullet), \tag{III.4.10.2}$$

where the target is a complex of S-modules representing $\tau^{\geq a}(S \otimes^{\mathrm{L}}_R M^\bullet)$ and vanishing in degrees $> b$. □

Definition III.4.11. Let M be an R-module. We say that M is *of π-finite type* if there exists $n \in \mathbb{N}$ and a π-surjective R-linear homomorphism $R^{\oplus n} \to M$.

This definition is a special case of III.6.1 below.

Lemma III.4.12. *Assume that R is Noetherian. Let M be an R-module.*

1. *If M is of π-finite type, then it is π-$[a,b]$-pseudo-coherent for any integers $a \leq 0 \leq b$. Conversely, if M is π-$[a,b]$-pseudo-coherent for some integers $a \leq 0 \leq b$, then M is of π-finite type.*
2. *If M is of π-finite type, then so are its subquotients. Conversely, if M admits a finite filtration of length l ([**Sta22**, 0121]) whose graded pieces are of π-finite type, then M is of π^l-finite type.*

Proof. (1) If M is of π-finite type, then there is a finitely generated R-submodule N of M such that $\pi M \subseteq N$. Since R is Noetherian, N is pseudo-coherent ([**Sta22**, 066E]). Hence, M is π-$[a,b]$-pseudo-coherent for any integers $a \leq 0 \leq b$. Conversely, if M is π-$[a,b]$-pseudo-coherent, we take a π-$[a,b]$-pseudo resolution $P^\bullet \to M[0]$. As a subquotient of a finitely generated R-module, $H^0(P^\bullet)$ is also finitely generated as R is Noetherian. Hence, $M = H^0(M[0])$ is of π-finite type.

(2) Let N be a finitely generated R-submodule of M such that $\pi M \subseteq N$. Let $M_0 \subseteq M_1$ be two R-submodules of M. Notice that $N \cap M_1$ is a finitely generated R-module as R is Noetherian. The conclusion follows from the π-surjectivity of $N \cap M_1 \to M_1/M_0$. Conversely, assume that there is a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M$ such that M_{i+1}/M_i is of π-finite type. Then, we see that M is of π^l-finite type by inductively using [**AG20**, 2.7.14.(ii)]. □

Proposition III.4.13. *Assume that R is Noetherian. Let $M^\bullet$ be a complex of R-modules.*

(1) *If $H^i(M^\bullet)$ is of π-finite type for any $i \geq a$ and if $M^i = 0$ for any $i > b$, then $M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*
(2) *If $M^\bullet$ is π-$[a,b]$-pseudo-coherent, then $H^i(M^\bullet)$ is of π-finite type for any $i \geq a$.*

Proof. (1) follows from III.4.12.(1) and III.4.8; and (2) follows from the same argument of III.4.12.(1). □

III.5. Glueing Sheaves up to Bounded Torsion

In this section, we fix a ring R and an element π of R.

III.5.1. Let E/C be a fibred site, $\mathcal{O} = (\mathcal{O}_\alpha)_{\alpha\in \mathrm{Ob}(C)}$ a sheaf of R-algebras over the total site E ([**SGA 4**$_{\mathrm{II}}$, VI.7.4.1]). We say that an $\mathcal{O}$-module $\mathcal{F} = (\mathcal{F}_\alpha)_{\alpha\in\mathrm{Ob}(C)}$ on E is *π-Cartesian*, if for every morphism $f : \beta \to \alpha$ in C, the induced map $f^*\mathcal{F}_\alpha \to \mathcal{F}_\beta$ is a π-isomorphism of $\mathcal{O}_\beta$-modules.

III.5.2. Let E be a category. Recall that a semi-representable object of E is a family $\{U_i\}_{i\in I}$ of objects of E. A morphism $\{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ of semi-representable objects of E is given by a map $\alpha : I \to J$ and for every $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ ([**Sta22**, 01G0]). Assume that E is a site ([**SGA 4**$_{\mathrm{I}}$, II.1.1.5]). For a semi-representable object $K = \{U_i\}_{i\in I}$ of objects of E, let $E_{/K} = \coprod_{i\in I} E_{/U_i}$ be the disjoint union of the localizations of E at U_i ([**Sta22**, 09WK]).

Let $r \in \mathbb{N}\cup\{\infty\}$. For an r-truncated simplicial semi-representable object $K_\bullet = (K_n)_{[n]\in\mathrm{Ob}(\Delta_{\leq r})}$ of E (where each K_n is a semi-representable object of E), we denote by $E_{/K_\bullet}$ the fibred site over the r-truncated simplicial category $\Delta_{\leq r}$ whose fibre over $[n]$ is $E_{/K_n}$ ([**Sta22**, 0D8A]). We denote by $\nu : E_{/K_\bullet} \to E$ the augmentation, and by $\nu_n : E_{/K_n} \to E$ the corresponding morphism of sites for any $n \in \mathbb{N}_{\leq r}$ ([**Sta22**, 0D8B]).

Proposition III.5.3. *Let E be a site where fibred products are representable, $\mathcal{O}$ a sheaf of R-algebras on E, $\{U_i \to X\}_{i\in I}$ a covering in E. Consider the 2-truncated Čech hypercovering ([**Sta22**, 01G6])*

$$K_\bullet = (\{U_i \times_X U_j \times_X U_k\}_{i,j,k\in I} \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \{U_i \times_X U_j\}_{i,j\in I} \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \{U_i\}_{i\in I}), \tag{III.5.3.1}$$

regarded as a 2-truncated simplicial semi-representable object of $E_{/X}$. Let $\mathcal{F}_\bullet = (\mathcal{F}_n)_{[n]\in\Delta_{\leq 2}}$ be a π-Cartesian $\mathcal{O}_{/K_\bullet}$-module over the 2-truncated simplicial ringed site $E_{/K_\bullet}$, and we put $\mathcal{F} = \nu_*\mathcal{F}_\bullet$ where $\nu : E_{/K_\bullet} \to E_{/X}$ is the augmentation. Then, the canonical map $\nu_0^*\mathcal{F} \to \mathcal{F}_0$ is a π^8-isomorphism.

