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Cosme Louart

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Cosme Louart. Random matrix theory and concentration of the measure theory for the study of high dimension data processing.. Signal and Image processing. Université Grenoble Alpes [2020-..], 2023. English. NNT : 2023GRALT011 . tel-04116888

HAL Id: tel-04116888

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THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES

École doctorale : EEATS - Electronique, Electrotechnique, Automatique, Traitement du Signal (EEATS)

Spécialité : Mathématiques appliquées

Unité de recherche : Grenoble Images Parole Signal Automatique

Théorie des matrices aléatoires et concentration de la mesure pour l'analyse d'algorithmes de traitement de données en grande dimension.

Random matrix theory and concentration of the measure theory for the study of high dimension data processing.

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Remerciements

Je remercie infiniment mes deux directeurs de thèse qui m'ont accompagné pendant ce long travail. Je remercie d'abord Romain qui m'a non seulement inculqué les normes et les usages du monde académique, mais m'a aussi énormément apporté sur le plan personnel. Je reviens continuellement à son enseignement d'une conception volontaire et heureuse du travail qui m'a aidé et m'aidera encore beaucoup pendant les périodes compliquées de mes recherches. Je regretterai toujours sa grande disponibilité et son efficacité légendaire. Il m'a maintes fois permis de redynamiser des égarements, démêler les incohérences ou finaliser en beauté un article avant la date limite. Il m'a permis non seulement de repousser mes limites intellectuelles mais aussi (et encore plus) mes limites physiques en me faisant découvrir les joies de la course en montagne, l'oxygène de haute altitude est sûrement pour quelque chose dans le bon déroulé de ma thèse. Merci aussi à Mohamed, qui a toujours tout fait pour me faciliter la vie dans mes rapports avec mon employeur (le CEA et l'équipe LIST que je remercie au passage de son hospitalité pendant mes passages à Paris). Mohamed m'a aussi donné de précieux conseils pour mes passages à l'oral, arrivant toujours à concentrer le message en des phrases simples et efficaces.

Je remercie tous les membres de mon jury de thèse qui ont eu la gentillesse de lire et commenter mon travail, plus particulièrement Philippe, mon arrière-grand-père suivant la généalogie thésard/directeur de thèse et rapporteur de ma thèse. À cause de plusieurs imperfections du manuscrit, il n'a pas pu mener à bien son projet courageux de tout relire dans les moindres détails, il m'a néanmoins permis de corriger une bonne partie du manuscrit qui aurait une tout autre forme sans son effort. I also warmly thank Radosław Adamczak who gave me valuable mathematical advice for the ultimate steps of my PhD.

Je remercie Gérard Besson et Pierre Comon responsables du suivi de ma thèse, ils ont été d'excellents interlocuteurs scientifiques.

Je remercie mes coreligionnaires de la Couillet's team avec qui j'ai pu faire et je continuerai à faire de belles collaborations Mohamed, Zhenyu, Malik, Xioyi. Je remercie tous les professeurs et chercheurs du GIPSA lab pour l'admirable atmosphère de camaraderie et de recherche qui y règne. Je remercie mes colocataires et collègues pour les repas et les pauses plus ou moins conventionnelles mais toujours revivifiantes passées ensemble. Merci à Priscilla, Pierre, Jeanne, Roza, Emmanuelle, Pedro, Imane, Raphaël, Cyprien, Kevin, Charles, Tayeb, Fateme, Ygit, Lorenzo, Reda, Hugues, Hugo, Victor.

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Introduction

a Concentration of the measure as an hypothesis on the data

A history of concentration

Let us describe here a phenomenon concerning distributions as diverse as the Gaussian distribution, the uniform distribution on the ball, on the discrete cube, and leading to extensively used statistical concentration inequalities like Hoeffding inequality, Bernstein inequality, Hanson Wright inequality... Although being lately mentioned in advanced statistical courses, *concentration of the measure* appears as a core justification for a lot of statistical phenomena and rightly deserves the increasing attention it gained in the last decades. Its appearance and first formalization is quite recent if we consider the long story of mathematics. Originally, Vitali Milman introduced this notion as an outburst of a geometrical result of isoperimetric inequalities on the sphere owned to Paul Lévy [Lév67] and brought up to date in [Mil71]. The formulation of the objective to bound the measure of ε -blowup of measurable sets [MS86] then opened the path to probabilistic development of this geometrical result. Some interpretation with Ricci curvature was brought by Gromov [GM83] and the active work of Talagrand [Tal88, Tal95] in the 80's provided a subtle understanding of independence under this theory. One further needs to mention the interpretation with log-Sobolev inequalities [Led05] and very recent work to give full characterization of dimension free concentration of product measure with transportation inequalities [GRST17, GRS⁺18].

Those fifty years of active research in the field brought very powerful, diverse and accurate concentration inequalities and our thesis make use of a very small amount of them. Our goal here is indeed not to contribute to the general theory of concentration of the measure but more modestly to employ some of its important results to deduce inferences on statistical object found in the theory of random matrices and useful for the understanding of basic machine learning methods.

Concentration of the measure in Statistics

One could regret that statistical course easily turn out into tedious catalogues of inequalities quite far from the mathematical aestheticism we cherish. It is somehow disappointing to see that the beautiful formalism issued from Lebesgue theory to express probability concept ends with such specific and scattered notions. We rather advocate here – when possible – for a standard approach to address statistical issue. The concentration of measure theory and the objects it naturally brings forwards seems well fitted to this role. To obtain optimal bounds, one often naturally needs to be slightly specific, it might however sometimes be relevant to lose accuracy at the benefit of generality, it gives the picture some reassuring unity and more importantly, it opens the path to multiple easily reachable side inferences.

To give a precise idea of the concentration inequalities we will be working with, let us start with a famous result of Talagrand. Given a random vector $X = (x_1, \dots, x_n) \in [0, 1]^n$ having independent entries, for any convex mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}$, 1-Lipschitz for the euclidean norm:

$$\forall t > 0 : \mathbb{P}(|f(X) - \mathbb{E}[f(X)]| \geq t) \leq Ce^{-ct^2}, \quad (1)$$

with $C = 2$ and $c = 1/4$. A same inequality is obtained for a Gaussian vector with independent, zero mean, unit variance entries – this time true for any 1-Lipschitz mappings f (no need to be convex). A straight forward proof of this result with non-optimal constants can be found in [Tao12, Theorem 2.1.12] that presents

in one page a very efficient justification provided by Maurey and Pisier. Since this kind of results are valid for any $t > 0$, one can then play on this parameter to give bounds on the moments of $f(X)$ with Fubini inequality (Proposition A.20). One could also consider other mappings f that would have different Lipschitz parameter. In a way it could be seen as a kind of Hoeffding inequality that would concern far more “observations” of the random vector than the mere sum of its entries.

In all our work this kind of inequality will serve both to express the hypotheses and the results. It will often be the starting point and the goal. We will develop to help us a wide range of tools that will explore the stability of the class of vectors satisfying this kind of inequality through concatenation, sum, product and some implicit formulation. But before developing these aspects principally needed for random matrix estimations, let us provide a geometric interpretation.

Geometric interpretation - notion of observable diameter

Let us consider the historical result of concentration of measure owned to Lévy and that states that any random vector X , uniformly distributed on the sphere $\sqrt{n}\mathbb{S}^{n-1}$ satisfies (1) for $C = 2$ and $c = \frac{1}{2}$ and for any 1-Lipschitz mappings (not just the convex mappings). We schematically represented some drawings of this distribution on Figure 1, and the images of these drawings through 1-Lipschitz mappings for the euclidean norm that are traditionally called *observations*. We choose for our illustration a normalized sum and the

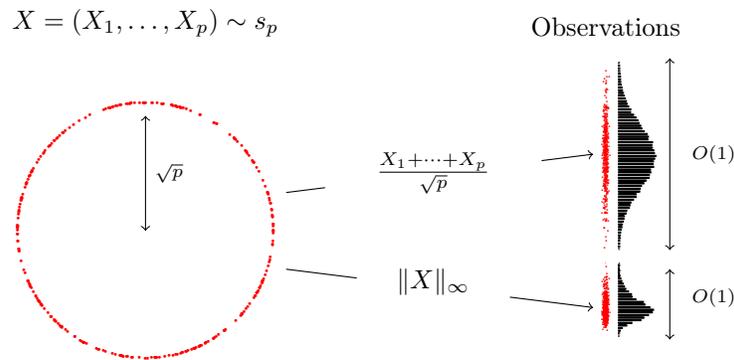


Figure 1: Schematic representation of the uniform distribution on the sphere $\sqrt{n}\mathbb{S}^{n-1}$ and some of its 1-Lipschitz observations. The observable diameter is $1/\sqrt{n}$ smaller than the diameter of the distribution.

infinite norm. What should be noted is that inequality (1) being independent with n we see that the two 1-Lipschitz functionals of the random vector X both have a standard deviation of order $O(1)$ (when n tends to infinity). However, the diameter of the distribution that one could define as being equal to $\mathbb{E}[\|X - X'\|]$, for X' , an independent copy of X , is of order $O(\sqrt{n})$. The essence of concentration of measure is exactly this big order difference between the distribution diameter and what Gromov called “observation diameter” and that we want to define here more simply as the quantity

$$D_o(X) \equiv \sup_{\substack{f: \mathbb{R}^n \rightarrow \mathbb{R} \\ \text{1-Lipschitz}}} \mathbb{E}[\|f(X) - f(X')\|], \quad (2)$$

for X' an independent copy of X . In this theory, the concentration of a vector can be strictly defined through the concentration of its observation that in a sense precipitate in \mathbb{R} the high dimensional concentrated structure.

Example of non concentrated random vectors

The presentation would be incomplete without counterexample. One may consider for instance the random vector $Z = [X, \dots, X] \in \mathbb{R}^n$ where $X \sim \mathcal{N}(0, 1)$. Here the metric diameter is of order $O(\sqrt{n})$, which is the

same as the diameter of the observation $\frac{1}{\sqrt{n}}(X + \dots + X)$ (the mapping $(z_1, \dots, z_n) \mapsto \frac{1}{\sqrt{n}}(z_1 + \dots + z_n)$ is 1-Lipschitz).

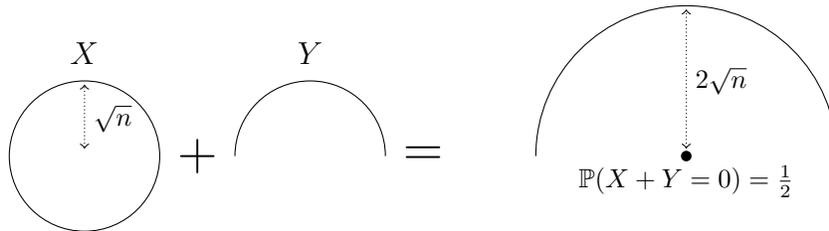


Figure 2: The sum of two concentrated random vectors can be non concentrated

Actually in this last example, any sub-sequence of the entries of Z would form a non concentrated random vector. For the sake of completeness, let us now be slightly more crafty and present as depicted on Figure 2, an example where X and Y are concentrated but not (X, Y) . This example is obtained considering a random vector X , uniformly distributed on the sphere $\sqrt{n}\mathbb{S}^{n-1}$ and an other random vector $Y \in \mathbb{R}^n$ defined as $Y = \phi(X)$ where, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\phi(x) = x$ if $x_1 \geq 0$ and $\phi(x) = -x$ otherwise. All linear observations of $X + Y$ are concentrated (it can be shown that the linear concentration is stable through summation) but this is not the case for all Lipschitz observations. The observation $\|X + Y\|$ has a one-out-of-two chance to be equal to 0 or to be equal to $2\sqrt{n}$ which means that:

$$\mathbb{P}(\|X\| - \|X'\| \geq \sqrt{n}) = \frac{1}{2}.$$

One can then show that the observable diameter is at least of order $O(\sqrt{n})$, which is the metric diameter of $X + Y$. This effect is due to the fact that f is clearly not Lipschitz, and Y in a sense “defies” X .

Relevant hypothesis for machine learning regular data sets

Apart from its structural flexibility that will be demonstrated in the first chapter of this manuscript, the theory of concentration of the measure also appears, for several aspects, to be well adapted to the study of regular machine learning problems. When X is Gaussian with zero mean and unit variance or when it is uniformly distributed on the sphere $\sqrt{n}\mathbb{S}^{n-1}$ (which is a very similar hypothesis thanks to a well known result of Poincaré) inequality (1) is satisfied for any 1-Lipschitz mapping. That means in particular that any 1-Lipschitz transformation $\Phi(X) \in \mathbb{R}^n$ of such vector satisfies the same inequality. This aspect is of capital importance because it opens the possibility to construct a lot of concentrated vectors that would not present independence relation between the entry or any symmetry relation as the one imposed by Gaussian distribution or uniform distribution on the sphere. In particular it allows us to believe that this concentration hypothesis can be adopted to a lot of diverse machine learning problems.

To give a partial but somehow convincing justification for the adoption of this assumption as a basis to machine learning theoretical inferences, we resort to the fundamental example of Generative adversarial neural network (GAN) that are able to construct realistic image as Lipschitz transformation of a Gaussian noise, demonstrating this way that some distribution of concentrated random vectors are very close to the data encountered in machine learning problems ([SLTC20]). Figure 3 presents some interesting samples of some of these artificially generated images.

Actually what seems to limit the range of application is the exponential concentration function on the right of the inequality in (1). It imposes that all the moments of the entries are bounded which exclude most of heavy tailed distribution from the scope of our study. Some inferences could be made with other concentration functions, but since these uses are quite marginal we did not take the time to explore these possibilities. Still, in Part III we study the performances of a robust estimation of scatter matrices of distribution presenting an exponentially concentrated component and an heavy tailed component (like Student distributions for instance). In that case, we adopt some light hypotheses on the first moments of the heavy tailed component to be able to infer some convergence bounds.



Figure 3: GAN generated images built from Gaussian noise with Lipschitz transformations. They follow a concentrated distribution.

More generally, it should be pointed out that because of the averaging effects it introduce, the concentration of measure phenomenon is perceptible in almost any machine learning algorithm and should therefore be kept in mind, the same way that data scientist keep constant concern of over-fitting to produce efficient algorithm. In particular concentration of the measure might explain how the number of weights in neural networks is related to the variability of the outputs and speed of exploration.

b Resort to Random Matrix Theory to predict performances

Performances related to first statistics of the data

When looking forward to express the performances of a given algorithm, one has to wonder with which elements of the problem he will be able to express the result. For modern image classification tasks for instance, to be able to accurately distinguish images of cats from images of dogs, it is not easy to assess what are the relevant elements of the distribution of cats and dogs on which depends the efficiency of the algorithm. The truth is that it relies on feature extraction which is peculiar to each kind of data, therefore attempts to formalize such tasks and, in a sense, “measure their difficulty” seems rather arduous.

Hopefully there exists still a wide range of algorithms whose performances is still under question and that appear to rely on the first statistics of the distributions, that is to say the expectation and the covariance of the different populations. These algorithms are the one that we targeted for our approach. This concerns mere ridge regression, but also kernel methods, support vector machine, and more elaborated regularized empirical risk minimization problems (see Chapter J in Part III).

A first reason to justify this ultimate dependence on the first statistics that we could put forward – although we could not rigorously apply it – is a result from [KGC18, NBA⁺18] that states that concentrated random vector projections have all the chances to behave like Gaussian vectors after projection on a deterministic or independent random vector.

Theorem 0.1 (CLT for concentrated vectors [Kla07, FGP07]). *Let $X \in \mathbb{R}^n$ be a random vector with $\mathbb{E}[X] = 0$ and $\mathbb{E}[XX^T] = I_p$, and ν be the uniform measure on the sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^p$ of radius 1. If X satisfies (1) for any 1-Lipschitz mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and for two constants $C, c > 0$ independent with n , then there exist three other constants $C, c, \kappa > 0$ independent with n and a set $\Theta \subset \mathcal{S}^{n-1}$ such that $\nu(\Theta) \geq 1 - \sqrt{n}Ce^{-c\sqrt{n}}$ and $\forall \theta \in \Theta$:*

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(\theta^T X \geq t) - G(t)| \leq \kappa n^{-1/4},$$

for $G(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-u^2/2} du$.

What prevents practical application of this theorem is the fact that one generally needs this central limit theorem for specific deterministic vectors θ that could sadly be part of the low probability set Θ^c . Therefore,

although one can note in practice that all the projections of concentrated random vectors have Gaussian distributions, there is no way to resort to this theorem.

Hopefully there exists an other rigorously established reason to justified why all the studied methods lead to a mere formula on the covariance. It is to be found in random matrix theory thanks to results similar to the free central limit theorem that states that sufficient independence between the entries leads to the convergence of the spectral measure to a law depending only on the two first statistics.

Random Matrix Theory for Machine Learning

The well known “curse of dimensionality” imposes to take into account both the number of data and the size of the data to provide rigorous estimations. This is the natural setting of random matrices hence the resort to this theory for a rigorous study of machine learning algorithm. The random matrices we study are the same as the one introduced by Wishart at the beginning of the 20st century [Wis28]. They write $\frac{1}{n}XX^T$ for given random matrices $X = (x_1, \dots, x_n)$ of independent columns that typically represent the different samples of a data set (Wishart was rather considering Matrices with independent Gaussian entries). The non centered sample covariance matrix $\frac{1}{n}XX^T$ naturally appears in a lot of supervised or unsupervised machine learning problem because it contains a lot of information regarding the relations between the means and the covariances of the different populations of the $(x_i)_{i \in [n]}$ but also about the “signals” that might be carried by several of the x_i ’s. In addition, it can be related to kernel matrices $\frac{1}{n}X^T X$ or $\frac{1}{n}\phi(X^T X)$ where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ acts entry-wise, as it has been shown in [CB16].

The core objective of the theory is the understanding of the law of the eigen values. Without looking at the mutual dependencies of the random eigen values, we will be particularly interested here in the mere spectral distribution

$$\frac{1}{p} \sum_{i=1}^p \delta_{\lambda_i},$$

where $\lambda_1, \dots, \lambda_p$ are the eigen values of $\frac{1}{n}XX^T$. The spectral distribution is traditionally estimated thanks to the Stiltjies transform $g : \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_p\} \rightarrow \mathbb{C}$ defined for any $z \in \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_p\}$ as:

$$g(z) \equiv \frac{1}{p} \sum_{i=1}^p \frac{1}{\lambda_i - z}.$$

However, the problems that interest us generally require to look at the projection on the eigen spaces of $\frac{1}{n}XX^T$. Insights on this matricial projection are provided thanks to the so-called resolvent $R(z) = (\frac{1}{n}XX^T - zI_p)^{-1}$ that satisfies in particular $g(z) = \frac{1}{p} \text{Tr}(R(z))$. Without going to far into details provided at the beginning of Part II, let us simply state that one needs at what point to estimate quantities of the form:

$$\text{Tr}(AR(z))$$

for deterministic matrices A satisfying $\|A\|_F \leq 1$ (where $\|\cdot\|_F$ denotes the Frobenius norm i.e. the natural euclidean norm on $\mathcal{M}_{p,n}$, also called the Hilbert-Schmidt norm). One of our important achievement is the expression of tight convergence bounds of the resolvent towards a so-called “deterministic equivalent” which is a deterministic matrix allowing to estimate linear forms of the resolvent. In particular we introduce a deterministic matrix $\tilde{R}(z)$ such that, given a constant $\varepsilon > 0$ independent with n , for all z at a distance bigger than ε to the spectrum of $\frac{1}{n}XX^T$, we have the concentration inequality:

$$\mathbb{P} \left(\left| \text{Tr}(A(R(z) - \tilde{R}(z))) \right| \geq t \right) \leq Ce^{-cnt^2} + Ce^{-cn},$$

for some constants $C, c > 0$ independent on n . It is easy to show that $\tilde{R}(z) = \mathbb{E}[R(z)]$ would satisfy this inequality, the main difficulty is actually to show the convergence of $\mathbb{E}[R(z)]$ towards a computable estimate. This computable deterministic equivalent is introduced thanks to a well known fixed point equation displayed in (3) below.

Natural hypotheses on the data matrix

We spend a lot of time on the choice of a good trade-off between the generality of the hypotheses and the efficiency of the result. Several settings were tried and for simplicity we will just present in this manuscript two of them. All over the manuscript, we will use the notations:

$$m_i = \mathbb{E}[x_i], \quad C_i \equiv \mathbb{E}[x_i x_i^T] \quad \text{and} \quad \Sigma_i \equiv C_i - m_i m_i^T.$$

First we assume that $p = p_n$ seen as a sequence of integer numbers depending on n satisfies $p \leq O(n)$. Some authors also require that $n \leq O(p)$ but we did not see the importance in our proof since the normalization of XX^T is done with n and not with p .

Second, we require $\sup_{i \in [n]} \|\mathbb{E}[x_i]\|$ to be bounded independently with n (and p). We provided some results for the case when the data x_1, \dots, x_n could share a finite number of signals of order $O(\sqrt{n})$ in [Lou21] but we eventually decided not to mention them here because the estimation of $\text{Tr}(AR(z))$ when $\|A\|_F \leq 1$ with good convergence bounds requires the hypothesis $\sup_{i \in [n]} \|\mathbb{E}[x_i]\| \leq O(1)$ mainly because of Proposition B.50

When the columns x_1, \dots, x_n follows a non bounded number of distributions, we needed to assume that for all $i \in [n]$, $\Sigma_i \geq \kappa I_p$, for some constant $\kappa > 0$. However we are not sure that this assumption is necessary.

Finally the most important hypothesis is of course the concentration of the data matrix X that can follow two kind of assumptions (i) the Lipschitz concentration where X satisfies (1) for any 1-Lipschitz mapping for the Frobenius norm and (ii) the convex concentration where X satisfies (1) only for 1-Lipschitz and convex mappings. One could have wished to take advantage of the independence of the data x_1, \dots, x_n to only adopt assumptions on the distributions of the x_i s and not on the whole matrix. One can indeed find in the literature some log-concave hypotheses that are stable through product of measures. But the natural optimal hypotheses should rather be found in [GRST17, GRS⁺18] where it is shown that some transportation inequalities ensures what they call “dimension free concentration inequalities”. More precisely, they give a necessary and sufficient condition on the laws of the x_i for the matrix X to be concentrated independently with n (with conditions provided both for Lipschitz and convex concentration). This result is of high importance, however, we will not make use of it for the sake of simplicity and merely assume the concentration of the matrix X as a whole.

c Notation and operation with concentration inequalities

Notations

Originally, concentration of the measure was introduced thanks to geometrical ideas concerning very general object expressed with measures and integrals signs rather than probabilities (with \mathbb{P}) and expectations (with \mathbb{E}). At that time, the resort to the conceptual formalism of probabilities would have been useless and inefficient since the theory was first conceived from a subtle control on the measurable sets. However, once the important theorem on fundamental concentrated class of distributions are introduced, it appears appropriate to let the formalism conceptually evolve in order to simplify calculus.

We meet in the study of concrete statistical problems (the best exemple might be the study of $(I_p - \frac{1}{n}XDX^T)^{-1}$ with X, D concentrated non independent in Part G) the constant resort to the kind of inequalities depicted in (1). This kind of equality admits diverse variants that are not valid for the same class of function f , for the same parameters $C, c > 0$ nor for the same concentration function $t \mapsto Ce^{-ct^2}$. For this reason, instead of rewriting each time the same heavy formulation or mentioning an equation reference with specifications on the precise setting, we pretentiously introduce a graphic personal notation that will hopefully simplify and clarify the inferences.

The central object when looking at concentration of the measure (that only concerns big dimension) is the observable diameter that is given by the parameter c in (1). The parameter C is generally – at least in our applications – a numerical constant that we maintain to be independent with the dimensions of the problem (mainly p and n). Therefore we chose a notation able to (i) provide the class of function f for which (1) is valid, (ii) display the concentration function and (iii) the observable diameter.

As an example (full rigorous precision is given in Part I, Definitions 9, 14 and 18 for respectively the Lipschitz, linear and convex concentration), if (1) is valid for any 1-Lipschitz mappings f , and c writes c'/σ_n^2

for a given sequence $\sigma = (\sigma_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ depending on n and a constant c' , independent with n , we note:

$$X \propto \mathcal{E}_2(\sigma).$$

Operations

To justify the choice and demonstrate the efficiency of such a notation, let us briefly present its combination with some of the operation on concentrated vectors. Let us consider $X, Y \in \mathbb{R}^n$ satisfying $X \propto \mathcal{E}_2(\sigma)$ and $Y \propto \mathcal{E}_2(\theta)$ for given sequences $\sigma = (\sigma_n)_{n \in \mathbb{N}}$ and $\theta = (\theta_n)_{n \in \mathbb{N}}$ of $\mathbb{R}_+^{\mathbb{N}}$. One can express the concentration of different operations:

- *The Lipschitz transformation* : given a sequence of λ_n -Lipschitz transformations $\Phi_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$:

$$\Phi(X) \propto \mathcal{E}_2(\lambda\sigma).$$

- *The sum*:

$$X + Y \propto \mathcal{E}_2(\sigma + \theta).$$

- *The product*. Given a product $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying $\forall x, y \in \mathbb{R}^n, \|xy\| \leq \|x\|\|y\|$, we have the concentration:

$$XY \propto \mathcal{E}_2(\mathbb{E}[\|Y\|]\sigma + \mathbb{E}[\|Y\|]\theta) + \mathcal{E}_1(\sigma\theta),$$

where $\mathcal{E}_1(\sigma)$ denotes a concentration function $t \mapsto Ce^{-ct/\sigma_n}$.

- *The argmin of a convex problem*. Given a deterministic \mathcal{C}^1 mapping $\Phi(X, Y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\forall x, y \in \mathbb{R}^n \times \mathbb{R}^p, \|\frac{\partial \phi}{\partial X}|_{(x,y)}\| \leq \nu$ and $\frac{\partial \phi}{\partial Y}|_{(x,y)}$ is a positive symmetric matrix satisfying $\frac{\partial \phi}{\partial Z}|_{(x,y)} \geq \kappa I_p$, for some sequences $\nu, \kappa \in \mathbb{R}_+^{\mathbb{N}}$, the unique solution to the equation:

$$\phi(X, Z) = 0, \quad Z \in \mathbb{R}^p$$

satisfies the concentration inequality $Z \propto \mathcal{E}_2(\frac{\sigma\nu}{\kappa})$

d Outline of the work

Concentration basics

The first part of the thesis aims at setting the ground for a concentration of measure approach to Machine learning algorithms. The definitions and basic properties are mainly taken from [Led05] and we added a lot of personal contribution that can be found in [LC18, LC22a]. The presentation first gives a lot of diverse inferences that can be obtained for random variables, we adapt them to the high dimension case in a second step with the introduction of three notions of concentration (Lipschitz, convex and linear) that are equivalent in low dimension. We then provide the concentration rules for the major operations that will be necessary in the study of machine learning algorithms. Mainly we look at norms, concatenations, products (and more importantly pseudo products allowing control on the variability towards each variable), solutions to convex minimization problems and contractive fixed point equations...

We display at the end of this presentation of concentration our personal approach to convex concentration with the introduction of the transversal concentration to lighten the convexity hypotheses required for an observation to concentrate under this rigid concentration setting. Our contribution in this last part is purely formal since most of the results (in particular the concentration of the product of convexly concentrated random vectors) was already known. It is however useful for the next part of the thesis that will set in particular the concentration and the estimation of the resolvent under convex concentration settings.

Concentration and estimation of the resolvent

The second part of the manuscript employs the different tools and notations of concentration of the measure to state the concentration and estimation of the resolvent, it mainly relies on the paper [LC22c]. If one introduces the notations:

$$\mathcal{I}^z : \Delta \mapsto \text{Diag}_{i \in [n]} \left(z - \text{Tr} \left(\Sigma_i \tilde{Q}^\Delta \right) \right) \quad \text{and} \quad \tilde{Q}^\Delta \equiv \left(I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\Delta} \right)^{-1}, \quad (3)$$

then the matrix $\tilde{Q}^{\tilde{\Lambda}^z}$, where $\tilde{\Lambda}^z$ satisfies $\tilde{\Lambda}^z = \mathcal{I}^z(\tilde{\Lambda}^z)$, is a deterministic equivalent of the matrix $Q^z \equiv (I_n - \frac{1}{zn} XX^T)^{-1}$. To be able to show the existence and uniqueness of $\tilde{\Lambda}^z$, but also to get efficient bounds on the concentration in the case where the columns of X are not identically distributed, one needs to introduce a semi-metric quite close to the hyperbolic metric and that writes for two diagonal matrices $D, D' \in \mathcal{D}_n(\mathbb{C})$:

$$d_s(D, D') = \left| \frac{D - D'}{\sqrt{\Im(D)\Im(D')}} \right|.$$

On a subset of $\mathcal{D}_n(\mathbb{C})$, the mapping \mathcal{I}^z is 1-Lipschitz for the semi metric d_s .

Quite interestingly, a very analog semi-metric defined on $\mathcal{D}_n(\mathbb{R}_+)$ (therefore without the imaginary parts) is used to prove in Part III, Chapter I the validity of some robust estimators of scatter matrices. That is why we present at the beginning of the part on the resolvent a full chapter on these two semi-metrics that share strong properties of stability.

With the semimetric d_s at hand it is possible after elaborate inferences to set the quasi-asymptotic convergence of the resolvent $R(z) \equiv (zI_p - \frac{1}{n} XX^T)^{-1}$ towards its deterministic equivalent for a parameter z being possibly close to the spectrum (but independent with n) and with convergence bounds provided in $O(1/\sqrt{n})$ for the Frobenius norm.

We then estimate the power of the resolvent that appears in the expression of the mean square error of the Ridge regression and of ELMs studied in Part III, Chapter H. Actually the estimation of the square is enough but we decided to put the estimation a little bit further since we found the formulas somehow inspiring.

We further provide a brief study of the resolvent under convex concentration hypotheses that was conducted in [Lou21]. The results are naturally weaker than what could be obtained under Lipschitz concentration hypotheses, in particular we could just obtain the linear concentration of Q but we provide some techniques to deduce similarly the concentration of matrices of the form QAX for given deterministic matrix $A \in \mathcal{M}_p$ or any $QP(XX^T)$ for a polynomial P having coefficient in \mathcal{M}_p .

We end this part with an important chapter relying on the previous estimation of the resolvent and some inferences on the concentration of the product of random vectors to set conditions that allow to estimate resolvents $(I_p - XDX^T)^{-1}$ when X and D are random and non independent. It requires a fine work of estimation that, to our knowledge, was not done before and that takes advantage of the short notations of concentration we introduced. Under certain condition on the dependence of D towards each of the columns x_i of X , it is possible to devise a deterministic equivalent for $(I_n - \frac{1}{n} XDX^T)^{-1}$. It relies on the estimation of $(I_n - \frac{1}{n} X\tilde{D}X^T)^{-1}$ for a deterministic matrix \tilde{D} , which is solved by the previous study of $(I_n - \frac{1}{n} XX^T)^{-1}$ when the columns of the x_1, \dots, x_n are not identically distributed. This estimation is at the core of the justification of the leave-one-out method exposed in Part III, Chapter J.

Applications to machine learning

The last part is dedicated to a selection of application in machine learning. Following the work presented in [LLC18], we first study the so-called “extreme learning machines” (ELM) where one wants to fits a label vector $Y = (y_1, \dots, y_n) \in \mathcal{M}_{d,n}$ that associates to each datum x_i a label $y_i \in \mathbb{R}^d$ that could characterize its class or a specific feature that one wants to estimate. The method consists in solving the problem:

$$\text{Minimize } \frac{1}{n} \sum_{i=1}^n \|\beta^T \sigma(W^T x_i) - y_i\|^2 + \gamma \|\beta\|_F^2, \quad \beta \in \mathbb{R}^q,$$

for given $\gamma > 0$, $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ applied entry-wise and $W \in \mathcal{M}_{p,q}$, a random matrix aiming at providing “random features” of the data X . This work does not take into account the randomness of X , it basically shows that the algorithm acts asymptotically like a kernel method whose performances closely depends on the matrix $\frac{1}{n}E[\sigma(W^T X)^T \sigma(W^T X)]$ that we express for different activation functions σ .

A second work that is taken from [LC22b] builds some bridge with heavy tailed statistics. The goal is to estimate a Scatter matrix which is defined as a surrogate for covariance matrices for distribution that does not admit second order moment. The data matrix we consider decomposes:

$$X = Z\sqrt{\tau} + m\mathbb{1}^T,$$

where $m \in \mathbb{R}^p$ is a deterministic mean, $Z \propto \mathcal{E}_2(1)$ has zero mean and τ might not admit a second order moment, we just need to assume that $\frac{1}{n} \sum_{i=1}^n \tau_i \leq O(1)$. The diagonal matrix τ is seen here like a noise and one wants to retrieve m and statistical properties of Z .

This is done thanks to the computation of \hat{C} , the M -estimator of the scatter matrix defined as unique solution to the equation:

$$\hat{C} = \phi(\hat{C}) \quad \text{where } \phi : M \mapsto \frac{1}{n} \sum_{i=1}^n u \left(\frac{1}{n} x_i^T (M + \gamma I_p)^{-1} x_i \right) x_i x_i^T,$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies $\lim_{x \rightarrow \infty} u(x) = 0$. The resolution of the problem starts from the remark that ϕ is 1-Lipschitz for the semi-metric d_s defined for any $D, D' \in \mathcal{D}_n(\mathbb{R}_+)$:

$$d_s(D, D') = \sup_{i \in [n]} \left| \frac{D_i - D'_i}{\sqrt{D_i D'_i}} \right|.$$

Then one can deduce properties that u has to satisfy to ensure existence, uniqueness and concentration of \hat{C} . We eventually provide an estimation of the the resolvent \hat{C} from which we can deduce the convergence of its spectral distribution.

The study of extreme learning machines is quite simple because β admits an explicit form $\frac{1}{n} \Sigma \left(\frac{1}{n} \Sigma^T \Sigma + \gamma I_n \right)^{-1} Y^T$. In a last work, we study the implicit formulation of solutions to empirical risk minimization that writes:

$$\text{Minimize } \frac{1}{n} \sum_{i=1}^n h_i(\beta^T x_i)^2 + \gamma \|\beta\|^2, \quad \beta \in \mathbb{R}^q,$$

where $h_1, \dots, h_n : \mathbb{R} \rightarrow \mathbb{R}$ are n deterministic convex mappings. We got our main inspiration from [Kar13] who introduces a technique called “leave-one-out” to be able to estimate properly the statistics of β . The idea is to set a relation between β and a vector β_{-i} that would be deprived of the contribution of x_i , then, replacing $\beta^T x_i$ with $\beta_{-i}^T x_i$, it should be possible to obtain, under Gaussian hypotheses on x_i , a pseudo-fixed point equation on the first statistics of the Gaussian random variable $\beta_{-i}^T x_i$.

We were able to set rigorously the leave-one-out method validity without Gaussian hypotheses and thanks to the estimations we did on the resolvent $(I_p - \frac{1}{n} X D X^T)^{-1}$ in Part II. However we are still not able to show the existence and uniqueness of the final fixed point equation providing estimates for the first statistics of β . We believe that the Gaussian hypotheses are not necessary and we could heuristically design a simpler fixed point equation that was verified empirically and that we hope to mathematically verify soon, with the widest hypotheses possible.

e Open questions, next steps

Our main efforts in this thesis were dedicated to the devise of a proper concentration of measure and random matrix framework to study machine learning algorithms. Although both theory were quite advanced, some work still needed to be done among which we can mention the estimation of the resolvent of matrices $\frac{1}{n} X D X^T$ with D and X random and non independent or the concentration of the solution of convex minimization problems and pseudo-product of random vectors. Most of the time was spent on up-stream theoretical

demonstrations, and as a consequence, practical results on machine learning performances sometimes lack thorough study of the possible improvements of the method. It would be however very interesting to compare the different methods and identify a standard approach leading to possible generalization, the same way that ELM asymptotically meet mere Kernel methods. This question is not easy in such an active field of research where the high rate of publication and the need to follow day-to-day technical development imposes a fast pace that undermine any attempt to unify a possible theory of Machine learning. Some replica methods already allowed to express the minimal classification error that an algorithm could reach for very simple linear Gaussian models [BKM⁺17, LM19]. We can then wonder if these inferences can generalize to more complex models, and the concentration of measure framework might be perfectly fitted to this task.

Naturally, the more elaborate an algorithm becomes, the harder it is to find a valid theoretical formalization. In particular, we do not envision clearly which random matrix theory inferences might be employed to solve modern machine learning problems evolving neural networks. Mainly because of the aforementioned reason that the performances depends on complex features very specific to the data and quite far from the traditional objects seen in random matrices. After working with classical deep learning neural networks one might have the impression that the main subtlety is to be found in the feature design (which could be more related to traditional sound processing, image processing...) and the dense layers at the end of the network just work as highly weighted regressors of low interest that merely try to approach multivariate mappings with piecewise linear mappings (in case of a ReLu activation).

The more advanced (and successful) methods relying on the mechanism of attention [VSP⁺17] look far more interesting. This mechanism is a natural improvement of the “blind” convolutions in image processing because it allows to find correlation inside the data relevant for the given task. The so-called “transformer neural networks”, after becoming state-of-the-art language processing algorithms, shew high performances on image processing and other diverse tasks thanks to an impressively flexible architecture able to adapt to a high variety of data. Their modular architecture and the simple scaling it allows seems perfectly fitted to interesting mathematical formalism that might require some information theory inferences.

Finally, if we do not look at the methods but at the data, an interesting development might be to try to adapt the tools we presented to sparse settings. It is quite fascinating to see that under sparse assumptions the spectral distribution of the sample covariance matrix still converges to a law but not to the Marcenko Pastur law. The asymptotic spectral distribution was computed with some physicist inferences thanks to so called “cavity method” [RCKT08] or “replica method” [Küh08]. More complex work was done to understand the behavior of the largest eigen values of Wigner-like sparse matrices [EKYY13, HLY20] and sample covariance matrices in [HLS19]. In the case of Wigner-like matrices it seems complicated to remove the hypothesis of independence between the entries but for sample covariances matrices, it would be interesting to keep our concentration of measure approach to understand sparse statistics appearing in a lot of practical examples (PCA with sparse data, adjacency matrix of directed graph). In particular, one would like to know what is the structural difference, in terms of concentration, that introduces a transition in the shape of the spectral distribution between non-sparse and sparse data.

In a field of research where the quantity is often provided at the expense of quality, rigorous theoretical insights are more than necessary to put some order and identify the meaningful axes of research. The task seems enormous but we hope that its theoretical attractiveness will bring data scientists to have a more sensible approach to their work.

Symbols

iif	“if and only if”
E	Typical normed vector space over \mathbb{R} , endowed with the norm $\ \cdot\ $.
E^*	Dual space of E (the set of linear maps from E to \mathbb{R}).
$\text{Conv}(A)$	Convex hull of $A \subset E$ (i.e., $\text{Conv}(A) = \bigcap \{C \subset E, C \text{ convex}, C \supset A\}$).
A_*	$A \setminus \{0\}$
A^c	Complementary of A . If $A \subset E$, $A^c = E \setminus A = \{x \in E, x \notin A\}$.
$\mathbb{1}_A$	Indicator function of $A \subset E$, $\mathbb{1}_A : E \rightarrow \{0, 1\}$ and $\mathbb{1}_A(x) = 1 \Leftrightarrow x \in A$. If A is not a set but an assertion (like $A = A(t) = (t \geq 1)$), $\mathbb{1}_A = 1$ if A is true and $\mathbb{1}_A = 0$ if A is false. If $\mathbb{1}$ presents no index, it designates a vector full of one with a convenient size for the context.
$\mathcal{F}(A, B)$	Set of mappings from A to B , $\mathcal{F}(A) = \mathcal{F}(A, B)$.
$\mathcal{L}(E, F)$	Given two vector spaces E, F , $\mathcal{L}(E, F)$ is the set of continuous linear mappings from E to F .
$\mathcal{A}(E, F)$	Given two vector spaces E, F , $\mathcal{A}(E, F)$ is the set of continuous affine mappings from E to F . If $f \in \mathcal{A}(E, F)$, $f - f(0) \in \mathcal{L}(E, F)$.
\mathbb{N}	Set of natural numbers
$[\mathfrak{n}]$	for $n \in \mathbb{N}$, $[\mathfrak{n}] = \{1, \dots, n\}$
\mathbb{R}	Set of real numbers ; $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$; $\mathbb{R}_- = -\mathbb{R}_+$. If $x \in \mathbb{R}^p$, one notes $[x]_i$, or more simply x_i , $1 \leq i \leq p$, the i^{th} entry of the vector x
$\lfloor x \rfloor$	Integer part of $x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{N}$ and verifies $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + 1 = \lceil x \rceil$.
\mathbb{S}^{p-1}	Sphere of \mathbb{R}^p ($\mathbb{S}^{p-1} = \{x \in \mathbb{R}^p \mid \ x\ = 1\}$).
i	Integer or complex number satisfying $i^2 = -1$
\mathbb{C}	Set of complex numbers.
$\Re(z), \Im(z)$	Real and Imaginary part of $z \in \mathbb{C}$, if $z = \Re(z) + i\Im(z)$ where $\Re(z), \Im(z) \in \mathbb{R}$
\mathbb{H}	Poincaré half plane $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$.
\bar{z}	conjugate of z , $\bar{z} = \Re(z) - \Im(z)i$
$\mathcal{M}_{p,n}$	Set of real matrices of size $p \times n$. If $M \in \mathcal{M}_{p,n}$, one notes $[M]_{i,i}$ or more simply $M_{i,j}$ the entry at the line i and column j . If $n = p$, we simply note $\mathcal{M}_p = \mathcal{M}_{p,p}$. Given a set A , $\mathcal{M}_{p,n}(A)$ is the set of matrices with entries in A
Diag	Diagonal operator. If $M \in \mathcal{M}_{p,n}$, $\text{Diag}(M) = (M_{i,i})_{1 \leq i \leq \min(p,n)}$; if $x \in \mathbb{R}^p$, $\text{Diag}_{q,n}(x) \in \mathcal{M}_{p,n}$, $[\text{Diag}_{p,n}(x)]_{i,i} = x_i$ if $1 \leq i \leq \min(p, q, n)$ and $[\text{Diag}_{p,n}(x)]_{i,j} = 0$ if $i \neq j$, for $1 \leq i \leq p, 1 \leq j \leq n$.
$\text{Sp}(M)$	Spectrum of the matrix M .
Tr	Trace operator on \mathcal{M}_p , $\forall M \in \mathcal{M}_p, \text{Tr } M = \sum_{i=1}^p M_{i,i}$.
\cdot^T	Transpose operator on \mathcal{M}_p , $\forall M \in \mathcal{M}_{p,n}, [M^T]_{i,j} = M_{j,i}, 1 \leq i \leq p, 1 \leq j \leq n$.
I_p	Identity matrix of \mathcal{M}_p , ($[I_p]_{i,j} = 0$ if $i \neq j$ and $[I_p]_{i,i} = 1, 1 \leq i, j \leq p$).
\mathfrak{S}_p	Set of permutations of $\{1, \dots, p\}$; $\mathfrak{S}_{p,n} = \mathfrak{S}_p \times \mathfrak{S}_n$.
\prec	Majorization relation, see Definition 19
x^\downarrow	Decreasing version of $x \in \mathbb{R}^p$, $\exists \sigma \in \mathfrak{S}_p$ such that $\forall i \in \{1, \dots, p\}, [x^\downarrow]_i = x_{\sigma(i)}$ and $[x^\downarrow]_i \leq [x^\downarrow]_{i-1}$ for $i \geq 2$.

\mathcal{S}_p	Set of symmetric matrices of $\mathcal{M}_p : S \in \mathcal{S}_n \Leftrightarrow S_{i,j} = S_{i,j}, 1 \leq i \leq p ; S \in \mathcal{S}_p^+ \Leftrightarrow \forall u \in \mathbb{R}^p, u^T S u \geq 0 ; S \in \mathcal{S}_p^- \Leftrightarrow -S \in \mathcal{S}_p^+$. Given $S_1, S_2 \in \mathcal{S}_p$, we say that S_1 is greater than S_2 and we note $S_1 \geq S_2$ if $S_1 - S_2 \in \mathcal{S}_p^+$.
$S^{1/2}$	Square root of the nonnegative symmetric matrix $S \in \mathcal{S}_p^+$ (with the diagonalization $S = P^T \Lambda P, P \in \mathcal{O}_p, \Lambda \in \mathcal{D}_p$, we define $S^{1/2} = P^T \Lambda^{1/2} P$ where $[\Lambda^{1/2}]_{i,i} = \Lambda_{i,i}^{1/2}$).
\mathcal{O}_p	Set of orthogonal matrices of $\mathcal{M}_p : P \in \mathcal{O}_{p,n} \Leftrightarrow P^{-1} = P^T ; \mathcal{O}_{p,n} = \mathcal{O}_p \times \mathcal{O}_n$.
$\mathcal{D}_{p,n}$	Set of diagonal matrices of $\mathcal{M}_{p,n} : D \in \mathcal{D}_{p,n} \Leftrightarrow D = \text{Diag}_{p,n}(\text{Diag}(D)) ; D \in \mathcal{D}_{p,n}^+ \Leftrightarrow D_{i,i} \geq 0, 1 \leq i \leq \min(p, n) ; D \in \mathcal{D}_{p,n}^- \Leftrightarrow -D \in \mathcal{D}_{p,n}^+$, we further note $\mathcal{D}_n^+ \equiv \mathcal{D}_n(\mathbb{R}_+)$. Given a set $A, \mathcal{D}_{p,n}(A)$ is the set of diagonal matrices with entries in A . When $n = p$, we simply note $\mathcal{D}_p = \mathcal{D}_{p,p}$.
\mathcal{P}_p	Set of permutation matrices of $\mathcal{M}_p : P \in \mathcal{P}_p \Leftrightarrow P \in \mathcal{O}_n$ and $(\exists \sigma \in \mathfrak{S}_p, P_{i,j} = 1 \Leftrightarrow \sigma(i) = j) ;$ we also define $\mathcal{P}_{p,n} = \{(U, V) \in \mathcal{P}_p \times \mathcal{P}_n \mid U I_{p,n} V^T = I_{p,n}\}$ where $I_{p,n} = \text{Diag}_{p,n}(\mathbb{1})$.
$\ \cdot\ _q$	ℓ_q -norm on \mathbb{R}^p for two integers $p, q \in \mathbb{N}_* ; \ x\ _q = (\sum_{i=1}^p x_i^q)^{1/q}$.
$\ \cdot\ $	Classical norm of the vector space E one is working on : if $E = \mathbb{R}^p$, the euclidean norm $\ \cdot\ _2$; if $E = M_{p,n}$, the spectral norm $(\forall M \in \mathcal{M}_{p,n} : \sup_{\ u\ =1} \ M u\)$.
$\ \cdot\ _1$	on \mathbb{R}^p , the ℓ_1 ; on $M_{p,n}$, the nuclear norm $(\forall M \in \mathcal{M}_{p,n}, \ M\ _1 = \text{Tr}((M M^T)^{1/2}))$.
$\ \cdot\ _F$	Frobenius norm, $\forall M \in \mathcal{M}_{p,n} : \ M\ _F = \sqrt{\text{Tr} M^T M} = \sqrt{\sum_{i=1}^p \sum_{j=1}^n M_{i,j} ^2}$.
$\ \cdot\ _*$	Dual norm on E^* . If $f \in E^*, \ f\ _* = \sup_{\ x\ \leq 1} f(x)$.
$\ \cdot\ _{\ell^\infty}$	Depends on the use, given p normes vector spaces $(E_1, \ \cdot\ _1), \dots, (E_p, \ \cdot\ _p)$, given any $x_1, \dots, x_p \in E_1 \times \dots \times E_p, \ (x_1, \dots, x_n)\ _{\ell^\infty} = \sup_{i \in [p]} \ x_i\ _i$.
$\ \cdot\ _d$	Diagonal euclidean norm. For $M \in \mathcal{M}_{n,p}, \ M\ _d = \sqrt{\sum_{i \in \min(n,p)} M_{i,i} ^2}$
$\ \cdot\ _\infty, \ \cdot\ _{\mathcal{B}(y_0, r)}$	Infinite norm and infinite semi-norm on subset. Given $f : E \rightarrow (F, \ \cdot\ _F), \ f\ _\infty = \sup_{x \in E} \ f(x)\ _F, \ f\ _{\mathcal{B}(y_0, r)} = \sup_{x \in \mathcal{B}(y_0, r)} \ f(x)\ _F$.
$\ \cdot\ _{\mathcal{L}}$	Lipschitz semi-norm. Given $f : (E, \ \cdot\ _E) \rightarrow (F, \ \cdot\ _F), \ f\ _{\mathcal{L}} = \sup_{x \neq y} \frac{\ f(x) - f(y)\ _F}{\ x - y\ _E}$.
$\ \cdot\ _{A,B}$	For $A \in \mathcal{M}_p, B \in \mathcal{M}_n, M \in \mathcal{M}_{p,n}, \ M\ _{A,B} \equiv \ A M B\ _F$
\mathcal{B}_t	Closed ball of E of size $t > 0, \mathcal{B}_t = \{0\}_t = \{x \in E \mid \ x\ \leq t\}$, when $t = 1$, we note $\mathcal{B} = \mathcal{B}_1$. We also use the notation $\mathcal{B}_{\ \cdot\ }(x, t) = \{x\}_t = \{y \in E \mid \ x - y\ \leq t\}$, the index $\ \cdot\ $ could be of course a distance d or simply unspecified when we implicitly consider the classical norm of E .
\mathbb{P}	We implicitly suppose that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where \mathcal{F} is a sigma-algebra of the set Ω and \mathbb{P} , a probability measure defined on the elements of \mathcal{F} . The random vectors we consider are then \mathbb{P} -measurable applications defined on Ω and taking value in normed vector spaces endowed with the Borel σ -algebra. In that setting, given a random vector $X \in E$ (i.e. $X : \Omega \rightarrow E$), for any Borel set $A \subset E$, we note $\mathbb{P}(X \in A) = \mathbb{P}(\{\omega \in \Omega, X(\omega) \in A\})$ and for any measurable function $f : E \mapsto \mathbb{R}$ and $t \in \mathbb{R}$, we note $\mathbb{P}(f(X) \geq t) = \mathbb{P}(\{\omega \in \Omega, f(X(\omega)) \geq t\}) \dots$
\mathbb{E}	The expectation operator. For any random vector $X \in E$ and any measurable function $f : E \rightarrow \mathbb{R}$, we define $\mathbb{E}[f(X)] = \int_\Omega f \circ X d\mathbb{P}$. When E has finite dimension, it is possible to define $\mathbb{E}X$ by integrating all the coordinates of X .
$X(\mathcal{A})$	Given a random vector X and an event $\mathcal{A} \subset \Omega, X(\mathcal{A}) \equiv \{x(\omega), \omega \in \mathcal{A}\}$.
a.s.	Almost surely, an event (i.e., an element of \mathcal{F}) A is true almost surely iff $\mathbb{P}(A) = 1$. We will often abusively mix up random vectors and classes of almost surely equal random vectors.
i.i.d.	“independent and identically distributed”.
$\sigma(X)$	If $X \in \mathcal{M}_{p,n}, \sigma(X) \in \mathbb{R}_+^{\min(p,n)}$ is the vector constituted of the singular values of X in increasing order (i.e. the eigenvalues of $(X X^T)^{1/2}$). If $X \in E$ is a random vector, $\sigma(X)$ is the σ -algebra generated by X (i.e., the σ -algebra of sets of Ω containing all the sets $X^{-1}(B)$ when B is a Borel set of E).

$\mathbb{E}[\cdot \cdot]$	Conditional expectation. Given a random variable $X \in \mathbb{R}$ \mathcal{F} -measurable and \mathcal{G} , a sub σ -algebra of \mathcal{F} , $\mathbb{E}[X \mathcal{G}]$ is the a \mathcal{G} -measurable random variable that satisfies for any additional \mathcal{G} -measurable random variable Y , $\mathbb{E}[Y\mathbb{E}[X \mathcal{G}]] = \mathbb{E}[YX]$. verifies for any \mathcal{G} -measurable random variable Y , $\mathbb{E}[YX] = \mathbb{E}[Y\mathbb{E}[X \mathcal{G}]]$. Given two random variables $X, Y \in \mathbb{R}$, $\mathbb{E}[X Y] = \mathbb{E}[X \sigma(Y)]$.
$\mathbb{P}(\cdot \cdot)$	Conditional probability. Given a Borel set $A \subset \mathbb{R}$ and a random variable $X \in \mathbb{R}$, $\mathbb{P}(A X) = \mathbb{E}[\mathbb{1}_A X]$.
\in, \pm	Concentration around a pivot or around a deterministic equivalent, see Definitions 3 and 14.
$O(\sigma), \leq, \geq$	see Definition 8.
\propto	Lipschitz concentration, see Definitions 2 and 9 respectively for random variables and random vectors.
\propto_c	Convex concentration, see Definition 17.
\propto^T	Transversal concentration, see Definition 20.
$D_o(X)$	Observable diameter of a random vector X , see (2)
$d\mu$	If μ is a probability law defined on E , for any function $f : E \rightarrow \mathbb{R}$, such that $\mu(f) = \int f d\mu = 1$, we note $f d\mu$ the measure verifying for all Borel set $B : f d\mu(B) = \int \mathbb{1}_B f d\mu$.
λ_p	Lebesgue measure on \mathbb{R}^p .
σ_p	Uniform measure on \mathbb{S}^p .
ν^p	Exponential measure. If $p = 1$ $\nu^1 = \frac{e^{\cdot}}{2} d\lambda_1$, for $p \geq 1$, $\nu^p = \nu^1 \otimes \dots \otimes \nu^1$ (p times).
β_q^p	Uniform measure on the ball $\mathcal{B}_{\ \cdot\ _q}$ of \mathbb{R}^p .
$\mathcal{N}(0, I_p)$	Distribution of Gaussian vectors of \mathbb{R}^p with zero mean and covariance I_p .
$\eta_{(E, \ \cdot\)}$	Norm degree, see Definition 16.
g	Stieltjes transform of a probability law μ on \mathbb{R} . Let $D \subset \mathbb{R}$ be the maximal Borel set such that $\mu(\mathbb{R} \setminus D) = 0$, then $\forall z \in \mathbb{C} \setminus D$, the Stieltjes transform is defined with the formula $g(z) = \int_D \frac{d\mu(w)}{w-z}$.
d_s	Stable semi metric. Given $z, z' \in \mathbb{R}_+$, $d_s(z, z') = \left \frac{z-z'}{\sqrt{zz'}} \right $, given $z, z' \in \mathbb{H}$, $d_s(z, z') = \left \frac{z-z'}{\sqrt{\Im(z)\Im(z')}} \right $, given $D, D' \in \mathcal{D}(A)$, with $A = \mathbb{R}_+$ or $A = \mathbb{H}$, $d_s(D, D') = \sup_{i \in [n]} (d_s(D_i, D'_i))$.
$d\phi _x$	Differential of ϕ taken on x , given $t \in \mathbb{R}$ tending to zero, $\phi(x+th) - \phi(x) = td\phi _x \cdot h + o(t)$, $d^2\phi _x = d\psi _x$ for $\psi : x \mapsto d\phi _x$.
$R_\gamma(S, \Delta), I_\gamma(S, \Delta)$	Given $S \in \mathcal{S}_p^n$, $\Delta \in \mathcal{D}_n^+$, $R_\gamma(S, \Delta) \equiv \left(\frac{1}{n} \sum_{i=1}^n \Delta_i S_i + \gamma I_p \right)^{-1} \in \mathcal{M}_p$, $I_\gamma(S, \Delta) \equiv \text{Diag} \left(\frac{1}{n} \text{Tr} (S_i R_\gamma(S, \Delta)) \right)_{1 \leq i \leq n} \in \mathcal{D}_n^+$, $\tilde{R}_z^C(\Delta) \equiv R_z \left(C, \frac{\Delta}{I_n + \Delta \Lambda^C} \right)$ and for $X \in \mathcal{M}_{p,n}$, $I^X(\Delta) \equiv I(X \cdot X^T, \Delta)$

Assumptions

Chapter F

Assumption 1. $p \leq O(n)$.

Assumption 2. $X \propto \mathcal{E}_2$.

Assumption 3. X has independent columns $x_1, \dots, x_n \in \mathbb{R}^p$.

Assumption 4. $\sup_{i \in [n]} \|\mu_i\| \leq O(1)$.

Assumption 5. $\inf_{i \in [n]} \Sigma_i \geq O(1)I_p$.

In Section F.2, we replace Assumption 2 with:

Assumption 2_c. $X \propto_c \mathcal{E}_2$.

Chapter H

Assumption 1 bis. $p \leq O(n)$, $m \leq O(n)$. In addition:

Assumption 2 bis. The matrix $W = (w_1, \dots, w_m)$ has independent identically distributed columns.

Assumption 3 bis. $W \propto \mathcal{E}_2$.

Assumption 4 bis. $\sup_{i \in [m]} \|\mathbb{E}[w_i]\| \leq O(1)$, $\|X\| < O(1)$ and $\|Y\|_\infty < O(1)$.

Assumption 6 (Function σ). The function σ is Lipschitz continuous with parameter λ_σ independent of n .

Chapter I

Assumptions 1, 4, 5 and:

Assumption 7. $u \in \mathcal{S}(\mathbb{R}^+)$ and there exists $u^\infty > 0$ such that $\forall t \in \mathbb{R}^+$, $u(t) \leq u^\infty$.

Assumption 2 ter. $Z \propto \mathcal{E}_2$.

Assumption 3 ter. The random vectors z_1, \dots, z_n are all independent.

Assumption 8. $\|m\| = O(1)$.

Assumption 9. $\frac{1}{n} \sum_{i=1}^n \tau_i \leq O(1)$.

Chapter J

Assumptions 1-5 and:

Assumption 10. The mappings $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in [n]$, are twice differentiable and $\sup_{i \in [n]} \|f_i\|_\infty$, $\sup_{i \in [n]} \|f'_i\|_\infty$, $\sup_{i \in [n]} \|h_i^{(3)}\|_\infty \leq O(1)$.

Part I

Concentration of the measure framework

The original theorem owed to Paul Pierre Levy and concerning the concentration of the uniform distribution σ_p on the sphere \mathbb{S}^p writes:

Theorem .2 (Normal concentration of σ_p , [Led05, Theorem 2.3]). *Given a degree $p \in \mathbb{N}$ and a random vector $Z \sim \sigma_p$, for any 1-Lipschitz function $f : \mathbb{R}^{p+1} \rightarrow \mathbb{R}$, we have the inequality:*

$$\mathbb{P}(|f(Z) - m_f| \geq t) \leq 2e^{-(p-1)t^2/2}, \quad (4)$$

where m_f is a median of $f(Z)$ verifying by definition $\mathbb{P}(f(Z) \geq m_f), \mathbb{P}(f(Z) \leq m_f) \geq \frac{1}{2}$.

There exist plenty of other distributions that verify this inequality like the uniform distribution on the ball, on $[0, 1]^n$, or the Gaussian distribution $\mathcal{N}(0, I_p)$ that are presented in [Led05] ; more generally, for Riemannian manifolds, the concentration can be interpreted as a positive lower bound on the Ricci curvature (see Gromov appendix in [MS86] or [Led05, Theorem 2.4]).

The concentration inequality verified on the sphere will structure our approach in a way that we will try to express our result with that form as often as possible to pursue Levy's idea; we will thus choose the short notation $f(Z) \in m_f \pm 2e^{-(p-1) \cdot^2/2}$ to express it in a simple way.

As Milman was the first to advocate it ([MS86]), the concentration of a random object appears to be an essential feature leading immediately to a lot of implications and controls on the object. We do not present here the historical introduction of concentration of the measure. Rather than the geometric, distribution-oriented approach, we directly adopt a probabilistic point of view on the notion. That does not mean that the presentation of the theory will be incomplete. On the contrary, we display here, almost always with their proofs, all the important theorems and propositions necessary to apprehend the theory and set rigorously Part II about the sample covariance. The usual approach to the concentration of measure is to start with geometric inequalities ruling high dimensional Banach spaces and then track from these powerful results some probabilistic properties on the real functionals, the "observable world". We propose here a reversed approach where we start from the probabilistic results on \mathbb{R} which offer us some interesting reasoning schemes all the same. Then, once the reader convinced by the direct computation improvements offered by the theory, we perform the fundamental step consisting in considering high dimensional concentration properties. Many of the results are derived from the complete presentation of the theory made by Ledoux in [Led05].

Chapter A

Concentration of a random variable

A.1 Definitions and first Properties

Given a random variable $X \in \mathbb{R}$, any function $t \mapsto \mathbb{P}(|X| \geq t)$ satisfies some important features that will form the definition of the notion of *concentration function* as a class of objects measuring the concentration of random variables (the same way as the modulus of continuity measure the continuity of mappings).

Definition 1 (Concentration function). *Any non-increasing and left continuous function $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$, is called a concentration function.*

Given a random variable Z and a concentration function α , we choose first to express the α -concentration of Z through the introduction of an independent copy Z' .

Definition 2 (Concentration of a random variable). *Given a concentration function α , the random variable Z is said to be α -concentrated, and we write $Z \propto \alpha$, iff for any independent copy Z' :*

$$\forall t > 0 : \mathbb{P}(|Z - Z'| \geq t) \leq \alpha(t). \tag{A.1}$$

In particular, if we often deal with concentration functions α that take values outside of $[0, 1]$, it is implicitly understood that we consider in the calculus $t \mapsto \min(\min(\alpha(u))_{u \leq t}, 1)$ instead of α .

In Definition 2, we see that α basically limits the variations of the random variable Z . The faster α decreases, the lower the variations of Z . Instinctively one might hope that this limitation of the variation would be equivalent to a concentration around some central quantity that will be called a *pivot* of the concentration in the following sense:

Definition 3 (Concentration around a pivot). *Given $a \in \mathbb{R}$, the random variable Z is said to be α -concentrated around the pivot a , and we write $Z \in a \pm \alpha$, iff :*

$$\forall t > 0 : \mathbb{P}(|Z - a| \geq t) \leq \alpha(t).$$

We call the parameter a a *pivot* because the concentration around a provides similar concentration around other values close to a . Given $\theta > 0$, let us denote τ_θ the operator defined on the set of concentration functions that verifies for any concentration function α :

$$\tau_\theta \cdot \alpha(t) = \begin{cases} 1 & \text{if } t \leq \theta \\ \alpha(t - \theta) & \text{if } t > \theta. \end{cases}$$

Then we have the simple Lemma:

Lemma A.1. *Given a random variable $Z \in \mathbb{R}$, a concentration function α , two real numbers $a, b \in \mathbb{R}$ and $\theta > 0$, we have the implication:*

$$\begin{cases} Z \in a \pm \alpha \\ \|a - b\| \leq \theta \end{cases} \implies Z \in b \pm \tau_\theta \cdot \alpha.$$

At first sight, there exists no pivot $a \in \mathbb{R}$ such that Definition 2 and Definition 3 are equivalent. We can however find an interesting relation considering the case $a = m_Z$ where m_Z is a median of Z , verifying by definition $\mathbb{P}(Z \geq m_Z) \geq 1/2$ and $\mathbb{P}(Z \leq m_Z) \geq 1/2$.

Proposition A.2 (Concentration around the median, from [Led05, Corollary 1.5]). *Given a random variable Z , a median m_Z of Z and a concentration function α , we have the implications :*

$$Z \propto \alpha \quad \Longrightarrow \quad Z \in m_Z \pm 2\alpha \quad \Longrightarrow \quad \exists a \in \mathbb{R} \mid Z \in a \pm 2\alpha \quad \Longrightarrow \quad Z \propto 4\alpha(\cdot/2)$$

where $\alpha(\cdot/2)$ is defined as being the function $t \mapsto \alpha(t/2)$.

We see here that if Z is α -concentrated, the tail of Z , i.e., the behavior of Z far from the median, is closely linked to the decreasing speed of α .

Proof. We need to consider the fact that $\mathbb{P}(Z = m_Z) = \epsilon$ may be non zero. Therefore there exist ϵ_1, ϵ_2 such that $\epsilon = \epsilon_1 + \epsilon_2$ and:

$$\mathbb{P}(Z < m_Z) = \frac{1}{2} - \epsilon_1 \qquad \mathbb{P}(Z > m_Z) = \frac{1}{2} - \epsilon_2.$$

Let us take $t > 0$. The first result follows from the inequalities:

$$\begin{aligned} \mathbb{P}(|Z - Z'| \geq t) &= \mathbb{P}(|Z - Z'| \geq t, Z' < m_Z) + \mathbb{P}(|Z - Z'| \geq t, Z' > m_Z) \\ &\quad + \mathbb{P}(|Z - m_Z| \geq t, Z' = m_Z) \\ &\geq (1/2 - \epsilon_1) \mathbb{P}(Z \geq t + Z' \mid Z' < m_Z) \\ &\quad + (1/2 - \epsilon_2) \mathbb{P}(Z' \geq t + Z \mid Z' > m_Z) \\ &\quad + \epsilon \mathbb{P}(|Z - m_Z| \geq t) \\ &\geq (1/2 - \epsilon) (\mathbb{P}(Z \geq t + m_Z) + \mathbb{P}(m_Z \geq t + Z)) \\ &\quad + \epsilon \mathbb{P}(|Z - m_Z| \geq t) \\ &= \frac{1}{2} \mathbb{P}(|Z - m_Z| \geq t). \end{aligned}$$

We prove the last implication setting that for any $a \in \mathbb{R}$:

$$\begin{aligned} \mathbb{P}(|Z - Z'| \geq t) &\leq \mathbb{P}(|Z - a + a - Z'| \geq t) \\ &\leq 2\mathbb{P}(|Z - a| \geq t/2). \end{aligned}$$

□

The reason why Definition 2 was presented firstly and thus given the importance of the naturally underlying definition, even though Definition 3 might seem more intuitive, lies in its immediate compatibility to the composition with any Lipschitz function and more generally with any uniformly continuous function. The uniform continuity of a function is sized by its *modulus of continuity*.

Definition 4 (Uniform continuity). *Any non decreasing function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous and null in 0 is called a modulus of continuity. Given two normed vector space $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, a function $f : E \rightarrow F$ is said to be continuous under the modulus of continuity ω if:*

$$\forall x, y \in E \quad : \quad \|f(x) - f(y)\|_F \leq \omega(\|x - y\|_E).$$

Of course a function admits a modulus of continuity iff it is uniformly continuous.

Lemma A.3. *Let us consider a random variable Z , a concentration function α , and a function $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous under a modulus of continuity ω . We allow ourselves to write ω^{-1} the pseudo inverse of ω defined (for ω bijective or not) as : $\omega^{-1}(w) = \inf\{t, \omega(t) \geq w\}$. We have the implication:*

$$Z \propto \alpha \quad \Longrightarrow \quad f(Z) \propto \alpha(\omega^{-1}(\cdot)).$$

Proof. It is straightforward to write:

$$\begin{aligned} \mathbb{P}(|f(Z) - f(Z')| \geq t) &\leq \mathbb{P}(\omega(|Z - Z'|) \geq t) \\ &\leq \mathbb{P}(|Z - Z'| \geq \omega^{-1}(t)) \leq \alpha(\omega^{-1}(t)), \end{aligned}$$

since for any $w, t > 0$, $\omega(t) \geq w \implies t \geq \omega^{-1}(w)$ by definition of the pseudo inverse. \square

The stability with respect to ω -continuous functions reflects the fact that a ω -continuous function contains the spreading of the distribution up to the modulus of continuity.

One property that we could expect from α -concentration is a stability towards the sum. Up to a multiplying factor of 2, this is the case :

Lemma A.4. *Given two random variables Z_1 and Z_2 and two concentration functions α, β , we have the implication:*

$$Z_1 \propto \alpha \text{ and } Z_2 \propto \beta \quad \implies \quad Z_1 + Z_2 \propto \alpha(\cdot/2) + \beta(\cdot/2).$$

Proof. Recall that Z'_1 and Z'_2 are two independent copies respectively of Z_1 and Z_2 . There is no reasons for Z'_1 to be independent of Z_2 (resp., Z'_2 of Z_1). The idea is to decompose the threshold $t > 0$ in $\frac{t}{2} + \frac{t}{2}$:

$$\begin{aligned} \mathbb{P}(|Z_1 + Z_2 - Z'_1 - Z'_2| \geq t) \\ \leq \mathbb{P}\left(|Z_1 - Z'_1| \geq \frac{t}{2}\right) + \mathbb{P}\left(|Z_2 - Z'_2| \geq \frac{t}{2}\right) \leq \alpha\left(\frac{t}{2}\right) + \beta\left(\frac{t}{2}\right). \end{aligned}$$

\square

Remark A.5. *This time the idea of the demonstration works the same with the setting of Definition 3, and we have:*

$$Z_1 \in a \pm \alpha \text{ and } Z_2 \in b \pm \beta \quad \implies \quad Z_1 + Z_2 \in a + b \pm \alpha(\cdot/2) + \beta(\cdot/2).$$

In a first approach, α -concentration performs badly with the product, as it requires a bound on both random variables involved which greatly reduces the number of possible applications.

