



# Trianguline variety and eigenvariety at points with non-regular Hodge-Tate weights

Zhixiang Wu

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# Trianguline variety and eigenvariety at points with non-regular Hodge-Tate weights

*Variété trianguline et variété de Hecke aux points de poids  
de Hodge-Tate non-réguliers*

## Thèse de doctorat de l'université Paris-Saclay

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**Benjamin SCHRAEN**, Professeur

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**Titre:** Variété trianguline et variété de Hecke aux points de poids de Hodge-Tate non-réguliers  
**Mots clés:** programme de Langlands  $p$ -adique, variété de Hecke, formes compagnons

**Résumé:** Cette thèse porte sur les formes automorphes  $p$ -adiques et l'aspect localement analytique du programme de Langlands  $p$ -adique. Nous étudions la géométrie locale de la variété trianguline et le problème des points compagnons sur la variété de Hecke aux points de poids de Hodge-Tate non-réguliers. Il s'agit d'une généralisation des travaux de Breuil-Hellmann-Schraen qui avaient traité le cas régulier. Nous établissons d'abord les modèles locaux et l'irréductibilité locale de la variété triangulines aux points génériques, points qui peuvent avoir des poids de Hodge-Tate non-réguliers et même non-entiers. Nous démontrons alors l'existence, dans le cas cristallin Frobe-

nus générique, des points compagnons de raffinement fixé mais de poids différents sur la variété de Hecke, en mettant en relation des propriétés partiellement classiques des formes automorphes  $p$ -adiques et certaines propriétés de cycles sur les modèles locaux. De plus, nous démontrons l'existence des points compagnons correspondant aux autres raffinements sur la variété de Hecke en utilisant un argument d'approximation. En conséquence, sous l'hypothèse de Taylor-Wiles, nous prouvons la conjecture de socle localement analytique de Breuil pour les représentations galoisiennes cristallines génériques de poids de Hodge-Tate non-réguliers.

**Titre:** Trianguline variety and eigenvariety at points with non-regular Hodge-Tate weights  
**Keywords:**  $p$ -adic Langlands program, eigenvariety, companion forms

**Abstract:** This thesis concerns  $p$ -adic automorphic forms and the locally analytic aspect of the  $p$ -adic Langlands program. We study the local geometry of the trianguline variety and the companion points problem on the eigenvariety at points with non-regular Hodge-Tate weights. This generalizes the works of Breuil-Hellmann-Schraen who have dealt with regular cases. We first establish the local models and local irreducibility for generic points on the trianguline variety, the points that might have non-regular or non-integral Hodge-Tate weights. Then in the crystalline Frobenius

generic cases, we obtain the existence of companion points with fixed refinements but with different weights on the eigenvariety, by relating partially classical properties of  $p$ -adic automorphic forms and certain properties of cycles on the local models. Furthermore, we can find companion points corresponding to other refinements on the eigenvariety using an approximation argument. As a result, under the Taylor-Wiles hypothesis, we prove Breuil's locally analytic socle conjecture for generic crystalline Galois representations with non-regular weights.

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*To my parents and my brother*





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# Chapter 1

## Introduction

This thesis studies the locally analytic aspect of the  $p$ -adic local Langlands program and  $p$ -adic automorphic forms. In [BHS19], Breuil-Hellmann-Schraen solved the locally analytic socle conjecture of Breuil in [Bre16, Bre15b] and determined all companion points on the eigenvariety of definite unitary groups in the generic crystalline cases under the Taylor-Wiles hypothesis. They developed a theory of a local model for the trianguline variety introduced in [BHS17b] and related the geometry of the local model and locally analytic representations of  $p$ -adic Lie groups. The results of Breuil-Hellmann-Schraen require that the Galois representations have regular Hodge-Tate(-Sen) weights. This thesis will remove the regularity assumptions for Hodge-Tate weights. We will build the local models for the trianguline variety at points with non-regular weights and prove the existence of all companion points on the eigenvariety and the locally analytic socle conjecture in non-regular generic crystalline cases under the Taylor-Wiles hypothesis.

This introductory chapter aims to provide a self-contained overview of the backgrounds ( $p$ -adic local Langlands program, automorphic forms of definite unitary groups, eigenvariety, etc.) of the main results in this thesis. No results in this chapter except those in §1.5 are due to the author.

In §1.1, we give a brief introduction on the  $p$ -adic local Langlands program of  $\mathrm{GL}_2(\mathbb{Q}_p)$ , developed by Breuil, Berger, Colmez, Paškūnas and many others, and the local-global compatibility results of Emerton.

In §1.2, we introduce automorphic forms for definite unitary groups and explain how to attach  $p$ -adic Galois representations to  $p$ -adic automorphic forms.

In §1.3, we construct the eigenvariety using the theory of locally analytic representations of  $p$ -adic Lie groups, i.e. via Emerton's Jacquet module functor. We also explain its relationship with trianguline Galois representations.

In §1.4, we present the conjectures of Breuil and Hansen on companion points and companion constituents, and the results of Breuil-Hellmann-Schraen.

In §1.5, we give a summary of the main results of this thesis and sketch some points of the proofs.

Let  $p$  be a prime number and  $n \geq 2$  be a positive integer. For a local or global field  $F$ , we write  $\mathcal{G}_F := \mathrm{Gal}(\overline{F}/F)$  for its absolute Galois group. We take  $L$  to be a sufficiently large finite extension of  $\mathbb{Q}_p$  as our coefficient field,  $\varpi_L$  to be a uniformizer of  $\mathcal{O}_L$  and let  $k_L$  be the residue field of  $L$ . We fix an embedding  $L \hookrightarrow \overline{\mathbb{Q}_p}$ .

### 1.1 The case of $\mathrm{GL}_2(\mathbb{Q}_p)$

The  $p$ -adic local Langlands correspondence for  $\mathrm{GL}_2(\mathbb{Q}_p)$  has now a satisfying answer. We start by recalling classical  $\ell$ -adic situations.

### 1.1.1 $\ell$ -adic local Langlands correspondence

The Langlands program seeks connections arising between representations of Galois groups of number fields and automorphic forms of Lie groups. Let  $K$  be a finite extension of  $\mathbb{Q}_p$  and  $\ell \neq p$  be another prime number. Let  $W_K$  be the Weil group of  $K$  which lies in a short exact sequence  $1 \rightarrow I_K \rightarrow W_K \rightarrow \mathbb{Z} \rightarrow 1$ , where  $I_K \subset \mathcal{G}_K$  is the inertia subgroup and is open in  $W_K$ . The  $\ell$ -adic local Langlands correspondences, proved by Harris-Taylor and Henniart, establish bijections  $\mathrm{LL}_\ell(r) \leftrightarrow r$  between irreducible admissible smooth  $\overline{\mathbb{Q}_\ell}$ -representations  $\mathrm{LL}_\ell(r)$  of  $\mathrm{GL}_n(K)$  and  $n$ -dimensional continuous Frobenius semi-simple  $\overline{\mathbb{Q}_\ell}$ -representations  $r$  of  $W_K$ , which are compatible with  $L$ -functions,  $\epsilon$ -factors and the local class field theory. By Grothendieck's  $\ell$ -adic monodromy, the previous continuous representations of  $W_K$  are equivalent to some Weil-Deligne representations: finite-dimensional smooth semisimple representations  $r$  of  $W_K$  with an action of a nilpotent operator  $N$  satisfying that  $r(\mathrm{Frob}_\ell)Nr(\mathrm{Frob}_\ell^{-1}) = q_K^{-1}N$  where  $q_K$  is the cardinal of the residue field of  $K$  and  $\mathrm{Frob}_\ell$  denotes some geometric Frobenius element in  $W_K$ .

### 1.1.2 $p$ -adic Galois representations

The world of  $p$ -adic Galois representations of the  $p$ -adic local field  $K$  are richer than the above  $\ell$ -adic cases, and are understood thanks to fundamental works of Fontaine. Fontaine introduced his period rings  $B_{\mathrm{cris}}, B_{\mathrm{st}}, B_{\mathrm{dR}}$  which are equipped with continuous actions of  $\mathcal{G}_K$  such that  $B_{\mathrm{cris}}^{\mathcal{G}_K} = B_{\mathrm{st}}^{\mathcal{G}_K} = K_0$ , where  $K_0$  denotes the maximal unramified subfield of  $K$ , and  $B_{\mathrm{dR}}^{\mathcal{G}_K} = K$ . Suppose that  $r$  is a continuous representation of  $\mathcal{G}_K$  over a finite-dimensional  $L$ -space  $V$ . The linear space  $D_*(r) := (V \otimes_{\mathbb{Q}_p} B_*)^{\mathcal{G}_K}$ , over  $K_0$  if  $*$  = cris/st or over  $K$  if  $*$  = dR, has dimension less than or equal to  $\dim_{\mathbb{Q}_p} V$ . Then the  $p$ -adic representation  $r$  is called crystalline/semistable (resp. de Rham) if  $D_*(r)$  has dimension exactly  $\dim_{\mathbb{Q}_p} V$  over  $K_0$  (resp.  $K$ ). Remark that crystalline representations are semistable and semistable representations are de Rham.

The linear spaces  $D_*(r)$  turns the study of de Rham Galois representations to the study of some simpler linear algebraic objects. The  $p$ -adic monodromy theorem, proved by Berger, André, Mebkhout and Kedlaya, states that a continuous representation  $r : \mathcal{G}_K \rightarrow \mathrm{GL}_n(L)$  is de Rham if and only if it is potentially semistable (i.e. there exists some finite extension  $K'$  of  $K$  such that the restriction  $r|_{\mathcal{G}_{K'}}$  is semistable). One can attach a potentially semistable representation  $r$  a filtered  $(\varphi, N, \mathcal{G}_K)$ -module as follows. Let  $K'$  be a finite extension of  $K$  such that  $r|_{\mathcal{G}_{K'}}$  is semistable. Then  $D_{\mathrm{st}}(r|_{\mathcal{G}_{K'}})$  is a free  $L \otimes_{\mathbb{Q}_p} K'_0$ -module with some semi-linear actions of a triple  $(\varphi, N, \mathcal{G}_K)$ , where the Frobenius operator  $\varphi$  and the nilpotent operator  $N$  act on  $B_{\mathrm{st}}$  satisfying  $N\varphi = p\varphi N$  and the action of  $\mathcal{G}_K$  factors through  $\mathrm{Gal}(K'/K)$ . Moreover, there is a  $\mathrm{Gal}(K'/K)$ -stable decreasing Hodge filtration on  $D_{\mathrm{dR}}(r|_{\mathcal{G}_{K'}}) = D_{\mathrm{st}}(r|_{\mathcal{G}_{K'}}) \otimes_{K'_0} K' = D_{\mathrm{dR}}(r) \otimes_K K'$  whose jumping degrees are minus of the Hodge-Tate weights.

Fontaine showed in [Fon94b] that one can recover potentially semistable Galois representations from the corresponding filtered  $(\varphi, N, \mathcal{G}_K)$ -modules. A filtered  $(\varphi, N, \mathcal{G}_K)$ -module  $D$  is said to be admissible if it arises from Galois representations (i.e.  $D = D_{\mathrm{st}}(r|_{\mathcal{G}_{K'}})$  for some potentially semistable  $r$ ). Fontaine conjectured that “weakly admissible implies admissible” which characterizes admissibility of filtered  $(\varphi, N, \mathcal{G}_K)$ -modules by certain explicit property defined in terms of Hodge polygons of the filtrations and Newton polygons of the  $\varphi$ -modules. This conjecture of Fontaine was proved by Colmez-Fontaine in [CF00].

The  $p$ -adic local Langlands program, initiated by Breuil, seeks to associate some continuous representations  $\Pi(r)$  of  $\mathrm{GL}_n(K)$  over  $p$ -adic fields to  $p$ -adic continuous representations  $r : \mathcal{G}_K \rightarrow \mathrm{GL}_n(L)$  which should be determined by and determine  $r$ . By Fontaine [Fon94b, Fon94a] or see [BS07, §4], there is an equivalence of categories between the category of  $(\varphi, N, \mathcal{G}_K)$ -modules (not filtered) and the category of Weil-Deligne representations. Denote the Weil-Deligne representation associated to  $D_{\mathrm{st}}(r|_{\mathcal{G}_{K'}})$  by  $\mathrm{WD}(r)$  (with coefficients in  $L$ ). Then we can attach to the semi-simplification of  $\mathrm{WD}(r)$  a smooth representation  $\mathrm{LL}_p \circ \mathrm{WD}(r)$  of  $\mathrm{GL}_n(K)$  on  $L$ -spaces via some

normalized local Langlands correspondence. However, in this way, the information of the Hodge filtration of  $D_{\mathrm{dR}}(r)$  is lost.

*Remark 1.1.1.* We describe  $\mathrm{WD}(r)$  for  $r : \mathcal{G}_K \rightarrow \mathrm{GL}_n(K)$  such that  $r|_{\mathcal{G}_{K'}}$  is semistable. Choose an embedding  $\tau'_0 : K'_0 \hookrightarrow L$ . Let  $\mathrm{WD}(r) := D_{\mathrm{st}}(r|_{\mathcal{G}_{K'}}) \otimes_{K'_0 \otimes_{\mathbb{Q}_p} L, \tau'_0 \otimes 1} L$  which is stable under the  $L$ -linear action of  $N$ . If  $x \in W_K$  and suppose that its image under  $W_K \rightarrow \mathbb{Z}$  is  $a$  and its image in  $\mathrm{Gal}(K'/K)$  is  $\bar{x}$ , then we require that  $x$  acts on  $\mathrm{WD}(r)$  via  $\bar{x}\varphi^{-fa}$ . Then  $\mathrm{WD}(r)$  is the Weil-Deligne representation associated to  $r$  and is independent of the choice of  $\tau_0$ .

### 1.1.3 Breuil's ideas

We now focus on  $n = 2$  and  $K = \mathbb{Q}_p$ . Assume that  $r : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(L)$  is de Rham with distinct Hodge-Tate weights  $h_1 < h_2$ . Let  $W(r) = \mathrm{Sym}^{h_2-h_1-1} L^2 \otimes \det^{h_1+1}$  be the algebraic representation of  $\mathrm{GL}_2/\mathbb{Q}_p$  of the highest weight  $(h_2, h_1+1)$ . The idea of Breuil in [Bre03b, Bre04] is to consider  $\mathrm{GL}_2(\mathbb{Q}_p)$ -invariant norms of the locally algebraic representation  $W(r) \otimes_L \mathrm{LL}_p \circ \mathrm{WD}(r)$  of  $\mathrm{GL}_2(\mathbb{Q}_p)$ , or equivalently  $\mathrm{GL}_2(\mathbb{Q}_p)$ -invariant  $\mathcal{O}_L$ -lattices (as unit balls for invariant norms) of the locally algebraic representation. Such tensor product of smooth representations and algebraic representations for  $\mathrm{GL}_2(\mathbb{Z}_p)$  already occurred for Breuil-Mézard conjecture [BM02]. Then the completion of the locally algebraic representations with respect to the  $\mathrm{GL}_2(\mathbb{Q}_p)$ -invariant norms are the unitary representations of  $\mathrm{GL}_2(\mathbb{Q}_p)$  over  $p$ -adic Banach space  $\Pi(r)$ . These unitary Banach representation  $\Pi(r)$  are the representations of  $\mathrm{GL}_2(\mathbb{Q}_p)$  that should be associated to  $p$ -adic Galois representations  $r$  in the  $p$ -adic local Langlands correspondence. The possibly non-uniqueness of equivalent classes of invariant norms inside one locally algebraic representation corresponds to possibly different  $r$  with the same associated Weil-Deligne representation and Hodge-Tate weights (but different Hodge filtrations).

Moreover, the integral structures of  $p$ -adic unitary Banach representations allow reductions modulo  $p$  so that the  $p$ -adic correspondence could be compatible with the mod- $p$  local Langlands correspondence of  $\mathrm{GL}_2(\mathbb{Q}_p)$ , a semi-simple version of which was given by Breuil in [Bre03a] based on the classification of smooth irreducible representations of  $\mathrm{GL}_2(\mathbb{Q}_p)$  over characteristic  $p$  fields by Barthel-Livné and Breuil.

### 1.1.4 Colmez's functor and $p$ -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

The  $p$ -adic local Langlands correspondence  $r \leftrightarrow \Pi(r)$  between 2-dimensional  $p$ -adic continuous representations of  $\mathcal{G}_{\mathbb{Q}_p}$  and certain  $p$ -adic unitary Banach representations of  $\mathrm{GL}_2(\mathbb{Q}_p)$  was defined for all  $r$ , not just for the above de Rham cases, by Colmez [Col10] using his “magical functor”. Write  $G = \mathrm{GL}_2(\mathbb{Q}_p)$ . We consider following categories in [Col10, §0.4]:

- $\mathrm{Rep}_{\mathrm{tors}} G$ : category of smooth admissible finite length  $\mathcal{O}_L[G]$ -modules that admit a central character,
- $\mathrm{Rep}_{\mathrm{tors}} \mathcal{G}_{\mathbb{Q}_p}$ : category of continuous representations of  $\mathcal{G}_{\mathbb{Q}_p}$  over finite length  $\mathcal{O}_L$ -modules,
- $\mathrm{Rep}_L G$ : category of  $L[G]$ -modules  $\Pi$  such that  $\Pi$  admits a  $G$ -stable  $\mathcal{O}_L$ -lattice  $\Pi^\circ$  which is  $p$ -adically complete and  $\Pi^\circ/p^k \Pi^\circ \in \mathrm{Rep}_{\mathrm{tors}} G$  for all  $k \in \mathbb{N}$ ,
- $\mathrm{Rep}_L \mathcal{G}_{\mathbb{Q}_p}$ : category of finite-dimensional continuous  $L$ -representations of  $\mathcal{G}_{\mathbb{Q}_p}$ .

Colmez constructed in [Col10] a covariant functor  $\mathbf{V} : \mathrm{Rep}_{\mathrm{tors}} G \rightarrow \mathrm{Rep}_{\mathrm{tors}} \mathcal{G}_{\mathbb{Q}_p}$  as well as its variant  $\mathbf{V} : \mathrm{Rep}_L G \rightarrow \mathrm{Rep}_L \mathcal{G}_{\mathbb{Q}_p}$ , which factors through the theory of  $(\varphi, \Gamma)$ -modules. Recall a  $(\varphi, \Gamma)$ -module with coefficient in  $\mathcal{O}_L$  (resp.  $L$ ) is a finite type module over  $\mathcal{O}_\mathcal{E} := \varprojlim_k \mathcal{O}_L/p^k[[T]][\frac{1}{T}]$  (resp.  $\mathcal{E} := \mathcal{O}_\mathcal{E}[\frac{1}{p}]$ ) equipped with continuous commuting semi-linear actions of an operator  $\varphi$  and the group  $\Gamma := \mathbb{Z}_p^\times$ . The theory of Fontaine-Wintenberger and Fontaine



gives a functor  $D \mapsto V(D)$  from the category of étale  $(\varphi, \Gamma)$ -modules over  $\mathcal{O}_{\mathcal{E}}$  (resp.  $\mathcal{E}$ ) to the category of continuous  $\mathcal{O}_L$ -representations (resp.  $L$ -representations) of  $\mathcal{G}_{\mathbb{Q}_p}$  of finite rank which induces an equivalence of categories. Colmez constructed in *loc. cit.* firstly a functor  $\Pi \mapsto \mathbf{D}(\Pi)$  from  $\text{Rep}_{\text{tors}} G$  to the category of étale  $(\varphi, \Gamma)$ -modules. Then  $V(\mathbf{D}(\Pi)) \in \text{Rep}_{\text{tors}} \mathcal{G}_{\mathbb{Q}_p}$  and  $\mathbf{V}(\Pi)$  is defined to be the twist of the Pontryagin dual of  $V(\mathbf{D}(\Pi))$  by the cyclotomic character. For  $\Pi \in \text{Rep}_L G$  with an  $G$ -invariant lattice  $\Pi^\circ$ , set  $\mathbf{V}(\Pi^\circ) := \varprojlim_k \mathbf{V}(\Pi^\circ/p^k \Pi^\circ)$ . Then  $\mathbf{V}(\Pi) := \mathbf{V}(\Pi^\circ)[\frac{1}{p}]$ .

**Remark 1.1.2.** The functor  $\mathbf{V}$  is exact and satisfies following properties. Suppose that  $\Pi \in \text{Rep}_{\text{tors}} G$  is an absolutely irreducible representation of  $G$  over  $k_L$ . Then  $\mathbf{V}(\Pi) = 0$  if  $\dim_{k_L} \Pi < \infty$ ,  $\dim_{k_L} \mathbf{V}(\Pi) \in \{1, 2\}$  if  $\dim_{k_L} \Pi = \infty$  and  $\dim_{k_L} \mathbf{V}(\Pi) = 2$  if and only if  $\Pi$  is not a subquotient of principal series representations over  $k_L$  (equivalently, if and only if  $\Pi$  is supersingular under the classification of Barthel-Livné), see [Col10, Thm. 0.10].

The following theorem is [CDP14, Thm. 1.1] (see also [Paš13]), which says that Colmez's functor  $\mathbf{V}$  realizes the  $p$ -adic local Langlands correspondence for  $G$ . It is showed in [CDP14], using [DS13], that an absolutely irreducible admissible  $L$ -Banach representation  $\Pi$  lies in  $\text{Rep}_L G$ . Such  $\Pi$  is said to be non-ordinary if it is not a subquotient of a unitary principal series representation.

**Theorem 1.1.3** (Colmez-Dospinescu-Paškūnas). *The functor  $\Pi \mapsto \mathbf{V}(\Pi)$  induces a bijection between isomorphism classes of absolutely irreducible admissible unitary  $L$ -Banach representations  $\Pi$  of  $\text{GL}_2(\mathbb{Q}_p)$  that are non-ordinary and 2-dimensional absolutely irreducible continuous  $L$ -representations of  $\mathcal{G}_{\mathbb{Q}_p}$ .*

Colmez constructed also directly representations  $\Pi(D)$  of  $\text{GL}_2(\mathbb{Q}_p)$  for étale  $(\varphi, \Gamma)$ -modules  $D$  that are free of rank 2 over  $\mathcal{O}_{\mathcal{E}}$  or  $\mathcal{E}$ , or for  $D = \mathbf{D}(\Pi)$  where  $\Pi \in \text{Rep}_{\text{tors}} G$  ([Col10, II.2.2, IV.4]). The construction of  $\Pi(D)$  is essentially inverse to the functor  $\mathbf{D}$ . If  $D$  is a  $(\varphi, \Gamma)$ -module of rank two over  $\mathcal{E}$ , then  $\mathbf{D}(\Pi(D))$  is the Tate dual of  $D$  ([Col10, Cor. IV.4.11]). And if  $D = \mathbf{D}(\Pi)$  where  $\Pi \in \text{Rep}_{\text{tors}} G$ , then  $\Pi(D)$  recovers  $\Pi$  up to finite length (over  $\mathcal{O}_L$ ) objects in  $\text{Rep}_{\text{tors}} G$  and a twist by a character ([Col10, Thm. IV.4.1]).

Let  $\epsilon : \mathcal{G}_{\mathbb{Q}_p} \rightarrow L^\times$  be the cyclotomic character of  $\mathcal{G}_{\mathbb{Q}_p}$ . If  $r : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(L)$  is a continuous representation and  $D$  is the  $(\varphi, \Gamma)$ -module over  $\mathcal{E}$  associated to  $r \otimes_L \epsilon$  under Fontaine's equivalence of categories, then we set  $\Pi(r) := \Pi(D)$  for the unitary  $L$ -Banach representation of  $\text{GL}_2(\mathbb{Q}_p)$  attached to  $r$ . When  $r$  is de Rham with distinct Hodge-Tate weights, the sub- $G$ -representation  $\Pi(r)^{\text{alg}}$  of locally algebraic vectors of  $\Pi(r)$  recovers  $W(r) \otimes_L L L_p \circ \text{WD}(r)^1$  in §1.1.3.

**Remark 1.1.4.** Colmez's functor uses a direct analogy between the  $(\varphi, \Gamma)$ -module structures and the actions of the mirabolic subgroup  $\begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ & 1 \end{pmatrix}$  of  $\text{GL}_2(\mathbb{Q}_p)$ :  $\Gamma = \mathbb{Z}_p^\times \sim \begin{pmatrix} \mathbb{Z}_p^\times & \\ & 1 \end{pmatrix}$ ,  $\mathcal{O}_{\mathcal{E}}/\varpi_L = k_L[[T]][\frac{1}{T}]$  where  $k_L[[T]] = k_L[[\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}]]$ , and  $\varphi$  is related to the matrix  $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$  of  $\text{GL}_2(\mathbb{Q}_p)$ .

### 1.1.5 Emerton's local-global compatibility for $\text{GL}_2(\mathbb{Q}_p)$

The above  $p$ -adic correspondence for  $\text{GL}_2(\mathbb{Q}_p)$  which factors through the theory of  $(\varphi, \Gamma)$ -modules is the “right” one largely because it is compatible with global situations, as shown by Emerton [Eme11]. In global situations, we can associate  $p$ -adic global Galois representations  $\mathcal{G}_{\mathbb{Q}} \rightarrow \text{GL}_2(\overline{\mathbb{Q}_p})$  for  $p$ -adic modular forms. Emerton introduced completed cohomology in [Eme06c] to  $p$ -adically interpolate classical modular forms. Let  $\mathbf{A}^\infty$  be the ring of finite adeles

<sup>1</sup>We follow the conventions in [Eme11, §3.2, §4.2] for  $p$ -adic local Langlands correspondence (without a twist of the cyclotomic character in the Galois side for Colmez's functor) and the normalization of the classical local Langlands correspondence.

and  $\mathbf{A}^{\infty,p}$  be the ring of adeles away from  $p$  and  $\infty$ . For open compact subgroups  $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$  and  $K^p \subset \mathrm{GL}_2(\mathbf{A}^{\infty,p})$ , let  $Y_{K_p K^p}$  be the modular curve over  $\mathbb{Q}$  of level  $K_p K^p$ . Its  $\mathbb{C}$ -points are the Riemann surfaces

$$\mathrm{GL}_2(\mathbb{Q}) \setminus ((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbf{A}^\infty)) / K_p K^p.$$

Fix a tame level  $K^p \subset \mathrm{GL}_2(\mathbf{A}^{\infty,p})$ . The completed cohomology of the tower of modular curves  $\cdots \rightarrow Y_{K_p K^p} \rightarrow \cdots \rightarrow Y_{\mathrm{GL}_2(\mathbb{Z}_p) K^p}$  where  $K_p$  runs over all open compact subgroups of  $\mathrm{GL}_2(\mathbb{Z}_p)$  is defined to be

$$\hat{H}^*(K^p)_{\mathcal{O}_L} := \varprojlim_n \varinjlim_{K_p} H_{\text{ét}}^*(Y_{K_p K^p, \overline{\mathbb{Q}}}, \mathcal{O}_L / \varpi^n \mathcal{O}_L).$$

We also let  $\hat{H}_{\mathcal{O}_L}^* := \varinjlim_{K^p} \hat{H}^1(K^p)_{\mathcal{O}_L}$  and  $\hat{H}_L^* := \hat{H}_{\mathcal{O}_L}^* \otimes_{\mathcal{O}_L} L$ . These completed cohomologies ( $\hat{H}^*(K^p)_{\mathcal{O}_L}$ ,  $\hat{H}_{\mathcal{O}_L}^*$  and  $\hat{H}_L^*$ ) are equipped with continuous commuting ‘‘Hecke’’ actions of  $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbf{A}^{\infty,p})$  and ‘‘Galois’’ actions of  $\mathcal{G}_{\mathbb{Q}}$ . Let  $\rho : \mathcal{G}_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(L)$  be a continuous, odd and absolutely irreducible representation which is unramified almost everywhere. For a prime number  $\ell$ , let  $\rho_\ell := \rho|_{\mathcal{G}_{\mathbb{Q}_\ell}}$ . The following theorem is [Eme11, Thm. 1.2.1].

**Theorem 1.1.5** (Emerton). *Under certain conditions on  $\rho$  in loc. cit. (for example, the mod  $\varpi_L$  reduction  $\bar{\rho}$  is absolutely irreducible, see [Eme11, Thm. 1.2.3]), there is a  $\mathrm{GL}_2(\mathbf{A}^\infty)$ -equivariant isomorphism:*

$$\Pi(\rho_p) \otimes_L \bigotimes_{\ell \neq p} {}' \mathrm{LL}_p(\rho_\ell) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}}}(\rho, \hat{H}_L^1) \quad (1.1.6)$$

where  $\mathrm{LL}_p(\rho_\ell)$  are smooth representations of  $\mathrm{GL}_2(\mathbb{Q}_\ell)$  given by the classical local Langlands correspondences in §1.1.1 and  $\Pi(\rho_p)$  is the unitary Banach representation of  $\mathrm{GL}_2(\mathbb{Q}_p)$  attached to  $\rho_p$  defined in the end of §1.1.4 via Colmez’s functor.

### 1.1.6 Fontaine-Mazur conjecture and locally algebraic representations

An important consequence of Emerton’s local-global compatibility results is his proof of lots of cases of the Fontaine-Mazur conjecture. The conjecture states that 2-dimensional geometric  $p$ -adic Galois representations of  $\mathcal{G}_{\mathbb{Q}}$  with distinct Hodge-Tate weights are attached to classical modular forms of weight  $\geq 2$  up to a twist ([FM95]). Recall, by Eichler-Shimura, that the systems of Hecke eigenvalues of classical modular cusp forms of weight  $k + 2 \geq 2$  appear in the étale cohomology

$$H^1(\mathrm{Sym}^k(R^1 \pi_* L)) := \varinjlim_{K_p, K^p} H_{\text{ét}}^1(Y_{K_p K^p, \overline{\mathbb{Q}}}, \mathrm{Sym}^k(R^1 \pi_* L))$$

where  $\pi : E \rightarrow Y_{K_p K^p}$  denotes the universal elliptic curves on the modular curves. These étale cohomologies  $H^1(\mathrm{Sym}^k(R^1 \pi_* L))$  are equipped with smooth  $\mathrm{GL}_2(\mathbf{A}^\infty)$ -actions and are realized in the completed cohomology via locally algebraic vectors by the following theorem ([Eme06a, Thm. 7.4.2]).

**Theorem 1.1.7** (Emerton). *There is a  $\mathrm{GL}_2(\mathbf{A}^\infty) \times \mathcal{G}_{\mathbb{Q}}$ -equivariant isomorphism*

$$\bigoplus_{k \geq 0, n \in \mathbb{Z}} H^1(\mathrm{Sym}^k(R^1 \pi_* L)) \otimes_L (\mathrm{Sym}^k L^2)^\vee \otimes_L \epsilon^n \xrightarrow{\sim} (\hat{H}_L^1)^{\mathrm{alg}} \quad (1.1.8)$$

where  $\epsilon = \epsilon \otimes \epsilon \circ \det$  is the one-dimensional  $p$ -adic cyclotomic character of  $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathcal{G}_{\mathbb{Q}}$ ,  $(\mathrm{Sym}^k L^2)^\vee$  are algebraic representations of  $\mathrm{GL}_2(\mathbb{Q}_p)$  and  $(\hat{H}_L^1)^{\mathrm{alg}}$  denotes the subspace of locally algebraic vectors with respect to actions of  $\mathrm{GL}_2(\mathbb{Z}_p)$ .

Suppose that  $\rho$  is as in (1.1.6) and that  $\rho_p$  is de Rham with distinct Hodge-Tate weights. If we know  $\Pi(\rho_p)^{\text{alg}}$  is non-zero in 1.1.6, then there are classical forms in the left-hand side of (1.1.8) associated with  $\rho$ . This is a prototype argument for the classicality in the proof of Fontaine-Mazur conjecture.

Kisin previously gave a proof in [Kis09b] for the Fontaine-Mazur conjecture via Breuil-Mézard conjecture, which also relies on the  $p$ -adic local Langlands correspondence for  $\text{GL}_2(\mathbb{Q}_p)$ .

### 1.1.7 $(\varphi, \Gamma)$ -modules over the Robba rings and locally analytic representations

The passage from an étale  $(\varphi, \Gamma)$ -module over  $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[\frac{1}{p}]$  to filtered  $(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$ -modules can factor through the theory of  $(\varphi, \Gamma)$ -modules over the Robba rings. On the other hand, the subspace  $\Pi^{\text{alg}}$  of locally algebraic vectors of an admissible Banach representation  $\Pi$  of  $\text{GL}_n(K)$  is contained in the subspace  $\Pi^{\text{an}}$  of locally  $(\mathbb{Q}_p)$ -analytic vectors of  $\Pi$ . We can hope that the locally analytic aspect of the  $p$ -adic local Langlands should relate the two objects ( $(\varphi, \Gamma)$ -modules over the Robba rings and locally analytic representations) directly.

Recall a vector  $v \in \Pi$  is locally analytic if the orbit map  $\text{GL}_n(\mathcal{O}_K) \rightarrow \Pi, g \mapsto gv$  is a locally analytic function, i.e.  $g \mapsto gv$  can be written as convergent power series with coefficients in  $\Pi$  locally for some charts of the  $p$ -adic manifold  $\text{GL}_n(\mathcal{O}_K)$ . The subspace  $\Pi^{\text{an}}$  is equipped with some topology so that  $\Pi^{\text{an}}$  is a locally analytic representation of  $\text{GL}_n(K)$  [ST02b, §3] and the inclusion  $\Pi^{\text{an}} \hookrightarrow \Pi$  is continuous. Even if  $\Pi^{\text{alg}}$  might be zero,  $\Pi^{\text{an}}$  is always dense in  $\Pi$  and the functor  $\Pi \rightarrow \Pi^{\text{an}}$  is exact by Schneider-Teitelbaum [ST03].

On the Galois side, let  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  be a continuous representation and let  $D(r)$  be the associated  $(\varphi, \Gamma_K)$ -module over  $\mathcal{E}_{L,K}$ . Here, we let  $K_{\infty} := K(\mu_{p^{\infty}})$  be the extension of  $K$  obtained by adding all  $p$ -th power roots of the unity,  $\Gamma_K := \text{Gal}(K(\mu_{p^{\infty}})/K)$  and  $\mathcal{E}_{L,K}$  is ring defined similarly as  $\mathcal{E}_L = \mathcal{E}_{L, \mathbb{Q}_p}$  but for general finite extensions  $K$  of  $\mathbb{Q}_p$ . Cherbonnier-Colmez proved in [CC98] that  $r$  is overconvergent, i.e. the  $(\varphi, \Gamma_K)$ -module  $D(r) = D^{\dagger}(r) \otimes_{\mathcal{E}_{L,K}^{\dagger}} \mathcal{E}_{L,K}$  can be defined over some subring  $\mathcal{E}_{L,K}^{\dagger}$  of  $\mathcal{E}_{L,K}$  of overconvergent elements (see also [Ber02, Thm. 0.1]). Let  $\mathcal{R}_{L,K}$  be the Robba ring of  $K$  over  $L$ . It is a ring also equipped with continuous commuting actions of  $\varphi$  and  $\Gamma_K$ . There is a ring homomorphism  $\mathcal{E}_{L,K}^{\dagger} \hookrightarrow \mathcal{R}_{L,K}$ . Let  $D_{\text{rig}}(r) := D^{\dagger}(r) \otimes_{\mathcal{E}_{L,K}^{\dagger}} \mathcal{R}_{L,K}$  be the  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}$  (cf. [Nak09, Lem. 1.30]) associated to  $r$ .

*Remark 1.1.9.* The bounded Robba ring  $\mathcal{R}_K^{\text{bd}}$  is the ring of functions  $f = \sum_{k \in \mathbb{N}} a_k T^k, a_k \in K'_0$  that converge on the rigid annulus  $s \leq |T|_p < 1$  for some  $0 < s < 1$  and are bounded, i.e.

$$\mathcal{R}_K^{\text{bd}} = \left\{ \sum_{k \in \mathbb{N}} a_k T^k, a_k \in K'_0 \mid \sup_{k \in \mathbb{N}} |a_k|_p < +\infty \wedge \exists 0 < s < 1, \lim_{n \rightarrow -\infty} |a_k|_p s^n = 0 \right\}$$

where  $K'_0$  is the maximal unramified extension of  $K$  inside  $K_{\infty}$  and  $|p|_p = p^{-1}$ .

The Robba ring  $\mathcal{R}_K$  which contains  $\mathcal{R}_K^{\text{bd}}$  is the unbounded version, i.e.

$$\mathcal{R}_K = \left\{ \sum_{k \in \mathbb{N}} a_k T^k, a_k \in K'_0 \mid \forall 0 < r < 1 \lim_{k \rightarrow +\infty} |a_k|_p r^k = 0 \wedge \exists 0 < s < 1, \lim_{k \rightarrow -\infty} |a_k|_p s^k = 0 \right\}.$$

And we let  $\mathcal{R}_{L,K} := L \otimes_{\mathbb{Q}_p} \mathcal{R}_K$ .

The ring  $\mathcal{E}_{L,K}$  is isomorphic to  $L \otimes_{\mathbb{Q}_p} \mathbf{B}_K$  where  $\mathbf{B}_K$  is in the notation of [Ber08a]. The ring  $\mathcal{E}_{L,K}^{\dagger}$  is isomorphic to  $L \otimes_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger}$  by choosing certain (in general non-canonically) element  $\pi_K \in \mathbf{B}_K$  so that the ring  $\mathbf{B}_K^{\dagger}$  is isomorphic to  $\mathcal{R}_K^{\text{bd}}$  by sending  $T$  to  $\pi_K$  ([Ber02, Prop. 1.4]).

Kedlaya constructed in [Ked04] (see also [Ked05, Ked08]) some slope filtration on the category of  $\varphi$ -modules over  $\mathcal{R}_K$ . And a  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_K$  is said to be étale if the underlying  $\varphi$ -module is pure of slope 0, equivalently if it has an étale model of a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_K^{\text{bd}}$ . By [Ked08, Prop. 1.5.5], the base change functor induces an equivalence between étale  $\varphi$ -modules over  $\mathcal{R}_K^{\text{bd}}$  and over  $\mathcal{R}_K$ . In summary, we have the following theorem.

**Theorem 1.1.10** (Fontaine-Wintenberger, Fontaine, Cherbonnier-Colmez, Berger, Kedlaya). *The functor  $r \mapsto D_{\text{rig}}(r)$  induces an equivalence of categories between the category of continuous finite-dimensional representations of  $\mathcal{G}_K$  over  $L$  and the category of étale  $(\varphi, \Gamma_K)$ -modules over the Robba ring  $\mathcal{R}_{L,K}$ .*

Moreover,  $p$ -adic Hodge theory invariants of  $r$  like  $D_{\text{dR}}(r)$ ,  $D_{\text{cris}}(r)$ , etc. can be extracted from  $D_{\text{rig}}(r)$ . Berger constructed in [Ber02] functors  $D \rightarrow W_{\text{dR}}^+(D)$  from  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_K$  to semi-linear  $B_{\text{dR}}^+$ -representations of  $\mathcal{G}_K$ . In the case  $D = D_{\text{rig}}(r)$ ,  $W_{\text{dR}}^+(D) = r \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$ . Berger constructed in [Ber08a] another functor  $D \mapsto W_e(D)$  from  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_K$  to continuous semi-linear  $B_e$ -representations of  $\mathcal{G}_K$  where  $B_e := B_{\text{cris}}^{\varphi=1}$  is a principal ideal domain. Berger showed that there exists an equivalence between  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_K$  and  $B$ -pairs  $(W_{\text{dR}}^+, W_e)$  (see [Nak09] for a  $\mathcal{R}_{L,K}$ -version). Berger also proved in [Ber08b] an equivalence between the category of “ $\Gamma_K$ -locally trivial”  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_K$  and the category of filtered  $(\varphi, N, \mathcal{G}_K)$ -modules, where étale  $(\varphi, \Gamma_K)$ -modules over the Robba ring correspond to admissible filtered  $(\varphi, N, \mathcal{G}_K)$ -modules.

A motivating example of the relationship between  $(\varphi, \Gamma)$ -modules over the Robba rings and locally analytic representations is that when  $n = 2$  and  $K = \mathbb{Q}_p$ , Colmez inserts  $D_{\text{rig}}(r)$  into his functor instead of the  $D(r)$ , then the output is the subspace  $\Pi(r)^{\text{an}}$  of locally analytic vectors of  $\Pi(r)$ , see [Col10, V].

### 1.1.8 Beyond $\text{GL}_2(\mathbb{Q}_p)$

The  $p$ -adic local Langlands correspondence is still largely conjectural beyond  $n = 1$  and the group  $\text{GL}_2(\mathbb{Q}_p)$ . With several attempts (e.g. [SV11, SVZ14, ZÍ18, Bre15a]), the full generalization of Colmez’s functor is still in mysterious. Moreover, the  $p$ -adic or mod- $p$  representation theory of  $p$ -adic Lie groups is much more difficult beyond  $\text{GL}_2(\mathbb{Q}_p)$ . For example, Breuil-Paşkūnas constructed in [BP12] (too) many supersingular representations for  $\text{GL}_2(K)$  when  $K$  is unramified  $\mathbb{Q}_p$  and  $K \neq \mathbb{Q}_p$  which already shattered the hope for a bijective mod- $p$  local Langlands for general groups. But one could still hope that  $p$ -adic local Langlands correspondence appear in geometry (e.g. [Sch18, Sch10, Sch11, DLB17, CDN20]) or in global theories, as suggested by  $\ell$ -adic cases and Emerton’s results for  $\text{GL}_2(\mathbb{Q}_p)$ . If  $\rho_p : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  can be globalized to be a restriction of a Galois representation  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  for a global number field  $F$  with  $K = F_v$  for some place  $v$  of  $F$  above  $p$ , the conjectural unitary  $p$ -adic Banach representation  $\Pi(\rho_p)$  attached to  $\rho_p$  should appear in some completed cohomology as that of modular curves (1.1.6). The mod- $p$  version of this approach can now determine invariants like Serre weights (e.g. [GLS15, GK14]) and even the mod- $p$  representations of  $\text{GL}_2(K)$  in certain cases e.g. [BHH<sup>+</sup>20, HW20, BHH<sup>+</sup>21]. For the locally analytic aspect, Breuil gives the locally analytic socle conjecture in [Bre16, Bre15b] describing part of the  $\text{GL}_n(K)$ -representations inside the  $\text{GL}_n(K)$ -socle of  $\Pi(\rho_p)^{\text{an}}$ . Breuil’s conjecture is verified in some cases when  $\rho_p$  is crystalline by Breuil-Hellmann-Schrae in [BHS19].

This thesis will work in the settings of [BHS19] and consider  $p$ -adic automorphic forms of definite unitary groups which should be the easiest situations in higher ranks.

## 1.2 Automorphic forms for definite unitary groups

The goal of this section is to define, for certain Galois representations  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  where  $F$  is a global number field, certain unitary Banach representations  $\Pi(\rho)$  of  $p$ -adic Lie groups associated with  $\rho$  cut out from the space of  $p$ -adic automorphic forms of definite unitary groups. We expect certain properties of  $\Pi(\rho)$  will be determined by the restriction of  $\rho$  to decomposition groups of places over  $p$ .

Let  $F$  be a quadratic imaginary extension of a totally real number field  $F^+$ . We will assume that the extension  $F/F^+$  is unramified. Let  $S_p$  be the set of all places of  $F^+$  above  $p$ . Assume

that any place  $v \in S_p$  splits as  $v = \widetilde{v}\widetilde{v}^c$  in  $F$  where  $\widetilde{v}^c$  denotes the image of  $v$  under the non-trivial element  $c \in \text{Gal}(F/F^+)$ . In this section, we fix an isomorphism  $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ .

### 1.2.1 Definite unitary groups

We introduce the unitary groups. Recall  $n \geq 2$ . Use also  $*$  for the complex conjugation  $x \mapsto x^* := c(x)$  for  $x \in \mathbb{C}$ . Let  $V = F^n$ , an  $n$ -dimensional  $F$ -space. An involution of second type  $(-)^*$  on  $B := \text{End}_F(V) = M_n(F)$  is an  $F^+$ -linear involution of  $B$  (i.e.  $(b_1 b_2)^* = b_2^* b_1^*$  and  $b^{**} = b$ ) coinciding with  $*$  on the center  $F$  of  $M_n(F)$ . Such involutions are equivalent to non-degenerate Hermitian forms on  $V$  (up to factors in  $(F^+)^{\times}$ ) which are bi-additive maps  $(-, -) : V \times V \rightarrow F$  such that  $(ax, by) = ab^*(x, y), \forall a, b \in F, x, y \in V$  and  $(x, y) = (y, x)^*$ . The non-degenerate Hermitian forms determine corresponding involutions by  $(bx, y) = (x, b^*y)$  for all  $b \in B, x, y \in V$ . Given a non-degenerate Hermitian form on  $V$ , we have the associated unitary group over  $F^+$  defined by

$$\mathbb{G}(R) = \{g \in B \otimes_{F^+} R \mid gg^* = \text{Id}\}$$

for all  $F^+$ -algebra  $R$ . For each  $F^+$ -embedding  $\iota : F \hookrightarrow F$ , the projection  $B \otimes_{F^+} F \twoheadrightarrow B \otimes_{F, \iota} F$  induces a splitting of  $\mathbb{G}$  over  $F$ , i.e.  $\mathbb{G} \times_{F^+} F \simeq \text{GL}_n/F$ .

*Remark 1.2.1.* If  $v$  is a real place of  $F^+$ ,  $\mathbb{G}_{F_v^+}$  is isomorphic to one of the usual unitary group  $U(p, q)$  of signature  $(p, q), p + q = n$  associated with the Hermitian form  $x_1 y_1^* + \cdots + x_p y_p^* - x_{p+1} y_{p+1}^* - \cdots - x_n y_n^*$  for  $x, y \in \mathbb{C}^n$ . The unitary group  $U(p, q)$  is definite (i.e.  $\mathbb{G}_{F_v^+}(\mathbb{R})$  is compact) if  $p = 0$  or  $n$ . For a finite place  $v$  that is inert in  $F^+$ ,  $\mathbb{G}_{F_v^+}$  is quasi-split if  $\mathbb{G}_{F_v^+}$  is isomorphic to the unitary group over  $F_v^+$  associated with the Hermitian form  $x_1 y_n^* + \cdots + x_i y_{n+1-i}^* + \cdots + x_n y_1^*$ . For an inert finite place  $v$ , there are two different isomorphic classes of unitary groups associated with  $F_v/F_v^+$  if  $2 \mid n$  and one if  $2 \nmid n$  ([Min11, §3.2.1]).

We take a unitary group  $\mathbb{G}$  over  $F^+$  that is definite (i.e.  $\mathbb{G}(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$  is compact) and quasi-split over all finite places (see the remark above). Such  $\mathbb{G}$  exists when  $2 \nmid n$  or  $4 \mid n[F^+ : \mathbb{Q}]$  by the Hasse principal ([Clo91, §2]). For any place  $v$  of  $F^+$ , we write  $G_v := \mathbb{G}(F_v^+)$  and  $G_{\infty} := \mathbb{G}(F^+ \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{v \text{ real}} G_v$ . As in [CHT08, §3.3] or [Tho12, §6], we can choose some order  $\mathcal{O}_B$  of  $B$  which is stable under the corresponding involution and are maximal for each place  $v$  of  $F^+$  that splits in  $F$ . Using the chosen order we can obtain a model  $\mathbb{G}$  over  $\mathcal{O}_{F^+}$  with the same notation. For each place  $v$  that splits in  $F$  and a place  $w \mid v$  of  $F$ , we choose an isomorphism  $i_w : G_v \xrightarrow{\sim} \text{GL}_n(F_w)$  which induces  $\mathbb{G}(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \text{GL}_n(\mathcal{O}_{F_w})$  and  $i_{w^c} = {}^t(c \circ i_w)^{-1}$ .

We also write  $G_p := \mathbb{G}(F_v^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) = \prod_{v \in S_p} G_v$ . For each  $v \in S_p$ , we choose a place  $\widetilde{v}$  of  $F$  over  $v$ . Then  $(i_{\widetilde{v}})_{v \in S_p}$  induces an isomorphism  $G_p \xrightarrow{\sim} \prod_{v \in S_p} \text{GL}_n(F_{\widetilde{v}})$  of  $p$ -adic Lie groups. Let  $K_p \subset G_p$  be the preimage of  $\prod_{v \in S_p} \text{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$  under the isomorphism.

### 1.2.2 Automorphic forms and Galois representations

We review the classical theory of automorphic forms of  $\mathbb{G}$ .

An automorphic representation  $\pi$  of  $\mathbb{G}(\mathbf{A}_{F^+})$  is an irreducible representation of  $\mathbb{G}(\mathbf{A}_{F^+})$  that appears in the space  $\mathcal{A}(\mathbb{G})$  of complex smooth  $G_{\infty}$ -finite functions on  $\mathbb{G}(F^+) \backslash \mathbb{G}(\mathbf{A}_{F^+})$  (cf. [BC09, §6.2.3]). Take an automorphic representation  $\pi$  of  $\mathbb{G}(\mathbf{A}_{F^+})$ . Then it is necessarily a restricted tensor product  $\pi = \otimes'_v \pi_v$  where for each place  $v$  of  $F^+$ ,  $\pi_v$  is an irreducible smooth representation of  $G_v$ . We write  $\pi_{\infty} := \otimes_{v \text{ real}} \pi_v$  and  $\pi^{\infty} := \otimes_{v \text{ finite}} \pi_v$ . Since  $G_{\infty}$  is compact,  $\pi_{\infty}$  is a finite-dimensional representation of  $G_{\infty}$  ([BC09, §6.7]). Take  $W_{\mathbb{C}} = \pi_{\infty}^{\vee}$  which is a finite-dimensional irreducible complex representation of  $G_{\infty}$  and let  $U = \prod_v U_v \subset \mathbb{G}(\mathbf{A}_{F^+}^{\infty})$  be an open compact subgroup such that  $(\pi^{\infty})^U \neq 0$ . Let  $\mathcal{A}(W_{\mathbb{C}})^U$  be the space of algebraic automorphic forms of level  $U$  and weight  $W_{\mathbb{C}}$  which are smooth functions  $f : \mathbb{G}(\mathbf{A}_{F^+}^{\infty}) \rightarrow W_{\mathbb{C}}$

such that  $f(gx) = g_\infty f(x)$  for all  $g = (g_\infty, g^\infty) \in \mathbb{G}(F^+) \subset G_\infty \times \mathbb{G}(\mathbf{A}_{F^+}^\infty)$  and are right  $U$ -invariant. Then  $(\pi^\infty)^U$  can be realized in  $\mathcal{A}(W_\mathbb{C})^U$ . Since  $\mathbb{G}(F^+) \backslash \mathbb{G}(\mathbf{A}_{F^+}^\infty)/U$  is a finite set,  $\mathcal{A}(W_\mathbb{C})^U$  is finite-dimensional for all  $U$ . Furthermore, we have a  $\mathbb{G}(\mathbf{A}_{F^+}^\infty)$ -equivariant isomorphism  $\mathcal{A}(W_\mathbb{C}) := \varinjlim_{U \subset \mathbb{G}(\mathbf{A}_{F^+}^\infty)} \mathcal{A}(W_\mathbb{C})^U \simeq \text{Hom}_{G_\infty}(W_\mathbb{C}^\vee, \mathcal{A}(\mathbb{G}))$  and under the isomorphism a decomposition

$$\mathcal{A}(\mathbb{G}) = \bigoplus_{W_\mathbb{C}} W_\mathbb{C}^\vee \otimes_\mathbb{C} \mathcal{A}(W_\mathbb{C}). \quad (1.2.2)$$

For each  $U = U_p U^p$  where  $U_p := \prod_{v \in S_p} U_v$ ,  $U^p := \prod_{v \notin S_p} U_v$ , we let  $S'_{U^p}$  be the set of all finite places of  $F^+$  that split in  $F$ , such that  $U_v$  is maximal and is not in  $S_p$ . We assume that  $U_v = \mathbb{G}(\mathcal{O}_{F_v^+})$  when  $v \in S'_{U^p}$ . For all places  $w$  of  $F$  over  $v \in S'_{U^p}$ , let  $T_{w,i}, i = 1, \dots, n$  be the Hecke operators associated to the pull-back of the double cosets

$$[\text{GL}_n(\mathcal{O}_{F_w}) \text{diag}(\varpi_{F_w} I_i, I_{n-i}) \text{GL}_n(\mathcal{O}_{F_w})]$$

under the isomorphism  $i_w : G_v \simeq \text{GL}_n(F_w)$  where  $I_i$  denotes the  $i$ -th identity matrices. Let  $\mathbb{T}_{\mathbb{Z}}^{S'_{U^p}} = \mathbb{Z}[T_{w,i} \mid w|v, v \in S'_{U^p}, i = 1, \dots, n]$  be the commutative  $\mathbb{Z}$ -algebra generated by  $T_{w,i}$ . Then  $\mathbb{T}_{\mathbb{Z}}^{S'_{U^p}}$  acts on  $\mathcal{A}(W_\mathbb{C})^U$  by the usual Hecke actions. The image of  $\mathbb{T}_{\mathbb{Z}}^{S'_{U^p}}$  in  $\text{End}_\mathbb{C}(\mathcal{A}(W_\mathbb{C})^U)$  generates a finite  $\mathbb{C}$ -algebra whose maximal ideals correspond to systems of Hecke eigenvalues  $\chi : \mathbb{T}_{\mathbb{Z}}^{S'_{U^p}} \rightarrow \mathbb{C}$  that appear in  $\mathcal{A}(W_\mathbb{C})^U$ . We will introduce how to attach  $p$ -adic Galois representations of  $\mathcal{G}_F = \text{Gal}(\overline{F}/F)$  for such systems of Hecke eigenvalues.

The representation  $W_\mathbb{C}$  of  $\mathbb{G}(F^+) = (\text{Res}_{F^+/\mathbb{Q}} \mathbb{G})(\mathbb{Q})$  can be defined over  $\overline{\mathbb{Q}}$  or even a number field contained in  $\overline{\mathbb{Q}}$  which splits  $\text{Res}_{F^+/\mathbb{Q}} \mathbb{G}$ . Extending the scalars to a large enough finite extension  $L$  of  $\mathbb{Q}_p$ , we obtain an algebraic representation  $W_L$  of  $G_p = (\text{Res}_{F^+/\mathbb{Q}} \mathbb{G})(\mathbb{Q}_p)$ . For an open compact subgroup  $U \subset \mathbb{G}(\mathbf{A}_{F^+}^\infty)$ , we write  $U_p := \prod_{v \in S_p} U_v$  and  $U^p := \prod_{v \notin S_p} U_v$ . We define the space  $S(U, W_L)$  to be the set of all functions  $f : \mathbb{G}(F^+) \backslash \mathbb{G}(\mathbf{A}_{F^+}^\infty) \rightarrow W_L$  such that  $f(xu_p u^p) = u_p^{-1} f(x)$  for all  $x \in \mathbb{G}(\mathbf{A}_{F^+}^\infty)$ ,  $u_p \in U_p$  and  $u^p \in U^p$ . Via the fixed isomorphism  $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ , there is a  $\mathbb{T}_{\mathbb{Z}}^{S'_{U^p}}$ -equivariant isomorphism

$$\begin{aligned} \mathcal{A}(W_\mathbb{C})^U &\xrightarrow{\sim} S(U, W_L) \otimes_L \overline{\mathbb{Q}}_p \\ f &\mapsto (G_p \times \mathbb{G}(\mathbf{A}^{\infty,p}) \rightarrow W_{\overline{\mathbb{Q}}_p} : (g_p, g^p) \mapsto g_p^{-1} f(g_p g^p)). \end{aligned} \quad (1.2.3)$$

Let  $\chi : \mathbb{T}_{\mathbb{Z}}^{S'_{U^p}} \rightarrow L$  be a system of Hecke eigenvalues that appear in  $S(U, W_L)$ . Then by [Gue11, Thm. 2.3], there exists a continuous semisimple Galois representation  $\rho_\chi : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  associated to  $\chi$  in the sense that for any place  $w$  of  $F$  above  $v \in S'_{U^p}$ ,  $\rho_{\chi,w} := \rho_\chi|_{\mathcal{G}_{F_w}}$  is unramified and the characteristic polynomial of  $\rho_\chi(\text{Frob}_w)$  is equal to

$$X^n + \dots + (-1)^i q_w^{\frac{i(i-1)}{2}} \chi(T_{w,i}) X^{n-i} + \dots + (-1)^n q_w^{\frac{n(n-1)}{2}} \chi(T_{w,n}) \quad (1.2.4)$$

where  $\text{Frob}_w \in \mathcal{G}_{F_w}$  is a geometric Frobenius element and  $q_w$  denotes the cardinal of the residue field of  $F_w$ . The condition above determines  $\rho_\chi$  uniquely (up to isomorphisms). We sketch how to obtain  $\rho_\chi$ . Let  $\pi = \pi_\infty \otimes \pi^\infty$  be an automorphic representation of  $\mathbb{G}(\mathbf{A}_{F^+})$  such that  $\pi_\infty = W_\mathbb{C}^\vee$  and  $(\pi^\infty)^U$  contains Hecke eigenforms of the eigenvalues  $\chi$  via (1.2.2) and (1.2.3). Let  $\text{BC}(\pi)$  be the base change from  $F^+$  to  $F$  of  $\pi$  by [Lab11, Cor. 5.3] which is a cohomological automorphic representation of  $\text{GL}_n(\mathbf{A}_F)$ . If  $w$  and  $w^c$  are places over some  $v \in S'_{U^p}$ , then the local component  $\text{BC}(\pi)_w \otimes \text{BC}(\pi)_{w^c}$  is a  $\text{GL}_n(F_w) \times \text{GL}_n(F_{w^c})$ -representation isomorphic to  $\pi_v \circ i_w^{-1} \otimes \pi_v \circ i_{w^c}^{-1}$ . The representation  $\text{BC}(\pi)$  is conjugate self-dual, i.e.  $\text{BC}(\pi)^\vee \circ c = \text{BC}(\pi)$  where  $(-)^\vee$  denotes the contragredient representation given by a twist of the involution  $g \mapsto {}^t g^{-1}$  of  $\text{GL}_n$ . In general, if  $\Pi = \otimes'_w \Pi_w$  is a cohomological cuspidal automorphic representation of  $\text{GL}_n(\mathbf{A}_E)$  where  $E$  is a CM field, then there exists a  $p$ -adic Galois representation  $\rho_\Pi : \mathcal{G}_E \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$

attached to  $\Pi$  where the characteristic polynomials of  $\rho_\Pi(\text{Frob}_w)$  and certain Hecke eigenvalues of  $\Pi_w$  are related similarly as in (1.2.4) for almost all finite places  $w$  of  $E$  (see for example [CH13, Sch15, HLTT16]). The Galois representation  $\rho_\chi$  is then the  $p$ -adic Galois representation attached to  $\text{BC}(\pi)$  ([Gue11, Thm. 2.3]). Moreover,  $\rho_\chi$  is polarized, i.e.  $\rho_\chi^\vee \simeq \rho_\chi^c \otimes \epsilon^{n-1}$  where  $\epsilon$  denotes the  $p$ -adic cyclotomic character.

Let  $\Pi = \otimes'_w \Pi_w$  be a cohomological cuspidal automorphic representation of  $\text{GL}_n(\mathbf{A}_E)$  and assume that  $\Pi$  is conjugate self-dual ( $\Pi^\vee \circ c \simeq \Pi$ ). Then the following local-global compatibility of the correspondence  $\Pi \rightarrow \rho_\Pi$  is already known. If  $w$  is a place of  $E$  and  $w \nmid p$ , then the Weil-Deligne representation associated with  $\rho_{\Pi,w}$  corresponds to  $\Pi_w$  via the classical local Langlands correspondence up to Frobenius semi-simplification and certain normalization (e.g. [Car12]). Moreover, if  $w$  is a place of  $E$  above  $p$ , then  $\rho_{\Pi,w}$  is de Rham and we have the same compatibility between  $\Pi_w$  and the Weil-Deligne representation attached to  $\rho_{\Pi,w}$  (§1.1.2) by [Car14]. For  $p$ -adic places  $w$  and such conjugate self-dual  $\Pi$ , the de Rhamness of  $\rho_{\Pi,w}$  follows from the constructions in most cases where  $\rho_\Pi$  were obtained from étale cohomologies of Shimura varieties of some non-definite unitary groups (e.g. [Clo90, Clo91]). The Hodge-Tate weights of  $\rho_\Pi$  (resp.  $\rho_\chi$ ) are regular and are determined explicitly by  $\Pi_\infty$  (resp.  $\pi_\infty$ ).

### 1.2.3 $p$ -adic automorphic forms and $p$ -adic Galois representations

The theory of  $p$ -adic automorphic or modular forms, pioneered by Serre, Katz, Hida, etc., was originated to study congruences (modulo  $p^N$ ,  $N \in \mathbb{N}_{\geq 1}$ ) between systems of Hecke eigenvalues. On the Galois side, the congruences correspond to  $p$ -adic deformations of Galois representations. There are possibly different ways to  $p$ -adically interpolate classical automorphic forms which are realized in possibly different cohomologies for general reductive groups. For the  $p$ -adic local Langlands correspondence, Emerton's approach seems the most suitable, as shown for  $\text{GL}_2/\mathbb{Q}$  (§1.1.5). Whatever, for the definite unitary groups which are compact at infinity, there are probably essentially no different choices. For example, different approaches in [Che04], [Eme06c], [AS08], etc. will lead to same eigenvarieties for our  $\mathbb{G}$ .

We fix a compact open subgroup  $U^p \subset \mathbb{G}(\mathbf{A}_{F^+}^{\infty,p})$ . The space of  $p$ -adic automorphic forms  $\widehat{S}(U^p, L)$  of the tame level  $U^p$  with coefficients in the  $p$ -adic field  $L$  is the space of all continuous functions  $f : \mathbb{G}(F^+) \setminus \mathbb{G}(\mathbf{A}_{F^+}^\infty)/U^p \rightarrow L$ . The space  $\widehat{S}(U^p, L)$  is equipped with an action of  $G_p$  by  $gf(a) = f(ag)$  for all  $g \in G_p$ . Moreover,  $\widehat{S}(U^p, L)$  can be identified with the 0-th degree completed cohomology  $\widehat{H}^0(U^p, L)$  of the tower  $(X_{U_p U^p})_{U_p}$  where  $U_p$  are open compact subgroups of  $G_p$  and  $X_{U_p U^p} := \mathbb{G}(F^+) \setminus \mathbb{G}(\mathbf{A}_{F^+}^\infty)/U_p U^p$  are discrete finite sets.

We will assume that  $U^p$  is sufficiently small in the sense that for any  $h \in \mathbb{G}(\mathbf{A}_{F^+}^\infty)$ , we have  $\mathbb{G}(F^+) \cap hK_p U^p h^{-1} = \{1\}$ . Under the assumption,  $\mathbb{G}(F^+) \setminus \mathbb{G}(\mathbf{A}_{F^+}^\infty)/U_p = \coprod_{i=1}^s s_i K_p$  for some integer  $s$ . Hence, as a  $K_p$ -representation,  $\widehat{S}(U^p, L)$  is equal to  $\mathcal{C}(K_p, L)^s$  where  $\mathcal{C}(K_p, L)$  denotes the space of all continuous functions on the  $p$ -adic manifold  $K_p$ .

If  $W_L$  is an algebraic representation of  $G_p$  over  $L$  as in §1.2.2, then we have an isomorphism  $S(U_p U^p, W_L) \simeq \text{Hom}_L(W_L^\vee, \widehat{S}(U^p, L))^{U_p}$ . Let  $U_p$  and  $W_L$  vary, there is a  $G_p \times \mathbb{T}_Z^{S'_{U^p}}$ -equivariant isomorphism (cf. [Eme06c, §3.1])

$$\widehat{S}(U^p, L)^{\text{la}lg} \otimes_L \mathbb{C} \simeq \bigoplus_{W_L} W_L^\vee \otimes_L \mathcal{A}(W_{\mathbb{C}})^{U^p} \quad (1.2.5)$$

where we have used (1.2.3) (thus the fixed isomorphism  $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$ ),  $G_p$  acts diagonally on the right-hand side,  $W_L$  runs over all algebraic representations of  $G_p$  (which all can be defined over  $L$ ) and  $\text{la}lg$  means the subspace of locally  $(\mathbb{Q}_p)$ -algebraic vectors with respect to the action of  $G_p$ .

Change the coefficient ring  $L$  in the definition of  $\widehat{S}(U^p, L)$ , we can obtain similarly the space of mod- $p$  automorphic forms  $\widehat{S}(U^p, k_L)$  and the integral version  $\widehat{S}(U^p, \mathcal{O}_L)$ . The unit ball

$\widehat{S}(U^p, \mathcal{O}_L)$  of  $\widehat{S}(U^p, L)$  makes  $\widehat{S}(U^p, L)$  into a unitary Banach representation of  $G_p$ . These spaces are also equipped with actions of the Hecke algebra  $\mathbb{T}_{\mathbb{Z}}^{S'_{U^p}}$ . We will attach Galois representations of  $\mathcal{G}_F$  below in corresponding coefficients for their systems of Hecke eigenvalues, using results for classical Hecke eigenvalues in §1.2.2.

We define in a first time the  $p$ -adic Hecke algebra. For each  $N \in \mathbb{N}_{\geq 1}$  and  $U_p$  an open compact subgroup of  $G_p$ , we let  $\mathbb{T}(U_p U^p, \mathcal{O}_L/\varpi^N)$  be the  $\mathcal{O}_L/\varpi^N$ -algebra generated by the image of  $\mathbb{T}_{\mathbb{Z}}^{S'_{U^p}}$  in  $\text{End}_{\mathcal{O}_L/\varpi^N}(H^0(X_{U_p U^p}, \mathcal{O}_L/\varpi^N))$ . Define  $\mathbb{T}(U^p, \mathcal{O}_L) := \varprojlim_{N, U_p} \mathbb{T}(U_p U^p, \mathcal{O}_L/\varpi^N)$ . Then  $\mathbb{T}(U^p, \mathcal{O}_L)$  is a Noetherian  $\mathcal{O}_L$ -algebra acting on  $\widehat{S}(U^p, \mathcal{O}_L) = \widehat{H}^0(U^p, \mathcal{O}_L)$ , as well as  $\widehat{H}^0(U^p, k_L)$  and  $\widehat{H}^0(U^p, L)$ . The ring  $\mathbb{T}(U^p, \mathcal{O}_L)$  is isomorphic to a finite product of complete local rings  $\prod_{\mathfrak{m}} \mathbb{T}(U^p, \mathcal{O}_L)_{\mathfrak{m}}$  where  $\mathfrak{m}$  runs over finitely many open maximal ideals of  $\mathbb{T}(U^p, \mathcal{O}_L)$  and  $\mathbb{T}(U^p, \mathcal{O}_L)_{\mathfrak{m}}$  denotes the localization of  $\mathbb{T}(U^p, \mathcal{O}_L)$  at  $\mathfrak{m}$ , cf. [Paš21]. Hence we have corresponding direct summands  $\widehat{S}(U^p, \mathcal{O}_L)_{\mathfrak{m}}$  (resp.  $\widehat{S}(U^p, L)_{\mathfrak{m}}$ ) of  $\widehat{S}(U^p, \mathcal{O}_L)$  (resp.  $\widehat{S}(U^p, L)$ ).

A maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}(U^p, \mathcal{O}_L)$  as above gives a system of mod- $p$  Hecke eigenvalues  $\mathbb{T}(U^p, \mathcal{O}_L) \rightarrow \mathbb{T}(U^p, \mathcal{O}_L)/\mathfrak{m}$  which appears in certain finite level  $H^0(X_{U_p U^p}, k_L)$  (cf. [Paš21, Lem. C.3]). Take the modulo  $p$  reduction of certain Galois representations in §1.2.2, we can attach a semi-simple mod- $p$  Galois representation  $\bar{\rho}_{\mathfrak{m}} : \mathcal{G}_F \rightarrow \text{GL}_n(\mathbb{T}(U^p, \mathcal{O}_L)/\mathfrak{m})$  for  $\mathfrak{m}$ .

From now on we fix a (non-Eisenstein) maximal ideal  $\mathfrak{m}^S$  of  $\mathbb{T}(U^p, \mathcal{O}_L)$  such that  $\bar{\rho} := \bar{\rho}_{\mathfrak{m}^S} : \mathcal{G}_F \rightarrow \text{GL}_n(k_L)$  (enlarge  $L$  suitably) is absolutely irreducible. Let  $S$  be the finite set of finite places of  $F^+$  consisting of all places in  $S_p$  and all places  $v$  such that  $U_v$  is not hyperspecial. Let  $R_{\bar{\rho}, S}$  be the completed  $\mathcal{O}_L$ -algebra parametrizing polarized deformations of  $\bar{\rho}$  that are unramified away from places above places in  $S$ . Then for every finite level  $U_p U^p$  such that  $H^0(X_{U_p U^p}, \mathcal{O}_L)_{\mathfrak{m}^S} \neq 0$ ,  $\bar{\rho}$  can be lifted to a representation  $\mathcal{G}_F \rightarrow \text{GL}_n(\mathbb{T}(U_p U^p, \mathcal{O}_L)_{\mathfrak{m}^S})$  assuming  $p \neq 2$  ([Tho12, Prop. 6.7]). From the universal property of  $R_{\bar{\rho}, S}$  we get surjections  $R_{\bar{\rho}, S} \twoheadrightarrow \mathbb{T}(U_p U^p, \mathcal{O}_L)_{\mathfrak{m}^S}$  and hence  $R_{\bar{\rho}, S} \twoheadrightarrow \mathbb{T}(U_p U^p, \mathcal{O}_L/\varpi^N)_{\mathfrak{m}^S}$  for  $N \in \mathbb{N}_{\geq 1}$ . Take the inverse limit over  $N$  and  $U_p$  we get a continuous surjection  $\psi : R_{\bar{\rho}, S} \twoheadrightarrow \mathbb{T}(U^p, \mathcal{O}_L)_{\mathfrak{m}^S}$  which allows us to attach Galois representation for any system of Hecke eigenvalues  $\mathbb{T}(U^p, \mathcal{O}_L)_{\mathfrak{m}^S} \rightarrow \overline{\mathbb{Q}}_p$ . Moreover,  $R_{\bar{\rho}, S}$  acts continuously on  $\widehat{S}(U^p, \mathcal{O}_L)_{\mathfrak{m}^S}$  and  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$  via  $\psi$  and the actions commute with that of  $G_p$ .

If  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  is a continuous representation that corresponds to a prime ideal  $\mathfrak{m}_{\rho}$  of  $R_{\bar{\rho}, S}[\frac{1}{p}]$ , we take  $\Pi(\rho) := \widehat{S}(U^p, L)_{\mathfrak{m}^S}[\mathfrak{m}_{\rho}]$  to be the subspace of  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$  consisting of elements that are annihilated by elements in  $\mathfrak{m}_{\rho}$ . The subspace  $\Pi(\rho)$  is a unitary Banach representation of  $G_p = \prod_{v \in S_p} \text{GL}_n(F_v)$  consisting of  $p$ -adic automorphic eigenforms inside  $\widehat{S}(U^p, L)$  associated with  $\rho$ . The hope is that  $\Pi(\rho)$  is a finite sum of copies of the unitary Banach representations of  $G_p$  attached to  $\rho_p := (\rho_{\bar{v}})_{v \in S_p}$  via the conjectural  $p$ -adic local Langlands correspondence, where  $\rho_{\bar{v}} := \rho|_{\mathcal{G}_{F_{\bar{v}}}}$ .

We already know that if  $\Pi(\rho)$  contains non-zero locally algebraic vectors, then  $\rho_{\bar{v}}$  are de Rham with regular Hodge-Tate weights since we see from (1.2.5) the corresponding systems of Hecke eigenvalues come from classical automorphic forms (1.2.2).

### 1.3 Emerton's Jacquet module and eigenvariety

We introduce eigenvarieties, the geometric families of finite slope  $p$ -adic automorphic eigenforms, following Emerton via his Jacquet module functor for locally analytic representations.

The family  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$  of  $p$ -adic automorphic forms in the last section satisfies a good finiteness property: as a unitary Banach representation of  $G_p$  it is admissible in the sense of [ST02a], i.e. the dual  $(\widehat{S}(U^p, L)_{\mathfrak{m}^S})'$  is a finitely generated  $L[[K_p]]$ -module where  $L[[K_p]] := L \otimes_{\mathcal{O}_L} \mathcal{O}_L[[K_p]]$  is the Iwasawa algebra. However, except for cases like  $n = 1$  or  $\text{GL}_2(\mathbb{Q}_p)$  where we have Colmez's functor (cf. [Eme11, Paš13]), currently a direct way to compare such big families with families of Galois representations, say  $R_{\bar{\rho}, S}$ , is out of reach. One reason is that the “fibers”  $\Pi(\rho)$  are



$G_p$ -representations and  $L[[K_p]]$  is non-commutative. Traditionally, one can restrict to families at finite levels (also at  $p$ ). One can also consider certain families of  $p$ -adic modular forms of level  $\Gamma_1(p^r N)$  where  $p \nmid N$  with possibly weights and  $r$  vary and look at the Hecke operators at  $p$  (the  $U_p$ -operators), e.g. [CM98]. This thesis will also restrict to certain subspace of  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}^{\text{an}}$ , namely the finite slope part.

First of all, let  $\widehat{S}(U^p, L)^{\text{an}}$  be the space of locally analytic functions

$$f : \mathbb{G}(F^+) \setminus \mathbb{G}(\mathbf{A}_{F^+}^\infty) / U^p \rightarrow L$$

which is also the subspace of locally  $(\mathbb{Q}_p)$ -analytic vectors of  $\widehat{S}(U^p, L)$  with respect to the action of  $G_p$ . It is a locally analytic representation of  $G_p$  stable under the action of  $\mathbb{T}_{\mathbb{Z}}^{S'U^p}$ .

### 1.3.1 Emerton's Jacquet module functor

Emerton [Eme06b] constructed his Jacquet module functors which are functors from locally analytic representations of a  $p$ -adic Lie group to locally analytic representations of its Levi subgroups.

We identify  $G_p = \prod_{v \in S_p} \text{GL}_n(F_{\tilde{v}})$ . Let  $N_{p,0}$  be the subgroup  $\prod_{v \in S_p} N(\mathcal{O}_{F_{\tilde{v}}})$  of  $G_p$  where  $N$  denotes the group of upper-triangular unipotent matrices. Let  $B_p = \prod_{v \in S_p} B(F_{\tilde{v}})$  be the Borel subgroup of upper-triangular matrices and let  $T_p = \prod_{v \in S_p} T(F_{\tilde{v}})$  be the diagonal torus. We consider the subspace  $\widehat{S}(U^p, L)^{\text{an}, N_{p,0}}$  of elements which are invariant under the action of  $N_{p,0}$ . Let  $T_p^+$  be the sub-monoid of  $T_p$  consisting of  $t \in T_p$  such that  $tN_{p,0}t^{-1} \subset N_{p,0}$ . Concretely,  $T_p^+$  is generated by the group  $T_{p,0} := \prod_{v \in S_p} T(\mathcal{O}_{F_{\tilde{v}}})$ , non-invertible elements  $t_{\tilde{v},i} := \text{diag}(\varpi_{F_{\tilde{v}}} I_i, I_{n-i}) \in T(F_{\tilde{v}})$  for  $i = 1, \dots, n-1$  and  $v \in S_p$  and  $t_{\tilde{v},n}, v \in S_p$ . There is a Hecke action of  $T_p^+$  on  $\widehat{S}(U^p, L)^{\text{an}, N_{p,0}}$  given by

$$U_t v = \frac{1}{|N_{p,0}/tN_{p,0}t^{-1}|} \sum_{n \in N_{p,0}/tN_{p,0}t^{-1}} ntv, \forall v \in \widehat{S}(U^p, L)^{\text{an}, N_{p,0}}, t \in T_p^+.$$

Then  $U_t v = tv$  if  $t \in T_{p,0}$  and  $U_{t_{\tilde{v},i}}$  acts by the above formula. Emerton's Jacquet module  $J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})$  of  $\widehat{S}(U^p, L)^{\text{an}}$  with respect to the Borel subgroup  $B_p$  is the finite slope part  $(\widehat{S}(U^p, L)^{\text{an}, N_{p,0}})_{\text{fs}}$  of  $\widehat{S}(U^p, L)^{\text{an}, N_{p,0}}$  with respect to the action of  $T_p^+$ . The formal definition is given by [Eme06b, Def. 3.2.1]:

$$J_{B_p}(\widehat{S}(U^p, L)^{\text{an}}) := \mathcal{L}_{b, T_p^+}(\mathcal{C}^{\text{an}}(\widehat{T}_{p,L}, L), \widehat{S}(U^p, L)^{\text{an}, N_{p,0}})$$

where  $\mathcal{L}_{b, T_p^+}$  denotes the space of continuous  $T_p^+$ -equivariant linear maps with the strong topology,  $\widehat{T}_{p,L}$  denotes the base change to  $L$  of the rigid space  $\widehat{T}_p$  over  $\mathbb{Q}_p$  which parametrizing continuous (equivalent locally  $\mathbb{Q}_p$ -analytic) characters of the abelian group  $T_p$  ([Eme17, §6.4], cf. [KPX14, §6.1]), and  $\mathcal{C}^{\text{an}}(\widehat{T}_{p,L}, L)$  denotes the space of rigid analytic functions on  $\widehat{T}_{p,L}$ .

*Remark 1.3.1.* The space of characters  $\widehat{\mathbb{Z}}$  of  $\mathbb{Z}$  is isomorphic to  $\mathbb{G}_m^{\text{an}}$  and  $\widehat{\mathbb{Z}}_p$  is a rigid open unit disk. And  $\widehat{T}_p = \widehat{T}_{p,0} \times \prod_{v \in S_p, i=1, \dots, n} \widehat{\mathbb{Z}}$ .

Then  $J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})$  is a locally analytic representation of  $T_p$  which justifies the name considering the Jacquet modules for smooth representations. And naively we may think the finite slope part as the part of  $\widehat{S}(U^p, L)^{\text{an}, N_{p,0}}$  where the action of  $T_p^+$  can be inverted and be extended to an action of  $T_p$ . In other words,  $J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})$  is the part where the operators  $U_{t_{\tilde{v},i}}, v \in S_p, i = 1, \dots, n$  have no zero eigenvalues. Concretely, if  $\underline{\delta} : T_p \rightarrow L^\times$  is a continuous character (so that  $\underline{\delta}(U_{t_{\tilde{v},i}}) \neq 0$ ), we have ([Eme06b, Prop. 3.2.12])

$$\text{Hom}_{T_p^+}(\underline{\delta}, \widehat{S}(U^p, L)^{\text{an}, N_{p,0}}) = \text{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})).$$

### 1.3.2 The eigenvariety

We use the Jacquet module  $J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})$  to construct the eigenvariety.

In Emerton's language, since  $\widehat{S}(U^p, L)^{\text{an}}$  is admissible,  $J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})$  is essentially admissible, which means that the strong dual  $J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})'$ , a topological module over  $\mathcal{C}^{\text{an}}(\widehat{T}_{p,L}, L)$ , defines and is the global section of a coherent sheaf over the quasi-Stein rigid space  $\widehat{T}_{p,L}$  ([Eme06b, Thm. 4.2.32][Eme17, Def. 6.4.9][Eme06c, Prop. 2.3.2]). The projection map from the support of this coherent sheaf to the weight space  $\mathcal{W} := \widehat{T}_{p,0}$  has discrete fibers ([Eme06b, 0.11]). This is actually Emerton's representation theoretic construction of some "spectral variety" and with little effort we will define the eigenvariety.

*Remark 1.3.2.* Hida [Hid86b, Hid86a] firstly constructed his ordinary families of  $p$ -adic modular forms using the  $U_p$ -operator (in our setting we can take  $U_p := \prod_{v \in S_p, i=1, \dots, n} U_{t_{\bar{v},i}}^2$ ). He picked out  $p$ -adic eigenforms whose  $U_p$ -eigenvalues are  $p$ -adic units (i.e. in  $\mathbb{Z}_p^\times$ ). Later, Coleman [Col97] constructed families of overconvergent  $p$ -adic modular eigenforms of finite slope (i.e. with non-zero  $U_p$ -eigenvalues) by developing a Fredholm-Riesz spectral theory for compact operators acting on certain Banach modules over Banach algebras. Finally, Coleman-Mazur constructed the eigencurve in [CM98], a rigid analytic space which parametrizing all finite slope overconvergent  $p$ -adic eigenforms of a fixed tame level.

In our setting, the compact operator  $U_p$  induces operators on certain Banach modules  $V_i$  related to  $(\widehat{S}(U^p, L)^{\text{an}, N_{p,0}})'$  over affinoid Banach algebras  $A_i$  where  $\text{Sp}(A_i) \subset \mathcal{W}$  are affinoid opens in the weight space. The spectral property of  $U_p$  guarantees that the finite slope part  $J_{B_p}(\widehat{S}(U^p, L)^{\text{an}})$  is well-behaved, cf. [BHS17b, §3.3].

Let  $\text{Spf}(R_{\bar{\rho},S})^{\text{rig}}$  be the rigid generic fiber of the formal scheme  $\text{Spf}(R_{\bar{\rho},S})$  in the sense of Berthelot [dJ95, §7]. A point of  $\text{Spf}(R_{\bar{\rho},S})^{\text{rig}}$  corresponds to a maximal ideal  $\mathfrak{m}$  of  $R_{\bar{\rho},S}[\frac{1}{p}]$  ([dJ95, Lem. 7.1.9]) and thus also corresponds to a deformation  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L')$  for some finite extension  $L'$  of  $L$  of  $\bar{\rho}$ . Recall  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$  is an  $R_{\bar{\rho},S}$ -module. The dual  $J_{B_p}(\widehat{S}(U^p, L)_{\mathfrak{m}^S})'$  of the Jacquet module localized at the non-Eisenstein maximal ideal  $\mathfrak{m}^S$  is the global section of a coherent sheaf over the quasi-Stein space  $\text{Spf}(R_{\bar{\rho},S})^{\text{rig}} \times \widehat{T}_{p,L}$  ([BHS17b, §3.1, §3.2]).

**Definition 1.3.3.** The eigenvariety  $Y(U^p, \bar{\rho})$  is the scheme-theoretic support of the coherent sheaf defined by  $J_{B_p}(\widehat{S}(U^p, L)_{\mathfrak{m}^S})'$  inside  $\text{Spf}(R_{\bar{\rho},S})^{\text{rig}} \times \widehat{T}_{p,L}$ .

The eigenvariety  $Y(U^p, \bar{\rho})$  is reduced in our case ([BHS17b, Cor 3.20]).

*Remark 1.3.4.* For the definition of the eigenvariety (not just localized at  $\mathfrak{m}^S$ ), one can (and usually) use  $\mathbb{T}_{\mathbb{Z}}^{S'U^p}$  instead of  $R_{\bar{\rho},S}$  which serves for recording Hecke eigenvalues away from  $p$ , e.g. [Buz07, Che04, Eme06c], see also [BHS17b, §3.3].

A point of  $\text{Spf}(R_{\bar{\rho},S})^{\text{rig}} \times \widehat{T}_{p,L}$  with the residue field  $L$  is given by a pair  $(\rho, \underline{\delta})$  where  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  corresponds to a maximal ideal  $\mathfrak{m}_\rho$  of  $R_{\bar{\rho},S}[\frac{1}{p}]$ ,  $\underline{\delta} = \prod_{v \in S_p} \underline{\delta}_v$  and for each  $v$ ,  $\underline{\delta}_v = (\delta_{v,1}, \dots, \delta_{v,n})$  is a continuous character of  $T(F_v^\times) = (F_v^\times)^n$ . From the definition of the eigenvariety, such a point  $(\rho, \underline{\delta})$  lies in  $Y(U^p, \bar{\rho})$  if and only if

$$\text{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\Pi(\rho)^{\text{an}})) \neq 0 \quad (1.3.5)$$

where recall  $\Pi(\rho)^{\text{an}} = \widehat{S}(U^p, L)_{\mathfrak{m}_\rho}^{\text{an}}$  is the locally analytic representation of  $G_p$  associated with  $\rho$ .

The eigenvariety  $Y(U^p, \bar{\rho})$  satisfies the following two remarkable properties as Coleman-Mazur eigencurves if  $U^p$  is sufficiently small (recall that  $\mathbb{G}(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$  is compact).

<sup>2</sup>We hope the reader will not be confused by the  $U_p$ -operator here and open compact subgroups  $U_p$  of  $G_p$  in §1.2.2

Firstly, the rigid space  $Y(U^p, \bar{\rho})$  is equidimensional of the same dimension with the weight space  $\mathcal{W}$ . And the weight map  $Y(U^p, \bar{\rho}) \rightarrow \mathcal{W} : (\rho, \underline{\delta}) \rightarrow \underline{\delta}|_{T_{p,0}}$  is locally (on the source) finite. This property is deduced from the general eigenvariety machinery in [Che04] and [Buz07].

Secondly, the classical points are Zariski dense in  $Y(U^p, \bar{\rho})$ . Here we say a point  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  is classical if  $\Pi(\rho)^{\text{lal}} \neq 0$ . For a classical point  $(\rho, \underline{\delta})$  of  $Y(U^p, \bar{\rho})$ , the Galois representation  $\rho$  is associated with classical automorphic forms of  $\mathbb{G}$  and  $\rho_{\tilde{v}}, v \in S_p$  are de Rham with regular Hodge-Tate weights. This density result is deduced from the first property above and a classicality criterion as of Coleman [Col96] that forms with finite “small slopes” are classical. The density also characterizes the eigenvariety as the Zariski closure of certain set of classical points so that different constructions (e.g. [Che04, Eme06c]) will lead to the same eigenvariety.

### 1.3.3 Trianguline representations and global triangulations

In the classical local Langlands correspondence, irreducible smooth representations with non-zero Jacquet modules are subquotients of parabolic inductions and are associated with Galois representations that are reducible. For locally analytic representations, one may wonder if we have  $J_{B_p}(\Pi(\rho)^{\text{an}}) \neq 0$ , whether the associated  $(\varphi, \Gamma)$ -modules  $D_{\text{rig}}(\rho_{\tilde{v}})$  over the Robba rings  $\mathcal{R}_{L, F_{\tilde{v}}}$  are reducible. The answer is yes thanks to the construction of the eigenvariety and the “global triangulation”. Even the converse should be true and is verified in  $\text{GL}_2(\mathbb{Q}_p)$ -cases [Eme11, §1.2.4].

We introduce the notion of the trianguline representation given by Colmez [Col08]. Firstly for a  $p$ -adic local field  $K$ , there is a bijection between continuous characters of  $K^\times$  and  $(\varphi, \Gamma_K)$ -modules of rank one. We write  $\mathcal{R}_{L, K}(\delta)$  for the  $(\varphi, \Gamma_K)$ -modules associated with a character  $\delta : K^\times \rightarrow L^\times$ . Then  $\mathcal{R}_{L, K}(\delta)$  is étale if and only if  $\delta$  is unitary, or equivalently  $\delta$  comes from a one-dimensional representation of  $\mathcal{G}_K$  via the local class field theory and then  $\mathcal{R}_{L, K}(\delta) = D_{\text{rig}}(\delta)$ .

**Definition 1.3.6** (Colmez). A continuous representation  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  is called trianguline if  $D_{\text{rig}}(r)$  admits a filtration by sub- $(\varphi, \Gamma_K)$ -modules

$$D_{\text{rig}}(r) = \text{Fil}_n \supset \cdots \supset \text{Fil}_i \supset \cdots \supset \text{Fil}_0 = \{0\} \quad (1.3.7)$$

such that for each  $1 \leq i \leq n$ ,  $\text{Fil}_i/\text{Fil}_{i-1} \simeq \mathcal{R}_{L, K}(\delta_i)$  for some character  $\delta_i$  of  $K^\times$ . In this case,  $\underline{\delta} = (\delta_i)_{1 \leq i \leq n}$  is called the parameter of the triangulation.

Notice that in the above definition, the submodules  $\text{Fil}_i$  are not required to be étale, thus not necessarily correspond to subrepresentations of  $\mathcal{G}_K$ . Trianguline representations include all semistable representations, hence all crystalline representations. And we will see, all the local  $p$ -adic Galois representations  $\rho_{\tilde{v}}, v \in S_p$  arising from  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  are trianguline up to enlarging coefficient fields.

We introduce the notion of locally algebraic characters. Let  $\Sigma = \text{Hom}(K, L)$  and assume  $|\Sigma| = [K : \mathbb{Q}_p]$ . For  $\mathbf{k} = (k_\tau)_{\tau \in \Sigma} \in \prod_{\tau \in \Sigma} \mathbb{Z}$ , we write  $z^{\mathbf{k}}$  for the character of  $K^\times$  sending  $z$  to  $\prod_{\tau \in \Sigma} \tau(z)^{k_\tau}$ . We say that a character  $\delta$  of  $K^\times$  is locally algebraic characters if  $\delta = z^{\mathbf{k}} \delta_{\text{sm}}$  for some  $\mathbf{k} \in \mathbb{Z}^\Sigma$  and a smooth character  $\delta_{\text{sm}}$  of  $K^\times$ .

Set  $\Sigma_v := \text{Hom}(F_{\tilde{v}}, L)$  for all  $v \in S_p$  and define  $\Sigma_p = \coprod_{v \in S_p} \Sigma_v$ . If  $\mathbf{k} = (\mathbf{k}_v)_{v \in S_p} = (k_{\tau,1}, \dots, k_{\tau,n})_{\tau \in \Sigma_p} \in (\mathbb{Z}^{\Sigma_p})^n$ , write  $z^{\mathbf{k}} := \prod_{v \in S_p} z^{\mathbf{k}_v}$  for the algebraic character of  $T_p = \prod_{v \in S_p} (F_{\tilde{v}}^\times)^n$ . Let  $\iota_v$  be the automorphism of  $\widehat{T}_{v,L}$

$$\iota_v : (\delta_{v,1}, \dots, \delta_{v,n}) \mapsto \delta_{B_v} \cdot (\delta_{v,1}, \dots, \delta_{v,i} \epsilon^{i-1}, \dots, \delta_{v,n} \epsilon^{n-1})$$

and let  $\iota = \prod_{v \in S_p} \iota_v : \widehat{T}_{p,L} \xrightarrow{\sim} \widehat{T}_{p,L}$ , where  $\delta_{B_v}$  is the smooth modulus character of  $B_v$  and  $\epsilon$  denotes the cyclotomic character. The following theorem is a corollary of the global triangulation results in [KPX14, Thm. 6.3.13][Liu15] and see also [Hel12].

**Theorem 1.3.8.** *Assume  $(\rho, \delta) \in Y(U^p, \bar{\rho})$ . Then there exists an algebraic character  $z^{\mathbf{k}} = (z^{\mathbf{k}_v})_{v \in S_p}$  of  $T_p = \prod_{v \in S_p} (F_v^\times)^n$  such that, up to enlarging the coefficient field,  $\rho_{\bar{v}}$  is trianguline of a parameter  $z^{\mathbf{k}_v} \iota_v^{-1}(\underline{\delta}_v)$  for all  $v \in S_p$ .*

On the eigencurve, such result was firstly noticed by Kisin in the language of “existence of crystalline periods” [Kis03]. We explain the meaning of the global triangulation. Take  $v \in S_p$ . The starting point is that there is a Zariski-dense subset  $Z \subset Y(U^p, \bar{\rho})$  such that for any  $z \in Z$ ,  $\rho_{\bar{v}, z}$  is trianguline of a parameter  $\iota_v^{-1}(\underline{\delta}_{v, z})$  (and the parameter satisfies certain regularity condition). Let  $\rho_v^{\text{univ}}$  be the universal representation on  $Y(U^p, \bar{\rho})$ . Then by [BC08], the associated  $(\varphi, \Gamma)$ -modules also vary as a family  $D_{\text{rig}}(\rho_v^{\text{univ}})$  over the Robba ring  $\mathcal{R}_{Y, K}$  of  $K$  over the rigid space  $Y := Y(U^p, \bar{\rho})$ . There are universal characters  $\delta_{v, 1}^{\text{univ}}, \dots, \delta_{v, n}^{\text{univ}} : F_v^\times \rightarrow \Gamma(Y, \mathcal{O}_Y)^\times$  pulled back from  $\widehat{T}_v$ . Then after replacing  $Y$  by a rigid space  $Y'$  with a birational proper map  $f : Y' \mapsto Y$ , the pullback of  $D_{\text{rig}}(\rho_v^{\text{univ}})$  admits a filtration of  $(\varphi, \Gamma)$ -sub-modules over  $\mathcal{R}_{Y', K}$ . Moreover, the global filtration of  $D_{\text{rig}}(\rho_v^{\text{univ}})$  interpolates the triangulations of  $D_{\text{rig}}(\rho_{\bar{v}})$  at points in  $f^{-1}(Z)$ , with graded pieces differing not much with  $\mathcal{R}_{Y', K}(\iota_v^{-1}(\underline{\delta}_v^{\text{univ}})_i)$  [KPX14, Cor. 6.3.10].

The dense subset  $Z \subset Y(U^p, \bar{\rho})$  of the trianguline points for the global triangulation above can be taken to be the set of certain classical points  $(\rho, \underline{\delta})$  of “small slope” or “non-critical” where  $\underline{\delta} \in J_{B_p}(\Pi(\rho)^{\text{lal}})$  and  $\rho_{\bar{v}}$  are crystalline (hence trianguline).

## 1.4 Conjectures on companion forms and locally analytic socles

Given  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$ , let  $W(\rho) := \{\underline{\delta} \in \widehat{T}_{p, L} \mid (\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})\}$ . Then  $W(\rho)$  is also the set of all characters of  $T_p$  that appear in  $J_{B_p}(\Pi(\rho)^{\text{an}})$ . It can be viewed as the set of weights of  $p$ -adic automorphic eigenforms with possibly different weights  $(\underline{\delta} |_{T_{p, 0}})$  and possibly different  $U_p$ -eigenvalues  $(\delta_v(U_{t_{\bar{v}, i}}), v \in S_p, i = 1, \dots, n)$  that are associated with a same Galois representation. We call these forms companion forms and the points on  $Y(U^p, \bar{\rho})$  with the same associated Galois representations the companion points. Since the  $G_p$ -representation  $\Pi(\rho)^{\text{an}}$  should be largely determined by  $\rho_p := (\rho_{\bar{v}})_{v \in S_p}$  via the conjectural  $p$ -adic local Langlands correspondence, the Jacquet module, hence the set  $W(\rho)$ , should be determined by  $D_{\text{rig}}(\rho_{\bar{v}})$  in some way. The recipe for the set  $W(\rho)$  is conjectured by Breuil [Bre16, Bre15b] in some cases and later by Hansen in general [HN17].

We start with some notation. Let  $\mathfrak{t} = \prod_{v \in S_p} \mathfrak{t}_v$ ,  $\mathfrak{b} = \prod_{v \in S_p} \mathfrak{b}_v$ ,  $\mathfrak{g} = \prod_{v \in S_p} \mathfrak{g}_v$  be the base change to  $L$  of the  $\mathbb{Q}_p$ -Lie algebras of  $T_p, B_p, G_p$  viewed also as affine spaces over  $L$ . For example  $\mathfrak{t}_v = \text{Lie}(T_v) \otimes_{\mathbb{Q}_p} L = (F_v^\times)^n \otimes_{\mathbb{Q}_p} L \simeq (L^{\Sigma_v})^n$ . Let  $\bar{B}_p$  be the opposite Borel with the Lie algebra  $\bar{\mathfrak{b}}$ . We identify the weight lattice  $X^*(\mathfrak{t})$  with  $(\mathbb{Z}^n)^{\Sigma_p}$  in the usual way. Recall  $\Sigma_v := \text{Hom}(F_v^\times, L)$  for all  $v \in S_p$  and  $\Sigma_p = \prod_{v \in S_p} \Sigma_v$ .

### 1.4.1 Companion points

Given  $\rho \in \text{Spf}(R_{\bar{\rho}, S})^{\text{rig}}$  such that  $\rho$  is attached to at least one point on the eigenvariety. There are already several restrictions on the set  $W(\rho)$ .

First, by Theorem 1.3.8, any character in  $W(\rho)$  has the form  $\iota(z^{\mathbf{k}} \underline{\delta})$  where  $z^{\mathbf{k}}$  is algebraic,  $\underline{\delta} = \prod_{v \in S_p} \underline{\delta}_v$  and  $\underline{\delta}_v, v \in S_p$  are parameters of some triangulations of  $\rho_{\bar{v}}, v \in S_p$  (after possibly extending  $L$ ).

Second, if  $\iota(\underline{\delta}) \in W(\rho)$ , then for any  $v$ , the weights  $\text{wt}(\delta_{v, i})$  of characters  $\delta_{v, i}, i = 1, \dots, n$  counted with multiplicities coincide with the Hodge-Tate-Sen weights of  $\rho_{\bar{v}}$ . Here for a local field  $K$  with  $\Sigma := \{K \hookrightarrow L\}$  as before and a continuous character  $\delta : K^\times \rightarrow L^\times$ , the weight  $\text{wt}(\delta) = (\text{wt}_\tau(\delta))_{\tau \in \Sigma}$  is the map  $x \mapsto \frac{d}{dt} \delta(\exp(tx))|_{t=0}$  in  $\text{Hom}_{\mathbb{Q}_p}(K, L)$  identified with  $K \otimes_{\mathbb{Q}_p} L = \prod_{\Sigma} L$ . For example, if  $\delta = z^{\mathbf{k}} \delta_{\text{sm}}$  for  $\mathbf{k} = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$  and  $\delta_{\text{sm}}$  is smooth, then  $\text{wt}_\tau(\delta) = k_\tau$ . And if  $\rho_{\bar{v}}$  is trianguline of a parameter  $\underline{\delta}_v = (\delta_{v, 1}, \dots, \delta_{v, n})$ , then the  $\tau$ -Hodge-Tate-Sen weights

of  $\rho_{\bar{v}}$  is the multiset  $\{\text{wt}_{\tau}(\delta_{\bar{v},i}), i = 1, \dots, n\}$ . The reason for the identification of weights (Hodge-Tate weights and weights of characters on the eigenvariety) is that both the weights of  $\delta_{v,i}$  and the Sen polynomials of  $\rho_{\bar{v}}$  vary analytically on the eigenvariety and the result holds for a Zariski dense subset, cf. [BHS17b, Prop. 2.9].

Finally, given  $\underline{\delta}$ , there is a partial order  $\geq$  on the set  $W(\underline{\delta}) = \prod_{v \in S_p} W(\underline{\delta}_v)$  where

$$W(\underline{\delta}_v) := \left\{ z^{\mathbf{k}_v} \underline{\delta}_v \mid \begin{array}{l} \mathbf{k}_v = (k_{\tau,i})_{\tau \in \Sigma_v, i=1, \dots, n} \in (\mathbb{Z}^{\Sigma_v})^n, \\ \{\text{wt}_{\tau}(\delta_{v,i}) \mid i = 1, \dots, n\} \\ = \{\text{wt}_{\tau}(\delta_{v,i}) + k_{\tau,i} \mid i = 1, \dots, n\}, \forall \tau \in \Sigma_v \end{array} \right\}$$

such that if  $\iota(\underline{\delta}') \in W(\rho)$  and  $\underline{\delta}'' \geq \underline{\delta}'$ , then  $\iota(\underline{\delta}'') \in W(\rho)$  (cf. [BHS17a, Thm. 5.5]). We say two characters  $\underline{\delta}'_v, \underline{\delta}''_v$  satisfies that  $\underline{\delta}'_v > \underline{\delta}''_v$  if we can find successive characters  $\underline{\delta}'_v = \underline{\delta}_v^{(0)}, \underline{\delta}_v^{(1)}, \dots, \underline{\delta}_v^{(m)} = \underline{\delta}''_v$  such that for each  $s = 1, \dots, m$ , there exists  $\tau \in \Sigma_v, 1 \leq i < j \leq n$  such that  $\text{wt}_{\tau}(\delta_{v,i}^{(s)}) - \text{wt}_{\tau}(\delta_{v,j}^{(s)}) \in \mathbb{Z}_{\leq 0}$  and  $\underline{\delta}_v^{(s-1)} = z^{\mathbf{k}_v} \underline{\delta}_v^{(s)}$  where  $k_{\tau,i} = \text{wt}_{\tau}(\delta_{v,j}^{(s)}) - \text{wt}_{\tau}(\delta_{v,i}^{(s)})$ ,  $k_{\tau,j} = \text{wt}_{\tau}(\delta_{v,i}^{(s)}) - \text{wt}_{\tau}(\delta_{v,j}^{(s)})$  and  $k_{\tau',i'} = 0$  if  $\tau' \neq \tau$  or  $i' \neq i, j$ . And we say  $\prod_{v \in S_p} \underline{\delta}'_v \geq \prod_{v \in S_p} \underline{\delta}''_v$  if and only if  $\underline{\delta}'_v \geq \underline{\delta}''_v$  for all  $v \in S_p$ . In some other language, if we identify the weights  $\text{wt}(\underline{\delta}')$  as points of  $\mathfrak{t}^*$  the dual of  $\mathfrak{t}$ , then  $\underline{\delta}' \geq \underline{\delta}''$  if and only if  $\iota(\underline{\delta}')\iota(\underline{\delta}'')^{-1} = \underline{\delta}'(\underline{\delta}'')^{-1}$  is algebraic and  $\text{wt}(\iota(\underline{\delta}'')) \uparrow \text{wt}(\iota(\underline{\delta}'))$  in the notation of the strong linkage principal in [Hum08, §5.1] with respect to  $\mathfrak{b}$ .

If  $\underline{\delta}_v$  is a parameter of some triangulation of  $\rho_{\bar{v}}$  (possibly after extending the scalar), let  $W_{\underline{\delta}_v}(\rho_{\bar{v}}) := \{\iota_v(\underline{\delta}'_v) \mid \underline{\delta}'_v \in W(\underline{\delta}_v), \underline{\delta}'_v \geq \underline{\delta}_v\}$ . Hansen's conjecture [HN17, Conj. 6.2.3] is the following.

**Conjecture 1.4.1** (Hansen). *Assume that the set  $W(\rho)$  of companion characters is non-empty. Then  $W(\rho)$  is equal to  $\prod_{v \in S_p} (\coprod_{\underline{\delta}_v} W_{\underline{\delta}_v}(\rho_{\bar{v}}))$  where  $\underline{\delta}_v$  runs over all parameters of triangulations of  $D_{\text{rig}}(\rho_{\bar{v}})$  for all  $v \in S_p$ .*

*Example 1.4.2.* We discuss examples of companion forms in the most classical situation. Suppose that  $N$  is an integer prime to  $p$ . Write  $q = e^{2\pi iz}$  for  $z$  in the Poincaré upper half-plane. Let  $f = q + \dots a_i q^i + \dots$  be a cusp modular form of level  $\Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})$  and of weight  $k \geq 1$ . We assume that  $f$  is an eigenform for Hecke operators  $T_{\ell}, \ell \nmid N$  (so  $T_p f = a_p f$ ). Let  $\alpha, \beta$  be two roots of the polynomial  $X^2 - a_p X + p^{k-1}$  and assume that  $\alpha \neq \beta, p\beta$ . Then  $f_{\alpha} = f(q) - \beta f(q^p)$  and  $f_{\beta} = f(q) - \alpha f(q^p)$  are two (old) forms of weight  $k$  and of level  $\Gamma_0(Np)$ . They are two  $p$ -stabilisations of  $f$  and are eigenvectors for the classical  $U_p$ -operator:  $U_p f_{\alpha} = \alpha f_{\alpha}, U_p f_{\beta} = \beta f_{\beta}$ . The forms  $f, f_{\alpha}, f_{\beta}$  are associated with the same  $p$ -adic Galois representation  $\rho = \rho_f$ . Moreover  $\alpha$  and  $\beta$  are also the two  $\phi$ -eigenvalues of  $D_{\text{cris}}(\rho_{f,p})$ . The two forms  $f_{\alpha}, f_{\beta}$  correspond to two different points on Coleman-Mazur's eigencurve. On the eigenvariety, the variant of the eigencurve defined by Emerton in [Eme06c, §4],  $f_{\alpha}$  and  $f_{\beta}$  correspond to two different points  $(\rho, \underline{\delta}_{\alpha})$  and  $(\rho, \underline{\delta}_{\beta})$  respectively. Explicitly, as characters of  $(\mathbb{Q}_p^{\times})^2$ ,

$$\underline{\delta}_{\alpha} = (\text{unr}(p^{-1}\alpha), \text{unr}(\beta)z^{2-k}), \underline{\delta}_{\beta} = (\text{unr}(p^{-1}\beta), \text{unr}(\alpha)z^{2-k}).$$

If  $k \geq 2$ , and  $v_p(\alpha) = k - 1$  (equivalently  $v_p(\beta) = 0$ ), it's possible that the character

$$\underline{\delta}'_{\alpha} := (\text{unr}(p^{-1}\alpha)z^{1-k}, \text{unr}(\beta)z)$$

also lies in  $W(\rho_f)$ , but  $\underline{\delta}'_{\alpha} \neq \underline{\delta}_{\alpha}, \underline{\delta}_{\beta}$ . This happens if and only if  $\rho_{f,p}$  splits as a sum of two characters  $z^{1-k}\text{unr}(\alpha) \oplus \text{unr}(\beta)$  of  $\mathcal{G}_{\mathbb{Q}_p}$  (see for examples [BE10, Ber14]).

In this case,  $\underline{\delta}_{\alpha}$  and  $\underline{\delta}'_{\alpha}$  correspond to the same refinement of  $\rho_{f,p}$ , but the point  $(\rho, \underline{\delta}'_{\alpha})$  on the eigenvariety is not related to classical modular forms (only to  $p$ -adic modular forms). After a twist,  $(\rho, \underline{\delta}'_{\alpha})$  corresponds to a point on the eigencurve associated to a finite slope  $p$ -adic modular form  $g$  of weight  $2 - k$ . Moreover,  $\rho_g \simeq \rho_f \otimes \chi_{\text{cyc}}^{k-1}$  and the form  $\theta^{k-1}g$ , where  $\theta^{k-1} = (q \frac{d}{dq})^{k-1}$  is the theta operator, corresponds to the same point with  $f_{\alpha}$  on the eigencurve.

### 1.4.2 Orlik-Strauch representations

The conjecture of companion points deals with the  $T_p$ -representations  $J_{B_p}(\Pi(\rho)^{\text{an}})$ . Breuil's locally analytic socle conjecture concerns the  $G_p$ -representation  $\Pi(\rho)^{\text{an}}$ . To state the conjecture, we introduce certain finite slope  $G_p$ -representations constructed by Orlik-Strauch in [OS15].

Recall  $K_p$  is a maximal compact open subgroup of  $G_p$  and  $T_p \subset \overline{B}_p$ . If  $H$  is a  $p$ -adic Lie group, let  $D(H)$  be the strong dual of the space  $\mathcal{C}^{\text{an}}(H, L)$  of locally analytic functions on  $H$  with values in  $L$ . Then  $D(H)$  is the algebra of distributions on  $H$  with products given by convolutions. When  $H$  is compact, Schneider-Teitelbaum show in [ST03] that  $D(H)$  is a Fréchet-Stein algebra ([ST03, Thm. 5.1]) and define a full abelian subcategory of coadmissible modules of the category of  $D(H)$ -module. In their definition, a locally analytic representation  $V$  of  $G_p$  is admissible if and only if the strong dual  $V'$  is coadmissible as a  $D(K_p)$ -module. And the duality induces an anti-equivalence of abelian categories between the category of admissible locally analytic representations of  $G_p$  and the category of  $D(G_p)$ -modules that are coadmissible as  $D(K_p)$ -modules.

Let  $\underline{\delta} = z^\lambda \underline{\delta}_{\text{sm}}$  be a locally algebraic character of  $T_p$  where  $\underline{\delta}_{\text{sm}}$  is a smooth character and  $\lambda \in X^*(\mathfrak{t})$  is an integral weight. Let  $(\text{Ind}_{\overline{B}_p}^{G_p} \underline{\delta} \delta_{B_p}^{-1})^{\text{an}}$  be the space of all locally analytic functions  $f : G_p \rightarrow L$  such that  $f(\bar{b}g) = (\underline{\delta} \delta_{B_p}^{-1})(\bar{b})f(g)$  for all  $\bar{b} \in \overline{B}_p$  where  $\delta_{B_p} = \prod_{v \in S_p} \delta_{B_v}$  is the smooth modulus character of  $B_p$  (restricted to  $T_p$ ). Let  $G_p$  act on  $(\text{Ind}_{\overline{B}_p}^{G_p} \underline{\delta} \delta_{B_p}^{-1})^{\text{an}}$  by right translations. Then  $(\text{Ind}_{\overline{B}_p}^{G_p} \underline{\delta} \delta_{B_p}^{-1})^{\text{an}}$  is a locally analytic principal series representation of  $G_p$ . The dual of  $(\text{Ind}_{\overline{B}_p}^{G_p} \underline{\delta} \delta_{B_p}^{-1})^{\text{an}}$  is equal to  $D(G_p) \otimes_{D(\overline{B}_p)} \underline{\delta}^{-1} \delta_{B_p}$ . Let  $D(\overline{B}_p, \mathfrak{g}) (= U(\mathfrak{g})D(\overline{B}_p))$  be the subalgebra of  $D(G_p)$  generated by  $D(\overline{B}_p)$  and  $U(\mathfrak{g})$ , where  $U(-)$  denotes the universal enveloping algebra. Then

$$D(G_p) \otimes_{D(\overline{B}_p)} \underline{\delta}^{-1} \delta_{B_p} = D(G_p) \otimes_{D(\overline{B}_p, \mathfrak{g})} (D(\overline{B}_p, \mathfrak{g}) \otimes_{D(\overline{B}_p)} z^{-\lambda} \underline{\delta}_{\text{sm}}^{-1} \delta_{B_p}).$$

Let  $\overline{M}(-\lambda) := U(\mathfrak{g}) \otimes_{\mathfrak{b}} (-\lambda)$  be the Verma module of the highest weight  $-\lambda$  with respect to  $\mathfrak{b}$  which lies in the BGG category  $\mathcal{O}^{\mathfrak{b}}$ . The locally finite  $\mathfrak{b}$ -action on  $\overline{M}(-\lambda)$  integrates to an action of  $\overline{B}_p$ , hence  $\overline{M}(-\lambda)$  becomes a  $D(\overline{B}_p, \mathfrak{g})$ -module. Then the  $D(\overline{B}_p, \mathfrak{g})$ -module  $D(\overline{B}_p, \mathfrak{g}) \otimes_{D(\overline{B}_p)} z^{-\lambda} \underline{\delta}_{\text{sm}}^{-1} \delta_{B_p}$  is equal to  $\overline{M}(-\lambda) \otimes_L \underline{\delta}_{\text{sm}}^{-1} \delta_{B_p}$  where  $U(\mathfrak{g})$  acts trivially on the second factor  $\underline{\delta}_{\text{sm}}^{-1} \delta_{B_p}$ . The Verma module has a unique irreducible quotient  $\overline{L}(-\lambda)$  in  $\mathcal{O}^{\mathfrak{b}}$  and its irreducible subquotients are of the form  $\overline{L}(-w \cdot \lambda)$  for some  $w$  in the Weyl group of  $\mathfrak{g}$  (identified with  $\prod_{v \in S_p} (S_n)^{\Sigma_v}$ ). Here  $w \cdot \lambda$  denotes the dot action  $w \cdot \lambda = w(\lambda + \rho) - \rho$  where  $\rho$  denotes the half sum of all positive roots of  $\mathfrak{g}$  with respect to  $\mathfrak{b}$ . Those  $\overline{L}(-w \cdot \lambda) \in \mathcal{O}^{\mathfrak{b}}$  also integrate to  $D(\overline{B}_p, \mathfrak{g})$ -modules. Hence  $D(G_p) \otimes_{D(\overline{B}_p)} \underline{\delta}^{-1} \delta_{B_p}$  admits subquotients  $D(G_p) \otimes_{D(\overline{B}_p, \mathfrak{g})} (\overline{L}(-w \cdot \lambda) \otimes \underline{\delta}_{\text{sm}}^{-1} \delta_{B_p})$ .

For an integral weight  $\lambda \in X^*(\mathfrak{t})$ , define

$$\mathcal{F}_{\overline{B}_p}^{G_p}(\overline{L}(-\lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}) := \left( D(G_p) \otimes_{D(\overline{B}_p, \mathfrak{g})} (\overline{L}(-\lambda) \otimes \underline{\delta}_{\text{sm}}^{-1} \delta_{B_p}) \right)'.$$

Then  $\mathcal{F}_{\overline{B}_p}^{G_p}(\overline{L}(-\lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1})$  is an admissible locally analytic representation of  $G_p$  and is a closed subrepresentation of  $(\text{Ind}_{\overline{B}_p}^{G_p} \underline{\delta} \delta_{B_p}^{-1})^{\text{an}}$ .

In general, let  $\overline{Q}_p$  be a parabolic subgroup of  $G_p$  such that  $\overline{B}_p \subset \overline{Q}_p \subset G_p$  with a Levi subgroup  $M_{\overline{Q}_p}$  containing  $T_p$  and the Lie algebra  $\overline{\mathfrak{q}}$  over  $L$ . Let  $\mathcal{O}_{\text{alg}}^{\overline{\mathfrak{q}}}$  be the subcategory of  $\mathcal{O}^{\overline{\mathfrak{q}}} \subset \mathcal{O}$  defined in [OS15, §2.5] with typical examples are  $\overline{L}(-\lambda)$  where  $\lambda \in X^*(\mathfrak{t})$  is a dominant weight for  $M_{\overline{Q}_p}$  with respect to  $M_{\overline{Q}_p} \cap B_p$ . Orlik-Strauch construct a functor  $(M, V) \mapsto \mathcal{F}_{\overline{Q}_p}^{G_p}(M, V)$  from  $M \in \mathcal{O}_{\text{alg}}^{\overline{\mathfrak{q}}}$  and smooth admissible representations  $V$  of  $M_{\overline{Q}_p}$  to admissible locally analytic representations of  $G_p$ . The functor is exact (on both the two arguments). If  $\overline{L}(-\lambda) \in \mathcal{O}_{\text{alg}}^{\overline{\mathfrak{q}}}$ , then

$\mathcal{F}_{\overline{B}_p}^{G_p}(\overline{L}(-\lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}) = \mathcal{F}_{\overline{Q}_p}^{G_p}(\overline{L}(-\lambda), (\text{Ind}_{\overline{B}_p \cap M_{\overline{Q}_p}}^{M_{\overline{Q}_p}} \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1})^{\text{sm}})$  where  $(-)^{\text{sm}}$  denotes the usual parabolic induction for smooth representations. Moreover, if  $(\text{Ind}_{\overline{B}_p}^{G_p} \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1})^{\text{sm}}$  is irreducible, then  $\mathcal{F}_{\overline{B}_p}^{G_p}(\overline{L}(-\lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1})$  is topologically irreducible for all  $\lambda \in X^*(\mathfrak{t})$ . See [Bre16, Thm. 2.3, (2.6)] for details.

### 1.4.3 Locally analytic socle conjecture

For representations  $\rho$  such that  $\rho_{\tilde{v}}, v \in S_p$  are de Rham with certain genericity condition, Breuil's locally analytic socle conjecture predicts all possible representations of the form  $\mathcal{F}_{\overline{Q}_p}^{G_p}(M, V)$  that should appear in the socle of  $\Pi(\rho)^{\text{an}}$  from the associated filtered  $(\varphi, N, \mathcal{G}_{F_{\tilde{v}}})$ -modules [Bre16, Bre15b].

#### 1.4.3.1 Refinements of generic crystabelline representations

We focus on the trianguline case of Breuil's conjecture with a stronger genericity assumption. Let  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  be a trianguline representation with a parameter  $\underline{\delta} = (\delta_1, \dots, \delta_n)$ , where  $K/\mathbb{Q}_p$  is a local field as in §1.3.3. If  $r$  is potentially semistable, by Berger [Ber08b], the trianguline filtration of  $D_{\text{rig}}(r)$  is equivalent to a full filtration of the filtered  $(\varphi, N, \mathcal{G}_K)$ -modules associated to  $r$ . Hence a triangulation of  $D_{\text{rig}}(r)$  determines and in fact is determined by a full filtration of the associated Weil-Deligne representation  $\text{WD}(r)$  in §1.1.2. Let  $\chi_1, \dots, \chi_n$  be the characters of  $W_K$  that appear in the semi-simplification of  $\text{WD}(r)$ . We make the following genericity assumption.

*Assumption 1.4.3.* For any  $i \neq j$ ,  $\chi_i^{-1} \chi_j \notin \{1, \text{unr}(p^f)\}$  where  $f = [k_K : \mathbb{F}_p]$  and for  $a \in L^\times$ ,  $\text{unr}(a)$  denotes the unramified character sending a geometric Frobenius to  $a$ .

The above genericity assumption implies that  $\text{WD}(r)$  is semisimple and  $N = 0$ . Since now  $\text{WD}(r)$  is a direct sum of characters of  $W_K$ , the corresponding  $(\varphi, N, \mathcal{G}_K)$ -module is also a direct sum of rank one objects, which implies that the action of  $\mathcal{G}_K \twoheadrightarrow \text{Gal}(K'/K)$  on the  $(\varphi, N, \mathcal{G}_K)$ -module factors through an abelian quotient. Hence  $r$  is crystabelline, i.e. we can assume  $K'$  is an abelian extension of  $K$  and  $r|_{\mathcal{G}_{K'}}$  is crystalline.

**Definition 1.4.4.** A refinement  $\mathcal{R}$  for a generic crystabelline representation  $r$  is a choice of an ordering  $(\chi_1, \dots, \chi_n)$  of the different characters of  $W_K$  that appear in  $\text{WD}(r)$ .

A refinement  $\mathcal{R} = \underline{\chi} = (\chi_1, \dots, \chi_n)$  induces a filtration on  $\text{WD}(r)$  (the  $i$ -th submodule is spanned by  $W_K$ -eigenspaces of characters  $\chi_1, \dots, \chi_i$ ). By Berger's dictionary [Ber08b], the refinements are in bijection with triangulations of  $D_{\text{rig}}(r)$ . We identify  $W_K$  with  $K^\times$  via the local reciprocity map normalized by sending a geometric Frobenius to  $\varpi_K$ . The parameter of the corresponding triangulation, as a character of  $(K^\times)^n$ , is equal to  $z^{w_{\mathcal{R}}(\mathbf{h})} \underline{\chi}$  where  $\mathbf{h} = (h_{\tau,1}, \dots, h_{\tau,n})_{\tau \in \Sigma} \in (\mathbb{Z}^n)^\Sigma$  denotes the Hodge-Tate weights of  $r$  such that  $h_{\tau,1} \leq \dots \leq h_{\tau,n}$  and  $w_{\mathcal{R}} \in (\mathcal{S}_n)^\Sigma$  is an element uniquely determined by  $\mathcal{R}$  up to stabilizers of  $\mathbf{h}$  in  $(\mathcal{S}_n)^\Sigma$ .

*Remark 1.4.5.* The element  $w_{\mathcal{R}}$  parametrizes the relative position of the trianguline filtration on  $D_{\text{dR}}(r) = D_{\text{dR}}(r|_{\mathcal{G}_{K'}})^{\mathcal{G}_K} = (D_{\text{st}}(r|_{\mathcal{G}_{K'}}) \otimes_{K'_0} K')^{\text{Gal}(K'/K)}$  and the Hodge filtration on  $D_{\text{dR}}(r)$ .

Then the Conjecture 1.4.1 can be formulated more explicitly for generic crystabelline representations, knowing all the parameters of triangulations.

**Conjecture 1.4.6.** Let  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  be a continuous representation corresponding to a point in  $\text{Spf}(R_{\overline{\rho}, S})^{\text{rig}}$ . Assume that for each  $v \in S_p$ ,  $\rho_{\tilde{v}}$  is generic crystabelline. If  $W(\rho) = \{\underline{\delta} \in \widehat{T}_{p,L} \mid (\rho, \underline{\delta}) \in Y(U^p, \overline{\rho})\} \neq \emptyset$ , then

$$W(\rho) = W(\rho_p) := \prod_{v \in S_p} \{\iota_v(z^{w_v(\mathbf{h}_v)} \underline{\chi}_v), w_v \geq w_{\mathcal{R}_v}, w_v \in (\mathcal{S}_n)^{\Sigma_v}\},$$

where  $\mathbf{h}_v$  denotes the Hodge-Tate weights of  $\rho_{\bar{v}}$  as above,  $\mathcal{R}_v = \underline{\chi}_v$  runs over all refinements of  $\rho_{\bar{v}}$ , and  $\geq$  denotes the usual Bruhat order on  $(\mathcal{S}_n)^{\Sigma_v}$ .

### 1.4.3.2 Breuil's conjecture

For  $\mathcal{R}_v, \underline{\chi}_v, w_v, \mathbf{h}_v$  as in Conjecture 1.4.6, denote by  $\underline{\delta}_{\mathcal{R}_v, w_v} = \iota_v(z^{w_v(\mathbf{h}_v)} \underline{\chi}_v)$  and for  $w = (w_v)_{v \in S_p}, \mathcal{R} = (\mathcal{R}_v)_{v \in S_p}$ , write  $\underline{\delta}_{\mathcal{R}, w} = \prod_{v \in S_p} \delta_{\mathcal{R}_v, w_v}$ . The smooth part  $\underline{\delta}_{\mathcal{R}, \text{sm}}$  of  $\underline{\delta}_{\mathcal{R}, w}$  is independent of  $w_v$  and is equal to  $\delta_{B_v}(\chi_1, \dots, \chi_i | \cdot |_{F_v}^{i-1}, \dots, \chi_n | \cdot |_{F_v}^{n-1})$  where  $|\cdot|_{F_v} = \text{unr}(p^{-f_v})$  is the usual valuation on  $F_v$ . Then  $(\text{Ind}_{B_v}^{G_v} \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_v}^{-1})^{\text{sm}}$  is irreducible under our genericity assumption and its isomorphic class is independent of the refinements  $\mathcal{R}_v$  by intertwining between smooth principal series. Let  $\lambda = (\lambda_{\tau, i})_{\tau \in \Sigma_p, i=1, \dots, n} \in X^*(\mathfrak{t}) = (\mathbb{Z}^n)^{\Sigma_p}$  defined by  $\lambda_{\tau, i} = h_{\tau, n+1-i} + i - 1$ . Then  $\underline{\delta}_{\mathcal{R}, w} = z^{w w_0 \cdot \lambda} \underline{\delta}_{\mathcal{R}, \text{sm}}$  for all  $w, \mathcal{R}$  where  $w_0$  denotes the longest element in  $\prod_{v \in S_p} (\mathcal{S}_n)^{\Sigma_v}$ . The following conjecture is Breuil's locally analytic socle conjecture in crystabelline cases.

**Conjecture 1.4.7 (Breuil).** *Let  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  be a continuous representation corresponding to a point in  $\text{Spf}(R_{\bar{\rho}, S})^{\text{rig}}$ . Assume that for each  $v \in S_p$ ,  $\rho_{\bar{v}}$  is generic crystabelline and  $W(\rho) = \{\underline{\delta} \in \hat{T}_{p, L} \mid (\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})\} \neq \emptyset$ . Then there is a  $G_p$ -embedding (possibly after enlarging  $L$ )*

$$\mathcal{F}_{B_p}^{G_p}(\bar{L}(-\lambda'), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}) \hookrightarrow \Pi(\rho)^{\text{an}}$$

for  $\lambda' \in X^*(\mathfrak{t})$  and a smooth character  $\underline{\delta}_{\text{sm}}$  of  $T_p$  if and only if the locally algebraic character  $z^{\lambda'} \underline{\delta}_{\text{sm}}$  is equal to some character  $\delta_{\mathcal{R}, w} = z^{w w_0 \cdot \lambda} \underline{\delta}_{\mathcal{R}, \text{sm}} \in W(\rho_p)$  where the set  $W(\rho_p)$  is defined in Conjecture 1.4.6.

We call those representations  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})$  that appear in  $\Pi(\rho)^{\text{an}}$  companion constituents. Conjecture 1.4.7 implies Conjecture 1.4.1 thanks to an adjunction formula of Breuil on Emerton's Jacquet modules in [Bre15b]. In our case, we have

$$\text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{M}(-w w_0 \cdot \lambda)^{\vee}, \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}), \Pi(\rho)^{\text{an}}) \simeq \text{Hom}_{T_p}(\underline{\delta}_{\mathcal{R}, w}, J_{B_p}(\Pi(\rho)^{\text{an}})) \quad (1.4.8)$$

where  $\bar{M}(-w w_0 \cdot \lambda)^{\vee}$  denotes the dual Verma module in  $\mathcal{O}^{\bar{b}}$ . By the exactness of Orlik-Strauch's functor and the knowledge of the subquotients of the dual Verma module, the irreducible  $G_p$ -subquotients of  $\mathcal{F}_{B_p}^{G_p}(\bar{M}(-w w_0 \cdot \lambda)^{\vee}, \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})$  are exactly  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w' w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})$  where  $w' \leq w$  and  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})$  is the unique irreducible quotient of  $\mathcal{F}_{B_p}^{G_p}(\bar{M}(-w w_0 \cdot \lambda)^{\vee}, \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})$ . In particular,

$$\text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}), \Pi(\rho)^{\text{an}}) \neq 0 \Rightarrow \text{Hom}_{T_p}(\underline{\delta}_{\mathcal{R}, w}, J_{B_p}(\Pi(\rho)^{\text{an}})) \neq 0. \quad (1.4.9)$$

Hence the appearance of the constituent  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})$  in  $\Pi(\rho)^{\text{an}}$  implies that the companion point  $(\rho, \underline{\delta}_{\mathcal{R}, w})$  appears on the eigenvariety  $Y(U^p, \bar{\rho})$ .

### 1.4.4 Results of Breuil-Hellmann-Schraen in regular cases

If  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  is a continuous representation with Hodge-Tate-Sen weights  $\mathbf{h} = (h_{\tau, 1}, \dots, h_{\tau, n})_{\tau \in \Sigma}$ , then we say the Hodge-Tate weights are regular if  $h_{\tau, i} \neq h_{\tau, j}$  for all  $\tau$  and  $i \neq j$ . We make the following Taylor-Wiles hypothesis, some of which are already used to associate Galois representations for  $p$ -adic automorphic forms.

*Assumption 1.4.10.* 1.  $p > 2$ ;



2.  $F$  is an unramified extension of  $F^+$ ;
3.  $\mathbb{G}$  is quasi-split at all finite places of  $F^+$ ;
4.  $U_v$  is hyperspecial at all places  $v$  of  $F^+$  that are inert in  $F$ ;
5.  $F$  contains no non-trivial  $\sqrt[p]{1}$  and the image of  $\bar{\rho} \mid_{\text{Gal}(\bar{F}/F(\sqrt[p]{1}))}$  is adequate, see [BHS19, Rem. 1.1].

Note that the last assumption on the image of  $\bar{\rho}$  is some “big image” condition, see e.g. [Tho12, §2]. The following theorem is proved in [BHS19].

**Theorem 1.4.11** (Breuil-Hellmann-Schraen). *Assume that  $U^p$  is small enough and the Taylor-Wiles hypothesis (Assumption 1.4.10). Then Conjecture 1.4.6 and Conjecture 1.4.7 are true for  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  such that for all  $v \in S_p$ ,  $\rho_{\bar{v}}$  are generic crystalline with regular Hodge-Tate weights.*

*Remark 1.4.12.* The above theorem is also proved by Ding in [Din19a] for  $n = 2$ . Moreover, Ding obtained results for companion points and companion constituents for generic trianguline but non-de Rham Galois representations [Din19a, Cor. 5.12] when  $n = 2$ .

*Remark 1.4.13.* Let me emphasize that the result of Breuil-Hellmann-Schraen (Theorem 1.4.11) already includes a general classicality result. Let  $\lambda = (\lambda_{\tau,i})_{\tau \in \Sigma_p, i=1, \dots, n}$  defined before Conjecture 1.4.7. Then the condition that Hodge-Tate weights are regular implies that  $\lambda$  is a dominant weight with respect to  $\mathfrak{b}$  (i.e.  $\lambda_{\tau,1} \geq \dots \geq \lambda_{\tau,n}, \forall \tau$ ). Then  $\bar{L}(-\lambda)$  integrates to a finite-dimensional algebraic representation of  $G_p$  and

$$\mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-\lambda), \delta_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}) \simeq \bar{L}(-\lambda) \otimes (\text{Ind}_{\bar{B}_p}^{G_p} \delta_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})^{\text{sm}}$$

is locally algebraic. The appearance of this companion constituent in  $\Pi(\rho)^{\text{an}}$  implies that  $\Pi(\rho)^{\text{alg}} \neq 0$ . Hence  $\rho$  is associated to classical automorphic forms by (1.2.5). Moreover, for a different refinement  $\mathcal{R}'$ , we have

$$\mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-\lambda), \delta_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}) \simeq \mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-\lambda), \delta_{\mathcal{R}', \text{sm}} \delta_{B_p}^{-1}) \quad (1.4.14)$$

which allows one to pass from one refinement to another. These arguments are not available when the Hodge-Tate weights are not regular.

The proof of Breuil-Hellmann-Schraen builds on their construction of the patched eigenvariety, the trianguline variety and the study of the local geometry of the trianguline variety.

#### 1.4.4.1 The trianguline variety

The trianguline variety, defined in [BHS17b] (see also [HS16]), which we will recall soon, is the local Galois theoretical variant of the eigenvariety. Similar construction was firstly given by Kisin in [Kis03].

Let  $K$  be a  $p$ -adic local field as in §1.4.1 and let  $\bar{r} : \mathcal{G}_K \rightarrow \text{GL}_n(k_L)$  be a continuous representation. Let  $R_{\bar{r}}$  be the framed deformation ring of  $\bar{r}$  and let  $\mathfrak{X}_{\bar{r}} := \text{Spf}(R_{\bar{r}})^{\text{rig}}$  be the rigid generic fiber over  $L$ . Let  $\mathcal{T}_L^n = \widehat{(K^\times)^n}_L$  be the rigid space over  $L$  parametrizing continuous characters  $\underline{\delta} : (K^\times)^n \rightarrow L^\times$ . In the analogy that the eigenvariety is the Zariski closure of trianguline classical points (cf. the end of §1.3.2), the trianguline variety  $X_{\text{tri}}(\bar{r})$  is defined to be the reduced Zariski closure in  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  of the subset

$$U_{\text{tri}}(\bar{r}) = \left\{ (r, \underline{\delta}) \in \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n \mid \begin{array}{l} r \text{ is trianguline of parameter } \underline{\delta}, \\ \delta_i / \delta_j \neq z^{-\mathbf{k}}, \epsilon z^{\mathbf{k}}, \forall i \neq j, \mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma \end{array} \right\}$$

where  $\epsilon$  denotes the cyclotomic character. The trianguline variety  $X_{\text{tri}}(\bar{r})$  is an equidimensional rigid space over  $L$ . Moreover the global triangulation applies and a similar result as Theorem 1.3.8 holds: for any point  $(r, \underline{\delta}) \in X_{\text{tri}}(\bar{r})$ ,  $r$  is trianguline of some parameter  $\underline{\delta}'$  such that  $\underline{\delta}'\underline{\delta}^{-1}$  is an algebraic character (after possibly extending the scalars). One can define and formulate parallel conjectures on companion points on the trianguline variety, see [BHS19, §4.2].

#### 1.4.4.2 The patched eigenvariety

The patching method of Taylor-Wiles, further developed by Kisin, is usually a key middle step to compare Hecke algebras and Galois deformation rings, and to prove automorphy lifting theorems. Rather than proving directly  $R = \mathbb{T}$ , the method enrich the Hecke side  $\mathbb{T}$  to a larger ring  $\mathbb{T}_\infty$  by patching together Hecke eigensystems of some infinitely many smaller tame levels at some auxiliary primes. On the Galois side, one replaces the global deformation ring  $R$  by  $R_\infty$ , a power series ring over some local Galois deformation ring  $R^{\text{loc}}$ . Then it might be easier to establish  $R_\infty = \mathbb{T}_\infty$  firstly.

In [CEG<sup>+</sup>16], the authors patched completed cohomologies, under the Taylor-Wiles assumption (Assumption 1.4.10), and obtained a patched Banach representation  $\Pi_\infty$  of  $G_p$  replacing  $\hat{S}(U^p, L)_{\mathfrak{m}^S}$  which is suitable to study  $p$ -adic local Langlands correspondence. Breuil-Hellmann-Schraen used  $\Pi_\infty$  to define the patched eigenvariety and compared it with the trianguline variety in [BHS17b].

For  $v \in S_p$ , let  $R_{\bar{\rho}_v}$  be (the reduced  $p$ -torsion free quotient of) the framed deformation ring of  $\bar{\rho}_v$  and let  $R_{\bar{\rho}_p} := \hat{\otimes}_{v \in S_p} R_{\bar{\rho}_v}$ . There is a complete Noetherian local ring

$$R_\infty = R_{\bar{\rho}_p} \hat{\otimes} R_{\bar{p}} \llbracket x_1, \dots, x_g \rrbracket$$

over  $\mathcal{O}_L$  where  $R_{\bar{p}}$  is some completed tensor product of certain deformation rings of  $\bar{\rho}_v$  for  $v \in S \setminus S_p$  and  $g$  is an integer. The action of  $R_{\bar{\rho}, S}$  on  $\hat{S}(U^p, L)_{\mathfrak{m}^S}$  factors through a quotient  $R_{\bar{\rho}, S}$ . There are surjections  $R_\infty \twoheadrightarrow R_\infty/\mathfrak{a} \twoheadrightarrow R_{\bar{\rho}, S}$  of rings which are compatible with local Galois deformations where  $\mathfrak{a} \subset R_\infty$  is an ideal. Then  $\Pi_\infty$  is a unitary Banach representation of  $G_p$  with an action of  $R_\infty$  and an isomorphism  $\Pi_\infty[\mathfrak{a}] \simeq \hat{S}(U^p, L)_{\mathfrak{m}^S}$  that is compatible with the actions of  $R_\infty/\mathfrak{a}$  and  $R_{\bar{\rho}, S}$ . In particular, if  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  corresponds to a map  $R_{\bar{\rho}, S} \rightarrow L$  and defines prime ideals  $\mathfrak{m}_\rho$  for  $R_\infty$  and  $R_{\bar{\rho}, S}$ , then

$$\Pi_\infty[\mathfrak{m}_\rho] = \hat{S}(U^p, L)_{\mathfrak{m}^S}[\mathfrak{m}_\rho] = \Pi(\rho).$$

This implies that in order to prove Conjecture 1.4.6 and Conjecture 1.4.7, it is enough to study  $\Pi_\infty$ .

Write  $\Pi_\infty^{\text{an}}$  for the subspace of locally  $R_\infty$ -analytic vectors in  $\Pi_\infty$  ([BHS17b, Déf. 3.2]). The patched eigenvariety  $X_p(\bar{\rho})$  is defined as the support of the coherent sheaf defined by the dual of Emerton's Jacquet module  $J_{B_p}(\Pi_\infty^{\text{an}})'$  inside

$$\text{Spf}(R_\infty)^{\text{rig}} \times \hat{T}_{p, L} \simeq \text{Spf}(R_{\bar{\rho}_p})^{\text{rig}} \times \text{Spf}(R_{\bar{p}} \llbracket x_1, \dots, x_g \rrbracket)^{\text{rig}} \times \hat{T}_{p, L} =: \mathfrak{X}_{\bar{\rho}_p} \times (\mathfrak{X}_{\bar{p}} \times \mathbb{U}^g) \times \hat{T}_{p, L}.$$

The eigenvariety  $Y(U^p, \bar{\rho})$  can be viewed as a closed subspace of the patched one  $X_p(\bar{\rho})$ .