Proof. For any $i,j,k \in I$, we denote by $f_i : U_i \to X$, $f_{ij} : U_i \times_X U_j \to X$, $f_{ijk} : U_i \times_X U_j \times_X U_k \to X$ the canonical morphisms, and denote by $\mathcal{G}_i$, $\mathcal{G}_{ij}$, $\mathcal{G}_{ijk}$ the restrictions of $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$ to U_i, $U_i \times_X U_j$, $U_i \times_X U_j \times_X U_k$ respectively. By definition ([**Sta22**, 09WM]), we have

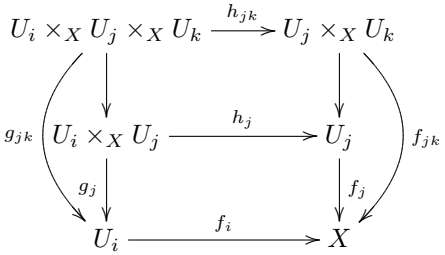
$$\mathcal{F} = \mathrm{Eq}(\prod_{j\in I} f_{j*}\mathcal{G}_j \rightrightarrows \prod_{j,k\in I} f_{jk*}\mathcal{G}_{jk}). \tag{III.5.3.2}$$

We need to show that the canonical map (note that the restriction functor f_i^* of sheaves commute with any limits as it admits a left adjoint $f_{i!}$),

$$f_i^*\mathcal{F} = \mathrm{Eq}(\prod_{j\in I} f_i^*f_{j*}\mathcal{G}_j \rightrightarrows \prod_{j,k\in I} f_i^*f_{jk*}\mathcal{G}_{jk}) \longrightarrow \mathcal{G}_i \tag{III.5.3.3}$$

given by composing the projection on the i-th component with the adjunction morphism $f_i^*f_{i*}\mathcal{G}_i \to \mathcal{G}_i$, is a π^8-isomorphism for any $i \in I$. Fixing $i \in I$, for any $j,k \in I$, we name some natural arrows as indicated in the following commutative diagram

(III.5.3.4)



Thus, we obtain the following commutative diagram of sheaves on $E_{/U_i}$,

(III.5.3.5)
$$\begin{array}{ccccccc}
g_{j*}g_j^*\mathcal{G}_i & \longrightarrow & g_{j*}\mathcal{G}_{ij} & \longleftarrow & g_{j*}h_j^*\mathcal{G}_j & \xleftarrow{\sim} & f_i^*f_{j*}\mathcal{G}_j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
g_{jk*}g_{jk}^*\mathcal{G}_i & \longrightarrow & g_{jk*}\mathcal{G}_{ijk} & \longleftarrow & g_{jk*}h_{jk}^*\mathcal{G}_{jk} & \xleftarrow{\sim} & f_i^*f_{jk*}\mathcal{G}_{jk} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
g_{k*}g_k^*\mathcal{G}_i & \longrightarrow & g_{k*}\mathcal{G}_{ik} & \longleftarrow & g_{k*}h_k^*\mathcal{G}_k & \xleftarrow{\sim} & f_i^*f_{k*}\mathcal{G}_k
\end{array}$$

where the three isomorphisms are the base change isomorphisms (cf. [**SGA 4**$_{\mathrm{III}}$, XII.4]). Since

(III.5.3.6)
$$\{g_{jk} : U_i \times_X U_j \times_X U_k \to U_i\}_{j,k\in I} \rightrightarrows \{g_j : U_i \times_X U_j \to U_i\}_{j\in I}$$

is a 1-truncated Čech hypercovering of U_i, the equalizer corresponding to the first column in (III.5.3.5) is equal to

(III.5.3.7)
$$\mathcal{G}_i = \mathrm{Eq}(\prod_{j\in I} g_{j*}g_j^*\mathcal{G}_i \rightrightarrows \prod_{j,k\in I} g_{jk*}g_{jk}^*\mathcal{G}_i)$$

by the sheaf property of $\mathcal{G}_i$ on $E_{/U_i}$. Since $\mathcal{F}_\bullet$ is π-Cartesian, the horizontal arrows in (III.5.3.5) are π^2-isomorphisms by III.3.2 (cf. II.7.3). Therefore, the morphisms between the equalizers corresponding to each column in (III.5.3.5) (see the second row of (III.5.3.8) in the following) are π^4-isomorphisms. In order to show that the canonical map $f_i^*\mathcal{F} \to \mathcal{G}_i$ (III.5.3.3) is a π^8-isomorphism, it remains to prove the square in the following natural diagram is commutative,

(III.5.3.8)
$$\begin{array}{ccccc}
 & & \mathcal{G}_i & \longleftarrow & f_i^*f_{i*}\mathcal{G}_i \\
 & & \downarrow{\scriptstyle \iota} & & \uparrow \\
\mathrm{Eq}(\prod g_{j*}g_j^*\mathcal{G}_i \rightrightarrows \prod g_{jk*}g_{jk}^*\mathcal{G}_i) & \longrightarrow & \mathrm{Eq}(\prod g_{j*}\mathcal{G}_{ij} \rightrightarrows \prod g_{jk*}\mathcal{G}_{ijk}) & \longleftarrow & \mathrm{Eq}(\prod f_i^*f_{j*}\mathcal{G}_j \rightrightarrows \prod f_i^*f_{jk*}\mathcal{G}_{jk})
\end{array}$$

where ι is the natural map making the left triangle commutative, and in each equalizer, j goes through I for the first product, and j, k go through I for the second product. Consider the commutative diagram for any $j, k \in I$,

(III.5.3.9)
$$\begin{array}{ccccc}
 & & U_i \times_X U_j \times_X U_k & & \\
 & \swarrow & \downarrow{\scriptstyle g_{jk}} & \searrow & \\
U_i \times_X U_j & & & & U_i \times_X U_k \xrightarrow{h_k} U_k \\
 & {\scriptstyle g_j}\searrow & \downarrow & \swarrow{\scriptstyle g_k} & \downarrow{\scriptstyle f_k} \\
 & & U_i & \xrightarrow{f_i} & X
\end{array}$$

from which we obtain the following natural commutative diagram

(III.5.3.10)
$$\begin{array}{ccccccc}
 & & g_{j*}\mathcal{G}_{ij} & & \mathcal{G}_i & \longleftarrow & f_i^*f_{i*}\mathcal{G}_i \\
 & \swarrow & \uparrow & \overset{\iota}{\swarrow} & & \overset{(1)}{\searrow} & \uparrow \\
g_{jk*}\mathcal{G}_{ijk} & \longleftarrow & \mathrm{Eq}(\prod g_{j*}\mathcal{G}_{ij} \rightrightarrows \prod g_{jk*}\mathcal{G}_{ijk}) & & & f_i^*f_{ik*}\mathcal{G}_{ik} & \longleftarrow \mathrm{Eq}(\prod f_i^*f_{j*}\mathcal{G}_j \rightrightarrows \prod f_i^*f_{jk*}\mathcal{G}_{jk}) \\
 & \searrow & \downarrow & \underset{\jmath}{\swarrow} & & \underset{(2)}{\searrow} & \downarrow \\
 & & g_{k*}\mathcal{G}_{ik} & \longleftarrow & g_{k*}h_k^*\mathcal{G}_k & \xleftarrow{\sim} & f_i^*f_{k*}\mathcal{G}_k
\end{array}$$