Lemma A.6. *Given two bounded random variables Z_1 and Z_2 such that $|Z_1| \leq K_1$ and $|Z_2| \leq K_2$ and two concentration functions α, β :*

$$Z_1 \propto \alpha \text{ and } Z_2 \propto \beta \quad \implies \quad Z_1 Z_2 \propto \alpha\left(\frac{\cdot}{2K_2}\right) + \beta\left(\frac{\cdot}{2K_1}\right).$$

Proof. Given $t > 0$:

$$\begin{aligned} \mathbb{P}(|Z_1 Z_2 - Z'_1 Z'_2| \geq t) &\leq \mathbb{P}\left(|Z_1(Z_2 - Z'_2)| \geq \frac{t}{2}\right) + \mathbb{P}\left(|(Z_1 - Z'_1)Z'_2| \geq \frac{t}{2}\right) \\ &\leq \mathbb{P}\left(|Z_2 - Z'_2| \geq \frac{t}{2K_1}\right) + \mathbb{P}\left(|Z_1 - Z'_1| \geq \frac{t}{2K_2}\right). \end{aligned}$$

\square

Actually, the setting of Definition 3 is more convenient here than the setting of Definition 2 because it allows us to only require one random variable to be bounded:

Lemma A.7. *Given two random variables Z_1 and Z_2 such that $|Z_1| \leq K_1$, two pivots $a, b \in \mathbb{R}$ and two concentration functions α, β , if $b \neq 0$ one has the implication:*

$$Z_1 \in a \pm \alpha \text{ and } Z_2 \in b \pm \beta \quad \implies \quad Z_1 Z_2 \in ab \pm \alpha\left(\frac{\cdot}{2|b|}\right) + \beta\left(\frac{\cdot}{2K_1}\right),$$

if $b = 0$, then the concentration of Z_1 and Z_2 implies that $Z_1 Z_2 \in 0 \pm \beta(\frac{\cdot}{K_1})$.

We may even go further and dispense with the bounding hypothesis to get in this case a slightly more complicated concentration form:

Proposition A.8. *Given two random variables Z_1 and Z_2 , two pivots $a, b \in \mathbb{R}$ and two concentration functions α, β such that $Z_1 \in a \pm \alpha$ and $Z_2 \in b \pm \beta$, if $a \neq 0$ and $b \neq 0$, then $Z_1 Z_2$ is concentrated around ab with :*

$$Z_1 Z_2 \in ab \pm \alpha \left(\sqrt{\frac{\cdot}{3}} \right) + \alpha \left(\frac{\cdot}{3|b|} \right) + \beta \left(\sqrt{\frac{\cdot}{3}} \right) + \beta \left(\frac{\cdot}{3|a|} \right).$$

If $a = 0$ and $b \neq 0$, we get $Z_1 Z_2 \in 0 \pm \alpha \left(\sqrt{\frac{\cdot}{2}} \right) + \alpha \left(\frac{\cdot}{2|b|} \right) + \beta \left(\sqrt{\frac{\cdot}{2}} \right)$ and if $a = b = 0$, $Z_1 Z_2 \in 0 \pm \alpha \left(\sqrt{\cdot} \right) + \beta \left(\sqrt{\cdot} \right)$.

Proof. We just prove the result for $a, b \neq 0$ since the other cases are simpler. The idea is to use the algebraic identity: $xy - ab = (x - a)(y - b) + a(y - b) + b(x - a)$ and the implication $xy \geq t \Rightarrow x \geq \sqrt{t}$ or $y \geq \sqrt{t}$ (for $x, y, t \geq 0$):

$$\begin{aligned} \mathbb{P}(|Z_1 Z_2 - ab| \geq t) &\leq \mathbb{P} \left(|Z_1 - a| \geq \sqrt{\frac{t}{3}} \right) + \mathbb{P} \left(|Z_2 - b| \geq \sqrt{\frac{t}{3}} \right) \\ &\quad + \mathbb{P} \left(|Z_1 - a| |b| \geq \frac{t}{3} \right) + \mathbb{P} \left(|Z_2 - b| |a| \geq \frac{t}{3} \right). \end{aligned}$$

□

In the case of the square of a random variable or even an integer power of any size, the concentration given by Proposition A.8 can simplify.

Proposition A.9. *Given $m \in \mathbb{N}$ and a random variable $Z \in a \pm \alpha$ with $a \in \mathbb{R}$ and α , one has the concentration:*

$$Z^m \in a^m \pm \alpha \left(\frac{\cdot}{2^m |a|^{m-1}} \right) + \alpha \left(\left(\frac{\cdot}{2} \right)^{\frac{1}{m}} \right).$$

Proof. Let us employ the algebraic identity:

$$Z^m = (Z - a + a)^m = \sum_{i=0}^m \binom{m}{i} a^{m-i} (Z - a)^i = a^m + a^m \sum_{i=1}^m \binom{m}{i} \left(\frac{Z - a}{a} \right)^i.$$

If $\left| \frac{Z-a}{a} \right| \leq 1$, for any $i \in \{1, \dots, m\}$, $\left| \frac{Z-a}{a} \right|^i \leq \left| \frac{Z-a}{a} \right|$ and conversely, if $\left| \frac{Z-a}{a} \right| \geq 1$, then $\left| \frac{Z-a}{a} \right|^i \leq \left| \frac{Z-a}{a} \right|^m$. This entails:

$$|Z^m - a^m| \leq (2|a|)^m \left(\left| \frac{Z-a}{a} \right| + \left| \frac{Z-a}{a} \right|^m \right) \tag{A.2}$$

and therefore:

$$\mathbb{P}(|Z^m - a^m| \geq t) \leq \mathbb{P} \left(|Z - a| \geq \frac{t}{2^m |a|^{m-1}} \right) + \mathbb{P} \left(|Z - a| \geq \left(\frac{t}{2} \right)^{\frac{1}{m}} \right).$$

□

We conclude this section by setting the continuity of the concentration property. We adopt the classical formalism for the convergence of a random variable (or vector).

Definition 5. *We say that a sequence of random variables (or random vectors) Z_n converges in law (or “in distribution” or “weakly”) to Z if for any real valued continuous function f with compact support:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(Z_n)] = \mathbb{E}[f(Z)].$$

Although we rather find in the literature the definition mentioning *continuous and bounded* functions, this equivalent definition relying on the class of continuous functions with compact support is more adapted to the convergence of concentration inequality (see the proof of Proposition A.11). We start with the preliminary lemma:

Lemma A.10 ([Ouv09], Proposition 14.17). *Let us consider a sequence of random variables Z_n , $n \in \mathbb{N}$, and a random variable Z with cumulative distribution functions respectively noted F_{Z_n} , $n \in \mathbb{N}$, and F_Z . The sequence $(Z_n)_{n \geq 0}$ converges in law to Z iff for any $t \in \mathbb{R}$ such that F_Z is continuous on t , $(F_{Z_n}(t))_{n \geq 0}$ converges to $F_Z(t)$.*

Proposition A.11. *Consider a sequence of random variables $(Z_n)_{n \geq 0}$ that converges in law to a random variable Z , a sequence of pivot $(a_n)_{n \geq 0}$ converging to a pivot $a \in \mathbb{R}$ and a sequence of concentration functions $(\alpha_n)_{n \geq 0}$ that point-wise converges to a continuous concentration function α . If we suppose that, for any $n \in \mathbb{N}$, $Z_n \in a_n \pm \alpha_n$ then $Z \in a \pm \alpha$.*

Proof. For any $n \in \mathbb{N}$, let us note $Y_n = |Z_n - a_n|$ and $Y = |Z - a|$. We wish to show first that $(Y_n)_{n \geq 0}$ converges in law to Y . Let us consider for that purpose a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support S and $\varepsilon > 0$. We know from the Heine-Cantor theorem that f is uniformly continuous, therefore there exists $\eta > 0$ such that:

$$|x - y| \leq \eta \implies |f(x) - f(y)| \leq \frac{\varepsilon}{2}.$$

Moreover, since $\lim_{n \rightarrow \infty} a_n = a$, there exists $n_0 > 0$ such that for any $n \geq n_0$, $|a_n - a| \leq \eta$. Eventually, if we introduce the function $g : x \rightarrow |x - a|$, we know that $f \circ g$ has a compact support $S' = a + S \cup a - S$, and thus there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$:

$$|\mathbb{E}[f(|Z_n - a|)] - \mathbb{E}[f(|Z - a|)]| = |\mathbb{E}[f \circ g(Z_n)] - \mathbb{E}[f \circ g(Z)]| \leq \frac{\varepsilon}{2}.$$

Therefore if we consider $n \geq \max(n_0, n_1)$:

$$\begin{aligned} |\mathbb{E}[f(Y_n)] - \mathbb{E}[f(Y)]| &\leq |\mathbb{E}[f(|Z_n - a_n|) - f(|Z_n - a|)]| \\ &\quad + |\mathbb{E}[f(|Z_n - a|) - f(|Z - a|)]| \leq \varepsilon. \end{aligned}$$

Then we know from Lemma A.10 that for any t such that the cumulative distribution function $F_Y = \mathbb{P}(|Z - a| \leq \cdot)$ is continuous around t :

$$\mathbb{P}(|Z - a| \geq t) = \lim_{n \rightarrow \infty} \mathbb{P}(|Z_n - a_n| \geq t) \leq \lim_{n \rightarrow \infty} \alpha_n(t) = \alpha(t).$$

Since α is continuous and $t \rightarrow \mathbb{P}(|Z - a| \geq t)$ is decreasing, we recover the preceding inequality for any $t > 0$, and $Z \in a \pm \alpha$. \square

In the setting of Definition 2, we can show the continuity of the concentration the same way introducing this time the random variables $Y_n = |Z_n - Z'_n|$ (Z'_n being a sequence of independent copies of Z_n). We present the next proposition without proof.

Proposition A.12. *In the setting of Proposition A.11, if for any $n \in \mathbb{N}$ $Z_n \propto \alpha_n$, then $Z \propto \alpha$.*

A.2 Exponential concentration

In [Led05], Ledoux defines a random variable as *normally concentrated* when the concentration function is of the form $Z \propto C e^{-(\cdot/\sigma)^2}$ for two given constants $C \geq 1$ and $\sigma > 0$. The form of the concentration function has no real importance on small dimensions since the result ensues from a mere bound on the Gaussian Q-function; it is however surprising that this form naturally appears for the uniform distribution on the sphere (see Theorem .2) and others in high dimensions.

In order to present a general picture that will be helpful later (when dealing with the concentration of products as already seen in Propositions A.8, A.9), let us include the general case of *q-exponential concentrations*.

Definition 6. Given $q > 0$, a random variable Z is said to be q -exponentially concentrated with head parameter $C \geq 1$ and tail parameter $\sigma > 0$ iff there exists a pivot $a \in \mathbb{R}$ such that $Z \in a \pm Ce^{-(\cdot/\sigma)^q}$.

Example A.13. A random variable Z following a Gaussian distribution with zero mean (i.e. zero median) and unit variance is 2-exponentially concentrated with a tail parameter equal to $\sqrt{2}$: $Z \in 0 \pm 2e^{-(\cdot)^2/2}$.

In practice, the random variables will depend on a random vector whose dimension, say p , tends to infinity, the tail parameter is then a function of p that represents the *asymptotic* speed of concentration since it has the same order as the standard deviation of Z . If the q -exponential concentration functions are employed reasonably, the asymptotic information can be transmitted from the head parameter to the tail parameter so that the head parameter would stay mainly uninformative and close to 1. For that purpose, the following lemma gives us an easy way to bound a given concentration to a close one with a head parameter equal to e .

Lemma A.14. Given $x, q > 0$ and $C \geq e$, we have the inequality:

$$\min(1, Ce^{-x}) \leq ee^{-x/2\log(C)}.$$

Proof. If $x \leq 2\log(C)$ the inequality is clear, and if $x \geq 2\log(C) \geq 2$, we deduce the result of the lemma from the equivalence

$$\log(C) - x \leq -\frac{x}{2\log(C)} \iff x \geq \frac{2\log(C)^2}{2\log(C) - 1},$$

since $\frac{2\log(C)^2}{2\log(C)-1} \leq \frac{2\log(C)^2}{\log(C)} \leq x$. □

To place ourselves under the hypotheses of Lemma A.14 and as it appears rather convenient in several propositions below we will suppose from now on that the tail parameter C is greater than e .

Exponential concentrations offer simple expressions of the concentration through shifting the pivot thanks to the following lemma.

Lemma A.15. Given the parameters $C \geq e$ and $q, \sigma, \theta > 0$:

$$\forall t > 0 : \tau_\theta \cdot Ce^{-(\cdot/\sigma)^q} \leq \max\left(e^{(\theta/\sigma)^q}, C\right) e^{-(\cdot/2\sigma)^q}.$$

Proof. When $t \geq 2\theta$, the increasing behavior of $t \mapsto t^q$ ensures that $(t - \theta)^q \geq (t/2)^q$ which implies the functional inequality on $[2\theta, \infty)$. When $t \leq 2\theta$:

$$\max\left(e^{(\theta/\sigma)^q}, C\right) e^{-(t/2\sigma)^q} \geq e^{(\theta/\sigma)^q} e^{-(2\theta/2\sigma)^q} \geq 1 \geq \tau_\theta \cdot Ce^{-(\cdot/\sigma)^q},$$

(by definition of τ_θ). □

This lemma, combined with lemma A.1, clarifies the notion of *tail parameters*. The next corollary shows that it can be seen as the diameter of a “black hole” centered around any pivot of the concentration, in a sense that each value inside this “black hole” can be considered as a satisfactory pivot for the concentration ; this will be called later, in the case of random vectors, the *observable diameter* of the distribution, following Gromov terminology [Gro79].

Corollary A.16. Given $C \geq e$ and three positive parameters $\sigma, \lambda, q > 0$, two real a and b such that $|a - b| \leq \lambda\sigma$ and a random variable Z , one has the implication:

$$Z \in a \pm Ce^{-(\cdot/\sigma)^q} \implies Z \in b \pm C' \exp\left(-\left(\frac{\cdot}{2\sigma}\right)^q\right),$$

where $C' = \max(C, e^{\lambda^q})$.

Note that the interesting aspect of the result is the independence of the head parameter $C' = \max(C, e^{\lambda^q})$ to the tail parameter σ . Moreover the tail parameter stays unmodified when $q < 1$.

We now have all the elements to show that, due to the high concentration of exponentially concentrated random vectors, every median plays a pivotal role among the different constants that can localize the concentration.

Proposition A.17 ([Led05, Proposition 1.8]). *Given a random variable Z , and a median m_Z of Z , if we suppose that $Z \in a \pm Ce^{-(\cdot/\sigma)^q}$ for a pivot $a \in \mathbb{R}$, then:*

$$Z \in m_Z \pm 2C \exp\left(-\left(\frac{\cdot}{2\sigma}\right)^q\right).$$

Proof. For some $\varepsilon > 0$, we choose $t_0 > \sigma(\log(2C) + \varepsilon)^{1/q}$. We know that $\mathbb{P}(|Z - a| \geq t_0) < \frac{1}{2}$ and consequently $|a - m_Z| \leq t_0$. Indeed if we suppose that $m_Z \geq a + t_0$, then:

$$1/2 \leq \mathbb{P}(Z \geq m_Z) \leq \mathbb{P}(Z - a \geq t_0) \leq \mathbb{P}(|Z - a| \geq t_0),$$

and we get the same absurd result if we suppose that $a \geq m_Z + t_0$. We can thus conclude thanks to Corollary A.16 (with $C' = \max(C, \exp(\frac{t_0}{\sigma^q})) = 2Ce^\varepsilon$), letting ε tend to zero. \square

The tails of q -exponentially concentrated random variables can be controlled rather easily and roughly thanks to the next lemma that is based on the same simple mathematical inequalities that lead to Corollary A.16.

Lemma A.18. *Given a random variable Z , two parameters $C \geq e$ and $\sigma > 0$ an exponent $q > 0$ and a pivot $a \in \mathbb{R}$, if $Z \in a \pm Ce^{-(\cdot/\sigma)^q}$ then:*

$$\forall t \geq 2|a| : \mathbb{P}(|Z| \geq t) \leq Ce^{-(t/2\sigma)^q}$$

(for any $t > 0$, $\mathbb{P}(|Z| \geq t) \leq Ce^{(|a|/\sigma)^q} e^{-(t/2\sigma)^q}$).

Remark A.19. [Led05, Proposition 1.7] *This lemma implies in particular that any exponentially concentrated random variable admits a bounded expectation. Thanks to Fubini identity, if $Z \in a \pm Ce^{-(\cdot/\sigma)^q}$, one has indeed:*

$$|\mathbb{E}[Z]| \leq \mathbb{E}[|Z|] \leq \int_{t=0}^{\infty} \mathbb{P}(|Z| \geq t) \leq 2|a| + \int_{t=2|a|}^{\infty} Ce^{-(t/2\sigma)^q} \leq \infty.$$

Very interestingly, exponential concentration is of great computation convenience to manage Hölder's inequality. For instance a general issue is to bound:

$$\mathbb{E}[(Z_1 - a_1)^{r_1} \cdots (Z_m - a_m)^{r_m}].$$

For any $\theta_1, \dots, \theta_m \in (0, 1)$ such that $\theta_1 + \dots + \theta_m = 1$, Hölder's inequality gives us directly:

$$\mathbb{E}[(Z_1 - a_1)^{r_1} \cdots (Z_m - a_m)^{r_m}] \leq \prod_{i=1}^m \left(\mathbb{E}|Z_i - a_i|^{\frac{r_i}{\theta_i}} \right)^{\theta_i}. \quad (\text{A.3})$$

As we will see in the next proposition, the quantities $\mathbb{E}|Z_i - a_i|^r$ can be bounded easily when $Z_i = a_i \pm Ce^{-(\cdot/\sigma)^q}$, and we will even show in the next proposition that the bounds on $\mathbb{E}|Z_i - a_i|^r$ for $r > 0$ can become a characterization (it is actually a *pseudo-characterization* since there is no equivalence) of q -exponential concentrations.

Proposition A.20 (Moment characterization of concentration, [Led05, Proposition 1.10]). *Given a random variable Z , a pivot $a \in \mathbb{R}$, two exponents $r, q > 0$, and two parameters $C \geq e$ and $\sigma > 0$, we have the implications :*

$$Z \propto Ce^{-(\cdot/\sigma)^q} \Rightarrow \forall r \geq q : \mathbb{E}[|Z - a|^r] \leq C \left(\frac{r}{q}\right)^{\frac{r}{q}} \sigma^r \Rightarrow Z \propto Ce^{-\frac{(\cdot/\sigma)^q}{e}}$$

and

$$Z \in a \pm Ce^{-(\cdot/\sigma)^q} \Rightarrow \forall r \geq q : \mathbb{E}[|Z - a|^r] \leq C \left(\frac{r}{q}\right)^{\frac{r}{q}} \sigma^r \Rightarrow Z \in a \pm Ce^{-\frac{(\cdot/\sigma)^q}{e}},$$

where $\Gamma : r \mapsto \int_0^\infty t^{r-1} e^{-t} dt$, (if $n \in \mathbb{N}$, $\Gamma(n+1) = n!$).

In both results, the first implication consists in bounding an expectation with a probability; it will involve the Fubini relation, giving for any positive random variable Z :

$$\mathbb{E}Z = \int_Z \left(\int_0^\infty \mathbb{1}_{t \leq Z} dt \right) dZ = \int_0^\infty \mathbb{P}(Z \geq t) dt,$$

where $\mathbb{1}_{t \leq Z}$ is equal to 1 if $t \leq Z$ and to 0 otherwise.

The second implication consists in bounding a probability with an expectation; it is a consequence of Markov's inequality, for any non decreasing function $f : \mathbb{R} \rightarrow \mathbb{R}$:

$$\mathbb{P}(Z \geq t) \leq \frac{\mathbb{E}f(Z)}{f(t)}.$$

These two key indications given we leave the proof to next subsection where we set a result treating more elaborate concentration function and omitting the factor $/e$ in the exponential power to set an equivalence relation (see Proposition A.30).

Inspiring from Proposition A.9 one can get with Proposition A.20 combined with Hölder inequality some interesting bound one the variations of the product of concentrated random variables.

Lemma A.21. *Given an integer $m \in \mathbb{N}$, m random variables Z_1, \dots, Z_m and m parameters a_1, \dots, a_m , $\sigma_1 > 0, \dots, \sigma_m > 0$ such that for all $i \in [m]$, $Z_i \propto a_i \pm Ce^{(\cdot/\sigma_i)^q}$:*

$$\mathbb{E} \left[|Z_1 \cdots Z_m - a_1 \cdots a_m|^l \right] \leq C 2^{ml} \left(\frac{ml}{q} \right)^{\frac{ml}{q}} (\sigma_1^l + |a_1|^l) \cdots (\sigma_m^l + |a_m|^l),$$

for some universal constant $C > 0$.

Proof. Let us start to write as in the proof of Proposition A.9:

$$\begin{aligned} Z_1 \cdots Z_m &= (Z_1 - a_1 + a_1) \cdots (Z_m - a_m + a_m) \\ &= \sum_{i=0}^m \sum_{0 \leq j_1 < \cdots < j_i \leq m} a_{j_1} \cdots a_{j_i} (Z_{\bar{j}_1} - a_{\bar{j}_1}) \cdots (Z_{\bar{j}_{m-i}} - a_{\bar{j}_{m-i}}) \end{aligned}$$

where $\{\bar{j}_1, \dots, \bar{j}_{m-i}\} \cup \{j_1, \dots, j_i\} = [m]$. For simplicity, let us introduce the notation $P_i^m = \{j_1, \dots, j_i \mid 1 \leq j_1 < \cdots < j_i \leq m\}$ and given $j = (j_1, \dots, j_i)$, $a_{\pi j} = a_{j_1} \cdots a_{j_i}$. We can first deduce, using the convexity of $t \mapsto t^l$:

$$\mathbb{E} \left[|Z_{\pi[m]} - a_{\pi[m]}|^l \right] \leq \left(\sum_{i=0}^{m-1} \#P_i^m \right)^l |a_{\pi[m]}|^l \sum_{i=0}^{m-1} \sum_{j \in P_i^m} \frac{1}{\sum_{i=0}^{m-1} \#P_i^m} \mathbb{E} \left[\left| \frac{Z - a}{|a|} \right|_{\pi \bar{j}}^l \right].$$

Hölder inequality and Proposition A.20 allows us to set that:

$$\mathbb{E} \left[|Z - a|_{\pi \bar{j}}^l \right] \leq C \left(\frac{(m-i)l}{q} \right)^{\frac{(m-i)l}{q}} \sigma_{j_1}^l \cdots \sigma_{j_{m-i}}^l,$$

and the identity $\sum_{i=0}^m \#P_i^m = 2^m$ allows us bound:

$$\begin{aligned} &\mathbb{E} \left[|Z_{\pi[m]} - a_{\pi[m]}|^l \right] \\ &\leq 2^{m(l-1)} \left(\frac{ml}{q} \right)^{\frac{ml}{q}} \sum_{i=0}^{m-1} \sum_{0 \leq j_1 < \cdots < j_i \leq m} |a_{j_1}|^l \cdots |a_{j_i}|^l C \sigma_{j_1}^l \cdots \sigma_{j_{m-i}}^l \\ &\leq C \left(\frac{ml}{q} \right)^{\frac{ml}{q}} 2^{m(l-1)} ((\sigma_1^l + |a_1|^l) \cdots (\sigma_m^l + |a_m|^l) - |a_m|^l). \end{aligned}$$

□

The main use of Lemma A.21 is to replace in expectation of a product estimation some elements with some of their pivots, in a lot of cases, we do not replace some of the random variable, that is where the following corollary can be useful.

Corollary A.22. *In the setting of Lemma A.21, let us consider an integer $k \in \mathbb{N}$ and a partition $\{\{j_1, \dots, j_k\}, \{\bar{j}_1, \dots, \bar{j}_{m-k}\}\}$ of $[m]$, we have the bound:*

$$\mathbb{E} \left[\left| Z_1 \cdots Z_m - Z_{j_1} \cdots Z_{j_k} a_{\bar{j}_1} \cdots a_{\bar{j}_{m-k}} \right|^l \right] \leq C 2^{ml} \left(\frac{ml}{q} \right)^{\frac{ml}{q}} (\sigma_1^l + |a_1|^l) \cdots (\sigma_m^l + |a_m|^l),$$

for some universal constant $C > 0$.

The following Lemma gives an alternative result to the aforementioned sufficiency of bounds on $\mathbb{E} [|Z - a|^r]$ (or on $\mathbb{E} [|Z - Z'|^r]$) for $r \in \mathbb{N}$.

Lemma A.23. *Given a random variable $Z \in \mathbb{R}_+$ and three parameters $C > e$, $q, \sigma > 0$ one has the implication:*

$$\forall m \in \mathbb{N}, \mathbb{E}[Z^m] \leq C \left(\frac{m}{q} \right)^{\frac{m}{q}} \sigma^m \quad \implies \quad \forall r \geq 0, \mathbb{E}[Z^r] \leq C e^{\frac{1}{e}} \left(\frac{r}{q} \right)^{\frac{r}{q}} \left(\sigma (2/\bar{q})^{\frac{1}{q}} \right)^r,$$

where $\bar{q} = \min(q, 1)$.

Proof. When $r \leq 1$, we already know thanks to Jensen's inequality, by concavity of $t \mapsto t^r$, that:

$$\mathbb{E}[Z^r] \leq (\mathbb{E}[Z])^r \leq C \left(\frac{1}{q} \right)^{\frac{r}{q}} \sigma^r \leq C e^{\frac{1}{e}} \left(\frac{r}{q} \right)^{\frac{r}{q}} \left(\frac{\sigma}{q^{\frac{1}{q}}} \right)^r,$$

since $\forall t > 0, t^t \geq \frac{1}{e^e}$.

When $r \geq 1$, one can invoke the well known result concerning ℓ^r norms, where in our case, $\|Z\|_{\ell^r} = \mathbb{E} [|Z|^r]^{1/r}$. Let us consider the general case where we are given $p_1, p_2 > 0$ such that $p_1 \leq r \leq p_2$ and we consider $\theta \in (0, 1)$ satisfying $1/r = \theta/p_1 + (1 - \theta)/p_2$. We then have the inequality :

$$\|Z\|_{\ell^r} \leq \|Z\|_{\ell^{p_1}}^\theta \|Z\|_{\ell^{p_2}}^{1-\theta}.$$

This implies:

$$\mathbb{E}[Z^r]^{\frac{1}{r}} \leq \left(C^{\frac{1}{p_1}} \sigma \left(\frac{p_1}{q} \right)^{\frac{1}{q}} \right)^\theta \left(C^{\frac{1}{p_2}} \sigma \left(\frac{p_2}{q} \right)^{\frac{1}{q}} \right)^{1-\theta} \leq \frac{C^{\frac{1}{r}} \sigma^{\frac{\theta}{q}} p_1^{\frac{\theta}{q}} p_2^{\frac{1-\theta}{q}}}{q^{\frac{1}{q}}}.$$

We would like to bound $p_1^{\frac{\theta}{q}} p_2^{\frac{1-\theta}{q}}$ with $r^{\frac{1}{q}}$. Unfortunately, for $\theta \in]0, 1[$, $p_1^\theta p_2^{1-\theta} > 1/(\theta/p_1 + (1 - \theta)/p_2) = r$ (this is due to the inequality of arithmetic and geometric means, itself a consequence of the concavity of the log function). However, for the particular setting under study:

$$\left(\frac{\theta}{p_1} + \frac{1-\theta}{p_2} \right) p_1^\theta p_2^{1-\theta} \leq \left(\frac{1}{p_2} + \frac{(p_2 - p_1)\theta}{p_1 p_2} \right) p_2 \leq 1 + \frac{p_2 - p_1}{p_1}.$$

As a consequence, taking $p_1 = \lfloor r \rfloor$ and $p_2 = \lceil r \rceil$, one obtains:

$$\mathbb{E} [|Z|^r] \leq C \sigma^r \left(\frac{2r}{q} \right)^{\frac{r}{q}}.$$

□

Remark A.24. *In Proposition A.20, we saw that if $Z \in a \pm C e^{-(\cdot/\sigma)^q}$ then $|\mathbb{E}Z - a| \leq \mathbb{E}|Z - a| \leq C \left(\frac{1}{q} \right)^{1/q} \sigma < \infty$. Therefore any q -exponentially concentrated random variable admits a finite expectation.*

Now that we know it exists, we are going to show that the expectation $\mathbb{E}Z$ plays the same pivotal role as any median.

Corollary A.25 ([Led05, Proposition 1.9]). *With the notations of the previous proposition, one has:*

$$Z \in a \pm Ce^{-(\cdot/\sigma)^q} \quad \implies \quad Z \in \mathbb{E}Z \pm e^{\frac{Cq}{q}} e^{-(\cdot/2\sigma)^q}.$$

Proof. We suppose that $Z \in Ce^{-(\cdot/\sigma)^q}$ for a pivot $a \in \mathbb{R}$. Proposition A.20 applied in the case $r = 1$ gives us $|a - \mathbb{E}Z| \leq C(\frac{1}{q})^{\frac{1}{q}} \sigma$. One can then invoke Corollary A.16, to get the concentration:

$$Z \in \mathbb{E}Z \pm C' e^{-(\cdot/2\sigma)^q},$$

with $C' = \max(C, e^{\frac{Cq}{q}})$. It is then interesting to note that the function $q \mapsto \frac{Cq}{q}$ has a minimum in $\frac{1}{\log(C)}$ where it takes the value $e \log C$. Then we see that

$$C = e^{\log C} \leq e^{e \log C} \leq e^{\frac{Cq}{q}},$$

and we can simplify the head parameter to obtain the result of the corollary. \square

If $Z \in a \pm Ce^{-(\cdot/\sigma)^p}$, we can then employ $\mathbb{E}Z$ as a pivot of Z to get a bound on the centered moments:

Corollary A.26. *With the notations of Corollary A.25, if we suppose that $Z \in a \pm Ce^{-(\cdot/\sigma)^q}$, then we have:*

$$\mathbb{E}|Z - \mathbb{E}Z|^r \leq e^{\frac{Cq}{q}} (2\sigma)^r \left(\frac{r}{q}\right)^{\frac{r}{q}}.$$

A.3 Multi-exponential regimes of exponential concentration

If one employs Proposition A.8 to exponentially concentrated random variables, one sees that the concentration function of the product lets appear different order exponential concentration functions. In particular:

Corollary A.27. *Given two random variables $X, Y \in \mathbb{R}$, if $X \in a \pm Ce^{-c(\cdot/\sigma)^q}$ and $Y \in b \pm Ce^{-ct^q}$, then:*

$$XY \in ab \pm 2Ce^{-c(\cdot/3\sigma \max(|a|, |b|))^q} + 2Ce^{-c(\cdot/3\sigma^2)^{q/2}}$$

This corollary can be proven thanks to an other lemma that will be of multiple use when dealing with random vectors and that allows us to set the concentration of a fonctionnal of a concentrated random variable $f(Z)$ when the variation on Z can be controlled with an other concentrated random variable. For instance for the product $Z_1 Z_2$ of Corollary A.27 the variation of $Z_1 Z_2$ towards Z_1 are bounded with $|Z_2|$. To bound the concentration of $\mathbb{P}(|Z_1 Z_2 - ab| \geq t)$ it is possible to decompose divide the probability in two parts with the introduction of the events $\mathcal{A}_u \equiv \{|Z_1|, |Z_2| \leq u\}$ and $\mathcal{A}_u^c = \{|Z_1|, |Z_2| > u\}$ that we can bound thanks to Lemma A.18 when $u \geq 2\mu_0 \equiv 2 \max(\mathbb{E}[|Z_1|], \mathbb{E}[|Z_2|])$. Depending on the values of t , one then just have to choose for u a good trade-off to retrieve a similar concentration to the one given by Corollary A.27.

The following lemma is provided for exponential concentration function, it could however be adapted to any other concentration function. The bound $\mathbb{P}(\mathcal{A}_u^c) \leq Ce^{-c(\mu/\sigma)^q}$ is however generally obtained thanks to Lemma A.18 controlling the tail of exponential concentrations.

Lemma A.28. *Let us consider a random variable X such that:*

$$\forall t > 0 : \quad \mathbb{P}(|X - X'| \geq t \mid \mathcal{A}_u) \leq Ce^{-c(t/\sigma u)^q},$$

where $(\mathcal{A}_u)_{u>0}$ is a family of events such that for some $\mu_0 > 0$:

$$\forall u \geq \mu_0 : \quad \mathbb{P}(\mathcal{A}_u^c) \leq Ce^{-c(u/\sigma)^q}$$

then we have the concentration:

$$\mathbb{P}(|X - X'| \geq t) \leq 2Ce^{-c(t/\sigma\mu_0)^q} + Ce^{-c(t/\sigma^2)^{q/2}}$$

Proof. Let us first consider $t \in [0, \mu_0^2]$, in that case $\mu_0 \geq t/\mu_0$, thus $e^{-c(u/\sigma)^q} \leq e^{-c(t/\mu_0\sigma)^q}$ and we can then bound:

$$\mathbb{P}(|X - X'| \geq t) \leq \mathbb{P}(|X - X'| \geq t \mid \mathcal{A}_{\mu_0}) + \mathbb{P}(\mathcal{A}_{\mu_0}^c) \leq 2Ce^{-c(t/\sigma\mu_0)^q}$$

When $t > \mu_0^2$ one still has:

$$\begin{aligned} \mathbb{P}(|X - X'| \geq t) &\leq \mathbb{P}(|X - X'| \geq t \mid \mathcal{A}_u) + \mathbb{P}(\mathcal{A}_u^c) \\ &\leq Ce^{-c(t/\sigma u)^q} + Ce^{-c(u/\sigma)^q} \end{aligned}$$

Then, if we chose $u \equiv \sqrt{t} \geq \mu_0$ by hypothesis on t , we obtain the inequality:

$$\mathbb{P}(|X - X'| \geq t) \leq Ce^{-(t/\sigma)^{q/2}}.$$

□

Extending this lemma to more complex decompositions where the variability of X depends on multiple variables through multiple events $\mathcal{A}_{u_1}, \dots, \mathcal{A}_{u_m}$, for each t , the trade-off choice between the variables u_1, \dots, u_m produces at the end what we call a “multiple regime of concentration”, and that can take the form (as in Theorem B.34):

$$Z \propto \max_{l \in [m]} C \exp\left(-c(\cdot/\sigma^l \mu^{(m-l)q/l})\right) \quad (\text{A.4})$$

where for all $l \in [m]$, we noted $\mu^{(m-l)} = \max\{\mu_{i_1} \cdots \mu_{i_{m-l}}, 0 \leq i_1 < \cdots < i_{m-l} < m\}$. In the case $m = 2$, we retrieve of course the result of Corollary A.27. In the multi-regime concentration (A.4), generally, σ , that represents the variation of the studied variables is generally lower than the μ_1, \dots, μ_m that represent the mean of the variables, therefore generally:

$$k \leq l \quad \implies \quad \sigma^l \mu^{(m-l)} \leq \sigma^k \mu^{(m-k)}.$$

One can then note that the exponential terms with the bigger tail (l large) have the smaller tail parameter. It is quite reassuring because if it was not the case, with a small modification of the head parameters, one could simply remove the low tail components. Next lemma presents an other relation that should respect different components of a canonical multi-regime concentration expression as defined in Definition 7.

Lemma A.29. *Given $c > 0$, $C \geq 1$, an integer $m \in \mathbb{N}$ and $2m$ parameters $q_1 > \cdots > q_m > 0$ and $\sigma_1 > \cdots > \sigma_m > 0$, we note for any $k < l \in \{0, \dots, m+1\}$:*

$$\begin{cases} t_{k,l} \equiv \left(\frac{\sigma_k^{q_k}}{\sigma_l^{q_l}}\right)^{\frac{1}{q_k - q_l}} & \text{if } 1 \leq k < l \leq m \\ t_{0,l} \equiv \sigma_l \left(\frac{\log C}{c}\right)^{\frac{1}{q}} & \text{if } l \leq m \\ t_{k,m+1} \equiv +\infty & \text{if } k \geq 1 \end{cases} \quad (\text{A.5})$$

then for any¹ $t \in [t_{l-1,l}, t_{l,l+1})$:

$$\max_{l \in m} (C \exp(-c(t/\sigma_l)^{q_l})) = C \exp(-c(t/\sigma_l)^{q_l}),$$

Therefore, given a subset $\{l_1, \dots, l_k\} \subset [m]$ such that $l_1 < \cdots < l_k$, $t_{0,l_1} < t_{l_1,l_2} < \cdots < t_{l_{k-1},l_k}$ and for all $i \in [k]$, $h \in \{l_i + 1, \dots, l_{i+1}\}$, $t_{l_i,k} \geq t_{l_i,i+1}$, we have the identity for any $t \in \mathbb{R}_+$:

$$\max_{l \in m} \left(C \exp\left(-c(t/\sigma^l)^{q/l}\right)\right) = \max_{i \in [k]} \left(C \exp\left(-c(t/\sigma^{l_i})^{q_i}\right)\right). \quad (\text{A.6})$$

¹It is implicitly understood that $[t_{l-1,l}, t_{l,l+1}) = \emptyset$ if $t_{l-1,l} > t_{l,l+1}$

Definition 7. Given $m \in \mathbb{N}_*$, a set of m couples of parameters $(q_1, \sigma_1), \dots, (q_m, \sigma_m) \in (\mathbb{R}^2)^m$ is called a sensible family of multi-regime parameters if:

- $q_1 > \dots > q_m$
- $\sigma_1 > \dots > \sigma_m$
- $\forall k, l \in [m], k < l:$

$$t_{k,k+1} \leq t_{k,l} \leq t_{l-1,l},$$

where we noted:

$$t_{k,l} \equiv \left(\frac{\sigma_k^{q_k}}{\sigma_l^{q_l}} \right)^{\frac{1}{q_k - q_l}}.$$

It is called a free family of multi-regime parameters if in addition $\forall k, l \in [m], k < l:$

$$t_{k-1,k} < t_{l-1,l}.$$

With this formalism, let us present a characterization with the moments of the concentration inequality (A.4).

Proposition A.30 (Moment characterization of multi-regime concentration). *Given an integer $m \in \mathbb{N}$, and a sensible family of $2m$ multi-regime parameters $\sigma_1, \dots, \sigma_m > 0$, $q_1, \dots, q_m > 0$, a random variable $Z \in \mathbb{R}$ satisfies the concentration:*

$$Z \propto \max_{l \in [m]} C \exp(-c(\cdot/\sigma_l)^{q_l})$$

for some constants $C, c > 0$ if and only if there exist two constants $C', c' > 0$ depending only on C, c such that for all $r > 0$, we have the bound:

$$\mathbb{E}[|Z - \mathbb{E}[Z]|^r] \leq C' \max_{l \in [m]} \left(\frac{r}{q_l c'} \right)^{\frac{r}{q_l}} \sigma_l^r. \quad (\text{A.7})$$

Proof. This proof is mainly a rewriting of [Led05, Proposition 1.10] with a fine study of the different concentration regimes. We start with the direct implication which is easier to prove. Assume that there exists two constants $C, c > 0$ such that:

$$\forall t > 0, \quad \mathbb{P}(|Z - \mathbb{E}[Z]| \geq t) \leq C \max_{l \in [m]} e^{-c(t/\sigma_l)^{q_l}}.$$

Given $r > 0$:

$$\begin{aligned} \mathbb{E}[|Z - \mathbb{E}[Z]|^r] &= \int_0^\infty \mathbb{P}(|Z - \mathbb{E}[Z]|^r \geq t) dt \\ &= \int_0^\infty r t^{r-1} \mathbb{P}(|Z - \mathbb{E}[Z]| \geq t) dt \\ &\leq \max_{l \in [m]} C r \int_0^\infty t^{r-1} e^{-(t/\sigma_l c^{-1/q_l})^{q_l}} dt \\ &= \max_{l \in [m]} C \left(\frac{\sigma_l}{c^{1/q_l}} \right)^r r \int_0^\infty t^{r-1} e^{-ct^{q_l}} dt, \end{aligned}$$

and, if we assume that $r \geq q_1$ ($\geq q_l$ for all $l \in [m]$):

$$r \int_0^\infty t^{r-1} e^{-t^{q_l}} dt = \frac{lr}{q_l} \int_0^\infty t^{\frac{r}{q_l}-1} e^{-t} dt = \frac{r}{q_l} \Gamma\left(\frac{r}{q_l}\right) \leq \left(\frac{r}{q_l}\right)^{\frac{r}{q_l}},$$

when $r < q_1$, one can still bound with Jensen's inequality (since $\frac{r}{q_1} \leq 1$):

$$\mathbb{E}[|Z - \mathbb{E}[Z]|^r] \leq \mathbb{E}[|Z - \mathbb{E}[Z]|^{q_1}]^{\frac{r}{q_1}} \leq C^{\frac{r}{q_1}} \max_{l \in [m]} l^{\frac{r}{q_1}} (c\sigma_l)^r \leq C^{\frac{r}{q_1}} m^m (c\sigma_1)^r.$$

Since $m^m, \max(C, 1) \leq O(1)$, we can choose cleverly our constants to set the first implication of the proposition.

Let us now assume (A.7). We deduce from Markov inequality and basic integration calculus that $\forall r > 0$:

$$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq t) \leq \frac{\mathbb{E}[|Z - \mathbb{E}[Z]|^r]}{t^r} \leq C \max_{l \in [m]} \left(\frac{r(\sigma_l/t)^{q_l}}{q_l c} \right)^{\frac{r}{q_l}}. \quad (\text{A.8})$$

Given any $k, l \in [m]$, we employ the notation of (A.5):

$$t_{k,l} \equiv \left(\frac{\sigma_k^{q_k}}{\sigma_l^{q_l}} \right)^{\frac{1}{q_k - q_l}}$$

and we note $t_l \equiv t_{l-1,l}$. We know in particular that $0 = t_0 \leq t_1 \leq \dots \leq t_{m+1} = \infty$. Given $l \in [m]$ and $t \in [t_l, t_{l+1}]$ if we chose $r = \frac{cq_m}{e} \left(\frac{t}{\sigma_l} \right)^{q_l}$, then, for all $k \in [m]$, we want to bound with a \mathcal{E}_l decay the quantity:

$$c_k(t) \equiv C \left(\frac{r}{cq_k} (\sigma_k/t)^{q_k} \right)^{\frac{r}{q_k}} = C \left(\frac{q_m}{eq_k} \left(\frac{\sigma_k^{q_k} \sigma_l^{-q_l}}{t^{q_k - q_l}} \right) \right)^{\frac{cq_m}{e} \left(\frac{t}{\sigma_l} \right)^{q_l}}$$

to be able to bound the concentration inequality (A.8).

If $k = l$, we have directly:

$$c_k(t) = c_l(t) = C \left(\frac{q_m}{eq_l} \right)^{\frac{cq_m}{e} \left(\frac{t}{\sigma_l} \right)^{q_l}} \leq C e^{-\frac{cq_m}{e} \left(\frac{t}{\sigma_l} \right)^{q_l}}.$$

If $k \leq l-1$, $q_k \geq q_l$ then $t_{k,l} \leq t_l \leq t$ which implies $1/t^{q_k - q_l} \leq 1/t_{k,l}^{q_k - q_l}$ and:

$$c_k(t) \leq \left(\frac{q_m}{eq_k} \left(\frac{\sigma_k^{q_k} \sigma_l^{-q_l}}{t_{k,l}^{q_k - q_l}} \right) \right)^{\frac{cq_m}{e} \left(\frac{t}{\sigma_l} \right)^{q_l}} \leq C e^{-\frac{cq_m}{e} \left(\frac{t}{\sigma_l} \right)^{q_l}}.$$

And the same way, when $k \geq l+1$, $q_k \leq q_l$, then $t \leq t_{l+1} \leq t_{l,k}$, and we can bound $1/t^{q_k - q_l} = t^{q_l - q_k} \leq t_{l,k}^{q_l - q_k}$ which allows us to conclude again that $c_k(t) \leq C e^{\frac{cq_m}{e} \left(\frac{t}{\sigma_l} \right)^{q_l}}$. When $t \in (0, t_1]$, choosing $r = \frac{q_m}{e} \left(\frac{t}{c\sigma_1} \right)^{q_1}$, we show the same way that $\forall k \in [m]$, $c_k(t) \leq C e^{-\frac{k}{m} \left(\frac{t}{c\sigma_1} \right)^{q_1}}$. We eventually obtain for all $t \in \cup_{0 \leq l \leq m} (t_l, t_{l+1}] \supset \mathbb{R}_*^+$:

$$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq t) \leq \max_{l \in [m]} C e^{-c' \left(\frac{t}{\sigma_l} \right)^{q_l}},$$

with $c' = \frac{cq_m}{e}$. This is the looked for concentration. \square

The previous development of q -exponential concentration primarily aims at providing a versatile and convenient "toolbox" (note that most introduced inequalities could have been enhanced, however to the expense of clarity) for the subsequent treatment of large dimensional random vectors, rather than random variables. This analysis will be performed through resorting to concentrated *functionals* of the random vectors, i.e., real images of through a mapping with different levels of regularity (linear, Lipschitz, convex...). For large dimensional vectors, one is mostly interested in the *order* of the concentration, thus the various constants appearing in most of the previous propositions and lemmas do not have any particular interest; of major interest instead is the independence of the concentration with respect to the random vector dimension, as observed for instance in Theorem .2 and that we will extend to other type of random vectors in what follows.

Chapter B

Lipschitz and linear concentration of a random vector

To generalize the notion of concentration to the case of a random vector of a normed vector space $(E, \|\cdot\|)$, one might be tempted to follow the idea of Definition 3 and say that a vector $Z \in E$ is α -concentrated if one has for instance $\mathbb{P}\left(\|Z - \tilde{Z}\| \geq t\right) \leq \alpha(t)$ for a deterministic vector \tilde{Z} , well chosen. This would describe a notion of a concentration *around a vector*.

However, this basic notion would not be compatible with the fundamental example of the uniform distribution on the sphere of radius \sqrt{p} presented in Theorem .2 or the Gaussian vectors of identity population covariance matrices. When the dimension grows, those random vectors concentrate around a growing sphere which is the exact opposite behavior of being concentrated around a point. Yet, they present strong dimension free concentration properties that we will progressively identify through the presentation of three fundamental notions:

1. The *Lipschitz concentration* which is the concentration of $f(Z) - f(Z')$ for any i.i.d. copy Z' of Z and any Lipschitz map $f : E \mapsto \mathbb{R}$.
2. The weaker notion of the *linear concentration* which is the concentration of $u(Z - \tilde{Z})$ for some deterministic vector $\tilde{Z} \in E$ (the so-called *deterministic equivalent*) and for any bounded linear form $u : E \mapsto \mathbb{R}$. For instance, we know from Theorem .2 that any random vector Z uniformly distributed on the sphere admits $\tilde{Z} = \mathbb{E}Z = 0$ as a deterministic equivalent. This means that most drawings of Z are close to the equator when the dimension grows.
3. We will further present in Chapter D the more complex notion of *convex concentration* which is the concentration of $f(Z) - f(Z')$ for any Lipschitz *and weakly convex* map $f : E \mapsto \mathbb{R}$. It is a notion weaker than the Lipschitz concentration and stronger than the linear concentration.

To discuss concentration of measure for high dimensional random vectors, we choose here to adopt the viewpoint of Levy families where the goal is to track the influence of the vector dimension over the concentration. Specifically, given a sequence of random vectors $(Z_p)_{p \geq \mathbb{N}}$ where each Z_p belongs to a space of dimension d_p (typically $d_p = p$ and \mathbb{R}^p , but our definition concerns also sequences of random variables for which $d_p = 1$), we wish to obtain inequalities of the form:

$$\forall p \in \mathbb{N}, \forall t > 0 : \mathbb{P}(|f_p(Z_p) - a_p| \geq t) \leq \alpha_p(t), \tag{B.1}$$

for a given sequence of concentration functions α_p ; $f_p : \mathbb{R}^{d_p} \rightarrow \mathbb{R}$ is a 1-Lipschitz function; and a_p is either a deterministic variable (typically $\mathbb{E}[f_p(Z_p)]$) or a random variable (for instance $f_p(Z'_p)$ with Z'_p an independent copy of Z_p). The sequences of random vectors $(Z_p)_{p \geq 0}$ satisfying inequality (B.1) for all sequences of 1-Lipschitz functions $(f_p)_{p \geq 0}$ are called *Levy families* or more simply *concentrated vectors* (with this denomination, we implicitly omit the dependence on p and abusively call “vectors” the *sequences* of random vectors of growing dimension).

To control the convergence speeds, we will extensively use the notation $O(a_n)$ in inequalities. We properly define this notation in the following definition.

Definition 8 (Definition and order relation of $O(\cdot)$). *Given two sequences $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+^{\mathbb{N}}$, we introduce the relation:*

$$a \mathcal{R} b \Leftrightarrow \exists \kappa, K > 0, \exists N \in \mathbb{N} \mid \forall n \geq N : \kappa a_n \leq b_n \leq K a_n$$

Then we note $O(\mathbb{R}_+^{\mathbb{N}})$ the quotient space $\mathbb{R}_+^{\mathbb{N}}/\mathcal{R}$, and given $a \in \mathbb{R}_+^{\mathbb{N}}$, we note $O(a) = \{b \in \mathbb{R}_+^{\mathbb{N}}, b \mathcal{R} a\}$.

The order relation “ \leq ” is defined on $O(\mathbb{R}_+^{\mathbb{N}})$ followingly:

$$O(a) \leq O(b) \quad \Leftrightarrow \quad \exists K > 0, \exists N \in \mathbb{N} \mid \forall n \geq N : a_n \leq K b_n,$$

the order relation “ \geq ” is defined the same way. It is easy to see that this order relation is independent of the represent of the class.

We extend these relations to relation between sequences of $\mathbb{R}_+^{\mathbb{N}}$ and classes of $O(\mathbb{R}_+^{\mathbb{N}})$ with the definition:

$$a \leq O(b) \Leftrightarrow O(a) \leq O(b) \quad \text{and} \quad a \geq O(b) \Leftrightarrow O(a) \geq O(b).$$

Concentrated vectors admitting an exponentially decreasing concentration function α_p or a combination of such functions are extremely flexible objects, more importantly they are directly given by some important theorem of the theory that we will present in next section.

B.1 Lipschitz concentration

We will work with normed (or semi-normed) vector spaces, although concentration of measure theory is classically developed in metric spaces. The presence of a norm (or a semi-norm¹) on the vector space is particularly important when establishing the concentration of a product of random vectors.

Definition/Proposition 9. *Given a sequence of normed vector spaces $(E_p, \|\cdot\|_p)_{p \geq 0}$, a sequence of random vectors² $(Z_p)_{p \geq 0} \in \prod_{p \geq 0} E_p$, a sequence of positive reals $(\sigma_p)_{p \geq 0} \in \mathbb{R}_+^{\mathbb{N}}$ and a parameter $q > 0$, we say that Z_p is Lipschitz q -exponentially concentrated with observable diameter of order $O(\sigma_p)$ iff one of the following three equivalent assertions is satisfied:³*

- $\exists C, c > 0 \mid \forall p \in \mathbb{N}, \forall 1\text{-Lipschitz } f : E_p \rightarrow \mathbb{R}, \forall t > 0 :$

$$\mathbb{P}(|f(Z_p) - f(Z'_p)| \geq t) \leq C e^{-c(t/\sigma_p)^q},$$

- $\exists C, c > 0 \mid \forall p \in \mathbb{N}, \forall 1\text{-Lipschitz } f : E_p \rightarrow \mathbb{R}, \forall t > 0 :$

$$\mathbb{P}(|f(Z_p) - m_f| \geq t) \leq C e^{-c(t/\sigma_p)^q},$$

- $\exists C, c > 0 \mid \forall p \in \mathbb{N}, \forall 1\text{-Lipschitz } f : E_p \rightarrow \mathbb{R}, \forall t > 0 :$

$$\mathbb{P}(|f(Z_p) - \mathbb{E}[f(Z_p)]| \geq t) \leq C e^{-c(t/\sigma_p)^q},$$

¹A semi-norm becomes a norm when it satisfies the implication $\|x\| = 0 \Rightarrow x = 0$.

²A random vector Z of E is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the normed vector space $(E, \|\cdot\|)$ (endowed with the Borel σ -algebra); one should indeed write $Z : \Omega \rightarrow E$, but we abusively simply denote $Z \in E$.

³Aside from the fact that they all give interesting interpretation of the concentration of a random vector, all three characterizations can be relevant, depending on the needs:

- the characterization with the independent copy is employed in Remark B.5 and in the proof of Theorem B.34;
- the characterization with the median is employed in the proof of Lemma B.4;
- the characterization with the expectation, likely the most intuitive, is used to establish Proposition B.50, Theorem G.1 and Lemma G.8.

where Z'_p is an independent copy of Z_p and m_f is a median⁴ of $f(Z_p)$; the mappings f are 1-Lipschitz for the norm (or semi-norm) $\|\cdot\|_p$. We denote in this case $Z_p \propto \mathcal{E}_q(\sigma_p)$ (or more simply $Z \propto \mathcal{E}_q(\sigma)$).

Remark B.1 (Multi-regime notation). *When several regimes of exponential concentration appear in the concentration function as in (A.4); for example when for any $f : E_p \rightarrow \mathbb{R}$, 1-Lipschitz one has the concentration:*

$$\mathbb{P}(|f(Z_p) - f(Z'_p)| \geq t) \leq C e^{-c(t/\sigma_p^{(1)})^{q_1}} + \dots + C e^{-c(t/\sigma_p^{(m)})^{q_m}},$$

one can note:

$$Z \propto \mathcal{E}_{q_1}(\sigma^{(1)}) + \dots + \mathcal{E}_{q_m}(\sigma^{(m)})$$

Remark B.2 (Quasi-asymptotic regime). *Most of our results will be expressed under the quasi-asymptotic regime where p is large. Sometimes, it will be natural to index the sequences of random vectors with two (or more) indices (e.g., the numbers of rows and columns for random matrices): in these cases, the quasi-asymptotic regime is not well defined since the different indices could have different convergence speeds. This issue is overcome with the extensive use of the notation $O(\sigma_t)$, where $t \in \Theta$ designates the (possibly multivariate) index. Given two sequences $(a_t)_{t \in \Theta}, (b_t)_{t \in \Theta} \in \mathbb{R}_+^\Theta$, we will denote $a_t \leq O(b_t)$ if there exists a constant $C > 0$ such that $\forall t \in \Theta, a_t \leq C b_t$ and $a_t \geq O(b_t)$ if $\forall t \in \Theta, a_t \geq C b_t$. For clarity, when dealing with a “constant” $K > 0$, we will often state that $K \leq O(1)$ and $K \geq O(1)$ (depending on the required control). For a concentrated random vector $Z_t \propto \mathcal{E}_q(\sigma_t)$, any sequence $(\nu_t)_{t \in \Theta} \in \mathbb{R}_+^\Theta$ such that $\sigma_t \leq O(\nu_t)$ is also an observable diameter of Z_t . When $\sigma_t \leq O(1)$, we simply write $Z_t \propto \mathcal{E}_q$.*

The equivalence between the three definitions is a consequence of the combination of Propositions A.2, A.17 Corollary A.16 and A.25). In the last item of Definition 9, the existence of the expectation of $f_p(Z_p)$ is guaranteed if any of the other two items holds as explained in Remark A.24.

In the last item of Definition 9, the existence of the expectation of $f_p(Z_p)$ is guaranteed if any of the other two items holds thanks to Remark A.19 (besides there always exists a median $m_{f_p} \in \mathbb{R}$). For the existence of the expectation of a random vector in a reflexive space or in a functional space of mappings having value in a reflexive space, refer to Definitions 12 and 13.

Remark B.3 (Metric versus normed spaces). *It is more natural, as done in [Led05], to introduce the notion of concentration in metric spaces, as one only needs to resort to Lipschitz mappings which merely require a metric structure on E . However, to express the concentration of a product (Theorem B.34), we will need to control the amplitude of concentrated vectors which is easily conducted under mere linear concentration assumptions when the metric is a norm.*

At one point of our study, it will be useful to invoke concentration for semi-norms in place of norms. Definition 9 is still consistent for these weaker objects.

When a concentrated vector $Z_p \propto \mathcal{E}_q(\sigma_p)$ takes values only on some subset $A_p \equiv Z_p(\Omega) \subset E_p$ (where Ω is the universe), it might be useful to be able to establish the concentration of observations $f_p(Z_p)$ where f_p is only 1-Lipschitz on A_p (and possibly non Lipschitz on $E_p \setminus A_p$). This would be an immediate consequence of Definition 9 if one were able to extend $f_p|_{A_p}$ into a mapping \tilde{f}_p Lipschitz on the whole vector space E_p ; but this is rarely possible. Yet, the observation $f_p(Z_p)$ does concentrate under the hypotheses of Definition 9.

Lemma B.4 (Concentration of locally Lipschitz observations). *Given a sequence of random vectors $Z_p : \Omega \rightarrow E_p$, satisfying $Z_p \propto \mathcal{E}_q(\sigma_p)$, for any sequence of mappings $f_p : E_p \rightarrow \mathbb{R}$, which are 1-Lipschitz on $Z_p(\Omega)$, we have the concentration $f_p(Z_p) \propto \mathcal{E}_q(\sigma_p)$.*

Proof. considering a sequence of median m_{f_p} of $f_p(Z_p)$ and the (sequence of) sets $S_p = \{f_p \leq m_{f_p}\} \subset E_p$, if we note for any $z \in E_p$ and $U \subset E_p, U \neq \emptyset, d(z, U) = \inf\{\|z - y\|, y \in U\}$, then we have for any $z \in A$ and $t > 0$:

$$\begin{aligned} f_p(z) \geq m_{f_p} + t & \implies d(z, S_p) \geq t \\ f_p(z) \leq m_{f_p} - t & \implies d(z, S_p^c) \geq t, \end{aligned}$$

⁴ $\mathbb{P}(f(Z_p) \geq m_f) \geq \frac{1}{2}$ and $\mathbb{P}(f(Z_p) \leq m_f) \geq \frac{1}{2}$.

since f_p is 1-Lipschitz on A . Therefore, since $z \mapsto d(z, S_p)$ and $z \mapsto d(z, S_p^c)$ are both 1-Lipschitz on E and both admit 0 as a median ($\mathbb{P}(d(Z_p, S_p) \geq 0) = 1 \geq \frac{1}{2}$ and $\mathbb{P}(d(Z_p, S_p) \leq 0) \geq \mathbb{P}(f_p(Z_p) \leq m_{f_p}) \geq \frac{1}{2}$),

$$\mathbb{P}(|f_p(Z_p) - m_{f_p}| \geq t) \leq \mathbb{P}(d(Z_p, S_p) \geq t) + \mathbb{P}(d(Z_p, S_p) \leq -t) \leq 2Ce^{-(t/c\sigma_p)}.$$

□

One could argue that, instead of Definition 9, we could have posed hypotheses on the concentration of Z_p on $Z_p(\Omega)$ only; however, we considered the present definition of concentration already quite complex as it stands. This locality aspect must be kept in mind: it will be exploited to obtain the concentration of products of random vectors.

Lemma B.4 is particularly interesting when working with conditioned variables.⁵

Remark B.5 (Concentration of conditioned vectors). *Given a (sequence of) random vectors $Z \propto \mathcal{E}_q(\sigma)$ and a (sequence of) events \mathcal{A} such that $\mathbb{P}(\mathcal{A}) \geq O(1)$, it is straightforward to show that $(Z \mid \mathcal{A}) \propto \mathcal{E}_q(\sigma)$, since there exist two constants $C, c > 0$ such that for any $p \in \mathbb{N}$ and any 1-Lipschitz mapping $f : E_p \rightarrow \mathbb{R}$, $\forall t > 0$:*

$$\begin{aligned} \mathbb{P}(|f(Z_p) - f(Z'_p)| \geq t \mid \mathcal{A}) &= \frac{1}{\mathbb{P}(\mathcal{A})} \mathbb{P}(|f(Z_p) - f(Z'_p)| \geq t, \mathcal{A}) \\ &\leq \frac{1}{\mathbb{P}(\mathcal{A})} \mathbb{P}(|f(Z_p) - f(Z'_p)| \geq t) \leq Ce^{-c(t/\sigma)^q}. \end{aligned}$$

This being said, Lemma B.4 allows us to obtain the same concentration inequality for any mapping $f : E_p \rightarrow \mathbb{R}$ 1-Lipschitz on $Z_p(\mathcal{A})$ (later on, we will sometimes abusively say that f is 1-Lipschitz on \mathcal{A}).

A simple but fundamental consequence of Definition 9 is that, as announced in the introduction, any Lipschitz transformation of a concentrated vector is also a concentrated vector. The Lipschitz coefficient of the transformation controls the concentration.

Proposition B.6 (Stability through Lipschitz mappings). *In the setting of Definition 9, given a sequence $(\lambda_p)_{p \geq 0} \in \mathbb{R}_{+}^{\mathbb{N}}$, a supplementary sequence of normed vector spaces $(E'_p, \|\cdot\|'_p)_{p \geq 0}$ and a sequence of λ_p -Lipschitz transformations $F_p : (E_p, \|\cdot\|_p) \rightarrow (E'_p, \|\cdot\|'_p)$, we have*

$$Z_p \propto \mathcal{E}_q(\sigma_p) \quad \Longrightarrow \quad F_p(Z_p) \propto \mathcal{E}_q(\lambda_p \sigma_p).$$

There exists a range of elemental concentrated random vectors, which may be found for instance in the monograph [Led05]. We recall below some of the major examples. In the following theorems, we only consider sequences of random vectors of the normed vector spaces $(\mathbb{R}^p, \|\cdot\|)$. For readability of the results, we will omit the index p .

Theorem B.7 (Fundamental examples of concentrated vectors). *The following sequences of random vectors are concentrated and satisfy $Z \propto \mathcal{E}_2$:*

- Z is uniformly distributed on the sphere $\sqrt{p}\mathbb{S}^{p-1}$.
- $Z \sim \mathcal{N}(0, I_p)$ has independent standard Gaussian entries.
- Z is uniformly distributed on the ball $\sqrt{p}\mathcal{B} = \{x \in \mathbb{R}^p, \|x\| \leq \sqrt{p}\}$.
- Z is uniformly distributed on $[0, \sqrt{p}]^p$.
- Z has the density $d\mathbb{P}_Z(z) = e^{-U(z)} d\lambda_p(z)$ where $U : \mathbb{R}^p \rightarrow \mathbb{R}$ is a positive functional with Hessian bounded from below by, say, cI_p with $c \geq O(1)$ and $d\lambda_p$ is the Lebesgue measure on \mathbb{R}^p .

Some fundamental results also give concentrations $Z \propto \mathcal{E}_1$ (when $Z \in \mathbb{R}^p$ has independent entries with density $\frac{1}{2}e^{-|\cdot|} d\lambda_1$, [Tal95]) or $Z \propto \mathcal{E}_q(p^{-\frac{1}{q}})$ (when $Z \in \mathbb{R}^p$ is uniformly distributed on the unit ball of the norm $\|\cdot\|_q$, [Led05]). .

The same way that an exponentially concentrated random variable admits an expectation (see Remark A.19), we are going to show in next section how the expectation of a random vector can be defined.

⁵Letting $X : \Omega \rightarrow E$ be a random vector and $\mathcal{A} \subset \Omega$ be a measurable subset of the universe Ω , $\mathcal{A} \in \mathcal{F}$, when $\mathbb{P}(\mathcal{A}) > 0$, the random vector $X \mid \mathcal{A}$ designates the random vector X conditioned with \mathcal{A} defined as the measurable mapping $(\mathcal{A}, \mathcal{F}_{\mathcal{A}}, \mathbb{P}/\mathbb{P}(\mathcal{A})) \rightarrow (E, \|\cdot\|)$ satisfying: $\forall \omega \in \mathcal{A}$, $(X \mid \mathcal{A})(\omega) = X(\omega)$. When there is no ambiguity, we will allow ourselves to designate abusively with the same notation “ \mathcal{A} ” the actual $\mathcal{A} \subset \Omega$ and the subset $X(\mathcal{A}) \subset E$.

B.2 Definition of the expectation in vector space

Thanks to the characterization of exponential concentration with a bound on moments (Proposition A.20), we already know that, if $Z \propto \mathcal{E}_q(\sigma)$, then for any Lipschitz mapping $f : \mathbb{E} \mapsto \mathbb{R}$, the functional $f(Z)$ admits an expectation $\mathbb{E}[f(Z)] = \int_E f(z) d\mathbb{P}(Z = z)$. This definition can be first generalized when E is a *reflexive space* (see Definition 11).

Given a normed vector space $(E, \|\cdot\|)$, we denote $(E', \|\cdot\|)$ the so-called “strong dual” of E , composed of the continuous linear forms of E for the norm $\|\cdot\|$. The norm $\|\cdot\|$ (written the same way as the norm on E for simplicity – no ambiguity being possible) is called the *strong norm* of E' and defined as follows.

Definition 10. *Given a normed vector space $(E, \|\cdot\|)$, the strong norm $\|\cdot\|$ is defined on E' as:*

$$\forall f \in E', \|f\| = \sup_{\|x\| \leq 1} |f(x)|.$$

The notion of reflexive vector space relies on the notion of “topological bidual” of E , denoted $(E'', \|\cdot\|)$ and defined by $E'' = (E')'$ with norm the strong norm of the dual of E' .

Definition 11 (Natural embedding, Reflexive space.). *The “natural embedding” of E into E'' is defined as the mapping:*

$$\begin{aligned} J : E &\longrightarrow E'' \\ x &\longmapsto (E' \ni f \mapsto f(x)). \end{aligned}$$

It can be shown that J is always one-to-one, but not always onto; when J is bijective, we say that E is reflexive.

If E is reflexive, then it can be identified with E'' (this is in particular the case of any vector space of finite dimension but also of any Hilbert space). One can then define the expectation of any concentrated vector $X \propto \mathcal{E}_q(\sigma)$ as follows:

Definition 12. *Given a random vector Z of a reflexive space $(E, \|\cdot\|)$, if the mapping $E' \ni f \mapsto \mathbb{E}[f(Z)] \in \mathbb{R}$ is continuous on E' , we define the expectation of Z as the vector:*

$$\mathbb{E}[Z] = J^{-1}(f \mapsto \mathbb{E}[f(Z)]). \quad (\text{B.2})$$

Remark B.8. *A reflexive space is a complete space (since it is in bijection with a dual space). It satisfies in particular the Picard Theorem which states that any contractive mapping $f : E \rightarrow E$ ($\forall x, y \in E, \|f(x) - f(y)\| \leq (1 - \varepsilon)\|x - y\|$ with $\varepsilon > 0$) admits a unique fixed point $y = f(y)$.*

Lemma B.9. *Given a reflexive space $(E, \|\cdot\|)$, a random vector $Z \in E$ and a continuous linear form $f \in E'$:*

$$f(\mathbb{E}[Z]) = \mathbb{E}[f(Z)].$$

Proof. It is just a consequence of the identity:

$$f(\mathbb{E}[Z]) = J(\mathbb{E}[Z])(f) = J(J^{-1}(f \mapsto \mathbb{E}[f(Z)]))(f) = \mathbb{E}[f(Z)].$$

□

Proposition B.10. *Given a reflexive space $(E, \|\cdot\|)$ and a random vector $Z \in E$, if $Z \propto \mathcal{E}_q(\sigma)$, then $\mathbb{E}[Z]$ can be defined with Definition 12.*

Proof. We just need to show that $f \mapsto \mathbb{E}[f(Z)]$ is continuous. There exists $K_p > 0$ such that $\mathbb{P}(\|Z_p\| \leq K_p) \geq \frac{1}{2}$, so that for any $f \in E'$, $\mathbb{P}(f(Z_p) \leq K_p \|f\|) \geq \frac{1}{2}$. Therefore, by definition, for any median m_f of $f(Z_p)$, $m_f \leq K_p \|f\|$. The two concentrations $f(Z) \in \mathbb{E}[f(Z)] \pm \mathcal{E}_q(\sigma)$ and $f(Z) \in m_f \pm \mathcal{E}_q(\sigma)$ then allows us to obtain a similar bound on $|\mathbb{E}[f(Z)]|$ which allows us to state that the mapping $f \mapsto |\mathbb{E}[f(Z)]|$ is continuous. □

It is still possible to define a notion of expectation when E is not reflexive but is a functional vector space having value on a reflexive space $(F, \|\cdot\|)$; for instance a subspace of F^G , for a given set G .

Definition 13. *Given reflexive space $(F, \|\cdot\|)$, a given set G , a subspace $E \subset F^G$, and a random vector $\phi \in E$, if for any $x \in F$, the mapping $F' \ni f \mapsto \mathbb{E}[f(\phi(x))]$ is continuous, we can define the expectation of ϕ as:*

$$\mathbb{E}[\phi] : x \mapsto \mathbb{E}[\phi(x)].$$

Remark B.11. *When the space $E \subset F^G$ is endowed with a norm $\|\cdot\|$ such that $(E, \|\cdot\|)$ is reflexive and $\forall x \in G, E \ni \phi \mapsto \phi(x)$ is continuous, then there is no ambiguity on the definitions. Indeed, if we note $\mathbb{E}_1[\phi]$ and $\mathbb{E}_2[\phi]$, respectively the expectation of ϕ given by Definition 12 and Definition 13, we can show for any $x \in F$ and any $f \in F'$:*

$$f(\mathbb{E}_1[\phi](x)) = \tilde{f}(\mathbb{E}_1[\phi]) = \mathbb{E}[\tilde{f}(\phi)] = \mathbb{E}[f(\phi(x))] = f(\mathbb{E}[\phi(x)]) = f(\mathbb{E}_2[\phi](x)),$$

where $\tilde{f} : E \ni \psi \rightarrow f(\psi(x))$ is a continuous linear form. Since this identity is true for any $f \in E'$, we know by reflexivity of E that $\forall x \in E: \mathbb{E}_1[\phi](x) = \mathbb{E}_2[\phi](x)$. This directly implies that $\mathbb{E}_1[\phi] = \mathbb{E}_2[\phi]$.