Let  $X_{\text{tri}}(\bar{\rho}_p) = \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_v)$  viewed as a closed subspace of  $\mathfrak{X}_{\bar{\rho}_p} \times \hat{T}_{p, L}$ . We extend the automorphism  $\iota$  of  $\hat{T}_{p, L}$  to  $X_{\text{tri}}(\bar{\rho}_p)$  by base change. Then the density of “small slope” points on  $X_p(\bar{\rho})$  and certain local-global compatibility imply that there is a closed embedding

$$X_p(\bar{\rho}) \hookrightarrow \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{p}} \times \mathbb{U}^g) \subset (\mathfrak{X}_{\bar{\rho}_p} \times \hat{T}_{p, L}) \times (\mathfrak{X}_{\bar{p}} \times \mathbb{U}^g).$$

Moreover, both  $X_p(\bar{\rho})$  and  $\iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{p}} \times \mathbb{U}^g)$  are equidimensional with the same dimension. Hence the patched eigenvariety  $X_p(\bar{\rho})$  is identified with a union of irreducible components of  $\iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{p}} \times \mathbb{U}^g)$  under the above closed embedding. This is a  $\mathbb{T}_\infty = R_\infty$ -kind result for irreducible components of the eigenvariety and the equalities will indeed hold locally around some points.

### 1.4.4.3 The cycles of companion constituents

The idea of Breuil-Hellmann-Schraen to prove Theorem 1.4.11 is to define cycles locally on  $X_p(\bar{\rho})$  which correspond to the appearance of companion constituents and to show that the expected cycles exist on the Galois side. The last step lies on the study of the local geometry of the trianguline variety via the construction of a local model. The spirit of the proof goes in a similar line with the case of mod- $p$  representations (i.e. weight part of Serre's conjecture and geometric Breuil-Mézard conjecture, [EG14]). We construct in this subsection the cycles of companion constituents.

Let  $x = (y, \underline{\delta}) = ((\rho_p = (\rho_{\bar{v}})_{v \in S_p}, \underline{\delta} = \prod_{v \in S_p} \delta_v), z) \in (\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p,L}) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  be a point as in Theorem 1.4.11, i.e. each  $\rho_{\bar{v}}$  is generic crystalline. Let  $\mathbf{h} = (\mathbf{h}_v)_{v \in S_p}$  be the Hodge-Tate weights of  $\rho_{\bar{v}}$  such that  $h_{\tau,1} \leq \dots \leq h_{\tau,n}$  for all  $\tau \in \Sigma_p$ , without assuming regularity, and define  $\lambda$  as before Conjecture 1.4.7. Then  $\lambda$  is dominant if and only if  $\mathbf{h}$  is regular. The properties mentioned in §1.4.1 and the discussions of triangulations of generic crystalline representations allow us to suppose that  $\underline{\delta} = z^\lambda \delta_{\mathcal{R},\text{sm}}$  for a refinement  $\mathcal{R} = (\mathcal{R}_{\bar{v}})_{v \in S_p}$  of  $\rho_p$ . Let  $\mathfrak{X}_\infty := \text{Spf}(R_\infty)^{\text{rig}}$ . Then  $r_x := (\rho_p, z)$  is a point on  $\mathfrak{X}_\infty$ . Let  $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x}$  be the completed local ring at  $r_x$ . We will construct cycles  $[L(w w_0 \cdot \lambda)]$  for companion constituents  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R},\text{sm}} \delta_{B_p}^{-1})$  for  $w \in \mathcal{S}_n^{\Sigma_p}$  inside  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ .

Let  $\mathfrak{m}_{r_x}$  be the ideal of  $R_\infty$  associated to  $r_x$ . We consider the associated locally analytic representation  $\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}$ . The starting point is an adjunction formula ([BHS19, Lem. 5.2.1])

$$\text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R},\text{sm}} \delta_{B_p}^{-1}), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) = (\text{Hom}_{U(\mathfrak{g})}(L(w w_0 \cdot \lambda), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}})^{N_{p,0}})^{T_p = \underline{\delta}_{\mathcal{R},\text{sm}}}.$$

To form cycles, we deform  $r_x$  and  $\underline{\delta}_{\mathcal{R},\text{sm}}$  in the right-hand side of the above formula. That is we consider the functor

$$M \mapsto \text{Hom}_{U(\mathfrak{g})}(M, \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}^\infty])^{N_{p,0}}[\mathfrak{m}_{\underline{\delta}_{\mathcal{R},\text{sm}}}^\infty]$$

for  $U(\mathfrak{g})$ -modules  $M \in \mathcal{O}$  where  $[\mathfrak{m}_{r_x}^\infty]$  denotes the subspace of elements that are killed by a power of  $\mathfrak{m}_{r_x}$  and  $\mathfrak{m}_{\underline{\delta}_{\mathcal{R},\text{sm}}}^\infty$  is the kernel of  $L[T_p] \rightarrow L$  induced by  $\underline{\delta}_{\mathcal{R},\text{sm}}$ . The functor is exact and the dual of the output for  $M$  is a finitely generated module over  $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x} = \varprojlim_i R_\infty[\frac{1}{p}]/\mathfrak{m}_y^i$ . We define the associated cycle  $[M]$  to be the support of the finitely generated module associated with  $M$  inside  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ . Then

$$[L(w w_0 \cdot \lambda)] \neq \emptyset \Rightarrow \text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R},\text{sm}} \delta_{B_p}^{-1}), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) \neq 0. \quad (1.4.15)$$

*Remark 1.4.16.* In [BHS19], the cycles in  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$  are defined to be elements in the free abelian group generated by the irreducible closed subschemes in  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ . In this thesis, it is enough to consider the cycles as topological closed subspaces in  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ , as we will simplify the relevant arguments in [BHS19] for the main theorem 1.5.1, even in regular case.

Let  $X_p(\bar{\rho})_{w w_0 \cdot \lambda}$  be the fiber of  $X_p(\bar{\rho}) \rightarrow \widehat{T}_{p,L} \xrightarrow{\text{wt}} \mathfrak{t}^*$  over  $w w_0 \cdot \lambda \in \mathfrak{t}^*$ . Let  $x_w = (y, \underline{\delta}_{\mathcal{R},w})$  be a companion point, which may or may not exist on  $X_p(\bar{\rho})$ , of  $x_{w_0}$  for  $w \in \mathcal{S}_n^{\Sigma_p}$ . Forgetting the characters we have maps

$$\text{Spec}(\widehat{\mathcal{O}}_{X_p(\bar{\rho})_{w w_0 \cdot \lambda}, x_w}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{X_p(\bar{\rho}), x_w}) \rightarrow \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$$

which are closed embeddings under our genericity assumption.

One input to study the cycles  $[M]$  defined above is that if we take the Verma module  $M(w w_0 \cdot \lambda)$ , then

$$[M(w w_0 \cdot \lambda)] = [\mathcal{M}_\infty \otimes_{\mathcal{O}_{X_p(\bar{\rho})}} \widehat{\mathcal{O}}_{X_p(\bar{\rho})_{w w_0 \cdot \lambda}, x_w}]$$

by (1.4.8) where  $\mathcal{M}_\infty$  is the coherent sheaf on  $X_p(\bar{\rho})$  defined by  $J_{B_p}(\Pi_\infty^{\text{an}})'$ . Moreover,  $[M(w w_0 \cdot \lambda)]$  is a union of  $[L(w' w_0 \cdot \lambda)]$  for  $w' \in \mathcal{S}_n^{\Sigma_p}$  such that  $L(w' w_0 \cdot \lambda)$  appear as subquotients of  $M(w w_0 \cdot \lambda)$ . Note that

$$\begin{aligned} [M(w w_0 \cdot \lambda)] \neq \emptyset &\Rightarrow \text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\overline{M}(-w w_0 \cdot \lambda)^\vee, \delta_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) \neq 0 \\ &\Leftrightarrow \text{Hom}_{T_p}(\delta_{\mathcal{R}, w}, \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) \neq 0 \\ &\Leftrightarrow x_w \in X_p(\bar{\rho}). \end{aligned} \tag{1.4.17}$$

Thus, one can prove the existence of the companion points on the patched eigenvariety or the appearance of the companion constituents in  $\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}$  by proving that the corresponding cycles are non-empty.

## 1.5 The main results

The main result of this thesis removes the regularity assumption on the Hodge-Tate weights in Breuil-Hellmann-Schraen's result (Theorem 1.4.11) for locally analytic socle conjecture and the existence of companion points in generic crystabelline cases.

**Theorem 1.5.1.** *Assume that the tame level  $U^p$  is small enough and assume the Taylor-Wiles hypothesis (Assumption 1.4.10). Then Conjecture 1.4.6 and Conjecture 1.4.7 are true for  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  such that for all  $v \in S_p$ ,  $\rho_v$  are generic crystalline.*

*Remark 1.5.2.* In contrast to the conjectures, the above theorem restricts to the crystalline cases. But there should be no essential difficulty for general generic crystabelline representations, as is noted in [BHS19, Rem. 4.2.4].

To prove the theorem, we first generalize the theory of local models for the trianguline variety of Breuil-Hellmann-Schraen to non-regular weights. Then we match the cycles on the patched eigenvariety with cycles on the local models using the relationship between partially classical companion constituents and partially de Rham Galois representations. This allows us to prove the existence of all companion points associated to the same refinements. For the companion points associated to other refinements in the non-regular cases, we approximate the non-regular points by regular points.

In the remaining part of this introduction, we explain those ideas in more details.

### 1.5.1 Local models of the trianguline variety with non-regular weights

Recall that  $L/\mathbb{Q}_p$  is a large enough coefficient field. Let  $G = \prod_{v \in S_p} (\text{Res}_{F_v/\mathbb{Q}_p} \text{GL}_{n/F_v}) \otimes_{\mathbb{Q}_p} L = \prod_{\Sigma_p} \text{GL}_{n/L}$ , a reductive group over  $L$ . Let  $B \subset G$  be the Borel subgroup of (products of) upper-triangular matrices and let  $T$  be (products of) the diagonal torus. Let  $P$  be a parabolic subgroup of  $G$  consisting of upper-block-triangular matrices and let  $P = M_P N_P$  be the Levi decomposition where  $M_P$  is the Levi subgroup containing  $T$ . We write  $\mathfrak{g}, \mathfrak{b}, \mathfrak{p}, \mathfrak{t}, \mathfrak{m}_P, \mathfrak{n}_P$  respectively for their Lie algebras viewed as affine spaces over  $L$ . Let  $W = \mathcal{S}_n^{\Sigma_p}$  be the Weyl group of  $G$  and  $W_P$  be the Weyl group of  $M_P$ .

The partial Grothendieck resolution  $\tilde{\mathfrak{g}}_P := G \times^P \mathfrak{p} \rightarrow \mathfrak{g}$  sends  $(g, \nu) \in G \times \mathfrak{p}$  to  $\text{Ad}(g)\nu$  where  $\text{Ad}$  denotes the adjoint action. We construct an algebraic variety over  $L$  by the product

$$X_P := \tilde{\mathfrak{g}}_B \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P.$$

As a closed subspace of  $\mathfrak{g} \times G/B \times G/P$  where  $G/B$  and  $G/P$  are flag varieties,

$$X_P = \{(\nu, g_1 B, g_2 P) \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{b}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}\}.$$

For  $w \in W/W_P$ , let  $X_{P,w}$  be the Zariski closure in  $X_P$  of the preimage under the map  $X_P \rightarrow G/B \times G/P$  of the  $G$ -orbit  $G(1, w) \subset G/B \times G/P$ . Then  $X_P$  is equidimensional with distinct irreducible components  $X_{P,w}$ ,  $w \in W/W_P$ . The following is the key geometric property of  $X_{P,w}$  for our applications.

**Theorem 1.5.3.** *For any  $w \in W/W_P$  and a point  $x \in X_{P,w}$ , the completion of the local ring  $\widehat{\mathcal{O}}_{X_{P,w},x}$  of  $X_{P,w}$  at  $x$  is irreducible.*

In other words,  $X_{P,w}$  is unibranch. If  $P = B$ , Breuil-Hellmann-Schraen proved that  $X_{B,w}$  is normal in [BHS19] which in particular implies that  $X_{B,w}$  is unibranch using the Cohen-Macaulay property of  $X_{B,w}$  proved by Bezrukavnikov-Riche in [BR12]. We prove that  $X_{P,w}$  is unibranch at  $x$  based on the normality of  $X_{B,w}$  and by showing that the fiber of the natural birational proper map  $X_{B,w} \rightarrow X_{P,w}$  over  $x$  is connected.

Now let  $y = (\rho_p = (\rho_{\bar{v}})_{v \in S_p}, \underline{\delta} = \prod_v \underline{\delta}_v)$  be an  $L$ -point on the trianguline variety  $X_{\text{tri}}(\bar{\rho}_p) = \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_{\bar{v}})$  such that the weights of  $\underline{\delta}$  are integers (equivalently the Hodge-Tate-Sen weights are integers) and  $\underline{\delta}$  satisfies the following genericity assumption.

**Assumption 1.5.4.** For each  $v \in S_p$ ,  $\underline{\delta}_v = (\delta_{v,1}, \dots, \delta_{v,n}) : (F_{\bar{v}}^\times)^n \rightarrow L^\times$ , we have  $\delta_{v,i} \delta_{v,j}^{-1} \neq z^{\mathbf{k}}, \epsilon z^{\mathbf{k}}$  for all  $i \neq j$  and  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$  where  $\epsilon$  denotes the cyclotomic character:  $\epsilon(z) = \prod_{\tau \in \Sigma_v} \tau(z)$  if  $z \in \mathcal{O}_{F_{\bar{v}}}^\times$  and  $\epsilon(\varpi_{F_{\bar{v}}}) = 1$ .

When  $\rho_{\bar{v}}$  are all crystabelline, the above assumption is equivalent to the genericity assumption 1.4.3. Let  $\mathbf{h} = (\mathbf{h}_v)_{v \in S_p} = (h_{\tau,1} \leq \dots \leq h_{\tau,n})_{\tau \in \Sigma_p}$  be the Hodge-Tate-Sen weights of  $\rho_{\bar{v}}$ ,  $v \in S_p$ . Since the Hodge-Tate-Sen weights are integers, the  $B_{\text{dR}}$ -representation

$$W_{\text{dR}}(D_{\text{rig}}(\rho_{\bar{v}})) = W_{\text{dR}}(\rho_{\bar{v}}) = B_{\text{dR}} \otimes_{\mathbb{Q}_p} \rho_{\bar{v}}$$

of  $\mathcal{G}_{F_{\bar{v}}}$  is almost de Rham under Fontaine's classification [Fon04]. Then

$$D_{\text{pdR}}(\rho_{\bar{v}}) = D_{\text{pdR}}(W_{\text{dR}}(D_{\text{rig}}(\rho_{\bar{v}}))) := (W_{\text{dR}}(\rho_{\bar{v}}) \otimes_{B_{\text{dR}}} B_{\text{dR}}[\log(t)])^{\mathcal{G}_{F_{\bar{v}}}}$$

is a finite free rank  $n$  module over  $L \otimes_{\mathbb{Q}_p} F_{\bar{v}} = \prod_{v \in \Sigma_v} L$  with a nilpotent operator  $\nu_{v,y}$  induced by the  $B_{\text{dR}}$ -derivation of  $B_{\text{dR}}[\log(t)]$  such that  $\nu(\log(t)) = -1$ . Moreover,  $\rho_{\bar{v}}$  is de Rham if and only if  $\nu_{v,y} = 0$  on  $D_{\text{pdR}}(\rho_{\bar{v}})$ . Recall  $W_{\text{dR}}^+(\rho_{\bar{v}}) = B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} \rho_{\bar{v}}$ . For  $i \in \mathbb{Z}$ , let

$$\text{Fil}^i D_{\text{pdR}}(\rho_{\bar{v}}) := (W_{\text{dR}}^+(\rho_{\bar{v}}) \otimes_{B_{\text{dR}}^+} t^i B_{\text{dR}}^+[\log(t)])^{\mathcal{G}_{F_{\bar{v}}}}$$

which define the Hodge filtration  $\text{Fil}^\bullet D_{\text{pdR}}(\rho_{\bar{v}})$  of  $D_{\text{pdR}}(\rho_{\bar{v}})$ .

Let  $P$  be the parabolic subgroup of  $G$  as in Theorem 1.5.3 such that  $W_P \subset W = S_n^{\Sigma_p}$  is the subgroup of the stabilizers of  $\mathbf{h}$ . Then the Hodge-Tate-Sen weights  $\mathbf{h}$  are regular if and only if  $P = B$ . Moreover, the Hodge filtrations  $(\text{Fil}^\bullet D_{\text{pdR}}(\rho_{\bar{v}}))_{v \in S_p}$  defines a point  $g_{2,y}P$  in the flag variety  $G/P$  after choosing bases for  $D_{\text{pdR}}(\rho_{\bar{v}})$ . On the other hand, the global triangulation as in Theorem 1.3.8 induces a unique full filtration on the  $(\varphi, \Gamma)$ -module  $D_{\text{rig}}(\rho_{\bar{v}})$  with the parameter a twist of  $\underline{\delta}_v$  by an algebraic character. The filtrations on  $D_{\text{rig}}(\rho_{\bar{v}})$  induce a full filtration of  $\prod_{v \in S_p} D_{\text{pdR}}(\rho_{\bar{v}})$  which gives a point  $g_{1,y}B \in G/B$ . The nilpotent operator  $\nu_y = \prod_{v \in S_p} \nu_{v,y}$  preserves the two filtration. Hence we get a point

$$y_{\text{pdR}} := (\nu_y, g_{1,y}B, g_{2,y}P) \in X_P.$$

Finally let  $w \in W/W_P$  be the unique element such that the weight  $\text{wt}(\underline{\delta}) \in (\mathbb{Z}^n)^{\Sigma_p}$  of the character  $\underline{\delta}$  coincides with  $w(\mathbf{h})$ .

With the above data, following Breuil-Hellmann-Schraen in the regular cases and noting that almost de Rham representations are stable under extensions and thus deformations, we have the following theorem on the local models of the trianguline variety.

**Theorem 1.5.5.** *Let  $y = (\rho_p, \underline{\delta})$  be an  $L$ -point of the trianguline variety  $X_{\text{tri}}(\bar{\rho}_p)$  such that  $\underline{\delta}$  satisfies the genericity assumption 1.5.4 and is locally algebraic. Let  $y_{\text{pdR}}$  be a point of  $X_P$  and  $w$  be the element of  $W/W_P$  associated with  $y$  as above. Then up to formally smooth maps, there is an isomorphism*

$$\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p), y} \simeq \widehat{\mathcal{O}}_{X_{P, w}, y_{\text{pdR}}}$$

*between the completed local rings of  $X_{\text{tri}}(\bar{\rho}_p)$  at  $y$  and that of  $X_{P, w}$  at  $y_{\text{pdR}}$ . In particular, the trianguline variety  $X_{\text{tri}}(\bar{\rho}_p)$  is irreducible at  $y$ .*

### 1.5.2 Cycles on the generalized Steinberg variety

Let  $x = (\iota(y), z) = ((\rho_p = (\rho_{\tilde{v}})_{v \in S_p}, \underline{\delta} = \prod_{v \in S_p} \underline{\delta}_v), z)$  be a point on the patched eigenvariety  $X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  with  $\rho_{\tilde{v}}$  generic crystalline as in §1.4.4.3. Write  $r_x = (\rho_p, z) \in \mathfrak{X}_{\infty} = \text{Spf}(R_{\infty})^{\text{rig}}$ . Assume also that  $\underline{\delta} = \underline{\delta}_{\mathcal{R}, w_0}$  for a refinement  $\mathcal{R}$  of  $\rho_p$  and  $w_0$  is the longest element in  $W$ . Write  $x_w = (r_x, \underline{\delta}_{\mathcal{R}, w}) \in \mathfrak{X}_{\infty} \times \widehat{T}_{p, L}$  for the companion points which may or may not lie in  $X_p(\bar{\rho})$ . Recall in §1.4.4.3, there are cycles (closed subspaces)  $[M(w w_0 \cdot \lambda)]$  and  $[L(w w_0 \cdot \lambda)]$  inside  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$  which are non-empty if and only if the corresponding companion points  $x_w$  lie in the eigenvariety (1.4.17) or the companion constituents appear in  $\Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}}$  (1.4.15).

We now define Galois-theoretical cycles  $\mathcal{Z}_{P, w}$  inside  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$  that will be compared with  $[L(w w_0 \cdot \lambda)]$ . The Galois cycles are pulled back from  $\text{Spec}(\widehat{\mathcal{O}}_{X_P, y_{\text{pdR}}})$  in Theorem 1.5.5 via the theory of the local models and the maps

$$\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x}) \rightarrow \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{\rho}_p}, \rho_p}) \hookleftarrow \text{Spec}(\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p), y}) \simeq \text{Spec}(\widehat{\mathcal{O}}_{X_{P, w_0}, y_{\text{pdR}}}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{X_P, y_{\text{pdR}}}) \quad (1.5.6)$$

where the fact that the map  $\text{Spec}(\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p), y}) \rightarrow \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{\rho}_p}, \rho_p})$  is a closed embedding is due to the genericity assumption.

Let  $\mathcal{N}$  be the subvariety of nilpotent matrices in  $\mathfrak{g}$  and let  $\mathfrak{n}_B, \mathfrak{n}_P$  be the nilradicals of  $\mathfrak{b}, \mathfrak{p}$ . The point  $y_{\text{pdR}} \in X_P$  lies in a subvariety, the generalized Steinberg variety, defined by

$$Z_P = \{(\nu, g_1 B, g_2 P) \in \mathcal{N} \times G/B \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_B, \text{Ad}(g_2^{-1})\nu \in \mathfrak{n}_P\}.$$

Similar to  $X_P$ , the algebraic variety  $Z_P$  is equidimensional with irreducible components  $Z_{P, w}$  parametrized by  $w \in W/W_P$ . Then  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w}, y_{\text{pdR}}})$  are closed subspaces in  $\text{Spec}(\widehat{\mathcal{O}}_{X_P, y_{\text{pdR}}})$  and are pulled back to closed subspaces  $\mathcal{Z}_{P, w}$  of  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$  via (1.5.6).

*Remark 1.5.7.* For simplicity we assume that  $z$  is a smooth point of  $\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g$  in this introduction. This assumption is not needed in the actual proof. And even with the assumption and considering only the underlying subsets, the cycles  $\mathcal{Z}_{P, w}$  defined here is not expected to coincide with  $[L(w w_0 \cdot \lambda)]$  in general. One should at least replace  $Z_{P, w}$  by characteristic cycles on  $Z_P$  of some  $G$ -equivariant  $\mathcal{D}$ -module on  $G/B \times G/P$  localized from the  $U(\mathfrak{g})$ -module  $L(w w_0 \cdot \lambda)$ .

The conjectures on companion points or companion constituents (Conjecture 1.4.6 and 1.4.7) are true on the local model side: the point  $y_{\text{pdR}} \in X_P$  lies in  $X_{P, w}$  (corresponding to that  $[M(w w_0 \cdot \lambda)] \neq \emptyset$ ) or lies in  $Z_{P, w}$  (corresponding to  $[L(w w_0 \cdot \lambda)] \neq \emptyset$ ) if and only if  $w \geq w_{\mathcal{R}}$  in  $W/W_P$ . The last assertion for points on  $X_P$  holds if  $\nu_y$  is zero, which is true since  $\rho_{\tilde{v}}$  are crystalline.

### 1.5.3 Partial classicality and partially de Rhamness

The aim is to prove that  $[L(w w_0 \cdot \lambda)] \neq \emptyset$  for  $w \geq w_{\mathcal{R}}$  under the assumption  $\mathcal{Z}_{P, w} \neq \emptyset$ . One input of the proof is that if the companion point  $x_w$  exists (for example  $x_{w_0}$  by our assumption), the cycle  $[M(w w_0 \cdot \lambda)]$  is non-empty and is roughly a copy of the cycle pulled back from the

local model defined by the subscheme  $\overline{X}_{P,w} := \{(\nu, g_1 B, g_2 P) \in X_{P,w} \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_B\}$  (cf. the end of §1.4.4.3). Hence one can compare the formula expressing  $[M(w w_0 \cdot \lambda)]$  as a union of  $[L(w' w_0 \cdot \lambda)]$  and as a union of  $\mathcal{Z}_{P,w'}$ . As closed subsets of  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ , we have

$$[M(w w_0 \cdot \lambda)] = \cup_{w \geq w' \geq w_{\mathcal{R}}} [L(w' w_0 \cdot \lambda)] = \cup_{w \geq w' \geq w_{\mathcal{R}}} \mathcal{Z}_{P,w'}.$$

An example to use the above formula is the first step of Breuil-Hellmann-Schraen's proof in the regular cases as the following. Let  $P = B, w = w_0$  and assume  $w_0 \neq w_{\mathcal{R}}$ , then  $L(\lambda)$  is a finite-dimensional algebraic representation of  $G_p$ . Moreover, the cycle  $[L(w_0 w_0 \cdot \lambda)]$  is by definition where  $\text{Hom}_{U(\mathfrak{g})}(L(\lambda), \Pi_\infty^{\text{an}}[\mathfrak{m}_r]) \neq 0$  for  $r$  varies in the infinitesimal neighbourhood of  $r_x \in \mathfrak{X}_\infty$ . Hence the cycle corresponds to the appearance of locally algebraic companion constituents in  $\Pi_\infty^{\text{an}}[\mathfrak{m}_r]$ . By some local-global compatibility result,  $[L(w_0 w_0 \cdot \lambda)]$  is contained in the locus where  $\rho_p = (\rho_{\tilde{v}})_{v \in S_p}$  are de Rham. However, the de Rham locus is the preimage of the subspace of  $Z_P$  defined by  $\nu = 0$  from the local models, which is exactly  $\mathcal{Z}_{P,w_0}$ . We conclude that  $[L(\lambda)] \subset \mathcal{Z}_{P,w_0}$ . Hence the existence of  $w' \neq w_0, w' \geq w_{\mathcal{R}}$  such that  $\mathcal{Z}_{P,w'} \neq \emptyset$  implies that there exists  $w' \neq w_0$  such that  $[L(w' w_0 \cdot \lambda)] \neq \emptyset$ . Then  $[M(w' w_0 \cdot \lambda)] \neq \emptyset$ , and the companion point  $x_{w'}$  exists and some companion constituent appears.

In the non-regular cases, no non-zero locally algebraic sub-representation exists in  $\Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}]$ . As a replacement, we use the so-called partially classical companion constituents or cycles, which on the Galois side correspond to partially de Rham  $(\varphi, \Gamma)$ -modules over the Robba rings. The word “classical” originally means classical automorphic forms which correspond to locally algebraic vectors in the completed cohomology.

Let  $Q = \prod_{v \in S_p} Q_v = \prod_{\tau \in \Sigma_p} Q_\tau \subset G = \prod_{\tau \in \Sigma_p} \text{GL}_{n/L}$  be a parabolic subgroup of upper-block-triangular matrices with the Levi subgroup  $M_Q$  containing  $T$ . Let  $\mathfrak{q} = \prod_{\tau \in \Sigma_p} \mathfrak{q}_\tau \subset \mathfrak{g}$  and  $\mathfrak{m}_Q = \prod_{\tau \in \Sigma_p} \mathfrak{m}_{Q_\tau}$  be their Lie algebras. The following theorem says that the appearance of partially classical constituents implies that the corresponding Galois representations are partially de Rham.

**Theorem 1.5.8.** *Let  $a = (r_a, \underline{\delta}) = ((\rho_p = (\rho_{\tilde{v}})_{v \in S_p}, \underline{\delta} = \prod_{v \in S_p} \underline{\delta}_v), z) \in \mathfrak{X}_\infty \times \widehat{T}_{p,L}$  be an  $L$ -point on the patched eigenvariety such that  $\underline{\delta}$  is locally algebraic and generic (Assumption 1.5.4). Let  $\underline{\delta}_{\text{sm}}$  be the smooth part of  $\underline{\delta}$ . For  $v \in S_p$ , let*

$$0 = \text{Fil}_0 D_{\text{rig}}(\rho_{\tilde{v}}) \subsetneq \cdots \subsetneq \text{Fil}_i D_{\text{rig}}(\rho_{\tilde{v}}) \subsetneq \cdots \subsetneq \text{Fil}_n D_{\text{rig}}(\rho_{\tilde{v}}) = D_{\text{rig}}(\rho_{\tilde{v}})$$

*be the unique triangulation of  $D_{\text{rig}}(\rho_{\tilde{v}})$  with a parameter a twist of  $\iota_v^{-1}(\underline{\delta}_v)$  by some algebraic character (cf. Theorem 1.3.8). Assume that  $\eta \in X^*(\mathfrak{t})$  is a dominant weight of  $M_Q$  and*

$$\text{Hom}_{G_p}(\mathcal{F}_{\overline{B}_p}^{G_p}(\overline{L}(-\eta), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_a}]) \neq 0. \quad (1.5.9)$$

*Then for  $\tau \in \Sigma_v, v \in S_p$  and if  $M_{Q_\tau} = \text{diag}(\text{GL}_{n_1}, \dots, \text{GL}_{n_t})$  where  $n_1 + \dots + n_t = n$ , for any  $1 \leq i \leq t$ , we have that the  $(\varphi, \Gamma_{F_{\tilde{v}}})$ -module*

$$\text{gr}_i^{Q_\tau} D_{\text{rig}}(\rho_{\tilde{v}}) := \text{Fil}_{n_1 + \dots + n_i} D_{\text{rig}}(\rho_{\tilde{v}}) / \text{Fil}_{n_1 + \dots + n_{i-1}} D_{\text{rig}}(\rho_{\tilde{v}})$$

*is  $\{\tau\}$ -de Rham, i.e. the nilpotent operator  $\nu_v$  is zero on  $D_{\text{pdR}}(\text{gr}_i^{Q_\tau} D_{\text{rig}}(\rho_{\tilde{v}})) \otimes_{L \otimes_{\mathbb{Q}_p} F_{\tilde{v}}, 1 \otimes \tau} L$ .*

We say that the points  $a$  or the Galois representations  $\rho_p = (\rho_{\tilde{v}})_{v \in S_p}$  are partially  $Q$ -de Rham if the conclusion of the above theorem is satisfied. Suppose now that  $\mathfrak{q} = \text{Lie}(Q_p) \otimes_{\mathbb{Q}_p} L$  for a parabolic subgroup  $Q_p$  of  $G_p$  with the Levi subgroup  $M_{Q_p}$ . If  $\eta$  is an integral dominant weight of  $M_Q$ , then

$$\text{Hom}_{G_p}(\mathcal{F}_{\overline{B}_p}^{G_p}(\overline{L}(-\eta), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_a}]) \neq 0 \Rightarrow \text{Hom}_{U(\mathfrak{m}_Q)}(L_{M_Q}(\eta), J_{Q_p}(\Pi_\infty^{\text{an}}[\mathfrak{m}_{r_a}])) \neq 0$$

where the locally analytic representation  $J_{Q_p}(\Pi_\infty^{\text{an}}[\mathfrak{m}_{r_a}])$  of  $M_{Q_p}$  is the parabolic version of Emerton's Jacquet module, and  $L_{M_Q}(\eta)$  is the finite-dimensional irreducible representation of  $\mathfrak{m}_{Q_p}$  of the highest weight  $\eta$  which integrates to an algebraic representation of  $M_{Q_p}$ . The theorem above shows that the existence of non-zero locally algebraic vectors in the parabolic Emerton's Jacquet module implies that the graded pieces of the so-called paraboline filtrations of  $(\varphi, \Gamma)$ -modules over the Robba rings are de Rham. Thus, the theorem is a natural generalization of the classical local-global compatibility result when  $Q = G$  (see the end of §1.2.3). And the proof of the results is by interpolating the results for classical points on the partial eigenvariety constructed by Ding in [Din19c] and by the global triangulation. When writing this part of the thesis, the author was informed that Breuil-Ding pursued similar ideas in non-trianguline cases, see [BD21].

Recall  $W_P$  is the stabilizer of the Hodge-Tate weights  $\mathbf{h}$  and  $\lambda = w_0(\mathbf{h}) + (0, \dots, n-1)_{\tau \in \Sigma_p}$ . Let  $\mathfrak{n}_Q$  be the nilradical of  $\mathfrak{q}$  so that  $\mathfrak{q} = \mathfrak{m}_Q + \mathfrak{n}_Q$ . The following theorem says that cycles  $\mathcal{Z}_{P,w}$  are partially  $Q$ -de Rham if  $ww_0 \cdot \lambda$  is a dominant weight of  $M_Q$ .

**Theorem 1.5.10.** *For each  $w \in W/W_P$ , the irreducible component  $\mathcal{Z}_{P,w}$  is contained in the subspace of  $\mathcal{Z}_P$  where  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q$  if and only if  $ww_0 \cdot \lambda$  is a dominant weight for  $\mathfrak{m}_Q$ .*

Combining Theorem 1.5.8 and Theorem 1.5.10, we can match certain  $\mathcal{Z}_{P,w}$  and  $[L(ww_0 \cdot \lambda)]$  in the proof of the main theorem 1.5.1. And we can prove the existence of all companion points  $(r_x, \underline{\delta}_{\mathcal{R},w})$ ,  $w \geq w_{\mathcal{R}}$  on  $X_p(\bar{\rho})$  for the refinement  $\mathcal{R}$  such that  $(r_x, \underline{\delta}_{\mathcal{R},w_0}) \in X_p(\bar{\rho})$ .

### 1.5.4 Companion points for different refinements

Let  $\mathcal{R}' \neq \mathcal{R}$  be another refinement. The previous step allows us to prove the existence of  $(r_x, \underline{\delta}_{\mathcal{R},w})$ ,  $w \geq w_{\mathcal{R}}$  in  $X_p(\bar{\rho})$  and in this step, we prove that  $(r_x, \underline{\delta}_{\mathcal{R}',w_0})$  is also in  $X_p(\bar{\rho})$ .

To explain the idea, we assume that  $n = 2$  and  $S_p = \{v\}$  consisting of a unique place and assume  $\Sigma_p = \{\tau_1, \tau_2\}$ . Suppose that the Hodge-Tate weights of  $\rho_{\tilde{v}}$  are  $h_{\tau_1,1} = h_{\tau_1,2} = 0$ ,  $0 = h_{\tau_2,1} < 1 = h_{\tau_2,2}$ . After the previous results, We can assume that the point  $x = (\rho_{\tilde{v}}, \iota(\delta_1, \delta_2), z) \in \mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p,L} \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  is in  $X_p(\bar{\rho})$  where  $y := (\rho_{\tilde{v}}, (\delta_1, \delta_2))$  is in the open subset  $U_{\text{tri}}(\bar{\rho}_p)$  of  $X_{\text{tri}}(\bar{\rho}_p)$  by taking  $w = w_{\mathcal{R}}$ . In other words,  $D_{\text{rig}}(\rho_{\tilde{v}}) = [\mathcal{R}_{L,F_{\tilde{v}}}(\delta_1) - \mathcal{R}_{L,F_{\tilde{v}}}(\delta_2)]$  is an extension of  $\mathcal{R}_{L,F_{\tilde{v}}}(\delta_2)$  by  $\mathcal{R}_{L,F_{\tilde{v}}}(\delta_1)$  which gives the triangulation of  $\rho_{\tilde{v}}$  of the parameter  $\underline{\delta}$ . In this example, we assume  $\delta_1 = \tau_2 \text{unr}(a)$  and  $\delta_2 = \text{unr}(b)$  where we write  $\tau_2$  for the character  $\tau_2 : K^\times \hookrightarrow L^\times$  and  $\text{unr}(a)$ ,  $\text{unr}(b)$  for the unramified characters of  $F_v^\times$  sending  $\varpi_{F_v}$  to  $a, b \in L^\times$ .

The idea is to use the results in the regular case. The strategy is to find  $y^i := (\rho_{\tilde{v}}^i, (\delta_1^i, \delta_2^i)) \in U_{\text{tri}}(\bar{\rho}_p)$  (hence  $D_{\text{rig}}(\rho_{\tilde{v}}^i) = [\mathcal{R}_{L,F_{\tilde{v}}}(\delta_1^i) - \mathcal{R}_{L,F_{\tilde{v}}}(\delta_2^i)]$ ) for  $i \in \mathbb{N}$  such that  $y^i$  have regular Hodge-Tate weights and converge to  $y$  in certain sense of rigid analytic geometry, and such that certain companion points  $(y^i)'$  of  $y^i$  converge to  $y' := (\rho_{\tilde{v}}, (\tau_2 \text{unr}(b), \text{unr}(a)))$  in  $X_{\text{tri}}(\bar{\rho}_p)$ . Then on the patched eigenvariety  $X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$ , the points  $(\iota(y^i), z)$  are in  $X_p(\bar{\rho})$  for  $i$  large using that  $X_{\text{tri}}(\bar{\rho}_p)$  is smooth at  $y$ . Their companion points  $(\iota((y^i)'), z)$  are on  $X_p(\bar{\rho})$  by the results of Breuil-Hellmann-Schraen in regular cases. Hence the companion point  $(\iota(y'), z)$ , as a limit of  $(\iota((y^i)'), z)$ , also exists on  $X_p(\bar{\rho})$  since the latter is closed in  $\iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$ . To achieve this plan, we seek points  $y^i = (\rho_{\tilde{v}}^i, (\delta_1^i, \delta_2^i))$  in  $U_{\text{tri}}(\bar{\rho}_p)$  near  $y$  that satisfy the following two conditions.

1.  $\rho_{\tilde{v}}^i$  are crystalline.
2.  $(\delta_1^i, \delta_2^i) = (\tau_1^{-i} \tau_2 \text{unr}(a^i), \text{unr}(b^i))$  where  $i \geq 1, a^i, b^i \in L^\times$ .

Then the companion points with dominant weights  $(y^i)' = (\rho_{\tilde{v}}^i, (\tau_2 \text{unr}(b^i), \tau_1^{-i} \text{unr}(a^i)))$  exist on  $X_{\text{tri}}(\bar{\rho}_p)$ . They converge to  $y'$  since  $(\tau_2 \text{unr}(b^i), \tau_1^{-i} \text{unr}(a^i)) = (\tau_2, \tau_2^{-1})(\text{unr}(b^i), \tau_2 \tau_1^{-i} \text{unr}(a^i))$  converge to  $(\tau_2, \tau_2^{-1})(\delta_2, \delta_1) = (\tau_2 \text{unr}(b), \text{unr}(a))$  by the assumption.

Now we explain how to find such extensions

$$D_{\text{rig}}(\rho_{\tilde{v}}^i) = [\mathcal{R}_{L,F_{\tilde{v}}}(\delta_1^i) - \mathcal{R}_{L,F_{\tilde{v}}}(\delta_2^i)]$$



near  $[\mathcal{R}_{L,F_v}(\delta_1) - \mathcal{R}_{L,F_v}(\delta_2)]$ . The condition (2) for  $(\delta_1^i, \delta_2^i)$  above is not hard to achieve. The problem is to get the crystalline (or de Rham) extensions with these non-dominant weights.

The following lemma gives a criterion of de Rhamness for these extensions.

**Lemma 1.5.11.** *Let  $(\eta_1, \eta_2) : (K^\times)^2 \rightarrow L^\times$  be locally algebraic characters and  $d \in \mathbb{Z}_{\geq 1}$  such that the weights satisfy  $\text{wt}_{\tau_1}(\eta_1) - \text{wt}_{\tau_1}(\eta_2) \in \mathbb{Z}_{\leq 0}$  and  $\text{wt}_{\tau_2}(\eta_1) - \text{wt}_{\tau_2}(\eta_2) = d$ . Assume that  $(\eta_1, \eta_2)$  is generic in the sense of Assumption 1.5.4. Then an extension  $[\mathcal{R}_{L,F_v}(\eta_1) - \mathcal{R}_{L,F_v}(\eta_2)]$  is de Rham if and only if the extension corresponds to a class in*

$$\text{Ker}(H_{(\varphi, \Gamma_{F_v})}^1(\mathcal{R}_{L,F_v}(\eta_1\eta_2^{-1})) \rightarrow H_{(\varphi, \Gamma_{F_v})}^1(t_{\tau_2}^{-d}\mathcal{R}_{L,F_v}(\eta_1\eta_2^{-1}))). \quad (1.5.12)$$

Here  $t_{\tau_2}$  is a generator of the sub- $(\varphi, \Gamma_{F_v})$ -module  $\mathcal{R}_{L,F_v}(\tau_2)$  of  $\mathcal{R}_{L,F_v}$ . Moreover, the kernel above has dimension 1.

Then the required extensions are picked out in the subspace of the moduli space of all extensions  $[\mathcal{R}_{L,F_v}(\eta_1) - \mathcal{R}_{L,F_v}(\eta_2)]$  cut out by the condition (1.5.12) and  $\text{wt}_{\tau_2}(\eta_1) - \text{wt}_{\tau_2}(\eta_2) = 1$ .

## 1.6 Outline of the thesis

Chapter 3 and Appendix A is taken from [Wu21]. We establish the local models for the trianguline variety (Theorem 1.5.5). Using the local models, we prove the existence of certain companion points with non-regular integral weights on the eigenvariety, as explained in §1.5.2 and §1.5.3.

Chapter 4 is [Wu22]. We complete the proof of Theorem 1.5.1, where we find all companion points on the eigenvariety with different refinements using the strategy explained in §1.5.4.

In Chapter 5, we generalize Theorem 1.5.5 by removing the integral assumption on Hodge-Tate-Sen weights, i.e. we will not require that the characters are locally algebraic. We establish the local models and local irreducibility for points on the trianguline variety with generic parameters. One needs to partition the Sen weights of the Galois representations by their mod  $\mathbb{Z}$  classes. The result is compatible with Conjecture 1.4.1 on companion points.

In Appendix B, we write the proof for the unibranched result of the local models at all points, completing the proof of Theorem 1.5.3. This is achieved by proving that certain Bott-Samelson-Demazure type resolutions have connected fibers (but for the unibranched result we still need the normality result in [BHS19]). The proof of Theorem 1.5.5 doesn't need this general result. However, the studies of the fibers are of some combinatorial interest.

## Chapter 2

# Résumé des résultats principaux

### 2.1 Conjecture sur le socle localement analytique

Soit  $F^+$  un corps de nombre totalement réel et  $S_p$  l'ensemble des places de  $F^+$  au-dessus de  $p$ . Soit  $F$  une extension imaginaire quadratique de  $F^+$  telle que chaque place de  $S_p$  est complètement décomposée dans  $F$ , soit  $n \geq 2$  un entier et soit  $\mathbb{G}$  un groupe unitaire totalement défini en  $n$  variables sur  $F^+$  et déployé par  $F$ . On fixe un sous-groupe compact ouvert  $U^p = \prod_{v \nmid p} U_v$  de  $\mathbb{G}(\mathbf{A}_{F^+}^{p\infty})$  et une extension finie  $L$  de  $\mathbb{Q}_p$  de corps résiduel  $k_L$ . Pour tout  $v \in S_p$ , soit  $\Sigma_v := \{\tau : F_v^+ \hookrightarrow L\}$  l'ensemble des plongements de  $F_v^+$  dans  $L$  et on suppose  $|\Sigma_v| = [F_v^+ : \mathbb{Q}_p]$ . Pour chaque  $v \in S_p$ , on fixe une place  $\tilde{v}$  de  $F$  au-dessus de  $v$  et on identifie  $F_v^+$  à  $F_{\tilde{v}}$ . L'espace des formes automorphes  $p$ -adiques sur  $\mathbb{G}$  de niveau modéré  $U^p$ , noté  $\hat{S}(U^p, L)$  est l'espace des fonctions continues  $\mathbb{G}(F^+) \backslash \mathbb{G}(\mathbf{A}_{F^+}^\infty)/U^p \rightarrow L$ . Soit  $G_p = \prod_{v \in S_p} G_v$  le groupe de Lie  $p$ -adique  $\mathbb{G}(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) = \prod_{v \in S_p} \mathbb{G}(F_v^+)$ . Soit  $B_p = \prod_{v \in S_p} B_v$  (resp.  $T_p = \prod_{v \in S_p} T_v$ ) le sous-groupe de Borel (resp. le tore maximal) de  $G_p \simeq \prod_{v \in S_p} \mathrm{GL}_n(F_v^+)$  des matrices triangulaires supérieures (resp. diagonales). Le groupe  $G_p$  agit sur  $\hat{S}(U^p, L)$  par translation à droite. Nous supposons de plus que  $p > 2$  et que  $\mathbb{G}$  est quasi-déployé en toute place finie de  $F^+$ . Soit  $\bar{F}$  une clôture algébrique de  $F$ . On fixe une représentation galoisienne (modulaire) absolument irréductible  $\bar{\rho} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_n(k_L)$  telle que  $\bar{\rho}$  est associée à un idéal maximal de algèbre de Hecke usuelle agissant sur  $\hat{S}(U^p, L)$ .

Suivant Emerton [Eme06c], une méthode pour construire les variétés de Hecke est d'utiliser le foncteur de Jacquet d'Emerton pour les représentations localement analytiques des groupes de Lie  $p$ -adiques. Il existe un espace analytique rigide sur  $L$  (notre variété de Hecke), noté  $Y(U^p, \bar{\rho})$ . Un point de  $Y(U^p, \bar{\rho})$  est une paire  $(\rho, \underline{\delta})$ , où  $\rho$  est une représentation  $p$ -adique continue de dimension  $n$  de  $\mathrm{Gal}(\bar{F}/F)$  et  $\underline{\delta}$  est un caractère continu de  $T_p$  qui apparaît dans  $J_{B_p}(\Pi(\rho)^{\mathrm{an}})$ . Ici  $\Pi(\rho)$  est la sous- $G_p$ -représentation de  $\hat{S}(U^p, L)$  associée à  $\rho$  découpée par l'idéal maximal associé à  $\rho$  de la "grosse" algèbre de Hecke,  $\Pi(\rho)^{\mathrm{an}}$  est le sous-espace de  $\Pi(\rho)$  des vecteurs localement analytiques qui est une représentation localement analytique admissible de  $G_p$ , et  $J_{B_p}(-)$  désigne le foncteur du module de Jacquet d'Emerton de sorte que  $J_{B_p}(\Pi(\rho)^{\mathrm{an}})$  est une représentation localement analytique du sous-groupe de Levi  $T_p$  de  $B_p$ .

#### 2.1.1 Raffinements des représentations cristallines génériques

Soit  $r : \mathcal{G}_K \rightarrow \mathrm{GL}_n(L)$  une représentation trianguline de paramètre  $\underline{\delta} = (\delta_1, \dots, \delta_n)$ , où  $K/\mathbb{Q}_p$  est un corps local. Si  $r$  est potentiellement semi-stable, d'après Berger [Ber08b], la filtration trianguline de  $D_{\mathrm{rig}}(r)$  est équivalente à une filtration complète des  $(\varphi, N, \mathcal{G}_K)$ -module filtré associé à  $r$ , et donc déterminée. et est en fait déterminée par, une filtration complète de la représentation de Weil-Deligne  $\mathrm{WD}(r)$ . Soient  $\chi_1, \dots, \chi_n$  les caractères de  $W_K$  qui apparaissent dans la semi-simplification de  $\mathrm{WD}(r)$ . Nous faisons l'hypothèse de généricité suivante.

**Hypothèse 2.1.1.** Pour tout  $i \neq j$ ,  $\chi_i^{-1}\chi_j \notin \{1, \text{unr}(p^f)\}$  où  $f = [k_K : \mathbb{F}_p]$  et pour  $a \in L^\times$ ,  $\text{unr}(a)$  désigne le caractère non ramifié qui envoie un Frobenius géométrique sur  $a$ .

L'hypothèse de généricité ci-dessus implique que  $\text{WD}(r)$  est semi-simple et  $N = 0$ . Puisque maintenant l'action de  $\mathcal{G}_K \twoheadrightarrow \text{Gal}(K'/K)$  sur le  $(\varphi, N, \mathcal{G}_K)$ -module se factorise par un quotient abélien,  $r$  est cristabelline, c'est-à-dire que nous pouvons supposer que  $K'$  est une extension abélienne de  $K$  et que  $r|_{\mathcal{G}_{K'}}$  est cristalline.

**Définition 2.1.2.** Un raffinement  $\mathcal{R}$  d'une représentation cristabelline générique  $r$  est un choix d'ordre  $(\chi_1, \dots, \chi_n)$  des différents caractères de  $W_K$  qui apparaissent dans  $\text{WD}(r)$ .

Un raffinement  $\mathcal{R} = \underline{\chi} = (\chi_1, \dots, \chi_n)$  induit une filtration sur  $\text{WD}(r)$  (le  $i$ -ième sous-module est engendré par les sous-espaces propres des caractères  $\chi_1, \dots, \chi_i$  de  $W_K$ ). D'après Berger [Ber08b], les raffinements sont en bijection avec les triangulations de  $D_{\text{rig}}(r)$ . Nous identifions  $W_K$  à  $K^\times$  via l'application de réciprocité locale normalisée en envoyant un Frobenius géométrique à  $\varpi_K$ . Le paramètre de la triangulation correspondante, en tant que caractère de  $(K^\times)^n$ , est égal à  $z^{w_{\mathcal{R}}(\mathbf{h})}\underline{\chi}$  où  $\mathbf{h} = (h_{\tau,1}, \dots, h_{\tau,n})_{\tau \in \Sigma} \in (\mathbb{Z}^n)^\Sigma$  désigne les poids de Hodge-Tate de  $r$  tels que  $h_{\tau,1} \leq \dots \leq h_{\tau,n}$  et  $w_{\mathcal{R}} \in (\mathcal{S}_n)^\Sigma$  est un élément uniquement déterminé par  $\mathcal{R}$  modulo le stabilisateur de  $\mathbf{h}$  dans  $(\mathcal{S}_n)^\Sigma$ .

**Remarque 2.1.3.** L'élément  $w_{\mathcal{R}}$  paramètre la position relative de la filtration trianguline sur  $D_{\text{dR}}(r) = D_{\text{dR}}(r|_{\mathcal{G}_{K'}})^{\mathcal{G}_K} = (D_{\text{st}}(r|_{\mathcal{G}_{K'}}) \otimes_{K'_0} K')^{\text{Gal}(K'/K)}$  et de la filtration de Hodge sur  $D_{\text{dR}}(r)$ .

Soit  $\Sigma_v := \text{Hom}(F_{\tilde{v}}, L)$  pour tout  $v \in S_p$  et définissons  $\Sigma_p = \coprod_{v \in S_p} \Sigma_v$ . Si  $\mathbf{k} = (\mathbf{k}_v)_{v \in S_p} = (k_{\tau,1}, \dots, k_{\tau,n})_{\tau \in \Sigma_p} \in (\mathbb{Z}^{\Sigma_p})^n$ , on a un caractère algébrique  $z^{\mathbf{k}} = \prod_{v \in S_p} z^{\mathbf{k}_v}$  de  $T_p$ . Soit  $\iota_v$  l'automorphisme de  $\hat{T}_{v,L}$  défini par

$$\iota_v : (\delta_{v,1}, \dots, \delta_{v,n}) \mapsto \delta_{B_v} \cdot (\delta_{v,1}, \dots, \delta_{v,i} \epsilon^{i-1}, \dots, \delta_{v,n} \epsilon^{n-1})$$

et notons  $\iota = \prod_{v \in S_p} \iota_v : \hat{T}_{p,L} \xrightarrow{\sim} \hat{T}_{p,L}$ , où  $\delta_{B_v}$  est le caractère module lisse de  $B_v$  et  $\epsilon$  désigne le caractère cyclotomique.

**Conjecture 2.1.4.** Soit  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  une représentation continue et  $(\rho, \bar{\rho}) \in Y(U^p, \bar{\rho})(L)$ . Supposons que pour chaque  $v \in S_p$ ,  $\rho_{\tilde{v}}$  est cristalline générique. Soit  $W(\rho) = \{\underline{\delta} \in \hat{T}_{p,L} \mid (\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})\}$ , alors

$$W(\rho) = W(\rho_p) := \prod_{v \in S_p} \{\iota_v(z^{w_v(\mathbf{h}_v)}\underline{\chi}_v), w_v \geq w_{\mathcal{R}_v}, w_v \in (\mathcal{S}_n)^{\Sigma_v}, \}$$

où  $\mathbf{h}_v$  désigne les poids Hodge-Tate de  $\rho_{\tilde{v}}$  comme ci-dessus,  $\mathcal{R}_v = \underline{\chi}_v$  sont tous les raffinements de  $\rho_{\tilde{v}}$ ,  $\geq$  désigne l'ordre de Bruhat de  $(\mathcal{S}_n)^{\Sigma_v}$ .

## 2.1.2 Conjecture de Breuil

Pour  $\mathcal{R}_v, \underline{\chi}_v, w_v, \mathbf{h}_v$  comme ci-dessus, notons  $\underline{\delta}_{\mathcal{R}_v, w_v} = \iota_v(z^{w_v(\mathbf{h}_v)}\underline{\chi}_v)$ , et posons  $\underline{\delta}_{\mathcal{R}, w} = \prod_{v \in S_p} \underline{\delta}_{\mathcal{R}_v, w_v}$  pour  $w = (w_v)_{v \in S_p}$ ,  $\mathcal{R} = (\mathcal{R}_v)_{v \in S_p}$ . La partie lisse  $\underline{\delta}_{\mathcal{R}_v, \text{sm}}$  de  $\underline{\delta}_{\mathcal{R}_v, w_v}$  est indépendante de  $w_v$  et est égale à  $\delta_{B_v}(\chi_1, \dots, \chi_i \cdot |\cdot|_{F_{\tilde{v}}}^{i-1}, \dots, \chi_n \cdot |\cdot|_{F_{\tilde{v}}}^{n-1})$  où  $|\cdot|_{F_{\tilde{v}}} = \text{unr}(p^{-f_v})$  est la valuation normalisée de  $F_{\tilde{v}}$ . La représentation  $(\text{Ind}_{\tilde{B}_v}^{G_v} \underline{\delta}_{\mathcal{R}_v, \text{sm}} \delta_{B_v}^{-1})^{\text{sm}}$  est irréductible sous notre hypothèse de généricité et sa classe d'isomorphisme est indépendante du choix de raffinements  $\mathcal{R}_v$ . Soit  $\lambda = (\lambda_{\tau,i})_{\tau \in \Sigma_p, i=1, \dots, n} \in X^*(\mathfrak{t}) = (\mathbb{Z}^n)^{\Sigma_p}$  défini par  $\lambda_{\tau,i} = h_{\tau, n+1-i} + i - 1$ . Alors  $\underline{\delta}_{\mathcal{R}, w} = z^{w w_0 \cdot \lambda} \underline{\delta}_{\mathcal{R}, \text{sm}}$  pour tout  $w, \mathcal{R}$  où  $w_0$  désigne l'élément le plus long de  $\prod_{v \in S_p} (\mathcal{S}_n)^{\Sigma_v}$ .

Pour  $\lambda \in X^*(\mathfrak{t})$ , on a une représentation localement analytique  $\mathcal{F}_{\tilde{B}_p}^{G_p}(\bar{L}(-\lambda'), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1})$  construite par Orlik-Strauch.

**Conjecture 2.1.5** (Breuil). *Soit  $\rho : \mathcal{G}_F \rightarrow \mathrm{GL}_n(L)$  une représentation continue et  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})(L)$ . Supposons que pour chaque  $v \in S_p$ ,  $\rho_{\bar{v}}$  est cristalline générique. Alors, il y a un  $G_p$ -plongement fermé*

$$\mathcal{F}_{B_p}^{G_p}(\bar{L}(-\lambda'), \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1}) \hookrightarrow \Pi(\rho)^{\mathrm{an}}$$

pour  $\lambda' \in X^*(t)$  et un caractère lisse  $\underline{\delta}_{\mathrm{sm}}$  de  $T_p$  si et seulement si le caractère localement algébrique  $z^{\lambda'} \underline{\delta}_{\mathrm{sm}}$  est égal à un caractère  $\delta_{\mathcal{R}, w} = z^{w w_0 \cdot \lambda} \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \in W(\rho_p)$  défini dans la Conjecture 2.1.4 où  $w \geq w_{\mathcal{R}}$ .

Nous appelons ces représentations  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{B_p}^{-1})$  qui apparaissent dans  $\Pi(\rho)^{\mathrm{an}}$  les constituants compagnons. La conjecture 2.1.5 est plus forte que la conjecture 2.1.4 grâce à une formule d'adjonction de Breuil concernant les modules de Jacquet d'Emerton dans [Bre15b]. Dans notre cas, nous avons

$$\mathrm{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{M}(-w w_0 \cdot \lambda)^{\vee}, \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{B_p}^{-1}), \Pi(\rho)^{\mathrm{an}}) \simeq \mathrm{Hom}_{T_p}(\underline{\delta}_{\mathcal{R}, w}, J_{B_p}(\Pi(\rho)^{\mathrm{an}})) \quad (2.1.6)$$

où  $\bar{M}(-w w_0 \cdot \lambda)^{\vee}$  désigne le module de Verma dual dans  $\mathcal{O}^{\bar{b}}$ . Par l'exactitude du foncteur d'Orlik-Strauch et la connaissance des sous-quotients du module de Verma dual, les sous-quotients irréductibles de  $\mathcal{F}_{B_p}^{G_p}(\bar{M}(-w w_0 \cdot \lambda)^{\vee}, \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{B_p}^{-1})$  sont exactement les  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w' w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{B_p}^{-1})$  où  $w' \leq w$  et  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{B_p}^{-1})$  est l'unique quotient irréductible de  $\mathcal{F}_{B_p}^{G_p}(\bar{M}(-w w_0 \cdot \lambda)^{\vee}, \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{B_p}^{-1})$ . En particulier,

$$\mathrm{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{B_p}^{-1}), \Pi(\rho)^{\mathrm{an}}) \neq 0 \Rightarrow \mathrm{Hom}_{T_p}(\underline{\delta}_{\mathcal{R}, w}, J_{B_p}(\Pi(\rho)^{\mathrm{an}})) \neq 0. \quad (2.1.7)$$

### 2.1.3 Résultats de Breuil-Hellmann-Schraen dans les cas réguliers

Si  $r : \mathcal{G}_K \rightarrow \mathrm{GL}_n(L)$  est une représentation galoisienne  $p$ -adique avec des poids de Hodge-Tate-Sen  $\mathbf{h} = (h_{\tau, 1}, \dots, h_{\tau, n})_{\tau \in \Sigma}$ , alors on dit que les poids de Hodge-Tate sont réguliers si  $h_{\tau, i} \neq h_{\tau, j}$  pour tout  $\tau$  et  $i \neq j$ . Nous faisons l'hypothèse, dite de Taylor-Wiles, suivante.

*Assumption 2.1.8.* 1.  $p > 2$  ;

2.  $F$  est une extension non ramifiée de  $F^+$  ;
3.  $\mathbb{G}$  est quasi-déployé en toutes les places finies de  $F^+$  ;
4.  $U_v$  est hyperspécial en toutes les places  $v$  de  $F^+$  qui sont inertes dans  $F$  ;
5.  $\sqrt[3]{1} \notin F$  et l'image de  $\bar{\rho} \mid_{\mathrm{Gal}(\bar{F}/F(\sqrt[3]{1}))}$  est adéquate, voir [BHS19, Rem. 1.1].

Notons que la dernière hypothèse sur l'image de  $\bar{\rho}$  est une condition de “grosse image”, voir [Tho12, §2]. Le théorème suivant est démontré dans [BHS19].

**Théorème 2.1.9** (Breuil-Hellmann-Schraen). *Supposons que  $U^p$  est suffisamment petit et l'hypothèse de Taylor-Wiles (Hypothèse 2.1.8). Alors Conjectures 2.1.4 et 2.1.5 sont vraies pour  $\rho : \mathcal{G}_F \rightarrow \mathrm{GL}_n(L)$  telle que pour tout  $v \in S_p$ ,  $\rho_{\bar{v}}$  est cristallin de poids de Hodge-Tate réguliers.*

*Remarque 2.1.10.* Le théorème ci-dessus est aussi prouvé par Ding dans [Din19a] pour  $n = 2$ . De plus, Ding a obtenu des résultats pour les points compagnons et les constituants compagnons pour les représentations galoisiennes génériques triangulines mais non de Rham [Din19a, Cor. 5.12] lorsque  $n = 2$ .

La démonstration de Breuil-Hellmann-Schraen repose sur leur construction de la variété de Hecke-Taylor-Wiles, la variété trianguline et l'étude de la géométrie locale de la variété trianguline.

### 2.1.4 La variété trianguline

La variété trianguline, définie dans [BHS17b], est la variante galoisienne locale de la variété de Hecke.

Soit  $K$  un corps local  $p$ -adique comme précédemment et soit  $\bar{r} : \mathcal{G}_K \rightarrow \mathrm{GL}_n(k_L)$  une représentation continue. Soit  $R_{\bar{r}}$  l'anneau de déformation cadré de  $\bar{r}$  et soit  $\mathfrak{X}_{\bar{r}} := \mathrm{Spf}(R_{\bar{r}})^{\mathrm{rig}}$  la fibre générique rigide sur  $L$  du schéma formel  $\mathrm{Spf}(R_{\bar{r}})$ . Soit  $\mathcal{T}_L^n = (\widehat{K^\times})^n_L$  l'espace rigide sur  $L$  paramétrant les caractères continus  $\underline{\delta} : (K^\times)^n \rightarrow L^\times$ . La variété trianguline  $X_{\mathrm{tri}}(\bar{r})$  est l'adhérence de Zariski réduite dans  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  du sous-ensemble

$$U_{\mathrm{tri}}(\bar{r}) = \left\{ (r, \underline{\delta}) \in \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n \mid \begin{array}{l} r \text{ is trianguline of parameter } \underline{\delta}, \\ \delta_i / \delta_j \neq z^{-\mathbf{k}}, \epsilon z^{\mathbf{k}}, \forall i \neq j, \mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma \end{array} \right\}$$

où  $\epsilon$  désigne le caractère cyclotomique. La variété trianguline  $X_{\mathrm{tri}}(\bar{r})$  est un espace rigide équidimensionnel sur  $L$ . De plus la triangulation globale s'applique: pour tout point  $(r, \underline{\delta}) \in X_{\mathrm{tri}}(\bar{r})$ ,  $r$  est une représentation trianguline d'un certain paramètre  $\underline{\delta}'$  tel que  $\underline{\delta}' \underline{\delta}^{-1}$  est un caractère algébrique.

### 2.1.5 La variété de Hecke-Taylor-Wiles

La méthode de patching de Taylor-Wiles, améliorée par Kisin, est généralement une étape intermédiaire clé pour comparer les algèbres de Hecke et les anneaux de déformation de Galois, et prouver la modularité. Plutôt que de prouver directement  $R = \mathbb{T}$ , la méthode enrichit le côté Hecke  $\mathbb{T}$  en un anneau plus grand  $\mathbb{T}_\infty$  en regroupant les systèmes propres de Hecke de certains une infinité de niveaux modérés et en remplaçant l'anneau de déformation global  $R$  par  $R_\infty$ , un anneau de série de puissances sur un anneau de déformation Galois local  $R^{\mathrm{loc}}$ . Ensuite, il peut être plus facile d'établir dans un premier temps  $R_\infty = \mathbb{T}_\infty$ .

Dans [CEG<sup>+</sup>16], les auteurs ont appliqué cette méthode à la cohomologie complétée et ont obtenu une représentation de Banach  $\Pi_\infty$  de  $G_p$  remplaçant  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$ . Breuil-Hellmann-Schraen ont utilisé  $\Pi_\infty$  pour définir la variété de Hecke-Taylor-Wiles et l'ont comparée à la variété trianguline dans [BHS17b].

Pour  $v \in S_p$ , soit  $R_{\bar{\rho}_v}$  (le quotient sans  $p$ -torsion réduit de) l'anneau de déformation cadré de  $\bar{\rho}_v$  et notons  $R_{\bar{\rho}_p} := \widehat{\bigotimes}_{v \in S_p} R_{\bar{\rho}_v}$ . Il existe un anneau local noethérien complet

$$R_\infty = R_{\bar{\rho}_p} \widehat{\otimes} R_{\bar{\rho}^p}[[x_1, \dots, x_g]]$$

sur  $\mathcal{O}_L$  où  $R_{\bar{\rho}^p}$  est le produit tensoriel complété de certains anneaux de déformation de  $\bar{\rho}_v$  pour  $v \in S \setminus S_p$  et  $g$  est un entier. L'action de  $R_{\bar{\rho}, S}$  sur  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$  se factorise par un quotient  $R_{\bar{\rho}, S}$  et il y a des surjections  $R_\infty \twoheadrightarrow R_\infty/\mathfrak{a} \twoheadrightarrow R_{\bar{\rho}, S}$  d'anneaux compatibles avec les déformations galoisiennes locales où  $\mathfrak{a} \subset R_\infty$  désigne un idéal. Alors, sous l'hypothèse de Taylor-Wiles (Hypothèse 2.1.8), il existe une représentation unitaire de Banach  $\Pi_\infty$  de  $G_p$  avec une action de  $R_\infty$  et un isomorphisme  $\Pi_\infty[\mathfrak{a}] \simeq \widehat{S}(U^p, L)_{\mathfrak{m}^S}$  compatible avec les actions de  $R_\infty/\mathfrak{a}$  et  $R_{\bar{\rho}, S}$ . En particulier, si  $\rho : \mathcal{G}_F \rightarrow \mathrm{GL}_n(L)$  correspond à une application  $R_{\bar{\rho}, S} \rightarrow L$  et définit des idéaux premiers  $\mathfrak{m}_\rho$  pour  $R_\infty$  et  $R_{\bar{\rho}, S}$ , alors on a

$$\Pi_\infty[\mathfrak{m}_\rho] = \widehat{S}(U^p, L)_{\mathfrak{m}^S}[\mathfrak{m}_\rho] = \Pi(\rho).$$

Donc, pour la Conjecture 2.1.4 et la Conjecture 2.1.5, il suffit d'étudier  $\Pi_\infty$ . Écrivons  $\Pi_\infty^{\mathrm{an}}$  pour le sous-espace des vecteurs localement  $R_\infty$ -analytiques dans  $\Pi_\infty$  ([BHS17b, Déf. 3.2]). La variété de Hecke-Taylor-Wiles  $X_p(\bar{\rho})$  est le support du dual du module Jacquet d'Emerton  $J_{B_p}(\Pi_\infty^{\mathrm{an}})'$  dans

$$\mathrm{Spf}(R_\infty)^{\mathrm{rig}} \times \widehat{T}_{p, L} \simeq \mathrm{Spf}(R_{\bar{\rho}_p})^{\mathrm{rig}} \times \mathrm{Spf}(R_{\bar{\rho}^p}[[x_1, \dots, x_g]])^{\mathrm{rig}} \times \widehat{T}_{p, L} = : \mathfrak{X}_{\bar{\rho}_p} \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g) \times \widehat{T}_{p, L}.$$

La variété de Hecke  $Y(U^p, \bar{\rho})$  peut être vue comme un sous-espace fermé de  $X_p(\bar{\rho})$ .

Soit  $X_{\text{tri}}(\bar{\rho}_p) = \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_v)$ , vu comme un sous-espace fermé de  $\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p,L}$ . On étend l'automorphisme  $\iota$  de  $\widehat{T}_{p,L}$  à  $X_{\text{tri}}(\bar{\rho}_p)$  par changement de base. Alors la densité de points de “petite pente” sur  $X_p(\bar{\rho})$  et une certaine compatibilité locale-globale pour  $\Pi_\infty$  impliquent qu'il y a un plongement fermé

$$X_p(\bar{\rho}) \hookrightarrow \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g) \subset (\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p,L}) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g).$$

De plus,  $X_p(\bar{\rho})$  et  $\iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  sont équidimensionnelles avec la même dimension et donc la variété de Hecke-Taylor-Wiles  $X_p(\bar{\rho})$  est identifiée à une union de composants irréductibles de  $\iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  par le plongement fermé ci-dessus. Il s'agit d'un résultat du type  $\mathbb{T}_\infty = R_\infty$  pour les composantes irréductibles de la variété de Hecke.

### 2.1.6 Les cycles des constituants compagnons

L'idée de Breuil-Hellmann-Schraen pour prouver le théorème 2.1.9 est de définir localement des cycles sur  $X_p(\bar{\rho})$  qui correspondent à l'apparition de constituants compagnons et de montrer que les cycles attendus existent du côté de Galois. La dernière étape repose sur l'étude de la géométrie locale de la variété trianguline via leur théorie d'un modèle local. L'esprit de la preuve est similaire au cas mod- $p$  (c'est-à-dire la partie de poids de la conjecture de Serre et la conjecture géométrique de Breuil-Mézard, [EG14]). Nous construisons dans cette sous-section les cycles des constituants compagnons.

Soit  $x = (y, \underline{\delta}) = ((\rho_p = (\rho_v)_{v \in S_p}, \underline{\delta} = \prod_{v \in S_p} \delta_v), z) \in (\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p,L}) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  un point comme dans le théorème 2.1.9, c'est-à-dire tel que chaque  $\rho_v$  soit générique cristalline. Soit  $\mathbf{h} = (\mathbf{h}_v)_{v \in S_p}$  les poids de Hodge-Tate de  $\rho_v$ , et définissons  $\lambda$  comme avant Conjecture 2.1.5. Alors  $\lambda$  est dominant si et seulement si  $\mathbf{h}$  est régulier. Les discussions sur les triangulations de représentations cristallines génériques permettent de supposer que  $\underline{\delta} = z^\lambda \delta_{\mathcal{R}, \text{sm}}$  pour un raffinement  $\mathcal{R} = (\mathcal{R}_v)_{v \in S_p}$  de  $\rho_p$ . Soit  $\mathfrak{X}_\infty := \text{Spf}(R_\infty)^{\text{rig}}$ . Alors  $r_x := (\rho_p, z)$  est un point sur  $\mathfrak{X}_\infty$ . Soit  $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x}$  l'anneau local complet en  $r_x$ . Nous allons définir des cycles  $[L(ww_0 \cdot \lambda)]$  associés aux constituants compagnons  $\mathcal{F}_{B_p}^{G_p}(\bar{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1})$  pour  $w \in \mathcal{S}_n^{\Sigma_p}$  dans  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ .

Soit  $\mathfrak{m}_{r_x}$  l'idéal de  $R_\infty$  associé à  $r_x$ , et on a la représentation localement analytique  $\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{un}}$ . Le point de départ est une formule d'adjonction ([BHS19, Lem. 5.2.1])

$$\text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) = (\text{Hom}_{U(\mathfrak{g})}(L(ww_0 \cdot \lambda), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}})^{N_{p,0}})^{T_p = \underline{\delta}_{\mathcal{R}, \text{sm}}}.$$

On considère le foncteur

$$M \mapsto \text{Hom}_{U(\mathfrak{g})}(M, \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}^\infty])^{N_{p,0}}[\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \text{sm}}}^\infty]$$

ou  $M \in \mathcal{O}$  est un  $U(\mathfrak{g})$ -module,  $[\mathfrak{m}_{r_x}^\infty]$  désigne le sous-espace des éléments qui sont annulés par une puissance de  $\mathfrak{m}_{r_x}$  et  $\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \text{sm}}}^\infty$  est le noyau de  $L[T_p] \rightarrow L$  induit par  $\underline{\delta}_{\mathcal{R}, \text{sm}}$ . Le foncteur est exact et le dual de l'image de  $M$  est un module de type fini sur  $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x} = \varprojlim_i R_\infty[\frac{1}{p}]/\mathfrak{m}_y^i$  qui permet de définir un cycle  $[M]$  dans  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ . On définit  $[M]$  comme le support du module de type fini associé à  $M$  dans  $\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x}$ . On a

$$[L(ww_0 \cdot \lambda)] \neq \emptyset \Rightarrow \text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\bar{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) \neq 0. \quad (2.1.11)$$

Soit  $X_p(\bar{\rho})_{ww_0 \cdot \lambda}$  la fibre de  $X_p(\bar{\rho}) \rightarrow \widehat{T}_{p,L} \xrightarrow{\text{wt}} \mathfrak{t}^*$  en  $ww_0 \cdot \lambda \in \mathfrak{t}^*$ . Soit  $x_w = (y, \underline{\delta}_{\mathcal{R}, w})$  un point compagnon de  $x = x_{w_0}$  pour  $w \in \mathcal{S}_n^{\Sigma_p}$ . En oubliant les caractères, on a des applications  $\text{Spec}(\widehat{\mathcal{O}}_{X_p(\bar{\rho})_{ww_0 \cdot \lambda}, x_w}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{X_p(\bar{\rho}), x_w}) \rightarrow \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$  qui sont des plongements fermés

sous notre hypothèse de généricité. Si nous prenons le module de Verma  $M(ww_0 \cdot \lambda)$ , alors  $[M(ww_0 \cdot \lambda)] = [J_{B_p}(\Pi_\infty^{\text{an}})' \otimes_{\mathcal{O}_{X_p(\bar{\rho})}} \widehat{\mathcal{O}}_{X_p(\bar{\rho})_{ww_0 \cdot \lambda, x_w}}]$  en utilisant (2.1.6). On peut exprimer  $[M(ww_0 \cdot \lambda)]$  comme une réunion de  $[L(w'w_0 \cdot \lambda)]$ ,  $w' \in \mathcal{S}_n^{\Sigma_p}$ . Notons que

$$\begin{aligned} [M(ww_0 \cdot \lambda)] \neq \emptyset &\Rightarrow \text{Hom}_{G_p}(\mathcal{F}_{\bar{B}_p}^{G_p}(\overline{M}(-ww_0 \cdot \lambda)^\vee, \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1}), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) \neq 0 \\ &\Leftrightarrow \text{Hom}_{T_p}(\underline{\delta}_{\mathcal{R}, w}, \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}) \neq 0 \\ &\Leftrightarrow x_w \in X_p(\bar{\rho}). \end{aligned} \tag{2.1.12}$$

Donc, on peut démontrer l'existence des points compagnons sur la variété de Hecke-Taylor-Wiles ou l'apparition des constituants compagnons dans  $\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}$  en prouvant que les cycles correspondants ne sont pas vides.

## 2.2 Les résultats principaux

Le résultat principal de cette thèse permet d'enlever l'hypothèse de régularité sur les poids de Hodge-Tate dans Théorème 2.1.9.

**Théorème 2.2.1.** *Supposons que le niveau modéré  $U^p$  est suffisamment petit et l'hypothèse de Taylor-Wiles (Hypothèse 2.1.8). Alors les Conjecture 2.1.4 et 2.1.5 sont vraies pour  $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(L)$  telles que pour tout  $v \in S_p$ ,  $\rho_{\bar{v}}$  sont cristallines génériques.*

Pour démontrer le théorème, nous généralisons d'abord la théorie des modèles locaux pour la variété trianguline de Breuil-Hellmann-Schraen aux poids non-réguliers. Ensuite, nous comparons les cycles sur la variété de Hecke-Taylor-Wiles avec les cycles sur les modèles locaux en utilisant la relation entre les constituants compagnons partiellement classiques et les représentations galoisiennes partiellement de Rham. Cela permet de prouver l'existence de tous les points compagnons associés aux mêmes raffinements. Pour les points compagnons associés à d'autres raffinements dans les cas non-réguliers, nous approximons les points non-réguliers par des points réguliers.

Dans la partie restante de cette introduction, nous expliquons ces idées en quelques détails.

### 2.2.1 Modèles locaux de la variété trianguline à poids non réguliers

Rappelons que  $L/\mathbb{Q}_p$  est un corps de coefficients suffisamment grand. Soit

$$G = \prod_{v \in S_p} (\text{Res}_{F_{\bar{v}}/\mathbb{Q}_p} \text{GL}_{n/F_{\bar{v}}}) \otimes_{\mathbb{Q}_p} L = \prod_{\Sigma_p} \text{GL}_{n/L}$$

groupe réductif. Soit  $B \subset G$  le sous-groupe de Borel des (produits des) matrices triangulaires supérieures et soit  $T$  les (produits du) tore diagonal. Soit  $P$  un sous-groupe parabolique de  $G$  de matrices triangulaires supérieures par bloc et soit  $P = M_P N_P$  la décomposition de Levi où  $M_P$  est le sous-groupe de Levi contenant  $T$ . On note respectivement  $\mathfrak{g}, \mathfrak{b}, \mathfrak{p}, \mathfrak{t}, \mathfrak{m}_P, \mathfrak{n}_P$  pour leurs algèbres de Lie vues comme des espaces affines sur  $L$ . Soit  $W = \mathcal{S}_n^{\Sigma_p}$  le groupe de Weyl de  $G$  et  $W_P$  le groupe de Weyl de  $M_P$ .

Nous avons la résolution partielle de Grothendieck  $\tilde{\mathfrak{g}}_P := G \times^P \mathfrak{p} \rightarrow \mathfrak{g}$  envoyant  $(g, \nu)$  sur  $\text{Ad}(g)\nu$  où  $\text{Ad}$  désigne l'action adjointe. On obtient alors une variété algébrique sur  $L$  en considérant le produit fibré

$$X_P := \tilde{\mathfrak{g}}_B \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P$$

ou encore comme un sous-espace fermé de  $\mathfrak{g} \times G/B \times G/P$ ,

$$X_P = \{(\nu, g_1 B, g_2 P) \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{b}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}\}.$$

Pour  $w \in W/W_P$ , on note  $X_{P,w}$  la clôture de Zariski dans  $X_P$  de la préimage sous l'application  $X_P \rightarrow G/B \times G/P$  de la  $G$ -orbite  $G(1, w) \subset G/B \times G/P$ . Alors on peut montrer que  $X_P$  est équidimensionnel avec des composantes irréductibles distinctes  $X_{P,w}, w \in W/W_P$ . Voici la propriété géométrique clé de  $X_{P,w}$  utile pour les applications.

**Théorème 2.2.2.** *Pour tout  $w \in W/W_P$  et tout point  $x \in X_{P,w}$ , la complétion de l'anneau local  $\hat{\mathcal{O}}_{X_{P,w},x}$  de  $X_{P,w}$  en  $x$  est irréductible.*

Autrement dit,  $X_{P,w}$  est unibranche. Si  $P = B$ , Breuil-Hellmann-Schraen ont démontré que  $X_{B,w}$  est normal dans [BHS19] ce qui implique que  $X_{B,w}$  est unibranche en utilisant la propriété de Cohen-Macaulay de  $X_{B,w}$  démontrée par Bezrukavnikov-Riche dans [BR12]. Nous prouvons que  $X_{P,w}$  est unibranche en  $x$  en utilisant la normalité de  $X_{B,w}$  et en montrant que la fibre de l'application naturelle propre birationnelle  $X_{B,w} \rightarrow X_{P,w}$  en  $x$  est connecté.

Soit maintenant  $y = (\rho_p = (\rho_{\bar{v}})_{v \in S_p}, \underline{\delta} = \prod_v \underline{\delta}_v)$  un  $L$ -point de la variété trianguline  $X_{\text{tri}}(\bar{\rho}_p) = \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_{\bar{v}})$  tel que les poids de  $\underline{\delta}$  soient des entiers (de manière équivalente, les poids de Hodge-Tate-Sen sont des entiers) et tel que  $\underline{\delta}$  satisfait l'hypothèse de généricité suivante.

**Hypothèse 2.2.3.** Pour chaque  $v \in S_p$ ,  $\underline{\delta}_v = (\delta_{v,1}, \dots, \delta_{v,n}) : (F_{\bar{v}}^\times)^n \rightarrow L^\times$ , nous avons  $\delta_{v,i} \delta_{v,j}^{-1} \neq z^{\mathbf{k}}, \epsilon z^{\mathbf{k}}$  pour tout  $i \neq j$  et  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$  où  $\epsilon$  désigne le caractère cyclotomique :  $\epsilon(z) = \prod_{\tau \in \Sigma_v} \tau(z)$  si  $z \in \mathcal{O}_{F_{\bar{v}}}^\times$  et  $\epsilon(\varpi_{F_{\bar{v}}}) = 1$ .

Lorsque les  $\rho_{\bar{v}}$  sont toutes cristabellines, l'hypothèse ci-dessus est équivalente à l'hypothèse de généricité 2.1.1. Soit  $\mathbf{h} = (\mathbf{h}_v)_{v \in S_p} = (h_{\tau,1} \leq \dots \leq h_{\tau,n}, \tau \in \Sigma_p)$  les poids de Hodge-Tate-Sen de  $\rho_{\bar{v}}, v \in S_p$ . Puisque les poids de Hodge-Tate-Sen sont des entiers, la  $B_{\text{dR}}$ -représentation  $W_{\text{dR}}(D_{\text{rig}}(\rho_{\bar{v}})) = W_{\text{dR}}(\rho_{\bar{v}}) = B_{\text{dR}} \otimes_{\mathbb{Q}_p} \rho_{\bar{v}}$  de  $\mathcal{G}_{F_{\bar{v}}}$  est presque de Rham selon la classification de Fontaine [Fon04]. Ainsi,

$$D_{\text{pdR}}(\rho_{\bar{v}}) = D_{\text{pdR}}(W_{\text{dR}}(D_{\text{rig}}(\rho_{\bar{v}}))) := (W_{\text{dR}}(\rho_{\bar{v}}) \otimes_{B_{\text{dR}}} B_{\text{dR}}[\log(t)])^{\mathcal{G}_{F_{\bar{v}}}}$$

est un module fini de rang  $n$  libre sur  $L \otimes_{\mathbb{Q}_p} F_{\bar{v}} = \prod_{v \in \Sigma_v} L$  avec un opérateur nilpotent  $\nu_{v,y}$  induit par la  $B_{\text{dR}}$ -dérivation de  $B_{\text{dR}}[\log(t)]$  telle que  $\nu(\log(t)) = -1$ . De plus,  $\rho_{\bar{v}}$  est de Rham si et seulement si  $\nu_{v,y} = 0$  sur  $D_{\text{pdR}}(\rho_{\bar{v}})$ . Soit  $P$  le sous-groupe parabolique de  $G$  comme dans Théorème 2.2.2 tel que  $W_P \subset W = \mathcal{S}_n^{\Sigma_p}$  soit le sous-groupe des stabilisateurs de  $\mathbf{h}$ . Alors les poids de Hodge-Tate-Sen  $\mathbf{h}$  sont réguliers si et seulement si  $P = B$ . On a de plus des filtrations Hodge-Tate  $\text{Fil}^\bullet D_{\text{pdR}}(\rho_{\bar{v}})$  de  $D_{\text{pdR}}(\rho_{\bar{v}})$  de sorte que  $(\text{Fil}^\bullet D_{\text{pdR}}(\rho_{\bar{v}}))_{v \in S_p}$  définit un point  $g_{2,y} P \in G/P$  après avoir choisi une base de  $D_{\text{pdR}}(\rho_{\bar{v}})$ . La triangulation globale alors induit une unique filtration complète sur le  $(\varphi, \Gamma)$ -module  $D_{\text{rig}}(\rho_{\bar{v}})$  de paramètre un produit de  $\underline{\delta}_v$  et de caractère algébrique, donc aussi un drapeau complet de  $\prod_{v \in S_p} D_{\text{pdR}}(\rho_{\bar{v}})$  qui donne un point  $g_{1,y} B \in G/B$ . L'opérateur nilpotent  $\nu_y = \prod_{v \in S_p} \nu_{v,y}$  préserve les deux filtrations. On obtient donc un point

$$y_{\text{pdR}} := (\nu_y, g_{1,y} B, g_{2,y} P) \in X_P.$$

Soit finalement  $w \in W/W_P$  l'unique élément tel que le poids  $\text{wt}(\underline{\delta}) \in (\mathbb{Z}^n)^{\Sigma_p}$  du caractère  $\underline{\delta}$  coïncide avec  $w(\mathbf{h})$ .

Avec les données ci-dessus, en suivant Breuil-Hellmann-Schraen dans les cas réguliers et en notant que les représentations presque de Rham sont stables sous les extensions et donc les déformations, on a le théorème suivant sur les modèles locaux de la variété trianguline.

**Théorème 2.2.4.** *Soit  $y = (\rho_p, \underline{\delta})$  un  $L$ -point de la variété trianguline  $X_{\text{tri}}(\bar{\rho}_p)$  tel que  $\underline{\delta}$  est localement algébrique et satisfait l'hypothèse de généricité 2.2.3. Soit  $y_{\text{pdR}}$  un point de  $X_P$  et  $w$  l'élément de  $W/W_P$  associé à  $y$  comme ci-dessus. Alors à des applications formellement lisses près, il existe un isomorphisme*

$$\hat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p), y} \simeq \hat{\mathcal{O}}_{X_{P,w}, y_{\text{pdR}}}$$

*d'anneaux locaux complétés de  $X_{\text{tri}}(\bar{\rho}_p)$  en  $y$  et de  $X_{P,w}$  en  $y_{\text{pdR}}$ . En particulier, la variété trianguline  $X_{\text{tri}}(\bar{\rho}_p)$  est irréductible en  $y$ .*



### 2.2.2 Cycles sur la variété de Steinberg généralisée

Soit  $x = (\iota(y), z) = ((\rho_p = (\rho_{\bar{v}})_{v \in S_p}, \underline{\delta} = \prod_{v \in S_p} \underline{\delta}_v), z)$  un point sur la variété de Hecke-Taylor-Wiles  $X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g)$  avec  $\rho_{\bar{v}}$  cristalline générique comme dans §2.1.6. Écrivons  $r_x = (\rho_p, z) \in \mathfrak{X}_{\infty} = \text{Spf}(R_{\infty})^{\text{rig}}$ . Supposons également que  $\underline{\delta} = \underline{\delta}_{\mathcal{R}, w_0}$  pour un raffinement  $\mathcal{R}$  de  $\rho_p$  et que  $w_0$  soit l'élément le plus long de  $W$ . Écrivons  $x_w = (r_x, \underline{\delta}_{\mathcal{R}, w}) \in \mathfrak{X}_{\infty} \times \widehat{T}_{p, L}$  pour les points compagnons qui possiblement ne sont pas dans  $X_p(\bar{\rho})$ . Rappelons que dans §2.1.6, nous avons défini des cycles  $[M(w w_0 \cdot \lambda)]$  et  $[L(w w_0 \cdot \lambda)]$  dans  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$ . Si les cycles sont non vides, alors les points compagnons correspondants  $x_w$  sont dans la variété de Hecke (voir 2.1.12) ou les constituants compagnons correspondants apparaissent dans  $\Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}}$  (voir 2.1.11).

Nous définissons maintenant les cycles de Galois  $\mathcal{Z}_{P, w}$  dans  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$  qui seront comparés à  $[L(w w_0 \cdot \lambda)]$  et sont issus de  $\text{Spec}(\widehat{\mathcal{O}}_{X_P, y_{\text{pdR}}})$  via la théorie des modèles locaux (théorème 2.2.4) et les applications

$$\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x}) \rightarrow \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{p}^p}, \rho_p}) \leftarrow \text{Spec}(\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p), y}) \simeq \text{Spec}(\widehat{\mathcal{O}}_{X_{P, w_0}, y_{\text{pdR}}}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{X_P, y_{\text{pdR}}}) \quad (2.2.5)$$

où le fait que l'application  $\text{Spec}(\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p), y}) \rightarrow \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{p}^p}, \rho_p})$  est un plongement fermé est dû à l'hypothèse de généricité.

Soit  $\mathcal{N}$  la sous-variété des matrices nilpotentes de  $\mathfrak{g}$  et soit  $\mathfrak{n}_B, \mathfrak{n}_P$  les radicaux de  $\mathfrak{b}, \mathfrak{p}$ . Le point  $y_{\text{pdR}} \in X_P$  appartient à une sous-variété, la variété généralisée de Steinberg, définie par

$$Z_P = \{(\nu, g_1 B, g_2 P) \in \mathcal{N} \times G/B \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_B, \text{Ad}(g_2^{-1})\nu \in \mathfrak{n}_P\}.$$

Semblable à  $X_P$ , la variété algébrique  $Z_P$  est équidimensionnelle avec des composantes irréductibles  $Z_{P, w}$  paramétrées par  $w \in W/W_P$ . Alors  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w}, y_{\text{pdR}}})$  définit un cycle (sous-espace fermé) dans  $\text{Spec}(\widehat{\mathcal{O}}_{X_P, y_{\text{pdR}}})$  et sont construits à partir de  $\mathcal{Z}_{P, w} \subset \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$  via (2.2.5).

La conjecture des points compagnons et des constituants compagnons (Conjecture 2.1.4 et 2.1.5) est vraie du côté de modèle local dans le sens où  $y_{\text{pdR}}$  existe dans  $X_{P, w}$  ou  $Z_{P, w}$  (équivalent à  $\mathcal{Z}_{P, w} \neq \emptyset$ ) si et seulement si  $w \geq w_{\mathcal{R}}$  dans  $W/W_P$ .

### 2.2.3 Propriété de partiellement classique et partiellement de Rham

Le but est de démontrer que  $[L(w w_0 \cdot \lambda)] \neq \emptyset$  pour  $w \geq w_{\mathcal{R}}$  sous l'hypothèse  $\mathcal{Z}_{P, w} \neq \emptyset$ . Le point de départ de la preuve est que si le point compagnon  $x_w$  existe (par exemple  $x_{w_0}$  selon notre hypothèse), le cycle  $[M(w w_0 \cdot \lambda)]$  est non vide et correspond au cycle du modèle local défini par le sous-schéma  $\overline{X}_{P, w} := \{(\nu, g_1 B, g_2 P) \in X_{P, w} \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_B\}$  (cf. la fin de §2.1.6). On peut exprimer  $[M(w w_0 \cdot \lambda)]$  comme une réunion de  $[L(w' w_0 \cdot \lambda)]$  et comme une réunion de  $\mathcal{Z}_{P, w'}$ . En tant que sous-ensembles fermés de  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$ , nous avons

$$[M(w w_0 \cdot \lambda)] = \cup_{w \geq w' \geq w_{\mathcal{R}}} [L(w' w_0 \cdot \lambda)] = \cup_{w \geq w' \geq w_{\mathcal{R}}} \mathcal{Z}_{P, w'}.$$

Un exemple d'utilisation de la formule ci-dessus est la première étape de la preuve de Breuil-Hellmann-Schraen dans les cas réguliers. Soit  $P = B$ ,  $w = w_0$  et supposons  $w_0 \neq w_{\mathcal{R}}$ , alors  $L(\lambda)$  est une représentation algébrique de dimension finie de  $G_p$ . De plus, le cycle  $[L(w_0 w_0 \cdot \lambda)]$  est, par définition, le lieu où  $\text{Hom}_{U(\mathfrak{g})}(L(\lambda), \Pi_{\infty}^{\text{an}}[\mathfrak{m}_{r_x}^{\infty}]) \neq 0$  et correspond à l'apparition de constituants compagnons localement algébriques. Par un résultat de compatibilité locale-globale,  $[L(w_0 w_0 \cdot \lambda)]$  est contenu dans le lieu où  $\rho_p = (\rho_{\bar{v}})_{v \in S_p}$  est de Rham. Cependant, le lieu de de Rham est la préimage du sous-espace de  $Z_P$  défini par  $\nu = 0$  à partir des modèles locaux, qui est exactement  $\mathcal{Z}_{P, w_0}$ . Nous concluons que  $[L(\lambda)] \subset \mathcal{Z}_{P, w_0}$ . D'où le fait que l'existence de  $w' \neq w_0, w' \geq w_{\mathcal{R}}$

tels que  $\mathcal{Z}_{P,w'} \neq 0$  implique qu'il existe  $w' \neq w_0$  tel que  $[L(w'w_0 \cdot \lambda)] \neq \emptyset$ . Alors  $[M(w'w_0 \cdot \lambda)] \neq \emptyset$ , le point compagnon  $x_{w'}$  existe et le constituant compagnon apparaît.

Dans les cas non-réguliers, aucune sous-représentation localement algébrique non nulle n'existe dans  $\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}$ . En remplacement, nous utilisons les constituants compagnons ou cycles partiellement classiques, qui du côté de Galois, correspondent à des  $(\varphi, \Gamma)$ -modules partiellement de Rham sur les anneaux de Robba. Le mot "classique" signifie des formes automorphes classiques qui correspondent à des vecteurs localement algébriques dans la cohomologie complétée.

Soit  $Q = \prod_{v \in S_p} Q_v = \prod_{\tau \in \Sigma_p} Q_\tau \subset G = \prod_{\tau \in \Sigma_p} \text{GL}_{n/L}$  un sous-groupe parabolique de matrices triangulaires supérieur par bloc avec le sous-groupe de Levi  $M_Q$  contenant  $T$ . Soit  $\mathfrak{q} = \prod_{\tau \in \Sigma_p} \mathfrak{q}_\tau \subset \mathfrak{g}$  et  $\mathfrak{m}_Q = \prod_{\tau \in \Sigma_p} \mathfrak{m}_{Q_\tau}$  leurs algèbres de Lie. Le théorème suivant dit que l'apparition de constituants partiellement classiques implique que les représentations galoisiennes correspondantes sont partiellement de Rham.

**Théorème 2.2.6.** *Soit  $x = (r_x, \underline{\delta}) = ((\rho_p = (\rho_{\tilde{v}})_{v \in S_p}, \underline{\delta} = \prod_{v \in S_p} \underline{\delta}_v), z) \in \mathfrak{X}_\infty \times \hat{T}_{p,L}$  un  $L$ -point sur la variété de Hecke-Taylor-Wiles tel que  $\underline{\delta}$  est localement algébrique et générique (hypothèse 2.2.3). Soit  $\underline{\delta}_{\text{sm}}$  la partie lisse de  $\underline{\delta}$ . Pour  $v \in S_p$ , soit*

$$0 = \text{Fil}_0 D_{\text{rig}}(\rho_{\tilde{v}}) \subsetneq \cdots \subsetneq \text{Fil}_i D_{\text{rig}}(\rho_{\tilde{v}}) \subsetneq \cdots \subsetneq \text{Fil}_n D_{\text{rig}}(\rho_{\tilde{v}}) = D_{\text{rig}}(\rho_{\tilde{v}})$$

*l'unique triangulation de  $D_{\text{rig}}(\rho_{\tilde{v}})$  de paramètre le produit de  $\iota_v^{-1}(\underline{\delta}_v)$  et un caractère algébrique. Supposons que  $\eta \in X^*(\mathfrak{t})$  est un poids dominant de  $M_Q$  et*

$$\text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\overline{L}(-\eta), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}]) \neq 0. \quad (2.2.7)$$

*Alors pour tout  $\tau \in \Sigma_v, v \in S_p$  et si  $M_{Q_\tau} = \text{diag}(\text{GL}_{n_1}, \dots, \text{GL}_{n_t})$  où  $n_1 + \dots + n_t = n$ , pour tout  $1 \leq i \leq t$ , on a que le  $(\varphi, \Gamma_{F_{\tilde{v}}})$ -module*

$$\text{gr}_i^{Q_\tau} D_{\text{rig}}(\rho_{\tilde{v}}) := \text{Fil}_{n_1+\dots+n_i} D_{\text{rig}}(\rho_{\tilde{v}}) / \text{Fil}_{n_1+\dots+n_{i-1}} D_{\text{rig}}(\rho_{\tilde{v}})$$

*est  $\{\tau\}$ -de Rham, c'est-à-dire que l'opérateur nilpotent  $\nu_v$  est nul sur*

$$D_{\text{pdR}}(\text{gr}_i^{Q_\tau} D_{\text{rig}}(\rho_{\tilde{v}})) \otimes_{L \otimes_{\mathbb{Q}_p} F_{\tilde{v}}, 1 \otimes \tau} L.$$

On dit que les points  $x$  (ou les représentations galoisiennes  $\rho_p = (\rho_{\tilde{v}})_{v \in S_p}$ ) sont partiellement  $Q$ -de Rham si la conclusion du théorème ci-dessus est satisfait. Supposons que  $\mathfrak{q} = \text{Lie}(Q_p) \otimes_{\mathbb{Q}_p} L$  pour un sous-groupe parabolique  $Q_p \subset G_p$  de Levi  $M_{Q_p}$ . Si  $\eta \in X^*(\mathfrak{t})$  est un poids qui est dominant de  $M_Q$ , alors

$$\text{Hom}_{G_p}(\mathcal{F}_{B_p}^{G_p}(\overline{L}(-\eta), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}]) \neq 0 \Rightarrow \text{Hom}_{U(\mathfrak{m}_Q)}(L_{M_Q}(\eta), J_{Q_p}(\Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}])) \neq 0$$

où la représentation localement analytique  $J_{Q_p}(\Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}])$  de  $M_{Q_p}$  est la version parabolique du module de Jacquet d'Emerton, et  $L_{M_Q}(\eta)$  est la représentation irréductible de dimension finie de  $\mathfrak{m}_{Q_p}$  du poids le plus élevé  $\eta$  qui s'intègre à une représentation algébrique de  $M_{Q_p}$ . Le théorème ci-dessus montre que l'existence de vecteurs localement algébriques non nuls dans le module parabolique de Jacquet d'Emerton implique que les gradués des filtrations paraboliques des  $(\varphi, \Gamma)$ -modules sur les anneaux de Robba sont de Rham. Ainsi, le théorème est une généralisation naturelle du résultat de compatibilité locale-globale classique lorsque  $Q = G$ . Et la preuve est basée sur les résultats pour les points classiques sur la variété de Hecke partielle construite par Ding dans [Din19c] et par la triangulation globale.

Rappelons que  $W_P$  est le stabilisateur des poids de Hodge-Tate  $\mathbf{h}$  et  $\lambda = w_0(\mathbf{h}) + (0, \dots, n-1)_{\tau \in \Sigma_p}$ . Soit  $\mathfrak{n}_Q$  le radical nil de  $\mathfrak{q}$  et  $\mathfrak{q} = \mathfrak{m}_Q + \mathfrak{n}_Q$ . Le théorème suivant dit que les cycles  $\mathcal{Z}_{P,w}$  sont partiellement  $Q$ -de Rham si  $ww_0 \cdot \lambda$  est un poids dominant de  $M_Q$ .

**Théorème 2.2.8.** *Pour chaque  $w \in W/W_P$ , la composante irréductible  $Z_{P,w}$  est contenue dans le sous-espace de  $Z_P$  où  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q$  si et seulement si  $ww_0 \cdot \lambda$  est un poids dominant pour  $\mathfrak{m}_Q$ .*

En combinant le théorème 2.2.6 et le théorème 2.2.8, on peut faire “correspondre” certains  $Z_{P,w}$  et  $[L(ww_0 \cdot \lambda)]$  dans la preuve du théorème principal. Et nous pouvons démontrer l’existence de tous les points compagnons  $(r_x, \underline{\delta}_{\mathcal{R},w}), w \geq w_{\mathcal{R}}$  sur  $X_p(\bar{\rho})$  pour le raffinement  $\mathcal{R}$  tel que  $(r_x, \underline{\delta}_{\mathcal{R},w_0}) \in X_p(\bar{\rho})$ .

## 2.2.4 Points compagnons pour différents raffinements

Soit  $\mathcal{R}' \neq \mathcal{R}$  un autre raffinement. L’étape précédente nous permet de démontrer l’existence de  $(r_x, \underline{\delta}_{\mathcal{R},w}), w \geq w_{\mathcal{R}}$  dans  $X_p(\bar{\rho})$  et dans cette étape, nous montrons que  $(r_x, \underline{\delta}_{\mathcal{R}',w_0})$  est également dans  $X_p(\bar{\rho})$ . Pour expliquer l’idée, nous supposons que  $n = 2$  et  $S_p = \{v\}$  consistent en une place unique et supposons  $\Sigma_p = \{\tau_1, \tau_2\}$ . Supposons que les poids Hodge-Tate de  $\rho_{\bar{v}}$  soient  $h_{\tau_1,1} = h_{\tau_1,2} = 0, 0 = h_{\tau_2,1} < 1 = h_{\tau_2,2}$ . D’après les résultats précédents, on peut supposer que le point  $x = (\rho_{\bar{v}}, \iota(\delta_1, \delta_2), z) \in \mathfrak{X}_{\bar{\rho}_p} \times \hat{T}_{p,L} \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  est dans  $X_p(\bar{\rho})$  où  $y := (\rho_{\bar{v}}, (\delta_1, \delta_2))$  est dans le sous-ensemble ouvert  $U_{\text{tri}}(\bar{\rho}_p)$  de  $X_{\text{tri}}(\bar{\rho}_p)$  en prenant  $w = w_{\mathcal{R}}$ . En d’autres termes,  $D_{\text{rig}}(\rho_{\bar{v}}) = [\mathcal{R}_{L,F_{\bar{v}}}(\delta_1) - \mathcal{R}_{L,F_{\bar{v}}}(\delta_2)]$  est une extension de  $\mathcal{R}_{L,F_{\bar{v}}}(\delta_2)$  par  $\mathcal{R}_{L,F_{\bar{v}}}(\delta_1)$  qui donne la triangulation de  $\rho_{\bar{v}}$  du paramètre  $\underline{\delta}$ . Dans cet exemple, nous supposons  $\delta_1 = \tau_2 \text{unr}(a)$  et  $\delta_2 = \text{unr}(b)$  où nous écrivons  $\tau_2$  pour le caractère  $\tau_2 : K^\times \hookrightarrow L^\times$  et  $\text{unr}(a), \text{unr}(b)$  pour les caractères non ramifiés de  $F_{\bar{v}}^\times$  envoyant  $\varpi_{F_{\bar{v}}}$  à  $a, b \in L^\times$ .

L’idée est d’utiliser les résultats dans le cas régulier. La stratégie consiste à trouver  $y^i := (\rho_{\bar{v}}^i, (\delta_1^i, \delta_2^i)) \in U_{\text{tri}}(\bar{\rho}_p)$  (donc  $D_{\text{rig}}(\rho_{\bar{v}}^i) = [\mathcal{R}_{L,F_{\bar{v}}}(\delta_1^i) - \mathcal{R}_{L,F_{\bar{v}}}(\delta_2^i)]$ ) pour  $i \in \mathbb{N}$  tel que  $y^i$  ont des poids de Hodge-Tate réguliers et convergent vers  $y$  au sens de la géométrie analytique rigide, et de sorte que certains points compagnons  $(y^i)'$  de  $y^i$  convergent vers

$$y' := (\rho_{\bar{v}}, (\tau_2 \text{unr}(b), \text{unr}(a)))$$

dans  $X_{\text{tri}}(\bar{\rho}_p)$ . Puis sur la variété de Hecke-Taylor-Wiles  $X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$ , les points  $(\iota(y^i), z)$  sont dans  $X_p(\bar{\rho})$  pour  $i$  grand en utilisant que  $X_{\text{tri}}(\bar{\rho}_p)$  est lisse en  $y$  et que leurs points compagnons  $(\iota((y^i)'), z)$  sont sur  $X_p(\bar{\rho})$  par les résultats de Breuil-Hellmann-Schraen dans des cas réguliers. Par conséquent, le point compagnon  $(\iota(y'), z)$ , comme une limite de  $(\iota((y^i)'), z)$ , existe également sur  $X_p(\bar{\rho})$  puisque ce dernier est fermé dans  $\iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$ . Pour réaliser ce plan, on cherche des points  $y^i = (\rho_{\bar{v}}^i, (\delta_1^i, \delta_2^i))$  dans  $U_{\text{tri}}(\bar{\rho}_p)$  près de  $y$  qui satisfont les deux conditions suivantes.

1.  $\rho_{\bar{v}}^i$  sont cristallins.
2.  $(\delta_1^i, \delta_2^i) = (\tau_1^{-i} \tau_2 \text{unr}(a^i), \text{unr}(b^i))$  où  $i \geq 1, a^i, b^i \in L^\times$ .

Alors les points compagnons avec des poids dominants  $(y^i)' = (\rho_{\bar{v}}^i, (\tau_2 \text{unr}(b^i), \tau_1^{-i} \text{unr}(a^i)))$  existent sur  $X_{\text{tri}}(\bar{\rho}_p)$  et convergent vers  $y'$  puisque

$$(\tau_2 \text{unr}(b^i), \tau_1^{-i} \text{unr}(a^i)) = (\tau_2, \tau_2^{-1})(\text{unr}(b^i), \tau_2 \tau_1^{-i} \text{unr}(a^i))$$

convergent vers  $(\tau_2, \tau_2^{-1})(\delta_2, \delta_1) = (\tau_2 \text{unr}(b), \text{unr}(a))$  par l’hypothèse.

Nous expliquons maintenant comment trouver des extensions  $D_{\text{rig}}(\rho_{\bar{v}}^i) = [\mathcal{R}_{L,F_{\bar{v}}}(\delta_1^i) - \mathcal{R}_{L,F_{\bar{v}}}(\delta_2^i)]$  proche de  $[\mathcal{R}_{L,F_{\bar{v}}}(\delta_1) - \mathcal{R}_{L,F_{\bar{v}}}(\delta_2)]$ . La condition (2) pour  $(\delta_1^i, \delta_2^i)$  ci-dessus n’est pas difficile à réaliser. Le problème est d’obtenir les extensions cristallines (ou de Rham) avec ces poids non dominants.

Le lemme suivant donne un critère de Rham pour ces extensions.

**Lemma 2.2.9.** *Soient  $(\eta_1, \eta_2) : (K^\times)^2 \rightarrow L^\times$  des caractères localement algébriques et  $d \in \mathbb{Z}_{\geq 1}$  tels que  $\text{wt}_{\tau_1}(\eta_1) - \text{wt}_{\tau_1}(\eta_2) \in \mathbb{Z}_{\leq 0}$  et  $\text{wt}_{\tau_2}(\eta_1) - \text{wt}_{\tau_2}(\eta_2) = d$ . Supposons que  $(\eta_1, \eta_2)$  est générique au sens de l'hypothèse 2.2.3. Alors une extension  $[\mathcal{R}_{L, F_{\bar{v}}}(\eta_1) - \mathcal{R}_{L, F_{\bar{v}}}(\eta_2)]$  est de Rham si et seulement si l'extension correspond à une classe dans*

$$\text{Ker}(H_{(\varphi, \Gamma_{F_{\bar{v}}})}^1(\mathcal{R}_{L, F_{\bar{v}}}(\eta_1 \eta_2^{-1})) \rightarrow H_{(\varphi, \Gamma_{F_{\bar{v}}})}^1(t_{\tau_2}^{-d} \mathcal{R}_{L, F_{\bar{v}}}(\eta_1 \eta_2^{-1}))). \quad (2.2.10)$$

Ici  $t_{\tau_2}$  est un générateur du sous- $(\varphi, \Gamma_{F_{\bar{v}}})$ -module  $\mathcal{R}_{L, F_{\bar{v}}}(\tau_2)$  de  $\mathcal{R}_{L, F_{\bar{v}}}$ . De plus, le noyau ci-dessus est de dimension 1.

Ensuite, les extensions requises sont sélectionnées dans le sous-espace de l'espace des modules de toutes les extensions  $[\mathcal{R}_{L, F_{\bar{v}}}(\eta_1) - \mathcal{R}_{L, F_{\bar{v}}}(\eta_2)]$  donné par condition (2.2.10) et  $\text{wt}_{\tau_2}(\eta_1) - \text{wt}_{\tau_2}(\eta_2) = 1$ .



## Chapter 3

# Local models of the trianguline variety and partially classical families

### 3.1 Introduction

Let  $p$  be a prime number. This paper concerns about  $p$ -adic automorphic forms of definite unitary groups and the locally analytic aspect of the  $p$ -adic local Langlands program. Its aim is to generalize several results of Breuil-Hellmann-Schraen in [BHS19] (local model for the trianguline variety, existence of companion points on the eigenvariety, locally analytic socle conjecture, etc.) to the cases when the Hodge-Tate weights are non-regular (i.e., not pairwise distinct).

#### 3.1.1 Companion points and main results

Let  $F^+$  be a totally real number field and  $S_p$  be the set of places of  $F^+$  above  $p$ . Let  $F$  be a quadratic imaginary extension of  $F^+$  such that every place in  $S_p$  splits in  $F$ ,  $n \geq 2$  be an integer and  $\mathbb{G}$  be a totally definite unitary group in  $n$  variables over  $F^+$  that is split over  $F$ . We fix an open compact subgroup  $U^p = \prod_{v \nmid p} U_v$  of  $\mathbb{G}(\mathbf{A}_{F^+}^{p\infty})$  and a finite extension  $L$  of  $\mathbb{Q}_p$  with residue field  $k_L$ . For all  $v \in S_p$ , let  $\Sigma_v := \{\tau : F_v^+ \hookrightarrow L\}$  and we assume  $|\Sigma_v| = [F_v^+ : \mathbb{Q}_p]$ . For each  $v \in S_p$ , we fix a place  $\tilde{v}$  of  $F$  above  $v$  and identify  $F_v^+ \simeq F_{\tilde{v}}^+$ . The space of  $p$ -adic automorphic forms on  $\mathbb{G}$  of tame level  $U^p$ , denoted by  $\hat{S}(U^p, L)$ , consists of continuous functions  $\mathbb{G}(F^+) \setminus \mathbb{G}(\mathbf{A}_{F^+}^\infty)/U^p \rightarrow L$ . Let  $G_p = \prod_{v \in S_p} G_v$  be the  $p$ -adic Lie group  $\mathbb{G}(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) = \prod_{v \in S_p} \mathbb{G}(F_v^+)$ . Let  $B_p = \prod_{v \in S_p} B_v$  (resp.  $T_p = \prod_{v \in S_p} T_v$ ) be the Borel subgroup (resp. the maximal torus) of  $G_p \simeq \prod_{v \in S_p} \mathrm{GL}_n(F_v^+)$  consisting of upper-triangular (resp. diagonal) matrices. Then  $G_p$  acts on  $\hat{S}(U^p, L)$  via right translations. We assume furthermore that  $p > 2$  and  $\mathbb{G}$  is quasi-split at all finite places of  $F^+$ . Let  $\bar{F}$  be an algebraic closure of  $F$ . We fix a (modular) absolutely irreducible Galois representation  $\bar{\rho} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_n(k_L)$  so that  $\bar{\rho}$  is associated with a maximal ideal of some usual Hecke algebra acting on  $\hat{S}(U^p, L)$ .

After Emerton [Eme06c], one way to construct eigenvarieties, rigid analytic varieties parameterizing finite slope overconvergent  $p$ -adic eigenforms, is using Emerton's Jacquet module functor for locally analytic representations of  $p$ -adic Lie groups. There exists a rigid space over  $L$  (our eigenvariety), denoted by  $Y(U^p, \bar{\rho})$ , on which a point is a pair  $(\rho, \underline{\delta})$ , where  $\rho$  is a  $p$ -adic continuous  $n$ -dimensional representation of  $\mathrm{Gal}(\bar{F}/F)$  and  $\underline{\delta}$  is a continuous character of  $T_p$  which appears in  $J_{B_p}(\Pi(\rho)^{\mathrm{an}})$ . Here  $\Pi(\rho)$  is the sub- $G_p$ -representation of  $\hat{S}(U^p, L)$  associated with  $\rho$  cut out by a prime ideal of certain Hecke algebra,  $\Pi(\rho)^{\mathrm{an}}$  is the subspace of  $\Pi(\rho)$  consisting of locally analytic vectors which is an admissible locally analytic representation of  $G_p$  and  $J_{B_p}(-)$  denotes the Emerton's Jacquet module functor so that  $J_{B_p}(\Pi(\rho)^{\mathrm{an}})$  is a locally analytic representation of the Levi subgroup  $T_p$  of  $B_p$ .

Take a point  $(\rho, \underline{\delta})$  on  $Y(U^p, \bar{\rho})$ . The problem of companion forms seeks to determine the set of characters  $\underline{\delta}'$  of  $T_p$ , denoted by  $W(\rho)$ , such that pairs  $(\rho, \underline{\delta}')$  appear on  $Y(U^p, \bar{\rho})$ . The existence of such *companion points*  $(\rho, \underline{\delta}')$  is closely related to the appearance of certain irreducible locally analytic representations of  $G_p$  explicitly determined by  $\underline{\delta}'$  and  $\rho$ , which we call *companion constituents*, inside  $\Pi(\rho)^{\text{an}}$ . The existence of such companion constituents is a special case of the locally analytic socle conjecture of Breuil [Bre16, Bre15b]. For  $v \in S_p$ , we let  $\rho_v := \rho|_{\text{Gal}(\overline{F_v^+}/F_v^+)}$ . The general recipe for  $W(\rho)$  has been conjectured by Hansen [HN17] which depends only on those local Galois representations  $\rho_v$  for  $v \in S_p$  and the notion of *trianguline representations* introduced by Colmez [Col08]. One could view the problem of companion forms or locally analytic socles as a locally analytic analogue of the weight part of Serre's modularity conjecture.

Let  $D_{\text{rig}}(\rho_v)$  be the étale  $(\varphi, \Gamma)$ -module over the Robba ring associated with  $\rho_v$  for  $v \in S_p$ . In the  $p$ -adic local Langlands program, locally analytic representations of  $p$ -adic Lie groups are expected to be related to  $(\varphi, \Gamma)$ -modules over the Robba rings which is the case for  $\text{GL}_2(\mathbb{Q}_p)$  by Colmez [Col10, V]. Beyond the foundational works of Kisin, Colmez and Emerton for  $\text{GL}_2(\mathbb{Q}_p)$  [Kis03, Col08, Eme11], we know in general and especially in our setting by the global triangulation results of Liu [Liu15] or Kedlaya-Porttharst-Xiao [KPX14] that the non-triviality of the Borel Emerton's Jacquet module  $J_{B_p}(\Pi(\rho)^{\text{an}})$  (i.e. in the finite slope case) implies that  $\rho_v$  is trianguline, i.e.  $D_{\text{rig}}(\rho_v)$  admits a full filtration

$$\text{Fil}^\bullet D_{\text{rig}}(\rho_v) : D_{\text{rig}}(\rho_v) = \text{Fil}^n D_{\text{rig}}(\rho_v) \supsetneq \cdots \supsetneq \text{Fil}^1 D_{\text{rig}}(\rho_v) \supsetneq \text{Fil}^0 D_{\text{rig}}(\rho_v) = \{0\} \quad (3.1.1)$$

of sub- $(\varphi, \Gamma)$ -modules such that the graded pieces are rank one  $(\varphi, \Gamma)$ -modules.

Under the Taylor-Wiles hypothesis on  $\bar{\rho}$ , Breuil-Hellmann-Schraen proved in [BHS19] the existence of all companion forms for regular generic crystalline points. In this paper, we generalize their results to non-regular generic crystalline points. To be precise, we take a point  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$ . We say  $\rho$  (or the point  $(\rho, \underline{\delta})$ ) is crystalline if for all  $v \in S_p$ ,  $\rho_v$  is crystalline. If  $\rho$  is crystalline, let  $(\varphi_{v,i})_{i=1,\dots,n}$  be the eigenvalues of  $\varphi^{f_v}$  where  $\varphi$  is the crystalline Frobenius acting on  $D_{\text{cris}}(\rho_v)$  and  $q_v = p^{f_v}$  is the cardinality of the residue field of  $F_v^+$ . Then we say  $\rho$  (or the point  $(\rho, \underline{\delta})$ ) is *generic* if for any  $v \in S_p$ ,  $\varphi_{v,i} \varphi_{v,j}^{-1} \notin \{1, q_v\}$  for  $i \neq j$ . Assume that  $\rho$  is generic crystalline. A refinement  $\mathcal{R}_v$  of  $\rho_v$  is a choice of an ordering of the pairwise distinct eigenvalues  $\varphi_{v,1}, \dots, \varphi_{v,n}$  and a refinement  $\mathcal{R} = (\mathcal{R}_v)_{v \in S_p}$  of  $\rho$  is a choice of a refinement  $\mathcal{R}_v$  for each  $v \in S_p$ . In fact the refinements  $\mathcal{R}_v$  correspond to triangulations of  $D_{\text{rig}}(\rho_v)$  as (3.1.1) by [Ber08b]. Then the conjectural set of characters  $W(\rho)$  admits a partition  $W(\rho) = \bigsqcup_{\mathcal{R}} W_{\mathcal{R}}(\rho)$  where  $W_{\mathcal{R}}(\rho) = \prod_{v \in S_p} W_{\mathcal{R}_v}(\rho_v)$  and each  $W_{\mathcal{R}_v}(\rho_v)$  is a finite set which can be explicitly described by  $\mathcal{R}_v$  and  $\rho$ . Remark that the partition of  $W(\rho)$  according to the refinements is also the partition under the equivalence relation that  $\underline{\delta} \sim \underline{\delta}'$  if and only if  $\underline{\delta}^{-1} \underline{\delta}'$  is a  $\mathbb{Q}_p$ -algebraic character of  $T_p$ . Our main theorem is the following.

**Theorem 3.1.2** (Theorem 3.4.20). *Assume that  $U^p$  is small enough and assume the Taylor-Wiles hypothesis (cf. §3.4.2):  $F$  is unramified over  $F^+$ ,  $F$  doesn't contain non-trivial  $p$ -th root of unity,  $U^p$  is hyperspecial at any finite place of  $F^+$  that is inert in  $F$  and  $\bar{\rho}(\text{Gal}(\bar{F}/F(\sqrt[p]{1})))$  is adequate. Let  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  be a point such that  $\rho$  is generic crystalline. Then there exists a refinement  $\mathcal{R}$  of  $\rho$  such that  $\underline{\delta} \in W_{\mathcal{R}}(\rho)$  and for any  $\underline{\delta}' \in W_{\mathcal{R}}(\rho)$ , the point  $(\rho, \underline{\delta}')$  exists on  $Y(U^p, \bar{\rho})$ . Moreover, all the companion constituents associated with  $W_{\mathcal{R}}(\rho)$  appear in  $\Pi(\rho)$ .*

**Remark 3.1.3.** In [BHS19], the above theorem was proved under the extra assumption that for each  $v \in S_p$ , the Hodge-Tate weights of  $\rho_v$  are regular (pairwise distinct). But a stronger version was proved in [BHS19]: in regular cases,  $(\rho, \underline{\delta}')$  exists on  $Y(U^p, \bar{\rho})$  for any refinement  $\mathcal{R}'$  of  $\rho$  and  $\underline{\delta}' \in W_{\mathcal{R}'}(\rho)$ . This stronger result is easy to get from Theorem 3.1.2 in regular cases using locally algebraic vectors in  $\Pi(\rho)$  and is not available in this paper for general crystalline points due to the non-existence of non-zero locally algebraic vectors in  $\Pi(\rho)$  when  $\rho$  is non-regular (the

non-existence can be seen using the results on infinitesimal characters in [DPS20]). See Remark 3.5.37 for a partial result. The existence of all companion points in generic non-regular crystalline cases will need other methods.

The method in [BHS19] was firstly replacing the eigenvariety  $Y(U^p, \bar{\rho})$  by a larger *patched eigenvariety*  $X_p(\bar{\rho})$  in [BHS17b, BHS17a] constructed from the patching module in [CEG<sup>+</sup>16]. The patching method allows us to reduce the study of the geometry of the patched eigenvariety to that of its local component, called *trianguline variety*, which parameterizes local trianguline Galois representations. Then Breuil-Hellmann-Schraen used a local model to describe the local geometry of the trianguline variety at certain points. We prove Theorem 3.1.2 by developing further the theory of local models. The major new inputs are the following two results.

Firstly, we construct local models of the trianguline variety for certain points with possibly non-regular Hodge-Tate weights and prove that the trianguline variety is irreducible at those points (Theorem 3.1.6). Those local models are algebraic varieties which are similar to the regular cases and reflect the phenomenon of the existence of companion points or companion constituents on the eigenvariety or in the space of  $p$ -adic automorphic forms.

Secondly, we show that for a general point  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  where  $\rho$  may not be de Rham above  $p$ , the existence of certain companion constituents, which we call *partially classical* constituents, will force the local Galois representations  $\rho_v, v \in S_p$  satisfy certain special properties for which we say  $\rho_v$  are *partially de Rham* (Theorem 3.1.8). The partially classical constituents are locally analytic representations of  $G_p$  which will give rise to the existence of certain *locally algebraic vectors* inside some non-Borel parabolic Emerton's Jacquet module  $J_{Q_p}(\Pi(\rho)^{\text{an}})$  of  $\Pi(\rho)^{\text{an}}$ , where  $Q_p$  is some parabolic subgroup of  $G_p$  containing  $B_p$  and is not equal to  $B_p$ . Let  $M_{Q_p}$  be the Levi subgroup of  $Q_p$  containing  $T_p$ . Then  $J_{Q_p}(\Pi(\rho)^{\text{an}})$  is a locally analytic representation of  $M_{Q_p}$  which, in analogue with the case of Borel Emerton's Jacquet module, should correspond to some so called (after Chenevier [Che11], see also [Ber17]) *paraboline filtrations* of  $D_{\text{rig}}(\rho_v)$

$$\text{Fil}_{Q_p}^\bullet D_{\text{rig}}(\rho_v) : D_{\text{rig}}(\rho_v) = \text{Fil}_{Q_p}^{t_v} D_{\text{rig}}(\rho_v) \supsetneq \cdots \supsetneq \text{Fil}_{Q_p}^1 D_{\text{rig}}(\rho_v) \supsetneq \text{Fil}_{Q_p}^0 D_{\text{rig}}(\rho_v) = \{0\}, v \in S_p$$

where the ranks of the graded pieces of the above filtrations should be sizes of the blocks of the Levi subgroup  $M_{Q_p}$ . Since we are always in the finite slope cases, we only focus on those paraboline filtrations that are sub-filtrations of the trianguline filtrations (3.1.1). This means that there exist integers  $0 = s_{v,0} < s_{v,1} < \cdots < s_{v,t_v-1} < s_{v,t_v} = n$  such that

$$\text{Fil}_{Q_p}^i D_{\text{rig}}(\rho_v) = \text{Fil}^{s_{v,i}} D_{\text{rig}}(\rho_v).$$

From  $(\varphi, \Gamma)$ -modules over the Robba rings one can always obtain semi-linear Galois representations over Fontaine's ring  $B_{\text{dR}}$  after Berger (e.g. [Ber08a]), thus we can define the de Rham property for  $(\varphi, \Gamma)$ -modules as for  $p$ -adic Galois representations. Our result then states that the appearance of certain locally algebraic vectors in  $J_{Q_p}(\Pi(\rho)^{\text{an}})$  implies that the graded pieces

$$\text{Fil}^{s_{v,i}} D_{\text{rig}}(\rho_v) / \text{Fil}^{s_{v,i-1}} D_{\text{rig}}(\rho_v)$$

are de Rham  $(\varphi, \Gamma)$ -modules for  $i = 1, \dots, t_v, v \in S_p$ . Recall that locally algebraic vectors in  $\widehat{S}(U^p, L)$  with respect to the action of  $G_p$  are  $p$ -adic avatars of algebraic regular automorphic forms which correspond to  $p$ -adic Galois representations that are de Rham over  $p$  with regular Hodge-Tate weights. Hence the result on partially classical constituents can be viewed as a form of generalization with some functoriality of the classical correspondence, beyond ordinary cases ([Din19b], etc.).

*Remark 3.1.4.* Partial de Rhamness as well as partial classicality was proposed by Ding in a narrow sense for 2-dimensional Galois representations [Din17a, Din17b, Din19a] and partial classicality was also mentioned by Ding for his partial eigenvariety for  $\text{GL}_n(\mathbb{Q}_p)$  [Din19c] which we will use. Our results combine and generalize both Ding's works.



Our key step (Proposition 3.4.13) to prove the existence of the companion constituents or companion points for a point  $x = (\rho, \underline{\delta})$  as in Theorem 3.1.2 goes roughly in the following way (see also §3.1.4, especially Example 3.1.11). It will also simplify the relevant arguments in [BHS19] even for the regular case. As in [BHS19], there are cycles (closed subspaces) passing through  $x$  on the eigenvariety which correspond to the appearance of companion constituents in  $\Pi(\rho)^{\text{an}}$ . By the second result above, those cycles corresponding to partially classical constituents (with respect to some parabolic subgroups) are partially de Rham which means that the corresponding Galois representations are partially de Rham. On the other hand, the local model also gives rise to cycles near  $x$  on the patched eigenvariety, which are expected to match those cycles corresponding to companion constituents. The point is that the partially de Rham properties are determined by the datum of local models, and it turns out that the partially de Rham cycles on the local models are exactly those cycles that should match partially classical constituents (Theorem 3.1.9). Then a finer study of the local models tells that there exist non-partially de Rham cycles passing through  $x$  which implies the existence of non-partially classical companion constituents inside  $\Pi(\rho)$ . In this way, we can obtain all companion constituents that can be seen by the local models (those constituents in Theorem 3.1.2).

In the remaining parts of this introduction, we give more details on the above results and their proofs.

### 3.1.2 Local models for the trianguline variety

We now explain our local results on the trianguline variety. For  $v \in S_p$ , the trianguline variety  $X_{\text{tri}}(\bar{\rho}_v)$  with respect to  $\bar{\rho}_v := \bar{\rho} \mid_{\text{Gal}(\overline{F_v^+}/F_v^+)}$  is a rigid analytic variety, a point of which is given by a pair  $(r, \underline{\delta})$  where  $r$  is a deformation of  $\bar{\rho}_v$  and  $\underline{\delta} = (\delta_i)_{1 \leq i \leq n}$  is a character of  $T_v = ((F_v^+)^{\times})^n$ , such that the subset of points  $(r, \underline{\delta})$ , where  $r$  is trianguline and  $\underline{\delta}$  corresponds to the graded pieces of certain trianguline filtration of  $r$  as (3.1.1), is Zariski dense.

We take an  $L$ -point  $x = (r, \underline{\delta})$  of  $X_{\text{tri}}(\bar{\rho}_v)$ . The weight  $\text{wt}(\delta_i)$  of each character  $\delta_i$  is a number in  $F_v^+ \otimes_{\mathbb{Q}_p} L \simeq \bigoplus_{\tau \in \Sigma_v} L$  and we write  $\text{wt}_{\tau}(\delta_i) \in L$  for the  $\tau$ -part of  $\text{wt}(\delta_i)$  for each  $\tau \in \Sigma$ . The multiset  $\{\text{wt}_{\tau}(\delta_i) \mid i \in \{1, \dots, n\}, \tau \in \Sigma_v\}$  is also the  $\tau$ -Sen weights of  $r$  (the generalized Hodge-Tate weights, counted with multiplicities). Then  $\underline{\delta}$  is locally  $\mathbb{Q}_p$ -algebraic if for all  $i = 1, \dots, n, \tau \in \Sigma_v, \text{wt}_{\tau}(\delta_i) \in \mathbb{Z}$ .

We say  $\underline{\delta}$  is *generic* if for any  $i \neq j$ , both  $\delta_i^{-1}\delta_j$  and  $\delta_i^{-1}\delta_j|_{\text{Norm}_{F_v^+/\mathbb{Q}_p}}$ , where  $|p|_p = p^{-1}$ , are not  $\mathbb{Q}_p$ -algebraic characters (i.e. not of the form  $z \mapsto \prod_{\tau \in \Sigma_v} \tau(z)^{k_{\tau}}$  where  $k_{\tau} \in \mathbb{Z}$  for every  $\tau \in \Sigma_v$ ).

In the case when  $\underline{\delta}$  is generic and locally  $\mathbb{Q}_p$ -algebraic,  $r$  is almost de Rham in the sense of Fontaine [Fon04]. Fontaine's theory associates  $r$  with a finite free  $F_v^+ \otimes_{\mathbb{Q}_p} L$ -module  $D_{\text{pdR}}(r)$  of rank  $n$  and a linear nilpotent operator  $N$  acting on  $D_{\text{pdR}}(r)$ . The space  $D_{\text{pdR}}(r)$  is equipped with two filtrations, both are stable under the action of  $N$ . One filtration is the Hodge filtration denoted by  $\text{Fil}_{\bullet}$ . Another filtration  $\mathcal{D}_{\bullet} := D_{\text{pdR}}(\text{Fil}^{\bullet} D_{\text{rig}}(r))$  comes from a trianguline filtration  $\text{Fil}^{\bullet} D_{\text{rig}}(r)$  on  $D_{\text{rig}}(r)$  determined by the point  $x$  (we emphasize that the functor  $D_{\text{pdR}}(-)$  is also defined for these  $(\varphi, \Gamma)$ -modules).

For  $\tau \in \Sigma_v$ , define  $D_{\text{pdR}, \tau}(r) := D_{\text{pdR}}(r) \otimes_{L \otimes_{\mathbb{Q}_p} F_v^+, 1 \otimes \tau} L$ ,  $\text{Fil}_{\tau, \bullet} := \text{Fil}_{\bullet} \otimes_{L \otimes_{\mathbb{Q}_p} F_v^+, 1 \otimes \tau} L$  and  $\mathcal{D}_{\tau, \bullet} := \mathcal{D}_{\bullet} \otimes_{L \otimes_{\mathbb{Q}_p} F_v^+, 1 \otimes \tau} L$ . Then for each  $\tau \in \Sigma_v$ ,  $\mathcal{D}_{\tau, \bullet} : \mathcal{D}_{\tau, 1} \subsetneq \dots \subsetneq \mathcal{D}_{\tau, n}$  is a complete flag of the  $L$ -space  $D_{\text{pdR}, \tau}(r)$ . The graded pieces of the Hodge filtration  $\text{Fil}_{\tau, \bullet}$  have  $L$ -dimensions that are equal to the multiplicities of the  $\tau$ -Sen weights.

Let  $G := \text{Res}_{F_v^+/\mathbb{Q}_p}(\text{GL}_{n/F_v^+}) \otimes_{\mathbb{Q}_p} L = \prod_{\tau \in \Sigma_v} \text{GL}_{n/L}$  be the algebraic group which acts on  $D_{\text{pdR}}(r) \simeq (F_v^+ \otimes_{\mathbb{Q}_p} L)^n \simeq \prod_{\tau \in \Sigma_v} L^n$ ,  $P$  be the standard parabolic subgroup of block upper-triangular matrices in  $G$  that is conjugate to the stabilizer subgroup of the Hodge filtration  $\text{Fil}_{\bullet}$ , and  $B$  the Borel subgroup of upper-triangular matrices of  $G$ . Let  $\mathfrak{g}$  (resp.  $\mathfrak{b}$ , resp.  $\mathfrak{p}$ ) be the Lie algebra of  $G$  (resp.  $B$ , resp.  $P$ ). The datum  $(N, \mathcal{D}_{\bullet}, \text{Fil}_{\bullet})$  associated with the point  $x$  can define a

point  $x_{\text{pdR}}$  of the following algebraic scheme

$$X_P := \{(\nu, g_1 B, g_2 P) \in \mathfrak{g} \times G/B \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{b}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}\} \quad (3.1.5)$$

where  $G/B, G/P$  are flag varieties and  $\text{Ad}$  denotes the adjoint action. Let  $W \simeq \prod_{\tau \in \Sigma_v} \mathcal{S}_n$  (resp.  $W_P$ ) be the Weyl group of  $G$  (resp. of the standard Levi subgroup of  $P$ ) where  $\mathcal{S}_n$  denotes the  $n$ -th symmetric group. Then  $X_P$  is equidimensional and its irreducible components  $X_{P,w}$  are parameterized by  $w \in W/W_P$  (see Definition 3.2.2). Let  $w = (w_\tau)_{\tau \in \Sigma_v} \in \prod_{\tau \in \Sigma_v} \mathcal{S}_n$  be an element such that  $\text{wt}_\tau(\delta_{w_\tau(1)}) \leq \dots \leq \text{wt}_\tau(\delta_{w_\tau(n)})$  for all  $\tau \in \Sigma_v$  and we use the same notation  $w$  to denote the image of  $w$  in  $W/W_P$ . The following theorem is proved in [BHS19] when  $P = B$ , i.e. when  $r$  has regular Hodge-Tate weights ( $\text{wt}_\tau(\delta_i) \neq \text{wt}_\tau(\delta_j)$  for all  $\tau \in \Sigma_v, i \neq j$ ).

**Theorem 3.1.6** (Theorem 3.3.17). *Let  $x = (r, \underline{\delta})$  be an  $L$ -point of  $X_{\text{tri}}(\bar{\rho}_v)$  such that  $\underline{\delta}$  is generic and locally  $\mathbb{Q}_p$ -algebraic. Then up to formally smooth morphisms of formal schemes, the completion  $\widehat{X}_{\text{tri}}(\bar{\rho}_v)_x$  of the trianguline variety  $X_{\text{tri}}(\bar{\rho}_v)$  at  $x$  is isomorphic to the completion  $\widehat{X}_{P,w,x_{\text{pdR}}}$  of  $X_{P,w}$  at  $x_{\text{pdR}}$ . Moreover, the trianguline variety  $X_{\text{tri}}(\bar{\rho}_v)$  is irreducible at  $x$ .*

The proof of Theorem 3.1.6 follows the strategy for regular cases in [BHS19]. The difficulty in our situation is to show that  $X_{P,w}$  is *unibranch* at  $x_{\text{pdR}}$ , i.e. the completion of the local ring of  $X_{P,w}$  at  $x_{\text{pdR}}$  is irreducible (Theorem 3.2.14). When  $P = B$ , it was proved by Bezrukavnikov-Riche in [BR12] that for  $w \in W$ ,  $X_{B,w}$  is Cohen-Macaulay and based on the Cohen-Macaulay result, Breuil-Hellmann-Schraen proved that  $X_{B,w}$  is normal in [BHS19] which in particular implies that  $X_{B,w}$  is unibranch. We prove that  $X_{P,w}$  is unibranch at  $x_{\text{pdR}}$  based on the normality of  $X_{B,w}$  (here  $w \in W$ ). There is a natural birational proper map  $f : X_{B,w} \rightarrow X_{P,w}$  of integral varieties. We can prove that the fiber  $f^{-1}(x_{\text{pdR}})$  is connected (Proposition 3.2.12). Since  $X_{B,w}$  is normal, the connectedness of the fiber is enough to establish the unibranch property that we need (Proposition 3.2.10). The important problem whether  $X_{P,w}$  (or  $X_{\text{tri}}(\bar{\rho}_v)$  at  $x$ ) is Cohen-Macaulay or normal remains unsolved.

### 3.1.3 Partially classical families and partial de Rhamness

We need some more notation to state the result on partially classical companion constituents. For a point  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$ ,  $\underline{\delta} = (\underline{\delta}_v)_{v \in S_p} = ((\delta_{v,i})_{i=1, \dots, n})_{v \in S_p}$  is a character of  $T_p = \prod_{v \in S_p} T_v \simeq \prod_{v \in S_p} ((F_v^+)^\times)^n$ . Let

$$\lambda = (\lambda_\tau)_{\tau \in \Sigma_v, v \in S_p} = ((\lambda_{\tau,i})_{i=1, \dots, n})_{\tau \in \Sigma_v, v \in S_p} := ((\text{wt}_\tau(\delta_i))_{i=1, \dots, n})_{\tau \in \Sigma_v, v \in S_p}$$

be the weight of  $\underline{\delta}$ . When  $\underline{\delta}$  is locally algebraic for which we mean that  $\lambda_{\tau,i} \in \mathbb{Z}$  for all  $\tau \in \Sigma_v, v \in S_p$ , Orlik-Strauch's theory [OS15] can construct a (generically irreducible) locally analytic representation  $\mathcal{L}(\lambda, \underline{\delta})$  of  $G_p$  from  $\lambda$  and (the smooth part of)  $\underline{\delta}$  which is a subquotient of some locally analytic principal series representation (cf. §3.4.3).