Indeed, the natural map $\jmath : f_i^*f_{ik*}\mathcal{G}_{ik} = f_i^*f_{i*}g_{k*}\mathcal{G}_{ik} \to g_{k*}\mathcal{G}_{ik}$ is defined by applying the adjunction morphism $f_i^*f_{i*} \to \mathrm{id}$ to $g_{k*}\mathcal{G}_{ik}$, and other natural arrows have appeared in the diagrams (III.5.3.5) and (III.5.3.8). Thus, the commutativity of (1) follows from applying the adjunction morphism $f_i^*f_{i*} \to \mathrm{id}$ to the canonical map $\mathcal{G}_i \to g_{k*}\mathcal{G}_{ik}$, and the commutativity of (2) follows from the

following natural commutative diagram

(III.5.3.11)
$$\begin{array}{ccccc} g_{k*}\mathcal{G}_{ik} & \xleftarrow{\jmath} & f_i^*f_{i*}g_{k*}\mathcal{G}_{ik} & = & f_i^*f_{k*}h_{k*}\mathcal{G}_{ik} \\ \uparrow & & \uparrow & & \uparrow \\ g_{k*}h_k^*\mathcal{G}_k & \longleftarrow & f_i^*f_{i*}g_{k*}h_k^*\mathcal{G}_k & = & f_i^*f_{k*}h_{k*}h_k^*\mathcal{G}_k \longleftarrow f_i^*f_{k*}\mathcal{G}_k \end{array}$$

where the vertical arrows are induced by the canonical map $h_k^*\mathcal{G}_k \to \mathcal{G}_{ik}$, and the composition of the second row is the base change isomorphism $f_i^*f_{k*}\mathcal{G}_k \xrightarrow{\sim} g_{k*}h_k^*\mathcal{G}_k$ by definition [**SGA 4**$_{\mathrm{III}}$, XII.(4.1)]. In particular, we see that the natural diagram extracted from (III.5.3.10),

(III.5.3.12)
$$\begin{array}{ccc} \mathcal{G}_i & \longleftarrow & f_i^*f_{i*}\mathcal{G}_i \\ \downarrow{\scriptstyle \iota} & & \uparrow \\ \mathrm{Eq}(\prod g_{j*}\mathcal{G}_{ij} \rightrightarrows \prod g_{jk*}\mathcal{G}_{ijk}) & & \mathrm{Eq}(\prod f_i^*f_{j*}\mathcal{G}_j \rightrightarrows \prod f_i^*f_{jk*}\mathcal{G}_{jk}) \\ \downarrow & & \downarrow \\ g_{k*}\mathcal{G}_{ik} & \longleftarrow & f_i^*f_{k*}\mathcal{G}_k \end{array}$$

is commutative for any $k \in I$. This shows that the diagram (III.5.3.8) is commutative, which completes the proof. □

Remark III.5.4. We expect a generalization to any 2-truncated hypercovering $K_\bullet$ of X as in [**Sta22**, 0D8E].

Example III.5.5. Let X be a quasi-compact and separated scheme, $K_0 = \{U_i \to X\}_{0\leq i\leq k}$ a finite open covering of X consisting of affine open subschemes. For any $n \in \mathbb{N}$, we define a semi-representable object of the Zariski site X_{Zar} of X,

(III.5.5.1)
$$K_n = \{U_{i_0} \cap \cdots \cap U_{i_n} \to X\}_{0\leq i_0,\cdots,i_n\leq k}.$$

These K_n naturally form a simplicial semi-representable object of X_{Zar}, $K_\bullet = (K_n)_{[n]\in \mathrm{Ob}(\Delta)}$, called the Čech hypercovering associated to K_0 of X. We put

(III.5.5.2)
$$X_n = \coprod_{0\leq i_0,\ldots,i_n\leq k} U_{i_0} \cap \cdots \cap U_{i_n}$$

which is a finite disjoint union of affine open subschemes of X, and denote by $\nu_n : X_n \to X$ the canonical morphism. It is clear that the site X_{Zar/K_n} is naturally equivalent to the Zariski site $X_{n,\mathrm{Zar}}$. We also obtain a simplicial affine scheme $X_\bullet = (X_n)_{[n]\in\mathrm{Ob}(\Delta)}$, and an augmentation $\nu : X_\bullet \to X$ (where we omit the subscript "Zar").

For any $\mathcal{O}_{X_\bullet}$-module $\mathcal{F}_\bullet$, we consider the ordered Čech complex $\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet)$, whose degree-$n$ term is the R-module ([**Sta22**, 01FG])

(III.5.5.3)
$$\check{C}^n_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet) = \prod_{0\leq i_0<\cdots<i_n\leq k} \mathcal{F}_n(U_{i_0} \cap \cdots \cap U_{i_n}).$$

In general, for any complex of $\mathcal{O}_{X_\bullet}$-modules $\mathcal{F}^\bullet_\bullet$, we consider the total complex of the ordered Čech complexes $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$, whose degree-$n$ term is the R-module (cf. [**Sta22**, 01FP])

(III.5.5.4)
$$\mathrm{Tot}^n(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)) = \bigoplus_{p+q=n} \prod_{0\leq i_0<\cdots<i_p\leq k} \mathcal{F}^q_p(U_{i_0} \cap \cdots \cap U_{i_p}).$$

Indeed, it depends only on the k-truncation $(\mathcal{F}^\bullet_n)_{[n]\in\mathrm{Ob}(\Delta_{\leq k})}$. If $\mathcal{F}^\bullet_\bullet$ is the pullback of a complex of quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}^\bullet$ (i.e. $\mathcal{F}^\bullet_\bullet = \nu^*\mathcal{F}^\bullet$), then there is an isomorphism in the derived category $\mathbf{D}(X)$ (cf. [**Sta22**, 01FK, 01FM, 0FLH]),

(III.5.5.5)
$$\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)) \xrightarrow{\sim} \mathrm{R}\Gamma(X, \mathcal{F}^\bullet).$$

Lemma III.5.6. *Under the assumptions in* III.5.5 *and with the same notation, for any quasi-coherent $\mathcal{O}_{X_\bullet}$-module $\mathcal{F}_\bullet$, $\mathcal{F} = \nu_*\mathcal{F}_\bullet$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. By definition, $\mathcal{F} = \mathrm{Eq}(\nu_{0*}\mathcal{F}_0 \rightrightarrows \nu_{1*}\mathcal{F}_1)$. As ν_i is quasi-compact and quasi-separated ($i = 0, 1$), $\nu_{i*}\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_X$-module. Hence, the equalizer $\mathcal{F}$ is also a quasi-coherent $\mathcal{O}_X$-module. □

Proposition III.5.7. *Under the assumptions in* III.5.5 *and with the same notation, let* $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, $\mathrm{sk}_r(X_\bullet) = (X_n)_{[n]\in \mathrm{Ob}(\Delta_{\leq r})}$ *the* r*-truncation of* $X_\bullet$, $\mathcal{F}_\bullet$ *a quasi-coherent* $\mathcal{O}_{\mathrm{sk}_r(X_\bullet)}$*-module of finite type. Assume that* $\mathcal{F}_\bullet$ *is* π*-Cartesian (cf.* III.5.1*). Then, there exists a quasi-coherent* $\mathcal{O}_X$*-module* $\mathcal{F}$ *of finite type and a* π^{25}*-isomorphism* $\nu^*\mathcal{F} \to \mathcal{F}_\bullet$.