Remark B.12. *Given a random mapping $\phi \in E \subset F^G$ with $(F, \|\cdot\|)$ reflexive, we can then deduce from Proposition B.10 that if for all $x \in G, \phi(x) \propto \mathcal{E}_q(\sigma)$ in $(F, \|\cdot\|)$ then we can define $\mathbb{E}[\phi]$. With a different formalism, we can endow F^G with the family of semi-norms $(\|\cdot\|_x)_{x \in G}$ defined for any $f \in F^G$ as $\|f\|_x = \|f(x)\|$; then if for all $x \in G, \phi \propto \mathcal{E}_q(\sigma)$ in $(F^G, \|\cdot\|_x)$, it is straightforward to set that $\mathbb{E}[\phi]$ is well defined. This is in particular the case if $E = \mathcal{B}(G, F)$ is the set of bounded mappings from G to F and $\phi \propto \mathcal{E}_q(\sigma)$ in $(F^G, \|\cdot\|_\infty)$ where for all $f \in F^G, \|f\|_\infty = \sup_{x \in G} \|f(x)\|$.*

The Lipschitz-concentrated vectors described in Definition 9 belong to the larger class of linearly concentrated random vectors that only requires the linear observations to concentrate. This ‘‘linear concentration’’ presents less stability properties than those described by Proposition B.6 but is still a relevant notion because:

1. although it must be clear that a concentrated vector Z is generally far from its expectation (for instance Gaussian vectors lie on an ellipse), it can still be useful to have some control on $\|Z - \mathbb{E}[Z]\|$ to express the concentration of product of vectors; linear concentration is a sufficient assumption for this control,
2. there are some examples (Proposition B.41 and B.48) where we can only derive linear concentration inequalities from a Lipschitz concentration hypothesis. In that case, we say that the Lipschitz concentration ‘‘degenerates’’ into linear concentration that appears as a ‘‘residual’’ concentration property.

These properties of linear concentration are discussed in depth in the next section.

B.3 Linear concentration and control on high order statistics

Definition 14 (Linearly concentrated vectors). *Given a sequence of normed vector spaces $(E_p, \|\cdot\|_p)_{p \geq 0}$, a sequence of random vectors $(Z_p)_{p \geq 0} \in \prod_{p \geq 0} E_p$, a sequence of deterministic vectors $(\tilde{Z}_p)_{p \geq 0} \in \prod_{p \geq 0} E_p$, a sequence of positive reals $(\sigma_p)_{p \geq 0} \in \mathbb{R}_+^{\mathbb{N}}$ and a parameter $q > 0$, Z_p is said to be q -exponentially linearly concentrated around the deterministic equivalent \tilde{Z}_p with an observable diameter of order $O(\sigma_p)$ iff there exist two constants $c, C > 0$ such that $\forall p \in \mathbb{N}$ and for any unit-normed linear form $f \in E'_p$ ($\forall p \in \mathbb{N}, \forall x \in E: |f(x)| \leq \|x\|$):*

$$\forall t > 0 : \mathbb{P}\left(\left|f(Z_p) - f(\tilde{Z}_p)\right| \geq t\right) \leq C e^{-(t/c\sigma_p)^q}.$$

When the property holds, we write $Z \in \tilde{Z} \pm \mathcal{E}_q(\sigma)$. If it is unnecessary to mention the deterministic equivalent, we will simply write $Z \in \mathcal{E}_q(\sigma)$. If we just need to control its amplitude, we can write $Z \in O(\theta) \pm \mathcal{E}_q(\sigma)$ when $\|\tilde{Z}_p\| \leq O(\theta_p)$. Of course we adopt also to linearly concentrated random vectors the multi regime notation defined in Remark B.1.

When $q = 2$, we retrieve the well known class of sub-Gaussian random vectors. We need this definition with generic q to prove Proposition B.41 which involves a weaker than \mathcal{E}_2 tail decay.

Of course linear concentration is stable through affine transformations.

Proposition B.13 (Stability through affine mappings). *Given two (sequences of) normed vector spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, a (sequence of) random vectors $Z \in E$, a (sequence of) deterministic vectors $\tilde{Z} \in E$ and a (sequence of) affine mappings $\phi : E \rightarrow F$ such that $\forall x \in E : \|\phi(x) - \phi(0)\|_F \leq \lambda\|x\|_E$:*

$$Z \in \tilde{Z} \pm \mathcal{E}_q(\sigma) \quad \implies \quad \phi(Z) \in \phi(\tilde{Z}) \pm \mathcal{E}_q(\lambda\sigma).$$

When the expectation can be defined, there exists an implication link between Lipschitz concentration (Definitions 9) and linear concentration (Definition 14).

Lemma B.14. *Given a normed space $(E, \|\cdot\|)$ and a random vector $Z \in E$ admitting an expectation (see Section B.2), we have the implication:*

$$Z \propto \mathcal{E}_q(\sigma) \quad \implies \quad Z \in \mathbb{E}[Z] \pm \mathcal{E}_q(\sigma).$$

This implication becomes an equivalence in law dimensional spaces (i.e. when the sequence index “ p ” is not linked to the dimension of the vector spaces E_p); then the distinction between linear concentration and Lipschitz concentration is not relevant anymore. To simplify the hypotheses, we will assume that the normed vector space does not change at all with p .

Proposition B.15. *Given a normed vector space of finite dimension $(E, \|\cdot\|)$, a sequence of random vectors $(Z_p)_{p \in \mathbb{N}} \in E^{\mathbb{N}}$ and a sequence of positive values $(\sigma_p)_{p \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, we have the equivalence:*

$$Z \propto \mathcal{E}_q(\sigma) \quad \iff \quad Z \in \mathbb{E}[Z] \pm \mathcal{E}_q(\sigma)$$

Proof. We already know from Lemma B.14 that $Z \propto \mathcal{E}_q(\sigma) \implies Z \in \mathbb{E}[Z] \pm \mathcal{E}_q(\sigma)$, so let us now assume that $Z \in \mathbb{E}[Z] \pm \mathcal{E}_q(\sigma)$. Let us note d , the dimension of E and introducing $(e_1, \dots, e_d) \in E^d$, a basis of d , we note $\|\cdot\|_{\ell_\infty}$, the norm defined for any $x = \sum_{i=1}^d x_i e_i$ as $\|x\|_{\ell_\infty} = \max_{i \in [d]} |x_i|$. There exists a constant α ($\alpha \leq O(1)$) such that for all $x \in E$, $\|x\| \leq \alpha\|x\|_{\ell_\infty}$ and therefore, one can bound for any 1-lipschitz mapping $f : E \rightarrow \mathbb{R}$ and any sequence of random vectors Z' , independent with Z :

$$\mathbb{P}(|f(Z) - f(Z')| \geq t) \leq \mathbb{P}(\|Z - Z'\| \geq t) \leq \mathbb{P}\left(\alpha \sup_{i \in [d]} |Z_i - Z'_i| \geq t\right) \leq dC e^{-(t/c\sigma\alpha)^q},$$

where $C, c > 0$ are two constants. Since $dC, c\alpha \leq O(1)$, we retrieve the Lipschitz concentration of Z . \square

Apart from Lipschitz concentration cases, Linear concentration is also provided by classical concentration inequalities like Bernstein’s or Hoeffding’s inequalities. It is what is called sub Gaussian random vectors (see [Ver18] for instance)

Example B.16. [Bernstein’s inequality, [Ver17, Theorem 2.8.2]] *Given p independent random variables $(Z_i)_{1 \leq i \leq p} \in \mathbb{R}^p$, and three parameters $C \geq e$ and $c, q > 0$, for any deterministic vector $a \in \mathbb{R}^p$, we have the implication:*

$$\forall i \in \{1, \dots, p\}, Z_i \in \mathbb{E}[Z_i] \pm \mathcal{E} \quad \implies \quad a^T Z \in a^T \mathbb{E}[Z] \pm \mathcal{E}_2(\|a\|) + \mathcal{E}_1(\|a\|_\infty),$$

where c is a numerical constant depending only on C and $C = (Z_1, \dots, Z_p)$. If we want to express it as a linear concentration, one can just obtain: $Z \in \mathbb{E}[Z] \pm \mathcal{E}_1$.

Example B.17. [Hoeffding’s inequality, [Ver17, Theorem 2.2.6]] *Given p independent random variables $Z_1, \dots, Z_p \in [0, 1]$, the random vector $Z = (Z_1, \dots, Z_p) \in \mathbb{R}^p$ is linearly concentrated and verifies $Z \in \mathbb{E}[Z] \pm \mathcal{E}_2$ in $(\mathbb{R}^p, \|\cdot\|)$.*

The next lemma is a formal expression of the assessment that “any deterministic vector located at a distance smaller than the observable diameter to a deterministic equivalent is also a deterministic equivalent”, it is a simple consequence of Lemma A.1.

Lemma B.18. *Given a random vector $Z \in E$, a deterministic vector $\tilde{Z} \in E$ such that $Z \in \tilde{Z} \pm \mathcal{E}_q(\sigma)$, we have the equivalence:*

$$Z \in \tilde{Z}' \pm \mathcal{E}_q(\sigma) \quad \iff \quad \|\tilde{Z} - \tilde{Z}'\| \leq O(\sigma)$$

With the notations of Lemma B.18, when $\|\tilde{Z}\| \leq O(\sigma)$, one can note that $0 \in E$ is also a deterministic equivalent of Z which lead to the following useful tool, principally applied to random variables.

Lemma B.19. *Given a random vectors $Z \in E$ satisfying: $Z \in O(\sigma) \pm \mathcal{E}_q(\sigma)$, any random vector $Y \in E$ satisfying $\|Y\| \leq \|Z\|$ also satisfies:*

$$Y \in O(\sigma) + \mathcal{E}_q(\sigma).$$

Proof. Since $Z \in 0 \pm \mathcal{E}_q(\sigma)$, there exists some constants $C, c > 0$ such that for any (sequence of) 1-Lipschitz mapping $f : E \rightarrow \mathbb{R}$:

$$\forall t > 0 : \mathbb{P}(|f(Y)| \geq t) \leq \mathbb{P}(|f(Z)| \geq t) \leq Ce^{-c(t/\sigma)^2}.$$

□

Definition 15 (Centered moments of random vectors). *Given a random vector $X \in \mathbb{R}^p$ and an integer $r \in \mathbb{N}$, we call the “ r^{th} centered moment of X ” the symmetric r -linear form $C_r^X : (\mathbb{R}^p)^r \rightarrow \mathbb{R}$ defined for any $u_1, \dots, u_r \in \mathbb{R}^p$ by*

$$C_r^X(u_1, \dots, u_p) = \mathbb{E} \left[\prod_{i=1}^r (u_i^T X - \mathbb{E}[u_i^T X]) \right].$$

When $r = 2$, the centered moment is the covariance matrix.

We define the operator norm of an r -linear form S of \mathbb{R}^p as

$$\|S\| \equiv \sup_{\|u_1\|, \dots, \|u_r\| \leq 1} S(u_1, \dots, u_p).$$

When S is symmetric, we employ the simpler formula $\|S\| = \sup_{\|u\| \leq 1} S(u, \dots, u)$. We then have the following characterization, consequence of Proposition A.20 and Lemma A.23.

Proposition B.20 (Moment characterization of linear concentration). *Given $q > 0$, a sequence of random vectors $X_p \in \mathbb{R}^p$, and a sequence of positive numbers $\sigma_p > 0$, we have the following equivalence:*

$$X \in \mathcal{E}_q(\sigma) \quad \iff \quad \exists C, c > 0, \forall p \in \mathbb{N}, \forall r \geq q : \|C_r^{X_p}\| \leq C \left(\frac{r}{q}\right)^{\frac{r}{q}} (c\sigma_p)^r$$

In particular, if we note $C = \mathbb{E}[XX^T] - \mathbb{E}[X]\mathbb{E}[X]^T$, the covariance of $X \in \mathcal{E}_q(\sigma)$, we see that $\|C\| \leq O(\sigma^2)$, if in addition $X \in O(\sigma) \pm \mathcal{E}_q(\sigma)$ (which means that $\|\mathbb{E}[X]\| \leq O(\sigma)$), then $\|\mathbb{E}[XX^T]\| \leq O(\sigma^2)$

With these results at hand, we are in particular in position to explain how a control on the norm can be deduced from a linear concentration hypothesis.

B.4 Control of the norm of linearly concentrated random vectors

Given a random vector $Z \in (E, \|\cdot\|)$, if $Z \in \tilde{Z} \pm \mathcal{E}_q(\sigma)$, the control of $\|Z - \tilde{Z}\|$ can be done easily when the norm $\|\cdot\|$ can be defined as the supremum on a set of linear forms; for instance when $(E, \|\cdot\|) = (\mathbb{R}^p, \|\cdot\|_\infty)$:

$\|x\|_\infty = \sup_{1 \leq i \leq p} e_i^T x$ (where (e_1, \dots, e_p) is the canonical basis of \mathbb{R}^p). We can then bound thanks to Lemma A.14:

$$\begin{aligned} \mathbb{P}\left(\|Z - \tilde{Z}\|_\infty \geq t\right) &= \mathbb{P}\left(\sup_{1 \leq i \leq p} e_i^T (Z - \tilde{Z}) \geq t\right) \\ &\leq \min\left(1, p \sup_{1 \leq i \leq p} \mathbb{P}\left(e_i^T (Z - \tilde{Z}) \geq t\right)\right) \\ &\leq \min\left(1, pCe^{-c(t/\sigma)^q}\right) \leq \max(C, e) \exp\left(-\frac{ct^q}{2\sigma^q \log(p)}\right), \end{aligned}$$

for some constants $c, C > 0$ ($C \leq O(1)$, $c \geq O(1)$).

To manage the infinity norm, the supremum is taken on a finite set $\{e_1, \dots, e_p\}$. Problems arise when this supremum must be taken on an infinite set. For instance, for the Euclidean norm, the supremum is taken over the whole unit ball $\mathcal{B}_{\mathbb{R}^p} \equiv \{u \in \mathbb{R}^p, \|u\| \leq 1\}$ since for any $x \in \mathbb{R}^p$, $\|x\| = \sup\{u^T x, \|u\| \leq 1\}$. This loss of cardinality control can be overcome if one introduces so-called ε -nets to discretize the ball with a net $\{u_i\}_{i \in I}$ (with I finite i.e. $|I| < \infty$) in order to simultaneously :

1. approach sufficiently the norm to ensure

$$\mathbb{P}\left(\|Z - \tilde{Z}\|_\infty \geq t\right) \approx \mathbb{P}\left(\sup_{i \in I} u_i^T (Z - \tilde{Z}) \geq t\right),$$

2. control the cardinality $|I|$ for the inequality

$$\mathbb{P}\left(\sup_{i \in I} u_i^T (Z - \tilde{Z}) \geq t\right) \leq |I| \mathbb{P}\left(u_i^T (Z - \tilde{Z}) \geq t\right)$$

not to be too loose.

In the case of \mathbb{R}^p , endowed with the euclidean norm, the number of ε -nets $|I|$ is of order $O(p)$, one can then show thanks to Lemma A.14 (more precisions are given in the proof of Proposition B.22) that there exist two constants $C, c > 0$ such that:

$$\mathbb{P}(\|Z - \tilde{Z}\| \geq t) \leq \max(C, e) \exp\left(-\frac{ct^q}{p\sigma^q}\right). \tag{B.3}$$

The approach with ε -nets in $(\mathbb{R}^p, \|\cdot\|)$ can be generalized to any normed vector space $(E, \|\cdot\|)$ when the norm can be written as a supremum through an identity of the kind :

$$\forall x \in E : \|x\| = \sup_{\substack{f \in H \\ \|f\| \leq 1}} f(x), \quad \text{with } H \subset E' \text{ and } \dim(\text{Vect}(H)) < \infty, \tag{B.4}$$

for a given $H \subset E'$ (for E' , the dual space of H) and with $\text{Vect}(H)$ the subspace of E' generated by H . Such a $H \subset E'$ exists in particular when $(E, \|\cdot\|)$ is a reflexive⁶ space [Jam57].

When $(E, \|\cdot\|)$ is of infinite dimension, it is possible to establish (B.4) for some $H \subset E$ when E is reflexive thanks to a result from [Jam57], or for some choice of semi-norms $\|\cdot\|$. We introduce the notion of *norm degree* which will help us adapt the concentration rate p appearing in the exponential term of concentration inequality (B.3) (concerning $(\mathbb{R}^p, \|\cdot\|)$) to other normed vector spaces.

Definition 16 (Norm degree). *Given a normed (or semi-normed) vector space $(E, \|\cdot\|)$, and a subset $H \subset E'$, the degree η_H of H is defined as:*

- $\eta_H \equiv \log(|H|)$ if H is finite,
- $\eta_H \equiv \dim(\text{Vect } H)$ if H is infinite.

⁶Introducing the mapping $J : E \rightarrow E''$ (where E'' is the bidual of E) satisfying $\forall x \in E$ and $\phi \in E'$: $J(x)(\phi) = \phi(x)$, the normed vector space E is said to be “reflexive” if J is onto.

If there exists a subset $H \subset E'$ such that (B.4) is satisfied, we denote $\eta(E, \|\cdot\|)$, or more simply $\eta_{\|\cdot\|}$, the degree of $\|\cdot\|$, defined as:

$$\eta_{\|\cdot\|} = \eta(E, \|\cdot\|) \equiv \inf \left\{ \eta_H, H \subset E' \mid \forall x \in E, \|x\| = \sup_{f \in H} f(x) \right\}.$$

Example B.21. We can give some examples of norm degrees:

- $\eta(\mathbb{R}^p, \|\cdot\|_\infty) = \log(p)$ ($H = \{x \mapsto e_i^T x, 1 \leq i \leq p\}$),
- $\eta(\mathbb{R}^p, \|\cdot\|) = p$ ($H = \{x \mapsto u^T x, u \in \mathcal{B}_{\mathbb{R}^p}\}$),
- $\eta(\mathcal{M}_{p,n}, \|\cdot\|) = n + p$ ($H = \{M \mapsto u^T M v, (u, v) \in \mathcal{B}_{\mathbb{R}^p} \times \mathcal{B}_{\mathbb{R}^n}\}$),
- $\eta(\mathcal{M}_{p,n}, \|\cdot\|_F) = np$ ($H = \{M \mapsto \text{Tr}(AM), A \in \mathcal{M}_{n,p}, \|A\|_F \leq 1\}$),
- $\eta(\mathcal{M}_{p,n}, \|\cdot\|_*) = np$ ($H = \{M \mapsto \text{Tr}(AM), A \in \mathcal{M}_{n,p}, \|A\| \leq 1\}$).⁷

Just to give some justification, if $E = \mathbb{R}^p$ or $E = \mathcal{M}_{p,n}$, the dual space E' can be identified with E through the representation with the scalar product. Given a subset $H' \subset E$ such that:

$$\forall x \in \mathbb{R}^p, \|x\|_\infty = \sup_{u \in H'} u^T x,$$

we can set that all $u \in H'$ satisfy $\|u\|_1 = \sum_{i=1}^p |u_i| \leq 1$ because if we note $u' = (\text{sign}(u_i))_{i \in [p]}$, we can bound $\|u\|_1 = u^T u' \leq \sup_{v \in H'} v^T u' \leq \|u'\|_\infty \leq 1$. Then, noting $H = \{e_1, \dots, e_p\}$, we know that $H \subset H'$, otherwise, if, say $e_i \notin H'$, then one could bound $\|e_i\|_\infty = \sup_{u \in H'} u^T e_i < 1$ (because if $\|u\|_1 \leq 1$ and $u \neq e_i$, then $u_i < 1$). Therefore $H \subset H'$ and it consequently reaches the minimum of $\eta_{H'}$. The value of the other norm indexes is justified with the same arguments.

Depending on the ambient vector space, one can employ one of these examples along with the following proposition to establish the concentration of the norm of a random vector. The following result is well known in the case of Euclidean norm of vectors or spectral norm of matrices, and the proof is almost a mere rewriting of [Tao12], Corollary 2.3.5.

Proposition B.22. Given a reflexive vector space $(E, \|\cdot\|)$ and a concentrated vector $Z \in E$ satisfying the linear concentration $Z \in \tilde{Z} \pm \mathcal{E}_q(\sigma)$:

$$\|Z - \tilde{Z}\| \propto \mathcal{E}_q\left(\eta_{\|\cdot\|}^{1/q} \sigma\right) \quad \text{and} \quad \mathbb{E} \left[\|Z - \tilde{Z}\| \right] \leq O\left(\eta_{\|\cdot\|}^{1/q} \sigma\right).$$

Proof. To prove the first implication, let us consider $H' \subset E^*$ such that $\forall x \in E, \|x\| = \sup_{f \in \mathcal{B}_{H'}} f(x)$. When H is finite, the bound is obtained the same as for the infinite norm. We know assume that H is infinite and we further note $H \equiv \text{Vect}(H')$. Given $\varepsilon \in (0, 1)$, a set $A \subset \mathcal{B}_H$ is said to be an ε -net of \mathcal{B}_H if $x, y \in A \Rightarrow \|x - y\| \geq \varepsilon$. We consider here $N_{1/2}$, a maximal $\frac{1}{2}$ -net of \mathcal{B}_H with respect to inclusion. We know that the balls of radius $\frac{1}{4}$ centered on the points of $N_{1/2}$ are all disjoint by hypothesis, and their volume is equal to $\mathcal{V}_{\mathcal{B}_H}/4^p$ (where $\mathcal{V}_{\mathcal{B}_H}$ is the volume of \mathcal{B}_H). Since they all belong to the ball of radius 2 and centered at the origin, we know that their number cannot exceed $8^{\dim(H)}$.

Besides, given a drawing of Z , there exists $f_0 \in \mathcal{B}_H$ such that $Z - \tilde{Z} = f_0(Z - \tilde{Z})$ (since \mathcal{B}_H is compact). Then there exists $f \in \mathcal{B}_H$ such that $\|f - f_0\|_*$ is bounded by $\frac{1}{2}$ (otherwise f_0 could be added to $N_{1/2}$). Furthermore :

$$\begin{aligned} \left\| Z - \tilde{Z} \right\| - f\left(Z - \tilde{Z}\right) &\leq \left| f\left(Z - \tilde{Z}\right) - f_0\left(Z - \tilde{Z}\right) \right| \\ &\leq \|f - f_0\|_* \left\| Z - \tilde{Z} \right\| \leq \frac{1}{2} \left\| Z - \tilde{Z} \right\|. \end{aligned}$$

⁷ $\|\cdot\|_*$ is the nuclear norm defined for any $M \in \mathcal{M}_{p,n}$ by $\|M\|_* = \text{Tr}(\sqrt{MM^T})$; it is the dual norm of $\|\cdot\|$, which means that for any $A, B \in \mathcal{M}_{p,n}$, $\text{Tr}(AB^T) \leq \|A\| \|B\|_*$. One must be careful that Proposition B.22 is rarely useful to bound the nuclear norm as explained in footnote 9.

Therefore:

$$\|Z - \tilde{Z}\| \leq 2 \sup \left\{ f \in N_{1/2} : f(Z - \tilde{Z}) \right\}$$

and this inequality being true for any drawing of Z , we have then by hypothesis the existence of two constants $C, c > 0$ such that:

$$\forall t > 0 : \mathbb{P} \left(\|Z - \tilde{Z}\| \geq t \right) \leq \sum_{N_{1/2}} C e^{-\frac{t^q}{c}} \leq 8^{\dim(H)} C e^{-\frac{t^q}{c}} \leq C' e^{-\frac{t^q}{c' \eta_{\|\cdot\|}}},$$

for some constant $C', c' > 0$, thanks to Lemma A.14. \square

Remark B.23. In Proposition B.22, if in addition, Z satisfies the Lipschitz concentration $Z \propto \mathcal{E}_q(\sigma)$, the norm satisfies the same concentration as it is a Lipschitz observation, and one gets⁸:

$$\|Z - \tilde{Z}\| \in O \left(\eta_{\|\cdot\|}^{1/q} \sigma \right) \pm \mathcal{E}_q(\sigma).$$

Example B.24. Given two random vectors $Z \in \mathbb{R}^p$ and $X \in \mathcal{M}_{p,n}$:

- if $Z \propto \mathcal{E}_2$ in $(\mathbb{R}^p, \|\cdot\|)$, then $\mathbb{E} \|Z\| \leq \|\mathbb{E}[Z]\| + O(\sqrt{p})$,
- if $X \propto \mathcal{E}_2$ in $(\mathcal{M}_{p,n}, \|\cdot\|)$, then $\mathbb{E} \|X\| \leq \|\mathbb{E}[X]\| + O(\sqrt{p+n})$,
- if $X \propto \mathcal{E}_2$ in $(\mathcal{M}_{p,n}, \|\cdot\|_F)$, then $\mathbb{E} \|X\| \leq \|\mathbb{E}[X]\|_F + O(\sqrt{pn})$.
- if $X \propto \mathcal{E}_2(\sqrt{\min(p,n)})$ in $(\mathcal{M}_{p,n}, \|\cdot\|_*)$, then $\mathbb{E} \|X\|_* \leq \|\mathbb{E}[X]\|_* + O(\sqrt{pn} \sqrt{\min(p,n)})$.⁹

Example B.25. Let us consider the semi norm $\|\cdot\|_d$ that will be useful later and that satisfies:

$$\forall M \in \mathcal{M}_n : \|M\|_d = \left(\sum_{i=1}^n M_{i,i}^2 \right)^{\frac{1}{2}} = \sup_{\substack{D \in \mathcal{D}_n \\ \|D\|_F \leq 1}} \text{Tr}(DM).$$

where \mathcal{D}_n is the set of diagonal matrices of \mathcal{M}_p , note that for all $D \in \mathcal{D}_n$, $\|D\|_d = \|D\|_F$, and for $A \in \mathcal{M}_p$, $\text{Tr}(AD) \leq \|A\|_d \|D\|_d$. We see directly that $\eta_{(\mathcal{M}_n, \|\cdot\|_d)} = \#\mathcal{D}_n = n$ and therefore for a given $X \in \mathcal{M}_n$ such that $X \propto \mathcal{E}_2$, we can bound $\mathbb{E} \|X\|_d \leq \|\mathbb{E}[X]\|_d + O(\sqrt{n})$.

Proposition B.22 is not always the optimal way to bound norms. For instance, given a vector $Z \in \mathbb{R}^p$ and a deterministic matrix $A \in \mathcal{M}_p$, if $Z \propto \mathcal{E}_q$, one is tempted to bound naively thanks to Proposition B.22:

- if $\|\mathbb{E}[Z]\| \leq O(p^{1/q})$, $\mathbb{E} \|AZ\| \leq \|A\| \mathbb{E} \|Z\| \leq O(\|A\| p^{\frac{1}{q}})$;
- if $\|\mathbb{E}[Z]\| \leq O(1)$, decomposing $A = P^T \Lambda Q$, where $P, Q \in \mathcal{O}_p$, $\Lambda = \text{Diag}(\lambda)$, $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p$ and setting $\tilde{Z} = (\tilde{Z}_1, \dots, \tilde{Z}_p) \equiv QZ$:

$$\mathbb{E} \|AZ\| = \mathbb{E} \|\Lambda QZ\| = \mathbb{E} \left[\sqrt{\sum_{i=1}^p \lambda_i^2 \tilde{Z}_i^2} \right] \leq \|\lambda\| \mathbb{E} [\|\tilde{Z}\|_\infty] \leq \|A\|_F O \left((\log p)^{\frac{1}{q}} \right).$$

Note indeed that $\tilde{Z} \propto \mathcal{E}_2$ and therefore $\mathbb{E} [\|\tilde{Z}\|_\infty] \leq \|\mathbb{E}[\tilde{Z}]\|_\infty + O \left((\log p)^{\frac{1}{q}} \right) \leq \|\mathbb{E}[Z]\| + O \left((\log p)^{\frac{1}{q}} \right)$.

⁸The notation $Z \in O(\theta) \pm \mathcal{E}_q(\sigma)$ was presented in Definition 14 for linearly concentrated vectors, it can be extended to concentrated random variables.

⁹One must be careful here that Theorem B.7 just provides concentration in the Euclidean spaces $(\mathbb{R}^p, \|\cdot\|)$ or $(\mathcal{M}_{p,n}, \|\cdot\|_F)$ from which one can deduce concentration in $(\mathbb{R}^p, \|\cdot\|_\infty)$ or $(\mathcal{M}_{p,n}, \|\cdot\|)$ since for all $x \in \mathbb{R}^p$, $\|x\|_\infty \leq \|x\|$ and for all $M \in \mathcal{M}_{p,n}$, $\|M\| \leq \|M\|_F$. However one cannot obtain a better bound than $\|M\|_* \leq \sqrt{\min(n,p)} \|M\|_F$: this for instance implies that a random matrix $X = (x_1, \dots, x_n)$ with x_1, \dots, x_n i.i.d. satisfying $\forall i \in [n]$, $x_i \sim \mathcal{N}(0, I_p)$ follows the concentration $X \propto \mathcal{E}_2(\sqrt{\min(p,n)})$ in $(\mathcal{M}_{p,n}, \|\cdot\|_*)$.

However, here, Proposition B.22 is suboptimal: one can reach a better bound thanks to the following lemma. We give a result for random vectors and random matrices, they are actually equivalent.

Lemma B.26. *Given a random vector $Z \in \mathcal{E}_q$ in $(\mathbb{R}^p, \|\cdot\|)$ such that $\|\mathbb{E}[Z]\| \leq O(1)$ and a deterministic matrix $A \in \mathcal{M}_p$:*

$$\mathbb{E}[\|AZ\|] \leq O(\|A\|_F).$$

and given a random matrix $X \in \mathcal{E}_2$ in $(\mathcal{M}_{p,n}, \|\cdot\|_F)$ such that $\|\mathbb{E}[X]\|_F \leq O(1)$ and a supplementary deterministic matrix $B \in \mathcal{M}_n$:

$$\mathbb{E}[\|AXB\|_F] \leq O(\|A\|_F\|B\|_F).$$

Proof. Denoting $\Sigma = \mathbb{E}[ZZ^T] = C_2^Z + \mathbb{E}[Z]\mathbb{E}[Z]^T$, we know from Proposition B.20 that $\|\Sigma\| \leq O(1)$; we can then bound with Jensen's inequality:

$$\mathbb{E}[\|AZ\|] \leq \sqrt{\mathbb{E}[Z^T A^T A Z]} = \sqrt{\mathbb{E}[\text{Tr}(\Sigma A^T A)]} \leq \sqrt{\|\Sigma\|} \|A\|_F \leq O(\|A\|_F).$$

The second result is basically the same. If we introduce $\tilde{X} \in \mathbb{R}^{pn}$ satisfying $\tilde{X}_{i(j-1)+j} = X_{i,j}$, we know that $\tilde{X} \propto \mathcal{E}_2$ like X (since $\|\tilde{X}\| = \|X\|_F$) and thanks to the previous result we can bound:

$$\mathbb{E}[\|AXB\|_F] = \mathbb{E}[\|A \otimes B \tilde{X}\|] \leq O(\|A \otimes B\|_F) = O(\|A\|_F\|B\|_F).$$

□

B.5 Linear concentration through finite and infinite sums

Independence is known to be a key elements to most of concentration inequalities. However, linear concentration behaves particularly well for the concatenation of random vectors whose dependence can not be disentangled.

The next proposition sets that the observable diameter for the ℓ^∞ norm remains unchanged through concatenation. Given a product $E \equiv \prod_{1 \leq i \leq m} E_i$, where $(E_1, \|\cdot\|_\infty), \dots, (E_m, \|\cdot\|_\infty)$ are m normed vector spaces we define the ℓ^∞ norm on E with the following identity:

$$(z_1, \dots, z_m) \in E : \|(z_1, \dots, z_m)\|_{\ell^\infty} = \sup_{1 \leq i \leq m} \|z_i\|_i. \quad (\text{B.5})$$

Proposition B.27. *Given two sequences $m \in \mathbb{N}^{\mathbb{N}}$ and $\sigma \in \mathbb{R}_+^{\mathbb{N}}$, a constant q , m sequences of normed vector spaces $(E_i, \|\cdot\|_i)_{1 \leq i \leq m}$, m sequences of deterministic vectors $\tilde{Z}_1 \in E_1, \dots, \tilde{Z}_m \in E_m$, and m sequences of random vectors $Z_1 \in E_1, \dots, Z_m \in E_m$ (possibly dependent) satisfying, for any $i \in \{1, \dots, m\}$, $Z_i \in \tilde{Z}_i \pm \mathcal{E}_q(\sigma)$, we have the concentration:*

$$(Z_1, \dots, Z_m) \in (\tilde{Z}_1, \dots, \tilde{Z}_m) \pm \mathcal{E}_q(\sigma), \quad \text{in } (E, \|\cdot\|_{\ell^\infty}).$$

In other word, the linear observable diameter of (Z_1, \dots, Z_m) can not be bigger than the observable diameter of (Z, \dots, Z) , where Z is chosen as the worse possible random vector satisfying the hypotheses of Z_1, \dots, Z_m . The same results hold of course if the different Z_1, \dots, Z_m follow the same combination of exponential concentration.

Remark B.28. *An example provided at the beginning of Section B.6 proves that this stability towards concatenation is not true for Lipschitz (nor convex concentration).*

Proof. Let us consider a linear function $u : E \rightarrow \mathbb{R}$, such that

$$\|u\|_\infty \equiv \sup_{\|z\|_\infty \leq 1} |u(z)| \leq 1.$$

Given $i \in [m]$, let us note $u_i : E_i \rightarrow \mathbb{R}$ the function defined as $u_i(z) = u((0, \dots, 0, z, 0, \dots, 0))$ (where z is in the i^{th} entry). For any $z \in E$, one can write:

$$u(z) = \sum_{i=1}^m n_i u'_i(z_i),$$

where $n_i \equiv \|u_i\| = \sup_{\|z\|_i \leq 1} u_i(z)$ and $u'_i = u_i/n_i$ ($\|u'_i\| = 1$). We have the inequality:

$$\sum_{i=1}^m n_i = \sum_{i=1}^m n_i \sup_{\|z_i\|_i \leq 1} u'_i(z_i) = \sup_{\|z\|_\infty \leq 1} u(z) \leq 1.$$

With this bound at hand, we plan to employ the characterization with the centered moments. Let us conclude thanks to Proposition A.20 and the convexity of $t \mapsto t^l$, for any $l \geq 1$:

$$\begin{aligned} \mathbb{E} \left[\left| u(Z) - u(\tilde{Z}) \right|^l \right] &\leq \mathbb{E} \left[\left(\sum_{i=1}^m n_i \left| u'_i(Z_i) - u'_i(\tilde{Z}_i) \right| \right)^l \right] \\ &\leq \left(\sum_{i=1}^m n_i \right)^l \mathbb{E} \left[\sum_{i=1}^m \frac{n_i}{\sum_{i=1}^m n_i} \left| u'_i(Z_i) - u'_i(\tilde{Z}_i) \right|^l \right] \\ &\leq \sup_{i \in [m]} \mathbb{E} \left[\left| u'_i(Z_i) - u'_i(\tilde{Z}_i) \right|^l \right] \leq C l^{\frac{1}{q}} \sigma^l. \end{aligned}$$

□

If we want to consider the concatenation of vectors with different observable diameters, it is more convenient to look at the concentration in a space $(\prod_{i=1}^m E_i, \ell^r)$, for any given $r > 0$, where, for any $(z_1, \dots, z_m) \in \prod_{i=1}^m E_i$:

$$\|(z_1, \dots, z_m)\|_{\ell^r} = \left(\sum_{i=1}^m \|z_i\|_i^r \right)^{1/r}.$$

Corollary B.29. *Given two constants $q, r > 0$, $m \in \mathbb{N}^{\mathbb{N}}$, $\sigma_1, \dots, \sigma_m \in (\mathbb{R}_+^{\mathbb{N}})^m$, m sequences of $(E_i, \|\cdot\|_i)_{1 \leq i \leq m}$, m sequences of deterministic vectors $\tilde{Z}_1 \in E_1, \dots, \tilde{Z}_m \in E_m$, and m sequences of random vectors $Z_1 \in E_1, \dots, Z_m \in E_m$ (possibly dependent) satisfying, for any $i \in \{1, \dots, m\}$, $Z_i \in \tilde{Z}_i \pm \mathcal{E}_2(\sigma_i)$, we have the concentration:*

$$(Z_1, \dots, Z_m) \in (\tilde{Z}_1, \dots, \tilde{Z}_m) \pm \mathcal{E}_2(\|\sigma\|_r), \quad \text{in } (E, \|\cdot\|_{\ell^r}),$$

Remark B.30. *When $E_1 = \dots = E_m = E$ in the setting of Corollary B.29, then for any vector $a = (a_1, \dots, a_m) \in \mathbb{R}_+^m$, we know that:*

$$\sum_{i=1}^m a_i Z_i \in \sum_{i=1}^m a_i \tilde{Z}_i \pm \mathcal{E}_2(|a|^T \sigma),$$

where $|a| = (|a_1|, \dots, |a_m|) \in \mathbb{R}_+^m$

Proof. We already know from Proposition B.27 that:

$$\left(\frac{Z_1}{\sigma_1}, \dots, \frac{Z_m}{\sigma_m} \right) \in \left(\frac{\tilde{Z}_1}{\sigma_1}, \dots, \frac{\tilde{Z}_m}{\sigma_m} \right) \pm \mathcal{E}_2, \quad \text{in } (E, \|\cdot\|_{\ell^\infty}).$$

Let us then consider the linear mapping:

$$\begin{aligned} \phi : (E, \|\cdot\|_{\ell^\infty}) &\longrightarrow (E, \|\cdot\|_{\ell^r}) \\ (z_1, \dots, z_m) &\longmapsto (\sigma_1 z_1, \dots, \sigma_m z_m), \end{aligned}$$

the Lipschitz character of ϕ is clearly $\|\sigma\|_r = (\sum_{i=1}^m \sigma_i^r)^{1/r}$, and we can deduce the concentration of $Z = \phi(\sigma_1 Z_1, \dots, \sigma_m Z_m)$. □

Corollary B.29 is very useful to set the concentration of infinite series of concentrated random variables. For that, we resort to Proposition A.11 that sets that the observable diameter of a limit of random vectors is equal to the limit of the observable vectors. Be careful that rigorously, there are two indexes, n coming from Definition 14 that only describes the concentration of sequences of random vectors, and m particular to this lemma that will tend to infinity. For clarity, we do not mention the index n in the following adaptation of Proposition A.11 to the case of concentrated random vectors.

Lemma B.31. *Given a sequence of random vectors $(Z_m)_{m \in \mathbb{N}} \in E^{\mathbb{N}}$, a sequence of positive reals $(\sigma_m)_{m \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ and a sequence of deterministic vectors $(\tilde{Z}_m)_{m \in \mathbb{N}} \in E^{\mathbb{N}}$ such that:*

$$Z_m \in \tilde{Z}_m \pm \mathcal{E}_2(\sigma_m),$$

if we assume that $(Z_m)_{m \in \mathbb{N}}$ converges in law¹⁰ when m tends to infinity to a random vector $(Z_\infty) \in E$, that $\sigma_m \xrightarrow[n \rightarrow \infty]{} \sigma_\infty$ and that $\tilde{Z}_m \xrightarrow[n \rightarrow \infty]{} \tilde{Z}_\infty$, then:

$$Z_\infty \in \tilde{Z}_\infty \pm \mathcal{E}_2(\sigma_\infty).$$

(The result also holds for Lipschitz and convex concentration but without deterministic equivalent)

Corollary B.32. *Given two constants $q, r > 0$, $\sigma_1, \dots, \sigma_m, \dots \in (\mathbb{R}_+^{\mathbb{N}})^{\mathbb{N}}$, a (sequences of) normed vector spaces $(E, \|\cdot\|)$, $\tilde{Z}_1, \dots, \tilde{Z}_m, \dots \in E^{\mathbb{N}}$ deterministic, and $Z_1, \dots, Z_m, \dots \in E^{\mathbb{N}}$ random (possibly dependent) satisfying, for any $n \in \mathbb{N}$, $Z_m \in \tilde{Z}_m \pm \mathcal{E}_2(\sigma_m)$. If we assume that $Z \equiv \sum_{n \in \mathbb{N}} Z_m$ is pointwise convergent¹¹, that $\sum_{m \in \mathbb{N}} \tilde{Z}_m$ is well defined and that $\sum_{n \in \mathbb{N}} \sigma_i \leq \infty$, then we have the concentration:*

$$\sum_{m \in \mathbb{N}} Z_m \in \sum_{m \in \mathbb{N}} \tilde{Z}_m \pm \mathcal{E}_2 \left(\sum_{m \in \mathbb{N}} \sigma_m \right), \text{ in } (E, \|\cdot\|),$$

Proof. We already know from Corollary B.29 that for all $m \in \mathbb{N}$:

$$\sum_{m=1}^M Z_m \in \sum_{m=1}^M \tilde{Z}_m \pm \mathcal{E}_2 \left(\sum_{m \in \mathbb{N}} \sigma_m \right), \text{ in } (E, \|\cdot\|).$$

Thus in order to employ Lemma B.31 let us note that for any bounded continuous mapping $f : E \rightarrow \mathbb{R}$, the dominated convergence theorem allows us to set that:

$$\mathbb{E} \left[f \left(\sum_{m=1}^M Z_m \right) \right] \xrightarrow{M \rightarrow \infty} \mathbb{E} \left[f \left(\sum_{m=1}^{\infty} Z_m \right) \right],$$

thus $(\sum_{m=1}^M Z_m)_{M \in \mathbb{N}}$ converges in law to $\sum_{m=1}^{\infty} Z_m$, which allows us to set the result of the corollary. \square

The concentration of infinite series directly implies the concentration of resolvents and other related operators (like $(I_n - X/\sqrt{p})^{-1}X^k$ for instance) as it will be exposed at the beginning of Part C.3 (see Theorem C.3). To be more precise, this approach to the resolvent with infinite sum is particularly interesting for convexly concentrated random matrices for which the product is only linearly concentrated (see Chapter D, Theorem D.21) Part II, that this corollary finds its relevancy under convex concentration hypotheses, where the linear concentration seems to be the best concentration property to obtain on the resolvent $Q^z = (zI_p - \frac{1}{n}XX^T)^{-1}$.

Returning to Lipschitz concentration, in order to control the concentration of the sum $X + Y$ or the product XY of two random vectors X and Y , a first step is to express the concentration of the concatenation (X, Y) . This last result is easily obtained for the class of linearly concentrated random vectors but a tight concentration of the product with good observable diameter is in general not accessible. In the class of Lipschitz concentrated vectors, the concentration of (X, Y) is far more involved, and assumptions of independence here play a central role (unlike for linear concentration).

¹⁰For any $n \in \mathbb{N}$, for any bounded continuous mapping $f : \prod_{m \geq 0} E_p \rightarrow \mathbb{R}^{\mathbb{N}}$:

$$\sup_{n \in \mathbb{N}} |\mathbb{E}[f(Z_{n,m})] - \mathbb{E}[f(Z_{n,\infty})]| \xrightarrow{m \rightarrow \infty} 0$$

¹¹For any $w \in \Omega$, $\sum_{m \in \mathbb{N}} \|Z_m(w)\| \leq \infty$ and we define $Z(w) \equiv \sum_{m \in \mathbb{N}} Z_m(w)$

B.6 Concentration of vector concatenation

One must be careful because, we saw in the introduction that the concatenation of two random vectors can be not concentrated. There exist two simple ways to obtain the concentration of (X, Y) . The first one follows from any identity $(X, Y) = \phi(Z)$ with Z concentrated and ϕ Lipschitz. It is also possible to deduce the concentration of (X, Y) from the concentration of X and Y when they are independent.

Proposition B.33 (Stability through independent concatenation). *[Led05, Proposition 1.11] Given $(E, \|\cdot\|)$ a sequence of normed vector spaces and two sequences of independent random vectors $X, Y \in E$, such that $X \propto \mathcal{E}_q(\sigma)$ and $Y \propto \mathcal{E}_r(\rho)$ (where $q, r > 0$ are two positive constants and $\sigma, \rho \in \mathbb{R}_+^{\mathbb{N}}$ are two sequences of positive reals), then:*

$$(X, Y) \propto \mathcal{E}_q(\sigma) + \mathcal{E}_r(\rho) \quad \text{in } (E^2, \|\cdot\|_{\ell^\infty}),$$

where, for all $x, y \in E^2$, $\|(x, y)\|_{\ell^\infty} = \max(\|x\|, \|y\|)$.¹²

Following our formalism, this means that there exist two positive constants $C, c > 0$ such that $\forall p \in \mathbb{N}$ and for any 1-Lipschitz function $f : (E_p^2, \|\cdot\|_{\ell^\infty}) \rightarrow (\mathbb{R}, |\cdot|)$, $\forall t > 0$:

$$\mathbb{P}(|f(X_p, Y_p) - f(X'_p, Y'_p)| \geq t) \leq C e^{-(t/c\sigma_p)^q} + C e^{-(t/c\rho_p)^r}.$$

The sum being a 2-Lipschitz operation (for the norm $\|\cdot\|_{\ell^\infty}$), the concentration of $X + Y$ is easily handled with Proposition B.6 and directly follows from the concentration of (X, Y) . For products of vectors, more work is required.

B.7 Concentration of generalized products of random vectors

To treat the product of vectors, we provide a general result of concentration of what could be called “*multilinearly m -Lipschitz mappings*” on normed vector spaces. Instead of properly defining this class of mappings we present it directly in the hypotheses of the theorem. Briefly, these mappings are multivariate functions which are Lipschitz on each variable, with a Lipschitz parameter depending on the product of the norms (or semi-norms) of the other variables and/or constants. To express the observable diameter of such an observation, one needs a supplementary notation.

Given a vector of parameters $(\nu_l)_{l \in [m]} \in \mathbb{R}_+^m$, for any $k \in [m]$, recall the notation given before the proof of Proposition A.30:

$$\nu^{(k)} \equiv \max_{1 \leq l_1 < \dots < l_k \leq m} \nu_{l_1} \cdots \nu_{l_k} = \nu_{(m-k+1)} \cdots \nu_{(m)},$$

where $\{\nu_{(l)}\}_{l \in [m]} = \{\nu_l\}_{l \in [m]}$ and $\nu_{(1)} \leq \dots \leq \nu_{(m)}$.

Theorem B.34 (Concentration of generalized product). *Given a constant m ($m \leq O(1)$), let us consider:*

- m (sequences of) normed vector spaces $(E_1, \|\cdot\|_1), \dots, (E_m, \|\cdot\|_m)$.
- m (sequences of) norms (or semi-norms) $\|\cdot\|'_1, \dots, \|\cdot\|'_m$, respectively defined on E_1, \dots, E_m .
- m (sequences of) random vectors $Z_1 \in E_1, \dots, Z_m \in E_m$ satisfying

$$Z \equiv (Z_1, \dots, Z_m) \propto \mathcal{E}_q(\sigma)$$

for some (sequence of) positive numbers $\sigma \in \mathbb{R}_+$, and for both norms¹³ $\|z_1, \dots, z_m\|_{\ell^\infty} = \sup_{i=1}^m \|z_i\|_i$ and $\|(z_1, \dots, z_m)\|'_{\ell^\infty} = \sup_{i=1}^m \|z_i\|'_i$ defined on $E = E_1 \times \dots \times E_m$.

¹²One could also have considered a number of equivalent norms such as $\|(x, y)\|_{\ell^1} = \|x\| + \|y\|$ or $\|(x, y)\|_{\ell^2} = \sqrt{\|x\|^2 + \|y\|^2}$.

¹³One just needs to assume the concentration $\|Z_i\|'_i \in \mu_i \pm \mathcal{E}_q(\sigma)$; the global concentration of Z_i for the norm (or semi-norm) $\|\cdot\|'_i$ is not required.

- a (sequence of) normed vector spaces $(F, \|\cdot\|)$, a (sequence of) mappings $\phi : E_1, \dots, E_m \rightarrow F$, such that $\forall (z_1, \dots, z_m) \in E_1 \times \dots \times E_m$ and $z'_i \in E_i$:

$$\|\phi(z_1, \dots, z_m) - \phi(z_1, \dots, z_{i-1}, z'_i, \dots, z_m)\| \leq \frac{\prod_{j=1}^m \max(\|z_j\|'_j, \mu_j)}{\max(\|z_i\|'_i, \mu_i)} \|z_i - z'_i\|_i.$$

where $\mu_i > 0$ is a (sequence of) positive reals such that $\mu_i \geq \mathbb{E}[\|Z_i\|'_i]$. We further assume $\mu_i \geq O(\sigma)$.¹⁴

Then we have the concentration :¹⁵

$$\phi(Z) \propto \max_{l \in [m]} \mathcal{E}_{q/l} \left(\sigma^l \mu^{(m-l)} \right) \quad (\text{B.6})$$

One can deduce from the moment characterization given in Proposition A.30 that, in the setting of Theorem B.34, the standard deviation (resp. the r^{th} centered moment with $r \leq O(1)$) of any 1-Lipschitz observation of $\phi(Z)$ is of order $O(\sigma \mu^{(m-1)})$ (resp. $O((\sigma \mu^{(m-1)})^r)$) thus the observable diameter, in the sense given by (2) in the introduction, is given by the first exponential decay, $\mathcal{E}_q(\sigma \mu^{(m-1)})$, which represents the guiding term of (B.6).

We give later a more general setting with Theorem B.46 where the variations of ϕ are not controlled with norms (or semi-norms) $(\|\cdot\|'_{j \in [m] \setminus \{i\}})$ but with concentrated variables. The setting is somehow more complex, but sometimes more easy to apply; the proof is morally the same.

One can relax the hypothesis “ m constant” and reach good concentration rates when m tends to infinity: this is done in Section B.9.

Proof. We use here again the notation $\mu_{(i)}$ of the ordered μ_1, \dots, μ_m . Let us consider Z'_1, \dots, Z'_m , m independent copies of Z_1, \dots, Z_m and a 1-Lipschitz (for the norm $\|\cdot\|$) mapping $f : F \rightarrow \mathbb{R}$. For simplicity, we will note $\phi(Z) \equiv \phi(Z_1, \dots, Z_m)$ (and $\phi(Z') = \phi(Z'_1, \dots, Z'_m)$). Given $t > 0$, we wish to bound:¹⁶

$$\mathbb{P}(|f(\phi(Z)) - f(\phi(Z'))| \geq t). \quad (\text{B.7})$$

The map $Z \mapsto f(\phi(Z))$ is not Lipschitz, unless Z is bounded. We thus decompose the probability argument into two events, one with bounded $\|Z\|'_{\ell^\infty}$ and $\|Z'\|'_{\ell^\infty}$ and the complementary with small probability. For all $i \in [m]$, since $\|Z\|'_i \in \mathbb{E}[\|Z\|'_i] \pm E_q(\sigma)$ and $\mathbb{E}[\|Z\|'_i] \leq \mu_i$ (and the same hold for Z'_i), we can then employ for all $t \geq \mu_i$ the identity:

$$\mathbb{P}(\|Z\|'_i \geq 2t) = \mathbb{P}(\|Z'\|'_i \geq 2t) \leq \mathbb{P}(\|\|Z\|'_i - \mathbb{E}[\|Z\|'_i]\| \geq 2t - \mu_i) \leq C e^{-(t/c\sigma)^q},$$

for two constants $C, c > 0$. The bounds on each of the $\|Z\|'_i$ and $\|Z'_i\|'_i$, $i \in [m]$, depend on the value of t . Let us note for all $i \in [m]$:

$$t_i \equiv \mu^{(m-i)} \mu_i^i = \mu^{(m-i+1)} \mu_i^{i-1} = \frac{(\mu^{(m-i+1)})^i}{(\mu^{(m-i)})^{i-1}} \quad \text{and} \quad t_{m+1} = \infty,$$

and remark that $t_1 \leq \dots \leq t_m$, (see Lemma B.36 for some details). The parameters t_1, \dots, t_m define the different regime of the concentration and will play the same role as in the proof of Proposition A.30. If $t \in [t_i, t_{i+1}]$, we decompose the probability (B.7) playing on the realization the events $\{Z \in B_i\} \cap \{Z' \in B_i\}$, where we defined:

$$B_i \equiv \left\{ z \in E_1 \times \dots \times E_m \mid \|z_1\|'_1 \leq 2 \left(\frac{t}{\mu^{(m-i)}} \right)^{\frac{1}{i}}; \dots; \|z_i\|'_i \leq 2 \left(\frac{t}{\mu^{(m-i)}} \right)^{\frac{1}{i}}; \|z_{i+1}\|'_{i+1} \leq 2\mu_{i+1}; \dots; \|z_m\|'_m \leq 2\mu_m \right\}.$$

Noting that for $j \in [m]$:

¹⁴This is a very light assumption: it is hard to find any practical example where $\mu_i \ll \sigma$, because, as pictured by Corollary B.35, μ_i will generally writes in our applications $C' \sigma \eta_{\|\cdot\|'_i}^{1/q}$, where η is a norm degree bigger than 1 by definition.

¹⁵Here since $m \leq O(1)$, taking the maximum over $l \in [m]$ is equivalent to taking the sum, up to a small change of the constants; we will thus indifferently write $\phi(Z) \propto \max_{l \in [m]} \mathcal{E}_{lq/m} \left((\sigma \mu^{(l-1)})^{\frac{m}{l}} \right)$ or $\phi(Z) \propto \sum_{l=1}^m \mathcal{E}_{lq/m} \left((\sigma \mu^{(l-1)})^{\frac{m}{l}} \right)$.

¹⁶We choose here to employ the characterization of the concentration with the independent copy, because, at some point of the proof, we restrict ourselves to an event \mathcal{A}_K and then $\mathbb{P}(|f(\phi(Z)) - f(\phi(Z'))| \geq t \mid \mathcal{A}_K)$ can be bounded directly from the concentration $(\phi(Z) \mid \mathcal{A}_K) \propto \mathcal{E}_q(K^{(m-1)}\sigma)$ resulting from Lemma B.4 and Remark B.5. To bound $\mathbb{P}(|f(\phi(Z)) - \mathbb{E}[f(\phi(Z))]| \geq t \mid \mathcal{A}_K)$, one would have needed to show first that $|\mathbb{E}[f(\phi(Z))] - \mathbb{E}[f(\phi(Z')) \mid \mathcal{A}_K]| \leq O(K^{(m-1)}\sigma)$ to then employ Lemma B.18.

- if $j \leq i$: $\left(\frac{t}{\mu^{(m-i)}}\right)^{\frac{1}{i}} \geq \left(\frac{t_i}{\mu^{(m-i)}}\right)^{\frac{1}{i}} = \mu_i \geq \mu_j$,
- if $j \geq i+1$: $\mu_j \geq \mu_{i+1} = \left(\frac{t_{i+1}}{\mu^{(m-i)}}\right)^{\frac{1}{i}} \geq \left(\frac{t}{\mu^{(m-i)}}\right)^{\frac{1}{i}}$,

we can first bound, on the first hand:

$$\begin{aligned}
\mathbb{P}(|f(\phi(Z)) - f(\phi(Z'))| \geq t, Z \in B_i^c \text{ or } Z' \in B_i^c) &\leq 2\mathbb{P}(Z \in B_i^c) \\
&\leq 2 \sum_{j=1}^{i-1} \mathbb{P}(\|Z_j\| \geq 2\mu_j) + 2 \sum_{k=i}^m \mathbb{P}\left(\|Z_k\| \geq 2 \left(\frac{t}{\mu^{(m-i)}}\right)^{\frac{1}{i}}\right) \\
&\leq 2mC \exp\left(-\frac{t/(c\sigma)^i}{\mu^{(m-i)}}\right)^{\frac{q}{i}}.
\end{aligned} \tag{B.8}$$

On the other hand, one can show that $f \circ \phi|_{B_i}$ is λ_i -Lipschitz with:

$$\lambda_i \equiv m2^{m-1} \left(\frac{t}{\mu^{(m-i)}}\right)^{\frac{i-1}{i}} \mu^{(m-i)}.$$

Therefore, following Remark B.5, since $\mathbb{P}(Z \in B_i) \geq O(1)$:

$$(f(\phi(Z)) \mid Z \in B_i) \propto \mathcal{E}_q(\lambda_i),$$

which allows us to bound (for all $t \in [t_i, t_{i-1}]$):

$$\mathbb{P}(|f(\phi(Z)) - f(\phi(Z'))| \geq t, Z, Z' \in B_i) \leq C \exp\left(-\frac{t}{(2^m c\sigma)^i \mu^{(m-i)}}\right)^{\frac{q}{i}}. \tag{B.9}$$

If $t \in (0, t_1] = (0, \mu^{(m)})$, we introduce:

$$B_0 \equiv \{z \in E_1 \times \cdots \times E_m \mid \|z_1\|'_1 \leq 2\mu_1; \dots; \|z_m\|'_m \leq 2\mu_m\},$$

one can still bound $\forall i \in [m]$, $\mu_i \geq \mu_1 = \frac{t_1}{\mu^{(m-1)}} \geq \frac{t}{\mu^{(m-1)}}$, and therefore $\mathbb{P}(Z \notin B_0) \leq mC e^{-\left(\frac{t}{c\sigma\mu^{(m-1)}}\right)^q}$, besides, $\phi|_{B_0}$ is $m2^{m-1}\mu^{(m-1)}$ -Lipschitz, therefore we retrieve:

$$\mathbb{P}(|f(\phi(Z)) - f(\phi(Z'))| \geq t, Z, Z' \in B_0) \leq C \exp\left(-t/2^m c\sigma\mu^{(m-1)}\right)^q.$$

Combining these bounds for $t \in (0, t_1]$ with (B.8) and (B.9), we obtain the result of the theorem (since $mC, 2^m c \leq O(1)$). \square

Corollary B.35. *In the setting of Theorem B.34, when $\forall i \in [m]$, $\|E[Z_i]\|'_i \leq O(\sigma\eta_{\|\cdot\|_i}^{1/q})$, we have the simpler concentration:*

$$\phi(Z) \propto \max_{l \in [m]} \mathcal{E}_{q/l} \left(\sigma^m \eta^{(m-l)} \right),$$

where we denoted $\eta = \left(\eta_{\|\cdot\|_1}^{1/q}, \dots, \eta_{\|\cdot\|_m}^{1/q}\right)$.

Proof. Proposition B.22 allows us to choose $\mu_i = C'\sigma\eta_{\|\cdot\|_i}^{1/q}$ for some constant $C' > 0$; we thus retrieve the result thanks to Theorem B.34. \square

It is possible to characterize the concentration provided by Theorem B.34 with the bound on the moments provided in Proposition A.30. One just has to show that the parameters of the concentration from a family of canonical multi-regime parameter as defined in Definition 7.

Lemma B.36. *Given two parameters $\sigma, q > 0$, an integer $m \geq 1$ and m parameters μ_1, \dots, μ_m such that for all $i \in [m]$, $\mu_i \geq \sigma$, if for any $l \in [m]$, we note $\sigma_l \equiv \sigma^l \mu^{(m-l)}$, then the sets of couples $(q, \sigma_1), (\frac{q}{2}, \sigma_2), \dots, (\frac{q}{m}, \sigma_m)$ forms a sensible family of canonical multi regime parameters.*

Proof. The inequalities $q > \frac{q}{2} > \dots > \frac{q}{m}$ and $\sigma_1 > \dots > \sigma_m$ are obvious. For simplicity, let us reorder the indexing of the argument of ϕ such that $(\mu_1, \dots, \mu_m) = (\mu_{(1)}, \dots, \mu_{(m)})$ (i.e., $\mu_1 \leq \dots \leq \mu_m$). One can then bound for any $k, l \in [m]$:

$$t_{k,l} = \left(\frac{(\mu^{(m-k)})^l}{(\mu^{(m-l)})^k} \right)^{\frac{1}{l-k}} = \left((\mu^{(m-l)})^{l-k} (\mu_{(k+1)} \dots \mu_{(l)})^l \right)^{\frac{1}{l-k}} \leq \mu^{(m-l)} \mu_{(l)}^l = \frac{(\mu^{(m-l+1)})^l}{(\mu^{(m-l)})^{l-1}} = t_{l-1,l}$$

And we show the same way:

$$t_{k,l} = \left((\mu^{(m-k)})^{l-k} (\mu_{(k+1)} \dots \mu_{(l)})^k \right)^{\frac{1}{l-k}} \geq \mu^{(m-k)} \mu_{(k+1)}^k = \frac{(\mu^{(m-k)})^{k+1}}{(\mu^{(m-k-1)})^k} = t_{k,k+1}.$$

□

Remark B.37 (Regime decomposition). *Lemma B.36 states that $(q, \sigma_1), \dots, (\frac{q}{m}, \sigma_m)$ is a family of sensible multi-regime parameters, however it could be non free (i.e. it could happen that $t_{l-1,l} = t_{l,l+1}$), in which case there would be less than m regimes.*

In particular, when $\mu_{(1)} = \dots = \mu_{(m)} \equiv \mu_0$,¹⁷ $\forall i \in [m]$, $t_i = \mu_0^m$. In this case, there are only two regimes and we can more simply write :

$$\Phi(Z) \propto \mathcal{E}_q(\sigma \mu_0^{m-1}) + \mathcal{E}_{\frac{q}{m}}(\sigma^m).$$

Let us give examples of “multilinear Lipschitz mappings” that would satisfy the hypotheses of Theorem B.34.

Example B.38 (Entry-wise product). *Letting \odot be the entry-wise product in \mathbb{R}^p defined as $[x \odot y]_i = x_i y_i$ (it is the Hadamard product for matrices), $\phi : (\mathbb{R}^p)^m \ni (x_1, \dots, x_m) \mapsto x_1 \odot \dots \odot x_m \in \mathbb{R}^p$ is multilinearly Lipschitz since we have for all $i \in [m]$:*

$$\|x_1 \odot \dots \odot x_{i-1} \odot (x_i - x'_i) \odot x_{i+1} \odot \dots \odot x_m\| \leq \left(\prod_{\substack{j=1 \\ j \neq i}}^m \|x_j\|_\infty \right) \|x_i - x'_i\|,$$

for all vectors $x_1, \dots, x_m, x'_1, \dots, x'_m \in \mathbb{R}^p$. As a practical case, if $(Z_1, \dots, Z_m) \propto \mathcal{E}_2$ and $\forall i \in [m]$, $\|\mathbb{E}[Z_i]\|_\infty \leq O(1)$, then Corollary B.35 and Remark B.37 imply:

$$Z_1 \odot \dots \odot Z_m \propto \mathcal{E}_2 \left(\log(p)^{\frac{m-1}{2}} \right) + \mathcal{E}_{\frac{2}{m}}.$$

In this case, the observable diameter of $Z_1 \odot \dots \odot Z_m$ is provided by $\mathcal{E}_2 \left(\log(p)^{\frac{m-1}{2}} \right)$: as such, under this very common setting, the entry-wise product has almost no impact on the rate of concentration.

Example B.39 (Matrix product). *The mapping $\phi : (\mathcal{M}_p)^q \ni (M_1, \dots, M_q) \mapsto M_1 \dots M_m \in \mathcal{M}_p$ is multilinearly Lipschitz since for all $i \in [m]$:¹⁸*

$$\|M_1 \dots M_{i-1} (M_i - M'_i) M_{i+1} \dots M_m\|_F \leq \left(\prod_{\substack{j=1 \\ j \neq i}}^m \|M_j\| \right) \|M_i - M'_i\|_F,$$

for all matrices $M_1, \dots, M_m, M'_1, \dots, M'_m \in \mathcal{M}_p$. Given m random matrices $X_1, \dots, X_m \in \mathcal{M}_p$ such that $(X_1, \dots, X_m) \propto \mathcal{E}_2$ and $\forall i \in [m]$, $\|\mathbb{E}[X_i]\| \leq O(\sqrt{p})$, Corollary B.35 implies:

$$X_1 \dots X_m \propto \mathcal{E}_2 \left(p^{\frac{m-1}{2}} \right) + \mathcal{E}_{\frac{2}{m}}.$$

In particular, for a “data” matrix¹⁹ $X = (x_1, \dots, x_n) \in \mathcal{M}_{p,n}$ satisfying $X \propto \mathcal{E}_2$ and $\mathbb{E}[\|X\|] \leq O(\sqrt{p+n})$,

¹⁷To be precise, it is sufficient to assume $\mu_{(2)} = \dots = \mu_{(m)}$, since $\mu_{(1)}$ never appears in the definition of the t_i for $i \in [m]$.

¹⁸One could have equivalently considered, for even $m \in \mathbb{N}$, the mapping $\phi : \mathcal{M}_{p,n}^m \rightarrow \mathcal{M}_{p,n}$ satisfying $\forall M_1, \dots, M_m \in \mathcal{M}_{p,n}$, $\phi(M_1, \dots, M_m) = M_1 M_2^T \dots M_{m-1} M_m^T$.

¹⁹That is, a matrix whose columns contain vectors of “data”, as per data science terminology.

the sample covariance matrix satisfies the concentration:

$$\frac{1}{n}XX^T \propto \mathcal{E}_2\left(\frac{\sqrt{p+n}}{n}\right) + \mathcal{E}_1\left(\frac{1}{n}\right),$$

which provides an observable diameter of order $O(1/\sqrt{n})$ when $p \leq O(n)$.

Example B.40 (Composition). We study here the concentration of the composition of random Lipschitz functions. Beyond the linear case, consider the composition of functions defined on $\mathcal{Lip}(\mathbb{R})$, the set of Lipschitz and bounded functions of \mathbb{R} . Given $f \in \mathcal{Lip}(\mathbb{R})$, we denote:

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| \qquad \|f\|_{\mathcal{L}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

($\|f\|_{\mathcal{L}}$ is not a norm but a semi-norm). The mapping $\phi : (\mathcal{Lip}(\mathbb{R}))^m \ni (f_1, \dots, f_m) \mapsto f_1 \circ \dots \circ f_m \in \mathcal{Lip}(\mathbb{R})$ is multilinearly Lipschitz and $\forall f_1 \dots f_m, f'_i \in \mathcal{Lip}(\mathbb{R})$:

$$\begin{aligned} \|(f_1 \circ \dots \circ f_m) - (f_1 \circ \dots \circ f_{i-1} \circ f'_i \circ f_{i+1} \circ \dots \circ f_m)\|_\infty \\ \leq \|f_1\|_{\mathcal{L}} \cdots \|f_{i-1}\|_{\mathcal{L}} \|f_i - f'_i\|_\infty. \end{aligned}$$

Therefore, assuming f_1, \dots, f_m all $O(1)$ -Lipschitz and satisfying the concentration $(f_1, \dots, f_m) \propto \mathcal{E}_q(\sigma)$, thanks to Theorem B.34, the following concentration holds :

$$f_1 \circ \dots \circ f_m \propto \mathcal{E}_q(\sigma) + \mathcal{E}_{q/m}(\sigma^m).$$

When $f_1 = \dots = f_m \equiv f$ and f is λ -Lipschitz with $1 - \lambda \geq O(1)$, we can follow the dependence of the concentration of $f \circ \dots \circ f$ on m thanks to Theorem B.52 that induces for any sequence of integers m the concentration inequality:

$$f^m \propto \mathcal{E}_q(\sigma(1 - \varepsilon)^m) + \mathcal{E}_{q/m}((\kappa\sigma)^m)$$

for some constants $\kappa, \varepsilon > 0$.

B.8 Generalized Hanson-Wright theorems and concentration of XDY^T

To give some more elaborate consequences of Theorem B.34, let us first provide a matricial version of the popular Hanson-Wright concentration inequality, [HW71]. It was already proven in [Ada15] for random vectors, which is a quite similar setting. The same Proposition can be set in convex concentration setting as it is done in Proposition D.6.

Proposition B.41 (Hanson-Wright, [Ada15]). Given two random matrices $X, Y \in \mathcal{M}_{p,n}$, assume that $(X, Y) \propto \mathcal{E}_2$ (say, in²⁰ $(\mathcal{M}_{p,n}^2, \|\cdot\|_{\ell^\infty})$) and $\|\mathbb{E}[X]\|_F, \|\mathbb{E}[Y]\|_F \leq O(1)$ (as $n, p \rightarrow \infty$). Then, for any deterministic matrix $A \in \mathcal{M}_p$, we have the linear concentration (in $(\mathcal{M}_{p,n}, \|\cdot\|_F)$):

$$Y^T AX \in \mathcal{E}_2(\|A\|_F) + \mathcal{E}_1(\|A\|).$$

This proposition, which provides a result in terms of linear concentration, points out an instability of the class of Lipschitz concentrated vectors which (here through products) degenerates into a mere linear concentration. This phenomenon fully justifies the introduction of the notion of linear concentration: it will occur again in Proposition B.48 and Lemma G.8. We present the proof directly here as it is a short and convincing application of Theorem B.34.

²⁰The norm $\|\cdot\|_{\ell^\infty}$ has been defined in Proposition B.33 providing the concentration of the concatenation of random vectors, here for any $A, B \in \mathcal{M}_{p,n}$, $\|(A, B)\|_{\ell^\infty} \equiv \max(\|A\|_F, \|B\|_F)$. A lot of equivalent norm satisfy the same concentration.