We fix  $v_0 \in S_p, \tau_0 \in \Sigma_{v_0}$  and a parabolic subalgebra  $\mathfrak{q}_{\tau_0}$  of block upper-triangular matrices of  $\mathfrak{gl}_n$ , the Lie algebra of  $\text{GL}_n/L$ . We assume that the standard Levi subalgebra  $\mathfrak{m}_{\tau_0}$  of  $\mathfrak{q}_{\tau_0}$  is isomorphic to  $\mathfrak{gl}_{s_1-s_0} \times \dots \times \mathfrak{gl}_{s_i-s_{i-1}} \times \dots \times \mathfrak{gl}_{s_t-s_{t-1}}$  where  $0 = s_0 < \dots < s_i < \dots < s_t = n$ . Suppose that  $\lambda_{\tau_0}$  is a dominant weight of  $\mathfrak{m}_{\tau_0}$ , or explicitly,  $\lambda_{\tau_0,a} \geq \lambda_{\tau_0,b}$  for every  $a \leq b, a, b \in [s_i + 1, s_{i+1}]$  and  $0 \leq i \leq t - 1$ . Let  $L_{\mathfrak{m}_{\tau_0}}(\lambda_{\tau_0})$  be the finite-dimensional irreducible  $\mathfrak{m}_{\tau_0}$ -representation of the highest weight  $\lambda_{\tau_0}$ . Let  $Q_p = \prod_{v \in S_p} Q_v$  be a standard parabolic subgroup of  $G_p$  such that its  $\tau_0$ -part Lie algebra  $\text{Lie}(Q_{v_0}) \otimes_{F_{v_0}^+, \tau_0} L$  is equal to  $\mathfrak{q}_{\tau_0}$ . Then the companion constituent  $\mathcal{L}(\lambda, \underline{\delta})$  is partially classical (with respect to  $Q_p$  and the set  $\{\tau_0\}$ ) in the sense that we have (by an adjunction formula, see §3.5.3)

$$\text{Hom}_{G_p}(\mathcal{L}(\lambda, \underline{\delta}), \Pi(\rho)^{\text{an}}) \neq 0 \quad \Rightarrow \quad \text{Hom}_{\mathfrak{m}_{\tau_0}}(L_{\mathfrak{m}_{\tau_0}}(\lambda_{\tau_0}), J_{Q_p}(\Pi(\rho)^{\text{an}})) \neq 0. \quad (3.1.7)$$

Assume that  $(\rho, \underline{\delta})$  is a point on  $Y(U^p, \bar{\rho})$  such that  $\underline{\delta}$  is locally algebraic and satisfies certain generic condition. Then  $D_{\text{rig}}(\rho_{v_0})$  is associated via the point  $(\rho, \underline{\delta})$  with a trianguline filtration  $\text{Fil}^\bullet D_{\text{rig}}(\rho_{v_0})$  as (3.1.1) which in turn leads to a filtration  $\mathcal{D}_{v_0, \bullet} = \prod_{\tau \in \Sigma_{v_0}} \mathcal{D}_{\tau, \bullet}$  of  $D_{\text{pdR}}(\rho_{v_0})$  with a nilpotent operator  $N_{v_0} = (N_\tau)_{\tau \in \Sigma_{v_0}}$  where every  $N_\tau$  keeps the filtration  $\mathcal{D}_{\tau, \bullet}$ . For  $i = 1, \dots, t$ , we let  $\text{gr}^{s_i} D_{\text{rig}}(\rho_{v_0}) := \text{Fil}^{s_i} D_{\text{rig}}(\rho_{v_0}) / \text{Fil}^{s_{i-1}} D_{\text{rig}}(\rho_{v_0})$  be the graded pieces of the paraboline sub-filtration  $\text{Fil}^{s^\bullet} D_{\text{rig}}(\rho_{v_0})$  corresponding to  $\mathfrak{m}_{\tau_0}$ . In the case of Galois representations, an almost de Rham representation  $\rho_v$  is de Rham if and only if the nilpotent operator on  $D_{\text{pdR}}(\rho_v)$  is zero. We can identify  $D_{\text{pdR}, \tau_0}(\text{gr}^{s_i} D_{\text{rig}}(\rho_{v_0})) := D_{\text{pdR}}(\text{gr}^{s_i} D_{\text{rig}}(\rho_{v_0})) \otimes_{L \otimes_{\mathbb{Q}_p} F_{v_0, 1}^+ \otimes_{\tau_0} L} L$  with  $\mathcal{D}_{\tau_0, s_i} / \mathcal{D}_{\tau_0, s_{i-1}}$  equipped with the restriction of the action of  $N_{\tau_0}$ . We say that the  $(\varphi, \Gamma)$ -module  $\text{gr}^{s_i} D_{\text{rig}}(\rho_{v_0})$  is  $\{\tau_0\}$ -partially de Rham if the restriction of the nilpotent operator  $N_{\tau_0}$  on  $\mathcal{D}_{\tau_0, s_i} / \mathcal{D}_{\tau_0, s_{i-1}}$  is zero.

**Theorem 3.1.8** (Theorem 3.5.28). *Let  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  be a point such that  $\underline{\delta}$  is locally algebraic and generic (Definition 3.4.2). If the  $\tau_0$ -part weight  $\lambda_{\tau_0}$  of  $\underline{\delta}$  is a dominant weight for  $\mathfrak{m}_{\tau_0}$  and  $\text{Hom}_{G_p}(\mathcal{L}(\lambda, \underline{\delta}), \Pi(\rho)^{\text{an}}) \neq 0$ , then for every  $1 \leq i \leq t$ , the graded piece  $\text{gr}^{s_i} D_{\text{rig}}(\rho_{v_0})$  is a  $\{\tau_0\}$ -partially de Rham  $(\varphi, \Gamma)$ -module.*

Theorem 3.1.8 is in fact a corollary of the global triangulation results and Ding's construction of partial eigenvarieties in [Din19c] which was based on the work of Hill-Loeffler [HL11]. Assume that  $\lambda_{\tau_0}$  is dominant for  $\mathfrak{m}_{\tau_0}$ . The partial eigenvariety of Ding, denoted by  $Y(U^p, \bar{\rho})(\lambda'_{\tau_0})$ , is a subvariety of  $Y(U^p, \bar{\rho})$  roughly consisting of points  $x = (\rho_x, \underline{\delta}_x) \in Y(U^p, \bar{\rho})$  such that  $\text{Hom}_{\mathfrak{m}'_{\tau_0}}(L_{\mathfrak{m}_{\tau_0}}(\lambda_{\tau_0}), J_{Q_p}(\Pi(\rho_x)^{\text{an}})) \neq \emptyset$  where  $\mathfrak{m}'_{\tau_0} := [\mathfrak{m}_{\tau_0}, \mathfrak{m}_{\tau_0}]$  is the derived subalgebra of  $\mathfrak{m}_{\tau_0}$ . Such construction forces that for any point  $(\rho_x, \underline{\delta}_x) \in Y(U^p, \bar{\rho})(\lambda'_{\tau_0})$ , the  $\tau_0$ -part weight  $\lambda_{x, \tau_0} = (\lambda_{x, \tau_0, i})_{i=1, \dots, n}$  of  $\underline{\delta}_x$  satisfies that  $\lambda_{x, \tau_0, a} - \lambda_{x, \tau_0, b} = \lambda_{\tau_0, a} - \lambda_{\tau_0, b}$  are non-negative integers independent of  $x$  for every  $a \leq b, a, b \in [s_i + 1, s_{i+1}]$  and  $0 \leq i \leq t - 1$ . Then the arguments of Berger-Colmez in [BC08] and the global triangulation show that the subset of points  $x \in Y(U^p, \bar{\rho})(\lambda'_{\tau_0})$  such that a suitable twist of  $\text{gr}^{s_i} D_{\text{rig}}(\rho_{x, v_0})$  is  $\{\tau_0\}$ -partially de Rham is Zariski closed. The feature of Ding's construction is that such constrain on the  $\tau_0$ -weights  $\lambda_{x, \tau_0}$  still allows  $\lambda_{x, \tau_0}$  to vary and to be dominant with respect to  $\mathfrak{g}_{\tau_0} := \mathfrak{gl}_n$  even if  $\lambda_{\tau_0}$  is not. The usual eigenvariety arguments imply that the subset of classical points, where there exist non-zero locally algebraic vectors in  $\Pi(\rho_x)$  and  $\underline{\delta}_x$  admit dominant weights, is Zariski dense in  $Y(U^p, \bar{\rho})(\lambda'_{\tau_0})$ . It follows from the classical local-global compatibility when  $\ell = p$  that classical points are de Rham ( $\rho_x$  are de Rham) where  $\text{gr}^{s_i} D_{\text{rig}}(\rho_{x, v_0})$  is automatically  $\{\tau_0\}$ -partially de Rham. Combining the Zariski dense and closed statements leads to Theorem 3.1.8.

### 3.1.4 Existence of companion constituents

The key observation to prove Theorem 3.1.2 is that Theorem 3.1.8 is reflected by the local models of the trianguline variety. There is a closed embedding  $Y(U^p, \bar{\rho}) \hookrightarrow X_p(\bar{\rho})$ . Here the patched eigenvariety  $X_p(\bar{\rho})$  is equidimensional and can be identified as a union of irreducible components of  $X_{\text{tri}}(\bar{\rho}_p) \times \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g$  where  $X_{\text{tri}}(\bar{\rho}_p) := \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_v)$ ,  $\mathbb{U}^g$  is an open polydisk and  $\mathfrak{X}_{\bar{\rho}^p}$  is certain tame part.

Let  $x = (\rho, \underline{\delta}) \in Y(U^p, \bar{\rho}) \subset X_p(\bar{\rho})$  be a generic crystalline point as in Theorem 3.1.2. Let  $G := \prod_{v \in S_p} \text{Res}_{F_v^+ / \mathbb{Q}_p}(\text{GL}_{n/F_v^+}) \otimes_{\mathbb{Q}_p} L$  and let  $W \simeq \prod_{v \in S_p, \tau \in \Sigma_v} \mathcal{S}_n$  be its Weyl group. A companion character in  $W_{\mathcal{R}}(\rho)$  is certain character  $\underline{\delta}_{\mathcal{R}, w}$  for some  $w \in W$  (Definition 3.4.12) with weight  $ww_0 \cdot \lambda$ . Here  $\lambda = (\lambda_\tau)_{\tau \in \Sigma_v, v \in S_p} \in \prod_{v \in S_p, \tau \in \Sigma_v} \mathbb{Z}^n$  is a “dominant” weight in the sense that  $\lambda_{\tau, i} - \lambda_{\tau, i+1} \geq -1$  for all  $\tau$  and  $1 \leq i \leq n - 1$  which is determined by the Hodge-Tate weights of  $\rho$ ,  $w_0$  is the longest element in  $W$  and  $ww_0 \cdot \lambda$  denotes the usual dot action. In [BHS19], for each companion constituent  $\mathcal{L}(ww_0 \cdot \lambda, \underline{\delta}_{\mathcal{R}, w})$ , there is an associated cycle  $[\mathcal{L}(ww_0 \cdot \lambda)]$  on  $X_p(\bar{\rho})$  in the infinitesimal neighbourhood of  $x$  such that  $[\mathcal{L}(ww_0 \cdot \lambda)] \neq \emptyset$  if and only if  $\text{Hom}_{G_p}(\mathcal{L}(ww_0 \cdot \lambda, \underline{\delta}_{\mathcal{R}, w}), \Pi(\rho)^{\text{an}}) \neq 0$ .

The idea of [BHS19] is to compare the cycles  $[\mathcal{L}(ww_0 \cdot \lambda)]$  with the cycles pulled back from *Steinberg varieties* via the theory of local models for  $X_{\text{tri}}(\bar{\rho}_p)$ . Let  $P = \prod_{v \in S_p} P_v$  be the standard parabolic subgroup of  $G$  where each  $P_v$  is the parabolic subgroup determined by the Hodge filtration of  $\rho_v$  as in Theorem 3.1.6 and let  $B$  be the standard Borel subgroup of  $G$ . Let  $\mathfrak{g}, \mathfrak{p}, \mathfrak{b}$  be the Lie algebras as before and let  $\mathfrak{u}$  be the nilpotent radical of  $\mathfrak{b}$ . The (generalized) Steinberg variety

$$Z_P := \{(\nu, g_1 B, g_2 P) \in \mathfrak{g} \times G/B \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{u}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}\}$$

is a subvariety of  $X_P$ . Let  $W_P$  be the Weyl group of the standard Levi subgroup of  $P$ . Then any  $w \in W_P$  fixes  $w_0 \cdot \lambda$  under the dot action. The irreducible components  $Z_{P,w}$  of  $Z_P$  are also parameterized by cosets  $w \in W/W_P$  (see §3.2.5 for details). Pulling back each  $Z_{P,w}$  defines a cycle  $\mathfrak{Z}_{P,w}$  on  $X_p(\bar{\rho})$ . The spirit of [BHS19] expects that  $\mathfrak{Z}_{P,w} \subset [\mathcal{L}(ww_0 \cdot \lambda)]$ .

Let  $\mathfrak{q} = \mathfrak{m}_Q + \mathfrak{n}_Q \subset \mathfrak{g}$  be the Lie algebra of a standard parabolic subgroup  $Q$  of upper-triangular block matrices of  $G$  where  $\mathfrak{n}_Q$  is the nilpotent radical and  $\mathfrak{m}_Q$  is the Levi factor of diagonal block matrices. Recall that partial de Rhamness means the vanishing of the nilpotent operator on the graded pieces of the parabolic sub-filtration which, in the notion of local models (3.1.5), is translated to that the entries of the upper-triangular matrix  $\text{Ad}(g_1^{-1})\nu$  in certain Levi diagonal blocks are zero. Hence if  $ww_0 \cdot \lambda$  is a dominant weight for  $\mathfrak{m}_Q$ , then Theorem 3.1.8 implies that the cycle  $[\mathcal{L}(ww_0 \cdot \lambda)]$  is contained in the locus pulled back from the subspace of  $Z_P$  cut out by the condition  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q$ . The following elementary result for which we state as a theorem is the counterpart on the local models.

**Theorem 3.1.9** (Theorem 3.2.26). *For each  $w \in W/W_P$ , the irreducible component  $Z_{P,w}$  is contained in the subspace of  $Z_P$  where  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q$  if and only if  $ww_0 \cdot \lambda$  is a dominant weight for  $\mathfrak{m}_Q$ .*

*Remark 3.1.10.* When  $P = B$ , we can replace the irreducible component  $Z_{P,w}$  in Theorem 3.1.9 by the characteristic variety associated with the  $G$ -equivariant  $\mathcal{D}$ -module of the localization of the irreducible  $U(\mathfrak{g})$ -module  $L(ww_0 \cdot 0)$  of the highest weight  $ww_0 \cdot 0$  (Proposition 3.2.32). This will give a more conceptual proof of the “if” part of the theorem. However, we do not need characteristic cycles in contrast to [BHS19] (our new argument will be simpler than that in *loc. cit.*, even for regular cases). Moreover, it is the “only if” part that will play a role.

We illustrate how Theorem 3.1.9 works in the proof of Theorem 3.1.2 and the difference between regular and non-regular cases by the following basic example.

*Example 3.1.11.* We assume  $n = 3$ ,  $x = (\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  and that  $\underline{\delta} = \underline{\delta}_{\mathcal{R}, w_0}$  has weight  $\lambda$  which is “dominant”. Take  $\tau_0 \in \Sigma_{v_0}, v_0 \in S_p$ . Assume  $\lambda_{\tau_0, 1} \geq \lambda_{\tau_0, 2}$  and that for any  $\tau \neq \tau_0$ ,  $\lambda_{\tau, 1} \geq \lambda_{\tau, 2} \geq \lambda_{\tau, 3}$ . Suppose that we are in the case when  $W_{\mathcal{R}}(\rho) = \{\underline{\delta}_{\mathcal{R}, w_0}, \underline{\delta}_{\mathcal{R}, sw_0}\}$  where  $s = (s_\tau)_{\tau \in \Sigma_v, v \in S_p}$  is a simple reflection such that  $s_\tau \cdot \lambda_\tau = \lambda_\tau$  if  $\tau \neq \tau_0$  and  $s_{\tau_0} \cdot (\lambda_{\tau_0, 1}, \lambda_{\tau_0, 2}, \lambda_{\tau_0, 3}) = (\lambda_{\tau_0, 2} - 1, \lambda_{\tau_0, 1} + 1, \lambda_{\tau_0, 3})$ . Then there is an equality of the underlying closed subspaces of cycles near  $x$ :

$$[\mathcal{L}(w_0 w_0 \cdot \lambda)] \cup [\mathcal{L}(sw_0 w_0 \cdot \lambda)] = \mathfrak{Z}_{P, w_0} \cup \mathfrak{Z}_{P, sw_0} \quad (3.1.12)$$

where both sides describe the fibers of the infinitesimal neighbourhood of  $x$  over the weight  $w_0 w_0 \cdot \lambda$ . The left-hand side of (3.1.12) comes from the construction of the eigenvariety using  $J_{B_p}(-)$ , and the knowledge of possible companion constituents for  $\rho$  in the situation. The right-hand side is provided by the local model where both  $\mathfrak{Z}_{P, w_0}$  and  $\mathfrak{Z}_{P, sw_0}$  are non-empty. By methods in [BHS19] and Theorem 3.1.6 we know  $[\mathcal{L}(w_0 w_0 \cdot \lambda)] \neq \emptyset$  (Proposition 3.4.17). We need to prove that  $[\mathcal{L}(sw_0 w_0 \cdot \lambda)] \neq \emptyset$  which will imply  $(\rho, \underline{\delta}_{\mathcal{R}, sw_0}) \in Y(U^p, \bar{\rho})$  and

$$\text{Hom}_{G_p}(\mathcal{L}(sw_0 w_0 \cdot \lambda, \underline{\delta}_{\mathcal{R}, sw_0}), \Pi(\rho)^{\text{an}}) \neq 0.$$

Firstly assume that the Hodge-Tate weights of  $\rho$  are regular. In this case  $\lambda_{\tau, 1} \geq \lambda_{\tau, 2} \geq \lambda_{\tau, 3}$  for all  $\tau$ . Hence  $\lambda$  is a dominant weight. The locally analytic representation of the form  $\mathcal{L}(\lambda, \delta_{\mathcal{R}, w_0})$

is locally algebraic and the cycle  $[\mathcal{L}(w_0w_0 \cdot \lambda)]$  is then contained in the de Rham locus. However,  $Z_{P,w_0}$  is equal to the locus where  $\nu = 0$  in  $Z_P = Z_B$ . From which we get  $[\mathcal{L}(w_0w_0 \cdot \lambda)] \subset \mathfrak{Z}_{P,w_0}$  and  $\mathfrak{Z}_{P,sw_0} \not\subset [\mathcal{L}(w_0w_0 \cdot \lambda)]$ . Hence  $[\mathcal{L}(sw_0w_0 \cdot \lambda)] \neq \emptyset$  by (3.1.12). This is the strategy used in [BHS19] for such situation.

Now assume that the  $\tau_0$ -Hodge-Tate weights of  $\rho_{v_0}$  are not regular and are equal to  $(2, 1, 1)$  so that  $\lambda_{\tau_0} = (2, 2, 3)$  and  $s_{\tau_0} \cdot \lambda_{\tau_0} = (1, 3, 3)$ . Theorem 3.1.8 implies that the cycle  $[\mathcal{L}(w_0w_0 \cdot \lambda)]$  is  $\{\tau_0\}$ -partially de Rham with respect to the standard Levi subalgebra  $\mathfrak{gl}_2 \times \mathfrak{gl}_1$  of  $\mathfrak{gl}_3$ . Since  $(1, 3, 3)$  is not dominant with respect to  $\mathfrak{gl}_2 \times \mathfrak{gl}_1$ , Theorem 3.1.9 tells that the cycle  $\mathfrak{Z}_{P,sw_0}$  is not fully contained in the  $\{\tau_0\}$ -partially de Rham locus (with respect to  $\mathfrak{gl}_2 \times \mathfrak{gl}_1$ ). Hence  $\mathfrak{Z}_{P,sw_0} \not\subset [\mathcal{L}(w_0w_0 \cdot \lambda)]$  which forces  $[\mathcal{L}(sw_0w_0 \cdot \lambda)] \neq \emptyset$  by (3.1.12).

*Remark 3.1.13.* The above strategy also allows obtaining certain companion points or constituents for non-de Rham trianguline representations. Theorem 3.1.9 suggests a partial classicality conjecture (a converse of Theorem 3.1.8) for almost de Rham representations with regular Hodge-Tate weights which is closely related to the locally analytic socle conjecture (Proposition 3.5.34).

### 3.1.5 Outline of the paper

We give a brief overview of the contents of the paper.

§3.2 studies the varieties appearing for the local models. The unibranched property of the local models is proved in §3.2.3 (Theorem 3.2.14). §3.2.5 contains the results on the generalized Steinberg varieties (Theorem 3.2.26). §3.2.6 is a complement of §3.2.5 to provide a point of view from geometric representation theory.

§3.3 establishes the local models for the trianguline variety in the non-regular cases. This part follows closely with [BHS19] in the regular cases. The first sections are devoted to recall and generalize the deformation theory of trianguline  $(\varphi, \Gamma)$ -modules. §3.3.6 transports the results on the Steinberg varieties in §3.2.5 to the trianguline variety via the local models.

§3.4 contains our main results on companion points and constituents and their proofs. §3.4.1 concerns the existence of local companion points on the trianguline variety. §3.4.2 is to recall the global settings and §3.4.3 recalls the theory of locally analytic representations and the definition of companion constituents. §3.4.4 is the core part where we prove the main theorems (Theorem 3.4.18 and Theorem 3.4.20). The key induction step is Proposition 3.4.13 which uses the results on Steinberg varieties in §3.2.5 and Theorem 3.4.10. The proof of Theorem 3.4.10 absorbs the results, postponed in §3.5, on the partially classical companion constituents.

§3.5 concerns the partially classical families and the partial classicality. A large effort (§3.5.2-§3.5.5) is for the construction of the partial eigenvarieties and studying their basic properties where most results have been obtained by Ding. We adapt his method in our setting to get the patched and more partial versions. The aim is to prove Theorem 3.5.28 in §3.5.6 on the partial de Rhamness of partially classical constituents which have been used in §3.4.4. §3.5.7 discusses the conjecture on partial classicality and several results for almost de Rham representations.

In Appendix A, we generalize certain result of Berger-Colmez in [BC08] on de Rham families of Galois representations to almost de Rham families of  $(\varphi, \Gamma)$ -modules (Proposition A.3.4). The result we get is stronger than what we need (in the proof of Proposition 3.5.26 and Theorem 3.4.10) and is possibly known to experts. We include a proof as it might be of use in the future.

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### 3.1.7 Notation

#### 3.1.7.1 Reductive groups

Let  $G$  denote a connected split reductive group over a field  $k$  with a maximal torus  $T$ , a Borel subgroup  $B$  containing  $T$  and the Levi decomposition  $B = TU$ . Write  $\bar{B}$  for the opposite Borel subgroup and  $\bar{U}$  for the unipotent radical of  $\bar{B}$ . Write  $R^+$  (resp.  $R$ , resp.  $R^-$ ) for the set of all positive roots with respect to  $B$  (resp. roots, resp. negative roots with respect to  $B$ ) of  $G$ . Write  $\Delta$  for the set of positive simple roots. For a root  $\alpha$ , denote by  $\alpha^\vee$  the corresponding coroot. We have in particular  $\langle \alpha^\vee, \alpha \rangle = 2$ , where  $\langle -, - \rangle$  denotes the pairing between the lattices of coweights  $X_*(T)$  and weights  $X^*(T)$  of  $T$ . Write  $W$  for the Weyl group of  $G$  and for  $\alpha \in R$ , denote by  $s_\alpha \in W$  for the corresponding reflection. Let  $S = \{s_\alpha \mid \alpha \in \Delta\}$  be the set of simple reflections. For every  $w \in W$ , we fix an element  $\dot{w} \in N_G(T)(k)$  that is sent to  $w$  via the isomorphism  $N_G(T)/T \simeq W$  where  $N_G(T)$  denotes the normalizer of  $T$  in  $W$ . We have  $\langle \alpha, s_\beta(\mu) \rangle = \langle s_\beta(\alpha), \mu \rangle$  for  $\alpha \in R, \beta \in \Delta, \mu \in X_*(T)$ .

We use fraktur letters  $\mathfrak{g}$  (resp.  $\mathfrak{b}$ , resp.  $\mathfrak{p}$ , resp.  $\mathfrak{t}$ , resp.  $\mathfrak{u}$ , resp.  $\bar{\mathfrak{u}}$ , etc.) for the Lie algebra of  $G$  (resp.  $B$ , resp.  $P$ , resp.  $T$ , resp.  $U$ , resp.  $\bar{U}$ , etc.). Denote by  $\text{Ad} : G \rightarrow \text{End}(\mathfrak{g})$  the adjoint representation. For a Lie algebra  $\mathfrak{g}$ , denote by  $U(\mathfrak{g})$  the universal enveloping algebra.

If  $P$  is a standard parabolic subgroup of  $G$  containing  $B$ , let  $P = M_P N_P$  be the standard Levi decomposition, where  $M_P$  is the standard Levi subgroup containing  $T$ . Let  $B_{M_P} = B \cap M_P$  and  $U_{M_P} = U \cap M_P$ . Let  $R_P \subset R$  be the set of roots of  $M_P$  and let  $R_P^+ = R^+ \cap R_P, R_P^- = R^- \cap R_P, \Delta_P = \Delta \cap R_P$ . Let  $\mathfrak{m}_P$  (resp.  $\mathfrak{n}_P$ , resp.  $\mathfrak{b}_{M_P}$ , resp.  $\mathfrak{u}_{M_P}$ ) be the Lie algebra of  $M_P$  (resp.  $N_P$ , resp.  $B_{M_P}$ , resp.  $U_{M_P}$ ). In particular,  $\mathfrak{n}_B = \mathfrak{u}$ .

Write  $W_P$  for the Weyl group of  $M_P$ . Let  $\text{lg}(-)$  denote the length of elements in  $W$  with respect to the set of simple reflections in  $S$ . We use the symbols  $\leq, \geq, <, >$  to denote the strong Bruhat order (resp. partial Bruhat order) on  $W$  (resp.  $W/W_P$ ) with respect to the Coxeter system  $(W, S)$  [BB06, §2.1, §2.5]. Write  $W^P$  for the set of elements  $w \in W$  that are the unique shortest elements in the cosets  $wW_P$  (cf. [BB06, §2.4]). Then  $W = W^P W_P$ . If  $w \in W$ , let  $w = w^P w_P$  be the unique decomposition such that  $w^P \in W^P, w_P \in W_P$  ([BB06, Prop. 2.4.4]). The map  $W \rightarrow W^P : w \mapsto w^P$  is order preserving ([BB06, Prop. 2.5.1]) and the partial order on  $W/W_P$  is induced by the order on  $W$  via the bijection  $W^P \leftrightarrow W/W_P$ . For  $w \in W/W_P$ , let  $\text{lg}_P(w) := \text{lg}(w^P)$  where  $w^P \in W^P \cap wW_P$ . When it is clear from the context, for  $w \in W$ , we use the same notation  $w$  to denote the coset  $wW_P \in W/W_P$ . Write  $w_0$  (resp.  $w_{P,0}$ ) for the longest element in  $W$  (resp.  $W_P$ ).

We write  $\overline{BwP/P}$  for the Schubert variety in the flag variety  $G/P$  corresponding to  $w \in W/W_P$ . It is the closure of the Schubert cell  $BwP/P$  in  $G/P$  (cf. [Jan07, II.13.8]).

A weight  $\lambda \in X^*(T)$  which is also viewed as a weight of  $\mathfrak{t}$  is said to be a dominant (resp. antidominant) weight for a standard Levi subgroup  $M_P$  or its Lie algebra  $\mathfrak{m}_P$  (with respect to  $B_{M_P}$  or  $\mathfrak{b}_{M_P}$ ) if  $\langle \alpha^\vee, \lambda \rangle \geq 0$  (resp.  $\langle \alpha^\vee, \lambda \rangle \leq 0$ ) for all  $\alpha \in \Delta_P$ .

The dot action is given by  $w \cdot \lambda = w(\lambda + \rho) - \rho$  for all  $w \in W$  and  $\lambda \in X^*(T)$  where  $\rho$  is the half sum of all positive roots.

#### 3.1.7.2 Local fields

Let  $K$  be a finite extension of  $\mathbb{Q}_p$  with a uniformizer  $\varpi_K$ . Write  $\mathcal{O}_K$  for the ring of integers of  $K$  and  $k_K$  for the residue field. Let  $\bar{K}$  be an algebraic closure of  $K$  and  $C$  be the completion of  $\bar{K}$ . Let  $K_0$  be the maximal unramified subfield of  $K$ . Write  $\mathcal{G}_K := \text{Gal}(\bar{K}/K)$  for its Galois group. Let  $B_{\text{dR}}^+ = B_{\text{dR}}^+(C), B_{\text{dR}} = B_{\text{dR}}^+[\frac{1}{t}]$  be Fontaine's de Rham period rings, where  $t$  is Fontaine's

$2\pi i$ . Let  $K(\mu_\infty)$  be the extension of  $K$  by adding all  $p$ -th power roots of unity, and we define  $\Gamma_K := \text{Gal}(K(\mu_\infty)/K)$ .

Take  $L$  a finite extension of  $\mathbb{Q}_p$  that splits  $K$ . Let  $\mathcal{C}_L$  denote the category of commutative local Artinian  $L$ -algebras with residue field  $L$ . If  $A \in \mathcal{C}_L$ , let  $\mathfrak{m}_A$  be its maximal ideal and the tensor product  $- \otimes_A L$  is always with respect to the map modulo  $\mathfrak{m}_A$ . Let  $\Sigma$  be the set of embeddings  $\tau : K \hookrightarrow L$ .

Write  $\mathcal{T}$  for the  $\mathbb{Q}_p$ -rigid analytic space parametrizing continuous characters of  $K^\times$  (cf. [KPX14, Exam. 6.1.5]) and  $\mathcal{T}_L = \mathcal{T} \times_{\mathbb{Q}_p} L$ . If  $A$  is an affinoid  $L$ -algebra, and  $\delta : K^\times \rightarrow A^\times$  is a continuous character, i.e. locally  $\mathbb{Q}_p$ -analytic, then define the weight  $\text{wt}(\delta) \in \text{Hom}_{\mathbb{Q}_p}(K, A) : x \mapsto \frac{d}{dt} \delta(\exp(tx))|_{t=0}$ . We identify  $\text{Hom}_{\mathbb{Q}_p}(K, A)$  with  $K \otimes_{\mathbb{Q}_p} A$  via the trace pairing of  $K$  and let  $\text{wt}_\tau(\delta)$  be the  $\tau$ -part of  $\text{wt}(\delta) \in K \otimes_{\mathbb{Q}_p} A = \prod_{\tau \in \Sigma} A$ .

Let  $\epsilon$  be the cyclotomic character of  $\mathcal{G}_K$  and we still use  $\epsilon$  to denote the character  $N_{K/\mathbb{Q}_p} |N_{K/\mathbb{Q}_p}|_{\mathbb{Q}_p}$  of  $K^\times$  where  $N_{K/\mathbb{Q}_p}$  is the norm map and  $|\cdot|_{\mathbb{Q}_p}$  is the standard valuation of  $\mathbb{Q}_p$ . For any  $a \in L^\times$ , let  $\text{unr}(a) : K^\times \rightarrow L^\times$  be the unramified character sending  $\varpi_K$  to  $a$ . If  $\mathbf{k} = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$ , write  $z^\mathbf{k}$  for the character  $K^\times \rightarrow L^\times : z \mapsto \prod_{\tau \in \Sigma} \tau(z)^{k_\tau}$ . If  $\underline{\delta} = (\delta_i)_{i \in I} : (K^\times)^I \rightarrow A^\times$  is a continuous character of  $(K^\times)^I$  for a finite set  $I$ , we write  $\text{wt}(\underline{\delta}) := (\text{wt}(\delta_i))_{i \in I} \in (K \otimes_{\mathbb{Q}_p} A)^I$  and similarly for  $\text{wt}_\tau(\underline{\delta})$ . If  $A$  is a finite local  $L$ -algebra and  $\delta : K^\times \rightarrow A^\times$ , then we say  $\delta$  is  $(\mathbb{Q}_p)$ -algebraic (resp. locally  $(\mathbb{Q}_p)$ -algebraic, resp. smooth) if  $\delta = z^\mathbf{k}$  for some  $\mathbf{k} \in \mathbb{Z}^\Sigma$  (resp.  $\text{wt}_\tau(\delta) \in \mathbb{Z} \subset A, \forall \tau \in \Sigma$ , resp.  $\text{wt}(\delta) = 0$ ). We say  $\underline{\delta} : (K^\times)^I \rightarrow A^\times$  is  $(\mathbb{Q}_p)$ -algebraic (resp. locally  $(\mathbb{Q}_p)$ -algebraic, resp. smooth) if  $\delta_i$  is  $(\mathbb{Q}_p)$ -algebraic (resp. locally  $(\mathbb{Q}_p)$ -algebraic, resp. smooth) for all  $i \in I$ .

If  $X$  is a rigid space, we write  $\mathcal{R}_{X,K}$  for the Robba ring of  $K$  over  $X$  ([KPX14, Def. 6.2.1], our notation follows [BHS19]) and if  $A$  is an affinoid algebra, write  $\mathcal{R}_{A,K} := \mathcal{R}_{\text{Sp}(A),K}$ . If  $\delta : K^\times \rightarrow \Gamma(X, \mathcal{O}_X)^\times$  is a continuous character, let  $\mathcal{R}_{X,K}(\delta)$  (or  $\mathcal{R}_{A,K}(\delta)$  if  $X = \text{Sp}(A)$ ) be the rank one  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{X,K}$  constructed in [KPX14, Cons. 6.2.4]. If  $D_X$  is a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{X,K}$ , set  $D_X(\delta) := D_X \otimes_{\mathcal{R}_{X,K}} \mathcal{R}_{X,K}(\delta)$ .

### 3.1.7.3 Miscellaneous

For a positive integer  $n$ , write  $\mathcal{S}_n$  for the  $n$ -th symmetric group.

If  $x$  is a point on  $X$ , a scheme locally of finite type over a field or a rigid analytic variety, then we denote by  $k(x)$  the residue field at  $x$ . Write  $X^{\text{red}}$  for the underlying reduced subspace.

If  $X$  is a scheme locally of finite type over a finite extension  $L$  of  $\mathbb{Q}_p$ , then we write  $X^{\text{rig}}$  for its rigid analytification ([Bos14, §5.4]). If  $R$  is a commutative Noetherian complete local ring over  $\mathcal{O}_L$  of residue field finite over  $k_L$ , then we denote by  $\text{Spf}(R)$  the formal scheme defined by  $R$  with its maximal ideal and we write  $\text{Spf}(R)^{\text{rig}}$  for its rigid generic fiber in the sense of Berthelot ([dJ95, §7]).

If  $R$  is a commutative ring, then write  $R^{\text{red}} := R/J$  for its nilreduction where  $J$  is the nilradical of  $R$ . If  $R$  is a commutative Noetherian local ring, denote by  $\hat{R}$  the completion of  $R$  with respect to the maximal ideal of  $R$ .

If  $Z$  is a topologically finitely generated abelian  $p$ -adic Lie group, then we write  $\hat{Z}$  for the rigid analytic space over  $\mathbb{Q}_p$  parameterizing continuous characters of  $Z$  (cf. [KPX14, Prop. 6.1.1]).

If  $\mathfrak{g}$  is a finite-dimensional Lie algebra over a field  $k$ , then we use the same notation  $\mathfrak{g}$  to denote the affine scheme over  $k$  such that  $\mathfrak{g}(A) = A \otimes_k \mathfrak{g}$  for any commutative  $k$ -algebra  $A$ .

If  $V$  is a module over a ring  $R$  and  $I$  is an ideal of  $R$ , then write  $V[I]$  for the subset  $\{v \in V \mid av = 0, \forall a \in I\}$ . If  $V$  is a vector space over a field with a linear action of a group  $G$ , then write  $V^G$  for the subspace  $\{v \in V \mid gv = v, \forall g \in G\}$ .

## 3.2 Unibranchness

In this section, we study some generalized version of the varieties built from Grothendieck's simultaneous resolution in [Ric08], [BR12] and [BHS19].

We fix a connected split reductive group  $G$  over a field  $k$  with characteristic that is very good for  $G$  ([KW13, Def. VI.1.6], we will only need the case when  $\text{char}(k) = 0$ ), with a maximal torus  $T$ , a Borel subgroup  $B = TU$  containing  $T$  and a standard parabolic subgroup  $P = M_P N_P$ . Let  $\mathfrak{g}$  (resp.  $\mathfrak{b}$ , resp.  $\mathfrak{p}$ , resp.  $\mathfrak{t}$ , etc.) be the Lie algebra of  $G$  (resp.  $B$ , resp.  $P$ , resp.  $T$ , etc.). Let  $W$  be the Weyl group of  $G$ .

### 3.2.1 The varieties

We shall define the varieties that we are going to study. Define the following schemes over  $k$ :

$$\begin{aligned} X_P &:= \{(\nu, g_1 B, g_2 P) \in \mathfrak{g} \times G/B \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{b}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}\} \\ Y_P &:= \{(\nu, gP) \in \mathfrak{b} \times G/P \mid \text{Ad}(g^{-1})\nu \in \mathfrak{p}\}. \end{aligned}$$

If  $P = B$ , then  $X_P$  is defined in [BHS19] and we denote by  $X := X_B, Y := Y_B$ . The scheme  $X_P$  (resp.  $Y_P$ ) is equipped with a left  $G$ -action (resp.  $B$ -action) given by  $g(\nu, g_1 B, g_2 P) = (\text{Ad}(g)\nu, gg_1 B, gg_2 P)$  for any  $g \in G, (\nu, g_1 B, g_2 P) \in X_P$  (resp.  $b(\nu, gP) = (\text{Ad}(b)\nu, bgP)$  for any  $b \in B, (\nu, gP) \in Y_P$ ). The morphism  $G \times^B Y_P \rightarrow X_P$  sending  $(g, (\nu, g_1 P))$  to  $(\text{Ad}(g)\nu, gB, gg_1 P)$  is an isomorphism, where the notation of  $G \times^B Y_P$  is taken from [Jan07, I.5.14]. Let  $\bar{U}$  be the opposite unipotent subgroup with respect to  $B$ . The projection  $G \rightarrow G/B$  is locally trivial:  $G$  is covered by open subsets of the form  $g\bar{U}B, g \in G$  and  $g\bar{U}B \simeq \bar{U} \times B$  as varieties. Hence  $X_P \simeq G \times^B Y_P$  is covered by open subschemes that are isomorphic to  $\bar{U} \times Y_P$ . Note that  $\bar{U}$  is smooth.

Suppose  $w \in W/W_P$ . Let  $U_{P,w} = \{(g_1 B, g_2 P) \in G/B \times G/P \mid g_1^{-1}g_2 P \in BwP/P\}$  be the equivariant partial Schubert cell in  $G/B \times G/P$ . Then  $U_{P,w}$  is a locally closed subscheme in  $G/B \times G/P$  of dimension  $\dim G - \dim B + \text{lg}_P(w)$  ([Jan07, II.13.8]). We let

$$V_{P,w} := \{(\nu, g_1 B, g_2 P) \in X_P \mid g_1^{-1}g_2 P \in BwP\} \quad (3.2.1)$$

be the preimage of  $U_{P,w}$  in  $X_P$  under the natural projection  $X_P \rightarrow G/B \times G/P$  and define  $V_{P,w}^Y := \{(\nu, gP) \in \mathfrak{b} \times BwP/P \mid \text{Ad}(g^{-1})\nu \in \mathfrak{p}\}$  similarly.

**Definition 3.2.2.** For every  $w \in W/W_P$ , let  $X_{P,w}$  (resp.  $Y_{P,w}$ ) be the Zariski closure of  $V_{P,w}$  in  $X_P$  (resp. the Zariski closure of  $V_{P,w}^Y$  in  $Y_P$ ) equipped with the reduced induced subscheme structure. When  $P = B$ , we write  $X_w$  (resp.  $Y_w$ ) for  $X_{B,w}$  (resp.  $Y_{B,w}$ ).

We define a variety  $\tilde{\mathfrak{g}}_P := \{(\nu, gP) \in \mathfrak{g} \times G/P \mid \text{Ad}(g^{-1})\nu \in \mathfrak{p}\} \simeq G \times^P \mathfrak{p}$  and denote by  $\tilde{\mathfrak{g}} := \tilde{\mathfrak{g}}_B$ . The projection to the first factor  $q_P : \tilde{\mathfrak{g}}_P \rightarrow \mathfrak{g}$  is the (partial) Grothendieck simultaneous resolution ([KW13, VI.8]). The scheme  $X_P$  is isomorphic to  $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P$  as in [BHS19] for  $X$ .

Define  $\text{pr}_P : X_P = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P \rightarrow \tilde{\mathfrak{g}}$  to be the projection to the first factor and  $\text{pr}_{P,w}$  to be its restriction to  $X_{P,w}$ . Similarly, we can define morphisms  $\text{pr}_P^Y$  and  $\text{pr}_{P,w}^Y$  which send  $(\nu, gP) \in Y_P \subset \mathfrak{b} \times G/P$  to  $\nu \in \mathfrak{b}$ . We let  $\mathfrak{g}^{\text{reg}} \subset \mathfrak{g}$  (resp.  $\mathfrak{g}^{\text{reg-ss}}$ ) be the open subscheme of  $\mathfrak{g}$  consisting of regular elements which by definition are those elements in  $\mathfrak{g}$  whose orbits under the adjoint action of  $G$  have the maximal possible dimension (resp. regular semisimple elements). Let  $\tilde{\mathfrak{g}}^{\text{reg}} := q_B^{-1}(\mathfrak{g}^{\text{reg}})$  (resp.  $\tilde{\mathfrak{g}}^{\text{reg-ss}} := q_B^{-1}(\mathfrak{g}^{\text{reg-ss}})$ ).

**Proposition 3.2.3.** 1. The scheme  $X_P$  (resp.  $Y_P$ ) is reduced, is a locally complete intersection, hence Cohen-Macaulay, and is equidimensional of dimension  $\dim G$  (resp.  $\dim B$ ). Its irreducible components are  $X_{P,w}$  (resp.  $Y_{P,w}$ ) for  $w \in W/W_P$ .



2. For each  $w \in W/W_P$ , the morphism  $\mathrm{pr}_{P,w} : X_{P,w} \rightarrow \tilde{\mathfrak{g}}$  (resp.  $\mathrm{pr}_{P,w}^Y : Y_{P,w} \rightarrow \mathfrak{b}$ ) is proper birational surjective and is an isomorphism over  $\tilde{\mathfrak{g}}^{\mathrm{reg}}$  (resp.  $\mathfrak{b}^{\mathrm{reg}} := \mathfrak{b} \cap \mathfrak{g}^{\mathrm{reg}}$ ).

*Proof.* The proof goes in the same way as that in [BHS19, §2.2] and we only give a sketch here. The fiber of the projection  $V_{P,w} \rightarrow U_{P,w}$  over a point  $(g_1 B, g_2 P) \in U_{P,w} \subset G/B \times G/P$  is

$$\{(\nu, g_1 B, g_2 P) \in X_P \mid \nu \in \mathrm{Ad}(g_1)\mathfrak{b} \cap \mathrm{Ad}(g_2)\mathfrak{p}\}$$

which is isomorphic to  $\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{p}$  as schemes where  $\dot{w} \in W^P$  is the shortest element in  $wW_P$ . The variety  $\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{p}$  is an affine space of dimension  $\dim B - \mathrm{lg}_P(w)$  ([BT72, Prop. 3.9 (ii)] or Lemma 3.2.7 below). Using [BHS19, Lem. 2.2.2] we see that  $V_{P,w}$  is a geometric vector bundle over  $U_{P,w}$  of total dimension  $\dim G$ . Hence  $X_{P,w}$  is irreducible of dimension  $\dim G$  for every  $w \in W/W_P$ . The scheme  $X_P$  is a union of the subsets  $X_{P,w}$ ,  $w \in W/W_P$  and is locally cut out by  $\dim G - \dim B + \dim G - \dim P$  equations from a smooth variety  $\mathfrak{g} \times G/B \times G/P$ . Thus  $X_P$  is locally of complete intersection, hence Cohen-Macaulay and equidimensional of dimension  $\dim G$ . The reducedness of  $X_P$  and (2) for  $X_P$  can be proved by the same arguments in the proof of [BHS19, Thm. 2.2.6] using Lemma 3.2.4 below to argue that each  $X_{P,w}$  contains one point in the fiber over any point of  $\tilde{\mathfrak{g}}^{\mathrm{reg-ss}}$  of the map  $\mathrm{pr}_P : X_P \rightarrow \tilde{\mathfrak{g}}$ . The proof of results for  $Y_P$  is similar or using the results for  $X_P$  together with the isomorphism  $G \times^B Y_P \simeq X_P$ .  $\square$

There is a natural proper surjective morphism of schemes  $q_{B,P} : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_P$ ,  $(\nu, gB) \mapsto (\nu, gP)$ . In fact, the surjectivity can be tested over an algebraically closed field and for closed points since the source and the target are both algebraic varieties. For any geometric point  $(\nu, g) \in \mathfrak{g} \times G$  such that  $\mathrm{Ad}(g^{-1})\nu \in \mathfrak{p}$ , one can always find an element  $h \in M_P$  such that  $\mathrm{Ad}(h^{-1}g^{-1})\nu \in \mathfrak{b}$ , then the point  $(\nu, ghB) \in X$  is sent to  $(\nu, gP) \in X_P$  by  $q_{B,P}$ . Now we have a factorization of  $q_B : \tilde{\mathfrak{g}} \xrightarrow{q_{B,P}} \tilde{\mathfrak{g}}_P \xrightarrow{q_P} \mathfrak{g}$ . The following lemma is a plain generalization of [BHS19, Prop. 2.1.1].

**Lemma 3.2.4.** *The morphism  $q_P : \tilde{\mathfrak{g}}_P \rightarrow \mathfrak{g}$  is proper and surjective. It is finite over  $\mathfrak{g}^{\mathrm{reg}}$  and is étale of degree  $|W/W_P|$  over  $\mathfrak{g}^{\mathrm{reg-ss}}$ .*

*Proof.* The properness of  $q_P$  comes from the factorization  $q_P : \tilde{\mathfrak{g}}_P \hookrightarrow \mathfrak{g} \times G/P \rightarrow \mathfrak{g}$  and the fact that the flag variety  $G/P$  is proper. Since  $q_B = q_P \circ q_{B,P}$  and  $q_B$  is surjective by [BHS19, Prop. 2.1.1],  $q_P$  must be surjective. Since  $q_B$  is quasi-finite over  $\mathfrak{g}^{\mathrm{reg}}$ , for any point  $s \in \mathfrak{g}^{\mathrm{reg}}$ , the fiber  $q_B^{-1}(s)$  is finite, hence the fiber  $q_P^{-1}(s) = q_{B,P}(q_B^{-1}(s))$  is finite using the fact that the map  $q_{B,P}$  is surjective. Hence  $q_P$  is quasi-finite over  $\mathfrak{g}^{\mathrm{reg}}$  and thus is also finite over  $\mathfrak{g}^{\mathrm{reg}}$  since  $q_P$  is proper. Let  $\mathfrak{t}^{\mathrm{reg}}$  be the open subscheme of  $\mathfrak{t}$  consisting of regular elements in the Lie algebra  $\mathfrak{t}$  of the torus  $T$ . By the proof of [KW13, Thm. VI.9.1] and the assumption that the characteristic of  $k$  is good for  $G$ , the morphism  $\mathfrak{t}^{\mathrm{reg}} \times G/T \rightarrow \tilde{\mathfrak{g}}^{\mathrm{reg-ss}} : (t, gT) \mapsto (\mathrm{Ad}(g)t, gB)$  is an isomorphism. The Weyl group  $W$  acts on the right on  $\mathfrak{t}^{\mathrm{reg}} \times G/T$  by  $\dot{w}(t, gT) = (\mathrm{Ad}(\dot{w}^{-1})t, g\dot{w}T)$  for  $w \in W$ . Then the composite map  $q'_B : \mathfrak{t}^{\mathrm{reg}} \times G/T \xrightarrow{\sim} \tilde{\mathfrak{g}}^{\mathrm{reg-ss}} \xrightarrow{q_B} \mathfrak{g}^{\mathrm{reg-ss}}$  is a Galois covering with Galois group  $W$ . Consider the morphism  $q'_{B,P} : \mathfrak{t}^{\mathrm{reg}} \times G/T \xrightarrow{\sim} q_B^{-1}(\mathfrak{g}^{\mathrm{reg-ss}}) \xrightarrow{q_{B,P}} q_P^{-1}(\mathfrak{g}^{\mathrm{reg-ss}})$ . One check that  $q'_{B,P}$  factors through  $(\mathfrak{t}^{\mathrm{reg}} \times G/T)/W_P$ , the étale sub-covering of  $q'_B$  associated with the subgroup  $W_P$ . We only need to verify that the induced morphism of varieties  $(\mathfrak{t}^{\mathrm{reg}} \times G/T)/W_P \rightarrow q_P^{-1}(\mathfrak{g}^{\mathrm{reg-ss}})$  is an isomorphism. We may assume  $k$  is algebraically closed. If two  $k$ -points  $(t_1, g_1), (t_2, g_2) \in (\mathfrak{t}^{\mathrm{reg}} \times G)(k)$  are sent to the same point in  $\tilde{\mathfrak{g}}_P$ , then  $\mathrm{Ad}(g_1)t_1 = \mathrm{Ad}(g_2)t_2$  and  $g_1^{-1}g_2 \in P(k)$ . Since  $t_1, t_2$  are regular, their centralizer in  $G$  is  $T$ . Comparing the centralizer of  $\mathrm{Ad}(g_1)t_1$  and  $\mathrm{Ad}(g_2)t_2$  we get  $g_2^{-1}g_1 \in N_G(T)(k) \cap P(k)$ . Thus the image of  $g_2^{-1}g_1$  in the Weyl group lies in  $W_P$ . Hence the map on  $k$ -points  $((\mathfrak{t}^{\mathrm{reg}} \times G/T)/W_P)(k) \rightarrow q_P^{-1}(\mathfrak{g}^{\mathrm{reg-ss}})(k)$  is a bijection. Now using an infinitesimal argument as in Step 2 and Step 5 of the proof of [KW13, Thm. VI.9.1], and notice that both  $(\mathfrak{t}^{\mathrm{reg}} \times G/T)/W_P$  (being an étale covering of the smooth variety  $\mathfrak{g}^{\mathrm{reg-ss}}$ ) and  $q_P^{-1}(\mathfrak{g}^{\mathrm{reg-ss}})$  (being an open subscheme of the smooth variety  $\tilde{\mathfrak{g}}_P$ ) are smooth varieties, we conclude that the map  $(\mathfrak{t}^{\mathrm{reg}} \times G/T)/W_P \rightarrow q_P^{-1}(\mathfrak{g}^{\mathrm{reg-ss}})$  is an isomorphism.  $\square$

We have a surjective proper morphism  $p_P : X = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \rightarrow X_P = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P$  from  $q_{B,P} : \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}_P$  by base change. For  $w \in W$ , we will use the same notation  $w$  to denote the image  $wW_P$  in  $W/W_P$  when it is clear from the context and write  $X_{P,w}, V_{P,w}, U_{P,w}$ , etc. for simplicity. The natural morphism  $G/B \times G/B \rightarrow G/B \times G/P$  sends  $U_{B,w}$  to  $U_{P,w}$  for every  $w \in W$ . Thus the open dense subscheme  $V_{B,w}$  of  $X_w$  is sent into the open dense subscheme  $V_{P,w}$  of  $X_{P,w}$  by  $p_P$ . Since  $X_w$  is a reduced closed subscheme of  $X$  and  $X_{P,w}$  is Zariski closed in  $X_P$ ,  $p_P$  sends each  $X_w$  into  $X_{P,w}$ . We let  $p_{P,w}$  be the restriction of  $p_P$  on  $X_w$ . Then there is a factorization  $\text{pr}_{B,w} : X_w \xrightarrow{p_{P,w}} X_{P,w} \xrightarrow{\text{pr}_{P,w}} \tilde{\mathfrak{g}}$ . Since both  $\text{pr}_{B,w}$  and  $\text{pr}_{P,w}$  are proper birational morphisms by Proposition 3.2.3, so is  $p_{P,w}$ . We have a similarly defined morphism  $p_{P,w}^Y : Y_w \rightarrow Y_{P,w}$  for every  $w \in W$  and we get a sequence of proper birational morphisms  $Y_w \xrightarrow{p_{P,w}^Y} Y_{P,w} \xrightarrow{\text{pr}_{P,w}^Y} \mathfrak{b}$ . Since  $X_{P,w}$  (resp.  $Y_{P,w}$ ) is irreducible, we have

**Proposition 3.2.5.** *For every  $w \in W$ , the morphism  $p_{P,w} : X_w \rightarrow X_{P,w}$  (resp.  $p_{P,w}^Y : Y_w \rightarrow Y_{P,w}$ ) is a proper birational surjection.*

*Remark 3.2.6.* Assume  $w \in W^P$ , then the map  $p_{P,w} : X_w \rightarrow X_{P,w}$  (reps.  $p_{P,w}^Y : Y_w \rightarrow Y_{P,w}$ ) induces an isomorphism of open subvarieties  $V_{B,w} \xrightarrow{\sim} V_{P,w}$  (resp.  $V_{B,w}^Y \xrightarrow{\sim} V_{P,w}^Y$ ): if  $\nu \in \mathfrak{b}$  and  $g = bw \in BwP$  satisfy  $\text{Ad}(g^{-1})\nu \in \mathfrak{p}$ , then  $\text{Ad}(g^{-1})\nu \in \mathfrak{b}$  by Lemma 3.2.7 below.

The following lemma is elementary (see [BT72, Prop. 3.9 (ii)]). But it is the combinatorial reason for several results in this section, therefore we include a proof here.

**Lemma 3.2.7.** *Let  $w \in W$ , then the following statements are equivalent:*

1.  $w \in W^P$ ;
2. If  $\nu \in \mathfrak{b}$ , then  $\text{Ad}(w)^{-1}\nu \in \mathfrak{p}$  if and only if  $\text{Ad}(w)^{-1}\nu \in \mathfrak{b}$ ;
3.  $\{\alpha \in R^+ \mid w(\alpha) \in R^-\} \cap R_P^+ = \emptyset$ ;
4.  $w(R_P^+) \subset R^+$ ;
5.  $\text{Ad}(w)(\mathfrak{m}_P \cap \mathfrak{u}) \subset \mathfrak{u}$ .

and for any  $w \in W$ ,  $\text{lg}_P(w) = |\{\alpha \in R^+ \setminus R_P^+ \mid w(\alpha) \in R^-\}|$ .

*Proof.* We have  $s_\alpha(R^+) = \{-\alpha\} \cup R^+ \setminus \{\alpha\}$  for every  $\alpha \in \Delta$  ([Hum12, Lem. 10.2.B]). Hence for  $\alpha \in \Delta$ ,  $|\{\alpha' \in R^+ \mid ws_\alpha(\alpha') \in R^-\}|$  is equal to  $|\{\alpha' \in R^+ \mid w(\alpha') \in R^-\}| - 1$  if and only if  $w(\alpha) \in R^-$ . Hence  $\text{lg}(ws_\alpha) = \text{lg}(w) - 1$  if and only if  $w(\alpha) \in R^-$ , from which we deduce (3)  $\Rightarrow$  (1). Conversely we assume (1). Then  $w(\Delta_P) \subset R^+$ . Hence (1)  $\Rightarrow$  (3). The equivalence between (3), (4), and (5) is trivial. The assertion (2) is equivalent to that  $w^{-1}(R^+) \cap R_P^- = \emptyset$  or  $w(R_P^-) \cap R^+ = \emptyset$  which is just (4) with a minus sign. Now if  $w \in W$  and we write  $w = w^P w_P$ ,  $w_P \in W_P$ . Since  $w_P(R^+ \setminus R_P^+) = R^+ \setminus R_P^+$ , we get  $\text{lg}(w^P) = \{\alpha \in R^+ \setminus R_P^+ \mid w^P(\alpha) \in R^-\} = \{\alpha \in R^+ \setminus R_P^+ \mid w(\alpha) \in R^-\}$ .  $\square$

We will also need the following lemma.

**Lemma 3.2.8.** *If  $w, w' \in W/W_P$ , then  $X_{P,w} \cap V_{P,w'} \neq \emptyset$  only if  $w \geq w'$  with respect to the Bruhat order on  $W/W_P$ .*

*Proof.* For  $w \in W/W_P$ , let  $\overline{U}_{P,w}$  be the closure of  $U_{P,w}$  in  $G/B \times G/P$ . As  $U_{P,w} \simeq G \times^B BwP/P$  under the isomorphism  $G/B \times G/P \xrightarrow{\sim} G \times^B G/P : (g_1B, g_2P) \mapsto (g_1, g_1^{-1}g_2P)$ , by [Jan07, I.5.21 (2)],  $\overline{U}_{P,w} \simeq G \times^B \overline{BwP/P}$ . For  $w \in W/W_P$ , write  $w^P \in wW_P$  for the shortest representative. Then by definition,  $w \geq w'$  in  $W/W_P$  if and only if  $w^P \geq (w')^P$  in  $W$ . Hence by [Jan07, II.13.8 (4)],  $\overline{Bw'P/P} \subset \overline{BwP/P}$  if and only if  $w \geq w'$  in  $W/W_P$ . In particular,

$Bw'P/P \subset \overline{BwP/P}$  if and only if  $w \geq w'$  in  $W/W_P$ . Since  $\overline{BwP/P}$  is  $B$ -invariant, we get  $\overline{BwP/P} = \cup_{w' \leq w} Bw'P/P$ . Hence  $\overline{U}_{P,w} = \cup_{w' \leq w} U_{P,w'}$ . Thus  $X_{P,w}$  is contained in the closed subspace

$$\{(\nu, g_1B, g_2P) \in X_P \mid (g_1B, g_2P) \in \overline{U}_{P,w}\} = \cup_{w' \leq w} V_{P,w'}$$

of  $X_P$  by definition. As  $V_{P,w'}, w' \in W/W_P$  are pairwise disjoint,  $X_{P,w} \cap V_{P,w'} = \emptyset$  if  $w'$  is not  $\leq w$  in  $W/W_P$ .  $\square$

### 3.2.2 Unibranchness

We recall the notion of unibranchness. A local ring  $R$  is called unibranch if the reduced reduction  $R^{\text{red}}$  is a domain and the integral closure  $R'$  of  $R^{\text{red}}$  in its field of fractions is local. We say that a locally Noetherian scheme  $S$  is unibranch at a point  $s \in S$  if the local ring  $\mathcal{O}_{S,s}$  is unibranch.

**Lemma 3.2.9.** *Let  $R$  be a reduced excellent Noetherian local ring, then  $R$  is unibranch if and only if its completion  $\widehat{R}$  with respect to the maximal ideal is irreducible.*

*Proof.* This is [Gro65, Sch. 7.8.3(vii)].  $\square$

**Proposition 3.2.10.** *Let  $f : Y \rightarrow X$  be a morphism of integral algebraic varieties over a field  $k$ . Assume that  $Y$  is normal and that  $f$  is proper and surjective. If  $x \in X$  is a point such that the fiber  $f^{-1}(x)$  of  $f$  over  $x$  is connected, then the local ring  $\mathcal{O}_{X,x}$  is unibranch and its completion  $\widehat{\mathcal{O}}_{X,x}$  with respect to the maximal ideal is irreducible.*

*Proof.* Let  $\nu : X^\nu \rightarrow X$  be the normalization of  $X$ . Since  $X$  is integral,  $X^\nu$  is integral and the morphism  $\nu$  is a finite surjection (see [Sta22, Tag 035Q] and [Sta22, Tag 035S]). Since  $Y$  is normal, there exists a unique factorization  $f : Y \xrightarrow{f'} X^\nu \xrightarrow{\nu} X$  (see also [Sta22, Tag 035Q]). Since  $f$  is proper and  $\nu$  is finite,  $f'$  is also proper ([Sta22, Tag 01W6]). The image of  $f'$  is then a closed subset of  $X^\nu$ . If  $f'$  is not dominant, the generic point of  $X^\nu$  is not in the image of  $f'$ , then the generic point of  $X$  is not in the image of  $f$ , which contradicts that  $f$  is surjective. Hence  $f'$  is surjective. We have morphisms

$$f^{-1}(x) \xrightarrow{f'} \nu^{-1}(x) \rightarrow \text{Spec}(k(x)),$$

where  $k(x)$  denotes the residue field at  $x \in X$ . Since the morphism  $f^{-1}(x) \xrightarrow{f'} \nu^{-1}(x)$  is surjective and  $f^{-1}(x)$  is connected, we get that  $\nu^{-1}(x)$  is connected. Now assume that  $x$  is contained in an affine open subset  $\text{Spec}(A)$  of  $X$ . Then  $\nu^{-1}(\text{Spec}(A)) = \text{Spec}(A')$  where  $A'$  is the integral closure of  $A$  in its field of fractions. Suppose that  $x$  corresponds to a prime ideal  $\mathfrak{p}$  of  $A$ . Then  $\nu^{-1}(x) = \text{Spec}(A'_\mathfrak{p}/\mathfrak{p}A'_\mathfrak{p})$  where  $A'_\mathfrak{p}$  is also the normalization of  $A_\mathfrak{p}$  in its field of fractions ([Sta22, Tag 0307]). As  $A'_\mathfrak{p}$  is finite over  $A_\mathfrak{p}$ , the fiber  $\text{Spec}(A'_\mathfrak{p}/\mathfrak{p}A'_\mathfrak{p})$  is finite over  $\text{Spec}(k(x))$  and thus is a finite union of discrete points as a topological space. The connectedness of the fiber means that  $\text{Spec}(A'_\mathfrak{p}/\mathfrak{p}A'_\mathfrak{p})$  consists of only one point. Hence there is only one prime ideal  $\mathfrak{p}'$  of  $A'_\mathfrak{p}$  which lies above  $\mathfrak{p}$ . Since finite morphisms are closed,  $\mathfrak{p}'$  is the unique maximal ideal of  $A'_\mathfrak{p}$  and thus  $A'_\mathfrak{p}$  is local. Since  $\mathcal{O}_{X,x}$  is excellent, we get  $\mathcal{O}_{X,x} = A_\mathfrak{p}$  is unibranch and  $\widehat{\mathcal{O}}_{X,x}$  is irreducible by [Gro65, Prop. 7.6.1] and [Gro65, Sch. 7.8.3(vii)].  $\square$

### 3.2.3 Connectedness of fibers over nilpotent elements

We establish the unibranch property for  $X_{P,w}$  (or  $Y_{P,w}$ ) at certain points using Proposition 3.2.10 and the normality of  $X_w$  ([BHS19, Thm. 2.3.6]).

Recall by Proposition 3.2.5, we have a sequence of birational proper surjective morphisms:

$$Y_w \xrightarrow{p_{P,w}^Y} Y_{P,w} \xrightarrow{\text{pr}_{P,w}^Y} \mathfrak{b}.$$

Let  $\mathcal{N}$  be the nilpotent subvariety of  $\mathfrak{g}$  consisting of nilpotent elements (cf. [KW13, VI.3]). Then  $\mathfrak{u} = \mathfrak{b} \cap \mathcal{N}$ . Denote by  $\text{pr}_w^Y := \text{pr}_{B,w}^Y : Y_w \rightarrow \mathfrak{b}$ .

**Proposition 3.2.11.** *If  $\nu \in \mathfrak{u} \subset \mathfrak{b}$  is a closed point and  $w \in W$ , then the closed subset  $(\text{pr}_w^Y)^{-1}(\nu)$  of  $Y_w$  is equal to*

$$\left\{ (\nu, gB) \in \mathfrak{b} \times \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b} \right\}.$$

*Proof.* The result is equivalent to that  $(\text{pr}_w^Y)^{-1}(\mathfrak{u}) = \left\{ (\nu, gB) \in \mathfrak{u} \times \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b} \right\}$  as closed subsets of  $Y_w$ . We have

$$G \times^B (\text{pr}_w^Y)^{-1}(\mathfrak{u}) = \{(\nu, g_1B, g_2B) \in \mathcal{N} \times G/B \times G/B \mid (\nu, g_1B, g_2B) \in X_w\} = \overline{X}_w$$

as closed subsets of  $X_w$  in the notation of [BHS19, §2.4]. The Steinberg variety

$$Z := \{(\nu, g_1B, g_2B) \in \mathcal{N} \times G/B \times G/B \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{u}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{u}\}$$

has irreducible components  $Z_{w'}, w' \in W$  which are the Zariski closures of the following subsets (see §3.2.5 for more details)

$$\{(\nu, g_1B, g_2B) \in \mathcal{N} \times G/B \times G/B \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{u}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{u}, g_1^{-1}g_2 \in Bw'B/B\}.$$

The union of the irreducible components  $Z_{w'}$  for  $w' \leq w$  of the Steinberg variety is then the closed subset

$$\begin{aligned} & \left\{ (\nu, g_1B, g_2B) \in \mathcal{N} \times G/B \times G/B \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{u}, \text{Ad}(g_2^{-1})\nu \in \mathfrak{u}, g_1^{-1}g_2 \in \overline{BwB/B} \right\} \\ &= G \times^B \left\{ (\nu, gB) \in \mathfrak{u} \times \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{u} \right\}, \end{aligned}$$

by the usual closure relation  $\overline{BwB/B} = \cup_{w' \leq w} Bw'B/B$  ([Jan07, II.13.7]). By [BHS19, Thm. 2.4.7] and the discussion after it, we have  $\overline{X}_w = \cup_{w' \leq w} Z_{w'}$ . Then we get

$$G \times^B (\text{pr}_w^Y)^{-1}(\mathfrak{u}) = G \times^B \left\{ (\nu, gB) \in \mathfrak{u} \times \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{u} \right\}$$

as closed subsets of  $X_w$ . Hence  $(\text{pr}_w^Y)^{-1}(\mathfrak{u}) = \left\{ (\nu, gB) \in \mathfrak{u} \times \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{u} \right\}$  as closed subsets of  $Y_w$ . Finally, we have an equality

$$\left\{ (\nu, gB) \in \mathfrak{u} \times \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{u} \right\} = \left\{ (\nu, gB) \in \mathfrak{u} \times \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b} \right\}$$

of closed subsets since the two sides are both closed subschemes in  $Y_w$  and contain same closed points.  $\square$

Now if  $w \in W$  is the *longest* element in  $wW_P$ , then  $\overline{BwB/B}$  is the preimage of  $\overline{BwP/P}$  via the natural projection  $G/B \rightarrow G/P$  (cf. [Jan07, II.13.8 (2)]). In particular,  $\overline{BwBP} = \overline{BwB}$ .

**Proposition 3.2.12.** *Assume that  $w \in W$  is the longest element in  $wW_P$ . If  $x = (\nu_x, g_xP) \in Y_{P,w} \subset \mathfrak{b} \times \overline{BwP/P}$  is a closed point such that  $\nu_x$  is nilpotent (i.e.  $\nu_x \in \mathfrak{u}$ ), then the fiber  $(p_{P,w}^Y)^{-1}(x)$  is connected.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccccccc}
(p_{P,w}^Y)^{-1}(x) & \hookrightarrow & (\mathrm{pr}_w^Y)^{-1}(\nu_x) & \longrightarrow & Y_w & \longrightarrow & \mathfrak{b} \times \overline{BwB/B} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
x & \longrightarrow & (\mathrm{pr}_{P,w}^Y)^{-1}(\nu_x) & \longrightarrow & Y_{P,w} & \longrightarrow & \mathfrak{b} \times \overline{BwP/P}
\end{array}$$

where each horizontal arrow is a closed embedding and each vertical arrow is surjective and projective. One sees that the formation of the varieties  $Y_{P,w}$  commutes with base change by fields: the formation of the varieties  $V_{P,w}^Y$  and  $Y_P$  commutes with base change by definition and after base change to a separable closure of  $k$ , the Zariski closure of  $V_{P,w}^Y$  with the reduced structure is still irreducible and descends (cf. [Bor12, Cor. AG.14.6]). And the fiber  $k(x) \times_{Y_{P,w} \times_k k(x)} (Y_w \times_k k(x)) = k(x) \times_{Y_{P,w} \times_k k(x)} (Y_{P,w} \times_k k(x)) \times_{Y_{P,w}} Y_w = k(x) \times_{Y_{P,w}} Y_w$ . Thus we may assume  $k(x) = k$  by base change. The composition

$$(\mathrm{pr}_w^Y)^{-1}(\nu_x) = \left\{ (\nu_x, gB) \in \mathfrak{b} \times \overline{BwB/B} \mid \mathrm{Ad}(g^{-1})\nu_x \in \mathfrak{b} \right\} \rightarrow (\mathrm{pr}_{P,w}^Y)^{-1}(\nu_x) \hookrightarrow \nu_x \times \overline{BwP/P}$$

can be identified with the morphism

$$\left\{ gB \in \overline{BwB/B} \mid \mathrm{Ad}(g^{-1})\nu_x \in \mathfrak{b} \right\} \rightarrow \overline{BwP/P} : gB \mapsto gP, \quad (3.2.13)$$

where we only consider the underlying reduced varieties and have used Proposition 3.2.11. To show that the fiber  $(p_{P,w}^Y)^{-1}(x)$  is connected, we only need to show that the morphism (3.2.13) has connected fibers. We pick a closed point  $g_x B$  in the fiber of  $g_x P$ . The fiber over  $g_x P$  is

$$\left\{ gB \in \overline{BwB/B} \mid \mathrm{Ad}(g^{-1})\nu_x \in \mathfrak{b}, g_x^{-1}g \in P/B \right\} \simeq \left\{ gB \in P/B \mid \mathrm{Ad}(g^{-1})(\mathrm{Ad}(g_x^{-1})\nu_x) \in \mathfrak{b} \right\}$$

since  $g \in P/B$  implies that  $g_x g \in \overline{BwBP/B} = \overline{BwB/B}$  by the assumption on  $w$ . To show that the latter is connected, we can assume that  $g_x$  is trivial and  $\nu_x \in \mathfrak{u}$  by replacing  $(\nu_x, g_x B)$  with  $(\mathrm{Ad}(g_x^{-1})\nu_x, B)$ . Assume that  $P = M_P N_P$  is the standard Levi decomposition and  $\mathfrak{p} = \mathfrak{m}_P + \mathfrak{n}_P$  where  $\mathfrak{m}_P$  (resp.  $\mathfrak{n}_P$ ) is the Lie algebra of  $M_P$  (resp.  $N_P$ ). Let  $B_{M_P} = B \cap M_P$ ,  $\mathfrak{b}_{M_P}$  be its Lie algebra and  $\mathfrak{u}_{M_P}$  be the variety of nilpotent elements in  $\mathfrak{b}_{M_P}$ . We have  $P/B \simeq M_P/B_{M_P}$  ([Jan07, II.1.8 (5)]). We can decompose  $\nu_x = m_x + n_x$  where  $m_x \in \mathfrak{u}_{M_P}$  and  $n_x \in \mathfrak{n}_P$ . Since  $\mathrm{Ad}(P)\mathfrak{n}_P \subset \mathfrak{n}_P$ , an element  $gB_{M_P} \in M_P/B_{M_P}$  satisfies  $\mathrm{Ad}(g^{-1})\nu_x \in \mathfrak{b}$  if and only if  $\mathrm{Ad}(g^{-1})m_x \in \mathfrak{b}_{M_P}$ . Hence there is an isomorphism

$$\left\{ gB \in P/B \mid \mathrm{Ad}(g^{-1})(\nu_x) \in \mathfrak{b} \right\} \simeq \left\{ gB_{M_P} \in M_P/B_{M_P} \mid \mathrm{Ad}(g^{-1})(m_x) \in \mathfrak{b}_{M_P} \right\}.$$

As a closed subspace of  $M_P/B_{M_P}$ , this is the Springer fiber: the fiber of the Springer resolution  $\tilde{\mathcal{N}}_{M_P} \rightarrow \mathcal{N}_{M_P}$  over the point  $m_x \in \mathcal{N}_{M_P}$  where  $\mathcal{N}_{M_P}$  is the nilpotent variety of  $\mathfrak{m}_P^1$ . The nilpotent variety is normal and, unlike the Grothendieck resolution, the Springer resolution is birational. Hence Springer fibers are connected by Zariski's main theorem (cf. [CG97, Rem. 3.3.26] or [Yun16, §1.4.1]).  $\square$

**Theorem 3.2.14.** *If  $x = (\nu, g_1 B, g_2 P) \in X_{P,w}$  (resp.  $x = (\nu, gP) \in Y_{P,w}$ ) is a closed point such that  $\nu$  is nilpotent, then the local ring  $\mathcal{O}_{X_{P,w},x}$  (resp.  $\mathcal{O}_{Y_{P,w},x}$ ) is unibranch and the completion  $\hat{\mathcal{O}}_{X_{P,w},x}$  (resp.  $\hat{\mathcal{O}}_{Y_{P,w},x}$ ) is irreducible.*

<sup>1</sup>The Springer fiber is the reduced subvariety associated with the subscheme  $\{gB_{M_P} \in M_P/B_{M_P} \mid \mathrm{Ad}(g^{-1})(m_x) \in \mathfrak{u}_{M_P}\}$  which shares the same underlying topological space with the subscheme  $\{gB_{M_P} \in M_P/B_{M_P} \mid \mathrm{Ad}(g^{-1})(m_x) \in \mathfrak{b}_{M_P}\}$  since they have the same closed points (cf. [Yun16, §1.2]).

*Proof.* Let  $w \in W$  such that  $w$  is the longest element in  $wW_P$ . Consider the surjective birational proper morphism  $p_{P,w} : X_w \rightarrow X_{P,w}$  (resp.  $p_{P,w}^Y : Y_w \rightarrow Y_{P,w}$ ) of integral varieties. The fiber  $p_{P,w}^{-1}(x)$  (resp.  $(p_{P,w}^Y)^{-1}(x)$ ) is connected by Proposition 3.2.12 and  $X_w$  (resp.  $Y_w$ ) is normal by [BHS19, Thm. 2.3.6]. Hence  $X_{P,w}$  (resp.  $Y_{P,w}$ ) is unibranch at  $x$  and the completion  $\hat{\mathcal{O}}_{X_{P,w},x}$  (resp.  $\hat{\mathcal{O}}_{Y_{P,w},x}$ ) is irreducible by Proposition 3.2.10.  $\square$

*Remark 3.2.15.* The above results are true in generality. The assumptions in Proposition 3.2.12 that  $\nu_x$  is nilpotent and  $w$  is the longest element in  $wW_P$  can be removed. Using a Bott-Samelson-Demazure type resolution for  $X_{P,w}$  in [Ric08, §1.7], one can show that for any  $w \in W$ , the fiber of  $p_{P,w} : X_w \rightarrow X_{P,w}$  over any point  $x \in X_{P,w}$  is connected. Thus by Proposition 3.2.12 and [BHS19, Thm. 2.3.6],  $X_{P,w}$  is unibranch at all points. One can see also from the proof of Proposition 3.2.12 that  $X_{P,w}$  is in fact geometrically unibranch ([Gro64, §23.2.1]). Moreover, Proposition 3.2.11 can be proved directly without using [BHS19, Thm. 2.4.7].

**Corollary 3.2.16.** *If  $x = (\nu, g_1 B, g_2 P) \in X_{P,w}$  is a closed point such that  $\nu$  is nilpotent, then the irreducible components of  $\text{Spec}(\hat{\mathcal{O}}_{X_{P,w},x})$  are  $\text{Spec}(\hat{\mathcal{O}}_{X_{P,w},x})$  for  $w \in W/W_P$  such that  $x \in X_{P,w}$ .*

*Proof.* Suppose that  $R$  is a reduced local excellent Noetherian ring with minimal prime ideals  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  such that every  $R/\mathfrak{p}_i$  is unibranch. By definition, the normalization  $R'$  of  $R$  is a product of  $(R/\mathfrak{p}_i)'$ , the normalizations of  $R/\mathfrak{p}_i$ . Since  $R/\mathfrak{p}_i$  is unibranch,  $(R/\mathfrak{p}_i)'$  is local and thus the number of maximal ideals of  $R'$  is  $m$ . By [Gro65, Sch. 7.8.3(vii)], minimal prime ideals of  $\hat{R}$  correspond bijectively to maximal ideals of  $R'$ . Hence there are exactly  $m$  minimal ideals of  $\hat{R}$ . Since the quotients  $\hat{R}/\mathfrak{p}_i \hat{R} = \widehat{R/\mathfrak{p}_i}$  of  $\hat{R}$  are integral, they correspond to all irreducible components of  $\text{Spec}(\hat{R})$ .  $\square$

### 3.2.4 The weight map

We prove some results for the weight map of  $X_{P,w}$ . Since the characteristic of  $k$  is very good for  $G$ , the ring morphisms  $S(\mathfrak{g}^*) \hookrightarrow S(\mathfrak{g}^*)^G \xrightarrow{\sim} S(\mathfrak{t}^*)^W$  induce a morphism  $\gamma_G : \mathfrak{g} \rightarrow \mathfrak{t}/W$  of  $k$ -schemes ([KW13, VI.8]). Applying this fact to the standard Levi subgroup  $M_P$  of  $P$  we get a map  $\gamma_{M_P} : \mathfrak{m}_P \rightarrow \mathfrak{t}/W_P$ . We define a map  $\kappa_P : \tilde{\mathfrak{g}}_P \rightarrow \mathfrak{t}/W_P : (\nu, gP) \mapsto \gamma_{M_P}(\overline{\text{Ad}(g^{-1})\nu})$  where  $\overline{\text{Ad}(g^{-1})\nu}$  denotes the image of  $\text{Ad}(g^{-1})\nu$  in  $\mathfrak{m}_P$  under the projection  $\mathfrak{p} \twoheadrightarrow \mathfrak{p}/\mathfrak{n}_P \xrightarrow{\sim} \mathfrak{m}_P$ . Let  $\kappa_1 : X_P \rightarrow \mathfrak{t}$  be the map sending  $(\nu, g_1 B, g_2 P)$  to the image of  $\text{Ad}(g_1^{-1})\nu$  in  $\mathfrak{t} = \mathfrak{b}/\mathfrak{u}$  and  $\kappa_2$  be the composition  $X_P = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P \rightarrow \tilde{\mathfrak{g}}_P \xrightarrow{\kappa_P} \mathfrak{t}/W_P$ . We have the following commutative diagram

$$\begin{array}{ccccc} \tilde{\mathfrak{g}} & \longrightarrow & \tilde{\mathfrak{g}}_P & \longrightarrow & \mathfrak{g} \\ \kappa_B \downarrow & & \kappa_P \downarrow & & \kappa_G = \gamma_G \downarrow \\ \mathfrak{t} & \longrightarrow & \mathfrak{t}/W_P & \longrightarrow & \mathfrak{t}/W \end{array}$$

where the horizontal arrows are natural projections. For  $i = 1, 2, w \in W/W_P$ , let  $\kappa_{i,w}$  be the restriction of  $\kappa_i$  to the closed subscheme  $X_{P,w}$ .

**Lemma 3.2.17.** *We have the following commutative diagram*

$$\begin{array}{ccc} X_{P,w} & \xrightarrow{\kappa_{2,w}} & \mathfrak{t}/W_P \\ \kappa_{1,w} \downarrow & \nearrow \alpha & \downarrow \\ \mathfrak{t} & \longrightarrow & \mathfrak{t}/W \end{array}$$

where the map  $\mathfrak{t} \rightarrow \mathfrak{t}/W$  and  $\mathfrak{t}/W_P \rightarrow \mathfrak{t}/W$  are natural projections and the map  $\alpha : \mathfrak{t} \rightarrow \mathfrak{t}/W_P$  is the composition of the map  $\text{Ad}(\dot{w}^{-1}) : \mathfrak{t} \rightarrow \mathfrak{t}$  with the projection  $\mathfrak{t} \rightarrow \mathfrak{t}/W_P$  (thus  $\alpha$  depends only on the class of  $w$  in  $W/W_P$ ).

*Proof.* This is a generalization of [BHS19, Lem. 2.3.4]. We only need to show  $\kappa_{2,w} = \alpha \circ \kappa_{1,w}$ . Since  $X_{P,w}$  is the closure of  $V_{P,w} = \{(\nu, g_1 B, g_2 P) \in X_P \mid g_1^{-1} g_2 \in BwP\}$  in  $X_P$ , we only need to verify  $\kappa_{2,w} = \text{Ad}(\dot{w}^{-1})\kappa_{1,w}$  when restricted to  $V_{P,w}$ . Let  $x = (\nu, g_1 B, g_2 P) \in X_P(S)$  for a  $k$ -algebra  $S$  and by replacing  $S$  by some fppf extension, we assume  $g_2 = g_1 \dot{w} \in G(S)$ . Then  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{b}(S)$  and  $\text{Ad}(\dot{w}^{-1})\text{Ad}(g_1^{-1})\nu \in \mathfrak{p}(S)$ . We assume  $w \in W^P$ . Then we have  $\text{Ad}(\dot{w}^{-1})\text{Ad}(g_1^{-1})\nu \in \mathfrak{b}(S)$  (cf. Lemma 3.2.7). The image of  $\text{Ad}(g_2^{-1})\nu$  in  $\mathfrak{m}_P(S) = \mathfrak{p}/\mathfrak{n}_P(S)$ , denoted by  $\overline{\text{Ad}(g_2^{-1})\nu}$ , lies in the subset  $\mathfrak{b}_{M_P}(S) = \mathfrak{m}_P(S) \cap \mathfrak{b}(S)$ . Let  $t$  denote the image of  $\overline{\text{Ad}(g_2^{-1})\nu}$  in  $\mathfrak{t}(S) = \mathfrak{t}_{M_P}(S) = \mathfrak{b}_{M_P}(S)/\mathfrak{u}_{M_P}(S)$ . Then  $\kappa_2(x) = \gamma_{M_P}(\overline{\text{Ad}(g_2^{-1})\nu})$  is the image of  $t$  in  $(\mathfrak{t}/W_P)(S)$  via the map  $\mathfrak{t} \rightarrow \mathfrak{t}/W_P$  (cf. [KW13, Thm. VI.8.3]). We have that  $\kappa_{1,w}(x)$  is the image of  $\text{Ad}(g_1^{-1})\nu$  in  $\mathfrak{t}(S) = \mathfrak{b}/\mathfrak{u}(S)$ , thus  $t = \text{Ad}(\dot{w}^{-1})\kappa_{1,w}(x)$ . Hence  $\kappa_{2,w} = \alpha \circ \kappa_{1,w}$ .  $\square$

Now let  $T_P := \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}/W_P$  and for all  $w \in W/W_P$ , let  $T_{P,w} = \{(z, \text{Ad}(\dot{w}^{-1})z) \mid z \in \mathfrak{t}\} \subset \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}/W_P$  be closed subschemes of  $T_P$ . Then  $T_{P,w} \simeq \mathfrak{t}$  is smooth for any  $w \in W/W_P$ . Similar to [BHS19, Lem. 2.5.1],  $T_P$  is equidimensional and  $\{T_{P,w} \mid w \in W/W_P\}$  is the set of irreducible components of  $T_P$ . We have a map  $(\kappa_1, \kappa_2) : X_P \rightarrow T_P$  and  $X_{P,w}$  is the unique irreducible component of  $X_P$  that dominates  $T_{P,w}$  by Lemma 3.2.17 (the dominance comes from the factorization  $\kappa_1 : X_{P,w} \xrightarrow{\text{pr}_{P,w}} \mathfrak{g} \xrightarrow{\kappa_B} \mathfrak{t} \simeq T_{P,w}$  and that  $\text{pr}_{P,w}$  is surjective by Proposition 3.2.3). Suppose that  $x = (\nu, g_1 B, g_2 P) \in X_P$  is a closed point such that  $\nu$  is nilpotent and let  $(0, 0) = (\kappa_1(x), \kappa_2(x)) \in T_P$ . If  $x \in X_{P,w} \subset X_P$  for some  $w \in W/W_P$ , we let  $\widehat{X}_{P,x}$  (resp.  $\widehat{X}_{P,w,x}$ , resp.  $\widehat{T}_{P,(0,0)}$ , resp.  $\widehat{T}_{P,w,(0,0)}$ ) be the completion of  $X_P$  at  $x$  (resp.  $X_{P,w}$  at  $x$ , resp.  $T_P$  at  $(0, 0)$ , resp.  $T_{P,w}$  at  $(0, 0)$ ). Since by Theorem 3.2.14, the structure ring of  $\widehat{X}_{P,w,x}$  is irreducible, using the same argument for [BHS19, Lem. 2.5.2], we get the following lemma.

**Lemma 3.2.18.** *The map  $\widehat{X}_{P,w',x} \hookrightarrow \widehat{X}_{P,x} \rightarrow \widehat{T}_{P,(0,0)}$  induced by the completions of the closed embedding  $X_{P,x} \hookrightarrow X_P$  and the map  $(\kappa_1, \kappa_2)$  factors through  $\widehat{T}_{P,w,(0,0)} \hookrightarrow \widehat{T}_{P,(0,0)}$  if and only if  $w' = w$  in  $W/W_P$ .*

### 3.2.5 Generalized Steinberg varieties

We shall study certain vanishing properties of irreducible components of generalized Steinberg varieties which might be well-known from the perspective of geometric representation theory (at least for the case when  $P = B$ , see §3.2.6). These vanishing properties will be the major new ingredients in the global applications of the local models for the trianguline variety.

We pick a standard parabolic subgroup  $Q = M_Q N_Q$  of  $G$  with Lie algebra  $\mathfrak{q} = \mathfrak{m}_Q + \mathfrak{n}_Q$  and Weyl group  $W_Q$ . Let  ${}^Q W$  be the set of elements  $w \in W$  such that  $w$  is the shortest element in the coset  $W_Q w$ . We consider the following scheme depending on the choice of the two parabolic subgroups  $P$  and  $Q$

$$Z_{Q,P} := \{(\nu, g_1 B, g_2 P) \in \mathcal{N} \times G/B \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}\}.$$

As there is an isomorphism  $\widetilde{\mathcal{N}} := \{(\nu, g) \in \mathcal{N} \times G/B \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b}\} \simeq G \times^B \mathfrak{u}$ , we can replace  $\mathcal{N}$  in the above definition by  $\mathfrak{g}$  (cf. [CG97, §3.2]). When  $Q = B$ ,  $Z_P := Z_{B,P}$  is some generalized Steinberg variety considered in [DR04]. We have a natural closed embedding

$$Z_{Q,P} \hookrightarrow Z_P$$

and generally,  $Z_{Q',P} \subset Z_{Q,P}$  if  $Q \subset Q'$ . For any  $w \in W_Q \backslash W/W_P$ , we let  $Z_{Q,P,w}$  be the Zariski closure of the subset  $H_{Q,P,w} := \{(\nu, g_1 B, g_2 P) \in Z_{Q,P} \mid g_1^{-1} g_2 \in QwP/P\}$  in  $Z_{Q,P}$  with the

reduced induced scheme structure. We write  $Z_{P,w} := Z_{B,P,w}$  for every  $w \in W/W_P$ . There is a unique shortest element  $w \in W$  in each double coset  $W_Q w W_P \in W_Q \backslash W/W_P$  and  $w \in W$  is the shortest element in  $W_Q w W_P$  if and only if  $w \in W^P \cap {}^Q W$  ([TSP<sup>+</sup>20, Prop. 2]).

**Proposition 3.2.19.** *1. The scheme  $Z_{Q,P,w}$  is irreducible and has dimension no more than  $\dim G - \dim T$ .*

*2.  $Z_P$  is equidimensional of dimension  $\dim G - \dim T$  with irreducible components  $Z_{P,w}, w \in W/W_P$ .*

*3. For any  $w \in W_Q \backslash W/W_P$ , the following statements are equivalent:*

- (a)  $Z_{Q,P,w} = Z_{P,w'}$  for some  $w' \in W^P$ ;*
- (b)  $\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} \subset \mathfrak{n}_Q$  (this condition is independent of the representative of  $w$  in  $W$ );*
- (c) if we take a representative  $w \in W^P \cap {}^Q W$ , then  $w_{Q,0}w \in W^P$ .*

*And if the above statements hold,  $w'W_P = w_{Q,0}wW_P$  where  $w \in W^P \cap {}^Q W$ .*

*4.  $Z_{Q,B}$  is equidimensional of dimension  $\dim G - \dim T$  with irreducible components*

$$Z_{B,w_{Q,0}w}, w \in {}^Q W.$$

*Proof.* Take a representative  $w \in W$  for  $w \in W_Q \backslash W/W_P$  and we write  $w$  instead of  $\dot{w}$  for simplicity.

(1) Let  $\overline{Z}_{Q,P} := \{(\nu, g_1 Q, g_2 P) \in \mathcal{N} \times G/Q \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}\}$  be a generalized Steinberg variety in [DR04]. Then  $Z_{Q,P} = \overline{Z}_{Q,P} \times_{G/Q} G/B$  and the natural morphism  $Z_{Q,P} \rightarrow \overline{Z}_{Q,P}$  is a locally trivial fibration of relative dimension  $\dim Q - \dim B$ . Let

$$\overline{H}_{Q,P,w} := \{(\nu, g_1 Q, g_2 P) \in \mathcal{N} \times G/Q \times G/P \mid \text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q, \text{Ad}(g_2^{-1})\nu \in \mathfrak{p}, g_1^{-1}g_2 \in QwP/P\}$$

and let  $\overline{Z}_{Q,P,w}$  be the Zariski closure of  $\overline{H}_{Q,P,w}$  in  $\overline{Z}_{Q,P}$ . Then  $H_{Q,P,w} = \overline{H}_{Q,P,w} \times_{G/Q} G/B$ .

We work as in Proposition 3.2.3 or [BHS19, Prop. 2.2.5]. The projection  $\overline{H}_{Q,P,w} \rightarrow G \cdot (Q, wP) \subset G/Q \times G/P$  is  $G$ -equivariant (with respect to the diagonal action of  $G$  on the double flag variety). The fiber over the point  $(Q, wP)$  is the affine space  $\mathfrak{n}_Q \cap \text{Ad}(w)\mathfrak{p}$ . The  $G$ -orbit  $G \cdot (Q, wP)$  is smooth, irreducible of dimension  $\dim G - \dim Q \cap wPw^{-1}$ . By Lemma [DR04, Lem. 2.3],  $\dim G - \dim Q \cap wPw^{-1} = \dim \mathfrak{n}_{wPw^{-1}} + \dim \mathfrak{n}_Q - \dim(\mathfrak{n}_Q \cap \mathfrak{n}_{wPw^{-1}}) = \dim \mathfrak{n}_P + \dim \mathfrak{n}_Q - \dim(\mathfrak{n}_Q \cap \text{Ad}(w)\mathfrak{n}_P)$ . Thus by [BHS19, Lem. 2.2.2],  $H_{Q,P,w}$  is a vector bundle over  $G \cdot (Q, wP) \times_{G/Q} G/B = \{(g_1 B, g_2 P) \in G/B \times G/P \mid g_1^{-1}g_2 \in QwP\}$  and is smooth of dimension  $(\dim G - \dim T) - \dim \mathfrak{n}_Q - \dim \mathfrak{u} + \dim \mathfrak{n}_P + \dim \mathfrak{n}_Q + \dim \mathfrak{n}_Q \cap \text{Ad}(w)\mathfrak{m}_P = (\dim G - \dim T) - (\dim \mathfrak{u} - \dim \mathfrak{n}_P - \dim \mathfrak{n}_Q \cap \text{Ad}(w)\mathfrak{m}_P) = (\dim G - \dim T) - (\dim \mathfrak{u} \cap \text{Ad}(w)\mathfrak{m}_P - \dim \mathfrak{n}_Q \cap \text{Ad}(w)\mathfrak{m}_P)$  (the last equality can be deduced from Lemma 3.2.7 (4) if we take  $w \in W^P$ ). Hence  $Z_{Q,P,w}$  is also irreducible of dimension  $(\dim G - \dim T) - (\dim \mathfrak{u} \cap \text{Ad}(w)\mathfrak{m}_P - \dim \mathfrak{n}_Q \cap \text{Ad}(w)\mathfrak{m}_P) \leq \dim G - \dim T$ .

(2) If  $Q = B$ , then  $\dim \mathfrak{u} \cap \text{Ad}(w)\mathfrak{m}_P - \dim \mathfrak{n}_Q \cap \text{Ad}(w)\mathfrak{m}_P = 0$ , the result follows.

(3) By the proof in (1), we see that the dimension of  $Z_{Q,P,w}$  is equal to  $\dim G - \dim T$  if and only if  $\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} \subset \mathfrak{n}_Q$ . This proves (a)  $\Leftrightarrow$  (b). In fact, for any  $w' \in W_Q$ ,  $\dim \text{Ad}(w'w)\mathfrak{m}_P \cap \mathfrak{u} = \dim(\mathfrak{u} - \mathfrak{n}_P) = \dim \text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u}$ . Thus if  $\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} \subset \mathfrak{n}_Q$ , then  $\text{Ad}(w'w)\mathfrak{m}_P \cap \mathfrak{u} \supset \text{Ad}(w')(\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u})$  of the same dimension and hence  $\text{Ad}(w'w)\mathfrak{m}_P \cap \mathfrak{u} = \text{Ad}(w')(\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u}) \subset \mathfrak{n}_Q$ .

Now we take  $w \in W^P$ . Then  $\text{Ad}(w)(\mathfrak{m}_P \cap \mathfrak{u}) \subset \mathfrak{u}$  by Lemma 3.2.7 and in this case, we have (similarly)  $\text{Ad}(w)(\mathfrak{m}_P \cap \overline{\mathfrak{u}}) \subset \overline{\mathfrak{u}}$  and therefore  $\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} = \text{Ad}(w)(\mathfrak{m}_P \cap \mathfrak{u})$ .

(b)  $\Rightarrow$  (c): Since  $w \in W^P$ , we get  $\text{Ad}(w)(\mathfrak{m}_P \cap \mathfrak{u}) = \text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} \subset \mathfrak{n}_Q$ . As for any  $w' \in W_Q$ ,  $\text{Ad}(w')\mathfrak{n}_Q = \mathfrak{n}_Q$ , we have  $\text{Ad}(w'w)(\mathfrak{m}_P \cap \mathfrak{u}) \subset \mathfrak{n}_Q \subset \mathfrak{u}$  and we conclude by Lemma 3.2.7 that  $w'w \in W^P$  for any  $w' \in W_Q$ .



(c)  $\Rightarrow$  (b): We have  $\text{Ad}(w_{Q,0})(\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u}) = \text{Ad}(w_{Q,0}w)(\mathfrak{m}_P \cap \mathfrak{u})$  is contained in  $\mathfrak{u}$  by Lemma 3.2.7. Since  $\text{Ad}(w_{Q,0})\mathfrak{n}_Q = \mathfrak{n}_Q$ ,  $\text{Ad}(w_{Q,0})(\mathfrak{m}_Q \cap \mathfrak{u}) = \mathfrak{m}_Q \cap \bar{\mathfrak{u}}$  and  $\mathfrak{u} = \mathfrak{m}_Q \cap \mathfrak{u} + \mathfrak{n}_Q$ , we get

$$\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} = \text{Ad}(w_{Q,0}w_{Q,0})(\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u}) \subset \text{Ad}(w_{Q,0})\mathfrak{u} \cap \mathfrak{u} = \mathfrak{n}_Q.$$

Assume above statements hold. We take  $w \in W^P \cap {}^QW$  in (c), then  $\text{lg}(w_{Q,0}w) = \text{lg}(w_{Q,0}) + \text{lg}(w)$ . Hence  $Bw_{Q,0}BwB = Bw_{Q,0}wB$  (cf. [Jan07, II.13.5 (7)]). As  $Q = \overline{Bw_{Q,0}B}$  ([Jan07, II.13.2 (6)]),  $Bw_{Q,0}wP = Bw_{Q,0}BwP \subset QwP$ . Similarly,  $QwP = \overline{Bw_{Q,0}B}BwBP \subset \overline{Bw_{Q,0}BwBP} = \overline{Bw_{Q,0}wP}$ . Let

$$H'_{Q,P,w} := \{(\nu, g_1B, g_2P) \in Z_{Q,P} \mid g_1^{-1}g_2 \in Bw_{Q,0}wP/P\}.$$

By the discussions in (1) and that  $Bw_{Q,0}wP$  is open dense in  $QwP$  (since  $Bw_{Q,0}wP$  is open dense in  $\overline{QwP} = \overline{Bw_{Q,0}wP}$  which contains  $QwP$ ), the Zariski closure of  $H'_{Q,P,w}$  is  $Z_{Q,P,w}$ . As  $H'_{Q,P,w} \subset Z_{P,w_{Q,0}w}$ , we get  $Z_{Q,P,w} \subset Z_{P,w_{Q,0}w}$ . Since  $\dim Z_{Q,P,w} = \dim Z_{P,w_{Q,0}w}$ , we conclude that  $Z_{Q,P,w} = Z_{P,w_{Q,0}w}$ .

(4) When  $P = B$ , we have  $\text{Ad}(w)\mathfrak{m}_B \cap \mathfrak{u} = \{0\} \subset \mathfrak{n}_Q$  for any  $w \in W_Q \setminus W$ , thus the result follows from (3).  $\square$

*Remark 3.2.20.* The result that the scheme  $\overline{Z}_{Q,B}$  in the proof of (1) of Proposition 3.2.19 (resp.  $Z_{B,P}$ ) is equidimensional with the irreducible components parameterized by  ${}^QW$  (resp.  $W^P$ ) is already known by [DR04, Thm. 4.1] (resp. [DR04, Thm. 3.1])

**Corollary 3.2.21.** *For any  $w \in W/W_P$ ,  $Z_{P,w}$  is contained in  $Z_{Q,P}$  if and only if  $wW_P = w_{Q,0}w_1W_P$  for some  $w_1 \in W^P \cap {}^QW$  such that  $w_{Q,0}w_1 \in W^P$ .*

*Proof.* Assume  $Z_{P,w} \subset Z_{Q,P}$ . Since  $Z_{Q,P} = \bigcup_{w' \in W_Q \setminus W/W_P} Z_{Q,P,w'}$  and each  $Z_{Q,P,w'}$  is irreducible, we get  $Z_{P,w} \subset Z_{Q,P,w'}$  for some  $w'$ . But  $Z_{Q,P,w'}$  has dimension no more than  $\dim Z_{P,w}$ . Hence  $Z_{P,w} = Z_{Q,P,w'}$ . Now the result follows from (3) of Proposition 3.2.19.  $\square$

**Corollary 3.2.22.** *Let  $x$  be a point of  $Z_{Q,B}$ , then there exists an irreducible component  $Z_{B,w}$  of  $Z_B$  such that  $x \in Z_{B,w}$  and  $Z_{B,w} \subset Z_{Q,B}$ .*

*Remark 3.2.23.* The above result for points on  $Z_{Q,B}$  doesn't hold in general for  $P \neq B$ . For example, if  $Q = G$ , then  $Z_{G,P} = Z_{G,P,e} \subsetneq Z_{P,w_0}$  if  $P \neq B$ .

**Definition 3.2.24.** 1. Let  $\mathbf{h} \in X_*(T)^{W_P}$  be an antidominant coweight (namely  $\mathbf{h} \in X_*(T)$ ,  $\langle \alpha, \mathbf{h} \rangle = 0, \forall \alpha \in \Delta_P$  and  $\langle \alpha, \mathbf{h} \rangle \leq 0, \forall \alpha \in \Delta$ ). We say  $\mathbf{h}$  is  $P$ -regular if  $\langle \alpha, \mathbf{h} \rangle < 0, \forall \alpha \in \Delta \setminus \Delta_P$ .

2. For  $\mathbf{h} \in X_*(T)$ , we say  $\mathbf{h}$  is strictly  $Q$ -dominant if  $\langle \alpha, \mathbf{h} \rangle > 0, \forall \alpha \in \Delta_Q$ .

**Lemma 3.2.25.** *If  $\mathbf{h} \in X_*(T)^{W_P}$  is  $P$ -regular antidominant, then the set of  $\alpha \in R$  such that  $\langle \alpha, \mathbf{h} \rangle < 0$  (resp.  $= 0$ , resp.  $> 0$ ) is  $R^+ \setminus R_P^+$  (resp.  $R_P$ , resp.  $R^- \setminus R_P^-$ ).*

**Theorem 3.2.26.** *For any  $w \in W/W_P$ ,  $Z_{P,w}$  is contained in  $Z_{Q,P}$  if and only if  $w(\mathbf{h})$  is strictly  $Q$ -dominant for some (or every)  $P$ -regular antidominant coweight  $\mathbf{h} \in X_*(T)^{W_P}$ .*

*Proof.* We take an arbitrary  $P$ -regular antidominant coweight  $\mathbf{h} \in X_*(T)^{W_P}$ . Take the representative  $w \in W^P$  for  $w \in W/W_P$  and write  $w$  for  $\dot{w}$ . The statement  $\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} \subset \mathfrak{n}_Q$  (which is implied by that  $Z_{P,w}$  is contained in  $Z_{Q,P}$  by (3) of Proposition 3.2.19) is equivalent to that  $w(R_P) \cap R_Q^+ = \emptyset$ , or  $w^{-1}(R_Q^+) \subset R \setminus R_P$ . Since  $\mathbf{h}$  is  $P$ -regular, for  $\alpha \in R$ ,  $\langle \alpha, \mathbf{h} \rangle \neq 0$  if and only if  $\alpha \notin R_P$ . Thus  $\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} \subset \mathfrak{n}_Q$  if and only if  $\langle w^{-1}(\alpha), \mathbf{h} \rangle = \langle \alpha, w(\mathbf{h}) \rangle \neq 0, \forall \alpha \in R_Q^+$ . On the other hand, if  $w(\mathbf{h})$  is strictly  $Q$ -dominant, then  $\langle \alpha, w(\mathbf{h}) \rangle > 0, \forall \alpha \in R_Q^+$ . We now only

need to prove that in the case when  $\langle \alpha, w(\mathbf{h}) \rangle \neq 0$  for all  $\alpha \in R_Q^+$ , we have  $w(\mathbf{h})$  is strictly  $Q$ -dominant if and only if  $w_{Q,0}w \in W^P \cap {}^QW$  by (3) of Proposition 3.2.19.

Since now  $\text{Ad}(w)\mathfrak{m}_P \cap \mathfrak{u} \subset \mathfrak{n}_Q$  and  $w \in W^P$ , we have  $w_{Q,0}w \in W^P$  as in the proof of (3.b)  $\Rightarrow$  (3.c) of Proposition 3.2.19. Let  $w_1 = w_{Q,0}w$ . Then  $w_1 \in {}^QW$  if and only if  $w_1^{-1} \in W^Q$ . The latter is equivalent to  $w_1^{-1}(R_Q^+) \subset R^+$  by Lemma 3.2.7. We calculate that  $\langle \alpha, w(\mathbf{h}) \rangle = \langle w_{Q,0}(\alpha), w_1(\mathbf{h}) \rangle = -\langle -w_{Q,0}(\alpha), w_1(\mathbf{h}) \rangle$  for every  $\alpha \in R_Q^+$ . As  $-w_{Q,0}(R_Q^+) = -R_Q^- = R_Q^+$  (cf. [Jan07, II.1.5]), we get that  $\langle \alpha, w(\mathbf{h}) \rangle \geq 0$  (resp.  $> 0$ ) for all  $\alpha \in R_Q^+$  if and only if  $\langle \alpha, w_1(\mathbf{h}) \rangle \leq 0$  (resp.  $< 0$ ) for all  $\alpha \in R_Q^+$ .

If  $w_1^{-1}(R_Q^+) \subset R^+$ , then  $\langle \alpha, w_1(\mathbf{h}) \rangle = \langle w_1^{-1}(\alpha), \mathbf{h} \rangle \leq 0$  for all  $\alpha \in R_Q^+$  since  $\mathbf{h}$  is  $P$ -regular antidominant. Thus  $\langle \alpha, w(\mathbf{h}) \rangle \geq 0$  for all  $\alpha \in R_Q^+$ . But we know  $\langle \alpha, w(\mathbf{h}) \rangle \neq 0$ , hence  $\langle \alpha, w(\mathbf{h}) \rangle > 0$  for any  $\alpha \in R_Q^+$ . Thus  $w(\mathbf{h})$  is strictly  $Q$ -dominant.

Conversely if  $\langle \alpha, w(\mathbf{h}) \rangle > 0$  for all  $\alpha \in R_Q^+$ , we get  $\langle w_1^{-1}(\alpha), \mathbf{h} \rangle < 0$  for all  $\alpha \in R_Q^+$ . Since  $\mathbf{h}$  is  $P$ -regular and antidominant, we get  $w_1^{-1}(\alpha) \in R^+ \setminus R_P^+, \forall \alpha \in R_Q^+$ . In particular,  $w_1^{-1}(R_Q^+) \subset R^+$ . Thus  $w_1 \in W^P \cap {}^QW$ .  $\square$

The following lemma will be very important for us.

**Lemma 3.2.27.** *If  $wW_P \neq w_0W_P$ , then there exists  $\alpha \in \Delta$  such that  $s_\alpha w > w$  in  $W/W_P$  and a standard parabolic subgroup  $Q$  of  $G$  satisfying both  $Z_{P,w} \not\subset Z_{Q,P}$  and  $Z_{P,s_\alpha w} \subset Z_{Q,P}$  (which implies that  $s_\alpha w(\mathbf{h})$  is strictly  $Q$ -dominant by Theorem 3.2.26 for every  $P$ -regular antidominant coweight  $\mathbf{h} \in X_*(T)^{W_P}$ ).*

*Proof.* We assume  $w \in W^P$ . We claim that we can take a simple root  $\alpha \in \Delta$  such that  $\alpha \in w(R^+ \setminus R_P^+)$  (or equivalently  $\langle \alpha, w(\mathbf{h}) \rangle < 0$ ). If two roots  $\alpha_1, \alpha_2 \notin R^+ \setminus R_P^+$  (equivalently  $\langle \alpha_i, \mathbf{h} \rangle \geq 0, i = 1, 2$ ), then  $\alpha_1 + \alpha_2 \notin R^+ \setminus R_P^+$ . Thus  $w^{-1}(R^+) \cap (R^+ \setminus R_P^+) = \emptyset$  if and only if  $w^{-1}(\Delta) \cap (R^+ \setminus R_P^+) = \emptyset$ . Assume that  $R^+ \cap w(R^+ \setminus R_P^+) = \emptyset$ . Then  $R^- \cap w_0w(R^+ \setminus R_P^+) = \emptyset$ . This is only possible if  $w_0w \in W_P$  (cf. Lemma 3.2.7) which contradicts our assumption. Hence we can take  $\alpha \in w(R^+ \setminus R_P^+) \cap \Delta$ . Since  $s_\alpha(R^+ \setminus \{\alpha\}) = R^+ \setminus \{\alpha\}$  and  $s_\alpha(\alpha) = -\alpha$ , we get  $\{\alpha' \in R^+ \setminus R_P^+ \mid s_\alpha w(\alpha') \in R^-\} = \{\alpha' \in R^+ \setminus R_P^+ \mid w(\alpha') \in R^-\} \amalg \{w^{-1}(\alpha)\}$ . Thus  $\text{lg}(s_\alpha w) = \text{lg}(w) + 1$  and  $s_\alpha w \in W^P$  by Lemma 3.2.7. Now take  $Q = B(\alpha) = \overline{B}s_\alpha\overline{B}$  the standard parabolic subgroup with  $R_Q = \{\alpha\}$ . Then  $w_{Q,0} = s_\alpha, w \in W^P \cap {}^QW$  and  $s_\alpha w \notin {}^QW$ . By Corollary 3.2.21,  $Z_{P,s_\alpha w} \subset Z_{Q,P}$  and  $Z_{P,w} \not\subset Z_{Q,P}$  (and now  $s_\alpha w(\mathbf{h})$  is strictly  $Q$ -dominant and  $w(\mathbf{h})$  is not).  $\square$

The projection  $p_P : X_B \rightarrow X_P$  induces a proper surjective morphism  $Z_B \rightarrow Z_P$ . Since  $p(H_{B,B,w}) \subset H_{B,P,w}$ , we see  $p_P$  sends  $Z_{B,w}$  to  $Z_{P,w}$  for any  $w \in W$ . When  $w \in W^P$ , the morphism  $H_{B,B,w} \rightarrow H_{B,P,w}$  is an isomorphism (cf. Remark 3.2.6) and  $p_P$  induces a proper birational surjection  $Z_{B,w} \rightarrow Z_{P,w}$  if and only if  $w \in W^P$  (for the only if part, see [DR04, Thm. 3.3]). For any  $w \in W/W_P$ , let  $\overline{X}_{P,w} := \kappa_{1,w}^{-1}(0)$  and  $\overline{X}_P := \kappa_1^{-1}(0)$  be the scheme-theoretic fiber over the zero weight. The underlying reduced space  $\overline{X}_{P,w}^{\text{red}}$  is contained in  $Z_P$ . It follows from the discussions after [BHS19, Thm. 2.4.7] that  $\overline{X}_{B,w} = \cup_{w' \leq w} Z_{B,w'}$  which we have used in the proof of Proposition 3.2.11. For  $w \in W/W_P$ , since  $p_{P,w} : \overline{X}_{B,w} \rightarrow X_{P,w}$  is surjective for any representative  $w \in W$ , we get  $\overline{X}_{P,w} = \cup_{w' \leq w, w' \in W/W_P} Z_{P,w'}$ .

We pick an arbitrary closed point  $x \in \overline{X}_P \subset X_P$  and assume  $x \in H_{B,P,w_x}$  for some  $w_x \in W/W_P$  (or equivalently  $x$  is in  $V_{P,w_x}$  which is defined in §3.2.1). We have always  $x \in Z_{P,w_x} \subset X_{P,w}$  for any  $w \in W/W_P, w \geq w_x$ .

Recall that if  $A$  is an excellent Noetherian local ring and  $\widehat{A}$  is the completion of  $A$  at the maximal ideal, then the set of irreducible components of  $\text{Spec}(\widehat{A})$  is the disjoint union of the sets of irreducible components of  $\text{Spec}(\widehat{A} \otimes_A A/\mathfrak{p}_i), i \in I$  where  $\{\mathfrak{p}_i, i \in I\}$  is the set of minimal prime ideals of  $A$  (to see this, use [Gro65, Prop. 7.6.1, Sch. 7.8.3(vii)] and that the normalization

of  $\text{Spec}(A)$  is the disjoint union of normalizations of its irreducible components, see [Sta22, Tag 035P]). Moreover,  $\text{Spec}(\widehat{A} \otimes_A A/\mathfrak{p}_i) = \text{Spec}(\widehat{A/\mathfrak{p}_i})$  is equidimensional with the same dimension as  $\text{Spec}(A/\mathfrak{p}_i)$  ([Gro65, Sch. 7.8.3(x)] and [Sta22, Tag 07NV]).

Since  $\text{Spec}(\mathcal{O}_{\overline{X}_P, x})$  is equidimensional with irreducible components  $\text{Spec}(\mathcal{O}_{Z_{P, w}, x})$  for  $w$  such that  $x \in Z_{P, w}$ ,  $\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_P, x})$  is equidimensional and its set of irreducible components is the disjoint union of the sets of irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w}, x})$  for  $w$  such that  $x \in Z_{P, w}$ . Similarly, the subspace  $\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_{P, w}, x})$  is equidimensional and its set of irreducible components is the disjoint union the sets of irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w'}, x})$  for  $w'$  such that  $x \in Z_{P, w'}$  and  $w \geq w'$  in  $W/W_P$ . In summary, we have, as topological spaces, for  $w \geq w_x$ :

$$\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_{P, w}, x}) = \bigcup_{x \in Z_{P, w'}, w \geq w'} \text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w'}, x}) \quad (3.2.28)$$

where each term in the right hand side is non-empty. The closed immersion  $Z_{Q, P} \hookrightarrow \overline{X}_P$  induces a closed immersion  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{Q, P}, x}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_P, x})$  after completion at  $x$ . The dimensions of irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{Q, P}, x})$  are no more than  $\dim G - \dim T$  and the set of all irreducible components of dimension  $\dim G - \dim T$  is the disjoint union of the sets of irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w}, x})$  with  $w \in W/W_P$  such that  $w(\mathbf{h})$  is strictly  $Q$ -dominant (Theorem 3.2.26).

Now we assume furthermore that the image of  $x$  in  $\mathfrak{g}$  is 0. Since  $Z_{P, w}$  is closed and contains the closed subset  $\{(0, g_1 B, g_2 P) \in \mathfrak{g} \times G/B \times G/P \mid g_1^{-1} g_2 \in BwP/P\}$ , we get

$$\{(0, g_1 B, g_2 P) \in \mathfrak{g} \times G/B \times G/P \mid g_1^{-1} g_2 \in \overline{BwP/P}\} \subset Z_{P, w}.$$

Hence in this case  $x \in Z_{P, w}$  if and only if  $w \geq w_x$  in  $W/W_P$ , and for  $w \geq w_x$ ,

$$\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_{P, w}, x}) = \bigcup_{w \geq w' \geq w_x} \text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w'}, x}) \quad (3.2.29)$$

where each term in the right hand side is non-empty. A more practical form of Lemma 3.2.27 is the following.

**Lemma 3.2.30.** *If the image of  $x \in \overline{X}_P$  in  $\mathfrak{g}$  is 0 and  $w_x W_P \neq w_0 W_P$ , then there exists  $\alpha \in \Delta$ ,  $s_\alpha w_x > w_x$  in  $W/W_P$  and a standard parabolic subgroup  $Q$  of  $G$  such that  $s_\alpha w_x(\mathbf{h})$  is strictly  $Q$ -dominant for every  $P$ -regular antidominant coweight  $\mathbf{h} \in X_*(T)^{W_P}$  and the space*

$$\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_{P, s_\alpha w_x}, x}) = \text{Spec}(\widehat{\mathcal{O}}_{Z_{P, s_\alpha w_x}, x}) \bigcup \text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w_x}, x}),$$

where  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, s_\alpha w_x}, x})$ ,  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w_x}, x}) \neq \emptyset$ , satisfies that  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, s_\alpha w_x}, x}) \subset \text{Spec}(\widehat{\mathcal{O}}_{Z_{Q, P}, x})$  and  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P, w_x}, x}) \not\subset \text{Spec}(\widehat{\mathcal{O}}_{Z_{Q, P}, x})$  (where all spaces are viewed as subspaces of  $\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_P, x}) \subset \text{Spec}(\widehat{\mathcal{O}}_{X_P, x})$ ).

### 3.2.6 Characteristic cycles

Contents in this subsection will not be used subsequently. We assume that  $k$  has characteristic 0 and keep the notation in the last section. Theorem 3.2.26 can be explained using geometric representation theory at least when  $P = B$  and is true if we replace  $Z_{B, w}$  by the *Kazhdan-Lusztig cycles* denoted by  $[\mathcal{L}(ww_0 \cdot 0)]$  in [BHS19, Thm. 2.4.7] (see below).

We assume  $P = B$ . For each weight  $\mu$  of  $\mathfrak{t}$ , we let  $M(\mu) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mu$  be the Verma module and let  $L(\mu)$  be the irreducible quotient of  $M(\mu)$ . Then for any  $w \in W$ , the localization

functor of Beilinson-Bernstein associates  $M(ww_0 \cdot 0)$  (resp.  $L(ww_0 \cdot 0)$ ) with a  $G$ -equivariant (regular holonomic)  $\mathcal{D}$ -module  $\mathfrak{M}(ww_0 \cdot 0)$  (resp.  $\mathfrak{L}(ww_0 \cdot 0)$ ) on  $G/B \times G/B$  ([BHS19, Rem. 2.4.3]). Let  $T^*(G/B \times G/B) \simeq \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$  be the cotangent bundle of  $G/B \times G/B$  and identify the Steinberg variety  $Z_B$  as a closed subscheme of  $T^*(G/B \times G/B)$ . Let  $Z^0(Z_B)$  be the free abelian group generated by the codimension 0 points in  $Z_B$ . The characteristic cycle  $[\mathfrak{M}]$  of a coherent  $\mathcal{D}$ -module  $\mathfrak{M}$  on  $G/B \times G/B$  is the associated cycle in  $Z^0(Z_B)$  of the characteristic variety  $\text{Ch}(\mathfrak{M})$ , the scheme-theoretic support of some  $\mathcal{O}_{T^*(G/B \times G/B)}$ -module  $\text{gr}(\mathfrak{M})$  constructed from  $\mathfrak{M}$  with respect to some good filtration ([BHS19, §2.4]). For  $w \in W$ , let  $[Z_{B,w}]$  be the cycles associated with the irreducible components  $Z_{B,w}$  which form a basis of  $Z^0(Z_B)$ . It follows from [BHS19, Thm. 2.4.7(iii)] that the coefficient of  $[Z_{B,w}]$  in  $[\mathfrak{L}(ww_0 \cdot 0)]$  is equal to 1. Hence Theorem 3.2.26 in this case ( $P = B$ ) can be deduced from the same statement replacing  $Z_{B,w}$  by  $[\mathfrak{L}(ww_0 \cdot 0)]$  (viewed as a union of irreducible components).

For a finitely generated  $U(\mathfrak{g})$ -module  $M$ , there exists a good filtration  $\{0\} = M_{-1} \subset M_0 \subset M_1 \subset \dots$  of  $M$  such that  $\mathfrak{g}M_i \subset M_{i+1}$  and the  $S(\mathfrak{g})$ -module  $\text{gr}(M) := \bigoplus_{i=0}^{\infty} M_i/M_{i-1}$  is finitely-generated, where  $S(\mathfrak{g})$  is the symmetric algebra of  $\mathfrak{g}$  (cf. [BB82, §4.1]). The associated variety  $V(M)$  is the support of  $\text{gr}(M)$  in  $\text{Spec}(S(\mathfrak{g})) = \mathfrak{g}^*$ , the dual space of  $\mathfrak{g}$ , and is independent of the choice of the good filtration (cf. *loc. cit.*). We only consider  $V(M)$  as an algebraic subset. Let  $Q = M_Q N_Q$  be a standard parabolic subgroup of  $G$  and  $L_{\mathfrak{m}_Q}(ww_0 \cdot 0)$  be a finite-dimensional irreducible representation of  $\mathfrak{m}_Q$  of the highest weight  $ww_0 \cdot 0$  for some  $w$  (which means that  $ww_0 \cdot 0$  is a dominant weight for  $\mathfrak{m}_Q$ ) inflated to a representation of  $\mathfrak{q}$ . Then  $M_Q(ww_0 \cdot 0) := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_{\mathfrak{m}_Q}(ww_0 \cdot 0)$  is a parabolic Verma module in the category  $\mathcal{O}^{\mathfrak{q}}$  ([Hum08, §9.4]) of the highest weight  $ww_0 \cdot 0$  and is in the principal block of the category  $\mathcal{O}$ . Let  $\mathfrak{q}^{\perp}$  be the subspace of  $\mathfrak{g}^*$  consisting of elements that vanish on  $\mathfrak{q}$ .

**Lemma 3.2.31.** *We have  $V(M_Q(ww_0 \cdot 0)) \subset \mathfrak{q}^{\perp}$ .*

*Proof.* We follow the proof of [BB82, Thm. 4.6, Cor. 4.7]. If  $w = w_0$ , then  $M_Q(0) = U(\mathfrak{g})/U(\mathfrak{q})$ , the result is obvious. In general,  $M_Q(ww_0 \cdot 0)$  is a subquotient of  $U(\mathfrak{g}) \otimes_{U([\mathfrak{q}, \mathfrak{q}])} W$  for some finite-dimensional  $\mathfrak{g}$ -module  $W$  where  $[\mathfrak{q}, \mathfrak{q}]$  denotes the commutator and the latter is equal to  $U(\mathfrak{g})/U([\mathfrak{q}, \mathfrak{q}]) \otimes W$  by the tensor identity ([Kna88, Prop. 6.5]). By [BB82, Lem. 4.1], we have  $V(U(\mathfrak{g})/U([\mathfrak{q}, \mathfrak{q}]) \otimes W) = V(U(\mathfrak{g})/U([\mathfrak{q}, \mathfrak{q}])) = [\mathfrak{q}, \mathfrak{q}]^{\perp}$ . Thus  $V(M_Q(ww_0 \cdot 0)) \subset [\mathfrak{q}, \mathfrak{q}]^{\perp}$ . Moreover  $V(M_Q(ww_0 \cdot 0)) \subset \mathfrak{b}^{\perp}$  since  $M_Q(ww_0 \cdot 0)$  is a subquotient of  $M(0)$ . Hence  $V(M_Q(ww_0 \cdot 0)) \subset \mathfrak{b}^{\perp} \cap [\mathfrak{q}, \mathfrak{q}]^{\perp} = \mathfrak{q}^{\perp}$ .  $\square$

Now we can prove a stronger version of Theorem 3.2.26. Remark that the statement “ $ww_0 \cdot 0$  is a dominant weight for  $\mathfrak{m}_Q$ ” is equivalent to the statement “ $w(\mathbf{h})$  is strictly  $Q$ -dominant for some (or every) regular antidominant coweight  $\mathbf{h} \in X_*(T)$ ”.

**Proposition 3.2.32.** *If  $L(ww_0 \cdot 0) \in \mathcal{O}^{\mathfrak{q}}$ , then the subset  $\text{Ch}(\mathfrak{L}(ww_0 \cdot 0))$  of  $Z_B$  is contained in  $Z_{Q,B}$ .*

*Proof.* By [Hum08, Prop. 9.3(e), §9.4],  $L(ww_0 \cdot 0)$  is a subquotient of  $M_Q(ww_0 \cdot 0)$ . By [HT07, Thm. 2.2.1(ii)], we only need to prove the same result replacing  $L(ww_0 \cdot 0)$  by  $M_Q(ww_0 \cdot 0)$ . Now let  $\mathfrak{M}_Q(ww_0 \cdot 0)$  be the localization of  $M_Q(ww_0 \cdot 0)$  on  $G/B \times G/B$  and  $\mathfrak{M}_Q(ww_0 \cdot 0)'$  be the corresponding  $\mathcal{D}$ -module on  $G/B$  which is the usual localization of  $M_Q(ww_0 \cdot 0)$ . Let  $q : \tilde{\mathcal{N}} = G \times^B \mathfrak{b}^{\perp} \rightarrow \mathfrak{g}^* : (g, v) \mapsto \text{Ad}(g)v$  be the moment map. By [Gin86, Prop. 8.1] and Lemma 3.2.31,  $q(\text{Ch}(\mathfrak{M}_Q(ww_0 \cdot 0)')) = \mathfrak{q}^{\perp}$ . As in the proof of [BHS19, Prop. 2.4.4], we get  $\text{Ch}(\mathfrak{M}_Q(ww_0 \cdot 0)) = G \times^B \text{Ch}(\mathfrak{M}_Q(ww_0 \cdot 0)')$  is contained in  $G \times^B q^{-1}(\mathfrak{q}^{\perp})$ . Under the usual identification  $\mathfrak{g}^* \simeq \mathfrak{g}$  given by the Killing form,  $\mathfrak{q}^{\perp}$  is identified with  $\mathfrak{n}_Q$ . One can check under the isomorphism [BHS19, (2.15)], we have  $G \times^B q^{-1}(\mathfrak{n}_Q) = Z_{Q,B}$ . Hence  $\text{Ch}(\mathfrak{L}(ww_0 \cdot 0)) \subset Z_{Q,B}$ .  $\square$

*Remark 3.2.33.* We discuss here some possible generalization of some results of cycles on  $Z_B$  in [BR12, §2.13, §2.14] and [BHS19, §2.4] for the generalized Steinberg varieties. There exists already a theory of localization for singular blocks in characteristic 0 ([BK15]). However, the characteristic cycles on  $Z_B$  can be produced via  $K$ -theory by [BR12, Prop. 2.14.2] and [BR12, Prop. 2.13.5]. The  $K$ -theory of generalized Steinberg variety  $Z_P$  is well-behaved ([DR14]) and can produce Kazhdan-Lusztig cycles on  $Z_P$  corresponding to elements in the Weyl group by roughly pushing forward the cycles  $[\mathfrak{M}(ww_0 \cdot 0)]$  and  $[\mathfrak{L}(ww_0 \cdot 0)]$  on  $Z_B$  via the map  $Z_B \rightarrow Z_P$  ([DR14, Thm. 2.1]). We do not know for general  $Z_P$  whether the similar formula as  $[\overline{X}_{B,w}] = [\mathfrak{M}(ww_0 \cdot 0)]$  ([BHS19, Prop. 2.4.6]) of cycles on  $Z_B$ , which was crucially used in [BHS19] and proved in [BR12, Prop. 2.14.2], holds in general for the cycles on  $Z_P$  from the  $K$ -theory. It is mentioned in [BR12, Rem. 2.14.3] that the previous formula for  $[\overline{X}_{B,w}]$  can be deduced by deformation arguments ([Gin86, §6] or [CG97, §7.3]). To get a generalized formula for  $[\overline{X}_{P,w}]$ , it seems that Cohen-Macaulayness of  $X_{P,w}$  would be needed, which is unknown to the author for  $P \neq B$ .

### 3.3 Local models for the trianguline variety

We apply the results of §3.2 to study local geometry of the trianguline variety at certain points, generalizing the results in [BHS19, §3].

We fix a finite extension  $K$  of  $\mathbb{Q}_p$  with a uniformizer  $\varpi_K$ . Let  $L$  be a finite extension of  $\mathbb{Q}_p$  that splits  $K$  with residue field  $k_L$  and set  $\Sigma = \text{Hom}(K, L)$ .

#### 3.3.1 Almost de Rham trianguline $(\varphi, \Gamma_K)$ -modules

We recall some basic notions for the deformation theory of trianguline  $(\varphi, \Gamma_K)$ -modules. For details and notation, see [BHS19, §3].

Let  $\text{Rep}_{\text{BdR}}(\mathcal{G}_K)$  (resp.  $\text{Rep}_{\text{BdR}^+}(\mathcal{G}_K)$ ) be the category of  $\text{BdR}$ -representations (resp.  $\text{BdR}^+$ -representations) of  $\mathcal{G}_K$  (free of finite rank, continuous for the natural topology and semi-linear). We have rings  $\text{B}_{\text{pdR}}^+ = \text{B}_{\text{dR}}^+[\log(t)]$  and  $\text{B}_{\text{pdR}} = \text{B}_{\text{dR}}[\log(t)]$  with the actions of  $\mathcal{G}_K$  satisfying that  $g(\log(t)) = \log(t) + \log(\epsilon(g))$  and  $\text{B}_{\text{pdR}}$  admits a  $\text{BdR}$ -derivative  $\nu_{\text{pdR}}$  such that  $\nu_{\text{pdR}}(\log(t)) = -1$  which preserves  $\text{B}_{\text{pdR}}^+$ .

If  $W$  is a  $\text{BdR}$ -representation of  $\mathcal{G}_K$ , then  $D_{\text{pdR}}(W) := (\text{B}_{\text{pdR}} \otimes_{\text{BdR}} W)^{\mathcal{G}_K}$  is a finite-dimensional  $K$ -vector space of dimension no more than  $\dim_{\text{BdR}} W$  with a linear nilpotent endomorphism  $\nu_W$ . In other words,  $D_{\text{pdR}}(W)$  is a  $K$ -representation of the additive algebraic group  $\mathbb{G}_a$ . The  $\text{BdR}$ -representation  $W$  is called *almost de Rham* if  $\dim_K D_{\text{pdR}}(W) = \dim_{\text{BdR}} W$ . A  $\text{BdR}^+$ -representation  $W^+$  is called almost de Rham if the  $\text{BdR}$ -representation  $W = W^+[\frac{1}{t}]$  is almost de Rham. The  $\text{BdR}^+$ -lattices which are stable under  $\mathcal{G}_K$  in an almost de Rham  $\text{BdR}$ -representation  $W$  are in bijection with filtrations of  $D_{\text{pdR}}(W)$  as  $K$ -representations of  $\mathbb{G}_a$  via  $W^+ \mapsto \text{Fil}_{W^+}^\bullet(D_{\text{pdR}}(W))$  where  $\text{Fil}_{W^+}^i(D_{\text{pdR}}(W)) := (t^i \text{B}_{\text{pdR}}^+ \otimes_{\text{BdR}} W^+)^{\mathcal{G}_K}$  for  $i \in \mathbb{Z}$  ([BHS19, Prop. 3.2.1]).

Let  $A \in \mathcal{C}_L$  be a local Artinian  $L$ -algebra with the maximal ideal  $\mathfrak{m}_A$ . Let  $\text{Rep}_{\text{pdR},A}(\mathcal{G}_K)$  be the category of almost de Rham  $\text{BdR}$ -representations  $W$  of  $\mathcal{G}_K$  together with a morphism of  $\mathbb{Q}_p$ -algebras  $A \rightarrow \text{End}_{\text{Rep}_{\text{BdR}}(\mathcal{G}_K)}(W)$  such that  $W$  is finite free over  $\text{BdR} \otimes_{\mathbb{Q}_p} A$ . Let  $\text{Rep}_{A \otimes_{\mathbb{Q}_p} K}(\mathbb{G}_a)$  be the category of pairs  $(V_A, \nu_A)$  where  $\nu_A$  is a nilpotent endomorphism of a finite free  $A \otimes_{\mathbb{Q}_p} K$ -module  $V_A$ . The functor  $D_{\text{pdR}}$  induces an equivalence of categories between  $\text{Rep}_{\text{pdR},A}(\mathcal{G}_K)$  and  $\text{Rep}_{A \otimes_{\mathbb{Q}_p} K}(\mathbb{G}_a)$  ([BHS19, Lemma. 3.1.4]).

We have the Robba ring  $\mathcal{R}_{A,K}$  of  $K$  with  $A$ -coefficients (cf. [KPX14, Def. 6.2.1]). A  $(\varphi, \Gamma_K)$ -module  $\mathcal{M}_A$  over  $\mathcal{R}_{A,K}[\frac{1}{t}]$  is defined to be a finite free  $\mathcal{R}_{A,K}[\frac{1}{t}]$ -module equipped with commuting semilinear actions of  $\varphi$  and  $\Gamma_K$  such that  $\mathcal{M}_A$  admits a  $(\varphi, \Gamma_K)$ -stable  $\mathcal{R}_K$ -lattice  $D_A$  which is a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_K$  with the actions of  $\varphi$  and  $\Gamma_K$  given by those of  $\mathcal{M}_A$ . Denote by  $\Phi\Gamma_{A,K}^+$  (resp.  $\Phi\Gamma_{A,K}$ ) the category of  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_{A,K}$  (resp. over  $\mathcal{R}_{A,K}[\frac{1}{t}]$ ).

A rank one  $(\varphi, \Gamma_K)$ -module  $\mathcal{M}_A$  over  $\mathcal{R}_{A,K}[\frac{1}{t}]$  is called of character type if  $\mathcal{M}_A$  is isomorphic to  $\mathcal{R}_{A,K}(\delta_A)[\frac{1}{t}]$  for some continuous character  $\delta_A : K^\times \rightarrow A^\times$  ([KPX14, Cons. 6.2.4]). A  $(\varphi, \Gamma_K)$ -module  $\mathcal{M}_A$  over  $\mathcal{R}_{A,K}[\frac{1}{t}]$  of rank  $n$  is called trianguline if there exists an increasing filtration  $\mathcal{M}_{A,\bullet} : \{0\} = \mathcal{M}_{A,0} \subset \mathcal{M}_{A,1} \subset \cdots \subset \mathcal{M}_{A,n} = \mathcal{M}_A$  of  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_{A,K}[\frac{1}{t}]$  such that  $\mathcal{M}_{A,i}/\mathcal{M}_{A,i-1}$  is a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{A,K}[\frac{1}{t}]$  of character type for every  $1 \leq i \leq n$  and the filtration  $\mathcal{M}_{A,\bullet}$  is called a triangulation of  $\mathcal{M}_A$ . Moreover, if  $\mathcal{M}_{A,i}/\mathcal{M}_{A,i-1} \simeq \mathcal{R}_{A,K}(\delta_{A,i})[\frac{1}{t}]$  for some characters  $\delta_{A,i} : K^\times \rightarrow A^\times, 1 \leq i \leq n$ , then we say  $\underline{\delta}_A = (\delta_{A,1}, \dots, \delta_{A,n})$  is a parameter of  $\mathcal{M}_A$ .

We have an exact functor  $W_{\text{dR}}^+(-)$  (resp.  $W_{\text{dR}}(-)$ ) from  $\Phi\Gamma_{A,K}^+$  (resp.  $\Phi\Gamma_{A,K}$ ) to  $\text{Rep}_{\text{B}_{\text{dR}}^+, A}(\mathcal{G}_K)$  (resp.  $\text{Rep}_{\text{B}_{\text{dR}}, A}(\mathcal{G}_K)$ ) ([Ber08a] and [BHS19, Lem. 3.3.5]). If  $\mathcal{M}_A \in \Phi\Gamma_{A,K}$  is trianguline with a parameter  $(\delta_{A,i})_{i=1, \dots, n}$  and if the characters  $\bar{\delta}_{A,i} := \delta_{A,i} \bmod \mathfrak{m}_A, 1 \leq i \leq n$  are locally algebraic, then  $W_{\text{dR}}(\mathcal{M}_A) \in \text{Rep}_{\text{B}_{\text{dR}}, A}(\mathcal{G}_K)$  and is a successive extension of rank one de Rham  $\text{B}_{\text{dR}}$ -representations of  $\mathcal{G}_K$  ([BHS19, Lem. 3.3.6]).

### 3.3.2 Groupoids

We recall the definitions of some groupoids over  $\mathcal{C}_L$  defined in [BHS19, §3].

Let  $D$  be a fixed  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}$  of rank  $n$ . Let  $\mathcal{M} = D[\frac{1}{t}]$ . We assume there exists and fix a triangulation  $\mathcal{M}_\bullet$  of  $\mathcal{M}$  of parameter  $\underline{\delta} = (\delta_1, \dots, \delta_n)$ . We assume that  $\delta_i$  is locally algebraic for any  $i = 1, \dots, n$ .

We let  $\mathcal{T}_0^n$  be the subset of  $\mathcal{T}_L^n$  that is the complement of characters  $(\delta_1, \dots, \delta_n)$  where  $\delta_i/\delta_j$  or  $\epsilon\delta_i/\delta_j$  is algebraic for some  $i \neq j$ . Let  $\mathcal{T}_0$  be the subset of  $\mathcal{T}_L$  which is the complement of the set of all  $L$ -points corresponding to characters of the form  $z^{\mathbf{k}}$  or  $\epsilon z^{\mathbf{k}}$  for some  $\mathbf{k} \in \mathbb{Z}^\Sigma$ . Remark that  $\mathcal{T}_0^n \neq (\mathcal{T}_0)^n$ .

We assume that the parameter  $\underline{\delta}$  of  $\mathcal{M}$  (which we have assumed to be locally algebraic) lies in  $\mathcal{T}_0^n(L)$ .

Let  $W^+ = W_{\text{dR}}^+(D) \in \text{Rep}_{\text{B}_{\text{dR}}^+, L}(\mathcal{G}_K), W := W_{\text{dR}}(\mathcal{M}) \in \text{Rep}_{\text{B}_{\text{dR}}, L}(\mathcal{G}_K)$ . Then  $W \in \text{Rep}_{\text{B}_{\text{dR}}, L}(\mathcal{G}_K)$  and  $W$  is filtered in  $\text{Rep}_{\text{B}_{\text{dR}}, L}(\mathcal{G}_K)$  with a filtration  $\mathcal{F}_\bullet : \mathcal{F}_1 = W_{\text{dR}}(\mathcal{M}_1) \subset \cdots \subset \mathcal{F}_n = W_{\text{dR}}(\mathcal{M}_n)$ . We fix an isomorphism  $\alpha : (L \otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{\text{pdR}}(W)$ .

The groupoid  $X_W$  over  $\mathcal{C}_L$  consists of triples  $(A, W_A, \iota_A)$  where  $A \in \mathcal{C}_L, W_A \in \text{Rep}_{\text{B}_{\text{dR}}, A}(\mathcal{G}_K)$  and  $\iota_A : W_A \otimes_A L \xrightarrow{\sim} W$ . A morphism  $(A, W_A, \iota_A) \rightarrow (B, W_B, \iota_B)$  in  $X_W$  is a morphism  $A \rightarrow B$  in  $\mathcal{C}_L$  and an isomorphism  $W_A \otimes_A B \xrightarrow{\sim} W_B$  compatible with  $\iota_A$  and  $\iota_B$ . The groupoid  $X_W^\square$  consists of  $(A, W_A, \iota_A, \alpha_A)$  where  $(A, W_A, \iota_A) \in X_W$  and  $\alpha_A : (A \otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{\text{pdR}}(W_A)$  such that  $\alpha_A$  modulo  $\mathfrak{m}_A$  coincides with  $\alpha$ . Similarly we have  $X_{W^+}, X_{W^+}^\square$  by replacing  $W, W_A$  with  $W^+, W_A^+$ . We have a forgetful morphism  $X_W^\square \rightarrow X_W$ .