Proof. If we also denote by $\nu : \mathrm{sk}_r(X_\bullet) \to X$ the augmentation, then $\mathcal{F}' = \nu_*\mathcal{F}_\bullet$ is a quasi-coherent $\mathcal{O}_X$-module by III.5.6 and the canonical morphism $\nu_0^*\mathcal{F}' \to \mathcal{F}_0$ is a π^8-isomorphism by III.5.3. We claim that the canonical morphism $\nu^*\mathcal{F}' \to \mathcal{F}_\bullet$ is a π^9-isomorphism. Indeed, for any integer $0 \leq n \leq r$ the morphism $\nu_n : X_n \to X$ is the composition of $\nu_0 : X_0 \to X$ with a projection $f : X_n \to X_0$ associated to a morphism $[0] \to [n]$ in Δ. Since $f^*\mathcal{F}_0 \to \mathcal{F}_n$ is a π-isomorphism by assumption, we see that $\nu_n^*\mathcal{F}' \to \mathcal{F}_n$ is a π^9-isomorphism.

For any $0 \leq i \leq k$, we denote by $\mathcal{G}_i$ the restriction of $\mathcal{F}_0$ to the component U_i, which is a quasi-coherent $\mathcal{O}_{X_i}$-module of finite type by assumption. By III.3.2.(1), there exists a π^{16}-isomorphism $\mathcal{G}_i \to \mathcal{F}'|_{U_i}$. Notice that $\mathcal{F}'$ is the filtered union of its quasi-coherent $\mathcal{O}_X$-submodules of finite type by [**EGA** $\mathrm{I}_{\mathrm{new}}$, 6.9.9]. Thus, there is a sufficiently large quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{F}$ of finite type such that the π^{16}-surjection $\mathcal{G}_i \to \mathcal{F}'|_{U_i}$ factors through $\mathcal{F}|_{U_i}$ for any $0 \leq i \leq k$. Thus, the inclusion $\mathcal{F} \subseteq \mathcal{F}'$ is π^{16}-surjective, which implies that the induced morphism $\nu^*\mathcal{F} \to \mathcal{F}_\bullet$ is a π^{25}-isomorphism. □

Lemma III.5.8. *Under the assumptions in* III.5.5 *and with the same notation, let* $\mathcal{F}_\bullet^\bullet \to \mathcal{G}_\bullet^\bullet$ *be a* π*-isomorphism of complexes of quasi-coherent* $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$*-modules. Then, the map*

$$\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{G}_\bullet^\bullet)) \tag{III.5.8.1}$$

is a π*-isomorphism. In particular, it is a* π^2*-quasi-isomorphism.*

Proof. By taking sections on an affine scheme, $\mathcal{F}_p^q(U_{i_0} \cap \cdots \cap U_{i_p}) \to \mathcal{G}_p^q(U_{i_0} \cap \cdots \cap U_{i_p})$ is still a π-isomorphism. This shows that (III.5.8.1) is a π-isomorphism. The second assertion follows from III.3.2. □

Lemma III.5.9. *Under the assumptions in* III.5.5 *and with the same notation, let* a *be an integer,* $\mathcal{F}_\bullet^\bullet \to \mathcal{G}_\bullet^\bullet$ *a morphism of complexes of quasi-coherent* $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$*-modules. Assume that for any* $0 \leq n \leq k$*, the map* $H^i(\mathcal{F}_n^\bullet) \to H^i(\mathcal{G}_n^\bullet)$ *is a* π*-isomorphism for any* $i > a$ *and* π*-surjective for* $i = a$*. Then, the map*

$$H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))) \longrightarrow H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{G}_\bullet^\bullet))) \tag{III.5.9.1}$$

is a $\pi^{4(k+1)}$*-isomorphism for any* $i > a + k$ *and* $\pi^{2(k+1)}$*-surjective for* $i = a + k$.

Proof. As taking the ordered Čech complex on $X_\bullet$ is an exact functor, we have $H^q(\check{C}^p_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)) = \check{C}^p_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}_\bullet^\bullet))$, where $H^q(\mathcal{F}_\bullet^\bullet) = (H^q(\mathcal{F}_n^\bullet))_{[n]\in \mathrm{Ob}(\Delta_{\leq k})}$ is a quasi-coherent $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$-module. Thus, there is a spectral sequence

$$E_2^{pq} = H^p(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}_\bullet^\bullet))) \Rightarrow H^{p+q}(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))), \tag{III.5.9.2}$$

which is convergent, since $\check{C}^p_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^q) = 0$ unless $0 \leq p \leq k$ ([**Sta22**, 0132]).

Let $\mathcal{K}_\bullet^\bullet$ be the cone of $\mathcal{F}_\bullet^\bullet \to \mathcal{G}_\bullet^\bullet$. The assumption implies that $H^i(\mathcal{K}_n^\bullet)$ is π^2-null for any $0 \leq n \leq k$ and $i \geq a$. The convergent spectral sequence (III.5.9.2) for $\mathcal{K}_\bullet^\bullet$ implies that for any $i \in \mathbb{Z}$, there is a finite filtration of length $\leq (k+1)$ on $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{K}_\bullet^\bullet)))$ whose graded pieces are subquotients of $E_2^{p,i-p}$ where $0 \leq p \leq k$. Since $E_2^{p,i-p}$ is π^2-null for any $i \geq a+k$, we see that $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{K}_\bullet^\bullet)))$ is $\pi^{2(k+1)}$-null for such i. The conclusion follows from the long exact sequence of cohomology groups associated to the distinguished triangle in $\mathbf{K}(R)$,

$$\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{G}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{K}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))[1]. \tag{III.5.9.3}$$

□

Proposition III.5.10. *Under the assumptions in* III.5.5 *and with the same notation, let* $a \leq b$ *be two integers,* $\mathcal{F}_\bullet^\bullet$ *a complex of quasi-coherent* $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$*-modules vanishing in degrees* $> b$*. Assume that*

(1) R *is Noetherian, and that*
(2) *the* R*-module* $H^p(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}_\bullet^\bullet)))$ *is of* π*-finite type for any* $0 \leq p \leq k$ *and* $q \geq a$ *(cf.* III.4.11*).*

Then, the complex of R*-modules* $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))$ *is* π^l*-*$[a+k, b+k]$*-pseudo-coherent for an integer* $l \geq 0$ *depending only on* $b - a$ *and* k.

PROOF. Consider the convergent spectral sequence (III.5.9.2). Notice that for any $0 \leq p \leq k$ and $q \geq a$, a subquotient of $E_2^{pq} = H^p(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}^\bullet_\bullet)))$ is of π-finite type by III.4.12.(2). Since there is a finite filtration of length $\leq (k+1)$ on $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)))$ whose graded pieces are subquotients of $E_2^{p,i-p}$ where $0 \leq p \leq k$. We see that $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)))$ is of π^{k+1}-finite type for any $i \geq a+k$ by III.4.12.(2). As $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$ vanishes in degrees $> b+k$ by definition, the conclusion follows from III.4.13.(1). □

III.6. Almost Coherent Modules

In this section, we fix a ring R with an ideal $\mathfrak{m}$ such that for any integer $l \geq 1$, the l-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$ (in particular, $\mathfrak{m} = \mathfrak{m}^2$, cf. [**GR03**, 2.1.6.(B)]). Let E be a site with a final object $*$, $\mathcal{O}$ a sheaf of R-algebras on E.