Proof. Let us first assume that $\|A\| \leq 1$ and consider a deterministic matrix $B \in \mathcal{M}_n$ such that $\|B\|_F \leq 1$. We then introduce the semi-norm $\|\cdot\|_{A,B}$ defined on $\mathcal{M}_{p,n}$ and satisfying for all $M \in \mathcal{M}_{p,n}$, $\|M\|_{A,B} \equiv \|AMB\|_F$. Note that for any $M, P \in \mathcal{M}_{p,n}$:

$$\mathrm{Tr}(BM^T AP) \leq \begin{cases} \|M\|_{A,B} \|P\|_F \\ \|M\|_F \|P\|_{A,B}. \end{cases}$$

Thanks to Lemma B.26, $\mathbb{E}[\|X\|_{A,B}], \mathbb{E}[\|Y\|_{A,B}] \leq O(\|A\|_F)$. Besides, since $(X, Y) \propto \mathcal{E}_q$ in $((\mathbb{R}^p)^2, \|\cdot\|_{\ell^\infty})$, and $\|A\|, \|B\| \leq 1$, we also know that $(X, Y) \propto \mathcal{E}_q$ in $((\mathbb{R}^p)^2, \|\cdot\|_{A,B,\ell^\infty})$ where $\|(M, P)\|_{A,B,\ell^\infty} = \max(\|M\|_{A,B}, \|P\|_{A,B})$. Therefore, the hypotheses of Theorem B.34 are satisfied for the norms $\|\cdot\|_{A,B}$ and $\|\cdot\|_F$ respectively replacing the norm $\|\cdot\|'$ and $\|\cdot\|$ we then deduce :

$$Y^T AX \in \mathcal{E}_q(\|A\|_F) + \mathcal{E}_{\frac{q}{2}},$$

(where we choose a tuple $\mu = (\mu_1, \mu_2) \in \mathbb{R}_+^2$ satisfying $\mu_1, \mu_2 \leq O(\|A\|_F)$)

If $\|A\| > 1$, one can still show that $\frac{1}{\|A\|} Y^T AX \in \mathcal{E}_q(\|A\|_F/\|A\|) + \mathcal{E}_{q/2}$ and retrieve the result thanks to Proposition B.6. \square

Remark B.42. For the reader information, we mention that in [Ada15], the concentration is even expressed on the random variable $\sup_{A \in \mathcal{A}} X^T AX$ where \mathcal{A} is a bounded set of matrices (and $X \in \mathbb{R}^p$).

Let us end this section with a useful consequence of Proposition B.41.

Corollary B.43. Given a deterministic matrix $A \in \mathcal{M}_p$ satisfying $\|A\|_F \leq 1$ and two random matrices $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in \mathcal{M}_{p,n}$ satisfying $(X, Y) \propto \mathcal{E}_2$ and $\sup_{i \in [n]} \|\mathbb{E}[x_i]\|, \|\mathbb{E}[y_i]\| \leq O(1)$ such that we have the concentration:

$$\|Y^T AX\|_d \in \mathcal{E}_2\left(\sqrt{\log(np)}\right) + \mathcal{E}_1.$$

If²¹, in addition, $\|A\|_* \leq O(1)$ or $\sup_{i \in [n]} \|\mathbb{E}[y_i x_i^T]\|_F \leq O(1)$ ²², then $\mathbb{E}[\|X^T AY\|_d] \leq O(\sqrt{n})$.

Remark B.44. Recall from Proposition B.20 that if $(x_i, y_i) \in \mathcal{E}_2$ and $\|\mathbb{E}[(x_i, y_i)]\|_F \leq O(1)$ (as under the hypotheses of Corollary B.43), $\|\mathbb{E}[y_i x_i^T]\| \leq O(1)$ and therefore, $\|\mathbb{E}[y_i x_i^T]\|_F \leq \sqrt{p} \|\mathbb{E}[y_i x_i^T]\| \leq O(\sqrt{p})$. In particular, when x_i is independent with y_i (and $\|\mathbb{E}[x_i]\|, \|\mathbb{E}[y_i]\| \leq O(1)$), we can bound $\|\mathbb{E}[y_i x_i^T]\|_F \leq \|\mathbb{E}[y_i]\| \|\mathbb{E}[x_i]\| \leq O(1)$.

Proof of Corollary B.43. Let us decompose $A = U^T \Lambda V$ with $\Lambda = \mathrm{Diag}(\lambda) \in \mathcal{D}_n$ and $U, V \in \mathcal{O}_p$, noting $\check{X} \equiv VX$ and $\check{Y} \equiv UY$ we have the identity:

$$\|Y^T AX\|_d \leq \sup_{\substack{D \in \mathcal{D}_n \\ \|D\|_F \leq 1}} \mathrm{Tr}(D \check{Y}^T \Lambda \check{X}) \leq \sup_{\|d\| \leq 1} d^T (\check{X} \odot \check{Y}) \lambda \leq \|\check{X} \odot \check{Y}\| \|\lambda\| \leq \|\check{X}\|_F \|\check{Y}\|_\infty,$$

and the same way, $\|Y^T AX\|_d \leq \|\check{X}\|_\infty \|\check{Y}\|_F$. Now we can bound thanks to Proposition B.22:

$$\mathbb{E}[\|\check{X}\|_\infty] \leq \|\mathbb{E}[\check{X}]\|_\infty + O(\sqrt{\log(pn)}) \leq \sup\{\|\mathbb{E}[x_i]\|, \|\mathbb{E}[y_i]\|, i \in [n]\} + O(\sqrt{\log(pn)}) \leq O(\sqrt{\log(pn)}),$$

and the same holds for $\mathbb{E}[\|\check{Y}\|_\infty]$. Therefore, applying Theorem B.34 to the mapping $\phi : (\check{X}, \check{Y}) \in \mathcal{M}_{p,n}^2 \mapsto \|\check{Y}^T \Lambda \check{X}\|_d$ with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_F$ respectively replacing the norms $\|\cdot\|'$ and $\|\cdot\|$, we obtain the looked for concentration.

²¹Actually, to bound the expectation we just need the concentration of each of the couples (x_i, y_i) but not of the matrix couple (X, Y) . Recall (see Section B.6) that the concentration of each the x_1, \dots, x_n does not imply the concentration of the whole matrix X , even if the columns are independent. To tackle this issue, some authors [PP09] require a logconcave distribution for all the columns because the product of logconcave distribution is also logconcave. However our assumptions are more general because they allow to take for x_i any $O(1)$ -Lipschitz transformation of a Gaussian vector which represents a far larger class of random vectors.

²²To be precise, one just needs $\sup_{i \in [n]} \|\mathbb{E}[x_i^T A y_i]\| \leq O(1)$.

To bound the expectation, we start with the identity $\|X^T AY\|_d = \sqrt{\sum_{i=1}^n (x_i^T Ay_i)^2}$, and we note that the hypotheses of Proposition B.41 are satisfied and therefore $x_i^T Ay_i \in \mathcal{E}_1$. Now, if $\|A\|_* \leq 1$, we can bound:

$$|\mathbb{E}[x_i^T Ay_i]| = \|\mathbb{E}[y_i x_i^T]\| \|A\|_* \leq O(1),$$

thanks to Proposition B.20 ($(x_i, y_i) \in \mathcal{E}_2$ and $\|\mathbb{E}[x_i] \mathbb{E}[y_i]^T\| \leq O(1)$). The same bound is true when $\|\mathbb{E}[y_i x_i^T]\|_F \leq O(1)$ because $|\mathbb{E}[x_i^T Ay_i]| \leq \|\mathbb{E}[y_i x_i^T]\|_F \|A\|_F$. As a consequence, $x_i^T Ay_i \in O(1) \pm \mathcal{E}_1$ (with the same concentration constants for all $i \in [n]$), and we can bound:

$$\mathbb{E} [\|X^T AY\|_d] \leq \sqrt{\sum_{i=1}^n \mathbb{E} [(x_i^T Ay_i)^2]} \leq O(\sqrt{n}).$$

□

Let us now give an example of application of Theorem B.34 when $m \geq 3$.

Considering three random matrices $X, Y \in \mathcal{M}_{p,n}$ and $D \in \mathcal{D}_n$ such that $(X, Y, D) \in \mathcal{E}_2$ and $\|\mathbb{E}[D]\|, \|\mathbb{E}[X]\|_F, \|\mathbb{E}[Y]\|_F \leq O(1)$ we wish to study the concentration of XDY^T . Theorem B.34 just allows us to obtain the concentration $XDY^T \in \mathcal{E}_2(n+p) + \mathcal{E}_1(\sqrt{n}) + \mathcal{E}_{2/3}$ since we cannot get a better bound than $\|XDY^T\|_F \leq \|X\| \|D\|_F \|Y\|$. However, considering some particular observations on XDY^T , it appears that the observable diameter can be smaller than n . Next Propositions reveal indeed that for any deterministic $u \in \mathbb{R}^p$ and $A \in \mathcal{M}_p$:

1. $XDY^T u$ is Lipschitz concentrated with an observable diameter of order $O(\|u\| \sqrt{(n+p) \log(n)})$
2. $\text{Tr}(AXDY^T)$ is concentrated with a standard deviation of order $O(\|A\|_F \sqrt{(n+p) \log(np)})$ if $\sup_{i \in [n]} \|\mathbb{E}[x_i y_i^T]\|_F \leq O(1)$ and $O(\|A\|_* \sqrt{(n+p) \log(np)})$ otherwise.

Proposition B.45. *Given three random matrices $X, Y = (y_1, \dots, y_n) \in \mathcal{M}_{p,n}$ and $D \in \mathcal{D}_n$ diagonal such that $(X, Y, D) \in \mathcal{E}_2$, $\|\mathbb{E}[X]\| \leq O(\sqrt{p+n})$ and $\|\mathbb{E}[D]\|, \sup_{i \in [n]} \|\mathbb{E}[y_i]\| \leq O(\sqrt{\log(n)})$, for any deterministic vector $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$:*

$$XDY^T u \in \mathcal{E}_2 \left(\sqrt{(p+n) \log n} \right) + \mathcal{E}_1 \left(\sqrt{p+n} \right) + \mathcal{E}_{2/3} \text{ in } (\mathbb{R}^p, \|\cdot\|).$$

Proof. The Lipschitz concentration of $XDY^T u$ is obtained thanks to the inequalities:

$$\|XDY^T v\| \leq \begin{cases} \|X\| \|D\| \|Y^T u\| \\ \|X\| \|D\|_F \|Y^T u\|_\infty. \end{cases}$$

Thanks to the bounds already presented in Example B.24 (the spectral norm $\|\cdot\|$ on \mathcal{D}_n is like the infinity norm $\|\cdot\|_\infty$ on \mathbb{R}^n), we then have:

- $\eta_X \equiv \mathbb{E}[\|X\|] \leq \|\mathbb{E}[X]\| + O(\sqrt{p+n}) \leq O(\sqrt{p+n})$,
- $\eta_D \equiv \mathbb{E}[\|D\|] \leq \|\mathbb{E}[D]\| + O(\sqrt{\log(n)}) \leq O(\sqrt{\log(n)})$,
- $\eta_{Y^T u} \equiv \mathbb{E}[\|Y^T u\|_\infty] \leq \sup_{i \in [n]} \|\mathbb{E}[y_i^T u]\| + O(\sqrt{\log(n)}) \leq O(\sqrt{\log(n)})$.

To obtain the result, we then employ Corollary B.35 for the tuple:

$$\eta = \left(O(\sqrt{n}), O(\sqrt{\log(np)}), O(\sqrt{\log(np)}) \right)$$

satisfying:

- $\eta^{(3-1)} = O(\max(\sqrt{(p+n) \log(n)}, \log(n))) \leq O(\sqrt{(p+n) \log(n)})$
- $\eta^{(3-2)} = O(\max(\sqrt{p+n}, \sqrt{\log(n)}) \leq O(\sqrt{p+n})$.

□

To express the concentration of $\text{Tr}(AXDY^T)$, it is convenient to introduce the following offshot of Theorem B.34 based on somehow elaborate but actually simpler assumptions.

Theorem B.46. *In the setting of Theorem B.34, let us assume that there exist a mapping for all $i \in [m]$ $\Psi_i : E_{+i} \rightarrow \mathbb{R}$, where $E_{+i} = E_1 \times \cdots \times E_m \times E_i$, such that $\forall i \in [n], \forall z = (z_1, \dots, z_m) \in E = E_1 \times \cdots \times E_m$ and $z'_i \in E_i$, we have the bound:*

$$\|\phi(z_1, \dots, z_m) - \phi(z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_m)\| \leq \Psi_i(z_{+i}) \|z_i - z'_i\|_i,$$

where $z_{+i} = (z_1, \dots, z_m, z'_i) \in E_{+i}$ and for any independent copy $Z' = (Z'_1, \dots, Z'_m)$ of Z , we have the concentration²³:

$$\Psi_i(Z_{+i}) \in O(\mu^{(m-1)}) \pm \max_{l \in [m-1]} \mathcal{E}_{q/l} \left(\sigma^l \mu^{(m-l-1)} \right)$$

where $Z_{+i} = (Z_1, \dots, Z_i, Z'_{i+1}, \dots, Z'_m, Z'_i) \in E_{+i}$, for a parameter vector $\mu \in \mathbb{R}_+^m$ such that $1 \leq \mu_{(1)} (\leq \mu_i, \forall i \in [m])$. Then:

$$\phi(Z) \propto \max_{l \in [m]} \mathcal{E}_{q/l} \left(\sigma^l \mu^{(m-l)} \right).$$

In the remainder, this result is sometimes applied in cases where:

$$\Psi_l(Z) \in O(\mu^{(m-1)}) \pm \max_{l \in [m-1]} \mathcal{E}_{q/l} \left(\frac{\sigma^l \mu^{(m-l)}}{\mu_{(m)}} \right),$$

as for instance in Proposition B.48 below. Although this corresponds to a stronger setting, the most important element, which provides the observable diameter of $\phi(Z)$, is the order of the expectation of $\Psi_l(Z)$ which still equals $O(\mu^{(m-1)})$.

Proof. We multiply, if necessary, $\mu_{(m)}$ by a constant in order to have for all $i \in [m]$: $\mathbb{E}[\Psi_i(Z_{+i})] \leq \mu^{(m-1)}$. With the same notation as before for $t \in [0, t_1]$:

$$\mathcal{A}_0 \equiv \left\{ \forall i \in [m] : \Psi_i(Z_{+i}) \leq 2\mu^{(m-1)} \right\},$$

then, since $\mu^{(m-1)} \geq t$, there exist two constants $C, c > 0$ such that we can bound:

$$\begin{aligned} \mathbb{P}(\mathcal{A}_0^c) &\leq m \sup_{i \in [m]} \mathbb{P} \left(|\Psi_i(Z_{+i}) - \mathbb{E}[\Psi_i(Z_{+i})]| \geq \mu^{(m-1)} \right) \\ &\leq m \sup_{\substack{i \in [m] \\ l \in [m-1]}} C \exp \left(- \left(\frac{\mu^{(m-1)}}{c \sigma^l \mu^{(m-l-1)}} \right)^{q/l} \right) \leq m \sup_{\substack{i \in [m] \\ l \in [m-1]}} C e^{-(t/c \sigma^l \mu^{(m-l-1)})^{q/l}}. \end{aligned}$$

Besides, since ϕ is $m\mu^{(m-1)}$ -Lipschitz on \mathcal{A}_0 , we can bound:

$$\mathbb{P}(|\phi(Z) - \phi(Z')| \geq t \mid \mathcal{A}_0) \leq C e^{-(t/mc\mu^{(m-1)})^q}.$$

When $t \in [t_i, t_{i+1}] = [\mu^{(m-i)} \mu_{(i)}^i, \mu^{(m-i)} \mu_{(i+1)}^i]$ for $i \in [m-1]$ or $t \in [t_m, \infty) = [\mu_{(m)}^m, \infty)$ for $i = m$, we rather work with the event:

$$\mathcal{A}_i \equiv \left\{ \forall i \in [m] : \Psi_i(Z_{+i}) \leq 2t \left(\frac{\mu^{(m-i)}}{t} \right)^{\frac{1}{i}} \right\}.$$

²³As explained in Definition 14, this expression means that $\Psi_l(Z, Z^{(i)}) \in \max_{l \in [m-1]} \mathcal{E}_{q/l} \left(\sigma^l \mu^{(m-l-1)} \right)$ and $\mathbb{E}[\Psi_l(Z, Z')] \leq O(\mu^{(m-1)})$.

On the first hand, since $t \left(\frac{\mu^{(m-i)}}{t} \right)^{\frac{1}{i}} \geq t_i^{\frac{i-1}{i}} (\mu^{(m-i)})^{\frac{1}{i}} \geq \mu^{(m-i)} \mu_{(i)}^{i-1} \geq \mu^{(m-1)}$, we can bound:

$$\begin{aligned} \mathbb{P}(A_i^c) &\leq m \sup_{j \in [m]} \mathbb{P} \left(|\Psi_j(Z_{+j}) - \mathbb{E}[\Psi_j(Z_{+j})]| \geq 2t \left(\frac{\mu^{(m-i)}}{t} \right)^{\frac{1}{i}} - \mu^{(m-1)} \right) \\ &\leq m \sup_{l \in [m-1]} C \exp \left(- \left(\frac{t}{c\sigma \mu^{(m-l-1)}} \left(\frac{\mu^{(m-i)}}{t} \right)^{\frac{1}{i}} \right)^q \right). \end{aligned}$$

Now, given $l \in [m-1]$ (and when possibly $i = m$ and $t \in [\mu_{(m)}^m, \infty)$):

- if $m-1 \geq l \geq i$, $\frac{t}{\mu^{(m-l-1)}} \left(\frac{\mu^{(m-i)}}{t} \right)^{\frac{1}{i}} \geq \frac{t}{\mu^{(m-l-1)} \mu_{(i+1)}} \geq \frac{t}{\mu^{(m-l)}}$,
- if $l \leq i-1$, $\frac{t}{\mu^{(m-l-1)}} \left(\frac{\mu^{(m-i)}}{t} \right)^{\frac{1}{i}} \geq \left(\frac{t}{\mu^{(m-l-1)}} \right)^{\frac{i-1}{i}} \geq \frac{t}{\mu^{(m-l)}}$,

since $\frac{t}{\mu^{(m-l-1)}} \geq \frac{t}{\mu^{(m-i)}} \geq \mu_{(i)}^i \geq 1$, by hypothesis. That allows us to bound:

$$\mathbb{P}(A_i^c) \leq m \sup_{l \in [m-1]} C e^{-(t/c\sigma^l \mu^{(m-1)})^{q/l}}. \quad (\text{B.10})$$

Besides, since ϕ is $mt \left(\frac{\mu^{(m-i)}}{t} \right)^{\frac{1}{i}}$ -Lipschitz on \mathcal{A}_i , we can bound:

$$\mathbb{P}(|\phi(Z) - \phi(Z')| \mid \mathcal{A}_i) \leq C e^{-(t/c^i m^i \mu^{(m-i)} \sigma^i)^{\frac{q}{i}}}. \quad (\text{B.11})$$

We then obtain our result combining (B.10) and (B.11) (recall that $c^m, m^m \leq O(1)$). \square

Remark B.47. *Theorem B.34 could be seen as an iterative consequence to Theorem B.46 as if we assume Theorem B.46 and Theorem B.34 up to $m = m_0 - 1$ and want to show its validity for $m = m_0$. We can show with the iteration hypothesis for $m = m_0 - 1$ that for all $i \in [m]$:*

$$\|Z_1\|'_1 \cdots \|Z_{i-1}\|'_{i-1} \|Z_{i+1}\|'_{i+1} \cdots \|Z_m\|'_m \in O(\mu_{-i}^{(m-1)}) + \sup_{l \in [m-1]} \mathcal{E}_l \left(\mu_{-i}^{(m-1-i)} \sigma^i \right),$$

where $\mu_{-i} \equiv \mu_1 \cdots \mu_{i-1} \mu_{i+1} \cdots \mu_m$. Now, since for all $k \in [m-1]$, $\mu_{-i}^{(k)} \leq O(\mu^{(k)})$, we retrieve the hypotheses of Theorem B.46 and we can prove Theorem B.34 for $m = m_0$.

The concentration of $\text{Tr}(AXDY^T)$ signifies a linear concentration of XDY^T , demonstrating as in Proposition B.41 the relevance of the notation of linear concentration. Note besides that this result can be seen as a weak offshot of Hanson-Wright concentration inequality if one takes $D = \sqrt{n}E_{1,1}$, where $[E_{1,1}]_{i,j} = 0$ for all $(i,j) \neq (1,1)$ and $[E_{1,1}]_{1,1} = 1$.

Proposition B.48. *Given three random matrices $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in \mathcal{M}_{p,n}$ and $D \in \mathcal{D}_n$ such that $(X, Y, D) \propto \mathcal{E}_2$, $\|\mathbb{E}[D]\|_F \leq O(\sqrt{n})$, $\|\mathbb{E}[X]\|_F, \|\mathbb{E}[Y]\|_F \leq O(1)$, we have the linear concentration²⁴:*

$$XDY^T \in \mathcal{E}_1(\sqrt{n}) + \mathcal{E}_{2/3} \quad \text{in } (\mathcal{M}_p, \|\cdot\|).$$

Before proving this corollary let us give a preliminary lemma of independent interest.

Lemma B.49. *Given two random matrices $X \in \mathcal{M}_{p,n}$ and $D \in \mathcal{D}_n$ such that $(X, D) \propto \mathcal{E}_2$, $\|\mathbb{E}[X]\| \leq O(1)$ and $\|\mathbb{E}[D]\|_F \leq O(\sqrt{n})$ and a deterministic matrix $A \in \mathcal{M}_{p,n}$ such that $\|A\|_F \leq 1$, we have the concentration:*

$$\|AXD\|_F \in O(\sqrt{n}) \pm \mathcal{E}_2(\sqrt{\log(np)}) + \mathcal{E}_1.$$

²⁴The estimation of $\mathbb{E}[XDY^T]$ is done in Proposition B.50

Proof. With the same decomposition $A = U^T \Lambda V$ and notation $\check{X} \equiv VX$ as in the proof of Corollary B.43, we have the identity:

$$\|AXD\|_F = \|\Lambda \check{X} D\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^p \lambda_j^2 \check{X}_{i,j}^2 D_i^2} \leq \|\check{X}\|_\infty \|D\|_F,$$

and besides, $\|AXD\|_F \leq \|\check{X}\| \|D\|$. Therefore, with some $\mu_1 \leq O(\mathbb{E}[\|\check{X}\|_\infty]) \leq O(\sqrt{\log(np)})$ and $\mu_2 \leq O(\mathbb{E}[\|D\|]) \leq O(\sqrt{\log(n)})$ we can employ Theorem B.34 to the mapping $\phi : (\check{X}, D) \in \mathcal{M}_{p,n} \times \mathcal{D}_n \rightarrow \|\Lambda \check{X} D\|_F$, to set:

$$\|AXD\|_F \propto \mathcal{E}_2(\sqrt{\log(np)}) + \mathcal{E}_1.$$

To bound the expectation, let us note that for all $i \in [n], j \in [p]$, $\check{X}_{i,j} \in O(1) \pm \mathcal{E}_2$ and $D_i \in O(O(\mathbb{E}[D_i])) \pm \mathcal{E}_2$, therefore, $\check{X}_{i,j} D_i \in O(\mathbb{E}[D_i]) \pm \mathcal{E}_2(\mathbb{E}[D_i] + 1) + \mathcal{E}_1$, and we can estimate:

$$\begin{aligned} \mathbb{E}[\check{X}_{i,j}^2 D_i^2] &= \mathbb{E}[(\check{X}_{i,j} D_i - \mathbb{E}[\check{X}_{i,j} D_i])^2] + \mathbb{E}[\check{X}_{i,j} D_i]^2 \\ &\leq O((1 + \mathbb{E}[D_i])^2) + \mathbb{E}[\check{X}_{i,j}^2] \mathbb{E}[(D_i - \mathbb{E}[D_i])^2] + \mathbb{E}[\check{X}_{i,j}^2] \mathbb{E}[D_i]^2 \\ &\leq O(1 + \mathbb{E}[D_i]^2) + O(1) + O(\mathbb{E}[D_i]^2), \end{aligned}$$

with constants independent of i, j . Finally, we can bound:

$$\mathbb{E}[\|AXD\|_F] \leq \sqrt{\sum_{i=1}^n \sum_{j=1}^p \lambda_j^2 \mathbb{E}[\check{X}_{i,j}^2 D_i^2]} \leq O\left(\sqrt{\left(\sum_{i=1}^n 1 + \mathbb{E}[D_i]^2\right) \left(\sum_{j=1}^p \lambda_j^2\right)}\right) \leq O(\sqrt{n})$$

since $\|\mathbb{E}[D]\|_F \leq O(\sqrt{n})$ and $\|\lambda\| = \|A\|_F \leq 1$. \square

Proof of Proposition B.48. Considering a deterministic matrix $A \in \mathcal{M}_{p,n}$, we will assume that $\|A\|_F \leq 1$ if $\sup_{i \in [n]} \|\mathbb{E}[y_i x_i^T]\|_F \leq O(1)$ (to show a concentration in $(\mathcal{M}_{p,n}, \|\cdot\|_F)$) and that $\|A\|_* \leq 1$ otherwise (to show a concentration in $(\mathcal{M}_{p,n}, \|\cdot\|)$). In both cases, Corollary B.43 and Lemma B.49 allows us to set:

$$\|Y^T A X\|_d, \|AXD\|_F, \|DY^T A\|_F \in O(\sqrt{n}) \pm \mathcal{E}_2(\sqrt{n}) + \mathcal{E}_1.$$

Besides, we can bound:

$$\text{Tr}(AXDY^T) \leq \begin{cases} \|AXD\|_F \|Y\|_F \\ \|DY^T A\|_F \|X\|_F \\ \|Y^T A X\|_d \|D\|_d, \end{cases}$$

which allows us to conclude thanks to Theorem B.46 applied with the parameter vector $\mu = (1, 1, \sqrt{n})$:

$$\text{Tr}(AXDY^T) \propto \mathcal{E}_2(\sqrt{n}) + \mathcal{E}_1(\sqrt{n}) + \mathcal{E}_{\frac{2}{3}} \propto \mathcal{E}_1(\sqrt{n}) + \mathcal{E}_{\frac{2}{3}}.$$

\square

In the setting of Proposition B.45, once one knows that XDY is concentrated it is natural to look for a simple deterministic equivalent. The next proposition help us for such a design. Note that the hypotheses are far lighter, in particular, we just need the linear concentration of D .

Proposition B.50. *Given three random matrices $D \in \mathcal{D}_n$, $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in \mathcal{M}_{p,n}$ and a deterministic matrix $\tilde{D} \in \mathcal{D}_n$, such that $D \in \tilde{D} \pm \mathcal{E}_2$ in $(\mathcal{D}_n, \|\cdot\|)$ and for all²⁵ $i \in [n]$, $(x_i, y_i) \propto \mathcal{E}_2$ and*

²⁵If we adopt the stronger assumptions $(X, Y) \propto \mathcal{E}_2$ in $(\mathcal{M}_{p,n}, \|\cdot\|)$ and $\|\mathbb{E}[X]\|_F, \|\mathbb{E}[Y]\|_F \leq O(1)$, we can show more directly thanks to Propositions B.41 and B.22:

$$\begin{aligned} & \left| \mathbb{E}[\text{Tr}(AXDY^T)] - \mathbb{E}[\text{Tr}(AX\mathbb{E}[D]Y^T)] \right| \\ &= \left| \mathbb{E} \left[\text{Tr} \left((Y^T A X - \mathbb{E}[Y^T A X]) (D - \mathbb{E}[D]) \right) \right] \right| \\ &\leq \sqrt{\mathbb{E} \left[\|Y^T A X - \mathbb{E}[Y^T A X]\|_d^2 \right]} \mathbb{E} \left[\|D - \mathbb{E}[D]\|_d^2 \right] \leq O(n) \end{aligned}$$

$\sup_{i \in [n]} \|\mathbb{E}[x_i]\|, \|\mathbb{E}[y_i]\| \leq O(1)$, we have the estimate :

$$\left\| \mathbb{E}[X D Y^T] - \mathbb{E}[X \mathbb{E}[D] Y^T] \right\|_F \leq O(n).$$

We can precise the estimation with supplementary assumptions:

- if $\|\tilde{D} - \mathbb{E}[D]\|_F \leq O(1)$ then $\left\| \mathbb{E}[X(D - \tilde{D})Y^T] \right\|_F \leq O\left(\sqrt{n \max(p, n)}\right)$
- if $\sup_{i \in [n]} \|\mathbb{E}[x_i y_i^T]\|_F \leq O(1)$ then $\left\| \mathbb{E}[X D Y^T] - \mathbb{E}[X \tilde{D} Y^T] \right\|_F \leq O(n)$.

Proof. Considering a deterministic matrix $A \in \mathcal{M}_p$, such that $\|A\|_F \leq 1$:

$$\begin{aligned} & \left| \mathbb{E}[\text{Tr}(A X D Y^T)] - \mathbb{E}[\text{Tr}(A X \mathbb{E}[D] Y^T)] \right| \\ & \leq \sum_{i=1}^n \left| \mathbb{E} [D_i x_i^T A y_i - \mathbb{E}[D_i] x_i^T A y_i] \right| \\ & = \sum_{i=1}^n \left| \mathbb{E} [(x_i^T A y_i - \mathbb{E}[x_i^T A y_i]) (D_i - \mathbb{E}[D_i])] \right| \\ & \leq \sum_{i=1}^n \sqrt{\mathbb{E} \left[|x_i^T A y_i - \mathbb{E}[x_i^T A y_i]|^2 \right] \mathbb{E} \left[|D_i - \mathbb{E}[D_i]|^2 \right]} \leq O(n) \end{aligned}$$

thanks to Hölder's inequality applied to the bounds given by Proposition A.30 (we know that $D_i \in \mathbb{E}[D_i] \pm \mathcal{E}_2$ and from Proposition B.41 that $x_i^T A y_i \in \mathbb{E}[x_i^T A y_i] \pm \mathcal{E}_2 + \mathcal{E}_1$; note that the concentration constants are the same for all $i \in [n]$).

Now, $|\mathbb{E}[x_i^T A y_i]| \leq \|A\|_F \|\mathbb{E}[y_i x_i^T]\|_F \leq O(\sqrt{p})$ thanks to Remark B.44 and if $\|\mathbb{E}[D] - \tilde{D}\|_F \leq O(1)$, we can bound:

$$\begin{aligned} \left| \mathbb{E} \left[\text{Tr} \left(A X \left(\mathbb{E}[D] - \tilde{D} \right) Y^T \right) \right] \right| & \leq \sum_{i=1}^n \left| \mathbb{E} [x_i^T A y_i] \left(\mathbb{E}[D_i] - \tilde{D}_i \right) \right| \\ & \leq \sup_{i \in [n]} |\mathbb{E} [x_i^T A y_i]| \sqrt{n} \left\| \mathbb{E}[D] - \tilde{D} \right\|_F \leq O(\sqrt{np}). \end{aligned}$$

If, $\|\mathbb{E}[D] - \tilde{D}\|_F$ is possibly of order far bigger than $O(1)$, but $\sup_{i \in [n]} \|\mathbb{E}[y_i x_i^T]\|_F \leq O(1)$, then $\sup_{i \in [n]} |\mathbb{E}[x_i^T A y_i]| \leq O(1)$, and we can still bound:

$$\left\| \mathbb{E} \left[\text{Tr} \left(A X \left(\mathbb{E}[D] - \tilde{D} \right) Y^T \right) \right] \right\|_F \leq n \sup_{i \in [n]} |\mathbb{E} [x_i^T A y_i]| \left\| \mathbb{E}[D] - \tilde{D} \right\| \leq O(n).$$

□

A non multi-linear application of Theorem B.34 is provided with the study of the resolvent $(I_p - \frac{1}{n} X D X^T)^{-1}$ with D diagonal, at the end of Part II.

B.9 Concentration of high order products

We give here some offshots of Theorem B.34 when m is a quasiasymptotic variable (thus a sequence of positive real values). The result of Theorem B.34 stays almost unmodified if m tends to infinity, one mainly needs a supplementary hypothesis.

Theorem B.51. *Under the hypotheses of Theorem B.34 but without the hypothesis that m is constant, if one further assumes that $(\log(m))^{\frac{1}{q}} \leq O\left(\frac{\mu(1)}{\sigma}\right)$, then there exists a constant $\kappa > 0$ such that we have the concentration:*

$$\phi(Z) \propto \max_{l \in [m]} \mathcal{E}_{q/l} \left(\kappa^m \sigma^l \mu^{(m-l)} \right).$$

Proof. One must be careful that this time m can tend to infinity. Given $i \in \{0\} \cup [m]$, we will employ again the variables $t_i = \mu^{(m-i)} \mu_{(i)}^i$ for $i \in [m]$, $t_0 = 0$ and $t_{m+1} = \infty$, but we note this time for all $i \in \{0\} \cup [m]$:

$$\mathcal{A}_i \equiv \left\{ \begin{aligned} &\|Z_1\|'_1, \|Z'_1\|'_1 \leq (\delta + 1) \left(\frac{t}{\mu^{(m-i)}} \right)^{\frac{1}{i}}; \dots; \|Z_i\|'_i, \|Z'_i\|'_i \leq (\delta + 1) \left(\frac{t}{\mu^{(m-i)}} \right)^{\frac{1}{i}}; \\ &\|Z_{i+1}\|'_{i+1}, \|Z'_{i+1}\|'_{i+1} \leq (\delta + 1) \mu_{i+1}; \dots; \|Z_m\|'_m, \|Z'_m\|'_m \leq (\delta + 1) \mu_m \end{aligned} \right\}$$

for some $\delta \geq 0$. As we saw in the proof of Theorem B.34, for all $1 \leq k \leq i < j \leq m$:

$$\forall t \leq t_{i+1} : \left(\frac{t/(c\sigma)^i}{\mu^{(m-i)}} \right)^{\frac{q}{i}} \leq \frac{\mu_j}{c\sigma} \quad \text{and} \quad \forall t \geq t_i : \left(\frac{t/(c\sigma)^i}{\mu^{(m-i)}} \right)^{\frac{1}{i}} \geq \frac{\mu_k}{c\sigma}.$$

Therefore, given $t \in [t_i, t_{i+1}]$, one obtains $\forall j \in \{i+1, \dots, m\}$:

$$\begin{aligned} \mathbb{P}(\|Z_j\|'_j \geq (\delta + 1) \mu_j) &\leq \mathbb{P}(\|Z_j\|'_j - \mathbb{E}[\|Z_j\|'_j] \geq \delta \mu_j) \\ &\leq C \exp\left(-\delta^q \left(\frac{\mu_j}{c\sigma}\right)^q\right) \leq C \exp\left(-\delta^q \left(\frac{t/(c\sigma)^i}{\mu^{(m-i)}}\right)^{\frac{q}{i}}\right), \end{aligned}$$

and the same way $\forall t \in [t_i, t_{i+1}]$ and $k \in [i]$:

$$\mathbb{P}\left(\|Z_k\|'_k \geq (\delta + 1) \left(\frac{t}{\mu^{(m-i)}}\right)^{\frac{q}{i}}\right) \leq C \exp\left(-\delta^q \left(\frac{t/(c\sigma)^i}{\mu^{(m-i)}}\right)^{\frac{q}{i}}\right).$$

By hypothesis, there exists a constant $K > 0$ such that $\forall i \in [m]$ and $t \in [t_i, t_{i+1}]$:

$$K \left(\frac{t/(c\sigma)^i}{\mu^{(m-i)}}\right)^{\frac{q}{i}} \geq K \left(\frac{\mu_i}{c\sigma}\right)^q \geq \log(m).$$

Therefore choosing $\delta = (K + 1)^{\frac{1}{q}}$:

$$\mathbb{P}(\mathcal{A}_i^c) \leq C \exp\left(\log m - (K + 1) \left(\frac{t/(c\sigma)^i}{\mu^{(m-i)}}\right)^{\frac{q}{i}}\right) \leq C \exp\left(-\left(\frac{t/(c\sigma)^i}{\mu^{(m-i)}}\right)^{\frac{q}{i}}\right),$$

and $\forall t \in (0, t_1]$:

$$\mathbb{P}(\mathcal{A}_0^c) \leq C \exp\left(\log m - (K + 1) \left(\frac{\mu_1}{c\sigma}\right)^q\right) \leq C \exp\left(\left(\frac{\mu_1}{c\sigma}\right)^q\right) \leq C \exp\left(-\left(\frac{t/c\sigma}{\mu^{(m-1)}}\right)^q\right).$$

Besides, for any $i \in \{0\} \cup [m]$ and $t \in [t_i, t_{i+1}]$, we can obtain in the asymptotic m case, the concentration inequality:

$$\mathbb{P}(|f(\phi(Z)) - f(\phi(Z'))| \geq t, Z \in \mathcal{A}_i) \leq C \exp\left(-\frac{t}{(m(1 + \delta)^m c\sigma)^i \mu^{(m-i)}}\right)^{\frac{q}{i}}.$$

Regrouping all the concentration inequalities for the different values of t and noting $\kappa \equiv (c + 1)(\delta + 1)e^{1/e} \leq O(1)$, we retrieve our result (note that $e^{1/e} \geq m^{1/m}$ for all $m > 0$). \square

When only one concentrated random vector is involved, one can obtain a better concentration inequality characterized by only two exponential regime (as explained in Remark B.37).

Theorem B.52. *With the hypotheses of Theorem B.34 in a setting where m is quasi-asymptotic, if $Z_1 = \dots = Z_m$, we do not have to assume anymore that $(\log(m))^{\frac{1}{q}} \leq O(\mathbb{E}[\|Z_i\|'_i]/\sigma)$ and if $\mu_1 = \dots = \mu_m \equiv \mu_0$, then, for any constant $\varepsilon > 0$ (i.e. such that $\varepsilon \geq O(1)$) we have the concentration:*

$$\phi(Z) \propto \mathcal{E}_q(m\sigma((1+\varepsilon)\mu_0)^{m-1}) + \mathcal{E}_{q/m}((\kappa\sigma)^m),$$

for some constant²⁶ $\kappa > 0$.

Note that when $1 - \mu_0 \geq O(1)$, one can choose ε sufficiently small such that $m((1+\varepsilon)\mu_0)^{m-1} \leq O(1)$ (when $m \rightarrow \infty$), then $\phi(Z) \propto \mathcal{E}_q(\sigma) + \mathcal{E}_{q/m}((\kappa\sigma)^m)$.

Proof. Let us note $K \equiv \max((1+\varepsilon)\mu, (t/m)^{1/m})$ and $\mathcal{A}_K \equiv \{\|Z\|' \geq K\}$, then, on the one hand $K - \mu \geq \frac{\varepsilon}{1+\varepsilon}$ and:

$$\mathbb{P}(\mathcal{A}_K^c) \leq \mathbb{P}(\|\|Z\|'\| - \mathbb{E}[\|Z\|'] \geq K - \mu) \leq Ce^{-\left(\frac{\frac{\varepsilon}{1+\varepsilon}K}{c\sigma}\right)^q} \leq Ce^{-\left(\frac{t}{m(\frac{1+\varepsilon}{\varepsilon}c\sigma)^m}\right)^{\frac{q}{m}}},$$

and on the second hand ϕ is mK^{m-1} -Lipschitz on \mathcal{A}_K thus:

$$\begin{aligned} \mathbb{P}(|f(\phi(Z)) - f(\phi(Z'))| \geq t \mid \mathcal{A}_K) &\leq Ce^{-(t/mc\sigma K^{m-1})^q} \\ &\leq Ce^{-(t/mc\sigma((1+\varepsilon)\mu)^{m-1})^q} + Ce^{-(t/m(c\sigma)^m)^{q/m}}. \end{aligned}$$

Therefore, we obtain the concentration:

$$\phi(Z) \propto \mathcal{E}_q(m\sigma((1+\varepsilon)\mu)^{m-1}) + \mathcal{E}_{q/m}\left(\left(ce^{1/e}\frac{1+\varepsilon}{\varepsilon}\sigma\right)^m\right), \quad (\text{B.12})$$

which provides us the wanted inequality since $ce^{1/e}\frac{1+\varepsilon}{\varepsilon} \leq O(1)$. \square

²⁶The presence of this constants basically imposes that if $\sigma \geq O(1)$, then the observable diameter of the $\mathcal{E}_{q/m}((\kappa\sigma)^m)$ decay can not tend to zero. A better inequality might be obtained, if one computes precisely the concentration constants, starting from a sharp concentration inequality on Z .

Chapter C

Concentration of solution to convex problems and contractive equations

This chapter aims at setting the concentration of the solution to convex problems like:

$$\text{Minimize } \Psi(Y), Y \in \mathbb{R}^p,$$

where $\Psi : \mathbb{R}^p \rightarrow \mathbb{R}$ is a random convex mapping. Although our results could be extended to cases where Ψ is only differentiable, we will assume in all this chapter that Ψ is \mathcal{C}^2 and strictly convex. This lower bound on the Hessian of Ψ is important since it will appear in the denominator of the observable diameter of Y .

In practical uses, the randomness of Ψ generally depends on a random vector $X \in F$ and the problem writes:

$$\text{Minimize } \psi(X)(Y), Y \in \mathbb{R}^p \tag{C.1}$$

for $\psi : F \rightarrow \mathcal{F}(\mathbb{R}^p, \mathbb{R}^p)$ deterministic and $X \in F$ satisfying some concentration inequality. As said above, in practical examples $d\psi(X)^2|_y$ is assumed to have a lower bound independent of X . One might desire to also have a bound limiting the variability due to X . It is generally not true, however we study in the first section this first example since the concentration of Y is then a simple consequence to the theorem of implicit functions and the observable diameter obtained is the same as the one that will be retrieved in more complex settings.

Note of course that Y minimizing (C.1) is also the solution to the equation:

$$\phi(X)(Y) = 0$$

where we noted for all $A, y \in F \times \mathbb{R}^p$ $\phi(A)(y) \equiv d\psi(A)|_y$. We will consider this associated problem in next section.

C.1 Bounded variability on X

In this section, we place ourselves in the case $F = \mathbb{R}^p$ and to employ next theorem to random convex problems, one has to choose:

$$\phi(X, Y) \equiv d\psi(X)|_Y \in \mathbb{R}^p$$

to retrieve the setting of the introduction of this section. Note that the partial derivatives $\frac{\partial \phi}{\partial X}$ and $\frac{\partial \phi}{\partial Y}$ respectively belong to $\mathcal{M}_{p,n}$ and \mathcal{M}_p .

Theorem C.1. *Let us consider a (sequence of) random vectors $X \in \mathbb{R}^n$ satisfying $X \propto \mathcal{E}_q$ and a deterministic \mathcal{C}^1 mapping $\phi : X \in \mathbb{R}^n \times Y \in \mathbb{R}^p \rightarrow \mathbb{R}^p$ such that $\forall A, y \in X(\Omega) \times \mathbb{R}^p$:*

- $\left\| \frac{\partial \phi}{\partial X} \Big|_{(A,y)} \right\| \leq \sigma,$

- $\frac{\partial \phi}{\partial Y} \Big|_{(A,y)}$ is a positive symmetric matrix satisfying $\frac{\partial \phi}{\partial Y} \Big|_{(A,y)} \geq \kappa I_p$,

for some (sequence of) positive parameters $\sigma, \kappa > 0$. Then, the unique solution to the equation:

$$\phi(X, Y) = 0, \quad Y \in \mathbb{R}^p$$

satisfies the concentration inequality $Y \propto \mathcal{E}_q\left(\frac{\sigma}{\kappa}\right)$.

Proof. The implicit function Theorem sets that there exists a \mathcal{C}^1 mapping $\theta : \mathbb{R}^n \rightarrow \mathbb{R}^p$ such that $Y = \theta(X)$ and:

$$d\theta|_X = \left(\frac{\partial \phi}{\partial Y} \Big|_{(X, \theta(Y))} \right)^{-1} \frac{\partial \phi}{\partial X} \Big|_{(X, \theta(Y))}.$$

One can then directly deduce the concentration of Y from the $O\left(\frac{\sigma}{\kappa}\right)$ -Lipschitz character of θ . \square

In more complex settings like the problems of robust regression that will be studied in Chapter J, the quantity $\left\| \frac{\partial \phi}{\partial X} \Big|_{(A,y)} \right\|$ does not admit a bound valid for all the drawings of X . One then has to take into account the dispersion of the different drawings of Y to be able to bound efficiently $\left\| \frac{\partial \phi}{\partial X} \Big|_{(A,y)} \right\|$. The solution presented here relies on a reformulation of convex problems as contractive fixed point equations more adapted to control the variability towards X .

C.2 From convex problem to a contractive fixed point equation

Starting from the equation

$$d\Psi|_Y = 0$$

one can introduce the mapping:

$$\Phi : Y \mapsto Y - \frac{d\Psi|_Y}{\|d^2\Psi\|_\infty},$$

we see that if for all $Y \in \mathbb{R}^p$, $d^2\Psi|_Y \geq \kappa I_p$ then Φ is contractive since:

$$\|d\Phi|_Y\| = \left\| I_p - \frac{d^2\Psi|_Y}{\|d^2\Psi\|_\infty} \right\| \leq 1 - \frac{\kappa}{\|d^2\Psi\|_\infty}$$

We will therefore from now on consider solutions $Y \in \mathbb{R}^p$ to fixed point equations:

$$Y = \Phi(Y),$$

where Φ is contractive.

We start with settings where Φ is affine and all the $\Phi^k(y)$ are linearly concentrated for deterministic y (Section C.3), then we study equation with general Φ with linearly and Lipschitz concentration hypotheses on $\Phi^k(y)$ for $k \leq O(\sqrt{\log n})$ (Section C.4), finally we precise this result for mappings Φ that are Lipschitz concentrated for the infinite norm on balls (Section C.5).

C.3 Concentration of solutions to quadratic problems (Φ affine)

Given two normed (or semi-normed) vector spaces $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, we denote $\mathcal{A}(E, F)$ the set of continuous affine mappings from E to F and we endow it with the norm:

$$\forall \phi \in \mathcal{A}(E, F) : \|\phi\|_{\mathcal{A}(E, F)} = \|\mathcal{L}(\phi)\|_{\mathcal{L}(E, F)} + \|\phi(0)\|_F$$

where $\mathcal{L}(\phi) = \phi - \phi(0)$ and $\|\mathcal{L}(\phi)\|_{\mathcal{L}(E, F)} = \sup_{x \in E, \|x\|_E \leq 1} \|\mathcal{L}(\phi)(x)\|_F$.

Example C.2. Let us consider as an introducing example the quadratic problem:

$$\text{Minimize: } \Psi(y) = \left\| \frac{1}{\sqrt{n}} Xy - z \right\|^2 + \|y\|^2, y \in \mathbb{R}^p$$

for a deterministic vector $z \in \mathbb{R}^p$ and where we merely assume that the columns of $X = (x_1, \dots, x_n)$ are independent Gaussian vectors satisfying for all $i \in [n]$, $x_i \sim \mathcal{N}(0, 1)$. We then note $\mathcal{A} \equiv \{\|X\| \leq 2\sqrt{n}\}$, we know that $\mathbb{P}(\mathcal{A}) \geq 1 - Ce^{-cn}$, therefore Remark B.5 allows us to deduce that $X \mid \mathcal{A} \propto \mathcal{E}_2$. If we note:

$$\Phi : y \mapsto y - \frac{1}{4} \left(\frac{1}{n} X X^T y + \frac{1}{\sqrt{n}} X z + y \right)$$

we know that:

$$\|\mathcal{L}(\Phi)\| = \left\| I_p - \frac{1}{4} \left(\frac{1}{n} X X^T + I_p \right) \right\| \leq \frac{3}{4}$$

and for all $k \in \mathbb{N}$:

$$\mathcal{L}(\Phi)^k(\Phi(0)) = \left(\frac{3}{4} \right)^k \left(I_p - \frac{1}{n} X X^T \right)^k \in \mathcal{E}_q \left(\left(\frac{3}{4} \right)^k \right) \quad \text{in } (E, \|\cdot\|),$$

The Theorem C.3 then allows us to show that $Y = \operatorname{argmin}_{y \in \mathbb{R}^p} \Psi(y)$ satisfies the linear concentration:

$$Y \mid \mathcal{A} \in \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right).$$

Let us now set the theorem.

Theorem C.3 (Concentration of the resolvent). *Given a (sequence of) reflexive vector space $(E, \|\cdot\|)$, let $\Phi \in \mathcal{A}(E)$ be a (sequence of) random mapping such that there exist two (sequences of) parameters $\sigma > 0$ and $\varepsilon \in (0, 1)$, such that $\|\mathcal{L}(\Phi)\| \leq 1 - \varepsilon$, and such that for all (sequences of) integer k :*

$$\mathcal{L}(\Phi)^k(\Phi(0)) \in \mathcal{E}_q(\sigma(1 - \varepsilon)^k) \quad \text{in } (E, \|\cdot\|)$$

Then the random equation

$$Y = \Phi(Y)$$

admits a unique solution satisfying the linear concentration:

$$Y = (Id_E - \mathcal{L}(\Phi))^{-1} \Phi(0) \in \mathcal{E}_q \left(\frac{\sigma}{\varepsilon} \right).$$

Proof. The vector Y is well defined and expresses:

$$Y = (Id_E - \mathcal{L}(\Phi))^{-1} \Phi(0) = \sum_{k=0}^{\infty} \mathcal{L}(\Phi)^k \Phi(0)$$

Now Corollary B.32 allows us to conclude from the concentrations $\mathcal{L}(\Phi)^k \Phi(0) \in \mathcal{E}_q(\sigma(1 - \varepsilon)^k)$ that:

$$\sum_{k=0}^{\infty} \mathcal{L}(\Phi)^k \Phi(0) \in \mathcal{E}_q \left(\sum_{k=0}^{\infty} \sigma(1 - \varepsilon)^k \right) = \mathcal{E}_q \left(\frac{\sigma}{\varepsilon} \right).$$

□

With Theorem C.3 at hand we can already prove the Linear concentration in the following theorem. The Lipschitz concentration will be provided later with Proposition C.5. minimizing problem of the form

$$\text{Minimize } y^T W y + \lambda Z y, \quad y \in \mathbb{R}^p,$$

C.4 When $\Phi^k(y)$ is linearly or Lipschitz concentrated for $k \leq \log(\eta_{(E, \|\cdot\|)})$ and y deterministic and bounded

When we can not get a decreasing observable diameter for the iterates of ϕ , or when ϕ is not affine, one needs a different approach that allows to treat, at the same time affine and non-affine mappings ϕ and extend very simply linear concentration inferences to Lipschitz concentration inferences. Given a normed vector space $(E, \|\cdot\|)$, we note $\mathcal{F}(E)$, the set of mappings from E to E . If f is bounded, we denote $\|f\|_\infty = \sup_{x,y \in E} \|f(x)\|$. For a Lipschitz mapping $f \in \mathcal{F}(E)$, we introduce the seminorm $\|\cdot\|_{\mathcal{L}}$ which provides the Lipschitz parameter and will play the role of $\|\mathcal{L}(\Phi)\|$ in Proposition C.4:

$$\|f\|_{\mathcal{L}} = \sup_{\substack{x,y \in E \\ x \neq y}} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

Proposition C.4. *Given a (sequence of) reflexive¹ vector space $(E, \|\cdot\|)$, we note η its norm degree (of course $\eta \geq O(1)$), we then consider $\Phi \in \mathcal{F}(E, E)$, a (sequence of) random Lipschitz mapping such that there exists a (sequence of) parameters $\varepsilon \in (0, 1)$ such that $\|\Phi\|_{\mathcal{L}} \leq 1 - \varepsilon$. Given an integer $k_0 \in \mathbb{N}$ such that $k_0 \geq \lceil -\frac{\log(\eta)}{q \log(1-\varepsilon)} \rceil$, noting \tilde{Y} the unique² (deterministic) solution to $\tilde{Y} = \mathbb{E}[\Phi^{k_0}(\tilde{Y})]$, if we further assume that³:*

$$\Phi^{k_0}(\tilde{Y}) \in \tilde{Y} \pm \mathcal{E}_q \left(\sum_{i=0}^{k_0} (1-\varepsilon)^i \sigma \right) \quad \text{in } (E, \|\cdot\|), \quad (\text{C.2})$$

for a given (sequence) of integer $\sigma > 0$, then, the random equation

$$y = \phi(y), \quad y \in E$$

admits a unique solution $Y \in E$ satisfying the linear concentration:

$$Y \in \tilde{Y} \pm \mathcal{E}_q \left(\frac{\sigma}{\varepsilon} \right).$$

Proof. The mapping $y \mapsto \Phi(y)$ is contractive, that proves the existence and uniqueness of Y (E is complete since it is reflexive – see Remark B.8).

Then let us try and bound $\|Y - \tilde{Y}\|$:

$$\begin{aligned} \|Y - \tilde{Y}\| &= \left\| \Phi^{k_0}(Y) - \mathbb{E} \left[\Phi^{k_0}(\tilde{Y}) \right] \right\| \leq \left\| \Phi^{k_0}(Y) - \Phi^{k_0}(\tilde{Y}) \right\| + \left\| \Phi^{k_0}(\tilde{Y}) - \mathbb{E} \left[\Phi^{k_0}(\tilde{Y}) \right] \right\| \\ &\leq \|\Phi\|_{\mathcal{L}}^{k_0} \|Y - \tilde{Y}\| + \left\| \Phi^{k_0}(\tilde{Y}) - \mathbb{E} \left[\Phi^{k_0}(\tilde{Y}) \right] \right\|. \end{aligned}$$

The inequality $\|\Phi^{k_0}\| \leq (1-\varepsilon)^{k_0}$, gives us the bound:

$$\|Y - \tilde{Y}\| \leq \frac{1}{1 - (1-\varepsilon)^{k_0}} \left\| \Phi^{k_0}(\tilde{Y}) - \mathbb{E} \left[\Phi^{k_0}(\tilde{Y}) \right] \right\|.$$

But we know from Proposition B.22 that:

$$\left\| \Phi^{k_0}(\tilde{Y}) - \mathbb{E} \left[\Phi^{k_0}(\tilde{Y}) \right] \right\| \in O \left(\frac{1}{\varepsilon} \eta^{1/q} \sigma (1 - (1-\varepsilon)^{k_0}) \right) \pm \mathcal{E}_q \left(\frac{1}{\varepsilon} \eta^{1/q} \sigma (1 - (1-\varepsilon)^{k_0}) \right)$$

(since $\sum_{i=0}^{k_0} (1-\varepsilon)^i = \frac{1-(1-\varepsilon)^{k_0}}{\varepsilon}$). which allows us to conclude thanks to Lemma B.19:

$$\|Y - \tilde{Y}\| \in O \left(\frac{\eta^{1/q} \sigma}{\varepsilon} \right) \pm \mathcal{E}_q \left(\frac{\eta^{1/q} \sigma}{\varepsilon} \right).$$

¹We suppose E reflexive to be able to define an expectation operator on the set of random vectors of E – see B.2

²The assumption on $\|\Phi\|_{\mathcal{L}}$ ensures the existence and uniqueness of \tilde{Y} .

³The term $\sum_{i=0}^{k_0} (1-\varepsilon)^i = \frac{1-(1-\varepsilon)^{k_0}}{\varepsilon}$ will appear naturally in application as pictured in Lemma C.8 below. In the application we will see below this concentration is true for any $k \in \mathbb{N}$.

Returning to our initial goal (the linear concentration of Y), we now bound, for $f \in \mathcal{L}(E)$ (we still have $\|\mathcal{L}(\Phi)\| \leq 1 - \varepsilon$):

$$\begin{aligned} \left| f(Y) - f(\tilde{Y}) \right| &\leq \left| f(\Phi^{k_0}(Y)) - f(\Phi^{k_0}(\tilde{Y})) \right| + \left| f(\Phi^{k_0}(\tilde{Y})) - f(\mathbb{E}[\Phi^{k_0}(\tilde{Y})]) \right| \\ &\leq (1 - \varepsilon)^{k_0} \|Y - \tilde{Y}\| + \left| f(\Phi^{k_0}(\tilde{Y})) - \mathbb{E}[f(\Phi^{k_0}(\tilde{Y}))] \right|. \end{aligned}$$

Further, noting that, with our choice of k_0 , $(1 - \varepsilon)^{k_0} = O(1/\eta^{1/q})$, we conclude again from the concentration of $\Phi^{k_0}(\tilde{Y})$ and Lemma B.19 that:

$$f(Y) \in f(\tilde{Y}) \pm \mathcal{E}_q\left(\frac{\sigma}{\varepsilon}\right),$$

thereby giving the sought-for concentration result. \square

Let us extend the result of Proposition C.4 to the case of Lipschitz concentration.

Proposition C.5. *In the setting of Proposition C.4, if we additionally assume that we have the Lipschitz concentration:*

$$\Phi^{k_0}(\tilde{Y}) \propto \mathcal{E}_q\left(\sum_{i=0}^{k_0} (1 - \varepsilon)^i \sigma\right) \quad \text{in } (E, \|\cdot\|).$$

then we have the Lipschitz concentration:

$$Y \propto \mathcal{E}_q\left(\frac{\sigma}{\varepsilon}\right),$$

Proof. We already know from Proposition C.4 that $Y \in \mathcal{E}_q(\sigma/\varepsilon)$. To show the Lipschitz concentration of Y , let us consider a Lipschitz map $f : E \rightarrow \mathbb{R}$ and introduce the mappings:

$$\begin{array}{ccc} U : E & \longmapsto & E \times \mathbb{R} \\ y & \longmapsto & (y, f(y)) \end{array} \qquad \begin{array}{ccc} V : E \times \mathbb{R} & \longmapsto & E \\ (y, t) & \longmapsto & y \end{array}$$

(note that $V \circ U = Id_E$). If we endow $E \times \mathbb{R}$ with the norm $\|\cdot\|_{\ell^\infty}$ satisfying $\forall (y, t) \in E \times \mathbb{R}, \|(y, t)\|_{\ell^\infty} = \max(\|y\|, |t|)$ then the mappings U and V are both 1-Lipschitz, consequently $\|U \circ \Phi \circ V\|_{\mathcal{L}} \leq 1 - \varepsilon$ and we can consider $\tilde{Z} \in E \times \mathbb{R}$ solution to:

$$\tilde{Z} = \mathbb{E} \left[U \circ \Phi^{k_0} \circ V(\tilde{Z}) \right],$$

by uniqueness of the solution to $y = \mathbb{E}[\Phi^{k_0}(y)]$, we know that $\tilde{Z} = (\tilde{Y}, \tilde{t})$ with $\tilde{t} \equiv \mathbb{E}[f(\Phi^{k_0}(\tilde{Y}))]$. Then:

$$(U \circ \Phi \circ V)^{k_0}(\tilde{Z}) = (\Phi^{k_0}(\tilde{Y}), f(\Phi^{k_0}(\tilde{Y}))) \propto \mathcal{E}_q\left(\sum_{i=0}^{k_0} (1 - \varepsilon)^i \sigma\right) \quad \text{in } (E \times \mathbb{R}, \|\cdot\|_{\ell^\infty})$$

by hypothesis. Therefore, one can deduce from Proposition C.4 the linear concentration of Z solution to the fixed point equation:

$$z = U \circ \Phi \circ V(z) \qquad z \in E \times \mathbb{R}; \tag{C.3}$$

from which one can deduce in particular, the concentration of the second component $f(Y)$. \square

C.5 When $\Phi \propto \mathcal{E}_2(\sigma)$ for the infinity norm

If one can assume the stronger hypothesis that Φ is concentrated as a random mapping and for the infinity norm, then we can infer the concentration of all the iterates of Φ . We provide here the result for a general setting where the mapping Φ is only bounded around a given point y_0 , for that, for any $r > 0$, we introduce the semi-norm $\|\cdot\|_{\mathcal{B}(y_0, r)}$ defined for any $f \in \mathcal{F}(E)$ and $y \in E$ as:

$$\|f\|_{\mathcal{B}(y_0, r)} = \sup_{\|y - y_0\| \leq r} \|f(y)\|$$

Theorem C.6. *Given a normed vector space $(E, \|\cdot\|)$ and a (sequence of) random mapping $\Phi : E \rightarrow E$ we assume that we are given a (sequence of) deterministic vector $y_0 \in E$ and three (sequences of) parameters $\sigma, \tau, \varepsilon > 0$ such that:*

- Φ is $(1 - \varepsilon)$ -Lipschitz
- $\|y_0 - \Phi(y_0)\| \leq \tau$
- $\Phi \propto \mathcal{E}_q(\sigma)$ in $(\mathcal{F}(E), \|\cdot\|_{\mathcal{B}(y_0, \tau/\varepsilon)})$,

then the random equation $Y = \Phi(Y)$ admits a unique solution $Y \in E$ satisfying the Lipschitz concentration:

$$Y \propto \mathcal{E}_q\left(\frac{\sigma}{\varepsilon}\right).$$

One can see in the proof of the theorem below that the required concentration is actually the concentration of $\Phi|_{\mathcal{B}(y_0, \frac{\tau}{\varepsilon})}$ to be able to employ Proposition C.5.

The following two lemmas allows us to apply Proposition C.5 to prove Theorem C.6 below.

Lemma C.7. *Any $(1 - \varepsilon)$ -Lipschitz mapping $\phi : E \rightarrow E$ is stable on any ball $\mathcal{B}(y_0, \frac{1}{\varepsilon}\|y_0 - \phi(y_0)\|)$, where $y_0 \in E$.*

Proof. Let us note $\tau \equiv \|y_0 - \phi(y_0)\|$ and for any $y \in \mathcal{B}(y_0, \frac{\tau}{\varepsilon})$, let us bound:

$$\begin{aligned} \|\Phi(y) - y_0\| &\leq \|\Phi(y) - \Phi(y_0)\| + \|\Phi(y_0) - y_0\| \\ &\leq (1 - \varepsilon)\|y - y_0\| + \|\Phi(y_0) - y_0\| \leq (1 - \varepsilon)\frac{\tau}{\varepsilon} + \tau = \frac{\tau}{\varepsilon} \end{aligned}$$

□

Lemma C.8. *Given a random mapping $\Phi : E \rightarrow E$, four (sequences of) parameters⁴ $\sigma, \varepsilon, \tau, \alpha > 0$ and a (sequence of) deterministic vector $y_0 \in E$, if we assume that:*

- $\|y_0 - \Phi(y_0)\| \leq \tau$,
- Φ is $(1 - \varepsilon)$ -Lipschitz,
- $\Phi \propto \mathcal{E}_q(\sigma)$ in $(\mathcal{F}(E, E), \|\cdot\|_{\mathcal{B}(y_0, \frac{\tau}{\varepsilon})})$,

then for any integer $k > 0$, we have the concentration:

$$\Phi^k \propto \mathcal{E}_q\left(\sum_{i=0}^k (1 - \varepsilon)^i \sigma\right) \quad \text{in } (\mathcal{F}(E), \|\cdot\|_{\mathcal{B}(y_0, \frac{\tau}{\varepsilon})}),$$

Proof. Thus, for any $f \in \Phi(\Omega)$ and for all $k \in \mathbb{N}$, $\Phi^k : \mathcal{B}(y_0, \frac{\tau}{\varepsilon}) \rightarrow \mathcal{B}(y_0, \frac{\tau}{\varepsilon})$, and we can bound for any supplementary mapping $g \in \Phi(\Omega)$:

$$\begin{aligned} \|f^k - g^k\|_{\mathcal{B}(y_0, \frac{\tau}{\varepsilon})} &\leq \sum_{i=1}^k (1 - \varepsilon)^{i-1} \sup_{y \in \mathcal{B}(y_0, \frac{\tau}{\varepsilon})} \|f(g^{k-i}(y)) - g(g^{k-i}(y))\| \\ &\leq \sum_{i=0}^k (1 - \varepsilon)^i \|f - g\|_{\mathcal{B}(y_0, \tau/\varepsilon)}. \end{aligned}$$

Thus the mapping $f \mapsto f^k$ is $\sum_{i=0}^k (1 - \varepsilon)^i$ -Lipschitz from $(\mathcal{F}(E), \|\cdot\|_{\mathcal{B}(y_0, \tau/\varepsilon)})$ to $(\mathcal{F}(E), \|\cdot\|_{\mathcal{B}(y_0, \tau)})$ and one can deduce the result of the Lemma from the concentration of Φ .

□

⁴Unlike in Theorem C.6, ε can here tend to zero.

Proof of Theorem C.6. Noting $k_0 = \lceil -\frac{\log(\eta)}{q \log(1-2\varepsilon)} \rceil$, let introduce \tilde{Y} the solution of the fixed point equation $\tilde{Y} = \mathbb{E}[\Phi^{k_0}](\tilde{Y})$. We know that:

$$\begin{aligned} \|\tilde{Y} - y_0\| &\leq \left\| \mathbb{E} \left[\Phi^{k_0}(\tilde{Y}) - \Phi^{k_0}(y_0) \right] \right\| + \|\mathbb{E}[\Phi^{k_0}(y_0)] - y_0\| \\ &\leq (1-\varepsilon)^{k_0} \|\tilde{Y} - y_0\| + \sum_{i=0}^{k_0-1} (1-\varepsilon)^i \|\Phi(y_0) - y_0\| \\ &\leq \frac{\sum_{i=0}^{k_0} (1-\varepsilon)^i \tau}{1 - (1-\varepsilon)^{k_0}} = \frac{\tau}{\varepsilon}, \end{aligned}$$

Therefore $\tilde{Y} \in \mathcal{B}(y_0, \frac{\tau}{\varepsilon})$ and we can besides deduce from the hypotheses of the theorem and Lemma C.8 that:

$$\Phi^{k_0}(\tilde{Y}) \propto \mathcal{E}_q \left(\sum_{i=0}^k (1-\varepsilon)^i \sigma \right).$$

We then see that the hypotheses of Proposition C.5 are satisfied and we can deduce the result of the theorem. \square

C.6 Concentration of minimizing solution Y to convex problems $\psi(X)(Y)$ with ψ deterministic and X concentrated

This study could be extended to a wide range and settings, but we now concentrate on a classical setting of Theorems C.4-C.6 where the randomness of ψ depends on a random vector $X \in F$ (for a normed vector space $(F, \|\cdot\|)$) and Y is the minimizing solution of:

$$\text{Minimize: } \psi(X)(Y)$$

for a given deterministic mapping $\psi : F \rightarrow \mathcal{F}(\mathbb{R}^p, \mathbb{R})$.

The issue is then, not only to show the concentration of Y but also the concentration of $(X, Y) \in F \times \mathbb{R}^p$ to be able to control the operations made on X and Y as needed in next section.

Corollary C.9. *Given a reflexive vector space $(F, \|\cdot\|_F)$, a random vector $X \in F$ satisfying $X \propto \mathcal{E}_q$ and a deterministic mapping $\psi : F \rightarrow \mathcal{F}(\mathbb{R}^p, \mathbb{R})$, we assume that for all X , that there exists a (sequence) of parameters $\kappa > 0$ such that $d^2(\psi(X)) \geq \kappa I_p$ (with the order relation on symmetric matrices). If we assume that there exists a (sequence of) vectors $y_0 \in \mathbb{R}^p$, such that:*

$$\delta \equiv \max \left(\sigma \|X\|, \|d\psi(X)|_{y_0}\| \right) \tag{C.4}$$

and the mapping $A \mapsto d(\psi(A))$ is $O(\sigma)$ -Lipschitz from $(F, \|\cdot\|_F)$ to⁵ $(\mathcal{F}(\mathbb{R}^p, \mathbb{R}^p), \|\cdot\|_{F, \mathcal{B}(y_0, \delta/\kappa)})$, then, $Y = \operatorname{argmin}_{y \in \mathbb{R}^p} \psi(X)(y)$ satisfies:

$$\left(\frac{\sigma}{\kappa} X, Y \right) \propto \mathcal{E}_2 \left(\frac{\sigma}{\kappa} \right).$$

Proof. We endow the vector space $E \equiv F \times \mathbb{R}^p$ with the norm $\|\cdot\|_{\ell^\infty}$ defined for any $(A, y) \in E$ as $\|(A, y)\|_{\ell^\infty} = \max(\|A\|_F, \|y\|)$. For any $K > 0$, we place ourselves on the event $\mathcal{A}_K \equiv \{\|d^2\psi(X)\|_{\mathcal{B}(y_0, \frac{\delta}{\kappa})} \leq K\}$. Then for any $B \in F$, we introduce the mapping:

$$\begin{aligned} \phi(B) : (F \times \mathbb{R}^p, \|\cdot\|_{\ell^\infty}) &\longrightarrow (F \times \mathbb{R}^p, \|\cdot\|_{\ell^\infty}) \\ (A, y) &\longmapsto \left(A + \frac{1}{K}(\sigma B - \kappa A), y - \frac{1}{K}d(\psi(B))|_{y_0} \right). \end{aligned}$$

⁵where for any $f \in \mathcal{F}(F, \mathbb{R})$, $\|f\|_{F, \mathcal{B}_{\mathbb{R}^p}(y_0, r)} = \sup\{\|f(y)\|_F, \|y - y_0\|_{\mathbb{R}^p} \leq r\}$

Noting $z_0 \equiv (0, y_0) \in F \times \mathbb{R}^p$, we can bound:

$$\|z_0 - \phi(X)(z_0)\| = \max \left(\frac{\sigma}{K} \|X\|, \frac{1}{K} \left\| d(\psi(X))|_{y_0} \right\| \right) = \frac{\delta}{K},$$

and of course, under \mathcal{A}_K , $\phi(X)$ is $(1 - \frac{\kappa}{K})$ -Lipschitz on $\mathcal{B}(z_0, \frac{\delta}{\kappa})$.

In addition, note that $B \mapsto \phi(B)$ is $O(\sigma/K)$ -Lipschitz from $(X(\mathcal{A}_K), \|\cdot\|_F)$ to $(\mathcal{F}(E, E), \|\cdot\|_{\mathcal{B}(z_0, \delta/\kappa)}^{\ell^\infty})$, therefore, it satisfies:

$$\phi(X) \mid \mathcal{A}_K \propto \mathcal{E}_q(\sigma/K) \text{ in } (\mathcal{F}(\mathbb{R}^p), \|\cdot\|_{\mathcal{B}(z_0, \delta/\kappa)}).$$

The random vector $(\frac{\sigma}{\kappa}X, Y)$ being the unique solution to the fixed point equation $z = \phi(X)(z)$, one can therefore employ Theorem C.6 to ϕ to set:

$$\left(\frac{\sigma}{\kappa}X, Y \right) \mid \mathcal{A}_K \propto \mathcal{E}_2 \left(\frac{\sigma}{\kappa} \right).$$

The concentration constants being independent of K , one can let K tend to infinity to get the expected result. □

Let us now try to replace from the hypotheses of Corollary C.10 assumption (C.4) by a weaker assumption.