The groupoid  $X_{W, \mathcal{F}_\bullet}$  over  $\mathcal{C}_L$  consists of  $(A, W_A, \mathcal{F}_{A,\bullet}, \iota_A)$  where  $(A, W_A, \iota_A) \in X_W$  and  $\mathcal{F}_{A,\bullet} = (\mathcal{F}_{A,i})_{i=1, \dots, n}$  is a filtration of  $W_A$  in  $\text{Rep}_{\text{B}_{\text{dR}}, A}(\mathcal{G}_K)$  such that  $\mathcal{F}_{A,i}/\mathcal{F}_{A,i-1}$  for  $i = 2, \dots, n$  and  $\mathcal{F}_{A,1}$  are  $A \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}}$ -modules free of rank one and  $\iota_A$  induces  $\mathcal{F}_{A,\bullet} \otimes_A L \xrightarrow{\sim} \mathcal{F}_\bullet$ . We let  $X_{W, \mathcal{F}_\bullet}^\square = X_{W, \mathcal{F}_\bullet} \times_{X_W} X_W^\square$  where the morphism  $X_{W, \mathcal{F}_\bullet} \rightarrow X_W$  is the obvious one.

The groupoid  $X_{\mathcal{M}, \mathcal{M}_\bullet}$  over  $\mathcal{C}_L$  consists of trianguline  $(\varphi, \Gamma_K)$ -modules  $\mathcal{M}_A$  over  $\mathcal{R}_{A,K}[\frac{1}{t}]$  for some  $A \in \mathcal{C}_L$  with a triangulation  $\mathcal{M}_{A,\bullet}$  of  $\mathcal{M}_A$  and an isomorphism  $j_A : \mathcal{M}_A \otimes_A L \xrightarrow{\sim} \mathcal{M}$  which is compatible with the filtrations.

The functor  $W_{\text{dR}}(-)$  induces a morphism  $X_{\mathcal{M}, \mathcal{M}_\bullet} \rightarrow X_{W, \mathcal{F}_\bullet}$ . By our generic assumption on  $\underline{\delta}$ , the morphism is formally smooth by [BHS19, Cor. 3.5.6] and is relatively representable ([BHS19, Lem. 3.5.3]).

Let  $X_D$  (resp.  $X_{\mathcal{M}}$ ) be the groupoid over  $\mathcal{C}_L$  of deformations of  $D$  (resp.  $\mathcal{M}$ ). Then essentially due to Berger's equivalence of  $B$ -pairs and  $(\varphi, \Gamma_K)$ -modules ([Ber08a]), the morphism induced by inverting  $t$  and the functors  $W_{\text{dR}}(-), W_{\text{dR}}^+(-)$

$$X_D \rightarrow X_{\mathcal{M}} \times_{X_W} X_{W^+}$$

is an equivalence of groupoids over  $\mathcal{C}_L$  ([BHS19, Prop. 3.5.1]).

Let  $X_{D,\mathcal{M}_\bullet} = X_D \times_{X_{\mathcal{M}}} X_{\mathcal{M},\mathcal{M}_\bullet}$ ,  $X_D^\square = X_D \times_{X_W} X_W^\square$  and  $X_{D,\mathcal{M}_\bullet}^\square = X_{D,\mathcal{M}_\bullet} \times_{X_W} X_W^\square$ . We let  $X_{W^+,\mathcal{F}_\bullet} = X_{W^+} \times_{X_W} X_{W,\mathcal{F}_\bullet}$  and  $X_{W^+,\mathcal{F}_\bullet}^\square = X_{W^+,\mathcal{F}_\bullet} \times_{X_W} X_W^\square$ . Then  $W_{\text{dR}}^+$  and  $W_{\text{dR}}$  induce morphisms

$$\begin{aligned} X_{D,\mathcal{M}_\bullet} &\rightarrow X_{W^+,\mathcal{F}_\bullet} \\ X_{D,\mathcal{M}_\bullet}^\square &\rightarrow X_{W^+,\mathcal{F}_\bullet}^\square \end{aligned}$$

which are formally smooth and relatively representable (as the base changes of  $X_{\mathcal{M},\mathcal{M}_\bullet} \rightarrow X_{W,\mathcal{F}_\bullet}$  up to equivalence [BHS19, Cor. 3.5.4]).

### 3.3.3 Representability

We start with some slight generalization of some results in [BHS19, §3.2] for cases of possibly non-regular Hodge-Tate weights.

We keep the notation in §3.3.2. If  $A \in \mathcal{C}_L$ ,  $W_A \in \text{Rep}_{\text{pdR},A}(\mathcal{G}_K)$  and  $W_A^+$  is a  $B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} A$ -lattice of  $W_A$ , set

$$\begin{aligned} D_{\text{pdR},\tau}(W_A) &:= D_{\text{pdR}}(W_A) \otimes_{A \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} A, \\ \text{Fil}_{W_A^+}^\bullet(D_{\text{pdR},\tau}(W_A)) &:= \text{Fil}_{W_A^+}^\bullet(D_{\text{pdR}}(W_A)) \otimes_{A \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} A, \\ \text{gr}_{\text{Fil}_{W_A^+}^\bullet}^i(D_{\text{pdR},\tau}(W_A)) &:= \text{Fil}_{W_A^+}^i(D_{\text{pdR},\tau}(W_A)) / \text{Fil}_{W_A^+}^{i+1}(D_{\text{pdR},\tau}(W_A)) \end{aligned}$$

for  $i \in \mathbb{Z}, \tau \in \Sigma$ .

Assume that for  $\tau \in \Sigma$ , the integers  $i$  such that  $\text{gr}_{\text{Fil}_{W^+}^\bullet}^i(D_{\text{pdR},\tau}(W)) \neq 0$  are

$$-k_{\tau,1} > \cdots > -k_{\tau,s_\tau}$$

for some positive integer  $s_\tau$  and we set  $m_{\tau,i} = \dim_L \text{gr}_{\text{Fil}_{W^+}^\bullet}^{-k_{\tau,i}}(D_{\text{pdR},\tau}(W))$  for  $1 \leq i \leq s_\tau$ . Then

$m_{\tau,1} + \cdots + m_{\tau,s_\tau} = n$  for each  $\tau \in \Sigma$ . We get (partial) flags  $D_{\text{pdR},\tau}(W) = \text{Fil}_{W^+}^{-k_{\tau,s_\tau}}(D_{\text{pdR},\tau}(W)) \supsetneq \cdots \supsetneq \text{Fil}_{W^+}^{-k_{\tau,1}}(D_{\text{pdR},\tau}(W)) \supsetneq \{0\}$  inside  $D_{\text{pdR},\tau}(W)$  for  $\tau \in \Sigma$ .

We set

$$G := \text{Res}_{K/\mathbb{Q}_p}(\text{GL}_{n/K}) \times_{\mathbb{Q}_p} L = \prod_{\tau \in \Sigma} \text{GL}_{n/L}.$$

Then  $G$  acts on  $D_{\text{pdR}}(W) = \prod_{\tau \in \Sigma} D_{\text{pdR},\tau}(W)$  via  $\alpha : (L \otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{\text{pdR}}(W)$ . We let  $P$  be the stabilizer of the filtration  $\text{Fil}_{W^+}^\bullet(D_{\text{pdR}}(W))$ . Then  $P = \prod_{\tau \in \Sigma} P_\tau$ , where  $P_\tau$  is the parabolic subgroup of  $\text{GL}_{n/L}$  which stabilizes the (partial) flag  $\text{Fil}_{W^+}^{-k_{\tau,\bullet}}(D_{\text{pdR},\tau}(W))$  via  $\alpha$ .

Recall we have the variety  $\tilde{\mathfrak{g}}_P = \{(\nu, gP) \in \mathfrak{g} \times G/P \mid \text{Ad}(g^{-1})\nu \in \mathfrak{p}\}$ . For any  $A \in \mathcal{C}_L$ ,  $A$ -points of the (partial) flag variety  $G/P = \prod_{\tau \in \Sigma} \text{GL}_{n/L}/P_\tau$  correspond to (partial) flags  $A^n = \text{Fil}_{\tau,s_\tau} \supsetneq \cdots \supsetneq \text{Fil}_{\tau,1} \supsetneq \text{Fil}_{\tau,0} = \{0\}$  where for each  $i = 1, \dots, s_\tau$ ,  $\text{Fil}_{\tau,i}/\text{Fil}_{\tau,i-1}$  is a free  $A$ -module of rank  $m_{\tau,i}$ . An  $A$ -point of  $\tilde{\mathfrak{g}}_P$  then corresponds to a (partial) flag  $(\text{Fil}_{\tau,i})_{\tau \in \Sigma, 1 \leq i \leq s_\tau}$  and a linear operator  $\nu_A = \prod_{\tau \in \Sigma} \nu_{A,\tau} \in \prod_{\tau \in \Sigma} \text{End}_A(A^n)$  which preserves the filtration. We thus have a point

$$x_{W^+} := \left( \alpha^{-1} \left( \left( \text{Fil}_{W^+}^{-k_{\tau,\bullet}}(D_{\text{pdR},\tau}(W)) \right)_{\tau \in \Sigma} \right), N_W := \alpha^{-1} \circ \nu_W \circ \alpha \right) \in \tilde{\mathfrak{g}}_P(L)$$

where  $\nu_W$  is the nilpotent operator acting on  $D_{\text{pdR}}(W)$ . Given  $(A, W_A^+, \iota_A, \alpha_A) \in X_{W^+}^\square$ , the  $A \otimes_{\mathbb{Q}_p} K$ -module  $D_{\text{pdR}}(W_A)$  is equipped with a filtration  $\text{Fil}_{W_A^+}^\bullet(D_{\text{pdR}}(W_A))$  together with an

$A \otimes_{\mathbb{Q}_p} K$ -linear nilpotent operator  $\nu_{W_A}$  which preserves the filtration. Via the isomorphism  $\alpha_A$ , these datum give rise to an  $A$ -point

$$\left( \alpha_A^{-1} \left( \left( \text{Fil}_{W_A^+}^{-k_{\tau}, \bullet} (D_{\text{pdR}, \tau}(W_A)) \right)_{\tau \in \Sigma} \right), N_{W_A} := \alpha_A^{-1} \circ \nu_{W_A} \circ \alpha_A \right)$$

of  $\tilde{\mathfrak{g}}_P$  as in the proof of the following proposition.

**Proposition 3.3.1.** *The groupoid  $X_{W^+}^{\square}$  is pro-representable and the functor*

$$(A, W_A^+, \iota_A, \alpha_A) \mapsto \left( \alpha_A^{-1} \left( \left( \text{Fil}_{W_A^+}^{-k_{\tau}, \bullet} (D_{\text{pdR}, \tau}(W_A)) \right)_{\tau \in \Sigma} \right), N_{W_A} \right)$$

*induces an isomorphism of functors between  $|X_{W^+}^{\square}|$  and  $\widehat{\mathfrak{g}}_P$ , the completion of  $\tilde{\mathfrak{g}}_P$  at  $x_{W^+}$ .*

*Proof.* The proof is essentially the same as [BHS19, Thm. 3.2.5]. For any  $A \in \mathcal{C}_L$ , by [BHS19, Lem. 3.2.2], the functor  $W_A^+ \rightarrow (D_{\text{pdR}}(W_A), \text{Fil}_{W_A^+}^{\bullet}(D_{\text{pdR}}(W_A)), \nu_{W_A})$  induces an equivalence between the category of almost de Rham  $A \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$ -representations of  $\mathcal{G}_K$  and the category of filtered  $A \otimes_{\mathbb{Q}_p} K$ -representations of  $\mathbb{G}_a$  (the definition before [BHS19, Lem. 3.2.2] should be that the graded pieces are projective  $A \otimes_{\mathbb{Q}_p} K$ -modules, see discussions below). If  $(A, W_A^+, \iota_A, \alpha_A) \in X_{W^+}^{\square}$ , by the proof of *loc. cit.*,  $\oplus_{i \in \mathbb{Z}} \text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^i (D_{\text{pdR}}(W_A))$  is projective over  $A \otimes_{\mathbb{Q}_p} K$  (which is equivalent to the condition that  $W_A^+$  is free over  $A \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$ ) and moreover for any finite  $A$ -module  $M$ , there is an isomorphism

$$M \otimes_A \text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^i (D_{\text{pdR}}(W_A)) \simeq \text{gr}_{\text{Fil}_{M \otimes_A W_A^+}^{\bullet}}^i (D_{\text{pdR}}(M \otimes_A W_A))$$

for each  $i \in \mathbb{Z}$ . We have decompositions  $D_{\text{pdR}}(W_A) = \oplus_{\tau \in \Sigma} D_{\text{pdR}, \tau}(W_A)$ ,  $\text{Fil}_{W_A^+}^i (D_{\text{pdR}}(W_A)) = \oplus_{\tau \in \Sigma} \text{Fil}_{W_A^+}^i (D_{\text{pdR}, \tau}(W_A))$  and  $\text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^i (D_{\text{pdR}}(W_A)) = \oplus_{\tau \in \Sigma} \text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^i (D_{\text{pdR}, \tau}(W_A))$  for  $i \in \mathbb{Z}$ . Then  $\text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^i (D_{\text{pdR}, \tau}(W_A))$  is free over  $A$  for each  $\tau \in \Sigma$  and  $i \in \mathbb{Z}$  since  $A$  is local. And we have ([BHS19, Cor. 3.2.3])

$$\text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^i (D_{\text{pdR}, \tau}(W_A)) \otimes_A L \simeq \text{gr}_{\text{Fil}_{W^+}^{\bullet}}^i (D_{\text{pdR}, \tau}(W)).$$

Thus  $\text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^i (D_{\text{pdR}, \tau}(W_A)) \neq 0$  if and only if  $i \in \{-k_{\tau, 1}, \dots, -k_{\tau, s_{\tau}}\}$  and  $\text{gr}_{\text{Fil}_{W_A^+}^{\bullet}}^{-k_{\tau, i}} (D_{\text{pdR}, \tau}(W_A))$  is free of rank  $m_{\tau, i}$  for  $i = 1, \dots, s_{\tau}$ . The datum

$$\left( \left( \text{Fil}_{W_A^+}^{-k_{\tau}, \bullet} (D_{\text{pdR}, \tau}(W_A)) \right)_{\tau \in \Sigma}, \nu_{W_A} \right)$$

together with  $\alpha_A, \iota_A$  up to isomorphisms is then equivalent to a morphism  $\text{Spec}(A) \rightarrow \tilde{\mathfrak{g}}_P$  whose image is in the infinitesimal neighbourhood of the  $L$ -point  $x_{W^+}$  of  $\tilde{\mathfrak{g}}_P$ .  $\square$

Recall that from the split group  $G$ , the parabolic subgroup  $P$  and a fixed Borel subgroup  $B = TU$  contained in  $P$ , we have defined a scheme  $X_P = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P$  in §3.2.1 whose irreducible components are  $X_{P, w}$  for  $w \in W/W_P$  where  $W$  denotes the Weyl group of  $G$ . We may assume that  $B$  is the group of upper-triangular matrices and  $T$  is the diagonal torus in  $G = \prod_{\tau \in \Sigma} \text{GL}_{n/L}$ . Let  $x$  be the  $L$ -point of the scheme  $X_P$  corresponding to

$$(\alpha^{-1} (D_{\text{pdR}}(\mathcal{F}_{\bullet})), \alpha^{-1} (\text{Fil}_{W^+}^{\bullet} (D_{\text{pdR}}(W))), N_W)$$

and let  $\widehat{X}_{P, x} = \text{Spf}(\widehat{\mathcal{O}}_{X_P, x})$  be the completion of  $X_P$  at  $x$ . The groupoid  $X_{W, \mathcal{F}_{\bullet}}^{\square}$  is pro-represented by the completion  $\widehat{\mathfrak{g}}$  of  $\tilde{\mathfrak{g}}$  at the point  $(\alpha^{-1} (D_{\text{pdR}}(\mathcal{F}_{\bullet})), N_W) \in \tilde{\mathfrak{g}}(L)$  ([BHS19, Cor. 3.1.9]). Then by the same proof as for Proposition 3.3.1 and [BHS19, Cor. 3.1.9, Cor. 3.5.8], we have the following generalization of [BHS19, Cor. 3.5.8].



**Proposition 3.3.2.** (1) The groupoid  $X_{W^+, \mathcal{F}_\bullet}^\square$  is pro-representable and the functor  $|X_{W^+, \mathcal{F}_\bullet}^\square|$  is pro-represented by  $\mathrm{Spf}(\widehat{\mathcal{O}}_{X_P, x})$ .  
 (2) The groupoid  $X_{D, \mathcal{M}_\bullet}^\square$  is pro-representable and the functor  $|X_{D, \mathcal{M}_\bullet}^\square|$  is pro-represented by a formal scheme which is formally smooth of relative dimension  $[K : \mathbb{Q}_p] \frac{n(n+1)}{2}$  over  $\widehat{X}_{P, x}$ .

The scheme  $\mathrm{Spec}(\widehat{\mathcal{O}}_{X_P, x})$  is equidimensional of dimension  $\dim \mathfrak{g} = [K : \mathbb{Q}_p]n^2$  with irreducible components  $\mathrm{Spec}(\widehat{\mathcal{O}}_{X_{P, w}, x})$  for  $w \in W/W_P$  such that  $x \in X_{P, w}(L)$  (Corollary 3.2.16). Let  $\widehat{X}_{P, w, x} = \mathrm{Spf}(\widehat{\mathcal{O}}_{X_{P, w}, x})$  for  $w \in W/W_P$  such that  $x \in X_{P, w}$ .

For any  $w \in W/W_P$ , we define  $X_{W^+, \mathcal{F}_\bullet}^{\square, w} = X_{W^+, \mathcal{F}_\bullet}^\square \times_{|X_{W^+, \mathcal{F}_\bullet}^\square|} \widehat{X}_{P, w, x}$  and let  $X_{W^+, \mathcal{F}_\bullet}^w$  be the subgroupoid of  $X_{W^+, \mathcal{F}_\bullet}$  which is the image of  $X_{W^+, \mathcal{F}_\bullet}^{\square, w}$  under the forgetful map  $X_{W^+, \mathcal{F}_\bullet}^\square \rightarrow X_{W^+, \mathcal{F}_\bullet}$ . Let  $X_{D, \mathcal{M}_\bullet}^w = X_{D, \mathcal{M}_\bullet} \times_{X_{W^+, \mathcal{F}_\bullet}} X_{W^+, \mathcal{F}_\bullet}^w$  and  $X_{D, \mathcal{M}_\bullet}^{\square, w} = X_{D, \mathcal{M}_\bullet}^\square \times_{X_{W^+, \mathcal{F}_\bullet}^\square} X_{W^+, \mathcal{F}_\bullet}^{\square, w}$ . Literally the same proof of [BHS19, Cor. 3.5.11] except that now we do not have the normalness result (but still have the irreducibility of  $\mathrm{Spec}(\widehat{\mathcal{O}}_{X_{P, w}, x})$  by Theorem 3.2.14) shows that

**Corollary 3.3.3.** For any  $w \in W/W_P$  such that  $x \in X_{P, w}(L)$ , the functor  $|X_{D, \mathcal{M}_\bullet}^{\square, w}|$  is pro-represented by a complete Noetherian local domain of residue field  $L$  and dimension  $[K : \mathbb{Q}_p](n^2 + \frac{n(n+1)}{2})$  and is formally smooth over  $\widehat{X}_{P, w, x}$ .

### 3.3.4 The weight map

We keep the notation in 3.3.3. We view the parameter  $\underline{\delta}$  of  $\mathcal{M}$  as an  $L$ -point of the rigid analytic space  $\mathcal{T}_L^n$ . The completion of  $\mathcal{T}_L^n$  at the point  $\underline{\delta}$  denoted by  $\widehat{\mathcal{T}}_{\underline{\delta}}^n$  pro-represents the functor from  $\mathcal{C}_L$  to deformations of the continuous character  $\underline{\delta} : (K^\times)^n \rightarrow L^\times$ . Given an object  $(A, \mathcal{M}_A, \mathcal{M}_{A, \bullet}, j_A) \in X_{\mathcal{M}, \mathcal{M}_\bullet}$ , there exists a unique parameter  $\underline{\delta}_A \in \mathcal{T}_L^n(A)$  of  $\mathcal{M}_A$  such that  $\delta_{A, i} \otimes_A L = \delta_i$  for  $i = 1, \dots, n$  by [BHS19, Lem. 3.3.4], which defines a morphism  $\omega_{\underline{\delta}} : X_{\mathcal{M}, \mathcal{M}_\bullet} \rightarrow \widehat{\mathcal{T}}_{\underline{\delta}}^n$  of groupoids over  $\mathcal{C}_L$ . Recall that  $\mathfrak{t}$  is the Lie algebra of  $T$ . Let  $\widehat{\mathfrak{t}}$  be the completion of  $\mathfrak{t}$  at  $0 \in \mathfrak{t}$ . Composing the morphism  $\mathrm{wt} - \mathrm{wt}(\underline{\delta}) : \widehat{\mathcal{T}}_{\underline{\delta}}^n \rightarrow \widehat{\mathfrak{t}}$  (cf. [BHS19, (3.16)]), we get a morphism  $(\mathrm{wt} - \mathrm{wt}(\underline{\delta})) \circ \omega_{\underline{\delta}} : X_{\mathcal{M}, \mathcal{M}_\bullet} \rightarrow \widehat{\mathfrak{t}}$  of groupoids over  $\mathcal{C}_L$ .

The map  $\kappa = \kappa_B : \widehat{\mathfrak{g}} \rightarrow \mathfrak{t}$  (§3.2.4) induces a map  $\widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{t}}$  by completion (since  $N_W$  is nilpotent). The map  $\kappa_{W, \mathcal{F}_\bullet} : X_{W, \mathcal{F}_\bullet}^\square \rightarrow |X_{W, \mathcal{F}_\bullet}^\square| \simeq \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{t}}$  factors through a map  $X_{W, \mathcal{F}_\bullet} \rightarrow \widehat{\mathfrak{t}}$  which is also denoted by  $\kappa_{W, \mathcal{F}_\bullet}$ . The composition  $X_{\mathcal{M}, \mathcal{M}_\bullet} \xrightarrow{W_{\mathrm{dR}}(-)} X_{W, \mathcal{F}_\bullet} \xrightarrow{\kappa_{W, \mathcal{F}_\bullet}} \widehat{\mathfrak{t}}$  coincides with the morphism  $(\mathrm{wt} - \mathrm{wt}(\underline{\delta})) \circ \omega_{\underline{\delta}}$  ([BHS19, Cor. 3.3.9]). Thus there is a morphism of groupoids over  $\mathcal{C}_L$

$$X_{\mathcal{M}, \mathcal{M}_\bullet} \rightarrow \widehat{\mathcal{T}}_{\underline{\delta}}^n \times_{\widehat{\mathfrak{t}}} X_{W, \mathcal{F}_\bullet}$$

which is formally smooth by [BHS19, Thm. 3.4.4].

### 3.3.5 The trianguline variety

We now prove our main results on the local models for the trianguline variety.

We fix a continuous representation  $\bar{r} : \mathcal{G}_K \rightarrow \mathrm{GL}_n(k_L)$  of  $\mathcal{G}_K$ . Let  $R_{\bar{r}}$  be the usual framed Galois deformation ring of  $\bar{r}$  which is a complete Noetherian local ring over  $\mathcal{O}_L$  with residue field  $k_L$  and let  $\mathfrak{X}_{\bar{r}}$  be the rigid analytic space over  $L$  associated with  $\mathrm{Spf}(R_{\bar{r}})$  (the rigid generic fiber in the sense of Berthelot, cf. [dJ95, §7]). Let  $\mathcal{T}_{\mathrm{reg}}^n$  be the Zariski open subset of  $\mathcal{T}_L^n$  which consists of points  $\underline{\delta} = (\delta_1, \dots, \delta_n)$  such that  $\delta_i/\delta_j \neq z^{-\mathbf{k}}, \epsilon z^{\mathbf{k}}$  for all  $i \neq j$  and  $\mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma$ . The trianguline variety  $X_{\mathrm{tri}}(\bar{r})$  defined to be the Zariski closure in  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  of the subset  $U_{\mathrm{tri}}(\bar{r}) =$

$\{(r, \underline{\delta}) \in \mathfrak{X}_{\bar{r}} \times \mathcal{T}_{\text{reg}}^n \mid r \text{ is trianguline of parameter } \underline{\delta}\}^2$  is a reduced rigid space over  $L$ , equidimensional of dimension  $n^2 + \frac{n(n+1)}{2}[K : \mathbb{Q}_p]$  with a Zariski open dense smooth subset  $U_{\text{tri}}(\bar{r})$  ([BHS17b, Thm. 2.6]).

Assume that  $x = (r, \underline{\delta}) \in X_{\text{tri}}(\bar{r}) \subset \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  is an  $L$ -point, then  $r$  is trianguline of some parameter  $\underline{\delta}' = (\delta'_i)_{1 \leq i \leq n}$  such that  $\delta_i^{-1} \delta'_i$  is an algebraic character of  $K^\times$  for all  $i$  after possibly enlarging the coefficient field ([KPX14, Thm. 6.3.13]). Let  $D = D_{\text{rig}}(r)$  be the étale  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}$  associated with  $r$  and  $\mathcal{M} = D[\frac{1}{t}]$ . We assume that  $\underline{\delta} \in \mathcal{T}_0^n$  is locally algebraic, then  $\mathcal{M}$  is equipped with a unique triangulation  $\mathcal{M}_\bullet$  of parameter  $\underline{\delta}$  ([BHS19, Prop. 3.7.1]) and thus we are in the situation of §3.3.2. We let  $W^+ = W_{\text{dR}}(D)$ ,  $W = W^+[\frac{1}{t}]$  and  $\mathcal{F}_\bullet = W_{\text{dR}}(\mathcal{M}_\bullet)$  as before. Assume that the  $\tau$ -Sen weights of  $r$  are integers  $h_{\tau,n} \geq \dots \geq h_{\tau,1}$  where each  $k_{\tau,s_\tau} > \dots > k_{\tau,1}$  appears in the sequence  $(h_{\tau,n}, \dots, h_{\tau,1})$  with multiplicities  $m_{\tau,s_\tau}, \dots, m_{\tau,1}$  and  $m_{\tau,s_\tau} + \dots + m_{\tau,1} = n$  for  $\tau \in \Sigma$ . The Hodge-Tate weights  $(k_{\tau,i})_{1 \leq i \leq s_\tau, \tau \in \Sigma}$  coincide with what we have defined in §3.3.3. We fix an isomorphism  $\alpha : (L \otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{\text{pdR}}(W)$  and let  $P$  be the parabolic subgroup of  $G$  defined in §3.3.3 determined by  $\text{Fil}_{W^+}^\bullet(D_{\text{pdR}}(W))$  and  $\alpha$ .

Let  $V$  be the representation of  $\mathcal{G}_K$  associated with  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  by forgetting the framing. Then there are groupoids  $X_r$  (resp.  $X_V$ ) over  $\mathcal{C}_L$  parameterizing liftings of  $r$  (resp. deformations of  $V$ ) ([BHS19, §3.6]). The functor  $D_{\text{rig}}$  induces an equivalence  $X_V \xrightarrow{\sim} X_D$  and there is an isomorphism of formal schemes  $X_r \simeq \widehat{\mathfrak{X}_{\bar{r},r}}$  where  $\widehat{\mathfrak{X}_{\bar{r},r}}$  is the completion of  $\mathfrak{X}_{\bar{r}}$  at the point  $r$  ([Kis09a, Lem. 2.3.3, Prop. 2.3.5]). Let  $\widehat{X_{\text{tri}}(\bar{r})}_x$  be the completion of the trianguline variety at the point  $x$ . Then we get a morphism  $\widehat{X_{\text{tri}}(\bar{r})}_x \rightarrow \widehat{\mathfrak{X}_{\bar{r},r}} \simeq X_r$  by projection.

Let  $X_{r,\mathcal{M}_\bullet} := X_r \times_{X_V} \times_{X_D} \times_{X_D} X_{D,\mathcal{M}_\bullet}$  where the map  $X_r \rightarrow X_V$  is forgetting the framing. Recall that the natural map  $X_{\mathcal{M},\mathcal{M}_\bullet} \rightarrow X_{\mathcal{M}}$  as well as its base change  $X_{r,\mathcal{M}_\bullet} \rightarrow X_r$  is a closed immersion since we have assumed  $\underline{\delta} \in \mathcal{T}_0^n$  ([BHS19, Prop. 3.4.6]).

**Proposition 3.3.4.** *The morphism  $\widehat{X_{\text{tri}}(\bar{r})}_x \rightarrow X_r$  factors through a closed immersion  $\widehat{X_{\text{tri}}(\bar{r})}_x \rightarrow X_{r,\mathcal{M}_\bullet}$ .*

*Proof.* This is [BHS19, Prop. 3.7.2] (based on [KPX14, Cor. 6.3.10]) and [BHS19, Prop. 3.7.3]. The original proof for [BHS19, Prop. 3.7.2] is not complete and we write a proof here. The argument will be needed for Theorem 3.4.10.

Pick an affinoid neighbourhood  $U$  of  $x$  in  $X_{\text{tri}}(\bar{r})$ . Let  $D_U$  be the  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{U,K}$  associated with the universal Galois representation of  $\mathcal{G}_K$  pulled back from  $\mathfrak{X}_{\bar{r}}$  (cf. [KPX14, Thm. 2.2.17]). By [KPX14, Cor. 6.3.10], there is a birational proper morphism  $f : \tilde{U} \rightarrow U$ , a filtration of sub- $(\varphi, \Gamma_K)$ -modules  $D_{\tilde{U},\bullet}$  over  $\mathcal{R}_{\tilde{U},K}$  of  $D_{\tilde{U}} := f^* D_U$  and invertible sheaves  $(\mathcal{L}_i)_{i=1,\dots,n}$  such that  $D_{\tilde{U},0} = 0, D_{\tilde{U},n} = D_{\tilde{U}}$  and there are inclusions  $D_{\tilde{U},i}/D_{\tilde{U},i-1} \hookrightarrow \mathcal{R}_{\tilde{U},K}(\delta_{\tilde{U},i}) \otimes_{\mathcal{O}_{\tilde{U}}} \mathcal{L}_i$  the cokernels of which are killed by  $t$  for  $i = 1, \dots, n$  where the characters  $\delta_{\tilde{U}}$  is the pullback of the character on  $\mathcal{T}_L^n$  via  $\tilde{U} \rightarrow U \subset X_{\text{tri}}(\bar{r}) \rightarrow \mathcal{T}_L^n$ .

Let  $R_r$  be the completion of  $R_{\bar{r}}[\frac{1}{p}]$  at the maximal ideal corresponding to  $r$  so that  $\widehat{\mathfrak{X}_{\bar{r},r}} \simeq \text{Spf}(R_r)$  and let  $R_{r,\mathcal{M}_\bullet}$  be the quotient of  $R_r$  such that  $R_{r,\mathcal{M}_\bullet}$  pro-represents  $X_{r,\mathcal{M}_\bullet}$ . Take an arbitrary point  $x' \in f^{-1}(x)$ . We firstly prove that the map  $R_r \rightarrow \hat{\mathcal{O}}_{U,x} \rightarrow \hat{\mathcal{O}}_{\tilde{U},x'}$  induced by  $\tilde{U} \rightarrow U \subset X_{\text{tri}}(\bar{r}) \rightarrow \mathfrak{X}_{\bar{r}}$  factors through the quotient  $R_{r,\mathcal{M}_\bullet}$ . Take  $A$  a local Artin  $L$ -algebra with residue field  $k(x')$  ( $A$  is a  $k(x')$ -algebra, [Sta22, Tag 0323]) with a composite  $x = \text{Sp}(k(x')) \rightarrow \text{Sp}(A) \rightarrow V$  where  $x \rightarrow \text{Sp}(A)$  corresponds to the reduction map  $A \rightarrow k(x')$ . Then the pullback along the map  $\text{Sp}(A) \rightarrow \tilde{U}$  of  $D_{\tilde{U},\bullet}[\frac{1}{t}]$  gives a triangulation  $\mathcal{M}_{A,\bullet}$  of  $D_A[\frac{1}{t}] = D_{\text{rig}}(r_A)[\frac{1}{t}]$  of parameter  $\underline{\delta}_A = \delta_{\tilde{U}} \otimes_{\mathcal{O}_{\tilde{U}}} A$  where  $r_A$  is the pullback of the universal Galois representation to  $A$  via the map  $R_r \rightarrow \hat{\mathcal{O}}_{\tilde{U},x'} \rightarrow A$ . Remark that the triangulation  $\mathcal{M}_{A,\bullet} \otimes_A k(x')$  of  $\mathcal{M}_A \otimes_A$

<sup>2</sup>Here a representation  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L')$  for some finite extension  $L'/\mathbb{Q}_p$  is said to be trianguline of character  $\underline{\delta}$  means that the  $(\varphi, \Gamma_K)$ -module  $D_{\text{rig}}(r)$  over  $\mathcal{R}_{L',K}$  admits a filtration of sub- $(\varphi, \Gamma_K)$ -modules whose graded pieces are isomorphic to  $\mathcal{R}_{L',K}(\delta_1), \dots, \mathcal{R}_{L',K}(\delta_n)$ .

$k(x')$  coincides with  $\mathcal{M}_\bullet \otimes_L k(x')$  where  $\mathcal{M}_\bullet$  is the unique triangulation on  $\mathcal{M} = D_{\text{rig}}(r) \left[ \frac{1}{t} \right]$  of parameter  $\underline{\delta}$  obtained by Lemma [BHS19, Prop. 3.7.1] (and by [KPX14, Thm. 4.4.3] and [BHS19, Lem. 3.4.3]). Let  $\tilde{A} := A \times_{k(x')} L \in \mathcal{C}_L$  be the subring of  $A$  consisting of elements whose reduction modulo the maximal ideal  $\mathfrak{m}_A$  of  $A$  lie in  $L$  (cf. [Sta22, Tag 08KG]). By writing out some  $L$ -bases of  $\tilde{A}$  and  $A$ , we see that the preimage of  $\mathcal{R}_{L,K} \left[ \frac{1}{t} \right] \subset \mathcal{R}_{k(x'),K} \left[ \frac{1}{t} \right]$  under the map  $\mathcal{R}_{A,K} \left[ \frac{1}{t} \right] \rightarrow \mathcal{R}_{k(x'),K} \left[ \frac{1}{t} \right]$  is  $\mathcal{R}_{\tilde{A},K} \left[ \frac{1}{t} \right]$ . As the reduction of  $r_A$  is  $r \otimes_L k(x')$  (resp. the reduction of  $\delta_{A,i} : \delta_A \otimes_A k(x')$  is  $\delta_i \otimes_L k(x')$ ),  $r_A$  (resp.  $\delta_{A,i}$ ) can be defined over  $\tilde{A}$  and we denote the model by  $r_{\tilde{A}}$  (resp.  $\delta_{\tilde{A},i}$ ) whose reduction modulo  $\mathfrak{m}_{\tilde{A}}$  is  $r$  (resp.  $\delta_i$ ). Then  $D_{\text{rig}}(r_A) \left[ \frac{1}{t} \right] = D_{\text{rig}}(r_{\tilde{A}}) \left[ \frac{1}{t} \right] \otimes_{\tilde{A}} A$ .

We need to show that the triangulation  $\mathcal{M}_{A,\bullet}$  also has a model  $\mathcal{M}_{\tilde{A},\bullet}$  over  $\mathcal{R}_{\tilde{A},K}$ . We extend the injection

$$\mathcal{M}_1 = \mathcal{R}_{L,K}(\delta_1) \left[ \frac{1}{t} \right] \hookrightarrow D_{\text{rig}}(r) \left[ \frac{1}{t} \right]$$

to an isomorphism  $\oplus_{i=1}^n \mathcal{R}_{L,K} \left[ \frac{1}{t} \right] \bar{e}_i \xrightarrow{\sim} D_{\text{rig}}(r) \left[ \frac{1}{t} \right]$  of  $\mathcal{R}_{L,K} \left[ \frac{1}{t} \right]$ -modules where  $\bar{e}_1$  is the image of a generator of the rank one free module  $\mathcal{R}_{L,K}(\delta_1) \left[ \frac{1}{t} \right]$ . Since the reduction of the injection  $\mathcal{R}_{A,K}(\delta_{A,1}) \left[ \frac{1}{t} \right] \hookrightarrow D_{\text{rig}}(r_A) \left[ \frac{1}{t} \right]$  is identified via pull back with  $\mathcal{R}_{L,K}(\delta_1) \left[ \frac{1}{t} \right] \otimes_L k(x') \hookrightarrow D_{\text{rig}}(r) \left[ \frac{1}{t} \right] \otimes_L k(x')$ , we can pick a lift  $e_1 \in D_{\text{rig}}(r_A) \left[ \frac{1}{t} \right]$  of  $\bar{e}_1 \otimes_L 1$  generating the image of the first injection. We can extend the injection to an isomorphism  $\mathcal{R}_{A,K} \left[ \frac{1}{t} \right]^n \simeq D_{\text{rig}}(r_A) \left[ \frac{1}{t} \right]$  of  $\mathcal{R}_{A,K} \left[ \frac{1}{t} \right]$ -modules and we may assume, after changing basis given by a matrix in  $\text{GL}_{n-1}(\mathcal{R}_{A,K} \left[ \frac{1}{t} \right])$ , the reduction of the extended basis  $e_2, \dots, e_n$  is equal to  $\bar{e}_2 \otimes_L 1, \dots, \bar{e}_n \otimes_L 1$  since we have a surjection  $\text{GL}_{n-1}(\mathcal{R}_{A,K} \left[ \frac{1}{t} \right]) \rightarrow \text{GL}_{n-1}(\mathcal{R}_{k(x'),K} \left[ \frac{1}{t} \right])$ . We also take a basis  $\tilde{e}_1, \dots, \tilde{e}_n$  of  $D_{\text{rig}}(r_{\tilde{A}}) \left[ \frac{1}{t} \right]$  such that the reduction modulo  $\mathfrak{m}_{\tilde{A}}$  of  $\tilde{e}_1, \dots, \tilde{e}_n$  is equal to  $\bar{e}_1, \dots, \bar{e}_n$ . The translation matrix  $M$  between the basis  $\tilde{e}_1 \otimes_{\tilde{A}} 1, \dots, \tilde{e}_n \otimes_{\tilde{A}} 1$  and  $e_1, \dots, e_n$  of  $D_{\text{rig}}(r_A) \left[ \frac{1}{t} \right]$  has trivial reduction modulo  $\mathfrak{m}_A$ . In particular,  $M \in \text{GL}_n(\mathcal{R}_{\tilde{A},K} \left[ \frac{1}{t} \right])$ . This means that we can choose  $\tilde{e}_i$  such that  $e_i = \tilde{e}_i \otimes_{\tilde{A}} 1$ . Then we see the element  $\tilde{e}_1$  defines an injection  $\mathcal{R}_{\tilde{A},K}(\delta_{\tilde{A},1}) \left[ \frac{1}{t} \right] \hookrightarrow D_{\text{rig}}(r_{\tilde{A}}) \left[ \frac{1}{t} \right]$  with the quotient a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{\tilde{A},K} \left[ \frac{1}{t} \right]$ . Applying the same argument on the quotient and by induction, we see that  $D_{\text{rig}}(r_{\tilde{A}}) \left[ \frac{1}{t} \right]$  admits a filtration  $\mathcal{M}_{\tilde{A},\bullet}$  such that  $\mathcal{M}_{\tilde{A},\bullet} \otimes_{\tilde{A}} A = \mathcal{M}_{A,\bullet}$  and  $\mathcal{M}_{\tilde{A},\bullet} \otimes_{\tilde{A}} L = \mathcal{M}_\bullet$ . Let  $\tilde{\mathcal{O}}_{\tilde{U},x'}$  be the subring  $\widehat{\mathcal{O}}_{\tilde{U},x'} \times_{k(x')} L$  of  $\widehat{\mathcal{O}}_{\tilde{U},x'}$ . The composite map  $R_r \rightarrow \widehat{\mathcal{O}}_{U,x} \rightarrow \widehat{\mathcal{O}}_{\tilde{U},x'} \rightarrow A$  factors through  $R_r \rightarrow \tilde{\mathcal{O}}_{\tilde{U},x'} \rightarrow \tilde{A}$  which gives rise the deformation  $r_{\tilde{A}}$  of  $r$ , i.e. an object  $(\tilde{A}, r_{\tilde{A}}) \in X_r$ . The discussion above shows that  $r_{\tilde{A}}$  admits a triangulation  $\mathcal{M}_{\tilde{A},\bullet}$  on  $D_{\text{rig}}(r_{\tilde{A}}) \left[ \frac{1}{t} \right]$  whose reduction modulo  $\mathfrak{m}_{\tilde{A}}$  is  $\mathcal{M}_\bullet$  and defines an object in  $X_{r,\mathcal{M}_\bullet}$ . Hence the morphism  $R_r \rightarrow \tilde{A}$ , as well as  $R_r \rightarrow \tilde{A} \rightarrow A$ , factors through  $R_{r,\mathcal{M}_\bullet}$ . This implies that the morphism  $R_r \rightarrow \widehat{\mathcal{O}}_{\tilde{U},x'}$  factors through the quotient  $R_{r,\mathcal{M}_\bullet}$ .

We now prove that the map  $R_r \rightarrow \widehat{\mathcal{O}}_{U,x}$  also factors through  $R_{r,\mathcal{M}_\bullet}$ . Otherwise assume there exists a non-zero element  $a$  in the kernel of  $R_r \rightarrow R_{r,\mathcal{M}_\bullet}$  such that the image of  $a$  in  $\widehat{\mathcal{O}}_{U,x}$  is not zero. As the morphism  $f : \tilde{U} \rightarrow U$  is proper and surjective, there is the Stein decomposition  $f : \tilde{U} \xrightarrow{f'} Z \xrightarrow{g} U$  such that  $g$  is a finite surjective morphism,  $f'$  is a surjective proper morphism and  $\mathcal{O}_Z \xrightarrow{\sim} f'_* \mathcal{O}_{\tilde{U}}$  ([BGR84, Prop. 9.6.3/5]). Then if we write  $U = \text{Sp}(A)$ ,  $Z = \text{Sp}(B)$ , the map  $A \rightarrow B$  induced by  $g$  is a finite injection (since  $U$  is reduced and  $g$  is a surjection). Let  $\mathfrak{m}$  be the maximal ideal of  $A$  corresponding to the point  $x$  and let  $\mathfrak{n}_1, \dots, \mathfrak{n}_m$  be the maximal ideals of  $B$  above  $\mathfrak{m}$ . Then there is an injection  $\hat{A}_{\mathfrak{m}} \hookrightarrow B \otimes_A \hat{A}_{\mathfrak{m}} \simeq \oplus_{i=1}^m \hat{B}_{\mathfrak{n}_i}$  ([Sta22, Tag 07N9]) where  $\hat{A}_{\mathfrak{m}}$  (resp.  $\hat{B}_{\mathfrak{n}_i}$ ) denotes the completion of  $A$  (resp.  $B$ ) with respect to the ideal  $\mathfrak{m}$  (resp.  $\mathfrak{n}_i$ ). Since the image of  $a$  in  $\hat{A}_{\mathfrak{m}}$  is not zero, the image of  $a$  in one of  $\hat{B}_{\mathfrak{n}_i}$  is not zero, which we may assume to be  $\hat{B}_{\mathfrak{n}_1}$ . Let  $z$  be the point on  $Z$  corresponding to  $\mathfrak{n}_1$ . By the theorem on formal functions ([BGR84, Thm. 9.6.3/2]),  $(f'_* \mathcal{O}_{\tilde{U}})_z^\wedge \xrightarrow{\sim} \varprojlim_s (\mathcal{O}_{\tilde{U}}/\mathfrak{n}_1^s)(\tilde{U})$ . Hence  $\hat{B}_{\mathfrak{n}_1} \xrightarrow{\sim} \varprojlim_s (\mathcal{O}_{\tilde{U}}/\mathfrak{n}_1^s)(\tilde{U})$ . Thus the image of  $a$  in  $(\mathcal{O}_{\tilde{U}}/\mathfrak{n}_1^s)(\tilde{U})$  is not zero for some  $s$ . It turns out that there exists  $x' \in (f')^{-1}(z)$  such that the image of  $a$  in  $\mathcal{O}_{\tilde{U},x'}/\mathfrak{n}_1^s$  is not zero ([BGR84, Cor. 9.4.2/7] and [BGR84, Prop. 9.5.3/1(iii)])

and hence the image of  $a$  in  $\widehat{C}_{n_1} := \varprojlim_s \mathcal{O}_{\widetilde{U},x'} / \mathfrak{m}_1^s$  is not zero where  $C := \mathcal{O}_{\widetilde{U},x'}$ . The completion of  $\widehat{C}_{n_1}$  with respect to the maximal ideal  $\mathfrak{m}_{x'} \widehat{C}_{n_1}$  is then equal to ([AM69, Prop. 10.12, 10.13] and that  $\mathcal{O}_{\widetilde{U},x'} / \mathfrak{m}_{x'}^s \mathcal{O}_{\widetilde{U},x'}$  is  $n_1$ -adically complete for any  $s$ )

$$\varprojlim_s \widehat{C}_{n_1} / \mathfrak{m}_{x'}^s \widehat{C}_{n_1} = \varprojlim_s C / \mathfrak{m}_{x'}^s C = \varprojlim_s \mathcal{O}_{\widetilde{U},x'} / \mathfrak{m}_{x'}^s \mathcal{O}_{\widetilde{U},x'} = \widehat{\mathcal{O}_{\widetilde{U},x'}}.$$

Hence the morphism  $\widehat{C}_{n_1} \rightarrow \widehat{\mathcal{O}_{\widetilde{U},x'}}$  is the completion of the Noetherian local ring  $\widehat{C}_{n_1}$  ([AM69, Thm. 10.26]) with respect to the maximal ideal, thus is injective ([AM69, Cor. 10.19]). We conclude that  $a$  is sent to a non-zero element in  $\widehat{\mathcal{O}_{\widetilde{U},x'}}$ . This contradicts that the morphism  $R_r \rightarrow \widehat{\mathcal{O}_{\widetilde{U},x'}}$  factors through the quotient  $R_{r,\mathcal{M}_\bullet}$  which we just proved! Thus we get the conclusion.  $\square$

We fix an isomorphism  $W \simeq (\mathcal{S}_n)^\Sigma$  by identifying  $n$ -tuples  $(a_{\tau,1}, \dots, a_{\tau,n})_{\tau \in \Sigma} \in (\mathbb{Z}^n)^\Sigma$  with the diagonal matrix  $\prod_{\tau \in \Sigma} \text{diag}(a_{\tau,1}, \dots, a_{\tau,n}) \in \mathfrak{g}$ . By [BHS17b, Prop. 2.9] (and the proof of [BHS19, Lem. 3.7.6]), the multisets of Sen weights of  $V$  are exactly the multisets  $\{\text{wt}_\tau(\delta_i) \mid i = 1, \dots, n\}, \tau \in \Sigma$ . Hence there exists a unique  $w = (w_\tau)_{\tau \in \Sigma} \in W/W_P$  such that

$$(h_{\tau,1}, \dots, h_{\tau,n}) = w_\tau^{-1}(\text{wt}_\tau(\delta_1), \dots, \text{wt}_\tau(\delta_n)) = (\text{wt}_\tau(\delta_{w_\tau(1)}), \dots, \text{wt}_\tau(\delta_{w_\tau(n)}))$$

for any  $\tau \in \Sigma$ . We denote by  $w \in W/W_P$  the element associated with the point  $x$  in this way.

Recall that there is a morphism  $(\kappa_1, \kappa_2) : X_P \rightarrow T_P = \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}/W_P$  in §3.2.4 which induces a morphism  $X_{D,\mathcal{M}_\bullet}^\square \rightarrow X_{W^+,\mathcal{F}_\bullet}^\square \rightarrow |X_{W^+,\mathcal{F}_\bullet}^\square| \xrightarrow{\sim} \widehat{X}_{P,x_{\text{pdR}}} \rightarrow \widehat{T}_{P,(0,0)}$  where  $x_{\text{pdR}}$  is the point in  $X_P(L)$  associated with the point  $x \in X_{\text{tri}}(\bar{r})(L)$  together with the fixed framing  $\alpha$  as in Proposition 3.3.2. The morphism  $X_{D,\mathcal{M}_\bullet}^\square \rightarrow \widehat{T}_{P,(0,0)}$  factors through a morphism  $X_{D,\mathcal{M}_\bullet} \rightarrow \widehat{T}_{P,(0,0)}$  (see the end of [BHS19, §3.5]). We let  $\Theta_x : \widehat{X}_{\text{tri}}(\bar{r})_x \rightarrow \widehat{T}_{P,(0,0)}$  be the composite map  $\widehat{X}_{\text{tri}}(\bar{r})_x \hookrightarrow X_{r,\mathcal{M}_\bullet} \rightarrow X_{V,\mathcal{M}_\bullet} \simeq X_{D,\mathcal{M}_\bullet} \rightarrow \widehat{T}_{P,(0,0)}$ . Recall that there are closed formal subschemes  $\widehat{T}_{P,w',(0,0)}$  of  $\widehat{T}_{P,(0,0)}$  for  $w' \in W/W_P$ .

**Proposition 3.3.5.** *The morphism  $\Theta_x$  factors through  $\widehat{T}_{P,w,(0,0)} \hookrightarrow \widehat{T}_{P,(0,0)}$ .*

*Proof.* Assume that  $x_A : \text{Spf}(A) \rightarrow \widehat{X}_{\text{tri}}(\bar{r})_x$  is an  $A$ -point for some  $A \in \mathcal{C}_L$ . Via the morphism  $\widehat{X}_{\text{tri}}(\bar{r})_x \rightarrow X_{D,\mathcal{M}_\bullet} \rightarrow X_{W^+,\mathcal{F}_\bullet}$ , the point  $x_A$  is associated with a  $B_{\text{dR}}^+ \otimes_{\mathbb{Q}_p} A$ -representation  $W_A^+$  of  $\mathcal{G}_K$ , a full filtration  $\mathcal{F}_{A,\bullet}$  of  $W_A = W_A^+[\frac{1}{t}]$  and a parameter  $\underline{\delta}_A = (\delta_{A,1}, \dots, \delta_{A,n})$  of the associated  $(\varphi, \Gamma_K)$ -module  $\mathcal{M}_A$  over  $\mathcal{R}_{A,K}[\frac{1}{t}]$ , which is a deformation of the datum  $(W^+, \mathcal{F}_\bullet, \underline{\delta})$ . We can choose a framing  $\alpha_A : (A \otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{\text{pdR}}(W_A)$  such that  $\alpha_A$  modulo  $\mathfrak{m}_A$  coincides with  $\alpha$ . Let  $x_{A,\text{pdR}}$  be the point

$$\left( \alpha_A^{-1}(D_{\text{pdR}}(\mathcal{F}_{A,\bullet})), \alpha_A^{-1} \left( \left( \text{Fil}_{W_A^+}^{-k_{\tau,\bullet}}(D_{\text{pdR},\tau}(W_A)) \right)_{\tau \in \Sigma} \right), N_{W_A} = \alpha_A^{-1} \circ \nu_{W_A} \circ \alpha_A \right) \in \widehat{X}_{P,x_{\text{pdR}}}(A)$$

corresponding to the element in  $X_{W^+,\mathcal{F}_\bullet}(A)$  given by  $x_A$ . Then  $\Theta_x$  sends  $x_A$  to

$$(\kappa_1(x_{A,\text{pdR}}), \kappa_2(x_{A,\text{pdR}})) \in (\mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}/W_P)(A)$$

whose reduction module  $\mathfrak{m}_A$  is  $(0,0)$  where  $\kappa_1, \kappa_2$  are defined in §3.2.4.

Explicitly,  $\kappa_1(x_{A,\text{pdR}}) \in \mathfrak{t}(A)$  equals to

$$(\nu_{W_{\text{dR}}}(\mathcal{R}_{A,K}(\delta_{A,1})[\frac{1}{t}]), \dots, \nu_{W_{\text{dR}}}(\mathcal{R}_{A,K}(\delta_{A,n})[\frac{1}{t}])) = (\text{wt}(\delta_{A,1}) - \text{wt}(\delta_1), \dots, \text{wt}(\delta_{A,n}) - \text{wt}(\delta_n)) \quad (3.3.6)$$

by [BHS19, Cor. 3.3.9] ( $\kappa_1$  and  $\kappa_{W,\mathcal{F}_\bullet}$  in §3.3.4 are both defined using  $\kappa_B : \widetilde{\mathfrak{g}} \rightarrow \mathfrak{t}$ ).

For each  $\tau \in \Sigma, i \in \mathbb{Z}$ , we let  $\mathrm{gr}^{\tau,i}(\nu_{W_A})$  be the restriction of  $\nu_{W_A}$  on  $\mathrm{gr}_{\mathrm{Fil}_{W_A}^+}^i(D_{\mathrm{pdR},\tau}(W_A))$  (see §3.3.3 for notation). Then

$$\kappa_2(x_{A,\mathrm{pdR}}) \in \mathfrak{t}/W_P(A) = \prod_{\tau \in \Sigma} \mathfrak{t}_{\tau}/W_{P_{\tau}}(A) = \prod_{\tau \in \Sigma} \prod_{i=1}^{s_{\tau}} \mathfrak{t}_{\tau,i}/W_{M_{\tau,i}}(A)$$

is (to simplify the notation we omit the identification  $\alpha_A$ )

$$\left( \gamma_{M_{\tau,1}} \left( \mathrm{gr}^{\tau,-k_{\tau,1}}(\nu_{W_A}) \right), \dots, \gamma_{M_{\tau,s_{\tau}}} \left( \mathrm{gr}^{\tau,-k_{\tau,s_{\tau}}}(\nu_{W_A}) \right) \right)_{\tau \in \Sigma} \quad (3.3.7)$$

where we decompose the Levi subalgebra  $\mathfrak{m}_{P_{\tau}} = \bigoplus_{i=1}^{s_{\tau}} \mathfrak{m}_{\tau,i}$  of  $\mathfrak{p}_{\tau}$ , where  $\mathfrak{p}_{\tau}$  denotes the Lie algebra of  $P_{\tau}$ , according to the projections to the graded pieces of the filtrations,  $M_{\tau,i}$  is the subgroup associated with  $\mathfrak{m}_{\tau,i}$ ,  $\mathfrak{t}_{\tau,i} = \mathfrak{t}_{\tau} \cap \mathfrak{m}_{\tau,i}$  and we identify every  $\mathrm{gr}^{\tau,-k_{\tau,i}}(\nu_{W_A})$  as an element in  $\mathfrak{m}_{\tau,i}$ . The map  $\gamma_{M_{\tau,i}} : \mathfrak{m}_{\tau,i}(A) \rightarrow (\mathfrak{t}_{\tau,i}/W_{M_{\tau,i}})(A)$  defined in §3.2.4 is no more than sending a matrix to (the coefficients of) its characteristic polynomial.

We need to prove that the  $\Theta_x(x_A) \in \widehat{T}_{P,w,(0,0)}(A)$ . By the definition of  $T_{P,w}$  in §3.2.4, we only need to verify the following equality

$$\kappa_2(x_{A,x_{\mathrm{pdR}}}) = w^{-1}(\kappa_1(x_{A,x_{\mathrm{pdR}}})) \quad (3.3.8)$$

in  $\mathfrak{t}/W_P(A)$ . The strategy is as in [BHS19, Lem. 3.7.4]. We will compare two factorizations of the Sen polynomial of  $W_A^+/tW_A^+$ . The first factorization (3.3.9) will be related to the Hodge filtrations and the second one (3.3.10) will be given by the trianguline filtrations.

First, by [BHS19, Lem. 3.7.5], the Sen polynomial of  $W_A^+/tW_A^+$  in

$$A \otimes_{\mathbb{Q}_p} K[Y] = \prod_{\tau \in \Sigma} (A \otimes_{\mathbb{Q}_p} K) \otimes_{A \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} A[Y] = \prod_{\tau \in \Sigma} A[Y]$$

is equal to

$$\begin{aligned} & \prod_{i \in \mathbb{Z}} \det \left( Y \mathrm{Id} + i \mathrm{Id} - \mathrm{gr}^i(\nu_{W_A}) \mid \mathrm{gr}_{\mathrm{Fil}_{W_A}^+}^i(D_{\mathrm{pdR}}(W_A)) \right) \\ &= \prod_{\tau \in \Sigma} \prod_{i=1}^{s_{\tau}} \det \left( Y \mathrm{Id} - k_{\tau,i} \mathrm{Id} - \mathrm{gr}^{\tau,-k_{\tau,i}}(\nu_{W_A}) \mid \mathrm{gr}_{\mathrm{Fil}_{W_A}^+}^{-k_{\tau,i}}(D_{\mathrm{pdR},\tau}(W_A)) \right). \end{aligned} \quad (3.3.9)$$

On the other hand, by [BHS19, Lem. 3.7.6], the Sen polynomial is

$$\prod_{i=1}^n (Y - \mathrm{wt}(\delta_{A,i})) = \prod_{\tau \in \Sigma} \prod_{i=1}^n (Y - \mathrm{wt}_{\tau}(\delta_i) - (\mathrm{wt}_{\tau}(\delta_{A,i}) - \mathrm{wt}_{\tau}(\delta_i))). \quad (3.3.10)$$

Comparing (3.3.9) and (3.3.10), we get an equality in  $A[Y]$  of the  $\tau$ -Sen polynomial of  $W_A^+/tW_A^+$  for each  $\tau \in \Sigma$ :

$$\begin{aligned} & \prod_{i=1}^n (Y - \mathrm{wt}_{\tau}(\delta_i) - (\mathrm{wt}_{\tau}(\delta_{A,i}) - \mathrm{wt}_{\tau}(\delta_i))) \\ &= \prod_{i=1}^{s_{\tau}} \det \left( Y \mathrm{Id} - k_{\tau,i} \mathrm{Id} - \mathrm{gr}^{\tau,-k_{\tau,i}}(\nu_{W_A}) \mid \mathrm{gr}_{\mathrm{Fil}_{W_A}^+}^{-k_{\tau,i}}(D_{\mathrm{pdR},\tau}(W_A)) \right). \end{aligned} \quad (3.3.11)$$

Modulo  $\mathfrak{m}_A$ , the right hand side of (3.3.11) calculates the  $\tau$ -Sen polynomial of  $W^+/tW^+$  in  $L[Y]$  in the usual way using the Hodge-Tate weights:

$$\begin{aligned} & \prod_{i=1}^{s_\tau} \det \left( Y \text{Id} - k_{\tau,i} \text{Id} - \text{gr}^{\tau, -k_{\tau,i}}(\nu_{W_A}) \mid \text{gr}_{\text{Fil}_{W_A^+}^\bullet}^{-k_{\tau,i}}(D_{\text{pdR}, \tau}(W_A)) \right) \\ & \equiv \prod_{i=1}^{s_\tau} \det \left( Y \text{Id} - k_{\tau,i} \text{Id} - \text{gr}^{\tau, -k_{\tau,i}}(\nu_W) \mid \text{gr}_{\text{Fil}_{W^+}^\bullet}^{-k_{\tau,i}}(D_{\text{pdR}, \tau}(W)) \right) \\ & = \prod_{i=1}^{s_\tau} (Y - k_{\tau,i})^{m_{\tau,i}} \end{aligned}$$

where the last equality uses the fact that  $\text{gr}^{\tau, -k_{\tau,i}}(\nu_W) \in \mathfrak{m}_{\tau,i}(L)$  is nilpotent (since  $\nu_W$  is nilpotent). Actually, for each  $i$ ,

$$\det \left( Y \text{Id} - k_{\tau,i} \text{Id} - \text{gr}^{\tau, -k_{\tau,i}}(\nu_{W_A}) \mid \text{gr}_{\text{Fil}_{W_A^+}^\bullet}^{-k_{\tau,i}}(D_{\text{pdR}, \tau}(W_A)) \right) \equiv (Y - k_{\tau,i})^{m_{\tau,i}} \pmod{\mathfrak{m}_A}. \quad (3.3.12)$$

As we have assumed that  $k_{\tau,i} \neq k_{\tau,j}$  if  $i \neq j$  and each  $\text{wt}_\tau(\delta_{A,j}) - \text{wt}_\tau(\delta_j)$  is in  $\mathfrak{m}_A$ , we get that  $\text{wt}_\tau(\delta_j) + (\text{wt}_\tau(\delta_{A,j}) - \text{wt}_\tau(\delta_j)) - k_{\tau,i} \in \mathfrak{m}_A$  if and only if  $\text{wt}_\tau(\delta_j) = k_{\tau,i}$ . Apply Lemma 3.3.14 below (where we take  $k_i = k_{\tau,i}$ ,  $m_i = m_{\tau,i}$  and  $a_{i,1}, \dots, a_{i,m_i}$  are those  $\text{wt}_\tau(\delta_j) + (\text{wt}_\tau(\delta_{A,j}) - \text{wt}_\tau(\delta_j))$  such that  $\text{wt}_\tau(\delta_j) = k_{\tau,i}$  for (3.3.11) and using (3.3.12), we get

$$\begin{aligned} & \prod_{\text{wt}_\tau(\delta_j) = k_{\tau,i}} (Y - \text{wt}_\tau(\delta_j) - (\text{wt}_\tau(\delta_{A,j}) - \text{wt}_\tau(\delta_j))) \\ & = \det \left( Y \text{Id} - k_{\tau,i} \text{Id} - \text{gr}^{\tau, -k_{\tau,i}}(\nu_{W_A}) \mid \text{gr}_{\text{Fil}_{W_A^+}^\bullet}^{-k_{\tau,i}}(D_{\text{pdR}, \tau}(W_A)) \right). \end{aligned}$$

Replace  $Y$  with  $Y + k_{\tau,i}$  in the above identity, we get

$$\prod_{\text{wt}_\tau(\delta_j) = k_{\tau,i}} (Y - (\text{wt}_\tau(\delta_{A,j}) - \text{wt}_\tau(\delta_j))) = \det \left( Y \text{Id} - \text{gr}^{\tau, -k_{\tau,i}}(\nu_{W_A}) \mid \text{gr}_{\text{Fil}_{W_A^+}^\bullet}^{-k_{\tau,i}}(D_{\text{pdR}, \tau}(W_A)) \right) \quad (3.3.13)$$

for any  $\tau \in \Sigma, i = 1, \dots, s_\tau$ .

In the following, we verify that (3.3.13) above for all  $\tau$  and  $i$  implies (and is equivalent to) the equality (3.3.8) which we want to prove.

Fix an arbitrary lift  $w = (w_\tau)_{\tau \in \Sigma}$  in  $W$  with the same notation for  $w \in W/W_P$ . The  $\tau$ -part of  $w^{-1}(\kappa_1(x_{A, x_{\text{pdR}}}))$  is (by (3.3.6))

$$\begin{aligned} & w_\tau^{-1}(\text{wt}(\delta_{A,1}) - \text{wt}(\delta_1), \dots, \text{wt}(\delta_{A,n}) - \text{wt}(\delta_n)) \\ & = (\text{wt}(\delta_{A, w_\tau(1)}) - \text{wt}(\delta_{w_\tau(1)}), \dots, \text{wt}(\delta_{A, w_\tau(n)}) - \text{wt}(\delta_{w_\tau(n)})) \end{aligned}$$

whose image in  $\mathfrak{t}_\tau/W_{P_\tau}(A) = \mathfrak{t}_{\tau,1}/W_{M_{\tau,1}}(A) \times \dots \times \mathfrak{t}_{\tau,s_\tau}/W_{M_{\tau,s_\tau}}(A)$  is (we use characteristic polynomials to denote the image of an element of  $\mathfrak{t}_{\tau,i}(A)$  in  $\mathfrak{t}_{\tau,i}/W_{M_{\tau,i}}(A)$ )

$$\left( \prod_{i=1}^{m_{\tau,1}} (Y - (\text{wt}_\tau(\delta_{A, w_\tau(i)}) - \text{wt}_\tau(\delta_{w_\tau(i)}))) , \dots , \prod_{i=n-m_{\tau,s_\tau}+1}^n (Y - (\text{wt}_\tau(\delta_{A, w_\tau(i)}) - \text{wt}_\tau(\delta_{w_\tau(i)}))) \right).$$

Recall  $w$  is chosen so that

$$(\text{wt}_\tau(\delta_{w_\tau(1)}), \dots, \text{wt}_\tau(\delta_{w_\tau(n)})) = (h_{\tau,1}, \dots, h_{\tau,n}) = (\underbrace{k_{\tau,1}, \dots, k_{\tau,1}}_{m_{\tau,1}}, \dots, \underbrace{k_{\tau,s_\tau}, \dots, k_{\tau,s_\tau}}_{m_{\tau,s_\tau}})$$

for all  $\tau \in \Sigma$ . Hence the  $\tau$ -part of  $w^{-1}(\kappa_1(x_{A,x_{\text{pdR}}}))$  can be furthermore rewritten as

$$\left( \prod_{\text{wt}_\tau(\delta_j)=k_{\tau,1}} (Y - (\text{wt}_\tau(\delta_{A,j}) - \text{wt}_\tau(\delta_j))) , \dots , \prod_{\text{wt}_\tau(\delta_j)=k_{\tau,s_\tau}} (Y - (\text{wt}_\tau(\delta_{A,j}) - \text{wt}_\tau(\delta_j))) \right).$$

Using (3.3.13), the above element is equal to

$$\begin{aligned} & \left( \det \left( Y \text{Id} - \text{gr}^{\tau, -k_{\tau,1}}(\nu_{W_A}) \right) , \dots , \det \left( Y \text{Id} - \text{gr}^{\tau, -k_{\tau,s_\tau}}(\nu_{W_A}) \right) \right) \\ &= \left( \gamma_{M_{\tau,1}} \left( \text{gr}^{\tau, -k_{\tau,1}}(\nu_{W_A}) \right) , \dots , \gamma_{M_{\tau,s_\tau}} \left( \text{gr}^{\tau, -k_{\tau,s_\tau}}(\nu_{W_A}) \right) \right) \end{aligned}$$

which is exactly the  $\tau$ -part of  $\kappa_2(x_{A,x_{\text{pdR}}})$  by (3.3.7). We conclude that

$$\kappa_2(x_{A,x_{\text{pdR}}}) = w^{-1}(\kappa_1(x_{A,x_{\text{pdR}}}))$$

in  $\mathfrak{t}/W_P(A)$ . Thus the image of  $x_A$  in  $\widehat{T}_{P,(0,0)}(A)$  is in  $\widehat{T}_{P,w,(0,0)}(A)$ . Hence the morphism  $\Theta_x$  factors through  $\widehat{T}_{P,w,(0,0)} \hookrightarrow \widehat{T}_{P,(0,0)}$ .  $\square$

**Lemma 3.3.14.** *Let  $k_1, \dots, k_s$  be pairwise different numbers in  $L$ . For each  $i = 1, \dots, s$ , assume that  $a_{i,1}, \dots, a_{i,m_i}$  are elements in  $A \in \mathcal{C}_L$  such that  $a_{i,j} - k_i \in \mathfrak{m}_A$  for any  $j = 1, \dots, m_i$ . For each  $i = 1, \dots, s$ , let  $P_i(Y) \in A[Y]$  be a monic polynomial of degree  $m_i$  such that  $P_i(Y) \equiv (Y - k_i)^{m_i} \pmod{\mathfrak{m}_A}$ . Assume that*

$$\prod_{i=1}^s \prod_{j=1}^{m_i} (Y - a_{i,j}) = \prod_{i=1}^s P_i(Y)$$

in  $A[Y]$ , then  $\prod_{j=1}^{m_i} (Y - a_{i,j}) = P_i(Y)$  for each  $i = 1, \dots, s$ .

*Proof.* Let  $F(Y) = \prod_{i=1}^s \prod_{j=1}^{m_i} (Y - a_{i,j})$ . Take any  $t \in \{1, \dots, s\}$ . We have  $\prod_{i=1}^s P_i(a_{t,1}) = F(a_{t,1}) = 0$ . If  $i \neq t$ , then  $P_i(a_{t,1}) \equiv (k_t - k_i)^{m_i} \pmod{\mathfrak{m}_A}$  is not in  $\mathfrak{m}_A$  since  $k_i \neq k_t$ . Hence  $P_t(a_{t,1}) \in A^\times$  if  $i \neq t$ . We get  $P_t(a_{t,1}) = 0$  in  $A$ . Hence  $P_t(Y) = (Y - a_{t,1})\tilde{P}_t(Y)$  where  $\tilde{P}_t(Y)$  is a monic polynomial and  $\tilde{P}_t(Y) \equiv (Y - k_t)^{m_t-1} \pmod{\mathfrak{m}_A}$ . Using the fact that if there is a monic polynomial  $G(Y) \in A[Y]$  such that  $G(Y) = (Y - a)G_1(Y) = (Y - a)G_2(Y)$  for some  $a \in A$  and  $G_1(Y), G_2(Y) \in A[Y]$ , then  $G_1(Y) = G_2(Y)$ , we get

$$\left( \prod_{i \neq t} \prod_{j=1}^{m_i} (Y - a_{i,j}) \right) \prod_{h=2}^{m_t} (Y - a_{t,h}) = \left( \prod_{i \neq t} P_i(Y) \right) \tilde{P}_t(Y).$$

Repeat the argument we find  $P_t(Y) = (Y - a_{t,1}) \cdots (Y - a_{t,m_t})$ .  $\square$

For  $w' \in W/W_P$ , we define  $X_{r,\mathcal{M}_\bullet}^{w'} = X_{r,\mathcal{M}_\bullet} \times_{X_{D,\mathcal{M}_\bullet}} X_{D,\mathcal{M}_\bullet}^{w'}$ . The functor  $|X_{r,\mathcal{M}_\bullet}|$  is pro-represented by a reduced equidimensional local ring  $R_{r,\mathcal{M}_\bullet}$  of dimension  $n^2 + [K : \mathbb{Q}_p] \frac{n(n+1)}{2}$  with minimal ideals  $\mathfrak{p}_{w'}, w' \in W/W_P$  such that  $R_{r,\mathcal{M}_\bullet}^{w'} := R_{r,\mathcal{M}_\bullet} / \mathfrak{p}_{w'}$  pro-represents  $|X_{r,\mathcal{M}_\bullet}^{w'}|$  (cf. [BHS19, Thm. 3.6.2 (i)(ii)], using Proposition 3.3.2, Corollary 3.3.3).

**Corollary 3.3.15.** *The closed immersion  $\widehat{X_{\text{tri}}(\bar{r})}_x \hookrightarrow X_{r,\mathcal{M}_\bullet}$  induces an isomorphism  $\widehat{X_{\text{tri}}(\bar{r})}_x \xrightarrow{\sim} X_{r,\mathcal{M}_\bullet}^w$ .*

*Proof.* The proof is the same with that of [BHS19, Cor. 3.7.8] using Proposition 3.3.5. By [BHS17b, §2.2] and discussions above, both  $\text{Spec}(\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{r}),x})$  and  $\text{Spec}(R_{r,\mathcal{M}_\bullet})$  are reduced equidimensional of dimension  $n^2 + [K : \mathbb{Q}_p] \frac{n(n+1)}{2}$ , thus the image of  $\text{Spec}(\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{r}),x}) \hookrightarrow \text{Spec}(R_{r,\mathcal{M}_\bullet})$

is a union of irreducible components of  $\text{Spec}(R_{r,\mathcal{M}_\bullet})$  of the form  $\text{Spec}(R_{r,\mathcal{M}_\bullet}^{w'})$  for some  $w' \in W/W_P$ . For any such  $w'$ , the morphism  $\text{Spec}(R_{r,\mathcal{M}_\bullet}^{w'}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{r}),x})$  induces a closed immersion  $X_{r,\mathcal{M}_\bullet}^{w'} \hookrightarrow \widehat{X_{\text{tri}}(\bar{r})}_x$  of formal schemes. The morphism  $\widehat{X_{\text{tri}}(\bar{r})}_x \hookrightarrow X_{r,\mathcal{M}_\bullet} \rightarrow \widehat{T}_{P,(0,0)}$  factors through  $\widehat{T}_{P,w,(0,0)} \hookrightarrow \widehat{T}_{P,(0,0)}$  by Proposition 3.3.5. Hence  $X_{r,\mathcal{M}_\bullet}^{w'} \rightarrow \widehat{T}_{P,(0,0)}$  factors through  $\widehat{T}_{P,w,(0,0)} \hookrightarrow \widehat{T}_{P,(0,0)}$ . We get that  $w' = w$  in  $W/W_P$  by Lemma 3.2.18 (cf. [BHS19, Thm. 3.6.2 (iii)]).  $\square$

Define  $X_{V,\mathcal{M}_\bullet}^{\square,w} := X_{V,\mathcal{M}_\bullet}^w \times_{X_{D,\mathcal{M}_\bullet}^w} X_{D,\mathcal{M}_\bullet}^{\square,w}$ ,  $X_{r,\mathcal{M}_\bullet}^{\square,w} := X_{r,\mathcal{M}_\bullet}^w \times_{X_{V,\mathcal{M}_\bullet}^w} X_{V,\mathcal{M}_\bullet}^{\square,w}$ . There are morphisms

$$\widehat{X_{\text{tri}}(\bar{r})}_x \xrightarrow{\sim} X_{r,\mathcal{M}_\bullet}^w \leftarrow X_{r,\mathcal{M}_\bullet}^{\square,w} \rightarrow X_{V,\mathcal{M}_\bullet}^{\square,w} \simeq X_{D,\mathcal{M}_\bullet}^{\square,w} \rightarrow X_{W^+, \mathcal{F}_\bullet}^{\square,w} \simeq \widehat{X}_{P,w,x_{\text{pdR}}} \quad (3.3.16)$$

where all morphisms are either isomorphisms or formally smooth and all groupoids are pro-representable. Let  $w_x \in W/W_P$  be the unique element such that  $x_{\text{pdR}} \in V_{P,w_x}$  (see (3.2.1)).

**Theorem 3.3.17.** *Let  $x = (r, \underline{\delta}) \in X_{\text{tri}}(\bar{r})(L)$  be a point such that  $\underline{\delta}$  is locally algebraic and both  $\delta_i/\delta_j$  and  $\epsilon\delta_i/\delta_j$  are not algebraic for any  $i \neq j$ . Then the trianguline variety  $X_{\text{tri}}(\bar{r})$  is irreducible at  $x$  and we have formally smooth morphisms  $\widehat{X_{\text{tri}}(\bar{r})}_x \leftarrow |X_{r,\mathcal{M}_\bullet}^{\square,w}| \rightarrow \widehat{X}_{P,w,x_{\text{pdR}}}$  of formal schemes. Moreover,  $w_x \leq w$  in  $W/W_P$ .*

*Proof.* The first assertions follow from Corollary 3.3.15 and (3.3.16). We remain to prove  $w_x \leq w$  in  $W/W_P$ . Since  $X_{r,\mathcal{M}_\bullet}^w \simeq \widehat{X_{\text{tri}}(\bar{r})}_x \neq \emptyset$ , we get  $\widehat{X}_{P,w,x_{\text{pdR}}} \neq \emptyset$ . Hence  $x_{\text{pdR}} \in X_{P,w}(L)$ . Since  $X_{P,w} \cap V_{P,w'} \neq \emptyset$  only if  $w \geq w'$  by Lemma 3.2.8, we get  $w \geq w_x$  in  $W/W_P$ .  $\square$

### 3.3.6 Partially de Rham cycles

We transport the results of cycles on the local models obtained in §3.2.5 to the trianguline variety. We continue to assume that  $x = (r, \underline{\delta}) \in X_{\text{tri}}(\bar{r})(L)$  is the point fixed in §3.3.5. Then we have a commutative diagram as in [BHS19, §4.3]:

$$\begin{array}{ccccc} X_{r,\mathcal{M}_\bullet}^w & \longleftarrow & X_{r,\mathcal{M}_\bullet}^{\square,w} & \longrightarrow & \widehat{X}_{P,w,x_{\text{pdR}}} \\ \downarrow & & \downarrow & & \downarrow \\ X_{r,\mathcal{M}_\bullet} & \longleftarrow & X_{r,\mathcal{M}_\bullet}^{\square} & \longrightarrow & \widehat{X}_{P,x_{\text{pdR}}} \\ \downarrow & \searrow \omega_{\underline{\delta}} & & & \downarrow \kappa_1 \\ X_r & & \widehat{\mathcal{T}}_{\underline{\delta}}^n & \xrightarrow{\text{wt} - \text{wt}(\underline{\delta})} & \widehat{\mathfrak{t}} \end{array}$$

where  $\omega_{\underline{\delta}}$  is the composite of the map  $\omega_{\underline{\delta}}$  defined in §3.3.4 for  $X_{\mathcal{M},\mathcal{M}_\bullet}$  with  $X_{r,\mathcal{M}} \rightarrow X_{\mathcal{M},\mathcal{M}_\bullet}$  and all horizontal arrows are formally smooth. It follows from the proof of [BHS19, Prop. 3.7.2] that the composite  $\widehat{X_{\text{tri}}(\bar{r})}_x \rightarrow X_{r,\mathcal{M}_\bullet} \xrightarrow{\omega_{\underline{\delta}}} \widehat{\mathcal{T}}_{\underline{\delta}}^n$  is the completion of the map  $\omega' : X_{\text{tri}}(\bar{r}) \hookrightarrow \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n \rightarrow \mathcal{T}_L^n$  at the points  $x$  and  $\underline{\delta}$  (cf. [BHS19, (3.30)]). Let  $X_{\text{tri}}(\bar{r})_{\text{wt}(\underline{\delta})}$  denote the fiber of the map  $\text{wt} \circ \omega' : X_{\text{tri}}(\bar{r}) \rightarrow \widehat{\mathfrak{t}}$  over  $\text{wt}(\underline{\delta})$ . Let  $\mathcal{R}_{r,\mathcal{M}_\bullet}, \mathcal{R}_{r,\mathcal{M}_\bullet}^w$  be the complete local rings that pro-represent the groupoids  $X_{r,\mathcal{M}_\bullet}, X_{r,\mathcal{M}_\bullet}^w$  and let  $\mathcal{R}_{r,\mathcal{M}_\bullet}^{\square}, \mathcal{R}_{r,\mathcal{M}_\bullet}^{\square,w}$  be the square versions. The above diagram of pro-representable groupoids corresponds to a diagram of spectra of complete local rings. Now take the fibers over  $0 \in \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{t},0})$ , we get the corresponding morphisms

$$\begin{array}{ccccc} \text{Spec}(\overline{\mathcal{R}}_{r,\mathcal{M}_\bullet}^w) & \longleftarrow & \text{Spec}(\overline{\mathcal{R}}_{r,\mathcal{M}_\bullet}^{\square,w}) & \longrightarrow & \text{Spec}(\widehat{\mathcal{O}}_{\widehat{X}_{P,w,x_{\text{pdR}}}}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(\overline{\mathcal{R}}_{r,\mathcal{M}_\bullet}) & \longleftarrow & \text{Spec}(\overline{\mathcal{R}}_{r,\mathcal{M}_\bullet}^{\square}) & \longrightarrow & \text{Spec}(\widehat{\mathcal{O}}_{\widehat{X}_{P,x_{\text{pdR}}}}) \\ \downarrow & & & & \\ \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{r}},r}) & & & & \end{array}$$



where  $\overline{X}_P$  and  $\overline{X}_{P,w}$  are defined in the end of §3.2.5. We know from §3.2.5 that the set of irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_P, x_{\text{pdR}}})$  is the disjoint union of the sets of irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P,w'}, x_{\text{pdR}}})$  for  $w' \in W/W_P$  such that  $x_{\text{pdR}} \in Z_{P,w'}$  and that  $\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_P, x_{\text{pdR}}})$  is equidimensional. By pullback and descent through formally smooth morphisms (cf. [Sta22, Tag 06HL]), we get that  $\text{Spec}(\overline{R}_{r, \mathcal{M}_\bullet})$  is equidimensional and there is a bijection between the irreducible components of  $\text{Spec}(\overline{R}_{r, \mathcal{M}_\bullet})$  and the irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_P, x_{\text{pdR}}})$ . For each  $w' \in W/W_P$ , we let  $\mathfrak{Z}_{w'}$  denote the union of irreducible components of  $\text{Spec}(\overline{R}_{r, \mathcal{M}_\bullet})$  that correspond to irreducible components of  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{P,w'}, x_{\text{pdR}}})$ . By base change, we get that  $\text{Spec}(\overline{R}_{r, \mathcal{M}_\bullet}^w)$  is a union of irreducible components of  $\text{Spec}(\overline{R}_{r, \mathcal{M}_\bullet})$  according to the formula (3.2.28).

**Remark 3.3.18.** In [BHS19, §4.3],  $\mathfrak{Z}_{w'}$  is defined to be a formal sum of irreducible components (cycles). Here we only consider  $\mathfrak{Z}_{w'}$  as a set-theoretic union of irreducible components or the underlying reduced subscheme which suffices for our applications. We also do not consider the Kazhdan-Lusztig cycles on  $\text{Spec}(\overline{R}_{r, \mathcal{M}_\bullet})$  defined as [BHS19, (4.7)]. See §3.2.6, especially Remark 3.2.33.

Take a standard parabolic subgroup  $Q = \prod_{\tau \in \Sigma} Q_\tau$  of  $G = \prod_{\tau \in \Sigma} \text{GL}_{n/L}$ . Suppose that for each  $\tau \in \Sigma$ , the standard Levi subgroup  $M_{Q_\tau}$  consists of diagonal block matrices of the form  $\text{GL}_{q_{\tau,1}/L} \times \cdots \times \text{GL}_{q_{\tau,t_\tau}/L}$  where  $q_{\tau,1} + \cdots + q_{\tau,t_\tau} = n$ . For  $i = 1, \dots, t_\tau$ , let  $\tilde{q}_i = q_{\tau,1} + \cdots + q_{\tau,i}$ . Let  $\tilde{q}_{\tau,0} = 0$ .

**Definition 3.3.19.** A pair  $(r_A, \mathcal{M}_{A,\bullet})$  where  $r_A$  is a continuous representation of  $\mathcal{G}_K$  of rank  $n$  over a finite-dimensional local  $L$ -algebra  $A$  and  $\mathcal{M}_{A,\bullet}$  is a triangulation of  $D_{\text{rig}}(r_A)[\frac{1}{t}]$  such that  $W_{\text{dR}}(D_{\text{rig}}(r_A)[\frac{1}{t}])$  is almost de Rham is said to be  $Q_\tau$ -de Rham if the nilpotent operator  $\nu_A$  on  $D_{\text{pdR},\tau}(W_{\text{dR}}(D_{\text{rig}}(r_A)[\frac{1}{t}]))$  vanishes when restricted to the graded pieces

$$D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{A,\tilde{q}_{\tau,i}})) / D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{A,\tilde{q}_{\tau,i-1}})), i = 1, \dots, t_\tau$$

of the sub-filtration of the full filtration  $D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{A,\bullet}))$ . A such pair  $(r_A, \mathcal{M}_{A,\bullet})$  is said to be  $Q$ -de Rham if it is  $Q_\tau$ -de Rham for all  $\tau \in \Sigma$ .

We have defined a closed subscheme  $Z_{Q,P}$  of  $\overline{X}_P$  in §3.2.5. Moreover, there is a closed immersion  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{Q,P}, x_{\text{pdR}}}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{\overline{X}_P, x_{\text{pdR}}})$  (we do not assume that  $\text{Spec}(\widehat{\mathcal{O}}_{Z_{Q,P}, x_{\text{pdR}}})$  is not empty). We define  $R_{r, \mathcal{M}_\bullet}^{\square, Q} := \overline{R}_{r, \mathcal{M}_\bullet}^{\square} \otimes_{\widehat{\mathcal{O}}_{\overline{X}_P, x_{\text{pdR}}}} \widehat{\mathcal{O}}_{Z_{Q,P}, x_{\text{pdR}}}$ . Let  $|X_{r, \mathcal{M}_\bullet}^{\square, Q}| := \text{Spf}(R_{r, \mathcal{M}_\bullet}^{\square, Q})$  and  $X_{r, \mathcal{M}_\bullet}^{\square, Q} := X_{r, \mathcal{M}_\bullet}^{\square} \times_{|X_{r, \mathcal{M}_\bullet}^{\square}|} |X_{r, \mathcal{M}_\bullet}^{\square, Q}|$ . Let  $X_{r, \mathcal{M}_\bullet}^Q$  be the image of the subgroupoid  $X_{r, \mathcal{M}_\bullet}^{\square, Q} \subset X_{r, \mathcal{M}_\bullet}^{\square}$  in  $X_{r, \mathcal{M}_\bullet}$ .

**Lemma 3.3.20.** 1. The full subgroupoid  $X_{r, \mathcal{M}_\bullet}^{\square, Q}$  of  $X_{r, \mathcal{M}_\bullet}^{\square}$  consists of objects

$$(A, r_A, \mathcal{M}_{A,\bullet}, j_A, \alpha_A)$$

where  $A \in \mathcal{C}_L$ ,  $r_A$  lifts  $r$  and  $(D_{\text{rig}}(r_A)[\frac{1}{t}], \mathcal{M}_{A,\bullet}, j_A, \alpha_A) \in X_{\mathcal{M}, \mathcal{M}_\bullet}^{\square}(A)$  such that  $j_A$  is the natural one and the pair  $(r_A, \mathcal{M}_{A,\bullet})$  is  $Q$ -de Rham.

2. The full subgroupoid  $X_{r, \mathcal{M}_\bullet}^Q$  of  $X_{r, \mathcal{M}_\bullet}$  consists of objects  $(A, r_A, \mathcal{M}_{A,\bullet}, j_A) \in X_{r, \mathcal{M}_\bullet}$  such that the pair  $(r_A, \mathcal{M}_{A,\bullet})$  is  $Q$ -de Rham. The inclusion  $X_{r, \mathcal{M}_\bullet}^Q \hookrightarrow X_{r, \mathcal{M}_\bullet}$  is relatively representable and is a closed immersion. Moreover,  $X_{r, \mathcal{M}_\bullet}^{\square, Q} = X_{r, \mathcal{M}_\bullet}^Q \times_{X_{r, \mathcal{M}_\bullet}} X_{r, \mathcal{M}_\bullet}^{\square}$ .

*Proof.* (1) Recall  $\alpha_A : (A \otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{\text{pdR}}(W_{\text{dR}}(D_{\text{rig}}(r_A)[\frac{1}{t}]))$ . Let  $(\nu, g_1 B) \in \widehat{\mathfrak{g}}(A)$  be the  $A$ -point corresponding to  $(A, W_{\text{dR}}(D_{\text{rig}}(r_A)[\frac{1}{t}]), W_{\text{dR}}(\mathcal{M}_{A,\bullet}), \iota_A, \alpha_A) \in X_{W, \mathcal{F}_\bullet}^{\square}$  via

$|X_{W,\mathcal{F}_\bullet}^\square| \simeq \widehat{\mathfrak{g}}$  in [BHS19, Cor. 3.1.9] induced by  $D_{\text{pdR}}$ . It follows from the definition that  $(r_A, \mathcal{M}_{A,\bullet})$  is  $Q$ -de Rham if and only if  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q(A)$ . Hence

$$(W_{\text{dR}}^+(D_{\text{rig}}(r_A)), W_{\text{dR}}(\mathcal{M}_{A,\bullet}), \iota_A, \alpha_A) \in |X_{W^+, \mathcal{F}_\bullet}^\square|(A)$$

satisfies that  $(r_A, \mathcal{M}_{A,\bullet})$  is  $Q$ -de Rham if and only if the corresponding object  $(\nu, g_1 B, g_2 P) \in \widehat{X}_{P, x_{\text{pdR}}}(A)$  under the isomorphism  $|X_{W^+, \mathcal{F}_\bullet}^\square| \simeq \widehat{X}_{P, x_{\text{pdR}}}$  in (1) of Proposition 3.3.2 satisfies  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{n}_Q(A)$ , in other words, if and only if  $(\nu, g_1 B, g_2 P) \in \widehat{Z}_{Q, P, x_{\text{pdR}}}(A)$ . Hence by definition,  $|X_{r, \mathcal{M}_\bullet}^{\square, Q}|$  is the subfunctor of  $|X_{r, \mathcal{M}_\bullet}^\square|$  sending  $A \in \mathcal{C}_L$  to the isomorphism classes of  $(r_A, \mathcal{M}_{A,\bullet}, j_A, \alpha_A)$  such that  $(r_A, \mathcal{M}_{A,\bullet})$  is  $Q$ -de Rham. The description for  $X_{r, \mathcal{M}_\bullet}^{\square, Q} = X_{r, \mathcal{M}_\bullet}^\square \times_{|X_{r, \mathcal{M}_\bullet}^\square|} |X_{r, \mathcal{M}_\bullet}^{\square, Q}|$  follows from the definition of the fiber product (see [Kis09a, (A.4)]).

(2) The condition for  $(A, r_A, \mathcal{M}_{A,\bullet}, j_A, \alpha_A) \in X_{r, \mathcal{M}_\bullet}^\square$  that  $(r_A, \mathcal{M}_{A,\bullet})$  is  $Q$ -de Rham in (1) is independent of the framing  $\alpha_A$ . Hence  $X_{r, \mathcal{M}_\bullet}^{\square, Q} = X_{r, \mathcal{M}_\bullet}^Q \times_{X_{r, \mathcal{M}_\bullet}^\square} X_{r, \mathcal{M}_\bullet}^\square$ . The other statements for  $X_{r, \mathcal{M}_\bullet}^Q$  are now obvious.  $\square$

Hence  $|X_{r, \mathcal{M}_\bullet}^Q|$  is pro-represented by a formal scheme  $\text{Spf}(R_{r, \mathcal{M}_\bullet}^Q)$  with a formally smooth morphism  $\text{Spf}(R_{r, \mathcal{M}_\bullet}^{\square, Q}) \rightarrow \text{Spf}(R_{r, \mathcal{M}_\bullet}^Q)$ . There is a closed immersion  $\text{Spec}(R_{r, \mathcal{M}_\bullet}^Q) \hookrightarrow \text{Spec}(\overline{R}_{r, \mathcal{M}_\bullet})$ . By Theorem 3.2.26 and above discussions,  $\mathfrak{Z}_{w'}$  is contained in  $\text{Spec}(R_{r, \mathcal{M}_\bullet}^Q)$  if and only if  $w'(\mathbf{h})$  is strictly  $Q$ -dominant where  $\mathbf{h} = (h_{\tau,1}, \dots, h_{\tau,n})_{\tau \in \Sigma}$  is the Hodge-Tate-Sen weights of  $r$  in §3.3.5.

### 3.4 Applications on companion points

In this section, we prove our main theorem (Theorem 3.4.18) on the existence of certain companion points on the eigenvariety as well as the appearance of related companion constituents in the space of  $p$ -adic automorphic forms (Proposition 3.4.17).

#### 3.4.1 Local companion points

We firstly generalize the result in [BHS19, §4.2] on the existence of all local companion points on the trianguline variety for generic crystalline representations. We continue to assume that  $K$  is a finite extension of  $\mathbb{Q}_p$  and  $L/\mathbb{Q}_p$  is a sufficiently large coefficient field as in §3.3.

Let  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  be a crystalline representation of  $\mathcal{G}_K$  which is a deformation of  $\bar{r} : \mathcal{G}_K \rightarrow \text{GL}_n(k_L)$  corresponding to an  $L$ -point on  $\mathfrak{X}_{\bar{r}}$ . Let  $D_{\text{cris}}(r)$  be the associated  $\varphi$ -module of rank  $n$  over  $L \otimes_{\mathbb{Q}_p} K_0$  equipped with a filtration  $\text{Fil}^\bullet D_{\text{dR}}(r)$  on  $D_{\text{dR}}(r) = (L \otimes_{\mathbb{Q}_p} K) \otimes_{L \otimes_{\mathbb{Q}_p} K_0} D_{\text{cris}}(r)$ . We assume that the Hodge-Tate weights of  $r$  are  $\mathbf{h} = (h_{\tau,1}, \dots, h_{\tau,n})_{\tau \in \Sigma}$  where each  $k_{\tau, s_\tau} > \dots > k_{\tau,1}$  appears in the sequence  $h_{\tau,n} \geq \dots \geq h_{\tau,1}$  with multiplicities  $m_{\tau, s_\tau}, \dots, m_{\tau,1}$  and  $m_{\tau, s_\tau} + \dots + m_{\tau,1} = n$  for  $\tau \in \Sigma$ . Let  $W_P$  be the stabilizer subgroup of  $\mathbf{h}$  under the action of  $W \simeq (S_n)^\Sigma$  as in §3.3.3 or §3.3.5. We fix an arbitrary embedding  $\tau_0 : K_0 \hookrightarrow L$ . After possibly enlarging  $L$ , we assume that the eigenvalues  $\varphi_1, \dots, \varphi_n$  of  $\varphi^{[K_0:\mathbb{Q}_p]}$  on  $L \otimes_{1 \otimes \tau_0, L \otimes_{\mathbb{Q}_p} K_0} D_{\text{cris}}(r)$  are all in  $L^\times$ . We say  $r$  is *generic* if the eigenvalues satisfy that  $\varphi_i \varphi_j^{-1} \notin \{1, p^{[K_0:\mathbb{Q}_p]}\}$  for all  $i \neq j$ . This generic assumption is independent of the choice of  $\tau_0$ .

We assume that  $r$  is generic and fix an ordering  $\underline{\varphi} := (\varphi_1, \dots, \varphi_n)$  of the eigenvalues, which is called a *refinement* of  $r$ , denoted by  $\mathcal{R}$ . For any  $w \in W/W_P$ , denote by  $z^{w(\mathbf{h})} \text{unr}(\underline{\varphi})$  the character

$$(z^{w(\mathbf{h})_1} \text{unr}(\varphi_1), \dots, z^{w(\mathbf{h})_n} \text{unr}(\varphi_n)) \in \mathcal{T}_L^n$$

of  $(K^\times)^n$  which lies in  $\mathcal{T}_0^n$  for any  $w \in W/W_P$  by our generic assumption on  $\underline{\varphi}$  (recall that  $\mathcal{T}_0^n$  is defined in the beginning of §3.3.2). The ordering  $\underline{\varphi}$  defines a filtration  $\text{Fil}_\bullet D_{\text{cris}}(r)$  on  $D_{\text{cris}}(r)$ ,

which, under Berger's dictionary ([Ber08b]), corresponds to a triangulation  $D_{\text{rig}}(r)_{\bullet} : D_{\text{rig}}(r)_1 \subset \cdots \subset D_{\text{rig}}(r)_n$  of  $D_{\text{rig}}(r) \in \Phi\Gamma_{L,K}^+$  (cf. [BHS17a, §2.2]) such that

$$D_{\text{rig}}(r)_i / D_{\text{rig}}(r)_{i-1} \simeq \mathcal{R}_{L,K}(z^{w_{\mathcal{R}}(\mathbf{h})_i} \text{unr}(\varphi_i)),$$

for all  $i = 1, \dots, n$  and certain  $w_{\mathcal{R}} \in W/W_P$  determined by  $\mathcal{R}$ . Moreover, the relative position of the filtration  $\text{Fil}_{\bullet} D_{\text{cris}}(r) \otimes_{K_0} K$  and the Hodge filtration  $\text{Fil}^{\bullet} D_{\text{dR}}(r)$  is parameterized by  $w_{\mathcal{R}} \in W/W_P$  (that is, if we choose a basis of  $D_{\text{dR}}(r)$ , the  $L$ -point in  $G/B \times G/P$  associated with the two filtrations lies in the Schubert cell  $U_{P,w_{\mathcal{R}}}$  defined in §3.2.1).

For  $w \in W/W_P$ , let  $x_w \in \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  be the point corresponding to the pair  $(r, z^{w(\mathbf{h})} \text{unr}(\varphi))$ . We let  $x = x_{w_0}$  be the dominant point and let  $w_x = w_{\mathcal{R}}$  (to make the notation agree with [BHS19, §4.2], the element  $w_x$  will coincide with the one in Theorem 3.3.17). Then  $x_{w_x} \in U_{\text{tri}}(\bar{r}) \subset X_{\text{tri}}(\bar{r})$  by the definition of  $U_{\text{tri}}(\bar{r})$ . If  $x_w \in X_{\text{tri}}(\bar{r})$ , then  $x_w$  satisfies the assumption in Theorem 3.3.17. Hence by Theorem 3.3.17,  $w \geq w_x$  in  $W/W_P$ . The converse is also true and the following theorem is a plain generalization of [BHS19, Thm. 4.2.3] which asserts that all expected local companion points of  $x_{w_0}$  exist on the trianguline variety (cf. [BHS19, Def. 4.2.1]). We repeat the proof here to introduce the notation that will be needed in the proof of Theorem 3.4.18.

**Theorem 3.4.1.** *The point  $x_w \in \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  is in  $X_{\text{tri}}(\bar{r})$  if and only if  $w \geq w_x$  in  $W/W_P$ .*

*Moreover, the set of points  $x' = (r, \underline{\delta}') \in \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  such that  $x'$  is in  $X_{\text{tri}}(\bar{r})$  is equal to*

$$\bigcup_{\mathcal{R}=\varphi} \{(r, z^{w(\mathbf{h})} \text{unr}(\varphi)), w \geq w_{\mathcal{R}}\}$$

where  $\mathcal{R}$  runs over all possible refinements of  $r$ .

*Proof.* We need a variant of Kisin's crystalline deformation space for irregular Hodge-Tate weights that is embedded into the trianguline variety as in [BHS17a, §2.2].

Let  $R_{\bar{r}}^{\mathbf{h}-\text{cr}}$  be the *framed* crystalline deformation ring of  $p$ -adic Hodge type determined by the Hodge-Tate weights  $\mathbf{h}$  in the sense of [Kis08] over  $\mathcal{O}_L$  (reduced and  $\mathbb{Z}_p$ -flat) and let  $\mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}$  be the rigid analytic generic fiber of  $\text{Spf}(R_{\bar{r}}^{\mathbf{h}-\text{cr}})$ . By [Kis08, Thm. 3.3.8],  $\mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}$  is smooth, equidimensional of dimension  $n^2 + \sum_{\tau \in \Sigma} \sum_{1 \leq i < j \leq s_{\tau}} m_{\tau,i} m_{\tau,j}$  over  $L$ . The beginning part of [BHS17a, §2.2] produces *mutatis mutandis* a rigid analytic space  $\tilde{\mathfrak{X}}_{\bar{r}}^{\mathbf{h}-\text{cr}} = \mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}} \times_{T_L^{\text{rig}}/\mathcal{S}_n} T_L^{\text{rig}}$  where  $T_L^{\text{rig}}$  is the rigid split torus over  $L$  of rank  $n$ ,  $\mathcal{S}_n$  is the symmetric group so that the quotient  $T_L^{\text{rig}}/\mathcal{S}_n$  parameterizes characteristic polynomials of the Frobenius on the Weil-Deligne representations associated with the crystalline deformations and  $\tilde{\mathfrak{X}}_{\bar{r}}^{\mathbf{h}-\text{cr}}$  parameterizes pairs  $(r, (\varphi_1, \dots, \varphi_n))$  where  $r \in \mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}$  and  $(\varphi_1, \dots, \varphi_n)$  is an ordering of the eigenvalues of the Frobenius on the Weil-Deligne representation associated with  $r$ . The same proof of [BHS17a, Lem. 2.2] shows that  $\tilde{\mathfrak{X}}_{\bar{r}}^{\mathbf{h}-\text{cr}}$  is reduced.

Let  $\mathfrak{Y}_{\bar{r}}^{\mathbf{h}-\text{cr}} \rightarrow \mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}$  be the  $\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n/K_0}) \times_{\mathbb{Q}_p} L$ -torsor of the trivialization of the underlying coherent sheaf of the universal filtered  $\varphi$ -modules over  $K_0 \times_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}}$  on  $\mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}$ . Then sending a crystalline representation with a trivialization of  $D_{\text{cris}}$  to its crystalline Frobenius and Hodge filtration defines a morphism  $f : \mathfrak{Y}_{\bar{r}}^{\mathbf{h}-\text{cr}} \rightarrow ((\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n/K_0}) \times_{\mathbb{Q}_p} L) \times_L G/P)^{\text{rig}}$  which is smooth. In fact, by [HH20, Prop. 8.17],  $\mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}$  is isomorphic to an open subspace  $\tilde{\mathcal{D}}_{\varphi,\mu}^{\text{ad},\text{adm},+}(\bar{r})$  of  $\tilde{\mathcal{D}}_{\varphi,\mu}^{\text{ad},\text{adm}}$  where  $\mathcal{D}_{\varphi,\mu}^{\text{ad}}$  is the quotient stack of the adic space associated with  $(\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n/K_0}) \times_{\mathbb{Q}_p} L) \times_L G/P$  by the action of  $\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n/K_0}) \times_{\mathbb{Q}_p} L$ ,  $\mathcal{D}_{\varphi,\mu}^{\text{ad},\text{adm}}$  is an open subspace of  $\mathcal{D}_{\varphi,\mu}^{\text{ad}}$  where there is a universal representation of  $\mathcal{G}_K$  on a vector bundle  $\mathcal{V}$  on  $\mathcal{D}_{\varphi,\mu}^{\text{ad},\text{adm}}$  and  $\tilde{\mathcal{D}}_{\varphi,\mu}^{\text{ad},\text{adm}}$  is the stack over  $\mathcal{D}_{\varphi,\mu}^{\text{ad},\text{adm}}$  trivializing  $\mathcal{V}$ . Then the morphism  $f$  induces a smooth morphism:

$$\tilde{f} : \tilde{\mathfrak{X}}_{\bar{r}}^{\mathbf{h}-\text{cr}} \times_{\mathfrak{X}_{\bar{r}}^{\mathbf{h}-\text{cr}}} \mathfrak{Y}_{\bar{r}}^{\mathbf{h}-\text{cr}} \rightarrow \left( (\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n/K_0}) \times_{\mathbb{Q}_p} L)^{\text{rig}} \times_{T_L^{\text{rig}}/\mathcal{S}_n} T_L^{\text{rig}} \right) \times_L (G/P)^{\text{rig}},$$

where the map  $(\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n/K_0}) \times_{\mathbb{Q}_p} L)^{\text{rig}} \rightarrow T_L^{\text{rig}}/\mathcal{S}_n$  is defined by [HH20, (9-1)]. The condition  $\varphi_i \varphi_j^{-1} \notin \{1, p^{[K_0, \mathbb{Q}_p]}\}$  for  $i \neq j$  cuts out a Zariski open subspace

$$\mathcal{D}^{\text{gen}} := \left( (\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n, K_0}) \times_{\mathbb{Q}_p} L)^{\text{rig}} \times_{T_L^{\text{rig}}/\mathcal{S}_n} T_L^{\text{rig}} \right)^{\text{gen}} \times_L (G/P)^{\text{rig}}$$

in the target of  $\tilde{f}$  and thus the inverse image of  $\mathcal{D}^{\text{gen}}$  under  $\tilde{f}$ , denoted by  $\tilde{Z}_{\bar{\tau}}^{\text{h-cr}}$ , is Zariski open dense in  $\tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}} \times_{\tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}}} \mathfrak{Y}_{\bar{\tau}}^{\text{h-cr}}$  (smooth morphisms are open). The subspace  $\tilde{Z}_{\bar{\tau}}^{\text{h-cr}}$  is invariant under the action of  $\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n/K_0}) \times_{\mathbb{Q}_p} L$  (change of bases) and thus descends along  $\tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}} \times_{\tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}}} \mathfrak{Y}_{\bar{\tau}}^{\text{h-cr}} \rightarrow \tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}}$  to a Zariski open dense subspace  $\widetilde{W}_{\bar{\tau}}^{\text{h-cr}}$  of  $\tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}}$ .

For any  $\varphi$ -module  $D$  of rank  $n$  over  $A \otimes_{\mathbb{Q}_p} K_0$  where  $A$  is an  $L$ -algebra, let

$$D_{\tau} := D \otimes_{L \otimes_{\mathbb{Q}_p} K_0, \text{id} \otimes \tau} L$$

for  $\tau \in \text{Hom}(K_0, L)$ . Let  $\sigma$  be the Frobenius automorphism of  $K_0$ . Then  $\varphi : D_{\tau_0 \circ \sigma^i} \rightarrow D_{\tau_0 \circ \sigma^{i+1}}$  where  $\tau_0 \circ \sigma^f = \tau_0 \circ \sigma^0 = \tau_0$ . Given a basis  $e_1, \dots, e_n$  of the  $A \otimes_{\mathbb{Q}_p} K_0$ -module, equivalent bases  $(e_i \otimes_{L \otimes_{\mathbb{Q}_p} K_0, \text{id} \otimes \tau} L)_{i=1, \dots, n}$  of  $D_{\tau}$  for each  $\tau \in \text{Hom}(K_0, L)$ , the matrix of  $\varphi$  is given by  $M = (M_{\tau})_{\tau \in \text{Hom}(K_0, L)} \in (\text{Res}_{K_0/\mathbb{Q}_p}(\text{GL}_{n, K_0}) \times_{\mathbb{Q}_p} L)(A)$  where  $M_{\tau}$  is the matrix of the morphism  $\varphi : D_{\tau} \rightarrow D_{\tau \circ \sigma}$  under the given bases. On  $\mathcal{D}^{\text{gen}}$ , the condition that  $\varphi_i \neq \varphi_j$  for  $i \neq j$  allows to define a Zariski closed subspace  $\mathcal{T} \subset \mathcal{D}^{\text{gen}}$  cut out by the condition that  $M_{\tau} = \text{diag}(a_{\tau, 1}, \dots, a_{\tau, n})$  are diagonal matrices where  $\prod_{\tau \in \text{Hom}(K_0, L)} a_{\tau, i} = \varphi_i$  for all  $i = 1, \dots, n$  (this corresponds to the choices of bases  $e_1, \dots, e_n$  of  $D$  such that  $\varphi^{[K_0, \mathbb{Q}_p]} e_i = \varphi_i e_i$ ). It is easy to see that  $\mathcal{T}$  is smooth over  $(G/P)^{\text{rig}}$ . Let  $\tilde{\mathfrak{Z}}_{\bar{\tau}}^{\text{h-cr}} := \tilde{f}^{-1}(\mathcal{T})$  be the inverse image of  $\mathcal{T}$  in  $\tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}} \times_{\tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}}} \mathfrak{Y}_{\bar{\tau}}^{\text{h-cr}}$ .

Then  $\tilde{\mathfrak{Z}}_{\bar{\tau}}^{\text{h-cr}} \rightarrow \widetilde{W}_{\bar{\tau}}^{\text{h-cr}}$  is a  $(\text{Res}_{K_0/\mathbb{Q}_p}(\mathbb{G}_m) \otimes_{\mathbb{Q}_p} L)^n$ -torsor corresponding to the trivialization of the  $\varphi$ -modules with the bases given by eigenvectors of  $\varphi^{[K_0, \mathbb{Q}_p]}$  as above (such bases exist locally because the morphism  $\varphi^{[K_0, \mathbb{Q}_p]} - \varphi_i$  between projective modules has cokernel and kernel of constant ranks over a reduced base). The map  $\tilde{\mathfrak{Z}}_{\bar{\tau}}^{\text{h-cr}} \rightarrow \mathcal{T} \rightarrow (G/P)^{\text{rig}}$  is also smooth. For any  $w \in W/W_P$ , descending along the map  $\tilde{\mathfrak{Z}}_{\bar{\tau}}^{\text{h-cr}} \rightarrow \widetilde{W}_{\bar{\tau}}^{\text{h-cr}}$  for the inverse image in  $\tilde{\mathfrak{Z}}_{\bar{\tau}}^{\text{h-cr}}$  of the Bruhat cell  $(BwP/P)^{\text{rig}}$ , where the inverse image is invariant under the action of  $(\text{Res}_{K_0/\mathbb{Q}_p}(\mathbb{G}_m) \otimes_{\mathbb{Q}_p} L)^n$ , is a Zariski locally closed subset in  $\widetilde{W}_{\bar{\tau}}^{\text{h-cr}}$ , denoted by  $\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}}$ . Let  $\overline{\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}}}$  be the Zariski closure of  $\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}}$  in  $\widetilde{W}_{\bar{\tau}}^{\text{h-cr}}$ . Then we have the usual closure relations  $\overline{\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}}} = \coprod_{w' \leq w} \widetilde{W}_{\bar{\tau}, w'}^{\text{h-cr}}$  by a similar argument as in the proof of [BHS19, Thm. 4.2.3] (using that smooth morphisms are open) and descent.

Hence a point  $x = (r, \varphi)$  with the refinement denoted by  $\mathcal{R}$  in  $\widetilde{W}_{\bar{\tau}}^{\text{h-cr}}$  lies in  $\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}}$  (resp.  $\overline{\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}}}$ ) if and only if  $w_{\mathcal{R}} = w$  (resp.  $w_{\mathcal{R}} \leq w$ ). For any  $w \in W/W_P$ , there is a morphism  $\iota_{\mathbf{h}, w} : \tilde{\mathfrak{X}}_{\bar{\tau}}^{\text{h-cr}} \times_L T_L^{\text{rig}} \rightarrow \tilde{\mathfrak{X}}_{\bar{\tau}} \times_L T_L^n$  that sends  $(r, \varphi)$  to  $(r, z^{w(\mathbf{h})} \text{unr}(\varphi))$ . We have  $\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}} \subset \iota_{\mathbf{h}, w}^{-1}(U_{\text{tri}}(\bar{r}))$  by [Ber08b]. Hence  $\overline{\widetilde{W}_{\bar{\tau}, w}^{\text{h-cr}}} \subset \iota_{\mathbf{h}, w}^{-1}(X_{\text{tri}}(\bar{r}))$  for any  $w \in W/W_P$ . Thus  $\iota_{\mathbf{h}, w}(\overline{\widetilde{W}_{\bar{\tau}, w_{\mathcal{R}}}^{\text{h-cr}}}) \subset X_{\text{tri}}(\bar{r})$  for any  $w \geq w_{\mathcal{R}}$  which finishes the proof of the first statement of the theorem.

To prove the last statement of the theorem, by the first statement, we only need to prove that if  $x' = (r, \varphi') \in X_{\text{tri}}(\bar{r})$ , then  $x' = x_w$  for some refinement  $\mathcal{R}$  of  $r$  and  $w \in W/W_P$ . This follows from the bijection between the triangulation of  $D_{\text{rig}}(r)$  and the refinements of  $r$ , [KPX14, Thm. 6.3.13] and [BHS17b, Prop. 2.9].  $\square$

### 3.4.2 $p$ -adic automorphic forms and eigenvarieties

We now turn to the global settings in [BHS19, §5] (see also [BHS17b, §2.4 & §3] or [BHS17a, §3]). We recall the basic notation and constructions. The reader should refer to *loc. cit.* for details. We assume from now on  $p > 2$ .

Let  $F^+$  be a totally real field and  $F/F^+$  be a CM quadratic extension such that any prime in  $S_p$ , the set of places of  $F^+$  above  $p$ , splits in  $F$ .

Let  $G$  be a unitary group in  $n \geq 2$  variables over  $F^+$  which is definite at all real places, split over  $F$  and quasi-split at all finite places.

We fix a tame level  $U^p = \prod_{v \notin S_p} U_v$  where for any finite place  $v \notin S_p$ ,  $U_v$  is an open compact subgroup of  $G(F_v^+)$ . We assume that  $U^p$  is small enough in the sense of [BHS17a, (3.9)]

Let  $S \supset S_p$  be a finite set of finite places of  $F^+$  that split in  $F$  and we require that every place of  $F^+$  that splits in  $F$  such that  $U_v$  is not maximal is contained in  $S$ . For each  $v \in S$ , we fix a place  $\tilde{v}$  of  $F$  above  $F^+$ .

We fix an isomorphism  $G \times_{F^+} F \simeq \mathrm{GL}_{n,F}$  which induces  $i_{\tilde{v}} : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\tilde{v}})$  for any  $v \in S_p$ .

Recall that  $L$  denotes the coefficient field, a finite extension of  $\mathbb{Q}_p$ . We assume that  $L$  is large enough such that  $|\mathrm{Hom}(F_{\tilde{v}}, L)| = [F_{\tilde{v}} : \mathbb{Q}_p]$  for all  $v \in S_p$ .

We define the set  $\Sigma_v := \mathrm{Hom}(F_{\tilde{v}}, L)$  for any  $v \in S_p$  and let  $\Sigma_p = \cup_{v \in S_p} \Sigma_v$ .

Denote by  $B_v$  (resp.  $\overline{B}_v$ , resp.  $T_v$ ) the subgroup of  $G_v := G(F_v^+)$  which is the preimage of the group of upper-triangular matrices (resp. lower-triangular matrices, resp. diagonal matrices) of  $\mathrm{GL}_n(F_{\tilde{v}})$  under  $i_{\tilde{v}}$  and let  $B_p = \prod_{v \in S_p} B_v$ ,  $\overline{B}_p = \prod_{v \in S_p} \overline{B}_v$  and  $T_p = \prod_{v \in S_p} T_v$ . Set  $G_p = \prod_{v \in S_p} G_v$ .

For  $v \in S_p$ , let  $\mathfrak{g}_v$  (resp.  $\mathfrak{b}_v$ , resp.  $\overline{\mathfrak{b}}_v$ , resp.  $\mathfrak{t}_v$ ) be the base change to  $L$  of the  $\mathbb{Q}_p$ -Lie algebra of  $G_v$  (resp.  $B_v$ , resp.  $\overline{B}_v$ , resp.  $T_v$ ). We define  $\mathfrak{g} = \prod_{v \in S_p} \mathfrak{g}_v$ ,  $\mathfrak{t} = \prod_{v \in S_p} \mathfrak{t}_v$ , etc. and for  $v \in S_p$ ,  $\tau \in \Sigma_v$ , set  $\mathfrak{g}_\tau = \mathfrak{g}_v \otimes_{F_{\tilde{v}} \otimes_{\mathbb{Q}_p} L, \tau \otimes \mathrm{id}} L$ ,  $\mathfrak{t}_\tau = \mathfrak{t}_v \otimes_{F_{\tilde{v}} \otimes_{\mathbb{Q}_p} L, \tau \otimes \mathrm{id}} L$ , etc..

Let  $\widehat{S}(U^p, L)$  (resp.  $\widehat{S}(U^p, \mathcal{O}_L)$ ) be the space of continuous functions  $G(F^+) \backslash G(\mathbf{A}_{F^+}^\infty) / U^p \rightarrow L$  (resp.  $G(F^+) \backslash G(\mathbf{A}_{F^+}^\infty) / U^p \rightarrow \mathcal{O}_L$ ) on which  $G_p$  acts via right translations. Then there is a Hecke algebra  $\mathbb{T}^S$  (a commutative  $\mathcal{O}_L$ -algebra, see [BHS17b, §2.4] for details) that acts on  $\widehat{S}(U^p, \mathcal{O}_L)$ . We fix a maximal ideal  $\mathfrak{m}^S$  of  $\mathbb{T}^S$  with residue field  $k_L$  (otherwise enlarging  $L$ ) such that

$$\widehat{S}(U^p, L)_{\mathfrak{m}^S} \neq 0$$

and that the associated Galois representation  $\overline{\rho} : \mathcal{G}_F \rightarrow \mathrm{GL}_n(k_L)$  is absolutely irreducible (i.e.  $\mathfrak{m}^S$  is non-Eisenstein) where  $\mathcal{G}_F := \mathrm{Gal}(\overline{F}/F)$ .

We assume furthermore the “standard Taylor-Wiles hypothesis”, that is we require that  $F$  is unramified over  $F^+$ ,  $F$  contains no non-trivial  $p$ -th root of unity,  $U_v$  is hyperspecial if the place  $v$  of  $F^+$  is inert in  $F$ , and  $\overline{\rho}(\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1})))$  is adequate (cf. [BHS19, Rem. 1.1]).

The Galois deformation ring  $R_{\overline{\rho}, S}$ , which parameterizes polarized deformations of  $\overline{\rho}$  unramified outside  $S$ , acts on  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$ . The subspace of locally  $\mathbb{Q}_p$ -analytic vectors of  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}$  for the action of  $G_p$ , denoted by  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}^{\mathrm{an}}$ , is a very strongly admissible locally  $\mathbb{Q}_p$ -analytic representation of  $G_p$  ([Eme07, Def. 0.12]). The eigenvariety  $Y(U^p, \overline{\rho})$  is defined to be the support of the coherent sheaf  $\left( J_{B_p}(\widehat{S}(U^p, L)_{\mathfrak{m}^S}^{\mathrm{an}}) \right)'$ , applying Emerton’s Jacquet functor (with respect to the parabolic subgroup  $B_p$  of  $G_p$ ) on  $\widehat{S}(U^p, L)_{\mathfrak{m}^S}^{\mathrm{an}}$  and then taking the continuous dual, on  $\mathrm{Spf}(R_{\overline{\rho}, S})^{\mathrm{rig}} \times \widehat{T}_{p, L}$ , where  $\widehat{T}_{p, L} = \prod_{v \in S_p} \widehat{T}_{v, L}$  denotes the base change to  $L$  of the  $\mathbb{Q}_p$ -rigid space parameterizing continuous characters of  $T_p = \prod_{v \in S_p} T_v$ .

Let  $R_\infty = \widehat{\bigotimes}_{v \in S} R'_{\overline{\rho}_v}[[x_1, \dots, x_g]]$  where for a place  $\tilde{v}$  of  $F$ ,  $R'_{\overline{\rho}_v}$  is the maximal reduced  $\mathbb{Z}_p$ -flat quotient of the local framed deformation ring of  $\overline{\rho}_{\tilde{v}} := \overline{\rho}|_{\mathcal{G}_{F_{\tilde{v}}}}$  over  $\mathcal{O}_L$  and  $g$  is certain determined integer. Then there is an  $\mathcal{O}_L$ -module  $M_\infty$  constructed in [CEG<sup>+</sup>16] (see [BHS17b, Thm. 3.5] and [BHS17a, §6]) equipped with actions of  $R_\infty$  and  $G_p$  so that  $\Pi_\infty := M'_\infty[\frac{1}{p}]$  is a  $R_\infty$ -admissible Banach representation of  $G_p$  ([BHS17b, Def. 3.1]). The *patched eigenvariety*  $X_p(\overline{\rho})$  is defined to be the support of the coherent sheaf  $\mathcal{M}_\infty := \left( J_{B_p}(\Pi_\infty^{R_\infty - \mathrm{an}}) \right)'$ , applying Emerton’s Jacquet functor on the subspace of locally  $R_\infty$ -analytic vectors of  $\Pi_\infty$  ([BHS17b, Def. 3.2]) and then taking the continuous dual, on  $\mathrm{Spf}(R_\infty)^{\mathrm{rig}} \times \widehat{T}_{p, L}$ . Then a point  $x = (r_x, \underline{\delta}) \in$

$\mathrm{Spf}(R_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L}$  lies in  $X_p(\bar{\rho})$  if and only if  $\mathrm{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\Pi_\infty^{R_\infty - \mathrm{an}}[\mathfrak{m}_{r_x}] \otimes_{k(r_x)} k(x))) \neq 0$  where  $\mathfrak{m}_{r_x}$  denotes the maximal ideal of  $R_\infty[\frac{1}{p}]$  corresponding to the point  $r_x \in \mathrm{Spf}(R_\infty)^{\mathrm{rig}}$  (cf. [BHS17b, Prop. 3.7]). Recall that  $X_p(\bar{\rho})$  is reduced ([BHS17b, Cor. 3.20]). The eigenvariety  $Y(U^p, \bar{\rho})$  is identified with a Zariski closed subspace of the patched one  $X_p(\bar{\rho})$ .

We denote by

$$\mathfrak{X}_{\bar{\rho}_p} := \mathrm{Spf}\left(\widehat{\bigotimes_{v \in S_p} R'_{\bar{\rho}_v}}\right)^{\mathrm{rig}}, \mathfrak{X}_{\bar{\rho}^p} := \mathrm{Spf}\left(\widehat{\bigotimes_{v \in S \setminus S_p} R'_{\bar{\rho}_v}}\right)^{\mathrm{rig}}, \mathbb{U}^g = \mathrm{Spf}(\mathcal{O}_L[[x_1, \dots, x_g]])^{\mathrm{rig}}$$

$$\text{and } \mathfrak{X}_\infty := \mathrm{Spf}(R_\infty)^{\mathrm{rig}} \simeq \mathfrak{X}_{\bar{\rho}_p} \times \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g.$$

Let  $X_{\mathrm{tri}}(\bar{\rho}_p) := \prod_{v \in S_p} X_{\mathrm{tri}}(\bar{\rho}_v)$  which by definition is a Zariski closed subspace of  $\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p,L}$ . Denote by  $\delta_{B_v} := |\cdot|_{F_v}^{n-1} \otimes \dots \otimes |\cdot|_{F_v}^{n-2i+1} \otimes \dots \otimes |\cdot|_{F_v}^{1-n}$  the smooth modulus character of  $B_v$  and  $\delta_{B_p} := \otimes_{v \in S_p} \delta_{B_v}$ . Let  $\iota_v$  be the automorphism

$$(\delta_{v,1}, \dots, \delta_{v,n}) \mapsto \delta_{B_v} \cdot (\delta_{v,1}, \dots, \delta_{v,i} \epsilon^{i-1}, \dots, \delta_{v,n} \epsilon^{n-1})$$

of  $\widehat{T}_{v,L}$  and let  $\iota = \prod_{v \in S_p} \iota_v : \widehat{T}_{p,L} \xrightarrow{\sim} \widehat{T}_{p,L}$ . We also use  $\iota$  to denote the automorphism of  $\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p,L} : (x, \underline{\delta}) \mapsto (x, \iota(\underline{\delta}))$ . Then the reduced closed subvariety  $X_p(\bar{\rho})$  of  $\mathfrak{X}_{\bar{\rho}_p} \times \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g \times \widehat{T}_{p,L}$  lies in the Zariski closed subspace  $\iota(X_{\mathrm{tri}}(\bar{\rho}_p)) \times \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g$  and is identified with a union of irreducible components of  $\iota(X_{\mathrm{tri}}(\bar{\rho}_p)) \times \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g$  ([BHS17b, Thm. 3.21]) any of which is of the form  $\iota(X) \times \mathfrak{X}^p \times \mathbb{U}^g$  where  $X$  (resp.  $\mathfrak{X}^p$ ) is an irreducible component of  $X_{\mathrm{tri}}(\bar{\rho}_p)$  (resp.  $\mathfrak{X}_{\bar{\rho}^p}$ ). An irreducible component  $X$  of  $X_{\mathrm{tri}}(\bar{\rho}_p)$  is said to be  $\mathfrak{X}^p$ -automorphic for an irreducible component  $\mathfrak{X}^p$  of  $\mathfrak{X}_{\bar{\rho}^p}$  if  $\iota(X) \times \mathfrak{X}^p \times \mathbb{U}^g$  is contained in  $X_p(\bar{\rho})$ .

**Definition 3.4.2.** 1. A character  $\underline{\delta} = (\delta_v)_{v \in S_p} \in \widehat{T}_{p,L}$  is called *generic* if for each  $v \in S_p$ ,  $\iota_v^{-1}(\delta_v) \in \mathcal{T}_{v,0}^n$ , where  $\mathcal{T}_{v,0}^n$  denotes the subset  $\mathcal{T}_0^n$  in §3.3.2 for characters of  $(F_v^\times)^n$ . Explicitly, we say  $\underline{\delta}$  is generic if  $\delta_{v,i} \delta_{v,j}^{-1} \cdot |\cdot|_{F_v}^{2i-2j} \epsilon^{j-i} \neq z^{\mathbf{k}}, \epsilon z^{\mathbf{k}}$  for any  $v \in S_p, i \neq j, \mathbf{k} \in \mathbb{Z}^{\Sigma_v}$ .

2. A point  $x = (r_x, \underline{\delta}) \in \mathfrak{X}_\infty \times \widehat{T}_{p,L}$  (or  $y = (\rho, \underline{\delta}) \in \mathrm{Spf}(R_{\bar{\rho},S})^{\mathrm{rig}} \times \widehat{T}_{p,L}$ ) is said to be generic if  $\underline{\delta}$  is generic.

### 3.4.3 Orlik-Strauch theory

We recall the theory of Orlik-Strauch on Jordan-Hölder factors of locally analytic principal series which will be the companion constituents in the locally analytic socles.

Let  $\mathcal{O}$  (resp.  $\overline{\mathcal{O}}$ ) be the BGG category of  $U(\mathfrak{g})$ -modules attached to the Borel subalgebra  $\mathfrak{b}$  (resp.  $\overline{\mathfrak{b}}$ ) ([Hum08, §1.1]). If  $M$  is in  $\overline{\mathcal{O}}_{\mathrm{alg}}$  ([Bre16, §2]) and  $V$  is a smooth representation of  $T_p$  over  $L$ , then Orlik-Strauch constructs a locally  $\mathbb{Q}_p$ -analytic representation  $\mathcal{F}_{\overline{B}_p}^{G_p}(M, V)$  of  $G_p$  ([OS15], see [Bre16, §2], [Bre15b, §2] and [BHS19, Rem. 5.1.2]). The functor  $\mathcal{F}_{\overline{B}_p}^{G_p}(-, -)$  is exact and contravariant (resp. covariant) in the first (resp. second) arguments (cf. [Bre15b, Thm. 2.2]). If  $\lambda = (\lambda_v)_{v \in S_p} = (\lambda_{\tau,1}, \dots, \lambda_{\tau,n})_{\tau \in \Sigma_v, v \in S_p} \in \prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$ , we let  $z^\lambda := \prod_{v \in S_p} z^{\lambda_v}$  be the algebraic character of  $T_p = \prod_{v \in S_p} T_v$  which satisfies that  $\mathrm{wt}_\tau((z^{\lambda_v})_i) = \lambda_{\tau,i}$  for every  $\tau \in \Sigma_v$ . Thus we may view  $\lambda$  as a weight of  $\mathfrak{t}$ . Assume that  $\underline{\delta} = (\delta_v)_{v \in S_p} \in \widehat{T}_{p,L}(L)$  is locally algebraic of weight  $\lambda$  (i.e.  $\mathrm{wt}(\underline{\delta}) := (\mathrm{wt}_\tau(\delta_{v,i}))_{i=1, \dots, n, \tau \in \Sigma_v, v \in S_p} \in \prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$  and  $\lambda = \mathrm{wt}(\underline{\delta})$ ) and we write  $\underline{\delta} = z^\lambda \underline{\delta}_{\mathrm{sm}}$  so that  $\underline{\delta}_{\mathrm{sm}}$  is a smooth character of  $T_p$ . We define

$$\mathcal{F}_{\overline{B}_p}^{G_p}(\underline{\delta}) := \mathcal{F}_{\overline{B}_p}^{G_p}\left(\left(U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} (-\lambda)\right)^\vee, \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1}\right)$$

where  $-\lambda$  is viewed as a weight of  $\bar{\mathfrak{b}}$  and  $(-)^{\vee}$  is the dual in  $\bar{\mathcal{O}}$  (cf. [Hum08, §3.2]). By [Bre15b, Thm. 4.3, Rem. 4.4],

$$\mathrm{Hom}_{G_p}(\mathcal{F}_{\bar{B}_p}^{G_p}(\underline{\delta}), \Pi^{\mathrm{an}}) \simeq \mathrm{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\Pi^{\mathrm{an}})) \quad (3.4.3)$$

for any very strongly admissible locally analytic representation  $\Pi^{\mathrm{an}}$  of  $G_p$  over  $L$ .

For such  $\lambda$ , let  $\bar{L}(\lambda)$  (resp.  $L(\lambda)$ ) be the irreducible  $U(\mathfrak{g})$ -module of the highest weight  $\lambda$  in  $\bar{\mathcal{O}}$  (resp. in  $\mathcal{O}$ ). Let  $W_{G_p} := \prod_{v \in S_p} (\mathcal{S}_n)^{\Sigma_v}$  be the Weyl group of  $\mathfrak{g} = \prod_v \mathfrak{g}_v = \prod_{\tau \in \Sigma_p} \mathfrak{g}_{\tau}$  acting naturally on  $\prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$  and we identify this action with the usual action of the Weyl group on weights of  $\mathfrak{t}$ . Let

$$\rho = \left( \frac{n-1}{2}, \dots, \frac{n-2i+1}{2}, \dots, \frac{1-n}{2} \right)_{\tau \in \Sigma_v, v \in S_p}$$

be the half sum of positive roots (with respect to  $\mathfrak{b}$ ) (the notation  $\rho$  will also be used to denote Galois representations when it will not confuse the reader according to the context). The dot action is given by  $w \cdot \mu = w(\mu + \rho) - \rho$  for any  $w \in W_{G_p}$  and  $\mu \in \prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$ .

We say  $\lambda \in \prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$  is *dominant* (resp. *anti-dominant*) (with respect to  $\mathfrak{b}$ ) if  $\lambda_{\tau,i} \geq \lambda_{\tau,i+1}, \forall \tau \in \Sigma_p, i = 1, \dots, n-1$  (resp.  $-\lambda$  is dominant). Now assume that  $\lambda \in \prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$  such that  $\lambda + \rho$  is dominant (so that  $\lambda$  is dominant in the sense of [Hum08, §3.5] with respect to  $\mathfrak{b}$ ). Let  $w_0 = (w_{v,0})_{v \in S_p} = ((w_{\tau,0})_{\tau \in \Sigma_v})_{v \in S_p}$  be the longest element in  $W_{G_p}$  and let  $W_{P_p} = \prod_{v \in S_p} W_{P_v}$  be the parabolic subgroup of  $W_{G_p}$  consisting of elements that fix  $w_0 \cdot \lambda$  under the dot action where  $P_p = \prod_{v \in S_p} P_v$  denotes the parabolic subgroup of  $\prod_{v \in S_p} (\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_{n/F_v}) \times_{\mathbb{Q}_p} L$  containing the Borel subgroup of upper-triangular matrices associated with  $W_{P_p}$ . Now  $-\lambda$  is dominant with respect to  $\bar{\mathfrak{b}}$  in the sense of [Hum08, §3.5] and an irreducible module  $\bar{L}(-\mu)$  is a subquotient of  $U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} (-ww_0 \cdot \lambda)$  if and only if  $\mu \uparrow ww_0 \cdot \lambda$  (cf. [Hum08, §5.1], the linkage relation  $\uparrow$  here is defined with respect to  $\mathfrak{b}$ ). One can prove that  $\mu \uparrow ww_0 \cdot \lambda$  if and only if  $-\mu = -w'w_0 \cdot \lambda$  for some  $w' \in W_{G_p}/W_{P_p}$  such that  $w' \leq w$  in  $W_{G_p}/W_{P_p}$ . Hence we conclude that the Jordan-Hölder factors of  $U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} (-ww_0 \cdot \lambda)$  are those  $\bar{L}(-w'w_0 \cdot \lambda)$  for  $w' \leq w$  in  $W_{G_p}/W_{P_p}$  (one can also use the fact that the translation functor  $T_{w_0 \cdot 0}^{w_0 \cdot \lambda}$  is exact ([Hum08, §7.1]) and  $T_{w_0 \cdot 0}^{w_0 \cdot \lambda} M(ww_0 \cdot 0) = M(ww_0 \cdot \lambda)$  for all  $w \in W_{G_p}$  where  $M(-)$  denotes the Verma modules with respect to  $\mathfrak{b}$ ,  $T_{w_0 \cdot 0}^{w_0 \cdot \lambda} L(ww_0 \cdot 0) = L(ww_0 \cdot \lambda)$  if  $w \in (W_{G_p})^{P_p}$  and  $T_{w_0 \cdot 0}^{w_0 \cdot \lambda} L(ww_0 \cdot 0) = 0$  if  $w \notin (W_{G_p})^{P_p}$  to reduce to regular cases, cf. [Hum08, Thm. 7.6, Thm. 7.9] or [Irv90, Prop. 2.1.1]).

For a locally algebraic character  $\underline{\delta} \in \hat{T}_{p,L}$  of weight  $\lambda$ , we define characters  $\underline{\delta}_w := z^{ww_0 \cdot \lambda} \underline{\delta}_{\mathrm{sm}}$  for  $w \in W_{G_p}/W_{P_p}$ . By the Orlik-Strauch theory ([Bre16, Thm. 2.3 & (2.6)]), if the smooth representation  $\mathrm{Ind}_{\bar{B}_p}^{G_p} \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1}$  is irreducible (which will be the case in our later discussions), the locally  $\mathbb{Q}_p$ -analytic representation  $\mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1})$  is irreducible. The Jordan-Hölder factors of  $\mathcal{F}_{\bar{B}_p}^{G_p}(\underline{\delta}_w)$  are those  $\mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-w'w_0 \cdot \lambda), \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1})$  where  $w' \leq w$  in  $W_{G_p}/W_{P_p}$  and  $\mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1})$  is the unique irreducible quotient of  $\mathcal{F}_{\bar{B}_p}^{G_p}(\underline{\delta}_w)$ .

### 3.4.4 The locally analytic socle conjecture

We prove our main results concerning the appearance of companion constituents in the completed cohomology and the existence of companion points on the eigenvariety in the situation of non-regular Hodge-Tate weights. The proofs of Proposition 3.4.17 and Theorem 3.4.18 follow essentially part of the proof of [BHS19, Thm. 5.3.3] with weakened assumptions. The new ingredients are in Theorem 3.4.10 and Proposition 3.4.13.

Let  $\lambda = (\lambda_{\tau,1}, \dots, \lambda_{\tau,n})_{\tau \in \Sigma_v, v \in S_p} \in \prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$  such that  $\lambda + \rho$  is dominant. Then we have the parabolic subgroup  $P_p$  of  $\prod_{v \in S_p} (\mathrm{Res}_{F_v/\mathbb{Q}_p} \mathrm{GL}_{n/F_v}) \times_{\mathbb{Q}_p} L$  associated with  $\lambda$  as in §3.4.3.

We firstly recall some representation theoretic results in [BHS19, §5.2] and the construction of certain “family of companion constituents” in the proof of [BHS19, Thm. 5.3.3].

We write  $\Pi_\infty^{\text{an}}$  for  $\Pi_\infty^{R_\infty - \text{an}}$ . Let  $U_p$  be the unipotent radical of  $B_p$  and  $U_0$  be a compact open subgroup of  $U_p$ . Suppose that  $\Pi^{\text{an}}$  is a very strongly admissible locally  $\mathbb{Q}_p$ -analytic representation of  $G_p$  over  $L$ . If  $M \in \mathcal{O}_{\text{alg}}$ , then  $M$  is equipped with an action of  $B_p$  and  $\text{Hom}_{U(\mathfrak{g})}(M, \Pi^{\text{an}})$  is equipped with a smooth action of  $B_p$ . Its space of  $U_0$ -invariants  $\text{Hom}_{U(\mathfrak{g})}(M, \Pi^{\text{an}})^{U_0}$  is then equipped with a Hecke action of  $T_p^+ := \{t \in T_p \mid tU_0t^{-1} \subset U_0\}$  given by ([BHS19, (5.9)])

$$f \mapsto t \cdot f := \delta_{B_p}(t) \sum_{u_0 \in U_0/tU_0t^{-1}} u_0 t f.$$

The starting point is the following adjunction formula. For any finite-dimensional smooth representation  $V$  of  $T_p$  over  $L$ , by [BHS19, Lem. 5.2.1] (which itself follows from [Bre15b, Prop. 4.2]), we have

$$\begin{aligned} \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \text{Hom}(M, L)^{\bar{u}^\infty}, V(\delta_{B_p}^{-1}) \right), \Pi^{\text{an}} \right) &= \text{Hom}_{T_p^+} \left( V, \text{Hom}_{U(\mathfrak{g})}(M, \Pi^{\text{an}})^{U_0} \right) \\ &= \text{Hom}_{T_p} \left( V, \left( \text{Hom}_{U(\mathfrak{g})}(M, \Pi^{\text{an}})^{U_0} \right)_{\text{fs}} \right). \end{aligned}$$

where  $(-)_{\text{fs}}$  denotes Emerton’s finite slope part functor ([Eme06b, Def. 3.2.1]) and  $\text{Hom}(M, L)^{\bar{u}^\infty}$  is the subspace of  $\text{Hom}(M, L)$  consisting of elements annihilated by a power of  $\bar{u}$ , base change to  $L$  of the  $\mathbb{Q}_p$ -Lie algebra of the unipotent radical of  $\overline{B}_p$  (see [Bre15b, §3]).