Definition III.6.1 ([**AG20**, 2.7.3])**.** Let M be an $\mathcal{O}$-module on E.

(1) We say that M is *almost zero* if it is π-null for any $\pi \in \mathfrak{m}$. We say that a morphism $f : M \to N$ of $\mathcal{O}$-modules is an *almost isomorphism* if its kernel and cokernel are almost zero.
(2) We say that M is *of π-finite type* for some element $\pi \in R$ if there exists a covering $\{U_i \to *\}_{i \in I}$ in E such that for any $i \in I$ there exist finitely many sections $s_1, \ldots, s_n \in M(U_i)$ such that the induced morphism of $\mathcal{O}|_{U_i}$-modules $\mathcal{O}^{\oplus n}|_{U_i} \to M|_{U_i}$ has π-null cokernel. We say that M is *of almost finite type* if it is of π-finite type for any $\pi \in \mathfrak{m}$.
(3) We say that M is *almost coherent* if M is of almost finite type, and if for any object U of E and any finitely many sections $s_1, \ldots, s_n \in M(U)$, the kernel of the induced morphism of $\mathcal{O}|_U$-modules $\mathcal{O}^{\oplus n}|_U \to M|_U$ is an $\mathcal{O}|_U$-module of almost finite type.

We refer to Abbes-Gros [**AG20**, 2.7, 2.8] for a more detailed study of almost coherent modules. They work in a slightly restricted basic setup for almost algebra [**AG20**, 2.6.1], but most of their arguments still work in our setup $(R, \mathfrak{m})$ by adding the following lemmas.

Lemma III.6.2. *Let M be an $\mathcal{O}$-module on E, $\pi_1, \pi_2 \in R$. If M is of π_i-finite type for $i = 1, 2$, then it is of $(x\pi_1 + y\pi_2)$-finite type for any $x, y \in R$. In particular, if there exists an integer $l \geq 1$ such that M is of π^l-finite type for any $\pi \in \mathfrak{m}$, then M is of almost finite type.*

PROOF. The problem is local on E. We may assume that there exist morphisms of $\mathcal{O}$-modules $f_i : \mathcal{O}^{\oplus n_i} \to M$ $(i = 1, 2)$ with π_i-null cokernels. Thus, the cokernel of $f_1 \oplus f_2 : \mathcal{O}^{\oplus n_1} \oplus \mathcal{O}^{\oplus n_2} \to M$ is killed by $x\pi_1 + y\pi_2$. The "in particular" part follows from the assumption that the ideal $\mathfrak{m}$ is generated by the subset $\{\pi^l \mid \pi \in \mathfrak{m}\}$. □

Lemma III.6.3. *Let M be an $\mathcal{O}$-module.*

(1) *Assume that there exists an integer $l \geq 1$ such that for any $\pi \in \mathfrak{m}$, there exists an almost coherent $\mathcal{O}$-module M_π and a π^l-isomorphism $M \to M_\pi$. Then, M is almost coherent.*
(2) *Assume that there exists an integer $l \geq 1$ such that for any $\pi \in \mathfrak{m}$, there exists an almost coherent $\mathcal{O}$-module M_π and a π^l-isomorphism $M_\pi \to M$. Then, M is almost coherent.*

PROOF. (1) The π^l-isomorphism $M \to M_\pi$ induces a π^{2l}-isomorphism $M_\pi \to M$ by III.3.2. Such an argument shows that (1) implies (2). We also see that M is of π^{3l}-finite type. Hence, M is of almost finite type by III.6.2. For any object U of E, and any morphism of $\mathcal{O}|_U$-modules $f : \mathcal{O}^{\oplus n}|_U \to M|_U$, consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \xrightarrow{f} & M \\ & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ker}(f_\pi) & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \xrightarrow{f_\pi} & M_\pi \end{array} \tag{III.6.3.1}$$

It is clear that $\mathrm{Ker}(f) \to \mathrm{Ker}(f_\pi)$ is a π^l-isomorphism. Since M_π is almost coherent by assumption, $\mathrm{Ker}(f_\pi)$ is of almost finite type. Hence, $\mathrm{Ker}(f)$ is of π^{3l}-finite type by the argument in the beginning. Thus, $\mathrm{Ker}(f)$ is of almost finite type by III.6.2. This verifies the almost coherence of M. □

We collect some basic properties about almost coherence that will be used in the rest of this article. Their proofs are essentially given in [**AG20**], and we only give a brief sketch here.

Proposition III.6.4 ([**AG20**, 2.7.16])**.** *Let $0 \longrightarrow M_1 \xrightarrow{u} M_2 \xrightarrow{v} M_3 \longrightarrow 0$ be an almost exact sequence of $\mathcal{O}$-modules on E. If two of M_1, M_2, M_3 are almost coherent, then so is the third.*

PROOF. Since almost isomorphisms preserve almost coherence by III.6.3, we may assume that $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact.

Assume that M_2 and M_3 are almost coherent. Then, M_1 is of almost finite type by [**AG20**, 2.7.14.(iii)] and III.6.2. Hence, M_1 is almost coherent as a submodule of an almost coherent $\mathcal{O}$-module M_2 by definition.

Assume that M_1 and M_2 are almost coherent. Then, M_3 is of almost finite type as a quotient of M_2. Let U be an object of E. We need to show that any homomorphism $f_3 : \mathcal{O}^{\oplus n}|_U \to M_3|_U$ has kernel of π^2-finite type for any $\pi \in \mathfrak{m}$ by III.6.2. The problem is local on E. Thus, we may take a π-surjection $f_1 : \mathcal{O}^{\oplus m}|_U \to M_1|_U$ and a lifting $f_3' : \mathcal{O}^{\oplus n}|_U \to M_2|_U$ of f_3. We put $f_2 = (f_1, f_3') : \mathcal{O}^{\oplus m+n}|_U \to M_2|_U$, and obtain a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^{\oplus m}|_U & \longrightarrow & \mathcal{O}^{\oplus m+n}|_U & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & M_1|_U & \longrightarrow & M_2|_U & \longrightarrow & M_3|_U & \longrightarrow & 0 \end{array} \tag{III.6.4.1}$$

The snake lemma shows that $\mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3)$ is π-surjective. Since $\mathrm{Ker}(f_2)$ is of almost finite type as M_2 is almost coherent, we see that $\mathrm{Ker}(f_3)$ is of π^2-finite type.