Corollary C.10. *under the hypotheses of Corollary C.9, if one further assumes that $\sigma \equiv \frac{1}{\sqrt{p}}$, $\eta_{F, \|\cdot\|} \leq O(\sqrt{p})$ and replaces (C.4) with:*

$$\delta_0 \equiv \max \left(\frac{1}{\sqrt{p}} \mathbb{E}[\|X\|], \mathbb{E} \left[\left\| d\psi(X)|_{y_0} \right\| \right] \right) \leq O(1),$$

and that for all $\delta > \delta_0$, the mapping $A \mapsto d(\psi(A))$ is $O(\delta/\sqrt{p})$ -Lipschitz from $(F, \|\cdot\|_F)$ to $(\mathcal{F}(\mathbb{R}^p, \mathbb{R}^p), \|\cdot\|_{F, \mathcal{B}(y_0, \delta/\kappa)})$, then one obtain the concentration:

$$\left(\frac{1}{\sqrt{p}\kappa}X, Y \right) \propto \mathcal{E}_2 \left(\frac{1}{\sqrt{p}\kappa} \right) + \mathcal{E}_1 \left(\frac{1}{p\kappa} \right).$$

Proof. We are going to use the bound resulting from Proposition B.22:

$$\forall \delta > 2\delta_0 : \quad \mathbb{P} \left(\frac{1}{\sqrt{p}} \|X\| \geq \delta \right) \leq C e^{-c p \delta^2} \quad \text{and} \quad \mathbb{P} \left(\left\| d\psi(X)|_{y_0} \right\| \geq \delta \right) \leq C e^{-c p \delta^2},$$

for some constants $C, c > 0$. Furthermore, placing ourselves on the event:

$$\mathcal{A}_\delta \equiv \left\{ \max \left(\frac{1}{\sqrt{p}} \|X\|, \left\| d\psi(X)|_{y_0} \right\| \right) \leq \delta \right\},$$

for $\delta > 0$ big enough, we know from Lemma B.4, Remark B.5 and Corollary C.9 that $\left(\frac{1}{\kappa\sqrt{p}}X, Y \right) \mid \mathcal{A}_\delta \propto \mathcal{E}_2(\delta/\sqrt{p}\kappa)$. Therefore there exist two constants $C', c' > 0$ such that for any 1-Lipschitz mapping $f : F \times \mathbb{R}^p \rightarrow \mathbb{R}$:

$$\mathbb{P} \left(\left| f \left(\left(\frac{1}{\kappa\sqrt{p}}X, Y \right) \right) - \mathbb{E} \left[f \left(\left(\frac{1}{\kappa\sqrt{p}}X, Y \right) \right) \right] \right| \geq t \mid \mathcal{A}_\delta \right) \leq C' e^{-c' p \kappa^2 t^2 / \delta^2}$$

Then we can deduce the concentration distinguishing the cases $t \leq \frac{\delta_0^2}{\kappa}$ and $t \geq \frac{\delta_0^2}{\kappa}$ and choosing in this last case $\delta = \sqrt{t/\kappa} \geq \delta_0$ (see Lemma A.28 for more precision). □

Chapter D

Convex Concentration

D.1 Definition and fundamental examples

With a combinatorial approach, Talagrand showed in the nineties that it is possible to find a weaker notion of concentration to apprehend the concentration of partly discrete distributions. In these cases, to be concentrated the “observation” not only needs to be Lipschitz but also to be quasiconvex, in the sense of the following definition.

Definition 17 (Quasiconvexity). *A function $f : E \rightarrow \mathbb{R}$ is said to be quasiconvex if for any real $t \in \mathbb{R}$, the set $\{z \in E : f(z) \leq t\} = \{f \leq t\}$ is convex.*

Remark D.1. *Quasiconvexity¹ concerns of course convex functions, but also any monotonous function supported on \mathbb{R} . More generally, given a convex function f and a non decreasing function g , the composition $g \circ f$ is quasiconvex.*

Definition 18. *Given a sequence of normed vector spaces $(E_n, \|\cdot\|_n)_{n \geq 0}$, a sequence of random vectors $(Z_n)_{n \geq 0} \in \prod_{n \geq 0} E_n$, a sequence of positive reals $(\sigma_n)_{n \geq 0} \in \mathbb{R}_+^{\mathbb{N}}$, we say that $Z = (Z_n)_{n \geq 1}$ is convexly concentrated with an observable diameter of order $O(\sigma_n)$ iff there exist two positive constants $C, c > 0$ such that $\forall n \in \mathbb{N}$ and for any 1-Lipschitz and quasi-convex function $f : E_n \rightarrow \mathbb{R}$ (for the norms $\|\cdot\|_n$)²,*

$$\forall t > 0 : \quad \mathbb{P}(|f(Z_n) - \mathbb{E}[f(Z_n)]| \geq t) \leq C e^{-c(t/\sigma_n)^2},$$

We write in that case $Z_n \propto_c \mathcal{E}_2(\sigma_n)$ (or more simply $Z \propto_c \mathcal{E}_2(\sigma)$).

We clearly have the implication:

$$Z \propto \mathcal{E}_q(\sigma) \quad \implies \quad Z \propto_c \mathcal{E}_q(\sigma) \quad \implies \quad Z \in \mathbb{E}[Z] \pm \mathcal{E}_q(\sigma)$$

and when $E = \mathbb{R}$ everything is equivalent to Definition 2.

The fundamental example of convex concentration that fully justifies the introduction of this notion was given by Talagrand and can write:

Theorem D.2 ([Tal95]). *A (sequence of) random vector $Z \in [0, 1]^n$ with independent entries satisfies $Z \propto_c \mathcal{E}_2$.*

A recent very interesting result of Huang and Tikhomirov [HT21] extends this kind of inequalities for random vectors with independent and subgaussian entries³:

¹The class of quasiconvex functions is rather interesting in the sense that it is wider than the class of merely convex functions but still verifies the property of the uniqueness of the minimum.

²In this inequality, one could have replaced the term “ $\mathbb{E}[f(Z_n)]$ ” by “ $f(Z'_n)$ ” (with Z'_n , an independent copy of Z_n) or by “ m_f ” (with m_f a median of $f(Z_n)$). All those three definitions are equivalent.

³The result of [HT21] is given with $\|\cdot\|_{\psi_2}$ norms, but we prefer to use here a well known interpretation in terms of concentration inequalities (see for instance [Ver18, Proposition 2.5.2].)

Theorem D.3 ([HT21], Theorem 1.3.). *Given a random vector $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ with independent components satisfying for all $i \in [n]$ and $C > 0$, $X_i \propto 2e^{-(t/\sigma)^2}$, then there exists two constants C, c independent with n and such that for all 1-Lipschitz and convex mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}$:*

$$\forall t > 0, \mathbb{P}(|f(X) - m_f| \geq t) \leq C \exp\left(-\frac{c(t/\sigma)^2}{\log\left(2 + \frac{cK^2n}{t^2}\right)}\right).$$

One should also mention the contributions of [GRST17, GRS⁺18] that give very interesting characterization of a vector $Z = (X_1, \dots, X_n)$ with independent i.i.d. entries satisfying $Z \propto_c \mathcal{E}_2$. To be more precise, the sufficient and necessary conditions for Z to satisfy respectively $Z \propto \mathcal{E}_2$ and $Z \propto_c \mathcal{E}_2$ are both expressed in [GRST17] with two transport inequalities on the law of the Z_1, \dots, Z_n (of course the transport inequality linked to the convex concentration is weaker); [GRS⁺18] then provides some more interpretable characterization of those transport inequalities.

Convex concentration is not so easy to extend to other distribution because the convexity (or the quasiconvexity) of a function is only defined for real-valued functions; indeed, most of the transformations between two vector spaces ruin the subtle structure of convexity. We can still slightly relax the hypothesis of independence in the theorem of Talagrand thanks to affine transformations:

Lemma D.4. *Given two vector spaces E, F and a quasiconvex (resp., convex) function $f : E \rightarrow \mathbb{R}$, for any affine function $g : F \rightarrow E$, the composition $f \circ g$ is also quasiconvex (resp., convex).*

We finish this first approach to convex concentration with a adaptation of Lemma B.4 and Remark B.5 to the convex concentration case:

Lemma D.5 (Concentration of locally convex mappings). *Given a sequence of positive numbers $\sigma_p > 0$, a sequence of random vector $Z_p \in E$ satisfying $Z \propto \mathcal{E}_2(\sigma)$, for any sequence of mappings $f_p : Z_p \rightarrow \mathbb{R}$ quasi-convex and 1-Lipschitz on $Z_p(\Omega)$, we have the concentration $f_p(Z_p) \propto \mathcal{E}_q(\sigma_p)$.*

Besides, for any (sequence of) convex subsets $A \subset E$, if there exists a constant $K > 0$ such that $\mathbb{P}(Z \in A) \geq K$ then:

$$(Z|Z \in A) \propto_c \mathcal{E}_2(\sigma).$$

Proof. The proof is the same as the one provided in Lemma B.4 except that this time, one needs the additional argument that since $S = \{f \leq m_f\}$ (for m_f , a median of f) is convex, the mappings $z \mapsto d(z, S)$ and $z \mapsto -d(z, S)$ are both quasi-convex thanks to the triangular inequality. \square

D.2 Degeneracy of convex concentration through product

Given two convexly concentrated random vectors $X, Y \in E$ satisfying $X, Y \propto_c \mathcal{E}_2(\sigma)$, the convex concentration of the couple $(X, Y) \propto_c \mathcal{E}_2(\sigma)$ is ensured if:

1. X and Y are independent
2. $(X, Y) = u(Z)$ with u affine and $Z \propto_c \mathcal{E}_2(\sigma)$.

We can then in particular state the concentration of $X + Y$ as it is a linear transformation of (X, Y) . For the product it is not as simple as for the Lipschitz concentration, in particular it seems like one cannot maintain the convex concentration through product, we can however deduce linear concentration, this is what we call degeneracy of convex concentration through product. Let us start with the Hanson Wright Theorem that was already stated by Adamczak for random vectors $X, Y \in \mathbb{R}^p$, we however found interesting to provide here a short proof based on our results for which the generalization to random matrices is straightforward. The demonstration is quite similar to Proposition B.41 based on the bound on $\mathbb{E}[\|AXB\|_F]$ given by Lemma B.26, but since we are dealing with convex concentration one needs in addition to decompose the observation into convex functionals.

Proposition D.6 ([Ada15]). *Given two random matrices $X, Y \in \mathcal{M}_{p,n}$ such that $(X, Y) \propto_c \mathcal{E}_2$ and $\|\mathbb{E}[X]\|_F, \|\mathbb{E}[Y]\|_F \leq O(1)$, for any $A \in \mathcal{M}_p$:*

$$Y^T A X \in \mathcal{E}_2(\|A\|_F) + \mathcal{E}_1(\|A\|).$$

Proof. Considering a deterministic matrix $B \in \mathcal{M}_{p,n}$ such that $\|B\|_F \leq 1$, the proof consists mainly in decomposing the mapping $X, Y \mapsto \text{Tr}(BY^T AX)$ into a sum of convex mappings. We start with the decomposition $A = U_A^T \Lambda V_A$ and $B = U_B^T \Gamma V_B$ to write:

$$\text{Tr}(BY^T AX) = \text{Tr}(U_B^T \Gamma V_B Y^T U_A^T \Lambda V_A X).$$

Let us then introduce $\tilde{X} = V_A X U_B^T$ and $\tilde{Y} = U_A Y V_B^T$, we know that $\tilde{X}, \tilde{Y} \propto \mathcal{E}_2$. We further decompose $\Lambda = \Lambda_+ - \Lambda_-$ and $\Gamma = \Gamma_+ - \Gamma_-$ with $\Lambda_+, \Lambda_-, \Gamma_+, \Gamma_- \geq 0$, to be able to decompose $\text{Tr}(BY^T AX)$ in positive quadratic forms. The following decomposition allows us to assume, without restriction that $\Lambda, \Gamma \geq 0$:

$$\text{Tr}(BY^T AX) = \text{Tr}(\Gamma_+ \tilde{Y}^T \Lambda_+ \tilde{X}) - \text{Tr}(\Gamma_+ \tilde{Y}^T \Lambda_- \tilde{X}) - \text{Tr}(\Gamma_- \tilde{Y}^T \Lambda_- \tilde{X}) + \text{Tr}(\Gamma_- \tilde{Y}^T \Lambda_+ \tilde{X}).$$

Then assuming that $\Lambda, \Gamma \geq 0$, we can decompose:

$$\text{Tr}(\Gamma \tilde{Y}^T \Lambda \tilde{X}) = \frac{1}{4} (\text{Tr}(\Gamma(\tilde{Y} + \tilde{X})^T \Lambda_+(\tilde{Y} + \tilde{X})) - \text{Tr}(\Gamma(\tilde{Y} - \tilde{X})^T \Lambda_+(\tilde{Y} - \tilde{X}))),$$

We are then left to show the concentration of the quadratic form $\text{Tr}(\Gamma X^T \Lambda X)$ (convex on X) for two diagonal matrices $\Lambda \in \mathcal{D}_p(\mathbb{R}_+)$ and $\Gamma \in \mathcal{D}_n(\mathbb{R}_+)$ and a random matrix $X \in \mathcal{M}_{p,n}$ satisfying $X \propto_c \mathcal{E}_2$. Let us bound:

$$\text{Tr}(\Gamma X^T \Lambda X) \leq \|X\|_F \|\Lambda X \Gamma\|_F.$$

We know from Lemma B.26 that $\mathbb{E}[\|\Lambda X \Gamma\|_F] \leq O(\|\Lambda\|_F)$ and from the concentration of the norm and Lemma A.18 that if $\|\Lambda\| \geq 1$, for all $t \geq 2\mathbb{E}[\|\Lambda X \Gamma\|_F]$:

$$\mathbb{P}(\|\Lambda X \Gamma\|_F \geq t) \leq C e^{-c(t/\|\Lambda\|_F)^2},$$

for some constants $C, c > 0$. One can then show thanks to Lemma A.28 the concentration:

$$\text{Tr}(\Gamma X^T \Lambda X) \propto \mathcal{E}_2(\|\Lambda\|_F) + \mathcal{E}_1.$$

We deduce the final result easily as in the proof of Proposition B.41 normalizing in the general case $\text{Tr}(\Gamma X^T \Lambda X)$ with $\|\Lambda\|$. \square

let us first consider the particular case of the entry-wise product in $E = \mathbb{R}^p$.

Theorem D.7. *Given a (sequence of) integer $m \in \mathbb{N}$ and a (sequence of) positive number $\sigma > 0$ such that $m \leq O(p)$, a (sequence of) m random vectors $X_1, \dots, X_m \in \mathbb{R}^p$, if we suppose that*

$$X \equiv (X_1, \dots, X_m) \propto_c \mathcal{E}_2(\sigma) \text{ in } ((\mathbb{R}^p)^m, \|\cdot\|_{\ell^\infty}),$$

(with the notation $\|\cdot\|_{\ell^\infty}$ defined in (B.5)) and that there exists a (sequence of) positive numbers $\kappa > 0$ such that $\forall i \in [m] : \|X_i\|_\infty \leq \kappa$, then:

$$X_1 \odot \dots \odot X_m \in \mathcal{E}_2((2e\kappa)^{m-1}\sigma) \text{ in } (\mathbb{R}^p, \|\cdot\|).$$

And if $X_1 = \dots = X_m = X$, the constant $2e$ is no more needed and we get the concentration $X^{\odot m} \in \mathcal{E}_2(\kappa^{m-1}\sigma)$.

Remark D.8. *If we replace the strong assumption $\forall i \in [m] : \|X_i\|_\infty \leq \kappa$, with the bound $\sup_{1 \leq i \leq m} \|\mathbb{E}[X_i]\|_\infty \leq O((\log p)^{1/q})$ we can still deduce a similar result to Example B.38, stating the existence of a constant $\kappa \leq O(1)$ such that:*

$$X_1 \odot \dots \odot X_m \in \mathcal{E}_2\left((\kappa\sigma)^m (\log(p))^{(m-1)/q}\right) + \mathcal{E}_{q/m}((\kappa\sigma)^m) \text{ in } (\mathbb{R}^p, \|\cdot\|).$$

We will use several time the following elementary result:

Lemma D.9. *Given a convex mapping $f : \mathbb{R} \rightarrow \mathbb{R}$, and a vector $a \in \mathbb{R}_+^p$, the mapping $F : \mathbb{R}^p \ni (z_1, \dots, z_p) \mapsto \sum_{i=1}^p a_i f(z_i) \in \mathbb{R}$ is convex (so in particular quasi-convex).*

To efficiently manage the concentration rate when multiplying a large number of random vectors, we will also need:

Lemma D.10. *Given m commutative or non commutative variables a_1, \dots, a_m of a given algebra, we have the identity:*

$$\sum_{\sigma \in \mathfrak{S}_m} a_{\sigma(1)} \cdots a_{\sigma(m)} = (-1)^m \sum_{I \subset [m]} (-1)^{|I|} \left(\sum_{i \in I} a_i \right)^m,$$

where $|I|$ is the cardinality of I .

Proof. The idea is to inverse the identity:

$$(a_1 + \cdots + a_m)^m = \sum_{J \subset I} \sum_{\{i_1, \dots, i_m\} = J} a_{i_1} \cdots a_{i_m},$$

thanks to the Rota formula (see [Rol06]) that sets for any mappings f, g defined on the set subsets of \mathbb{N} and having values in a commutative group (for the sum):

$$\forall I \subset \mathbb{N}, f(I) = \sum_{J \subset I} g(J) \quad \iff \quad \forall I \subset \mathbb{N}, g(I) = \sum_{J \subset I} \mu_{\mathcal{P}(\mathbb{N})}(J, I) f(J),$$

where $\mu_{\mathcal{P}(\mathbb{N})}(J, I) = (-1)^{|I \setminus J|}$ is an analog of the Moëbus function for the order relation induced by the inclusions in $\mathcal{P}(\mathbb{N})$. In our case, for any $J \subset [m]$, if we set:

$$f(J) = \left(\sum_{i \in J} a_i \right)^m \quad \text{and} \quad g(J) = \sum_{\{i_1, \dots, i_m\} = J} a_{i_1} \cdots a_{i_m},$$

we see that for any $I \subset [m]$, $f(I) = \sum_{J \subset I} g(J)$, therefore taking the Rota formula in the case $I = [m]$, we obtain the result of the Lemma (in that case, $\mu_{\mathcal{P}(\mathbb{N})}(J, I) = (-1)^{m-|J|}$ and $\sum_{\{i_1, \dots, i_m\} = I} a_{i_1} \cdots a_{i_m} = \sum_{\sigma \in \mathfrak{S}_m} a_{\sigma(1)} \cdots a_{\sigma(m)}$). \square

Proof of Theorem D.7. Let us first assume that all the X_i are equal to a vector $Z \in \mathbb{R}^p$. Considering $a = (a_1, \dots, a_p) \in \mathbb{R}^p$, we want to show the concentration of $a^T Z^{\odot m} = \sum_{i=1}^p a_i z_i^m$ where z_1, \dots, z_p are the entries of Z .

The mapping $p_m : x \mapsto x^m$ is not quasi-convex when m is odd, therefore, in that case we decompose it into the difference of two convex mappings $p_m(z) = p_m^+(z) - p_m^-(z)$ where:

$$p_m^+ : z \mapsto \max(z^m, 0) \quad \text{and} \quad p_m^- : z \mapsto -\min(z^m, 0), \quad (\text{D.1})$$

(say that, if m is even, then we set $p_m^+ = p_m$ and $p_m^- : z \mapsto 0$). For the same reasons, we decompose $\phi_a^+ : z \mapsto a^T p_m^+(z)$ and $\phi_a^- : z \mapsto a^T p_m^-(z)$ into:

$$\phi_a^+ = \phi_{|a|}^+ - \phi_{|a|-a}^+ \quad \text{and} \quad \phi_a^- = \phi_{|a|}^- - \phi_{|a|-a}^-$$

(for $|a| = (|a_i|)_{1 \leq i \leq p}$), so that:

$$a^T Z^{\odot m} = \phi_{|a|}^+(Z) - \phi_{|a|-a}^+(Z) - \phi_{|a|}^-(Z) + \phi_{|a|-a}^-(Z)$$

becomes a combination of quasi-convex functionals of Z . We now need to measure their Lipschitz parameter. Let us bound for any $z \in \mathbb{R}^p$:

$$\left| \phi_{|a|}^+(z) \right| = \sum_{i=1}^n |a_i| |z_i|^m \leq \|a\| \|z\| \|z\|_{\infty}^{m-1},$$

and the same holds for $\phi_{|a|-a}^+$, $\phi_{|a|}^-$ and $\phi_{|a|-a}^-$. Note then that $\phi_{|a|}^+$, $\phi_{|a|-a}^+$, $\phi_{|a|}^-$ and $\phi_{|a|-a}^-$ are all $\|a\| \kappa^{m-1}$ -Lipschitz to conclude on the concentration of $X^{\odot m}$.

Now, if we assume that the X_1, \dots, X_m are different, we employ Lemma D.10 in this commutative case to write ($|\mathfrak{S}_m| = m!$):

$$(X_1 \odot \dots \odot X_m) = \frac{(-1)^m}{m!} \sum_{I \subset [m]} (-1)^{|I|} \left(\sum_{i \in I} X_i \right)^{\odot m}. \quad (\text{D.2})$$

Therefore, the sum $(\mathbb{R}^p)^I \ni z_1, \dots, z_{|I|} \mapsto \sum_{i \in I} z_i \in \mathbb{R}^p$ being m -Lipschitz for the norm $\|\cdot\|_\infty$, we know that $\forall I \subset [m]$, $\sum_{i \in I} X_i \prec_c \mathcal{E}_2(m\sigma)$, and $\|\sum_{i \in I} X_i\|_\infty \leq \kappa m$, therefore, $(\sum_{i \in I} X_i)^{\odot m} \in \mathcal{E}_2(m^m \kappa^{m-1} \sigma)$. We can then exploit Proposition B.27 to obtain

$$\left(\left(\sum_{i \in I} X_i \right)^{\odot m} \right)_{I \subset [m]} \in \mathcal{E}_2(m^m \kappa^{m-1} \sigma) \quad \text{in } \left((\mathbb{R}^p)^{2^m}, \|\cdot\|_{\ell^\infty} \right),$$

(note that $\#\{I \subset [m]\} = 2^m$) Thus summing the 2^m concentration inequalities, we can conclude from Equation (D.2), and the Stirling formula $\frac{m^m}{m!} = \frac{e^m}{\sqrt{2\pi m}} + O(1)$ that:

$$(X_1 \odot \dots \odot X_m) \in \mathcal{E}_2((2e\kappa)^{m-1} \sigma).$$

□

To set the concentration of the product of matrices, it is interesting to introduce the notion of transversal concentration that will help us to simplify the approach to convexity by reducing the set of concentration with equivalence relations.

D.3 Transversal Convex concentration

Let us introduce the group $\mathcal{O}_{p,n} = \mathcal{O}_p \times \mathcal{O}_n$ where \mathcal{O}_m , $m \in \mathbb{N}$, is the orthogonal group of matrices of \mathcal{M}_m ; it acts on $\mathcal{M}_{p,n}$ following the formula:

$$\text{for } (U, V) \in \mathcal{O}_{p,n}, M \in \mathcal{M}_{p,n} : (U, V) \cdot M = U M V^T.$$

The function $f : R \in \mathcal{M}_{p,n} \mapsto \text{Tr } \phi(R)$ is $\mathcal{O}_{p,n}$ -invariant, in the sense that $\forall (U, V) \in \mathcal{O}_{p,n}, \forall R \in \mathcal{M}_{p,n}$, $f((U, V) \cdot R) = f(R)$. A result originally owed to Chandler Davis in [Dav57], and that can be found in a more general setting in [GH05], gives us the hint that such an invariance can help us showing that f is quasiconvex.

To present this theorem, we note $\mathcal{D}_{p,n}^+$ the set of nonnegative diagonal matrices of $\mathcal{M}_{p,n}$:

$$\mathcal{D}_{p,n}^+ = \left\{ (M_{i,j})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq n}} \in \mathcal{M}_{p,n} \mid i \neq j \Leftrightarrow M_{i,j} = 0 \text{ and } \forall i \in \{1, \dots, d\} : M_{i,i} \geq 0 \right\},$$

where $d = \min(p, n)$.

Theorem D.11. *If a $\mathcal{O}_{p,n}$ -invariant function $f : \mathcal{M}_{p,n} \rightarrow \mathbb{R}$ is quasiconvex on $\mathcal{D}_{p,n}^+$, then it is quasiconvex on the whole set $\mathcal{M}_{p,n}$.*

The original Davis' theorem is only concerned with symmetric matrices and we could not find any proof of this theorem for the case of rectangle matrices although it is not so different (it was also aiming at proving the *convexity* of f and not its *quasiconvexity* – but it is actually simpler to treat). We provide here a rigorous proof of Theorem D.11 with the help of some results borrowed from [Bha97]. Let us first present basic notions to set the theorem. Given a vectorial space E and a group G acting on E , for any subset $A \subset E$, we note $G \cdot A = \{g \cdot a, g \in G, a \in A\}$. We say that a set T is *transversal* if $G \cdot T = E$ and we say that a function f is G -invariant if $\forall x \in E, \forall g \in G, f(g \cdot x) = f(x)$. In the same vein, we say that a set $A \subset E$ is G -invariant if $G \cdot A = A$ and that it is G -invariant in T if $A \subset T$ and $G \cdot A \cap T = A$. Given $U \subset T$, we note $L_T^G(U)$ the smallest convex subset of T containing U and G -invariant in T . We give here an adaptation of one of the result of Grabovsky and Hijab to the case of quasiconvex functions.

Theorem D.12 (cf. [GH05], Theorem 4). *Let us consider a vector space E , a group G acting on E , and a convex and transversal subset $T \subset E$. We suppose that for any $U \subset T$, the set $G \cdot L_T^G(U)$ is convex; we call this property the convexity conservation of G from T . Then any G -invariant function f is quasiconvex iff its restriction to T is quasiconvex.*

Intuitively, this theorem states that the quasiconvexity (or the mere convexity as in [GH05]) of any function G -invariant is “transversal” to the action of G when G preserves the convexity from T . A curious reader might be interested in simplifying the convexity conservation property as we presented it from a transversal subset $T \subset E$ to a mere conservation of the convexity of any convex subset $U \subset E$ (i.e., for any convex set $U \subset E$, $f(U)$ is convex). This would be indeed an hypothesis more than sufficient for the result of the theorem. However, in practice, and in particular for the applications we want to consider, it cannot be verified.

Proof. Let us note $f|_T$ the restriction of f on T . Given any $t \in \mathbb{R}$, we know that the set $\{f|_T \leq t\} = \{x \in T, f(x) \leq t\}$ is convex. Then the set $\{f \leq t\} = G \cdot \{f|_T \leq t\} = G \cdot L_T^G(\{f|_T \leq t\})$ is also convex thanks to the convexity conservation of G from T . \square

To simplify the application of Theorem D.12, Grabowsky and Hijab provide us with a useful property:

Proposition D.13 (Convexity conservation, [GH05], Theorem 3). *With the notations of Theorem D.12, if for any $x, y \in T$, the set $G \cdot L_T^G(\{x, y\})$ is convex, then G conserves the convexity from T .*

Proof. Let us consider $U \subset T$, and two points $x, y \in G \cdot L_T^G(U)$. There exists $x^*, y^* \in T$ such that $x, y \in G \cdot \{x^*, y^*\}$. We know by hypothesis that the set $G \cdot L_T^G(\{x^*, y^*\})$ is convex, and moreover, it contains x and y . Therefore, any element of the segment $[x, y]$ is also in $L_T^G(\{x^*, y^*\}) \subset L_T^G(U)$. \square

In the case of a matrix concentration, Theorem D.12 can be applied to the transversal set of nonnegative diagonal matrices $\mathcal{D}_{p,n}^+$ for the action of the group $\mathcal{O}_{p,n}$. The transversal character of $\mathcal{D}_{p,n}^+$ is a consequence of the singular value decomposition. Indeed, for any matrix $M \in \mathcal{M}_{p,n}$, there exists $(U, V) \in \mathcal{O}_{p,n}$ such that

$$M = U\Sigma V^T \quad \text{with } \Sigma = \text{Diag}_{p,n}(\sigma_i(M))_{1 \leq i \leq d},$$

where $d = \min(p, n)$, $\sigma_i(M)$ is the i^{th} singular value of M and the notation $\text{Diag}_{p,n}(a_i)$ represents an element of $\mathcal{D}_{p,n}^+$ having the values a_i on the diagonal.

To prove Theorem D.11, let us characterize the sets $L(X) = L_{\mathcal{D}_{p,n}^+}^{\mathcal{O}_{p,n}}(\{X\})$ when $X \in \mathcal{D}_{p,n}^+$. We know that $\mathcal{D}_{p,n}^+$ is invariant under the action of the subgroup of permutations $\mathcal{P}_{p,n}$. Given a subset U of a vector space, we note $\text{Conv}(U)$ the convex hull of U .

Proposition D.14. *Given $X \in \mathcal{D}_{p,n}$, $L(X) = \text{Conv}(\mathcal{P}_{p,n} \cdot \{X\})$.*

Proof. We know from the uniqueness of the singular value decomposition that for any $U \subset \mathcal{D}_{p,n}^+$, $(\mathcal{O}_{p,n} \cdot U) \cap \mathcal{D}_{p,n}^+ = \mathcal{P}_{p,n} \cdot U$. Consequently, since the convexity is stable under the action of $\mathcal{P}_{p,n}$ (they are linear transformations), $L(X) = \text{Conv}(\mathcal{P}_{p,n} \cdot \{X\})$. \square

Here the tools of *majorization* as presented for instance in [Bha97] are perfectly adapted to the description of $\text{Conv}(\mathcal{S}_{p,n} \cdot \{x\})$ that we identify with $\text{Conv}(\mathcal{P}_{p,n} \cdot \{X\}) = L(X)$. Given a vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let us note $x^\downarrow = (x_1^\downarrow, \dots, x_d^\downarrow)$, a decreasing ordered version of x ($x_1^\downarrow \geq \dots \geq x_d^\downarrow$ and $\exists \sigma \in \mathcal{S}_d \mid x^\downarrow = \sigma \cdot x$).

Definition 19. *Given two vectors $x, y \in \mathbb{R}^d$, $d \in \mathbb{N}$, we say that y is majorized by x and we note $y \prec x$ iff :*

$$\forall k \in \{1, \dots, d\} : \sum_{i=1}^k y_i^\downarrow \leq \sum_{i=1}^k x_i^\downarrow \quad \text{and} \quad \sum_{i=1}^d x_i = \sum_{i=1}^d y_i.$$

Majorization offers a complete characterization of $\text{Conv}(\mathcal{S}_d \cdot \{x\})$, $x \in \mathbb{R}^d$:

Theorem D.15 ([Bha97], Theorem II.1.10). *Given a vectors $x \in \mathbb{R}^d$:*

$$\{y \in \mathbb{R}^d, y \prec x\} = \text{Conv}(\mathcal{S}_d \cdot \{x\}).$$

Majorization appears to be the perfect tool to control the singular decomposition of a sum of matrices as we will see in Theorem D.17. This is the core argument to justify the convexity conservation of $\mathcal{O}_{p,n}$ from $\mathcal{D}_{p,n}^+$ that we need to prove Theorem D.11. Let us first give an intermediate result that we originally owe to Schur and whose proof can be found in [MOA11] or [Bha97].

Proposition D.16 (Schur's Theorem, B.1. in [MOA11]). *Given a symmetric matrix $S \in \mathcal{M}_p$, we have the majorization $\text{Diag}(S) \prec \sigma(S)$.*

This proposition entails a kind of triangular inequality for the set of singular values.

Theorem D.17 ([Bha97], Exercise II.1.15). *Given two matrices $A, B \in \mathcal{M}_{p,n}$, $\sigma(A+B) \prec \sigma(A) + \sigma(B)$.*

Of course, it is important that the vectors $\sigma(A)$ and $\sigma(B)$ are both ordered when we sum them.

Proof. Given a symmetric matrix $S \in \mathcal{M}_q$, there exists $(U_S, V_S) \in \mathcal{O}_{p,n}$ such that $U_S A V_S^T = \text{Diag}_{p,n} \sigma(S)$, thus:

$$\sum^k \text{Diag}(U_S S V_S^T) = \sum^k \sigma(S),$$

where given $x \in \mathbb{R}^d$ and $k \in \{1, \dots, d\}$, $\sum^k x = \sum_{i=1}^k x_i^\downarrow$. Besides, since $\sigma(USV^T) = \sigma(S)$, we know from Proposition D.16 that:

$$\sum^k \sigma(S) = \sup_{(U,V) \in \mathcal{O}_{p,n}} \sum^k \text{Diag}(USV^T). \quad (\text{D.3})$$

Now, if we suppose that we are given a general matrix $A \in \mathcal{M}_{p,n}$ and $(U_A, V_A) \in \mathcal{O}_{p,n}$ such that $U_A A V_A^T = \text{Diag}_{p,n} \sigma(A)$. If we introduce the matrices:

$$\tilde{A} = \begin{pmatrix} (0) & A \\ A^T & (0) \end{pmatrix} \in \mathcal{M}_{p+n}, \quad \text{and} \quad P = \begin{pmatrix} U & (0) \\ (0) & V \end{pmatrix} \in \mathcal{M}_{p+n},$$

we have the identity:

$$P \tilde{A} P^T = \begin{pmatrix} (0) & D \\ D & (0) \end{pmatrix} \in \mathcal{M}_{p+n}, \quad \text{with } D = \text{Diag}_{p,n}(\sigma(A)).$$

Depending on the relation between p and n , we introduce the invertible matrices:

$$\text{if } d = n : Q = \begin{pmatrix} I_d & (0) & I_d \\ (0) & I_d & (0) \\ -I_d & (0) & I_d \end{pmatrix} \quad \text{and if } d = p : Q = \begin{pmatrix} I_d & I_d & (0) \\ -I_d & I_d & (0) \\ (0) & (0) & I_d \end{pmatrix},$$

then if $d = n$, $QP\tilde{A}(PQ)^T = \text{Diag}(\sigma(A), 0 \cdots 0, -\sigma(A))$ and if $d = p$, $QP\tilde{A}(PQ)^T = \text{Diag}(\sigma(A), -\sigma(A), 0 \cdots 0)$. Thus in both cases, we obtain a diagonalisation of \tilde{A} that allows us to generalize the identity (D.3) for any matrix $A \in \mathcal{M}_{p,n}$ and with $0 \leq k \leq d$. The supremum of a sum being lower than the sum of a supremum, for any pair of matrices $A, B \in \mathcal{M}_{p,n}$:

$$\sigma(A+B) \prec \sigma(A) + \sigma(B).$$

□

Now that the picture is clearer, we can prove Theorem D.11:

Proof of Theorem D.11. To employ Theorem D.12, let us show the convexity conservation property of $\mathcal{O}_{p,n}$ from $\mathcal{D}_{p,n}^+$. Inspired by Proposition D.13, we consider two non-negative diagonal matrices $X, Y \in \mathcal{D}_{p,n}^+$, and we note $L(X, Y) = L_{\mathcal{O}_{p,n}}^{\mathcal{D}_{p,n}^+}(\{X, Y\})$. We want to show that $\mathcal{O}_{p,n} \cdot L(X, Y)$ is convex.

Noting $x = \text{Diag } X$ and $y = \text{Diag } Y$, let us first show that $L(X, Y) = K(x, y)$, with:

$$K(x, y) = \{Z \in \mathcal{D}_{p,n}^+, \text{Diag } Z \prec \lambda x + (1 - \lambda)y, 1 \leq \lambda \leq 1\}.$$

We know from Theorem D.15 that for any $U \in \mathcal{D}_{p,n}^+$, $L(U) = \{\text{Diag } Z \prec \text{Diag } U\}$ and therefore, since for any $t \in [0, 1]$, $tX + (1 - t)Y \in L(X, Y)$, we obtain the first inclusion $K(x, y) \subset L(X, Y)$.

To prove the converse inclusion, let us show that $K(x, y)$ is convex (we already know that $\mathcal{O}_{p,n} \cdot K(x, y) \cap \mathcal{D}_{p,n}^+ = \mathcal{P}_{p,n} \cdot K(x, y) = K(x, y)$ by definition of the relation \prec).

We consider $A, B \in K(x, y)$, $t \in [0, 1]$ and we set $C = tA + (1 - t)B$. Therefore, We know that there exists $t_z, t_w \in [0, 1]$ such that $\text{Diag } A \prec \lambda x + (1 - \lambda)y$ and $\text{Diag } B \prec \mu x + (1 - \mu)y$, therefore :

$$\text{Diag } C \prec (t\lambda + (1 - t)\mu) x + (t(1 - \lambda) + (1 - t)(1 - \mu)) y \in [x, y].$$

In conclusion, since $X, Y \in K(x, y)$ and $K(x, y)$ is $\mathcal{P}_{p,n}$ -invariant and convex we recover the second inclusion $K(x, y) \subset L(X, Y)$.

Thus we are left to show that $\mathcal{O}_{p,n} \cdot K(x, y)$ is convex. We consider this time $A, B \in \mathcal{O}_{p,n} \cdot K(x, y)$, $t \in [0, 1]$, and we introduce $C = tA + (1 - t)B$. We know from Theorem D.17 that:

$$\sigma(C) \prec t\sigma(A) + (1 - t)\sigma(B),$$

and as we saw before, that implies that $\text{Diag } \sigma(C) \in K(x, y)$. We can then conclude with the relations

$$C \in \mathcal{O}_{p,n} \cdot \{\text{Diag } \sigma(C)\} \subset \mathcal{O}_{p,n} \cdot K(x, y) = \mathcal{O}_{p,n} \cdot L(X, Y).$$

We can apply Proposition D.13 to get the hypothesis of Theorem D.12 that entails Theorem D.11 in our setting. \square

Let us define the subgroup of row permutations $\mathcal{P}_p = \{U \in \mathcal{O}_p \mid U_{i,j} \in \{0, 1\}, 1 \leq i, j \leq p\}$ and the subgroup of full permutations:

$$\mathcal{P}_{p,n} = \{(U, V) \in \mathcal{P}_p \times \mathcal{P}_n \mid UI_{p,n}V^T = I_{p,n}\} \subset \mathcal{O}_{p,n},$$

where $I_{p,n} \in \mathcal{D}_{p,n}$ is a matrix full of ones on the diagonal. Given a matrix $A \in \mathcal{M}_{p,n}$, we note $\text{Diag}(A) = (A_{1,1}, \dots, A_{d,d})$, the vector composed of its diagonal terms (recall that $q = \min(p, n)$). It is tempting to identify the set $\mathcal{D}_{p,n}^+$ with \mathbb{R}_+^d , and the actions of $\mathcal{P}_{p,n}$ on a diagonal matrix A to the actions of the group of permutation \mathfrak{S}_d on the vector $\text{Diag}(A)$, where we define the action of \mathfrak{S}_d on \mathbb{R}^d as:

$$\forall \tau \in \mathfrak{S}_d, \forall x \in \mathbb{R}^d : \tau \cdot x = (x_{\tau(1)}, \dots, x_{\tau(d)}).$$

With these considerations in mind, we see that a direct interesting consequence of Theorem D.11 is that there exists a link between the convex concentration of a matrix X and the convex concentration of the vector of its singular values. Recall that the sequence of singular values verifies $\sigma_1(A) \geq \dots \geq \sigma_d(A) \geq 0$ and for $1 \leq i \leq d$, the i^{th} singular value of M can be defined as:

$$\sigma_i(A) = \max_{\substack{F \subset \mathbb{R}^n \\ \dim F \geq i}} \min_{\substack{x \in F \\ \|x\|=1}} \|Mx\| = \min_{\substack{F \subset \mathbb{R}^n \\ \dim F \geq n-i+1}} \max_{\substack{x \in F \\ \|x\|=1}} \|Mx\|, \quad (\text{D.4})$$

where the subsets F of \mathbb{R}^d on which is computed the optimization are subspaces of \mathbb{R}^n . We introduce the convenient function σ mapping a matrix to the ordered sequence of its singular values:

$$\begin{aligned} \sigma : \mathcal{M}_{p,n} &\rightarrow \mathbb{R}_+^d \\ M &\mapsto (\sigma_1(M), \dots, \sigma_d(M)). \end{aligned}$$

To formalize this transfer of concentration between a matrix X and $\sigma(X)$ let us introduce a new notion of concentration in a vector space E : the *convex concentration transversally to the action of a group G acting on E* .

Definition 20. Given a normal vector space E , a group G acting on E , a concentration function α and a random vector $Z \in E$, we say that Z is convexly α -concentrated transversally to the action of G and we note $Z \propto_G^T \alpha$ iff for any 1-Lipschitz, quasiconvex and G -invariant function $f : E \rightarrow \mathbb{R}$, $f(Z)$ is α concentrated.

Remark D.18. In the setting of Definition 20, we have the induction chain:

$$Z \propto \alpha \implies Z \propto_c \alpha \implies Z \propto_G^T \alpha.$$

Definition 20 is perfectly adapted to the next theorem which is very similar to [Led05, Corollary 8.21] concerning the eigenvalues of a random symmetric matrix).

Theorem D.19. Given a normal vector space E , a concentration function α and a random matrix $X \in \mathcal{M}_{p,n}$, we have the equivalence:

$$X \propto_{\mathcal{O}_{p,n}}^T \alpha \iff \sigma(X) \propto_{\mathfrak{S}_d}^T \alpha,$$

where $d = \min(p, n)$.

Let us first show the Lipschitz character of σ , it is a well known result that can be found for instance in [GL96]:

Lemma D.20 (Theorem 8.1.15 in [GL96]). *The function σ is 1-Lipschitz.*

Proof. Given $M, H \in \mathcal{M}_{p,n}$, and $i \in \{1, \dots, d\}$ (where as before, $d = \min(p, n)$), we know from formula (D.4) that:

$$\begin{aligned} \lambda_i(A + H) &= \min_{\substack{F \subset \mathbb{R}^n \\ \dim F \geq n-i+1}} \max_{\substack{x \in F \\ \|x\|=1}} \|(A + H)x\| \\ &\leq \min_{\substack{F \subset \mathbb{R}^n \\ \dim F \geq n-i+1}} \left(\max_{\substack{x \in F \\ \|x\|=1}} \|Ax\| + \max_{\substack{x \in F \\ \|x\|=1}} \|Hx\| \right) \\ &\leq \min_{\substack{F \subset \mathbb{R}^n \\ \dim F \geq n-i+1}} \max_{\substack{x \in F \\ \|x\|=1}} \|Ax\| + \max_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \|Hx\| \leq \lambda_i(A) + \lambda_1(H), \end{aligned}$$

and the same way, we can show that $\lambda_i(A + H) \geq \lambda_i(A) - \lambda_n(H)$. Therefore, we get:

$$|\lambda_i(A + H) - \lambda_i(A)| \leq \max(\lambda_1(H), \lambda_n(H)) \leq \|H\|.$$

□

Proof of Theorem D.19. It is a simple corollary of Theorem D.11. If $X \propto_{\mathcal{O}_{p,n}}^T \alpha$, and given a 1-Lipschitz, convex and \mathfrak{S}_n -invariant function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, one can introduce the function \tilde{F} defined as:

$$\begin{aligned} \tilde{F} : \mathcal{M}_{p,n} &\longrightarrow \mathbb{R} \\ M &\longmapsto f(\sigma(M)). \end{aligned}$$

The function F is 1-Lipschitz thanks to Lemma D.20 and $\mathcal{O}_{p,n}$ -invariant because of the uniqueness of the singular decomposition. Besides we can identify the set $\mathcal{D}_{p,n}^+$ with \mathbb{R}^d and introduce a function $\tilde{f} : \mathcal{D}_{p,n}^+ \rightarrow \mathbb{R}$ verifying $\tilde{f}(\text{diag}_{p,n}(x)) = f(x)$ for $x \in \mathbb{R}^d$. In that case, since f is \mathfrak{S}_d -invariant and convex, \tilde{f} is also convex and since $\tilde{f} = F|_{\mathcal{D}_{p,n}^+}$, Theorem D.11 allows us to set that F is also convex. Therefore, the random variable $f(\sigma(X)) = F(X)$ is α -concentrated by hypothesis on X .

Reciprocally, let us suppose that we are given a random matrix $X \in \mathcal{M}_{p,n}$ such that $\sigma(X) \propto_{\mathfrak{S}_d}^T \alpha$, and let us consider a 1-Lipschitz, convex and $\mathcal{O}_{p,n}$ -invariant function $F : \mathcal{M}_{p,n} \rightarrow \mathbb{R}$. The restriction $F|_{\mathcal{D}_{p,n}^+}$ is also 1 Lipschitz, convex and $\mathcal{P}_{p,n}$ -invariant. Thus, with the same identification as before between $\mathcal{D}_{p,n}^+$ and \mathbb{R}^d , we can assert by hypothesis that $F(X) = \tilde{F}|_{\mathcal{D}_{p,n}^+}(\sigma(X))$ is α -concentrated (we defined $\tilde{F}|_{\mathcal{D}_{p,n}^+}(x) = F|_{\mathcal{D}_{p,n}^+}(\text{diag}_{p,n}(x))$, it is a \mathfrak{S}_d -invariant function). □

Theorem D.19 can be rather powerful to set the concentration of symmetric functionals of random singular values and playing with this symmetry we can show interesting properties of concentration on products.

The result of concentration of a product of matrices convexly concentrated was already proven in [MS11] but since their formulation is slightly different, we give a new proof below with the formulation required for the study of the resolvent.

Theorem D.21 ([MS11], Theorem 1). *Let us consider three sequences $m \in \mathbb{N}^{\mathbb{N}}$ and $\sigma, \kappa \in \mathbb{R}_+^{\mathbb{N}}$, and a sequence of m random matrices $X_1 \in \mathcal{M}_{n_0, n_1}, \dots, X_m \in \mathcal{M}_{n_{m-1}, n_m}$, satisfying⁴:*

$$(X_1, \dots, X_m) \propto_c \mathcal{E}_2(\sigma) \text{ in } (\mathcal{M}_{n_0, n_1} \times \dots \times \mathcal{M}_{n_{m-1}, n_m}, \|\cdot\|_F),$$

In the particular case where $X_1 = \dots = X_n \equiv X$, it is sufficient⁵ to assume that $X \propto_c \mathcal{E}_2(\sigma)$ in $(\mathcal{M}_n, \|\cdot\|_F)$. If there exist a sequence of positive values $\kappa > 0$ such that $\forall i \in [m], \|X_i\| \leq \kappa$, then the product is concentrated for the nuclear norm:

$$X_1 \cdots X_m \in \mathcal{E}_2(\kappa^{m-1} \sigma \sqrt{n_0 + \dots + n_m}) \quad \text{in } (\mathcal{M}_{n_0, n_m}, \|\cdot\|_*),$$

where, for any $M \in \mathcal{M}_{n_0, n_m}$, $\|M\|_* = \text{Tr}(\sqrt{MM^T})$ (it is the dual norm of the spectral norm).

Remark D.22. *The hypothesis $\|X\| \leq \kappa$ might look quite strong, however in classical settings where $X \propto \mathcal{E}_2$ and $\|\mathbb{E}[X]\| \leq O(\sqrt{n})$ it has been shown that there exist three constants $C, c, K > 0$ such that $\mathbb{P}(\|X\| \geq K\sqrt{n}) \leq Ce^{-cn}$. Placing ourselves on the event $\mathcal{A} = \{\|X\| \leq K\sqrt{n}\}$, we can then show from Lemma D.5 below that:*

$$((X/\sqrt{n})^m \mid \mathcal{A}) \in \mathcal{E}_2(K^{m-1}/\sqrt{n}) \quad \text{and} \quad \mathbb{P}(\mathcal{A}^c) \leq Ce^{-cn},$$

(here $\sigma = 1/\sqrt{n}$ and $\kappa = K$). The same inferences hold for the concentration of $(XX^T/(n+p))^m$.

Proof of Theorem D.21. Let us start to study the case where $X_1 = \dots = X_m \equiv X \in \mathcal{M}_n$ and $X \propto \mathcal{E}_2$ in $(\mathcal{M}_n, \|\cdot\|_F)$. We know from Theorem D.19 that:

$$\sigma(X) \propto_{\mathfrak{S}_p}^T \mathcal{E}_2,$$

and therefore, as a \sqrt{n} -Lipschitz linear observation of $\sigma(X)^{\odot m} \in \mathcal{E}_2(\kappa^{m-1}\sigma)$ (see Theorem D.7), $\text{Tr}(X^m)$ follows the concentration:

$$\text{Tr}(X^m) = \sum_{i=1}^p \sigma_i(X)^m \in \mathcal{E}_2(\sqrt{n}\kappa^{m-1}\sigma).$$

Now, we consider the general setting where we are given m matrices X_1, \dots, X_m , a deterministic matrix $A \in \mathcal{M}_{n, n_0}$ satisfying $\|A\| \leq 1$, and we want to show the concentration of $\text{tr}(AX_1, \dots, X_m)$. First note that we stay in the hypotheses of the theorem if we replace X_1 with AX_1 , we are thus left to show the concentration of $\text{Tr}(X_1 \cdots X_m)$. We can not employ again Lemma D.10 without a strong hypothesis of commutativity on the matrices X_1, \dots, X_n . Indeed, one could not have gone further than a concentration on the whole term $\sum_{\sigma \in \mathfrak{S}_p} \text{Tr}(X_{\sigma(1)} \cdots X_{\sigma(m)})$. However, we can still introduce the random matrix

$$Y = \begin{pmatrix} 0 & X_{m-1} & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & X_1 \\ X_m & & & & 0 \end{pmatrix} \quad \text{then} \quad Y^m = \begin{pmatrix} 0 & X_1^m & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & X_3^2 \\ X_2^1 & & & & 0 \end{pmatrix},$$

where for $i, j \in \{2, \dots, m-1\}$, $X_i^j \equiv X_i X_{i+1} \cdots X_m X_1 \cdots X_j$. Since $Y \in \mathcal{M}_{n_0 + \dots + n_m}$ satisfies $Y \propto \mathcal{E}_2(\sigma)$ and $\|Y\| \leq \kappa$, the first part of the proof provides the concentration $Y^m \in \mathcal{E}_2(\kappa^{m-1} \sigma \sqrt{n_0 + \dots + n_m})$ in $(\mathcal{M}_n, \|\cdot\|_*)$ which directly implies the concentration of $X_1^m = X_1 \cdots X_m$. \square

⁴The norm $\|\cdot\|_F$ is defined on $\mathcal{M}_{n_0, n_1} \times \dots \times \mathcal{M}_{n_{m-1}, n_m}$ by the identity:

$$\|(M_1, \dots, M_p)\|_F = \sqrt{\|M_1\|_F^2 + \dots + \|M_m\|_F^2}$$

⁵Be careful that $X \propto_c \mathcal{E}_2(\sigma)$ does not imply that $(X, \dots, X) \propto_c \mathcal{E}_2(\sigma)$, it is only true when $(\mathcal{M}_n)^m$ is endowed with the norm $\|\cdot\|_{F, \ell^\infty}$, satisfying for any $M = (M_1, \dots, M_m) \in (\mathcal{M}_n)^m$, $\|M\|_{F, \ell^\infty} = \sup_{1 \leq i \leq m} \|M_i\|_F$

Part II

Concentration of the resolvent of sample covariance matrix

Considering a non centered sample covariance matrix $\frac{1}{n}XX^T$, where $X = (x_1, \dots, x_n) \in \mathcal{M}_{p,n}$ is the data matrix, we denote $\text{Sp}(\frac{1}{n}XX^T)$ the spectrum of $\frac{1}{n}XX^T$. The spectral distribution of $\frac{1}{n}XX^T$, denoted $\mu \equiv \frac{1}{p} \sum_{\lambda \in \text{Sp}(\frac{1}{n}XX^T)} \delta_\lambda$, is classically studied through its Stieltjes transform expressed as:

$$g : \mathbb{C} \setminus \text{Sp} \left(\frac{1}{n}XX^T \right) \longrightarrow \mathbb{C}$$

$$z \longmapsto \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}.$$

The relevance of the Stieltjes transform has been extensively justified in some seminal works [MP67, Sil86] by the Cauchy integral that provides for any analytical mapping f defined on a neighborhood of a subset $B \subset \text{Sp}(\frac{1}{n}XX^T)$ the identity:

$$\int_B f(\lambda) d\mu(\lambda) = \frac{1}{2i\pi} \oint_{\gamma} f(z) g(z) dz,$$

where $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \text{Sp}(\frac{1}{n}XX^T)$ is a closed path on which f is defined and whose interior I_γ satisfies $I_\gamma \cap \text{Sp}(\frac{1}{n}XX^T) = B \cap \text{Sp}(\frac{1}{n}XX^T)$. But we can go further and approximate linear functionals of the eigenvectors thanks to the resolvent. If we denote E_B the random eigenspace associated to the eigenvalues of $\frac{1}{n}XX^T$ belonging to B , and Π_B the orthogonal projector on E_B , then for any deterministic matrix $A \in \mathcal{M}_p$:

$$\text{Tr}(\Pi_B A) = \frac{1}{2i\pi} \int_{\gamma} \text{Tr}(AR(z)) dz \quad \text{with } R(z) \equiv \left(\frac{1}{n}XX^T - zI_p \right)^{-1}. \quad (\text{D.5})$$

The matrix $R(z)$ is commonly called the resolvent of $\frac{1}{n}XX^T$. It satisfies in particular that, for all $z \in \mathbb{C} \setminus \text{Sp}(\frac{1}{n}XX^T)$, $g(z) = \frac{1}{p} \text{Tr}(R(z))$. It thus naturally becomes the central element of the study of the spectral distribution. One of the first tasks in random matrix theory is to devise a so called “deterministic equivalent” for $R(z)$ ([HLN07a]), that we will denote here $\tilde{R}(z)$. Specifically, we look for a deterministic matrix computable from the first statistics of our problem and close to $\mathbb{E}[R(z)]$. In particular, a main result of random matrix theory sets that this deterministic equivalent only expresses with means and covariances of the x_i ’s. Two questions then arise:

1. How close to $\mathbb{E}[R(z)]$ is $R(z)$?
2. What does this notion of closeness really mean ?

The first questions relate to concentrations properties on $R(z)$ that arise from concentration properties on X . The study of random matrices originally studied with i.i.d. entries ([MP67], [Yin86]), mere Gaussian hypotheses ([BKV96]), or with weaker hypotheses concerning the first moments of the entries (supposed to be independent or at least independent up to an affine transformation). Some more recent works showed the concentration of the spectral distribution of Wishart or Wigner matrices with bounding assumption on the entries of the random matrix under study –or at least a linear transformation of it– allows to employ Talagrand results in [GZ00, GL09] that also treat some log-concave hypotheses allowing to relax some independence assumptions as it is done in [AC15]. One can also find very light hypotheses on the quadratic functionals of the columns ([BZ08]), improved in [Yas16] or on the norms of the columns and the rows ([Ada11]). In the present work, we do not consider the case of what is called convexly concentrated random vectors in [VW14, MS11, Ada11] as it is done in [GZ00] because it requires a different approach (see Section F.2).

The Lipschitz concentration hypothesis we assume on X in a sense “propagates” to the resolvent that then satisfies, for all deterministic matrices A such that $\|A\|_F \equiv \sqrt{\text{Tr}(AA^T)} \leq 1$ and for all z not too close to the spectrum of $\frac{1}{n}XX^T$:

$$\mathbb{P}(|\text{Tr}(A(R(z) - \mathbb{E}[R(z)]))| \geq t) \leq Ce^{-cnt^2} + Ce^{-cn}, \quad (\text{D.6})$$

for some numerical constants C, c independent of n, p . We see from (D.6) the important benefit gained with our concentration hypothesis on X : it provides simple *quasi-asymptotic* results on the convergence of the

resolvent with speed rates, while most of the results on random matrices are classically expressed in the limiting regime $n, p \rightarrow \infty$.

The condition $\|A\|_F \leq 1$ answers our second question: a specificity of our approach is to control the convergence of the resolvent with the Frobenius norm at a speed of order $O(1/\sqrt{n})$. The concentration inequality (D.6) means that all linear forms of $R(z)$, which are 1-Lipschitz⁶ for the Frobenius norm, have a standard deviation of order $O(1/\sqrt{n})$; this is crucial to be able to estimate quantities expressed in (D.5). Generally, the Stieltjes transform $g(z) = -\frac{1}{p} \text{Tr}(R(z))$ is classically the only studied linear forms of the resolvent (it is $1/\sqrt{p}$ -Lipschitz so its standard deviation is of order $O(1/\sqrt{pn})$ which is a classical result although not exactly under a concentration of measure assumption) or projections on deterministic vectors $u^T R(z)u$, for which only the concentration in spectral norm with a speed of order $O(1/\sqrt{n})$ is needed.

Those remarks gain a real importance when we are able to estimate the expectation of $R(z)$ with a deterministic equivalent (that we can compute). We look for a closeness relation in Frobenius norm:

$$\left\| \mathbb{E}[R(z)] - \tilde{R}(z) \right\|_F \leq O\left(\frac{1}{\sqrt{n}}\right).$$

We may then replace in (D.6) the term “ $\mathbb{E}[R(z)]$ ” by $\tilde{R}(z)$ which we are able to compute from the expectations and covariances of the columns x_1, \dots, x_n .

Note that we do not assume that the columns are identically distributed: in particular, the means and covariances can be all different (although they have to satisfy some boundedness properties expressed in Assumptions 4 and 5 at the beginning of Chapter F). This remark may be related to the studies made of matrices X with a variance profile: but this is here even more general because the laws of the columns are not solely defined from their means and covariances (although the spectral distribution of $\frac{1}{n}XX^T$ just depends on these quantities).

The extension of Marcenko Pastur result to non-identically distributed columns was well known; one can cite for instance [HLN07b, WCDS12, Yin20] treating a very similar problem but with different assumptions of concentration (just some moments of the entries need to be bounded) the result is then given in the form of a limit (not a concentration inequality), [KA16] showing that no eigenvalues lie outside of the support with Gaussian hypotheses and [DKL22] imposing some weak isotropic conditions on the different covariances. We will follow the proof scheme of [HLN07b] where the authors introduce two consecutive deterministic equivalents, the first one depending on the expectations of $\Lambda^z = z - \frac{1}{n}x_i^T Q_{-i}^z x_i$, and the second one being expressed through fixed point equation that approximate those quantities. We can indeed estimate the expectations of the quantities $\Lambda_1^z, \dots, \Lambda_n^z$ with a diagonal matrix⁷ $\tilde{\Lambda}^z = \text{Diag}(\tilde{\Lambda}_i^z)_{i \in [n]} \in \mathcal{D}_n(\mathbb{C})$ obtained after successive iteration of the following equation (quite different from the one presented in [HLN07b]):

$$\forall i \in [n]: \tilde{\Lambda}_i^z = z - \frac{1}{n} \text{Tr}\left(\Sigma_i \tilde{Q}^{\tilde{\Lambda}^z}\right) \quad \text{with} \quad \tilde{Q}^{\tilde{\Lambda}^z} \equiv \left(I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\tilde{\Lambda}_i^z}\right)^{-1}, \quad (\text{D.7})$$

in which $\Sigma_i = \mathbb{E}[x_i x_i^T]$.

The difficulties are (i) to prove the existence and uniqueness of $\tilde{\Lambda}^z$ and (ii) to ensure some stability properties⁸ on this equation eventually allowing us to assert that $\|\mathbb{E}[R(z)] - \tilde{R}(z)\|_F \leq O(1/\sqrt{n})$, where $\tilde{R}(z) \equiv \frac{1}{z} \tilde{Q}^{\tilde{\Lambda}^z}$. The existence and uniqueness of similar equations has already proven thanks to complex analysis justification (normal family theorem in [HLN07b] and Vitali’s theorem in [WCDS12, Yin20]), those approaches allows to extend convergence properties for some z to the whole convex half plane \mathbb{H} , however we are not only looking for an asymptotic result but for a quasi asymptotic result, meaning that we want precise convergence bounds for n and p big but not infinite.

The two aforementioned difficulties disappear with the introduction of a convenient semi-metric⁹ d_s on which the fixed point equation satisfied by $\tilde{\Lambda}^z$ is contractive, leading (after still some work since a semi-metric

⁶or λ -Lipschitz with $\lambda \leq O(1)$

⁷The interest to resort to a diagonal matrix of \mathcal{M}_n rather than to a vector of \mathbb{R}^n will be clearer later – mainly to employ Proposition B.50 in a natural formalism.

⁸Conceptually, it means that if we have a diagonal matrix $L \in \mathcal{M}_n$ satisfying $\forall i \in [n]: L_i \equiv z - \frac{1}{n} \text{Tr}\left(\Sigma_i \tilde{Q}^L\right)$ then L is “close” to $\tilde{\Lambda}^z$.

⁹A semi metric is defined as a metric that does not satisfy the triangular inequality.

is not as easy to treat as if d_s were a true metric) to existence, uniqueness and stability properties. This semi-metric, quite similar to the one already introduced in [LC22b] to study robust estimators, is defined for any $D, D' \in \mathcal{D}_n(\mathbb{H})$ as:

$$d_s(D, D') = \left\| \frac{D - D'}{\sqrt{\Im(D)\Im(D')}} \right\|.$$

This semi-metric appears as a central object in random matrix theory, one can indeed point out the fact that any Stieltjes transform is 1-Lipschitz under this semi-metric (see Proposition E.18). It relates to Poincaré metric, the hyperbolic metric writes indeed $d_{\mathbb{H}}(z, z') = \cosh(d_s(z, z')^2 - 1)$. Apart from the book [EH70] that provided a groundwork for the introduction of such a metric, this approach already gained some visibility in [HFS07, KLV13] or in a context closer to ours for the study of Wishart matrices and in a squared form in [AKE16]. We prefer an expression proportional to $\|D - D'\|$ because it is more adapted to comparison with classical distance on $\mathcal{D}_n(\mathbb{H})$, as it is done in Proposition E.23 that subsequently provides bounds to the convergence speed. Let us outline that once the appropriate semi-metric is identified, contractivity properties are not sufficient to prove the existence and uniqueness to (D.7): one also needs to introduce the correct space over which the mapping is contractive ($\mathcal{D}_{I^z} \equiv \{D \in \mathcal{D}_n(\mathbb{H}), \frac{D}{z} \in \mathcal{D}_n(\mathbb{H})\}$).

Chapter E

Stable semi metric

E.1 Definition and first properties

The stable semi-metric which we define here is a convenient object which allows us to set Banach-like fixed point theorems. It has a crucial importance to prove the existence and uniqueness of \hat{C} but also to obtain some random matrix identities on \hat{C} , such as the estimation of its limiting spectral distribution.

Definition 21. We call the stable semi-metric on $\mathcal{D}_n^+ = \{D \in \mathcal{D}_n, \forall i \in [n], D_i > 0\}$ the function:

$$\forall \Delta, \Delta' \in \mathcal{D}_n^+ : d_s(\Delta, \Delta') \equiv \left\| \frac{\Delta - \Delta'}{\sqrt{\Delta \Delta'}} \right\|. \quad (\text{E.1})$$

In particular, this semi-metric can be defined on \mathbb{R}^+ , identifying \mathbb{R}^+ with \mathcal{D}_1^+ .

The function d_s is not a metric because it does not satisfy the triangular inequality, one can see for instance that:

$$d_s(4, 1) = \frac{3}{2} > \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = d_s(4, 2) + d_s(2, 1) \quad (\text{E.2})$$

More precisely, for any $x, z \in \mathbb{R}_+$ such that $x < z$, if one differentiates twice the mapping $g : y \rightarrow \frac{(xy-x)^2}{xy} + \frac{(z-y)^2}{xy}$, one obtains:

$$g'(y) = \frac{1}{y} - \frac{y}{x^2} + \frac{1}{z} - \frac{z}{x^3} \quad \text{and} \quad g''(y) = \frac{3y}{x^3} + \frac{3z}{x^3} > 0,$$

which proves that g is strictly convex on $[x, z]$ and therefore it admits a minimum y_0 on $]x, z[$ (since $g(x) = g(z)$). In particular, one can bound:

$$d_s(x, z) > \sqrt{d_s(x, y_0)^2 + d_s(y_0, z)^2}$$

One can however sometimes palliate this weakness when needed thanks to the following inequality proved in Section E.5.

Proposition E.1 (Pseudo triangular inequality). *Given $x, z, y \in \mathbb{R}^+$:*

$$|x - y| \leq |y - z| \quad \implies \quad d_s(x, y) \leq d_s(x, z).$$

In addition, for any $p \in \mathbb{N}^$ and $y_1, \dots, y_{p-1} \in \mathbb{R}^+$, we have the inequalities¹:*

$$d_s(x, y_1) + \dots + d_s(y_{p-1}, z) \geq d_s\left(x^{\frac{1}{p}}, z^{\frac{1}{p}}\right) \geq d_s(x, z)^{1/p}$$

and the left inequality turns into an equality in the case $y_i = x^{\frac{p-i}{p}} z^{\frac{i}{p}}$ for $i \in \{1, \dots, p-1\}$.

¹The mapping $x \mapsto x^p$ is not Lipschitz for the semi-metric d_s (unlike $x \mapsto x^{\frac{1}{p}}$).

Proof. For a given integer $p \geq 1$, let us differentiate the mapping:

$$\begin{aligned} f_p : \quad \mathbb{R}_+^{p-1} &\longrightarrow \mathbb{R} \\ (y_1, \dots, y_{p-1}) &\longmapsto \frac{y_1 - x}{\sqrt{y_1 x}} + \dots + \frac{z - y_{p-1}}{\sqrt{z y_{p-1}}} \end{aligned}$$

one can compute for any $y_1, \dots, y_{p-1} \in \mathbb{R}^+$ and $i \in [p-1]$:

$$\frac{\partial f_p(y_1, \dots, y_{p-1})}{\partial y_i} = \frac{1}{2} \frac{1}{\sqrt{y_i y_{i-1}}} \left(1 + \frac{y_{i-1}}{y_i} \right) - \frac{1}{2} \frac{1}{\sqrt{y_{i+1} y_i}} \left(1 + \frac{y_{i+1}}{y_i} \right) \quad (\text{E.3})$$

(where y_0 and y_p designate respectively x and z) In particular, when $p = 1$, for any $y \geq x > 0$:

$$\frac{\partial}{\partial y} \left(\frac{y-x}{\sqrt{yx}} \right) = \frac{1}{2} \frac{1}{\sqrt{xy}} \left(1 + \frac{x}{y} \right) \geq 0$$

which proves the first result of the proposition. Now if we assume that $y \leq x \leq z$:

$$d_s(x, y) + d_s(y, z) \geq d_s(x, z),$$

and the same inequality holds if one assumes that $x \leq z \leq y$. Returning to the setting of the proposition, we can therefore place ourselves in the open space:

$$\mathcal{U}_{x,z}^p = \{(y_1, \dots, y_p) \in \mathbb{R}_+^{p-1}, x < y_1 < \dots < y_{p-1} < z\}.$$

If one fixes $x, z \in \mathbb{R}^+$, then $f_p(x, y_1, \dots, y_{p-1}, z) = d_s(x, y_1) + \dots + d_s(y_{p-1}, z)$ is minimum for y_1, \dots, y_{p-1} satisfying:

$$\frac{1}{\sqrt{y_i y_{i-1}}} \left(1 - \frac{y_{i-1}}{y_i} \right) = \frac{1}{\sqrt{y_{i+1} y_i}} \left(1 - \frac{y_i}{y_{i+1}} \right)$$

which is equivalent to $y_i = \sqrt{y_{i-1} y_{i+1}}$. Noting $\tilde{x} = \log(x)$, $\tilde{y}_1 = \log(y_1)$, \dots , $\tilde{y}_n = \log(y_n)$, $\tilde{z} = \log(z)$, we see that this identity writes $\tilde{y}_i = \frac{1}{2}(\tilde{y}_{i-1} + \tilde{y}_{i+1})$, which implies $\tilde{y}_i = \tilde{x} + \frac{i}{p}(\tilde{z} - \tilde{x})$, or in other words:

$$y_i = x^{\frac{p-i}{p}} z^{\frac{i}{p}}.$$

In that case:

$$d_s(y_i, y_{i+1}) = \left| \frac{x^{\frac{p-i}{2p}} z^{\frac{i}{2p}}}{x^{\frac{p-i-1}{2p}} z^{\frac{i+1}{2p}}} - \frac{x^{\frac{p-i-1}{2p}} z^{\frac{i+1}{2p}}}{x^{\frac{p-i}{2p}} z^{\frac{i}{2p}}} \right| = \left| \frac{x^{\frac{1}{2p}}}{z^{\frac{1}{2p}}} - \frac{z^{\frac{1}{2p}}}{x^{\frac{1}{2p}}} \right| = d_s \left(x^{\frac{1}{p}}, z^{\frac{1}{p}} \right),$$

and the same holds for $d_s(x, y_1)$ and $d_s(y_{p-1}, z)$.

The last inequality is just a consequence of the concavity of $t \rightarrow t^{1/p}$:

$$d_s \left(x^{\frac{1}{p}}, z^{\frac{1}{p}} \right) = \frac{z^{\frac{1}{p}} - x^{\frac{1}{p}}}{(xz)^{\frac{1}{2p}}} = \frac{\frac{1}{p} \int_0^{z-x} (t+x)^{\frac{1-p}{p}} dt}{(xz)^{\frac{1}{2p}}} \leq \frac{\frac{1}{p} \int_0^{z-x} t^{\frac{1-p}{p}} dt}{(xz)^{\frac{1}{2p}}} = \left(\frac{z-x}{(xz)^{\frac{1}{2}}} \right)^{\frac{1}{p}} = d_s(x, z)^{\frac{1}{p}}.$$

□

The semi-metric d_s is called stable due to its many interesting stability properties.