In particular, for any point  $y = (r_y, \underline{\delta}) \in \mathfrak{X}_\infty \times \widehat{T}_{p,L}$  with the corresponding maximal ideal  $\mathfrak{m}_{r_y}$  for  $r_y$  such that  $\underline{\delta}$  is locally algebraic of weight  $ww_0 \cdot \lambda$  and has smooth part  $\underline{\delta}_{\text{sm}}$ , we have

$$\begin{aligned} \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \overline{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_y}] \otimes_{k(r_y)} k(y) \right) \\ = \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \left( \text{Hom}_{U(\mathfrak{g})}(L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_y}] \otimes_{k(r_y)} k(y))^{U_0} \right)_{\text{fs}} \right). \end{aligned} \quad (3.4.4)$$

Recall by (3.4.3),  $y \in X_p(\bar{\rho})$  if and only if  $\text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p}(\underline{\delta}_w), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_y}] \otimes_{k(r_y)} k(y) \right) \neq 0$ , and there is an injection

$$\begin{aligned} \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \overline{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_y}] \otimes_{k(r_y)} k(y) \right) \\ \hookrightarrow \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p}(\underline{\delta}_w), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_y}] \otimes_{k(r_y)} k(y) \right) \end{aligned} \quad (3.4.5)$$

induced by the quotient  $U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} (-ww_0 \cdot \lambda) \twoheadrightarrow \overline{L}(-ww_0 \cdot \lambda)$  for any  $w \in W_{G_p}$ .

For any  $w \in W_{G_p}$ , let  $X_p(\bar{\rho})_{ww_0 \cdot \lambda}$  be the fiber of the composite map  $X_p(\bar{\rho}) \rightarrow \widehat{T}_{p,L} \xrightarrow{\text{wt}} \mathfrak{t}^*$  over  $ww_0 \cdot \lambda \in \mathfrak{t}^*$ . Here  $\mathfrak{t}^*$  denotes the rigid space associated with  $\text{Hom}_L(\mathfrak{t}, L)$  and the map  $\text{wt}$  sends a character of  $T_p$  to its weight. Since  $J_{B_p}(\Pi_\infty^{\text{an}}) = ((\Pi_\infty^{\text{an}})^{U_0})_{\text{fs}}$ , the quotient  $U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} ww_0 \cdot \lambda \twoheadrightarrow L(ww_0 \cdot \lambda)$  induces a closed immersion as (3.4.5)

$$\begin{aligned} \left( \text{Hom}_{U(\mathfrak{g})}(L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} \right)_{\text{fs}} &\hookrightarrow \left( \text{Hom}_{U(\mathfrak{t})}(ww_0 \cdot \lambda, \Pi_\infty^{\text{an}}[\mathfrak{u}])^{U_0} \right)_{\text{fs}} \\ &= \text{Hom}_{U(\mathfrak{t})}(ww_0 \cdot \lambda, J_{B_p}(\Pi_\infty^{\text{an}})) \end{aligned} \quad (3.4.6)$$

which is compatible with actions of  $R_\infty[\frac{1}{\ell}]$  and  $T_p$  (see **Step 8** of the proof of [BHS19, Thm. 5.3.3] for more details on the topology). The continuous dual  $\text{Hom}_{U(\mathfrak{t})}(ww_0 \cdot \lambda, J_{B_p}(\Pi_\infty^{\text{an}}))'$  of the target is the global section of the coherent sheaf  $\mathcal{M}_\infty \otimes_{\mathcal{O}_{X_p(\bar{\rho})}} \mathcal{O}_{X_p(\bar{\rho})_{ww_0 \cdot \lambda}}$  over the quasi-Stein space  $X_p(\bar{\rho})_{ww_0 \cdot \lambda}$  (as a closed subspace of the quasi-Stein space  $\text{Spf}(R_\infty)^{\text{rig}} \times \widehat{T}_{p,L}$ , cf. [Eme17, Def. 2.1.17] and [ST03, §3]). Then the continuous dual  $\left( \text{Hom}_{U(\mathfrak{g})}(L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} \right)_{\text{fs}}'$  of



the closed subspace corresponds to a coherent sheaf, denoted by  $\mathcal{L}_{ww_0 \cdot \lambda}$ , on  $X_p(\bar{\rho})_{ww_0 \cdot \lambda}$  and the continuous dual of (3.4.6) gives a surjection of coherent sheaves

$$\mathcal{M}_\infty \otimes_{\mathcal{O}_{X_p(\bar{\rho})}} \mathcal{O}_{X_p(\bar{\rho})_{ww_0 \cdot \lambda}} \twoheadrightarrow \mathcal{L}_{ww_0 \cdot \lambda}$$

on  $X_p(\bar{\rho})_{ww_0 \cdot \lambda}$ . Let  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$  be the schematic support of  $\mathcal{L}_{ww_0 \cdot \lambda}$  which is a Zariski-closed subspace of  $X_p(\bar{\rho})_{ww_0 \cdot \lambda}$ . Let  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}^{\text{red}}$  be the underlying reduced analytic subvariety of  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$  (cf. [BGR84, §9.5.3]).

*Remark 3.4.7.* We warn the reader that the subspace  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$  of  $X_p(\bar{\rho})$  should not be confused with any subspace of the non-patched eigenvariety  $Y(\bar{\rho}, U^p)$ . Note that when  $L(ww_0 \cdot \lambda)$  is a finite-dimensional representation of  $\mathfrak{g}$ ,  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$  is similar to a “partial eigenvariety” that will be defined in §3.5.4 (by taking  $Q_p = G_p, J = \Sigma_p$ ), but with more restriction on the weights of characters. And in this case one can prove that  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$  is equidimensional by the usual arguments for the eigenvarieties, cf. Lemma 3.5.14. We don’t know whether one should expect that  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$  is equidimensional when  $ww_0 \cdot \lambda$  is no longer dominant.

Let  $\mathfrak{m}_{\underline{\delta}_{\text{sm}}}$  be the kernel of the morphism  $L[T_p] \rightarrow L$  of  $L$ -algebras given by a smooth character  $\underline{\delta}_{\text{sm}}$ . Then for any  $L$ -point  $y = (r_y, z^{ww_0 \cdot \lambda} \underline{\delta}_{\text{sm}}) \in X_p(\bar{\rho})_{ww_0 \cdot \lambda} \subset \mathfrak{X}_\infty \times \widehat{T}_{p,L}$ ,

$$\begin{aligned} \mathcal{L}_{ww_0 \cdot \lambda} \otimes_{\mathcal{O}_{X_p(\bar{\rho})_{ww_0 \cdot \lambda}}} \widehat{\mathcal{O}}_{X_p(\bar{\rho})_{ww_0 \cdot \lambda}, y} &= \varprojlim_{s, t \in \mathbb{N}} \left( \text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} \right)'_{\text{fs}} / \left( \mathfrak{m}_{r_y}^s, \mathfrak{m}_{\underline{\delta}_{\text{sm}}}^t \right) \\ &= \varprojlim_{s, t \in \mathbb{N}} \left( \text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} [\mathfrak{m}_{r_y}^s][\mathfrak{m}_{\underline{\delta}_{\text{sm}}}^t] \right)' \\ &= \left( \varprojlim_{s, t \in \mathbb{N}} \text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} [\mathfrak{m}_{r_y}^s][\mathfrak{m}_{\underline{\delta}_{\text{sm}}}^t] \right)' \end{aligned} \quad (3.4.8)$$

where each

$$\text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} [\mathfrak{m}_{r_y}^s][\mathfrak{m}_{\underline{\delta}_{\text{sm}}}^t] = \text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty[\mathfrak{m}_{r_y}^s]^{\text{an}})^{U_0} [\mathfrak{m}_{\underline{\delta}_{\text{sm}}}^t]$$

is finite-dimensional (using that  $\Pi_\infty^{\text{an}}$  is  $R_\infty$ -admissible, the finiteness comes from the related property of Emerton’s Jacquet module, see [BHS19, Lem. 5.2.4]).

We denote by  $\text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} [\mathfrak{m}_{r_y}^\infty][\mathfrak{m}_{\underline{\delta}_{\text{sm}}}^\infty]$  for the last term in the bracket in (3.4.8). Thus

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \bar{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_y}] \right) \neq 0$$

if and only if (by (3.4.4))

$$\text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} [\mathfrak{m}_{r_y}][\mathfrak{m}_{\underline{\delta}_{\text{sm}}}^\infty] \neq 0$$

if and only if  $y \in Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$  which is equivalent to that

$$\text{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_\infty^{\text{an}})^{U_0} [\mathfrak{m}_{r_y}^\infty][\mathfrak{m}_{\underline{\delta}_{\text{sm}}}^\infty] \neq 0.$$

Next, we take  $y = ((\rho_p, \underline{\delta}), z) \in Y_p(\bar{\rho})_{ww_0 \cdot \lambda} \subset X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g)$  a generic  $L$ -point (Definition 3.4.2) and  $w = (w_v)_{v \in S_p} = (w_\tau)_{\tau \in \Sigma_p, v \in S_p} \in W_{G_p}/W_{P_p} = \prod_{v \in S_p} (\mathcal{S}_n)^{\Sigma_v}/W_{P_v}$  such that  $\text{wt}(\underline{\delta}) = ww_0 \cdot \lambda$ . Write  $\rho_p = (\rho_v)_{v \in S_p} \in \prod_{v \in S_p} \text{Spf}(R_{\bar{\rho}_v})^{\text{rig}}$ . Let  $\mathbf{h} = (\mathbf{h}_v)_{v \in S_p} = (\mathbf{h}_\tau)_{\tau \in \Sigma_p} = (h_{\tau,1}, \dots, h_{\tau,n})_{\tau \in \Sigma_p}$  be the anti-dominant Hodge-Tate-Sen weights of  $\rho_p$  where we view  $\mathbf{h}$  as a coweight of  $\mathfrak{t}$  and “anti-dominant” means that  $h_{\tau,1} \leq \dots \leq h_{\tau,n}, \forall \tau \in \Sigma_p$ . Then  $w_0 \cdot \lambda = (h_{\tau,1}, \dots, h_{\tau,i} + i - 1, \dots, h_{\tau,n} + n - 1)_{\tau \in \Sigma_p} \in \prod_{v \in S_p} (\mathbb{Z}^n)^{\Sigma_v}$ . The stabilizer subgroup of  $\mathbf{h}$  under the usual action of  $W_{G_p} = (\mathcal{S}_n)^{\Sigma_p}$  is  $W_{P_p}$ .

By Theorem 3.3.17 and the generic assumption on  $\underline{\delta}$ ,  $(\rho_p, \iota^{-1}(\underline{\delta}))$  lies in a unique irreducible component  $X$  of  $X_{\text{tri}}(\bar{\rho}_p)$ . The union of irreducible components of  $X_p(\bar{\rho})$  that pass through  $y$  are of the form  $\iota(X) \times (\cup_{i \in I} \mathfrak{X}_i^p) \times \mathbb{U}^g$  where  $\mathfrak{X}_i^p \times \mathbb{U}^g$ ,  $i \in I$  are some irreducible components of  $\mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g$ . Then  $\iota^{-1}$  induces a closed immersion  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}^{\text{red}} \hookrightarrow X_{\text{tri}}(\bar{\rho}_p)_{w(\mathbf{h})} \times \mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g$  where  $X_{\text{tri}}(\bar{\rho}_p)_{w(\mathbf{h})}$  denotes the fiber of the composite  $X_{\text{tri}}(\bar{\rho}_p) \rightarrow \hat{T}_{p,L} \xrightarrow{\text{wt}} \mathfrak{t}^*$  over  $w(\mathbf{h})$ . Hence  $\iota$  induces a surjection

$$\hat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p)_{w(\mathbf{h})}, \iota^{-1}(\rho_p, \underline{\delta})} \hat{\otimes}_L \hat{\mathcal{O}}_{\mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g, z} \twoheadrightarrow \hat{\mathcal{O}}_{Y_p(\bar{\rho})_{ww_0 \cdot \lambda, y}^{\text{red}}}.$$

Recall in §3.3.5, for each  $v \in S_p$ ,  $D_{\text{rig}}(\rho_{\bar{v}})[\frac{1}{t}]$  is equipped with a unique triangulation  $\mathcal{M}_{\bar{v}, \bullet}$  associated with the point  $(\rho_{\bar{v}}, \iota_v^{-1}(\underline{\delta}_v)) \in X_{\text{tri}}(\bar{\rho}_{\bar{v}})$  and we have an isomorphism  $\widehat{X_{\text{tri}}(\bar{\rho}_{\bar{v}})}_{(\rho_{\bar{v}}, \iota_v^{-1}(\underline{\delta}_v))} \simeq X_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{w_v}$  by Corollary 3.3.15. In §3.3.6, we have  $\hat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_{\bar{v}})_{w_v(\mathbf{h}_{\bar{v}})}, \iota_v^{-1}(\rho_{\bar{v}}, \underline{\delta}_v)} \simeq \bar{R}_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{w_v}$ . Hence  $\hat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p)_{w(\mathbf{h})}, \iota^{-1}(\rho_p, \underline{\delta})} \simeq \bar{R}_{\rho_p, \mathcal{M}_{\bullet}}^w := \hat{\otimes}_{v \in S_p} \bar{R}_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{w_v}$ . Define  $R_{\rho_p} := \hat{\otimes}_{v \in S_p} R_{\rho_{\bar{v}}}$ . We get a composite map

$$R_{\rho_p} \rightarrow \bar{R}_{\rho_p, \mathcal{M}_{\bullet}}^w \simeq \hat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p)_{w(\mathbf{h})}, \iota^{-1}(\rho_p, \underline{\delta})} \rightarrow \hat{\mathcal{O}}_{Y_p(\bar{\rho})_{ww_0 \cdot \lambda, y}^{\text{red}}} \quad (3.4.9)$$

Let  $Q_p = \prod_{v \in S_p} \prod_{\tau \in \Sigma_v} Q_{\tau}$  be a standard parabolic subgroup of  $\prod_{v \in S_p} (\text{Res}_{F_{\bar{v}}/\mathbb{Q}_p} \text{GL}_{n/F_{\bar{v}}}) \times_{\mathbb{Q}_p} L = \prod_{\tau \in \Sigma_p} \text{GL}_{n/L}$  (with the Borel subgroup of upper-triangular matrices and the maximal torus the diagonal matrices). For  $v \in S_p$ , the statement “ $w_v(\mathbf{h}_{\bar{v}})$  is strictly  $Q_v$ -dominant” (Definition 3.2.24) is equivalent to “ $w_v w_{v,0} \cdot \lambda_v$  is a dominant weight for the standard Levi subgroup of  $Q_v$ ”. Let  $R_{\rho_p, \mathcal{M}_{\bullet}}^{Q_p} := \hat{\otimes}_{v \in S_p} R_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{Q_v}$  where the latter is defined in the end of §3.3.6, roughly speaking, parametrizing trianguline deformations of  $\rho_{\bar{v}}$  that are  $Q_v$ -de Rham. There are closed immersions

$$\text{Spec}(R_{\rho_p, \mathcal{M}_{\bullet}}^{Q_p}) \hookrightarrow \text{Spec}(\bar{R}_{\rho_p, \mathcal{M}_{\bullet}}) \hookrightarrow \text{Spec}(R_{\rho_p}).$$

**Theorem 3.4.10.** *If  $ww_0 \cdot \lambda$  is a dominant weight for the standard Levi subgroup of  $Q_p$ , then the morphism (3.4.9):  $R_{\rho_p} \rightarrow \hat{\mathcal{O}}_{Y_p(\bar{\rho})_{ww_0 \cdot \lambda, y}^{\text{red}}}$  factors through  $R_{\rho_p, \mathcal{M}_{\bullet}}^{Q_p}$ .*

*Proof.* The proof is based on Theorem 3.5.28.

We have a closed immersion  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}^{\text{red}} \hookrightarrow X_p(\bar{\rho})$ . We argue for a fixed  $v \in S_p$ . Let  $D := D_{\text{rig}}(\rho_{\bar{v}}^{\text{univ}})$  be the  $(\varphi, \Gamma_{F_{\bar{v}}})$ -module over  $\mathcal{R}_{X_p(\bar{\rho}), F_{\bar{v}}}$  associated with the universal Galois representation  $\rho_{\bar{v}}^{\text{univ}}$  of  $\mathcal{G}_{F_{\bar{v}}}$  pulled back from  $\mathfrak{X}_{\bar{p}^p}$  (cf. [Liu15, Def. 2.12]). By [BHS17b, Thm. 3.19] and [KPX14, Cor. 6.3.10], there is a birational proper surjective morphism  $f : X' \rightarrow X_p(\bar{\rho})$ , a filtration of sub- $(\varphi, \Gamma_{F_{\bar{v}}})$ -modules  $D_{X', \bullet}$  over  $\mathcal{R}_{X', F_{\bar{v}}}$  of  $D_{X'} := f^* D$  such that for any point  $y' \in X'$ , the base change  $D_{y', \bullet}[\frac{1}{t}]$  is a triangulation of  $D_{y'}[\frac{1}{t}] = D_{\text{rig}}(\rho_{y', \bar{v}})[\frac{1}{t}]$  of parameter  $\delta_{y', v, 1}, \dots, \delta_{y', v, n}$  where we use the same notations with different subscripts to denote the pullback of the representation  $\rho_{\bar{v}}^{\text{univ}}$ , characters from  $\hat{T}_v$ , etc.. Let  $Y'$  be the underlying reduced analytic subspace of  $f^{-1}(Y_p(\bar{\rho})_{ww_0 \cdot \lambda}^{\text{red}})$ . We pick an arbitrary affinoid neighbourhood  $V$  of an arbitrary point  $y' \in f^{-1}(y)$  in  $Y'$ . By Theorem 3.5.28 below, the definition of  $Y_p(\bar{\rho})_{ww_0 \cdot \lambda}$ , (3.4.4), and the assumption that  $ww_0 \cdot \lambda$  is dominant with respect to the standard Levi of  $Q_p$ , we have that for any point  $y'' \in V$ ,  $D_{y''}[\frac{1}{t}]$  with the triangulation  $D_{y'', \bullet}[\frac{1}{t}]$  is  $Q_v$ -de Rham (Definition 3.3.19).

We firstly prove that the map  $R_{\rho_{\bar{v}}} \rightarrow \hat{\mathcal{O}}_{Y', y'}$  factors through the quotient  $R_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{Q_v}$ . The proof is similar to that of Proposition 3.3.4. Take  $A$  a local Artin  $L$ -algebra with residue field  $k(y')$  and a composite  $y' = \text{Sp}(k(y')) \rightarrow \text{Sp}(A) \rightarrow V$ . The pullback  $D_{A, \bullet}$  along the map  $\text{Sp}(A) \rightarrow V$  of the global triangulation  $D_{V, \bullet} = D_{X', \bullet} \times_{X'} V$  gives a triangulation  $\mathcal{M}_{A, \bullet} := D_{A, \bullet}[\frac{1}{t}]$  of  $D_{\text{rig}}(\rho_{A, \bar{v}})[\frac{1}{t}]$  of parameter  $\delta_{A, v, 1}, \dots, \delta_{A, v, n}$ . Since the filtration  $D_{X', \bullet}$  is a strictly trianguline filtration on a Zariski open dense subset of  $X'$ , and the Sen polynomials vary analytically, the Sen weights of  $D_{V, i}$  are fixed integers (the weights of  $\delta_{V, v, 1}, \dots, \delta_{V, v, i}$ ). Hence we can apply Theorem A.3.4 below for each  $D_{V, i}$ , and we get finite projective  $\mathcal{O}_V$ -modules

written by  $D_{\text{pdR}}(D_{V,i})$  (in short for  $D_{\text{pdR}}(W_{\text{dR}}(D_{V,i})) = \prod_{\tau \in \Sigma_v} D_{\text{pdR},\tau}(W_{\text{dR}}(D_{V,i}))$ ) with  $\mathcal{O}_V$ -linear operators  $\nu_V$ . Check the proof of Theorem A.3.4, the filtration  $D_{V,\bullet}$  induces natural maps  $D_{\text{pdR}}(D_{V,i}) \rightarrow D_{\text{pdR}}(D_{V,i+1})$ . By Theorem A.3.4 again,  $D_{\text{pdR}}(D_{V,\bullet})$  are specialized to the  $k(y'')$ -filtration  $D_{\text{pdR}}(W_{\text{dR}}(D_{y'',\bullet}))$  for any  $y'' \in V$ . Since  $V$  is reduced, the sheaves  $D_{\text{pdR}}(D_{V,i})$  form a saturated filtration of  $D_{\text{pdR}}(D_V)$ , i.e.,  $D_{\text{pdR}}(D_{V,i})$  is mapped injectively into  $D_{\text{pdR}}(D_{V,i+1})$  and the graded pieces  $D_{\text{pdR}}(D_{V,i+1})/D_{\text{pdR}}(D_{V,i})$  are locally free of rank one. Moreover, for any  $\tau \in \Sigma_v$ ,  $1 \leq i < i' \leq n$  and  $y'' \in V$ ,

$$(D_{\text{pdR},\tau}(D_{V,i'})/D_{\text{pdR},\tau}(D_{V,i})) \otimes_{\mathcal{O}_V} k(y'') \simeq D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{y'',i'}/\mathcal{M}_{y'',i})).$$

Let  $0 = s_{\tau,0} < \dots < s_{\tau,i} < \dots < s_{\tau,t_\tau} = n$  be integers such that the Levi subgroup of  $Q_\tau$  is  $\text{GL}_{s_{\tau,1}-s_{\tau,0}} \times \dots \times \text{GL}_{s_{\tau,i}-s_{\tau,i-1}} \times \dots \times \text{GL}_{s_{\tau,t_\tau}-s_{\tau,t_\tau-1}}$ . Since  $D_{y''}[\frac{1}{t}]$  is  $Q_v$ -de Rham for any  $y'' \in V$ ,  $\nu_{y''} = \nu_V \otimes_{\mathcal{O}_V} k(y'')$  acts as zero on  $D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{y'',s_{\tau,i}}/\mathcal{M}_{y'',s_{\tau,i-1}}))$  for  $1 \leq i \leq t_\tau$  and  $\tau \in \Sigma_v$ . As  $V$  is reduced, it follows that  $\nu_V$  itself is zero on the graded pieces  $D_{\text{pdR},\tau}(D_{V,s_{\tau,i}})/D_{\text{pdR},\tau}(D_{V,s_{\tau,i-1}})$ . By Theorem A.3.4, the same assertion holds for the base change  $\nu_A$  of  $\nu_V$  on  $D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{A,s_{\tau,i}}/\mathcal{M}_{A,s_{\tau,i-1}}))$ . By Definition 3.3.19,  $D_{\text{rig}}(\rho_{A,\tilde{v}})[\frac{1}{t}]$  with the triangulation is  $Q_v$ -de Rham (Definition 3.3.19).

We prove that the composite  $R_{\rho_{\tilde{v}}} \rightarrow \hat{\mathcal{O}}_{Y',y'} \rightarrow A$  factors through a map  $R_{\rho_{\tilde{v}}} \rightarrow R_{\rho_{\tilde{v}},\mathcal{M}_{\tilde{v},\bullet}}^{Q_v} \rightarrow A$ . Let  $\tilde{A} := A \times_{k(y')} L$  be the subring of  $A$  consisting of elements whose reduction modulo the maximal ideal  $\mathfrak{m}_A$  of  $A$  lie in  $L$ . Then the map  $R_{\rho_{\tilde{v}}} \rightarrow A$  factors through  $R_{\rho_{\tilde{v}}} \rightarrow \tilde{A} \subset A$  automatically. By similar arguments as in the proof of Proposition 3.3.4, there exists a model  $\rho_{\tilde{A},\tilde{v}}$  (resp.  $\delta_{A,v,i}$ , resp.  $\mathcal{M}_{\tilde{A},\bullet}$ ) of  $\rho_{A,\tilde{v}}$  (resp.  $\delta_{\tilde{A},v,i}$ , resp.  $\mathcal{M}_{A,\bullet}$ ) over  $\tilde{A}$  whose reduction modulo  $\mathfrak{m}_{\tilde{A}}$  is  $\rho_{y,\tilde{v}}$  (resp.  $\delta_{y,v,i}$ , resp.  $\mathcal{M}_{y,\tilde{v},\bullet}$ ). This implies that the map  $R_{\rho_{\tilde{v}}} \rightarrow \hat{\mathcal{O}}_{Y',y'} \rightarrow A$  factors through  $R_{\rho_{\tilde{v}}} \rightarrow R_{\rho_{\tilde{v}},\mathcal{M}_{\tilde{v},\bullet}}^{Q_v} \rightarrow \tilde{A} \subset A$ . Moreover,  $D_{\text{rig}}(\rho_{A,\tilde{v}})[\frac{1}{t}] = D_{\text{rig}}(\rho_{\tilde{A},\tilde{v}})[\frac{1}{t}] \otimes_{\tilde{A}} A$  and  $\mathcal{M}_{\tilde{A},\bullet} \otimes_{\tilde{A}} A = \mathcal{M}_{A,\bullet}$ . It follows from the exactness and the functoriality of  $D_{\text{pdR}}(W_{\text{dR}}(-))$  that there exist isomorphisms  $D_{\text{pdR}}(W_{\text{dR}}(\mathcal{M}_{\tilde{A},\bullet})) \otimes_{\tilde{A}} A \simeq D_{\text{pdR}}(W_{\text{dR}}(\mathcal{M}_{A,\bullet}))$  of finite free  $A$ -modules with  $A$ -linear nilpotent operators (cf. the proof of [BHS19, Lem. 3.1.4], writing  $A$  as the cokernel of finite free  $\tilde{A}$ -modules). Since  $\tilde{A} \rightarrow A$  is injective, the vanishing of  $\nu_A = \nu_{\tilde{A}} \otimes_{\tilde{A}} A$  on the finite free  $A$ -modules  $D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{A,s_{\tau,i}}/\mathcal{M}_{A,s_{\tau,i-1}})) = D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{\tilde{A},s_{\tau,i}}/\mathcal{M}_{\tilde{A},s_{\tau,i-1}})) \otimes_{\tilde{A}} A$  implies the vanishing of  $\nu_{\tilde{A}}$  on  $D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{\tilde{A},s_{\tau,i}}/\mathcal{M}_{\tilde{A},s_{\tau,i-1}}))$ . This means that  $\rho_{\tilde{A},\tilde{v}}$  with the filtration  $\mathcal{M}_{\tilde{A},\bullet}$  is also  $Q_v$ -de Rham. The definition of  $R_{\rho_{\tilde{v}},\mathcal{M}_{\tilde{v},\bullet}}^{Q_v}$  and Lemma 3.3.20 implies that the map  $R_{\rho_{\tilde{v}}} \rightarrow \tilde{A}$  factors through  $R_{\rho_{\tilde{v}},\mathcal{M}_{\tilde{v},\bullet}}^{Q_v} \rightarrow \tilde{A}$ .

By taking  $A = \mathcal{O}_{Y',y'}/\mathfrak{m}_{\mathcal{O}_{Y',y'}}^j$  for all  $j \in \mathbb{N}$ , we conclude that the map  $R_{\rho_{\tilde{v}}} \rightarrow \hat{\mathcal{O}}_{Y',y'}$  factors through the quotient  $R_{\rho_{\tilde{v}},\mathcal{M}_{\tilde{v},\bullet}}^{Q_v}$ .

Now the argument in the last part of the proof of Proposition 3.3.4, using the surjectivity and properness of  $f : Y' \rightarrow Y_p(\bar{\rho})_{ww_0,\lambda}^{\text{red}}$  and the reducedness of  $Y_p(\bar{\rho})_{ww_0,\lambda}^{\text{red}}$ , shows that the map  $R_{\rho_{\tilde{v}}} \rightarrow \hat{\mathcal{O}}_{Y_p(\bar{\rho})_{ww_0,\lambda,y}^{\text{red}}}$  also factors through  $R_{\rho_{\tilde{v}},\mathcal{M}_{\tilde{v},\bullet}}^{Q_v}$ .  $\square$

*Remark 3.4.11.* We expect that the morphism  $R_{\rho_p} \rightarrow \hat{\mathcal{O}}_{Y_p(\bar{\rho})_{ww_0,\lambda,y}}$  as in Theorem 3.4.10 factors through  $R_{\rho_p,\mathcal{M}_{\bullet}}^{Q_p}$ . However, we don't know how to prove this stronger result.

From now on we focus on generic crystalline points. A point  $\rho_p = (\rho_{\tilde{v}})_{v \in S_p} \in \mathfrak{X}_{\bar{\rho}_p}(L)$  is said to be *generic crystalline* if for each  $v \in S_p$ ,  $\rho_{\tilde{v}} : \mathcal{G}_{F_{\tilde{v}}} \rightarrow \text{GL}_n(L)$  is generic crystalline (in the sense in §3.4.1). If  $\rho_p$  is generic, a refinement  $\mathcal{R} = (\mathcal{R}_{\tilde{v}})_{v \in S_p}$  of  $\rho_p$  is a choice of a refinement  $\mathcal{R}_{\tilde{v}}$  of  $\rho_{\tilde{v}}$  for each  $v \in S_p$ .

Suppose that  $\rho_p \in \mathfrak{X}_{\bar{\rho}_p}(L)$  is a generic crystalline point with refinements  $\mathcal{R}_{\tilde{v}}$  given by orderings  $\varphi_{\tilde{v}} \in (L^\times)^n$  and that  $\mathbf{h}_{\tilde{v}}$  is the anti-dominant Hodge-Tate weights of  $\rho_{\tilde{v}}$  for  $v \in S_p$ . Recall for each  $v \in S_p$ , there exists  $w_{\mathcal{R}_{\tilde{v}}} \in (\mathcal{S}_n)^{\Sigma_v}/W_{P_v}$  such that  $z^{w_{\mathcal{R}_{\tilde{v}}}(\mathbf{h}_{\tilde{v}})} \text{unr}(\varphi_{\tilde{v}})$  is a parameter of  $\rho_{\tilde{v}}$

and then the point  $(\rho_{\tilde{v}}, z^{w_{\mathcal{R}_{\tilde{v}}}(\mathbf{h}_{\tilde{v}})} \text{unr}(\varphi_{\tilde{v}}))$  is in  $U_{\text{tri}}(\bar{\rho}_{\tilde{v}}) \subset X_{\text{tri}}(\bar{\rho}_{\tilde{v}})$  (see §3.4.1). We continue to assume  $\lambda_{\tau, i} = h_{\tau, n+1-i} + i - 1, \forall \tau \in \Sigma_p, i = 1, \dots, n$ .

**Definition 3.4.12.** For  $w = (w_v)_{v \in S_p} \in W_{G_p}/W_{P_p}$ , let  $\underline{\delta}_{\mathcal{R}, w}$  be the character  $\iota(z^{w(\mathbf{h})} \text{unr}(\varphi))$  of  $T_p$  where  $z^{w(\mathbf{h})} \text{unr}(\varphi)$  is the abbreviation for the character  $\prod_{v \in S_p} z^{w_v(\mathbf{h}_{\tilde{v}})} \text{unr}(\varphi_{\tilde{v}})$  of  $T_p = \prod_{v \in S_p} T_v$ .

Then for each  $w \in W_{G_p}/W_{P_p}$ ,  $\underline{\delta}_{\mathcal{R}, w}$  is generic and  $w w_0 \cdot \lambda$  is the weight of  $\underline{\delta}_{\mathcal{R}, w}$ . The smooth part  $\underline{\delta}_{\mathcal{R}, \text{sm}} = z^{-w w_0 \cdot \lambda} \underline{\delta}_{\mathcal{R}, w}$  is independent of  $w$ . We write  $w_{\mathcal{R}} := (w_{\mathcal{R}_{\tilde{v}}})_{v \in S_p} \in W_{G_p}/W_{P_p}$ .

We already know that the point  $y_{w_{\mathcal{R}}} := (\rho_p, \iota^{-1}(\underline{\delta}_{\mathcal{R}, w_{\mathcal{R}}}))$  is in  $U_{\text{tri}}(\bar{\rho}_p)$ . If we assume Conjecture 3.23 of [BHS17b] (which follows from a general automorphy lifting conjecture by [BHS17b, Prop. 3.27]), we could get that the irreducible component of  $X_{\text{tri}}(\bar{\rho}_p)$  passing through  $y_{w_{\mathcal{R}}}$  is  $\mathfrak{X}^p$ -automorphic for any irreducible component  $\mathfrak{X}^p$  of  $\mathfrak{X}_{\bar{\rho}^p}$ . Hence there should exist a point  $((\rho_p, \underline{\delta}_{\mathcal{R}, w_{\mathcal{R}}}), z) \in X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$ . Then by [BHS17a, Thm. 5.5], together with the discussion on local companion points in §3.4.1, we could expect  $x_w := ((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z) \in X_p(\bar{\rho})$  if and only if  $w \geq w_{\mathcal{R}}$  in  $W_{G_p}/W_{P_p}$ .

We will not consider the automorphy lifting anymore in this paper. Rather, our aim (Theorem 3.4.18) is to prove that all companion points  $x_w, w \geq w_{\mathcal{R}}$  are in  $X_p(\bar{\rho})$  under the assumption that there exists  $w' \in W_{G_p}/W_{P_p}$  such that  $x_{w'}$  is in  $X_p(\bar{\rho})$ . The assumption will always imply  $x_{w_0} \in X_p(\bar{\rho})$  by [BHS17a, Thm. 5.5].

The following proposition is the key new step to achieve the existence of companion points, where Theorem 3.4.10 is used. In the proof of Theorem 3.4.18, we will use some induction and deformation arguments to reduce the existence of more general companion points on  $X_p(\bar{\rho})$  to the special situation considered in Proposition 3.4.13 below.

**Proposition 3.4.13.** *Assume that the points  $((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z) \in \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  are in  $X_p(\bar{\rho})(L)$  for any  $w > w_{\mathcal{R}}$  in  $W_{G_p}/W_{P_p}$  where  $\rho_p$  is generic crystalline and  $\mathcal{R}$  is a refinement of  $\rho_p$  as above. If  $w_{\mathcal{R}} W_{P_p} \neq w_0 W_{P_p}$ , then  $((\rho_p, \underline{\delta}_{\mathcal{R}, w_{\mathcal{R}}}), z) \in X_p(\bar{\rho})(L)$ .*

*Proof.* We will prove

$$\text{Hom}_{U(\mathfrak{g})}(L(w_{\mathcal{R}} w_0 \cdot \lambda), \Pi_{\infty}^{\text{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \text{sm}}}^{\infty}] \neq 0$$

where  $r_x := (\rho_p, z) \in \mathfrak{X}_{\infty}$ . This will imply (by (3.4.4))

$$\text{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \bar{L}(-w_{\mathcal{R}} w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1} \right), \Pi_{\infty}^{\text{an}}[\mathfrak{m}_{r_x}] \right) \neq 0$$

and by (3.4.3) and (3.4.5), imply furthermore that  $((\rho_p, \underline{\delta}_{\mathcal{R}, w_{\mathcal{R}}}), z) \in X_p(\bar{\rho})$ .

For each  $v \in S_p$ , we still write  $w_{\mathcal{R}_{\tilde{v}}}$  for the shortest representative of  $w_{\mathcal{R}_{\tilde{v}}}$  in  $(\mathcal{S}_n)^{\Sigma_v}$  and write  $w_{\mathcal{R}} = (w_{\mathcal{R}_{\tilde{v}}})_{v \in S_p} \in W_{G_p}$ . We may assume that there exists and fix a place  $v$  such that  $w_{\mathcal{R}_{\tilde{v}}}$  is not in the coset  $w_{v,0} W_{P_v}$  since  $w_{\mathcal{R}} W_{P_p} \neq w_0 W_{P_p}$ . Then by Lemma 3.2.27, there exists a simple root  $\alpha$  of  $(\text{Res}_{F_{\tilde{v}}/\mathbb{Q}_p} \text{GL}_{n/F_{\tilde{v}}}) \times_{\mathbb{Q}_p} L$  and a standard parabolic subgroup  $Q = \prod_{v \in S_p} Q_v = \prod_{\tau \in \Sigma_p} Q_{\tau}$  of  $\prod_{v \in S_p} (\text{Res}_{F_{\tilde{v}}/\mathbb{Q}_p} \text{GL}_{n/F_{\tilde{v}}}) \times_{\mathbb{Q}_p} L$  such that the element  $w' := s_{\alpha} w_{\mathcal{R}} = (w'_{v'})_{v' \in S_p} \in W_{G_p}$  where  $w'_{v'} = w_{\mathcal{R}_{\tilde{v}'}}$  if  $v' \neq v$  and  $w'_v = s_{\alpha} w_{\mathcal{R}_{\tilde{v}}}$  satisfies that  $\text{lg}_{P_p}(w') = \text{lg}_{P_p}(w_{\mathcal{R}}) + 1, w'_v(\mathbf{h}_{\tilde{v}})$  is strictly  $Q_v$ -dominant and  $w_{\mathcal{R}_{\tilde{v}}}(\mathbf{h}_{\tilde{v}})$  is *not* strictly  $Q_v$ -dominant (Definition 3.2.24). By our assumption, the point  $x_{w'} := ((\rho_p, \underline{\delta}_{\mathcal{R}, w'}), z)$  is in  $X_p(\bar{\rho})$ .

In particular,  $\mathcal{M}_{\infty} \otimes_{\mathcal{O}_{X_p(\bar{\rho})}} \hat{\mathcal{O}}_{X_p(\bar{\rho})_{w' w_0 \cdot \lambda, x_{w'}}} \neq 0$ . Equivalently, since  $\mathcal{M}_{\infty}$  is defined using  $J_{B_p}(\Pi_{\infty}^{\text{an}}) = ((\Pi_{\infty}^{\text{an}})^{U_0})_{\text{fs}}$ , by taking dual and arguing as in (3.4.8), we get

$$\text{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} w' w_0 \cdot \lambda, \Pi_{\infty}^{\text{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \text{sm}}}^{\infty}] \neq 0,$$

(see also [BHS19, (5.16), (5.18)]).

If

$$\mathrm{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \overline{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{\overline{B}_p}^{-1} \right), \Pi_{\infty}^{\mathrm{an}}[\mathfrak{m}_{r_x}] \right) \neq 0,$$

then by (3.4.3) and (3.4.5), the point  $((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z)$  will appear in  $X_p(\overline{\rho})$ . Then  $(\rho_p, \iota^{-1}(\underline{\delta}_{\mathcal{R}, w})) \in X_{\mathrm{tri}}(\overline{\rho}_p)$ . This is possible only if  $w \geq w_{\mathcal{R}}$  by Theorem 3.4.1. Hence the irreducible constituents of  $\mathcal{F}_{\overline{B}_p}^{G_p}(\underline{\delta}_{\mathcal{R}, w'})$  that may appear as subrepresentations in  $\Pi_{\infty}^{\mathrm{an}}[\mathfrak{m}_{r_x}]$  are

$$\mathcal{F}_{\overline{B}_p}^{G_p} \left( \overline{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \mathrm{sm}} \delta_{\overline{B}_p}^{-1} \right)$$

where  $w' \geq w \geq w_{\mathcal{R}}$  in  $W_{G_p}/W_{P_p}$ , i.e.  $w = w'$  or  $w = w_{\mathcal{R}}$ . And for any  $w \leq w', w \notin \{w', w_{\mathcal{R}}\}$ ,

$$\mathrm{Hom}_{U(\mathfrak{g})} (L(ww_0 \cdot \lambda), \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}] = 0.$$

The functor  $M \mapsto \mathrm{Hom}_{U(\mathfrak{g})} (M, \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}]$  is exact on the category  $\mathcal{O}$ . This fact comes essentially from that the dual of  $\Pi_{\infty}$  is a finite projective  $S_{\infty}[[K_p]][\frac{1}{p}]$ -module and that the functors of taking generalized eigenspace of compact operators are exact. See discussions before [BHS19, (5.21)] (and the arguments in [BHS17a, Thm. 5.5] of proving firstly similar results for ideals of  $S_{\infty}[\frac{1}{p}]$ ).

Hence we get an exact sequence

$$\begin{aligned} 0 &\rightarrow \mathrm{Hom}_{U(\mathfrak{g})} (L(w'w_0 \cdot \lambda), \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}] \\ &\rightarrow \mathrm{Hom}_{U(\mathfrak{g})} (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} w'w_0 \cdot \lambda, \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}] \\ &\rightarrow \mathrm{Hom}_{U(\mathfrak{g})} (L(w_{\mathcal{R}}w_0 \cdot \lambda), \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}] \rightarrow 0. \end{aligned}$$

To prove  $\mathrm{Hom}_{U(\mathfrak{g})} (L(w_{\mathcal{R}}w_0 \cdot \lambda), \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}] \neq 0$ , by the above exact sequence, we only need to show that

$$\mathrm{Hom}_{U(\mathfrak{g})} (L(w'w_0 \cdot \lambda), \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}] \neq \mathrm{Hom}_{U(\mathfrak{g})} (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} w'w_0 \cdot \lambda, \Pi_{\infty}^{\mathrm{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}][\mathfrak{m}_{\underline{\delta}_{\mathcal{R}, \mathrm{sm}}}^{\infty}]$$

or equivalently by taking dual as in (3.4.8) to show that the map

$$\mathcal{M}_{\infty} \otimes_{\mathcal{O}_{X_p(\overline{\rho})}} \widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}} \twoheadrightarrow \mathcal{L}_{w'w_0 \cdot \lambda} \otimes_{\mathcal{O}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda}}} \widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}} \quad (3.4.14)$$

is not an isomorphism.

We prove it by contradiction. Assume that (3.4.14) is an isomorphism. The action of  $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}$  on the right-hand side factors through

$$\widehat{\mathcal{O}}_{Y_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}} = \mathcal{O}_{Y_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}} \otimes_{\mathcal{O}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}}} \widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}}.$$

Thus by Lemma 3.4.16 below the  $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}$ -module in the right hand side of (3.4.14) has support whose underlying reduced subspace is  $\mathrm{Spec}(\widehat{\mathcal{O}}_{Y_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}}^{\mathrm{red}})$ . Since  $\mathcal{O}_{X_p(\overline{\rho})}$  acts faithfully on  $\mathcal{M}_{\infty}$ , the support of  $\mathcal{M}_{\infty} \otimes_{\mathcal{O}_{X_p(\overline{\rho})}} \mathcal{O}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}$  is set-theoretically equal to  $X_p(\overline{\rho})_{w'w_0 \cdot \lambda}$  ([Sta22, Tag 00L3]) with the underlying reduced subspace  $X_p(\overline{\rho})_{w'w_0 \cdot \lambda}^{\mathrm{red}}$ . By Lemma 3.4.16 below, the underlying reduced subscheme of the support of the  $\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}$ -module in the left hand side of (3.4.14) is  $\mathrm{Spec}(\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}}^{\mathrm{red}})$ . Thus we have

$$\mathrm{Spec}(\widehat{\mathcal{O}}_{X_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}}^{\mathrm{red}}) = \mathrm{Spec}(\widehat{\mathcal{O}}_{Y_p(\overline{\rho})_{w'w_0 \cdot \lambda, x_{w'}}}^{\mathrm{red}}).$$

By Theorem 3.4.10 and the above equality, the map

$$R_{\rho_p} \rightarrow \widehat{\mathcal{O}}_{X_{\text{tri}}(\bar{\rho}_p)_{w'(\mathbf{h})}, \iota^{-1}(\rho_p, \underline{\delta}_{\mathcal{R}, w'})} \widehat{\otimes}_L \widehat{\mathcal{O}}_{\mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g, z} \rightarrow \widehat{\mathcal{O}}_{X_p(\bar{\rho})_{w'w_0 \cdot \lambda}^{\text{red}}, x_{w'}}$$

factors through  $R_{\rho_p, \mathcal{M}_\bullet}^{Q_p}$ . We now prove this is not possible. Assume that  $\iota(X) \times \mathfrak{X}^p \times \mathbb{U}^g$  is an irreducible component of  $X_p(\bar{\rho})$  passing through  $x_{w'}$  where  $X$  is the unique irreducible component of  $X_{\text{tri}}(\bar{\rho}_p)$  passing through  $\iota^{-1}(\rho_p, \underline{\delta}_{\mathcal{R}, w'})$  and  $\mathfrak{X}^p$  is an irreducible (reduced) component of  $\mathfrak{X}_{\bar{p}^p}$ . Let  $X_{w'(\mathbf{h})}$  be the fiber of  $X$  over the weight  $w'(\mathbf{h})$  and let  $X_{w'(\mathbf{h})}^{\text{red}}$  be the underlying reduced subspace. Then  $\iota(X_{w'(\mathbf{h})}^{\text{red}}) \times \mathfrak{X}^p \times \mathbb{U}^g$  is a reduced subspace of  $X_p(\bar{\rho})_{w'w_0 \cdot \lambda}^{\text{red}}$  and the map

$$R_{\rho_p} \rightarrow \widehat{\mathcal{O}}_{X_{w'(\mathbf{h})}^{\text{red}}, \iota^{-1}(\rho_p, \underline{\delta}_{\mathcal{R}, w'})} \widehat{\otimes}_L \widehat{\mathcal{O}}_{\mathfrak{X}^p \times \mathbb{U}^g, z}$$

factors through  $R_{\rho_p, \mathcal{M}_\bullet}^{Q_p}$ . Hence the map  $R_{\rho_p} \rightarrow \widehat{\mathcal{O}}_{X_{w'(\mathbf{h})}^{\text{red}}, \iota^{-1}(\rho_p, \underline{\delta}_{\mathcal{R}, w'})}$  factors through  $R_{\rho_p, \mathcal{M}_\bullet}^{Q_p}$ .

Denote by  $y_{w'_v} := (\rho_{\bar{v}}, z_{w'_v(\mathbf{h}_{\bar{v}})}^{\text{unr}}(\varphi_{\bar{v}}))$  the point on  $X_{\text{tri}}(\bar{\rho}_{\bar{v}})_{w'_v(\mathbf{h}_{\bar{v}})}$  and  $X_v$  the irreducible component passing the point with  $X_{w'_v(\mathbf{h}_{\bar{v}})}$  the fiber of  $X_v$  over the weight  $w'_v(\mathbf{h}_{\bar{v}})$ . We get that the morphism  $R_{\rho_{\bar{v}}} \twoheadrightarrow R_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{w'_v} \twoheadrightarrow \widehat{\mathcal{O}}_{X_{w'_v(\mathbf{h}_{\bar{v}})}^{\text{red}}, y_{w'_v}}$  factors through  $R_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{Q_v}$ , i.e.

$$\text{Spec}(\widehat{\mathcal{O}}_{X_{w'_v(\mathbf{h}_{\bar{v}})}^{\text{red}}, y_{w'_v}}) \subset \text{Spec}(R_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{Q_v}). \quad (3.4.15)$$

By discussions in §3.3.6 and (3.2.29), the underlying topological space  $\text{Spec}(\widehat{\mathcal{O}}_{X_{w'_v(\mathbf{h}_{\bar{v}})}^{\text{red}}, y_{w'_v}})$ , which as a topological space is equal to  $\text{Spec}(\overline{R}_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{w'_v})$ , is a union of non-empty cycles denoted by  $\mathfrak{Z}_{w_{\mathcal{R}_{\bar{v}}}}$  and  $\mathfrak{Z}_{w'_v}$  in §3.3.6. By our choice of  $Q_v$ ,  $w_{\mathcal{R}_{\bar{v}}}(\mathbf{h}_{\bar{v}})$  is not strictly  $Q_v$ -dominant. Then by Lemma 3.2.30 or the discussion in the end of §3.3.6,  $\mathfrak{Z}_{w_{\mathcal{R}_{\bar{v}}}}$  is not contained in  $\text{Spec}(R_{\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet}}^{Q_v})$ , this contradicts (3.4.15)!  $\square$

**Lemma 3.4.16.** *Let  $A$  be an excellent Noetherian ring,  $\mathfrak{m}$  be a maximal ideal of  $A$ ,  $J$  be the nilradical of  $A$ , and  $M$  be a finite  $A$ -module with a faithful action of  $A$ . Let  $\widehat{A}$  be the  $\mathfrak{m}$ -adic completion of  $A$ .*

1.  $\widehat{A}/\widehat{J} = \widehat{A}/\widehat{J}$  is the nilreduction of  $\widehat{A}$ .
2.  $\widehat{A}$  acts on  $\widehat{M} := \widehat{A} \otimes_A M$  faithfully and the underlying reduced scheme of the support of  $\widehat{M}$  is  $\text{Spec}(\widehat{A}/\widehat{J})$ .

*Proof.* (1) The sequence  $0 \rightarrow \widehat{J} \rightarrow \widehat{A} \rightarrow \widehat{A}/\widehat{J} \rightarrow 0$  is exact ([AM69, Prop. 10.12]). Moreover,  $\widehat{A}/\widehat{J}$  is reduced ([Sta22, Tag 07NZ]) and  $\widehat{J}$  is nilpotent. Hence  $\widehat{J}$  is the nilradical of  $\widehat{A}$ .

(2) We have a natural injection  $A \hookrightarrow \text{Hom}_A(M, M)$  of finite  $A$ -modules. Tensoring with  $\widehat{A}$ , we get an injection  $\widehat{A} \hookrightarrow \text{Hom}_A(M, M) \otimes_A \widehat{A} = \text{Hom}_{\widehat{A}}(\widehat{M}, \widehat{M})$  ([GD60, Cor. 0.7.3.4]) of  $\widehat{A}$ -modules by the flatness of  $\widehat{A}$  over  $A$  ([AM69, Prop. 10.14]). The injection means that  $\widehat{A}$  acts faithfully on  $\widehat{M}$ . Hence the support of  $\widehat{M}$  as a  $\widehat{A}$ -module is  $\text{Spec}(\widehat{A})$  and the underlying reduced subscheme is  $\text{Spec}(\widehat{A}/\widehat{J})$  by (1).  $\square$

The remaining steps mainly rely on the local irreducibility of the trianguline variety at generic points and the crystalline deformation spaces introduced in the proof of Theorem 3.4.1. The following proposition reduces the existence of companion constituents in the generic crystalline cases to the existence of companion points.

**Proposition 3.4.17.** *Let  $x = ((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z) \in X_p(\bar{\rho})(L) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  be a point such that  $\rho_p$  is generic crystalline with a refinement  $\mathcal{R}$ ,  $w \in W_{G_p}/W_{P_p}$  and the weight of  $\underline{\delta}_{\mathcal{R}, w}$  (Definition 3.4.12) is  $w w_0 \cdot \lambda$ . Let  $r_x$  be the image of  $x$  in  $\mathfrak{X}_{\infty}$  and  $\mathfrak{m}_{r_x}$  be the corresponding maximal ideal of  $R_{\infty}[\frac{1}{p}]$ . Then*

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1} \right), \Pi_{\infty}^{\text{an}}[\mathfrak{m}_{r_x}] \right) \neq 0.$$

*Proof.* The method is similar to **Step 8** and **Step 9** in the proof of [BHS19, Thm. 5.3.3]. We will firstly prove the result for the case when  $w = w_{\mathcal{R}}$  which will be a consequence of (3.4.3) and Theorem 3.4.1. For general points, notice that the appearance of the companion constituents is equivalent to that  $x$  is in the Zariski closed subspace  $Y_p(\bar{\rho})_{w w_0 \cdot \lambda}$  of  $X_p(\bar{\rho})$  constructed in the beginning of §3.4.4. Thus, it suffices to prove that  $x$  lies in the closure of generic crystalline points in  $Y_p(\bar{\rho})_{w w_0 \cdot \lambda}$  satisfying  $w = w_{\mathcal{R}}$ . This can be achieved using the variants of the crystalline deformation space defined in the proof of Theorem 3.4.1.

First, assume  $w W_{P_p} = w_{\mathcal{R}} W_{P_p}$  and  $\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1} \right), \Pi_{\infty}^{\text{an}}[\mathfrak{m}_{r_x}] \right) = 0$ , then there exists  $w' \in W_{G_p}/W_{P_p}$  and  $w' < w$  in  $W_{G_p}/W_{P_p}$  such that

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \bar{L}(-w' w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{B_p}^{-1} \right), \Pi_{\infty}^{\text{an}}[\mathfrak{m}_{r_x}] \right) \neq 0$$

since  $\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p}(\underline{\delta}_{\mathcal{R}, w}), \Pi_{\infty}^{\text{an}}[\mathfrak{m}_{r_x}] \right) \neq 0$ . Therefore  $\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p}(\underline{\delta}_{\mathcal{R}, w'}), \Pi_{\infty}^{\text{an}}[\mathfrak{m}_{r_x}] \right) \neq 0$ . By (3.4.3), the point  $((\rho_p, \underline{\delta}_{\mathcal{R}, w'}), z) \in (\mathfrak{X}_{\bar{\rho}_p} \times \hat{T}_{p, L}) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  is in  $X_p(\bar{\rho})$ . Then the point  $(\rho_p, z^{w'(\mathbf{h})} \text{unr}(\varphi))$  is in  $X_{\text{tri}}(\bar{\rho}_p)$ . This is not possible by Theorem 3.3.17. Hence the conclusion holds in the case when  $w = w_{\mathcal{R}}$ .

In general, by Theorem 3.3.17 and the generic assumption on  $\rho_p$ ,  $(\rho_p, z^{w(\mathbf{h})} \text{unr}(\varphi))$  lies in a unique irreducible component  $X$  of  $X_{\text{tri}}(\bar{\rho}_p)$ . We assume that  $x$  lies in an irreducible component of  $X_p(\bar{\rho})$  of the form  $\iota(X) \times \mathfrak{X}^p \times \mathbb{U}^g$  for an irreducible component  $\mathfrak{X}^p$  of  $\mathfrak{X}_{\bar{\rho}^p}$ .

In the proof of Theorem 3.4.1, we have constructed crystalline deformation spaces  $\widetilde{W}_{\bar{\rho}_v, w_v}^{\mathbf{h}_{\bar{v}} - \text{cr}}$  (resp. the closure  $\overline{\widetilde{W}_{\bar{\rho}_v, w_v}^{\mathbf{h}_{\bar{v}} - \text{cr}}}$ ), which roughly parameterizing the pairs  $(\rho_v, \mathcal{R}_v)$  of generic crystalline deformations with refinements satisfying that  $w_{\mathcal{R}_v} = w_v$  (resp.  $w_{\mathcal{R}_v} \leq w_v$ ). We have also defined morphisms  $\iota_{\mathbf{h}_{\bar{v}}, w_v} : \widetilde{W}_{\bar{\rho}_v, w_v}^{\mathbf{h}_{\bar{v}} - \text{cr}} \rightarrow X_{\text{tri}}(\bar{\rho}_v)$  in the end of the proof of Theorem 3.4.1 sending  $(\rho_v, \mathcal{R}_v = \varphi_v)$  to  $(\rho_v, z^{w_v(\mathbf{h}_{\bar{v}})} \text{unr}(\varphi_v))$ . Let  $\iota_{\mathbf{h}, w} := \prod_{v \in S_p} \iota_{\mathbf{h}_{\bar{v}}, w_v} : \prod_{v \in S_p} \widetilde{W}_{\bar{\rho}_v, w_v}^{\mathbf{h}_{\bar{v}} - \text{cr}} \rightarrow \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_v) = X_{\text{tri}}(\bar{\rho}_p)$ . Then the point  $(\rho_p, z^{w(\mathbf{h})} \text{unr}(\varphi))$  is in the image of  $\iota_{\mathbf{h}, w}$  since  $w \geq w_{\mathcal{R}}$ . Denote by  $\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} = \prod_{v \in S_p} \widetilde{W}_{\bar{\rho}_v, w_v}^{\mathbf{h}_{\bar{v}} - \text{cr}}$  and  $\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} = \prod_{v \in S_p} \widetilde{W}_{\bar{\rho}_v, w_v}^{\mathbf{h}_{\bar{v}} - \text{cr}}$ .

We take an affinoid neighbourhood  $U$  of  $(\rho_p, z^{w(\mathbf{h})} \text{unr}(\varphi))$  in  $X$  and pick a small open affinoid  $V \subset \iota_{\mathbf{h}, w}^{-1}(U)$  such that  $(\rho_p, z^{w(\mathbf{h})} \text{unr}(\varphi)) \in \iota_{\mathbf{h}, w}(V)$ . Then  $\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} \cap V$  is Zariski open dense in  $V$ . Points in  $\iota \circ \iota_{\mathbf{h}, w}(\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} \cap V) \times \mathfrak{X}^p \times \mathbb{U}^g \subset X_p(\bar{\rho})_{w w_0 \cdot \lambda} \cap (\iota(X) \times \mathfrak{X}^p \times \mathbb{U}^g)$  are generic crystalline. Hence by the discussion above for the case  $w = w_{\mathcal{R}}$ , we have  $\iota \circ \iota_{\mathbf{h}, w}(\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} \cap V) \times \mathfrak{X}^p \times \mathbb{U}^g \subset Y_p(\bar{\rho})_{w w_0 \cdot \lambda}$ , i.e.

$$(\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} \cap V) \times \mathfrak{X}^p \times \mathbb{U}^g \subset ((\iota \circ \iota_{\mathbf{h}, w}) \times \text{id} \times \text{id})^{-1}(Y_p(\bar{\rho})_{w w_0 \cdot \lambda}).$$

Since  $Y_p(\bar{\rho})_{w w_0 \cdot \lambda}$  is Zariski closed in  $X_p(\bar{\rho})$ ,  $((\iota \circ \iota_{\mathbf{h}, w}) \times \text{id} \times \text{id})^{-1}(Y_p(\bar{\rho})_{w w_0 \cdot \lambda}) \cap (V \times \mathfrak{X}^p \times \mathbb{U}^g)$  is Zariski closed in  $V \times \mathfrak{X}^p \times \mathbb{U}^g$ . Hence  $V \times \mathfrak{X}^p \times \mathbb{U}^g \subset ((\iota \circ \iota_{\mathbf{h}, w}) \times \text{id} \times \text{id})^{-1}(Y_p(\bar{\rho})_{w w_0 \cdot \lambda})$ . This implies  $((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z) \in Y_p(\bar{\rho})_{w w_0 \cdot \lambda}$ .  $\square$

Now we can prove our main theorem.

**Theorem 3.4.18.** *Assume that there exists  $w' \in W_{G_p}/W_{P_p}$  such that the point*

$$((\rho_p, \underline{\delta}_{\mathcal{R}, w'}), z) \in \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g)$$

*is in  $X_p(\bar{\rho})(L)$  where  $\rho_p$  is generic crystalline and  $\mathcal{R}$  is a refinement of  $\rho_p$ . Then  $((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z) \in X_p(\bar{\rho})(L)$  if and only if  $w \geq w_{\mathcal{R}}$  in  $W_{G_p}/W_{P_p}$ .*

*Proof.* The “only if” part follows from Theorem 3.4.1.

By [BHS17a, Thm. 5.5] and the assumption  $((\rho_p, \underline{\delta}_{\mathcal{R}, w'}), z) \in X_p(\bar{\rho})$ , we can assume

$$((\rho_p, \underline{\delta}_{\mathcal{R}, w_0}), z) \in X_p(\bar{\rho}).$$

We prove the “if” part by descending induction on the integers  $\ell \leq \lg_{P_p}(w_0)$  for the following induction hypothesis:

$\mathcal{H}_\ell$  : if  $y_{w_0} = ((\rho_p, \underline{\delta}_{\mathcal{R}, w_0}), z) \in X_p(\bar{\rho})$  is a generic crystalline point of Hodge-Tate weights  $\mathbf{h}$  with a refinement  $\mathcal{R}$ , then for any  $w$  such that  $w \geq w_{\mathcal{R}}$  and  $\lg_{P_p}(w) \geq \ell$ , the point  $y_w := ((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z) \in \iota(X_{\text{tri}}(\bar{\rho}_p)) \times \mathfrak{X}_{\bar{p}^p} \times \mathbb{U}^g$  is in  $X_p(\bar{\rho})$ .

For  $\ell = \lg_{P_p}(w_0)$ , there is nothing to prove. We now assume  $\mathcal{H}_\ell$  and prove  $\mathcal{H}_{\ell-1}$ .

Firstly, the assertion of  $\mathcal{H}_{\ell-1}$  holds under  $\mathcal{H}_\ell$  automatically for any generic crystalline  $y_{w_0}$  and  $w$  if  $\lg_{P_p}(w_{\mathcal{R}}) \geq \ell$ . By Proposition 3.4.13,  $\mathcal{H}_\ell$  implies that if  $y_{w_0}$  as above is in  $X_p(\bar{\rho})$  and  $\lg_{P_p}(w_{\mathcal{R}}) = \ell - 1$ , then  $y_{w_{\mathcal{R}}} \in X_p(\bar{\rho})$ . By Proposition 3.4.17,  $y_{w_{\mathcal{R}}} \in Y_p(\bar{\rho})_{w_{\mathcal{R}} w_0 \cdot \lambda}$  (Proposition 3.4.13 and 3.4.17 are proved for  $L$ -points, but the equivalent statements for companion constituents can be proved in the same way after enlarging the coefficient field  $L$ ).

Thus, the assertion of  $\mathcal{H}_{\ell-1}$  holds at least for generic crystalline points  $y_{w_0}$  such that  $\lg_{P_p}(w_{\mathcal{R}}) = \ell - 1$ . For more general crystalline generic points  $y_{w_0}$ ,  $w \geq w_{\mathcal{R}}$  and  $\lg_{P_p}(w) \geq \ell - 1$ , we will show that  $y_w$  lies in the closure of points  $y'_w = ((\rho'_p, \underline{\delta}_{\mathcal{R}', w}), z)$ , the companion points of generic crystalline points of the form  $y'_{w_0} = ((\rho'_p, \underline{\delta}_{\mathcal{R}', w_0}), z) \in X_p(\bar{\rho})$  such that  $\lg_{P_p}(w_{\mathcal{R}'}) = \ell - 1$ . Since  $y'_{w_0}$  are in  $X_p(\bar{\rho})$  by the previous discussions,  $y_w$  will be also in  $X_p(\bar{\rho})$ .

We take an arbitrary  $y_{w_0} := ((\rho_p, \underline{\delta}_{\mathcal{R}, w_0}), z) \in X_p(\bar{\rho})$  as in the hypothesis  $\mathcal{H}_{\ell-1}$ , we need to prove that for any  $w$  such that  $w \geq w_{\mathcal{R}}$ ,  $\lg_{P_p}(w) = \ell - 1$ , we have  $y_w \in X_p(\bar{\rho})$ . We may assume  $\lg_{P_p}(w_{\mathcal{R}}) \leq \ell - 1$  and by proving the equivalent statement on companion constituents, we may assume  $y_{w_0}$  is an  $L$ -point. Recall as in the proof of Theorem 3.4.1 or Proposition 3.4.17, for each  $v \in S_p$ , there is a variant of crystalline deformation space  $\iota_{\mathbf{h}_{\tilde{v}}, w_v, 0} : \widetilde{W}_{\bar{\rho}_{\tilde{v}}}^{\mathbf{h}_{\tilde{v}} - \text{cr}} \hookrightarrow X_{\text{tri}}(\bar{\rho}_{\tilde{v}})$  with the image consisting of generic crystalline points with weights  $w_{v,0}(\mathbf{h}_{\tilde{v}})$  and for each  $w \in (\mathcal{S}_n)^{\Sigma_v}/W_{P_v}$ , there exists a Zariski locally closed subset  $\widetilde{W}_{\bar{\rho}_{\tilde{v}}, w_v}^{\mathbf{h}_{\tilde{v}} - \text{cr}}$  and its closure  $\overline{\widetilde{W}_{\bar{\rho}_{\tilde{v}}, w_v}^{\mathbf{h}_{\tilde{v}} - \text{cr}}} = \bigcup_{w'_v \leq w_v} \widetilde{W}_{\bar{\rho}_{\tilde{v}}, w'_v}^{\mathbf{h}_{\tilde{v}} - \text{cr}}$  with an injection  $\iota_{\mathbf{h}_{\tilde{v}}, w_v} : \overline{\widetilde{W}_{\bar{\rho}_{\tilde{v}}, w_v}^{\mathbf{h}_{\tilde{v}} - \text{cr}}} \hookrightarrow X_{\text{tri}}(\bar{\rho}_{\tilde{v}})$  sending  $(\rho_{\tilde{v}}, \underline{\varphi}_{\tilde{v}})$  to  $(\rho_{\tilde{v}}, z^{w_v(\mathbf{h}_{\tilde{v}})} \text{unr}(\underline{\varphi}_{\tilde{v}}))$ . The image of  $\iota_{\mathbf{h}_{\tilde{v}}, w_v}$  consists of generic crystalline points of the weight  $w_v(\mathbf{h}_{\tilde{v}})$  such that the relative positions of the trianguline filtrations and the Hodge filtrations are parameterized by some  $w'_v \leq w_v$  in  $(\mathcal{S}_n)^{\Sigma_v}/W_{P_v}$ . We let  $\iota_{\mathbf{h}, w_0} := \prod_{v \in S_p} \iota_{\mathbf{h}_{\tilde{v}}, w_v, 0}$  (resp.  $\iota_{\mathbf{h}, w} := \prod_{v \in S_p} \iota_{\mathbf{h}_{\tilde{v}}, w_v}$ ) be the embedding of  $\widetilde{W}_{\bar{\rho}_p}^{\mathbf{h} - \text{cr}} := \prod_{v \in S_p} \widetilde{W}_{\bar{\rho}_{\tilde{v}}}^{\mathbf{h}_{\tilde{v}} - \text{cr}}$  (resp.  $\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} := \prod_{v \in S_p} \widetilde{W}_{\bar{\rho}_{\tilde{v}}, w_v}^{\mathbf{h}_{\tilde{v}} - \text{cr}}$ ) into  $X_{\text{tri}}(\bar{\rho}_p)$ .

Take  $w \in W_{G_p}/W_{P_p}$  such that  $w \geq w_{\mathcal{R}}$  and  $\lg_{P_p}(w) = \ell - 1$ . Let  $\iota(X) \times \mathfrak{X}^p \times \mathbb{U}^g$  be an irreducible component of  $X_p(\bar{\rho})$  passing through  $y_{w_0}$  where  $X$  is the irreducible component of  $X_{\text{tri}}(\bar{\rho}_p)$  passing through  $\iota^{-1}(\rho_p, \underline{\delta}_{\mathcal{R}, w_0})$  and  $\mathfrak{X}^p$  is an irreducible component of  $\mathfrak{X}_{\bar{p}^p}$ . We take an affinoid open neighbourhood  $U$  of the point  $\iota^{-1}(\rho_p, \underline{\delta}_{\mathcal{R}, w_0})$  in  $X$ . Since  $w_{\mathcal{R}} \leq w$ , the point  $\iota^{-1}(\rho_p, \underline{\delta}_{\mathcal{R}, w_0}) = (\rho_p, z^{w_0(\mathbf{h})} \text{unr}(\underline{\varphi}))$  is in  $\iota_{\mathbf{h}, w_0}(\overline{\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}}})$ . Hence the intersection  $V := \widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}} \cap \iota_{\mathbf{h}, w_0}^{-1}(U)$  is Zariski open dense in the affinoid  $\overline{V} := \overline{\widetilde{W}_{\bar{\rho}_p, w}^{\mathbf{h} - \text{cr}}} \cap \iota_{\mathbf{h}, w_0}^{-1}(U)$  (we take  $U$  small enough so that  $\iota_{\mathbf{h}, w_0}^{-1}(U) \cap \widetilde{W}_{\bar{\rho}_p}^{\mathbf{h} - \text{cr}} = \iota_{\mathbf{h}, w_0}^{-1}(U) \cap \widetilde{\mathfrak{X}}_{\bar{\rho}_p}^{\mathbf{h} - \text{cr}}$  where  $\widetilde{\mathfrak{X}}_{\bar{\rho}_p}^{\mathbf{h} - \text{cr}} := \prod_{v \in S_p} \widetilde{\mathfrak{X}}_{\bar{\rho}_{\tilde{v}}}^{\mathbf{h}_{\tilde{v}} - \text{cr}}$ ) and  $(\rho_p, \underline{\varphi}) \in \overline{V}$ . Then  $(\iota \circ \iota_{\mathbf{h}, w_0}(V)) \times \mathfrak{X}^p \times \mathbb{U}^g$  is a subset in  $X_p(\bar{\rho})$ . Any point in the subset satisfies the condition in  $\mathcal{H}_{\ell-1}$  and for these points,  $w_{\mathcal{R}} = w$ ,  $\lg_{P_p}(w_{\mathcal{R}}) = \ell - 1$ . Hence their companion



points  $(\iota \circ \iota_{h,w}(V)) \times \mathfrak{X}^p \times \mathbb{U}^g$  is contained in  $X_p(\bar{\rho})$  by the discussion in the beginning of the proof where we have used Proposition 3.4.13 and  $\mathcal{H}_\ell$ . We get that the preimage of  $X_p(\bar{\rho})$  under the map  $(\iota \circ \iota_{h,w}) \times \text{id} \times \text{id} : \widetilde{W}_{\bar{\rho},w}^{\mathbf{h}-\text{cr}} \times \mathfrak{X}^p \times \mathbb{U}^g \rightarrow \iota(X_{\text{tri}}(\bar{\rho}_p)) \times \mathfrak{X}^p \times \mathbb{U}^g$  contains the Zariski closure of  $V \times \mathfrak{X}^p \times \mathbb{U}^g$  inside  $\bar{V} \times \mathfrak{X}^p \times \mathbb{U}^g$  which is equal to  $\bar{V} \times \mathfrak{X}^p \times \mathbb{U}^g$ . This means that the point  $y_w = (\iota(\rho_p, z^{w(\mathbf{h})} \text{unr}(\varphi)), z)$  is in  $X_p(\bar{\rho})$ . Hence  $\mathcal{H}_{\ell-1}$  holds.  $\square$

*Remark 3.4.19.* In the proof of [BHS19, Thm. 5.3.3], results in Theorem 3.4.18 were obtained under the assumption, in addition to regular weights, that  $z$  is in the smooth locus of  $\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g$  which is certainly expected to be not necessary. Our proof realizes this expectation!

Finally, we state the theorem for  $p$ -adic automorphic forms. Recall that there is a closed embedding  $Y(U^p, \bar{\rho}) \hookrightarrow X_p(\bar{\rho})$  from the eigenvariety to the patched one and remark that we always assume all the hypotheses in §3.4.2. Proposition 3.4.17 and Theorem 3.4.18 immediately imply the following theorem.

**Theorem 3.4.20.** *Let  $\rho : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_n(L)$  be a continuous representation such that  $\rho_p = (\rho|_{\mathcal{G}_{F_v}})_{v \in S_p}$  is generic crystalline. Assume that  $\rho$  corresponds to a point  $(\rho, \underline{\delta}_{\mathcal{R},w'}) \in Y(U^p, \bar{\rho})(L) \subset \text{Spf}(R_{\bar{\rho},S})^{\text{rig}} \times_L \widehat{T}_{p,L}$  where  $\mathcal{R}$  is a refinement of  $\rho_p$  and  $w' \in W_{G_p}/W_{P_p}$ . Let  $\mathfrak{m}_\rho$  be the maximal ideal of  $R_{\bar{\rho},S}[\frac{1}{p}]$  corresponding to  $\rho$ . Let  $\lambda$  be the weight of  $\underline{\delta}_{\mathcal{R},w_0}$ . Then  $(\rho, \underline{\delta}_{\mathcal{R},w}) \in Y(U^p, \bar{\rho})$  if and only if  $w \geq w_{\mathcal{R}}$  in  $W_{G_p}/W_{P_p}$ , and for all  $w \geq w_{\mathcal{R}}$ ,*

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \bar{L}(-ww_0 \cdot \lambda), \underline{\delta}_{\mathcal{R},\text{sm}} \delta_{B_p}^{-1} \right), \widehat{S}(U^p, L)_{\mathfrak{m}_S}^{\text{an}}[\mathfrak{m}_\rho] \right) \neq 0.$$

*Proof.* Recall the action of  $R_{\bar{\rho},S}$  on  $\widehat{S}(U^p, L)_{\mathfrak{m}_S}$  factors through a quotient  $R_{\bar{\rho},S}$ . And there is an ideal  $\mathfrak{a}$  of  $R_\infty$ , a surjection  $R_\infty/\mathfrak{a}R_\infty \twoheadrightarrow R_{\bar{\rho},S}$  and an isomorphism

$$\widehat{S}(U^p, L)_{\mathfrak{m}_S} \simeq \Pi_\infty[\mathfrak{a}]$$

that is compatible with the action of  $R_\infty$  and  $R_{\bar{\rho},S}$  on the two sides.

Suppose that under the closed embedding  $Y(U^p, \bar{\rho}) \hookrightarrow X_p(\bar{\rho})$ ,  $(\rho, \underline{\delta}_{\mathcal{R},w'})$  is sent to the point  $x = ((\rho_p, \underline{\delta}_{\mathcal{R},w'}), z) \in X_p(\bar{\rho})$ . Let  $r_x = (\rho_p, z)$  and let  $\mathfrak{m}_{r_x}$  be the maximal ideal of  $R_\infty[\frac{1}{p}]$  corresponding to  $r_x$ . Then  $\mathfrak{m}_{r_x}$  contains  $\mathfrak{a}$ . Hence there is an isomorphism of  $G_p$ -representations

$$\widehat{S}(U^p, L)_{\mathfrak{m}_S}^{\text{an}}[\mathfrak{m}_\rho] \simeq \Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}].$$

Note that  $(\rho, \underline{\delta}_{\mathcal{R},w}) \in Y(U^p, \bar{\rho})$  is equivalent to  $\text{Hom}_{T_p}(\underline{\delta}_{\mathcal{R},w}, J_{B_p}(\widehat{S}(U^p, L)_{\mathfrak{m}_S}^{\text{an}}[\mathfrak{m}_\rho])) \neq 0$ . Hence the assertions of the theorem follow from similar statements replacing  $\widehat{S}(U^p, L)_{\mathfrak{m}_S}^{\text{an}}[\mathfrak{m}_\rho]$  by  $\Pi_\infty^{\text{an}}[\mathfrak{m}_{r_x}]$ , which are true by Proposition 3.4.17 and Theorem 3.4.18.  $\square$

## 3.5 The partial eigenvariety

In this section, we use Ding's partial eigenvariety ([Din19c]) to prove some general result on the relationship between partially classical finite slope locally analytic representations and partially de Rham properties of trianguline  $(\varphi, \Gamma)$ -modules (Theorem 3.5.28) which has been used for Theorem 3.4.10. Most results around the partial eigenvariety in this section except for those in §3.5.6 and §3.5.7 are essentially due to Ding, and we adapt his results for the patching module.

### 3.5.1 Notation

We keep the notation and assumptions in §3.4.2 and §3.4.3.

For each  $v \in S_p$ , let  $Q_v$  be a standard parabolic subgroup of  $\text{GL}_{n/F_v}$  (not  $\text{Res}_{F_v/\mathbb{Q}_p}(\text{GL}_{n/F_v}) \times_{\mathbb{Q}_p} L$ !) containing the Borel subgroup of upper-triangular matrices. Write  $Q_v = M_{Q_v} N_{Q_v}$  for the

standard Levi decomposition where  $M_{Q_v}$  is the standard Levi subgroup containing the diagonal torus and  $N_{Q_v}$  is the unipotent radical. For an algebraic reductive group  $H$ , we use the notation  $H'$  to denote the derived subgroup of  $H$ . We also use the same notation  $Q_v, M_{Q_v}, N_{Q_v}, M'_{Q_v}$  to denote the groups of  $F_{\bar{v}}$ -points which are identified with subgroups of the  $p$ -adic Lie group  $G_v$  via  $i_{\bar{v}} : G(F_v^+) \simeq \mathrm{GL}_n(F_{\bar{v}})$ . We have  $p$ -adic Lie groups  $Q_p := \prod_{v \in S_p} Q_v, M_{Q_p} := \prod_{v \in S_p} M_{Q_v}, M'_{Q_p} := \prod_{v \in S_p} M'_{Q_v}$ , etc.. Assume for each  $v \in S_p$ , the standard Levi  $M_{Q_v}$  is the group of diagonal block matrices of  $\mathrm{GL}_n/F_{\bar{v}}$  of the form  $\mathrm{GL}_{q_{v,1}/F_{\bar{v}}} \times \cdots \times \mathrm{GL}_{q_{v,t_v}/F_{\bar{v}}}$  where  $n = q_{v,1} + \cdots + q_{v,t_v}$ . We let  $\tilde{q}_{v,i} = \sum_{j=1}^i q_{v,j}$  for any  $v, 1 \leq i \leq t_v$  and let  $\tilde{q}_{v,0} = 0$ . Let  $B_{Q_p} = B_p \cap M_{Q_p}, \bar{B}_{Q_p} = \bar{B}_p \cap M_{Q_p}$  and write  $B_{Q_p} = T_p U_{Q_p}$  (resp.  $\bar{B}_{Q_p} = T_p \bar{U}_{Q_p}$ ) for the Levi decomposition of  $B_{Q_p}$  (resp.  $\bar{B}_{Q_p}$ ).

We have  $p$ -adic Lie subgroups  $T'_{Q_p} := T_p \cap M'_{Q_p}, B'_{Q_p} := T'_{Q_p} U_{Q_p}$  and  $Z_{M_{Q_p}}$  the center of  $M_{Q_p}$ .

Recall  $\mathfrak{g} = \prod_{v \in S_p} \prod_{\tau \in \Sigma_v} \mathfrak{g}_{\tau}$  and similarly we let  $\mathfrak{m}_{Q_p} = \prod_{v \in S_p} \prod_{\tau \in \Sigma_v} \mathfrak{m}_{Q_v, \tau}$  (resp.  $\mathfrak{m}'_{Q_p} = \prod_{v \in S_p} \prod_{\tau \in \Sigma_v} \mathfrak{m}'_{Q_v, \tau}$ , resp.  $\mathfrak{t}'_{Q_p} = \prod_{v \in S_p} \prod_{\tau \in \Sigma_v} \mathfrak{t}'_{Q_v, \tau}$ , resp.  $\mathfrak{b}'_{Q_p} = \prod_{v \in S_p} \prod_{\tau \in \Sigma_v} \mathfrak{b}'_{Q_v, \tau}$ , resp.  $\mathfrak{z}_{M_{Q_p}} = \prod_{v \in S_p} \prod_{\tau \in \Sigma_v} \mathfrak{z}_{M_{Q_v, \tau}}$ , etc.) be the base change to  $L$  of the  $\mathbb{Q}_p$ -Lie algebra of the  $p$ -adic Lie group  $M_{Q_p}$  (resp.  $M'_{Q_p}$ , resp.  $T'_{Q_p}$ , resp.  $B'_{Q_p}$ , resp.  $Z_{M_{Q_p}}$ , etc.). We have  $\mathfrak{t} = \mathfrak{t}'_{Q_p} \times \mathfrak{z}_{M_{Q_p}}, \mathfrak{m}_{Q_p} = \mathfrak{m}'_{Q_p} \times \mathfrak{z}_{M_{Q_p}}$  and the morphism  $Z_{M_{Q_p}} \times M'_{Q_p} \rightarrow M_{Q_p}$  is locally an isomorphism.

We pick an arbitrary nonempty subset  $J$  of  $\Sigma_p$  and set  $J_v = J \cap \Sigma_v$  for  $v \in S_p$ . We let  $\mathfrak{m}'_{Q_p, J} := \prod_{v \in S_p} \prod_{\tau \in J_v} \mathfrak{m}'_{Q_v, \tau}, \mathfrak{t}'_{Q_p, J} := \prod_{v \in S_p} \prod_{\tau \in J} \mathfrak{t}'_{Q_v, \tau}$ , etc.. We will only need the case when  $|J| = 1$  but adding this extra assumption will not simplify the notation.

We fix a uniform pro- $p$  normal subgroup  $H_p = \prod_{v \in S_p} H_v$  of the maximal compact subgroup  $K_p = \prod_{v \in S_p} K_v = \prod_{v \in S_p} i_{\bar{v}}^{-1}(\mathrm{GL}_n(\mathcal{O}_{F_{\bar{v}}}))$  of  $G_p$  where each  $H_v$  is good  $F_{\bar{v}}$ -analytic with an Iwahori decomposition as in [Eme06b, Def. 4.1.3]. Let  $U_{Q_p, 0} = U_{Q_p} \cap H_p, M_{Q_p, 0} = M_{Q_p} \cap H_p, N_{Q_p, 0} = N_{Q_p} \cap H_p, T'_{Q_p, 0} = T'_{Q_p} \cap H_p$ , etc. and define  $U_{Q_v, 0}$ , etc. similarly. Let  $Z_{M_{Q_p}}^+ = \prod_{v \in S_p} Z_{M_{Q_v}}^+$  (resp.  $T_p^+ = \prod_{v \in S_p} T_v^+$ , resp.  $T_{M_{Q_p}}^+ = \prod_{v \in S_p} T_{M_{Q_v}}^+$ ) be the submonoid of  $Z_{M_{Q_p}}$  (resp.  $T_p$ , resp.  $T_p$ ) consisting of elements  $t$  such that  $tN_{Q_p, 0}t^{-1} \subset N_{Q_p, 0}$  (resp.  $tN_{B_p, 0}t^{-1} \subset N_{B_p, 0}$ , resp.  $tU_{Q_p, 0}t^{-1} \subset tU_{Q_p, 0}$ ). We use the notation  $(-)_\mathrm{fs}$  to denote Emerton's finite slope part functor [Eme06b, Def. 3.2.1] with respect to one of the submonoids  $Z_{M_{Q_p}}^+, T_p^+$  or  $T_{M_{Q_p}}^+$  of  $T_p$  where the exact meaning will be clear from the context.

Recall that  $\Pi_\infty$  is the patched representation of  $G_p$  and  $\Pi_\infty^{\mathrm{an}}$  denotes the subspace of locally  $R_\infty$ -analytic vectors of  $\Pi_\infty$ . We have an integer  $q$ , a ring  $S_\infty$  in [BHS17b, §3.2] and we fix an isomorphism  $S_\infty \simeq \mathcal{O}_L[[\mathbb{Z}_p^q]]$ . If  $H$  is a group, we denote by  $\tilde{H} := H \times \mathbb{Z}_p^q$ . Then  $\Pi_\infty$  is equipped with an action of  $\tilde{G}_p$  from the action of  $S_\infty \rightarrow R_\infty$  ([BHS17b, §3.1]). Since the patching module  $M_\infty$  is finite projective over  $S_\infty[[K_p]]$  ([BHS17b, Thm. 3.5]), it is finite free over  $S_\infty[[H_p]]$  as the ring  $S_\infty[[H_p]]$  is local. Hence  $\Pi_\infty|_{\tilde{H}_p} \simeq \mathcal{C}(\tilde{H}_p, L)^m$  for some integer  $m$  where  $\mathcal{C}(\tilde{H}_p, L)$  denotes the space of continuous functions over  $\tilde{H}_p$  with coefficients in  $L$ . By [BHS17b, Prop. 3.4],  $J_{Q_p}(\Pi_\infty^{\mathrm{an}})$  is an essentially admissible locally  $\mathbb{Q}_p$ -analytic representation ([Eme17, Def. 6.4.9]) of  $\mathbb{Z}_p^s \times M_{Q_p}$  for some surjection  $\mathcal{O}_L[[\mathbb{Z}_p^s]] \twoheadrightarrow R_\infty$  where  $J_{Q_p}$  is the Emerton's Jacquet module functor with respect to the parabolic subgroup  $Q_p$ . By definition ([Eme06b, Def. 3.4.5]),  $J_{Q_p}(\Pi_\infty^{\mathrm{an}})$  is the finite slope part of  $\Pi_\infty^{\mathrm{an}, N_{Q_p, 0}}$  with respect to the action of the submonoid  $Z_{M_{Q_p}}^+$  of  $Z_{M_{Q_p}}$ .

### 3.5.2 The partial Emerton-Jacquet module functor

We recall the notion of locally  $\Sigma_p \setminus J$ -analytic representations introduced in [Sch10, §2] (also see [Din17a, §6.1] or [Din19a, Appendix B]) and the partial Emerton-Jacquet module defined in [Din19c, §2.2.2].

Suppose that  $V$  is a locally  $\mathbb{Q}_p$ -analytic representation of  $M_{Q_p} = \prod_{v \in S_p} M_{Q_v}$  over  $L$ . A

vector  $v \in V$  is called locally  $\Sigma_p \setminus J$ -analytic with respect to the derived subgroup  $M'_{Q_p}$  if the differential of the locally analytic function on  $M'_{Q_p} : g \mapsto gv$  at the identity  $e \in M'_{Q_p}$ , which *a priori* lies in  $\text{Hom}_{\mathbb{Q}_p}(T_e M'_{Q_p}, V) = \text{Hom}_L(\mathfrak{m}'_{Q_p}, V)$ , vanishes on  $\mathfrak{m}'_{Q_p, J} = \prod_{v \in S_p} \prod_{\tau \in J_v} \mathfrak{m}'_{Q_v, \tau}$ . Remark that since the adjoint action of  $M'_{Q_v}$  on  $\mathfrak{m}'_{Q_v}$  is  $F_v$ -linear,  $\text{Ad}(M'_{Q_p})\mathfrak{m}'_{Q_p, J} = \mathfrak{m}'_{Q_p, J}$ .

We fix a tuple  $\tilde{\lambda}_J = (\tilde{\lambda}_\tau)_{\tau \in J} = (\tilde{\lambda}_{\tau, 1}, \dots, \tilde{\lambda}_{\tau, n})_{\tau \in J} \in (\mathbb{Z}^n)^J$  such that  $\tilde{\lambda}_{\tau, i} \geq \tilde{\lambda}_{\tau, j}$  for all  $i \geq j, \tau \in J$ . We identify  $\tilde{\lambda}_J$  with an element in  $\mathfrak{t}^* := \text{Hom}_L(\mathfrak{t}, L)$  that vanishes on  $\prod_{\tau \in \Sigma_p \setminus J} \mathfrak{t}_\tau$ . Denote by  $\lambda'_J$  the image of  $\tilde{\lambda}_J$  in  $(\mathfrak{t}'_{Q_p})^* := \text{Hom}_L(\mathfrak{t}'_{Q_p}, L)$ . Then there exists a unique algebraic representation of  $\prod_{v \in S_p} \text{Res}_{F_v/\mathbb{Q}_p} (M'_{Q_v}) \otimes_{\mathbb{Q}_p} L$  over  $L$  with the highest weight  $\lambda'_J$ . Let  $L_{M'_{Q_p}}(\lambda'_J)$  be the associated  $\mathbb{Q}_p$ -algebraic representation of the  $p$ -adic Lie group  $M'_{Q_p}$  over  $L$  via the embedding  $M'_{Q_p} = \prod_{v \in S_p} M'_{Q_v}(F_v) \hookrightarrow \prod_{v \in S_p} \text{Res}_{F_v/\mathbb{Q}_p} (M'_{Q_v})_L(L)$ . The  $U(\mathfrak{m}'_{Q_p})$ -module  $L_{M'_{Q_p}}(\lambda'_J)$  is the unique irreducible quotient of  $U(\mathfrak{m}'_{Q_p}) \otimes_{U(\mathfrak{b}'_{Q_p})} \lambda'_J$  or  $U(\mathfrak{m}'_{Q_p, J}) \otimes_{U(\mathfrak{b}'_{Q_p, J})} \lambda'_J$  (elements in  $\mathfrak{m}'_{Q_v, \tau}$  act as zero on the module if  $\tau \notin J_v$ ). We equip  $L_{M'_{Q_p}}(\lambda'_J)$  with an action of  $\mathfrak{m}_{Q_p} = \mathfrak{m}'_{Q_p} \oplus \mathfrak{z}_{M_{Q_p}}$  where the action of  $\mathfrak{z}_{M_{Q_p}}$  is given by  $\tilde{\lambda}_J$  and denote by  $L_{M_{Q_p}}(\tilde{\lambda}_J)$  for the  $\mathbb{Q}_p$ -algebraic  $M_{Q_p}$ -representation on  $L_{M'_{Q_p}}(\lambda'_J)$ . Let  $L_{M_{Q_p}}(\tilde{\lambda}_J)' := \text{Hom}_L(L_{M_{Q_p}}(\tilde{\lambda}_J), L)$  be the usual dual representation of  $M_{Q_p}$ .

If  $V$  is a locally  $\mathbb{Q}_p$ -analytic representation of  $M_{Q_p}$  over  $L$ , let  $(V \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J)')^{\Sigma_p \setminus J\text{-an}}$  be the closed  $L$ -subspace of  $V \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J)'$  generated by locally  $\Sigma_p \setminus J$ -analytic vectors with respect to the diagonal action of  $M'_{Q_p}$ . Then  $(V \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J)')^{\Sigma_p \setminus J\text{-an}}$  is a locally  $\Sigma_p \setminus J$ -analytic representation of  $M'_{Q_p}$  in the sense of [Sch10, Def. 2.4] and is stable under the action of  $M_{Q_p}$ . We have

$$\text{Hom}_{U(\mathfrak{m}'_{Q_p, J})}(L_{M_{Q_p}}(\tilde{\lambda}_J), V) \simeq (L_{M_{Q_p}}(\tilde{\lambda}_J)' \otimes_L V)^{\Sigma_p \setminus J\text{-an}}$$

as topological representations of  $M_{Q_p}$  (cf. [Din17a, Rem. 6.1.5]) where the action on the left hand side is the natural one as in [BHS19, §5.2].

Now assume that  $V$  is an essentially admissible locally  $\mathbb{Q}_p$ -analytic representation of  $\mathbb{Z}_p^s \times G_p$ . We will take  $V = \Pi_\infty^{\text{an}} = \Pi_\infty^{R_\infty\text{-an}}$  and the action of  $\mathbb{Z}_p^s$  is given by  $\mathcal{O}_L[[\mathbb{Z}_p^s]] \twoheadrightarrow R_\infty$  or  $s = q$  and  $\mathcal{O}_L[[\mathbb{Z}_p^q]] \xrightarrow{\sim} S_\infty \rightarrow R_\infty$ . We let  $\mathbb{Z}_p^s$  act trivially on  $L_{M_{Q_p}}(\tilde{\lambda}_J)$ . Then  $J_{Q_p}(V)$  is an essentially admissible representation of  $\mathbb{Z}_p^s \times M_{Q_p}$  by [Eme06b, Thm. 4.2.32]. We define

$$J_{Q_p}(V)_{\lambda'_J} := \text{Hom}_{U(\mathfrak{m}'_{Q_p, J})}(L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(V)) \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J)$$

equipped with the diagonal action of  $\mathbb{Z}_p^s \times M_{Q_p}$ . There is a natural  $\mathbb{Z}_p^s \times M_{Q_p}$ -equivariant morphism:

$$\text{Hom}_{U(\mathfrak{m}'_{Q_p, J})}(L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(V)) \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J) \rightarrow J_{Q_p}(V) : f \otimes v \mapsto f(v). \quad (3.5.1)$$

**Lemma 3.5.2.** *The representation  $J_{Q_p}(V)_{\lambda'_J}$  of  $\mathbb{Z}_p^s \times M_{Q_p}$  depends only on  $\lambda'_J$  (in particular on  $J$ , but not on the lift  $\tilde{\lambda}_J$ ). The morphism (3.5.1) is a closed embedding and identifies  $J_{Q_p}(V)_{\lambda'_J}$  with a closed  $\mathbb{Z}_p^s \times M_{Q_p}$ -sub-representation of  $J_{Q_p}(V)$ .*

*Proof.* For the injection, we can apply [Din17a, Prop. 6.1.3] with respect to  $M'_{Q_p}$ . The last assertion follows from the same arguments in [Din17b, Cor. B.2] and we prove it now. Since  $J_{Q_p}(V)$  is an essentially admissible representation of  $\mathbb{Z}_p^s \times M_{Q_p}$ ,  $J_{Q_p}(V) \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J)' \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J)$  is also an essentially admissible representation of  $\mathbb{Z}_p^s \times M_{Q_p}$  by Lemma 3.5.3 below. By [Eme17,

Prop. 6.4.11], the closed sub-representation  $J_{Q_p}(V)_{\lambda'_J}$  is an essentially admissible representation of  $\mathbb{Z}_p^s \times M_{Q_p}$ . Thus the injection  $J_{Q_p}(V)_{\lambda'_J} \hookrightarrow V$  of essentially admissible representations of  $\mathbb{Z}_p^s \times M_{Q_p}$  is a closed embedding by *loc. cit.*  $\square$

**Lemma 3.5.3.** *Assume that  $V$  is an essentially admissible locally analytic representation over  $L$  of a locally analytic group  $G$  where the center  $Z$  of  $G$  is topologically finitely generated. If  $W$  is a finite-dimensional locally analytic representation over  $L$  of  $G$  on which the action of  $Z$  is trivial, then  $V \otimes_L W$  is also essentially admissible.*

*Proof.* We have  $(V \otimes_L W)' \simeq V' \otimes_L W'$  as topological vector spaces. Since  $V$  is essentially admissible, by definition, there exists a covering of  $\widehat{Z}$  by open affinoids  $\widehat{Z}_1 \subset \widehat{Z}_2 \subset \dots$  and a sequence  $\mathbb{H}_0 \supset \mathbb{H}_1 \supset \mathbb{H}_2 \dots$  of rigid analytic open subgroups with respect to a compact open subgroup  $H = \mathbb{H}_0(\mathbb{Q}_p)$  of  $G$  as in [Eme17, §5.2] such that the strong dual  $V'$ , as a coadmissible module over the Fréchet-Stein algebra  $\mathcal{C}^{\text{an}}(\widehat{Z}, L) \widehat{\otimes}_L D(H, L)$ , is isomorphic to  $\varprojlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_n, L) \widehat{\otimes}_L D(\mathbb{H}_n^\circ, H) \widehat{\otimes}_{\mathcal{C}^{\text{an}}(\widehat{Z}, L) \widehat{\otimes}_L D(H, L)} V'$  ([Eme17, Thm. 1.2.11, Def. 6.4.9]). Here  $L$  denotes the coefficient field,  $\mathcal{C}^{\text{an}}(\widehat{Z}, L)$  (resp.  $\mathcal{C}^{\text{an}}(\widehat{Z}_n, L)$ ) is the algebra of rigid analytic functions on  $\widehat{Z}$  (resp.  $\widehat{Z}_n$ ) ([Eme17, Def. 2.1.18]),  $D(H, L)$  is the algebra of locally  $\mathbb{Q}_p$ -analytic distributions on  $H$  and  $D(\mathbb{H}_n^\circ, H)$  is the strong dual of  $\mathbb{H}_n^\circ$ -analytic functions on  $H$  as [Eme06b, (4.1.2)]. The isomorphism  $\mathcal{C}^{\text{an}}(\widehat{Z}, L) \widehat{\otimes}_L D(H, L) \simeq \varprojlim_n \mathcal{C}^{\text{an}}(\widehat{Z}_n, L) \widehat{\otimes}_L D(\mathbb{H}_n^\circ, H)$  defines a weak Fréchet-Stein structure on  $\mathcal{C}^{\text{an}}(\widehat{Z}, L) \widehat{\otimes}_L D(H, L)$  ([Eme17, Def. 1.2.6, Lem. 1.1.29]). We write  $A = \mathcal{C}^{\text{an}}(\widehat{Z}, L) \widehat{\otimes}_L D(H, L)$  and  $A_n = \mathcal{C}^{\text{an}}(\widehat{Z}_n, L) \widehat{\otimes}_L D(\mathbb{H}_n^\circ, H)$ .

The Dirac distribution  $L[H] (\subset D(H, K))$  ([ST02b, §2, §3]), as well as  $L[Z] (\subset D(Z, L) \subset \mathcal{C}^{\text{an}}(\widehat{Z}, L))$  ([Eme17, Prop. 6.4.6]), acts on  $(V \otimes_L W)' = V' \otimes_L W'$  diagonally where  $L[Z]$  acts trivially on the second factor  $W'$  since  $Z$  acts trivially on  $W'$ .

Denote by  $\rho$  the action of  $H$  on  $W'$ . There are two ring homomorphisms  $\alpha : L[H] \rightarrow L[H] \otimes_L \text{End}_L(W')$ ,  $h \mapsto h \otimes 1$ ,  $\beta : L[H] \rightarrow L[H] \otimes_L \text{End}_L(W')$ ,  $h \mapsto h \otimes \rho(h)$  and a map  $\gamma : L[H] \otimes_L \text{End}_L(W') \rightarrow L[H] \otimes_L \text{End}_L(W') : h \otimes m \mapsto h \otimes \rho(h)m$  such that  $\beta = \gamma \circ \alpha$ . By [HL11, Prop. 4.4], for each (large enough)  $n$ ,  $\alpha, \beta, \gamma$  can be extended uniquely to continuous maps  $\alpha_n, \beta_n : D(\mathbb{H}_n^\circ, H) \rightarrow D(\mathbb{H}_n^\circ, H) \otimes_L \text{End}_L(W')$  and  $\gamma_n : D(\mathbb{H}_n^\circ, H) \otimes_L \text{End}_L(W') \rightarrow D(\mathbb{H}_n^\circ, H) \otimes_L \text{End}_L(W')$  such that  $\beta_n = \gamma_n \circ \alpha_n$ . Taking the complete tensor product with  $\mathcal{C}^{\text{an}}(\widehat{Z}_n, L)$ , we get similar maps  $\alpha'_n, \beta'_n : A_n \rightarrow A_n \otimes_L \text{End}_L(W')$  and  $\gamma'_n : A_n \otimes_L \text{End}_L(W') \rightarrow A_n \otimes_L \text{End}_L(W')$  extending the maps  $\alpha' = \text{id} \otimes_L \alpha, \beta' = \text{id} \otimes_L \beta : L[Z] \otimes_L L[H] \rightarrow L[Z] \otimes_L L[H] \otimes_L \text{End}_L(W')$  and  $\gamma' = \text{id} \otimes_L \gamma$ .

The tensor product  $U_n := (A_n \widehat{\otimes}_A V') \otimes_L W'$  is naturally an  $A_n \otimes_L \text{End}_L(W')$ -module where  $A_n$ , as well as  $L[Z] \otimes_L L[H]$ , acts on the second factor  $W'$  trivially. Then as in [HL11, Prop. 4.6], the map  $\beta'_n : A_n \rightarrow A_n \otimes_L \text{End}_L(W')$  equips  $U_n$  a twisted action of  $A_n$  extending the diagonal action of  $L[Z] \otimes_L L[H]$  on  $U_n$ . Since  $U_n$  is a finitely generated  $A_n \otimes_L \text{End}_L(W')$ -module and  $A_n \otimes_L \text{End}_L(W)$  is a finitely generated  $A_n$ -module with respect to the action of  $A_n$  via both  $\alpha'_n$  and  $\beta'_n$  by [HL11, Cor. 4.5], we get that  $U_n$  is a finitely generated  $A_n$ -module and we have an isomorphism  $A_n \widehat{\otimes}_{A_{n+1}} U_{n+1} \simeq U_n$  with the twisted actions. Hence  $\varprojlim_n U_n$  is a coadmissible module over the Fréchet-Stein algebra  $A$  ([Eme17, Def. 1.2.8]). The action of  $A$  extends the action of  $L[Z] \otimes_L L[H]$  on  $(V \otimes W)'$  via the isomorphism  $V' \otimes_L W' \simeq \varprojlim_n U_n$  of topological vector spaces. Such extension of the action of  $L[Z] \otimes_L L[H]$  to  $A$  on  $(V \otimes_L W)'$  is unique by [Eme17, Prop. 6.4.7(ii)] and by that  $L[H]$  is dense in  $D(H, L)$  which acts continuously on  $(V \otimes W)'$  by [ST02b, Lem. 3.1, Cor. 3.4], thus coincides with the usual action of  $A$  on  $(V \otimes_L W)'$ . We get that  $V \otimes_L W$  is an essentially admissible representation of  $G$ .  $\square$

**Lemma 3.5.4.** *There is an isomorphism*

$$J_{B_{Q_p}} \left( J_{Q_p}(V)_{\lambda'_J} \right) \simeq J_{B_{Q_p}} \left( \text{Hom}_{U(\mathfrak{m}'_{Q_p, J})} \left( L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(V) \right) \right) \otimes_L \tilde{\lambda}_J$$

*of essentially admissible locally  $\mathbb{Q}_p$ -analytic representations of  $\mathbb{Z}_p^s \times T_p$ .*

*Proof.* This can be proved by arguments in [Din19c, Cor. 2.11] together with [Din17a, Lem. 7.2.12]. Remark that  $L_{M_{Q_p}}(\tilde{\lambda}_J)$  is  $J$ -algebraic in the sense of [Din17a, §6.1.1] and  $L_{M_{Q_p}}(\tilde{\lambda}_J)^{U_{Q_p,0}} = L_{M_{Q_p}}(\tilde{\lambda}_J)^{u_{Q_p,J}}$ .  $\square$

**Lemma 3.5.5.** *We have*

$$\left( \text{Hom}_{U(\mathfrak{m}'_{Q_p,J})} \left( L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(V) \right)^{U_{Q_p,0}} \right)_{\text{fs}} \simeq \left( \left( V^{N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J - \text{an}, U_{Q_p,0}} \right)_{\text{fs}},$$

where the finite slope part is taken with respect to the Hecke action of  $T_{M_{Q_p}}^+$  for the left hand side and that of  $T_p^+$  for the right hand side.

*Proof.* By [Eme06b, Prop. 3.2.9],  $\left( V^{N_{Q_p,0}} \right)_{\text{fs}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \simeq \left( V^{N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)_{\text{fs}}$ . Since the action of  $Z_{M_{Q_p}}^+$  commutes with  $\mathfrak{m}'_{Q_p,J}$ , by [Eme06b, Prop. 3.2.11], we get

$$\text{Hom}_{U(\mathfrak{m}'_{Q_p,J})} \left( L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(V) \right) = \left( \left( V^{N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J - \text{an}} \right)_{\text{fs}}. \quad (3.5.6)$$

Now the arguments of [HL11, Thm. 5.3(2)] or [Din19c, Lem. 2.18] using [Eme06b, Prop. 3.2.4(ii)] shows that

$$\left( \left( \left( V^{N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J - \text{an}} \right)^{U_{Q_p,0}} \right)_{\text{fs}} \simeq \left( \left( V^{N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J - \text{an}, U_{Q_p,0}} \right)_{\text{fs}}.$$

Combining the isomorphism above with (3.5.6), we get the desired isomorphism.  $\square$

### 3.5.3 An adjunction formula

We prove an adjunction formula for the partial Emerton-Jacquet module functor based on [BHS19, Lem. 5.2.1].

Suppose that  $\Pi^{\text{an}}$  is a very strongly admissible locally  $\mathbb{Q}_p$ -analytic representation over  $L$  of  $G_p$  and  $\underline{\delta} = z^\lambda \underline{\delta}_{\text{sm}}$  where  $z^\lambda$  is the  $\mathbb{Q}_p$ -algebraic character of  $T_p$  of weight  $\lambda = (\lambda_\tau)_{\tau \in \Sigma_p}$  and  $\underline{\delta}_{\text{sm}}$  is a smooth character of  $T_p$ . We write  $\lambda = \lambda_J + \lambda_{\Sigma_p \setminus J}$  according to the decomposition  $\mathfrak{t}^* = \mathfrak{t}_J^* \oplus \mathfrak{t}_{\Sigma_p \setminus J}^*$  which means that  $\lambda_J$  (resp.  $\lambda_{\Sigma_p \setminus J}$ ) vanishes on  $\mathfrak{t}_{\Sigma_p \setminus J}$  (resp.  $\mathfrak{t}_J$ ). We assume that the image of  $\lambda$  in  $(\mathfrak{t}'_{Q_p,J})^*$  is equal to  $\lambda'_J$  in §3.5.2. Then (see §3.4.3)

$$\text{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\Pi^{\text{an}})) = \text{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \left( U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} (-\lambda) \right)^\vee, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right)$$

The  $U(\mathfrak{q})$ -module  $U(\mathfrak{q}) \otimes_{U(\bar{\mathfrak{b}})} \lambda$  admits a quotient

$$\begin{aligned} L_J(\lambda) &:= \left( \otimes_{v \in S_v, \tau \in J_v} L_{\mathfrak{m}_{Q_v, \tau}}(\lambda_\tau) \right) \otimes \left( \otimes_{v \in S_v, \tau \notin J_v} U(\mathfrak{m}_{Q_v, \tau}) \otimes_{U(\mathfrak{b}_{Q_v, \tau})} \lambda_\tau \right) \\ &= L_{M_{Q_p}}(\lambda_J) \otimes_L M_{\mathfrak{m}_{Q_p}}(\lambda_{\Sigma_p \setminus J}). \end{aligned} \quad (3.5.7)$$

where  $M_{\mathfrak{m}_{Q_p}}(\lambda_{\Sigma_p \setminus J}) := U(\mathfrak{m}_{Q_p, \Sigma_p \setminus J}) \otimes_{U(\mathfrak{b}_{Q_p, \Sigma_p \setminus J})} \lambda_{\Sigma_p \setminus J}$  and  $L_{M_{Q_p}}(\lambda_J)$  is defined as  $L_{M_{Q_p}}(\tilde{\lambda}_J)$  (only with a possibly different action of  $\mathfrak{z}_{M_{Q_p}, J}$ ). Thus there is an injection (see the beginning of §3.4.4 for  $(-)^{\bar{u}^\infty}$ )

$$\begin{aligned} &\text{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \text{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\bar{u}^\infty}, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right) \\ &\hookrightarrow \text{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \left( U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} (-\lambda) \right)^\vee, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right). \end{aligned} \quad (3.5.8)$$

Recall there is a closed immersion  $J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \hookrightarrow J_{Q_p}(\Pi^{\text{an}})$  in Lemma 3.5.2, which induces a closed embedding

$$J_{B_{Q_p}} \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right) \hookrightarrow J_{B_p}(\Pi^{\text{an}})$$

by [Eme06b, Lem. 3.4.7(iii)] and by that  $J_{B_p}(\Pi^{\text{an}}) \simeq J_{B_{Q_p}}(J_{Q_p}(\Pi^{\text{an}}))$  ([HL11, Thm. 5.3]).

**Proposition 3.5.9.** *There exists an isomorphism*

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \text{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\overline{u}^\infty}, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right) \xrightarrow{\sim} \text{Hom}_{T_p} \left( \underline{\delta}, J_{B_{Q_p}} \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right) \right)$$

such that the following diagram commutes

$$\begin{array}{ccc} \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \text{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\overline{u}^\infty}, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right) & \xrightarrow{\sim} & \text{Hom}_{T_p} \left( \underline{\delta}, J_{B_{Q_p}} \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right) \right) \\ \downarrow (3.5.8) & & \downarrow \\ \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \left( U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} (-\lambda) \right)^\vee, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right) & \xrightarrow{\sim} & \text{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\Pi^{\text{an}})) \end{array}$$

where the right vertical arrow is induced by (3.5.1).

*Proof.* By [BHS19, Lem. 5.2.1], [Eme06b, Prop. 3.4.9] and that  $U(\mathfrak{n}_{Q_p})$  acts trivially on  $L_J(\lambda)$ , we have

$$\begin{aligned} & \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \text{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\overline{u}^\infty}, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right) \\ &= \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{g})} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), \Pi^{\text{an}} \right)^{N_{B_p,0}} \right) \\ &= \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( L_J(\lambda), \Pi^{\text{an}}[\mathfrak{n}_{Q_p}] \right)^{N_{B_p,0}} \right) \\ &= \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( L_J(\lambda), \Pi^{\text{an}, N_{Q_p,0}} \right)^{U_{Q_p,0}} \right) \\ &= \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( L_J(\lambda), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p,0}} \right) \end{aligned}$$

where the last equations are given by similar arguments as in the proof of Lemma 3.5.5. Similarly using  $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} U(\mathfrak{q}) \otimes_{U(\mathfrak{b})} \lambda$  and that  $U(\mathfrak{n}_{Q_p})$  acts trivially on  $U(\mathfrak{q}) \otimes_{U(\mathfrak{b})} \lambda = U(\mathfrak{m}_{Q_p}) \otimes_{U(\mathfrak{b}_{Q_p})} \lambda$ , we get

$$\begin{aligned} & \text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \left( U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{b}})} (-\lambda) \right)^\vee, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi^{\text{an}} \right) \\ &= \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( U(\mathfrak{m}_{Q_p}) \otimes_{U(\mathfrak{b}_{Q_p})} \lambda, J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p,0}} \right) \end{aligned}$$

Thus the injection (3.5.8) corresponds to the injection

$$\begin{aligned} & \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( L_J(\lambda), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p,0}} \right) \\ & \hookrightarrow \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( U(\mathfrak{m}_{Q_p}) \otimes_{U(\mathfrak{b}_{Q_p})} \lambda, J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p,0}} \right). \end{aligned}$$

Firstly, we have

$$\begin{aligned}
& \text{Hom}_{U(\mathfrak{m}_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \\
&= \text{Hom}_{U(\mathfrak{m}'_{Q_p, J}) \otimes U(\mathfrak{z}_{M_{Q_p}, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \\
&= \text{Hom}_{U(\mathfrak{z}_{M_{Q_p}, J})} \left( 1, \text{Hom}_{U(\mathfrak{m}'_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right) \right)^{U_{Q_p, 0}} \\
&= \text{Hom}_{U(\mathfrak{z}_{M_{Q_p}, J})} \left( 1, \text{Hom}_{U(\mathfrak{m}'_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \right) \\
&= \text{Hom}_{U(\mathfrak{t}_{Q_p, J})} \left( 1, \text{Hom}_{U(\mathfrak{m}'_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \right)
\end{aligned}$$

where 1 denotes the trivial module of the universal envelope algebras and the last equality comes from that the action of  $\mathfrak{t}'_{Q_p, J}$  on  $\text{Hom}_{U(\mathfrak{m}'_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}}$  is already trivial. Similar arguments as in Lemma 3.5.4 replacing  $\tilde{\lambda}_J$  there by  $\lambda_J$  gives

$$\begin{aligned}
& \text{Hom}_{U(\mathfrak{t}_{Q_p, J})} \left( 1, \text{Hom}_{U(\mathfrak{m}'_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \right) \\
&= \text{Hom}_{U(\mathfrak{t}_{Q_p, J})} \left( \lambda_J, \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right)^{U_{Q_p, 0}} \right). \tag{3.5.10}
\end{aligned}$$

We get

$$\text{Hom}_{U(\mathfrak{m}_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \simeq \text{Hom}_{U(\mathfrak{t}_{Q_p, J})} \left( \lambda_J, \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right)^{U_{Q_p, 0}} \right). \tag{3.5.11}$$

Hence

$$\begin{aligned}
& \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( L_J(\lambda), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \\
&= \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( L_{M_{Q_p}}(\lambda_J) \otimes_L M_{\mathfrak{m}_{Q_p}}(\lambda_{\Sigma_p \setminus J}), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \\
&= \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( M_{\mathfrak{m}_{Q_p}}(\lambda_{\Sigma_p \setminus J}), L_{M_{Q_p}}(\lambda_J)' \otimes J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \\
&= \text{Hom}_{U(\mathfrak{t})} \left( \lambda_{\Sigma_p \setminus J}, \text{Hom}_{U(\mathfrak{m}_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right) [\mathfrak{u}_{Q_p}] \right)^{U_{Q_p, 0}} \\
&= \text{Hom}_{U(\mathfrak{t})} \left( \lambda_{\Sigma_p \setminus J}, \text{Hom}_{U(\mathfrak{m}_{Q_p, J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \right) \\
&\stackrel{(3.5.11)}{=} \text{Hom}_{U(\mathfrak{t})} \left( \lambda_{\Sigma_p \setminus J}, \text{Hom}_{U(\mathfrak{t}_{Q_p, J})} \left( \lambda_J, \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right)^{U_{Q_p, 0}} \right) \right) \\
&= \text{Hom}_{U(\mathfrak{t})} \left( \lambda, \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right)^{U_{Q_p, 0}} \right). \tag{3.5.12}
\end{aligned}$$

Finally, we get

$$\begin{aligned}
& \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{m}_{Q_p})} \left( L_J(\lambda), J_{Q_p}(\Pi^{\text{an}}) \right)^{U_{Q_p, 0}} \right) \\
&\stackrel{(3.5.12)}{=} \text{Hom}_{T_p} \left( \underline{\delta}_{\text{sm}}, \text{Hom}_{U(\mathfrak{t})} \left( \lambda, \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right)^{U_{Q_p, 0}} \right) \right) \\
&= \text{Hom}_{T_p} \left( \underline{\delta}, \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right)^{U_{Q_p, 0}} \right) \\
&= \text{Hom}_{T_p} \left( \underline{\delta}, J_{B_{Q_p}} \left( J_{Q_p}(\Pi^{\text{an}})_{\lambda'_J} \right) \right)
\end{aligned}$$

and similarly

$$\begin{aligned}
& \mathrm{Hom}_{T_p} \left( \underline{\delta}_{\mathrm{sm}}, \mathrm{Hom}_{U(\mathfrak{m}_{Q_p})} \left( U(\mathfrak{m}_{Q_p}) \otimes_{U(\mathfrak{b}_{Q_p})} \lambda, J_{Q_p}(\Pi^{\mathrm{an}}) \right)^{U_{Q_p,0}} \right) \\
&= \mathrm{Hom}_{T_p} \left( \underline{\delta}_{\mathrm{sm}}, \mathrm{Hom}_{U(\mathfrak{t})} \left( \lambda, J_{Q_p}(\Pi^{\mathrm{an}})[\mathfrak{u}_{Q_p}] \right)^{U_{Q_p,0}} \right) \\
&= \mathrm{Hom}_{T_p} \left( \underline{\delta}_{\mathrm{sm}}, \mathrm{Hom}_{U(\mathfrak{t})} \left( \lambda, J_{Q_p}(\Pi^{\mathrm{an}})^{U_{Q_p,0}} \right) \right) \\
&= \mathrm{Hom}_{T_p} \left( \underline{\delta}, J_{Q_p}(\Pi^{\mathrm{an}})^{U_{Q_p,0}} \right) \\
&= \mathrm{Hom}_{T_p} \left( \underline{\delta}, J_{B_p}(\Pi^{\mathrm{an}}) \right).
\end{aligned}$$

The commutativity of the diagram in the statement of the proposition can be checked by the following commutative diagram and by comparing with (3.5.1) and (3.5.10)

$$\begin{array}{ccc}
\mathrm{Hom} \left( \lambda_J, L_{M_{Q_p}}(\lambda_J)' \otimes J_{Q_p}(\Pi^{\mathrm{an}}) \otimes L_{M_{Q_p}}(\lambda_J)^{U_{Q_p,0}} \right) & \longrightarrow & \mathrm{Hom} \left( \lambda_J, J_{Q_p}(\Pi^{\mathrm{an}}) \right) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{U(\mathfrak{m}_{Q_p,J})} \left( L_{M_{Q_p}}(\lambda_J), J_{Q_p}(\Pi^{\mathrm{an}}) \right) & \longrightarrow & \mathrm{Hom}_{U(\mathfrak{m}_{Q_p,J})} \left( U(\mathfrak{m}_{Q_p,J}) \otimes_{U(\mathfrak{b}_{Q_p,J})} \lambda_J, J_{Q_p}(\Pi^{\mathrm{an}}) \right)
\end{array}$$

where we identify  $L_{M_{Q_p}}(\lambda_J)^{U_{Q_p,0}} = \lambda_J$ .  $\square$

### 3.5.4 The partial eigenvariety

We use the partial Emerton-Jacquet functor to define the partial eigenvariety and use the usual eigenvariety machinery to obtain its basic properties.

Since  $J_{B_{Q_p}} \left( J_{Q_p}(\Pi_{\infty}^{\mathrm{an}})_{\lambda'_J} \right)$  is an essentially admissible locally analytic representation of  $\mathbb{Z}_p^s \times T_p$ , the continuous dual  $J_{B_{Q_p}} \left( J_{Q_p}(\Pi_{\infty}^{\mathrm{an}})_{\lambda'_J} \right)'$  is the global section of a coherent sheaf over the quasi-Stein space  $\mathrm{Spf}(R_{\infty})^{\mathrm{rig}} \times \widehat{T}_{p,L}$ . We define the *partial eigenvariety*  $X_p(\bar{\rho})(\lambda'_J)$  to be the scheme-theoretic support of the coherent sheaf associated with  $J_{B_{Q_p}} \left( J_{Q_p}(\Pi_{\infty}^{\mathrm{an}})_{\lambda'_J} \right)'$  on  $\mathrm{Spf}(R_{\infty})^{\mathrm{rig}} \times \widehat{T}_{p,L}$ .

The closed embedding in Lemma 3.5.2 induces a closed embedding  $J_{B_{Q_p}} \left( J_{Q_p}(\Pi_{\infty}^{\mathrm{an}})_{\lambda'_J} \right) \hookrightarrow J_{B_p}(\Pi_{\infty}^{\mathrm{an}})$  by [Eme06b, Lem. 3.4.7(iii)] and [HL11, Thm. 5.3]. Taking the continuous dual and then taking supports, we get a closed embedding  $X_p(\bar{\rho})(\lambda'_J) \hookrightarrow X_p(\bar{\rho})$  of rigid analytic spaces over  $L$ . Let  $X_p(\bar{\rho})_{\lambda'_J}$  (resp.  $(\widehat{T}_{p,L})_{\lambda'_J}$ ) be the fiber of  $X_p(\bar{\rho})$  (resp.  $\widehat{T}_{p,L}$ ) over  $\lambda'_J$  via the map  $X_p(\rho) \rightarrow \widehat{T}_{p,L} \xrightarrow{\mathrm{wt}} \mathfrak{t}^* \rightarrow (\mathfrak{t}'_{Q_p,J})^*$  where  $\mathfrak{t}^* \rightarrow (\mathfrak{t}'_{Q_p,J})^*$  is the restriction map. Since the action of  $\mathfrak{t}'_{Q_p,J}$  on  $\mathrm{Hom}_{U(\mathfrak{m}'_{Q_p,J})} \left( L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(\Pi_{\infty}^{\mathrm{an}}) \right)^{U_{Q_p,0}}$  as well as on

$$J_{B_{Q_p}} \left( \mathrm{Hom}_{U(\mathfrak{m}'_{Q_p,J})} \left( L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(\Pi_{\infty}^{\mathrm{an}}) \right) \right)$$

is zero by [Eme06b, Prop. 3.2.11], by Lemma 3.5.4, we have a commutative diagram:

$$\begin{array}{ccccc}
X_p(\bar{\rho})(\lambda'_J) & \hookrightarrow & X_p(\bar{\rho})_{\lambda'_J} & \hookrightarrow & X_p(\rho) \\
& \searrow & \downarrow & & \downarrow \\
& & (\widehat{T}_{p,L})_{\lambda'_J} & \hookrightarrow & \widehat{T}_{p,L}
\end{array}$$



where horizontal arrows are closed immersions.

If  $x = (y, \underline{\delta}) \in \mathrm{Spf}(R_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L}$  is a point in  $X_p(\bar{\rho})_{\lambda'_J}$  with  $\mathfrak{m}_y$  the maximal ideal of  $R_\infty[\frac{1}{p}]$  corresponding to  $y$ , then equivalently  $\mathrm{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_y] \otimes_{k(y)} k(x))) \neq 0$  and  $\mathrm{wt}(\underline{\delta})'_J$ , the image of the weight  $\mathrm{wt}(\underline{\delta})$  of  $\underline{\delta}$  in  $(\mathfrak{t}'_{Q_p,J})^*$ , is equal to  $\lambda'_J$ . Assume that  $\lambda := \mathrm{wt}(\underline{\delta}) \in (\mathbb{Z}^n)^{\Sigma_p}$  is integral and let  $\underline{\delta}_{\mathrm{sm}} := z^{-\lambda} \underline{\delta}$ . Then by (3.5.8), there is an injection

$$\begin{aligned} & \mathrm{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \mathrm{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\bar{u}^\infty}, \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty[\mathfrak{m}_y]^{\mathrm{an}} \otimes_{k(y)} k(x) \right) \\ & \hookrightarrow \mathrm{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \left( U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} (-\lambda) \right)^\vee, \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty[\mathfrak{m}_y]^{\mathrm{an}} \otimes_{k(y)} k(x) \right) \neq 0. \end{aligned}$$

**Proposition 3.5.13.** *Let  $x = (y, \underline{\delta}) \in X_p(\bar{\rho})_{\lambda'_J} \subset \mathrm{Spf}(R_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L}$  be a point with  $\lambda = \mathrm{wt}(\underline{\delta}) \in (\mathbb{Z}^n)^{\Sigma_p}$ . Then  $x$  is in  $X_p(\bar{\rho})(\lambda'_J)$  if and only if*

$$\mathrm{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \mathrm{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\bar{u}^\infty}, \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty[\mathfrak{m}_y]^{\mathrm{an}} \otimes_{k(y)} k(x) \right) \neq 0.$$

*Proof.* The point  $x$  is in  $X_p(\bar{\rho})(\lambda'_J)$  if and only if

$$\mathrm{Hom}_{T_p} \left( \underline{\delta}, J_{B_{Q_p}} \left( J_{Q_p}(\Pi_\infty^{\mathrm{an}})_{\lambda'_J} \right) [\mathfrak{m}_y] \otimes_{k(y)} k(x) \right) \neq 0.$$

By the left exactness of Jacquet module functors ([Eme06b, Lem. 3.4.7(iii)]) and the definition of  $J_{Q_p}(\Pi_\infty^{\mathrm{an}})_{\lambda'_J}$ , we have  $J_{B_{Q_p}} \left( J_{Q_p}(\Pi_\infty^{\mathrm{an}})_{\lambda'_J} \right) [\mathfrak{m}_y] \otimes_{k(y)} k(x) = J_{B_{Q_p}} \left( J_{Q_p}(\Pi_\infty^{\mathrm{an}}[\mathfrak{m}_y])_{\lambda'_J} \right) \otimes_{k(y)} k(x)$ . Since  $\Pi_\infty[\mathfrak{m}_y]$  is an admissible Banach representation of  $G_p$  over  $k(y)$ , we get  $\Pi_\infty[\mathfrak{m}_y]^{\mathrm{an}} = \Pi_\infty[\mathfrak{m}_y]^{R_\infty - \mathrm{an}} = \Pi_\infty^{\mathrm{an}}[\mathfrak{m}_y]$  by [BHS17b, (3.3), Prop. 3.8] (cf. [BHS17b, Prop. 3.7]). Hence  $x \in X_p(\bar{\rho})(\lambda'_J)$  if and only if

$$\mathrm{Hom}_{T_p} \left( \underline{\delta}, J_{B_{Q_p}} \left( J_{Q_p}(\Pi_\infty[\mathfrak{m}_y]^{\mathrm{an}})_{\lambda'_J} \right) \otimes_{k(y)} k(x) \right) \neq 0.$$

Now the result follows by applying Proposition 3.5.9.  $\square$

We now study the eigenvariety  $X_p(\bar{\rho})(\lambda'_J)$  in a standard way as in [BHS17b, §3.3] or [Din19c, §2.4].

For a uniform pro- $p$  Lie group  $H$  which is a product of locally  $F_v^-$ -analytic groups such as  $H_p$ , we let  $\mathcal{C}^{\mathrm{la}}(H, L)$  be the space of locally  $\mathbb{Q}_p$ -analytic functions on  $H$  with coefficients in  $L$ . For every integer  $h \geq 1$ , let  $r_h = \frac{1}{p^{h-1}(p-1)}$ . Let  $\mathcal{C}^{(h)}(H, L)$  be the subspace of  $\mathcal{C}^{\mathrm{la}}(H, L)$  defined in [CD14, Def. IV.1]. Then  $\mathcal{C}^{(h)}(H, L)$  is a Banach space over  $L$  and  $\mathcal{C}^{\mathrm{la}}(H, L) = \varinjlim_h \mathcal{C}^{(h)}(H, L)$ . Let  $\mathcal{C}^{\Sigma_p \setminus J - \mathrm{an}}(H, L) := \mathcal{C}^{\mathrm{la}}(H, L)^{\Sigma_p \setminus J - \mathrm{an}}$  (resp.  $\mathcal{C}^{\Sigma_p \setminus J - \mathrm{an}, (h)}(H, L) := \mathcal{C}^{(h)}(H, L)^{\Sigma_p \setminus J - \mathrm{an}}$ ) be the space of locally  $\Sigma_p \setminus J$ -analytic functions in  $\mathcal{C}^{\mathrm{la}}(H, L)$  (resp.  $\mathcal{C}^{(h)}(H, L)$ ) with respect to the action of  $H$ . Remark that the notion of locally  $\Sigma_p \setminus J$ -analytic functions can be defined for general products of locally  $F_v^-$ -analytic manifolds without referring to group actions, see [Sch10, Def. 2.1]. Then  $\mathcal{C}^{\Sigma_p \setminus J - \mathrm{an}}(H, L) = \varinjlim_h \mathcal{C}^{\Sigma_p \setminus J - \mathrm{an}, (h)}(H, L)$  ([Eme17, Prop. 1.1.41]). Let  $D(H, L) = \mathcal{C}^{\mathrm{an}}(H, L)'$  be the strong dual. Recall for any  $h$ , there is a closed immersion of Banach algebras  $D_{p^{-r_h}}(H, L) \hookrightarrow D_{< p^{-r_h}}(H, L)$  defined in [ST03, §4] and note  $D_{< p^{-r_h}}(H, L) = \mathcal{C}^{(h)}(H, L)'$  is the strong dual. Set  $D^{\Sigma_p \setminus J - \mathrm{an}}(H, L) := \mathcal{C}^{\Sigma_p \setminus J - \mathrm{an}}(H, L)' = \varprojlim_h D_{< p^{-r_h}}^{\Sigma_p \setminus J - \mathrm{an}}(H, L)$  where

$$D_{< p^{-r_h}}^{\Sigma_p \setminus J - \mathrm{an}}(H, L) := \left( \mathcal{C}^{\Sigma_p \setminus J - \mathrm{an}, (h)}(H, L) \right)'.$$

Let  $D_{p^{-r_h}}^{\Sigma_p \setminus J - \mathrm{an}}(H, L)$  be the completion of  $D^{\Sigma_p \setminus J - \mathrm{an}}(H, L)$  with respect to the quotient norm from the norm on  $D(H, L)$  induced via  $D(H, L) \hookrightarrow D_{p^{-r_h}}(H, L)$  (cf. [Sch10, §2.5]). Hence

we have a quotient  $D_{p^{-r_h}}(H, L) \twoheadrightarrow D_{p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L)$ . As  $\mathcal{C}^{\Sigma_p \setminus J\text{-an}, (h)}(H, L)$  is dense in  $\mathcal{C}^{\Sigma_p \setminus J\text{-an}}(H, L)$ , we get an injection  $D_{p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L) \hookrightarrow D_{<p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L)$ . The norm on the target is the quotient norm from the norm on  $D_{<p^{-r_h}}(H, L)$  where the surjectivity of

$$D_{<p^{-r_h}}(H, L) \rightarrow D_{<p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L)$$

follows from the Hahn-Banach theorem ([Sch13, Prop. 9.2]) and the norm on  $D_{<p^{-r_h}}(H, L)$  is the same as that on  $D_{p^{-r_h}}(H, L)$  when restricted to  $D(H, L)$  ([ST03, P162]). Hence we get a closed embedding

$$D_{p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L) \hookrightarrow D_{<p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L).$$

In conclusion, we have the following diagram of morphisms of Banach algebras over  $L$

$$\begin{array}{ccccc} D_{<p^{-r_{h+1}}}(H, L) & \hookrightarrow & D_{p^{-r_h}}(H, L) & \hookrightarrow & D_{<p^{-r_h}}(H, L) \\ \Downarrow & & \Downarrow & & \Downarrow \\ D_{<p^{-r_{h+1}}}^{\Sigma_p \setminus J\text{-an}}(H, L) & \hookrightarrow & D_{p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L) & \hookrightarrow & D_{<p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(H, L) \end{array}$$

where each horizontal arrow is an injection and each vertical arrow is a surjection.

If  $H$  is moreover abelian, we let  $\widehat{H}_L$  be the rigid analytic space over  $L$  parameterizing locally  $\mathbb{Q}_p$ -analytic characters of  $H$ . The rigid analytic space

$$\widehat{T}'_{Q_p, 0, \Sigma_p \setminus J, L} := \varinjlim_h \mathrm{Sp}(D_{p^{-r_h}}^{\Sigma_p \setminus J\text{-an}}(T'_{Q_p, 0}, L))$$

over  $L$  parameterizing locally  $\Sigma_p \setminus J$ -analytic characters of  $T'_{Q_p, 0}$  is strictly quasi-Stein, smooth and equidimensional ([Din17a, Prop. 6.1.13, Prop. 6.1.14]) and is a closed analytic subspace of  $\widehat{T}'_{Q_p, 0, L}$ . By [Sch10, Prop. 2.18] (cf. [Din17a, §6.1.4]), the following commutative diagram is Cartesian

$$\begin{array}{ccc} \widehat{T}'_{Q_p, 0, \Sigma_p \setminus J, L} & \hookrightarrow & \widehat{T}'_{Q_p, 0, L} \\ \downarrow \mathrm{wt} & & \downarrow \mathrm{wt} \\ (\mathfrak{t}'_{Q_p, \Sigma_p \setminus J})^* & \hookrightarrow & (\mathfrak{t}'_{Q_p})^* \end{array}$$

where  $\mathfrak{t}'_{Q_p, \Sigma_p \setminus J} \simeq \mathfrak{t}'_{Q_p} / \mathfrak{t}'_{Q_p, J}$  (thus there is a closed embedding  $(\mathfrak{t}'_{Q_p, \Sigma_p \setminus J})^* \hookrightarrow (\mathfrak{t}'_{Q_p})^*$ ).

We pick an element  $z \in T_p^+$  such that  $\cap_{i \geq 1} z^i N_{B_p, 0} z^{-i}$  only consists of the identity element. Assume that  $z H_p z^{-1}$  is normalized by  $N_{B_p, 0}$ . Denote by  $\mathcal{W}_0 := \widehat{Z}_{M_{Q_p}, 0, L} \times \widehat{T}'_{Q_p, 0, \Sigma_p \setminus J, L} = \mathrm{Spf}(S_\infty)^{\mathrm{rig}} \times \widehat{Z}_{M_{Q_p}, 0, L} \times \widehat{T}'_{Q_p, 0, \Sigma_p \setminus J, L}$ .

**Lemma 3.5.14.** *There exists an admissible covering of  $\mathcal{W}_0$  by open affinoids  $\mathrm{Sp}(B_1) \subset \mathrm{Sp}(B_2) \subset \dots \subset \mathrm{Sp}(B_h) \subset \dots$  and for any  $h \geq 1$ , there exist*

- an orthonormalizable Banach  $B_h$ -module  $V_h$ ,
- a compact  $B_h$ -map  $z_h : V_h \rightarrow V_h$ , continuous  $B_h$ -maps  $\alpha_h : V_h \rightarrow V_{h+1} \widehat{\otimes}_{B_{h+1}} B_h$  and  $\beta_h : V_{h+1} \widehat{\otimes}_{B_{h+1}} B_h \rightarrow V_h$  such that  $z_h = \beta_h \circ \alpha_h$  and  $\alpha_h \circ \beta_h = z_{h+1} \otimes 1_{B_h}$ , and
- an isomorphism of topological  $D(\widehat{Z}_{M_{Q_p}, 0, L}) \widehat{\otimes}_L D^{\Sigma_p \setminus J\text{-an}}(T'_{Q_p, 0}, L)$ -modules

$$\left( \left( \Pi_\infty^{\mathrm{an}, N_{Q_p}, 0} \otimes_{L_{M_{Q_p}}} (\widetilde{\lambda}_J)' \right)^{\Sigma_p \setminus J\text{-an}, U_{Q_p}, 0} \right)' \simeq \varprojlim_h V_h$$

such that the action of the operator  $\pi_z$  induced by the Hecke action of  $z$  on the left hand side coincides with the action of  $(z_h)_{h \in \mathbb{N}}$  on the right hand side.

We can summarize the datum in the following commutative diagram:

$$\begin{array}{ccccccc}
 \left( \left( \Pi_{\infty}^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J-\text{an}, U_{Q_p,0}} \right)' & \succ & \cdots & \longrightarrow & V_{h+1} & \succ & V_{h+1} \hat{\otimes}_{B_{h+1}} B_h \xrightarrow{\beta_h} V_h \\
 \pi_z \downarrow & & & & \downarrow z_{h+1} & & \downarrow z_h \\
 & & & & & \downarrow z_{h+1} \otimes 1_{B_h} & \swarrow \alpha_h \\
 \left( \left( \Pi_{\infty}^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J-\text{an}, U_{Q_p,0}} \right)' & \succ & \cdots & \longrightarrow & V_{h+1} & \succ & V_{h+1} \hat{\otimes}_{B_{h+1}} B_h \xrightarrow{\beta_h} V_h
 \end{array}$$

*Proof.* We have  $\Pi_{\infty}|_{\tilde{H}_p} \simeq \mathcal{C}(\tilde{H}_p, L)^m$  and  $\Pi_{\infty}^{\text{an}} = \Pi_{\infty}^{S_{\infty}-\text{an}} \simeq \mathcal{C}^{\text{la}}(\tilde{H}_p, L)^m$  by [BHS17b, Prop. 3.8]. Since  $\tilde{H}_p = \overline{N}_{Q_p,0} \times \tilde{M}_{Q_p,0} \times N_{Q_p,0}$ , we have

$$\Pi_{\infty}^{\text{an}} \simeq \left( \mathcal{C}^{\text{la}}(\overline{N}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{\text{la}}(\tilde{M}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{\text{la}}(N_{Q_p,0}, L) \right)^m. \quad (3.5.15)$$

Thus

$$\Pi_{\infty}^{\text{an}, N_{Q_p,0}} \simeq \left( \mathcal{C}^{\text{la}}(\overline{N}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{\text{la}}(\tilde{M}_{Q_p,0}, L) \right)^m.$$

By the twisting lemma ([Din19c, Lem. 2.19]), we have an isomorphism of  $\tilde{M}_{Q_p,0}$ -representations

$$\begin{aligned}
 \mathcal{C}^{\text{la}}(\tilde{M}_{Q_p,0}, L) \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)'|_{\tilde{M}_{Q_p,0}} &\xrightarrow{\sim} \mathcal{C}^{\text{la}}(\tilde{M}_{Q_p,0}, L)^{m'} \\
 f \otimes v &\mapsto (g \mapsto f(g)gv)
 \end{aligned} \quad (3.5.16)$$

for  $m' = \dim_L L_{M_{Q_p}}(\tilde{\lambda}_J)'$ . Hence

$$\begin{aligned}
 \Pi_{\infty}^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' &\simeq \left( \mathcal{C}^{\text{la}}(\overline{N}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{\text{la}}(\tilde{M}_{Q_p,0}, L) \otimes_L L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^m \\
 &\simeq \left( \mathcal{C}^{\text{la}}(\overline{N}_{Q_p,0}, L) \hat{\otimes}_L \left( \mathcal{C}^{\text{la}}(\tilde{M}_{Q_p,0}, L) \right)^{m'} \right)^m \\
 &\simeq \left( \mathcal{C}^{\text{la}}(\overline{N}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{\text{la}}(\tilde{Z}_{M_{Q_p,0}}, L) \hat{\otimes}_L \mathcal{C}^{\text{la}}(M'_{Q_p,0}, L) \right)^r,
 \end{aligned}$$

where  $r = mm'$ . Then

$$\Pi_{\infty}^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \simeq \varinjlim_h \left( \mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \hat{\otimes}_L \mathcal{C}^{(h)}(M'_{Q_p,0}, L) \right)^r.$$

Hence by [Eme17, Prop. 1.1.41], we get

$$\begin{aligned}
 &\left( \Pi_{\infty}^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J-\text{an}} \\
 &= \varinjlim_h \left( \mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \hat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J-\text{an}, (h)}(M'_{Q_p,0}, L) \right)^r \\
 &= \varinjlim_h \mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L)^r \hat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \hat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J-\text{an}, (h)}(\overline{U}_{Q_p,0}, L) \\
 &\quad \hat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J-\text{an}, (h)}(T'_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J-\text{an}, (h)}(U_{Q_p,0}, L).
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\left( \Pi_{\infty}^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J-\text{an}, U_{Q_p,0}} \\
 &= \varinjlim_h \mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L)^r \hat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \hat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J-\text{an}, (h)}(\overline{U}_{Q_p,0}, L) \hat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J-\text{an}, (h)}(T'_{Q_p,0}, L).
 \end{aligned}$$

We let  $W_h = (\mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L)^r \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h)}(\overline{U}_{Q_p,0}, L))'$ . By [Eme17, Prop. 1.1.22, Prop. 1.1.32], we get

$$\begin{aligned} \left( \left( \Pi_\infty^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J - \text{an}, U_{Q_p,0}} \right)' &= \varprojlim_h W_h \widehat{\otimes}_L \left( \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h)}(T'_{Q_p,0}, L) \right)' \\ &= \varprojlim_h W_h \widehat{\otimes}_L D_{< p^{-r_h}}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L D_{< p^{-r_h}}^{\Sigma_p \setminus J - \text{an}}(T'_{Q_p,0}, L). \end{aligned} \quad (3.5.17)$$

Set  $B_h = D_{p^{-r_h}}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L D_{p^{-r_h}}^{\Sigma_p \setminus J - \text{an}}(T'_{Q_p,0}, L)$  and  $V_h := W_h \widehat{\otimes}_L B_h$  which is an orthonormalizable Banach  $B_h$ -module by definition [Buz07, §2]. Then

$$\begin{aligned} V_h &= \left( \left( \mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h)}(\overline{U}_{Q_p,0}, L) \right)^r \widehat{\otimes}_L \mathcal{C}^{(h+1)}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h+1)}(T'_{Q_p,0}, L) \right)' \\ &\quad \widehat{\otimes}_{D_{< p^{-r_{h+1}}}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L D_{< p^{-r_{h+1}}}^{\Sigma_p \setminus J - \text{an}}(T'_{Q_p,0}, L)} B_h. \end{aligned}$$

As the map

$$W_{h+1} \widehat{\otimes}_L D_{< p^{-r_{h+1}}}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L D_{< p^{-r_{h+1}}}^{\Sigma_p \setminus J - \text{an}}(T'_{Q_p,0}, L) \rightarrow W_h \widehat{\otimes}_L D_{< p^{-r_h}}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L D_{< p^{-r_h}}^{\Sigma_p \setminus J - \text{an}}(T'_{Q_p,0}, L)$$

factors through  $V_h$ , we conclude that  $\left( \left( \Pi_\infty^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \right)^{\Sigma_p \setminus J - \text{an}, U_{Q_p,0}} \right)' = \varprojlim_h V_h$ .