Assume that M_1 and M_3 are almost coherent. Then, M_2 is of almost finite type by [**AG20**, 2.7.14.(ii)] and III.6.2. Let U be an object of E. We need to show that any homomorphism $f_2 : \mathcal{O}^{\oplus n}|_U \to M_2|_U$ has kernel of π^2-finite type for any $\pi \in \mathfrak{m}$ by III.6.2. The problem is local on E. Thus, we may take a π-surjection $\mathcal{O}^{\oplus m}|_U \to \mathrm{Ker}(v \circ f_2)$ as $\mathrm{Ker}(v \circ f_2)$ is of almost finite type (M_3 is almost coherent). Thus, we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{O}^{\oplus m}|_U & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \longrightarrow & M_3|_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \| & & \\ 0 & \longrightarrow & M_1|_U & \xrightarrow{u} & M_2|_U & \xrightarrow{v} & M_3|_U & \longrightarrow & 0 \end{array} \tag{III.6.4.2}$$

By diagram chasing, we see that $\mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2)$ is π-surjective. Since $\mathrm{Ker}(f_1)$ is of almost finite type as M_1 is almost coherent, we see that $\mathrm{Ker}(f_2)$ is of π^2-finite type. □

Corollary III.6.5 ([**AG20**, 2.7.17])**.** *For any morphism $f : M \to N$ of almost coherent $\mathcal{O}$-modules,* $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ *and* $\mathrm{Coker}(f)$ *are almost coherent.*

Corollary III.6.6. *Assume that $\mathcal{O}$ is almost coherent as an $\mathcal{O}$-module. Then, any cohomology group of a complex of finite free $\mathcal{O}$-modules is almost coherent.*

PROOF. A finite free $\mathcal{O}$-module is almost coherent by III.6.4. Thus, a cohomology group of a complex of finite free A-modules is almost coherent by III.6.5. □

Proposition III.6.7 ([**AG20**, 2.8.7])**.** *Let $X = \mathrm{Spec}(A)$ be an affine scheme over R, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, $\pi \in \mathfrak{m}$. Then, the $\mathcal{O}_X$-module $\mathcal{F}$ is of π-finite type (resp. of almost finite type, almost coherent) on the Zariski site of X if and only if the A-module $\mathcal{F}(X)$ is of π-finite type (resp. of almost finite type, almost coherent) on the trivial site of a single point.*

PROOF. It is clear that the statement for "of π-finite type" implies that for "of almost finite type" and thus implies that for "almost coherent". It remains to show that for any $\pi \in \mathfrak{m}$ and any finitely many elements $f_1, \ldots, f_n \in A$ generating A as an ideal, an A-module M is of π-finite type if and only if the A_{f_i}-module M_{f_i} is of π-finite type for any $1 \leq i \leq n$. The necessity is obvious. For the sufficiency, we write $M = \bigcup_{\lambda \in \Lambda} M_\lambda$ as a filtered union of its A-submodules of finite type. There exists $\lambda_0 \in \Lambda$ large enough such that $M_{\lambda_0, f_i} \to M_{f_i}$ is a π-isomorphism for any $1 \leq i \leq n$. Hence, $M_{\lambda_0} \to M$ is a π-isomorphism, which completes the proof. □

Lemma III.6.8 (cf. [**Kie72**, 2.2])**.** *Let $k \in \mathbb{N}$, $X_\bullet = (X_n)_{[n] \in \mathrm{Ob}(\Delta_{\leq k})}$ a k-truncated simplicial affine scheme over R, $M_\bullet$ a quasi-coherent $\mathcal{O}_{X_\bullet}$-module. Assume that the $\mathcal{O}_{X_\bullet}$-modules $\mathcal{O}_{X_\bullet}$ and $M_\bullet$ are almost coherent. Then, for any $\pi \in \mathfrak{m}$, there exists a π-exact sequence of quasi-coherent $\mathcal{O}_{X_\bullet}$-modules*

$$\cdots \longrightarrow F_\bullet^{-1} \longrightarrow F_\bullet^0 \longrightarrow M_\bullet, \tag{III.6.8.1}$$

such that F_n^i is a finite free $\mathcal{O}_{X_n}$-module for any $i \leq 0$ and $0 \leq n \leq k$.

PROOF. Firstly, we construct $F^0_\bullet$. For each $0 \leq n \leq k$, we take a finite free $\mathcal{O}_{X_n}$-module N_n and a π-surjection $h_n : N_n \to M_n$ (as M_n is of π-finite type, cf. III.6.7). We put

$$F^0_n = \bigoplus_{0\leq m\leq k} \bigoplus_{\alpha\in \mathrm{Hom}_\Delta([m],[n])} \alpha^* N_m. \tag{III.6.8.2}$$

It forms naturally a finite free $\mathcal{O}_{X_\bullet}$-module $F^0_\bullet$. There is a natural morphism $F^0_n \to M_n$ defined on the (m, α)-component by the composition

$$\alpha^* N_m \xrightarrow{\alpha^*(h_m)} \alpha^* M_m \longrightarrow M_n. \tag{III.6.8.3}$$

It induces a π-surjective homomorphism of $\mathcal{O}_{X_\bullet}$-modules $F^0_\bullet \to M_\bullet$ (cf. the proof of [**Kie72**, 2.2]). Notice that its kernel $M^{-1}_\bullet$ is also a quasi-coherent $\mathcal{O}_{X_\bullet}$-module which is almost coherent by III.6.5. Thus, we can apply the previous procedure to $M^{-1}_\bullet$ and we construct $F^i_\bullet$ inductively for any $i \leq 0$. □

III.7. Proof of the Main Theorem

This section is devoted to proving the following theorem.

Theorem III.7.1. *Let R be a ring with an ideal $\mathfrak{m}$ such that for any integer $l \geq 1$, the l-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$. Consider a flat proper morphism of finite presentation $f : X \to S$ between R-schemes. Assume that $\mathcal{O}_X$ and $\mathcal{O}_S$ are almost coherent as modules over themselves. Then, for any quasi-coherent and almost coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any $q \in \mathbb{N}$, $\mathrm{R}^q f_*\mathcal{M}$ is a quasi-coherent and almost coherent $\mathcal{O}_S$-module.*

PROOF. The problem is local on S. Thus, we may assume that $S = \mathrm{Spec}(A)$ is affine. Since f is quasi-compact and quasi-separated, $\mathrm{R}^q f_*\mathcal{M}$ is a quasi-coherent $\mathcal{O}_S$-module for any $q \in \mathbb{N}$. Thus, it remains to prove that $H^q(X, \mathcal{M})$ is an almost coherent A-module by III.6.7. We write X as a finite union of affine open subschemes $X = \bigcup_{0\leq i\leq k} U_i$ for some $k \in \mathbb{N}_{\geq 2}$, and consider the k-truncated Čech hypercovering $X_\bullet = (X_n)_{[n]\in \mathrm{Ob}(\Delta_{\leq k})}$ (cf. III.5.5), where for any $0 \leq n \leq k$,

$$X_n = \coprod_{0\leq i_0,\dots,i_n\leq k} U_{i_0} \cap \cdots \cap U_{i_n}, \tag{III.7.1.1}$$

which is a finite disjoint union of affine open subschemes of X as X is separated. Let $\nu : X_\bullet \to X$ denote the augmentation.