Property E.2. Given $\Delta, \Delta' \in \mathcal{D}_n^+$ and $\Lambda \in \mathcal{D}_n^+$:

$$d_s(\Lambda \Delta, \Lambda \Delta') = d_s(\Delta, \Delta') \quad \text{and} \quad d_s(\Delta^{-1}, \Delta'^{-1}) = d_s(\Delta, \Delta').$$

Property E.3. Given four diagonal matrices $\Delta, \Delta', D, D' \in \mathcal{D}_n^+$:

$$d_s(\Delta + D, \Delta' + D') \leq \max(d_s(\Delta, \Delta'), d_s(D, D')).$$

To prove this property one needs two elementary results.

Lemma E.4. *Given four positive numbers $a, b, \alpha, \beta \in \mathbb{R}^+$:*

$$\sqrt{ab} + \sqrt{\alpha\beta} \leq \sqrt{(a+\alpha)(b+\beta)} \quad \text{and} \quad \frac{a+\alpha}{b+\beta} \leq \max\left(\frac{a}{b}, \frac{\alpha}{\beta}\right)$$

Proof. For the first result, we deduce from the inequality $4ab\alpha\beta \leq (a\alpha + b\beta)^2$:

$$\left(\sqrt{ab} + \sqrt{\alpha\beta}\right)^2 = ab + \alpha\beta + 2\sqrt{ab\alpha\beta} \leq ab + \alpha\beta + a\beta + b\alpha = (a+\alpha)(b+\beta)$$

For the second result, we simply bound:

$$\frac{a+\alpha}{b+\beta} \leq \frac{a}{b} \frac{b}{b+\beta} + \frac{\alpha}{\beta} \frac{\beta}{b+\beta} \leq \max\left(\frac{a}{b}, \frac{\alpha}{\beta}\right) \left(\frac{b}{b+\beta} + \frac{\beta}{b+\beta}\right) = \max\left(\frac{a}{b}, \frac{\alpha}{\beta}\right)$$

□

Proof of Property E.3. For any $\Delta, \Delta', D, D' \in \mathcal{D}_n^+$, there exists $i_0 \in [n]$ such that:

$$\begin{aligned} d_s(\Delta + D, \Delta' + D') &= \frac{|\Delta_{i_0} - \Delta'_{i_0} + D_{i_0} - D'_{i_0}|}{\sqrt{(\Delta_{i_0} + D_{i_0})(\Delta'_{i_0} + D'_{i_0})}} \\ &\leq \frac{|\Delta_{i_0} - \Delta'_{i_0}| + |D_{i_0} - D'_{i_0}|}{\sqrt{\Delta_{i_0}\Delta'_{i_0} + \sqrt{D_{i_0}D'_{i_0}}}} \leq \max\left(\frac{|\Delta_{i_0} - \Delta'_{i_0}|}{\sqrt{\Delta_{i_0}\Delta'_{i_0}}}, \frac{|D_{i_0} - D'_{i_0}|}{\sqrt{D_{i_0}D'_{i_0}}}\right) \end{aligned}$$

thanks to Lemma E.4. □

E.2 Stable class

Definition 22 (Stable class). *The set of 1-Lipschitz functions for the stable semi-metric is called the stable class. We denote it:*

$$\mathcal{S}(\mathcal{D}_n^+) \equiv \{f : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+ \mid \forall \Delta, \Delta' \in \mathcal{D}_n^+, \Delta \neq \Delta' : d_s(f(\Delta), f(\Delta')) \leq d_s(\Delta, \Delta')\}.$$

The elements of $\mathcal{S}(\mathcal{D}_n^+)$ are called the stable mappings.

Let us then provide the properties which justify why we call $\mathcal{S}(\mathcal{D}_n^+)$ a *stable class*: this class indeed satisfies far more stability properties than the usual Lipschitz mappings (for a given norm). Those stability properties are direct consequences to Properties E.2 and E.3.

Property E.5. *Given $\Lambda, \Gamma \in \mathcal{D}_n^+$ and $f, g \in \mathcal{S}(\mathcal{D}_n^+)$:*

$$(\Delta \mapsto \Lambda f(\Gamma\Delta)) \in \mathcal{S}(\mathcal{D}_n^+), \quad \frac{1}{f} \in \mathcal{S}(\mathcal{D}_n^+), \quad f \circ g \in \mathcal{S}(\mathcal{D}_n^+), \quad f + g \in \mathcal{S}(\mathcal{D}_n^+).$$

E.3 The sub-monotonic class

The stable class has a very simple interpretation when $n = 1$. Given a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ we introduce two characteristic functions $f_j, f_\times : \mathbb{R}^+ \rightarrow \mathbb{R}^+$:

$$f_j : x \mapsto \frac{f(x)}{x} \quad \text{and} \quad f_\times : x \mapsto xf(x).$$

Property E.6. *A function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a stable mapping if and only if f_j is non-increasing and f_\times is non-decreasing.*

Proof. Let us consider $x, y \in \mathbb{R}^+$, such that, say, $x \leq y$. We suppose in a first time that $f_{/}$ is non-increasing and that f_{\times} is non-decreasing. We know that $\frac{f(x)}{x} \geq \frac{f(y)}{y}$, and subsequently:

$$f(y) - f(x) \leq \frac{f(y)}{y}(y - x) \quad \text{and} \quad f(y) - f(x) \leq \frac{f(x)}{x}(y - x) \quad (\text{E.4})$$

The same way, since $f(x)x \leq f(y)y$ we also have the inequalities:

$$f(x) - f(y) \leq \frac{f(y)}{x}(y - x) \quad \text{and} \quad f(x) - f(y) \leq \frac{f(x)}{y}(y - x) \quad (\text{E.5})$$

Now if $f(y) \geq f(x)$, we can take the root of the product of the two inequalities of (E.4) and if $f(y) \leq f(x)$, we take the root of the product of the two inequalities of (E.5), to obtain, in both cases:

$$|f(x) - f(y)| \leq \sqrt{\frac{f(y)f(x)}{xy}} |x - y|$$

That means that $f \in \mathcal{S}(\mathbb{R}^+)$.

Conversely, if we now suppose that $f \in \mathcal{S}(\mathbb{R}^+)$, we then use the bound:

$$|f(y) - f(x)| \leq \sqrt{\frac{f(y)f(x)}{xy}} (y - x).$$

First, if $f(x) \leq f(y)$, then $f(x)x \leq f(y)y$ and we can bound:

$$f(y) - f(x) \leq \max\left(\frac{f(x)}{x}, \frac{f(y)}{y}\right) (y - x) \leq \max\left(\left(\frac{y}{x} - 1\right) f(x), \left(1 - \frac{x}{y}\right) f(y)\right)$$

which directly implies $\frac{f(y)}{y} \leq \frac{f(x)}{x}$. Second, if $f(x) \geq f(y)$, $\frac{f(x)}{x} \geq \frac{f(y)}{y}$ and we can then bound in the same way:

$$f(x)xy - f(y)xy \leq \max(xf(x)(y - x), (y - x)yf(y))$$

which implies $xf(x) \leq yf(y)$. In both cases ($f(x) \leq f(y)$ and $f(y) \leq f(x)$), and we see that $f_{/}(x) \geq f_{/}(y)$ and $f_{\times}(x) \leq f_{\times}(y)$, proving the result. \square

This property allows us to understand directly that the stability of a function is a local behavior. We then conclude straightforwardly that the supremum or the infimum of stable mappings is also stable.

Corollary E.7. *Given a family of stable mappings $(f_{\theta})_{\theta \in \Theta} \in \mathcal{S}(\mathbb{R}^+)^{\Theta}$, for a given set Θ , the mappings $\sup_{\theta \in \Theta} f_{\theta}$ and $\inf_{\theta \in \Theta} f_{\theta}$ are both stable.*

Given $f : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$, we can introduce by analogy to the case of mappings on \mathbb{R}^+ , the mappings $f_{/}, f_{\times} : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$ defined with:

$$f_{/} : \Delta \mapsto \text{Tr}\left(\frac{f(\Delta)}{\Delta}\right) \quad \text{and} \quad f_{\times} : \Delta \mapsto \text{Tr}(\Delta f(\Delta))$$

Inspiring from Property E.6, one can then define:

Definition 23 (Sub-monotonuous class). *A mapping $f : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$ is said to be sub-monotonuous if and only if f_{\times} is non decreasing and $f_{/}$ is non-increasing, we note this class of mappings $\mathcal{S}_m(\mathcal{D}_n^+)$.*

Remark E.8. *We know from Property E.6 that $\mathcal{S}(\mathbb{R}_+) = \mathcal{S}_m(\mathbb{R}_+)$ but for $n > 1$, none of the classes $\mathcal{S}_m(\mathcal{D}_n^+)$ and $\mathcal{S}(\mathcal{D}_n^+)$ contains strictly the other one. On the first hand, introducing:*

$$f : \mathcal{D}_2^+ \longrightarrow \mathcal{D}_2^+ \\ \Delta \longmapsto \text{Diag}\left(\frac{1}{\Delta_2}, \frac{1}{\Delta_1}\right),$$

we see that $f \in \mathcal{S}(\mathcal{D}_n^+)$ but $f \notin \mathcal{S}_m(\mathcal{D}_n^+)$ (for $\Delta = \text{Diag}(1, 2)$ and $\Delta' = \text{Diag}(2, 2)$, $\Delta \leq \Delta'$ but $\text{Tr}(f(\Delta)\Delta) = \frac{5}{2} > 2 = \text{Tr}(f(\Delta')\Delta')$). On the other hand, the mapping:

$$g : \mathcal{D}_2^+ \longrightarrow \mathcal{D}_2^+ \\ \Delta \longmapsto \text{Diag} \left(\frac{\Delta_1 \Delta_2}{1 + \Delta_2}, 1 \right)$$

is in $\mathcal{S}_m(\mathcal{D}_n^+)$ because:

$$\left\{ \begin{array}{l} \frac{\partial g_i}{\partial \Delta_1} = \frac{2\Delta_1 \Delta_2}{1 + \Delta_2} \geq 0 \\ \frac{\partial g_i}{\partial \Delta_2} = \frac{\Delta_1^2}{(1 + \Delta_2)^2} + 1 \geq 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \frac{\partial g_j}{\partial \Delta_1} = 0 \leq 0 \\ \frac{\partial g_j}{\partial \Delta_2} = \frac{1}{(1 + \Delta_2)^2} - \frac{1}{\Delta_2^2} \leq 0. \end{array} \right.$$

However, we can see that g is not stable if we introduce the diagonal matrices $\Delta = \text{Diag}(1, 2)$ and $\Delta' = \text{Diag}(2, 3)$ since we have then:

$$d_s(g(\Delta), g(\Delta')) = d_s \left(\frac{2}{3}, \frac{3}{2} \right) = \frac{5}{6} > \frac{1}{\sqrt{2}} = \max \left(\frac{|3-2|}{\sqrt{6}}, \frac{|2-1|}{\sqrt{2}} \right) = d_s(\Delta, \Delta')$$

E.4 Fixed Point theorem for stable and sub-monotonic mappings

The Banach fixed point theorem states that a contracting function on a complete space admits a unique fixed point. The extension of this result to contracting mappings on \mathcal{D}_n^+ , for the semi-metric d_s , is not obvious: first because d_s does not verify the triangular inequality and second because the completeness needs to be proven. The completeness of the semi-metric space (\mathcal{D}_n^+, d_s) is left in Section E.5 since we will not need it. Let us start with a first bound.

Lemma E.9. *Given a mapping $f : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$, contracting for the semi metric d_s , any sequence of diagonal matrices $(\Delta_n)_{n \in \mathbb{N}}$ satisfying $\Delta^{(p+1)} = f(\Delta^{(p)})$ is bounded from below and above and satisfies for all $p \in \mathbb{N}$ and $i \in [n]$:*

$$\exp \left(-\frac{\lambda d_s(\Delta^{(1)}, \Delta^{(0)})}{2(1-\lambda)} \right) \Delta_i^{(1)} \leq \Delta_i^{(p)} \leq \exp \left(\frac{\lambda d_s(\Delta^{(1)}, \Delta^{(0)})}{2(1-\lambda)} \right) \Delta_i^{(1)}$$

Proof. Noting $\lambda > 0$, the Lipschitz parameter of f for the semi-metric d_s , let us first show that, for all $i \in [n]$:

$$\forall p \in \mathbb{N}, \sqrt{\frac{\Delta^{(p+1)}_i}{\Delta^{(p)}_i}} \leq 1 + \lambda^p d_s(\Delta^{(1)}, \Delta^{(0)}). \quad (\text{E.6})$$

When $\Delta_i^{(p+1)} \leq \Delta_i^{(p)}$, it is obvious and when $\Delta^{(p+1)} \geq \Delta^{(p)}$ the contractivity of f allows us to set $d_s(\Delta^{(p+1)}, \Delta^{(p)}) \leq \lambda^p d_s(\Delta^{(1)}, \Delta^{(0)})$, which implies:

$$\sqrt{\frac{\Delta_i^{(p+1)}}{\Delta_i^{(p)}}} \leq \sqrt{\frac{\Delta_i^{(p)}}{\Delta_i^{(p+1)}}} + \lambda^p d_s(\Delta^{(1)}, \Delta^{(0)}) \leq 1 + \lambda^p d_s(\Delta^{(1)}, \Delta^{(0)}).$$

Multiplying (E.6) for all $p \in \{1, \dots, P\}$, we obtain:

$$\begin{aligned} \sqrt{\frac{\Delta_i^{(P)}}{\Delta_i^{(1)}}} &\leq 1 \leq \prod_{p=1}^P \left(1 + \lambda^p d_s(\Delta^{(1)}, \Delta^{(0)}) \right) = \exp \left(\sum_{p=1}^P \log \left(1 + \lambda^p d_s(\Delta^{(1)}, \Delta^{(0)}) \right) \right) \\ &\leq \exp \left(\sum_{p=1}^P \lambda^p d_s(\Delta^{(1)}, \Delta^{(0)}) \right) \leq \exp \left(\frac{\lambda d_s(\Delta^{(1)}, \Delta^{(0)})}{1-\lambda} \right). \end{aligned}$$

With a similar approach, we can eventually show the result of the lemma. \square

Let us now present two fixed point results that will justify the definition of the robust scatter matrix C but also of the deterministic diagonal matrix U introduced in Section I.1.

Theorem E.10. *Any mapping $f : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$, contracting for the stable semi-metric d_s , admits a unique fixed point $\Delta^* \in \mathcal{D}_n(\mathbb{R}^+ \cup \{0\})$ satisfying $\Delta^* = f(\Delta^*)$.*

Proof. We cannot repeat exactly the proof of the Banach fixed point theorem since d_s does not satisfy the triangular inequality. Noting $\lambda \in (0, 1)$ the parameter such that $\forall \Delta, \Delta' \in \mathcal{D}_n^+, d_s(f(\Delta), f(\Delta')) \leq \lambda d_s(\Delta, \Delta')$, we show that the sequence $(\Delta^{(k)})_{k \geq 0}$ satisfying:

$$\Delta^{(0)} = I_n \quad \text{and} \quad \forall k \geq 1 : \Delta^{(k)} = f(\Delta^{(k-1)})$$

is a Cauchy sequence in $(\overline{\mathcal{D}_n^+}, \|\cdot\|)$, where $\overline{\mathcal{D}_n^+} \equiv \mathcal{D}_n(\mathbb{R}^+ \cup \{0\})$.

We know from Lemma E.9 that there exists $\delta > 0$ such that $\forall p \in \mathbb{N}, \|\Delta^{(p)}\| \leq \delta$. One can then bound for any $p \in \mathbb{N}$:

$$\left\| \Delta^{(p+1)} - \Delta^{(p)} \right\| \leq \delta d_s(\Delta^{(p+1)}, \Delta^{(p)}) \leq \lambda^p \delta d_s(\Delta^{(1)}, \Delta^{(0)}).$$

Therefore, thanks to the triangular inequality (in $(\mathcal{D}_n^+, \|\cdot\|)$), for any $n \in \mathbb{N}$:

$$\begin{aligned} \left\| \Delta^{(p+n)} - \Delta^{(p)} \right\| &\leq \left\| \Delta^{(p+n)} - \Delta^{(p+n-1)} \right\| + \dots + \left\| \Delta^{(p+1)} - \Delta^{(p)} \right\| \\ &\leq \frac{\delta d_s(\Delta^{(1)}, \Delta^{(0)})}{1 - \lambda} \lambda^p \xrightarrow{p \rightarrow \infty} 0. \end{aligned}$$

That allows us to conclude that $(\Delta^{(p)})_{p \in \mathbb{N}}$ is a Cauchy sequence, and therefore that it converges to a diagonal matrix $\Delta^* \equiv \lim_{p \rightarrow \infty} \Delta^{(p)} \in \overline{\mathcal{D}_n^+}$ which is complete (closed in a complete set). But since $\Delta^{(p)}$ is bounded from below and above thanks to Lemma E.9, we know that $\Delta^* \in \mathcal{D}_n^+$. By contractivity of f , it is clearly unique. \square

It is possible to relax a bit the contracting hypotheses on f if one supposes that f is monotonic. We express rigorously this result in next theorem, but it will not be employed in our applications in Chapter I about the robust estimation of scatter matrix since we preferred to assume u bounded to obtain the contracting properties of the fixed point satisfied by $\hat{\Delta}$.

Theorem E.11. *Let us consider a weakly monotonic mapping $f : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$ bounded from below and above. If we suppose that f is stable and verifies:*

$$\forall \Delta, \Delta' \in \mathcal{D}_n^+ : d_s(f(\Delta), f(\Delta')) < d_s(\Delta, \Delta') \quad (\text{E.7})$$

then there exists a unique fixed point $D \in \mathcal{D}_n^+$ satisfying $\Delta^ = f(\Delta^*)$.*

E.5 Supplementary inferences on the stable semi-metric and topological properties

Remark E.12. *Not all the stable mappings admit a continuous continuation on $\overline{\mathcal{D}_n^+}$. To construct a counter example, for any $n \in \mathbb{N}$, let us note*

- $e_n : x \mapsto \frac{3}{2} - 2^{n-1}x$ (it satisfies $e_n(1/2^n) = 1$ and $e_n(3/2^{n+1}) = \frac{3}{4}$),
- $d_n : x \mapsto 2^{n-1}x$ (it satisfies $d_n(1/2^{n-1}) = 1$ and $e_n(3/2^{n+1}) = \frac{3}{4}$),
- $v_n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying for all $x \in \mathbb{R}^+$, $v_n(x) = \max(e_n, d_n)$ (in particular, $v_n(2^n) = v_n(2^{n-1}) = 1$),
- $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying for all $x \in \mathbb{R}^+$, $f(x) = \inf_{n \in \mathbb{N}} v_n(x)$.

We know from Property E.6 that for all $n \in \mathbb{N}$, d_n is stable and that e_n is stable on $[0, 3/2^{n+1}]$ (where $x \mapsto xe_n(x)$ is non decreasing) which eventually allows us to set that f is stable, thanks to Corollary E.7.

However f does not admit continuous continuation on 0 since:

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{2^n}\right) = 1 \neq \frac{3}{4} = \lim_{n \rightarrow \infty} f\left(\frac{3}{2^n}\right).$$

Lemma E.13. Any Cauchy sequence of (\mathcal{D}_n^+, d_s) is bounded from below and above (in \mathcal{D}_n^+).

Proof. Considering a Cauchy sequence of diagonal matrices $\Delta^{(k)} \in \mathcal{D}_n^+$, we know that there exists $K \in \mathbb{N}$ such that:

$$\forall p, q \geq K, \forall i \in \{1, \dots, n\} : |\Delta_i^{(p)} - \Delta_i^{(q)}| \leq \sqrt{\Delta_i^{(p)} \Delta_i^{(q)}}.$$

For $k \in \mathbb{N}$, let us introduce the indexes $i_M^k, i_m^k \in \mathbb{N}$, satisfying:

$$\Delta_{i_M^k}^{(k)} = \max\left(\Delta_i^{(k)}, 1 \leq i \leq n\right) \quad \text{and} \quad \Delta_{i_m^k}^{(k)} = \min\left(\Delta_i^{(k)}, 1 \leq i \leq n\right).$$

If we suppose that there exists a subsequence $(\Delta_{i_M^k}^{(\phi(k))})_{k \geq 0}$ such that $\Delta_{i_M^k}^{(\phi(k))} \xrightarrow[k \rightarrow \infty]{} \infty$, then

$$\sqrt{\Delta_{i_M^k}^{(\phi(k))}} \leq \sqrt{\Delta_{i_M^k}^{(N)}} + \frac{\Delta_{i_M^k}^{(\phi(k))}}{\sqrt{\Delta_{i_M^k}^{(\phi(k))}}} \xrightarrow[k \rightarrow \infty]{} \sqrt{\Delta_{i_M^k}^{(N)}} < \infty$$

which is absurd. Therefore $(\Delta_{i_M^k}^{(k)})_{k \geq 0}$ and thus also $(\Delta^{(k)})_{k \geq 0}$ are bounded from above. For the lower bound, we consider the same way a subsequence $(\Delta_{i_m^k}^{(\psi(k))})_{k \geq 0}$ such that $\Delta_{i_m^k}^{(\psi(k))} \xrightarrow[k \rightarrow \infty]{} 0$. We have:

$$\Delta_{i_M^k}^{(\phi(k))} \geq \Delta_{i_m^k}^{(N)} - \sqrt{\Delta_{i_m^k}^{(N)} \Delta_{i_m^k}^{(\phi(k))}} \xrightarrow[k \rightarrow \infty]{} \sqrt{\Delta_{i_m^k}^{(N)}} > 0$$

which is once again absurd. \square

Property E.14. The semi-metric space (\mathcal{D}_n^+, d_s) is complete.

Proof. Given a Cauchy sequence of diagonal matrices $\Delta^{(k)} \in \mathcal{D}_n^+$, we know from the preceding lemma that there exists $\delta_M, \delta_m \in \mathbb{R}^+$ such that $\forall k \geq 0 : \delta_m I_n \leq \Delta^{(k)} \leq \delta_M I_n$. Thanks to the Cauchy hypothesis:

$$\forall \varepsilon > 0, \exists K \geq 0 \mid \forall p, q \geq K : \forall i \in \{1, \dots, n\} : \left| \Delta_i^{(p)} - \Delta_i^{(q)} \right| \leq \varepsilon \delta_M$$

and, as a consequence, $(\Delta^{(k)})_{k \geq 0}$ is a Cauchy sequence in the complete space $(\mathcal{D}_n^{0,+}, \|\cdot\|)$: it converges to a matrix $\Delta^{(\infty)} \in \mathcal{D}_n^{0,+}$. Moreover, $\Delta^{(\infty)} \geq \delta_m I_n$ (as any $\Delta^{(k)}$) for all $k \in \mathbb{N}$, so that $\Delta^{(\infty)} \in \mathcal{D}_n^+$ and we are left to showing that $\Delta^{(k)} \xrightarrow[k \rightarrow \infty]{} \Delta^{(\infty)}$ for the semi-metric d_s . It suffices to write:

$$d_s(D^{(k)}, D^{(\infty)}) = \left\| \frac{D^{(k)} - D^{(\infty)}}{\sqrt{D^{(k)} D^{(\infty)}}} \right\| \leq \delta_m \left\| D^{(k)} - D^{(\infty)} \right\| \xrightarrow[k \rightarrow \infty]{} 0.$$

\square

Proof of Theorem E.11. We first suppose that f is non-decreasing. As before, let us consider $\delta_M, \delta_m \in \mathbb{R}^+$ such that $\forall \Delta \in \mathcal{D}_n^+ : \delta_m I_n \leq f(\Delta) \leq \delta_M I_n$. The sequence $(\Delta^{(k)})_{k \geq 0}$ satisfying $\Delta^{(0)} = \Delta_m I_n$, and for all $k \geq 1$, $\Delta^{(k)} = f(\Delta^{(k-1)})$ is a non-decreasing sequence bounded superiorly with δ_M , thus it converges to $\Delta^* \in \mathcal{D}_n^+$ and $\Delta^* = f(\Delta^*)$. This fixed point is clearly unique thanks to (E.7).

Now if f is non-increasing then $\Delta \mapsto f^2(\Delta)$ is non-decreasing and bounded inferiorly and superiorly thus it admits a unique fixed point $\Delta^* \in \mathcal{D}_n^+$ satisfying $\Delta^* = f^2(\Delta^*)$. We can deduce that $f(\Delta^*) = f^2(f(\Delta^*))$ which implies by uniqueness of the fixed point that $f(\Delta^*) = \Delta^*$ and the uniqueness of such a Δ^* is again a consequence of (E.7). \square

E.6 Complex stable semi-metric and some consequences in random matrix theory

Let us for any $D, D' \in \mathcal{D}_n(\mathbb{H})$ as:

$$d_s(D, D') = \sup_{1 \leq i \leq n} \frac{|D_i - D'_i|}{\sqrt{\Im(D_i)\Im(D'_i)}}$$

(it lacks the triangular inequality to be a true metric). This semi-metric is introduced to set Banach-like fixed point theorems.

Definition 24. Given $\lambda > 0$, we denote $\mathcal{C}_s^\lambda(\mathcal{D}_n(\mathbb{H}))$ (or more simply \mathcal{C}_s^λ when there is no ambiguity), the class of functions $f : \mathcal{D}_n(\mathbb{H}) \rightarrow \mathcal{D}_n(\mathbb{H})$, λ -Lipschitz for the semi-metric d_s ; i.e. satisfying for all $D, D' \in \mathcal{D}_n(\mathbb{H})$:

$$d_s(f(D), f(D')) \leq \lambda d_s(D, D').$$

when $\lambda < 1$, we say that f is contracting for the semi-metric d_s .

Proposition E.15. Given three parameters $\alpha, \lambda, \theta > 0$ and two mappings $f \in \mathcal{C}_s^\lambda$ and $g \in \mathcal{C}_s^\theta$,

$$\frac{-1}{f} \in \mathcal{C}_s^\lambda, \quad \alpha f \in \mathcal{C}_s^\lambda, \quad f \circ g \in \mathcal{C}_s^{\lambda\theta}, \quad \text{and} \quad f + g \in \mathcal{C}_s^{\max(\lambda, \theta)}.$$

The stability towards the sum is a mere adaptation of Property E.3

Lemma E.16. Given four diagonal matrices $\Delta, \Delta', D, D' \in \mathcal{D}_n(\mathbb{H})$:

$$d_s(\Delta + D, \Delta' + D') \leq \max(d_s(\Delta, \Delta'), d_s(D, D')).$$

Proof of Lemma E.16. For any $\Delta, \Delta', D, D' \in \mathcal{D}_n(\mathbb{H})$, there exists $i_0 \in [n]$ such that:

$$\begin{aligned} d_s(\Delta + D, \Delta' + D') &= \frac{|\Delta_{i_0} - \Delta'_{i_0} + D_{i_0} - D'_{i_0}|}{\sqrt{\Im(\Delta_{i_0} + D_{i_0})\Im(\Delta'_{i_0} + D'_{i_0})}} \\ &\leq \frac{|\Delta_{i_0} - \Delta'_{i_0}| + |D_{i_0} - D'_{i_0}|}{\sqrt{\Im(\Delta_{i_0})\Im(\Delta'_{i_0})} + \sqrt{\Im(D_{i_0})\Im(D'_{i_0})}} \\ &\leq \max\left(\frac{|\Delta_{i_0} - \Delta'_{i_0}|}{\sqrt{\Im(\Delta_{i_0})\Im(\Delta'_{i_0})}}, \frac{|D_{i_0} - D'_{i_0}|}{\sqrt{\Im(D_{i_0})\Im(D'_{i_0})}}\right) \end{aligned}$$

thanks to Lemma E.4. □

We can now present our fixed point theorem that has been demonstrated once again in [LC22b]:

Theorem E.17 ([LC22b] Theorem 3.13). Given a subset \mathcal{D}_f of $\mathcal{D}_n(\mathbb{H})$ and a mapping $f : \mathcal{D}_f \rightarrow \mathcal{D}_f$ with an imaginary part bounded from above and below (in \mathcal{D}_f), if it is furthermore contracting for the stable semi-metric d_s on \mathcal{D}_f , then there exists a unique fixed point $\Delta^* \in \mathcal{D}_f$ satisfying $\Delta^* = f(\Delta^*)$.

We will now employ the semi-metric d_s indifferently on diagonal matrices $\mathcal{D}_n(\mathbb{H})$ or vectors of \mathbb{H}^n or more simply with variables of \mathbb{H} as in next proposition.

Proposition E.18. All the Stieltjes transforms are 1-Lipschitz for the semi-metric d_s on \mathbb{H} .

Proof. We consider a Stieltjes transform $g : z \rightarrow \int \frac{d\mu(t)}{t-z}$ for a given measure μ on \mathbb{R} . Given $z, z' \in \mathbb{H}$, we can bound thanks to Cauchy-Schwarz inequality:

$$\begin{aligned} |g(z) - g(z')| &\leq \left| \int \frac{z' - z}{(t-z)(t-z')} d\mu(t) \right| \leq \left| \frac{z' - z}{\sqrt{\Im(z)\Im(z')}} \right| \left| \int \frac{\sqrt{\Im(z)\Im(z')}}{(t-z)(t-z')} d\mu(t) \right| \\ &\leq \left| \frac{z' - z}{\sqrt{\Im(z)\Im(z')}} \right| \sqrt{\int \frac{\Im(z)}{|t-z|^2} d\mu(t)} \sqrt{\int \frac{\Im(z')}{|t-z'|^2} d\mu(t)} \\ &= \sqrt{\Im(g(z))\Im(g(z'))} d_s(z, z') \end{aligned}$$

□

We did not find any particular use of this proposition (the idea could be to solve $g(z) = z$) but the stability it introduces looks interesting.

We deduce directly from Lemma E.16 that if $f, g : \mathcal{D}_n(\mathbb{H}) \rightarrow \mathcal{D}_n(\mathbb{H})$ are λ -Lipschitz for d_s then $f + g$ is also λ -Lipschitz. This property gives a very fast proof to show the existence and uniqueness of solutions to the equations studied in [AKE16] (however their proof is not much longer). We start with a preliminary proposition:

Proposition E.19. *Given a matrix $S \in \mathcal{M}_{n,p}(\mathbb{R}_+)$, $z \mapsto Sz$ goes from \mathbb{H}^p to \mathbb{H}^n and it is 1-Lipschitz for the semi-metric d_s .*

Proof. For any $z \in \mathbb{H}^p$, $\Im(Sz) = S\Im(z) \in \mathbb{R}_+^n$ since all the entries of S are positive. If we denote s_1, \dots, s_n , the columns of S , we can decompose $Sz = \sum_{i=1}^n s_i \pi_i(z)$, where $\pi_i(z) = z_i$. Each mapping $s_i \pi_i$ is 1-Lipschitz for d_s , since we have for any $z, z' \in \mathbb{H}^p$:

$$d_s(s_i \pi_i(z), s_i \pi_i(z')) = \sup_{j \in [p]} \left| \frac{[s_i]_j z_i - [s_i]_j z'_i}{\sqrt{\Im([s_i]_j z_i) \Im([s_i]_j z'_i)}} \right| = \frac{z_i - z'_i}{\sqrt{\Im(z_i) \Im(z'_i)}},$$

therefore, as a sum of 1-Lipschitz operators, we know that $z \rightarrow Sz$ is also 1-Lipschitz for d_s . \square

Proposition E.20. *Given any $a \in \mathbb{R}^n$ and any matrix² $S \in \mathcal{M}_n(\mathbb{R}_+)$, and $z \in \mathbb{H}$, the equation:*

$$-\frac{1}{m} = z\mathbb{1} + a + Sm$$

admits a unique solution $m \in \mathbb{C}_+^n$.

Proof. Let us introduce $I : x \rightarrow z\mathbb{1} + a - S\frac{\mathbb{1}}{x}$. To employ Theorem E.17, let us first show that the imaginary part of $I(x)$ is bounded from below and above for all $x \in \mathbb{H}^n$. Given $x \in \mathbb{H}^n$ we see straightforwardly that $\Im(I(x)) \geq \Im(z)$, we can furthermore bound:

$$\Im(I(x)) \leq \Im(z)\mathbb{1} + S \frac{\Im(x)}{|x|^2} \leq \Im(z)\mathbb{1} + S \frac{\mathbb{1}}{\Im(x)} \leq \left(\Im(z)I_p + \frac{1}{\Im(z)}S \right) \mathbb{1} \equiv \kappa_I \mathbb{1}.$$

Besides, we already know from Proposition E.15 that I is 1-Lipschitz for d_s but we need a Lipschitz parameter lower than 1. Given $x, y \in \mathbb{H}^n$ we can bound thanks to Proposition E.15 and E.19:

$$|I(x) - I(y)| \leq \left| S \left(\frac{1}{x} - \frac{1}{y} \right) \right| \leq \sqrt{\Im \left(S \left(\frac{1}{x} \right) \right) \Im \left(S \left(\frac{1}{y} \right) \right)} d_s(x, y)$$

that implies that the Lipschitz parameter of I is lower than:

$$\sqrt{\left(1 - \frac{\Im(z)}{\Im(I(x))} \right) \left(1 - \frac{\Im(z)}{\Im(I(y))} \right)} \leq 1 - \frac{\Im(z)}{\kappa_I} < 1$$

We conclude then with Theorem E.17 that there exists a unique $x \in \mathbb{H}^n$ such that $x = I(x)$, from which we deduce the existence and uniqueness of $m = \frac{1}{x}$. \square

E.7 Stability of the stable semi-metric towards perturbations

A classical problem is to be able to bound the difference between two solution of two contractive equations for the stable semi-metric. This is possible when one of the equation is a small perturbation of the other one. For the absolute value on \mathbb{R} for instance, if one is given two $1 - \varepsilon$ Lipschitz mapping $f, g : \mathbb{R} \rightarrow \mathbb{R}$, and two fixed point $x, y \in \mathbb{R}$ satisfying:

$$x = f(x) \qquad \text{and} \qquad y = g(y),$$

²Note that unlike in [AKE16], we do not suppose that S is symmetric.

then one can bound thanks to the triangular inequality:

$$|x - y| \leq |f(x) - f(y)| + |f(y) - g(y)| \leq (1 - \varepsilon)|x - y| + |f(y) - g(y)|, \quad (\text{E.8})$$

which implies $|x - y| \leq \frac{|f(y) - g(y)|}{\varepsilon}$. When working with the semi-metric d_s , the triangular inequality is not valid (see (E.2)), one therefore needs more elaborated inferences displayed below. Note that the contractiveness of g is actually completely useless.

Proposition E.21. *Given a diagonal matrices $\Gamma \in \mathcal{D}_n(\mathbb{H})$, a mapping $f : \mathcal{D}_n(\mathbb{H}) \rightarrow \mathcal{D}_n(\mathbb{H})$ λ -Lipschitz³ for the semi-metric d_s with $\lambda < 1$ and admitting the fixed point $\tilde{\Gamma} = f(\tilde{\Gamma})$, we have the bound:*

$$d(\Gamma, \tilde{\Gamma}) \leq \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\| / (1 - \lambda - \lambda d(\mathfrak{S}(\Gamma), \mathfrak{S}(f(\Gamma))))$$

The same result is valid replacing \mathbb{H} with \mathbb{R}_+ and removing the symbols \mathfrak{S} .

This proposition is a consequence of the following elementary result that we set for the semi-metric defined on $\mathcal{D}_n(\mathbb{H})$ since it needs more justification than to set the result on $\mathcal{D}_n(\mathbb{R}_+)$ (which merely expresses removing the symbols \mathfrak{S} in the inequality).

Lemma E.22. *Given three diagonal matrices $\Gamma^1, \Gamma^2, \Gamma^3 \in \mathcal{D}_n(\mathbb{H})$:*

$$\left\| \frac{\Gamma^3}{\sqrt{\mathfrak{S}(\Gamma^1)}} \right\| \leq \left\| \frac{\Gamma^3}{\sqrt{\mathfrak{S}(\Gamma^2)}} \right\| (1 + d_s(\mathfrak{S}(\Gamma^1), \mathfrak{S}(\Gamma^2))).$$

Proof. Let us simply bound:

$$\begin{aligned} \left\| \frac{\Gamma^3}{\sqrt{\mathfrak{S}(\Gamma^1)}} \right\| &\leq \left\| \frac{\Gamma^3}{\sqrt{\mathfrak{S}(\Gamma^2)}} \right\| + \left\| \frac{\Gamma^3 \left(\sqrt{\mathfrak{S}(\Gamma^2)} - \sqrt{\mathfrak{S}(\Gamma^1)} \right)}{\sqrt{\mathfrak{S}(\Gamma^2)\mathfrak{S}(\Gamma^1)}} \right\| \\ &\leq \left\| \frac{\Gamma^3}{\sqrt{\mathfrak{S}(\Gamma^2)}} \right\| + \left\| \frac{\Gamma^3}{\sqrt{\mathfrak{S}(\Gamma^2)}} \right\| \left\| \frac{\mathfrak{S}(\Gamma^2) - \mathfrak{S}(\Gamma^1)}{\sqrt{\mathfrak{S}(\Gamma^1)} \left(\sqrt{\mathfrak{S}(\Gamma^2)} + \sqrt{\mathfrak{S}(\Gamma^1)} \right)} \right\|, \end{aligned}$$

we can then conclude with the bound $\sqrt{\mathfrak{S}(\Gamma^1)} \left(\sqrt{\mathfrak{S}(\Gamma^2)} + \sqrt{\mathfrak{S}(\Gamma^1)} \right) \geq \sqrt{\mathfrak{S}(\Gamma^1)\mathfrak{S}(\Gamma^2)}$. \square

Proof of Proposition E.23. Let us simply bound thanks to Lemma E.22:

$$\begin{aligned} d_s(\tilde{\Gamma}, \Gamma) &\leq \left\| \frac{\tilde{\Gamma} - f(\Gamma)}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\| + \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\| \\ &\leq d_s(\tilde{\Gamma}, f(\Gamma)) (1 + d_s(\mathfrak{S}(\Gamma), \mathfrak{S}(f(\Gamma)))) + \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\| \\ &\leq \lambda d_s(\tilde{\Gamma}, \Gamma) (1 + d_s(\mathfrak{S}(\Gamma), \mathfrak{S}(f(\Gamma)))) + \left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\| \leq \frac{\left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\|}{1 - \lambda - \lambda d_s(\mathfrak{S}(\Gamma), \mathfrak{S}(f(\Gamma)))} \end{aligned}$$

since $d_s(\mathfrak{S}(\Gamma), \mathfrak{S}(f(\Gamma))) \leq o(1) \leq \frac{1-\lambda}{2\lambda}$ for s sufficiently big. \square

³Actually, f does not need to be λ -Lipschitz on the whole set $\mathcal{D}_n(\mathbb{H})$, but we need to be able to bound:

$$d(f(\tilde{\Gamma}), f(\Gamma)) \leq \lambda d(\tilde{\Gamma}, \Gamma)$$

Proposition E.21 will be employed to set asymptotic results when $n \rightarrow \infty$ and to show continuity properties with a parameter $t \rightarrow 0$ in Chapters F and I.

Proposition E.23. *Let us consider a family of mappings of $\mathcal{D}_{n_s}(\mathbb{H})$, $(f^s)_{m \in \mathbb{N}}$, each f^s being λ -Lipschitz for the semi-metric d_s with $\lambda < 1$ and admitting the fixed point $\tilde{\Gamma}^s = f^s(\tilde{\Gamma}^s)$ and a family of diagonal matrices Γ^s . If one assumes that⁴ $d_s(\mathfrak{S}(\Gamma^s), \mathfrak{S}(f^s(\Gamma^s))) \leq o_{s \rightarrow \infty}(1)$, then:*

$$d_s(\Gamma^s, \tilde{\Gamma}^s) \leq O_{s \rightarrow \infty} \left(\left\| \frac{f^s(\Gamma^s) - \Gamma^s}{\sqrt{\mathfrak{S}(\tilde{\Gamma}^s)\mathfrak{S}(\Gamma^s)}} \right\| \right)$$

The same result is valid replacing \mathbb{H} with \mathbb{R}_+ and removing the symbols \mathfrak{S} .

Proof. It suffices to bound for s sufficiently big:

$$\frac{\left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\|}{1 - \lambda - \lambda d_s(\mathfrak{S}(\Gamma), \mathfrak{S}(f(\Gamma)))} \leq O \left(\left\| \frac{f(\Gamma) - \Gamma}{\sqrt{\mathfrak{S}(\tilde{\Gamma})\mathfrak{S}(\Gamma)}} \right\| \right)$$

since $d_s(\mathfrak{S}(\Gamma), \mathfrak{S}(f(\Gamma))) \leq o(1) \leq \frac{1-\lambda}{2\lambda}$ when s tends to infinity. □

⁴Usually the notations $O(1)$ and $o(1)$ are used for quasi-asymptotic studies when n tends to infinity but in this proposition, the relevant parameter is s , thus $d_s(f^s(\Gamma^s), \Gamma^s) \leq o_{s \rightarrow \infty}(1)$ means that for all $K > 0$, there exists $S \in \mathbb{N}$ such that for all $s \geq S$, $d_s(f^s(\Gamma^s), \Gamma^s) \leq K$.

Chapter F

Statistical study of the resolvent

Denoting $\lambda_1 \geq \dots \geq \lambda_p \geq 0$ the p eigenvalues of $\frac{1}{n}XX^T$ the spectral distribution is defined followingly:

$$\mu = \frac{1}{p} \sum_{i=1}^p \delta_{\lambda_i}.$$

F.1 Concentration and estimation of the resolvent

We work here under the formalisme of Levy families where the random matrix under study, $X \in \mathcal{M}_{p,n}$ is seen as a sequence depending on n and p is also a sequence depending on n but that should satisfy the following relation:

Assumption 1. $p \leq O(n)$.

We need of course a concentration hypothesis on X :

Assumption 2. $X \propto \mathcal{E}_2$.

A third natural and fundamental hypothesis is to assume that the n columns of $X = (x_1, \dots, x_n)$ are independent. Again, we do not assume that x_1, \dots, x_n are identically distributed: we can possibly have n different distributions for the columns of X .

Assumption 3. X has independent columns $x_1, \dots, x_n \in \mathbb{R}^p$.

Let us note for simplicity, for any $i \in [n]$:

$$\mu_i \equiv \mathbb{E}[x_i] \qquad \Sigma_i \equiv \mathbb{E}[x_i x_i^T] \qquad \text{and} \qquad C_i \equiv \Sigma_i - \mu_i \mu_i^T.$$

It is easy to deduce from Assumption 2 (see Proposition B.20) that there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$, $\|C_i\| \leq O(1)$. But to have the best convergence bounds, we also need to impose:¹

Assumption 4. $\sup_{i \in [n]} \|\mu_i\| \leq O(1)$.

We conclude with a last assumption which seems important to precisely approximate the support of the spectral distribution.² Although we are unsure of its importance, our line of arguments could not avoid it; it is nonetheless quite a weak constraint in view of the practical use of our result.

Assumption 5. $\inf_{i \in [n]} \Sigma_i \geq O(1)I_p$.

¹In [LLC18], we only supposed that $\|\mu_i\| \leq O(\sqrt{n})$, however, then we only had an approximation of the resolvent with the spectral norm, here we will provide a similar convergence result with the Frobenius norm.

²the values of the Stieltjes distribution $g(z)$ can be approximated for $z \in \mathbb{C}$ sufficiently far from the real axis or for $z \in \mathbb{R}$ sufficiently far from the support without this assumption.

1.1 Resorting to a “concentration zone” for Q^z

Before studying the matricial case, let us first place ourselves in \mathbb{R} . We consider $X \in \mathbb{R}$, a Gaussian random variable with zero mean and variance equal to σ^2 ($X \sim \mathcal{N}(0, \sigma^2)$). In particular, although we work with unidimensional variables, there can still be a possible dependence on n , and we can write $X \propto \mathcal{E}_2$. The random variable $Q \equiv 1/(1 - X)$ is only defined if $X \neq 1$ and its law f_Q can be computed on $\mathbb{R} \setminus \{1\}$ and satisfies:

$$f_Q(q) = \frac{e^{-(1-\frac{1}{q})^2/\sigma^2}}{\sqrt{2\pi\sigma q^2}}.$$

Thus Q is clearly not exponentially concentrated (when $q \rightarrow \infty$, $f_Q(q) \sim \frac{e^{-1/\sigma^2}}{q^2}$ therefore the expectation of Q is not even defined). However, if σ is small enough (at least $\sigma \leq o(1)$), it can be interesting to consider the event $\mathcal{A}_Q \equiv \{X \leq \frac{1}{2}\}$ satisfying $\mathbb{P}(\mathcal{A}_Q^c) \leq Ce^{-1/2\sigma^2}$. The mapping $f : z \mapsto \frac{1}{1-z}$ being 4-Lipschitz on $(-\infty, \frac{1}{2}]$, one sees that $(Q | \mathcal{A}_Q) \in \mathcal{E}_2$. Following this setting, in the matricial cases, we also need to place ourselves in a concentration zone \mathcal{A}_Q where the fixed point Q is defined; sufficiently small to retrieve an exponential concentration with $Q | \mathcal{A}_Q$ but large enough to be highly probable.

The same resort to a *concentration zone* will take place for the resolvent matrix, for that purpose, we introduce in this section an event of high probability, \mathcal{A}_Q , under which the eigen values of $\frac{1}{zn}XX^T$ are far from 1, for all $z \in S^\varepsilon$. Let us start with a bound on $\|X\|$, Assumption 4 leads us to:

$$\|\mathbb{E}[X]\| \leq \sqrt{n} \sup_{1 \leq i \leq n} \|\mathbb{E}[x_i]\| \leq O(\sqrt{n})$$

then, we deduce from Example B.24 applied to Assumption 2 that $\mathbb{E}[\|X\|] \leq \|\mathbb{E}[X]\| + O(\sqrt{n}) \leq O(\sqrt{n})$. Now, introducing a constant $\varepsilon > 0$ such that $O(1) \leq \varepsilon \leq O(1)$ and $\nu > 0$ defined with:

$$\sqrt{\nu} \equiv \mathbb{E} \left[\frac{1}{\sqrt{n}} \|X\| \right] \leq O(1),$$

we denote:

$$\mathcal{A}_\nu \equiv \left\{ \frac{1}{n} \|XX^T\| \leq \nu + \varepsilon \right\}. \quad (\text{F.1})$$

Since $\|X\|/\sqrt{n} \in \nu \pm \mathcal{E}_2(1/\sqrt{n})$, we know that there exist two constants $C, c > 0$ such that $\mathbb{P}(\mathcal{A}_\nu^c) \leq Ce^{-cn}$. The mapping $M \rightarrow \frac{1}{n}MM^T$ is $O(1/\sqrt{n})$ Lipschitz on $X(\mathcal{A}_\nu) \subset \mathcal{M}_{p,n}$ and therefore, thanks to Lemma B.4 and Remark B.5:

$$\frac{1}{n}XX^T | \mathcal{A}_\nu \propto \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right)$$

Let us note $\sigma : \mathcal{M}_p \rightarrow \mathbb{R}^p$, the mapping that associates to any matrix the sequence of its eigen values in decreasing order. It is well known (see [GL96, Theorem 8.1.15]) that σ is 1-Lipschitz (from $(\mathcal{M}_p, \|\cdot\|_F)$ to $(\mathbb{R}^p, \|\cdot\|)$) and therefore if we denote $\lambda_1, \dots, \lambda_p$ the eigen values of $\frac{1}{n}XX^T$ such that $\lambda_1 \geq \dots \geq \lambda_n$, we have the concentration:

$$(\lambda_1, \dots, \lambda_p) | \mathcal{A}_\nu \in (\mathbb{E}_{\mathcal{A}_\nu}[\lambda_1], \dots, \mathbb{E}_{\mathcal{A}_\nu}[\lambda_p]) \pm \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right).$$

Given a set $T \subset \mathbb{C}$, we note for any $\varepsilon > 0$, $T^\varepsilon = \{z \in \mathbb{C}, \exists t \in T, |z - t| \leq \varepsilon\}$. Now, let us denote

$$S \equiv \{\mathbb{E}_{\mathcal{A}_\nu}[\lambda_1], \dots, \mathbb{E}_{\mathcal{A}_\nu}[\lambda_p]\},$$

(with the expectation taken on \mathcal{A}_ν). We show in the next lemma that the event

$$\mathcal{A}_Q \equiv \mathcal{A}_\nu \cap \{\forall i \in [\min(p, n)] : \lambda_i \in S^{\varepsilon/2}\}$$

has an overwhelming probability.

Lemma F.1. *There exist $C, c > 0$ such that $\forall n \in \mathbb{N} \mathbb{P}(\mathcal{A}_Q^c) \leq Ce^{-cn}$.*

Proof of Lemma F.1. Starting from the identity $S^{\frac{\varepsilon}{2}} \cup_{i \in [p]} \left[\mathbb{E}_{\mathcal{A}_\nu}[\lambda_i] - \frac{\varepsilon}{2}, \mathbb{E}_{\mathcal{A}_\nu}[\lambda_i] + \frac{\varepsilon}{2} \right]$, we see that $S^{\frac{\varepsilon}{2}}$ is a union of, say, d intervals of \mathbb{R} . There exists $2d$ indexes $i_1 \leq \dots \leq i_d$ and $j_1 \leq \dots \leq j_d$ in $[\min(p, n)]$ such that:

$$S^{\frac{\varepsilon}{2}} = \bigcup_{k \in [d]} \left[\mathbb{E}_{\mathcal{A}_\nu}[\lambda_{i_k}] - \frac{\varepsilon}{2}, \mathbb{E}_{\mathcal{A}_\nu}[\lambda_{j_k}] + \frac{\varepsilon}{2} \right].$$

Since $\mathbb{E}_{\mathcal{A}_\nu}[\lambda_{i_1}] \geq 0$ and $\mathbb{E}_{\mathcal{A}_\nu}[\lambda_{j_d}] \leq \sqrt{\nu} \pm O(e^{-cn}) \leq O(1)$, we can bound:

$$\varepsilon d \leq \mathbb{E}_{\mathcal{A}_\nu}[\lambda_{j_d}] + \varepsilon \leq O(1),$$

That implies in particular that $d \leq O(1)$ because $\varepsilon \geq O(1)$. We can then bound thanks to the concentration of the $2d$ random variables $\lambda_{i_1}, \dots, \lambda_{i_d}$ and $\lambda_{j_1}, \dots, \lambda_{j_d}$:

$$\begin{aligned} \mathbb{P}(\mathcal{A}_Q^c) &= \mathbb{P}\left(\exists k \in [d], |\lambda_{i_k} - \mathbb{E}_{\mathcal{A}_\nu}[\lambda_{i_k}]| > \frac{\varepsilon}{2} \text{ or } |\lambda_{j_k} - \mathbb{E}_{\mathcal{A}_\nu}[\lambda_{j_k}]| > \frac{\varepsilon}{2}\right) \\ &\leq \sum_{k=1}^d \left(\mathbb{P}\left(|\lambda_{i_k} - \mathbb{E}_{\mathcal{A}_\nu}[\lambda_{i_k}]| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(|\lambda_{j_k} - \mathbb{E}_{\mathcal{A}_\nu}[\lambda_{j_k}]| > \frac{\varepsilon}{2}\right) \right) \leq 2dCe^{-cn\varepsilon^2/4}, \end{aligned}$$

where C and c are the two constants appearing in the concentration inequality of $(\lambda_1, \dots, \lambda_p) = \sigma\left(\frac{1}{n}XX^T\right)$. \square

Now, let us remark that the spectrum of $\frac{1}{n}XX^T$ is closely related to the spectrum of $\frac{1}{n}X^TX$ via the equivalence:

$$\lambda \in \text{Sp}\left(\frac{1}{n}XX^T\right) \setminus \{0\} \iff \lambda \in \text{Sp}\left(\frac{1}{n}X^TX\right) \setminus \{0\}.$$

As a consequence, one of the two matrices has $|n - p|$ supplementary zeros in its spectrum. Since those zeros do not carry any specific information about the distribution of $\frac{1}{n}XX^T$, we try to remove them from the study to re-establish the symmetry. The $\min(p, n)$ first entries of $\sigma\left(\frac{1}{n}XX^T\right)$ and $\sigma\left(\frac{1}{n}X^TX\right)$ are the same (and some of them can cancel), we thus naturally introduce the set:

$$S_{-0} \equiv \{\mathbb{E}_{\mathcal{A}_Q}[\lambda_i], i \in [\min(p, n)]\},$$

Be careful that if $0 \in \text{Sp}\left(\frac{1}{n}XX^T\right) \cap \text{Sp}\left(\frac{1}{n}X^TX\right)$ then $0 \in S_{-0}$.

To avoid the issue on zero, instead of studying, as it is usually done, the resolvent $\left(\frac{1}{n}XX^T - zI_p\right)^{-1}$, we rather look at:

$$Q^z \equiv \left(I_p - \frac{1}{zn}XX^T\right)^{-1},$$

that has the advantage of satisfying $\|Q^z\| \leq O(1)$, for all $z \in S_{-0}$, which will allow us to show the concentration of Q^z even for z close to zero when $n < p$ (and $0 \notin S_{-0}$).

One of the objective of the study was to establish converging bounds when z is at a constant distance from S_{-0} . We were not able to set it for all our results (we failed with Proposition F.20 where we could only set the result for z at a constant distance from S) however, we hope to be able to set the general result after further research, we thus keep, when accessible, general formulations with S_{-0} .

1.2 Concentration of the resolvent $Q^z = \left(I_n - \frac{1}{nz}XX^T\right)^{-1}$.

Given a matrix $A \in \mathcal{M}_{p,n}(\mathbb{C})$, we denote $|A| = \sqrt{A\bar{A}^T}$ ($A\bar{A}^T$ is a nonnegative Hermitian matrix). With a simple diagonalization procedure, one can show for any Hermitian matrix A the simple characterization of the spectrum of $|A|$:

$$\text{Sp}(|A|) = \{|\lambda|, \lambda \in \text{Sp}(A)\}, \tag{F.2}$$

note also that for any real vector $u \in \mathbb{R}^p$:

$$|u^T Au| \leq u^T |A| u. \quad (\text{F.3})$$

It is possible to go further than the mere bound $|Q^z| \leq O(1)$ when z is close to 0 and $p \leq n$, we are then able to show that $|Q^z| \leq O(|z|/(|z|+1))$. When $p \geq n$ no such bound is true and it is then convenient to rather look at the coresolvent $\check{Q}^z \equiv (I_p - \frac{1}{nz} X^T X)^{-1}$ that satisfies in that regime $|\check{Q}^z| \leq O(|z|/(|z|+1))$. To formalize this approach, we introduce two quantities that will appear in our convergence speeds:

$$\kappa_z \equiv \begin{cases} \frac{|z|}{1+|z|} & \text{if } p \leq n \\ 1 & \text{if } n \leq p \end{cases} \quad \check{\kappa}_z \equiv \begin{cases} 1 & \text{if } p \leq n \\ \frac{|z|}{1+|z|} & \text{if } n \leq p \end{cases}$$

note that both of them are bounded by 1 but they can tend to zero with $|z|$ depending on the sign of $p - n$, besides:

$$\kappa_z \check{\kappa}_z = \frac{|z|}{1+|z|}. \quad (\text{F.4})$$

Note than in our formalism, the parameter z , as most quantities of our manuscript, is varying with n (we do not assume that $O(1) \leq |z| \leq O(1)$ like ε . It is not a "constant").

Lemma F.2. *Under \mathcal{A}_Q , Q^z and \check{Q}^z can be defined on 0 and for any³ $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$:*

$$O(\kappa_z) I_p \leq |Q^z| \leq O(\kappa_z) I_p \quad \text{and} \quad O(\check{\kappa}_z) I_p \leq |\check{Q}^z| \leq O(\check{\kappa}_z) I_p$$

(for the classical order relation on hermitian matrices).

Proof. We can diagonalize the nonnegative symmetric matrix: $\frac{1}{n} X X^T = P D P^T$, with $D = \text{Diag}(\lambda_1, \dots, \lambda_p)$ and $P \in \mathcal{O}_p$, an orthogonal matrix. There exists $q \in [p]$ such that $\lambda_{q+1} = \dots = \lambda_p = 0$, and for all $i \leq q$, $\lambda_i \neq 0$ (possibly $q = p$ or $q < \min(p, n)$). Then, if we denote $P_0 \in \mathcal{M}_{p, p-q}$ the matrix composed of the $p - q$ last columns of P , P_+ the matrix composed of the rest of the q columns, and $D_+ = \text{Diag}(\lambda_1, \dots, \lambda_q)$, we can decompose:

$$Q^z = P \begin{pmatrix} z I_p \\ z I_p - D \end{pmatrix} P^T = P_0 P_0^T + P_+ \begin{pmatrix} z I_q \\ z I_q - D_+ \end{pmatrix} P_+^T.$$

Since $P_0 P_0^T = 0$ and $P_+ P_+^T = 0$, we have: $|Q^z|^2 = P_0 P_0^T + P_+ \begin{pmatrix} |z|^2 I_q \\ |z I_q - D_+|^2 \end{pmatrix} P_+^T$. We can bound:

$$O\left(\frac{|z|}{1+|z|}\right) \leq \frac{|z|}{|z| + \nu + \varepsilon} \leq P_+ \left(\frac{|z| I_q}{|z I_q - D_+|} \right) P_+^T \leq \frac{|z|}{d(z, S_{-0}) - \frac{\varepsilon}{2}} \leq O\left(\frac{|z|}{1+|z|}\right)$$

Therefore, in all cases ($p \leq n$ or $p \geq n$) $O(1) \leq |Q^z| \leq O(1)$. However, when $p \leq n$, we can precise those bounds:

- if $\mathbb{E}_{\mathcal{A}_Q}[\lambda_{\min(p, n)}] > \frac{\varepsilon}{2}$, then P_0 is empty, $P_+ = P$ and we see that $O\left(\frac{|z|}{1+|z|}\right) \leq |Q^z| \leq O\left(\frac{|z|}{1+|z|}\right)$.
- if $\mathbb{E}_{\mathcal{A}_Q}[\lambda_{\min(p, n)}] \leq \frac{\varepsilon}{2}$, the bound $d(z, S_{-0}) \geq \varepsilon$ implies that $|z| \geq \frac{\varepsilon}{2}$ and $O(\kappa_z) = O\left(\frac{|z|}{1+|z|}\right) \leq O(1) \leq O(\kappa_z)$, therefore:

$$O(\kappa_z) \leq O(\min(1, \kappa_z)) I_p \leq |Q^z| \leq O(1 + \kappa_z) I_p \leq O(\kappa_z).$$

The inequalities on \check{Q}^z are proven the same way. □

³In theory, both the parameter z and the set S_{-0}^ε depends on our asymptotic parameter n , so one should rigorously write $z_n \in S_{-0}^\varepsilon(n)$

Proposition F.3. *Given $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$, we have the concentrations $Q^z \mid \mathcal{A}_Q \propto \mathcal{E}_2(\kappa_z/\sqrt{n})$ in $(\mathcal{M}_p, \|\cdot\|_F)$ and $\check{Q}^z \mid \mathcal{A}_Q \propto \mathcal{E}_2(\check{\kappa}_z/\sqrt{n})$ in $(\mathcal{M}_n, \|\cdot\|_F)$.*

Proof. Noting $\Phi : \mathcal{M}_{p,n} \rightarrow \mathcal{M}_p(\mathbb{C})$ and $\check{\Phi} : \mathcal{M}_{p,n} \rightarrow \mathcal{M}_n(\mathbb{C})$ defined as:

$$\Phi(M) = \left(I_p - \frac{MM^T}{zn} \right)^{-1} \quad \text{and} \quad \check{\Phi}(M) = \left(I_n - \frac{M^T M}{zn} \right)^{-1},$$

it is sufficient to show that Φ (resp. $\check{\Phi}$) is $O(\kappa_z/\sqrt{n})$ -Lipschitz (resp. $O(\check{\kappa}_z/\sqrt{n})$ -Lipschitz) on $\mathcal{M}_{n,p}^{\mathcal{A}_Q} \equiv X(\mathcal{A}_Q)^4$. For any $M \in \mathcal{M}_{n,p}^{\mathcal{A}_Q}$ and any $H \in \mathcal{M}_{p,n}$, we can bound $\|M\| \leq (\nu + \varepsilon)\sqrt{n} \leq O(\sqrt{n})$ and:

$$\|d\Phi|_M \cdot H\|_F = \left\| \Phi(M) \frac{1}{nz} (MH^T + HM^T) \Phi(M) \right\|_F$$

We can now distinguish two different cases:

- if $p \leq n$ then:

$$\|d\Phi|_M \cdot H\|_F \leq O\left(\frac{\|H\|_F |z|}{(1+|z|)^2 \sqrt{n}}\right) \leq O\left(\frac{\|H\|_F |z|/\sqrt{n}}{1+|z|}\right).$$

- if $p \geq n$, we know that $\|\Phi(M)\| \leq O(\kappa_z) \leq O(1)$ and $\|\check{\Phi}(M)\| \leq O(\check{\kappa}_z) \leq O(\frac{|z|}{1+|z|})$. We employ then the classical identity:

$$\Phi(M)M = M^T \check{\Phi}(M) \tag{F.5}$$

to be able to bound:

$$\begin{aligned} \|d\Phi|_M \cdot H\|_F &\leq \frac{2(\nu + \varepsilon)\|H\|_F}{|z|\sqrt{n}} \|\check{\Phi}(M)\| \|\Phi(M)\| \\ &\leq O\left(\frac{\|H\|_F}{(1+|z|)\sqrt{n}}\right) \leq O\left(\frac{\|H\|_F}{\sqrt{n}}\right) \end{aligned}$$

Thus, in all cases, under $X(\mathcal{A}_Q)$, Q^z is a $O(\kappa_z/\sqrt{n})$ -Lipschitz transformation of $X \propto \mathcal{E}_2(1)$ and as such, it satisfies the concentration inequality of the proposition thanks to Lemma B.4 and Remark B.5. The same holds for \check{Q}^z . \square

Remark F.4. *The preceding proof partly relies on the assertion $\|X\| \leq \sqrt{n}$ available under the event \mathcal{A}_Q . However one does not need it to be able to prove that Q is a Lipschitz transformation of X since one can bound the element QX/\sqrt{n} as a whole thanks to the formulas:*

$$\frac{1}{zn} Q^z X X^T = Q^z - I_n \quad \text{and} \quad \frac{1}{zn} \check{Q}^z X^T X = \check{Q}^z - I_n. \tag{F.6}$$

On the one hand:

$$\left\| \frac{1}{\sqrt{n}} Q^z X \right\| \leq \sqrt{|z| \left\| \frac{1}{zn} Q^z X X^T Q^z \right\|} \leq \sqrt{|z| \|(Q^z)^2 - Q^z\|} \leq O(\sqrt{|z|(\kappa_z + \kappa_z^2)}) \leq O(\sqrt{|z|\kappa_z})$$

On the other hand, (F.5) gives us:

$$\left\| \frac{1}{\sqrt{n}} Q^z X \right\| = \left\| \frac{1}{\sqrt{n}} X \check{Q}^z \right\| \leq \sqrt{|z| \left\| \frac{1}{zn} Q^z X^T X Q^z \right\|} \leq O(\sqrt{|z|\check{\kappa}_z})$$

⁴ $\mathcal{M}_{n,p}^{\mathcal{A}_Q} \subset \{M \in \mathcal{M}_{n,p}, \frac{1}{n}\|MM^T\| \leq \nu + \varepsilon\}$

1.3 A first deterministic equivalent

One is often merely working with linear functionals of Q^z , and since Proposition F.3 implies that $Q^z | \mathcal{A}_Q \in \mathbb{E}_{\mathcal{A}_Q} Q^z \pm \mathcal{E}_2$, one naturally wants to estimate the expectation $\mathbb{E}_{\mathcal{A}_Q}[Q^z]$.

In [LC18] is provided a deterministic equivalent $\tilde{Q}^z \in \mathcal{M}_p(\mathbb{C})$ satisfying $\|\mathbb{E}[Q^z] - \tilde{Q}^z\| \leq O(1/\sqrt{n})$ for any $z \in \mathbb{R}^-$, we are going to show below a stronger result,

- with a Frobenius norm replacing the spectral norm,
- for any complex $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$,
- for random vectors x_1, \dots, x_n having possibly different distributions (it was assumed in [LC18] that there was a finite number of classes)

An efficient approach, developed in particular in [Sil86, SB95] is to look for a deterministic equivalent of Q^z depending on a deterministic diagonal matrix $\Delta \in \mathbb{R}^n$ and having the form:

$$\tilde{Q}^\Delta = (I_p - \Sigma^\Delta)^{-1} \quad \text{where} \quad \Sigma^\Delta \equiv \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\Delta_i} = \frac{1}{n} \mathbb{E}[X \Delta^{-1} X^T]. \quad (\text{F.7})$$

One can then express the difference with the expectation of Q^z under \mathcal{A}_Q , $\mathbb{E}_{\mathcal{A}_Q}[Q^z]$ followingly:

$$\begin{aligned} \mathbb{E}_{\mathcal{A}_Q}[Q^z] - \tilde{Q}^\Delta &= \mathbb{E}_{\mathcal{A}_Q} \left[Q^z \left(\frac{1}{zn} X X^T - \Sigma^\Delta \right) \tilde{Q}^\Delta \right] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\mathcal{A}_Q} \left[Q^z \left(\frac{x_i x_i^T}{z} - \frac{\Sigma_i}{\Delta_i} \right) \tilde{Q}^\Delta \right]. \end{aligned}$$

To pursue the estimation of the expectation, one needs to control the dependence between Q^z and x_i . For that purpose, one uses classically the Schur identities:

$$Q^z = Q_{-i}^z + \frac{1}{zn} \frac{Q_{-i}^z x_i x_i^T Q_{-i}^z}{1 - \frac{1}{zn} x_i^T Q_{-i}^z x_i} \quad \text{and} \quad Q^z x_i = \frac{Q_{-i}^z x_i}{1 - \frac{1}{zn} x_i^T Q_{-i}^z x_i}, \quad (\text{F.8})$$

for $Q_{-i}^z = (I_n - \frac{1}{zn} X_{-i} X_{-i}^T)^{-1}$ (recall that $X_{-i} = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \in \mathcal{M}_{p,n}$). The Schur identities can be seen as simple consequences to the so called ‘‘resolvent identity’’ that can be generalized to any, possibly non commuting, square matrices $A, B \in \mathcal{M}_p$ with the identity:

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \quad \text{or} \quad A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1} \quad (\text{F.9})$$

(it suffices to note that $A(A^{-1} + A^{-1}(B - A)B^{-1})B = I_p$).

Introducing the notation:

$$\Lambda^z \equiv \text{Diag}_{1 \leq i \leq n} \left(z - \frac{1}{n} x_i^T Q_{-i}^z x_i \right), \quad \text{satisfying:} \quad \forall i \in [n] : Q^z x_i = \frac{z}{\Lambda^z} Q_{-i}^z x_i,$$

one can express thanks to the independence between Q_{-i}^z and x_i :

$$\begin{aligned} \mathbb{E}_{\mathcal{A}_Q}[Q^z] - \tilde{Q}^\Delta &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\mathcal{A}_Q} \left[Q_{-i}^z \left(\frac{x_i x_i^T}{\Lambda_i^z} - \frac{\Sigma_i}{\Delta_i} \right) \tilde{Q}^\Delta \right] + \frac{1}{n} \sum_{i=1}^n \frac{1}{\Delta_i} \mathbb{E}_{\mathcal{A}_Q} \left[(Q_{-i}^z - Q^z) \Sigma_i \tilde{Q}^\Delta \right] \\ &= \varepsilon_1 + \delta_1 + \delta_2 + \varepsilon_2 \end{aligned} \quad (\text{F.10})$$

$$\text{with :} \quad \begin{cases} \varepsilon_1 = \frac{1}{n} \mathbb{E}_{\mathcal{A}_Q} \left[\sum_{i=1}^n Q_{-i}^z x_i \left(\frac{\Delta_i - \Lambda_i^z}{\Lambda_i^z \Delta_i} \right) x_i^T \tilde{Q}^\Delta \right] = \frac{1}{n} \mathbb{E}_{\mathcal{A}_Q} \left[Q^z X \left(\frac{\Delta - \Lambda^z}{z \Delta} \right) X^T \tilde{Q}^\Delta \right] \\ \delta_1 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\mathcal{A}_Q} \left[Q_{-i}^z \left(\frac{x_i x_i^T - \mathbb{E}_{\mathcal{A}_Q}[x_i x_i^T]}{\Delta_i} \right) \tilde{Q}^\Delta \right] \\ \delta_2 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\mathcal{A}_Q} \left[Q_{-i}^z \left(\frac{\mathbb{E}_{\mathcal{A}_Q}[x_i x_i^T] - \Sigma_i}{\Delta_i} \right) \tilde{Q}^\Delta \right] \\ \varepsilon_2 = -\frac{1}{zn^2} \sum_{i=1}^n \frac{1}{\Delta_i} \mathbb{E}_{\mathcal{A}_Q} \left[Q^z x_i x_i^T Q_{-i}^z \Sigma_i \tilde{Q}^\Delta \right], \end{cases}$$

where we recall that $Q^z - Q_{-i}^z = \frac{1}{nz} Q^z x_i x_i^T Q_{-i}^z$. From this decomposition, one is enticed into choosing, in a first step $\Delta \approx \mathbb{E}_{\mathcal{A}_Q}[\Lambda^z] \in \mathcal{D}_n(\mathbb{C})$ so that ε_1 would be small. We will indeed take for Δ , the deterministic diagonal matrix:

$$\hat{\Lambda}^z \equiv \text{Diag} \left(z - \frac{1}{n} \text{Tr}(\Sigma_i \mathbb{E}_{\mathcal{A}_Q}[Q^z]) \right)_{1 \leq i \leq n} \in \mathcal{D}_n(\mathbb{C}).$$

Lemma F.5. *Given $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$, $(\Lambda^z | \mathcal{A}_Q) \propto \mathcal{E}_2(\kappa_z/\sqrt{n})$ in $(\mathcal{D}_n(\mathbb{C}), \|\cdot\|)$.*

Proof. Inspiring from the proof of Proposition F.3, one can show easily that for all $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$, the mapping $X \rightarrow \Lambda^z = \text{Diag}_{i \in [n]}(z - \frac{1}{n} x_i^T Q_{-i}^z x_i)$ is a $O(\kappa_z/\sqrt{n})$ -Lipschitz transformation from $(\mathcal{M}_{p,n}, \|\cdot\|_F)$ to $(\mathcal{D}_n, \|\cdot\|)$ under \mathcal{A}_Q (since then $\|x_i\| \leq O(\sqrt{n})$). \square

Putting the Schur identities (F.8), the relation (F.5) and the formula (F.11)

$$\frac{1}{zn} \check{Q}^z X X^T = \check{Q}^z - I_n, \quad (\text{F.11})$$

together one obtains:

$$\frac{z}{\Lambda^z} = \text{Diag}_{i \in [n]} \left(\frac{1}{1 - \frac{1}{zn} x_i^T Q_{-i}^z x_i} \right) = I_n + \frac{1}{zn} \text{Diag}(X^T Q^z X) = \text{Diag}(\check{Q}^z) \quad (\text{F.12})$$

To be able to use Proposition B.50 with $\hat{\Lambda}^z$, one further needs:

Lemma F.6. $\|\mathbb{E}_{\mathcal{A}_Q}[\Lambda^z] - \hat{\Lambda}^z\| \leq O(\kappa_z/n)$.