We now track the action of the element  $z$ . The action of  $z$  sends  $(\Pi_\infty)_{\tilde{H}_p}^{(h)}$  to  $(\Pi_\infty)_{z\tilde{H}_p z^{-1}}^{(h)}$  where  $(\Pi_\infty)_{\tilde{H}_p}^{(h)}$  is the subspace of  $\Pi_\infty^{\text{an}}$  defined in [CD14, IV.D]. By [BHS17a, Lem. 5.2 & proof of Lem. 5.3], the action of  $z$  on  $\Pi_\infty^{\text{an}}$  sends  $(\Pi_\infty)_{\tilde{H}_p}^{(h)}$  to

$$\mathcal{C}^{(h-1)}(\overline{N}_{Q_p,0}, L)^m \widehat{\otimes}_L \mathcal{C}^{(h-1)}(\overline{U}_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(T'_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{\text{la}}(U_{Q_p,0}, L) \otimes \mathcal{C}^{\text{la}}(N_{Q_p,0}, L)$$

in term of the isomorphism (3.5.15).

We assume that an homeomorphism  $\mathbb{Z}_p^u \simeq M_{Q_p,0}$  is chosen so that the matrix coefficients of elements in  $M_{Q_p,0}$  are overconvergent analytic functions on  $\mathbb{Z}_p^u$  (for example, we can choose coordinates of  $U_{Q_p,0}$  and  $\overline{U}_{Q_p,0}$  as a product of root groups where each is identified with some  $\mathcal{O}_{F_v}$  and coordinates of  $T_{p,0}$  as products of some  $1 + \varpi_{F_v}^t \mathcal{O}_{F_v}$  with  $t$  large enough. Then the matrix coefficients are in the ring generated by polynomials and functions of the form  $\exp(\varpi_{F_v}^t x)$ ,  $x \in \mathcal{O}_{F_v}$ ). Since the action of  $g$  and  $g^{-1}$  on  $L_{M_{Q_p}}(\tilde{\lambda}_J)'$ , for  $g \in M_{Q_p,0}$ , are given by polynomial functions on the matrix coefficients of  $g, g^{-1}$  compositing with embeddings in  $\Sigma_p$ , the matrix coefficients of the twisting isomorphism (3.5.16) in the basis given by one of that of  $L_{M_{Q_p}}(\tilde{\lambda}_J)'$  are overconvergent analytic functions on  $\mathbb{Z}_p^u$ . By a similar argument as in Lemma 3.5.18 below, for any  $h$  large enough (which we may assume from now on), there is an isomorphism

$$\left( (\Pi_\infty)_{\tilde{H}_p}^{(h)} \right)^{N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)' \simeq \mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L)^r \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(M'_{Q_p,0}, L)$$

and similarly the isomorphism (3.5.16) sends

$$\mathcal{C}^{(h-1)}(\overline{N}_{Q_p,0}, L)^m \widehat{\otimes}_L \mathcal{C}^{(h-1)}(\overline{U}_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(T'_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{\text{la}}(U_{Q_p,0}, L) \widehat{\otimes}_L L_{M_{Q_p}}(\tilde{\lambda}_J)'$$

to

$$\mathcal{C}^{(h-1)}(\overline{N}_{Q_p,0}, L)^r \widehat{\otimes}_L \mathcal{C}^{(h-1)}(\overline{U}_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p,0}}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(T'_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{\text{la}}(U_{Q_p,0}, L).$$

Hence the action of  $z$  on  $\left((\Pi_\infty)^{(h)}_{\tilde{H}_p}\right)^{N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)'$  sends, after (3.5.16),

$$\mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L)^r \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p},0}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(M'_{Q_p,0}, L)$$

to

$$\mathcal{C}^{(h-1)}(\overline{N}_{Q_p,0}, L)^r \widehat{\otimes}_L \mathcal{C}^{(h-1)}(\overline{U}_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p},0}, L) \widehat{\otimes}_L \mathcal{C}^{(h)}(T'_{Q_p,0}, L) \widehat{\otimes} \mathcal{C}^{\text{la}}(U_{Q_p,0}, L).$$

Finally, we conclude that the Hecke action of  $z$ , denoted by  $\pi_z$ , on  $\left(\Pi_\infty^{\text{an}, N_{Q_p,0}} \otimes L_{M_{Q_p}}(\tilde{\lambda}_J)'\right)^{\Sigma_p \setminus J - \text{an}, U_{Q_p,0}}$  induces a map sending

$$\left(\mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h)}(\overline{U}_{Q_p,0}, L)\right)^r \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p},0}, L) \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h)}(T'_{Q_p,0}, L)$$

to

$$\left(\mathcal{C}^{(h-1)}(\overline{N}_{Q_p,0}, L) \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h-1)}(\overline{U}_{Q_p,0}, L)\right)^r \widehat{\otimes}_L \mathcal{C}^{(h)}(\tilde{Z}_{M_{Q_p},0}, L) \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h)}(T'_{Q_p,0}, L).$$

Taking the dual of the above map we see that  $\pi_z$  induces a morphism  $\alpha_{h-1} : V_{h-1} \rightarrow V_h \widehat{\otimes}_{B_h} B_{h-1}$ . We also have a morphism  $\beta_h = \beta'_h \otimes 1_{B_h} : V_{h+1} \widehat{\otimes}_{B_{h+1}} B_h \rightarrow V_h$  where  $\beta'_h$  is induced by the dual of the compact map

$$\mathcal{C}^{(h)}(\overline{N}_{Q_p,0}, L)^r \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h)}(\overline{U}_{Q_p,0}, L) \rightarrow \mathcal{C}^{(h+1)}(\overline{N}_{Q_p,0}, L)^r \widehat{\otimes}_L \mathcal{C}^{\Sigma_p \setminus J - \text{an}, (h+1)}(\overline{U}_{Q_p,0}, L).$$

Hence  $\beta_h$  is a compact map of Banach  $B_h$ -modules. We put  $z_h = \beta_h \circ \alpha_h$ . Then  $z_{h+1} \otimes 1_{B_h} = \alpha_h \circ \beta_h$ .  $\square$

**Lemma 3.5.18.** *Assume that  $g$  is an overconvergent analytic function over  $\mathbb{Z}_p$ . Then there exists  $C$  such that for any  $h > C$  and  $f \in \mathcal{C}^{(h)}(\mathbb{Z}_p, L)$ , we have  $gf \in \mathcal{C}^{(h)}(\mathbb{Z}_p, L)$ .*

*Proof.* By definition ([CD14, Def. IV.1]),

$$\mathcal{C}^{(h)}(\mathbb{Z}_p, L) = \left\{ \sum_{n \in \mathbb{N}} a_n \binom{x}{n} \mid \lim_{n \rightarrow +\infty} (v_p(a_n) - r_h n) = +\infty \right\}$$

is a Banach space with valuation  $v^{(h)}\left(\sum_{n \in \mathbb{N}} a_n \binom{x}{n}\right) = \inf_n (v_p(a_n) - r_h n)$ . Let  $f = \sum_{n \in \mathbb{N}} a_n \binom{x}{n} \in \mathcal{C}^{(h)}(\mathbb{Z}_p, L)$ . We compute  $x \left(\sum_{n \in \mathbb{N}} a_n \binom{x}{n}\right) = \sum_{n \in \mathbb{N}} a_n x \binom{x}{n} = \sum_{n \in \mathbb{N}} a_n \left((n+1) \binom{x}{n+1} + n \binom{x}{n}\right) = \sum_{n \geq 1} n(a_n + a_{n-1}) \binom{x}{n}$ . We have

$$\lim_{n \rightarrow \infty} (v_p(n(a_n + a_{n-1})) - r_h n) \geq \lim_{n \rightarrow +\infty} \min \{(v_p(a_n) - r_h n), (v_p(a_{n-1}) - r_h(n-1) - r_h)\} = +\infty$$

and

$$\begin{aligned} v^{(h)}\left(x \sum_{n \in \mathbb{N}} a_n \binom{x}{n}\right) &= \inf_n (v_p(n(a_n + a_{n-1})) - r_h n) \\ &\geq \inf_n \min \{(v_p(a_n) - r_h n), (v_p(a_{n-1}) - r_h(n-1) - r_h)\} \\ &\geq \inf_n (v_p(a_n) - r_h n) - r_h = v^{(h)}(f) - r_h. \end{aligned}$$

We get  $x^n f$  lies in  $\mathcal{C}^{(h)}(\mathbb{Z}_p, L)$  and  $v^{(h)}(x^n f) \geq v^{(h)}(f) - r_h n$  for any  $n$ . Since  $g$  is analytic, we can assume  $g = \sum_{n \in \mathbb{N}} b_n x^n$ . Then  $v^{(h)}(b_n x^n f) \geq v^{(h)}(f) + (v_p(b_n) - r_h n)$ . Since  $g$  is overconvergent and  $r_h \rightarrow 0$  if  $h \rightarrow \infty$ , for  $h$  large enough, we have  $\lim_{n \rightarrow +\infty} v^{(h)}(f) + (v_p(b_n) - r_h n) = +\infty$ . Hence  $gf = \sum_{n \in \mathbb{N}} b_n x^n f$  converges in the Banach space  $\mathcal{C}^{(h)}(\mathbb{Z}_p, L)$ .  $\square$

We denote by  $\mathcal{W}_{\lambda'_J} := \mathrm{Spf}(S_\infty)^{\mathrm{rig}} \times \widehat{T}_{M_{Q_p},0,L} \times (\widehat{T}'_{Q_p,0,L})_{\lambda'_J}$  where  $(\widehat{T}'_{Q_p,0,L})_{\lambda'_J}$  is the fiber of  $\widehat{T}'_{Q_p,0,L}$  over  $\lambda'_J$  via the map  $\widehat{T}'_{Q_p,0,L} \xrightarrow{\mathrm{wt}} (\mathfrak{t}')^* \xrightarrow{\mathrm{res}} (\mathfrak{t}'_{Q_p,J})^*$ . There is an isomorphism  $\mathcal{W}_0 \rightarrow \mathcal{W}_{\lambda'_J} : x \mapsto xz^{\tilde{\lambda}_J}$  where  $z^{\tilde{\lambda}_J}$  is the character of  $T_{p,0}$  on  $L_{M_{Q_p}}(\tilde{\lambda}_J)^{U_{Q_p,0}}$  in Lemma 3.5.4. The restriction of characters from  $T_p$  to  $T_{p,0}$  induces a morphism  $\omega_{X(\lambda'_J)} : X_p(\bar{\rho})(\lambda'_J) \rightarrow \mathcal{W}_{\lambda'_J}$ . The eigenvariety machinery in [Buz07] and [Che04] leads to the following basic result on the partial eigenvariety.

**Proposition 3.5.19.** *The partial eigenvariety  $X_p(\bar{\rho})(\lambda'_J)$  is equidimensional and for any point in  $X_p(\bar{\rho})(\lambda'_J)$ , there exists an open affinoid neighborhood  $U$  such that there exists an affinoid open subset  $W$  of  $\mathcal{W}_{\lambda'_J}$  satisfying that the restriction of  $\omega_{X(\lambda'_J)}$  to any irreducible component of  $U$  is finite and surjective onto  $W$ . Moreover, the image of an irreducible component of  $X_p(\bar{\rho})(\lambda'_J)$  is a Zariski open subset of  $\mathcal{W}_{\lambda'_J}$ .*

*Proof.* The result can be proved by a slight modification of the proofs in [BHS17b, §3.3] replacing  $J_{B_p}(\Pi_\infty^{\mathrm{an}})$  (resp.  $\mathcal{W}_\infty$ ) in *loc. cit.* with the module  $J_{B_{Q_p}} \left( \mathrm{Hom}_{U(\mathfrak{m}'_{Q_p,J})} \left( L_{M_{Q_p}}(\tilde{\lambda}_J), J_{Q_p}(\Pi_\infty^{\mathrm{an}}) \right) \right)$  (resp.  $\mathcal{W}_0$ ) using Lemma 3.5.5 and Lemma 3.5.14 and then applying Lemma 3.5.4 to obtain the results for  $X_p(\bar{\rho})(\lambda'_J)$  and  $\mathcal{W}_{\lambda'_J}$ .  $\square$

### 3.5.5 Density of classical points

We prove the density of de Rham points on the partial eigenvariety which will be the input for the application of the partial eigenvariety in next subsection.

Suppose  $x = (y, \underline{\delta}) \in X_p(\bar{\rho}) \subset \mathrm{Spf}(R_\infty)^{\mathrm{rig}} \times \widehat{T}_{p,L}$  is a point such that  $\lambda = \mathrm{wt}(\underline{\delta})$  is in  $(\mathbb{Z}^n)^{\Sigma_p}$  and is dominant with respect to  $\mathfrak{b}$ . The  $U(\mathfrak{g})$ -module  $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$  (and its quotient  $U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda)$  in §3.5.3) admits a unique irreducible quotient  $L(\lambda)$ , and hence

$$\mathcal{F}_{\overline{B}_p}^{G_p} \left( \mathrm{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\bar{u}^\infty}, \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1} \right),$$

as well as  $\mathcal{F}_{\overline{B}_p}^{G_p}(\underline{\delta})$ , admits a locally  $\mathbb{Q}_p$ -algebraic quotient  $\mathcal{F}_{\overline{B}_p}^{G_p} \left( L(\lambda)', \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1} \right)$  which is isomorphic to  $\mathcal{F}_{\overline{B}_p}^{G_p} \left( L(\lambda)', (\mathrm{Ind}_{\overline{B}_p}^{G_p} \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1})^{\mathrm{sm}} \right) \simeq L(\lambda) \otimes (\mathrm{Ind}_{\overline{B}_p}^{G_p} \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1})^{\mathrm{sm}}$  (cf. [BHS17b, §3.5]). We say that  $x$  is a *classical point* if

$$\mathrm{Hom}_{G_p} \left( L(\lambda) \otimes (\mathrm{Ind}_{\overline{B}_p}^{G_p} \underline{\delta}_{\mathrm{sm}} \delta_{B_p}^{-1})^{\mathrm{sm}}, \Pi_\infty[\mathfrak{m}_y]^{\mathrm{an}} \otimes_{k(y)} k(x) \right) \neq 0.$$

*Remark 3.5.20.* Our definition of classical points differs from [BHS17b, Def. 3.15] because we will consider points that are crystabelline rather than only crystalline.

**Proposition 3.5.21.** *The subset of classical points in  $X_p(\bar{\rho})(\lambda'_J)$  is Zariski dense. Moreover, for any point  $x = (y, \underline{\delta}) \in X_p(\bar{\rho})(\lambda'_J)$  such that  $\underline{\delta}$  is locally algebraic and any irreducible component  $X$  of  $X_p(\bar{\rho})(\lambda'_J)$  such that  $x \in X$ , there is an affinoid open subset  $U$  of  $X$  containing  $x$  such that the subset of classical points is Zariski-dense in  $U$ .*

*Proof.* The proof follows that of [BHS17b, Thm. 3.19] and [Din19c, Thm. 3.12]. For each  $v \in S_p$ ,  $1 \leq i \leq n$ , we let  $\gamma_{\tilde{v},i}$  be the element  $\mathrm{diag}(\underbrace{\varpi_{\tilde{v}}, \dots, \varpi_{\tilde{v}}}_i, 1, \dots, 1) \in \mathrm{GL}_n(F_{\tilde{v}})$  where  $\varpi_{\tilde{v}}$  is a fixed uniformizer of  $F_{\tilde{v}}$ .

By Proposition 3.5.19, we can pick a covering by open affinoids of any irreducible component of  $X_p(\bar{\rho})(\lambda'_J)$  such that  $\omega_{X(\lambda'_J)}$  sends such open affinoids surjectively and finitely onto open affinoids of  $\mathcal{W}_{\lambda'_J}$ . For any  $x = (y, \underline{\delta}) \in X_p(\bar{\rho})(\lambda'_J)$  with  $\underline{\delta}$  locally algebraic, we pick a such affinoids  $U \subset X_p(\bar{\rho})(\lambda'_J)$  and let  $W = \omega_{X(\lambda'_J)}(U)$ . We prove that the subset of classical points is Zariski dense in  $U$ .

Since  $U$  is affinoid, the rigid analytic functions  $(y, \underline{\delta}) \mapsto \underline{\delta}_v \delta_{B_v}^{-1}(\gamma_{\tilde{v}, i})$  are bounded on  $U$  and thus there exists a constant  $C > 0$  such that  $C \leq |\underline{\delta}_v \delta_{B_v}^{-1}(\gamma_{\tilde{v}, i})|_p$  over  $U$  for any  $v \in S_p, i = 1, \dots, n$  where  $|\cdot|_p$  is the  $p$ -adic absolute value such that  $|p|_p = \frac{1}{p}$ .

If  $\underline{\delta} \in (\widehat{T}_{Q_p, 0, L})_{\lambda'_J} = \widehat{Z}_{M_{Q_p, 0, L}} \times (\widehat{T}'_{Q_p, 0, L})_{\lambda'_J}$  and  $\lambda = \text{wt}(\underline{\delta})$ , then for any  $v \in S_p, \tau \in J_v$ , we have  $\lambda_{\tau, \tilde{q}_{v, i+j}} - \lambda_{\tau, \tilde{q}_{v, i+j+1}} = \tilde{\lambda}_{\tau, \tilde{q}_{v, i+j}} - \tilde{\lambda}_{\tau, \tilde{q}_{v, i+j+1}} \geq 0$  for  $1 \leq j \leq q_{v, i+1} - 1$  and  $0 \leq i \leq t_v - 1$ . We pick a constant  $C' > 0$  such that  $C' |\tau(\varpi_{\tilde{v}})|_p^{-1-C'} > 1$  for any  $v \in S_p, \tau \in \Sigma_v$ . Then the set of points in  $(\widehat{T}_{Q_p, 0, L})_{\lambda'_J}$  with integral dominant weights (i.e.  $\lambda \in (\mathbb{Z}^n)^{\Sigma_p}$  and  $\lambda_{\tau, i} \geq \lambda_{\tau, j}, \forall \tau \in \Sigma_p, i \leq j$ ) satisfying the following conditions

$$\lambda_{\tau, i} - \lambda_{\tau, i+1} > C', \forall i \in \{1, \dots, n-1\} \quad \text{if } v \in S_p, \tau \notin J_v \quad (3.5.22)$$

$$\lambda_{\tau, i} - \lambda_{\tau, i+1} > C', \forall i \in \{\tilde{q}_{v, 1}, \dots, \tilde{q}_{v, t_v-1}\} \quad \text{if } v \in S_p, \tau \in J_v \quad (3.5.23)$$

accumulates ([BHS17b, Def. 2.2]) at locally algebraic characters in  $(\widehat{T}_{Q_p, 0, L})_{\lambda'_J}$ .

Let  $Z$  be the subset of  $W$  consisting of points such that the images in  $(\widehat{T}_{Q_p, 0, L})_{\lambda'_J}$  via the map  $W \subset \mathcal{W}_{\lambda'_J} = \text{Spf}(S_\infty)^{\text{rig}} \times (\widehat{T}_{Q_p, 0, L})_{\lambda'_J} \rightarrow (\widehat{T}_{Q_p, 0, L})_{\lambda'_J}$  are locally algebraic and satisfy (3.5.22) and (3.5.23). Note that the last map is smooth, hence open. Since  $U$  contains a point with locally algebraic character, so is  $W$ . Hence  $Z$  is Zariski dense in  $W$ .

We claim that the set of dominant points in  $U$  satisfying the conditions (3.5.22) and (3.5.23) (i.e.  $\omega_{X(\lambda'_J)}^{-1}(Z)$ ) is Zariski dense in  $U$ . Otherwise, by the irreducibility of  $U$ , the Zariski closure of  $\omega_{X(\lambda'_J)}^{-1}(Z)$  in  $U$  has dimension strictly less than that of  $U$ . But the image of the closure of  $\omega_{X(\lambda'_J)}^{-1}(Z)$  in  $W$  is a closed subset containing  $Z$ , which must equal to  $W$  and hence shares the same dimension with  $U$ . This contradicts the assertion on the dimension of the Zariski closure of  $\omega_{X(\lambda'_J)}^{-1}(Z)$ . Hence the claim holds (this is the argument in the proof of [BHS17b, Thm. 3.19]).

We will show that any point in  $\omega_{X(\lambda'_J)}^{-1}(Z)$  is classical which allows us to conclude that the set of classical points is Zariski-dense in  $U$ . The Zariski density of the classical points in the whole  $X_p(\bar{\rho})(\lambda'_J)$  then follows from the fact that the set of locally algebraic characters is Zariski dense in  $(\widehat{T}_{Q_p, 0, L})_{\lambda'_J}$  and the last assertion in Proposition 3.5.19.

Now we assume that  $x = (y, \underline{\delta}) \in U \subset X_p(\bar{\rho})(\lambda'_J) \subset \text{Spf}(R_\infty)^{\text{rig}} \times \widehat{T}_{p, L}$  is a point such that the weight  $\lambda$  of  $\underline{\delta}$  is integral dominant and satisfies (3.5.22) and (3.5.23). We prove that  $x$  is a classical point.

Remark that the conditions (3.5.22) and (3.5.23) are some “small slope” conditions and the proof of the classicality will be essentially the same with the usual case (i.e., for points on  $X_p(\bar{\rho})$  as in [BHS17b, Thm. 3.19]). However, we cannot directly cite the proof of [BHS17b, Thm. 3.19]. This is because that under the restriction of the weights on the partial eigenvariety  $X_p(\bar{\rho})(\lambda'_J)$ , the points that satisfy the full “small slope” condition as in [BHS17b, (3.11)] can not be Zariski dense. The condition (3.5.22) and (3.5.23) here is only some *weaker* “small slope” condition. To prove the classicality, one will need furthermore to use the fact that  $x$  is “partially classical”, i.e.,  $x$  lies in the partial eigenvariety  $X_p(\bar{\rho})(\lambda'_J)$ . Ding has already proved such results in special cases in [Din17a] and [Din19c]. Since a direct reference is not available for our situation, we would like to write down the details of the proof below.

Without loss of generality, we assume that the residue field of  $x$  is  $L$ . By Proposition 3.5.13, there is a non-zero morphism

$$\mathcal{F}_{B_p}^{G_p} \left( \text{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\bar{u}^\infty}, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right) \hookrightarrow \Pi_\infty[\mathfrak{m}_y]^{\text{an}}.$$

By definition (see §3.5.3),

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda) = \left( \otimes_{v \in S_v, \tau \in \Sigma_v \setminus J_v} U(\mathfrak{g}_\tau) \otimes_{U(\mathfrak{b}_\tau)} \lambda_\tau \right) \otimes \left( \otimes_{v \in S_v, \tau \in J_v} U(\mathfrak{g}_\tau) \otimes_{U(\mathfrak{q}_\tau)} L_{\mathfrak{m}_{Q_v, \tau}}(\lambda_\tau) \right).$$

Hence the irreducible subquotients of  $U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda)$  are  $L(w w_0 \cdot \lambda) = \otimes_{\tau \in \Sigma_p} L(w_\tau w_{\tau,0} \cdot \lambda_\tau)$  where  $w = (w_\tau)_{\tau \in \Sigma_p}$ ,  $w_0 = (w_{\tau,0})_{\tau \in \Sigma_p}$  such that if  $\tau \in J_v$ , then  $L(w_\tau w_{\tau,0} \cdot \lambda_\tau)$  is a subquotient of  $U(\mathfrak{g}_\tau) \otimes_{U(\mathfrak{q}_\tau)} L_{\mathfrak{m}_{Q_v, \tau}}(\lambda_\tau)$ . In particular, the weight  $w_\tau w_{\tau,0} \cdot \lambda_\tau$  is  $\mathfrak{m}_{Q_v, \tau}$ -dominant (with respect to  $\mathfrak{b}_{Q_v, \tau}$ ) by [Hum08, Prop. 9.3(e)]. We fix one such  $w$  as above and assume  $\text{Hom}_{G_p} \left( \mathcal{F}_{\overline{B}_p}^{G_p} \left( \overline{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty[\mathfrak{m}_y]^{\text{an}} \right) \neq 0$ . We need prove  $w = w_0$ .

For each  $v \in S_p$ , let  $P_{w,v} = M_{P_{w,v}} N_{P_{w,v}}$  be the standard parabolic subgroup of upper-triangular block matrices in  $\text{GL}_{n/F_v}$  with the standard Levi decomposition such that  $P_{w,v}$  is maximal for  $w_v w_{v,0} \cdot \lambda$  (i.e the opposite  $\overline{P}_{w,v}$  is the maximal parabolic subgroup such that  $\overline{L}(-w_v w_{v,0} \cdot \lambda_v) \in \mathcal{O}^{\overline{\mathfrak{p}}_{w,v}}$ ). We also use the same notation  $P_{w,v} = M_{P_{w,v}} N_{P_{w,v}}$  for the associated  $p$ -adic Lie groups and let  $P_{w,p} = \prod_{v \in S_p} P_{w,v}$ , etc.. Any irreducible constituent of

$$\mathcal{F}_{\overline{B}_p}^{G_p} \left( \overline{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right) \simeq \mathcal{F}_{\overline{P}_{w,p}}^{G_p} \left( \overline{L}(-w w_0 \cdot \lambda), (\text{Ind}_{\overline{B}_p \cap M_{w,p}}^{M_{w,p}} \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1})^{\text{sm}} \right)$$

has the form  $\mathcal{F}_{\overline{P}_{w,p}}^{G_p} \left( \overline{L}(-w w_0 \cdot \lambda), \pi_{M_{P_{w,p}}} \right)$  for some irreducible constituent  $\pi_{M_{P_{w,p}}}$  of the smooth induction  $(\text{Ind}_{\overline{B}_p \cap M_{w,p}}^{M_{w,p}} \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1})^{\text{sm}}$  (cf. [Bre16, Thm 2.3(ii)(iii), (2.6)]). The central character of  $\pi_{M_{P_{w,p}}}$  is  $\underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}$ . Since at least one of such irreducible constituents appears in  $\Pi_\infty[\mathfrak{m}_y]$ , by [Bre16, Cor. 3.5], we get that  $z^{w w_0 \cdot \lambda} \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1}(z) \in \mathcal{O}_L$  for any  $z$  lies  $Z_{M_{P_{w,p}}}^+$  where  $Z_{M_{P_{w,p}}}$  is the center of  $M_{P_{w,p}}$ . Equivalently, for any  $v \in S_p$ ,

$$z^{w_v w_{v,0} \cdot \lambda_v} \underline{\delta}_{v, \text{sm}} \delta_{B_v}^{-1}(z) \in \mathcal{O}_L \quad (3.5.24)$$

for any  $z \in Z_{M_{P_{w,v}}}^+$ .

Firstly assume that there exist  $v \in S_p, \tau \in \Sigma_v \setminus J_v$  such that  $w_\tau \neq w_{\tau,0}$ . Then  $P_{w,v} \neq G_v$ . By (the proof of) [BHS17b, Prop. 5.4], there exists  $i_v \in \{1, \dots, n\}$  such that  $\gamma_{\tilde{v}, i_v} \in Z_{M_{P_{w,v}}}^+$  and  $|\gamma_{\tilde{v}, i_v}^{w_\tau w_{\tau,0} \cdot \lambda_\tau - \lambda_\tau}|_p \geq |\tau(\varpi_{\tilde{v}})|_p^{-1 - \min_i(\lambda_{\tau, i} - \lambda_{\tau, i+1})}$ . Since for any  $\tau' \in \Sigma_v, \lambda_{\tau'} + \rho_{\tau'}$  is strictly dominant and  $w_{\tau'} w_{\tau',0} \cdot \lambda_{\tau'} - \lambda_{\tau'} = w_{\tau'} w_{\tau',0}(\lambda_{\tau'} + \rho_{\tau'}) - (\lambda_{\tau'} + \rho_{\tau'})$ , we have  $|\gamma_{\tilde{v}, i_v}^{w_{\tau'} w_{\tau',0} \cdot \lambda_{\tau'} - \lambda_{\tau'}}|_p \geq 1$ . Hence by (3.5.22) we get

$$|\gamma_{\tilde{v}, i_v}^{w_v w_{v,0} \cdot \lambda_v} \underline{\delta}_{v, \text{sm}} \delta_{B_v}^{-1}(\gamma_{\tilde{v}, i_v})|_p = |\gamma_{\tilde{v}, i_v}^{w_v w_{v,0} \cdot \lambda_v - \lambda_v} \underline{\delta}_v \delta_{B_v}^{-1}(\gamma_{\tilde{v}, i_v})|_p \geq C |\tau(\varpi_{\tilde{v}})|_p^{-1 - \min_i(\lambda_{\tau, i} - \lambda_{\tau, i+1})} > 1$$

which contradicts (3.5.24).

Now we assume  $w_\tau = w_{\tau,0}$  for every  $\tau \notin J$ . Then for any  $v \in S_p, P_{w,v} \supset Q_v$ . Assume  $w_\tau \neq w_{\tau,0}$  for some  $v \in S_p, \tau \in J_v$ . By (the proof of) [Din19c, Lem. 3.18], there exists  $i_v \in \{\tilde{q}_{v,1}, \dots, \tilde{q}_{v,t_v-1}\}$  such that  $\gamma_{\tilde{v}, i_v} \in Z_{M_{P_{w,v}}}^+$  and  $|\gamma_{\tilde{v}, i_v}^{w_\tau w_{\tau,0} \cdot \lambda_\tau - \lambda_\tau}|_p \geq |\tau(\varpi_{\tilde{v}})|_p^{-1 - \min_i(\lambda_{\tau, \tilde{q}_{v,i}} - \lambda_{\tau, \tilde{q}_{v,i+1}})}$ . As in the previous step, by (3.5.23), we have

$$|\gamma_{\tilde{v}, i_v}^{w_v w_{v,0} \cdot \lambda_v} \underline{\delta}_{v, \text{sm}} \delta_{B_v}^{-1}(\gamma_{\tilde{v}, i_v})|_p = |\gamma_{\tilde{v}, i_v}^{w_v w_{v,0} \cdot \lambda_v - \lambda_v} \underline{\delta}_v \delta_{B_v}^{-1}(\gamma_{\tilde{v}, i_v})|_p \geq C |\tau(\varpi_{\tilde{v}})|_p^{-1 - \min_i(\lambda_{\tau, \tilde{q}_{v,i}} - \lambda_{\tau, \tilde{q}_{v,i+1}})} > 1$$

which also contradicts (3.5.24).

Therefore we conclude  $w = w_0$  and the point is classical.  $\square$

**Proposition 3.5.25.** *If  $(y, \underline{\delta}) = ((\rho_{\tilde{v}})_{v \in S_p}, z, \underline{\delta}) \in X_p(\bar{\rho}) \subset \mathfrak{X}_{\bar{\rho}_p} \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}_g) \times \widehat{T}_{p,L}$  is a classical point, then  $\rho_{\tilde{v}}$  is de Rham for any  $v \in S_p$ .*

*Proof.* This is the local-global compatibility result of [CEG<sup>+</sup>16, Lemma. 4.31]. Since  $x$  is classical, after possibly enlarging  $L$ , there is an injection  $L(\lambda) \otimes \pi_{\text{sm}} \hookrightarrow \Pi_\infty[\mathfrak{m}_y]$ , where  $\mathfrak{m}_y$  is the corresponding maximal ideal of  $R_\infty[\frac{1}{p}]$  and  $\pi_{\text{sm}}$  is some smooth representation of  $G_p$ . Let  $\Omega$  be the Bernstein component containing  $\pi_{\text{sm}}$ ,  $(J, \lambda_{\text{sm}})$  be a semisimple Bushnell-Kutzko type for



$\Omega$ , where  $J$  is compact open subgroup of  $G_p$  and  $\lambda_{\text{sm}}$  is an irreducible smooth representation of  $J$ , and  $\tau = (\tau_{\tilde{v}})_{v \in S_p}$  be the corresponding inertia type ([CEG<sup>+</sup>16, §3.3, §3.4]). After possibly enlarging  $L$ , we assume that  $\lambda_{\text{sm}}$  is defined over  $L$  and then by the type theory, there is a  $J$ -injection  $L(\lambda) \otimes \lambda_{\text{sm}} \hookrightarrow L(\lambda) \otimes \pi_{\text{sm}} \hookrightarrow \Pi_{\infty}[\mathfrak{m}_y]$  (for the rationality problem, cf. [CEG<sup>+</sup>16, §3.13], especially (3.15) in *loc. cit.*). We fix a  $J$ -stable  $\mathcal{O}_L$ -lattice  $\lambda^{\circ}$  of  $L(\lambda) \otimes \lambda_{\text{sm}}$ , then  $\text{Hom}_J(\lambda^{\circ}, \Pi_{\infty}[\mathfrak{m}_y]) \neq 0$ . Thus  $\text{Hom}_J(\lambda^{\circ}, \Pi_{\infty}^{\circ}[\mathfrak{m}_y]) \neq 0$  here  $\mathfrak{m}_y$  is viewed as an ideal of  $R_{\infty}$  that doesn't contain  $p$  and  $\Pi_{\infty}^{\circ}$  is the unit ball of  $\Pi_{\infty}$ . By the Schikhof duality ([CEG<sup>+</sup>16, §1.8]), we get that  $\mathfrak{m}_y$  is in the support of  $M_{\infty}(\lambda^{\circ})$  where  $M_{\infty}(\lambda^{\circ}) := \text{Hom}_{\mathcal{O}_L[[J]]}^{\text{cont}}(M_{\infty}, (\lambda^{\circ})')$  is finite free over  $S_{\infty}$  (cf. [CEG<sup>+</sup>16, Lem. 4.30]), thus finite over  $R_{\infty}$ . The methods in the proof of [CEG<sup>+</sup>16, Lem. 4.17 (1)] together with the classical local-global compatibility when  $\ell = p$  show that the action of  $R_{\infty}$  on  $M_{\infty}(\lambda^{\circ})$  factors through  $R_{\infty} \otimes_{\widehat{\otimes}_{v \in S_p} R'_{\rho_{\tilde{v}}}} \widehat{\otimes}_{v \in S_p} R_{\rho_{\tilde{v}}}^{\mathbf{h}_{\tilde{v}}, \tau_{\tilde{v}}}$  where  $R_{\rho_{\tilde{v}}}^{\mathbf{h}_{\tilde{v}}, \tau_{\tilde{v}}}$  is the framed potentially semi-stable deformation ring of Kisin ([Kis08, Thm. 2.7.6]) of inertia type  $\tau_{\tilde{v}}$  and Hodge-Tate weights  $\mathbf{h}_{\tilde{v}}$  associated with  $\lambda_v$ . This implies that  $\rho_{\tilde{v}}$  is potentially semi-stable for any  $v \in S_p$ .  $\square$

### 3.5.6 Partially de Rham trianguline $(\varphi, \Gamma)$ -modules

In this subsection, we use the density of de Rham points on the partial eigenvariety and the global triangulation to prove that points on the partial eigenvariety are “partially de Rham”.

Let  $x = ((\rho_{x,p}, \iota(\underline{\delta}_x), z_x)) = ((\rho_{x,\tilde{v}}, \iota_v(\underline{\delta}_{x,v}))_{v \in S_p}, z_x) \in \iota \left( \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_{\tilde{v}}) \right) \times (\mathfrak{X}_{\bar{p}^p} \times \mathbb{U}_g)$  be a point in  $X_p(\bar{\rho})$  (for notation see §3.4.2), and assume that  $\underline{\delta}_x$  is locally algebraic and  $\delta_{x,v} \in \mathcal{T}_{v,0}^n, \forall v \in S_p$  (Definition 3.4.2, or  $\iota(\underline{\delta}_x)$  is generic). Then  $\rho_{x,\tilde{v}}$  is an almost de Rham representation of  $\mathcal{G}_{F_{\tilde{v}}}$  (cf. §3.3.1). The global triangulation (on  $X_{\text{tri}}(\bar{\rho}_{\tilde{v}})$ ) implies that the  $(\varphi, \Gamma_{F_{\tilde{v}}})$ -module  $\mathcal{M}_{x,\tilde{v}} := D_{\text{rig}}(\rho_{x,\tilde{v}})[\frac{1}{t}]$  over  $\mathcal{R}_{k(x), F_{\tilde{v}}}[\frac{1}{t}]$  is equipped with a unique triangulation of parameter  $\underline{\delta}_{x,v}$  ([BHS19, Prop. 3.7.1]), denoted by  $\{0\} = \mathcal{M}_{x,\tilde{v},0} \subsetneq \cdots \subsetneq \mathcal{M}_{x,\tilde{v},n} = \mathcal{M}_{x,\tilde{v}}$ . Then for  $\tau \in \Sigma_v$ ,  $D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{x,\tilde{v}}))$  is equipped with a filtration  $D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{x,\tilde{v},\bullet}))$  of vector spaces over  $k(x)$  together with a nilpotent linear operator  $\nu_{x,\tau}$  which keeps the filtration. Recall that  $M_{Q_v}$  is the group of diagonal block matrices of the form  $\text{GL}_{q_{v,1}/F_{\tilde{v}}} \times \cdots \times \text{GL}_{q_{v,t_v}/F_{\tilde{v}}}$ . For  $i \in \{1, \dots, t_v\}$ , we let  $\nu_{x,\tau,i}$  be the action of  $\nu_{x,\tau}$  on

$$D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{x,\tilde{v},\tilde{q}_{v,i}})) / D_{\text{pdR},\tau}(W_{\text{dR}}(\mathcal{M}_{x,\tilde{v},\tilde{q}_{v,i-1}})).$$

Recall  $Q_v$  is a standard parabolic subgroup of  $\text{GL}_{n/F_{\tilde{v}}}$ . We denote by  $Q_{\tau} := Q_v \otimes_{F_{\tilde{v}},\tau} L$ . Then  $\nu_{x,\tau,i} = 0$  for all  $i = \{1, \dots, t_v\}$  if and only if  $\rho_{x,\tilde{v}}$  with the filtration  $\mathcal{M}_{x,\tilde{v},\bullet}$  is  $Q_{\tau}$ -de Rham (Definition 3.3.19).

**Proposition 3.5.26.** *Let  $x$  be a point as above. If  $x$  is in  $X_p(\bar{\rho})(\lambda'_J) \subset X_p(\bar{\rho})$ , then  $\rho_{x,\tilde{v}}$  with the filtration  $\mathcal{M}_{x,\tilde{v},\bullet}$  is  $Q_{\tau}$ -de Rham for any  $\tau \in J_v, v \in S_p$ .*

*Proof.* We fix  $i \in \{1, \dots, t_v\}, v \in S_p, \tau \in J_v$ . Consider the closed immersion  $X_p(\bar{\rho})(\lambda'_J) \hookrightarrow X_p(\bar{\rho})$  in §3.5.4. Let  $D_{\text{rig}}(\rho_{\tilde{v}}^{\text{univ}})$  be the  $(\varphi, \Gamma_{F_{\tilde{v}}})$ -module over  $\mathcal{R}_{X_p(\bar{\rho}), F_{\tilde{v}}}$  associated with the universal Galois representation  $\rho_{\tilde{v}}^{\text{univ}}$  of  $\mathcal{G}_{F_{\tilde{v}}}$  ([Liu15, Def. 2.12]). By Lemma 3.5.27 below and [BHS17b, Thm. 3.19], there is a birational proper morphism  $f : X' \rightarrow X_p(\bar{\rho})$ , a  $(\varphi, \Gamma_{F_{\tilde{v}}})$ -module  $M'$  over  $\mathcal{R}_{X', F_{\tilde{v}}}$  such that for any  $x' \in X'$  with  $\delta_{x',v} \in \mathcal{T}_{v,0}^n$  (we use the same notation  $\delta_v$ , etc. with different subscripts to denote the pull back of the character  $\delta_v$ , etc. from  $X_{\text{tri}}(\bar{\rho}_p)$ ), an isomorphism of  $(\varphi, \Gamma_{F_{\tilde{v}}})$ -module over  $\mathcal{R}_{k(x'), F_{\tilde{v}}}[\frac{1}{t}]$  :  $M'_{x'}[\frac{1}{t}] \simeq \mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i}} / \mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i-1}}$  where  $\mathcal{M}_{x',\tilde{v},\bullet}$  denotes the unique filtration of  $D_{\text{rig}}(\rho_{x',\tilde{v}})[\frac{1}{t}]$  of parameter  $\delta_{x',v,1}, \dots, \delta_{x',v,n}$ . We know  $X'$  is reduced. Since the Sen polynomial varies analytically and the set of strictly trianguline points is Zariski dense, we get that the  $\tau$ -Sen polynomial of  $M'_{x'}$  is equal to  $\prod_{j=\tilde{q}_{v,i-1}+1}^{\tilde{q}_{v,i}} (T - \text{wt}_{\tau}(\delta_{x',v,j}))$  for any  $x' \in X'$ . Let  $X''$  be the preimage of  $X_p(\bar{\rho})(\lambda'_J)$  under  $f$ . Then  $f|_{X''}$  is

still proper ([BGR84, §9.6.2]). Let  $M'_{X''}$  be the pullback of  $M'$  to  $X''$  and let  $M'_{X''}(\delta_{X'',v,\tilde{q}_v}^{-1}) := M'_{X''} \otimes_{\mathcal{R}_{X'',F_{\tilde{v}}}} \mathcal{R}_{X'',F_{\tilde{v}}}(\delta_{X'',v,\tilde{q}_v}^{-1})$ . Thus for any  $x' \in X''$ , the  $\tau$ -Sen weights of  $M'_{x'}(\delta_{x',v,\tilde{q}_v}^{-1})$ :

$$\begin{aligned} & (\text{wt}_{\tau}(\delta_{x',v,\tilde{q}_{v,i-1}+1}) - \text{wt}_{\tau}(\delta_{x',v,\tilde{q}_v}), \dots, \text{wt}_{\tau}(\delta_{x',v,\tilde{q}_{v,i-1}+q_{v,i}}) - \text{wt}_{\tau}(\delta_{x',v,\tilde{q}_v})) \\ &= \left( (\tilde{\lambda}_{\tau,\tilde{q}_{v,i-1}+1} - \tilde{q}_{v,i-1}) - (\tilde{\lambda}_{\tau,\tilde{q}_v} - \tilde{q}_{v,i} + 1), \dots, 0 \right) \end{aligned}$$

are all certain fixed integers (see §3.5.4). Applying Proposition A.3.4 in Appendix A, we conclude that the subset of points  $x' \in X''$  such that  $M'_{x'}[\frac{1}{t}](\delta_{x',v,\tilde{q}_v}^{-1})$  is  $\tau$ -de Rham, the points such that the nilpotent operator vanishes on  $D_{\text{pdR},\tau} \left( W_{\text{dR}}(M'_{x'}(\delta_{x',v,\tilde{q}_v}^{-1})[\frac{1}{t}]) \right)$ , is Zariski closed in  $X''$ . We denote this subset by  $Y$ . Then  $f(Y)$  is an analytic closed subset of  $X_p(\bar{\rho})(\lambda'_J)$  ([BGR84, Prop. 9.6.3/3]).

We pick an affinoid  $U$  of  $X_p(\bar{\rho})(\lambda'_J)$  containing  $x$  as in Proposition 3.5.21 so that the classical points is Zariski dense in  $U$ . By shrinking  $U$  and its image in  $\mathcal{W}_{\lambda'_J}$  suitably, we can assume that for any point  $x' \in U$ ,  $\delta_{x',v} \in \mathcal{T}_{v,0}^n$  (this is possible since  $\mathcal{T}_{v,0}^n$  is Zariski open in the space of characters of  $(F_{\tilde{v}}^{\times})^n$ ). Suppose that  $x'$  is a point in  $U$  such that  $\delta_{x',v}$  is locally algebraic and  $x'' \in f^{-1}(x')$ . We have  $\mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i}}/\mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i-1}} \otimes_{k(x')} k(x'') \simeq \mathcal{M}_{x'',\tilde{v},\tilde{q}_{v,i}}/\mathcal{M}_{x'',\tilde{v},\tilde{q}_{v,i-1}}$ . Since  $W_{\text{dR}}(\mathcal{R}_{k(x'),F_{\tilde{v}}}(\delta_{x',v,\tilde{q}_v}^{-1})[\frac{1}{t}])$  is trivial ([BHS19, Lem. 3.3.7]) and the functor  $W_{\text{dR}}(-)$  is tensor functorial, we get that  $\mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i}}/\mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i-1}}$  is  $\tau$ -de Rham if and only if

$$(\mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i}}/\mathcal{M}_{x',\tilde{v},\tilde{q}_{v,i-1}})(\delta_{x',v,\tilde{q}_v}^{-1})$$

is  $\tau$ -de Rham if and only if  $x' \in f(Y)$ . Then by Proposition 3.5.25,  $f(Y) \cap U$  contains a Zariski dense subset of  $U$  (classical points in  $U$ ). Furthermore,  $f(Y) \cap U$  is Zariski closed in  $U$  ([BGR84, Prop. 9.5.3/2]). Thus  $U \subset f(Y)$ . Hence  $\mathcal{M}_{x,\tilde{v},\tilde{q}_{v,i}}/\mathcal{M}_{x,\tilde{v},\tilde{q}_{v,i-1}}$  is  $\tau$ -de Rham.  $\square$

**Lemma 3.5.27.** *Let  $X$  be a reduced analytic rigid space over  $L$  and  $M$  is a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{X,K}$  of rank  $n$  where  $K$  is a local field over  $\mathbb{Q}_p$ . We assume that there exists a Zariski dense subset  $X_{\text{alg}}$  of  $X$  such that  $M$  is densely pointwise strictly trianguline ([KPX14, Def. 6.3.2]) with respect to a parameter  $\delta_{X,1}, \dots, \delta_{X,n} : K^{\times} \rightarrow \Gamma(X, \mathcal{O}_X)^{\times}$  and the subset  $X_{\text{alg}}$ . We assume furthermore that if  $x \in X_{\text{alg}}$ , then  $\delta_x \in \mathcal{T}_{\text{reg}}^n$  (see §3.3.5). Then for any  $0 \leq a < b \leq n$ , there exists a birational proper map  $f : X' \rightarrow X$  and a  $(\varphi, \Gamma_K)$ -module  $M'$  over  $\mathcal{R}_{X',K}$  such that, let  $\delta_{X',1}, \dots, \delta_{X',n}$  be the pull back of characters  $\delta_{X,1}, \dots, \delta_{X,n}$ , the following statements hold.*

(1) *The set of points  $x \in X'$  such that  $M'_x \simeq \text{fil}_b((f^*M)_x)/\text{fil}_a((f^*M)_x)$ , where  $\text{fil}_{\bullet}((f^*M)_x)$  is the unique strictly trianguline filtration on  $(f^*M)_x$  of parameter  $\delta_{x,1}, \dots, \delta_{x,n}$ , contains  $f^{-1}(X_{\text{alg}})$  and is Zariski open dense in  $X'$ .*

(2) *Suppose  $x \in X'$  such that  $\delta_x \in \mathcal{T}_0^n$ . Assume that  $A$  is a finite-dimensional local  $L$ -algebra with residue field  $k(x)$  and a map  $\text{Sp}(A) \rightarrow X'$  with image  $x$ , then the pull back  $(f^*M)_{A[\frac{1}{t}]}$  is trianguline with a unique triangulation  $\text{fil}_{\bullet}((f^*M)_{A[\frac{1}{t}]})$  of parameter  $\delta_{A,1}, \dots, \delta_{A,n}$  in the sense of §3.3.1. Moreover there is an isomorphism  $M'_A[\frac{1}{t}] \simeq \text{fil}_b((f^*M[\frac{1}{t}])_A)/\text{fil}_a((f^*M[\frac{1}{t}])_A)$  and  $M'_A[\frac{1}{t}]$  has a parameter  $\delta_{A,a+1}, \dots, \delta_{A,b}$ .*

*Proof.* By [KPX14, Cor. 6.3.10], after replacing  $X$  (resp.  $X_{\text{alg}}$ ) by some  $X'$  (resp.  $f^{-1}(X_{\text{alg}})$ ), we may assume that  $M$  admits a filtration  $\text{fil}_{\bullet}M$  of  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_{X,K}$  satisfying the requirement (1) and (2) in *loc. cit.*. The uniqueness of the triangulation of given parameter in (2) is by [BHS19, Lem. 3.4.3 & Prop. 3.4.6].

If  $a = 0$ , we can take the submodule  $M' := \text{fil}_bM$  of  $M$ . Then (1) of the Lemma is satisfied by the choice. The existence of a triangulation of  $M_A[\frac{1}{t}]$  in (2) follows from (2) in [KPX14, Cor. 6.3.10] which is of parameter  $\delta_{A,1}, \dots, \delta_{A,n}$  and it is by our choice that  $M'_A[\frac{1}{t}] \simeq \text{fil}_b(M_A[\frac{1}{t}])$ .

By replacing  $M$  with  $\text{fil}_bM$ , we may assume  $b = n$ . We need roughly pick a “quotient”  $(\varphi, \Gamma_K)$ -modules of  $M$  (on some  $X'$ ) rather than a submodule as in the previous step. For a

$(\varphi, \Gamma_K)$ -module  $M_Y$  over  $\mathcal{R}_{Y,K}$  for a rigid space  $Y$ , we let  $M_Y^\vee := \text{Hom}_{\mathcal{R}_{Y,K}}(M, \mathcal{R}_{Y,K})$  be the dual  $(\varphi, \Gamma_K)$ -module of  $M$ . The dual functor  $(\cdot) \mapsto (\cdot)^\vee$  of  $(\varphi, \Gamma_K)$ -modules commutes with base change, is exact on short exact sequences of  $(\varphi, \Gamma_K)$ -modules and sends  $\mathcal{R}_{Y,K}(\delta)$  to  $\mathcal{R}_{Y,K}(\delta^{-1})$  for any continuous character  $\delta : K^\times \rightarrow \Gamma(Y, \mathcal{O}_Y)^\times$  (cf. [KPX14, Con. 6.2.4]). Then the  $(\varphi, \Gamma_K)$ -module  $M^\vee$  over  $\mathcal{R}_{X,K}$  is densely pointwise strictly trianguline with respect to parameters  $\delta_{X,n}^{-1}, \dots, \delta_{X,1}^{-1}$  by the assumption and [KPX14, Prop. 6.2.8]. By [KPX14, Cor. 6.3.10], after replacing  $X$  (resp.  $X_{\text{alg}}$ ) by some  $X'$  (resp.  $f^{-1}(X_{\text{alg}})$ ), we may assume that  $M^\vee$  admits a filtration  $\text{fil}_\bullet M^\vee$  of  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_{X,K}$  such that for any point  $x$  in a Zariski open dense subset  $Z$  containing  $X_{\text{alg}}$ ,  $(\text{fil}_\bullet M^\vee)_x$  is a strictly triangulation of  $M_x^\vee$  with parameters  $\delta_{x,n}^{-1}, \dots, \delta_{x,1}^{-1}$  and each  $(\text{fil}_i M^\vee / \text{fil}_{i-1} M^\vee)[\frac{1}{t}]$  is isomorphic to  $\mathcal{R}_{X,K}(\delta_{n-i+1}^{-1})[\frac{1}{t}]$  up to a line bundle. We set  $M' = (\text{fil}_{n-a} M^\vee)^\vee$ . Then for  $x \in Z$ ,  $M'_x \simeq M_x / \text{fil}_a M_x$  is trianguline of parameter  $\delta_{x,a+1}, \dots, \delta_{x,n}$ . For  $x \in X$  and  $A$  satisfying the condition in (2) of the lemma, we get by the construction that  $(\text{fil}_{n-a} M^\vee)_A[\frac{1}{t}]$  is trianguline of parameter  $\delta_{A,n}^{-1}, \dots, \delta_{A,a+1}^{-1}$  and  $M'_A[\frac{1}{t}]$  is trianguline of parameter  $\delta_{A,n}^{-1}, \dots, \delta_{A,1}^{-1}$ . Taking dual, we get that  $M_A[\frac{1}{t}]$  is trianguline of parameters  $\delta_{A,1}, \dots, \delta_{A,n}$  and  $M'_A[\frac{1}{t}]$  is trianguline of parameters  $\delta_{A,a+1}, \dots, \delta_{A,n}$ .  $\square$

Then we immediately get the main result of this section.

**Theorem 3.5.28.** *Let  $x = ((\rho_p, \underline{\delta}), z) \in X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}_g)$  be a point such that  $\underline{\delta}$  is locally algebraic and generic. Let  $\lambda = \text{wt}(\underline{\delta})$  and  $\underline{\delta}_{\text{sm}}$  be the smooth part of  $\underline{\delta}$ . Let  $y$  be the image of  $x$  in  $\mathfrak{X}_\infty$ . If for some  $v \in S_p, \tau \in \Sigma_v, \lambda_\tau$  is  $\mathfrak{m}_{Q_v, \tau}$ -dominant (with respect to  $\mathfrak{b}_{Q_v, \tau}$ ) and we have*

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \bar{L}(-\lambda), \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty^{\text{an}}[\mathfrak{m}_y] \otimes_{k(y)} k(x) \right) \neq 0,$$

*then  $\rho_{\tilde{v}}$  with the unique triangulation on  $D_{\text{rig}}(\rho_{\tilde{v}})[\frac{1}{t}]$  of parameter  $\delta_{v,1}, \dots, \delta_{v,n}$  is  $Q_\tau$ -de Rham.*

*Proof.* We take  $J = \{\tau\} \subset \Sigma_p$ . The irreducible  $U(\mathfrak{g})$ -module  $L(\lambda)$  of the highest weight  $\lambda$  is the unique quotient of  $U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda)$  (for the notation, see §3.5.3) by [Hum08, §9.4]. Thus by the functoriality of  $\mathcal{F}_{\bar{B}_p}^{G_p}(-, -)$ , we get

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \text{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(\lambda), L \right)^{\bar{u}^\infty}, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty^{\text{an}}[\mathfrak{m}_y] \otimes_{k(y)} k(x) \right) \neq 0.$$

Hence by Proposition 3.5.13,  $x \in X_p(\bar{\rho})(\lambda'_J)$ . By Proposition 3.5.26,  $\rho_{\tilde{v}}$  with the triangulation is  $Q_\tau$ -de Rham.  $\square$

### 3.5.7 Conjectures on partial classicality and locally analytic socle

We state a conjecture on partial classicality of almost de Rham Galois representations and discuss its relationship with the locally analytic socle conjecture. We only state the conjecture for the patched eigenvariety where the local model is available since in this special case the conjecture is more accessible and the converse of the conjecture is known to some extent (Theorem 3.5.28). We also give some partial results.

We use the notation before Theorem 3.4.10. Let  $x = ((\rho_p, \underline{\delta}), z) \in X_p(\bar{\rho})(L) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  be a generic point with integral weights. Let  $\mathbf{h}$  be the Hodge-Tate weights of  $\rho_p$  and  $\lambda$  be the weight of  $\underline{\delta}$ . We assume that  $\lambda + \rho$  is dominant (with respect to  $\mathfrak{b}$ ). There are the companion points  $x_w = ((\rho_p, \underline{\delta}_w), z) \in (\mathfrak{X}_{\bar{\rho}_p} \times \hat{T}_{p,L}) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  for  $w \in W_{G_p}$  where  $\underline{\delta}_w$  is defined in the end of §3.4.3. Let  $r_x$  be the image of  $x$  in  $\mathfrak{X}_\infty$  and  $\mathfrak{m}_{r_x}$  be the corresponding maximal ideal of  $R_\infty[\frac{1}{p}]$ . For  $v \in S_p$ , as  $\iota_v^{-1}(\underline{\delta}_v) \in \mathcal{T}_{v,0}^n$ ,  $\mathcal{M}_{\tilde{v}} := D_{\text{rig}}(\rho_{\tilde{v}})[\frac{1}{t}]$  is trianguline with a unique triangulation  $\mathcal{M}_{\tilde{v}, \bullet}$  of parameter  $\iota_v^{-1}(\underline{\delta}_v)$ . Let  $Q_p = \prod Q_v$  be a standard parabolic subgroup of  $G_p$  as in §3.5.1 and let  $Q_\tau$  be the base change to  $L$  of the standard parabolic subgroup of  $\text{GL}_{n/F_{\tilde{v}}}$  via

$\tau : F_{\tilde{v}} \rightarrow L$  for all  $\tau \in \Sigma_v, v \in S_p$ . Recall that  $J_{Q_p}(\Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}})$  is a locally analytic representation of  $M_{Q_p}$ . Take a non-empty subset  $J \subset \Sigma_p$  and let  $J_v = J \cap \Sigma_v$  for all  $v \in S_p$ . Following [Din17a, §6.1], a vector  $v \in J_{Q_p}(\Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}})$  is called  $J$ -classical if there is a finite-dimensional algebraic  $\mathfrak{m}_{Q_p, J}$ -module  $V$  and a  $\mathfrak{m}_{Q_p, J}$ -equivariant map  $V \rightarrow J_{Q_p}(\Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}})$  such that the image of the map contains  $v$ .

We say that the Hodge-Tate weights  $\mathbf{h}$  are regular if  $h_{\tau, i} \neq h_{\tau, j}$  for all  $\tau \in \Sigma_p, i \neq j$ . We now state the conjecture on partial classicality.

**Conjecture 3.5.29.** *Assume that  $\mathbf{h}$  is regular and the pair  $(\rho_{\tilde{v}}, \mathcal{M}_{\tilde{v}, \bullet})$  as above is  $Q_{\tau}$ -de Rham for all  $\tau \in J_v, v \in S_p$  (Definition 3.3.19), then  $J_{Q_p}(\Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}})$  contains non-zero  $J$ -classical vectors.*

**Remark 3.5.30.** The assumption that  $\mathbf{h}$  is regular is necessary: if  $\rho_p$  is de Rham with non-regular Hodge-Tate weights,  $Q_p = G_p$  and  $J = \Sigma_p$ , then there exists no non-zero locally algebraic vector in  $\Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}}$  by the local-global compatibility or [DPS20].

Next, we formulate a weak version of the locally analytic socle conjecture for the point  $x$ . Recall that  $\text{Spf}(R_{\rho_p, \mathcal{M}_{\bullet}})$  is isomorphic to  $\widehat{X}_{P_p, x_{\text{pdR}}}$  up to formally smooth morphisms where  $X_{P_p}$  is the variety defined in §3.2 with respect to the standard parabolic subgroup  $P_p$  determined by  $\mathbf{h}$  or  $\lambda$  of the algebraic group  $\prod_{v \in S_p} \text{Res}_{F_{\tilde{v}}/\mathbb{Q}_p}(\text{GL}_{n/F_{\tilde{v}}}) \times_{\mathbb{Q}_p} L$  as in §3.4.3 and  $x_{\text{pdR}} = (x_{\text{pdR}, \tilde{v}})_{v \in S_p}$  is a point on  $X_{P_p}$  associated with  $(\rho_p, \mathcal{M}_{\bullet}) = (\rho_{\tilde{v}}, \mathcal{M}_{\tilde{v}, \bullet})_{v \in S_p}$  as in §3.3.5. By §3.3.6,  $\text{Spec}(\overline{R}_{\rho_p, \mathcal{M}_{\bullet}}) \subset \text{Spec}(R_{\rho_p})$  is a union of cycles of the form  $\mathfrak{Z}_w$  for  $w \in W_{G_p}/W_{P_p}$ . Moreover,  $\mathfrak{Z}_w \neq \emptyset$  if and only if  $x_{\text{pdR}} \in Z_{P_p, w}$ . Let  $w_x \in W_{G_p}/W_{P_p}$  be the element as before parameterizing the relative position between the Hodge filtration and the trianguline filtration. Then by discussions in §3.2.5,  $\mathfrak{Z}_w \neq \emptyset$  only if  $w \geq w_x$  in  $W_{G_p}/W_{P_p}$ . In general, there exists  $w \geq w_x$  such that  $\mathfrak{Z}_w = \emptyset$  (for example, if  $\mathbf{h}$  is regular and  $\rho_p$  is not de Rham, then  $\mathfrak{Z}_{w_0} = \emptyset$ ).

**Conjecture 3.5.31.** *If  $\mathfrak{Z}_w \neq \emptyset$ , then*

$$\text{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \overline{L}(-w w_0 \cdot \lambda), \delta_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}} \right) \neq 0.$$

**Remark 3.5.32.** This is only a weak form of the locally analytic socle conjecture as we explain below.

For simplicity, we assume that  $\mathbf{h}$  is regular and  $z$  is in the smooth locus of  $\mathfrak{X}_{\overline{p}^p} \times \mathbb{U}^g$ . As in the proof of [BHS19, Thm. 5.3.3] or Proposition 3.4.13, the dual of

$$\text{Hom}_{U(\mathfrak{g})} (L(w w_0 \cdot \lambda), \Pi_{\infty}^{\text{an}})^{U_0} [\mathfrak{m}_{r_x}^{\infty}] [\mathfrak{m}_{\delta_{\text{sm}}}^{\infty}]$$

is a module over  $\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x}$  via the map  $\text{Spec}(\widehat{\mathcal{O}}_{X_p(\overline{p})_{w w_0 \cdot \lambda}, x_w}) \hookrightarrow \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$  (remark that  $\text{Spec}(\widehat{\mathcal{O}}_{X_p(\overline{p})_{w w_0 \cdot \lambda}, x_w})$  can be empty), denoted by  $\mathcal{L}(w w_0 \cdot \lambda)$  in [BHS19, (5.22)]. Abusing the notation, let  $[\mathcal{L}(w w_0 \cdot \lambda)]$  be the set-theoretic support of  $\mathcal{L}(w w_0 \cdot \lambda)$  in  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$ . For  $w' \in W_{G_p}$ , we let  $\mathfrak{Z}'_{w'} \subset \text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x})$  be the preimage of the closed subset  $\mathfrak{Z}_{w'}$  in  $\text{Spec}(R_{\rho_p})$  via the formally smooth morphism  $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X}_{\infty}, r_x}) \rightarrow \text{Spec}(R_{\rho_p})$ . Now let  $\mathfrak{C}_{w'}$  be the union of  $\mathfrak{Z}'_{w''}, w'' \in W_{G_p}$  such that  $a_{w', w''} \geq 1$  where  $a_{w', w''}$  is the coefficient appeared in [BHS19, (2.16)] (with respect to the algebraic group  $\prod_{v \in S_p} \text{Res}_{F_{\tilde{v}}/\mathbb{Q}_p}(\text{GL}_{n/F_{\tilde{v}}}) \times_{\mathbb{Q}_p} L$ ) so that  $\mathfrak{C}_{w'}$  is the underlying set of the Kazhdan-Lusztig cycle appeared in [BHS19, (5.24)] with the same notation. Then  $\mathfrak{Z}'_{w'} \subset \mathfrak{C}_{w'}$  and  $\mathfrak{Z}'_{w''} \subset \mathfrak{C}_{w'}$  only if  $w' \geq w''$  in  $W_{G_p}$  ([BHS19, Thm. 2.4.7(iii)]).

In the spirit of the analogue between the subsets (or even real cycles)  $[\mathcal{L}(w w_0 \cdot \lambda)]$  and  $\mathfrak{C}_w$ , Conjecture 3.5.31 expects  $\mathfrak{Z}'_w \subset [\mathcal{L}(w w_0 \cdot \lambda)]$ . In general, it is possible that there exists some  $w' > w$  such that  $\mathfrak{Z}'_w \subset \mathfrak{C}_{w'}$  ([BHS19, Rem. 2.4.5]). Hence we also expect that  $\mathfrak{Z}'_w \subset [\mathcal{L}(w' w_0 \cdot \lambda)]$  and thus if  $\mathfrak{Z}'_w \neq \emptyset$ , we expect that

$$\text{Hom}_{G_p} \left( \mathcal{F}_{B_p}^{G_p} \left( \overline{L}(-w' w_0 \cdot \lambda), \delta_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_{\infty}[\mathfrak{m}_{r_x}]^{\text{an}} \right) \neq 0.$$

But now it may happen that  $\mathfrak{Z}'_{w'} = \emptyset$  (this will not happen if  $\rho_p$  is de Rham, see (3.2.29)). Hence Conjecture 3.5.31 should not predict all the possible companion constituents in general.

However, Theorem 3.4.10 imposes some restriction for elements  $w' \in W_{G_p}$  such that  $\mathfrak{Z}'_w \subset [\mathcal{L}(w'w_0 \cdot \lambda)]$  just as the restriction for the characteristic cycles in Proposition 3.2.32.

*Remark 3.5.33.* The weak conjecture on locally analytic socles (Conjecture 3.5.31) still implies the existence of the companion points on the eigenvariety: since we have  $\mathfrak{Z}_{w_x} \neq \emptyset$  by definition, we get  $x_{w_x} \in X_p(\bar{\rho})$  if Conjecture 3.5.31 is true, which implies that  $x_w \in X_p(\bar{\rho})$  for all  $w \geq w_x$  in  $W_{G_p}/W_{P_p}$  by [BHS17a, Thm. 5.5].

**Proposition 3.5.34.** *Conjecture 3.5.31 implies Conjecture 3.5.29.*

*Proof.* Now  $\mathbf{h}$  is regular. Let  $B_p = \prod_{\tau \in \Sigma_p} B_\tau$  be the standard Borel subgroup of upper-triangular matrices in  $\prod_{v \in S_p} \text{Res}_{F_{\bar{v}}/\mathbb{Q}_p}(\text{GL}_n/F_{\bar{v}}) \times_{\mathbb{Q}_p} L$ . Since  $(\rho_{\bar{v}}, \mathcal{M}_{\bar{v}, \bullet})$  is  $Q_\tau$ -de Rham for  $\tau \in J_v, v \in S_p$ , by definition, we have  $x_{\text{pdR}} \in Z_{\tilde{Q}_p, P_p}$  where  $\tilde{Q}_p := \prod_{\tau \in J} Q_\tau \prod_{\tau \notin J} B_\tau$  and  $Z_{\tilde{Q}_p, P_p}$  is defined in §3.2.5. By Corollary 3.2.22, there exists  $w \in W_{G_p}$  such that  $x_{\text{pdR}} \in Z_{B_p, w}$  and  $Z_{B_p, w} \subset Z_{\tilde{Q}_p, B_p}$ . Take one such  $w$ , then  $\mathfrak{Z}_w \neq \emptyset$ . By Theorem 3.2.26,  $ww_0 \cdot \lambda$  is a dominant weight for the Lie algebra of the standard Levi subgroup of  $\tilde{Q}_p$ . In particular,  $L(ww_0 \cdot \lambda)$  is the irreducible quotient of  $U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(ww_0 \cdot \lambda)$  where  $L_J(ww_0 \cdot \lambda)$  is defined via (3.5.7). The statement of Conjecture 3.5.31 and the exactness of the functor  $\mathcal{F}_{\bar{B}_p}^{G_p}$  imply that

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \text{Hom} \left( U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_J(ww_0 \cdot \lambda), L \right)^{\bar{u}^\infty}, \underline{\delta}_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}} \right) \neq 0.$$

By the beginning part of the proof of Proposition 3.5.9, we get

$$\text{Hom}_{U(\mathfrak{m}_{Q_p})} (L_J(ww_0 \cdot \lambda), J_{Q_p}(\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}})) \neq 0.$$

Then the non-zero image of any non-zero  $U(\mathfrak{m}_{Q_p})$ -equivariant map  $L_J(ww_0 \cdot \lambda) \rightarrow J_{Q_p}(\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}})$  gives rise to non-zero  $J$ -classical vectors since  $L_{M_{Q_p}}(\lambda_J)$  in (3.5.7) is a finite-dimensional representation of  $\mathfrak{m}_{Q_p, J}$ .  $\square$

**Corollary 3.5.35.** *Conjecture 3.5.29 is true if  $\rho_{\bar{v}}$  is crystalline for every  $v \in S_p$ .*

*Proof.* By [BHS19, Thm. 5.3.3], Conjecture 3.5.31 is true for crystalline points. Hence Conjecture 3.5.29 is true for such points by Proposition 3.5.34.  $\square$

*Remark 3.5.36.* The method in the proof of Proposition 3.4.13 based on Theorem 3.4.10 can be used to give non-trivial evidence for Conjecture 3.5.31 beyond de Rham cases. To convince the reader, we sketch an example here. Let's keep the notation and assumptions in Conjecture 3.5.31 and Remark 3.5.32 and assume  $w_x = e$  is the identity in  $W_{G_p}$ . Hence  $\mathfrak{Z}'_e \neq \emptyset$ . By the (dual of) exact sequences as [BHS19, (5.21)], writing  $M(w_0w_0 \cdot \lambda)$  as a successive extension of subquotients  $L(ww_0 \cdot \lambda), w \in W_{G_p}$ , we get an equality of topological spaces inside  $\text{Spec}(\hat{\mathcal{O}}_{\mathfrak{X}_\infty, r_x})$ :

$$\text{Spec}(\hat{\mathcal{O}}_{X_p(\bar{\rho})_{w_0w_0 \cdot \lambda, x_{w_0}}}) = \text{Supp}(\mathcal{M}_\infty \otimes_{\mathcal{O}_{X_p(\bar{\rho})}} \hat{\mathcal{O}}_{X_p(\bar{\rho})_{w_0w_0 \cdot \lambda, x_{w_0}}}) = \bigcup_{w \in W_{G_p}} [\mathcal{L}(ww_0 \cdot \lambda)].$$

On the other hand, by the theory of the local model (3.2.28), we have

$$\text{Spec}(\hat{\mathcal{O}}_{X_p(\bar{\rho})_{w_0w_0 \cdot \lambda, x_{w_0}}}) = \bigcup_{w \in W_{G_p}} \mathfrak{Z}'_w$$

Hence we get

$$\bigcup_{w \in W_{G_p}} [\mathcal{L}(ww_0 \cdot \lambda)] = \bigcup_{w \in W_{G_p}} \mathfrak{Z}'_w.$$

A priori, in the above equality, we know only  $\mathfrak{Z}'_e \neq \emptyset$ . As  $\mathbf{h}$  is antidominant, we see that any irreducible component of  $\mathfrak{Z}'_e$  is not contained in  $\text{Spec}(R_{\rho_p, \mathcal{M}_\bullet}^{\tilde{Q}_p})$  for any standard parabolic subgroup  $\tilde{Q}_p$  of  $\prod_{v \in S_p} \text{Res}_{F_{\bar{v}}/\mathbb{Q}_p}(\text{GL}_n/F_{\bar{v}}) \times_{\mathbb{Q}_p} L$  unless  $\tilde{Q}_p$  is the Borel subgroup by Theorem 3.2.26. But for any  $w \neq e$ , the cycle  $[\mathcal{L}(ww_0 \cdot \lambda)]$  is contained in  $\text{Spec}(R_{\rho_p, \mathcal{M}_\bullet}^{\tilde{Q}_p})$  for some non-Borel  $\tilde{Q}_p$  by Theorem 3.4.10. Hence any component of  $\mathfrak{Z}'_e$  is not contained in  $[\mathcal{L}(ww_0 \cdot \lambda)]$  for any  $w \neq e$ . Since  $\mathfrak{Z}'_e$  is equidimensional with the same dimension as  $\text{Spec}(\hat{\mathcal{O}}_{X_p(\bar{\rho})_{w_0 w_0 \cdot \lambda, x_{w_0}}})$ , we must have  $\mathfrak{Z}'_e \subset [\mathcal{L}(w_0 \cdot \lambda)]$ . Hence  $[\mathcal{L}(w_0 \cdot \lambda)] \neq \emptyset$  which implies that

$$\text{Hom}_{G_p} \left( \mathcal{F}_{\bar{B}_p}^{G_p} \left( \bar{L}(-w_0 \cdot \lambda), \delta_{\text{sm}} \delta_{B_p}^{-1} \right), \Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}} \right) \neq 0.$$

Then  $x_e \in X_p(\bar{\rho})$ . By [BHS17a, Thm. 5.5],  $x_w \in X_p(\bar{\rho})$  for all  $w \in W_{G_p}$ .

*Remark 3.5.37.* Theorem 3.4.18 only concerns the existence of all companion constituents that can be seen via the local models. In general, there exist companion points  $x' = ((\rho_p, \underline{\delta}'), z) \in X_p(\bar{\rho}) \subset \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  of  $x$  such that  $\underline{\delta}^{-1} \underline{\delta}'$  is not an algebraic character. In the situation of Theorem 3.4.18,  $x'$  is one of points  $((\rho_p, \underline{\delta}_{\mathcal{R}', w'}), z)$  for some other choice of refinements  $\mathcal{R}'$ . The existence of all companion points of all refinements on  $X_p(\bar{\rho})$  for regular crystalline points is obtained by the existence of locally algebraic constituents of the form  $L(\lambda) \otimes (\text{Ind}_{\bar{B}_p}^{G_p} \delta_{\text{sm}} \delta_{B_p}^{-1})^{\text{sm}}$  in  $\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}$  and the intertwining between smooth principal series (see the beginning of [BHS19, §5.3] for details). In the non-regular crystalline cases, there exist no such locally algebraic constituents in the socle of  $\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}}$ . Still, in some special cases, one can obtain the existence of some companion points of other refinements using the locally algebraic representations of  $M_{Q_p}$  that appear in  $J_{Q_p}(\Pi_\infty[\mathfrak{m}_{r_x}]^{\text{an}})$  (cf. [Din19c, Prop. 2.14]).



## Chapter 4

# Companion points on the eigenvariety with non-regular weights

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### 4.1 Introduction

Let  $p$  be a prime number, then there exists pairs of ( $p$ -adic) elliptic modular eigenforms  $(f, g)$  of level  $\Gamma_0(p) \cap \Gamma_1(N)$  for some  $p \nmid N$  such that  $f$  and  $g$  share the same eigenvalues for Hecke operators  $T_\ell$  when  $\ell \nmid pN$  (i.e.  $f$  and  $g$  are associated with the same  $p$ -adic Galois representation), but have *different* non-zero eigenvalues for the  $U_p$ -operator. The results on the existence of such *companion forms* for  $p$ -adic or mod- $p$  modular forms, as of Gross in [Gro90], have many significant applications. For example, Buzzard and Taylor ([BT99], and see [Buz03]) use Gross's results to prove the classicality of overconvergent  $p$ -adic weight one modular forms (hence certain cases of the Artin's conjecture), and their methods have been successfully generalized for Hilbert modular forms of parallel weight one, e.g., [Pil17], [PS16] and [Sas19].

In [HN17], Hansen made a conjecture on the existence of all companion forms for finite slope overconvergent  $p$ -adic automorphic forms of general  $\mathrm{GL}_n$  in the language of determining the set of *companion points* on the eigenvariety that are associated with the same  $p$ -adic Galois representation but with possibly different  $U_p$ -eigenvalues or weights. Similar to the weight part of Serre's modularity conjecture, the recipes for companion forms are given by the  $p$ -adic local Galois representations. In fact, the conjecture on companion points is closely related to Breuil's locally analytic socle conjecture in [Bre16][Bre15b] from the point of view of the local-global compatibility in the locally analytic aspect of the  $p$ -adic local Langlands program.

We will work in the setting of definite unitary groups as Breuil. Let  $F$  be a quadratic imaginary extension of a totally real field  $F^+$ . Let  $S_p$  be the set of places of  $F^+$  that divide  $p$ . We assume that each  $v \in S_p$  splits in  $F$ . Let  $G$  be a definite unitary group of rank  $n \geq 2$  over  $F^+$  that is split over  $F$  (so that  $G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) \simeq \prod_{v \in S_p} \mathrm{GL}_n(F_v^+)$ ). Then an eigenvariety  $Y(U^p, \bar{\rho})$  of  $G$ , of certain tame level  $U^p$  and localized at a modular absolutely irreducible  $\bar{\rho} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ , is a rigid analytic space parametrizing pairs  $(\rho, \underline{\delta})$  where  $\rho : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$  are continuous representations which lift  $\bar{\rho}$  and  $\underline{\delta} = (\delta_v)_{v \in S_p} = (\delta_{v,i})_{v \in S_p, i=1, \dots, n} : \prod_{v \in S_p} ((F_v^+)^{\times})^n \rightarrow \bar{\mathbb{Q}}_p^{\times}$  is a continuous character such that  $\rho$  is associated with a finite slope overconvergent  $p$ -adic automorphic form of  $G$  which has “weight”  $\underline{\delta} \mid \prod_{v \in S_p} (\mathcal{O}_{F_v^+}^{\times})^n$  and has “ $U_p$ -eigenvalues”  $\prod_{j=1}^i \delta_{v,j}(\varpi_{F_v^+})$  for  $v \in S_p, i = 1, \dots, n$  where  $\varpi_{F_v^+}$  denotes some uniformizers.

Recall an algebraic character of  $F_v^+$  has the form  $(F_v^+)^{\times} \rightarrow \bar{\mathbb{Q}}_p^{\times} : z \mapsto \prod_{\tau: F_v^+ \hookrightarrow \bar{\mathbb{Q}}_p} \tau(z)^{k_{\tau}}$  for some  $k_{\tau} \in \mathbb{Z}$ . Now take a point  $x = (\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  and assume that  $\rho_v := \rho \mid_{\mathrm{Gal}(\bar{F}_v/F_v)}$  is



crystalline for all  $v \in S_p$  where  $\tilde{v} \mid v$  is a place of  $F$  chosen for each  $v \in S_p$ . Then  $\underline{\delta}$  is locally algebraic, i.e.  $\underline{\delta} = \underline{\delta}_{\text{alg}} \underline{\delta}_{\text{sm}}$  where each  $\delta_{\text{alg},v,i}$  is algebraic and  $\delta_{\text{sm},v,i}$  is smooth. A companion point  $(\rho, \delta')$  of  $x$  falls in one of the following two types:

- (a)  $\underline{\delta}'_{\text{alg}} \neq \underline{\delta}_{\text{alg}}$  but  $\underline{\delta}'_{\text{sm}} = \underline{\delta}_{\text{sm}}$  (different “weights”);
- (b)  $\underline{\delta}'_{\text{sm}} \neq \underline{\delta}_{\text{sm}}$  (different “ $U_p$ -eigenvalues up to some normalizations”).

Our main theorem is the following.

**Theorem 4.1.1** (Theorem 4.5.2). *Suppose that  $x = (\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  is a point such that  $\rho_v$  is generic crystalline (see §4.5.2 for the generic condition) for all  $v \in S_p$ . Assume the tame level is sufficiently small and the usual Taylor-Wiles hypothesis (Assumption 4.5.1). Then all the companion points of  $x$  in the conjecture of Hansen or Breuil appear on  $Y(U^p, \bar{\rho})$ .*

The above theorem was already proved by Breuil-Hellmann-Schraen in [BHS19] under the assumption that the Hodge-Tate weights of each  $\rho_v$  are regular (i.e. pairwise different). In [Wu21], the author removed the regular assumption on the Hodge-Tate weights, but only was able to prove the existence of all companion points of the type (a) above in the non-regular cases. The task of this paper is to find all companion points of type (b) for non-regular points. These are companion points corresponding to different triangulation (refinements) of the trianguline (crystalline) Galois representations.

The proof of our theorem is motivated by some arguments in ordinary cases and will use the known results in both regular and non-regular cases. In ordinary cases, modularity lifting theorems were proved for ordinary families of Galois representations that will specialize to companion points with possibly non-regular weights. In our finite slope/trianguline cases, for a non-regular point  $x$  as in Theorem 4.1.1, the naive strategy is to find a sequence of points  $x^i$  on  $Y(U^p, \bar{\rho})$  with regular Hodge-Tate weights such that  $x = \varinjlim_i x^i$  and certain companion points  $(x^i)'$  of  $x^i$ , which will exist on  $Y(U^p, \bar{\rho})$  using [BHS19], satisfy that  $(x^i)'$  converge to a point  $x'$  on  $Y(U^p, \bar{\rho})$  and that  $x'$  is a companion point of  $x$  of type (b).

The actual proof is Galois-theoretical. Using patching methods [CEG<sup>+</sup>16] and the patched eigenvariety [BHS17b], we can reduce the task to find those nearby regular  $x^i$  to a similar problem on the *trianguline variety* in [BHS17b], the local Galois-theoretical eigenvariety. Those  $x^i = (\rho^i, \underline{\delta}^i)$  (now  $\rho^i$  are representations of local Galois groups) are found by studying some “crystalline/de Rham” loci on the moduli space of trianguline  $(\varphi, \Gamma)$ -modules in the proof of [BHS17b, Thm. 2.6] and  $\rho^i$  will be the Galois representations corresponding to certain étale trianguline  $(\varphi, \Gamma)$ -modules of parameter  $\underline{\delta}^i$  (the étaleness will be achieved by the results of Hellmann in [Hel16]). The key example is the case  $n = 2$ .

*Remark 4.1.2.* Our proof for the existence of companion points of type (b) will not use directly the theory of local models of the trianguline variety in [BHS19] and [Wu21]. However, it is the existence of all companion points of type (a) in [Wu21], which used the local models, that allows us to keep working in the smooth locus of the trianguline variety consisting of points  $(\rho, \underline{\delta})$  where  $\rho$  is trianguline of parameter  $\underline{\delta}$ .

*Remark 4.1.3.* As a corollary of Theorem 4.1.1, we can determine all the companion constituents (certain locally analytic representations of  $\prod_{v \in S_p} \text{GL}_n(F_v^+)$ ) in the Hecke-isotypic part of the completed cohomology of  $G$  associated with generic crystalline Galois representations in Breuil’s locally analytic socle conjecture (Corollary 4.5.4). Since there are no locally algebraic constituents in the non-regular cases, the existence of all of these companion constituents could be a replacement in the automorphic side to compare with de Rhamness of the Galois side in the  $p$ -adic Langlands correspondence in this particular non-regular situation.

*Remark 4.1.4.* For non-regular weights, the existence of all companion points will not lead directly to a classicality result of Hecke eigensystems in contrast with [BT99] since classical automorphic

representations for definite unitary groups will have regular weights. The results of this paper might be able to be adapted for Hilbert modular forms and have applications in classicality of  $p$ -adic Hilbert modular forms with non-regular and possibly non-parallel weights.

The paper is organized as follows. In §4.2 and §4.3, we collect some (presumably known) results on  $(\varphi, \Gamma)$ -modules over the Robba rings. In §4.4, we find the companion points on the trianguline variety. In §4.5, we apply the local results in §4.4 to the global settings and prove the main theorem.

#### 4.1.1 Notation

We will use the notation in [Wu21, §1.7]. Let  $K$  be a finite extension of  $\mathbb{Q}_p$  and  $L/\mathbb{Q}_p$  be a large enough coefficient field such that  $\Sigma := \{\tau : K \hookrightarrow L\}$  has size  $[K : \mathbb{Q}_p]$ . Let  $C$  be the completion of an algebraic closure of  $K$ . We have the Robba ring  $\mathcal{R}_{L,K}$  of  $K$  over  $L$  defined in [KPX14, Def. 6.2.1]. Let  $t \in \mathcal{R}_{L,K}$  denote Fontaine's  $2\pi i$  and  $t = u \prod_{\tau \in \Sigma} t_\tau$  for some  $u \in \mathcal{R}_{L,K}^\times$  (see [KPX14, Not. 6.2.7] for details). For  $\mathbf{k} = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$ , write  $t^{\mathbf{k}} = \prod_{\tau \in \Sigma} t_\tau^{k_\tau}$ . If  $\delta : K^\times \rightarrow L^\times$  is a continuous character, let  $\mathcal{R}_{L,K}(\delta)$  be the associated rank one  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_{L,K}$  in [KPX14, Cons. 6.2.4] where  $\Gamma_K = \text{Gal}(K(\mu_\infty)/K)$ . Then  $t^{\mathbf{k}}\mathcal{R}_{L,K} = \mathcal{R}_{L,K}(z^{\mathbf{k}})$  where  $z^{\mathbf{k}}$  denotes the character  $z \mapsto \prod_{\tau \in \Sigma} \tau(z)^{k_\tau}$ . If  $a \in L^\times$ , then denote by  $\text{unr}(a)$  the unramified character of  $K^\times$  sending a uniformizer of  $K$  to  $a$ . Let  $\mathcal{T}_L$  be the rigid space over  $L$  parametrizing continuous characters of  $K^\times$  and  $\mathcal{T}_0 \subset \mathcal{T}_L$  be the complement of the subset of characters  $\delta$  such that  $\delta$  or  $\epsilon\delta^{-1}$  is algebraic. Here  $\epsilon$  is the character  $\text{Norm}_{K/\mathbb{Q}_p}|\text{Norm}_{K/\mathbb{Q}_p}|_{\mathbb{Q}_p}$  of  $K^\times$ . We can define  $\tau$ -part  $\text{wt}_\tau(\delta)$  of the weight  $\text{wt}(\delta)$  of  $\delta$  (see [Wu21, §1.7.2]). The cyclotomic character of  $\mathcal{G}_K := \text{Gal}(\overline{K}/K)$  has Hodge-Tate weights one. We fix an integer  $n \geq 2$ .

## 4.2 Cohomology of $(\varphi, \Gamma_K)$ -modules

We collect some results of the cohomology of  $(\varphi, \Gamma_K)$ -modules (of character type). We fix  $\delta : K^\times \rightarrow L^\times$  to be a continuous character. Recall if  $D$  is a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}$ , then  $H_{\varphi, \Gamma_K}^i(D[\frac{1}{t}]) = \varinjlim_{m \rightarrow +\infty} H_{\varphi, \Gamma_K}^i(t^{-m}D)$ ,  $i = 0, 1, 2$  ([BHS19, (3.11)]).

**Lemma 4.2.1.** *If  $\delta \in \mathcal{T}_0$ , then  $\dim_L H_{\varphi, \Gamma_K}^i(\mathcal{R}_{L,K}(\delta)) = 0$  for  $i = 0, 2$  and  $\dim_L H_{\varphi, \Gamma_K}^1(\mathcal{R}_{L,K}(\delta)) = [K : \mathbb{Q}_p]$ .*

*Proof.* [KPX14, Prop. 6.2.8]. □

Recall in [BHS19, §3.3] we have a functor  $W_{\text{dR}}$  (resp.  $W_{\text{dR}}^+$ ) sending a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}[\frac{1}{t}]$  (resp.  $\mathcal{R}_{L,K}$ ) to an  $L \otimes_{\mathbb{Q}_p} B_{\text{dR}}$ -representations (resp.  $L \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$ -representation) of  $\mathcal{G}_K$ .

**Lemma 4.2.2.** *If  $\delta \in \mathcal{T}_0$  and is locally algebraic, then*

$$H_{\varphi, \Gamma_K}^1(\mathcal{R}_{L,K}(\delta)[\frac{1}{t}]) \xrightarrow{\sim} H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{R}_{L,K}(\delta)[\frac{1}{t}])).$$

*Proof.* [BHS19, Lem. 3.4.2] □

**Proposition 4.2.3.** *For  $\mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma$ ,  $i = 0, 1$ ,*

$$\dim_L H^i(t^{-\mathbf{k}}\mathcal{R}_{L,K}(\delta)/\mathcal{R}_{L,K}(\delta)) = |\{\tau \in \Sigma \mid k_\tau \geq 1, \text{wt}_\tau(\delta) \in \{1, \dots, k_\tau\}\}|.$$

*Proof.* This follows from [Ked09, Appendix A] and [Nak09, Lem. 2.16] (and some other well-known results: [Liu07, Thm.4.7, Cor. 4.8] and the comparison in [Nak09, Prop. 2.2], or a generalized version [Nak14, Thm. 5.11]). □

**Corollary 4.2.4.** For  $\mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma$ , and  $\delta \in \mathcal{T}_0$ , then

$$\begin{aligned} & \dim_L \operatorname{Ker} \left( H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}}\mathcal{R}_{L,K}(\delta)) \right) \\ &= \dim_L \operatorname{Coker} \left( H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}}\mathcal{R}_{L,K}(\delta)) \right) \\ &= |\{\tau \in \Sigma \mid k_\tau \geq 1, \operatorname{wt}_\tau(\delta) \in \{1_\tau, \dots, k_\tau\}\}|. \end{aligned}$$

**Corollary 4.2.5.** If  $\delta \in \mathcal{T}_0$  is locally algebraic and  $\operatorname{wt}_\tau(\delta) \leq 0$  for all  $\tau \in \Sigma$ , then the natural maps  $\mathcal{R}_{L,K}(\delta) \hookrightarrow t^{-\mathbf{k}}\mathcal{R}_{L,K}(\delta)$  induce isomorphisms  $H_{\varphi, \gamma_K}^i(\mathcal{R}_{L,K}(\delta)) \xrightarrow{\sim} H_{\varphi, \gamma_K}^i(t^{-\mathbf{k}}\mathcal{R}_{L,K}(\delta))$  for all  $i = 0, 1, 2, \mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma$ .

If  $\mathbf{k} \in \mathbb{Z}^\Sigma$ , write  $\mathbf{k}^\sharp \in \mathbb{Z}^\Sigma$  where  $k_\tau^\sharp = k_\tau$  if  $k_\tau \geq 1$  and  $k_\tau^\sharp = 0$  otherwise.

**Proposition 4.2.6.** Assume that  $\delta \in \mathcal{T}_0$  is locally algebraic with weights  $\mathbf{k} \in \mathbb{Z}^\Sigma$ . Then the image of an element  $x \in H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta))$  in  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta)[\frac{1}{t}])$  is 0 if and only if

$$x \in \operatorname{Ker} \left( H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}^\sharp}\mathcal{R}_{L,K}(\delta)) \right).$$

*Proof.* We have  $\operatorname{wt}_\tau(\delta z^{-\mathbf{k}^\sharp}) \leq 0$  for all  $\tau \in \Sigma$ . Thus by Corollary 4.2.5,  $H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}^\sharp}\mathcal{R}_{L,K}(\delta)) \simeq H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta)[\frac{1}{t}])$ .  $\square$

### 4.3 A crystalline criterion

We need some criterion to guarantee that the points on the trianguline variety we will find in the next section are crystalline. For the definition of de Rham or crystalline  $(\varphi, \Gamma_K)$ -modules, see [HS16, Def. 2.5]. We say a trianguline  $(\varphi, \Gamma_K)$ -module of parameter  $\underline{\delta} = (\delta_1, \dots, \delta_n)$  is generic if  $\delta_i \delta_j^{-1} \in \mathcal{T}_0$  for all  $i \neq j$  (or  $\underline{\delta} \in \mathcal{T}_0^n$  in the notation of [Wu21, §3.2], remark that  $\mathcal{T}_0^n \neq (\mathcal{T}_0)^n$ ). Recall that a locally algebraic character  $\delta : K^\times \rightarrow L^\times$  is crystalline (or semi-stable) if and only if the smooth part  $\delta_{\text{sm}}$  is unramified (see [KPX14, Exam. 6.2.6]).

**Lemma 4.3.1.** If  $D$  is a generic trianguline  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}$  of parameter  $\underline{\delta}$  such that all  $\delta_i$  are crystalline, then  $D$  is a crystalline  $(\varphi, \Gamma_K)$ -module if and only if  $D$  is de Rham.

*Proof.* This follows from the proof of [HS16, Cor. 2.7(i)]. Assume  $D$  is de Rham. As  $D$  is a successive extension of crystalline  $(\varphi, \Gamma_K)$ -modules,  $D$  is semi-stable (by [Ber08b], see also the arguments in [Ber02, §6.1]). By the generic assumption, the monodromy must be trivial. Hence  $D$  is crystalline.  $\square$

**Lemma 4.3.2.** Let  $D$  be a trianguline  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}$  of rank  $n$  with the trianguline filtration  $\operatorname{Fil}_\bullet D$  such that  $\operatorname{Fil}_i D / \operatorname{Fil}_{i-1} D \simeq \mathcal{R}_{L,K}(\delta_i)$  for  $i = 1, \dots, n$ . Fix  $i_0 \in \{1, \dots, n-1\}$  and let  $D_0 = \operatorname{Fil}_{i_0} D$  and  $D_1 = D / \operatorname{Fil}_{i_0} D$ . Assume that  $\underline{\delta}$  is locally algebraic and let  $\lambda = (\lambda_{\tau,i})_{\tau \in \Sigma, i=1, \dots, n} = \operatorname{wt}(\underline{\delta}) \in (\mathbb{Z}^\Sigma)^n$ . Assume that for every  $\tau \in \Sigma$ ,  $\lambda_{\tau,i} < \lambda_{\tau,j}$  if  $i > i_0 \geq j$  and that both  $D_0$  and  $D_1$  are de Rham, then  $D$  is de Rham.

*Proof.* This is a generalization of [HS16, Prop. 2.6]. We need to prove that  $\dim_L W_{\text{dR}}(D)^{\mathcal{G}_K} = n[K : \mathbb{Q}_p]$ . For  $\tau \in \Sigma$ , let  $k_\tau = \max_{i > i_0} \lambda_{\tau,i}$ . Then  $\dim_L W_{\text{dR}}^+(t^{-\mathbf{k}} D_0)^{\mathcal{G}_K} = 0$  as the Hodge-Tate weights of  $t^{-\mathbf{k}} D_0$  are positive and  $t^{-\mathbf{k}} D_0$  is de Rham. We have an exact sequence

$$0 \rightarrow W_{\text{dR}}^+(t^{-\mathbf{k}} D)^{\mathcal{G}_K} \rightarrow W_{\text{dR}}^+(t^{-\mathbf{k}} D_1)^{\mathcal{G}_K} \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}^+(t^{-\mathbf{k}} D_0)).$$

The Hodge-Tate weights of  $t^{-\mathbf{k}} D_0$  are  $\geq 1$ , hence  $H^1(\mathcal{G}_K, W_{\text{dR}}^+(t^{-\mathbf{k}} D_0)) = 0$  by [Nak14, Cor. 5.6] (we have that  $H^1(\mathcal{G}_K, C(i)) = 0$  for  $i \neq 0$  by [Fon04, Prop. 2.15(ii)]). We get

$\dim_L W_{\text{dR}}^+(t^{-\mathbf{k}}D)^{\mathcal{G}_K} = \dim_L W_{\text{dR}}^+(t^{-\mathbf{k}}D_1)^{\mathcal{G}_K} = (n - i_0)[K : \mathbb{Q}_p]$  since  $t^{-\mathbf{k}}D_1$  is de Rham with non-positive Hodge-Tate weights. As  $D_0$  is de Rham,  $\dim_L W_{\text{dR}}(D_0)^{\mathcal{G}_K} = i_0[K : \mathbb{Q}_p]$ . Since  $W_{\text{dR}}^+(t^{-\mathbf{k}}D_0)[\frac{1}{t}] \cap W_{\text{dR}}^+(t^{-\mathbf{k}}D) = W_{\text{dR}}^+(t^{-\mathbf{k}}D_0)$ , we have  $W_{\text{dR}}(D_0)^{\mathcal{G}_K} \cap W_{\text{dR}}^+(t^{-\mathbf{k}}D)^{\mathcal{G}_K} = W_{\text{dR}}^+(t^{-\mathbf{k}}D_0)^{\mathcal{G}_K} = \{0\}$ . Then  $W_{\text{dR}}^+(t^{-\mathbf{k}}D)^{\mathcal{G}_K}$  and  $W_{\text{dR}}(D_0)^{\mathcal{G}_K}$  span an  $n[K : \mathbb{Q}_p]$ -dimensional  $L$ -subspace in  $W_{\text{dR}}(D)^{\mathcal{G}_K}$ .  $\square$

The above lemma will be used in the following form later.

**Proposition 4.3.3.** *Assume that  $D$  is a trianguline  $(\varphi, \Gamma_K)$ -module of rank  $n$  over  $\mathcal{R}_{L,K}$  with the trianguline filtration  $\text{Fil}_\bullet D$  such that  $\text{Fil}_i D / \text{Fil}_{i-1} D \simeq \mathcal{R}_{L,K}(\delta_i)$  for  $i = 1, \dots, n$ . Fix  $i_0 \in \{1, \dots, n-1\}$  and let  $D_0 = \text{Fil}_{i_0-1} D$ ,  $D_1 = \text{Fil}_{i_0+1} D / D_0$  and  $D_2 = D / \text{Fil}_{i_0+1} D$ . Let  $\lambda = \text{wt}(\delta)$  and assume that  $\delta$  is locally algebraic. Assume that for every  $\tau \in \Sigma$ ,  $\lambda_{\tau,i} > \lambda_{\tau,i+1}$  if  $i \neq i_0$ ,  $\lambda_{\tau,i} > \lambda_{\tau,i_0}$ ,  $\lambda_{\tau,i_0+1}$  if  $i < i_0$ , and  $\lambda_{\tau,i} < \lambda_{\tau,i_0}$ ,  $\lambda_{\tau,i_0+1}$  if  $i > i_0 + 1$ . If  $D_1$  is de Rham, then  $D$  is de Rham.*

## 4.4 Critical points hunting

Let  $\bar{r} : \mathcal{G}_K \rightarrow \text{GL}_n(k_L)$  be a continuous representation. We firstly recall some constructions around the trianguline variety  $X_{\text{tri}}(\bar{r})$  in [BHS17b, §2.2]. In the first parts of this section, we will only need the Zariski open dense subset  $U_{\text{tri}}(\bar{r}) \subset X_{\text{tri}}(\bar{r})$ .

### 4.4.1 The trianguline variety

Let  $\mathcal{T}_{\text{reg}}^n$  be the Zariski open subset of  $\mathcal{T}_L^n$  consisting of characters  $\underline{\delta} = (\delta_i)_{i=1, \dots, n}$  such that  $\delta_i \delta_j^{-1} \neq z^{-\mathbf{k}}, \epsilon z^{\mathbf{k}}$  for  $i \neq j$  and  $\mathbf{k} \in \mathbb{Z}_{\geq 0}^\Sigma$ . There are rigid spaces  $\mathcal{S}_n^\square(\bar{r}) \rightarrow \mathcal{S}_n$  over  $\mathcal{T}_{\text{reg}}^n$  in the proof of [BHS17b, Thm. 2.6] (and in [HS16, §2.2]) which will be used later, and we recall below.

The space  $\mathcal{S}_n$  represents the functor sending a reduced rigid space  $X$  over  $L$  to the isomorphic classes of quadruples  $(D_X, \text{Fil}_\bullet D_X, \nu_X, \underline{\delta}_X)$  where  $D_X$  is a  $(\varphi, \Gamma_K)$ -module of rank  $n$  over  $\mathcal{R}_{X,K}$  where  $\mathcal{R}_{X,K}$  denotes the Robba ring of  $K$  over  $X$ ,  $\text{Fil}_\bullet D_X$  is a filtration of sub- $(\varphi, \Gamma_K)$ -modules of  $D_X$  which are locally direct summands as  $\mathcal{R}_{X,K}$ -modules,  $\underline{\delta}_X \in \mathcal{T}_{\text{reg}}^n(X)$  and  $\nu_X : \text{Fil}_i D_X / \text{Fil}_{i-1} D_X \simeq \mathcal{R}_{X,K}(\delta_i)$  (we omit the subscripts of the spaces for the universal characters to simplify the notation). There are obvious morphisms  $\mathcal{S}_n \rightarrow \mathcal{S}_{n-1} \times_L \mathcal{T}_L \rightarrow \mathcal{T}_{\text{reg}}^{n-1} \times_L \mathcal{T}_L \subset \mathcal{T}_L^n$ . Let  $U \subset \mathcal{S}_{n-1} \times_L \mathcal{T}_L$  be the preimage of  $\mathcal{T}_{\text{reg}}^n$  which is Zariski open in  $\mathcal{S}_{n-1} \times_L \mathcal{T}_L$  and let  $D_U$  be the  $(\varphi, \Gamma)$ -modules over  $U$  pulled back from the universal one on  $\mathcal{S}_{n-1}$ . Then  $\mathcal{S}_n \simeq \text{Spec}^{\text{an}}(\text{Sym}^\bullet(\mathcal{E}xt_{\varphi, \gamma_K}^1(\mathcal{R}_{U,K}(\delta_n), D_U)^\vee))$  is a geometric vector bundle over  $U$  where  $\mathcal{E}xt_{\varphi, \gamma_K}^1(\mathcal{R}_{U,K}(\delta_n), D_U) \simeq H_{\varphi, \gamma_K}^1(D_U(\delta_n^{-1}))$  is a locally free sheaf on  $U$  of rank  $(n-1)[K : \mathbb{Q}]$  ([HS16, Prop. 2.3]) and the notion  $\text{Spec}^{\text{an}}$  is taken from [Con06, Thm. 2.2.5]. It follows from induction that the map  $\mathcal{S}_n \rightarrow \mathcal{T}_{\text{reg}}^n \subset \mathcal{T}_L^n$  is smooth.

Let  $\mathcal{S}_n^{\text{adm}} \subset \mathcal{S}_n$  be the open subset (as adic spaces) of the admissible locus, which comes from a rigid space, and let  $\mathcal{S}_n^{\square, \text{adm}} \rightarrow \mathcal{S}_n^{\text{adm}}$  be the  $\text{GL}_n$ -torsor trivializing the universal Galois representation over  $\mathcal{S}_n^{\text{adm}}$ . Let  $\mathcal{S}_n^{\square, \text{adm}, +} \subset \mathcal{S}_n^{\square, \text{adm}}$  be the admissible open subset where the universal framed representation  $\mathcal{G}_K \rightarrow \text{GL}_n(\Gamma(\mathcal{S}_n^{\square, \text{adm}}, \mathcal{O}_{\mathcal{S}_n^{\square, \text{adm}}}))$  factors through  $\mathcal{G}_K \rightarrow \text{GL}_n(\Gamma(\mathcal{S}_n^{\square, \text{adm}}, \mathcal{O}_{\mathcal{S}_n^{\square, \text{adm}}}^+))$ . We denote by  $\mathcal{S}_n^\square(\bar{r})$  the admissible open subset of  $\mathcal{S}_n^{\square, \text{adm}, +}$  where the reduction  $\mathcal{G}_K \rightarrow \text{GL}_n(\Gamma(\mathcal{S}_n^{\square, \text{adm}}, \mathcal{O}_{\mathcal{S}_n^{\square, \text{adm}, +}}^+ / \mathcal{O}_{\mathcal{S}_n^{\square, \text{adm}, +}}^{++}))$  coincides with  $\bar{r}$  (see also the discussion before [HH20, Prop. 8.17]). The map  $\kappa : \mathcal{S}_n^\square(\bar{r}) \rightarrow \mathcal{S}_n \rightarrow \mathcal{T}_L^n$  is also smooth.

Let  $R_{\bar{r}}$  (over  $\mathcal{O}_L$ ) be the framed deformation ring of  $\bar{r}$  and let  $\mathfrak{X}_{\bar{r}} := \text{Spf}(R_{\bar{r}})^{\text{rig}}$  be the rigid generic fiber (we follow the notation in [BHS19] and [Wu21] rather than [BHS17b]). We have a subset

$$U_{\text{tri}}(\bar{r}) := \{ (r, \underline{\delta}) \in \mathfrak{X}_{\bar{r}} \times \mathcal{T}_{\text{reg}}^n \mid r \text{ is trianguline of a parameter } \underline{\delta} \}$$

of  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  (see [BHS17b, §2.2]). The image of  $\mathcal{S}_n^\square(\bar{r}) \rightarrow \mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  is equal to  $U_{\text{tri}}(\bar{r})$  and the trianguline variety  $X_{\text{tri}}(\bar{r})$  is the Zariski closure of  $U_{\text{rig}}(\bar{r})$  in  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  with the reduced induced structure. The open subset  $U_{\text{tri}}(\bar{r})$  of  $X_{\text{tri}}(\bar{r})$  is smooth and the map  $\pi_{\bar{r}} : \mathcal{S}_n^\square(\bar{r}) \rightarrow U_{\text{tri}}(\bar{r}) \subset X_{\text{tri}}(\bar{r})$  is smooth [BHS17b, Thm. 2.6].

#### 4.4.2 Some “de Rham” locus

We will define some subspace  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r}) \subset \mathcal{S}_n^\square(\bar{r})$  where the criteria in the last section will apply for certain points on it.

We fix datum  $i_0 \in \{1, \dots, n-1\}$ , a subset  $J \subset \Sigma$  and  $\mathbf{k}_J = (k_\tau)_{\tau \in J} \in \mathbb{Z}_{\geq 1}^J$ . We allow  $J$  to be  $\emptyset$  or  $\Sigma$ . Let  $\mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  be the subset of characters  $\underline{\delta} \in \mathcal{T}_{\text{reg}}^n$  such that  $\text{wt}_\tau(\delta_{i_0} \delta_{i_0+1}^{-1}) = k_\tau$  for all  $\tau \in J$  and  $\delta_{i_0} \delta_{i_0+1}^{-1} \in \mathcal{T}_0$ . Let  $\mathfrak{t}$  be the base change to  $L$  of the  $\mathbb{Q}_p$ -Lie algebra of  $(K^\times)^n$  and view its dual  $\mathfrak{t}^*$  as the affine space of weights and we have a weight map  $\text{wt} : \mathcal{T}_L^n \rightarrow \mathfrak{t}^*$ . Let  $\mathfrak{t}_{(i_0,\mathbf{k}_J)}^*$  be the subspace of points  $(\lambda_{\tau,i})_{\tau \in \Sigma, i=1, \dots, n}$  such that  $\lambda_{\tau,i_0} - \lambda_{\tau,i_0+1} = k_\tau$  for all  $\tau \in J$ .

**Lemma 4.4.1.** *The rigid space  $\mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  is smooth reduced equidimensional of dimension  $[K : \mathbb{Q}_p]n - |J|$  and is étale over  $\mathfrak{t}_{(i_0,\mathbf{k}_J)}^*$ .*

*Proof.* This follows from [Din17a, Prop. 6.1.13].  $\square$

Consider the universal  $(\varphi, \Gamma_K)$ -modules  $D_X$  and  $\text{Fil}_\bullet D_X$  over  $X = \mathcal{S}_n \times_{\mathcal{T}_L^n} \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  or  $\mathcal{S}_n^\square(\bar{r}) \times_{\mathcal{T}_L^n} \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  pulled back from  $\mathcal{S}_n$ . The extension

$$0 \rightarrow \mathcal{R}_{X,K}(\delta_{X,i_0}) \rightarrow \text{Fil}_{i_0+1} D_X / \text{Fil}_{i_0-1} D_X \rightarrow \mathcal{R}_{X,K}(\delta_{X,i_0+1}) \rightarrow 0$$

together with the trivialization  $\nu_X$  defines a section  $s_X$  in

$$\mathcal{E}xt_{\varphi,\gamma_K}^1(\mathcal{R}_{X,K}(\delta_{X,i_0+1}), \mathcal{R}_{X,K}(\delta_{X,i_0})) \simeq H_{\varphi,\gamma_K}^1(\mathcal{R}_{X,K}(\delta_{X,i_0} \delta_{X,i_0+1}^{-1})).$$

By the main result of [KPX14], both  $H_{\varphi,\gamma_K}^1(\mathcal{R}_{X,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  and  $H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  are coherent sheaves on  $X$ . We define the subspace  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}$  or  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r})$  to be the vanishing locus on  $X = \mathcal{S}_n \times_{\mathcal{T}_L^n} \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  or  $\mathcal{S}_n^\square(\bar{r}) \times_{\mathcal{T}_L^n} \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  of the image of  $s_X$  under the natural map

$$H_{\varphi,\gamma_K}^1(\mathcal{R}_{X,K}(\delta_{i_0} \delta_{i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_{i_0} \delta_{i_0+1}^{-1})).$$

The vanishing loci are Zariski closed subspaces as  $H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  is locally free (by Lemma 4.4.2 below).

Then the image of  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r})$  in  $U_{\text{tri}}(\bar{r})$  consists of  $x = (r, \underline{\delta})$  such that  $\text{wt}_\tau(\delta_{i_0} \delta_{i_0+1}^{-1}) = k_\tau$  for  $\tau \in J$ ,  $\delta_{i_0} \delta_{i_0+1}^{-1} \in \mathcal{T}_0$  and the extension (the condition will be independent of the trivialization of  $\mathcal{R}_{k(x),K}(\delta_i)$ )

$$0 \rightarrow \mathcal{R}_{k(x),K}(\delta_{i_0}) \rightarrow \text{Fil}_{i_0+1} D_{\text{rig}}(r) / \text{Fil}_{i_0-1} D_{\text{rig}}(r) \rightarrow \mathcal{R}_{k(x),K}(\delta_{i_0+1}) \rightarrow 0$$

corresponds to an element in  $H_{\varphi,\gamma_K}^1(\mathcal{R}_{k(x),K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  which lies in the kernel of

$$H_{\varphi,\gamma_K}^1(\mathcal{R}_{k(x),K}(\delta_{i_0} \delta_{i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{k(x),K}(\delta_{i_0} \delta_{i_0+1}^{-1})).$$

Remark that the image of  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r})$  in  $U_{\text{tri}}(\bar{r})$  is a locus where  $\text{Fil}_{i_0+1} D_{\text{rig}}(r) / \text{Fil}_{i_0-1} D_{\text{rig}}(r)$  is de Rham if  $\delta_{i_0}$  and  $\delta_{i_0+1}$  are locally algebraic (cf. Lemma 4.2.2). Using this property and Proposition 4.3.3 we will pick out certain crystalline points with regular Hodge-Tate weights in the image of  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r})$  in  $U_{\text{tri}}(\bar{r})$  (Proposition 4.4.10). The following lemmas (particularly Lemma 4.4.3) show that the geometry of  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r})$  is good and will be important for the purpose.

**Lemma 4.4.2.** *Let  $X$  be a reduced rigid space over  $L$  and  $\delta_X : K^\times \rightarrow \Gamma(X, \mathcal{O}_X)^\times$  be a continuous character. Assume that for any  $x \in X$ , we have  $\delta_x \in \mathcal{T}_0$  and  $\text{wt}_\tau(\delta_x) = k_\tau$  for all  $\tau \in J$ . Then the coherent sheaves  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{X,K}(\delta_X))$ ,  $H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_X))$ , as well as*

$$\text{Ker}(H_{\varphi, \gamma_K}^1(\mathcal{R}_{X,K}(\delta_X)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_X)))$$

and

$$\text{Coker}(H_{\varphi, \gamma_K}^1(\mathcal{R}_{X,K}(\delta_X)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_X)))$$

are finite projective over  $X$  of rank  $|\Sigma|, |\Sigma|, |J|, |J|$  respectively and their formation commutes with arbitrary base change.

*Proof.* We write  $\text{Ker}(\delta_X)$  or  $\text{Coker}(\delta_X)$  for the kernel or the cokernel of the map

$$H_{\varphi, \gamma_K}^1(\mathcal{R}_{X,K}(\delta_X)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_X))$$

for simplicity.

For any  $x \in X$ ,  $\dim_{k(x)} H_{\varphi, \gamma_K}^1(\mathcal{R}_{k(x),K}(\delta_x)) = \dim_{k(x)} H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{k(x),K}(\delta_x)) = |\Sigma|$  by Lemma 4.2.1 and  $\dim_{k(x)} \text{Ker}(\delta_x) = \dim_{k(x)} \text{Coker}(\delta_x) = |J|$  by Corollary 4.2.4 and our assumptions on  $\delta_x$ . The fact that  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{k(x),K}(\delta_x))$  and  $H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{k(x),K}(\delta_x))$  are locally free and commute with base change of the form  $\text{Sp}(k(x)) \rightarrow X$  for  $x \in X$  follows from [HS16, Prop. 2.3]. Thus for any  $x \in X$ ,

$$\begin{aligned} & \text{Coker}(\delta_X) \otimes_{\mathcal{O}_X} k(x) \\ & \simeq \text{Coker}(H_{\varphi, \gamma_K}^1(\mathcal{R}_{X,K}(\delta_X)) \otimes_{\mathcal{O}_X} k(x) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_X)) \otimes_{\mathcal{O}_X} k(x)) \\ & \simeq \text{Coker}(H_{\varphi, \gamma_K}^1(\mathcal{R}_{k(x),K}(\delta_x)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{k(x),K}(\delta_x))) \\ & \simeq \text{Coker}(\delta_x). \end{aligned}$$

Thus  $\text{Coker}(\delta_X)$  has constant rank, hence is projective by [KPX14, Lem. 2.1.8 (1)], and commutes with base change of the form  $\text{Sp}(k(x)) \rightarrow X$  for  $x \in X$ .

Let  $\text{Im}(\delta_X)$  be the image of the map  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{X,K}(\delta_X)) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_X))$ . Then we have  $\dim_{k(x)} \text{Im}(\delta_x) = |\Sigma| - |J|$  for any  $x \in X$ . By the exact sequence

$$0 \rightarrow \text{Im}(\delta_X) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{X,K}(\delta_X)) \rightarrow \text{Coker}(\delta_X) \rightarrow 0$$

and  $\text{Tor}_{\mathcal{O}_X}^1(k(x), \text{Coker}(\delta_X)) = 0$ , we get

$$0 \rightarrow \text{Im}(\delta_X) \otimes_{\mathcal{O}_X} k(x) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{k(x),K}(\delta_x)) \rightarrow \text{Coker}(\delta_x) \rightarrow 0$$

for any  $x \in X$ . Hence  $\text{Im}(\delta_X) \otimes_{\mathcal{O}_X} k(x) \simeq \text{Im}(\delta_x)$  for any  $x \in X$  and  $\text{Im}(\delta_X)$  is finite projective of rank  $|\Sigma| - |J|$ . Repeat the argument using the exact sequence

$$0 \rightarrow \text{Ker}(\delta_X) \rightarrow H_{\varphi, \gamma_K}^1(\mathcal{R}_{X,L}(\delta_X)) \rightarrow \text{Im}(\delta_X) \rightarrow 0$$

and that  $\text{Tor}_{\mathcal{O}_X}^1(k(x), \text{Im}(\delta_X)) = 0$ , we see  $\text{Ker}(\delta_X) \otimes_{\mathcal{O}_X} k(x) \simeq \text{Ker}(\delta_x)$  and  $\text{Ker}(\delta_X)$  is finite projective of rank  $|J|$ .

The statement for general base changes, which we will not essentially need, follows from [KPX14, Lem. 4.1.5, Thm. 4.4.3 (2)] and the locally-freeness of those sheaves over the base  $X$ .  $\square$

**Lemma 4.4.3.** *The morphism  $\kappa : \mathcal{S}_{n, (i_0, \mathbf{k}_J)}^\square(\bar{r}) \rightarrow \mathcal{T}_{(i_0, \mathbf{k}_J)}^n$  is smooth.*

*Proof.* The diagram

$$\begin{array}{ccc}
\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r}) & \rightarrow & \mathcal{S}_{n,(i_0,\mathbf{k}_J)} \\
\downarrow & & \downarrow \\
\mathcal{S}_n^\square(\bar{r}) & \longrightarrow & \mathcal{S}_n
\end{array}$$

is Cartesian. Hence the map  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r}) \rightarrow \mathcal{S}_{n,(i_0,\mathbf{k}_J)}$  is smooth. The map  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^\square(\bar{r}) \rightarrow \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  factors through  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)} \rightarrow \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$ . Therefore, we only need to prove that  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)} \rightarrow \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  is smooth. In §4.4.1, we have maps  $\mathcal{S}_i \rightarrow \mathcal{S}_{i-1} \times_L \mathcal{T}_L$ . We can define  $\mathcal{S}_{i_0+1,(i_0,\mathbf{k})}$  replacing  $n$  by  $i_0 + 1$ . We have  $\mathcal{T}_{(i_0,\mathbf{k}_J)}^n = \mathcal{T}_{\text{reg}}^n \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1}$ . The section  $s_{\mathcal{S}_n \times_{\mathcal{T}_L^n} \mathcal{T}_{(i_0,\mathbf{k}_J)}^n}$  is the pullback of the section  $s_{\mathcal{S}_{i_0+1} \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1}}$  via  $\mathcal{S}_n \times_{\mathcal{T}_L^n} \mathcal{T}_{(i_0,\mathbf{k}_J)}^n \rightarrow \mathcal{S}_{i_0+1} \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1}$  since the definition of  $s_X$  only involves  $\text{Fil}_{i_0+1} D_X$  and  $\delta_{i_0}, \delta_{i_0+1}$ . Thus the diagram

$$\begin{array}{ccc}
\mathcal{S}_{n,(i_0,\mathbf{k}_J)} & \rightarrow & \mathcal{S}_{i_0+1,(i_0,\mathbf{k}_J)} \\
\downarrow & & \downarrow \\
\mathcal{S}_n & \longrightarrow & \mathcal{S}_{i_0+1}
\end{array}$$

is Cartesian. As each  $\mathcal{S}_i \rightarrow \mathcal{S}_{i-1} \times_L \mathcal{T}_L$  is smooth (as a geometric vector bundle over a Zariski open subset of the image), we see so is, by base change,  $\mathcal{S}_{i,(i_0,\mathbf{k}_J)} \rightarrow \mathcal{S}_{i-1,(i_0,\mathbf{k}_J)} \times_L \mathcal{T}_L \rightarrow \mathcal{T}_{(i_0,\mathbf{k}_J)}^i \subset \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i-1} \times_L \mathcal{T}_L$  for  $i \geq i_0 + 2$  if the result is true for  $i_0 + 1$ . Thus, we reduce to the case when  $n = i_0 + 1$ .

We consider the map

$$\mathcal{S}_{i_0+1,(i_0,\mathbf{k}_J)} \rightarrow \mathcal{S}_{i_0+1} \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1} \rightarrow (\mathcal{S}_{i_0} \times_L \mathcal{T}_L) \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1}.$$

Since the map  $\mathcal{S}_{i_0} \rightarrow \mathcal{T}_L^{i_0}$  is smooth, so is  $(\mathcal{S}_{i_0} \times_L \mathcal{T}_L) \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1} \rightarrow \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1}$ . Write  $V$  for  $(\mathcal{S}_{i_0} \times_L \mathcal{T}_L) \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1}$ . We only need to prove that  $\mathcal{S}_{i_0+1,(i_0,\mathbf{k}_J)}$  is a geometric vector bundle over  $V$  which will imply all we need.

Recall that  $\mathcal{S}_{i_0+1} \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1} \simeq \text{Spec}^{\text{an}}(\text{Sym}^\bullet(H_{\varphi,\gamma_K}^1(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1})))^\vee)$  where  $\text{Fil}_{i_0} D_V$  is the universal one pulled back from  $\mathcal{S}_{i_0}$  and  $\delta_{i_0+1}$  is the character pulled back from  $\mathcal{T}_{(i_0,\mathbf{k}_J)}^{i_0+1}$ . Consider the kernel of the following composite of morphisms of coherent sheaves on  $V$

$$H_{\varphi,\gamma_K}^1(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(\mathcal{R}_{V,K}(\delta_{i_0} \delta_{i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{V,K}(\delta_{i_0} \delta_{i_0+1}^{-1})) \quad (4.4.4)$$

where  $\mathcal{R}_{V,K}(\delta_{i_0}) = \text{Fil}_{i_0} D_V / \text{Fil}_{i_0-1} D_V$ . We denote the kernel (resp. cokernel) of the above composite morphism by  $\text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))$  (resp.  $\text{Coker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))$ ).

We claim that  $\text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))$  is locally free of rank  $(i_0 - 1)|\Sigma| + |J|$  and

$$\text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1})) \otimes_{\mathcal{O}_V} k(x) \simeq \text{Ker}(\text{Fil}_{i_0} D_x(\delta_{x,i_0+1}^{-1}))$$

for any  $x \in V$ . If  $i_0 = 1$ , this follows from Lemma 4.4.2. Now we assume  $i_0 - 1 \geq 1$ . The sheaves  $H_{\varphi,\gamma_K}^1(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))$ ,  $H_{\varphi,\gamma_K}^1(\mathcal{R}_{V,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  and  $H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{V,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  are locally free of ranks  $i_0|\Sigma|$ ,  $|\Sigma|$  and  $|\Sigma|$  respectively and commute with base change. The morphism

$$H_{\varphi,\gamma_K}^1(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(\mathcal{R}_{V,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$$

is surjective since for any  $x \in V$ ,  $H_{\varphi,\gamma_K}^2(\text{Fil}_{i_0-1} D_x(\delta_{x,i_0+1}^{-1})) = 0$  (see [HS16, Prop. 2.3]). Hence

$$\text{Coker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1})) = \text{Coker}(H_{\varphi,\gamma_K}^1(\mathcal{R}_{V,K}(\delta_{i_0} \delta_{i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{V,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))).$$

By Lemma 4.4.2, we get  $\text{Coker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))$  is locally free of rank  $|J|$  and for any point  $x \in V$ ,  $\text{Coker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1})) \otimes_{\mathcal{O}_V} k(x) \simeq \text{Coker}(\text{Fil}_{i_0} D_x(\delta_{x,i_0+1}^{-1}))$ . Repeat the last step of the proof of Lemma 4.4.2, we get the desired claim.

The injection  $\text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1})) \hookrightarrow H_{\varphi, \gamma_K}^1(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))$  of projective coherent sheaves induces a surjection

$$H_{\varphi, \gamma_K}^1(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))^\vee \twoheadrightarrow \text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))^\vee$$

which by [Con06, Thm. 2.2.5] induces a closed embedding

$$\text{Spec}^{\text{an}}(\text{Sym}^\bullet(\text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))^\vee)) \hookrightarrow \mathcal{S}_{i_0+1} \times_{\mathcal{T}_L^{i_0+1}} \mathcal{T}_{(i_0, \mathbf{k}_J)}^{i_0+1}.$$

The left-hand side is a geometric vector bundle over  $V$  by the previous discussion, and we remain to prove that  $\text{Spec}^{\text{an}}(\text{Sym}^\bullet(\text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))^\vee))$  coincides with  $\mathcal{S}_{i_0+1, (i_0, \mathbf{k}_J)}$ . The statement is local and trivial. We write a proof below.

We may take an affinoid open  $W = \text{Sp}(A) \subset V$  and assume that the sheaves in (4.4.4) are free over  $W$ . Then since all the modules are projective, we may take a basis  $e_1, \dots, e_{i_0|\Sigma|}$  of  $H_{\varphi, \gamma_K}^1(\text{Fil}_{i_0} D_W(\delta_{i_0+1}^{-1}))$  and assume that the surjection

$$H_{\varphi, \gamma_K}^1(\text{Fil}_{i_0} D_W(\delta_{i_0+1}^{-1})) \twoheadrightarrow H_{\varphi, \gamma_K}^1(\mathcal{R}_{W, K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$$

corresponds to projection to the subspace  $\langle e_1, \dots, e_{|\Sigma|} \rangle$  (equivalently choose a split of the surjection). We assume that  $e'_1, \dots, e'_{|\Sigma|}$  is a basis of  $H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{W, K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$ . As the cokernel and the kernel of the map  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{W, K}(\delta_{i_0} \delta_{i_0+1}^{-1})) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{W, K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  are locally free, we may, after possibly shrinking  $W$ , assume that the morphism is given by sending  $e_{|J|+1}, \dots, e_{|\Sigma|}$  to  $e'_{|J|+1}, \dots, e'_{|\Sigma|}$  and sending  $e_1, \dots, e_{|J|}$  to 0. Let  $e_1^\vee, \dots, e_{i_0|\Sigma|}^\vee$  be the dual basis. Then

$$\text{Spec}^{\text{an}}(\text{Sym}^\bullet(H_{\varphi, \gamma_K}^1(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))^\vee)) \text{ resp. } \text{Spec}^{\text{an}}(\text{Sym}^\bullet(\text{Ker}(\text{Fil}_{i_0} D_V(\delta_{i_0+1}^{-1}))^\vee))$$

are covered by

$$W_N := \text{Sp}(A\langle p^N e_1^\vee, \dots, p^N e_{i_0|\Sigma|}^\vee \rangle) \text{ resp. } \text{Sp}(A\langle p^N e_1^\vee, \dots, p^N e_{|J|}^\vee, p^N e_{|\Sigma|+1}^\vee, \dots, p^N e_{i_0|\Sigma|}^\vee \rangle)$$

where  $N \in \mathbb{N}$ . The tautological section  $s_{W_N}$  of the sheaf

$$H_{\varphi, \gamma_K}^1(\mathcal{R}_{W_N, K}(\delta_{i_0} \delta_{i_0+1}^{-1})) = \mathcal{O}_{W_N} e_1 \oplus \dots \oplus \mathcal{O}_{W_N} e_{|\Sigma|}$$

is given by  $e_1^\vee e_1 + \dots + e_{|\Sigma|}^\vee e_{|\Sigma|}$ . Thus the image of  $s_{W_N}$  in

$$H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{W_N, K}(\delta_{i_0} \delta_{i_0+1}^{-1})) = \mathcal{O}_{W_N} e'_1 \oplus \dots \oplus \mathcal{O}_{W_N} e'_{|\Sigma|}$$

is given by  $e_{|J|+1}^\vee e'_{|J|+1} + \dots + e_{|\Sigma|}^\vee e'_{|\Sigma|}$ . Hence the vanishing locus is cut out by  $e_{|J|+1}^\vee = \dots = e_{|\Sigma|}^\vee = 0$  and coincides with  $\text{Sp}(A\langle p^N e_1^\vee, \dots, p^N e_{|J|}^\vee, p^N e_{|\Sigma|+1}^\vee, \dots, p^N e_{i_0|\Sigma|}^\vee \rangle)$ .  $\square$

#### 4.4.3 Nearby critical crystalline points

Now we fix  $x = (r, \underline{\delta}) \in U_{\text{tri}}(\bar{r})(L) \subset (\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n)(L)$ . We assume that  $r$  is crystalline and  $\underline{\delta} = z^\lambda \text{unr}(\underline{\varphi}) := (z^{\lambda_i} \text{unr}(\varphi_i))_{i=1, \dots, n}$  for some  $\lambda = (\lambda_{\tau, i})_{\tau \in \Sigma, i=1, \dots, n} \in (\mathbb{Z}^n)^\Sigma$  and  $\underline{\varphi} \in (L^\times)^n$ . Assume furthermore that  $\varphi_i \varphi_j^{-1} \notin \{1, q\}$  for all  $i \neq j$  where  $q$  is the cardinal of the residue field of  $\mathcal{O}_K$ . This means that  $r$  is generic in the sense of [Wu21, §4.1] or the beginning of §4.5.2, and  $\underline{\delta} \in \mathcal{T}_0^n$ .



We continue to fix  $i_0 \in \{1, \dots, n-1\}$ . Let  $J := \{\tau \in \Sigma \mid \lambda_{\tau, i_0} \geq \lambda_{\tau, i_0+1} + 1\}$  and let  $\mathbf{k}_J = (k_\tau)_{\tau \in J} := (\lambda_{\tau, i_0} - \lambda_{\tau, i_0+1})_{\tau \in J}$ . We have a filtration  $\text{Fil}_\bullet D_{\text{rig}}(r)$  such that  $\text{Fil}_i D_{\text{rig}}(r)/\text{Fil}_{i-1} D_{\text{rig}}(r) \simeq \mathcal{R}_{L,K}(\delta_i)$ . Since  $r$  is de Rham, so are  $D_{\text{rig}}(r)$  and the subquotient  $\text{Fil}_{i_0+1} D_{\text{rig}}(r)/\text{Fil}_{i_0-1} D_{\text{rig}}(r)$ . The extension

$$0 \rightarrow \mathcal{R}_{L,K}(\delta_{i_0}) \rightarrow \text{Fil}_{i_0+1} D_{\text{rig}}(r)/\text{Fil}_{i_0-1} D_{\text{rig}}(r) \rightarrow \mathcal{R}_{L,K}(\delta_{i_0+1}) \rightarrow 0$$

defines an element in  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$  (up to  $L^\times$ ) which lies in the kernel of

$$H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta_{i_0} \delta_{i_0+1}^{-1})) \rightarrow H_{\varphi, \gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{L,K}(\delta_{i_0} \delta_{i_0+1}^{-1}))$$

by Lemma 4.2.2, Proposition 4.2.6, [BHS19, Lem. 3.3.7, Lemma 3.4.2], the isomorphism

$$H^1(\mathcal{G}_K, L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}}) \simeq \text{Ext}_{\text{Rep}_{L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}}}(\mathcal{G}_K)}^1(L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}}, L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}})$$

and the following lemma.

**Lemma 4.4.5.** *Let  $W$  be an  $L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}}$ -representation of  $\mathcal{G}_K$  which is an extension  $0 \rightarrow L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}} \rightarrow W \rightarrow L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}} \rightarrow 0$  as representations of  $\mathcal{G}_K$ . Then  $W$  is trivial (i.e.  $W \simeq (L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}})^2$ ) if and only if the extension splits.*

*Proof.* If the extension splits, then  $W$  is trivial. Conversely, if  $W$  is trivial, then  $\dim_L W^{\mathcal{G}_K} = 2[K : \mathbb{Q}_p]$  and we have an exact sequence of  $L \otimes_{\mathbb{Q}_p} K$ -modules  $0 \rightarrow L \otimes_{\mathbb{Q}_p} K \rightarrow W^{\mathcal{G}_K} \rightarrow L \otimes_{\mathbb{Q}_p} K \rightarrow 0$ . The extension splits and we may choose a section  $L \otimes_{\mathbb{Q}_p} K \rightarrow W^{\mathcal{G}_K}$  which induces a section  $L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}} \rightarrow W$  of  $L \otimes_{\mathbb{Q}_p} \text{B}_{\text{dR}}$ -representations.  $\square$

Thus  $x$  lies in the image of  $\mathcal{S}_{n, (i_0, \mathbf{k}_J)}^\square(\bar{r})$ . Recall the following diagram.

$$\begin{array}{ccc} \mathcal{S}_{n, (i_0, \mathbf{k}_J)}^\square(\bar{r}) & \subset & \mathcal{S}_n^\square(\bar{r}) \\ \pi_{\bar{r}} \swarrow & & \searrow \kappa \\ x \in U_{\text{tri}}(\bar{r}) & \xrightarrow{\omega'} & \mathcal{T}_{(i_0, \mathbf{k}_J)}^n \subset \mathcal{T}_L^n. \end{array}$$

To state our main result of this section, we make some preparations in rigid geometry.

**Definition 4.4.6.** Let  $A$  and  $B$  be two subsets of a rigid space  $X$  over  $L$ . Then we say that  $A$  *quasi-accumulates* at  $B$  if for every point  $b \in B$  and every affinoid open neighbourhood  $Y$  of  $b$ ,  $A \cap Y \neq \emptyset$  (compare with [BHS17b, Def. 2.2]).

**Lemma 4.4.7.** *If  $A$  and  $B$  are two subsets of a rigid space  $X$ , then  $A$  quasi-accumulates at  $B$  if and only if for any  $b \in B$  and any affinoid open neighbourhood  $Y$  of  $b$ ,  $b$  lies in the Zariski closure of  $Y \cap A$  in  $Y$ . In particular, if  $A$  quasi-accumulates at  $B$ , then  $B$  is contained in the Zariski closure of  $A$  in  $X$ .*

*Proof.* We prove by contradiction. Assume that  $A$  quasi-accumulates at  $B$  and there exists an affinoid neighbourhood  $Y$  of  $b \in B$  such that  $b$  is not in the Zariski closure  $\overline{Y \cap A}$  in  $Y$ . Since Zariski open subsets in an affinoid are admissible open ([Bos14, Cor. 5.1.9]), there exists an affinoid neighbourhood  $Y' \subset Y \setminus \overline{Y \cap A}$  of  $b$ . Then  $Y' \cap A = \emptyset$ , this contradicts the assumption.  $\square$

**Lemma 4.4.8.** *Let  $Y \hookrightarrow X$  be a closed immersion of rigid analytic spaces over  $L$ . Let  $Z$  be a subset of  $Y$  and  $y \in Y$  be a point. Then  $Z$  quasi-accumulates at  $y$  in  $X$  if and only if  $Z$  quasi-accumulates at  $y$  in  $Y$ .*

*Proof.* The problem is local and we may assume  $X = \mathrm{Sp}(A)$ ,  $Y = \mathrm{Sp}(B)$  and  $B = A/I$  for an ideal  $I$ . Assume that  $Z$  quasi-accumulates at  $y$  in  $X$ . We only need to prove that for any affinoid neighbourhood  $Y'$  of  $y$  in  $Y$ , there exists an affinoid neighbourhood  $Y'' \subset Y'$  such that  $Y''$  has the form  $X' \cap Y$  for some affinoid neighbourhood  $X'$  of  $y$  in  $X$ . As affinoid subdomains are open in the canonical topology ([Bos14, Prop. 3.3.19]) and Weierstrass domains form a basis of the canonical topology ([Bos14, Lem. 3.3.8]), we may assume that  $Y''$  has the form  $\{x \in Y \mid |f_i(x)| \leq 1\}$  for  $f_1, \dots, f_m \in A/I$ . We may choose lifts  $\tilde{f}_1, \dots, \tilde{f}_m$  for  $f_1, \dots, f_m$  in  $A$ . Then  $\{x \in X \mid |f_i(x)| \leq 1\} \cap Y = \{x \in Y \mid |f_i(x)| \leq 1\}$ .  $\square$

We say a character  $\underline{\delta} = (\delta_1, \dots, \delta_n) \in \mathcal{T}_L^n$  of  $(K^\times)^n$  is crystalline if for any  $1 \leq i \leq n$ ,  $\delta_i|_{\mathcal{O}_K^\times} = z^{\mathbf{k}'_i}$  for some  $\mathbf{k}'_i \in \mathbb{Z}^\Sigma$ .