We fix an element $\pi \in \mathfrak{m}$ and an integer $a < -(q + k + 2)$ in the following, and take a sequence of quasi-coherent $\mathcal{O}_{X_\bullet}$-modules by III.6.8,

$$0 \longrightarrow \mathcal{F}^a_\bullet \longrightarrow \cdots \longrightarrow \mathcal{F}^{-1}_\bullet \longrightarrow \mathcal{F}^0_\bullet \longrightarrow \mathcal{M}_\bullet = \nu^*\mathcal{M}, \tag{III.7.1.2}$$

such that $\mathcal{F}^\bullet_n \to \mathcal{M}_n[0]$ is a π-$[a, 0]$-pseudo resolution (cf. III.4.1) for any $0 \leq n \leq k$. In other words, for any $a \leq i \leq 0$ and $0 \leq n \leq k$,

(1) $\mathcal{F}^i_n$ is a finite free $\mathcal{O}_{X_n}$-module, and
(2) $H^i(\mathcal{F}^\bullet_n)$ is π-null for any $a < i < 0$, and $H^0(\mathcal{F}^\bullet_n) \to \mathcal{M}_n$ is a π-isomorphism.

For any morphism $\alpha : [m] \to [n]$ in $\Delta_{\leq k}$ (regarded also as a morphism $X_n \to X_m$), we denote by $C^\bullet_\alpha$ the cone of the induced map $\alpha^*\mathcal{F}^\bullet_m \to \mathcal{F}^\bullet_n$ of complexes of finite free $\mathcal{O}_{X_n}$-modules.

Lemma III.7.2. *For any morphism $\alpha : [m] \to [n]$ in $\Delta_{\leq k}$, there exists a homomorphism of finite free $\mathcal{O}_{X_n}$-modules $s^i_\alpha : C^i_\alpha \to C^{i-1}_\alpha$ for any $i \geq a + 1$ such that*

$$\pi^{-4a} \cdot \mathrm{id}_{C^i_\alpha} = s^{i+1}_\alpha \circ \mathrm{d}^i_\alpha + \mathrm{d}^{i+1}_\alpha \circ s^i_\alpha, \tag{III.7.2.1}$$

where $\mathrm{d}^i_\alpha : C^i_\alpha \to C^{i+1}_\alpha$ is the differential map.

PROOF. We firstly note that $C^i_\alpha = \mathcal{F}^i_n \oplus \alpha^*\mathcal{F}^{i+1}_m$ is a finite free $\mathcal{O}_{X_n}$-module. In particular, $C^\bullet_\alpha$ vanishes outside $[a - 1, 0]$. By definition, $H^i(\mathcal{F}^\bullet_n) \to H^i(\mathcal{M}_n[0])$ is a π-isomorphism for any $i > a$. Notice that the induced map $\alpha^*H^i(\mathcal{M}_m[0]) \to H^i(\mathcal{M}_n[0])$ is an isomorphism since $\mathcal{M}_\bullet = \nu^*\mathcal{M}$. Thus, the induced map $\alpha^*H^i(\mathcal{F}^\bullet_m) \to H^i(\mathcal{F}^\bullet_n)$ is a π^2-isomorphism of $\mathcal{O}_{X_n}$-modules, which implies that $H^i(C^\bullet_\alpha)$ is π^4-null for any $i > a$. Thus, the conclusion follows directly from III.3.3 (cf. (III.3.3.4)). □

Now we write A as a filtered union of finitely generated $\mathbb{Z}$-subalgebras $A = \mathrm{colim}_{\lambda\in\Lambda} A_\lambda$. By [**EGA** IV$_3$, 8.5.2, 8.8.2, 8.10.5, 11.2.6], there exists an index $\lambda_0 \in \Lambda$ such that $\pi \in A_{\lambda_0}$,

(1) a flat proper morphism of finite presentation $f_{\lambda_0} : X_{\lambda_0} \to S_{\lambda_0} = \mathrm{Spec}(A_{\lambda_0})$ whose base change along $S \to S_{\lambda_0}$ is f,
(2) affine open subschemes $U_{\lambda_0,i}$ $(0 \leq i \leq k)$ of X_{λ_0} whose base change along $S \to S_{\lambda_0}$ is U_i,

(3) a complex of finite free $\mathcal{O}_{X_{\lambda_0,\bullet}}$-modules $0 \to \mathcal{F}^a_{\lambda_0,\bullet} \to \cdots \to \mathcal{F}^{-1}_{\lambda_0,\bullet} \to \mathcal{F}^0_{\lambda_0,\bullet} \to 0$ whose pullback along $S \to S_{\lambda_0}$ is $\mathcal{F}^\bullet_\bullet$ (where $X_{\lambda_0,\bullet}$ is the k-truncated Čech hypercovering associated to $X_{\lambda_0,0} = \coprod_{0\le i\le k} U_{\lambda_0,i} \to X_{\lambda_0}$),

(4) a homomorphism of finite free $\mathcal{O}_{X_{\lambda_0,n}}$-modules $s^i_{\lambda_0,\alpha} : C^i_{\lambda_0,\alpha} \to C^{i-1}_{\lambda_0,\alpha}$ for any $i \ge a+1$ and any morphism $\alpha : [m] \to [n]$ in $\Delta_{\le k}$, such that

$$\pi^{-4a} \cdot \mathrm{id}_{C^i_{\lambda_0,\alpha}} = s^{i+1}_{\lambda_0,\alpha} \circ \mathrm{d}^i_{\lambda_0,\alpha} + \mathrm{d}^{i+1}_{\lambda_0,\alpha} \circ s^i_{\lambda_0,\alpha}, \tag{III.7.2.2}$$

where $C^\bullet_{\lambda_0,\alpha}$ is the cone of $\alpha^*\mathcal{F}^\bullet_{\lambda_0,m} \to \mathcal{F}^\bullet_{\lambda_0,n}$, and $\mathrm{d}^i_{\lambda_0,\alpha} : C^i_{\lambda_0,\alpha} \to C^{i+1}_{\lambda_0,\alpha}$ is the differential map.

We note that X_{λ_0} and S_{λ_0} are Noetherian.