This lemma that seems quite simple actually requires three preliminary results whose main aim is to show that Q_{-i}^z is close to Q^z . Let us first try and bound Λ^z thanks to (F.12).

Lemma F.7. *Given $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$:*

$$O(|z|) I_n \leq O\left(\frac{|z|}{\check{\kappa}_z}\right) I_n \leq |\Lambda^z| \leq O\left(\frac{|z|}{\check{\kappa}_z}\right) I_n \leq O(1 + |z|) I_n$$

Proof. The inequalities provided in (F.2), (F.3) and Lemma F.2 imply:

$$O(\check{\kappa}_z) \leq \inf_{i \in [n]} \text{Sp}(\check{Q}^z) \leq |\text{Diag}(\check{Q}^z)| = |\text{Diag}_{i \in [n]}(e_i^T \check{Q}^z e_i)| \leq \sup_{i \in [n]} \text{Sp}(\check{Q}^z) \leq O(\check{\kappa}_z)$$

where e_1, \dots, e_n are the n vectors of the canonical basis of \mathbb{R}^n ($e_i \in \mathbb{R}^n$ is full of 0 except in the i^{th} entry where there is a 1). One can then deduce the result of the lemma thanks to (F.12). \square

Then to be able to neglect the dependence relation between x_i and X under \mathcal{A}_Q we introduce the following lemma.

Lemma F.8 (Independence under \mathcal{A}_Q). *Given two mappings $f : \mathbb{R}^p \rightarrow \mathcal{M}_p$ and $g : \mathcal{M}_{p,n} \rightarrow \mathcal{M}_p$ such that under \mathcal{A}_Q , $\|f(x_i)\|_F \leq O(\kappa_f)$ and $\|g(X_{-i})\| \leq O(\kappa_g)$, we can approximate:*

$$\|\mathbb{E}_{\mathcal{A}_Q}[f(x_i)g(X_{-i})] - \mathbb{E}_{\mathcal{A}_Q}[f(x_i)]\mathbb{E}_{\mathcal{A}_Q}[g(X_{-i})]\|_F \leq O(\kappa_f \kappa_g e^{-cn}),$$

for some constant $c \geq O(1)$.

Proof. Let us continue $f|_{x_i(\mathcal{A}_Q)}$ and $g|_{X_{-i}(\mathcal{A}_Q)}$ respectively on \mathbb{R}^p and on $\mathcal{M}_{p,n}$ defining for any $x \in \mathbb{R}^p$ and $M \in \mathcal{M}_{p,n}$:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in x_i(\mathcal{A}_Q) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{g}(M) = \begin{cases} g(M) & \text{if } M \in X_{-i}(\mathcal{A}_Q) \\ 0 & \text{otherwise} \end{cases}$$

Let us estimate:

$$\begin{aligned}\mathbb{E}_{\mathcal{A}_Q} [f(x_i)g(X_{-i})] &= \frac{\mathbb{E} [\mathbb{1}_{X \in X(\mathcal{A}_Q)} f(x_i)g(X_{-i})]}{\mathbb{P}(\mathcal{A}_Q)} \\ &= \mathbb{E} \left[\mathbb{1}_{X \in X(\mathcal{A}_Q)} \tilde{f}(x_i) \tilde{g}(X_{-i}) \right] + \frac{1 - \mathbb{P}(\mathcal{A}_Q)}{\mathbb{P}(\mathcal{A}_Q)} \mathbb{E} \left[\tilde{f}(x_i) \tilde{g}(X_{-i}) \mathbb{1}_{X \in X(\mathcal{A}_Q)} \right].\end{aligned}$$

the far right hand term cancels since \mathcal{A}_Q has a probability bigger than $1 - Ce^{-cn}$ for some constants $C, c > 0$:

$$\frac{1 - \mathbb{P}(\mathcal{A}_Q)}{\mathbb{P}(\mathcal{A}_Q)} \mathbb{E} \left[\tilde{f}(x_i) \tilde{g}(X_{-i}) \mathbb{1}_{X \in X(\mathcal{A}_Q)} \right] \leq \frac{Ce^{-cn}}{1 - Ce^{-cn}} \mathbb{E} \left[\left| \tilde{f}(x_i) \tilde{g}(X_{-i}) \right| \right] \leq O(\kappa_f \kappa_g e^{-cn}).$$

For all $\omega \in \Omega$, we besides know that:

$$\mathbb{1}_{X \in X(\mathcal{A}_Q)}(\omega) = \mathbb{1}_{X(\omega) \in X(\mathcal{A}_Q)}(\omega) \leq \mathbb{1}_{x_i \in x_i(\mathcal{A}_Q)}(\omega) \mathbb{1}_{X_{-i} \in X_{-i}(\mathcal{A}_Q)}(\omega),$$

and the inequality $\mathbb{1}_{\mathcal{A}_Q}(\omega) \neq \mathbb{1}_{x_i \in x_i(\mathcal{A}_Q)}(\omega) \mathbb{1}_{X_{-i} \in X_{-i}(\mathcal{A}_Q)}(\omega)$ only happens for $\omega \in \mathcal{A}_Q^c$. We can then bound:

$$\left| \mathbb{E} \left[(\mathbb{1}_{X \in \mathcal{A}_Q} - \mathbb{1}_{x_i \in x_i(\mathcal{A}_Q)} \mathbb{1}_{X_{-i} \in X_{-i}(\mathcal{A}_Q)}) \tilde{f}(x_i) \tilde{g}(X_{-i}) \right] \right| \leq \kappa_f \kappa_g \mathbb{E} [\mathbb{1}_{\mathcal{A}_Q^c}] \leq O(\kappa_f \kappa_g e^{-cn}),$$

which allows us to set⁵ thanks to the independence between X_{-i} and x_i :

$$\begin{aligned}\mathbb{E}_{\mathcal{A}_Q} [f(x_i)g(X_{-i})] &= \mathbb{E} \left[\mathbb{1}_{x_i \in x_i(\mathcal{A}_Q)} \tilde{f}(x_i) \mathbb{1}_{X_{-i} \in X_{-i}(\mathcal{A}_Q)} \tilde{g}(X_{-i}) \right] + O_{\|\cdot\|_F}(\kappa_f \kappa_g e^{-cn}) \\ &= \mathbb{E} [\tilde{f}(x_i)] \mathbb{E} [\tilde{g}(X_{-i})] + O_{\|\cdot\|_F}(\kappa_f \kappa_g e^{-cn}) \\ &= \mathbb{E}_{\mathcal{A}_Q} [f(x_i)] \mathbb{E}_{\mathcal{A}_Q} [g(X_{-i})] + O_{\|\cdot\|_F}(\kappa_f \kappa_g e^{-cn})\end{aligned}$$

□

As it was done in Proposition F.3, one can show that $u^T Q_{-i}^z x_i$ and $u^T Q_{-i}^z x_i$ are both κ_z -Lipschitz transformations of $X \times \mathcal{E}_2$ and deduce:

Lemma F.9. *Given a deterministic vector $u \in \mathbb{R}^p$, we have the concentration:*

$$u^T Q_{-i}^z x_i \mid \mathcal{A}_Q \in O(\kappa_z) \pm \mathcal{E}_2(\kappa_z)$$

Proof. As it was done in Proposition F.3, one can show that $u^T Q_{-i}^z x_i$ is a κ_z -Lipschitz transformations of $X \times \mathcal{E}_2$ and deduce the concentration. Besides, one can bound thanks to Lemma F.8 and our assumptions:

$$\left| \mathbb{E}_{\mathcal{A}_Q} [u^T Q_{-i}^z x_i] \right| \leq |u^T \mathbb{E}_{\mathcal{A}_Q} [Q_{-i}] m_i| + O\left(\frac{\kappa_z}{n}\right) \leq O(\kappa_z)$$

□

Lemma F.10. *Given $z \in \mathbb{C} \setminus S_{\leq 0}^{\varepsilon}$, for all $i \in [n]$, $\|\mathbb{E}_{\mathcal{A}_Q} [Q^z - Q_{-i}^z]\| \leq O(\frac{\kappa_z}{n})$, the same way, one has $\|\mathbb{E}_{\mathcal{A}_Q} [|Q^z|^2 - |Q_{-i}^z|^2]\| \leq O(\frac{\kappa_z^2}{n})$.*

Proof. As we saw with Lemma F.8, we can consider that under \mathcal{A}_Q , X_{-i} and x_i are almost independent. We now omit the exponent “ z ” on Q and Q_{-i} for simplicity. Given two deterministic vectors $u, v \in \mathbb{R}^p$, we can deduce from Lemma F.9 and the estimation of the product of concentrated random variables given in Lemma A.21:

$$\begin{aligned}\left| u^T \mathbb{E}_{\mathcal{A}_Q} [Q - Q_{-i}] v \right| &\leq \left| \mathbb{E}_{\mathcal{A}_Q} \left[\frac{u^T Q_{-i} x_i x_i^T Q_{-i} v}{n \Lambda_i} \right] \right| \\ &\leq \frac{\tilde{\kappa}_z}{n |z|} \mathbb{E}_{\mathcal{A}_Q} [|u^T Q_{-i} x_i| |x_i^T Q_{-i} v|] \leq O\left(\frac{\tilde{\kappa}_z \kappa_z}{n |z|}\right) \leq O\left(\frac{\kappa_z}{n}\right).\end{aligned}$$

⁵for $x_p, y_p \in \mathcal{M}_p$ and $(a_p)_{p \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ the notation $x_p = y_p + O_{\|\cdot\|_F}(a_p)$ signifies that $\|x_p - y_p\|_F \leq O(a_p)$

thanks to the formula $\kappa_z \check{\kappa}_z = \frac{|z|}{1+|z|}$.

For the difference of the squares, let us bound with the same justifications:

$$\begin{aligned} & |u^T \mathbb{E}_{\mathcal{A}_Q} [|Q|^2 - |Q_{-i}|^2] v| = |u^T \mathbb{E}_{\mathcal{A}_Q} [\bar{Q}Q - \bar{Q}_{-i}Q_{-i}] v| \\ &= \left| \mathbb{E}_{\mathcal{A}_Q} \left[\frac{u^T Q_{-i} x_i x_i^T \bar{Q} Q_{-i} v}{n \Lambda_i} \right] \right| + \left| \mathbb{E}_{\mathcal{A}_Q} \left[\frac{u^T \bar{Q}_{-i} Q_{-i} x_i x_i^T \bar{Q}_{-i} v}{n \Lambda_i} \right] \right| \\ &\leq O \left(\frac{\check{\kappa}_z \kappa_z^3}{|z| n} \right) \leq O \left(\frac{\kappa_z^2}{n} \right), \end{aligned}$$

□

Proof of Lemma F.6. The proof of the concentration of Λ^z was already presented before the lemma. We are just left to bound:

$$\begin{aligned} & \left\| \hat{\Lambda}^z - \mathbb{E}_{\mathcal{A}_Q} [\Lambda^z] \right\| \\ & \leq \frac{1}{n} \sup_{i \in [n]} (|\text{Tr}(\Sigma_i \mathbb{E}_{\mathcal{A}_Q} [Q - Q_{-i}])| + |\text{Tr}((\Sigma_i - \mathbb{E}_{\mathcal{A}_Q} [x_i x_i^T]) \mathbb{E}_{\mathcal{A}_Q} [Q_{-i}])| \\ & \quad + |\text{Tr}(\mathbb{E}_{\mathcal{A}_Q} [x_i x_i^T] \mathbb{E}_{\mathcal{A}_Q} [Q_{-i}] - \mathbb{E}_{\mathcal{A}_Q} [x_i x_i^T Q_{-i}])|) \leq O \left(\frac{\kappa_z}{n} \right) \end{aligned} \quad (\text{F.13})$$

with the same arguments as in the proof of Lemma F.10. □

One can directly deduce from the concentration of Λ^z and Lemma F.7 a bound on its expectation $\hat{\Lambda}^z$.

Lemma F.11. *Given $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$:*

$$O(|z|) I_n \leq O \left(\frac{|z|}{\check{\kappa}_z} \right) I_n \leq \left| \hat{\Lambda}^z \right| \leq O \left(\frac{|z|}{\check{\kappa}_z} \right) I_n \leq O(1 + |z|) I_n$$

Proof. One already know from Lemma F.7 that $O \left(\frac{|z|}{\check{\kappa}_z} \right) \leq |\Lambda^z| \leq O \left(\frac{|z|}{\check{\kappa}_z} \right)$ then it suffices to bound thanks to (F.13): $\left\| \hat{\Lambda}^z - \mathbb{E}_{\mathcal{A}_Q} [\Lambda^z] \right\| \leq O \left(\frac{\kappa_z}{n} \right) \leq O \left(\frac{|z|}{n \check{\kappa}_z} \right)$. □

We can now prove the main result of this subsections that allows us to set that $\tilde{Q}^{\hat{\Lambda}^z}$ is a deterministic equivalent of Q^z (thanks to Lemma B.18).

Proposition F.12. *Given $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$:*

$$\left\| \tilde{Q}^{\hat{\Lambda}^z} \right\| \leq O(\kappa_z) \quad \text{and} \quad \left\| \mathbb{E}_{\mathcal{A}_Q} [Q^z] - \tilde{Q}^{\hat{\Lambda}^z} \right\|_F \leq O \left(\frac{\kappa_z}{\sqrt{n}} \right).$$

To prove this proposition, we will bound the different elements of the decomposition (F.10). To bound ε_1 , we will need the following lemma. The concentration is quit sharp since we have:

$$\min(\kappa_z, \check{\kappa}_z) = \kappa_z \check{\kappa}_z = \frac{|z|}{1+|z|}$$

Lemma F.13. *Given $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$, under \mathcal{A}_Q ,*

$$Q^z X = X^T \tilde{Q}^z \mid \mathcal{A}_Q \propto \mathcal{E}_2 \left(\frac{|z|}{1+|z|} \right)$$

and $\forall i \in [n]$, $\|\mathbb{E}_{\mathcal{A}_Q} [Q^z x_i]\| \leq O \left(\frac{|z|}{1+|z|} \right)$.

Proof. We follow the steps of the proof of Proposition F.3. Depending on the sign of $p - n$, it is more convenient to work with the expression $Q^z X$ (when $p \leq n$) or with $X^T \bar{Q}^z$ (when $p \geq n$). We just treat here the case $p \leq n$ and therefore look at the variations of the mapping $\Psi : \mathcal{M}_{p,n} \rightarrow \mathcal{M}_{p,n}(\mathbb{C})$ defined as:

$$\Psi(M) = \frac{1}{z} \left(I_p - \frac{MM^T}{zn} \right)^{-1} M.$$

to show the concentration of $Q^z X = \Psi(X)$. For all $H, M \in \mathcal{M}_{n,p}^{\mathcal{A}_Q} \equiv X(\mathcal{A}_Q)$ (and with the notation $\Phi(M) = \left(I_p - \frac{MM^T}{zn} \right)^{-1}$ given in the proof of Proposition F.3):

$$\begin{aligned} \|d\Psi|_M \cdot H\| &\leq \left\| \Psi(M) \frac{1}{nz} (MH^T + HM^T) \Psi(M) M \right\| + \|\Psi(M) H\| \\ &\leq O\left(\frac{|z|\|H\|}{(1+|z|)^2}\right) + O\left(\frac{|z|\|H\|}{1+|z|}\right) \leq O(\kappa_z \check{\kappa}_z \|H\|). \end{aligned}$$

Thus, under \mathcal{A}_Q , Ψ is $O(\kappa_z \check{\kappa}_z)$ -Lipschitz (for the Frobenius norm) and therefore $Q^z X \propto \mathcal{E}_2(\kappa_z \check{\kappa}_z)$.

To control the norm of $\mathbb{E}_{\mathcal{A}_Q}[Q^z x_i]$, let us employ Schur identities (F.8) and bound for any deterministic $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$:

$$|\mathbb{E}_{\mathcal{A}_Q}[u^T Q^z x_i]| = |z| \left| \mathbb{E}_{\mathcal{A}_Q} \left[\frac{u^T Q_{-i}^z x_i}{\Lambda_i^z} \right] \right| \leq |z| \mathbb{E}_{\mathcal{A}_Q}[|u^T Q_{-i}^z x_i|] \frac{\check{\kappa}_z}{|z|} \leq O(\kappa_z \check{\kappa}_z),$$

thanks to Lemmas F.9 and F.7. □

Proof of Proposition F.12. Let us note for simplicity $\kappa_{\bar{Q}} \equiv \|\tilde{Q}^{\hat{\Lambda}^z}\|$. Looking at decomposition (F.10) we can start with the bound:

$$\|\varepsilon_1\|_F = \left\| \frac{1}{zn} \mathbb{E}_{\mathcal{A}_Q} \left[Q^z X \left(\hat{\Lambda}^z - \Lambda^z \right) \left(\hat{\Lambda}^z \right)^{-1} X^T \right] \tilde{Q}^{\Lambda^z} \right\|_F \leq O\left(\frac{\kappa_{\bar{Q}} \kappa_z^2 \check{\kappa}_z^2}{|z|^2 \sqrt{n}}\right) \leq O\left(\frac{\kappa_{\bar{Q}}}{\sqrt{n}}\right)$$

obtained with the bound $\frac{1}{\hat{\Lambda}^z} \leq O\left(\frac{\check{\kappa}_z}{|z|}\right) \leq O(1)$ given by Lemma F.11 and applying Proposition B.50 with the hypotheses:

- $X | \mathcal{A}_Q \propto \mathcal{E}_2$ and $\|\mathbb{E}_{\mathcal{A}_Q}[x_i]\| \leq O(1)$ Assumption 2,
- $Q^z X | \mathcal{A}_Q \propto \mathcal{E}_2(\kappa_z \check{\kappa}_z)$ and $\|\mathbb{E}_{\mathcal{A}_Q}[Q^z x_i]\| \leq O(\kappa_z \check{\kappa}_z)$ given by Lemma F.13,
- $\Lambda^z | \mathcal{A}_Q \in \mathbb{E}_{\mathcal{A}_Q}[\Lambda^z] \pm \mathcal{E}_2\left(\frac{\kappa_z}{\sqrt{n}}\right)$ in $(\mathcal{D}_n, \|\cdot\|)$ given by Lemma F.5,
- $\|\mathbb{E}_{\mathcal{A}_Q}[\Lambda^z] - \hat{\Lambda}^z\|_F \leq O(\kappa_z/\sqrt{n})$ thanks to Lemma F.6

Second, for any matrix $A \in \mathcal{M}_p(\mathbb{C})$ satisfying $\|A\|_F \leq 1$, let us bound thanks to Cauchy-Schwarz inequality:

$$\begin{aligned} |\text{Tr}(A\varepsilon_2)| &\leq \sqrt{\frac{1}{|z|^2 n^2} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^z X |\hat{\Lambda}^z|^{-2} X^T \bar{Q}^z \bar{A}^T \right) \right]} \\ &\quad \cdot \sqrt{\frac{1}{n^2} \sum_{i=1}^n \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(\tilde{Q}^{\hat{\Lambda}^z} \Sigma_i Q_{-i}^z x_i x_i^T Q_{-i}^z \Sigma_i \bar{Q}^{\hat{\Lambda}^z} \right) \right]} \\ &\leq O\left(\frac{\kappa_z^4 \check{\kappa}_z^4}{|z|^2} \sqrt{\frac{\|A\|_F^2 \sup_{i \in [n]} \text{Tr}(\Sigma_i^3) \kappa_{\bar{Q}}^2}{n}}\right) \leq O\left(\frac{\kappa_{\bar{Q}}}{\sqrt{n}}\right) \end{aligned}$$

thanks to the bounds provided by our assumptions, and Lemmas F.2, F.11 and F.13.

Third, we bound easily with Lemma F.8 the quantity:

$$\begin{aligned} \|\mathrm{Tr}(A\delta_1)\| &\equiv \frac{1}{n} \sum_{i=1}^n \frac{1}{|\hat{\Lambda}_i^z|} \left\| \mathrm{Tr} \left(\tilde{Q}^{\hat{\Lambda}^z} A \mathbb{E}_{\mathcal{A}_Q} [Q_{-i}^z x_i x_i^T] \right) \right. \\ &\quad \left. - \mathrm{Tr} \left(\tilde{Q}^{\hat{\Lambda}^z} A \mathbb{E}_{\mathcal{A}_Q} [Q_{-i}^z] \mathbb{E}_{\mathcal{A}_Q} [x_i x_i^T] \right) \right\| \leq O \left(\frac{\kappa_{\tilde{Q}}}{\sqrt{n}} \right). \end{aligned}$$

And we can bound $\|\mathrm{Tr}(A\delta_2)\| \leq O \left(\frac{\kappa_{\tilde{Q}}}{n} \right)$ since $\|\Sigma_i - \mathbb{E}_{\mathcal{A}_Q} [x_i x_i^T]\| \leq O \left(\frac{1}{n} \right)$ (as explained in the proof of Lemma F.10).

Taking the supremum on $A \in \mathcal{M}_{p,n}(\mathbb{C})$ and putting the bounds on $\|\varepsilon_1\|_F$, $\|\varepsilon_2\|_F$, $\|\delta_1\|_F$ and $\|\delta_2\|_F$ together, we obtain:

$$\left\| \mathbb{E}_{\mathcal{A}_Q} [Q^z] - \tilde{Q}^{\hat{\Lambda}^z} \right\|_F \leq O \left(\frac{\kappa_{\tilde{Q}}}{\sqrt{n}} \right)$$

So, in particular $\kappa_{\tilde{Q}} \equiv \left\| \tilde{Q}^{\hat{\Lambda}^z} \right\| \leq \left\| \mathbb{E}_{\mathcal{A}_Q} [Q^z] \right\| + O \left(\frac{\kappa_{\tilde{Q}}}{\sqrt{n}} \right)$, which implies that $\kappa_{\tilde{Q}} \leq O(\kappa_z)$ as $\left\| \mathbb{E}_{\mathcal{A}_Q} [Q^z] \right\|$ since $\frac{1}{\sqrt{n}} = o(1)$. We obtain then directly the second bound of the proposition. \square

1.4 Definition of the second deterministic equivalent thanks to the semi-metric d_s

Proposition F.12 slightly simplified the problem because while we initially had to estimate the expectation of the whole matrix Q^z , now, we just need to approach the diagonal matrix $\hat{\Lambda}^z \equiv \mathrm{Diag}(z - \frac{1}{n} \mathrm{Tr}(\Sigma_i Q_{-i}^z))_{i \in [n]}$. One might be tempted to introduce from the pseudo identity (where \tilde{Q} was defined in (F.7)):

$$\hat{\Lambda}_i^z = z - \frac{1}{n} \mathrm{Tr}(\Sigma_i \mathbb{E}_{\mathcal{A}_Q} [Q^z]) \approx z - \frac{1}{n} \mathrm{Tr}(\Sigma_i \tilde{Q}^{\hat{\Lambda}^z}) \quad (\text{F.14})$$

a fixed point equation whose solution would be a natural estimate for $\hat{\Lambda}^z$. This equation is given in Theorem F.18: we chose $\tilde{\Lambda}^z$ satisfying

$$\tilde{\Lambda}_i^z = z - \frac{1}{n} \mathrm{Tr} \left(\Sigma_i \tilde{Q}^{\tilde{\Lambda}^z} \right).$$

We are now going to prove that $\tilde{\Lambda}^z$ is well defined for any $z \in \mathbb{H}$ (where we recall that $\mathbb{H} \equiv \{z \in \mathbb{C}, \Im(z) > 0\}$) to prove Theorem F.18. Introducing the mapping:

$$\forall L \in \mathcal{D}_n(\mathbb{H}) : \mathcal{I}^z(L) \equiv zI_n - \mathrm{Diag} \left(\frac{1}{n} \mathrm{Tr} \left(\Sigma_i \tilde{Q}^L \right) \right)_{1 \leq i \leq n},$$

we want to show that \mathcal{I}^z admits a unique fixed point. For that purpose, we are going to employ the Theorem E.17 on the mapping \mathcal{I}^z ($z \in \mathbb{H}$), that is, as we will see in Proposition F.15 below, contractive for semi-metric d_s presented in last chapter and defined for any $D, D' \in \mathcal{D}_n(\mathbb{H})$ as:

$$d_s(D, D') = \sup_{1 \leq i \leq n} \frac{|D_i - D'_i|}{\sqrt{\Im(D_i) \Im(D'_i)}}.$$

We first need to restrict our study on a subset of $\mathcal{D}_n(\mathbb{H})$:

$$\mathcal{D}_{\mathcal{I}^z} \equiv \{D \in \mathcal{D}_n(\mathbb{H}), D/z \in \mathcal{D}_n(\mathbb{H})\}$$

Lemma F.14. *For any $z \in \mathbb{H}$, $\mathcal{I}^z(\mathcal{D}_{\mathcal{I}^z}) \subset \mathcal{D}_{\mathcal{I}^z}$.*

Proof. Considering $z \in \mathbb{H}$, and $L \in \mathcal{D}_{\mathcal{I}^z}$ and $i \in [n]$, the decomposition $\tilde{Q}^L = \left(I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Re(L_j)\Sigma_i}{|L_j|^2} - \frac{i}{n} \sum_{i=1}^n \frac{\Im(L_j)\Sigma_i}{|L_j|^2} \right)^{-1}$ allows us to set thanks to the resolvent identity (F.9):

$$\begin{aligned} \Im(\mathcal{I}^z(L)_i) &= \Im(z) + \frac{1}{2in} \operatorname{Tr} \left(\Sigma_i \left(\tilde{Q}^L + \bar{\tilde{Q}}^L \right) \right) \\ &= \Im(z) + \frac{1}{n} \operatorname{Tr} \left(\Sigma_i \tilde{Q}^L \sum_{i=1}^n \frac{\Im(L_j)\Sigma_i}{|L_j|^2} \bar{\tilde{Q}}^L \right) > 0 \end{aligned}$$

(since $\tilde{Q}^L \Sigma_i \bar{\tilde{Q}}^L$ is a non negative Hermitian matrix). The same way, one can show:

$$\Im(\mathcal{I}^z(L)_i/z) = \frac{1}{n|z|^2} \operatorname{Tr} \left(\Sigma_i \tilde{Q}^L \left(\Im(z) + \sum_{i=1}^n \frac{\Im(L_j/z)\Sigma_i}{|L_j/z|^2} \right) \bar{\tilde{Q}}^L \right) > 0$$

□

Let us now express the Lipschitz parameter of \mathcal{I}^z for the semi metric d_s .

Proposition F.15. *For any $z \in \mathbb{H}$, the mapping \mathcal{I}^z is 1-Lipschitz for the semi-metric d_s on $\mathcal{D}_{\mathcal{I}^z}$ and satisfies for any $L, L' \in \mathcal{D}_{\mathcal{I}^z}$:*

$$d_s(\mathcal{I}^z(L), \mathcal{I}^z(L')) \leq \sqrt{(1 - \phi(z, L))(1 - \phi(z, L'))} d_s(L, L'),$$

where for any $w \in \mathbb{H}$ and $L \in \mathcal{D}_{\mathcal{I}^z}$:

$$\phi(w, L) = \frac{\Im(w)}{\sup_{1 \leq i \leq n} \Im(\mathcal{I}^w(L))_i} \in (0, 1).$$

Proof. Let us bound for any $L, L' \in \mathcal{D}_{\mathcal{I}^z}$:

$$\begin{aligned} |\mathcal{I}^z(L)_i - \mathcal{I}^z(L')_i| &= \frac{1}{n} \operatorname{Tr} \left(\Sigma_i \tilde{Q}^L \left(\frac{1}{n} \sum_{j=1}^n \frac{L_j - L'_j}{L_j L'_j} \Sigma_j \right) \bar{\tilde{Q}}^{L'} \right) \\ &= \frac{1}{n} \operatorname{Tr} \left(\Sigma_i \tilde{Q}^L \left(\frac{1}{n} \sum_{j=1}^n \frac{L_j - L'_j}{\sqrt{\Im(L_j)\Im(L'_j)}} \frac{\sqrt{\Im(L_j)\Im(L'_j)}}{L_j L'_j} \Sigma_j \right) \bar{\tilde{Q}}^{L'} \right) \\ &\leq d_s(L, L') \sqrt{\frac{1}{n} \operatorname{Tr} \left(\Sigma_i \tilde{Q}^L \left(\frac{1}{n} \sum_{i=1}^n \frac{\Im(L_j)\Sigma_i}{|L_j|^2} \right) \bar{\tilde{Q}}^L \right)} \\ &\quad \cdot \sqrt{\frac{1}{n} \operatorname{Tr} \left(\Sigma_i \bar{\tilde{Q}}^{L'} \left(\frac{1}{n} \sum_{i=1}^n \frac{\Im(L'_j)\Sigma_i}{|L'_j|^2} \right) \tilde{Q}^{L'} \right)} \\ &\leq d_s(L, L') \sqrt{(\Im(\mathcal{I}^z(L)_i) - \Im(z)) (\Im(\mathcal{I}^z(L')_i) - \Im(z))}, \end{aligned} \tag{F.15}$$

thanks to Cauchy-Schwarz inequality and the identity

$$0 \leq \frac{1}{n} \operatorname{Tr} \left(\Sigma_i \tilde{Q}^L \left(\sum_{i=1}^n \frac{\Im(L'_j)\Sigma_i}{|L'_j|^2} \right) \bar{\tilde{Q}}^z(L) \right) = \Im(\mathcal{I}^z(L)_i) - \Im(z)$$

issued from the proof of Lemma F.14. Dividing both sides of (F.15) by $\sqrt{(\Im(\mathcal{I}^z(L)_i) - \Im(z)) (\Im(\mathcal{I}^z(L')_i) - \Im(z))}$, we retrieve the wanted Lipschitz parameter.

□

The contractivity of \mathcal{I}^z is not fully stated in the previous proposition because, the Lipschitz parameter depends on the values of L, L' and we want a bound uniform on $\mathcal{D}_{\mathcal{I}^z}$. This will be done thanks to the two lemmas.

Lemma F.16 (Commutation between inversion and modulus of matrices). *Given an invertible matrix $M \in \mathcal{M}_p$, $|M^{-1}| = |M|^{-1}$ and for any $K > 0$:*

$$\Im M^{-1} \geq KI_p \text{ or } \Re M^{-1} \geq KI_p \quad \implies \quad |M| \leq \frac{1}{K} I_p.$$

Proof. We have the identity:

$$|M^{-1}|^2 = M^{-1} \bar{M}^{-1} = (\bar{M}M)^{-1} = (|M|^2)^{-1}.$$

then we take the square root on both sides to obtain the first identity (recall that the modulus of a matrix is a non negative hermitian matrix). Now let us assume that $\Im(M^{-1}) \geq K$, we know that:

$$\begin{aligned} |M^{-1}|^2 &= \Im(M^{-1})\Im(M^{-1})^T + \Re(M^{-1})\Re(M^{-1})^T \\ &\quad - i\Re(M^{-1})\Im(M^{-1})^T + i\Im(M^{-1})\Re(M^{-1})^T, \end{aligned}$$

is a nonnegative hermitian matrix satisfying for all $x \in \mathbb{C}^p$ (the cross terms cancel):

$$\begin{aligned} \bar{x}^T |M^{-1}|^2 x &= \bar{x}^T \Im(M^{-1})\Im(M^{-1})^T x + \bar{x}^T \Re(M^{-1})\Re(M^{-1})^T x \\ &\geq \bar{x}^T \Im(M^{-1})\Im(M^{-1})^T x \geq K^2 \|x\|^2. \end{aligned}$$

Thus the lower eigen value of $|M^{-1}| = |M|^{-1}$ is bigger than K and therefore $|M| \leq \frac{1}{K} I_p$. □

Lemma F.17. *Given $L \in \mathcal{D}_{I^z}$, we can bound:*

$$\Im(z)I_n \leq |\mathcal{I}^z(L)| \leq O\left(|z| + \frac{|z|}{\Im(z)}\right) I_n$$

and:

$$O\left(\frac{\Im(z)}{1 + \Im(z)}\right) I_p \leq |Q^{I^z(L)}| \leq \frac{|z|I_p}{\Im(z)}.$$

Proof. The lower bound of $\mathcal{I}^z(L)$ is immediate (see the proof of Lemma F.14). If $L \in \mathcal{D}_{I^z}$, then we know that $L/z \in \mathcal{D}_n(\mathbb{H})$, and therefore, noting that:

$$\Im\left((\tilde{Q}^L/z)^{-1}\right) = \Im\left(zI_p - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{L_i/z}\right) = \Im(z)I_p + \frac{1}{n} \sum_{i=1}^n \frac{\Im(L_i/z)\Sigma_i}{|L_i/z|^2} \geq \Im(z)I_p,$$

we can deduce from Lemma F.16 that $|\tilde{Q}^L/z| \leq \frac{1}{\Im(z)}$ and thus $\|\tilde{Q}^L\| \leq \frac{|z|}{\Im(z)}$ which gives us directly the upper bound on $\mathcal{I}^z(L)$ since $\text{Tr}(\Sigma_i) \leq O\left(|z| + \frac{|z|}{\Im(z)}\right)$.

Finally, we can bound:

$$\left\| I_n - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\mathcal{I}^z(L)_i} \right\| \leq 1 + \frac{1}{n} \sum_{i=1}^n \frac{\|\Sigma_i\|}{|\Im(\mathcal{I}^z(L)_i)|} \leq 1 + O\left(\frac{1}{\Im(z)}\right),$$

which provides the lower bound on $|\tilde{Q}^{\mathcal{I}^z(L)}|$ since $O\left(\frac{\Im(z)}{1 + \Im(z)}\right) \leq \frac{1}{1 + O(\frac{1}{\Im(z)})}$. □

Theorem F.18. *Given n nonnegative symmetric matrices $\Sigma_1, \dots, \Sigma_n \in \mathcal{M}_p$, for all $z \in \mathbb{H}$, the equation:*

$$\forall i \in [n], L_i = z - \frac{1}{n} \text{Tr} \left(\Sigma_i \left(I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{L_i} \right)^{-1} \right) \quad (\text{F.16})$$

admits a unique solution $L \in \mathcal{D}_n(\mathbb{H})$ that we denote $\tilde{\Lambda}^z$.

Proof. On the domain $\mathcal{I}^z(\mathcal{D}_{\mathcal{I}^z})$, the mapping \mathcal{I}^z is bounded and contracting for the semi-metric d_s thanks to Proposition F.15 and Lemma F.17. The hypotheses of Theorem E.17 are thus satisfied, and we know that there exists a unique diagonal matrix $\tilde{\Lambda}^z \in I^z(\mathcal{D}_{I^z})$ such that $I^z(\tilde{\Lambda}^z) = \tilde{\Lambda}^z$. There can not exist a second diagonal matrix $\Lambda' \in \mathcal{D}_n(\mathbb{H})$ such that $\Lambda' = I^z(\Lambda')$ because then Proposition F.15 (true on the whole domain $\mathcal{D}_n(\mathbb{H})$) would imply that $d_s(\Lambda', \tilde{\Lambda}^z) < d_s(\Lambda', \tilde{\Lambda}^z)$. \square

We end this section with an interesting result on $\tilde{\Lambda}^z$ that will however not have any use for our main results.

Lemma F.19. $\sup_{i \in [n]} |\tilde{\Lambda}_i^z| \leq O(1 + |z|)$.

Proof. If we assume that $\forall i \in [n], |\tilde{\Lambda}_i^z| \geq 2\nu$, then we deduce that $\frac{1}{n} \sum_{i=1}^n \frac{1}{|\tilde{\Lambda}_i^z|} \leq \frac{1}{2\nu}$ and $|\frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\tilde{\Lambda}_i^z}| \leq \frac{1}{2}$, and therefore, $|\tilde{Q}^{\tilde{\Lambda}^z}| \leq 2$. As a consequence, $\forall i \in [n]$:

$$|\tilde{\Lambda}_i^z| \leq |z| + \frac{1}{n} \text{Tr} \left(\Sigma_i |\tilde{Q}^{\tilde{\Lambda}^z}| \right) \leq |z| + \frac{2p\nu}{n}$$

We can conclude that:

$$\sup_{i \in [n]} |\tilde{\Lambda}_i^z| \leq \max \left(\frac{\nu}{2}, |z| + \frac{2p\nu}{n} \right) \leq O(1 + |z|).$$

\square

1.5 Convergence of $\hat{\Lambda}^z$ towards $\tilde{\Lambda}^z$

To show the convergence of $\hat{\Lambda}^z$ towards $\tilde{\Lambda}^z = I^z(\tilde{\Lambda}^z)$, we need Proposition E.23 bounding the distance to a fixed point of a contracting mapping for the semi-metric d_s . This sets what we call the stability of the equation. First allowing us to bound $\|\hat{\Lambda}^z - \tilde{\Lambda}^z\|$, it will then be employed to show the continuity of $z \mapsto \tilde{\Lambda}^z$. In the former application, the convergence parameter is n , while in the latter application it is a parameter $t \in \mathbb{C}$ in the neighbourhood of 0.

To be employ Proposition E.23 on the matrices $\tilde{\Gamma}^n = \tilde{\Lambda}^z$ and $\Gamma^n = \hat{\Lambda}^z$ and on the mapping $f^n = \mathcal{I}^z$, we first need to set the following proposition. Unfortunately, we need to assume here that $d(z, S^\varepsilon) \geq O(1)$ (and not $d(z, S_{-0}^\varepsilon) \geq O(1)$).

Proposition F.20. *Given $z \in \mathbb{C} \setminus S^\varepsilon$:*

$$d_s \left(\mathfrak{S}(\mathcal{I}^z(\hat{\Lambda}^z)), \mathfrak{S}(\hat{\Lambda}^z) \right) \leq O \left(\frac{1}{n} \right)$$

Remark F.21. *The bound $O(\frac{1}{n})$ comes from the fact that for any $z \in \mathbb{C} \setminus S^\varepsilon$, if $n < p$ then $0 \in S$ and $|z| \geq \varepsilon$ and if $n \geq p$ then $\kappa_z = \frac{|z|}{1+|z|}$. Therefore, in all cases $\frac{\kappa_z}{|z|} \leq O(1)$.*

To prove Proposition F.20 but also to show later that the mapping \mathcal{I}^z is contracting, we will need:

Lemma F.22. *Given $z \in \mathbb{C} \setminus S_{-0}^\varepsilon$:*

$$\mathfrak{S}(z) \leq \inf_{i \in [n]} \mathfrak{S} \left(\hat{\Lambda}_i^z \right) \leq \sup_{i \in [n]} \mathfrak{S} \left(\hat{\Lambda}_i^z \right) \leq O(\mathfrak{S}(z))$$

Proof. The lower bound is obvious since for all $i \in [n]$:

$$\mathfrak{S}(\hat{\Lambda}_i^z) = \mathfrak{S}(z) + \frac{\mathfrak{S}(z)}{|z|^2 n^2} \mathbb{E}_{\mathcal{A}_Q} [\text{Tr} (Q_{-i}^z X_{-i} X_{-i}^T \bar{Q}_{-i}^z \Sigma_i)] \geq \mathfrak{S}(z).$$

The upper bound is obtained thanks to the bound, valid under \mathcal{A}_Q , $Q_{-i}^z X_{-i} / |z| \sqrt{n} \leq O(1)$ provided by Lemma F.13. \square

Proof of Proposition F.20. We can bound thanks to Lemma F.22:

$$d_s\left(\Im(\mathcal{I}^z(\hat{\Lambda}^z)), \Im(\hat{\Lambda}^z)\right) = \sup_{1 \leq i \leq n} \left| \frac{\frac{1}{n} \operatorname{Tr}\left(\Sigma_i \Im\left(\tilde{Q}^{\hat{\Lambda}^z} - \mathbb{E}_{\mathcal{A}_Q}[Q^z]\right)\right)}{\sqrt{\Im(\hat{\Lambda}_i^z) \Im(\mathcal{I}^z(\hat{\Lambda}^z)_i)}} \right| \leq O\left(\frac{1}{\sqrt{n}} \left\| \frac{\Im(\tilde{Q}^{\hat{\Lambda}^z})}{\Im(z)} - \frac{\Im(\mathbb{E}_{\mathcal{A}_Q}[Q^z])}{\Im(z)} \right\|_F\right),$$

since $\|\Sigma_i\|_F \leq O(\sqrt{n})$. The identity $\frac{1}{z} X X^T \bar{Q}^z = \bar{Q}^z - I_n$ allows us to write:

$$\frac{\Im(\mathbb{E}_{\mathcal{A}_Q}[Q^z])}{\Im(z)} = \frac{1}{\Im(z)} \mathbb{E}_{\mathcal{A}_Q} \left[Q^z \left(\frac{\Im(z)}{n|z|^2} X X^T \right) \bar{Q}^z \right] = \frac{1}{z} \mathbb{E}_{\mathcal{A}_Q} [|Q^z|^2 - Q^z] (= \frac{1}{z} \mathbb{E}_{\mathcal{A}_Q} [|Q^z|^2 - \bar{Q}^z]). \quad (\text{F.17})$$

We already know how to estimate $\mathbb{E}_{\mathcal{A}_Q}[Q^z]$ thanks to Proposition F.12, we are thus left to estimate $\mathbb{E}_{\mathcal{A}_Q}[|Q^z|^2]$. We do not give the full justifications that are closely similar to those presented in the proof of Proposition F.12 – mainly application of Schur identities (F.8), Proposition B.50 and Lemmas F.5, F.10. To complete this estimation, we consider a deterministic matrix $A \in \mathcal{M}_p$, and we estimate:

$$\begin{aligned} & \operatorname{Tr}\left(A \mathbb{E}_{\mathcal{A}_Q} \left[\bar{Q}^z \left(Q^z - \tilde{Q}^{\hat{\Lambda}^z} \right) \right]\right) \\ &= \frac{1}{n} \sum_{i=1}^n \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\bar{Q}^z Q^z \left(\frac{x_i x_i^T}{z} - \frac{\Sigma_i}{\hat{\Lambda}_i^z} \right) \tilde{Q}^{\hat{\Lambda}^z} \right] \right) \\ &= \frac{1}{n} \sum_{i=1}^n \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\bar{Q}^z Q^z x_i x_i^T \tilde{Q}^{\hat{\Lambda}^z}}{\hat{\Lambda}_i^z} \right] \right) - \frac{1}{n} \sum_{i=1}^n \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\bar{Q}^z Q^z \Sigma_i \tilde{Q}^{\hat{\Lambda}^z}}{\hat{\Lambda}_i^z} \right] \right) \\ &= \frac{1}{n} \sum_{i=1}^n \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\bar{Q}^z x_i x_i^T |Q^z|^2 \tilde{Q}^{\hat{\Lambda}^z}}{\hat{\Lambda}_i^z} \right] \right) - \frac{1}{n^2} \sum_{i=1}^n \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\bar{Q}^z x_i x_i^T |Q^z|^2 x_i x_i^T \tilde{Q}^{\hat{\Lambda}^z}}{z \hat{\Lambda}_i^z} \right] \right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\bar{Q}^z x_i x_i^T \Sigma_i \tilde{Q}^{\hat{\Lambda}^z}}{\hat{\Lambda}_i^z} \right] \right) + O\left(\frac{\kappa_z}{\sqrt{n}}\right) \\ &= -\mathbb{E}_{\mathcal{A}_Q} \left[\frac{1}{n} \operatorname{Tr} \left(A \bar{Q}^z X \frac{\Delta^z}{z \hat{\Lambda}_i^z} X^T \tilde{Q}^{\hat{\Lambda}^z} \right) \right] + O\left(\frac{\kappa_z}{\sqrt{n}}\right) \end{aligned}$$

with the introduction of the notation:

$$\Delta^z \equiv \operatorname{Diag}_{i \in [n]} \left(\frac{1}{n} x_i^T |Q^z|^2 x_i \right).$$

The random diagonal matrix Δ^z being a $O(\kappa_z^2/\sqrt{n})$ Lipschitz transformation of X for the spectral norm $\|\cdot\|$ on \mathcal{D}_n , we know that $\Delta^z | \mathcal{A}_Q \in \hat{\Delta}^z \pm \mathcal{E}_2(\kappa_z/\sqrt{n})$, where we noted $\hat{\Delta}^z \equiv \mathbb{E}_{\mathcal{A}_Q}[\Delta^z]$. One can then pursue the estimation, thanks again to Proposition B.50:

$$\begin{aligned} \operatorname{Tr}\left(A \mathbb{E}_{\mathcal{A}_Q} \left[\bar{Q}^z \left(Q^z - \tilde{Q}^{\hat{\Lambda}^z} \right) \right]\right) &= \frac{1}{n} \sum_{i=1}^n \frac{\hat{\Delta}_i^z}{\hat{\Lambda}_i^z} \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\bar{Q}^z x_i x_i^T \tilde{Q}^{\hat{\Lambda}^z}}{\hat{\Lambda}_i^z} \right] \right) + O\left(\frac{\kappa_z}{\sqrt{n}}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{\hat{\Delta}_i^z}{|\hat{\Lambda}_i^z|^2} \operatorname{Tr} \left(A \mathbb{E}_{\mathcal{A}_Q} \left[\bar{Q}^z x_i \Sigma_i \tilde{Q}^{\hat{\Lambda}^z} \right] \right) + O\left(\frac{\kappa_z}{\sqrt{n}}\right) \end{aligned}$$

Now, taking the expectation under \mathcal{A}_Q on the identity valid for any $i \in [n]$:

$$\Im(\hat{\Lambda}_i^z) = \Im(z) - \frac{1}{n} \Im(x_i^T Q^z x_i) = \Im(z) \left(1 + \frac{1}{nz} (x_i^T |Q^z|^2 x_i - x_i^T Q^z x_i) \right) = \frac{\Im(z)}{z} (\Delta_i^z + \Lambda_i^z), \quad (\text{F.18})$$

we deduce that $\hat{\Delta}^z = \frac{z}{\Im(z)}\Im(\hat{\Lambda}^z) - \hat{\Lambda}^z + O(\kappa_z/n)$. Therefore, with the identity $\mathbb{E}_{\mathcal{A}_Q}[\bar{Q}^z \tilde{Q}^{\hat{\Lambda}^z}] = |\tilde{Q}^{\hat{\Lambda}^z}|^2 + O_{\|\cdot\|_F}(\kappa_z/\sqrt{n})$, we can estimate (see the proof of Lemma F.14 for the identification of $\Im(\text{Tr}(\tilde{Q}^{\hat{\Lambda}^z}))$):

$$\begin{aligned} \text{Tr}(A\mathbb{E}_{\mathcal{A}_Q}[|Q^z|^2]) &= \frac{1}{n} \sum_{i=1}^n \left(\frac{z}{\Im(z)} \frac{\Im(\hat{\Lambda}_i^z)}{|\hat{\Lambda}_i^z|^2} - \frac{1}{\hat{\Lambda}_i^z} \right) \text{Tr}(A\mathbb{E}_{\mathcal{A}_Q}[\bar{Q}_{-i}^z \Sigma_i \tilde{Q}^{\hat{\Lambda}^z}]) \\ &\quad + \text{Tr}(A|\tilde{Q}^{\hat{\Lambda}^z}|^2) + O\left(\frac{\kappa_z}{\sqrt{n}}\right) \\ &= \frac{z}{\Im(z)}\Im(\text{Tr}(A\tilde{Q}^{\hat{\Lambda}^z})) + \frac{1}{n} \text{Tr}(A\tilde{Q}^{\hat{\Lambda}^z}) + O\left(\frac{\kappa_z}{\sqrt{n}}\right) \end{aligned}$$

and therefore, we can deduce from Proposition F.12 and (F.17):

$$\frac{\Im(\text{Tr}(AQ^z))}{\Im(z)} = \frac{1}{z} \text{Tr}(A\mathbb{E}_{\mathcal{A}_Q}[|Q^z|^2 - Q^z]) = \frac{\Im(\text{Tr}(A\tilde{Q}^{\hat{\Lambda}^z}))}{\Im(z)} + O\left(\frac{\kappa_z}{|z|\sqrt{n}}\right).$$

we can bound thanks to identity F.17 and inequality $\frac{\kappa_z}{|z|} \leq O(1)$ given by Remark F.21:

$$\left\| \frac{\Im(\mathbb{E}_{\mathcal{A}_Q}[Q^z]) - \Im(\tilde{Q}^{\hat{\Lambda}^z})}{\Im(z)} \right\|_F \leq O\left(\frac{\kappa_z}{|z|\sqrt{n}}\right) \leq O\left(\frac{1}{\sqrt{n}}\right),$$

and we retrieve the result of the proposition. \square

We now have all the elements to show the convergence of $\hat{\Lambda}^z$ to $\tilde{\Lambda}^z$. Here again, we need to assume as in Proposition F.20 that $z \in \mathbb{C} \setminus S^\varepsilon$.

Proposition F.23. *For any $z \in \mathbb{H}$ such that $d(z, S) \geq \varepsilon$:*

$$\|\hat{\Lambda}^z - \tilde{\Lambda}^z\| \leq O\left(\frac{\kappa_z}{n}\right) \quad \text{and} \quad O\left(\frac{|z|}{\kappa_z}\right) \leq |\tilde{\Lambda}^z| \leq O\left(\frac{|z|}{\kappa_z}\right)$$

Proof. We already know from Proposition F.20 that $d_s(\Im(\mathcal{I}^z(\hat{\Lambda}^z)), \Im(\hat{\Lambda}^z)) \leq o(1)$. Besides, the Lipschitz parameter λ of \mathcal{I}^z on the set $\{\tilde{\Lambda}^z, \hat{\Lambda}^z, n \in \mathbb{N}\}$ is such that $1 - \lambda \geq O(1)$. Recall indeed from Proposition F.15 that:

$$\lambda \leq \sqrt{\left(1 - \frac{\Im(z)}{\sup_{i \in [n]} \Im(\tilde{\Lambda}_i^z)}\right) \left(1 - \frac{\Im(z)}{\sup_{i \in [n]} \Im(\hat{\Lambda}_i^z)}\right)} \leq \sqrt{1 - O(1)} \leq 1 - O(1),$$

thanks to Lemma F.22. Therefore, we can employ Proposition E.23 to set that:

$$\left\| \frac{\hat{\Lambda}^z - \tilde{\Lambda}^z}{\sqrt{\Im(\hat{\Lambda}^z)\Im(\tilde{\Lambda}^z)}} \right\| = d_s(\hat{\Lambda}^z, \tilde{\Lambda}^z) \leq O\left(\left\| \frac{\hat{\Lambda}^z - \mathcal{I}^z(\hat{\Lambda}^z)}{\sqrt{\Im(\hat{\Lambda}^z)\Im(\tilde{\Lambda}^z)}} \right\|\right),$$

which implies thanks to Lemma F.22 that:

$$\|\hat{\Lambda}^z - \tilde{\Lambda}^z\| \leq O\left(\sqrt{\frac{\sup_{i \in [n]} \Im(\tilde{\Lambda}_i^z)}{\inf_{i \in [n]} \Im(\tilde{\Lambda}_i^z)}} \|\hat{\Lambda}^z - \mathcal{I}^z(\hat{\Lambda}^z)\|\right).$$

We reach here the only point of the whole proof where we will employ Assumption 5. It is to set that:

$$\begin{aligned} \inf_{i \in [n]} \Im(\tilde{\Lambda}_i^z) &= \Im(z) + \inf_{i \in [n]} \sum_{j=1}^n \frac{\Im(\tilde{\Lambda}_j^z)}{n|\tilde{\Lambda}_j^z|^2} \text{Tr}(\Sigma_i \tilde{Q}^{\hat{\Lambda}^z} \Sigma_j \tilde{Q}^{\tilde{\Lambda}^z}) \\ &\geq \Im(z) + \sum_{j=1}^n \frac{\Im(\tilde{\Lambda}_j^z)}{n^z |\tilde{\Lambda}_j^z|^2} O(\text{Tr}(\tilde{Q}^{\hat{\Lambda}^z} \Sigma_j \tilde{Q}^{\tilde{\Lambda}^z})) \geq O\left(\sup_{i \in [n]} \Im(\tilde{\Lambda}_i^z)\right) \end{aligned}$$

As a conclusion:

$$\|\hat{\Lambda}^z - \tilde{\Lambda}^z\| \leq O\left(\|\hat{\Lambda}^z - \mathcal{I}^z(\hat{\Lambda}^z)\|\right) \leq O\left(\frac{1}{\sqrt{n}}\|\hat{Q} - \tilde{Q}^{\hat{\Lambda}^z}\|\right) \leq O\left(\frac{\kappa_z}{n}\right).$$

we can further deduce that Λ^z and $\tilde{\Lambda}^z$ have the same upper and lower bound of order $O(|z|/\tilde{\kappa}_z)$ since $\kappa_z \leq O(|z|/\tilde{\kappa}_z)$. \square

1.6 Concentration and final estimation of the resolvent

The estimation of Q^z is a simple consequence of the convergence of $\hat{\Lambda}^z$ towards $\tilde{\Lambda}^z$.

Corollary F.24. *For any $z \in \mathbb{C} \setminus S_\varepsilon$, $\|\tilde{Q}^{\tilde{\Lambda}^z}\| \leq O(\kappa_z)$ and:*

$$\left\|\mathbb{E}_{\mathcal{A}_Q}[Q^z] - \tilde{Q}^{\tilde{\Lambda}^z}\right\|_F \leq O\left(\frac{\kappa_z}{\sqrt{n}}\right)$$

Proof. We already know from Proposition F.12 that $\left\|\mathbb{E}_{\mathcal{A}_Q}[Q^z] - \tilde{Q}^{\hat{\Lambda}^z}\right\|_F \leq O(\kappa_z/\sqrt{n})$, thus we are left to bound:

$$\begin{aligned} \left\|\tilde{Q}^{\hat{\Lambda}^z} - \tilde{Q}^{\tilde{\Lambda}^z}\right\|_F &\leq \left\|\tilde{Q}^{\hat{\Lambda}^z} \left(\frac{1}{n} \sum_{i=1}^n \frac{\hat{\Lambda}_i^z - \tilde{\Lambda}_i^z}{\hat{\Lambda}_i^z \tilde{\Lambda}_i^z} \Sigma_i\right) \tilde{Q}^{\tilde{\Lambda}^z}\right\|_F \\ &\leq \sup_{i \in [n]} \left| \frac{\hat{\Lambda}_i^z - \tilde{\Lambda}_i^z}{\hat{\Lambda}_i^z \tilde{\Lambda}_i^z} \right| \|\Sigma_i\|_F \left\|\tilde{Q}^{\hat{\Lambda}^z}\right\| \left\|\tilde{Q}^{\tilde{\Lambda}^z}\right\| \\ &\leq O\left(\frac{\kappa_z^3 \kappa_z^2 \sqrt{p}}{|z|^2 n}\right) \leq O\left(\frac{\kappa_z}{\sqrt{n}}\right) \end{aligned}$$

(thanks to Lemma F.7, F.2, F.19 and Proposition F.23) We can then deduce that:

$$\left\|\tilde{Q}^{\tilde{\Lambda}^z}\right\|_F \leq \left\|\tilde{Q}^{\hat{\Lambda}^z} - \tilde{Q}^{\tilde{\Lambda}^z}\right\|_F + \left\|\tilde{Q}^{\hat{\Lambda}^z}\right\|_F \leq O(\kappa_z). \quad \square$$

The concentration of Q^z naturally implies the concentration of $R : z \mapsto \frac{1}{z}Q^z$ around $\tilde{R} : z \mapsto \frac{1}{z}\tilde{Q}^{\tilde{\Lambda}^z}$ that for integration purpose, we set with the semi-norm $\|\cdot\|_{F, S^\varepsilon}$, defined for every $f \in \mathcal{F}(\mathbb{C}, \mathcal{M}_p)$ as:

$$\|f\|_{F, S^\varepsilon} = \sup_{z \in \mathbb{C} \setminus S^\varepsilon} \|f(z)\|_F.$$

Theorem F.25. $R \mid \mathcal{A}_Q \in \tilde{R} \pm \mathcal{E}_2(1/\sqrt{n})$ in $(\mathcal{M}_{p,n}^{\mathbb{C} \setminus S^\varepsilon}, \|\cdot\|_{F, S^\varepsilon})$.

Proof of Theorem F.25. We saw in the proof of Proposition F.3 that the mapping $R \in (\mathcal{F}(\mathbb{C} \setminus S^\varepsilon), \|\cdot\|_{F, S^\varepsilon})$ defined, under \mathcal{A}_Q , for any $z \in \mathbb{C} \setminus S^\varepsilon$ as $R(z) = -\frac{1}{z}Q^z$ has a Lipschitz parameter bounded by:

$$\sup_{z \in \mathbb{C} \setminus S^\varepsilon} O\left(\frac{\kappa_z}{|z|\sqrt{n}}\right) = O\left(\frac{1}{\sqrt{n}}\right)$$

Thanks to the bound $\frac{\kappa_z}{|z|} \leq O(1)$ justified in Remark F.21. As a $O(1/\sqrt{n})$ -Lipschitz transformation of $X \propto \mathcal{E}_2$, $(R \mid \mathcal{A}_Q) \propto \mathcal{E}_2(1/\sqrt{n})$, we can then conclude thanks to Corollary F.24 with the bound:

$$\begin{aligned} \left\|\mathbb{E}_{\mathcal{A}_Q}[R] - \tilde{R}\right\|_{F, S^\varepsilon} &\leq \sup_{z \in \mathbb{C} \setminus S^\varepsilon} \frac{1}{|z|} \left\|\mathbb{E}_{\mathcal{A}_Q}[Q^z] - \tilde{Q}^{\tilde{\Lambda}^z}\right\| \\ &\leq \sup_{z \in \mathbb{C} \setminus S^\varepsilon} O\left(\frac{\kappa_z}{\sqrt{n}|z|}\right) \leq O\left(\frac{1}{\sqrt{n}}\right) \end{aligned} \quad \square$$

The projections on deterministic vectors provide us with good estimates on isolated eigenvectors, but a concentration in spectral norm would have been sufficient for this kind of result. A key consequence of Theorem F.25 lies in the accurate estimates of projections on high dimensional subspaces $F \subset \mathbb{R}^p$ it provides; this is shown⁶ in Figure F.1 that depicts some of these projections with increasing numbers of classes⁷. Given $k \in \mathbb{N}$, we consider $B \equiv \{\theta_1, \dots, \theta_k\} \subset \mathbb{R}$, a (random) subset of k eigenvalues of $\frac{1}{n}XX^T$, E_B the eigenspace associated to those eigenvalues and Π_B and Π_F , respectively the orthogonal projection on E_B and F . If one can construct⁸ a deterministic path γ such that $\{\mathbb{E}[\theta_i], i \in [k]\}^\varepsilon \subset \gamma \subset \mathbb{C} \setminus S_{-0}^\varepsilon$ then we can bound (since $\|\Pi_F\|_F = \sqrt{\dim(F)} \leq O(\sqrt{p})$):

$$\mathbb{P} \left(\left| \frac{1}{p} \text{Tr}(\Pi_F \Pi_A) - \frac{1}{2ip\pi} \oint_\gamma \text{Tr}(\Pi_F \tilde{R}(z)) dz \right| \geq t \right) \leq Ce^{-cnp t^2} + Ce^{-cn},$$

Although it is not particularly needed for practical use, we are now going to show that the mapping $\tilde{g} : z \rightarrow \frac{1}{p} \text{Tr}(\tilde{R}(z))$ is a Stieltjes transform that converges towards \tilde{g} on $\mathbb{C} \setminus S^\varepsilon$.

1.7 Approach with the Stieltjes formalism

We present here some arguably well known results (see [KA16] for instance) about the Stieltjes transform of the eigen value distribution that allows to get some interesting inferences about its support. We start with an interesting identity that gives a direct link between the Stieltjes transforms g and \tilde{g} with the diagonal matrix Λ^z and $\tilde{\Lambda}^z$. From the equality $Q^z - \frac{1}{nz}XX^TQ^z = I_p$, and the Schur identities (F.8) we can deduce that:

$$\begin{aligned} g(z) &= -\frac{1}{pz} \text{Tr}(Q^z) = -\frac{1}{z} - \frac{1}{npz^2} \sum_{i=1}^n x_i^T Q^z x_i \\ &= -\frac{1}{z} - \frac{1}{npz} \sum_{i=1}^n \frac{x_i^T Q_{-i} x_i}{\Lambda_i^z} = \frac{1}{z} \left(\frac{n}{p} - 1 \right) - \frac{1}{p} \sum_{i=1}^n \frac{1}{\Lambda_i^z} = g^{\Lambda^z}(z), \end{aligned}$$

with the notation g^{D^z} , defined for any mapping $D : \mathbb{H} \ni z \mapsto D^z \in \mathcal{D}_n(\mathbb{H})$ as⁹:

$$g^{D^z} : z \mapsto \frac{1}{z} \left(\frac{n}{p} - 1 \right) - \frac{1}{p} \sum_{i=1}^n \frac{1}{D^z}.$$

Interestingly enough, if we denote $\tilde{g} \equiv g^{\tilde{\Lambda}}$, then one can rapidly check that we have the equality $\tilde{g} = -\frac{1}{pz} \text{Tr}(\tilde{Q}^{\tilde{\Lambda}^z})$. To show that \tilde{g} is a Stieltjes transform, we will employ the following well known theorem that can be found for instance in [Bol97]:

Theorem F.26. *Given an analytic mapping $f : \mathbb{H} \rightarrow \mathbb{H}$, if $\lim_{y \rightarrow +\infty} -iyf(iy) = 1$ then f is the Stieltjes transform of a probability measure μ that satisfies the two reciprocal formulas:*

- $f(z) = \int \frac{\mu(d\lambda)}{\lambda - z}$,
- for any continuous point¹⁰ $a < b$: $\mu([a, b]) = \lim_{y \rightarrow 0^+} \frac{1}{\pi} \int_a^b \Im(f(x + iy)) dx$.

If, in particular, $\forall z \in \mathbb{H}$, $zf(z) \in \mathbb{H}$, then $\mu(\mathbb{R}^-) = 0$ and f admits an analytic continuation on $\mathbb{C} \setminus (\mathbb{R}_+ \cup \{0\})$.

⁶It must be noted that the setting of Figure F.1 does not exactly fall under our hypotheses (since here $\|\mathbb{E}[x_i]\| \geq O(\sqrt{p})$), as the amplitude of the signals must be sufficiently large for the resulting eigenvalues to isolate from the bulk of the distribution when the number of classes is high ($\sqrt{p} \approx 14$ is not so large). However, even in this extreme setting the prediction are good.

⁷The number of classes is the number of different distributions that can follow the column vectors of X

⁸The possibility to construct such a path and the condition on B for its existence, related to the notion of clusters, is a very interesting question that we do not address in this study.

⁹The Stieltjes transform of the spectral distribution of $\frac{1}{n}X^T X$ is $\tilde{g} = \tilde{g}^{\Lambda^z}$, where for all $D : \mathbb{H} \ni z \mapsto D^z \in \mathcal{D}_n(\mathbb{H})$, $\tilde{g}^{D^z} : z \mapsto -\frac{1}{p} \sum_{i=1}^n \frac{1}{D^z}$

¹⁰We can add the property $\forall x \in \mathbb{R}$, $\mu(\{x\}) = \lim_{y \rightarrow 0^+} y \Im(f(x + iy))$, here for μ to be continuous in a, b , we need $\mu(\{a\}) = \mu(\{b\}) = 0$

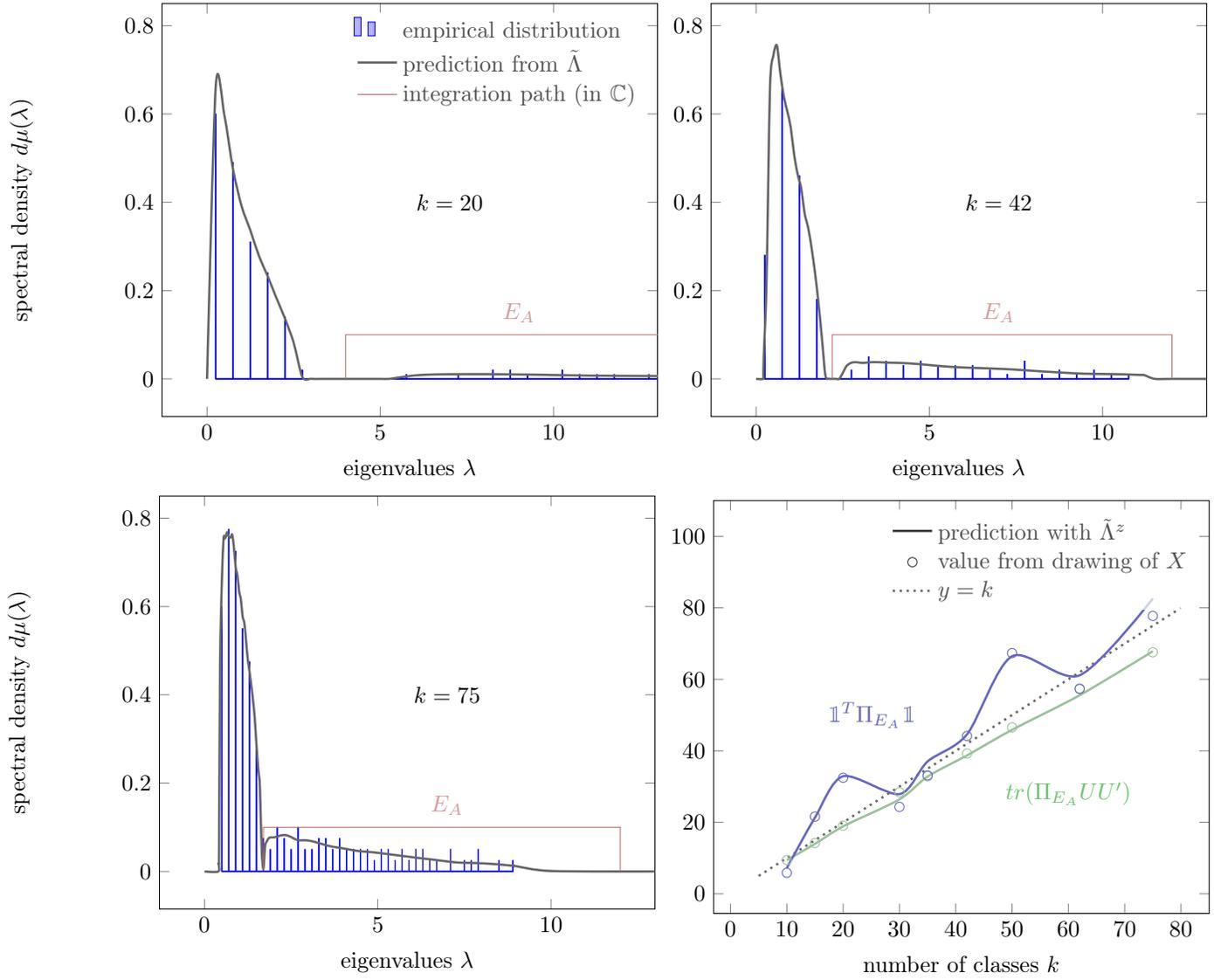


Figure F.1: Prediction of the alignment of the signals in the data towards the eigen space of the biggest eigen values of $\frac{1}{n}XX^T$ for $p = 200$, $n = kn_k$ where $n_k = 20$ and k is the number of classes taking the values $k = 10, 15, 20, 30, 35, 42, 50, 62, 75$. The signals u_1, \dots, u_k are drawn randomly and independently following a law $\mathcal{N}(0, I_p)$, and we let $U = (u_1, \dots, u_k) \in \mathcal{M}_{p,k}$. Then, for all $j \in [k]$ and $l \in [20]$, $x_{j+ln_k} \sim \mathcal{N}(u_j, I_p)$. **(top and bottom left)** representation of the spectral distribution of $\frac{1}{n}XX^T$ and its prediction with $\tilde{\Lambda}^z$. **(bottom right)** representation (with marks) of the quantities $\text{Tr}(\Pi_{E_A} UU^T)$ and $\mathbb{1}^T \Pi_{E_A} \mathbb{1}$ and their prediction (smooth line) with the integration of, respectively, $\Im(\frac{1}{z\pi} \text{Tr}(\tilde{Q}^{\tilde{\Lambda}^z}))$ and $\Im(\frac{1}{z\pi} \mathbb{1}^T \tilde{Q}^{\tilde{\Lambda}^z} \mathbb{1})$ on the path drawn in red on the other graphs. The line $y = k$ represents instances of $\text{Tr}(\Pi_A) = k$ that can be approximated integrating $\frac{1}{z\pi} \text{Tr}(\tilde{Q}^{\tilde{\Lambda}^z})$. The projection $\text{Tr}(\Pi_{E_A} UU^T)$ is a little lower than k because the eigen vectors associated to the highest eigen values of $\frac{1}{n}XX^T$ are not perfectly aligned with the signal due to the randomness of X .