**Lemma 4.4.9.** *Let  $C$  be a positive integer. Then the set of crystalline characters  $\underline{\delta} \in \mathcal{T}_L^n$  such that, if we write  $\lambda = \mathrm{wt}(\underline{\delta})$ ,  $\lambda_{\tau,i} - \lambda_{\tau,i+1} > C$  if  $i \neq i_0$ ,  $\lambda_i - \lambda_{i_0} > C$ ,  $\lambda_i - \lambda_{i_0+1} > C$  if  $i < i_0$ ,  $\lambda_i - \lambda_{i_0} < -C$ ,  $\lambda_i - \lambda_{i_0+1} < -C$  if  $i > i_0 + 1$ ,  $\lambda_{\tau,i_0} = \lambda_{\tau,i_0+1}$  if  $\tau \in J$  and  $\lambda_{\tau,i_0+1} - \lambda_{\tau,i_0} > C$  quasi-accumulates at the trivial character in  $\mathcal{T}_L^n$*

*Proof.* Let  $q = p^{[K_0:\mathbb{Q}_p]}$ , where  $K_0$  is the maximal unramified subfield of  $K$ , and  $d = |\Sigma|$ . Take a uniformizer  $\varpi_K$  of  $K$ . We prove that for any character  $\delta \in \mathcal{T}_L$  such that  $\delta(\varpi_K) = 1$ , the set  $\{\delta^{p^N(q-1)}, N \in \mathbb{N}\}$  quasi-accumulates at the trivial character. For some  $m$  large, we have  $\mathcal{O}_K^\times = \mathbb{Z}_p^d \times \mu(\mathcal{O}_K) \times \mathbb{Z}/(q-1)$  where  $\mathbb{Z}_p^d / \exp(\varpi_K^m \mathcal{O}_K)$  is finite and  $\mu(\mathcal{O}_K)$  denotes the  $p$ -power roots of unity in  $\mathcal{O}_K$  (see [Neu13, Prop. II.5.7]). We only need to consider  $\mathbb{Z}_p^d = \mathbb{Z}_p e_1 \oplus \dots \oplus \mathbb{Z}_p e_n$  since the characters  $\delta^{p^N(q-1)}$  are trivial on the torsion subgroups of  $\mathcal{O}_K^\times$  and  $\varpi_K^{\mathbb{Z}}$  when  $N$  is large. The space  $\widehat{\mathbb{Z}_p^d} = \mathbb{U}^d$  which parametrizes characters of  $\mathbb{Z}_p^d$  is the open polydisk in  $d$  variables  $T_1, \dots, T_d$  by sending a character  $\delta$  to  $(\delta(e_1) - 1, \dots, \delta(e_d) - 1)$ . Then  $\delta^{p^N}$  is sent to  $(\delta(e_1)^{p^N} - 1, \dots, \delta(e_d)^{p^N} - 1)$ . For any  $x \in C$  such that  $|x - 1|_p < 1$ , where  $|\cdot|_p$  denotes the standard valuation,  $\lim_{N \rightarrow \infty} |x^{p^N} - 1|_p = \lim_{N \rightarrow \infty} |\sum_{1 \leq i \leq p^N} \binom{p^N}{i} (x - 1)^i|_p = 0$ . Hence for any  $\epsilon > 0$  and  $N$  large enough, we have  $(\delta(e_1)^{p^N} - 1, \dots, \delta(e_d)^{p^N} - 1) \in \overline{\mathbb{B}}(0, \epsilon)^d := \{x \in \mathbb{U}^d \mid |T_1(x)|_p \leq \epsilon, \dots, |T_d(x)|_p \leq \epsilon\}$ . Any affinoid neighbourhood of 0 in  $\overline{\mathbb{B}}(0, \frac{1}{p})^d$  contains a Weierstrass subdomain of the form  $\{x \in \overline{\mathbb{B}}(0, \frac{1}{p})^d \mid |f_1(x)|_p \leq 1, \dots, |f_m(x)|_p \leq 1\}$  for some  $f_1, \dots, f_m \in L\langle p^{-1}T_1, \dots, p^{-1}T_d \rangle$  by [Bos14, Lem. 3.3.8, Prop. 3.3.19]. Since  $|f_i(0)|_p \leq 1$ , there exists  $\epsilon > 0$  satisfying that for all  $(x_1, \dots, x_d) \in C^d$  such that  $|p^{-1}x_i|_p < \epsilon$  for all  $i = 1, \dots, d$ ,  $|f_i(x_1, \dots, x_n)|_p \leq 1$ . Hence  $\overline{\mathbb{B}}(0, \epsilon)^d \subset \{x \in \overline{\mathbb{B}}(0, \frac{1}{p})^d \mid |f_1(x)|_p \leq 1, \dots, |f_m(x)|_p \leq 1\}$ . Therefore, we have  $\{\delta^{p^N(q-1)}, N \in \mathbb{N}\}$  quasi-accumulates at the trivial character.

We write  $x_\tau$  for the character that sends  $x \in \mathcal{O}_K^\times$  to  $\tau(x)$  and  $\varpi_K$  to 1. For  $i \neq i_0, i_0 + 1$ , let  $\delta_i = \prod_{\tau \in \Sigma} x_\tau^{-i}$ . Let  $\delta_{i_0} = \prod_{\tau \in J} x_\tau^{-i_0} \prod_{\tau \notin J} x_\tau^{-i_0-1}$  and  $\delta_{i_0+1} = \prod_{\tau \in J} x_\tau^{-i_0} \prod_{\tau \notin J} x_\tau^{-i_0}$ . Let  $\underline{\delta} = (\delta_1, \dots, \delta_n)$ . Then  $\underline{\delta}$ , as well as its powers, is crystalline. The set  $\{\underline{\delta}^{p^N(q-1)} \mid N \in \mathbb{N}\}$  quasi-accumulates at the trivial character by a similar proof as above and  $\underline{\delta}^{p^N(q-1)}$  satisfies the requirements for the weights if  $N$  is large.  $\square$

Finally we can prove the main local results. Write  $z^{\mathbf{k}_J}$  for the character  $K^\times \rightarrow L^\times : z \mapsto \prod_{\tau \in J} \tau(z)^{k_\tau}$ .

**Proposition 4.4.10.** *Let  $X$  be an affinoid open neighbourhood of  $x$  in  $U_{\mathrm{tri}}(\bar{r})$ .*

1. *There exists a subset  $Z \subset X$  that quasi-accumulates at  $x$  and such that for every  $z = (r_z, \underline{\delta}_z) \in Z$ ,*

(a)  *$z$  lies in the image of  $\mathcal{S}_{n, (i_0, \mathbf{k}_J)}^\square(\bar{r}) \subset \mathcal{S}_n^\square$ ,*

(b)  *$\underline{\delta}_z \in \mathcal{T}_0^n$  is crystalline, and*

(c) if we write  $\lambda_z$  for  $\text{wt}(\underline{\delta}_z)$ , then for every  $\tau \in \Sigma$ ,  $\lambda_{z,\tau,i} > \lambda_{z,\tau,i+1}$  if  $i \neq i_0$ ,  $\lambda_{z,\tau,i} > \lambda_{z,\tau,i_0}$ ,  $\lambda_{z,\tau,i_0} > \lambda_{z,\tau,i_0+1}$  if  $i < i_0$ ,  $\lambda_{z,\tau,i} < \lambda_{z,\tau,i_0}$ ,  $\lambda_{z,\tau,i_0+1} < \lambda_{z,\tau,i}$  if  $i > i_0+1$ , and  $\lambda_{z,\tau,i_0} < \lambda_{z,\tau,i_0+1}$  if  $\tau \notin J$ .

2. Every point in  $Z$  is generic crystalline and regular (i.e.  $\lambda_{z,\tau,i} \neq \lambda_{z,\tau,j}$  for all  $i \neq j, \tau \in \Sigma$ ).

3. Let  $\zeta$  be the automorphism of  $\mathcal{T}_L^n$  sending  $\underline{\delta}' = (\delta'_1, \dots, \delta'_n)$  to

$$(\delta'_1, \dots, \delta'_{i_0-1}, \delta'_{i_0+1} z^{\mathbf{k}_J}, \delta'_{i_0} z^{-\mathbf{k}_J}, \delta'_{i_0+2}, \dots, \delta'_n).$$

Use also the notation  $\zeta$  to denote the automorphism of  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n : (r, \underline{\delta}') \mapsto (r, \zeta(\underline{\delta}'))$ . Then  $\zeta(Z)$  is a subset of  $X_{\text{tri}}(\bar{r})$  and quasi-accumulates at  $\zeta(x)$  in  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$ . As a consequence,  $\zeta(x)$  lies in  $X_{\text{tri}}(\bar{r})$  and  $\zeta(Z)$  quasi-accumulates at  $\zeta(x)$  in  $X_{\text{tri}}(\bar{r})$ .

*Proof.* (1) By the definition of quasi-accumulation, we only need to verify that there exists one point  $z \in X$  for an arbitrary affinoid open neighbourhood  $X \subset U_{\text{tri}}(\bar{r})$  satisfying the condition (a), (b) and (c). Let  $\pi_{\bar{r}}^{-1}(X)$  be the preimage of  $X$  in  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^{\square}(\bar{r})$ . Then  $\pi_{\bar{r}}^{-1}(X)$  is admissible open. We only need to prove that there exists a point  $z' \in \pi_{\bar{r}}^{-1}(X)$  such that  $\kappa(z') = \omega'(\pi_{\bar{r}}(z')) \in \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  satisfies the conditions in (b) and (c). As  $\kappa : \mathcal{S}_{n,(i_0,\mathbf{k}_J)}^{\square}(\bar{r}) \rightarrow \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  is smooth by Lemma 4.4.3, the image  $\kappa(\pi_{\bar{r}}^{-1}(X))$ , which contains  $\underline{\delta}$ , contains an admissible open subset of  $\mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  that contains  $\underline{\delta}$  by [Bos14, Cor. 9.4.2]. Then the result follows from that the set of points  $\underline{\delta}' \in \mathcal{T}_{(i_0,\mathbf{k}_J)}^n$  that satisfy (b) and (c) quasi-accumulates at  $\underline{\delta}$  by Lemma 4.4.9 (since  $\underline{\delta} \in \mathcal{T}_0^n$  and  $\mathcal{T}_L^n \rightarrow \mathcal{T}_0^n : \underline{\delta}' \mapsto \underline{\delta}\underline{\delta}'$  is an isomorphism).

(2) Assume  $z = (r_z, \underline{\delta}_z) \in Z$  as in (1). By (c), the  $\tau$ -weights of  $\underline{\delta}_z$  are pairwise different, thus the Sen weights of  $r_z$  are regular. Since  $(r_z, \underline{\delta}_z)$  lies in the image of  $\mathcal{S}_{n,(i_0,\mathbf{k}_J)}^{\square}(\bar{r})$ , the extension

$$0 \rightarrow \mathcal{R}_{k(z),K}(\delta_{z,i_0}) \rightarrow \text{Fil}_{i_0+1} D_{\text{rig}}(r_z) / \text{Fil}_{i_0-1} D_{\text{rig}}(r_z) \rightarrow \mathcal{R}_{k(z),K}(\delta_{z,i_0+1}) \rightarrow 0$$

corresponds to an element (up to  $L^\times$ ) in the kernel of

$$H_{\varphi,\gamma_K}^1(\mathcal{R}_{k(z),K}(\delta_{z,i_0} \delta_{z,i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(t^{-\mathbf{k}_J} \mathcal{R}_{k(z),K}(\delta_{z,i_0} \delta_{z,i_0+1}^{-1}))$$

and in particular, in the kernel of

$$H_{\varphi,\gamma_K}^1(\mathcal{R}_{k(z),K}(\delta_{z,i_0} \delta_{z,i_0+1}^{-1})) \rightarrow H_{\varphi,\gamma_K}^1(\mathcal{R}_{k(z),K}(\delta_{z,i_0} \delta_{z,i_0+1}^{-1})[\frac{1}{t}]).$$

Since  $\delta_{z,i_0}, \delta_{z,i_0+1}$  are both locally algebraic, we get that  $\text{Fil}_{i_0+1} D_{\text{rig}}(r_z) / \text{Fil}_{i_0-1} D_{\text{rig}}(r_z)[\frac{1}{t}]$  is a direct sum of de Rham  $(\varphi, \Gamma_K)$ -modules over  $\mathcal{R}_{k(z),K}[\frac{1}{t}]$  by [BHS19, Lem. 3.3.7]. Hence the  $(\varphi, \Gamma_K)$ -module  $\text{Fil}_{i_0+1} D_{\text{rig}}(r_z) / \text{Fil}_{i_0-1} D_{\text{rig}}(r_z)$  over  $\mathcal{R}_{k(z),K}$  is de Rham. By Proposition 4.3.3 and the condition of weights in (c),  $r_z$  is de Rham. By (b) and Lemma 4.3.1,  $r_z$  is generic crystalline.

(3) Let  $z = (\rho_z, \underline{\delta}_z) \in Z \subset U_{\text{tri}}(\bar{r})$ . Then  $\underline{\delta}_z = z^{\lambda_z} \text{unr}(\underline{\varphi}_z)$  for a refinement  $\underline{\varphi}_z = (\varphi_{z,1}, \dots, \varphi_{z,n})$  where  $\lambda_z$  is as in (c) (we abuse the notation  $z$  for a point and the character). Let  $\underline{\varphi}'_z$  be the refinement such that  $\varphi'_{z,i} = \varphi_{z,i}$  if  $i \neq i_0, i_0+1$  and  $\varphi'_{z,i_0} = \varphi_{z,i_0+1}, \varphi'_{z,i_0+1} = \varphi_{z,i_0}$ . Let  $\lambda_z^{\text{dom}}$  be the weight such that  $\lambda_{z,\tau,i}^{\text{dom}} = \lambda_{z,\tau,i}$  if  $i \neq i_0, i_0+1$  or if  $\tau \in J$  and let  $\lambda_{z,\tau,i_0}^{\text{dom}} = \lambda_{z,\tau,i_0+1}, \lambda_{z,\tau,i_0+1}^{\text{dom}} = \lambda_{z,\tau,i_0}$  if  $\tau \notin J$ . Then  $\lambda_z^{\text{dom}}$  is dominant and differs from  $\lambda_z$  by permutations. It is easy to verify that  $z^{\lambda_z^{\text{dom}}} \text{unr}(\underline{\varphi}'_z) = \zeta(z^{\lambda_z} \text{unr}(\underline{\varphi}_z))$ . By [BHS19, Thm. 4.2.3], all the companion points of  $z$  exist on  $X_{\text{tri}}(\bar{r})$ . In particular the dominant point  $(r_z, z^{\lambda_z^{\text{dom}}} \text{unr}(\underline{\varphi}'_z))$  corresponding to the refinement  $\underline{\varphi}'_z$  is on  $X_{\text{tri}}(\bar{r})$ . Let  $z$  vary, we see  $\zeta(Z) \subset X_{\text{tri}}(\bar{r})$ . Since  $Z$  quasi-accumulates at  $x$  in  $U_{\text{tri}}(\bar{r}) \subset X_{\text{tri}}(\bar{r})$ ,  $Z$  quasi-accumulates at  $x$  in  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$  by Lemma 4.4.8. Since  $\zeta$  is an automorphism,  $\zeta(Z)$  quasi-accumulates at  $\zeta(x)$  in  $\mathfrak{X}_{\bar{r}} \times \mathcal{T}_L^n$ . As  $\zeta(Z) \subset X_{\text{tri}}(\bar{r})$ , we see  $\zeta(Z)$  quasi-accumulates at  $\zeta(x) \in X_{\text{tri}}(\bar{r})$  by Lemma 4.4.7.  $\square$

## 4.5 Companion points on the eigenvariety

We now prove the existence of all companion points for generic crystalline points on the eigenvariety. We recall the definition of the eigenvariety for definite unitary groups in [BHS19, §5.1] or [BHS17a, §3.1].

### 4.5.1 The eigenvariety

Let  $F$  be a quadratic imaginary extension of a totally real field  $F^+$ . Let  $S_p$  be the set of places of  $F^+$  that divide  $p$ . We assume that each  $v \in S_p$  splits in  $F$  and for every  $v \in S_p$ , we choose a place  $\tilde{v}$  of  $F$  above  $v$ . Let  $G$  be a definite unitary group of rank  $n \geq 2$  over  $F^+$  that is split over  $F$  so that  $G_p = \prod_{v \in S_p} G_v := G(F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p) \simeq \prod_{v \in S_p} \mathrm{GL}_n(F_{\tilde{v}})$  (we fix an isomorphism  $G \times_{F^+} F \simeq \mathrm{GL}_{n/F}$ ). Let  $B_p = \prod_{v \in S_p} B_v$  be the subgroup of upper triangular matrices in  $G_p$  and let  $T_p = \prod_{v \in S_p} T_v \subset B_p$  be the diagonal torus. Let  $U^p = \prod_{v \nmid p} U_v$  be a sufficiently small (see [BHS17a, (3.9)]) open compact subgroup of  $G(\mathbf{A}_{F^+}^{p\infty})$ . Write  $\hat{S}(U^p, L) := \{f : G(F^+) \backslash G(\mathbf{A}_{F^+}^\infty)/U^p \rightarrow L, \text{ continuous}\}$ , where  $L/\mathbb{Q}_p$  is a large enough finite extension with the residue field  $k_L$ . Let  $G_p$  act by right translations on  $\hat{S}(U^p, L)$ . Let  $S$  be a finite set of places of  $F^+$  that split in  $F$  which contains all split places  $v \notin S_p$  such that  $U_v$  is not maximal and also contains  $S_p$ . The space  $\hat{S}(U^p, L)$  is also endowed with some usual action of (away from  $S$ ) Hecke operators and one can talk about the  $p$ -adic representations of  $\mathcal{G}_F := \mathrm{Gal}(\overline{F}/F)$  associated with Hecke eigenvalues that appear in  $\hat{S}(U^p, L)$ . We fix a modular absolutely irreducible  $\bar{\rho} : \mathcal{G}_F \rightarrow \mathrm{GL}_n(k_L)$  and write  $\hat{S}(U^p, L)_{\bar{\rho}} \neq 0$  for the localization of  $\hat{S}(U^p, L)$  at the non-Eisenstein maximal ideal of the Hecke algebra over  $\mathcal{O}_L$  associated with  $\bar{\rho}$  (see [BHS17b, §2.4] for details). We assume the following “standard Taylor-Wiles hypothesis”.

*Assumption 4.5.1.* 1.  $p > 2$ ;

2.  $F$  is an unramified extension of  $F^+$ ;

3.  $G$  is quasi-split at all finite places of  $F^+$ ;

4.  $U_v$  is hyperspecial at all places  $v$  of  $F^+$  that are inert in  $F$ ;

5.  $F$  contains no non-trivial  $\sqrt[p]{1}$  and the image of  $\bar{\rho} \mid_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$  is adequate, see [BHS19, Rem. 1.1].

Let  $R_{\bar{\rho}, S}$  be the deformation ring of polarized deformations of  $\bar{\rho}$  that are unramified outside  $S$ . This is a Noetherian complete local ring over  $\mathcal{O}_L$  with residue field  $k_L$ . We have an action of  $R_{\bar{\rho}, S}$  over  $\hat{S}(U^p, L)_{\bar{\rho}}$  which factors through the Hecke actions and commutes with that of  $G_p$  (for details, see also [BHS17b, §2.4]). Let  $\mathrm{Spf}(R_{\bar{\rho}, S})^{\mathrm{rig}}$  denote the rigid generic fiber of the formal scheme  $\mathrm{Spf}(R_{\bar{\rho}, S})$  in the sense of Berthelot, cf. [dJ95, §7]. Let  $\hat{T}_p$  be the rigid space over  $\mathbb{Q}_p$  parametrizing continuous characters of  $T_p$  and we write  $\hat{T}_{p, L}$  for its base change. Denote by  $\hat{S}(U^p, L)_{\bar{\rho}}^{\mathrm{an}}$  the subspace of  $\hat{S}(U^p, L)_{\bar{\rho}}$  consisting of  $\mathbb{Q}_p$ -locally analytic vectors under the action of  $G_p$ . Then  $\hat{S}(U^p, L)_{\bar{\rho}}^{\mathrm{an}}$  is a locally analytic representation of  $G_p$  and if we apply Emerton’s Jacquet module functor with respect to  $B_p$ ,  $J_{B_p}(\hat{S}(U^p, L)_{\bar{\rho}}^{\mathrm{an}})$  becomes an essentially admissible locally analytic representation of  $T_p$  ([Eme17, Def. 6.4.9]). The dual  $J_{B_p}(\hat{S}(U^p, L)_{\bar{\rho}}^{\mathrm{an}})'$  defines a coherent sheaf on the quasi-Stein space  $\mathrm{Spf}(R_{\bar{\rho}, S})^{\mathrm{rig}} \times \hat{T}_{p, L}$ . We define the eigenvariety  $Y(U^p, \bar{\rho})$  to be the scheme-theoretical support of the coherent sheaf defined by  $J_{B_p}(\hat{S}(U^p, L)_{\bar{\rho}}^{\mathrm{an}})'$  in  $\mathrm{Spf}(R_{\bar{\rho}, S})^{\mathrm{rig}} \times \hat{T}_{p, L}$ . An  $L$ -point  $(\rho, \underline{\delta}) \in \mathrm{Spf}(R_{\bar{\rho}, S})^{\mathrm{rig}} \times \hat{T}_{p, L}$  is in  $Y(U^p, \bar{\rho})$  if and only if

$$\mathrm{Hom}_{T_p}(\underline{\delta}, J_{B_p}(\hat{S}(U^p, L)_{\bar{\rho}}[\mathfrak{m}_{\rho}]^{\mathrm{an}})) \neq 0$$

where  $\mathfrak{m}_\rho$  is the maximal ideal of  $R_{\bar{\rho},S}[\frac{1}{p}]$  corresponding to  $\rho$  and  $\widehat{S}(U^p, L)_{\bar{\rho}}[\mathfrak{m}_\rho]$  denotes the subspace of elements in  $\widehat{S}(U^p, L)_{\bar{\rho}}$  annihilated by  $\mathfrak{m}_\rho$ .

### 4.5.2 The companion points

We will give the description of all companion points for a generic crystalline point. Suppose  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})(L)$ . Let  $\rho_v := \rho|_{\mathcal{G}_{F_{\bar{v}}}}$  for  $v \in S_p$ . Set  $\Sigma_v := \{\tau : F_{\bar{v}} \hookrightarrow L\}$  for  $v \in S_p$  and  $\Sigma_p := \cup_{v \in S_p} \Sigma_v$ . Assume that for each  $v \in S_p$ ,  $\rho_v$  is crystalline. Then we have  $\varphi$ -modules  $D_{\text{cris}}(\rho_v)$  over  $L \otimes_{\mathbb{Q}_p} F_{\bar{v},0}$ , where  $F_{\bar{v},0}$  is the maximal unramified subfield of  $F_{\bar{v}}$ . Take  $\tau_{v,0} \in \Sigma_v$ . Then  $\varphi^{[F_{\bar{v},0}:\mathbb{Q}_p]}$  acts linearly on  $D_{\text{cris}}(\rho_v) \otimes_{L \otimes_{\mathbb{Q}_p} F_{\bar{v},0}, 1 \otimes \tau_{v,0}} L$ . Let  $\{\varphi_{v,1}, \dots, \varphi_{v,n}\}$  be the multiset of eigenvalues of  $\varphi^{[F_{\bar{v},0}:\mathbb{Q}_p]}$  which is independent of the choice of  $\tau_{v,0}$ . We say that  $\rho$  is *generic crystalline* if  $\varphi_{v,i} \varphi_{v,j}^{-1} \notin \{1, p^{[F_{\bar{v},0}:\mathbb{Q}_p]}\}$  for any  $i \neq j$  and  $v \in S_p$ . A *refinement*  $\mathcal{R} = (\mathcal{R}_v)_{v \in S_p}$  for the generic crystalline representation  $\rho$  is a choice of an ordering  $\mathcal{R}_v : \underline{\varphi}_v = (\varphi_{v,1}, \dots, \varphi_{v,n})$  of the  $n$  different eigenvalues for all  $v \in S_p$ . Thus,  $\rho$  has  $(n!)^{|S_p|}$  different refinements.

Let  $|\cdot|_{F_{\bar{v}}}$  be the norm of  $F_{\bar{v}}$  such that  $|p|_{F_{\bar{v}}} = p^{-[F_{\bar{v}}:\mathbb{Q}_p]}$ . Denote by  $\delta_{B_v}$  the smooth character  $|\cdot|_{F_{\bar{v}}}^{n-1} \otimes \dots \otimes |\cdot|_{F_{\bar{v}}}^{n-2i+1} \otimes \dots \otimes |\cdot|_{F_{\bar{v}}}^{1-n}$  of  $T_v \simeq (F_{\bar{v}}^\times)^n$  and  $\delta_{B_p} = \otimes_{v \in S_p} \delta_{B_v}$  the character of  $T_p$ . We define an automorphism  $\iota = (\iota_v)_{v \in S_p}$  of  $\widehat{T}_{p,L} = \prod_{v \in S_p} \widehat{T}_{v,L}$  given by  $\iota_v((\delta_{v,1}, \dots, \delta_{v,n})) = \delta_{B_v}(\delta_{v,1}, \dots, \delta_{v,i} \epsilon^{i-1}, \dots, \delta_{v,n} \epsilon^{n-1})$  where  $\epsilon$  denotes the cyclotomic characters.

Let  $\mathbf{h} = (\mathbf{h}_\tau)_{\tau \in \Sigma_p} = (h_{\tau,1}, \dots, h_{\tau,n})_{\tau \in \Sigma_p}$  where  $h_{\tau,1} \leq \dots \leq h_{\tau,n}$  are the  $\tau$ -Hodge-Tate weights of  $\rho_v$  if  $\tau \in \Sigma_v$ . Let  $\mathcal{S}_n$  be the  $n$ -th symmetric group and act on the  $n$ -tuples  $(h_{\tau,1}, \dots, h_{\tau,n})$  in standard ways. For  $w = (w_v)_{v \in S_p} = (w_\tau)_{\tau \in \Sigma_p}$  with  $w_\tau \in (\mathcal{S}_n)^{\Sigma_p}$ , define a character  $\underline{\delta}_{\mathcal{R},w} := \left( \iota_v \left( z^{w_v(\mathbf{h}_v)} \text{unr}(\varphi_v) \right) \right)_{v \in S_p}$  of  $T_p$ . Let  $W_{P_p} = (W_{P_\tau})_{\tau \in \Sigma_p}$  be the subgroup of  $(\mathcal{S}_n)^{\Sigma_p}$  consisting of permutations that fix  $\mathbf{h}$ . Here  $P_\tau$  denotes the parabolic subgroup of block upper-triangular matrices in  $\text{GL}_n$  with the Weyl group (of its Levi subgroup) identified with  $W_{P_\tau}$ . Set  $D_{\text{dR},\tau}(\rho_v) := D_{\text{dR}}(\rho_v) \otimes_{L \otimes_{\mathbb{Q}_p} F_{\bar{v},1} \otimes \tau} L$ . If we choose a basis  $(e_1, \dots, e_n)$  of  $D_{\text{dR},\tau}(\rho_v)$  of eigenvectors of  $\varphi^{[F_{\bar{v},0}:\mathbb{Q}_p]}$  with eigenvalues  $(\varphi_{v,1}, \dots, \varphi_{v,n})$ , then the Hodge-Tate filtration on  $D_{\text{dR},\tau}(\rho_v)$  corresponds to a point on the flag variety  $\text{GL}_n/P_\tau$  which lies in some Bruhat cell  $B_\tau w_{\mathcal{R}_\tau} P_\tau / P_\tau$  for some  $w_{\mathcal{R}_\tau} \in \mathcal{S}_n / W_{P_\tau}$  and  $w_{\mathcal{R}_\tau}$  is independent of scaling of the eigenvectors. Here  $\mathcal{R}$  signifies the refinement  $\underline{\varphi}$ . Let  $w_{\mathcal{R}} = (w_{\mathcal{R}_\tau})_{\tau \in \Sigma_p} \in (\mathcal{S}_n)^{\Sigma_p} / W_{P_p}$ .

Define a subset of points of  $\widehat{T}_{p,L}$

$$W(\rho) := \{\underline{\delta}_{\mathcal{R},w} \mid w \in (\mathcal{S}_n)^{\Sigma_p} / W_{P_p}, w \geq w_{\mathcal{R}}, \mathcal{R} \text{ is a refinement of } \rho\}$$

where  $\geq$  denotes the usual Bruhat order on  $\mathcal{S}_n$  (or its quotient). Notice that there is a natural partition  $W(\rho) = \coprod_{\mathcal{R}} W_{\mathcal{R}}(\rho)$  and  $W(\rho)$  depends only on  $\rho_v, v \in S_p$ .

By the control of the companion points on the trianguline variety in the generic crystalline cases ([BHS19, §4.2] and [Wu21, §4.1]), we have an inclusion  $\{\underline{\delta}' \mid (\rho, \underline{\delta}') \in Y(U^p, \bar{\rho})\} \subset W(\rho)$ . Below is our main theorem.

**Theorem 4.5.2.** *Let  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})(L)$  be a generic crystalline point as above and recall that we have assumed the Taylor-Wiles hypothesis (Assumption 4.5.1). Then*

$$W(\rho) \subset \{\underline{\delta}' \mid (\rho, \underline{\delta}') \in Y(U^p, \bar{\rho})\}.$$

*Proof.* We need the patched eigenvariety in [BHS17b, §3.2]. For  $v \in S_p$ , let  $R'_{\bar{\rho}_v} / \mathcal{O}_L$  be the maximal reduced  $\mathbb{Z}_p$ -flat quotient of the framed deformation ring of  $\bar{\rho}_v$ . We can similarly define  $R'_{\bar{\rho}_v}$  for  $v \in S \setminus S_p$ . Let  $K_p = \prod_{v \in S_p} \text{GL}_n(\mathcal{O}_{F_{\bar{v}}}) \subset \prod_{v \in S_p} \text{GL}_n(F_{\bar{v}}) \simeq G_p$ . Recall that under the Taylor-Wiles assumption, there are some positive integers  $g$  and  $q$ , a patching module  $M_\infty$  in [CEG<sup>+</sup>16] over the ring  $R_\infty = \widehat{\otimes}_{v \in S} R'_{\bar{\rho}_v}[[x_1, \dots, x_g]]$ , an  $\mathcal{O}_L$ -morphism  $S_\infty := \mathcal{O}_L[[y_1, \dots, y_q]] \rightarrow R_\infty$  and a surjection  $R_\infty / \mathfrak{a} \rightarrow R_{\bar{\rho},S}$  of completed local rings over  $\mathcal{O}_L$  where  $\mathfrak{a} = (y_1, \dots, y_q)$ , such

that  $M_\infty$  is a finite projective  $S_\infty[[K_p]]$ -module and  $\Pi_\infty := \text{Hom}_{\mathcal{O}_L}^{\text{cont}}(M_\infty, L)$  is a  $R_\infty$ -admissible Banach representation of  $G_p$  with an isomorphism  $\Pi_\infty[\mathfrak{a}] \simeq \widehat{S}(U^p, L)_{\bar{\rho}}$  that is compatible with the actions of  $R_\infty/\mathfrak{a}$  and  $R_{\bar{\rho}, S}$  (the action of  $R_{\bar{\rho}, S}$  factors through the quotient  $R_{\bar{\rho}, S}$ ). Write  $\Pi_\infty^{\text{an}}$  for the subspace of locally  $R_\infty$ -analytic vectors in  $\Pi_\infty$  ([BHS17b, Déf. 3.2]). The patched eigenvariety  $X_p(\bar{\rho})$  is the support of  $J_{B_p}(\Pi_\infty^{\text{an}})'$  inside  $\text{Spf}(R_\infty)^{\text{rig}} \times \widehat{T}_{p, L} \simeq \text{Spf}(\widehat{\otimes}_{v \in S_p} R'_{\bar{\rho}_v})^{\text{rig}} \times \text{Spf}(\widehat{\otimes}_{v \in S \setminus S_p} R'_{\bar{\rho}_v})^{\text{rig}} \times \text{Spf}(\mathcal{O}_L[[x_1, \dots, x_g]])^{\text{rig}} \times \widehat{T}_{p, L} =: \mathfrak{X}_{\bar{\rho}_p} \times \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g \times \widehat{T}_{p, L}$ . By [BHS17b, Thm. 3.21, §4.1], we have closed embeddings

$$Y(U^p, \bar{\rho}) \hookrightarrow X_p(\bar{\rho}) \hookrightarrow \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g) \subset \mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p, L} \times \mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g$$

where  $X_{\text{tri}}(\bar{\rho}_p) = \prod_{v \in S_p} X_{\text{tri}}(\bar{\rho}_v)$  and  $\iota$  is extended to an automorphism of  $\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p, L}$  by base change. Moreover,  $X_p(\bar{\rho})$  is equidimensional and is identified with a union of irreducible components of  $\iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  under the above closed embedding. By the argument as in the first steps of the proof of [BHS19, Thm. 5.3.3], we are reduced to prove the lemma below.

**Lemma 4.5.3.** *Assume that a point  $((\rho_p = (\rho_v)_{v \in S_p}, \iota(\underline{\delta})), y) \in \iota(X_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$  is in  $X_p(\bar{\rho})(L)$  where each  $\rho_v, v \in S_p$  is generic crystalline. Then  $((\rho_p, \underline{\delta}_{\mathcal{R}, w}), z) \in X_p(\bar{\rho})(L)$  if and only if  $w \geq w_{\mathcal{R}}$  in  $(S_n)^{\Sigma_p}/W_{P_p}$  where  $\mathcal{R}$  denotes refinements of  $\rho_p$ .*

Now we prove the lemma. By [Wu21, Thm. 4.10], we may assume

$$((\rho_p = (\rho_v)_{v \in S_p}, \iota(\underline{\delta})), z) \in \iota(U_{\text{tri}}(\bar{\rho}_p)) \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g).$$

Suppose that  $\underline{\delta}$  corresponds to a refinement  $\underline{\varphi} = (\varphi_v)_{v \in S_p}$ . We need to prove that the companion points for other refinements exist on the eigenvariety. We only need to prove the existence of companion points for an arbitrary refinement  $\underline{\varphi}'$  such that  $\underline{\varphi}'_v = \underline{\varphi}_v$  for all  $v \neq v_0$  for some  $v_0$  and  $\varphi'_{v_0}$  is the refinement permuting  $\varphi_{v_0, i_0}$  and  $\varphi_{v_0, i_0+1}$ . By Proposition 4.4.10, there exists a subset  $Z \subset X_{\text{tri}}(\bar{\rho}_{v_0})$  that quasi-accumulates at  $(\rho_{v_0}, \underline{\delta}_{v_0})$  consisting of generic regular crystalline points and their local companion points  $\zeta(Z)$  quasi-accumulates at  $\zeta((\rho_{v_0}, \underline{\delta}_{v_0}))$  which is a local companion point of  $(\rho_{v_0}, \underline{\delta}_{v_0})$  for the refinement  $\underline{\varphi}'_{v_0}$ . Since  $U_{\text{tri}}(\bar{\rho}_p)$  is smooth at  $(\rho_p, \underline{\delta})$ , we may assume every  $(z, (\rho_v, \underline{\delta}_v)_{v \neq v_0}), z \in Z$  is contained in the same irreducible component of  $X_{\text{tri}}(\rho_p)$  with  $((\rho_v, \underline{\delta}_v)_{v \in S_p})$ . In particular for any  $z \in Z$ ,  $(\iota(z, (\rho_v, \underline{\delta}_v)_{v \neq v_0}), y) \in X_p(\bar{\rho})$ .

For  $z = (\rho_{v_0, z}, \underline{\delta}_{v_0, z}) \in Z$ , let  $z_{\text{dom}} = (\rho_{v_0, z}, \underline{\delta}_{v_0, z, \text{dom}})$  be the companion point of  $z$  with dominant weight in  $X_{\text{tri}}(\bar{\rho}_{v_0})$  corresponding to the same refinement as  $z$ , denoted by  $\underline{\varphi}_{v_0, z}$ . By [BHS17a, Thm. 5.5], the companion points  $(\iota(z_{\text{dom}}, (\rho_v, \underline{\delta}_v)_{v \neq v_0}), y)$  are in  $X_p(\bar{\rho})$ . Let  $\mathfrak{m}_{r_z}$  be the maximal ideal of  $R_\infty$  corresponding to  $r_z = ((\rho_{v_0, z}, (\rho_v)_{v \neq v_0}), y) \in \text{Spf}(R_\infty)^{\text{rig}}$ . By the classicality (which follows from applying [Wu21, Prop. 4.9] for  $(\iota(z_{\text{dom}}, (\rho_v, \underline{\delta}_v)_{v \neq v_0}), y)$ , but essentially [BHS17a, Thm. 3.9] is enough for us, and the classicality is only partial for  $v_0$ ), there exist companion constituents of the form (as a representation of  $G_p = \prod_{v \in S_p} G_v$ )

$$(\text{Alg}(z) \otimes \text{PS}(\underline{\varphi}_{v_0, z}))^{\widehat{\otimes}_{v \neq v_0} \Pi(\underline{\delta}_v)}$$

inside  $\Pi_\infty[\mathfrak{m}_{r_z}]^{\text{an}}$ . Here  $\Pi(\underline{\delta}_v)$  are certain locally analytic representations of  $G_v$  constructed by Orlik-Strauch in [OS15] determined by  $\underline{\delta}_v$ ,  $\text{Alg}(z)$  is a finite dimensional algebraic representation of  $G_v$  determined by the weight of  $\underline{\delta}_{v_0, z, \text{dom}}$  and  $\text{PS}(\underline{\varphi}_{v_0, z})$  is the smooth induction

$$(\text{Ind}_{\bar{B}_v}^{G_v} \otimes_{i=1}^n \text{unr}(\varphi_{v_0, z, i} q_{v_0}^{1-i}))^{\text{sm}},$$

where  $q_{v_0}$  is the cardinal of the residue field of  $F_v^+$ . Let  $\underline{\varphi}'_{v_0, z}$  be the refinement of  $\rho_{v_0, z}$  from  $\underline{\varphi}_{v_0, z}$  produced by exchanging only  $\varphi_{v_0, z, i_0}$  and  $\varphi_{v_0, z, i_0+1}$ . As in the discussions in the beginning of [BHS19, §5.3], we have the intertwining

$$(\text{Ind}_{\bar{B}_v}^{G_v} \otimes_{i=1}^n \text{unr}(\varphi_{v_0, z, i} q_{v_0}^{1-i}))^{\text{sm}} \simeq (\text{Ind}_{\bar{B}_v}^{G_v} \otimes_{i=1}^n \text{unr}(\varphi'_{v_0, z, i} q_{v_0}^{1-i}))^{\text{sm}}$$

of smooth principal series. Hence the companion constituent  $(\text{Alg}(z) \otimes \text{PS}(\varphi'_{\underline{v}_0, z})) \hat{\otimes}_{v \neq v_0} \Pi(\underline{\delta}_v)$  also appears in  $\Pi_\infty[\mathfrak{m}_{r_z}]^{\text{an}}$ . This implies that some companion point of  $(\iota(z), (\rho_v, \underline{\delta}_v)_{v \neq v_0}, y)$  corresponding to the other refinement changing  $\varphi_{v_0, z}$  to  $\varphi'_{v_0, z}$  is in  $X_p(\bar{\rho})$ . By [Wu21, Thm. 4.10], the companion points  $(\iota(\zeta(z), (\rho_v, \underline{\delta}_v)_{v \neq v_0}), y)$  are in  $X_p(\bar{\rho})$  and quasi-accumulate at the point  $(\iota(\zeta((\rho_{v_0}, \underline{\delta}_{v_0})), (\rho_v, \underline{\delta}_v)_{v \neq v_0}), y)$  in  $\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p, L} \times (\mathfrak{X}_{\bar{\rho}^p} \times \mathbb{U}^g)$ . Hence

$$(\iota(\zeta((\rho_{v_0}, \underline{\delta}_{v_0})), (\rho_v, \underline{\delta}_v)_{v \neq v_0}), y) \in X_p(\bar{\rho})$$

by Lemma 4.4.8 and Lemma 4.4.7.  $\square$

### 4.5.3 Locally analytic socle conjecture

Let  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})(L)$  be generic crystalline as before. We write  $\lambda = (\lambda_\tau)_{\tau \in \Sigma_p, v \in S_p} \in (\mathbb{Z}^n)^{\Sigma_p}$  where  $\lambda_\tau = (\lambda_{\tau, 1}, \dots, \lambda_{\tau, n}) := (h_{\tau, n}, \dots, h_{\tau, i} + n - i, \dots, h_{\tau, 1} + n - 1)$ . We identify the base change to  $L$  of the  $\mathbb{Q}_p$ -Lie algebra of  $G_p$  with  $\mathfrak{g} := \prod_{\tau \in \Sigma_p} \mathfrak{gl}_{n/L}$ . Let  $\bar{\mathfrak{b}} = \prod_{\tau \in \Sigma_p} \bar{\mathfrak{b}}_\tau$  be the Borel subalgebra of  $\mathfrak{g}$  of lower triangular matrices and  $\mathfrak{t} = \prod_{\tau \in \Sigma_p} \mathfrak{t}_\tau$  be the Cartan subalgebra of diagonal matrices. We view  $\lambda$  as a weight of  $\mathfrak{t}$  and extend it to  $\bar{\mathfrak{b}}$ . For a weight  $\mu$  of  $\mathfrak{t}$ , let  $\bar{L}(\mu)$  be the irreducible  $\mathfrak{g}$ -module with the highest weight  $\mu$  in the BGG category attached to  $\bar{\mathfrak{b}}$ . For a refinement  $\mathcal{R}$  of  $\rho$ , we write  $\underline{\delta}_{\mathcal{R}, \text{sm}}$  for the smooth part of  $\delta_{\mathcal{R}, w}$ , that is  $\underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{\mathcal{R}, w}^{-1}$  is an algebraic character of  $T_p$ . Notice that  $\underline{\delta}_{\mathcal{R}, \text{sm}}$  is independent of  $w$ . Let  $\bar{B}_p$  be the opposite Borel subgroup of  $B_p$  in  $G_p$ . Recall by Orlik-Strauch's theory [OS15], we have topologically irreducible admissible locally analytic representations  $\mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{\bar{B}_p}^{-1})$ , see e.g. [Wu21, §4.3]. Here  $w_0$  is the longest element in  $\mathcal{S}_n^{\Sigma_p}$  and  $w w_0 \cdot \lambda$  denotes the usual dot action. By [Wu21, Prop. 4.9], we have the following corollary of Theorem 4.5.2 on the locally analytic socle conjecture.

**Corollary 4.5.4.** *Under the assumptions and notation of Theorem 4.5.2, there is an injection*

$$\mathcal{F}_{\bar{B}_p}^{G_p}(\bar{L}(-w w_0 \cdot \lambda), \underline{\delta}_{\mathcal{R}, \text{sm}} \delta_{\bar{B}_p}^{-1}) \hookrightarrow \widehat{S}(U^p, L)_{\bar{\rho}}[\mathfrak{m}_\rho]^{\text{an}} \quad (4.5.5)$$

of locally analytic representations of  $G_p$  for all refinements  $\mathcal{R}$  of  $\rho$  and  $w \in \mathcal{S}_n^{\Sigma_p} / W_{P_p}, w \geq w_{\mathcal{R}}$ .

*Remark 4.5.6.* Assuming that, in the situation of Theorem 4.5.2 and Corollary 4.5.4, the Hodge-Tate weights of  $\rho_v$  satisfy that  $h_{\tau, i} \neq h_{\tau, j}$  for all  $i \neq j$  and  $\tau \in \Sigma_p$ , then there exists a finite length admissible locally analytic representation  $\Pi(\rho_p)^{\text{fs}} := \hat{\otimes}_{v \in S_p} \Pi(\rho_v)^{\text{fs}}$  of  $G_p$  in [BH20] such that the  $G_p$ -socle of  $\Pi(\rho_p)^{\text{fs}}$  coincides with the finite direct sum of pairwise non-isomorphic irreducible admissible locally analytic representations of  $G_p$  that are isomorphic to one of those in the left-hand side of (4.5.5) and there exists an injection  $\Pi(\rho_p)^{\text{fs}} \hookrightarrow \widehat{S}(U^p, L)_{\bar{\rho}}[\mathfrak{m}_\rho]^{\text{an}}$  ([BH20, Thm. 1.1]). The representation  $\Pi(\rho_p)^{\text{fs}}$  is called the “finite slope part” since it is constructed from principal series (thus has Jordan-Hölder factors of the type of Orlik-Strauch). Using Corollary 4.5.4, similar result still holds without the regular assumption on Hodge-Tate weights. One just need to notice that [BH20, Prop. 4.8] is proved without any assumption on the regularity of weights and we can define  $\Pi(\rho_v)^{\text{fs}}$  in non-regular cases in the same way as [BH20, Def. 5.7]. Then the proof of [BH20, Thm. 5.12] applies with minor modifications.

## Chapter 5

# Trianguline variety at points with non-integral weights

The local models in [BHS19] and §3.3 for the trianguline variety are based on the theory of almost de Rham  $B_{dR}$ -representations of Fontaine in [Fon04], i.e. the classification of semilinear  $B_{dR}$ -representations of  $\mathcal{G}_K = \text{Gal}(\overline{K}/K)$  for a finite extension  $K$  of  $\mathbb{Q}_p$  such that the Hodge-Tate-Sen weights of their  $B_{dR}^+$ -lattices are integers. However, Fontaine has classified all  $B_{dR}$ -representations of  $\mathcal{G}_K$  in *loc. cit.* according to the  $\mathcal{G}_K$ -orbits of the Sen weights in  $\overline{K}/\mathbb{Z}$ . This suggests that a general description of the trianguline variety at general points with possibly non-locally algebraic parameters is possible. And it turns out that the theory of almost de Rham representations is essentially enough for our study in general trianguline cases.

The main theorem of this chapter is Theorem 5.5.3. The difference with Theorem 1.5.5 is that Theorem 5.5.3 does not require that the Sen weights are integers.

We use the notation in §3.3. Let  $\Sigma = \{\tau : K \hookrightarrow L\}$  and  $L$  be a large enough coefficient field such that  $\Sigma$  has size  $[K : \mathbb{Q}_p]$ . Let  $C$  be the  $p$ -adic completion of an algebraic closure  $\overline{K}$  of  $K$ .

### 5.1 Decomposition of $L \otimes_{\mathbb{Q}_p} C$ -representations

We study the decomposition of  $L \otimes_{\mathbb{Q}_p} C$ -representations according to the Sen weights. In this section, we fix  $\tau : K \hookrightarrow L$ .

Let  $V_C$  be a continuous semi-linear  $L \otimes_{\tau, K} C$ -representation of  $\mathcal{G}_K$  that is finite free of rank  $n$  over  $L \otimes_{\tau, K} C$ . Let  $\text{Rep}_{L \otimes_{\tau, K} C}(\mathcal{G}_K)$  be the category of continuous semi-linear  $L \otimes_{\tau, K} C$ -representation of  $\mathcal{G}_K$ . Then  $D_{\text{Sen}}^{K_\infty}(V_C) = (V_C^{H_K})_f$  (in the notation of [Fon04, §2.1]) is a free  $L \otimes_{\tau, K} K_\infty$ -module of rank  $n$  (see remark below) equipped with a semi-linear action of  $\Gamma_K$  and a linear Sen operator  $\nabla$  whose characteristic polynomial  $P(T)$  lies in  $L \otimes_{\tau, K} K[T] = L[T]$ . The zeros of the Sen polynomial  $P(T)$  is called the (Hodge-Tate-)Sen weights of  $V_C$ .

*Remark 5.1.1.* A priori, in [Fon04],  $D_{\text{Sen}}^{K_\infty}(V_C)$  is only known to be free over  $K_\infty$ . To show the freeness over  $L \otimes_{\tau, K} K_\infty$ , we can use the argument of [Nak09, Lem. 1.30]. Take  $L' = L \cap K_\infty$  and let  $I = \{\tau' : L' \hookrightarrow L\}$  be all the  $K$ -embeddings from  $L'$  to  $L$ . Then  $L \otimes_{\tau, K} K_\infty = \prod_{\tau' \in I} L \otimes_{\tau', L'} K_\infty$  (we can assume that  $L/\mathbb{Q}_p$  is Galois firstly). Now each  $L \otimes_{\tau', L'} K_\infty$  is a field and we need to show that the  $L \otimes_{\tau', L'} K_\infty$ -dimension of  $D_{\text{Sen}}^{K_\infty}(V_C) \otimes_{L \otimes_{\tau, K} L', 1 \otimes \tau'} L$  is independent of  $\tau'$ . This follows from that  $\Gamma_K$  acts transitively on  $I$ . We may also use the general theory in [BC08, §3.3].

We assume that all the Sen weights of  $V_C$  are in  $L$ . If  $a \in L$  is a Sen weight of  $V_C$ , we write  $D_{\text{Sen}}^{K_\infty}(V_C)_a$  for the generalized eigenspace of  $\nabla$  inside  $V_C$  for the eigenvalue  $a$ . Since  $\nabla$  commutes with the actions of  $L \otimes_{\tau, K} K_\infty$  and  $\Gamma_K$ ,  $D_{\text{Sen}}^{K_\infty}(V_C)_a$  is a semilinear representation of  $\Gamma_K$  over  $L \otimes_{\tau, K} K_\infty$  (i.e. free over  $L \otimes_{\tau, K} K_\infty$  by the above remark) and is a direct summand of



$D_{\text{Sen}}^{K_\infty}(V_C)$ . Let  $V_{C,a}$  be the semilinear  $L \otimes_{\tau,K} C$ -representation of  $\mathcal{G}_K$  generated by  $D_{\text{Sen}}^{K_\infty}(V_C)_a$  in  $V_C$ . Let  $\mathcal{A}$  be the set of distinct Sen weights of  $V_C$ . Recall that we have assumed that  $\mathcal{A} \subset L$ . Since  $V_C = D_{\text{Sen}}^{K_\infty}(V_C) \otimes_{K_\infty} C$  and  $D_{\text{Sen}}^{K_\infty}(V_C) = \prod_{a \in \mathcal{A}} D_{\text{Sen}}^{K_\infty}(V_C)_a$ , we get that  $V_C = \prod_{a \in \mathcal{A}} V_{C,a}$  as a decomposition of  $L \otimes_{\tau,K} C$ -representations of  $\mathcal{G}_K$ . Each  $V_{C,a}$  has the single Hodge-Tate weight  $a$  possibly with multiplicities.

If  $V \in \text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)$ , one can define the continuous group cohomologies  $H^i(\mathcal{G}_K, V)$  as usual ([Fon04, §1.1]). By checking definitions of extensions and group cohomologies using cocycles, there are isomorphisms of  $L$ -spaces

$$H^0(\mathcal{G}_K, V) = \text{Hom}_{\text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)}(L \otimes_{\tau,K} C, V)$$

and

$$H^1(\mathcal{G}_K, V) = \text{Ext}_{\text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)}^1(L \otimes_{\tau,K} C, V).$$

**Lemma 5.1.2.** *Assume that  $V_1, V_2 \in \text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)$ . Suppose that  $V_1, V_2$  share no same Sen weights and all their Sen weights are in  $L$ . Then*

$$\text{Hom}_{\text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)}(V_1, V_2) = 0, \text{Ext}_{\text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)}^1(V_1, V_2) = 0.$$

*Proof.* Let  $V_i^* = \text{Hom}_{L \otimes_{\tau,K} C}(V, L \otimes_{\tau,K} C)$  be the dual representation. Then one can check  $D_{\text{Sen}}^{K_\infty}(V_i^*) = D_{\text{Sen}}^{K_\infty}(V_i)^*$  with Sen operator  $-{}^t\nabla$  where  ${}^t\nabla$  denotes the transpose map. And  $D_{\text{Sen}}^{K_\infty}(V_1^* \otimes_{L \otimes_{\tau,K} C} V_2) = D_{\text{Sen}}^{K_\infty}(V_1)^* \otimes_{L \otimes_{\tau,K} C} D_{\text{Sen}}^{K_\infty}(V_2)$  with Sen operator  $-{}^t\nabla \otimes 1 + 1 \otimes \nabla$ . In particular, none of the Sen weights of  $V_1^* \otimes_{L \otimes_{\tau,K} C} V_2$  is zero. Then as a  $C$ -representation of  $\mathcal{G}_K$ ,  $V_1^* \otimes_{L \otimes_{\tau,K} C} V_2$  has no zero Sen weight. We get  $H^0(\mathcal{G}_K, V_1^* \otimes_{L \otimes_{\tau,K} C} V_2) = H^1(\mathcal{G}_K, V_1^* \otimes_{L \otimes_{\tau,K} C} V_2) = 0$  by [Fon04, Prop. 2.15].  $\square$

*Remark 5.1.3.* It is possible that  $V_1$  and  $V_2$  as in the above lemma have Sen weights that are in the same  $\mathcal{G}_K$ -orbits in  $L$ . In this case,  $\text{Hom}_{\text{Rep}_C(\mathcal{G}_K)}(V_1, V_2) \neq 0$  and  $V_1^* \otimes_C V_2$  should admit zero Sen weights.

If  $a \in \mathcal{A}$  such that there exists  $V(a) \in \text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)$  of rank one (over  $L \otimes_{\tau,K} C$ ) with a unique Sen weight  $a$ , then  $V_{C,a} \otimes_{L \otimes_{\tau,K} C} V(a)^*$  has only Sen weights zero. And  $L \otimes_{\tau,K} C$ -representations with weights zero can be classified as in [Fon04]. By [Fon04, Prop. 2.15]

$$\text{Ext}_{\text{Rep}_{L \otimes_{\tau,K} C}(\mathcal{G}_K)}^1(L \otimes_{\tau,K} C, L \otimes_{\tau,K} C) = H^1(\mathcal{G}_K, L \otimes_{\tau,K} C)$$

has dimension one over  $L$ .

*Remark 5.1.4.* If  $a \in L$  is the  $\tau$ -part weight of a continuous character  $\delta : K^\times \rightarrow L^\times$ , then  $(W_{\text{dR}}^+(\mathcal{R}_{L,K}(\delta))/t) \otimes_{L \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} L$  has weight  $a$  (cf. [KPX14, Lem. 6.2.12]). In general, there might exist  $a \in L$  such that all simple  $L \otimes_{\tau,K} C$ -representations of  $\mathcal{G}_K$  with the unique Sen weight  $a$  have ranks (over  $L \otimes_{\tau,K} C$ ) strictly greater than one. We give below an example of such  $a$  when  $L = K = \mathbb{Q}_p$  using Fontaine's classification.

We change, for the moment, the choice of  $K_\infty$  and  $\Gamma_K$  to follow [Fon04]. Let  $K_\infty$  be the unique  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}_p$ , contained in  $\mathbb{Q}_p(\zeta_{p^\infty})$ . Let  $\Gamma = \text{Gal}(K_\infty/K)$ ,  $\Gamma_r \simeq p^{r+1}\mathbb{Z}_p \subset \Gamma \xrightarrow[\log(\chi_{\text{cyc}})]{\sim} p\mathbb{Z}_p$ , and  $K_r := K_\infty^{\Gamma_r}$ . Let  $\eta : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \Gamma \rightarrow \mathbb{Q}/\mathbb{Z}$  sending a generator  $\gamma_0 = p$  of  $\Gamma = p\mathbb{Z}_p$  to  $\frac{1}{p^r}$ .

Let  $r$  be the smallest integer such that  $v_p(p^{r+1}a) > \frac{1}{p-1}$ . Let  $\chi_a : \mathcal{G}_{K_r} \rightarrow \Gamma_r \rightarrow \mathbb{Q}_p^\times$  be the character sending  $\gamma \in p^{r+1}\mathbb{Z}_p \simeq \Gamma_r$  to  $\exp(a\gamma)$ . Let  $K_\infty[a]$  be a simple semilinear  $K_\infty$ -subrepresentation of  $\Gamma$  inside  $N_\infty[a] := K_\infty \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p[\Gamma] \otimes_{\mathbb{Q}_p[\Gamma_r]} \chi_a)$ . The simple  $C$ -representation  $C[a]$  of  $\mathcal{G}_{\mathbb{Q}_p}$  with weight  $a$  is  $C \otimes_{K_\infty} K_\infty[a]$ .

It's enough to find  $a$  such that  $r \geq 1$  and  $N_\infty[a]$  is already a simple  $K_\infty$ -representation of  $\Gamma$ . Let  $b = \exp(p^{r+1}a)$ . By [Fon04, Prop. 2.13],  $\mathrm{End}_{\mathrm{Rep}_{K_\infty}(\Gamma)}(N_\infty[a])$  is the central simple algebra  $\Lambda_{\mathbb{Q}_p}(\eta, b)$  over  $\mathbb{Q}_p$  of dimension  $(p^r)^2$  generated over  $K_r$  by an element  $c$  such that  $c^{p^r} = b$  and  $cu = \gamma_0(u)c$  for  $u \in K_r$ . Moreover, the dimension of  $K_\infty[a]$  is equal to  $p^s$  where  $p^{2s}$  is the dimension of the division algebra over  $\mathbb{Q}_p$  that is equal to  $\Lambda_{\mathbb{Q}_p}(\eta, b)$  inside  $\mathrm{Br}(\mathbb{Q}_p)$ .

Hence we only need to find  $a \in \mathbb{Q}_p$  such that  $\Lambda_{\mathbb{Q}_p}(\eta, b)$  is a division algebra. We can choose firstly  $r \geq 1$  and  $b \in 1 + p\mathbb{Z}_p^\times$  and then take  $a = p^{-1-r} \log(b)$ . For example, we can take  $r = 1$  and  $b = 1 + p$ . Then  $\Lambda_{\mathbb{Q}_p}(\eta, b)$  has dimension  $p^2$  over  $\mathbb{Q}_p$ . If  $\Lambda_{\mathbb{Q}_p}(\eta, b)$  is not a division algebra, then  $\Lambda_{\mathbb{Q}_p}(\eta, b) \simeq M_p(\mathbb{Q}_p)$ . However  $\Lambda_{\mathbb{Q}_p}(\eta, b)$  is not trivial in  $\mathrm{Br}(\mathbb{Q}_p)$  since  $b \notin \mathrm{Nm}_{K_1/\mathbb{Q}_p}(K_1) = p^{\mathbb{Z}}[\mathbb{F}_p^\times](1 + p^2\mathbb{Z}_p)$ .

## 5.2 Decomposition of $L \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ -representations

We decompose  $L \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ -representations according to the Sen weights in  $L$  modulo  $\mathbb{Z}$ . In this section, we still fix  $\tau : K \hookrightarrow L$ .

Let  $W^+$  be an  $L \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}^+$ -representation of  $\mathcal{G}_K$ . By the definition in [BHS19, §3.2],  $W^+$  is a finite free  $L \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}^+$ -module. Let  $W = W^+[\frac{1}{t}]$ . Let  $n \in \mathbb{N}$  be the rank of  $W^+$  over  $L \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ . For each  $\tau \in \Sigma$ , let  $W_\tau^+ := W^+ \otimes_{L \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} L$  and similarly for  $W_\tau$ .

We have the  $L \otimes_{\mathbb{Q}_p} C$ -representation  $W^+/t$  or the  $L \otimes_{\tau, K} C$ -representation  $W_\tau^+/t$  of  $\mathcal{G}_K$ . Let  $\mathcal{A}_\tau$  be the set of Hodge-Tate-Sen weights of  $W_\tau^+/t$ . In the notation of the last section,  $W_\tau^+/t = \bigoplus_{a \in \mathcal{A}_\tau} (W_\tau^+/t)_a$ . Assume that  $\mathcal{A}_\tau \subset L$ .

Let  $\overline{\mathcal{A}}_\tau := (\mathcal{A}_\tau + \mathbb{Z})/\mathbb{Z} \subset L/\mathbb{Z}$  and for  $a \in \mathcal{A}_\tau$ , write  $\overline{a} = a + \mathbb{Z}$  for its class in  $\overline{\mathcal{A}}_\tau$  and define  $(W_\tau^+/t)_{\overline{a}} := \bigoplus_{a \in \overline{a}} (W_\tau^+/t)_a$ . Set  $d_{\tau, \overline{a}} := \dim_L (W_\tau^+/t)_{\overline{a}}$ . Then  $d_{\tau, \overline{a}}$  is the sum of multiplicities of all  $\tau$ -Sen weights of  $W_\tau^+$  that lie in  $a + \mathbb{Z}$ .

We have a functor  $D_{\mathrm{pdR}}(-) : W_\tau \mapsto (W_\tau \otimes_{\mathrm{B}_{\mathrm{dR}}} \mathrm{B}_{\mathrm{dR}}[\log(t)])^{\mathcal{G}_K}$ . The output  $D_{\mathrm{pdR}}(W_\tau)$  is a finite-dimensional  $K$ -vector space with a nilpotent operator  $\nu_{W_\tau}$  and an action of  $L$ . Then  $D_{\mathrm{pdR}}(W_\tau)$  is a finite-dimensional  $L$ -space with an  $L$ -linear nilpotent operator  $\nu_{W_\tau}$ . We have an injection of  $\nu$ -graded  $\mathcal{G}_K$ -representations ([Fon04, Thm. 4.1])

$$D_{\mathrm{pdR}}(W_\tau) \otimes_K \mathrm{B}_{\mathrm{dR}}[\log(t)] \hookrightarrow W_\tau \otimes_{\mathrm{B}_{\mathrm{dR}}} \mathrm{B}_{\mathrm{dR}}[\log(t)]$$

which induces, by taking  $\nu = 0$  parts, an injection

$$(D_{\mathrm{pdR}}(W_\tau) \otimes_K \mathrm{B}_{\mathrm{dR}}[\log(t)])^{\nu_{W_\tau} \otimes 1 + 1 \otimes \nu} \hookrightarrow W_\tau.$$

Let  $W_{\tau, \overline{0}}$  be the image of the above map.

**Lemma 5.2.1.** *The underlying  $L \otimes_{\tau, K} \mathrm{B}_{\mathrm{dR}}$ -module of the semilinear  $\mathrm{B}_{\mathrm{dR}}$ -representation  $W_{\tau, \overline{0}}$  of  $\mathcal{G}_K$  is finite free of the rank  $d_{\tau, \overline{0}}$ .*

*Proof.* Since  $D_{\mathrm{pdR}}(W_\tau)$  is free over  $L$ , the first assertion follows from the equivalence in [BHS19, Lem. 3.1.4]. The sub- $\mathrm{B}_{\mathrm{dR}}$ -module  $W_{\tau, \overline{0}}$  is a direct summand as  $\mathrm{B}_{\mathrm{dR}}$  is a field. We only need to prove that  $W_{\tau, \overline{0}}$  is free of rank  $[L : K]d_{\tau, \overline{0}}$  over  $\mathrm{B}_{\mathrm{dR}}$ . The latter is the sum of multiplicities of integral Sen weights of  $W_\tau^+/t$  as  $C$ -representations (not  $L \otimes_{\tau, K} C$ ). Hence we can forget the  $L$ -actions and the result follows from the classification of  $\mathrm{B}_{\mathrm{dR}}$ -representations in [Fon04, §3.7], or go back to the proof of [Fon04, Thm. 3.13] using [Fon04, Lem. 3.14].  $\square$

If  $\delta : K^\times \rightarrow L^\times$  is a continuous character, let  $W_{\mathrm{dR}}^+(\delta) := W_{\mathrm{dR}}^+(\mathcal{R}_{L, K}(\delta))$ . Assume that  $\mathrm{wt}(\delta) = (k_\tau)_{\tau \in \Sigma} \in L^n$ . Then  $W_{\mathrm{dR}}^+(\delta)_\tau$  is a rank one  $L \otimes_{\tau, K} \mathrm{B}_{\mathrm{dR}}^+$ -representation of  $\mathcal{G}_K$  with the Sen weight  $k_\tau$ , see the end of [Nak09, §1.4]. Then  $W_{\mathrm{dR}}(\delta)_\tau = W_{\mathrm{dR}}^+(\delta)_\tau[\frac{1}{t}]$  is the unique  $L \otimes_{\tau, K} \mathrm{B}_{\mathrm{dR}}$ -representation of  $\mathcal{G}_K$  or rank one which admits an  $L \otimes_{\tau, K} \mathrm{B}_{\mathrm{dR}}^+$ -lattice with Sen weight in  $k_\tau + \mathbb{Z}$  (if  $W'_\tau$  has  $L \otimes_{\tau, K} \mathrm{B}_{\mathrm{dR}}^+$  lattices of weight in  $\mathbb{Z}$  and is of rank one over  $L \otimes_{\tau, K} \mathrm{B}_{\mathrm{dR}}$ , then  $W'_\tau \simeq L \otimes_{\tau, K} \mathrm{B}_{\mathrm{dR}}$  since  $(W'_\tau)^{\mathcal{G}_K} \neq 0$ ).

*Remark 5.2.2.* There exists a unique up to isomorphisms rank one  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -representation of weight 0, using the uniqueness of the associated filtered rank one  $L$ -modules ([BHS19, Lem. 3.2.2]). The  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -representation  $W_{\text{dR}}^+(\delta)_\tau$  is also the unique rank one representation of  $\mathcal{G}_K$  with Sen weight  $k_\tau$ .

From now on, we assume the following assumption.

*Assumption 5.2.3.* Assume that for any  $\tau \in \Sigma, a \in \mathcal{A}_\tau$ , there exists a continuous character  $\delta : K^\times \rightarrow L^\times$  such that  $\text{wt}_\tau(\delta) = a$ .

This assumption will be satisfied if  $W^+ = W_{\text{dR}}^+(\delta)$  for a continuous character  $\delta : K^\times \rightarrow L^\times$ . For  $\bar{a} \in \bar{\mathcal{A}} \subset L$  and  $\delta$  such that  $\text{wt}_\tau(\delta) = a$ , we let  $W_{\tau, \bar{a}}$  be the image of the map

$$\begin{aligned} (D_{\text{pdR}}(W_\tau \otimes_{L \otimes_{\tau, K} B_{\text{dR}}} W_{\text{dR}}(\delta)_\tau^*) \otimes_K B_{\text{dR}}[\log(t)])^{\nu=0} \otimes_{L \otimes_{\tau, K} B_{\text{dR}}} W_{\text{dR}}(\delta)_\tau \\ \hookrightarrow W_\tau \otimes_{L \otimes_{\tau, K} B_{\text{dR}}} W_{\text{dR}}(\delta)_\tau^* \otimes_{L \otimes_{\tau, K} B_{\text{dR}}} W_{\text{dR}}(\delta)_\tau \xrightarrow{\sim} W_\tau. \end{aligned}$$

We write  $W(\delta^{-1})_\tau$  for  $W_\tau \otimes_{L \otimes_{\tau, K} B_{\text{dR}}} W_{\text{dR}}(\delta)_\tau^*$ . As in the Lemma 5.2.1 when  $\bar{a} = \mathbb{Z}$ , we get that  $W_{\tau, \bar{a}}$  is free of rank  $d_{\tau, \bar{a}}$  over  $L \otimes_{\tau, K} B_{\text{dR}}$  and admits  $\mathcal{G}_K$ -invariant lattices over  $L \otimes_{\tau, K} B_{\text{dR}}^+$ . Hence we have a map

$$\bigoplus_{\bar{a} \in \bar{\mathcal{A}}_\tau} W_{\tau, \bar{a}} \rightarrow W_\tau \quad (5.2.4)$$

which is an injection since

$$\text{Hom}_{\text{Rep}_{L \otimes_{\tau, K} B_{\text{dR}}}}(\mathcal{G}_K)(W'_{\tau, \bar{a}_1}, W'_{\tau, \bar{a}_2}) = H^0(\mathcal{G}_K, (W'_{\tau, \bar{a}_1})^* \otimes_{L \otimes_{\tau, K} B_{\text{dR}}} W'_{\tau, \bar{a}_2}) = 0$$

for  $\bar{a}_1 \neq \bar{a}_2$  and  $\mathcal{G}_K$ -invariant sub- $L \otimes_{\tau, K} B_{\text{dR}}$ -modules  $W'_{\tau, \bar{a}_i} \subset W_{\tau, \bar{a}_i}$  (see also Lemma 5.2.6 below). Note that  $W'_{\tau, \bar{a}_i} \in \text{Rep}_{L \otimes_{\tau, K} B_{\text{dR}}}(\mathcal{G}_K)$  by arguments in Remark 5.1.1 and that  $\mathcal{G}_K$  acts transitively on  $K$ -embeddings  $L \hookrightarrow B_{\text{dR}}$ , and  $W'_{\tau, \bar{a}_i}$  admits  $\mathcal{G}_K$ -invariant  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -lattices by [BHS19, Rem. 3.13] and a twist. By considering the  $B_{\text{dR}}$ -dimensions, we get that (5.2.4) induces a decomposition

$$W_\tau = \bigoplus_{\bar{a} \in \bar{\mathcal{A}}_\tau} W_{\tau, \bar{a}}.$$

For each  $a \in \bar{\mathcal{A}}$ , let  $W_{\tau, \bar{a}}^+ = W_{\tau, \bar{a}} \cap W_\tau^+$ . Then it is easy to see that  $W_{\tau, \bar{a}}^+$  is a  $\mathcal{G}_K$ -stable  $B_{\text{dR}}^+$ -lattices of  $W_{\tau, \bar{a}}$ , saturated in  $W_\tau^+$  and is an  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -module. Hence  $W_{\tau, \bar{a}}^+ \in \text{Rep}_{L \otimes_{\tau, K} B_{\text{dR}}^+}(\mathcal{G}_K)$ . The  $L \otimes_{\tau, K} C$ -map  $W_{\tau, \bar{a}}^+/tW_{\tau, \bar{a}}^+ \rightarrow W_\tau^+/t$  is an injection with image in  $(W_\tau^+/t)_{\bar{a}}$ , hence induces an isomorphism  $W_{\tau, \bar{a}}^+/tW_{\tau, \bar{a}}^+ \xrightarrow{\sim} (W_\tau^+/t)_{\bar{a}}$  considering  $C$ -dimensions. Then the  $B_{\text{dR}}^+$ -map  $\bigoplus_{\bar{a} \in \bar{\mathcal{A}}_\tau} W_{\tau, \bar{a}}^+ \rightarrow W_\tau^+$  is a surjection modulo  $t$  and the source and the target have the same  $B_{\text{dR}}^+$ -rank, thus is an isomorphism. Finally, we get the decomposition

$$W_\tau^+ = \bigoplus_{\bar{a} \in \bar{\mathcal{A}}_\tau} W_{\tau, \bar{a}}^+.$$

*Remark 5.2.5.* There is a more direct way to see the decomposition of  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -representations. For  $W_\tau^+$ , one can define  $D_{\text{diff}}^{K_\infty, +}(W_\tau^+)$  as [Fon04, §3.3] which is a  $L \otimes_{\tau, K} K_\infty[[t]]$ -module with a semilinear action of  $\Gamma_K$  and from  $\Gamma_K$ , there is an  $L \otimes_{\tau, K} K_\infty$ -linear operator  $\nabla$  on  $D_{\text{diff}}^{K_\infty, +}(W_\tau^+)$  such that  $\nabla(t^a) = at^a$ , etc. (see also §A.2). For each  $i \in \mathbb{N}$ , we have a decomposition

$$D_{\text{diff}}^{K_\infty, +}(W_\tau^+)/t^i = \bigoplus_{\bar{a} \in \bar{\mathcal{A}}_\tau} (D_{\text{diff}}^{K_\infty, +}(W_\tau^+)/t^i)_{\bar{a}}$$

as  $L \otimes_{\tau, K} K_\infty$ -modules considering generalized eigenspaces of  $\nabla$ . Each  $(D_{\text{diff}}^{K_\infty, +}(W_\tau^+)/t^i)_{\bar{a}}$  is stable under the action of  $\Gamma_K$ . Moreover,  $(D_{\text{diff}}^{K_\infty, +}(W_\tau^+)/t^i)_{\bar{a}}$  is stable under the action of  $t$  since  $\bar{a} = a + \mathbb{Z}$ . Hence  $D_{\text{diff}}^{K_\infty, +}(W_\tau^+)_{\bar{a}} := \varprojlim_i (D_{\text{diff}}^{K_\infty, +}(W_\tau^+)/t^i)_{\bar{a}}$  is an  $L \otimes_{\tau, K} K_\infty[[t]]$ -semilinear representation of  $\Gamma_K$  and we have a decomposition

$$D_{\text{diff}}^{K_\infty, +}(W_\tau^+) = \bigoplus_{\bar{a} \in \bar{\mathcal{A}}_\tau} D_{\text{diff}}^{K_\infty, +}(W_\tau^+)_{\bar{a}}.$$

**Lemma 5.2.6.** *Let  $W_1^+, W_2^+$  be  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -representations of  $\mathcal{G}_K$  such that their Sen weights are in  $L$  and  $W_1, W_2$  share no same Sen weights mod  $\mathbb{Z}$ . Then  $\text{Hom}_{\text{Rep}_{L \otimes_{\tau, K} B_{\text{dR}}^+}(\mathcal{G}_K)}(W_1^+, W_2^+) = 0$  and  $\text{Ext}_{\text{Rep}_{L \otimes_{\tau, K} B_{\text{dR}}^+}(\mathcal{G}_K)}^1(W_1^+, W_2^+) = 0$ . Moreover if  $W_i = W_i^+[\frac{1}{t}]$ , then similar results hold for  $W_1$  and  $W_2$  in  $\text{Rep}_{L \otimes_{\tau, K} B_{\text{dR}}}(\mathcal{G}_K)$ .*

*Proof.* By considering  $W^+ := (W_1^+)^* \otimes_{L \otimes_{\tau, K} B_{\text{dR}}^+} W_2^+$ , the proof is similar to that of Lemma 5.1.2, using [Nak14, Cor. 5.6] which follows from the fact that ([Nak14, Lem. 5.5])

$$H^i(\mathcal{G}_K, W^+) \simeq \varprojlim_j H^i(\mathcal{G}_K, W^+ / t^j W^+)$$

and  $H^i(\mathcal{G}_K, t^j W^+ / t^{j+1} W^+) = 0$  for all  $j \in \mathbb{Z}, i = 0, 1$  if  $W^+$ , hence all  $t^j W^+ / t^{j+1} W^+$ , has no integral Sen weights.  $\square$

The above lemma implies that the decompositions of  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -representations of  $\mathcal{G}_K$  according to the mod  $\mathbb{Z}$ -classes of Sen weights in  $L$  we obtained are natural.

### 5.3 Deformation of $L \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$ -representations

We fix  $W^+, W_\tau^+, \mathcal{A} = (\mathcal{A}_\tau)_{\tau \in \Sigma}$  as in the last section. Let  $A \in \mathcal{C}_L$ , i.e.  $A$  is a local Artinian algebra over  $L$  such that  $A/m_A = L$ , where  $m_A$  is the maximal ideal of  $A$ .

Let  $(A, W_A^+, \iota_A) \in X_{W^+}$  where  $W_A^+ \in \text{Rep}_{A \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+}(\mathcal{G}_K)$  is a semilinear representation of  $\mathcal{G}_K$  free of rank  $n$  over  $A \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$  together with an isomorphism  $\iota_A : W_A^+ \otimes_A L \xrightarrow{\sim} W^+$ . Let  $W_{A, \tau}^+ = W_A^+ \otimes_{L \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} L$  as before. Then  $\iota_A : W_{A, \tau}^+ \otimes_A L \xrightarrow{\sim} W_\tau^+$ . The mod  $\mathbb{Z}$  classes of the Sen weights of the  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -representation  $W_{A, \tau}^+$  are in  $\overline{\mathcal{A}}_\tau$  since  $W_{A, \tau}^+$  is a successive extension of subquotients of  $W_\tau^+$ . Hence we have also a decomposition of  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -representations of  $\mathcal{G}_K$

$$W_{A, \tau}^+ = \bigoplus_{a \in \mathcal{A}_\tau} W_{A, \tau, \bar{a}}^+.$$

Moreover, each  $W_{A, \tau, \bar{a}}^+$  is stable under the action of  $A$  and hence is a projective  $A \otimes_{\tau, K} B_{\text{dR}}^+$ -module. The map  $\iota_A : W_{A, \tau}^+ / m_A \xrightarrow{\sim} W_\tau^+$  must induce isomorphisms  $W_{A, \tau, \bar{a}}^+ / m_A \xrightarrow{\sim} W_{\tau, \bar{a}}^+$  by Lemma 5.2.6. By the following lemma,  $W_{A, \tau, \bar{a}}^+$  is free of rank  $d_{\tau, \bar{a}}$  over  $A \otimes_{\tau, K} B_{\text{dR}}^+$ .

**Lemma 5.3.1.** *Suppose that  $R$  is a ring and  $m$  is a nilpotent ideal of  $R$ . If  $M$  is a finite projective  $R$ -module such that  $M/m$  is free of rank  $r$  over  $R/m$ , then  $M$  is also free of rank  $r$  over  $m$ .*

*Proof.* Let  $e_1, \dots, e_r$  be elements of  $M$  that lift a basis of  $M/m$  over  $R/m$  which induces a map  $f : R^r \rightarrow M$ . Then  $(M/f(R^r))/m = 0$ . By Nakayama's lemma,  $f$  is a surjection. Let  $K$  be the kernel of  $f$ . Then  $K$  is finitely generated as a direct summand of  $R^r$ . Using that  $\text{Tor}_1^R(M, R/m) = 0$ , we get  $K/mK = 0$ . Hence  $K = 0$ .  $\square$

Suppose that there is a trianguline filtration  $\mathcal{F}_\bullet : \{0\} \subsetneq W_1 \subsetneq \dots \subsetneq W_i \subsetneq \dots \subsetneq W_n = W$  of  $L \otimes_{\mathbb{Q}_p} B_{\text{dR}}$ -representations of  $\mathcal{G}_K$  of a parameter  $\underline{\delta} = (\delta_1, \dots, \delta_n) : (K^\times)^n \mapsto L^\times$ , i.e.  $W_i/W_{i-1} = W_{\text{dR}}(\delta_i)$  for  $i = 1, \dots, n$ . For  $\tau \in \Sigma$ , write the  $\tau$ -part by  $W_{i, \tau}$ . Each  $W_{i, \tau}$  decomposes as  $W_{i, \tau} = \bigoplus_{\bar{a} \in \overline{\mathcal{A}}_\tau} W_{i, \tau, \bar{a}}$ . For each  $i$ , there exists  $\bar{a} \in \overline{\mathcal{A}}_\tau$  such that  $W_{i, \tau}/W_{i-1, \tau} = W_{i, \tau, \bar{a}}/W_{i-1, \tau, \bar{a}}$  and  $\text{wt}_\tau(\delta_i) \in a + \mathbb{Z}$ . Hence  $\mathcal{F}_\bullet$  induces trianguline filtrations  $\mathcal{F}_{\tau, \bar{a}, \bullet} : \{0\} \subseteq W_{1, \tau, \bar{a}} \subseteq \dots \subseteq W_{i, \tau, \bar{a}} \subseteq \dots \subseteq W_{n, \tau, \bar{a}} = W_{\tau, \bar{a}}$  of  $W_{\tau, \bar{a}}$ . We rewrite  $\mathcal{F}_{\tau, \bar{a}, \bullet}$  as  $\{0\} \subsetneq \mathcal{F}_{1, \tau, \bar{a}} \subsetneq \dots \subsetneq \mathcal{F}_{i, \tau, \bar{a}} \subsetneq \dots \subsetneq \mathcal{F}_{d_{\tau, \bar{a}}, \tau, \bar{a}} = W_{\tau, \bar{a}}$ . It is clear that we can recover  $\mathcal{F}_\bullet$  from  $\mathcal{F}_{\tau, \bar{a}, \bullet}$  where  $\tau \in \Sigma, \bar{a} \in \overline{\mathcal{A}}_\tau$ .

Recall we have groupoids  $X_{W^+, \mathcal{F}_\bullet} = X_{W^+} \times_{X_W} X_{W, \mathcal{F}_\bullet}$  in [BHS19, §3.1]. Assume that we have  $(A, W_A^+, \iota_A, \mathcal{F}_{A, \bullet}) \in X_{W^+, \mathcal{F}_\bullet}$ , i.e.  $\mathcal{F}_{A, \bullet} : \{0\} \subsetneq W_{A, 1} \subsetneq \dots \subsetneq W_{A, i} \subsetneq \dots \subsetneq W_{A, n} = W_A$

is a trianguline filtration whose graded pieces are rank one  $A \otimes_{\mathbb{Q}_p} B_{\text{dR}}$ -representations of  $\mathcal{G}_K$  such that  $\iota_A$  induces  $W_{A,i} \otimes_A L \xrightarrow{\sim} W_i$ . By the discussions above (replacing  $B_{\text{dR}}^+$  by  $B_{\text{dR}}$ ), for each  $i$ ,  $W_{A,i,\tau} = \bigoplus_{\bar{a} \in \bar{\mathcal{A}}_\tau} W_{A,i,\tau,\bar{a}}$ . For each  $i$ ,  $W_{A,i,\tau}/W_{A,i-1,\tau}$  is a deformation of  $W_{i,\tau}/W_{i-1,\tau} = W_{i,\tau,\bar{a}}/W_{i-1,\tau,\bar{a}}$  for  $a$  such that  $\text{wt}_\tau(\delta_i) \in a + \mathbb{Z}$  and is equal to  $W_{A,i,\tau,\bar{a}}/W_{A,i-1,\tau,\bar{a}}$ . Thus  $\mathcal{F}_{A,\bullet}$  induces trianguline filtrations  $\mathcal{F}_{A,\tau,\bar{a},\bullet}$  on  $W_{A,\tau,\bar{a}}$ .

We define groupoids  $X_{W_{\tau,\bar{a}},\mathcal{F}_{\tau,\bar{a},\bullet}}$  over  $\mathcal{C}_L$  similarly as  $X_{W^+,\mathcal{F}_\bullet}$  replacing  $L \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$  by  $L \otimes_{\tau,K} B_{\text{dR}}^+$ . For each  $\tau, \bar{a}$ , the map  $(A, W_A^+, \iota_A, \mathcal{F}_A) \mapsto (A, W_{A,\tau,\bar{a}}^+, \iota_{A,\tau,\bar{a}}, \mathcal{F}_{\tau,\bar{a},\bullet})$  defines a morphism  $X_{W^+,\mathcal{F}_\bullet} \rightarrow X_{W_{\tau,\bar{a}},\mathcal{F}_{\tau,\bar{a},\bullet}}$  of groupoids over  $\mathcal{C}_L$ .

**Lemma 5.3.2.** *The morphism  $X_{W^+,\mathcal{F}_\bullet} \rightarrow \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} X_{W_{\tau,\bar{a}},\mathcal{F}_{\tau,\bar{a},\bullet}}$ , where the product is taken over  $\mathcal{C}_L$ , induces an equivalence of categories over  $\mathcal{C}_L$ .*

*Proof.* The functor is fully faithful by Lemma 5.2.6. To show that the functor is essentially surjective, we suppose that we have  $(A, W_{A,\tau,\bar{a}}^+, \iota_{A,\tau,\bar{a}}, \mathcal{F}_{A,\tau,\bar{a},\bullet})$  for each  $\bar{a}, \tau$ . We take the direct sum  $(\bigoplus_{\tau,\bar{a}} W_{A,\tau,\bar{a}}^+)$  which defines an object in  $X_{W^+}$  and we can arrange the filtration  $\mathcal{F}_{A,\bullet} = (W_{A,i})_{i=1,\dots,n}$  by  $W_{A,i} = \bigoplus_{\tau,\bar{a}} W_{A,i,\tau,\bar{a}}$  where  $W_{A,i,\tau,\bar{a}} = \mathcal{F}_{A,\tau,\bar{a},j_{i,\bar{a}}}$  where  $j_{i,\bar{a}} = |\{1 \leq j \leq i, \text{wt}(\delta_j)_\tau \in a + \mathbb{Z}\}|$ .  $\square$

We fix  $a \in \bar{a} + \mathbb{Z}$  and a character  $\delta_{\tau,a} : K^\times \rightarrow L^\times$  such that  $\text{wt}_\tau(\delta_{\tau,a}) = a \in L$  for each  $\bar{a} \in \bar{\mathcal{A}}_\tau$ . Write  $W_{\text{dR}}^+(\delta_{\tau,a})_{A,\tau}$  for the rank one  $A \otimes_{\tau,K} B_{\text{dR}}^+$ -representation  $W_{\text{dR}}^+(\mathcal{R}_{A,K}(\delta_{\tau,a}))_\tau$  of  $\mathcal{G}_K$  for  $A \in \mathcal{C}_L$ . If  $W_{A,\tau,\bar{a}}^+ \in X_{W_{\tau,\bar{a}}^+}$ , then  $W_{A,\tau,\bar{a}}^+$  has only Sen weights in  $a + \mathbb{Z}$ . Hence  $W_{A,\tau,\bar{a}}^+(\delta_{\tau,a}^{-1}) := W_{A,\tau,\bar{a}}^+ \otimes_{A \otimes_{\tau,K} B_{\text{dR}}^+} W_{\text{dR}}^+(\delta_{\tau,a}^{-1})_{A,\tau}$  is an almost de Rham  $A \otimes_{\tau,K} B_{\text{dR}}^+$ -representation of  $\mathcal{G}_K$ . For  $W_{A,\tau,\bar{a}} = W_{A,\tau,\bar{a}}^+[\frac{1}{t}]$ , we define

$$D_{\text{pdR},\tau,\bar{a}}(W_{A,\tau,\bar{a}}) := D_{\text{pdR}}(W_{A,\tau,\bar{a}}^+(\delta_{\tau,a}^{-1})[\frac{1}{t}])$$

which is a finite free  $A$ -module of rank  $d_{\tau,\bar{a}}$  with an  $A$ -linear operator  $\nu_{A,\tau,\bar{a}}$ . Moreover, the  $A$ -module  $D_{\text{pdR},\tau,\bar{a}}(W_{A,\tau,\bar{a}})$  is equipped with a (Hodge-Tate) filtration

$$\text{Fil}_{W_{A,\tau,\bar{a}}^+}^i D_{\text{pdR},\tau,\bar{a}}(W_{A,\tau,\bar{a}}) = (W_{A,\tau,\bar{a}}^+(\delta_{\tau,a}^{-1}) \otimes_{B_{\text{dR}}^+} t^i B_{\text{dR}}^+[\log(t)])^{\mathcal{G}_K}, i \in \mathbb{N}$$

by finite free sub- $A$ -modules ([BHS19, Lem. 3.2.2]).

If we replace the above  $a$  by  $a' = a + k, k \in \mathbb{Z}$  and  $\delta_{\tau,a}$  by  $\delta'_{\tau,a'}$  where  $\text{wt}_\tau(\delta'_{\tau,a'}) = a' \in L$ , then  $D_{\text{pdR},\tau,\bar{a}}(W_{A,\tau,\bar{a}})$  is unchanged (since  $W_{\text{dR}}(\delta)_\tau \simeq L \otimes_{\tau,K} B_{\text{dR}}$  if  $\text{wt}_\tau(\delta) = 0$ ) and the partial flag given by the Hodge filtration is also not changed with only degrees of the filtration shifted (see Remark 5.2.2).

We fix a framing  $\alpha_{\tau,\bar{a}} : L^n \simeq D_{\text{pdR},\tau,\bar{a}}(W_{\tau,\bar{a}})$ . Then we can define framed version groupoids  $X_{W_{\tau,\bar{a}}}^\square, X_{W_{\tau,\bar{a}},\mathcal{F}_{\tau,\bar{a},\bullet}}^\square$ , etc. over  $\mathcal{C}_L$  as in [BHS19, §3.1] or §3.3.2. For each  $\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau$ , we let  $G_{\tau,\bar{a}} := \text{GL}_{d_{\tau,\bar{a}}}$  and  $B_{\tau,\bar{a}} \subset G_{\tau,\bar{a}}$  be the group of upper-triangular matrices and  $T_{\tau,\bar{a}} \subset B_{\tau,\bar{a}}$  be the diagonal matrices. Assume that the Sen weights of  $W_{\tau,\bar{a}}^+$  are  $\mathbf{h}_{\tau,\bar{a}} = a + (h_{\tau,\bar{a},1}, \dots, h_{\tau,\bar{a},d_{\tau,\bar{a}}})$  where  $h_{\tau,\bar{a},1} \leq \dots \leq h_{\tau,\bar{a},d_{\tau,\bar{a}}}$  are integers. According to the regularity of  $\mathbf{h}_{\tau,\bar{a}}$ , there is a standard parabolic subgroup  $P_{\tau,\bar{a}}$  containing  $B_{\tau,\bar{a}}$  as in §3.3.3 whose Weyl group  $W_{P_{\tau,\bar{a}}} \subset W_{G_{\tau,\bar{a}}} = S_{d_{\tau,\bar{a}}}$  is the subgroup of the stabilizers of  $\mathbf{h}_{\tau,\bar{a}}$ . We have defined an algebraic variety  $X_{P_{\tau,\bar{a}}}$  in §3.2.1 associated with  $P_{\tau,\bar{a}} \subset G_{\tau,\bar{a}}$ . Then the framing  $\alpha_{\tau,\bar{a}}$ , the  $L$ -space  $D_{\text{pdR},\tau,\bar{a}}(W_{\tau,\bar{a}})$  with the nilpotent operator  $\nu_{\tau,\bar{a}}$ , the filtration  $D_{\text{pdR},\tau,\bar{a}}(\mathcal{F}_{\tau,\bar{a},\bullet})$  and the filtration  $\text{Fil}_{W_{\tau,\bar{a}}^+}^\bullet D_{\text{pdR},\tau,\bar{a}}(W_{\tau,\bar{a}})$  defines an  $L$ -point  $x_{\tau,\bar{a}}$  of  $X_{P_{\tau,\bar{a}}}$  as in §3.3.3 (note that  $\mathbf{h}_{\tau,\bar{a}} - a \in \mathbb{Z}^{d_{\tau,\bar{a}}}$  is now the Sen weights of  $W_{\tau,\bar{a}}^+(\delta_{\tau,a}^{-1})$ ). Twisting every thing by  $W_{\text{dR}}^+(\delta_{\tau,a}^{-1})_{A,\tau}$  and applying Proposition 3.3.1 and Proposition 3.3.2, we get the following proposition.

**Proposition 5.3.3.** *The groupoid  $X_{W_{\tau, \bar{a}}, \mathcal{F}_{\tau, \bar{a}}, \bullet}^\square$  is pro-representable and the functor  $|X_{W_{\tau, \bar{a}}, \mathcal{F}_{\tau, \bar{a}}, \bullet}^\square|$  is pro-represented by  $\widehat{X}_{P_{\tau, \bar{a}}, x_{\tau, \bar{a}}} = \mathrm{Spf}(\widehat{\mathcal{O}}_{X_{P_{\tau, \bar{a}}, x_{\tau, \bar{a}}}})$ .*

Let  $G = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} G_{\tau, \bar{a}}$ ,  $B = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} B_{\tau, \bar{a}}$ ,  $P = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} P_{\tau, \bar{a}}$ ,  $W_P = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} W_{P_{\tau, \bar{a}}}$  and  $T = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} T_{\tau, \bar{a}}$ . Then  $X_P = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} X_{P_{\tau, \bar{a}}}$ . Let  $\alpha = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} \alpha_{\tau, \bar{a}}$ . We can define the framed version  $X_{W^+, \mathcal{F}_\bullet}^\square$  in an obvious way and using Lemma 5.3.2 and the above proposition, we get the following representability result.

**Proposition 5.3.4.** *The groupoid  $X_{W^+, \mathcal{F}_\bullet}^\square$  is pro-representable and the functor  $|X_{W^+, \mathcal{F}_\bullet}^\square|$  is pro-represented by  $\widehat{X}_{P, x} = \mathrm{Spf}(\widehat{\mathcal{O}}_{X_{P, x}})$ .*

Let  $\mathfrak{t}, \mathfrak{g}, \mathfrak{b}, \mathfrak{p}$  be the lie algebras of  $T, G, B, P$ . By §3.2.4, there is a weight map  $\kappa_1 : X_P \rightarrow \mathfrak{t}$  sending  $(\nu, g_1 B, g_2 P) \in X_P$  to the image of  $\mathrm{Ad}(g^{-1})\nu$  under the map  $\mathfrak{b} \rightarrow \mathfrak{t}$ . Since  $\nu = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} \nu_{\tau, \bar{a}}$  is nilpotent, the image of  $x$  in  $\mathfrak{t}$  is zero. Thus we get a map  $X_{W^+, \mathcal{F}_\bullet}^\square \rightarrow \widehat{\mathfrak{t}}$  where  $\widehat{\mathfrak{t}}$  is the completion of  $\mathfrak{t}$  at zero which descends to  $X_{W^+, \mathcal{F}_\bullet} \rightarrow \widehat{\mathfrak{t}}$  and in fact factors through  $\kappa_{W, \mathcal{F}_\bullet} : X_{W, \mathcal{F}_\bullet} \rightarrow \widehat{\mathfrak{t}}$  as in §3.3.4.

## 5.4 Trianguline $(\varphi, \Gamma_K)$ -modules

Let  $D$  be a  $(\varphi, \Gamma_K)$ -module of rank  $n$  over  $\mathcal{R}_{L, K}$  and let  $\mathcal{M} = D[\frac{1}{t}]$  which is a  $(\varphi, \Gamma_K)$ -module of rank  $n$  over  $\mathcal{R}_{L, K}[\frac{1}{t}]$ . Let  $\mathcal{M}_\bullet : \{0\} \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_i \subsetneq \cdots \subsetneq \mathcal{M}_n = \mathcal{M}$  be a trianguline filtration of  $\mathcal{M}$  of a parameter  $\underline{\delta} = (\delta_1, \dots, \delta_n)$ . Then  $\mathcal{M}_i / \mathcal{M}_{i-1} = \mathcal{R}_{L, K}(\delta)[\frac{1}{t}]$ . Let  $W^+ = W_{\mathrm{dR}}^+(D)$ ,  $W = W_{\mathrm{dR}}(\mathcal{M})$  and  $\mathcal{F}_\bullet = W_{\mathrm{dR}}(\mathcal{M}_\bullet)$ .

Recall the groupoid  $X_{\mathcal{M}, \mathcal{M}_\bullet}$  consists of objects  $(A, \mathcal{M}_A, j_A, \mathcal{M}_{A, \bullet})$  where  $A \in \mathcal{C}_L$ ,  $\mathcal{M}_A$  is a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{A, K}[\frac{1}{t}]$ ,  $\mathcal{M}_{A, \bullet}$  is a trianguline filtration of  $\mathcal{M}_A$  where each  $\mathcal{M}_{A, i} / \mathcal{M}_{A, i-1}$  is of character type ([BHS19, Def. 3.3.2]) and  $j_A : \mathcal{M}_{A, \bullet} \otimes_A L \xrightarrow{\sim} \mathcal{M}_\bullet$ .

The functor  $W_{\mathrm{dR}}(-)$  induces a morphism  $X_{\mathcal{M}, \mathcal{M}_\bullet} \rightarrow X_{W, \mathcal{F}_\bullet}$ . Given  $(A, \mathcal{M}_A, j_A, \mathcal{M}_{A, \bullet}) \in X_{\mathcal{M}, \mathcal{M}_\bullet}$ , there exists a unique continuous character  $\underline{\delta}_A = (\delta_{A, 1}, \dots, \delta_{A, n}) : (K^\times)^n \rightarrow A^\times$  such that  $\mathcal{M}_{A, i} / \mathcal{M}_{A, i-1} \simeq \mathcal{R}_{A, K}(\delta_{A, i})[\frac{1}{t}]$  and  $\delta_{A, i} \otimes_A L = \delta_i$  for  $i = 1, \dots, n$  by [BHS19, , Lem. 3.3.4]. The map  $\omega_{\underline{\delta}} : (A, \mathcal{M}_A, j_A, \mathcal{M}_{A, \bullet}) \mapsto (A, \underline{\delta}_A)$  defines a morphism  $X_{\mathcal{M}, \mathcal{M}_\bullet} \rightarrow \widehat{\mathcal{T}}_{\underline{\delta}}^n$  where  $\mathcal{T}^n$  is the space of continuous characters of  $(K^\times)^n$ .

We identify  $\mathfrak{t}$  with the weight space  $\prod_{\tau \in \Sigma} \mathbb{A}^{n, \mathrm{an}}$  of characters of  $(K^\times)^n$  in the following way. Recall  $\mathfrak{t} = \prod_{\tau \in \Sigma, \bar{a} \in \bar{\mathcal{A}}_\tau} \mathfrak{t}_{\tau, \bar{a}}$  where  $\mathfrak{t}_{\tau, \bar{a}} = \mathrm{Lie}(T_{\tau, \bar{a}})$ . For each  $\tau, \bar{a}$ , let  $j_{\tau, \bar{a}, 1} < \cdots < j_{\tau, \bar{a}, d_{\tau, \bar{a}}}$  be the numbers  $i$  in  $\{1, \dots, n\}$  such that  $\mathrm{wt}_\tau(\delta_i) \in a + \mathbb{Z}$ . Then we identify  $(\lambda_{\tau, \bar{a}, i})_{\tau, \bar{a}, 1 \leq i \leq d_{\tau, \bar{a}}} \in \mathfrak{t}$  with  $\prod_{\tau \in \Sigma} \sum_{\bar{a} \in \bar{\mathcal{A}}_\tau} \sum_{i=1}^{d_{\tau, \bar{a}}} \lambda_{\tau, \bar{a}, i} e_{\tau, j_{\tau, \bar{a}, i}} \in \prod_{\tau \in \Sigma} \mathbb{A}^{n, \mathrm{an}}$  where  $e_{\tau, i}, i = 1, \dots, n$  are the usual basis of  $\mathbb{A}^{n, \mathrm{an}}$ . Then  $\mathrm{wt}(\underline{\delta}) \in \mathfrak{t}$ .

**Lemma 5.4.1.** *The diagram*

$$\begin{array}{ccc} X_{\mathcal{M}, \mathcal{M}_\bullet} & \xrightarrow{\omega_{\underline{\delta}}} & \widehat{\mathcal{T}}_{\underline{\delta}}^n \\ \downarrow & & \downarrow \mathrm{wt}(-) - \mathrm{wt}(\underline{\delta}) \\ X_{W, \mathcal{F}_\bullet} & \xrightarrow{\kappa_{W, \mathcal{F}_\bullet}} & \widehat{\mathfrak{t}} \end{array}$$

*commutes.*

*Proof.* Take  $(A, \mathcal{M}_A, j_A, \mathcal{M}_{A, \bullet}) \in X_{\mathcal{M}, \mathcal{M}_\bullet}$ . Suppose that its image under  $\omega_{\underline{\delta}}$  is  $\underline{\delta}_A$  and its image in  $X_{W, \mathcal{F}_\bullet}$  is  $x = (W_A, \mathcal{F}_{A, \bullet}, \iota_A)$ . Then  $\mathcal{F}_{A, i} / \mathcal{F}_{A, i-1} = W_{\mathrm{dR}}(\delta_{A, i}) = \prod_{\tau \in \Sigma} W_{\mathrm{dR}}(\delta_{A, i})_\tau$ . For any  $1 \leq i \leq d_{\tau, \bar{a}}$ ,  $\mathcal{F}_{A, \tau, \bar{a}, i} / \mathcal{F}_{A, \tau, \bar{a}, i-1} \simeq W_{\mathrm{dR}}(\delta_{A, j_{\tau, \bar{a}, i}})_\tau$  by the construction of  $\mathcal{F}_{A, \tau, \bar{a}, \bullet}$ . Apply

(the proof of) [BHS19, Lem. 3.3.6] to  $W_{\text{dR}}(\delta_{A,i})_\tau \otimes_{A \otimes_{\tau,K} B_{\text{dR}}} W_{\text{dR}}(\delta_{\tau,a}^{-1})_{A,\tau}$  (which is almost de Rham), the  $t_{\tau,\bar{a}}(A)$  part of  $\kappa_{W,\mathcal{F}_\bullet}(x)$  is equal to

$$(\text{wt}_\tau(\delta_{A,j_{\tau,\bar{a}},i} \otimes_L \delta_{\tau,a}^{-1}) - \text{wt}_\tau(\delta_{j_{\tau,\bar{a}},i} \otimes_L \delta_{\tau,a}^{-1}))_{i=1,\dots,d_{\tau,\bar{a}}} = (\text{wt}_\tau(\delta_{A,j_{\tau,\bar{a}},i}) - \text{wt}_\tau(\delta_{j_{\tau,\bar{a}},i}))_{i=1,\dots,d_{\tau,\bar{a}}}.$$

Hence  $\kappa_{W,\mathcal{F}_\bullet}(x) = \text{wt}(\underline{\delta}_A) - \text{wt}(\underline{\delta})$ .  $\square$

We are going to show that the morphism  $X_{\mathcal{M},\mathcal{M}_\bullet} \rightarrow \widehat{\mathcal{T}}_\delta^n \times_{\widehat{\mathcal{T}}} X_{\mathcal{M},\mathcal{M}_\bullet}$  of groupoids over  $\mathcal{C}_L$  is formally smooth under the generic condition as in the integral cases ([BHS19, Thm. 3.4.4]).

Recall  $\mathcal{T}_0$  is the space of continuous characters of  $K^\times$  that are not of the form  $z^{\mathbf{k}}$  or  $\epsilon z^{\mathbf{k}}$  for some  $\mathbf{k} \in \mathbb{Z}^\Sigma$ . From now on we assume that  $\underline{\delta} \in \mathcal{T}_0^n$  is generic, i.e.  $\delta_i \delta_j^{-1} \notin \mathcal{T}_0$  for all  $i \neq j$ .

**Lemma 5.4.2.** *If  $\delta : K^\times \rightarrow L$  is a continuous character such that  $\delta \in \mathcal{T}_0$ , then the map*

$$H_{\varphi,\gamma_K}^1(\mathcal{R}_{L,K}(\delta)[\tfrac{1}{t}]) \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\delta))$$

*induced by  $W_{\text{dR}}(-)$  is surjective.*

*Proof.* If  $\text{wt}_\tau(\delta) \in \mathbb{Z}$  for all  $\tau \in \Sigma$ , the result is proved in [BHS19, Lem. 3.4.2]. In the extreme case that  $\text{wt}_\tau(\delta) \notin \mathbb{Z}$  for all  $\tau \in \Sigma$ , the right-hand side is zero by Lemma 5.2.6.

We still follow the proof of [BHS19, Lem. 3.4.2]. The two side of the above map is not changed if we replace  $\delta$  by  $\delta z^{-\mathbf{k}}$  where  $\mathbf{k} = (k_\tau)_{\tau \in \Sigma} \in \mathbb{N}^\Sigma$  such that  $\text{wt}_\tau(\delta) \leq 0$  if  $\text{wt}_\tau(\delta) \in \mathbb{Z}$ . By [BHS19, Lem. 3.3.3] or Corollary 4.2.4, we have  $H_{\varphi,\gamma_K}^1(\mathcal{R}_{L,K}(\delta)[\tfrac{1}{t}]) = H_{\varphi,\gamma_K}^1(\mathcal{R}_{L,K}(\delta))$ .

Let  $W(\delta) = (W_e(\delta), W_{\text{dR}}^+(\delta))$  be the  $L$ - $B$  pair associated to  $\delta$ . Let  $k = |\{\tau \in \Sigma \mid \text{wt}_\tau(\delta) \in \mathbb{Z}\}|$ . Then  $\dim_L H^1(\mathcal{G}_K, W_{\text{dR}}(\delta)) = k$  by Lemma 5.2.6. As  $\dim_L H^1(\mathcal{G}_K, W(\delta)) = |\Sigma|$ , we only need to prove that the dimension of  $H_g^1(\mathcal{G}_K, W(\delta)) := \ker(H^1(\mathcal{G}_K, W(\delta)) \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\delta)))$  over  $L$  is  $|\Sigma| - k$ . By [Nak09, Prop. 2.11], we reduce to prove that the dimension of the kernel

$$\ker(H^1(\mathcal{G}_K, W(\epsilon\delta^{-1})) \rightarrow H^1(\mathcal{G}_K, W_e(\epsilon\delta^{-1})))$$

has dimension  $k$ . As in the proof of [BHS19, Lem. 3.4.2], the map

$$H^1(\mathcal{G}_K, W(\epsilon\delta^{-1})) \rightarrow H^1(\mathcal{G}_K, W_e(\epsilon\delta^{-1}))$$

factors through

$$H^1(\mathcal{G}_K, W(\epsilon\delta^{-1})) \rightarrow H^1(\mathcal{G}_K, W(z^{-\mathbf{k}'}\epsilon\delta^{-1})) \rightarrow H^1(\mathcal{G}_K, W_e(\epsilon\delta^{-1}))$$

where  $\mathbf{k}' = (k'_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$  such that  $\text{wt}_\tau(z^{-\mathbf{k}'}\epsilon\delta^{-1}) \leq 0$  if  $\text{wt}_\tau(z^{-\mathbf{k}'}\epsilon\delta^{-1}) \in \mathbb{Z}$ . Since we have assumed that all integral weights of  $\delta$  are negative,  $\text{wt}_\tau(\epsilon\delta^{-1}) \geq 1$  if  $\text{wt}_\tau(\epsilon\delta^{-1}) \in \mathbb{Z}$ . By Corollary 4.2.4 again, the kernel of the map  $H^1(\mathcal{G}_K, W(\epsilon\delta^{-1})) \rightarrow H^1(\mathcal{G}_K, W(z^{-\mathbf{k}'}\epsilon\delta^{-1}))$  has dimension  $k$ . Thus we only need to prove that the map

$$H^1(\mathcal{G}_K, W(z^{-\mathbf{k}'}\epsilon\delta^{-1})) \rightarrow H^1(\mathcal{G}_K, W_e(z^{-\mathbf{k}'}\epsilon\delta^{-1}))$$

is an isomorphism.

Let  $\delta' = z^{-\mathbf{k}'}\epsilon\delta^{-1}$  whose integral weights are all negative. By [Nak09, §2.1], we have a long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{G}_K, W(\delta')) &\rightarrow H^0(\mathcal{G}_K, W_e(\delta')) \oplus H^0(\mathcal{G}_K, W_{\text{dR}}^+(\delta')) \rightarrow H^0(\mathcal{G}_K, W_{\text{dR}}(\delta')) \\ &\rightarrow H^1(\mathcal{G}_K, W(\delta')) \rightarrow H^1(\mathcal{G}_K, W_e(\delta')) \oplus H^1(\mathcal{G}_K, W_{\text{dR}}^+(\delta')) \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\delta')) \\ &\rightarrow H^2(\mathcal{G}_K, W(\delta')) \end{aligned}$$

By the generic assumption, [Nak09, Prop. 2.14] and the Tate duality, we have  $H^i(\mathcal{G}_K, W(\delta')) = 0$  if  $i = 0, 2$ . And the map  $H^i(\mathcal{G}_K, W_{\text{dR}}^+(\delta')) \rightarrow H^i(\mathcal{G}_K, W_{\text{dR}}(\delta'))$  are isomorphisms by Lemma 5.1.2 and by the assumption of negative integral weights (since

$$H^i(\mathcal{G}_K, t^{-k-1}W_{\text{dR}}^+(\delta')/t^{-k}W_{\text{dR}}^+(\delta')) = 0$$

for all  $i = 1, 2$  and  $k \geq 0$ ). Hence  $H^1(\mathcal{G}_K, W(\delta')) \xrightarrow{\sim} H^1(\mathcal{G}_K, W_e(\delta'))$ .  $\square$

**Lemma 5.4.3.** *Assume that  $\delta_A : K^\times \rightarrow A^\times$  is a continuous character where  $A \in \mathcal{C}_L$  and  $\delta_A \otimes_A L \in \mathcal{T}_0$ . Then  $H_{\varphi, \gamma_K}^0(\mathcal{G}_K, \mathcal{R}_{A,K}(\delta_A)[\frac{1}{t}]) = H_{\varphi, \gamma_K}^2(\mathcal{G}_K, \mathcal{R}_{A,K}(\delta_A)[\frac{1}{t}]) = 0$  and the map  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{A,K}(\delta_A)[\frac{1}{t}]) \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\delta_A))$  is surjective.*

*Proof.* The assertions that  $H_{\varphi, \gamma_K}^0(\mathcal{G}_K, \mathcal{R}_{A,K}(\delta_A)[\frac{1}{t}]) = H_{\varphi, \gamma_K}^2(\mathcal{G}_K, \mathcal{R}_{A,K}(\delta_A)[\frac{1}{t}]) = 0$  is exactly [BHS19, Lem. 3.4.3(i)]. Since  $\mathcal{R}_{A,K}(\delta_A)[\frac{1}{t}]$  is a successive extension of  $\mathcal{R}_{A,K}(\delta_L)[\frac{1}{t}]$  where  $\delta_L = \delta_A \otimes_A L$ , we can prove the surjectivity of  $H^1$  by induction on the rank of a  $(\varphi, \Gamma)$ -module  $\mathcal{M}$  over  $\mathcal{R}_{L,K}[\frac{1}{t}]$  which is a successive extension of  $\mathcal{R}_{A,K}(\delta_L)[\frac{1}{t}]$ . The proof of [BHS19, Lem. 3.4.3(i)] imply that  $H_{\varphi, \gamma_K}^i(\mathcal{M}) = 0$  for  $i = 0, 2$  for such  $\mathcal{M}$ . Write a short exact sequence

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{R}_{A,K}(\delta_L)[\frac{1}{t}] \rightarrow 0$$

and assume that the map  $H_{\varphi, \gamma_K}^1(\mathcal{M}') \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}'))$  is a surjection. Then we have a map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{\varphi, \gamma_K}^1(\mathcal{M}') & \longrightarrow & H_{\varphi, \gamma_K}^1(\mathcal{M}) & \longrightarrow & H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta_L)[\frac{1}{t}]) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(\mathcal{G}_K, W_{\text{dR}}(\delta_L)) & \rightarrow & H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}')) & \rightarrow & H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M})) & \rightarrow & H^1(\mathcal{G}_K, W_{\text{dR}}(\delta_L)) & \rightarrow & H^2(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}')) \end{array}$$

between long exact sequences. By the previous lemma and the induction hypothesis, the map  $H_{\varphi, \gamma_K}^1(\mathcal{M}') \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}'))$  and  $H_{\varphi, \gamma_K}^1(\mathcal{R}_{L,K}(\delta_L)[\frac{1}{t}]) \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\delta_L))$  are surjections. The map  $0 \rightarrow H^2(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}'))$  is an injection. Then by a diagram chasing (the five lemma), the map  $H_{\varphi, \gamma_K}^1(\mathcal{M}) \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}))$  is also a surjection.  $\square$

**Proposition 5.4.4.** *The morphism  $X_{\mathcal{M}, \mathcal{M}_\bullet} \rightarrow \widehat{\mathcal{T}}_{\delta}^n \times_{\mathfrak{t}} X_{\mathcal{M}, \mathcal{M}_\bullet}$  of groupoids over  $\mathcal{C}_L$  is formally smooth if  $\delta$  is generic.*

*Proof.* The proof is mostly the same with that of [BHS19, Thm. 3.4.4] using previous lemmas. Let  $A \twoheadrightarrow B$  be a surjection in  $\mathcal{C}_L$ ,  $\delta_A \in \widehat{\mathcal{T}}_{\delta}^n(A)$  and  $(W_A, \mathcal{F}_{A, \bullet}, \iota_A) \in X_{\mathcal{M}, \mathcal{M}_\bullet}(A)$ ,  $(W_B, \mathcal{F}_{B, \bullet}, \iota_B) = (W_A, \mathcal{F}_{A, \bullet}, \iota_A) \otimes_A B$  is the image under  $W_{\text{dR}}(-)$  of  $(\mathcal{M}_B, \mathcal{M}_{B, \bullet}, j_B) \in X_{\mathcal{M}, \mathcal{M}_\bullet}$  with the parameter  $\delta_B = \delta_A \otimes_A B$  such that  $\delta_B \otimes_B L = \delta$ . Fix  $1 \leq i \leq n$ , suppose that we have  $\{0\} \subsetneq \mathcal{M}_{A,1} \subsetneq \cdots \subsetneq \mathcal{M}_{A,i-1}$  with  $\mathcal{M}_{A,j}/\mathcal{M}_{A,j-1} = \mathcal{R}_{A,K}(\delta_{A,j})[\frac{1}{t}]$ . We need to prove that the map, as [BHS19, (3.23)],

$$\begin{aligned} & H_{\varphi, \gamma_K}^1(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1})) \rightarrow \\ & H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1}))) \times_{H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}_{B,i-1}(\delta_{B,i}^{-1})))} H_{\varphi, \gamma_K}^1(\mathcal{M}_{B,i-1}(\delta_{B,i}^{-1})). \end{aligned}$$

is surjective. Using [BHS19, Lem. 3.4.5], we need to prove that the map

$$H_{\varphi, \gamma_K}^1(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1})) \rightarrow H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1}))) \quad (5.4.5)$$

is surjective,

$$H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}_{B,i-1}(\delta_{B,i}^{-1}))) = H^1(\mathcal{G}_K, W_{\text{dR}}(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1}))) \otimes_A B \quad (5.4.6)$$



and

$$H^1_{\varphi, \gamma_K}(\mathcal{M}_{B,i-1}(\delta_{B,i}^{-1})) = H^1_{\varphi, \gamma_K}(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1})) \otimes_A B. \quad (5.4.7)$$

Now  $\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1})$  is a successive extension of  $\mathcal{R}_{L,K}(\delta_{A,j}\delta_{A,i}^{-1})[\frac{1}{t}]$ ,  $j < i$ , hence is a successive extension of rank one  $(\varphi, \Gamma_K)$ -modules of the form  $\mathcal{R}_{L,K}(\delta')[\frac{1}{t}]$  where  $\delta' \in \mathcal{T}_0$ . By the proof of Lemma 5.4.3, we get that (5.4.5) is surjective.

To show (5.4.7), notice that also by the proof of Lemma 5.4.3, both  $H^0_{\varphi, \gamma_K}$  and  $H^2_{\varphi, \gamma_K}$  vanish for  $\mathcal{M}_{B,i-1}(\delta_{B,i}^{-1})$  and  $\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1})$ , hence the cohomologies commute with base change, see the proof of [BHS19, Thm. 3.4.4].

Finally, we need to prove (5.4.7). Since  $W_{\text{dR}}(-)$  is exact, we have  $W_{\text{dR}}(\mathcal{M}_{B,i-1}(\delta_{B,i}^{-1})) = W_{\text{dR}}(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1})) \otimes_A B$ . Write  $W'_A = W_{\text{dR}}(\mathcal{M}_{A,i-1}(\delta_{A,i}^{-1}))$ ,  $W'_B = W_A \otimes_A B$  and  $W'_L = W_A \otimes_A L$ . By the discussions in §5.2, we have a decomposition  $W'_L = \bigoplus_{\tau \in \Sigma, \bar{a} \in \overline{\mathcal{A}}'_\tau} W'_{L, \tau, \bar{a}}$  where  $\overline{\mathcal{A}}'_\tau$  are mod  $\mathbb{Z}$  classes of the  $\tau$ -Sen weights of certain  $L \otimes_{\tau, K} B_{\text{dR}}^+$ -lattices of  $W'_{L, \tau} = W'_L \otimes_{L \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} L$ . By §5.3, we have corresponding decompositions  $W'_* = \bigoplus_{\tau \in \Sigma, \bar{a} \in \overline{\mathcal{A}}'_\tau} W'_{*, \tau, \bar{a}}$  where  $*$  =  $A, B$  and  $W'_{*, \tau, \bar{a}} \otimes_* L = W'_{L, \tau, \bar{a}}$ . Then it is easy to see the maps  $W'_{A, \tau, \bar{a}} \otimes_A B \rightarrow W'_{B, \tau, \bar{a}}$  are isomorphisms. If  $\bar{a} \neq \mathbb{Z}$ , then  $H^1(\mathcal{G}_K, W'_{*, \tau, \bar{a}}) = 0$  by Lemma 5.2.6. Hence it is enough to prove  $H^1(\mathcal{G}_K, W'_{A, \tau, \bar{0}} \otimes_A B) = H^1(\mathcal{G}_K, W'_{A, \tau, \bar{0}}) \otimes_A B$ . Now  $W'_{A, \tau, \bar{0}}$  are almost de Rham and the result can be proved as in the proof of [BHS19, Thm. 3.4.4].  $\square$

Recall now  $\mathcal{M} = D[\frac{1}{t}]$ ,  $W^+ = W_{\text{dR}}^+(D)$ . By [BHS19, Prop. 3.5.1], the morphism  $X_D \rightarrow X_{\mathcal{M}} \times_{X_W} X_{W^+}$  is an equivalence. The proof of [BHS19, Lem. 3.5.3] doesn't need that the parameter of  $\mathcal{M}_\bullet$  is locally algebraic. Let  $X_{D, \mathcal{M}_\bullet} = X_D \times_{X_{\mathcal{M}}} X_{\mathcal{M}, \mathcal{M}_\bullet}$  and  $X_{D, \mathcal{M}_\bullet}^\square = X_{D, \mathcal{M}_\bullet} \times_{X_{W^+, \mathcal{F}_\bullet}} X_{W^+, \mathcal{F}_\bullet}^\square$ .

**Corollary 5.4.8.** *The morphism  $X_{D, \mathcal{M}_\bullet}^\square \rightarrow X_{W^+, \mathcal{F}_\bullet}^\square$  is formally smooth and relatively representable. The groupoid  $X_{D, \mathcal{M}_\bullet}^\square$  is pro-representable and the functor  $|X_{D, \mathcal{M}_\bullet}^\square|$  is pro-represented by a formal scheme which is formally smooth of relative dimension  $[K : \mathbb{Q}_p] \frac{n(n+1)}{2}$  over  $\widehat{X}_{P, x} = \text{Spf}(\widehat{\mathcal{O}}_{X_P, x})$  where  $x \in X_P$  are datum associated to  $(W^+, \mathcal{F}_\bullet, \alpha)$  in the end of §5.3.*

*Proof.* With Proposition 5.3.4 and Proposition 5.4.4, the corollary can be proved in the same way as for [BHS19, Cor. 3.5.8], except for the relative dimensions. For the dimensions, notice that the proof of [BHS19, Prop. 3.5.7] still holds whenever  $\underline{\delta} \in \mathcal{T}_0^n$  and by the pro-representability of  $|X_{W, \mathcal{F}_\bullet}^\square|$  in our general situation,  $X_{\mathcal{M}, \mathcal{M}_\bullet}^\square$  is pro-representable and  $|X_{\mathcal{M}, \mathcal{M}_\bullet}^\square|$  is pro-represented by a formally smooth Noetherian complete local ring. The dimension of  $|X_{\mathcal{M}, \mathcal{M}_\bullet}^{\text{ver}}|$  in *loc. cit.* is  $n + [K : \mathbb{Q}_p] \frac{n(n+1)}{2}$  and the relative dimension of  $X_{\mathcal{M}, \mathcal{M}_\bullet}^\square \rightarrow X_{\mathcal{M}, \mathcal{M}_\bullet}$  is  $\dim G$ . Hence the dimension of  $X_{\mathcal{M}, \mathcal{M}_\bullet}^\square$  is  $\dim G + \dim |X_{\mathcal{M}, \mathcal{M}_\bullet}^{\text{ver}}| - n = \dim G + [K : \mathbb{Q}_p] \frac{n(n+1)}{2}$ , from which we can get the relative dimension.  $\square$

Let  $W$  be the Weyl group of  $G$  and recall  $W_P \subset W$  is the parabolic subgroup. By §3.2, the irreducible components of  $X_P$  are  $X_{P, w}$ ,  $w \in W/W_P$  which are unibranch. Let  $\widehat{X}_{P, w, x} = \text{Spf}(\widehat{\mathcal{O}}_{X_P, w, x})$  be the completion of  $X_{P, w}$  at  $x$ . As in §3.3.3, we define subgroupoids  $X_{D, \mathcal{M}}^\square$  of  $X_{D, \mathcal{M}}$  for  $w \in W/W_P$  by

$$X_{D, \mathcal{M}_\bullet}^{\square, w} = X_{D, \mathcal{M}_\bullet}^\square \times_{|X_{W^+, \mathcal{F}_\bullet}^\square|} \widehat{X}_{P, w, x}$$

where if  $x \notin X_{P, w}$ ,  $X_{D, \mathcal{M}_\bullet}^{\square, w}$  is empty. Then we have subgroupoids  $X_{D, \mathcal{M}_\bullet}^w$  of  $X_{D, \mathcal{M}_\bullet}$  such that the morphism  $X_{D, \mathcal{M}_\bullet}^{\square, w} \rightarrow X_{D, \mathcal{M}_\bullet}^w$  is formally smooth.

Recall in §3.2.4, we have a map  $(\kappa_1, \kappa_2) : X_P \rightarrow T_P := \mathfrak{t} \times_{\mathfrak{t}/W_P} \mathfrak{t}/W_P$  and irreducible components  $T_{P, w}$  of  $T_P$  such that the induced map  $\widehat{X}_{P, w', x} \rightarrow \widehat{T}_{P, (0,0)}$  factors through  $\widehat{T}_{P, w, (0,0)}$  if and only if  $w' = w$  in  $W/W_P$ . The composition  $X_{D, \mathcal{M}_\bullet}^\square \rightarrow \widehat{X}_{P, x} \rightarrow \widehat{T}_{P, (0,0)}$  factors through  $X_{D, \mathcal{M}_\bullet} \rightarrow \widehat{T}_{P, (0,0)}$ .

## 5.5 The trianguline variety

Suppose that  $x = (r, \underline{\delta}) \in X_{\text{tri}}(\bar{r})(L) \subset (\mathfrak{X}_{\bar{r}} \times \mathcal{T}^n)(L)$  where  $\underline{\delta} \in \mathcal{T}_0^n$  is generic and  $r : \mathcal{G}_K \rightarrow \text{GL}_n(L)$  is continuous. Let  $V$  be the underlying representation of  $r$  forgetting the framing. Let  $D = D_{\text{rig}}(V)$ . In [BHS19, §3.6], we have the formally smooth map  $X_r \rightarrow X_V$  and an equivalence  $X_V \xrightarrow{\sim} X_D$  induced by  $D_{\text{rig}}(-)$ .

Let  $\mathcal{M}, W^+, W$  be the datum associated to  $D$  as in §5.4. By [BHS19, Prop. 3.7.2],  $\mathcal{M}$  admits a unique triangulation  $\mathcal{M}_{\bullet}$  of a parameter  $\underline{\delta}$ . By [BHS19, Prop. 3.4.6], the map  $X_{r, \mathcal{M}_{\bullet}} = X_r \times_{X_D} X_{D, \mathcal{M}_{\bullet}} \rightarrow X_r \simeq \widehat{\mathfrak{X}_{\bar{r}, r}}$  is a closed immersion. By [BHS19, Prop. 3.7.2] (see also Proposition 3.3.4 or [BD21, A.4]), the map  $\widehat{X_{\text{tri}}(\bar{r})}_x \rightarrow \widehat{\mathfrak{X}_{\bar{r}, r}}$  factors as

$$\widehat{X_{\text{tri}}(\bar{r})}_x \hookrightarrow X_{r, \mathcal{M}_{\bullet}} \hookrightarrow X_r \simeq \widehat{\mathfrak{X}_{\bar{r}, r}}$$

and is a closed immersion.

Let  $\mathbf{h} = \prod_{\tau \in \Sigma} \prod_{\bar{a} \in \bar{\mathcal{A}}_{\tau}} (a + h_{\tau, \bar{a}, 1}, \dots, a + h_{\tau, \bar{a}, d_{\tau, \bar{a}}})$  be the Sen weights of  $W^+$  as in §5.3 where  $h_{\tau, \bar{a}, 1} \leq \dots \leq h_{\tau, \bar{a}, d_{\tau, \bar{a}}}$  are integers (the Sen weights are in  $L$  because they coincide with weights of  $\underline{\delta}$ , [BHS17b, Prop. 2.9]). We have defined the groups  $G, P, T$  and the Weyl groups  $W, W_P$  that are associated to  $\mathbf{h}$ . Let  $x_{\text{pdR}} \in X_P$  be the point associated to  $(W^+, \mathcal{F}_{\bullet})$  and some framings  $\alpha_{\tau, \bar{a}}$  of  $D_{\text{pdR}, \tau, \bar{a}}(W_{\tau, \bar{a}})$  in Proposition 5.3.4 (but we change to the notation  $x_{\text{pdR}}$  here).

Note that  $W$  acts on  $\mathfrak{t} = \prod_{\tau, \bar{a}} L^{d_{\tau, \bar{a}}}$  and  $W_P$  is the stabilizer of  $\mathbf{h}$ . Let  $w \in W/W_P$  be the element such that  $\text{wt}(\underline{\delta}) = w(\mathbf{h})$  where  $\text{wt}(\underline{\delta}) \in \mathfrak{t}$  by the identification in the beginning of §5.4.

**Proposition 5.5.1.** *The morphism  $\widehat{X_{\text{tri}}(\bar{r})}_x \hookrightarrow X_{r, \mathcal{M}_{\bullet}} \rightarrow X_{D, \mathcal{M}_{\bullet}} \rightarrow \widehat{T}_{P, (0,0)}$  factors through  $\widehat{T}_{P, w, (0,0)}$ .*

*Proof.* The proof has no essential difference with that of Proposition 3.3.5. Assume that we have a deformation  $(W_A^+, \mathcal{F}_{A, \bullet})$  of  $(W^+, \mathcal{F}_{\bullet})$  which comes from  $\widehat{X_{\text{tri}}(\bar{r})}_x$ . The Sen polynomial  $P_{W_A^+}(Y)$  of  $W_A^+$  is equal to

$$\prod_{\tau \in \Sigma} P_{W_{A, \tau}^+}(Y) = \prod_{\tau \in \Sigma} \prod_{i=1}^n (Y - \text{wt}_{\tau}(\delta_{A, i})) \in \prod_{\tau \in \Sigma} A[Y]$$

by [BHS19, Lem. 3.7.6]. Write  $W_{A, \tau}^+ = \bigoplus_{\bar{a} \in \bar{\mathcal{A}}_{\tau}} W_{A, \tau, \bar{a}}^+$ . Then  $P_{W_{A, \tau}^+}(Y) = \prod_{\bar{a} \in \bar{\mathcal{A}}_{\tau}} P_{W_{A, \tau, \bar{a}}^+}(Y)$ . Recall in the beginning of §5.4,  $j_{\tau, \bar{a}, 1} < \dots < j_{\tau, \bar{a}, d_{\tau, \bar{a}}}$  are the numbers  $i$  such that  $\text{wt}_{\tau}(\delta_i) \in \bar{a}$ . Mod  $m_A$ , since all roots of  $P_{W_{\tau, \bar{a}}^+}(Y)$  are in  $a + \mathbb{Z}$ , we get  $P_{W_{\tau, \bar{a}}^+}(Y) = \prod_{i=1}^{d_{\tau, \bar{a}}} (Y - \text{wt}_{\tau}(\delta_{j_{\tau, \bar{a}}, i}))$ . By Lemma 3.3.14, using that different  $P_{W_{\tau, \bar{a}}^+}(Y)$  share no same roots for different  $\bar{a}$ , we have

$$P_{W_{A, \tau, \bar{a}}^+}(Y) = \prod_{i=1}^{d_{\tau, \bar{a}}} (Y - \text{wt}_{\tau}(\delta_{A, j_{\tau, \bar{a}}, i}))$$

for all  $\tau, \bar{a}$ . Now  $W_{A, \tau, \bar{a}}^+(\delta_{\tau, \bar{a}}^{-1})$  has Hodge-Tate weights  $h_{\tau, \bar{a}, 1} \leq \dots \leq h_{\tau, \bar{a}, d_{\tau, \bar{a}}}$  and the Sen polynomial  $\prod_{i=1}^{d_{\tau, \bar{a}}} (Y + a - \text{wt}_{\tau}(\delta_{A, j_{\tau, \bar{a}}, i}))$ . Let  $w_{\tau, \bar{a}}$  be the component of  $w \in W/W_P$  in  $W_{\tau, \bar{a}}/W_{P_{\tau, \bar{a}}}$  where  $W_{\tau, \bar{a}}$  denotes the Weyl group of  $G_{\tau, \bar{a}} = \text{GL}_{d_{\tau, \bar{a}}}$ . The proof of Proposition 3.3.5 shows that the image of the point in  $X_{P_{\tau, \bar{a}}}(A)$  corresponding to  $(W_{A, \tau, \bar{a}}^+, \mathcal{F}_{\tau, \bar{a}, \bullet}, \alpha_{\tau, \bar{a}})$  in  $T_{P, \tau, \bar{a}} := \mathfrak{t}_{\tau, \bar{a}} \times_{\mathfrak{t}_{\tau, \bar{a}}/W_{\tau, \bar{a}}} \mathfrak{t}_{\tau, \bar{a}}/W_{P_{\tau, \bar{a}}}$  lies in the component corresponding to  $w_{\tau, \bar{a}}$ . And we get the desired result by taking the product over  $\tau$  and  $\bar{a}$ .  $\square$

The following results can be proved in the same way as in §3.3.5 (and we can define similarly  $X_{r, \mathcal{M}_{\bullet}}^w$  as the image of  $X_{r, \mathcal{M}_{\bullet}}^{\square, w} = X_{r, \mathcal{M}_{\bullet}} \times_{X_{D, \mathcal{M}_{\bullet}}} X_{D, \mathcal{M}_{\bullet}}^{\square, w}$  under the morphism  $X_{r, \mathcal{M}_{\bullet}}^{\square} \rightarrow X_{r, \mathcal{M}_{\bullet}}$ ).

**Corollary 5.5.2.** *The closed immersion  $\widehat{X_{\text{tri}}(\bar{r})}_x \hookrightarrow X_{r, \mathcal{M}_\bullet}$  induces an isomorphism*

$$\widehat{X_{\text{tri}}(\bar{r})}_x \xrightarrow{\sim} X_{r, \mathcal{M}_\bullet}^w.$$

Let  $w_x \in W/W_P$  be the unique element such that  $x_{\text{pdR}} \in V_{P, w_x}$  (see (3.2.1)).

**Theorem 5.5.3.** *Let  $x = (r, \underline{\delta}) \in X_{\text{tri}}(\bar{r})(L)$  be a point such that  $\underline{\delta} \in \mathcal{T}_0^n$  is generic. Then the trianguline variety  $X_{\text{tri}}(\bar{r})$  is irreducible at  $x$  and we have formally smooth morphisms*

$$\widehat{X_{\text{tri}}(\bar{r})}_x \leftarrow |X_{r, \mathcal{M}_\bullet}^{\square, w}| \rightarrow \widehat{X}_{P, w, x_{\text{pdR}}}$$

of formal schemes. Moreover,  $w_x \leq w$  in  $W/W_P$ .

## 5.6 An application

Let  $(r, \underline{\delta}) \in X_{\text{tri}}(\bar{r})(L)$  be as in the last section and  $\mathcal{M}_\bullet$  be the associated trianguline filtration of  $D_{\text{rig}}(r)[\frac{1}{t}]$ . If  $\underline{\delta}' = (\delta'_1, \dots, \delta'_n)$  is another parameter of  $\mathcal{M}$ , then  $H_{\varphi, \gamma_K}^0(\mathcal{R}_{L, K}(\delta_i(\delta'_1)^{-1})[\frac{1}{t}]) \neq 0$  for some  $i$  (otherwise  $H_{\varphi, \gamma_K}^0(\mathcal{M}(\delta'_1)^{-1}) = 0$ ). Hence  $\delta_i(\delta'_1)^{-1}$  is algebraic by [BHS19, Lem. 3.3.4]. By the generic assumption on  $\underline{\delta}$ , we get that the short exact sequence  $0 \rightarrow \mathcal{M}_{i-1} \rightarrow \mathcal{M}_i \rightarrow \mathcal{R}_{L, K}(\delta_i)[\frac{1}{t}] \rightarrow 0$  must split if the  $i$  above is not 1. An argument by induction shows that  $\underline{\delta}' \in \mathcal{T}_0^n$  is also generic. Hence we can say a trianguline representation  $r$  is generic if one of (equivalently all) the trianguline parameters of  $r$  is generic. The following corollary is a consequence of Theorem 5.5.3.

**Corollary 5.6.1.** *Let  $(\rho, \underline{\delta}) \in Y(U^p, \bar{\rho})$  be a point on the eigenvariety in the setting of §1.4. Assume that  $\rho_{\bar{v}}$  are generic for all places  $v \in S_p$ . Then  $\underline{\delta}$  is contained in the set of characters conjectured by Hansen determined by  $\rho_p = (\rho_{\bar{v}})_{v \in S_p}$ .*

*Proof.* We fix a lifting  $\rho^{\text{univ}} : \mathcal{G}_F \rightarrow \text{GL}_n(R_{\bar{\rho}, S})$  for the universal deformation of  $\bar{\rho} : \mathcal{G}_F \rightarrow \text{GL}_n(k_L)$ . Recall  $R_{\bar{\rho}_p} = \widehat{\otimes}_{v \in S_p} R_{\bar{\rho}_{\bar{v}}}$  where  $R_{\bar{\rho}_{\bar{v}}}$  are certain framed deformation rings. The universal properties for the liftings  $\rho_{\bar{v}}^{\text{univ}}, v \in S_p$  induce a map  $R_{\bar{\rho}_p} \rightarrow R_{\bar{\rho}, S}$ . On the rigid generic fiber, we get  $\text{Spf}(R_{\bar{\rho}, S})^{\text{rig}} \rightarrow \mathfrak{X}_{\bar{\rho}_p}$ . Hence there is a map  $f : Y(U^p, \bar{\rho}) \hookrightarrow \text{Spf}(R_{\bar{\rho}, S})^{\text{rig}} \times \widehat{T}_{p, L} \rightarrow \mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p, L} : (\rho, \underline{\delta}) \mapsto (\rho_p = (\rho_{\bar{v}})_{v \in S_p}, \underline{\delta})$ . Recall  $\iota(X_{\text{tri}}(\bar{\rho}_p))$  is a closed subspace of  $\mathfrak{X}_{\bar{\rho}_p} \times \widehat{T}_{p, L}$ . Hence the preimage  $f^{-1}(\iota(X_{\text{tri}}(\bar{\rho}_p)))$  is a closed analytic subset of  $Y(U^p, \bar{\rho})$ . However, the subset of strictly dominant generic crystalline very classical points, in the sense of [BHS17b, Thm. 3.19], is Zariski dense in  $Y(U^p, \bar{\rho})$ . This Zariski dense subset is mapped into  $\iota(X_{\text{tri}}(\bar{\rho}_p))$ . Thus  $f^{-1}(\iota(X_{\text{tri}}(\bar{\rho}_p))) = Y(U^p, \bar{\rho})$  or  $f(Y(U^p, \bar{\rho})) \subset \iota(X_{\text{tri}}(\bar{\rho}_p))$ . Then this corollary follows from the statement of  $w \geq w_x$  for  $x \in X_{\text{tri}}(\bar{\rho}_p)$  in Theorem 5.5.3.  $\square$

One can also wonder about possible companion constituents from the local models. For this purpose, we generalize below the construction of locally analytic representations in [OS15] to non-integral weights.

Let  $K/\mathbb{Q}_p$  be the local field as before. Let  $\mathfrak{g}_n$  be the base change to  $L$  of the  $\mathbb{Q}_p$ -Lie algebra of  $\text{GL}_n(K)$  and let  $\mathfrak{b}_n \subset \mathfrak{g}_n$  (resp.  $\mathfrak{t}_n \subset \mathfrak{b}_n$ ) be that of  $B_n(K)$  (resp.  $T_n(K)$ ) where  $B_n$  is the group of upper-triangular (resp. diagonal) matrices of  $\text{GL}_n$ .

If  $\lambda \in \prod_{\tau \in \Sigma} L^n$  is a weight of  $\mathfrak{t}_n$ , possibly non-integral, we have the Verma module  $M(\lambda) = U(\mathfrak{g}_n) \otimes_{U(\mathfrak{b}_n)} \lambda$  in the BGG category  $\mathcal{O}$  and also the irreducible quotient  $L(\lambda)$  of  $M(\lambda)$ . One has a reflection group  $W_{[\lambda]}$  in [Hum08, §3.4]. Let  $W_{[\lambda], P}$  be the subgroup of  $W_{[\lambda]}$  consisting of elements  $w$  such that  $w \cdot \lambda = \lambda$ . Let  $w_0$  be the longest element of  $W_{[\lambda]}$ . We assume that  $\lambda$  is dominant in the sense of [Hum08, §3.5]. Same as the integral cases, by [Hum08, §5.1], we have  $L(w'w_0 \cdot \lambda)$  is a subquotient of  $M(w_0w' \cdot \lambda)$  if and only if  $w' \leq w$  in  $W_{[\lambda]}/W_{[\lambda], P}$ .