Lemma III.7.3. *For any morphism $\alpha : [m] \to [n]$ in $\Delta_{\le k}$, the induced map of coherent $\mathcal{O}_{X_{\lambda_0,n}}$-modules $\alpha^* H^i(\mathcal{F}^\bullet_{\lambda_0,m}) \to H^i(\mathcal{F}^\bullet_{\lambda_0,n})$ is a π^{-8a}-isomorphism for any $i \ge a+2$.*

Proof. It follows directly from the relation (III.7.2.2). □

Lemma III.7.4. *For any $i \ge a+2$, there exists a coherent $\mathcal{O}_{X_{\lambda_0}}$-module $\mathcal{G}^i_{\lambda_0}$ and a π^{-200a}-isomorphism $\nu^*_{\lambda_0}\mathcal{G}^i_{\lambda_0} \to H^i(\mathcal{F}^\bullet_{\lambda_0,\bullet})$, where $\nu_{\lambda_0} : X_{\lambda_0,\bullet} \to X_{\lambda_0}$ is the augmentation.*

Proof. It follows by applying directly III.5.7 to the coherent $\mathcal{O}_{X_{\lambda_0,\bullet}}$-module $H^i(\mathcal{F}^\bullet_{\lambda_0,\bullet})$, whose condition is satisfied by III.7.3. □

Lemma III.7.5. *The A_{λ_0}-module $H^i(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, H^j(\mathcal{F}^\bullet_{\lambda_0,\bullet})))$ is of π^{-400a}-finite type for any $0 \le i \le k$ and $j \ge a+2$.*

Proof. Notice that by (III.5.5.5) and III.7.4, we have

$$H^i(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \nu^*_{\lambda_0}\mathcal{G}^j_{\lambda_0})) = H^i(X_{\lambda_0}, \mathcal{G}^j_{\lambda_0}), \tag{III.7.5.1}$$

which is an A_{λ_0}-module of finite type, since X_{λ_0} is proper over the Noetherian scheme $\mathrm{Spec}(A_{\lambda_0})$. Thus, $H^i(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, H^j(\mathcal{F}^\bullet_{\lambda_0,\bullet})))$ is of π^{-400a}-finite type by III.5.8 and III.7.4. □

Lemma III.7.6. *The complex of A_{λ_0}-modules $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet}))$ is π^l-$[a+k+2, k]$-pseudo-coherent for an integer $l \ge 0$ depending only on a and k.*

Proof. It follows directly from III.5.10 whose conditions are satisfied by III.7.5. □

Lemma III.7.7. *The complex of A-modules $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$ is π^l-$[a+k+2, k]$-pseudo-coherent for an integer $l \ge 0$ depending only on a and k.*

Proof. By III.4.10 and III.7.6, $\tau^{\ge(a+k+2)}(A \otimes^{\mathrm{L}}_{A_{\lambda_0}} \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet})))$ is represented by a π^l-$[a+k+2, k]$-pseudo-coherent complex of A-modules for an integer $l \ge 0$ depending only on a and k. Since $f_{\lambda_0} : X_{\lambda_0} \to S_{\lambda_0}$ is flat, $A \otimes^{\mathrm{L}}_{A_{\lambda_0}} \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet}))$ is also represented by $A \otimes_{A_{\lambda_0}} \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet})) = \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$. Hence, $\tau^{\ge(a+k+2)}(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$ is π^l-$[a+k+2, k]$-pseudo-coherent for an integer $l \ge 0$ depending only on a and k by III.4.4. The conclusion follows from applying III.4.3.(2) to $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)) \to \tau^{\ge(a+k+2)}(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$. □

Lemma III.7.8. *The complex of A-modules $\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{M}_\bullet)$ is π^l-$[a+k+2, k]$-pseudo-coherent for an integer $l \ge 0$ depending only on a and k.*

Proof. Since $\mathcal{F}^\bullet_\bullet \to \mathcal{M}_\bullet[0]$ is a π-$[a, 0]$-pseudo resolution, the map $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))) \to H^i(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{M}_\bullet))$ is a $\pi^{4(k+1)}$-isomorphism for any $i \ge a+k+1$ by III.5.9. The conclusion follows from III.4.3.(1) and III.7.7. □

Recall that we have taken $a < -(q+k+2)$ in the beginning. Since $\mathrm{R}\Gamma(X, \mathcal{M})$ is represented by the ordered Čech complex $\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{M}_\bullet)$, we see that $H^q(X, \mathcal{M})$ is π^l-isomorphic to an almost coherent A-module for an integer $l \ge 0$ depending only on a and k by III.6.6 and III.7.8. Since l is independent of the choice of $\pi \in \mathfrak{m}$, the A-module $H^q(X, \mathcal{M})$ is almost coherent by III.6.3, which completes the proof of our main theorem III.7.1. □

III.8. Remark on Abbes-Gros' Construction of the Relative Hodge-Tate Spectral Sequence

III.8.1. Let K be a complete discrete valuation field of characteristic 0 with algebraically closed residue field of characteristic $p > 0$. Let $(f, g) : (X'^{\triangleright} \to X') \to (X^{\circ} \to X)$ be a morphism of open immersions of quasi-compact and quasi-separated schemes over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$. Consider the following conditions:

(1) The associated log schemes $(X', \mathscr{M}_{X'})$ and $(X, \mathscr{M}_X)$ endowed with the compactifying log structures are adequate in the sense of [**AGT16**, III.4.7] (which holds for instance if the open immersions $X'^{\triangleright} \to X'$ and $X^{\circ} \to X$ are semi-stable over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$, cf. I.10.11).
(2) The morphism of log schemes $(X', \mathscr{M}_{X'}) \to (X, \mathscr{M}_X)$ is smooth and saturated.
(3) The morphism of schemes $g : X' \to X$ is projective.

Under these assumptions, Abbes-Gros proved Faltings' main p-adic comparison theorem in the relative case for the morphism (f, g) [**AG20**, 5.7.4], and constructed a relative Hodge-Tate spectral sequence [**AG20**, 6.7.5] (for an explicit local version, see [**AG20**, 6.9.6] and I.1.4). We explain that their proof and construction are still valid if we replace the assumption (3) by the following assumption

(3)' The morphism of schemes $g : X' \to X$ is proper.

III.8.2. The assumption on the projectivity of g has been only used in the proof of [**AG20**, 5.3.31]. There, they encountered a Cartesian diagram of schemes

$$\begin{array}{ccc} \overline{X}'^{(\infty)} & \longrightarrow & X' \\ {\scriptstyle g^{(\infty)}}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{X}^{(\infty)} & \longrightarrow & X \end{array} \tag{III.8.2.1}$$

where $\overline{X}^{(\infty)}$ is an $\mathcal{O}_{\overline{K}}$-scheme such that $\mathcal{O}_{\overline{X}^{(\infty)}}$ and $\mathcal{O}_{\overline{X}'^{(\infty)}}$ are almost coherent as modules over themselves ([**AG20**, 5.3.5.(ii)]), and a quasi-coherent and almost coherent $\mathcal{O}_{\overline{X}'^{(\infty)}}$-module $\mathscr{G}$. For proving the almost coherence of $\mathrm{R}^i g_*^{(\infty)} \mathscr{G}$, they applied [**AG20**, 2.8.18] where the assumption on the projectivity of g has been used.

Now we replace the assumption (3) by the assumption (3)', by replacing [**AG20**, 2.8.18] by our main theorem III.7.1. Indeed, the morphism g is flat by the assumption (2) (cf. [**Kat89**, 4.5]), proper by the assumption (3)', of finite presentation by the assumptions (3)' and (1) (as X is locally Noetherian). Hence, so is the morphism $g^{(\infty)}$ by base change. Therefore, we deduce the almost coherence of $\mathrm{R}^i g_*^{(\infty)} \mathscr{G}$ from our main theorem III.7.1.

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