Take  $\mu$  to be the weight of a continuous character  $\underline{\delta} : T_n(K) \rightarrow L$ . Then  $M(\mu)$  can be equipped with a structure of a  $D(B_n(K), \mathfrak{g}_n)$ -module  $M(\delta) := D(B_n(K), \mathfrak{g}_n) \otimes_{D(B_n(K))} \delta$ . The  $D(\mathrm{GL}_n(K))$ -module  $D(\mathrm{GL}_n(K)) \otimes_{D(B_n(K), \mathfrak{g}_n)} M(\delta)$  is a dual of certain locally analytic principal series (§1.4.2).

We show that  $L(\mu)$  is also a  $D(B_n(K), \mathfrak{g}_n)$ -module. We fix an ordering of positive roots  $\alpha_1, \dots, \alpha_m$  of  $\mathfrak{b}_n$  and let  $y_1, \dots, y_m$  be the standard basis of  $\bar{\mathfrak{b}}_n$  corresponding to negative roots  $-\alpha_1, \dots, -\alpha_m$  (see [Hum08, §0.1]). Then  $M(\delta)$  is spanned by  $y_1^{i_1} \cdots y_m^{i_m} v_\mu$  where  $v_\mu$  denotes a highest weight vector and  $i_1, \dots, i_m \in \mathbb{N}$ . On the space spanned by  $y_1^{i_1} \cdots y_m^{i_m} v_\mu$ ,  $T_n(K)$  acts by  $\underline{\delta} \alpha_1^{-i_1} \cdots \alpha_m^{-i_m}$  where we view  $\alpha_i$  as algebraic characters of  $T_n(K)$ . Hence if  $\alpha$  is a sum of positive roots, the weight space  $M(\delta)_{\mu-\alpha}$  corresponding to  $\mu - \alpha$  is stable under the action of  $D(T_n(K))$  and is equal to the space  $M(\delta)_{\underline{\delta}\alpha^{-1}}$  where  $D(T_n(K))$  acts by the character  $\underline{\delta}\alpha^{-1}$ . Let  $N$  be the kernel of  $M(\mu) \rightarrow L(\mu)$ . Since  $N \in \mathcal{O}$ ,  $N$  can be written as a direct sum of its weight spaces  $N_{\mu'} \subset M_{\mu'}$ . Thus  $N$  is stable under the action of  $D(T_n(K))$ . Moreover, the kernel  $N$  is  $N_n(K)$ -invariant as in the proof of [OS15, Lem. 3.2] where  $N_n$  denotes the unipotent radical of  $B_n$ . Hence  $N \rightarrow M(\underline{\delta})$  is also a  $D(B_n(K), \mathfrak{g}_n)$ -map. Then the quotient  $L(\lambda)$  is a  $D(B_n(K), \mathfrak{g}_n)$ -module and we denote it by  $L(\underline{\delta})$ .

The dual of  $D(\mathrm{GL}_n(K)) \otimes_{D(B_n(K), \mathfrak{g}_n)} L(\underline{\delta})$  is a subquotient of some locally analytic principal series of  $\mathrm{GL}_n(K)$ . If we could know when these subquotients are topologically irreducible, then the results of this chapter would help to make conjectures on the socle of the locally analytic representations associated to trianguline representations with possibly non-integral Hodge-Tate-Sen weights.



# Appendix A

## Families of almost de Rham $(\varphi, \Gamma_K)$ -modules

We show that the method of Berger-Colmez in [BC08] to construct de Rham families for Galois representations can be generalized to construct partially almost de Rham families for  $(\varphi, \Gamma_K)$ -modules.

### A.1 Preliminary

Let  $K$  be a finite extension of  $\mathbb{Q}_p$  with a uniformizer  $\varpi_K$ ,  $\overline{K}$  be an algebraic closure of  $K$  and  $C$  be the completion of  $\overline{K}$ . Let  $K_0$  be the maximal unramified (over  $\mathbb{Q}_p$ ) subfield of  $K$ . Let  $K_\infty$  be the extension of  $K$  by adding all  $p$ -th power roots of unity,  $K_m = K(\mu_{p^m})$  for  $m \geq 1$ . We set  $\Gamma_K := \text{Gal}(K_\infty/K)$ ,  $\Gamma_{K_m} := \text{Gal}(K_\infty/K_m)$  and  $H_K := \text{Gal}(\overline{K}/K_\infty)$ . Let  $\epsilon : \Gamma_K \rightarrow \mathbb{Z}_p^\times$  be the cyclotomic character.

Let  $A$  be an affinoid algebra over  $\mathbb{Q}_p$ . We keep the notation of the Robba rings and  $(\varphi, \Gamma_K)$ -modules in §3.1.7. Recall we have Robba rings  $\mathcal{R}_{A,K}^r$  which is denoted by  $\mathcal{R}_A^r(\pi_K)$  in [KPX14, Def. 2.2.2] for any  $r > 0$  less than certain constant.

Recall a  $\varphi$ -module over  $\mathcal{R}_{A,K}^r$  is a finite projective module  $M^r$  over  $\mathcal{R}_{A,K}^r$  equipped with an isomorphism  $\varphi^* M^r \xrightarrow{\sim} M^{r/p} = M^r \otimes_{\mathcal{R}_{A,K}^r} \mathcal{R}_{A,K}^{r/p}$  where  $\varphi : \mathcal{R}_{A,K}^r \rightarrow \mathcal{R}_{A,K}^{r/p}$  and a  $\varphi$ -module over  $\mathcal{R}_{A,K}$  is the base change of a  $\varphi$ -module over some  $\mathcal{R}_{A,K}^{r_0}$  ([KPX14, Def. 2.2.6]). A  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{A,K}$  is the base change of a  $\varphi$ -module equipped with a commuting semi-linear continuous action of  $\Gamma_K$  over some  $\mathcal{R}_{A,K}^{r_0}$  ([KPX14, Def. 2.2.12]).

We recall some constructions from  $(\varphi, \Gamma_K)$ -modules.

Suppose that  $D_A$  is a  $(\varphi, \Gamma)$ -module over  $\mathcal{R}_{A,K}$  of rank  $n$ . By definition,  $D_A$  is the base change of a  $(\varphi, \Gamma)$ -module  $D_A^{r_0}$  over  $\mathcal{R}_{A,K}^{r_0} = \mathcal{R}_{\mathbb{Q}_p,K}^{r_0} \hat{\otimes}_{\mathbb{Q}_p} A$  ([Eme17, Prop. 1.1.29]) for some  $r_0$  small enough. For any  $r$ , let  $m(r)$  be the minimal integer such that  $p^{m(r)-1}[K(\mu_\infty) : K_0(\mu_\infty)]r \geq 1$ . Then there exists a continuous  $\Gamma_K$ -equivariant injection  $\iota_{m(r)} : \mathcal{R}_{\mathbb{Q}_p,K}^r \hookrightarrow K_{m(r)}[[t]]$  (e.g. [Ber08a, §1.2]). As in [BC08, §4.3], we define  $A \hat{\otimes}_{\mathbb{Q}_p} K_m[[t]] := \varprojlim_k A \hat{\otimes}_{\mathbb{Q}_p} (K_m[[t]]/t^k)$  for any  $m \geq m(r_0)$  and we will always assume  $m \geq m(r_0)$  in the remaining part of this appendix.

Let  $D_{\text{dif}}^{K_m,+}(D_A) := A \hat{\otimes}_{K_m}[[t]] \otimes_{\iota_{m(r)}, \mathcal{R}_{A,K}^r} D_A^r$  be the “localization” if  $m = m(r)$  and  $r \leq r_0$  where  $D_A^r = D_A^{r_0} \otimes_{\mathcal{R}_{A,K}^{r_0}} \mathcal{R}_{A,K}^r$ . Then  $D_{\text{dif}}^{K_m,+}(D_A)$  is a finite projective  $A \hat{\otimes}_{K_m}[[t]]$ -module with a semi-linear continuous action of  $\Gamma_K$ . Here the continuity means that for each  $s \geq 1$  the action of  $\Gamma_K$  on  $D_{\text{dif}}^{K_m,+}(D_A)/t^s D_{\text{dif}}^{K_m,+}(D_A)$  is continuous. In particular, for any  $x \in D_{\text{dif}}^{K_m,+}(D_A)/t^s D_{\text{dif}}^{K_m,+}(D_A)$ ,  $\lim_{\gamma \in \Gamma_K, \gamma \rightarrow 1} \gamma(x) = x$  in the finite  $A$ -Banach module

$$D_{\text{dif}}^{K_m,+}(D_A)/t^s D_{\text{dif}}^{K_m,+}(D_A).$$

Write  $D_{\text{dif}}^{K_m}(D_A) := D_{\text{dif}}^{K_m,+}(D_A)[\frac{1}{t}]$ .

Let  $D_{\text{Sen}}^{K_m}(D_A) := D_{\text{dif}}^{K_m,+}(D_A)/tD_{\text{dif}}^{K_m,+}(D_A)$ . Then  $\varphi$  induces  $\Gamma_K$ -equivariant isomorphisms (cf. [Liu15, §2.3])

$$\begin{aligned} D_{\text{Sen}}^{K_m}(D_A) \otimes_{K_m} K_{m+1} &\xrightarrow{\sim} D_{\text{Sen}}^{K_{m+1}}(D_A), \\ D_{\text{dif}}^{K_m,+}(D_A) \otimes_{K_m[[t]]} K_{m+1}[[t]] &\xrightarrow{\sim} D_{\text{dif}}^{K_{m+1},+}(D_A). \end{aligned}$$

If  $B$  is an affinoid algebra with a morphism  $A \rightarrow B$ , denote by  $D_B := M \hat{\otimes}_A B$  the  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{B,K}$ . Then by the construction, there is a natural isomorphism  $D_{\text{dif}}^{K_m,+}(D_B) \simeq D_{\text{dif}}^{K_m,+}(D_A) \hat{\otimes}_A B$ . If  $x$  is a point on  $\text{Sp}(A)$ , we let  $D_x = D_A \otimes_A k(x)$  be the  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{k(x),K}$ . The following lemma is not essential.

**Lemma A.1.1.** *There exists a finite admissible covering  $\{\text{Sp}(A_i)\}$  of  $\text{Sp}(A)$  such that each  $D_{\text{dif}}^{K_m,+}(D_{A_i})$  (resp.  $D_{\text{Sen}}^{K_m}(D_{A_i})$ ) is a free  $A_i \hat{\otimes}_{\mathbb{Q}_p} K_m[[t]]$ -module (resp.  $A_i \otimes_{\mathbb{Q}_p} K_m$ -module).*

*Proof.* We have  $A \hat{\otimes}_{\mathbb{Q}_p} K_m[[t]] = \varprojlim_k A \hat{\otimes}_{\mathbb{Q}_p} (K_m[[t]]/t^k) = \varprojlim_k (A \hat{\otimes}_{\mathbb{Q}_p} K_m)[[t]]/t^k = K_m \otimes_{\mathbb{Q}_p} A[[t]]$  (cf. [Eme17, Prop. 1.2.5]) is  $t$ -adically complete. Thus the finite projective module  $D_{\text{dif}}^{K_m,+}(D_A) \hat{\otimes}_A A_i$  over  $A_i \hat{\otimes}_{\mathbb{Q}_p} K_m[[t]]$  is free if and only if  $D_{\text{Sen}}^{K_m}(D_A) \hat{\otimes}_A A_i$  over  $A_i \otimes_{\mathbb{Q}_p} K_m$  is free. Therefore we only need work for  $D_{\text{Sen}}^{K_m}(D_A)$ .

Let  $\mathfrak{m}$  be a maximal ideal of  $A$  corresponding to a point  $x \in \text{Sp}(A)$ . Since  $D_x^r$  is free over  $\mathcal{R}_{k(x),K}^r$  ([Ber08a, Prop. 1.1.1]),  $D_{\text{Sen}}^{K_m}(D_x)$  is free over  $k(x) \otimes_{\mathbb{Q}_p} K_m$ . As  $D_{\text{Sen}}^{K_m}(D_A) \otimes_A A_{\mathfrak{m}}$  is finite projective over  $A_{\mathfrak{m}} \otimes_{\mathbb{Q}_p} K_m$  and  $\mathfrak{m} \otimes_{\mathbb{Q}_p} K_m$  is contained in the Jacobson radical of  $A_{\mathfrak{m}} \otimes_{\mathbb{Q}_p} K_m$ ,  $D_{\text{Sen}}^{K_m}(D_A) \otimes_A A_{\mathfrak{m}}$  is free over  $A_{\mathfrak{m}} \otimes_{\mathbb{Q}_p} K_m$  by Nakayama's lemma. We choose a morphism  $(A \otimes K_m)^n \rightarrow D_{\text{Sen}}^{K_m}(D_A)$  of  $A \otimes_{\mathbb{Q}_p} K_m$ -modules (after possibly shrinking  $\text{Spec}(A)$ ) such that it induces an isomorphism  $(A_{\mathfrak{m}} \otimes_{\mathbb{Q}_p} K_m)^n \rightarrow D_{\text{Sen}}^{K_m}(D_A) \otimes_A A_{\mathfrak{m}}$  of  $A_{\mathfrak{m}} \otimes_{\mathbb{Q}_p} K_m$ -modules after tensoring  $A_{\mathfrak{m}}$ . Let  $C_1$  (resp.  $C_2$ ) be the kernel (resp. cokernel) of this morphism. Then  $C_1 \otimes_A A_{\mathfrak{m}} = C_2 \otimes_A A_{\mathfrak{m}} = 0$ . Since  $A$  is Noetherian, we can pick a finite set  $\{x_i\}$  of generators of  $C_1$  and  $C_2$  over  $A$ . For any  $x_i$ , there exists an element  $a_i \in A$  such that  $a_i x_i = 0$  and  $a_i \notin \mathfrak{m}$ . Let  $A_S$  be the localization of  $A$  by inverting all  $a_i$ . Then  $C_1 \otimes_A A_S = C_2 \otimes_A A_S = 0$ . Thus the morphism  $(A_S \otimes_{\mathbb{Q}_p} K_m)^n \rightarrow D_{\text{Sen}}^{K_m}(D_A) \otimes_A A_S$  of  $A_S \otimes_{\mathbb{Q}_p} K_m$ -modules is an isomorphism. Now since Zariski open subsets are admissible open and every Zariski covering is admissible ([Bos14, §5.1 Cor. 9]) and  $\text{Spec}(A)$  is quasi-compact, we can find a finite covering of  $\text{Sp}(A)$  by affinoid subdomains  $\text{Sp}(A_i)$  such that  $D_{\text{Sen}}^{K_m}(D_A) \otimes_A A_i$  is free.  $\square$

Thus from now on we assume that  $D_{\text{Sen}}^{K_m(r_0)}(D_A)$  is a free  $A \otimes_{\mathbb{Q}_p} K_{m(r_0)}$ -module. Since the action of  $\Gamma_K$  on  $D_{\text{Sen}}^{K_m}(D_A)$  is continuous, there exists  $m_1 > m(r_0)$  such that  $\|M_\gamma - I\| < 1$  for any  $\gamma \in \Gamma_{K_{m_1}}$  where  $M_\gamma$  denotes the matrix of  $\gamma$  on  $D_{\text{Sen}}^{K_m(r_0)}(D_A)$  with coefficients in  $A \otimes_{\mathbb{Q}_p} K_{m(r_0)}$  with respect to some basis and the operator norm  $\|\cdot\|$  of operators on the finite Banach  $A$ -module  $D_{\text{Sen}}^{K_m(r_0)}(D_A)$  is equivalent to the matrix norm given by norms of coefficients (cf. [KPX14, Prop. 2.2.14], [Eme17, Prop. 1.2.4]). Then for any  $\gamma \in \Gamma_{K_{m_1}}$ , the Sen operator  $\nabla : \log(\gamma)/\log(\epsilon(\gamma)) = -\frac{1}{\log(\epsilon(\gamma))} \sum_{i=1}^{\infty} \frac{(1-\gamma)^i}{i}$  on  $D_{\text{Sen}}^{K_{m_1}}(D_A)$  converges and acts  $A \otimes_{\mathbb{Q}_p} K_{m_1}$ -linearly on  $D_{\text{Sen}}^{K_{m_1}}(D_A)$ . The characteristic polynomial of the Sen operator lies in  $(A \otimes_{\mathbb{Q}_p} K_{m_1}[T])^{\Gamma_K} = A \otimes_{\mathbb{Q}_p} K[T]$  (cf. [Fon04, Prop. 2.5] or [KPX14, Def. 6.2.11]). We take  $m_0 \geq m_1$  such that for any  $\gamma \in \Gamma_{K_{m_0}}$ ,  $\|\log(\epsilon(\gamma))\nabla\| < \|p\|^{\frac{1}{p-1}}$ . Then for any  $m > m_0$ ,  $\gamma \in \Gamma_{K_m}$ , the action of the  $A \otimes_{\mathbb{Q}_p} K_m$ -linear operator  $\gamma$  on  $D_{\text{Sen}}^{K_m}(D_A) = K_m \otimes_{K_{m(r_0)}} D_{\text{Sen}}^{K_m(r_0)}(D_A)$  can be recovered by  $\gamma = \exp(\log(\epsilon(\gamma))\nabla) = \sum_{i=0}^{\infty} \frac{(\log(\epsilon(\gamma))\nabla)^i}{i!}$ . Later, we will need to take  $m > m_0$  to deduce the final result on  $K$ . We remark that the same formula for  $\gamma \in \Gamma_{K_m}$  may not hold on  $D_{\text{dif}}^{K_m,+}(D_A)$  in general.

From now on, we assume that  $A$  is moreover an  $L$ -algebra where  $L$  is a finite extension of  $\mathbb{Q}_p$  that splits  $K$ . We denote by  $\Sigma$  the set of all embeddings of  $K$  in  $L$ . For  $\tau \in \Sigma$ , we set  $D_{\text{Sen}, \tau}^{K_m}(D_A) := D_{\text{Sen}}^{K_m}(D_A) \otimes_{A \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} A$  (resp.  $D_{\text{dif}, \tau}^{K_m, +}(D_A) := D_{\text{dif}}^{K_m, +}(D_A) \otimes_{A \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} A$ ) which is a free  $A \otimes_K K_m$ -module (resp.  $A \widehat{\otimes}_K K_m[[t]]$ -module) and is stable under the action of  $\Gamma_K$  since  $\Gamma_K$  commutes with  $A \otimes_{\mathbb{Q}_p} K$ .

## A.2 Almost de Rham representations

We start with the cases when  $A$  is a finite  $L$ -algebra. We will give certain characteristic properties of almost de Rham  $B_{\text{dR}}$ -representations on the level of  $D_{\text{dif}}^{K_m, +}(D_A)$  (Proposition A.2.2) as is done for de Rham representations in [BC08, Prop. 5.2.1].

Recall that the ring  $B_{\text{pdR}} = B_{\text{dR}}[\log(t)]$  is equipped with a  $B_{\text{dR}}$ -derivation  $\nu_{\text{pdR}}(\log(t)) = -1$ . We pick a topological generator  $\gamma_K \in \Gamma_K$  (and  $\gamma_{K_m} \in \Gamma_{K_m}$ ). Then  $\gamma_K(\log(t)) = \log(t) + \log(\epsilon(\gamma_K))$ . Let  $\nabla$  be the operator  $\log(\gamma_K^{p^s})/\log(\epsilon(\gamma_K^{p^s}))$  for  $s$  large enough acting on  $D_{\text{dif}}^{K_m, +}(D_A)$  in the sense of [Fon04, Prop. 3.7]. Then  $\nabla$  acts  $K_m$ -linearly,  $\nabla(t^a) = at^a$  for any  $a \in \mathbb{Z}$  and  $\nabla(ax) = a\nabla(x) + \nabla(a)x$  for any  $a \in K_m((t))$ . Formally let  $\nabla(\log(t)^a) = a\log(t)^{a-1}$ ,  $\forall a \in \mathbb{N}$  as in [Fon04, §3.8]. The following lemma will be useful.

**Lemma A.2.1.** *Let  $B \in \{K_m((t)), K((t))\}$  and  $W$  be a continuous semi-linear  $B$ -representation of  $\Gamma_K$ . Then  $(W \otimes_B B[\log(t)])^{\nabla=0}$  can be identified with  $W^{\nabla=\text{nil}}$ , the space of elements  $x \in W$  such that  $\nabla$  acts nilpotently on  $x$ . The identification restricts to a bijection between  $(W \otimes_B B[\log(t)])^{\Gamma_K}$  and  $W^{(\gamma_K-1)=\text{nil}}$ , the space of elements  $x \in W$  such that  $\gamma_K - 1$  acts nilpotently on  $x$ . Moreover, under the identification, the action of the  $B$ -derivation  $\nu_{\text{pdR}}$  on  $(W \otimes_B B[\log(t)])^{\Gamma_K}$  is the action of  $\nabla$  on  $W^{(\gamma_K-1)=\text{nil}}$ .*

*Proof.* Assume  $\sum_{i=0}^N a_i \log(t)^i \in (W \otimes_B B[\log(t)])^{\nabla=0}$  where  $a_i \in W$ . Then

$$\begin{aligned} 0 &= \nabla \left( \sum_{i=0}^N a_i \log(t)^i \right) \\ &= \nabla(a_0) + \sum_{i=1}^N (\nabla(a_i) \log(t)^i + ia_i \log(t)^{i-1}) \\ &= \nabla(a_N) \log(t)^N + \sum_{i=0}^{N-1} ((i+1)a_{i+1} + \nabla(a_i)) \log(t)^i. \end{aligned}$$

Hence  $a_i = (-1)^i \frac{1}{i!} \nabla^i(a_0)$  and  $\nabla^{N+1}(a_0) = 0$ . The map  $(W \otimes_B B[\log(t)])^{\nabla=0} \rightarrow W^{\nabla=\text{nil}} : \sum_{i=0}^N a_i \log(t)^i \mapsto a_0$  is then a bijection with the inverse map  $a \mapsto \sum_{i=0}^N \frac{(-1)^i}{i!} \nabla^i(a_0) \log(t)^i$  if  $\nabla^{N+1}(a) = 0$ .

Now assume  $\sum_{i=0}^N a_i \log(t)^i \in (W \otimes_B B[\log(t)])^{\gamma_K=1}$  where  $a_i \in W$  and  $a_N \neq 0$ . Then

$$\begin{aligned} \sum_{i=0}^N a_i \log(t)^i &= \sum_{i=0}^N \gamma_K(a_i) (\log(t) + \log(\epsilon(\gamma_K)))^i \\ &= \sum_{i=0}^N \left( \sum_{j=i}^N \binom{j}{i} \gamma_K(a_j) \log(\epsilon(\gamma_K))^{j-i} \right) \log(t)^i. \end{aligned}$$

Thus  $a_i = \sum_{j=i}^N \binom{j}{i} \log(\epsilon(\gamma_K))^{j-i} \gamma_K(a_j)$  for each  $i$ . We get  $a_N = \gamma_K(a_N)$ . We now prove



$(\gamma_K - 1)^i a_{N-i+1} = 0$  by descending induction. Assume this is true for all  $i > i_0$ . Then

$$\begin{aligned} (\gamma_K - 1)a_{i_0} &= \gamma_K(a_{i_0}) - a_{i_0} = \gamma_K(a_{i_0}) - \sum_{j=i_0}^N \binom{j}{i_0} \log(\epsilon(\gamma_K))^{j-i_0} \gamma_K(a_j) \\ &= - \sum_{j=i_0+1}^N \binom{j}{i_0} \log(\epsilon(\gamma_K))^{j-i_0} \gamma_K(a_j). \end{aligned}$$

By the induction hypothesis,  $(\gamma_K - 1)^{N-i_0} \gamma_K(a_j) = \gamma_K(\gamma_K - 1)^{N-i_0}(a_j) = 0$  for any  $j > i_0$ . Thus  $(\gamma_K - 1)^{N-i_0+1} a_{i_0} = 0$  which finishes the induction step. Hence  $\gamma_K - 1$  acts nilpotently on  $a_0$ .

Conversely, assume we are given  $a_0$  such that  $(\gamma_K - 1)^{N+1} a_0 = 0$ . Then  $\nabla^{N+1}(a_0) = 0$  since  $\nabla = \log(\gamma_K^s) / \log(\epsilon(\gamma_K^s)) = \log(\gamma_K) / \log(\epsilon(\gamma_K)) = - \sum_{i=1}^N \frac{(1-\gamma_K)^i}{i \log(\epsilon(\gamma_K))}$  on  $a_0$  where  $s$  is any large integer (since  $\nabla$  is defined using [Fon04, Prop. 3.7], the equality should be verified modulo  $t^s$  for all  $s$  and notice that  $N$  is independent of  $s$ ). We now verify that  $x = \sum_{i=0}^N \frac{(-1)^i}{i!} \nabla^i(a_0) \log(t)^i$  is fixed by  $\gamma_K$ . Since  $\gamma_K$  commutes with  $\nabla$ ,  $\gamma_K - 1$  and  $\nabla$  act nilpotently on each  $\nabla^i(a_0)$  as well as  $\log(t)^i$  and  $\nabla^i(a_0) \log(t)^i$  (since  $(\gamma_K - 1)(a \log(t)^i) = (\gamma_K - 1)(a) \log(t)^i + \sum_{j=0}^{i-1} \binom{i}{j} \log(\epsilon(\gamma_K))^{i-j} \gamma_K(a) \log(t)^j$ ). Thus  $\gamma_K(x) = \exp(\log(\epsilon(\gamma_K)) \nabla) x = \left(1 + \sum_{i=1}^{\infty} \frac{\log(\epsilon(\gamma_K))^i \nabla^i}{i!}\right) x = x$  where the first two identities come from the formal calculation using  $\nabla = \log(\gamma_K) / \log(\epsilon(\gamma_K))$  on  $x$  and the last identity comes from that  $\nabla(x) = 0$  by  $\nabla^{N+1}(a_0) = 0$  and the computation in first step.

The last assertion follows from that

$$\nu_{\text{pdR}} \left( \sum_{i=0}^N \frac{(-1)^i}{i!} \nabla^i(x) \log(t)^i \right) = \sum_{i=0}^{N-1} \frac{(-1)^i}{i!} \nabla^i(\nabla(x)) \log(t)^i$$

if  $\nabla^{N+1}(x) = 0$ . □

The following proposition generalizes [BC08, Prop. 5.2.1] using essentially the same technique in *loc. cit.* and will be applied for  $m > m_0$ .

**Proposition A.2.2.** *Assume that  $D_{\text{dif}}^{K_m,+}$  is a continuous semilinear  $K_m[[t]]$ -representation of  $\Gamma_{K_m}$  of rank  $n$  and  $D_{\text{dif}}^{K_m} = D_{\text{dif}}^{K_m,+}[\frac{1}{t}]$ . Assume that each eigenvalue of  $\gamma_{K_m}$  on  $D_{\text{Sen}}^{K_m} := D_{\text{dif}}^{K_m,+} / t D_{\text{dif}}^{K_m,+}$  is equal to  $\epsilon(\gamma_{K_m})^c$  for some integer  $c \in [a, b]$  where  $a, b \in \mathbb{Z}$ . Then the following statements hold.*

1. *There exists a  $\Gamma_{K_m}$ -invariant  $K_m[[t]]$ -lattice  $N$  inside  $D_{\text{dif}}^{K_m}$  such that  $(\gamma_{K_m} - 1)^l N \subset tN$  for some  $l \geq 1$ . And we have  $t^{-a} D_{\text{dif}}^{K_m,+} \subset N \subset t^{-b} D_{\text{dif}}^{K_m,+}$ .*
2.  *$D_{\text{pdR}}^{K_m} := (D_{\text{dif}}^{K_m} \otimes_{K_m} K_m[\log(t)])^{\Gamma_{K_m}}$  is an  $n$ -dimensional  $K_m$ -space.*

Moreover, the number  $l$  can be taken to be  $(b - a + 1)n$ .

*Proof.* We firstly prove that (1) implies (2). Consider operators  $\alpha_k := \prod_{i=1}^k f_i(\gamma_{K_m})$  for  $k \geq 1$  where we take polynomials  $f_i(T) = (T - 1)^l h_i(T) + 1 = (\epsilon(\gamma_{K_m})^{-i} T - 1)^l g_i(T)$  for  $i \geq 1$  thanks to Lemma A.2.4 below. Since  $(\gamma_{K_m} - 1)^l$  is trivial on  $N/tN$ ,  $\alpha_k$  is the identity on  $N/tN$ . We set  $\alpha_0(x) = x$ . We now prove by induction that  $(\gamma_{K_m} - 1)^l \alpha_k(x) \in t^{k+1}N$  and  $\alpha_k(x) - \alpha_{k+1}(x) \in t^{k+1}N$  for any  $x \in N$  and  $k \geq 0$ . If  $k = 0$ , the result is easy. Assume that the result is true for  $k$ . We have  $(\frac{\gamma_{K_m}}{\epsilon(\gamma_{K_m})^{k+1}} - 1)(t^{k+1}x) = t^{k+1}\gamma_{K_m}(x) - t^{k+1}x = t^{k+1}(\gamma_{K_m} - 1)(x)$  for any  $x \in D_{\text{dif}}^{K_m}$ . Then  $(\gamma_{K_m} - 1)^l \alpha_{k+1}(x) = f_{k+1}(\gamma_{K_m})(\gamma_{K_m} - 1)^l \alpha_k(x) = g_{k+1}(\gamma_{K_m})(\frac{\gamma_{K_m}}{\epsilon(\gamma_{K_m})^{k+1}} - 1)^l (t^{k+1} \frac{(\gamma_{K_m} - 1)^l \alpha_k(x)}{t^{k+1}}) = g_{k+1}(\gamma_{K_m}) t^{k+1} (\gamma_{K_m} - 1)^l (\frac{(\gamma_{K_m} - 1)^l \alpha_k(x)}{t^{k+1}}) \in$

$t^{k+2}N$  since  $\gamma_{K_m}(t^{k+2}N) \subset t^{k+2}N$  and we have  $(\gamma_{K_m} - 1)^l (\frac{(\gamma_{K_m}-1)^l \alpha_k(x)}{t^{k+1}}) \in tN$  by the induction hypothesis and the assumption. Then  $\alpha_{k+1}(x) - \alpha_{k+2}(x) = h_{k+2}(\gamma_{K_m})(\gamma_{K_m} - 1)^l \alpha_{k+1}(x) \in t^{k+2}N$  which finishes the induction. Hence when  $k \rightarrow +\infty$ , the sequence  $\alpha_k(x)$  converges to an element in  $N$  denoted by  $\alpha(x)$ . Moreover  $(\gamma_{K_m} - 1)^l \alpha(x) = 0$  and the  $K_m$ -map  $\alpha : N \rightarrow N$  is the identity modulo  $t$ . Hence the image of  $\alpha$ , denoted by  $N_{\text{pdR}}$ , has  $K_m$ -dimension at least  $\dim N/tN = n$ . Applying Lemma A.2.1 for  $\Gamma_{K_m}$ , the space  $N_{\text{pdR}}$  can be viewed as a subspace of  $D_{\text{dif}}^{K_m}$ . Assume that  $N'_{\text{pdR}}$  is a  $\Gamma_{K_m}$ -invariant finite-dimensional  $K_m$ -subspace of  $(D_{\text{dif}}^{K_m})^{(\gamma_{K_m}-1)\text{-nil}}$  that is stable under  $\Gamma_{K_m}$  and contains  $N_{\text{pdR}}$ . Then the  $K_m[[t]]$ -lattice  $N'$  generated by  $N'_{\text{pdR}}$  has full rank  $n$ , stable under the action of  $\Gamma_{K_m}$  and satisfies  $(\gamma_{K_m} - 1)^{l'} N' \subset tN'$  for some  $l' \geq 1$ . Repeat the previous argument of  $N, l$  for  $N', l'$ , we find there exists a  $K_m$ -map  $\alpha' : N' \rightarrow N'$  which is the identity over  $N'/tN'$  and  $N'_{\text{pdR}}$ . One can also prove that  $\alpha'(tN') = 0$  by showing that  $\alpha'_k(tN') \subset t^{k+1}N'$  for any  $k$  (since  $f_{k+1}(\gamma_{K_m})(t^{k+1}N') = g_{k+1}(\gamma_{K_m})(\frac{\gamma_{K_m}}{\epsilon(\gamma_{K_m})^{k+1}} - 1)^{l'}(t^{k+1}N') \subset t^{k+1}(\gamma_{K_m} - 1)^{l'}N' \subset t^{k+2}N'$ ). Thus  $N'_{\text{pdR}} = \alpha'(N'_{\text{pdR}}) = \alpha'(N'/tN')$  has  $K_m$ -dimension  $n$ . Hence the dimension of  $D_{\text{pdR}}^{K_m}$  is no more than  $n$  (this also follows from general theory, cf. [Fon04, Thm. 3.22(ii) & Prop. 2.1]). Thus  $D_{\text{pdR}}^{K_m}$  is identified with  $N_{\text{pdR}}$  via Lemma A.2.1. This finishes the proof of (2).

To prove (1), we have the following claim.

**Claim A.2.3.** *Under our assumption, there exists  $l$  such that*

$$\prod_{i=a}^{2b-a} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l D_{\text{dif}}^{K_m,+} \subset t^{b-a+1} D_{\text{dif}}^{K_m,+}.$$

*Proof of Claim A.2.3.* Let  $F_k(T) = \prod_{i=a+k}^{b+k} (T - \epsilon(\gamma_{K_m})^i)^n$ . The characteristic polynomial of  $\gamma_{K_m}$  on the  $K_m$ -space  $t^k D_{\text{dif}}^{K_m,+} / t^{k+1} D_{\text{dif}}^{K_m,+} = t^k D_{\text{Sen}}^{K_m,+}$  divides  $F_k(T)$  for any  $k \in \mathbb{N}$  by the assumption. In particular,  $F_k(\gamma_{K_m}) t^k D_{\text{dif}}^{K_m,+} \subset t^{k+1} D_{\text{dif}}^{K_m,+}$ . Hence

$$F_{b-a}(\gamma_{K_m}) \cdots F_0(\gamma_{K_m}) D_{\text{dif}}^{K_m,+} \subset t^{b-a+1} D_{\text{dif}}^{K_m,+}.$$

We take  $l = (b - a + 1)n$ . Then  $F_{b-a}(T) \cdots F_0(T)$  divides  $\prod_{i=a}^{2b-a} (T - \epsilon(\gamma_{K_m})^i)^l$ . Hence  $\prod_{i=a}^{2b-a} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l D_{\text{dif}}^{K_m,+} \subset t^{b-a+1} D_{\text{dif}}^{K_m,+}$ .  $\square$

Now we prove (1). We let  $N$  be the sub- $K_m[[t]]$ -lattice generated by

$$t^{-k} \prod_{i \in [a, 2b-a] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l x$$

for all  $x \in D_{\text{dif}}^{K_m,+}$  and  $k \in [a, b]$ . Then  $N \subset t^{-b} D_{\text{dif}}^{K_m,+}$  by definition and  $\gamma_{K_m}(N) \subset N$ . We have  $t^a N \supset \{\prod_{i \in [a, 2b-a] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l x \mid x \in D_{\text{dif}}^{K_m,+}, k \in [a, b]\}$ . The image of  $\{\prod_{i \in [a, 2b-a] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l x \mid x \in D_{\text{dif}}^{K_m,+}, k \in [a, b]\}$  in  $D_{\text{Sen}}^{K_m}$  is equal to  $\{\prod_{i \in [a, 2b-a] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l x \mid x \in D_{\text{Sen}}^{K_m}, k \in [a, b]\}$ . By our assumption the characteristic polynomial of  $\gamma_{K_m}$  on  $D_{\text{Sen}}^{K_m}$  is of the form  $\prod_{i \in [a, b]} (x - \epsilon(\gamma_{K_m})^i)^{n_i}$  for some  $n_i \geq 0$ . The polynomials  $\prod_{i \in [a, b] \setminus \{k\}} (T - \epsilon(\gamma_{K_m})^i)^l, k \in [a, b]$  share no common zero and generate the constant polynomial 1 by Hilbert's Nullstellensatz. Hence  $\{\prod_{i \in [a, b] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l x \mid x \in D_{\text{Sen}}^{K_m}, k \in [a, b]\}$  generates  $D_{\text{Sen}}^{K_m}$  as a  $K_m$ -space. Moreover, we have assumed that  $(T - \epsilon(\gamma_{K_m})^i)$  is prime to the characteristic polynomial of  $\gamma_{K_m}$  on  $D_{\text{Sen}}^{K_m}$  if  $i \notin [a, b]$ . Thus  $\prod_{i \in [b+1, 2b-a]} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l$  is a bijection on  $D_{\text{Sen}}^{K_m}$ . Hence the image of  $\{\prod_{i \in [a, 2b-a] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l x \mid x \in D_{\text{dif}}^{K_m,+}, k \in$

$[a, b]$  in  $D_{\text{Sen}}^{K_m}$  generates  $D_{\text{Sen}}^{K_m}$  which implies that  $t^a N \supset D_{\text{dif}}^{K_m, +}$  by Nakayama's lemma. Finally,

$$\begin{aligned} (\gamma_{K_m} - 1)^l t^{-k} \prod_{i \in [a, 2b-a] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l D_{\text{dif}}^{K_m, +} &= t^{-k} \prod_{i \in [a, 2b-a]} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l D_{\text{dif}}^{K_m, +} \\ &\subset t^{b-a+1-k} D_{\text{dif}}^{K_m, +} \quad (\text{Claim A.2.3}) \\ &\subset t^{b+1-k} N \\ &\subset tN \end{aligned}$$

for any  $k \in [a, b]$ . Since  $(\gamma_{K_m} - 1)^l tN \subset tN$  and  $N/tN$  is generated under  $K_m$  by the image of  $t^{-k} \prod_{i \in [a, 2b-a] \setminus \{k\}} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l D_{\text{dif}}^{K_m, +}$ ,  $k \in [a, b]$ , we conclude that  $(\gamma_{K_m} - 1)^l N \subset tN$ .  $\square$

**Lemma A.2.4.** *For any  $k \in \mathbb{Z} \setminus \{0\}$ , there exists a polynomial  $f_k(T) \in K[T]$  such that  $f_k(T) = (T - 1)^l h_k(T) + 1$  and  $f_k(T) = (\epsilon(\gamma_{K_m})^{-k} T - 1)^l g_k(T)$  for some polynomials  $h_k(T), g_k(T) \in K[T]$ .*

*Proof.* The ideal  $((T - 1)^l, (\epsilon(\gamma_{K_m})^{-k} T - 1)^l) = (1)$  by Hilbert's Nullstellensatz since  $\epsilon(\gamma_{K_m})^k \neq 1$  for  $k \neq 0$ . Hence we can find  $h_k(T), g_k(T) \in K[T]$  such that  $(\epsilon(\gamma_{K_m})^{-k} T - 1)^l g_k(T) - (T - 1)^l h_k(T) = 1$ .  $\square$

**Lemma A.2.5.** *Assume that  $A$  is a finite extension of  $L$  and all the Sen weights (the roots of the Sen polynomial) of  $D_A$  are in  $\mathbb{Z}$  and  $m > m_0$ . Then we have*

$$\begin{aligned} \left( D_{\text{dif}}^{K_m}(D_A) \otimes_{K_m} K_m[\log(t)] \right)^{\Gamma_K} &= D_{\text{pdR}}(W_{\text{dR}}(D_A)), \\ \left( D_{\text{dif}}^{K_m}(D_A) \otimes_{K_m} K_m[\log(t)] \right)^{\Gamma_{K_m}} &= K_m \otimes_K D_{\text{pdR}}(W_{\text{dR}}(D_A)). \end{aligned}$$

*Proof.* By definition,  $W_{\text{dR}}^+(D_A) = D_A^r \otimes_{\mathcal{R}_{\mathbb{Q}_p, K, \ell_m}^r} B_{\text{dR}}^+$  (the  $m$  here is large enough since  $\varphi^* D_A^r \simeq D_A^{r/p}$ , cf. [Ber08a, Prop. 2.2.6(2)]) is a free  $A \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+$ -module of rank  $n$ . The map  $B_{\text{dR}}^+ \otimes_{K_m[[t]]} D_{\text{dif}}^{K_m, +}(D_A) \rightarrow W_{\text{dR}}^+(D_A)$  induced by the injection  $K_m[[t]] \hookrightarrow B_{\text{dR}}^+$  is a  $\mathcal{G}_K$ -equivariant isomorphism and we have a similar statement for  $D_{\text{dif}}^{K_\infty, +}(D_A) = K_\infty \otimes_{K_m} D_{\text{dif}}^{+, K_m}(D_A)$ . Now let  $X_f$  be the  $K_\infty[[t]]$ -module associated with the  $B_{\text{dR}}^+$ -representation  $W_{\text{dR}}^+(D_A)$  in [Fon04, Thm. 3.6]. Then  $X_f$  contains  $D_{\text{dif}}^{K_\infty, +}(D_A)$  by *loc. cit.* (our convention on  $\Gamma_K$  differs from that in [Fon04], however the results in *loc. cit.* apply if we firstly replace  $K$  by some finite extension in  $K_\infty$  and then descent if needed). Modulo  $t$  and by definition,  $X_f/t = ((W_{\text{dR}}^+(D_A)/t)^{H_K})_f = ((C \otimes_{K_\infty} D_{\text{Sen}}^{K_\infty}(D_A))^{H_K})_f$ , where  $C \otimes_{K_\infty} D_{\text{Sen}}^{K_\infty}(D_A)$  is a semi-linear  $C$ -representation of  $\mathcal{G}_K$  on which  $\mathcal{G}_K$  acts on  $D_{\text{Sen}}^{K_\infty}(D_A)$  via  $\Gamma_K$  and  $((C \otimes_{K_\infty} D_{\text{Sen}}^{K_\infty}(D_A))^{H_K})_f$  denotes the union of all finite-dimensional  $K$ -subspace that is stable under  $\Gamma_K$  of  $C \otimes_{K_\infty} D_{\text{Sen}}^{K_\infty}(D_A)$  as in [Fon04, Thm. 2.4]. Thus  $((C \otimes_{K_\infty} D_{\text{Sen}}^{K_\infty}(D_A))^{H_K})_f$  contains  $D_{\text{Sen}}^{K_\infty}(D_A)$  and has the same  $K_\infty$ -rank with  $D_{\text{Sen}}^{K_\infty}(D_A)$ . Hence  $X_f/t$  is identified with  $D_{\text{Sen}}^{K_\infty}(D_A)$  in  $W_{\text{dR}}^+(D_A)/t$ . Then the inclusion  $D_{\text{dif}}^{K_\infty, +}(D_A) \hookrightarrow X_f$  is a surjection. Hence  $X_f = D_{\text{dif}}^{K_\infty, +}(D_A)$  in  $W_{\text{dR}}^+(D_A)$  and  $\Delta_{\text{dR}}(W_{\text{dR}}(D_A)) = D_{\text{dif}}^{K_\infty}(D_A) = D_{\text{dif}}^{K_\infty, +}(D_A)[\frac{1}{t}]$  in the notation of [Fon04, Thm. 3.12].

Now by the assumption on Sen weights, the  $A \otimes_{\mathbb{Q}_p} B_{\text{dR}}$ -representation  $W_{\text{dR}}(D_A)$  is almost de Rham. We have

$$D_{\text{pdR}}(W_{\text{dR}}(D_A)) = (W_{\text{dR}}(D_A) \otimes_{B_{\text{dR}}} B_{\text{dR}}[\log(t)])^{\mathcal{G}_K} = \left( (W_{\text{dR}}(D_A) \otimes_{B_{\text{dR}}} B_{\text{dR}}[\log(t)])^{H_K} \right)^{\Gamma_K}.$$

If  $a \in D_{\text{dif}}^{K_\infty}(D_A)$  such that  $\nabla^N(a) = 0$  for some  $N$ , then the sub- $K_\infty$ -space spanned by  $\nabla^i(a)$  is finite-dimensional and stable under  $\nabla$ . Thus  $(D_{\text{dif}}^{K_\infty}(D_A) \otimes_{K_\infty} K_\infty[\log(t)])^{\nabla=0} =$

$D_{\mathrm{dR},\infty}(W_{\mathrm{dR}}(D_A))$  by [Fon04, Prop. 3.25]. Since  $W_{\mathrm{dR}}(D_A)$  is almost de Rham,  $D_{\mathrm{dR},\infty}(W_{\mathrm{dR}}(D_A)) = K_\infty \otimes_K D_{\mathrm{pdR}}(W_{\mathrm{dR}}(D_A))$  by [Fon04, §3.6]. The identification

$$K_\infty \otimes_K D_{\mathrm{pdR}}(W_{\mathrm{dR}}(D_A)) = \left( D_{\mathrm{dif}}^{K_\infty}(D_A) \otimes_{K_\infty} K_\infty[\log(t)] \right)^{\nabla=0} \quad (\text{A.2.6})$$

of  $K_\infty$ -subspaces in  $(W_{\mathrm{dR}}(D_A) \otimes_{B_{\mathrm{dR}}} B_{\mathrm{dR}}[\log(t)])^{H_K}$  is  $\Gamma_K$ -equivariant. Since  $m \geq m_0$ , eigenvalues of  $\gamma_{K_m} = \exp(\log(\epsilon(\gamma_{K_m}))\nabla)$  on  $D_{\mathrm{Sen}}^{K_m}(D_A)$  are of the form  $\epsilon(\gamma_{K_m})^i$  for  $i \in \mathbb{Z}$ . By Proposition A.2.2, the dimension of  $\left( D_{\mathrm{dif}}^{K_m}(D_A) \otimes_{K_m} K_m[\log(t)] \right)^{\Gamma_{K_m}}$  over  $K_m$  is equal to  $n[A : \mathbb{Q}_p]$ . By Hilbert 90, the dimension of the  $K$ -space  $\left( D_{\mathrm{dif}}^{K_m}(D_A) \otimes_{K_m} K_m[\log(t)] \right)^{\Gamma_K}$  is equal to  $n[A : \mathbb{Q}_p]$ . Taking  $\Gamma_K$ -invariants on the two sides of (A.2.6) and counting dimensions, we get  $D_{\mathrm{pdR}}(W_{\mathrm{dR}}(D_A)) = \left( D_{\mathrm{dif}}^{K_\infty}(D_A) \otimes_{K_\infty} K_\infty[\log(t)] \right)^{\Gamma_K} = \left( D_{\mathrm{dif}}^{K_m}(D_A) \otimes_{K_m} K_m[\log(t)] \right)^{\Gamma_K}$ .  $\square$

We may consider some more general cases. Suppose that  $A$  is a local Artinian  $L$ -algebra with residue field  $L'$  finite over  $L$ . We assume that all the  $\tau$ -Sen weights of  $D_{L'}$  are integers for a fixed  $\tau \in \Sigma$  which means that we do not require other Sen weights to be integers. As a  $(\varphi, \Gamma_K)$ -module over  $\mathcal{R}_{L,K}$ , the  $\tau$ -Sen weights of  $D_A$  are all integers since  $D_A$  is a successive extension of  $D_{L'}$ . Recall

$$D_{\mathrm{pdR},\tau}(W_{\mathrm{dR}}(D_A)) := D_{\mathrm{pdR}}(W_{\mathrm{dR}}(D_A)) \otimes_{A \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} A.$$

Thus

$$\begin{aligned} D_{\mathrm{pdR},\tau}(W_{\mathrm{dR}}(D_A)) &= \left( \left( D_{\mathrm{dif}}^{K_\infty}(D_A) \otimes_{A \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} A \right) \otimes_{K_\infty} K_\infty[\log(t)] \right)^{\Gamma_K} \\ &= \left( D_{\mathrm{dif},\tau}^{K_\infty}(D_A) \otimes_{K_\infty} K_\infty[\log(t)] \right)^{\Gamma_K} \end{aligned}$$

as both the actions of  $A$  and  $K$  commute with  $\Gamma_K$ . By Proposition A.2.2 and Hilbert 90, we get

$$D_{\mathrm{pdR},\tau}(W_{\mathrm{dR}}(D_A)) = \left( D_{\mathrm{dif},\tau}^{K_m}(D_A) \otimes_{K_m} K_m[\log(t)] \right)^{\Gamma_K}$$

has  $L$ -rank  $n \dim_L(A)$  if  $m > m_0$ . An argument of [BHS19, Lem. 3.1.4] shows that  $D_{\mathrm{pdR},\tau}(W_{\mathrm{dR}}(D_A))$  is flat over  $A$  and thus free of rank  $n$  over  $A$ .

### A.3 A family of almost de Rham $(\varphi, \Gamma_K)$ -modules

We assume that  $A$  is an affinoid algebra over  $L$  and show that  $D_{\mathrm{pdR}}(W_{\mathrm{dR}}(D_x))$ ,  $x \in \mathrm{Sp}(A)$  form a family under certain condition.

We take  $m > m_0$  and fix  $\tau \in \Sigma$ . After possibly shrinking  $A$ , we assume that  $D_{\mathrm{dif},\tau}^{K_m,+}(D_A)$  is a free  $A \hat{\otimes}_K K_m[[t]]$ -module of rank  $n$  with a continuous semilinear action of  $\Gamma_K$ . We assume that for any point  $x \in \mathrm{Sp}(A)$ , the  $\tau$ -Sen weights, by definition the eigenvalues of  $\nabla$  on  $D_{\mathrm{Sen},\tau}^{K_m}(D_x)$ , are integers and lie in  $[a, b]$  where  $a, b \in \mathbb{Z}$  is independent of  $x$ . If  $A_0$  is a finite-dimensional local  $L$ -algebra with a morphism  $A \rightarrow A_0$ , then  $D_{\mathrm{pdR},\tau}(W_{\mathrm{dR}}(D_{A_0}))$  is a finite free  $A_0$ -module of rank  $n$  equipped with an  $A_0$ -linear nilpotent operator  $\nu_{A_0}$  by discussions in the end of last section.

**Lemma A.3.1.** *If the  $\tau$ -Sen polynomial of  $D_{\mathrm{Sen},\tau}^{K_m}(D_A)$  is  $\prod_{i \in I} (T - a_i)^{n_i} \in \mathbb{Z}[T]$  where  $a_i$  are integers and  $a_i \neq a_j$  if  $i \neq j$ , then we have  $\prod_{i \in I} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^{a_i})^{n_i} D_{\mathrm{dif},\tau}^{K_m,+}(D_A) \subset t D_{\mathrm{dif},\tau}^{K_m,+}(D_A)$ .*

*Proof.* By the Cayley-Hamilton theorem,  $\prod_i (\nabla - a_i)^{n_i} = 0$  on  $D_{\mathrm{Sen},\tau}^{K_m}(D_A)$ . Since the polynomials  $(T - a_i)^{n_i}$ ,  $i \in I$  are prime to each other in  $\mathbb{Q}[T]$ , we can focus on each generalized eigenspace

after possibly shrinking  $\text{Spec}(A)$  and reduce to the case when  $\nabla$  is nilpotent on  $D_{\text{Sen},\tau}^{K_m}(D_A)$ , i.e.  $\nabla^n = 0$  on  $D_{\text{Sen},\tau}^{K_m}(D_A)$ . Since  $m > m_0$ ,  $\gamma_{K_m} = \exp(\log(\epsilon(\gamma_{K_m}))\nabla) = \sum_{i=0}^{\infty} \frac{(\log(\epsilon(\gamma_{K_m}))\nabla)^i}{i!}$  on  $D_{\text{Sen},\tau}^{K_m}(D_A)$ . We get that  $(\gamma_{K_m} - 1)^n = 0$  on  $D_{\text{Sen},\tau}^{K_m}(D_A)$ .  $\square$

**Lemma A.3.2.** *There exists an integer  $l_0$  such that  $\prod_{i \in [a,b]} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^{l_0} D_{\text{dif},\tau}^{K_m,+}(D_A) \subset tD_{\text{dif},\tau}^{K_m,+}(D_A)$ .*

*Proof.* Let  $J$  be the nilradical of  $A$ . We have  $D_{\text{dif},\tau}^{K_m,+}(D_{A/J}) = D_{\text{dif},\tau}^{K_m,+}(D_A)/JD_{\text{dif},\tau}^{K_m,+}(D_A)$ . The  $(\varphi, \Gamma_K)$ -module  $D_{A/J}$  satisfies the condition in Lemma A.3.1 at least on each connected component of  $\text{Spec}(A/J)$ . Thus there exists an integer  $l'$  such that

$$\prod_{i \in [a,b]} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^{l'} D_{\text{dif},\tau}^{K_m,+}(D_A) \subset tD_{\text{dif},\tau}^{K_m,+}(D_A) + JD_{\text{dif},\tau}^{K_m,+}(D_A).$$

Since  $A$  is Noetherian,  $J$  is finitely generated and there is an integer  $N$  such that  $J^N = 0$ . Then we can take  $l_0 = Nl'$ .  $\square$

**Lemma A.3.3.** *Let  $A$  be a Noetherian ring over  $\mathbb{Q}_p$  and  $J$  be its nilradical. If  $P(T) \in A[T]$  is a polynomial such that the image of  $P(T)$  in  $A/J[T]$  is in  $\mathbb{Q}_p[T]$  and has no factor  $(T-1)$  in  $\mathbb{Q}_p[T]$ , then for any  $l \geq 1$ , there exist  $G_1(T), G_2(T) \in A[T]$  such that  $1 = G_1(T)P(T) + G_2(T)(T-1)^l$  in  $A[T]$ .*

*Proof.* Denote by  $\bar{P}(T)$  the image of  $P(T)$  in  $\mathbb{Q}_p(T) \subset A/J[T]$ . By Hilbert's Nullstellensatz, there exist  $G'_1(T), G'_2(T) \in \mathbb{Q}_p[T]$  such that  $1 = G'_1(T)\bar{P}(T) + G'_2(T)(T-1)^l$ . Thus  $G'_1(T)P(T) + G'_2(T)(T-1)^l = 1 - H(T)$  where  $H(T) \in J[T]$ . There exists  $N$  such that  $H(T)^N = 0$ . Hence  $1 - H(T)$  is invertible in  $A[T]$ . We let  $G_i(T) = G'_i(T)(1 - H(T))^{-1}$  for  $i = 1, 2$ .  $\square$

The proof of the following proposition follows that of [BC08, Thm. 5.3.2].

**Proposition A.3.4.** *Assume that for every  $x \in \text{Sp}(A)$ , all the roots of the  $\tau$ -Sen polynomial of  $D_x$  are in  $\mathbb{Z} \cap [a, b]$ . Then there exists a finite projective  $A$ -module  $D_{\text{pdR},\tau}(W_{\text{dR}}(D_A))$  of rank  $n$  equipped with a nilpotent  $A$ -linear operator  $\nu_A$  such that there is a natural isomorphism of pairs*

$$(D_{\text{pdR},\tau}(W_{\text{dR}}(D_A)) \otimes_A A_0, \nu_A \otimes_A A_0) \simeq (D_{\text{pdR},\tau}(W_{\text{dR}}(D_{A_0})), \nu_{A_0})$$

for any finite-dimensional local  $L$ -algebra and map  $\text{Sp}(A_0) \rightarrow \text{Sp}(A)$ .

*Proof.* For  $k \geq 1$ , let

$$\beta_k = \left( \prod_{j=b-a+1}^{b-a+k} f_j(\gamma_{K_m}) \right) \prod_{i=a-b, i \neq 0}^{b-a} \left( \frac{\gamma_{K_m} - \epsilon(\gamma_{K_m})^i}{1 - \epsilon(\gamma_{K_m})^i} \right)^l$$

where  $f_j(T)$  is chosen by Lemma A.2.4 and  $l = (b-a+1)l_0$  is determined by Lemma A.3.2. Since  $m > m_0$ , by Lemma A.3.2 and the argument for Claim A.2.3,

$$\prod_{i=a-b}^{b-a+k} (\gamma_{K_m} - \epsilon(\gamma_{K_m})^i)^l t^{-b} D_{\text{dif},\tau}^{K_m,+}(D_A) \subset t^{1+k-a} D_{\text{dif},\tau}^{K_m,+}(D_A)$$

for  $k \geq 0$ . Since  $(T - \epsilon(\gamma_{K_m})^{b-a+i})^l \mid f_{b-a+i}(T)$  for any  $i \geq 1$ , we get that if  $x \in t^{-b} D_{\text{dif},\tau}^{K_m,+}(D_A)$ , then  $(\gamma_{K_m} - 1)^l \beta_k(x) \in t^{k+1-a} D_{\text{dif},\tau}^{K_m,+}(D_A)$ . Hence if  $x \in t^{-b} D_{\text{dif},\tau}^{K_m,+}(D_A)$ ,  $\beta_k(x) - \beta_{k+1}(x) \in t^{k+1-a} D_{\text{dif},\tau}^{K_m,+}(D_A)$  by the condition in Lemma A.2.4. Thus  $\beta := \lim_{k \rightarrow +\infty} \beta_k$  defines an  $A \otimes_K$

$K_m$ -linear map  $t^{-b}D_{\text{dif},\tau}^{K_m,+}(D_A) \rightarrow t^{-b}D_{\text{dif},\tau}^{K_m,+}(D_A)$ . Let  $M_A^{K_m}$  be the image of  $\beta$ . Then  $M_A^{K_m}$  is stable under the action of  $\Gamma_K$  and  $(\gamma_{K_m} - 1)^l M_A^{K_m} = 0$ . Since the map  $\left(\prod_{i=a-b, i \neq 0}^{b-a} \frac{\gamma_{K_m} - \epsilon(\gamma_{K_m})^i}{1 - \epsilon(\gamma_{K_m})^i}\right)^l$  is an isomorphism on  $M_A^{K_m}$  and  $f_{b-a+k}(\gamma_{K_m}) = (\gamma_{K_m} - 1)^l h_{b-a+k}(\gamma_{K_m}) + 1$  is the identity on  $M_A^{K_m}$  for  $k \geq 1$ ,  $\beta$  induces an automorphism of  $M_A^{K_m}$ . The image of the characteristic polynomial of  $\gamma_{K_m} = \exp(\log(\epsilon(\gamma_{K_m}))\nabla)$  on  $t^{-k}D_{\text{dif},\tau}^{K_m,+}(D_A)/t^{-k+1}D_{\text{dif},\tau}^{K_m,+}(D_A)$  in  $A^{\text{red}}[T]$  is prime to  $(T - 1)^l$  if  $k \notin [a, b]$  (the roots have the form  $\epsilon(\gamma_{K_m})^{i-k}$  for some  $i \in [a, b]$ ). Thus by Lemma A.3.3, we know there exists no non-zero  $(\gamma_{K_m} - 1)$ -nilpotent element in  $t^{-k}D_{\text{dif},\tau}^{K_m,+}(D_A)/t^{-k+1}D_{\text{dif},\tau}^{K_m,+}(D_A)$  for any  $k \notin [a, b]$ . Hence  $M_A^{K_m} \cap t^{1-a}D_{\text{dif},\tau}^{K_m,+}(D_A) = \{0\}$ . We get that the natural  $A \otimes_K K_m$ -module morphism  $M_A^{K_m} \rightarrow t^{-b}D_{\text{dif},\tau}^{K_m,+}(D_A)/t^{-a+1}D_{\text{dif},\tau}^{K_m,+}(D_A)$  is an injection. The decomposition

$$\beta : M_A^{K_m} \hookrightarrow t^{-b}D_{\text{dif},\tau}^{K_m,+}(D_A)/t^{-a+1}D_{\text{dif},\tau}^{K_m,+}(D_A) \xrightarrow{\beta} M_A^{K_m}$$

implies that  $M_A^{K_m}$  is a finite projective  $A \otimes_K K_m$ -module as a direct summand of a finite free  $A \otimes_K K_m$ -module. For any finite-dimensional local Artinian  $L$ -algebra  $A_0$  with  $\text{Sp}(A_0) \rightarrow \text{Sp}(A)$ ,  $\beta$  also acts on  $t^{-b}D_{\text{dif},\tau}^{K_m,+}(D_{A_0}) = t^{-b}D_{\text{dif},\tau}^{K_m,+}(D_A) \otimes_A A_0$  by extending the scalars since  $D_{\text{dif},\tau}^{K_m,+}(D_{A_0}) = D_{\text{dif},\tau}^{K_m,+}(D_A) \otimes_A A_0$  satisfies the same result as  $A$  for the same  $l_0$  in Lemma A.3.2. We have that  $\beta$  is also an automorphism on  $M_{A_0}^{K_m}$  which is defined to be the  $(\gamma_{K_m} - 1)$ -nilpotent elements in  $D_{\text{dif},\tau}^{K_m}(D_{A_0})$  (which is contained in  $t^{-b}D_{\text{dif},\tau}^{K_m,+}(D_{A_0})$  by Proposition A.2.2). The image of  $\beta$  on  $t^{-b}D_{\text{dif},\tau}^{K_m}(D_{A_0})$  is  $M_{A_0}^{K_m} \otimes_A A_0$ . Hence  $M_A^{K_m} \otimes_A A_0$  contains and thus is equal to  $M_{A_0}^{K_m}$  since  $M_A^{K_m} \otimes_A A_0$  is  $(\gamma_{K_m} - 1)$ -nilpotent.

The  $A \otimes_K K_m$ -linear action of  $\nabla = \log(\gamma_{K_m})/\log(\epsilon(\gamma_{K_m}))$  on  $M_A^{K_m}$  is defined since  $\gamma_{K_m} - 1$  is nilpotent on  $M_A^{K_m}$ . We now set  $D_{\text{pdR},\tau}^{K_m}(W_{\text{dR}}(D_A)) := \left(M_A^{K_m} \otimes_{K_m} K_m[\log(t)]\right)^{\Gamma_{K_m}}$  which is equipped with an  $A \otimes_K K_m$ -linear nilpotent operator  $\nu_A^{K_m}$  via  $\nu_{\text{pdR}}$  in the usual way. By the same method in the proof of Lemma A.2.1, there is an isomorphism

$$\left(D_{\text{pdR},\tau}^{K_m}(W_{\text{dR}}(D_A)), \nu_A^{K_m}\right) \simeq \left(M_A^{K_m}, \nabla\right)$$

of  $A \otimes_K K_m$ -modules with nilpotent operators which is *not* compatible with the action of  $\Gamma_K/\Gamma_{K_m}$ . For each  $A_0$  as before, via the identification in Lemma A.2.1,  $M_{A_0}^{K_m}$  is identified with

$$\left(M_{A_0}^{K_m} \otimes_{K_m} K_m[\log(t)]\right)^{\Gamma_{K_m}} = \left(D_{\text{dif},\tau}^{K_m}(D_{A_0}) \otimes_{K_m} K_m[\log(t)]\right)^{\Gamma_{K_m}} = D_{\text{pdR},\tau}(W_{\text{dR}}(D_{A_0})) \otimes_K K_m$$

by Lemma A.2.5 since  $m > m_0$ . Hence

$$(D_{\text{pdR},\tau}^{K_m}(W_{\text{dR}}(D_A)) \otimes_A A_0, \nu_A^{K_m} \otimes_A A_0) \simeq (D_{\text{pdR},\tau}(W_{\text{dR}}(D_{A_0})) \otimes_K K_m, \nu_{A_0} \otimes_K K_m)$$

which is compatible with the action of  $\Gamma_K$ .

Now the result follows from the descent in [BC08, Prop. 2.2.1] by setting

$$(D_{\text{pdR},\tau}(W_{\text{dR}}(D_A)), \nu_A) := \left(D_{\text{pdR},\tau}^{K_m}(W_{\text{dR}}(D_A))^{\Gamma_K/\Gamma_{K_m}}, \nu_A^{K_m}|_{D_{\text{pdR},\tau}(W_{\text{dR}}(D_A))}\right)$$

which is also isomorphic to  $\left((M_A^{K_m})^{(\gamma_K - 1)\text{-nil}}, \nabla\right)$  by the same arguments in Lemma A.2.1.  $\square$



## Appendix B

# Connected fibers of Bott-Samelson-Demazure type resolutions

Recall in the setting of §3.2, we have a reductive group  $G$  over a field  $k$  and a standard parabolic subgroup  $P$  of  $G$ . For  $w$  an element in the Weyl group  $W$  of  $G$ , we defined a scheme  $Y = Y_B$  as well as  $Y_P$  in the beginning of §3.2.1. We use  $W_P$  to denote the Weyl group of the standard Levi subgroup of  $P$ . Then we have varieties  $Y_w$  or  $Y_{P,w}$  (as well as  $X_w, X_{P,w}$ ) for  $w \in W$  or  $w \in W/W_P$ . We know  $Y_w$  are normal for all  $w$  by [BHS19]. This appendix will prove that for  $w \in W$ , the fibers of the natural map  $Y_w \rightarrow Y_{P,w}$  are connected. This will imply that  $Y_{P,w}$  are unibranch at all the points, as claimed in Remark 3.2.15. We will use notation of §3.2.

### B.1 Example of simple reflections

In this computational section, we compute  $Y_{P,w}$  when  $w = s_0 = s_{\alpha_0} \notin W_P$  is a simple reflection corresponding to a positive simple root  $\alpha_0$  of  $B$ . This is already done in [Ric08, §1.4] but the result is crucial for the sequel, thus we would like to repeat it in our notations. We may assume that  $G$  is semisimple and simply-connected.

Let  $f : Y_P \rightarrow G/P$  be the natural projection. Notice that the closed subscheme  $g^{-1}(\overline{Bs_0P/P})$  is covered by locally closed subschemes  $V_{P,s_0}^Y$  and  $V_{P,e}^Y$  (defined after (3.2.1)), where  $e$  is the identity element in  $W$ . Hence  $f^{-1}(\overline{Bs_0P/P}) = Y_{P,e} \cup Y_{P,s_0}$ . The projective line  $\overline{Bs_0P/P}$  is covered by two affine lines  $U_{-\alpha_0}P/P$  and  $s_0U_{-\alpha_0}P/P$  where  $U_{-\alpha_0}$  is given by [Spr10, Prop. 8.1.1].

We use variable  $x$  to denote the coordinate of the line  $U_{-\alpha_0}P/P$  via  $u_{-\alpha_0} : \text{Spec}(k[x]) \xrightarrow{\sim} U_{-\alpha_0}$ . Choose  $e_\alpha$  for some basis of the root spaces corresponding to roots  $\alpha$  in the Lie algebra  $\mathfrak{g}$  and  $\omega_\alpha \in \mathfrak{t}$  be fundamental coweights such that  $\beta(\omega_\alpha) = \delta_{\alpha\beta}$  for positive simple roots  $\alpha, \beta \in \Delta$  (cf. [Hum08, §0.6]). Then  $\mathfrak{b} = \bigoplus_{\alpha \in \Delta} k\omega_\alpha \oplus \bigoplus_{\alpha \in R^+} ke_\alpha$ . Let variable  $t_\alpha, y_\beta$  denote the coefficients of elements  $b = \sum_{\alpha \in \Delta} t_\alpha \omega_\alpha + \sum_{\alpha \in R^+} y_\alpha e_\alpha \in \mathfrak{b}$  such that the coordinate ring  $R$  of  $\mathfrak{b}$  is  $k[t_\alpha, y_\beta \mid \alpha \in \Delta, \beta \in R^+]$ .

If  $\beta$  is a positive root and  $\beta \neq \alpha_0$ , we have  $\text{Ad}(u_{-\alpha_0}(x))e_\beta \in \bigoplus_{j \in \mathbb{N}, \beta - j\alpha_0 \in R} Re_{\beta - j\alpha_0}$  for any  $x \in R$  (cf. [Spr10, §8.2.3]). If  $\alpha \in \Delta$  and  $\alpha \neq \alpha_0$ , then  $\text{Ad}(u_{-\alpha_0}(-x))\omega_\alpha = \omega_\alpha$  as  $[\omega_\alpha, e_{-\alpha_0}] = 0$  since  $\alpha_0(\omega_\alpha) = 0$ . Hence if  $\beta \in R^+ \setminus \{\alpha_0\}$ , then  $\text{Ad}(u_{-\alpha_0}(-x))e_\beta \in \mathfrak{b} \subset \mathfrak{g}$  (write  $\beta$  as a linear combination of positive simple roots with non-negative coefficients, then the coefficients of  $\beta - j\alpha_0$  in basis  $\Delta$  are all non-positive only if  $\beta \in \mathbb{Z}\alpha$  since the root system is reduced, see [Mil17, Thm. 21.11(b)]). Considering the subgroup  $G_\alpha \simeq \text{SL}_2$  corresponding to  $\alpha$  as in [Spr10, §7.1, §8.1.1], one computes that  $\text{Ad}(u_{-\alpha_0}(-x))e_{\alpha_0} = e_{\alpha_0} + xh_{\alpha_0} - x^2e_{-\alpha_0}$  where  $h_{\alpha_0} \in \mathfrak{t}$  such



that  $\alpha(h_{\alpha_0}) = 2$  (as in [Hum08, §0.1]). Moreover  $\text{Ad}(u_{-\alpha_0}(-x))h_{\alpha_0} = h_{\alpha_0} - \alpha(h_{\alpha_0})e_{-\alpha_0}$ . Hence,  $\text{Ad}(u_{-\alpha_0}(-x))\omega_{\alpha_0} = \omega_{\alpha_0} - \alpha(\omega_{\alpha_0})e_{-\alpha_0} = \omega_{\alpha_0} - e_{-\alpha_0}$ . Hence if  $b = \sum_{\alpha \in \Delta} t_{\alpha}\omega_{\alpha} + \sum_{\alpha \in R^+} y_{\alpha}e_{\alpha}$ , the element  $\text{Ad}(u_{-\alpha_0}(-x))(b)$  is in  $\mathfrak{b}$  if and only if  $-xt_{\alpha_0} - x^2y_{\alpha_0} = 0$ .

Therefore the open subscheme  $f^{-1}(U_{-\alpha_0}P/P)$  is isomorphic to the closed subscheme of  $\text{Spec}(R \otimes_k k[x])$  defined by the ideal  $(-xt_{\alpha_0} - x^2y_{\alpha_0})$ . There are two irreducible components of  $f^{-1}(U_{-\alpha_0}P/P)$ :  $x = 0$  and  $y_{\alpha_0}x + t_{\alpha_0} = 0$ , where the component  $x = 0$  is an open dense subscheme of  $Y_{P,e}$  and the component  $y_{\alpha_0}x + t_{\alpha_0} = 0$  is an open dense subscheme of  $Y_{P,s_0}$ . One may also consider  $f^{-1}(s_0U_{-\alpha_0}P/P)$  and find that  $Y_{P,s_0} \simeq \text{Proj}(R[x, y]/(y_{\alpha_0}x + t_{\alpha_0}y))$  the blowup of  $\mathfrak{b}$  with respect to the ideal  $(y_{\alpha_0}, t_{\alpha_0})$ . The closed subscheme  $Y_{P,s_0} \setminus V_{P,s_0}^Y = Y_{P,s_0} \cap Y_{P,e}$  is the preimage of the point  $P/P \in \overline{Bs_0P/P}$  under the map  $f|_{Y_{P,s_0}}$ .

We see  $Y_{P,s_0} \cap Y_{P,e} = \{(b, gP) \in Y_P \mid t_{\alpha_0} = 0, gP = P\}$  and  $Y_{P,s_0} \cap \{(b, gP) \in Y_P \mid t_{\alpha_0} = 0\} = \{(b, gP) \in Y_P \mid t_{\alpha_0} = 0, gP \in \overline{Bs_1P/P}\}$ . In more down to earth description, assume that the  $K$ -point  $(\sum_{\alpha \in \Delta} t_{\alpha}\omega_{\alpha} + \sum_{\alpha \in R^+} y_{\alpha}e_{\alpha}, gP) \in (\mathfrak{b} \times \overline{Bs_0P/P})(K)$  is in  $Y_{P,s_0}(K)$  for some integral domain  $K$ , then we must be in one of the following cases:

1.  $t_{\alpha_0} \neq 0$  and  $y_{\alpha_0} \neq 0$ , then  $gP \in \overline{Bs_0P/P}(K)$  is uniquely determined by  $b$  and we can write  $gP = u_{-\alpha_0}(t_{\alpha_0}/y_{\alpha_0})P$ ;
2.  $t_{\alpha_0} \neq 0$  and  $y_{\alpha_0} = 0$ , then  $gP = s_0P$ ;
3.  $t_{\alpha_0} = 0$  and  $y_{\alpha_0} \neq 0$ , then  $gP = P$ ;
4.  $t_{\alpha_0} = 0$  and  $y_{\alpha_0} = 0$ , then  $g$  can be any element in  $\overline{Bs_0P/P}$ .

One can check directly that we have the following lemma.

**Lemma B.1.1.** *If  $(\nu, gP) \in Y_{P,s_0}$ , then  $\text{Ad}(g^{-1})\nu$  shares the same Levi factor with  $\text{Ad}(\dot{s}_0^{-1})\nu$ .*

*Proof.* Note that in general this is a consequence of [BHS19, Lem. 2.3.4].  $\square$

## B.2 The resolutions

Let  $\text{pr}' : X = \tilde{\mathfrak{g}} \times_{\tilde{\mathfrak{g}}} \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$  be the projection to the second factor and denote its restriction to  $X_w$  by  $\text{pr}'_w := \text{pr}'|_{X_w} : X_w \rightarrow \tilde{\mathfrak{g}}$ . By symmetry the morphism  $\text{pr}'_w$  is also proper and birational. Now assume  $w \in W$  and  $w = s_1 \cdots s_m$  is a reduced expression of  $w$  with  $s_1, \dots, s_m$  simple reflections. We define

$$X(s_1, \dots, s_m) := X_{s_1} \times_{\tilde{\mathfrak{g}}} \cdots \times_{\tilde{\mathfrak{g}}} X_{s_m},$$

where the morphisms in the fiber products are given in the following diagram:

$$\begin{array}{ccccc} X_{s_1} & & \cdots & & X_{s_n} \\ & \searrow \text{pr}'_{s_1} & & \searrow \text{pr}'_{s_n} & \\ & & \tilde{\mathfrak{g}} & & \tilde{\mathfrak{g}} \end{array}$$

Then the scheme  $X(s_1, \dots, s_m)$  can also be described as

$$X(s_1, \dots, s_m) = \{(\nu, g_0, \dots, g_m) \in \mathfrak{g} \times G/B \times \cdots \times G/B \mid (\nu, g_{i-1}, g_i) \in X_{s_i}, \forall 1 \leq i \leq m\}.$$

We also define a scheme  $Y'(s_1, \dots, s_m) :=$

$$\{(\nu, g_1, \dots, g_m) \in (\mathfrak{b} \times \overline{Bs_1B} \times \cdots \times \overline{Bs_mB}) \mid (\text{Ad}((g_1 \cdots g_{i-1})^{-1})\nu, g_iB) \in Y_{s_i}, \forall 1 \leq i \leq m\},$$

where we set  $g_0 = 1$ . The variety is equipped with a right action of the group  $B^m$ : if  $(b_1, \dots, b_m) \in B^m$ ,  $(\nu, g_1, \dots, g_m)(b_1, \dots, b_m) := (\nu, g_1b_1, \dots, b_{m-1}^{-1}g_mb_m)$ . Then we define

$$Y(s_1, \dots, s_m) := Y'(s_1, \dots, s_m)/B^m$$

be the quotient of  $Y'(s_1, \dots, s_m)$  by the action of  $B^m$ . We need to show that the quotient  $Y(s_1, \dots, s_m)$  is a scheme and we need to introduce the Bott-Samelson-Demazure variety.

Let  $\text{BSD}(s_1, \dots, s_m) := \overline{Bs_1B} \times^B \dots \times^B \overline{Bs_mB/B}$  be the Bott-Samelson-Demazure variety associated with the sequence  $(s_1, \dots, s_m)$  of simple reflections  $W$ . The notation  $\overline{Bs_1B} \times^B \dots \times^B \overline{Bs_mB/B}$  should be understood to be the quotient of the variety  $\overline{Bs_1B} \times \dots \times \overline{Bs_mB}$  by the  $B^m$ -action given by  $(g_1, \dots, g_m)(b_1, \dots, b_m) = (g_1b_1, \dots, b_{m-1}^{-1}gb_m)$ . The quotient map is locally trivial (cf. [Jan07, §13.4]). Recall that  $\text{BSD}(s_1, \dots, s_m)$  is smooth and contains an open dense subscheme  $\text{BSD}^\circ(s_1, \dots, s_m) := Bs_1B \times^B \dots \times^B (Bs_mB/B)$  replacing Bruhat varieties by cells. We have natural morphisms of varieties  $r = r_{(s_1, \dots, s_m)} : \text{BSD}(s_1, \dots, s_m) \rightarrow \overline{BwB}$  sending  $(g_1, \dots, g_m)$  to  $g_1 \dots g_m \in \overline{BwB/B}$  which induces an isomorphism of open subschemes  $Bs_1B \times^B \dots \times^B (Bs_mB/B) \xrightarrow{\sim} BwB/B$  and the product morphisms  $r$  are rational resolutions of the Bruhat varieties (see [Jan07, II.13.6.] and [BK07, §3.4]). Let

$$\partial\text{BSD}(s_1, \dots, s_m) := \text{BSD}(s_1, \dots, s_m) \setminus \text{BSD}^\circ(s_1, \dots, s_m)$$

be the complement of  $\text{BSD}^\circ(s_1, \dots, s_m)$  in  $\text{BSD}(s_1, \dots, s_m)$ . Then  $\partial\text{BSD}(s_1, \dots, s_m)$  is a divisor of  $\text{BSD}(s_1, \dots, s_m)$  and consists of  $m$  irreducible components:

$$\partial\text{BSD}(s_1, \dots, s_m) = \cup_{1 \leq i \leq m} \overline{Bs_1B} \times^B \dots \times^B (\overline{Bs_iB} \setminus Bs_iB) \times^B \dots \times^B \overline{Bs_mB/B}.$$

We have a map  $Y(s_1, \dots, s_m) \rightarrow \mathfrak{b} \times \text{BSD}(s_1, \dots, s_m) : (\nu, g_1, \dots, g_m) \mapsto (\nu, g_1 \dots g_m)$ .

**Lemma B.2.1.** *The morphism  $i : Y(s_1, \dots, s_m) \rightarrow \mathfrak{b} \times \text{BSD}(s_1, \dots, s_m)$  identifies  $Y(s_1, \dots, s_m)$  as a closed subscheme of  $\mathfrak{b} \times \text{BSD}(s_1, \dots, s_m)$ .*

*Proof.* If  $m = 1$ , the results hold by computations in §B.1 and  $Y(s_1)$  is a closed subvariety of  $\mathfrak{b} \times \text{BSD}(s_1)$  locally cut out by one equation. The closed subscheme  $Y'(s_1) \hookrightarrow \mathfrak{b} \times \overline{Bs_1B}$  is also locally cut out by one equation since the projection  $\overline{Bs_1B} \rightarrow \overline{Bs_1B/B}$  is a local trivial bundle. The scheme  $Y'(s_1, \dots, s_m)$  is a closed subscheme of  $\mathfrak{b} \times \overline{Bs_1B} \times \dots \times \overline{Bs_mB}$  cut out by relations  $(\text{Ad}((g_1, \dots, g_{i-1})^{-1})\nu, g_i) \in Y'(s_i), 1 \leq i \leq m$ . As the quotient map  $\mathfrak{b} \times \overline{Bs_1B} \times \dots \times \overline{Bs_mB} \rightarrow \mathfrak{b} \times \text{BSD}(s_1, \dots, s_m)$  is locally trivial, the closed subscheme  $Y'(s_1, \dots, s_m)$  of  $\mathfrak{b} \times \overline{Bs_1B} \times \dots \times \overline{Bs_mB}$  descends to a closed subscheme  $Y(s_1, \dots, s_m)$  and the quotient morphism  $Y'(s_1, \dots, s_m) \rightarrow Y(s_1, \dots, s_m)$  is locally trivial.  $\square$

**Remark B.2.2.**  $Y(s_1, \dots, s_m)$  is not smooth when  $G = \text{GL}_3$  and  $w$  is the longest element in the Weyl group of  $G$ .

The scheme  $Y(s_1, \dots, s_m)$  is equipped with a left  $B$ -action:

$$b(\nu, g_1, \dots, g_m) \mapsto (\text{Ad}(b)\nu, bg_1, \dots, g_i, \dots, g_m)$$

for  $b \in B$  and respectively  $X(s_1, \dots, s_m)$  is equipped with a left  $G$ -action:

$$g(\nu, g_0, \dots, g_m) \mapsto (\text{Ad}(g)\nu, gg_0, \dots, gg_m)$$

for  $g \in G$ . Then the map

$$\begin{aligned} G \times^B Y(s_1, \dots, s_m) &\rightarrow X(s_1, \dots, s_m) \\ (g, (\nu, g_1, \dots, g_m)) &\mapsto (\text{Ad}(g)\nu, g, gg_1, \dots, gg_1 \dots g_m) \end{aligned}$$

is an isomorphism of  $G$ -schemes. Similarly  $X(s_1, \dots, s_m)$  admits an open subvariety

$$X^\circ(s_1, \dots, s_m) := V_{s_1} \times_{\tilde{g}} \dots \times_{\tilde{g}} V_{s_m} \simeq G \times^B Y^\circ(s_1, \dots, s_m).$$

We have natural morphisms  $r^X : X(s_1, \dots, s_m) \rightarrow X$  sending  $(\nu, g_0, \dots, g_m)$  to  $(\nu, g_0, g_m)$  and  $r^Y : Y(s_1, \dots, s_m) \rightarrow Y$  sending  $(\nu, g_1, \dots, g_m)$  to  $(\nu, g_1 \dots g_m)$ .

In the remaining part of the appendix, we will focus on  $Y, Y_w, Y_{P,w}$ , etc. and omit the discussions for  $X, X_w, X_{P,w}$ , etc..

**Lemma B.2.3.** *The restriction of  $r^Y$  to the open subscheme  $Y^\circ(s_1, \dots, s_m)$  induces an isomorphism*

$$Y^\circ(s_1, \dots, s_m) \xrightarrow{\sim} V_{P,w}^Y.$$

*Proof.* We have an isomorphism  $BSD^\circ(s_1, \dots, s_m) \xrightarrow{\sim} BwB/B$  and

$$\begin{aligned} Y^\circ(s_1, \dots, s_m) \\ = \{(\nu, g_1, \dots, g_m) \in \mathfrak{b} \times Bs_1B \times^B \dots \times^B Bs_mB \mid \text{Ad}(g_i^{-1} \dots g_1^{-1})\nu \in \mathfrak{b}, \forall 1 \leq i \leq m\}. \end{aligned}$$

Thus we need to verify that  $\text{Ad}(g_i^{-1} \dots g_1^{-1})\nu \in \mathfrak{b}, \forall 1 \leq i \leq m$ , where  $g_i \in Bs_iB$  for all  $1 \leq i \leq m$ , if and only if  $\text{Ad}((g_1 \dots g_m)^{-1})\nu \in \mathfrak{b}$ . Since  $s_1 \dots s_m$  is a reduced expression of  $w$ , the expressions  $s_1 \dots s_i$  is reduced for all  $i$ . Hence we have  $Bs_1 \dots s_iB = Bs_1B \dots Bs_iB$ . We can then reduce to the case when  $(g_1, \dots, g_m) = (\dot{s}_1, \dots, \dot{s}_m)$ . Assume  $\nu = \sum_{\alpha \in \Delta} t_\alpha \omega_\alpha + \sum_{\beta \in R^+} y_\beta e_\beta$  is in  $\mathfrak{b}$ . Then  $\text{Ad}((\dot{s}_1 \dots \dot{s}_i)^{-1})\nu \in \mathfrak{b}$  if and only if  $y_\beta = 0$  for all  $\beta \in A_i$  where  $A_i := \{\beta \in R^+ \mid s_i \dots s_1(\beta) \in R^-\}$  (cf. [Spr10, 9.2.1]). But we have  $A_1 \subset \dots \subset A_m$  since the number  $|A_i| = i$  is the length of  $s_1 \dots s_i$  (if  $x = A_{i-1}$  is not contained in  $A_i$  and if  $s_i = s_\alpha$  for a positive simple root  $\alpha$ , then  $x = (s_{i-1} \dots s_1)^{-1}(-\alpha) \in R^+$  and  $A_i = A_{i-1} \setminus \{(s_{i-1} \dots s_1)^{-1}(-\alpha)\}$ ).  $\square$

**Proposition B.2.4.** *The morphism  $r$  and  $r^Y$  are proper, and  $r$  (resp.  $r^Y$ ) send  $X(s_1, \dots, s_m)$  (resp.  $Y(s_1, \dots, s_m)$ ) onto  $X_w$ , the closure of  $V_w$  in  $X$  (resp.  $Y_w$  the closure of  $V_w^Y$  in  $Y$ )<sup>1</sup>.*

*Proof.* We prove by induction that the ideal generalized by relations  $(\text{Ad}(g_{k-1}^{-1} \dots g_0^{-1})\nu, g_k) \in Y'(s_k), 1 \leq k \leq i$  is locally generalized by  $i$  elements in  $\mathcal{O}_{\mathfrak{b} \times \overline{Bs_1B} \times \dots \times \overline{Bs_mB}}$  for each  $1 \leq i \leq m$ . Assume the result is true for  $i-1$ , modulo relations given by  $(\text{Ad}(g_{k-1}^{-1} \dots g_0^{-1})\nu, g_k) \in Y'(s_k), 1 \leq k \leq i-1$ , we see  $\nu' := \text{Ad}((g_1, \dots, g_{i-1})^{-1})\nu$  lies in  $\mathfrak{b}$ . Thus we only need to add the relation  $(\nu', g_i) \in Y'(s_i)$  which is locally given by one equation (after modulo the relations  $\leq i-1$ ). Hence  $Y'(s_1, \dots, s_m)$  is a closed subvariety of  $\mathfrak{b} \times \overline{Bs_1B} \times \dots \times \overline{Bs_mB}$  locally cut out by  $m$  equations.

We now prove by induction on the length of  $w$  that the image of  $r$  lies in  $X_w$ . The case  $\text{lg}(w) = 1$  is clear. Now suppose that  $w = s_1 \dots s_m$  is a reduced expression of  $w$  and let  $w' = s_2 \dots s_m, s = s_1$ . Like for  $Y(s_1, \dots, s_m)$ , we can similarly define a variety  $Y(s, w')$  such that  $G \times^B Y(s, w') = X(s, w') := X_s \times_{\tilde{\mathfrak{g}}} X_{w'}$ . Then there exists a morphism  $X(s_2, \dots, s_m) \twoheadrightarrow X_{w'}$  by the induction hypothesis, as well as  $X(s_1, \dots, s_m) \rightarrow X(s_1, w')$ . So we only need to show that the image of  $X(s, w')$  in  $X$ , under the map induced by  $\overline{BsB} \times^B \overline{Bw'B} \rightarrow \overline{BwB}$ , is in  $X_{sw'}$ . The open subset  $X(s, w')^\circ := V_s \times_{\tilde{\mathfrak{g}}} V_{w'}$  is isomorphic to  $V_w$  as in Lemma B.2.3. Since the map  $X(s, w') \rightarrow X$  is proper, we only need to verify that  $X(s, w')^\circ$  is dense in  $X(s, w')$ . The subscheme  $Y(s, w')$  is locally cut out by one equation from  $\overline{Bs_1B} \times^B Y(w')$ , as in the discussion in the beginning of the proof. Moreover  $\overline{Bs_1B} \times^B Y(w')$  is integral of dimension  $\dim \mathfrak{b} + 1$ . Hence all irreducible components of  $Y(s, w')$  have dimension no less than that of  $Y(w')$  by Krull's principal ideal theorem. So we reduced to prove that  $X(s, w') \setminus X(s, w')^\circ$  has strictly less dimension than that of  $X(w')$ . The complement is the union of closed subspaces of  $((\nu, (g_1B, g_2B)), (\nu, (g_2B, g_3B))) \in X(s_1, w')$  where  $g_1B = g_2B$  or  $(\nu', g_2^{-1}g_3B) \in Y_{w'} \setminus V_{w'}^Y$  where  $\nu' = \text{Ad}(g_2)^{-1}\nu$  (for an arbitrary representative  $g_2$  of  $g_2B$ ). Suppose that  $C$  is an irreducible component in the complement where  $g_1B = g_2B$ , then by the example in §B.1,  $\kappa(\nu') \in \mathfrak{t}^s$  where  $\kappa : \mathfrak{b} = \mathfrak{t} + \mathfrak{n} \rightarrow \mathfrak{t}$  is the projection. Hence the intersection of  $C$  with the open dense subset  $V_{w'}$  have dimension less or equal to  $\dim \mathfrak{g} - 1$  by [BHS19, Lem. 2.3.1]. Thus  $C$  has dimension at most  $\dim \mathfrak{g} - 1$ . Now suppose that the image of an irreducible component

<sup>1</sup>We expect that  $Y(s_1, \dots, s_m)$  is irreducible, normal and locally of complete intersection when  $s_1 \dots s_m$  is a reduced expression, however we don't know proof. There was a mistake in the proof of the previous version of this proposition where we claimed the irreducibility. We thank Valentin Hernandez for pointing out it.

$C \subset X(s, w') \setminus X(s, w')^\circ$  satisfies that the image of  $C$  in  $X_{w'}$  is in the complement of  $V_{w'}$  and assume that the image is an irreducible component  $C' \subset X_{w'} \setminus V_{w'}$ . The locus in  $C$  where the dimension jump by one from  $C'$  to  $C$  (the locus where the fibers of  $C \rightarrow C'$  have dimension 1) is in the preimage of the non-regular locus of  $\mathfrak{g}$  by the example in B.1. But by the proof of [BHS19, Thm. 2.3.6],  $C'$  intersects with the regular locus, hence the non-regular locus of  $C'$  has dimension less or equal to  $\dim C' - 1$ . We get  $\dim C \leq \dim C'$ . Hence the complement has strictly less dimension than  $X(s, w')$  and we conclude that  $X(s, w')$  is irreducible with the open dense subset  $X(s, w')^\circ$ . Then the image of  $X(s, w')$  in  $X$  is  $X_w$  since the map is proper.  $\square$

Denote  $r_{(s_1, \dots, s_m)}$  (resp.  $r_{(s_1, \dots, s_m)}^Y$ ) the morphism  $r : X(s_1, \dots, s_m) \rightarrow X_w$  (resp.  $r^Y : Y(s_1, \dots, s_m) \rightarrow Y_w$ ). Then  $r_{(s_1, \dots, s_m)}$  and  $r_{(s_1, \dots, s_m)}^Y$  are proper surjections, and the composite morphisms

$$\begin{aligned} r_{P, (s_1, \dots, s_m)} &:= p_{P, w} \circ r_{(s_1, \dots, s_m)} : X(s_1, \dots, s_m) \rightarrow X_{P, w} \\ r_{P, (s_1, \dots, s_m)}^Y &:= p_{P, w}^Y \circ r_{(s_1, \dots, s_m)}^Y : Y(s_1, \dots, s_m) \rightarrow Y_{P, w} \end{aligned}$$

are also proper. If  $w = w^P \in W$ , then  $r_{P, (s_1, \dots, s_m)}$  (resp.  $r_{P, (s_1, \dots, s_m)}^Y$ ) induces an isomorphism  $X^\circ(s_1, \dots, s_m) \xrightarrow{\sim} V_{P, w}$  (resp.  $Y^\circ(s_1, \dots, s_m) \xrightarrow{\sim} V_{P, w}^Y$ ).

### B.3 Connectedness of fibers

We aim to show that the fibers of the morphism  $r_{(s_1, \dots, s_m)} : Y(s_1, \dots, s_m) \rightarrow Y_w$  are connected. Let  $(\nu, g) \in Y_w$ . Let  $\pi_{(s_1, \dots, s_m)}^b : Y(s_1, \dots, s_m) \rightarrow \mathfrak{b}$  and  $\pi_w : Y_w \rightarrow \mathfrak{b}$  be the natural projections (to the first factors).

The following lemma serves as a motivation and a basic example.

**Lemma B.3.1.** *If  $\nu$  is a regular element in  $\mathfrak{b}$ , then  $(\pi_{(s_1, \dots, s_m)}^b)^{-1}(\nu)$  contains only one point.*

*Proof.* This is true when  $w = s$  is a simple reflection by [BHS19, Thm 2.2.6] since  $Y(s) = Y_s$ . This implies that for any simple reflection  $s$ , if  $(\nu, g) \in Y_s$  and  $\nu$  is regular, then  $gB$  is uniquely determined by  $\nu$ . An easy induction shows that the fiber of  $Y(s_1, \dots, s_m)$  over  $\nu$  is also a single point for any  $\nu$  regular. This could be explained in an elementary way which motivates the remaining arguments in this section. We assume that  $G = \mathrm{GL}_n$ . By the computations in the simple reflection cases of  $Y_s$ , we should show that if  $\nu = (\nu_{ij})_{i \leq j} \in \mathfrak{b}$  is regular and  $\nu_{ii} = \nu_{i+1, i+1}$  for some  $i$ , then  $\nu_{i, i+1}$  is not 0. To check the property for regular elements, by lemma B.3.4 below, we can conjugate  $\nu$  by an element in  $B$  such that  $\nu'_{ij} = 0$  if  $\nu_{ii} \neq \nu_{jj}$  and  $\nu'_{i, i+1} = \nu_{i, i+1}$  if  $\nu_{ii} = \nu_{i+1, i+1}$ . Then using some permutations, we can reduce to the case that the diagonals of  $\nu$  are all same. Then we can check the property is invariant under the conjugations of elementary matrices  $I_n + xE_{ab}$ ,  $b > a$ , diagonal torus and no nontrivial Weyl group element can conjugate such  $\nu$  into  $\mathfrak{b}$ . The property is then true for all such regular  $\nu$  thanks to the Bruhat decompositions and Jordan normal forms.  $\square$

Recall for a subset  $I$  of the set of the positive simple roots  $S$ , we have the corresponding standard Levi subgroup  $M_I$  of  $G$  containing the fixed diagonal torus  $T$  with  $I$  the set of positive simple roots of  $M_I$ . Let  $W_I$  be the Weyl group of  $M_I$  and let  $W^I \subset W$  be the set of shortest representatives in  $W$  of  $W/W_I$ . Then for each  $w \in W^P = W/W_P$ , we have a Levi subgroup  $M_{I, w} := \mathrm{Ad}(w)(M_I)$  containing  $T$ . Let  $Z_{I, w}$  be the center of  $M_{I, w}$ . Then  $Z_{I, w} = \mathrm{Ad}(w)(Z_I)$ . Let  $\mathfrak{m}_{I, w}$  (resp.  $\mathfrak{z}_{I, w}$ ) be the Lie algebra of  $M_I$  (resp.  $Z_{I, w}$ ). As  $\mathfrak{z}_{I, e} = \{\lambda \in \mathfrak{t} \mid \langle \alpha, \lambda \rangle = 0, \forall \alpha \in I\}$ , we get  $\mathfrak{z}_{I, w} = \{\lambda \in \mathfrak{t} \mid \langle w(\alpha), \lambda \rangle = 0, \forall \alpha \in I\}$ .

**Lemma B.3.2.** *If  $w \in W^I$ , then  $w(I) \subset R^+$ .*

*Proof.* This is a special case of (4) of Lemma 3.2.7.  $\square$

It is possible that there exists different pairs  $(I, w \in W^I)$  gives the same Levi subgroup  $M_{I,w}$ . For example if  $n = 3$  and  $S = \{\alpha_1, \alpha_2\}$ , then  $M_{\{\alpha_2\}, s_1} = M_{\{\alpha_1\}, s_2}$ .

**Lemma B.3.3.** *The two Levi subgroup  $M_{I,w}, M_{I',w'}$  of  $G$  are the same if and only if  $w(I) = w'(I')$ .*

*Proof.* The subset  $w(I)$  of  $R^+$  is the set of positive simple roots of  $M_{I,w}$  with respect to the Borel subgroup  $B \cap M_{I,w}$ . Hence the subset  $w(I)$  of  $R^+$  determines and is uniquely determined by  $M_{I,w}$ .  $\square$

Given  $\lambda \in \mathfrak{t}$ , let  $M_{I,w}$  be the maximal Levi subgroup containing  $T$  such that  $\lambda \in \mathfrak{z}_{I,w}$ . In other words,  $M_{I,w}$  is the centralizer of  $\lambda$  and the roots of the Levi subgroup are exactly  $\alpha \in R$  such that  $\langle \alpha, \lambda \rangle = 0$ . Let  $B_{I,w} = M_{I,w} \cap B$ . Let  $\mathfrak{b}_{I,w}$  be the Lie algebra of  $B_{I,w}$ .

We conjugate  $\nu \in \mathfrak{b}$  in a “good” form by the following lemma.

**Lemma B.3.4.** *Assume  $\nu \in \mathfrak{b}$  with  $\lambda \in \mathfrak{t}$  the factor of  $\nu$  under the Levi decomposition of  $\mathfrak{b}$ . Assume that the centralizer of  $\lambda$  is  $M_{I,w}$  for some  $I \subset S$  and  $w \in W^I$  as above. Then there exists  $g \in B$  such that  $\text{Ad}(g^{-1})\nu \in \mathfrak{b}_{I,w}$ .*

*Proof.* We prove by induction on the length  $i$  of  $\alpha \in R^+$  that there exists  $g \in B$  such that if we write  $\text{Ad}(g^{-1})\nu = \lambda + \sum_{\alpha \in R^+} x_\alpha e_\alpha$  where  $e_\alpha$  is a basis of the nilpotent radical of  $\mathfrak{b}$ , then  $x_\alpha = 0$  if  $\langle \alpha, \lambda \rangle \neq 0$  and  $\text{lg}(\alpha) \leq i$ .

Recall  $\text{Ad}(u_\alpha(-x))e_\beta \in e_\beta + \oplus_{\alpha' = i\alpha + j\beta, i, j > 0} k e_{\alpha'}$  for  $\alpha, \beta \in R^+$ . Take  $\alpha \in R^+$ . If we consider the subgroup  $\text{SL}_2$  corresponding to  $\alpha$  (see [Spr10, §8.1]), we find  $\text{Ad}(u_\alpha(-x))\lambda = \lambda + x\langle \alpha, \lambda \rangle e_\alpha$ . Hence  $\text{Ad}(u_\alpha(-x))\nu = \nu + x\langle \alpha, \lambda \rangle e_\alpha + \sum_{\alpha' = i\alpha + j\beta, i, j > 0} c_{\alpha'} e_{\alpha'}$  for some  $c_{\alpha'}$ . If  $\langle \alpha, \lambda \rangle \neq 0$ , we can take  $x$  such that the coefficient of  $e_\alpha$  of  $\text{Ad}(u_\alpha(-x))\nu$  is zero and the coefficients of  $e_{\alpha'}$ , where  $\alpha'$  has length strictly less than that of  $\alpha$  or equal to that of  $\alpha$  but  $\alpha \neq \alpha'$ , are the same with that of  $\nu$ .  $\square$

We should consider the fiber

$$F_{(s_1, \dots, s_m)}(\nu) := (\pi_{(s_1, \dots, s_m)}^{\mathfrak{b}})^{-1}(\nu) \subset \text{BSD}(s_1, \dots, s_m)$$

for  $\nu \in \mathfrak{b}$ . Notice that if  $(g_1, \dots, g_m) \in F_{(s_1, \dots, s_m)}(\nu)$ , then for each  $i$ , the Levi factor  $\lambda_i$  of  $\text{Ad}((g_1 \cdots g_i)^{-1})\nu$  in  $\mathfrak{t}$  is the same as that of  $\text{Ad}((s_1 \cdots s_i)^{-1})\nu$  by Lemma B.1.1.

We take  $(I, w_\nu)$  for  $\lambda = \lambda_0$  as in the previous lemma. Since  $w \in W^I$ ,  $B_{I,w} = \text{Ad}(w)(B_{I,e}) \subset B$  and we have a closed embedding  $M_{I,w}/B_{I,w} \hookrightarrow G/B$  induced by  $M_{I,w} \subset G$ .

We can define  $Y_{M_{I,w}}(t_1, \dots, t_l)$  with respect to  $B_{I,w}$  where  $t_1, \dots, t_l \in S_w := wSw^{-1}$  are simple reflections of the Weyl group  $W_{I,w} := wW_Iw^{-1}$  of  $M_{I,w}$ . We remark from now on, when we write  $M_{I,w}$ ,  $w$  is always in  $W^I$ .

**Lemma B.3.5.** *The morphism  $(\nu, g_1, \dots, g_l) \rightarrow (\text{Ad}(\dot{s}^{-1})\nu, \dot{s}^{-1}g_1\dot{s}, \dots, \dot{s}^{-1}g_l\dot{s})$  induces an isomorphism of closed subschemes  $Y_{M_{I,w}}(t_1, \dots, t_l) \xrightarrow{\sim} Y_{M_{I, s^{-1}w}}(s^{-1}t_1s, \dots, s^{-1}t_ls)$  of  $\mathfrak{b} \times (G \times^B \dots \times^B G/B)$  if  $s$  is a simple reflection in  $S$  that is not in  $W_{I,w}$ .*

*Proof.* We remark that  $s^{-1}B_{I,w}s = B_{I, s^{-1}w} \subset B$  in this situation since  $s^{-1}w \in W^I$  (otherwise  $s^{-1}w = ws'$  for some reflection  $s' \in W_I$  by [BB06, Cor. 2.5.2], then  $s = ws'w^{-1} \in W_{I,w}$ , but this is not possible by our choice of  $s$ , see also the next lemma). Thus  $Y_{M_{I,w}}(t_1, \dots, t_l)$  and  $Y_{M_{I, s^{-1}w}}(s^{-1}t_1s, \dots, s^{-1}t_ls)$  are closed subschemes of  $\mathfrak{b} \times (G \times^B \dots \times^B G/B)$  and the maps are well defined. The isomorphism is true because there is no substantial difference between the two side except changing the embedding of  $M_I$  into  $G$ .  $\square$

We assume now  $\nu \in \mathfrak{b}_{I, w_\nu}$  (see Lemma B.3.4). We let  $T$  be the sequence of  $s_i$  (as a subsequence of  $(s_1, \dots, s_m)$ ) such that  $\lambda_i := \text{Ad}(s_i \cdots s_1)(\lambda)$  is equal to  $\lambda_{i-1}$  and we rename elements in the sequence  $T$  by  $t_1, \dots, t_l$  (in the original order). We also rename elements in  $K := (s_1, \dots, s_m) \setminus T$  in the original order by  $u_1, \dots, u_{m-l}$ . For each  $i = 1, \dots, l$ , let  $h(i)$  be the minimal number in  $\{1, \dots, m-l\}$  such that  $u_{h(i)}$  appears after  $t_i$  in the sequence  $(s_1, \dots, s_m)$ . For each  $N \in \{1, \dots, m\}$ , let  $T^N, K^N$  be the truncated sequences consisting of elements in  $(s_1, \dots, s_N)$ . Let  $t_i^N := ((\prod_{j \geq h(i), u_j \in K^N} u_j)^{-1} t_i (\prod_{j \geq h(i), u_j \in K^N} u_j)) \in W$  if  $t_i \in T^N$ . Let  $w_N := u_1 \cdots u_{|K^N|}$ .

We need some combinatorial lemmas. Let  $R_I^+$  (resp.  $R_I^-$ ) be the set of positive (resp. negative) roots of  $M_I$ . For any  $w \in W$ , we write  $w = w^I w_I$  where  $w_I \in W_I$  and  $w^I \in W^I$ . Then by Lemma 3.2.7, for  $w \in W$ ,  $\text{lg}_I(w) := \text{lg}(w^I)$  is equal to the size of the set  $\{\alpha \in R^+ \setminus R_I^+ \mid w(\alpha) \in R^-\}$ . And  $\text{lg}(w_I) = \text{lg}(w) - \text{lg}_I(w) = |\{\alpha \in R_I^+ \mid w(\alpha) \in R^-\}| = |\{\alpha \in R_I^+ \mid w(\alpha) \in R_I^-\}|$  since  $R_I = R_I^+ \cup R_I^-$  is the set of roots of  $M_I$  and is stable under the action of  $W_I$ .

**Lemma B.3.6.** *Let  $w \in W^I$  and  $s$  be a simple reflection in  $W$ . If  $w^{-1}sw$  is in  $W_I$ , then  $w^{-1}sw$  is also a simple reflection in  $W_I$ . If  $w^{-1}sw \notin W_I$ , then  $sw \in W^I$ .*

*Proof.* Assume  $w^{-1}sw \in W_I$ . We need to prove that  $\text{lg}(w^{-1}sw) = 1$ . We have  $\text{lg}_I(w^{-1}sw) = 0$  since  $w^{-1}sw \in W_I$ . Thus  $\text{lg}(w^{-1}sw) = |\{\alpha \in R_I^+ \mid w^{-1}sw(\alpha) \in R_I^-\}|$ . Since  $w \in W^I$ , we have  $w(R_I^+) \subset R^+$ ,  $w(R_I^-) \subset R^-$ . Thus if  $\alpha \in R_I^+$ , such that  $w^{-1}sw(\alpha) \in R_I^-$ , then  $w(\alpha) \in R^+$  and  $s(w(\alpha)) \in R^-$ . Such  $\alpha$  is unique if it exists as  $s$  is a simple reflection in  $W$ .

Now assume  $w^{-1}sw \notin W_I$ . If  $\text{lg}(sw) \geq \text{lg}(w)$ , we get  $sw \in W^I$  by [BB06, Cor. 2.5.2]. If  $\text{lg}(sw) \leq \text{lg}(w)$ , then by [BB06, Prop. 1.4.2 (iii)],  $\text{lg}(w) = \text{lg}(sw) + 1$ . Write  $w = ssw = s(sw)^I (sw)_I$ . Since  $w \in W^I$  and  $(sw)_I \in W_I$ , we get  $\text{lg}(w) = \text{lg}_I(w) \leq \text{lg}(s(sw)^I) \leq \text{lg}((sw)^I) + 1$ . Hence  $\text{lg}((sw)^I) \geq \text{lg}(sw)$ . The equality must hold and we get  $sw \in W^I$ .  $\square$

**Lemma B.3.7.** *Assume that  $w \in W^I$ ,  $s_1 \cdots s_l$  is reduced expression in  $W$  and each  $w^{-1}s_i w$  lies in  $W_I$ . Then  $\text{lg}(w^{-1}s_1 \cdots s_l w) = l$ .*

*Proof.* We have  $\text{lg}(s_1 \cdots s_l) = \{\alpha \in w(R^+) \mid s_1 \cdots s_l(\alpha) \in w(R^-)\}$ . By the condition  $w^{-1}s_i w \in W_I$ , the unique positive root that is sent to  $R^-$  by  $s_i$  is in  $w(R_I^+)$ . Hence the action of  $s_i$  keeps the sets  $R^+ \setminus w(R_I^+)$ ,  $R^- \setminus w(R_I^-)$  and  $w(R_I)$ . If  $\alpha \in R^+$  is sent to  $R^-$  by  $s_1 \cdots s_l$ , then  $\alpha \notin R^+ \setminus w(R_I^+)$ . Thus  $l = \text{lg}(s_1 \cdots s_l) \leq \text{lg}(w^{-1}s_1 \cdots s_l w)$  and the equality must hold since by the previous lemma  $w^{-1}s_i w$  are simple reflections.  $\square$

**Lemma B.3.8.** *The sequence  $(t_i^N)_{t_i \in T^N}$  is a sequence of simple reflections in  $W_{I, w_N^{-1}w_\nu}$  and  $t_1^N \cdots t_{|T^N|}^N$  is a reduced expression in  $W_{I, w_N^{-1}w_\nu}$  of  $w_N^{-1}s_1 \cdots s_N$ . Moreover,  $w_N^{-1}w_\nu \in W^I$ .*

*Proof.* We prove by induction on  $N$ . Assume the results hold for  $N-1$ , then  $(t_i^{N-1})_{t_i \in T^{N-1}}$  is a sequence of simple reflections of  $W_{I, w_{N-1}^{-1}w_\nu}$  and is a reduced expression.

If  $s_N \in T$ , then  $K^N = K^{N-1}$ ,  $w_{N-1} = w_N$ ,  $t_i^{N-1} = t_i^N$  if  $t_i \in T^{N-1}$ . Hence  $M_{I, w_{N-1}^{-1}w_\nu} = M_{I, w_N^{-1}w_\nu}$ . The subgroup  $W_{I, w_N^{-1}w_\nu} = w_N^{-1}W_{I, w_\nu}w_N$  consists exactly of elements in  $W$  that fix  $\text{Ad}(w_N^{-1})(\lambda) = \text{Ad}(s_N \cdots s_1)(\lambda)$  and  $s_N$  fixes it by definition. By Lemma B.3.6,  $s_N$  is a simple reflection in  $W_{I, w_N^{-1}w_\nu}$ . We need to prove that  $\text{lg}(w_N^{-1}s_1 \cdots s_N w_N^{-1}w_\nu) = |T^N|$ . Note that

$$\begin{aligned} N &= \text{lg}(s_1 \cdots s_N) = \{\alpha \in R^+ \mid s_1 \cdots s_N(\alpha) \in R^-\} \\ &= \{\alpha \in w_N^{-1}w_\nu(R_I^+) \mid s_1 \cdots s_N(\alpha) \in R^-\} + \{\alpha \in R^+ \setminus w_N^{-1}w_\nu(R_I^+) \mid s_1 \cdots s_N(\alpha) \in R^-\}. \end{aligned}$$

Since  $s_N = t_{|T^N|}^N$  is a simple reflection and  $w_N^{-1}w_N s_N w_N^{-1}w_\nu$  is a simple reflection in  $W_I$ , we see by the argument in the previous lemmas that  $s_N$  keeps  $R^+ \setminus w_N^{-1}w_\nu(R_I^+)$ . We get

$$\{\alpha \in R^+ \setminus w_N^{-1}w_\nu(R_I^+) \mid s_1 \cdots s_N(\alpha) \in R^-\} = \{\alpha \in R^+ \setminus w_N^{-1}w_\nu(R_I^+) \mid s_1 \cdots s_{N-1}(\alpha) \in R^-\}.$$

Hence

$$\{\alpha \in w_N^{-1}w_\nu(R_I^+) \mid s_1 \cdots s_N(\alpha) \in R^-\} = \{\alpha \in w_N^{-1}w_\nu(R_I^+) \mid s_1 \cdots s_{N-1}(\alpha) \in R^-\} + 1.$$

Since  $w_N^{-1}s_1 \cdots s_N$  and  $w_N^{-1}s_1 \cdots s_{N-1}$  are in  $W_{I, w_N^{-1}w_\nu}$ , their actions keep  $w_N^{-1}w_\nu(R_I)$ . Thus

$$\begin{aligned} & \{\alpha \in w_N^{-1}w_\nu(R_I^+) \mid s_1 \cdots s_N(\alpha) \in R^-\} \\ &= \{\alpha \in w_N^{-1}w_\nu(R_I^+) \mid w_N^{-1}s_1 \cdots s_N(\alpha) \in w_N^{-1}(R^- \cap w_\nu(R_I))\} \\ &= \{\alpha \in w_N^{-1}w_\nu(R_I^+) \mid w_N^{-1}s_1 \cdots s_N(\alpha) \in w_N^{-1}w_\nu(R_I^-)\} \end{aligned}$$

where we have used that  $w_\nu(R_I^\pm) \subset R^\pm$ . But using that  $w_N^{-1}s_1 \cdots s_N$  is in  $W_{I, w_N^{-1}w_\nu}$  again, we have

$$\lg(w_N^{-1}s_1 \cdots s_N w_N^{-1}w_\nu) = \{\alpha \in w_N^{-1}w_\nu(R_I^+) \mid w_N^{-1}s_1 \cdots s_N(\alpha) \in w_N^{-1}w_\nu(R_I^-)\}.$$

Hence  $\lg(w_N^{-1}s_1 \cdots s_N w_N^{-1}w_\nu) = \lg(w_N^{-1}s_1 \cdots s_{N-1} w_N^{-1}w_\nu) + 1$  and the results hold for  $N$  by the induction assumptions.

If  $s_N \in K^N$ , then  $M_{I, w_N^{-1}w_\nu} = s_N M_{I, w_{N-1}^{-1}w_\nu} s_N^{-1}$  and  $t_i^N = s_N t_i^{N-1} s_N^{-1}$ . By definition,  $s_N$  doesn't fix  $\text{Ad}(w_{N-1}^{-1})(\lambda)$  which means that  $s_N$  is not in  $W_{I, w_{N-1}^{-1}w_\nu}$ . By the induction hypothesis,  $w_{N-1}^{-1}w_\nu \in W^I$ . By Lemma B.3.8,  $w_N^{-1}w_\nu = s_N w_{N-1}^{-1}w_\nu$  is in  $W^I$ .  $\square$

We let  $F_N = F_{(s_1, \dots, s_N)}(\nu)$  (resp.  $F_{w_N} = F_{M_{I, w_N^{-1}w_\nu}, (t_1^N, \dots, t_{|T_N|}^N)}(\text{Ad}(w_N^{-1})\nu)$ ) be the fiber of  $Y(s_1, \dots, s_N)$  over  $\nu \in \mathfrak{b}_{I, w_\nu}$  (resp. the fiber of  $Y_{M_{I, w_N^{-1}w_\nu}}(t_1^N, \dots, t_{|T_N|}^N)$  over  $\text{Ad}(w_N^{-1})(\nu)$ ). We let  $F_{N, B} \subset \mathfrak{b} \times (G \times^B \cdots \times^B G)$  be the preimage of  $F_N$  and similarly for  $F_{w_N, B_{I, w_N^{-1}w_\nu}}$ .

**Proposition B.3.9.** *There exists a morphism  $h_N : F_{w_N, B_{I, w_N^{-1}w_\nu}} \rightarrow F_{N, B}$  of the underlying topological spaces which induces an isomorphism after modulo  $B$  of the fiber of  $Y(s_1, \dots, s_N)$  over  $\nu \in \mathfrak{b}_{I, w_\nu}$  and the fiber of  $Y_{M_{I, w_N^{-1}w_\nu}}(t_1^N, \dots, t_{|T_N|}^N)$  over  $\text{Ad}(\dot{w}_N^{-1})(\nu)$  where  $\dot{w}_N = \dot{u}_1 \cdots \dot{u}_{|K^N|}$  and such that the map  $m_N : G \times^B \cdots \times^B G \rightarrow G$  of taking the products satisfying that  $m_N \circ h_N(g) = \dot{w}_N m_N(g)$  for any  $g \in F_{w_N, B_{I, w_N^{-1}w_\nu}}$ .*

*Proof.* We prove by induction on  $N$ .

We start by  $N = 1$ . Firstly assume that  $s_1$  fixes  $\lambda$ , the Levi factor of  $\nu$ . Then  $s_1 \in W_{I, w_\nu}$  and we have an isomorphism  $\overline{B_{I, w_\nu} s_1 B_{I, w_\nu}} / B_{I, w_\nu} \xrightarrow{\sim} \overline{B s_1 B} / B$ . Hence  $g \in \overline{B_{I, w_\nu} s_1 B_{I, w_\nu}}$  such that  $(\nu, g B_{I, w_\nu}) \in Y_{M_{I, w_\nu}}(s_1)$  if and only if  $(\nu, g B) \in Y(s_1)$  by the explicit computations in §B.1. Assume that  $s_1$  doesn't fix  $\lambda$ , there exists a unique  $g \in \overline{B s_1 B} / B$  such that  $(\nu, g B) \in Y(s_1)$  by the explicit example. Since  $\nu \in \mathfrak{b}_{I, w_\nu}$ , we have  $g = s_1 B$ . Then the result follows from that  $Y_{M_{I, w_\nu}}(e) = \mathfrak{b}_{I, w_\nu}$  by the definition.

Now we assume there exists an embedding  $h_{N-1} : F_{w_{N-1}, B_{I, w_{N-1}^{-1}w_\nu}} \rightarrow F_{N-1, B}$  which induces an isomorphism  $F_{w_{N-1}} \xrightarrow{\sim} F_{N-1}$  and such that  $m_{N-1} \circ h_{N-1}(g) = \dot{w}_{N-1}^{-1} m_{N-1}(g)$  for any  $g \in F_{w_{N-1}, B_{I, w_{N-1}^{-1}w_\nu}}$ . Let  $a_{N-1}(g) := \text{Ad}((m_{N-1}(g))^{-1})\nu$  (resp.  $a_{w_{N-1}}(g) := \text{Ad}((m_{N-1}(g))^{-1})(\text{Ad}(\dot{w}_{N-1}^{-1})\nu)$  for any  $g \in F_{N-1, B}$  (resp.  $g \in F_{w_{N-1}, B_{I, w_{N-1}^{-1}w_\nu}}$ ). Then

$$F_{N, B} = \{(g_0, g) \in F_{N-1, B} \times^B G \mid (a_{N-1}(g_0), g B) \in Y(s_N)\}.$$

Assume  $s_N \in T^N$ . Then  $w_N = w_{N-1}$ ,  $t_i^N = t_i^{N-1}$  and  $s_N \in W_{I, w_{N-1}^{-1}w_\nu}$ . We have

$$\begin{aligned} & F_{w_N, B_{I, w_N^{-1}w_\nu}} \\ &= \{(g_0, g) \in F_{w_{N-1}, B_{I, w_{N-1}^{-1}w_\nu}} \times^{B_{I, w_{N-1}^{-1}w_\nu}} M_{I, w_N^{-1}w_\nu} \mid (a_{w_{N-1}}(g_0), g B_{I, w_N^{-1}w_\nu}) \in Y_{M_{I, w_N^{-1}w_\nu}}(s_N)\}. \end{aligned}$$

The Levi factor of  $a_{N-1}(h_{N-1}(g_0))$  is  $s_{N-1} \cdots s_1(\lambda)$  and  $a_{N-1}(h_{N-1}(g_0)) = a_{w_{N-1}}(g_0)$  lies in  $\mathfrak{b}_{I, w_{N-1}^{-1} w_\nu}$  if  $g_0 \in F_{w_{N-1}, B_{I, w_{N-1}^{-1} w_\nu}}$ . We know for any  $\nu' \in \mathfrak{b}_{I, w_{N-1}^{-1} w_\nu}$ ,  $g \in M_{I, w_{N-1}^{-1} w_\nu}$ ,  $(\nu', gB_{I, w_{N-1}^{-1} w_\nu})$  is in  $Y_{M_{I, w_{N-1}^{-1} w_\nu}}(s_N)$  if and only if  $(\nu', gB) \in Y(s_N)$  by the case of  $N = 1$  or explicit computations. Thus we can define a morphism  $f : F_{w_N, B_{I, w_{N-1}^{-1} w_\nu}} \rightarrow F_{N, B}$  by sending  $g_0 \in F_{w_{N-1}, B_{I, w_{N-1}^{-1} w_\nu}}$  to the one in  $F_{N-1, B}$  via the map  $h_{N-1}$  and the natural embedding  $M_{I, w_{N-1}^{-1} w_\nu} \hookrightarrow G$ . This is in fact an isomorphism modulo  $B$ : the map is compatible with respect to the projections  $F_N \rightarrow F_{N-1}$  and  $F_{w_N} \rightarrow F_{w_{N-1}}$ , the base spaces  $h_{N-1} : F_{w_{N-1}} \xrightarrow{\sim} F_{N-1}$  and the morphism restricted to the fibers are all isomorphisms by the  $N = 1$  cases.

Assume  $s_N \in K^N$ . Since  $s_N$  doesn't fix  $\text{Ad}(w_{N-1}^{-1})(\lambda)$ , we know there is an isomorphism  $F_{N-1} \simeq F_N$  and if  $\nu' \in \mathfrak{b}_{I, w_{N-1}^{-1} w_\nu}$ , then  $(\nu', gB) \in Y_{s_N}$  if and only if  $gB = s_N B$  (as  $s_N \notin M_{I, w_{N-1}^{-1} w_\nu}$ ). Therefore, we still have a map

$$F_{w_{N-1}, B_{I, w_{N-1}^{-1} w_\nu}} \rightarrow F_{N, B} = \{(g_0, g) \in F_{N-1, B} \times^B G \mid (a_{N-1}(g_0), gB) \in Y(s_N)\}$$

sending  $g_0 \in F_{w_{N-1}, B_{I, w_{N-1}^{-1} w_\nu}}$  to  $(h_{N-1}(g_0), \dot{s}_N)$ . One can similarly verify that this is an isomorphism modulo  $B$ . Now the conjugation of  $\dot{s}_N$  induces an isomorphism  $F_{w_N, B_{I, w_{N-1}^{-1} w_\nu}} \simeq F_{w_{N-1}, B_{I, w_{N-1}^{-1} w_\nu}}$  as in Lemma B.3.5 and one can check the composite map  $F_{w_N, B_{I, w_{N-1}^{-1} w_\nu}} \rightarrow F_{N, B}$  satisfies our requirement.  $\square$

**Lemma B.3.10.** *Assume  $\nu \in \mathfrak{b}$  with  $\lambda$  the Levi factor of  $\nu$  in  $\mathfrak{t}$ . Assume that  $\lambda$  lies in the center of  $\mathfrak{g}$  (i.e.  $M_{I, w_\nu} = G$ ), then  $\pi_w^{-1}(\nu) = \{g \in \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b}\}$  and  $(\pi_{(s_1, \dots, s_m)}^\mathfrak{b})^{-1}(\nu) = \{(g_1, \dots, g_m) \in \overline{Bs_1B} \times^B \cdots \times^B \overline{Bs_mB/B} \mid \text{Ad}((g_1 \cdots g_i)^{-1})\nu \in \mathfrak{b}, \forall i\}$  as closed subspaces.*

*Proof.* The assertion that  $(\pi_{(s_1, \dots, s_m)}^\mathfrak{b})^{-1}(\nu)$  is equal to

$$\{(g_1, \dots, g_m) \in \overline{Bs_1B} \times^B \cdots \times^B \overline{Bs_mB/B} \mid \text{Ad}((g_1 \cdots g_i)^{-1})\nu \in \mathfrak{b}, \forall i\}$$

follows from the example when  $m = 1$  (§B.1) and by induction. We prove the morphism from

$$\{(g_1, \dots, g_m) \in \overline{Bs_1B} \times^B \cdots \times^B \overline{Bs_mB/B} \mid \text{Ad}((g_1 \cdots g_i)^{-1})\nu \in \mathfrak{b}, \forall i\}$$

to

$$\{g \in \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b}\}$$

is surjective. We can assume  $g = w'$  with  $w' = \dot{s}'_1 \cdots \dot{s}'_k$  being a reduced expression such that  $\text{Ad}(g^{-1})\nu \in \mathfrak{b}$ . Then there exists  $i_1 < \cdots < i_k$  such that  $s_{i_j} = \dot{s}'_j$  since  $w' \leq w$ . Moreover,  $\text{Ad}((\dot{s}'_{i_1} \cdots \dot{s}'_{i_j})^{-1})\nu \in \mathfrak{b}$  for all  $j$  by Lemma B.2.3. Let  $g' = (g_1, \dots, g_m)$  with  $g_{i_j} = \dot{s}_{i_j}$  for all  $j$  and  $g_i = e$  otherwise. Then  $g' \in \{(g_1, \dots, g_m) \in \overline{Bs_1B} \times^B \cdots \times^B \overline{Bs_mB/B} \mid \text{Ad}((g_1 \cdots g_i)^{-1})\nu \in \mathfrak{b}, \forall i\}$  and is sent to  $g$ .  $\square$

**Remark B.3.11.** The identification  $\pi_w^{-1}(\nu) = \{g \in \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b}\}$  for all  $\nu$  nilpotent means that  $\overline{X}_w$  contains  $Z_{w'}$  for all  $w' \leq w$  in the notation of [BHS19, §2.4].

**Proposition B.3.12.** *The fiber of the morphism  $r_{(s_1, \dots, s_m)} : Y(s_1, \dots, s_m) \rightarrow Y_w$  over any point  $(\nu, gB) \in Y_w$  such that the Levi factor  $\lambda$  of  $\nu$  lies in the center of  $\mathfrak{g}$  is connected.*

*Proof.* We prove the fibers of the map

$$\begin{aligned} \pi : \{(g_1, \dots, g_m) \in \overline{Bs_1B} \times^B \cdots \times^B \overline{Bs_mB/B} \mid \text{Ad}((g_1 \cdots g_i)^{-1})\nu \in \mathfrak{b}, \forall i\} \\ \rightarrow \{g \in \overline{BwB/B} \mid \text{Ad}(g^{-1})\nu \in \mathfrak{b}\} \end{aligned}$$



are connected. We follow the proof in the Schubert varieties cases ([BK07, Prop 3.2.1]). We prove by induction on  $m$ . If there is a unique  $g_1 \in \overline{Bs_1B}/B$  such that  $\text{Ad}(g_1^{-1})\nu \in \mathfrak{b}$ . Then the fibers of  $r_{(s_1, \dots, s_m)}$  are the same as the fibers of

$$\begin{aligned} \{(g_2, \dots, g_m) \in \overline{Bs_2B} \times^B \dots \times^B \overline{Bs_mB}/B \mid \text{Ad}((g_2 \dots g_i)^{-1})(\text{Ad}(g_1^{-1})\nu) \in \mathfrak{b}\} \\ \rightarrow \{g \in \overline{Bs_1wB}/B \mid \text{Ad}(g^{-1})(\text{Ad}(g_1^{-1})\nu) \in \mathfrak{b}\} \end{aligned}$$

which are connected by our induction hypothesis. Now if  $g_1$  is not unique, then for any  $g_1 \in \overline{Bs_1B}/B$ ,  $(\nu, g_1B) \in Y_{s_1}$  by the computations in §B.1. Consider the map

$$f : \{(g_1, \dots, g_m) \in \overline{Bs_1B} \times^B \dots \times^B \overline{Bs_mB}/B \mid \text{Ad}((g_1 \dots g_i)^{-1})\nu \in \mathfrak{b}\} \rightarrow \overline{Bs_1B}/B$$

which restricts to a map

$$f' : r_{(s_1, \dots, s_m)}^{-1}(g) \rightarrow \overline{Bs_1B}/B.$$

For any  $g_1 \in \overline{Bs_1B}/B$ ,  $f'^{-1}(g_1)$  is the fiber of the map

$$\begin{aligned} \{(g_2, \dots, g_m) \in \overline{Bs_2B} \times^B \dots \times^B \overline{Bs_mB}/B \mid \text{Ad}((g_2 \dots g_i)^{-1})(\text{Ad}(g_1^{-1})\nu) \in \mathfrak{b}\} \\ \rightarrow \{g \in \overline{Bs_1wB}/B \mid \text{Ad}(g^{-1})(\text{Ad}(g_1^{-1})\nu) \in \mathfrak{b}\} \end{aligned}$$

over  $g_1^{-1}g$  (for any choice of  $g_1 \in g_1B$ ) which is connected if it is not empty (by the induction hypothesis). So we only need to prove that  $f'$  has connected image.

Note that the fiber  $f'^{-1}(g_1)$  is not empty if and only if  $(\text{Ad}(g_1^{-1})\nu, g_1^{-1}g) \in Y_{s_1w}$  if and only if  $g_1^{-1}g \in \overline{Bs_1wB}$  by Lemma B.3.10. Assume that the image of  $f'$  contains at least two different point and we prove that the image is in fact the whole projective line. Without loss of generality, we can assume  $g = w'$  for some  $w' \leq w$ . The subset  $\{g_1 \in \overline{Bs_1B}/B \mid g_1^{-1}w' \in \overline{Bs_1wB}/B\}$  is stable under the conjugation action of the maximal torus  $T$  of  $G$  on  $\mathbb{P}^1 = \overline{Bs_1B}/B$  since  $Btg_1^{-1}wt^{-1}B = Btg_1^{-1}t^{-1}wB$  for any  $t \in T$ . Thus the image of  $f'$  contains two fixed points under the action of the torus:  $B, s_1B \in \overline{Bs_1B}/B$ . Hence  $s_1w', w' \in \overline{Bs_1wB}$ . If  $s_1w' > w'$ , since  $s_1w' \leq s_1w$ ,  $\overline{Bs_1B}w' \subset \overline{Bs_1B}Bw'B \subset \overline{Bs_1wB}$ . If  $s_1w' < w'$ , then  $\overline{Bs_1B}w' \subset \overline{Bs_1Bs_1s_1w'} \subset \overline{Bs_1B}Bs_1w'B \subset \overline{Bw'B} \subset \overline{Bs_1wB}$ . In both cases,  $g_1^{-1}g \in \overline{Bs_1wB}$  for all  $g_1 \in \overline{Bs_1B}$ , thus the conclusion.  $\square$

**Lemma B.3.13.** *The fibers of the map*

$$\begin{aligned} \pi : \{(g_1, \dots, g_m) \in \overline{Bs_1B} \times^B \dots \times^B \overline{Bs_mB}/B \mid \text{Ad}((g_1 \dots g_i)^{-1})\nu \in \mathfrak{b}, \forall i\} \\ \rightarrow \{g \in \overline{BwP}/P \mid \text{Ad}(g^{-1})\nu \in \mathfrak{p}\} \end{aligned}$$

are connected if the Levi factor of  $\nu$  in  $\mathfrak{t}$  lies in the center of  $\mathfrak{g}$ . Here we don't assume  $w \in W^P$  but still assume  $s_1 \dots s_m$  is a reduced expression of  $w$  in  $W$ .

*Proof.* We prove in the same way by induction on  $m$ . If  $m = 1$  and  $s_m \notin W_P$ , the result is true since  $\overline{Bs_mB}/B \simeq \overline{Bs_mP}/P$ . If  $m = 1$ ,  $s_m \in W_P$ , then  $\{gB \in \overline{Bs_mB}/B \mid \text{ad}(g^{-1})\nu \in \mathfrak{b}\}$  is mapped into one point  $P \in G/P$  and the space  $\{g \in \overline{Bs_mB}/B \mid \text{ad}(g^{-1})\nu \in \mathfrak{b}\}$  is connected (it is one point or a projective line). Now for arbitrary  $m$ , take any  $g \in G$  such that  $\text{Ad}(g^{-1})\nu \in \mathfrak{p}$ . We can assume for all  $hB \in \overline{Bs_1B}/B$ ,  $\text{ad}(h^{-1})\nu \in \mathfrak{b}$  using the induction hypothesis. Let  $I = \{hB \in \overline{Bs_1B}/B \mid h^{-1}g \in \overline{Bs_1wP}/P\}$  where  $s_1w = s_2 \dots s_m$ . We can safely assume  $g = w'$  where  $w'$  is shortest in  $w'W_P$ . If  $I$  contains at least two point, since  $I$  is stable under the action of the diagonal torus  $T$ , we can assume  $s_1w', w' \in \overline{Bs_1wP}/P$ . If  $s_1w' > w'$ ,  $\overline{Bs_1B}w' \subset \overline{Bs_1B}Bw'B \subset \overline{Bs_1w'B}$ . Since  $s_1w' \in \overline{Bs_1wP}/P$ ,  $s_1w' \leq s_1w$  in the Bruhat order of  $W/W_P$ . Thus  $\overline{Bs_1B}w' \subset \overline{Bs_1w'P} \subset \overline{Bs_1wP}$ . If  $w' > s_1w'$ ,  $\overline{Bs_1B}w' \subset \overline{Bs_1Bs_1s_1w'} \subset \overline{Bs_1B}Bs_1w'B \subset \overline{Bw'B} \subset \overline{Bs_1wP}$ . Then we can argue as in the previous lemma.  $\square$

*Remark B.3.14.* The above lemma can also be proved using Proposition 3.2.12 at least in special cases.

**Theorem B.3.15.** *Fibers of the morphism  $r_{(s_1, \dots, s_m)}^Y : Y(s_1, \dots, s_m) \rightarrow Y_w$  are connected.*

*Proof.* Pick a point  $(\nu, gB) \in Y_w$ , then we can assume  $\nu$  has the good form by Lemma B.3.4:  $\nu$  corresponds to some Levi subgroup  $M_{I, w_\nu}$  and  $\nu \in \mathfrak{b}_{I, w_\nu}$ . Since we have the morphisms  $\pi_{(s_1, \dots, s_m)}^{\mathfrak{b}} : Y(s_1, \dots, s_m) \rightarrow Y_w \rightarrow \mathfrak{b}$ , the fiber over  $(\nu, gB) \in Y_w$  is the fiber of the map  $F_{(s_1, \dots, s_m)}(\nu) \rightarrow G/B$  over  $gB$  which is isomorphic to the fiber of  $F_{M_{I, w_m^{-1}w_\nu}, (t_1^m, \dots, t_l^m)}(\text{Ad}(\dot{w}_m^{-1})\nu)$  over  $\dot{w}_m^{-1}gB_{I, w_m^{-1}w_\nu}$  for some representative  $g \in \dot{w}_m M_{I, w_m^{-1}w_\nu}$  by Proposition B.3.9 which is also the fiber of  $Y_{M_{I, w_m^{-1}w_\nu}}(t_1^m, \dots, t_l^m)$  over

$$(\text{Ad}(\dot{w}_m^{-1})\nu, \dot{w}_m^{-1}gB_{I, w_m^{-1}w_\nu}).$$

Since the Levi factor of  $\text{Ad}(\dot{w}_m^{-1})\nu$  in  $\mathfrak{t}$  is fixed by  $W_{I, w_m^{-1}w_\nu}$ , the fiber is connected by applying Proposition B.3.12 for the group  $M_{I, w_m^{-1}w_\nu}$  and by Lemma B.3.8  $\square$

**Theorem B.3.16.** *Fibers of the composite map  $Y(s_1, \dots, s_m) \rightarrow Y_w \rightarrow Y_{P, w}$  are connected.*

*Proof.* Pick a point  $(\nu, gP) \in Y_{P, w}$ , then we can assume  $\nu$  has the good form by Lemma B.3.4:  $\nu$  corresponds to some Levi subgroup  $M_{I, w_\nu}$  and  $\nu \in \mathfrak{b}_{I, w_\nu}$ . Since we have the morphisms  $Y(s_1, \dots, s_m) \rightarrow Y_{P, w} \rightarrow \mathfrak{b}$ , the fiber over  $(\nu, gP) \in Y_{P, w}$  is the fiber of the map  $F_{(s_1, \dots, s_m)} \rightarrow G/B \rightarrow G/P$  over  $gP$  which is isomorphic to the fiber of the map

$$F_{M_{I, w_m^{-1}w_\nu}, (t_1^m, \dots, t_l^m)}(\text{Ad}(\dot{w}_m^{-1})\nu) \rightarrow M_{I, w_m^{-1}w_\nu}/B_{I, w_m^{-1}w_\nu} \rightarrow M_{I, w_m^{-1}w_\nu}/P_{I, w_m^{-1}w_\nu}$$

over  $\dot{w}_m^{-1}gP_{I, w_m^{-1}w_\nu}$  by Proposition B.3.9, where  $P_{I, w_m^{-1}w_\nu} := M_{I, w_m^{-1}w_\nu} \cap P$ . The image of  $F_{M_{I, w_m^{-1}w_\nu}, (t_1^m, \dots, t_l^m)}(\text{Ad}(\dot{w}_m^{-1})\nu) \rightarrow M_{I, w_m^{-1}w_\nu}/P_{I, w_m^{-1}w_\nu}$  is

$$\{gP_{I, w_m^{-1}w_\nu} \in \overline{B_{I, w_m^{-1}w_\nu} w_m^{-1}w P_{I, w_m^{-1}w_\nu} / P_{I, w_m^{-1}w_\nu}} | \text{Ad}(g^{-1})\nu \in \mathfrak{p}_{w_m^{-1}w_\nu}\}$$

by Lemma B.3.10 (since the Levi factor of  $\text{Ad}(\dot{w}_m^{-1})\nu$  in  $\mathfrak{t}$  is invariant under the action of  $W_{I, w_m^{-1}w_\nu}$ ) and this map has connected fibers by Lemma B.3.13 and Lemma B.3.8.  $\square$

**Corollary B.3.17.** *Fibers of the map  $Y_w \rightarrow Y_{P, w}$  are all connected.*



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