The first hypothesis to verify in order to employ Theorem F.26 is the analyticity of \tilde{g} , that originates from the analyticity of $z \rightarrow \tilde{\Lambda}^z$. We can show with limiting arguments that $\tilde{\Lambda}^z$ is analytical as a limit of a sequence of analytical mappings. However, although it is slightly more laborious, we prefer here to prove the analyticity from the original definition. First let us show the continuity with Proposition E.23.

Proposition F.27. *The mapping $z \mapsto \tilde{\Lambda}^z$ is continuous on \mathbb{H} .*

Proof. Given $z \in \mathbb{H}$, we consider a sequence $(t_s)_{s \in \mathbb{N}} \in \{w \in \mathbb{C} \mid w + z \in \mathbb{H}\}$ such that $\lim_{s \rightarrow \infty} t_s = 0$. Let us verify the assumption of Proposition E.23 where for all $s \in \mathbb{N}$, $f^s = I^{z+t_s}$, $\tilde{\Gamma}^s = \tilde{\Lambda}^{z+t_s}$ and $\Gamma^s = \tilde{\Lambda}^z$ (it does not depend on s). We already know from Proposition F.15 that f^s are all contracting for the stable semi-metric with a Lipschitz parameter $\lambda < 1$ that can be chosen independent from s for s big enough. Let us express for any $s \in \mathbb{N}$ and any $i \in [n]$:

$$f^s(\Gamma^s)_i - \Gamma_i^s = I^{z+t_s}(\tilde{\Lambda}^z)_i - \tilde{\Lambda}_i^z = t_s \quad (\text{F.19})$$

Noting that for s sufficiently big, $\Im(I^{z+t_s}(\tilde{\Lambda}^z)) = \Im(t_s) + \Im(\tilde{\Lambda}^z) \geq \frac{\Im(\tilde{\Lambda}^z)}{4} \geq \frac{\Im(z)}{4}$, we see that $d_s(\Im(f^s(\Gamma^s)_i), \Im(\Gamma_i^s)) \leq \frac{4|\Im(t_s)|}{\Im(z)} \xrightarrow{s \rightarrow \infty} 0$. Therefore, the assumptions of Proposition E.23 are satisfied and we can conclude that there exists $K > 0$ such that for all $s \in \mathbb{N}$:

$$\left\| \frac{\tilde{\Lambda}^{z+t_s} - \tilde{\Lambda}^z}{\sqrt{\Im(\tilde{\Lambda}^{z+t_s})\Im(\tilde{\Lambda}^z)}} \right\| \leq \frac{K|t_s|}{\inf_{i \in [n]} \sqrt{\Im(\tilde{\Lambda}^{z+t_s})\Im(\tilde{\Lambda}^z)}} \leq \frac{2K|t_s|}{\Im(z)}.$$

Besides, we can also bound:

$$\sqrt{\Im(\tilde{\Lambda}^{z+t_s})} \leq \frac{2\sqrt{\Im(\tilde{\Lambda}^z)}}{\Im(z)} (\Im(\tilde{\Lambda}^z) + Kt_s) \leq O(1),$$

That directly implies that $\|\tilde{\Lambda}^{z+t_s} - \tilde{\Lambda}^z\| \leq O(t_s) \xrightarrow{s \rightarrow \infty} 0$, and consequently, $z \mapsto \tilde{\Lambda}^z$ is continuous on \mathbb{H} . \square

Let us now show that $z \mapsto \tilde{\Lambda}^z$ is differentiable. Employing again the notation $f^t = \mathcal{I}^{z+t}$, we can decompose (noting for $D \in \mathcal{D}_n$, $R(D) \equiv (zI_p - \frac{1}{n} \sum_{i=1}^n)$):

$$\begin{aligned} (\tilde{\Lambda}^{z+t} - \tilde{\Lambda}^z) &= (f^t(\tilde{\Lambda}^{z+t}) - f^t(\tilde{\Lambda}^z) + f^t(\tilde{\Lambda}^z) - f^0(\tilde{\Lambda}^z)) \\ &= \text{Diag}_{i \in [n]} \left(\frac{1}{n} \text{Tr} \left(\Sigma_i \tilde{Q}^{\tilde{\Lambda}^{z+t}} \frac{1}{n} \sum_{j=1}^n \frac{\tilde{\Lambda}_j^{z+t} - \tilde{\Lambda}_j^z}{\tilde{\Lambda}_j^{z+t} \tilde{\Lambda}_j^z} \Sigma_j \tilde{Q}^{\tilde{\Lambda}^z} \right) \right) + tI_n \end{aligned}$$

Now, if we introduce the vector $a(t) = (\tilde{\Lambda}_i^z - \tilde{\Lambda}_i^{z+t})_{1 \leq i \leq n} \in \mathbb{R}^n$, and for any $D, D' \in \mathcal{D}_n(\mathbb{H})$, the matrix:

$$\Psi(D, D') = \left(\frac{1}{n} \frac{\text{Tr}(\Sigma_i R(D) \Sigma_j R(D'))}{D_j D'_j} \right)_{1 \leq i, j \leq n} \in \mathcal{M}_n$$

We have the equation:

$$a(t) = \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z+t})a(t) + t\mathbb{1}. \quad (\text{F.20})$$

To be able to solve this equation we need:

Lemma F.28. *Given any $z, z' \in \mathbb{H}$, $I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})$ is invertible.*

Proof. We are going to show the injectivity of $I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})$. Let us introduce a vector $x \in \mathbb{R}^n$ such that $x = \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})x$. We can bound thanks to Cauchy-Schwartz inequality, with similar calculus as in the proof

of Proposition F.15:

$$\begin{aligned} |x_i| &= \left| \frac{1}{n} \operatorname{Tr} \left(\Sigma_i R(\tilde{\Lambda}^z) \sum_{j=1}^n \frac{x_j \Sigma_j}{\sqrt{\Im(\tilde{\Lambda}^z) \Im(\tilde{\Lambda}^{z'})}} \frac{\sqrt{\Im(\tilde{\Lambda}^z) \Im(\tilde{\Lambda}^{z'})}}{\tilde{\Lambda}_j^z \tilde{\Lambda}_j^{z'}} R(\tilde{\Lambda}^{z'}) \right) \right| \\ &\leq \sup_{j \in [n]} \left| \frac{x_j}{\sqrt{\Im(\tilde{\Lambda}^z) \Im(\tilde{\Lambda}^{z'})}} \right| \sqrt{\Im(\tilde{\Lambda}_i^z) - \Im(z)} \sqrt{\Im(\tilde{\Lambda}_i^{z'}) - \Im(z')} \end{aligned}$$

therefore, if we denote $\|x\|_{\tilde{\Lambda}} \equiv \sup_{i \in [n]} \left| \frac{x_j}{\sqrt{\Im(\tilde{\Lambda}^{z'}) \Im(\tilde{\Lambda}^{z'})}} \right|$, we have then the bound:

$$\|x\|_{\tilde{\Lambda}^{z'}, \tilde{\Lambda}^{z'}} \leq \|x\|_{\tilde{\Lambda}^{z'}, \tilde{\Lambda}^{z'}} \sqrt{(1 - \phi(z, \tilde{\Lambda}^{z'}))(1 - \phi(z, \tilde{\Lambda}^{z'}))}$$

which directly implies that $x = 0$ since we know that $\phi(z, \tilde{\Lambda}^{z'}) = \frac{\Im(w)}{\sup_{1 \leq i \leq n} \Im(I^w(\tilde{\Lambda}^{z'})_i)} \in (0, 1)$. \square

From the continuity of $(z, z') \mapsto \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})$, and the limit $\lim_{x \rightarrow \infty} \|\Psi(\tilde{\Lambda}^x, \tilde{\Lambda}^x)\| = 0$ (see the proof of Proposition F.33), we can deduce as a side result from Lemma F.28:

Lemma F.29. *Given any $z, z' \in \mathbb{H}$, $\|\Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})\| < 1$*

The continuity of $z \mapsto \tilde{\Lambda}^z$ given by Proposition F.27 and the continuity of the inverse operation on matrices (around $I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^z)$ which is invertible), allows us to let t tend to zero in the equation

$$\frac{1}{t} a(t) = (I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z+t}))^{-1} \mathbb{1},$$

to obtain:

Proposition F.30. *The mapping $z \mapsto \tilde{\Lambda}^z$ is analytic on \mathbb{H} , and satisfies:*

$$\frac{\partial \tilde{\Lambda}^z}{\partial z} = \operatorname{Diag} \left((I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^z))^{-1} \mathbb{1} \right)$$

We can then conclude first that for all $i \in [n]$, the mappings $z \rightarrow \frac{1}{\tilde{\Lambda}_i^z}$ are Stieltjes transforms.

Proposition F.31. *For all $i \in [n]$, there exists a distribution $\tilde{\mu}_i$ with support on \mathbb{R}_+ whose Stieltjes transform is $z \mapsto -\frac{1}{\tilde{\Lambda}_i^z}$*

Proof. We just check the hypotheses of Theorem F.26. We already know that $z \mapsto -\frac{1}{\tilde{\Lambda}_i^z}$ is analytical thanks to Proposition F.30 and the lower bound $\tilde{\Lambda}_i^z \geq \Im(z) > 0$. Besides, $\forall z \in \mathbb{H}$:

$$\Im \left(-\frac{1}{\tilde{\Lambda}_i^z} \right) = \frac{\Im(\tilde{\Lambda}_i^z)}{|\tilde{\Lambda}_i^z|^2} > 0 \quad \text{and} \quad \Im \left(-\frac{z}{\tilde{\Lambda}_i^z} \right) = \frac{\Im(\tilde{\Lambda}_i^z/z)}{|\tilde{\Lambda}_i^z/z|} > 0,$$

since $\tilde{\Lambda}^z \in \mathcal{D}_{I_z}$. Finally recalling from Lemma F.17 that for all $y \in \mathbb{R}_+$, $\|\tilde{Q}^{\tilde{\Lambda}^{iy}}\| \leq \frac{|iy|}{\Im(iy)} = 1$, we directly see that for all $j \in [n]$:

$$\frac{\tilde{\Lambda}_j^{iy}}{iy} = 1 + \frac{1}{iyn} \operatorname{Tr}(\Sigma_j \tilde{Q}^{\tilde{\Lambda}^{iy}}) \xrightarrow{y \rightarrow +\infty} 1.$$

we can then conclude with Theorem F.26. \square

We can then deduce easily that \tilde{g} is also a Stieltjes transform with an interesting characterization of its measure with the $\tilde{\mu}_i$ (defined in Proposition F.31).

Proposition F.32. *The mapping \tilde{g} is the Stieltjes transform of the measure:*

$$\tilde{\mu} = \left(\frac{n}{p} - 1\right) \delta_0 + \frac{1}{p} \sum_{i=1}^n \tilde{\mu}_i,$$

where δ_0 is the Dirac measure on 0 (if $p > n$, then the measures $\tilde{\mu}_1, \dots, \tilde{\mu}_n$ contains Dirac weights on zero that cancel the term $-\frac{p-n}{p}\delta_0$).

Recall that $\tilde{\mu}$, satisfies $\tilde{g}(z) = \int_0^{+\infty} \frac{1}{\lambda-z} d\tilde{\mu}(\lambda)$, and let us denote \tilde{S} , its support. This formula implies that \tilde{g} is analytic on $\mathbb{C} \setminus \tilde{S}$ and that for all $z \in \mathbb{C} \setminus \tilde{S}$, $\tilde{g}(\bar{z}) = \overline{\tilde{g}(z)}$. To precise the picture let us provide a result of compactness of \tilde{S} .

Proposition F.33. *The measure $\tilde{\mu}$ has a compact support $\tilde{S} \subset \mathbb{R}_+$ and $\sup \tilde{S} \leq O(1)$.*

Proof. We are going to show that for x sufficiently big, $\lim_{y \rightarrow 0^+} \Im(g(x+iy)) = 0$, which will allow us to conclude thanks to the relation between $\tilde{\mu}$ and \tilde{g} given in Theorem F.26. Considering $z = x+iy \in \mathbb{H}$, for $x, y \in \mathbb{R}$ and such that

$$x \geq x_0 \equiv \max \left(\frac{8}{n} \sup_{i \in [n]} \text{Tr}(\Sigma_i), 4\nu \right)$$

let us show first that $\forall i \in [n]$, $\Re(\tilde{\Lambda}_i^z) \geq \frac{x_0}{2}$. This is a consequence of the fact that \mathcal{I}^z is stable on $\mathcal{A} \equiv \mathcal{D}_n(\{t \geq \frac{x}{2}\} + i\mathbb{R}_+^*) \cap \mathcal{D}_{I^z}$. Indeed, given $L \in \mathcal{A}$

$$\Re((\tilde{Q}^L)^{-1}) = I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Re(\Lambda_i) \Sigma_i}{|L_i|^2} \geq I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\Re(L_i)} \geq \frac{1}{2}$$

and as we already know, since \mathcal{D}_{I^z} , $\Im((\tilde{Q}^L)^{-1}) \geq 0$, therefore, $\|\tilde{Q}^L\| \leq 2$. We can then bound:

$$\begin{aligned} \Re(\mathcal{I}^z(L)_i) &= x - \frac{1}{n} \text{Tr} \left(\Sigma_i \tilde{Q}^L \left(1 - \frac{1}{n} \sum_{j=1}^n \frac{\Re(L_j) \Sigma_j}{|L_j|^2} \right) \tilde{Q}^L \right) \\ &\geq x - \frac{4}{n} \text{Tr}(\Sigma_i) \geq \frac{x}{2}. \end{aligned}$$

Thus as a limit of elements of \mathcal{A} , $\tilde{\Lambda}^z \in \mathcal{A}$, and $\forall i \in [n]$, $\Re(\tilde{\Lambda}_i^z) \geq \frac{x}{2}$.

Besides, let us bound:

$$\begin{aligned} \Im(\tilde{\Lambda}_i^z) &= y + \frac{1}{n} \text{Tr} \left(\Sigma_i R(\tilde{\Lambda}^z) \frac{1}{n} \sum_{j=1}^n \frac{\Im(\tilde{\Lambda}_j^z) \Sigma_j}{|\tilde{\Lambda}_j^z|^2} \tilde{R}(\tilde{\Lambda}^z) \right) \\ &\leq y + \frac{4}{n} \text{Tr}(\Sigma_i) \left\| \frac{1}{n} \sum_{j=1}^n \Sigma_j \right\| \sup_{j \in [n]} \frac{\Im(\tilde{\Lambda}_j^z)}{\Re(\tilde{\Lambda}_j^z)^2}. \end{aligned}$$

We can further bound $\left\| \frac{1}{n} \sum_{j=1}^n \Sigma_j \right\| \leq \nu$, since $\frac{1}{n} \|XX^T\| = (\|X\|/\sqrt{n})^2 \in \nu \pm \mathcal{E}_1(1/\sqrt{n})$ and therefore $\left\| \frac{1}{n} \sum_{j=1}^n \Sigma_j \right\| \leq \mathbb{E}[\frac{1}{n} \|XX^T\|] \leq \nu + O(1/\sqrt{n}) \leq 2\nu$. Besides $\Re(\tilde{\Lambda}_j^z)^2 \geq \frac{xx_0}{4}$, we can eventually bound $\sup_{j \in [n]} \Im(\tilde{\Lambda}_j^z) \leq y + \frac{2\nu}{x} \sup_{j \in [n]} \Im(\tilde{\Lambda}_j^z)$, which implies, for x sufficiently big:

$$\sup_{j \in [n]} \Im(\tilde{\Lambda}_j^z) \leq \frac{y}{1 - \frac{2\nu}{x}} \xrightarrow{y \rightarrow 0^+} 0.$$

We can then conclude letting y tend to 0 in the formulation $\tilde{g} = g^{\tilde{\Lambda}}$:

$$\Im(\tilde{g}(x+iy)) = \frac{y}{x^2 + y^2} \left(\frac{n}{p} - 1 \right) + \frac{1}{p} \sum_{i=1}^n \frac{\Im(\tilde{\Lambda}_i^z)}{\Re(\tilde{\Lambda}_i^z)^2 + \Im(\tilde{\Lambda}_i^z)^2} \xrightarrow{y \rightarrow 0^+} 0$$

□

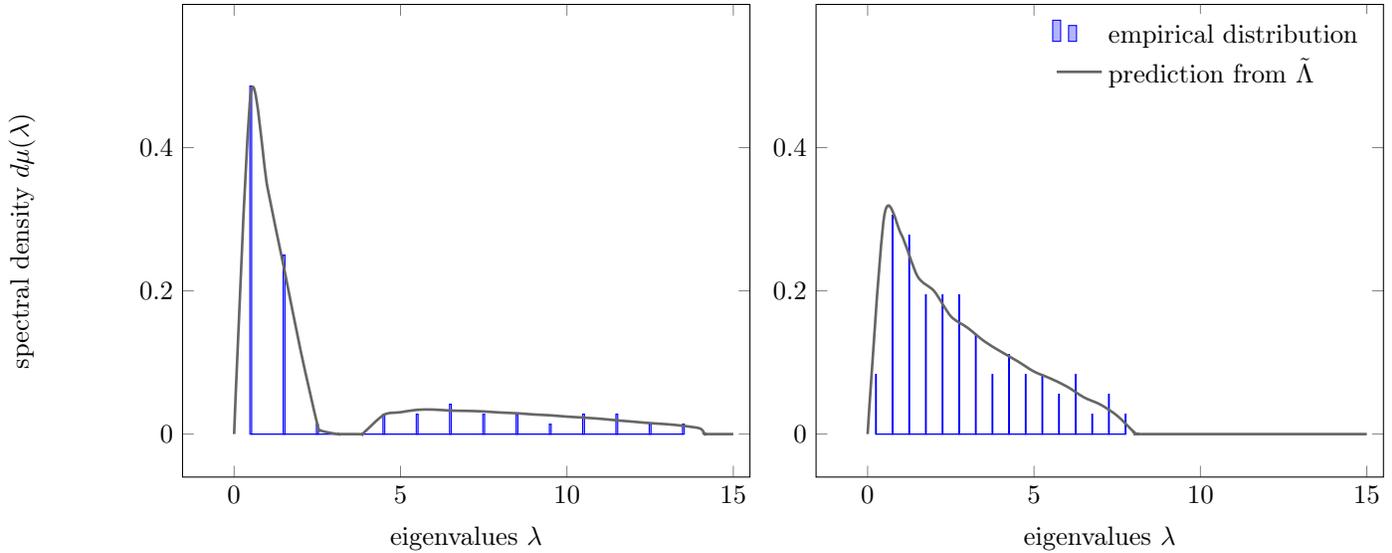


Figure F.2: Spectral distribution of $\frac{1}{n}XX^T$ and its deterministic estimate obtained from $\tilde{\Lambda}$ for $n = 160$ and $p = 80$. Introducing P an orthogonal matrix chosen randomly and $\Sigma \in \mathcal{D}_p$ such that for $j \in \{1, \dots, 20\}$, $\Sigma_j = 1$ and for $j \in \{21, \dots, 80\}$, $\Sigma_j = 8$, we chose **(left)** $\forall i \in [n]$, $x_i \sim \mathcal{N}(0, \Sigma)$ and **(right)** $\forall i \in [n]$, $x_i \sim \mathcal{N}(0, \Sigma_i)$, where $\Sigma_1 = \Sigma$ and $\Sigma_{i+1} = P^T \Sigma_i P$ for all $i \in [n]$. The histograms would have been similar for any other concentrated vectors x_1, \dots, x_n having the same covariances and comparable observation diameter (see Definition 9)

We end this section with the proof of the convergence of the Stieltjes transform of the spectral distribution g towards \tilde{g} .

We introduce the semi-norm $\|\cdot\|_{S^\varepsilon}$, defined for any $f \in \mathcal{F}(\mathbb{C})$ as:

$$\|f\|_{S^\varepsilon} = \sup_{z \in \mathbb{C} \setminus S^\varepsilon} |f(z)|.$$

Theorem F.34. $g \mid \mathcal{A}_Q \in \tilde{g} \pm \mathcal{E}_2(1/\sqrt{pn})$ in $(\mathbb{C} \setminus S^\varepsilon, \|\cdot\|_{S^\varepsilon})$.

Proof. We know from remark F.21 that $O(\frac{\kappa_z}{z}) \leq O(1)$ and therefore we can show as in the proof of Proposition F.3 that the mapping g defined for any $z \in \mathbb{C} \setminus (S^\varepsilon \cup \{0\})$ with the identity $g(z) = -\frac{1}{pz} \text{Tr}(Q^z)$ is, under \mathcal{A}_Q a $O(1/\sqrt{pn})$ -Lipschitz transformation of X , thus $(g \mid \mathcal{A}_Q) \propto \mathcal{E}_2(1/\sqrt{pn})$ in $(\mathcal{F}(\mathbb{C}), \|\cdot\|_{S^\varepsilon})$. We can then conclude thanks to the bound:

$$\begin{aligned} \sup_{z \in \mathbb{C} \setminus S^\varepsilon} |\mathbb{E}_{\mathcal{A}_Q}[g(z)] - \tilde{g}(z)| &\leq \sup_{z \in \mathbb{C} \setminus S^\varepsilon} \frac{1}{zp} \left| \text{Tr} \left(\mathbb{E}_{\mathcal{A}_Q}[Q^z] - \tilde{Q}^{\tilde{\Lambda}^z} \right) \right| \\ &\leq \sup_{z \in \mathbb{C} \setminus S^\varepsilon} O \left(\frac{\kappa_z}{|z|\sqrt{np}} \right) \leq O \left(\frac{1}{\sqrt{np}} \right). \end{aligned}$$

□

In particular, thanks to the Cauchy identity, for any analytical mapping $f : \mathbb{C} \rightarrow \mathbb{C}$, since the integration on bounded paths of $\mathbb{C} \setminus S^\varepsilon$ is Lipschitz for the norm $\|\cdot\|_{S^\varepsilon}$, we can approximate:

$$\mathbb{P} \left(\left| \int f(\lambda) d\mu(\lambda) - \frac{1}{2i\pi} \oint_\gamma f(z) g(z) dz \right| \geq t \right) \leq Ce^{-cnp t^2} + Ce^{-cn},$$

for any closed path $S^\varepsilon \subset \gamma \subset \mathbb{C} \setminus S^\varepsilon$ with length l_γ satisfying $l_\gamma \leq O(1)$. There exists a correspondence between a distribution and its Stieltjes transform: denoting μ the spectral distribution of $\frac{1}{n}XX^T$, we have

indeed for any real $a < b$:

$$\mu([a, b]) = \lim_{y \rightarrow 0} \frac{1}{\pi} \int_a^b \Im(g(x + iy)) dx.$$

As seen on Figure F.2, this measure is naturally close to $\tilde{\mu}$ defined for any real $a < b$ as $\tilde{\mu}([a, b]) = \lim_{y \rightarrow 0} \frac{1}{\pi} \int_a^b \Im(\tilde{g}(x + iy)) dx$, it can indeed be shown that the Kolmogorov distance between those two measures tends to zero as stated in [Cho22].

F.2 Statistical study of the resolvent with convex concentration hypotheses

We want in this section to extend the result of Section F.1 to the case of convexly concentrated matrices. Recall that convexly concentrated random vectors class is not stable through Lipschitz maps, we just set it was stable through affine transformations. As a consequence, in this setting, the concentration of the resolvent $Q^z = (I_n - X/z)^{-1}$ is no more a mere consequence of a bound on its differential on $X \in \mathcal{M}_p$. Still, as first shown by [GZ00], it is possible to obtain concentration properties on the sum of Lipschitz functionals of the eigen values. Here we pursue the study, looking at *linear* concentration properties of Q for which concentration inequalities are only satisfied by 1-Lipschitz *and linear* functionals f . The well known identity

$$\frac{1}{p} \sum_{\lambda \in \text{Sp}(X)} f(\lambda) = -\frac{1}{2i\pi} \oint_{\gamma} \frac{f(z)}{z} \text{Tr}(Q^z) dz, \quad (\text{F.21})$$

is true for any analytical mapping f defined on the interior of a path $\gamma \in \mathbb{C}$ containing the spectrum of X (or any limit of such mappings), therefore, our results on the concentration of Q^z concern in particular the quantities studied in [GZ00].

The linear concentration of the resolvent is proven thanks to Theorem C.3, expressing it as a sum $Q^z = \frac{1}{z} \sum_{i=1}^{\infty} (X/z)^i$. The linear concentration of the powers of X was justified in Theorem D.21. We call this weakening of the concentration property “the degeneracy of the convex concentration through multiplication”. The linear concentration of the resolvent is though sufficient for most practical applications that rely on an estimation of the Stieltjes transform $m(z) = \frac{1}{z^p} \text{Tr}(Q^z)$ or on projections on Q^z .

We still work under Assumptions 1-5, we just adapt Assumption 2 to the convex concentration setting and assume instead:

Assumption 2_c (Convex concentration). $X \propto_c \mathcal{E}_2$.

We also keep the same notations $S = \{\mathbb{E}_{\mathcal{A}_\nu}[\lambda_1], \dots, \mathbb{E}_{\mathcal{A}_\nu}[\lambda_p]\}$ and

$$\mathcal{A}_Q \equiv \left\{ \forall i \in [p], \sigma_i \left(\frac{1}{n} X X^T \right) \in S^{\varepsilon/2} \right\},$$

for some $\varepsilon \geq O(1)$. the concentration of $\sigma(X)/\sqrt{n} \in \mathbb{E}[\sigma(X)] \pm \mathcal{E}_2(1/\sqrt{n})$, allows us to set¹¹ as in Lemma F.1 that there exist two constants $C, c > 0$ such that $\mathbb{P}(\mathcal{A}_Q^c) \leq C e^{-cn\varepsilon^2}$.

We can beside deduce from Lemma D.5 that $(X | \mathcal{A}_Q) \propto_c \mathcal{E}_2$.

Placing ourselves under the event \mathcal{A}_Q , let us first show that the resolvent $Q^z \equiv (I_p - \frac{1}{nz} X X^T)^{-1}$ is concentrated if z has a big enough modulus. Be careful that the following concentration is expressed for the nuclear norm (for any deterministic matrix $A \in \mathcal{M}_p$ such that $\|A\| \leq O(1)$, $\text{Tr}(A Q^z) \in \mathcal{E}_2$). The next proposition is just provided as a first direct application of Theorem C.3, a stronger result is provided in Proposition F.36.

Proposition F.35. *Given $z \in \mathbb{C}$ such that $|z| \geq \nu + \varepsilon$:*

$$(Q^z | \mathcal{A}_Q) \in \mathcal{E}_2 \quad \text{in } (\mathcal{M}_p, \|\cdot\|_*).$$

¹¹In Lemma F.1, the proof is conducted for Lipschitz concentration hypotheses on X . However, since only the linear concentration of $\sigma(X)$ is needed, the justification is the same in a context of convex concentration thanks to Theorem D.19.

Proof. We know from Lemma D.5 that $(X \mid \mathcal{A}_Q) \propto_c \mathcal{E}_2$ and from Theorem D.21 that (here $\kappa = \nu + \frac{\varepsilon}{2} \leq O(1)$, $\sigma = 1/\sqrt{n}$ and $p = O(n)$):

$$\text{Under } \mathcal{A}_Q: \quad \left(\frac{1}{n} X X^T \right)^m \in \mathcal{E}_2 \left(\left(\nu + \frac{\varepsilon}{2} \right)^m \sqrt{m} \right) \quad \text{in } (\mathcal{M}_p, \|\cdot\|_*).$$

Let us then note that $(\nu + \frac{\varepsilon}{2})^m \sqrt{m} = O\left((\nu + \frac{3\varepsilon}{4})^m\right)$ and for $z \in \mathbb{C}$ satisfying our hypotheses: $(\nu + \frac{3\varepsilon}{4})/|z| \leq 1 - \frac{\varepsilon}{4(\nu + \varepsilon)}$. We can then deduce from Corollary B.32 that under \mathcal{A}_Q :

$$Q^z = \frac{1}{z} \left(I_p - \frac{1}{zn} X X^T \right)^{-1} = \frac{1}{z} \sum_{i=1}^{\infty} \left(\frac{1}{zn} X X^T \right)^i \in \mathcal{E}_2 \left(\frac{4}{\varepsilon} (\nu + \varepsilon) \right).$$

□

Let us now try to study the concentration of Q^z when z gets close to the spectrum.

Proposition F.36. *For all $z \in \mathbb{C} \setminus S^\varepsilon$:*

$$(Q^z \mid \mathcal{A}_Q) \in \mathcal{E}_2(\kappa_z) \quad \text{in } (\mathcal{M}_p, \|\cdot\|_*),$$

Proof. Proposition F.35 already set the result for $|z| \geq \nu + \varepsilon \equiv \rho$, therefore, let us now suppose that $|z| \leq \rho$.

With the notation $M \equiv \left(\Im(z)^2 + \left(\Re(z) - \frac{1}{n} X X^T \right)^2 \right)^{-1} = |\frac{1}{z} Q^z|^2$, let us decompose:

$$\frac{1}{z} Q^z = \left(\Re(z) - \frac{1}{n} X X^T \right) M - \Im(z) i M. \quad (\text{F.22})$$

We can then deduce the linear concentration of M with the same justifications as previously thanks to the Taylor decomposition:

$$M = \frac{|z|^2}{\rho^2} \sum_{m=0}^{\infty} \left(1 - \frac{\Im(z)^2}{\rho^2} - \frac{\left(\Re(z) - \frac{1}{n} X X^T \right)^2}{\rho^2} \right)^m.$$

Indeed, $\|\Re(z) I_p - \frac{1}{n} X X^T\| \leq d(\Re(z), S)$ and $d(z, S)^2 = \Im(z)^2 + d(\Re(z), S)^2 \leq \rho$ thus:

$$\left\| 1 - \frac{\Im(z)^2}{\rho^2} - \frac{1}{\rho^2} \left(\Re(z) I_p - \frac{1}{n} X X^T \right)^2 \right\| \leq 1 - \frac{d(z, S)^2}{\rho^2} \leq 1 - \frac{\varepsilon^2}{\rho^2} < 1.$$

We therefore deduce from (F.22) that:

$$(Q^z \mid \mathcal{A}_Q) \in \mathcal{E}_2 \left(\frac{|z|}{\varepsilon^2} \left(|\Im(z)| + |\Re(z)| + \nu + \frac{\varepsilon}{2} \right) \right) = \mathcal{E}_2(\kappa_z).$$

□

One can then follow the lines of study made for the Lipschitz concentration case to show that Q^z is also concentrated around $\tilde{Q}^{\tilde{\Lambda}^z}$ under convex concentration hypotheses. Although Proposition F.36 gives us a concentration of Q^z in nuclear norm, we will estimate $\mathbb{E}[Q^z]$ with the Frobenius norm as in the Lipschitz concentration case.

Proposition F.37. *For any $z \in \mathbb{C} \setminus S^\varepsilon$ in convex concentration setting:*

$$\left\| \mathbb{E}[Q^z] - \tilde{Q}^{\tilde{\Lambda}^z} \right\|_F \leq O \left(\frac{\kappa^z}{\sqrt{n}} \right).$$

The Proof of the Theorem is done the same way as in the Lipschitz case, it just relies on the Lemmas F.40 and F.38 below.

Lemma F.38. *Under \mathcal{A}_Q , for any $z \in \mathbb{C} \setminus S^\varepsilon$ and any $i \in [n]$:*

$$\|\mathbb{E}[Q^z - Q_{-i}^z]\| \leq O\left(\frac{1}{n}\right).$$

The proof is the same as the one of Lemma F.10 and relies on the bound on Λ_i^z given by Lemma F.7 and the concentration of $u^T Q_{-i}^z x_i$ provided in next lemma.

Lemma F.39. *For any $z \in \mathbb{C} \setminus S^\varepsilon$, any $i \in [n]$ and any $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$:*

$$(u^T Q_{-i}^z x_i \mid \mathcal{A}_Q) \in O(1) \pm \mathcal{E}_2.$$

Proof. We do not care about the independence issues brought by \mathcal{A}_Q . Let us simply bound for any $t > 0$ and under \mathcal{A}_Q :

$$\begin{aligned} & \mathbb{P}\left(|u^T Q_{-i}^z x_i - \mathbb{E}[u^T Q_{-i}^z x_i]| \geq t\right) \\ & \leq \mathbb{P}\left(|u^T Q_{-i}^z(x_i - \mu_i)| \geq \frac{t}{2}\right) + \mathbb{P}\left(|u^T(Q_{-i}^z - \mathbb{E}[Q_{-i}^z])\mu_i| \geq \frac{t}{2}\right) \\ & \leq \mathbb{E}\left[Ce^{-cnt^2/\|Q_{-i}\|^2}\right] + Ce^{-cnt^2} \leq 2Ce^{-c'tt^2}, \end{aligned}$$

for some constants $C, c, c' > 0$. Besides, we can bound:

$$|\mathbb{E}[u^T Q_{-i}^z x_i]| = |u^T \mathbb{E}[Q_{-i}^z] m_i| \leq O(1),$$

thanks to Lemma F.2. □

The last Lemma is then important to employ similar results as Proposition B.50 as in the proof of Proposition F.12.

Lemma F.40. *For any $z \in \mathbb{C} \setminus S^\varepsilon$:*

$$\forall i \in [n], \Lambda_i^z \mid \mathcal{A}_Q \in \tilde{\Lambda}_i^z \pm \mathcal{E}_2(\kappa_z/\sqrt{n}) + \mathcal{E}_1(\kappa_z/n).$$

Besides, for any deterministic matrix $A \in \mathcal{M}_p$:

$$(x_i^T A Q^z x_i \mid \mathcal{A}_Q) \in \text{Tr}(\Sigma_i A \mathbb{E}[Q^z]) \pm \mathcal{E}_2(\kappa_z \|A\|_F) + \mathcal{E}_1(\kappa_z \|A\|).$$

Proof. We will first show the concentration of $\frac{1}{n} x_i^T A Q_{-i}^z x_i$ for any deterministic matrix $A \in \mathcal{M}_p$ that will in particular imply the concentration of Λ_i^z . Recall that under \mathcal{A}_Q , $\|X\| \leq O(1)$ and $\|Q^z\| \leq \kappa_z$. Given $i \in [n]$, we want to bound:

$$\begin{aligned} & |x_i^T A Q_{-i}^z x_i - \text{Tr}(\Sigma_i A \mathbb{E}[Q_{-i}^z])| \\ & \leq |x_i^T A Q_{-i}^z x_i - \text{Tr}(\Sigma_i A Q_{-i}^z)| + |\text{Tr}(\Sigma_i A (Q_{-i}^z - \mathbb{E}[Q_{-i}^z]))|. \end{aligned}$$

Now we know that, for X_{-i} fixed, we can bound thanks to Proposition D.6:

$$\begin{aligned} \mathbb{P}\left(|x_i^T A Q_{-i}^z x_i - \text{Tr}(\Sigma_i^T A Q_{-i}^z)| \geq t\right) & \leq \mathbb{E}\left[Ce^{-c(t/\|Q_{-i}^z\| \|A\|_F)^2} + Ce^{-ct/\|Q_{-i}^z\| \|A\|}\right] \\ & \leq Ce^{-c'(t/\|A\|_F \kappa_z)^2} + Ce^{-c't/\|A\| \kappa_z}, \end{aligned}$$

for some constants $C, c, c' > 0$, thanks to Lemma F.2.

Besides, we know from Proposition F.36 and Lemma B.18 that $Q_{-i}^z \in \mathbb{E}[Q^z] \pm \mathcal{E}_2(1/\sqrt{n})$ in $(\mathcal{M}_p, \|\cdot\|_*)$, which allows us to bound:

$$\mathbb{P}\left(|\text{Tr}(\Sigma_i A Q_{-i}^z) - \text{Tr}(\Sigma_i A \mathbb{E}[Q^z])| \geq t\right) \leq Ce^{-ct^2/\|A\|_F^2}$$

for some constants $C, c > 0$, since $\|\Sigma_i\| \leq O(1)$. Putting the two concentration inequalities together, we obtain the concentration of $\frac{1}{n} x_i^T A Q_{-i}^z x_i$.

Now the identity $\frac{1}{n} x_i^T A Q^z x_i = \frac{1}{n} \frac{x_i^T A Q_{-i}^z x_i}{\Lambda_i^z}$, and the bounds $\Lambda_i^z \geq O(|z|/\kappa_z)$ and $\frac{1}{n} x_i^T A Q_{-i}^z x_i \leq \kappa_z \rho$ allows us to deduce the concentration of $\frac{1}{n} x_i^T A Q^z x_i$ thanks to Lemma A.6 setting the concentration of the product of concentrated variables. □

F.3 The deterministic equivalents of the powers of the resolvent

Our aim here is to find deterministic equivalent for all the powers $(Q^z)^m$, for $m \in \mathbb{N}$ and for $z \in \mathbb{C}$ satisfying $d(z, S^\varepsilon) \geq O(1)$ for a given constant $\varepsilon > 0$ ($\varepsilon \geq O(1)$). For simplicity, we will omit the exponent z of the resolvent. We continue to work under Assumptions 1-5 we thus won't recall them in the coming lemmas and propositions. The main goal of this section is to display the formulas given in Proposition F.47, because they might be seen in other contexts thereby allowing to interesting analogies. The coming proofs are quite laborious and do not present all the required justification mainly to avoid complex inferences of little interest.

In the concentration inequalities, it is interesting to keep track of the dependence on m to use it later for a quasi-asymptotic formulations, we will thus assume sometimes that $m \in \mathbb{N}^{\mathbb{N}}$ is a sequence of integer (as n and p). Since it is costless to devise directly a deterministic equivalent for $QA_1QA_2 \cdots A_{m-1}Q$ where A_1, \dots, A_{m-1} are all matrices of unit spectral norm we will consider this general case. Considering from now on $m-1$ matrices A_1, \dots, A_{m-1} , where $\forall i \in [m]$, $\|A_i\| \leq 1$ and we note for any matrix $M \in \mathcal{M}_p$, $k \in [m]$:

$$M^{A_i^k} = \begin{cases} \text{if } k < l : I_p \\ \text{if } k = l : M \\ \text{if } k > l : MA_lMA_{l+1} \dots A_{k-1}M \end{cases} .$$

We further note for formulation simplicity $A_m = I_p$.

Proposition F.41. *Given a sequence of integer $m \in \mathbb{N}^{\mathbb{N}}$:*

$$Q^{A_i^m} \mid \mathcal{A}_Q \in \mathbb{E}[Q^{A_i^m}] \pm \mathcal{E}_2 \left(\frac{m\kappa_z^m}{\sqrt{n}} \right) \quad \text{in} \quad (\mathcal{M}_p, \|\cdot\|_F)$$

Proof. The result is proven the same way as Proposition F.3, employing the bound on Q^z given by Lemma F.2. \square

We now look for a computable deterministic equivalent of $Q^{A_i^m}$. Let us first provide a useful formula to start to disentangle the dependence between $Q^{A_i^m}$ and x_i ; it is a consequence of the Schur formulas (F.8)

Lemma F.42. *For any $m \in \mathbb{N}_*$:*

$$Q^{A_i^m} = Q_{-i}^{A_i^m} + \frac{1}{\Lambda_i^z n} \sum_{l=1}^m Q_{-i}^{A_i^l} x_i x_i^T Q_{-i} A_l Q^{A_i^{m-l}}$$

where $\mathcal{L}_k^m = \{(l_1, \dots, l_k) \in \mathbb{N}^k \mid 1 \leq l_1 < \dots < l_k \leq m\}$ and $\mathcal{L}_0^m = \{()\}$ contains only the 0-tuple (and recall that $A_m = I_p$).

Proof. The result is just a consequence of the following telescoping sum decomposition (where we set $A_0, A_m = I_p$):

$$\begin{aligned} Q^{A_i^m} - Q_{-i}^{A_i^m} &= \sum_{l=1}^m Q_{-i}^{A_i^{l-1}} A_{l-1} Q^{A_i^m} - Q_{-i}^{A_i^l} A_l Q^{A_i^{m-l}} \\ &= \sum_{l=1}^m Q_{-i}^{A_i^{l-1}} A_{l-1} (Q - Q_{-i}) A_l Q^{A_i^{m-l}} = \frac{1}{\Lambda_i^z n} \sum_{l=1}^m Q_{-i}^{A_i^l} x_i x_i^T Q_{-i} A_l Q^{A_i^{m-l}}. \end{aligned}$$

\square

Lemma F.43. *Given a deterministic vector $u \in \mathbb{R}^p$, for any $k, l \in \mathbb{N}$ such that $1 \leq k < l < m$:*

$$u^T Q_{-i}^{A_i^k} x_i \mid \mathcal{A}_Q \in O(\kappa_z^{l-k+1}) \pm \mathcal{E}_2((c\kappa_z)^{l-k+1}),$$

for some constant $c > 1$.

Corollary F.44. $\|\mathbb{E}_{\mathcal{A}_Q}[Q_{-i}^{A^m} - Q_{-i}^{A^1}]\| = O\left(\frac{(c\kappa_z)^m}{n}\right)$ for a constant $c > 1$.

Proof. We do not use directly the second identity of Lemma F.42 because it would let a term m^m appear in the bound. Instead of the first result of Lemma F.42, one could have stated the identity:

$$Q_{-i}^{A^1} = Q_{-i}^{A^m} + \frac{1}{\hat{\Lambda}_i^z n} \sum_{l=1}^m Q_{-i}^{A^{l-1}} A_{l-1} Q_{-i} x_i x_i^T Q_{-i}^{A_l^m},$$

(recall the notation $A_0 = I_p$). Putting those two identities together one obtains:

$$\begin{aligned} Q_{-i}^{A^1} &= Q_{-i}^{A^m} + \frac{1}{\hat{\Lambda}_i^z n} \sum_{l=1}^m Q_{-i}^{A_l^1} x_i x_i^T Q_{-i}^{A_l^m} \\ &\quad + \frac{1}{(\hat{\Lambda}_i^z n)^2} \sum_{l=1}^m \sum_{k=l+2}^m Q_{-i}^{A_l^1} x_i x_i^T Q_{-i} A_l Q_{-i}^{A_{l+1}^{k-1}} A_{k-1} Q_{-i} x_i x_i^T Q_{-i}^{A_k^m} \\ &\quad + \frac{1}{(\hat{\Lambda}_i^z n)^2} \sum_{l=1}^m Q_{-i}^{A_l^1} x_i x_i^T Q_{-i} A_l Q_{-i} x_i x_i^T Q_{-i}^{A_{l+1}^m} \end{aligned} \tag{F.23}$$

Thus, knowing from Lemmas F.43 and F.2 that given any deterministic vector $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$, any $l \in [m]$ and any $k \in \{l+2, \dots, m\}$ we have:

- $u^T Q_{-i}^{A_l^1} x_i \mid \mathcal{A}_Q \in O(\kappa_z^l) \pm \mathcal{E}_2((c\kappa_z)^l)$
- $\frac{1}{|\hat{\Lambda}_i^z n|} \left| x_i^T Q_{-i} A_l Q_{-i}^{A_{l+1}^{k-1}} A_{k-1} Q_{-i} x_i \right| \leq \frac{\kappa_z^{k-l+1} \tilde{\kappa}_z^2}{|z|n} \leq O\left(\frac{\kappa_z^{k-l}}{n}\right),$

one can deduce from the estimation of the product of concentrated random variables given in Lemma A.21 that for all $u, v \in \mathbb{R}^p$ such that $\|u\|, \|v\| \leq 1$:

$$u^T \mathbb{E}_{\mathcal{A}_Q} \left[Q_{-i}^{A^m} - Q_{-i}^{A^1} \right] v \leq O\left(\frac{(c\kappa_z)^m}{n}\right) + O\left(\frac{(c\kappa_z)^{k-l-1}}{n^2}\right) \leq O\left(\frac{(c\kappa_z)^m}{n}\right).$$

□

Before starting the estimation of $\mathbb{E}_{\mathcal{A}_Q}[Q_{-i}^{A^1}]$, we still need a result analogous to Lemma F.13.

Lemma F.45. Given $k, l \in \mathbb{N}$ such that $1 \leq k < l < m$:

$$Q_{-i}^{A^l} X \mid \mathcal{A}_Q \propto \mathcal{E}_2((c\kappa_z)^{l-k+1}),$$

for some constant $c > 1$.

Proof. The concentration is really proven the same way as in the proof of Proposition F.3 (and the fact that there exists a constant $c > 0$ such that for all $i \in [m]$, $i \leq c^i$).

To bound $\|\mathbb{E}_{\mathcal{A}_Q}[Q_{-i}^{A^k} x_i]\|$, we use the identity (F.23) and bound for any deterministic vector $u \in \mathbb{R}^p$ such that $\|u\| \leq 1$:

$$\begin{aligned} \left| \mathbb{E}_{\mathcal{A}_Q} \left[u^T Q_{-i}^{A^l} x_i \right] \right| &= \left| \mathbb{E}_{\mathcal{A}_Q} \left[u^T Q_{-i}^{A^k} x_i \right] \right| + \frac{1}{\hat{\Lambda}_i^z n} \sum_{j=k}^l \left| \mathbb{E}_{\mathcal{A}_Q} \left[u^T Q_{-i}^{A_j^k} x_i x_i^T Q_{-i}^{A_j^l} v \right] \right| \\ &\quad + \left| \frac{1}{(\hat{\Lambda}_i^z n)^2} \sum_{j=k}^l \mathbb{E}_{\mathcal{A}_Q} \left[u^T Q_{-i}^{A_j^k} x_i \right] \mathbb{E}_{\mathcal{A}_Q} \left[x_i^T Q_{-i} A_j Q_{-i} x_i \right] \mathbb{E}_{\mathcal{A}_Q} \left[x_i^T Q_{-i}^{A_{j+1}^l} v \right] \right| \\ &\quad + \left| \frac{1}{(\hat{\Lambda}_i^z n)^2} \sum_{j=k}^l \sum_{o=j+2}^l \mathbb{E}_{\mathcal{A}_Q} \left[u^T Q_{-i}^{A_j^k} x_i \right] \mathbb{E}_{\mathcal{A}_Q} \left[x_i^T Q_{-i} A_j Q_{-i}^{A_{j+1}^{o-1}} A_{o-1} Q_{-i} x_i \right] \mathbb{E}_{\mathcal{A}_Q} \left[x_i^T Q_{-i}^{A_o^l} v \right] \right| \\ &\quad + O\left((c\kappa)^{l-k+1} + \frac{(l-k)(c\kappa)^{l-k+1} \tilde{\kappa}_z}{n|z|} \left(1 + \frac{\kappa_z \tilde{\kappa}_z (l-k)}{|z|n} \right) \right) \leq O((c\kappa_z)^{l-k+1}), \end{aligned}$$

for some constant $c > 0$.

□

Proposition F.46. *With the notation:*

$$T^{A_l^m} \equiv \text{Diag}_{i \in [m]} \left(\frac{1}{n \tilde{\Lambda}_i^z} x_i^T Q_{-i}^{A_l^m} x_i \right),$$

one can estimate for any (sequence of) integer m and any set of deterministic matrices $A_1, \dots, A_{m-1} \in \mathcal{M}_p^{m-1}$:

$$\left\| Q^{A_1^m} - Q^{A_1^{m-1}} A_{m-1} \tilde{Q} - \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^{m-1} \frac{1}{\tilde{\Lambda}_i^z} \sum_{l \in L_k^{m-1}} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q_{-i}^{A_l^1} \Sigma_i \tilde{Q} \right) \right] \tilde{T}_i^{A_l^2} \dots \tilde{T}_i^{A_l^m} \right\|_F \leq O \left(\frac{(c\kappa_z)^m}{\sqrt{n}} \right),$$

where for any $k, l \in [m], k < l, \tilde{T}_i^{A_l^k}$ is a deterministic matrix satisfying $\|\tilde{T}_i^{A_l^m} - \tilde{T}_i^{A_l^k}\|_F \leq O((c\kappa_z)^{l-k+1}/\sqrt{n})$.

Proof. Given a deterministic matrix $A \in \mathcal{M}_p$ such that $\|A\|_F \leq 1$, let us try and estimate $\mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A_1^m} A_{m-1} (Q - \tilde{Q}) \right) \right]$, we allow ourselves not to display all the steps of the calculus since similar inferences were already displayed in the proofs of Propositions F.12 and F.20. For simplicity, we note \tilde{Q} instead of $\tilde{Q}^{\tilde{\Lambda}^z}$.

$$\begin{aligned} & \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A_1^{m-1}} A_{m-1} (Q - \tilde{Q}) \right) \right] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\mathcal{A}_Q} \left[\frac{1}{z} \text{Tr} \left(A Q^{A^m} x_i x_i^T \tilde{Q} \right) \right] - \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\text{Tr} \left(A Q^{A^m} \Sigma_i \tilde{Q} \right)}{\tilde{\Lambda}_i^z} \right] \\ &= \frac{1}{n} \sum_{i=1}^n a_i + b_i + c_i \end{aligned}$$

where for all $i \in [n]$:

$$\begin{aligned} a_i &= \mathbb{E}_{\mathcal{A}_Q} \left[\frac{\tilde{\Lambda}_i^z - \Lambda_i^z}{\tilde{\Lambda}_i^z \tilde{\Lambda}_i^z} \text{Tr} \left(A Q^{A^{m-1}} A_{m-1} Q_{-i} x_i x_i^T \tilde{Q} \right) \right] \\ b_i &= \frac{1}{\tilde{\Lambda}_i^z} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A \left(Q^{A_1^{m-1}} - Q_{-i}^{A_1^{m-1}} \right) A_{m-1} Q_{-i} x_i x_i^T \tilde{Q} \right) \right] \\ c_i &= \frac{1}{\tilde{\Lambda}_i^z} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A \left(Q_{-i}^{A^m} - Q^{A^m} \right) \Sigma_i \tilde{Q} \right) \right] \end{aligned}$$

Let us first apply Proposition B.50 with the hypotheses:

- $\tilde{Q} X \mid \mathcal{A}_Q \propto \mathcal{E}_2(\kappa_z)$ and $\|\mathbb{E}[\tilde{Q} x_i]\| \leq O(1)$ thanks to Assumption 4, 2,
- $Q^{A_1^m} X \mid \mathcal{A}_Q \propto \mathcal{E}_2((c\kappa_z)^m)$ and $\|\mathbb{E}_{\mathcal{A}_Q}[Q^{A_1^m} x_i]\| \leq O((c\kappa_z)^m)$ given by Lemma F.45,
- $\Lambda^z \mid \mathcal{A}_Q \in \mathbb{E}_{\mathcal{A}_Q}[\Lambda^z] \pm \mathcal{E}_2\left(\frac{\kappa^z}{\sqrt{n}}\right)$ in $(\mathcal{D}_n, \|\cdot\|)$ given by Lemma F.5,
- $\|\mathbb{E}_{\mathcal{A}_Q}[\Lambda^z] - \hat{\Lambda}^z\|_F \leq O(\kappa^z/\sqrt{n})$ thanks to Lemma F.6,

to set:

$$\left| \frac{1}{n} \sum_{i=1}^n a_i \right| = \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A^m} X \frac{\tilde{\Lambda}^z - \Lambda^z}{\tilde{\Lambda}^z} X^T \tilde{Q} \right) \right] \leq O \left(\frac{\check{\kappa}_z (c\kappa_z)^{m+2}}{|z| \sqrt{n}} \right) \leq \left(\frac{(c\kappa_z)^{m+1}}{\sqrt{n}} \right)$$

The bound on c_i is just a consequence of Corollary F.44:

$$|c_i| \leq \kappa_z \left\| \mathbb{E}_{\mathcal{A}_Q} \left[Q_{-i}^{A^l} - Q^{A^l} \right] \right\| = O \left(\frac{\kappa_z^m}{n} \right),$$

Because it relies on similar justifications, we will directly replace Λ_i^z with $\tilde{\Lambda}_i^z$ in the coming estimations.

The quantities a_i , $i \in [n]$, are not negligible as the two others, we need to evaluate it. We can again inspire from Lemma F.42 and write:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n b_i &= \frac{1}{z \tilde{\Lambda}_i^z n^2} \sum_{i=1}^n \sum_{l=1}^{m-1} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A^l} x_i x_i^T Q_{-i}^{A^l} x_i x_i^T \tilde{Q} \right) \right] \\ &= \frac{1}{z n^2} \sum_{l=1}^{m-1} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A^l} X T X^T \tilde{Q} \right) \right], \end{aligned}$$

(since $\tilde{\kappa}_z \kappa_z / |z| \leq 1$). Noting that $T^{A^l} \propto \mathcal{E}_2 \left(\frac{(c\kappa_z)^{m-l}}{\sqrt{n}} \right)$ in $(\mathcal{D}_n, \|\cdot\|)$, one can use one more time Proposition B.50 to obtain for any deterministic diagonal matrix \tilde{T}^{A^l} such that $\|\mathbb{E}_{\mathcal{A}_Q}[T^{A^l}] - \tilde{T}^{A^l}\|_F \leq O \left(\frac{(c\kappa_z)^{m-l+1}}{\sqrt{n}} \right)$:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n b_i &= \frac{1}{nz} \sum_{l=1}^{m-1} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A^l} X \tilde{T}^{A^l} X^T \tilde{Q} \right) \right] + O \left(\frac{(c\kappa_z)^m}{\sqrt{n}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{1}{\tilde{\Lambda}_i^z} \sum_{l=1}^{m-1} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A^{l-1}} A_l Q_{-i} x_i \tilde{T}_i^{A^l} x_i^T \tilde{Q} \right) \right] + O \left(\frac{(c\kappa_z)^m}{\sqrt{n}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{1}{\tilde{\Lambda}_i^z} \sum_{l=1}^{m-1} \mathbb{E}_{\mathcal{A}_Q} \left[x_i^T \tilde{Q} A Q_{-i}^{A^l} x_i \tilde{T}_i^{A^l} \right] \\ &\quad + \frac{1}{nz \tilde{\Lambda}_i^z} \sum_{i=1}^n \sum_{l=1}^{m-1} \sum_{k=1}^{l-1} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q^{A^k} x_i x_i Q_{-i}^{A^k} x_i \tilde{T}_i^{A^l} x_i^T \tilde{Q} \right) \right] + O \left(\frac{(c\kappa_z)^m}{\sqrt{n}} \right), \end{aligned}$$

With the same resort to Proposition B.50, on can show iteratively that:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n b_i &= \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^{m-1} \frac{1}{\tilde{\Lambda}_i^z} \sum_{l \in L_k^{m-1}} \mathbb{E}_{\mathcal{A}_Q} \left[x_i^T \tilde{Q} A Q_{-i}^{A^{l_1}} x_i \tilde{T}_i^{A^{l_1}} \dots \tilde{T}_i^{A^{l_k}} \right] + O \left(\frac{(c\kappa_z)^m}{\sqrt{n}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^{m-1} \frac{1}{\tilde{\Lambda}_i^z} \sum_{l \in L_k^{m-1}} \mathbb{E}_{\mathcal{A}_Q} \left[\text{Tr} \left(A Q_{-i}^{A^{l_1}} \Sigma_i \tilde{Q} \right) \right] \tilde{T}_i^{A^{l_1}} \dots \tilde{T}_i^{A^{l_k}} + O \left(\frac{(c\kappa_z)^m}{\sqrt{n}} \right), \end{aligned}$$

for some constant $c' > 0$ (each time one uses the proposition, the bounds of the estimation are the same and we use it $\sum_{k=1}^{m-1} \binom{k}{m-1} k \leq (m-1)2^{m-1}$). \square

Given two integers $k, l > 0$, $h \leq l$ and a tuple $\alpha = (\alpha_{-h}, \dots, \alpha_0, \alpha_1, \dots, \alpha_{l-1}) \in [n]^{l+h}$, let us note:

$$\Psi_\alpha^h \equiv \mathbb{E}_{\mathcal{A}_Q} \left[\frac{1}{n} \text{Tr} \left(\tilde{\Sigma}_{\alpha_{-h}} \tilde{Q} \dots \tilde{\Sigma}_{\alpha_{-1}} \tilde{Q} \tilde{\Sigma}_{\alpha_0} Q \dots \tilde{\Sigma}_{\alpha_{l-1}} Q \right) \right],$$

where for any $i \in [n]$, we noted $\tilde{\Sigma}_i \equiv \Sigma_i / \tilde{\Lambda}_i^z$. We further introduce the tensor of shape $n \times \dots \times n$ (l times) $\Phi_l^h \equiv (\Phi_\alpha^h)_{\alpha \in [n]^l}$. Given $h > 0$ and replacing in the last estimation A with $\frac{1}{n} \tilde{\Sigma}_{\alpha_{1-h}} \tilde{Q} \dots \tilde{\Sigma}_{\alpha_{-1}} \tilde{Q} \tilde{\Sigma}_{\alpha_0}$ satisfying:

$$\left\| \frac{1}{n} \tilde{\Sigma}_{\alpha_{1-h}} \tilde{Q} \dots \tilde{\Sigma}_{\alpha_{-1}} \tilde{Q} \tilde{\Sigma}_{\alpha_0} \right\|_F \leq O \left(\frac{1}{\sqrt{n}} \right),$$

we know that for any $\alpha = (\alpha_{-h}, \dots, \alpha_0, \alpha_1, \dots, \alpha_{m-1}) \in [n]^{m+h}$:

$$\Psi_\alpha^h - \Psi_\alpha^{h+1} = \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^{m-1} \sum_{l \in L_k^{m-1}} \Psi_{\alpha_{-h}, i}^{h+1} \Psi_{\alpha_{l_1}, i}^0 \dots \Psi_{\alpha_k}^0 + O \left(\frac{(c\kappa_z)^m}{n} \right),$$

where for any $k, l \in \{h, m-1\}$, $k < l$, $\alpha_k^l = \alpha_k, \dots, \alpha_{l-1}$.

Inspiring from this equation, we are going to introduce with next proposition a set of tensors that will allow us to approximate $\mathbb{E}[Q^{A^m}]$.

Proposition F.47. *There exists a unique sequence of tensors Ψ satisfying:*

$$\left\{ \begin{array}{l} \forall m > 2, h \in \mathbb{N}, \alpha = (\alpha_{-h}, \dots, \alpha_{m-1}) \in [n]^{m+h} : \\ \tilde{\Psi}_\alpha^h - \tilde{\Psi}_\alpha^{h+1} = \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^{m-1} \sum_{l \in L_k^{m-1}} \tilde{\Psi}_{\alpha_{-h}, i}^{h+1} \tilde{\Psi}_{\alpha_{l_1}, i}^0 \cdots \tilde{\Psi}_{\alpha_{l_k}, i}^0, \\ \forall l \in [n], \alpha \in [n]^l : \tilde{\Psi}_\alpha^{l-1} = \tilde{\Psi}_\alpha^l = \frac{1}{n} \text{Tr} \left(\tilde{\Sigma}_{\alpha_1} \tilde{Q} \cdots \tilde{\Sigma}_{\alpha_l} \tilde{Q} \right). \end{array} \right.$$

Proof. We already know that for any $i \in [n]$:

$$\tilde{\Psi}_{(i)}^0 = \tilde{\Psi}_{(i)}^1 = \frac{1}{n} \text{Tr}(\tilde{\Sigma}_i \tilde{Q})$$

Let us now assume that there exists $m > 1$ such that for any $k \in [m-1]$, and any $h \in \{0, \dots, k\}$, and any $\alpha \in [n]^k$, $\tilde{\Psi}_\alpha^h$ is well defined and we know how to compute it. Besides, for any $\alpha \in [m]^n$, we also know how to compute $\tilde{\Psi}_\alpha^{m-1} = \tilde{\Psi}_\alpha^m$. We then further assume that there exist $h \in [m]$ such that we know how to compute any $\tilde{\Psi}_\alpha^k$ for $k < h$. If $h \geq 1$, then we see from the iteration formula that $\tilde{\Psi}_\alpha^h$ expresses as a sum of computable elements, the only issue raises when $h = 0$, then $\tilde{\Psi}_\alpha^0$ are appearing on both side of the equality. To invoke the invertibility of $\tilde{\Psi}_2^2$, let us first consider:

$$\tilde{\Psi}_\alpha^0 = \tilde{\Psi}_\alpha^1 + \frac{1}{n} \sum_{i=1}^n \sum_{l=1}^{m-1} \tilde{\Psi}_{\alpha_0, i}^1 \tilde{\Psi}_{\alpha_l, i}^0 + \frac{1}{n} \sum_{i=1}^n \sum_{k=2}^{m-1} \sum_{l \in L_k^{m-1}} \tilde{\Psi}_{\alpha_{-h}, i}^1 \cdots \tilde{\Psi}_{\alpha_{l_k}, i}^0$$

The right hand term that raise concern is $\sum_{i=1}^n \tilde{\Psi}_{\alpha_0, i}^1 \tilde{\Psi}_{\alpha_l, i}^0$. Since $\tilde{\Psi}_{\alpha_1, i}^0 = \tilde{\Psi}_{i, \alpha_1}^0$ and $\tilde{\Psi}_2^1 = \tilde{\Psi}_2^2$, we can rewrite it $(\tilde{\Psi}_2^2 \tilde{\Psi}_m^0)_\alpha$ (where the right product of a matrix and a tensor is a classical matricial product on the first variable of the tensor). We can bound for any $\alpha \in [n]^2$:

$$\left| \tilde{\Psi}_\alpha^2 \right| \leq \frac{\|\Sigma_{\alpha_1}\| \|\Sigma_{\alpha_2}\|}{|z|^2},$$

therefore, for $|z|$ sufficiently big, $(I_n - \tilde{\Psi}_2^2)$ is invertible and one has the identity:

$$\tilde{\Psi}_m^0 = (I_n - \tilde{\Psi}_2^2)^{-1} \cdot \left(\tilde{\Psi}_\alpha^1 + \frac{1}{n} \sum_{i=1}^n \sum_{l=2}^{m-1} \tilde{\Psi}_{\alpha_0, i}^1 \tilde{\Psi}_{\alpha_l, i}^0 \right) \quad (\text{F.24})$$

$$+ \frac{1}{n} \sum_{i=1}^n \sum_{k=2}^{m-1} \sum_{l \in L_k^{m-1}} \tilde{\Psi}_{\alpha_0, i}^1 \tilde{\Psi}_{\alpha_{l_1}, i}^0 \cdots \tilde{\Psi}_{\alpha_{l_k}, i}^0 \Big)_{(\alpha_0, \dots, \alpha_{m-1}) \in [n]^m}. \quad (\text{F.25})$$

Complex analysis inferences should then allow us to set the uniqueness for all the values of z . \square

Once $\tilde{\Psi}_m^h$ is defined for all $m \in \mathbb{N}$, $h \in [m]$, one can define iteratively for any sequence of deterministic matrices A_{-h}, \dots, A_{m-1} followingly:

$$\left\{ \begin{array}{l} \forall m > 2, h \in \mathbb{N}, \forall A_{-h}, \dots, A_{m-1} \in \mathcal{M}_p^{m+h} : \\ \tilde{\Psi}_{A_{-h}, \dots, A_{m-1}}^h - \tilde{\Psi}_{A_{-h}, \dots, A_{m-1}}^{h+1} \\ = \frac{1}{n} \sum_{i=1}^n \sum_{k=1}^{m-1} \sum_{l \in L_k^{m-1}} \tilde{\Psi}_{A_{-h}, \dots, A_{l_1-1}, \Sigma_i}^{h+1} \tilde{\Psi}_{A_{l_1}, \dots, A_{l_2-1}, \Sigma_i}^0 \cdots \tilde{\Psi}_{A_{l_k}, \dots, A_{m-1}, \Sigma_i}^0, \\ \forall l \in [n], A_1, \dots, A_l \in \mathcal{M}_p^l : \tilde{\Psi}_{A_{-h}, \dots, A_{m-1}}^{l-1} = \tilde{\Psi}_{A_{-h}, \dots, A_{m-1}}^l = \frac{1}{n} \text{Tr} \left(A_1 \tilde{Q} \cdots A_{l-1} \tilde{Q} A_l \tilde{Q} \right). \end{array} \right.$$

One should need a proper proof to justify it, but we will just explain that the existence and uniqueness is proven iteratively considering sequences of matrices $(A_1, \dots, A_i, \tilde{\Sigma}_{\alpha_{i+1}}, \dots, \tilde{\Sigma}_{\alpha_m})$ with $(\alpha_{i+1}, \dots, \alpha_m) \in [n]^{m-i}$ and i going from 1 to m .

We can finally properly estimate $Q^{A_1^m}$. We express this result in the case m constant, but one could probably get a similar result for an asymptotic m . The proof is just a laborious use of the explicit formulation (F.24).

Proposition F.48. *Given a constant integer $m \in \mathbb{N}$ and m matrices A_1, \dots, A_m satisfying $\|A_i\|_F \leq 1$, we can estimate:*

$$\mathrm{Tr}(A_m Q^{A_1^m}) \mid \mathcal{A}_Q \in \tilde{\Psi}_{A_1, \dots, A_m}^0 \pm \mathcal{E}_2 \left(\frac{\kappa_z}{\sqrt{n}} \right).$$

Chapter G

Concentration of the resolvent with random diagonal term

We study here the concentration of a resolvent $Q = (I_p - \frac{1}{n}XDXT)^{-1}$ with Assumptions 1-5 for X and D (in particular D is random). Among other use, this object appears when studying robust regression [EKBB⁺13, MLC19a]. In several settings, robust regression can be expressed by the following fixed point equation:

$$\beta = \frac{1}{n} \sum_{i=1}^n f(x_i^T \beta) x_i, \quad \beta \in \mathbb{R}^p, \quad (\text{G.1})$$

where β is the weight vector performing the regression (to classify data, for instance). It was then shown in [SLTC21] that the estimation of the expectation and covariance of β (and therefore, of the performances of the algorithm) rely on an estimation of Q , with $D = \text{Diag}(f'(x_i^T \beta))$. To obtain a sharp concentration on Q (as it is done in Theorem G.1 below), one has to understand the dependence between Q and x_i , for all $i \in [n]$. This is performed with the notation, given for any $M = (m_1, \dots, m_n) \in \mathcal{M}_{p,n}$ or any $\Delta = \text{Diag}_{i \in [n]}(\Delta_i) \in \mathcal{D}_n$:

- $M_{-i} = (m_1, \dots, m_{i-1}, 0, m_{i+1}, \dots, m_n) \in \mathcal{M}_{p,n}$,
- $\Delta_{-i} = \text{Diag}(\Delta_1, \dots, \Delta_{i-1}, 0, \Delta_{i+1}, \dots, \Delta_n) \in \mathcal{D}_n$.

Theorem G.1. *Given a positive random diagonal matrix $D \in \mathcal{D}_n^+$ and a random matrix $X = (x_1, \dots, x_n)$, in the regime¹ $p \leq O(n)$ and under the assumptions:*

- $(X, D) \propto \mathcal{E}_2$,
- all the columns x_1, \dots, x_n are independent,
- $O(1) \leq \sup_{i \in [n]} \|\mathbb{E}[x_i]\| \leq O(1)$,
- for all $i \in [n]$, there exists a random positive diagonal matrix $D^{(i)} \in \mathcal{D}_n^+$, independent of x_i , such that $\sup_{i \in [n]} \|D_{-i} - D_{-i}^{(i)}\|_F \leq O(1)$,
- there exist² three constants $\kappa, \kappa_D, \varepsilon > 0$ ($\varepsilon \geq O(1)$ and $\kappa, \kappa_D \leq O(1)$), such that $\|X\| \leq \sqrt{n}\kappa$, $\|D^{(i)}\|, \|D\| \leq \kappa_D$ and $\kappa^2 \kappa_D \leq 1 - \varepsilon$,

¹It is not necessary to assume that $p \leq O(n)$ but it simplifies the concentration result (if $p \gg n$, the concentration is not as good, but it can still be expressed).

²The assumptions $\|X\|/\sqrt{n}$ bounded and $\kappa^2 \kappa_D \leq 1 - \varepsilon$ might look a bit strong (since it is not true for matrices with i.i.d. Gaussian entries), it is indeed enough to assume that $\mathbb{E}[\|X\|] \leq O(\sqrt{n})$ and introduce a parameter $z > 0$ to study the behavior of $(zI_p - \frac{1}{n}XDXT)^{-1}$ when z is far from the spectrum of $\frac{1}{n}XDXT$ – as it is done in Section F.1. We however preferred here to make a relatively strong hypothesis not to have supplementary notations and proof precautions, that might have blurred the message.

the resolvent $Q \equiv (I_p - \frac{1}{n}XDXT)^{-1}$ follows the concentration

$$Q \in \mathbb{E}[Q] \pm \mathcal{E}_{3/2} \left(\sqrt{\frac{\log n}{n}} \right) \quad \text{in } (\mathcal{M}_p, \|\cdot\|_F).$$

Inspiring from the formulation of the deterministic equivalent introduced in Chapter F, we introduce the following notation that will help us to express the deterministic equivalent of Q . Given $\delta, D \in \mathcal{M}_n$, we note:

$$\tilde{Q}^\delta(D) \equiv \left(I_p - \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\frac{D_i}{1 + \delta D_i} \right] \Sigma_i \right)^{-1},$$

(where we recall that $\forall i \in [n]$, $\Sigma_i \equiv \mathbb{E}[x_i x_i^T]$).

Theorem G.2. *For any diagonal matrix $D' \in \mathcal{D}_n$, the fixed point equation:*

$$\delta = \text{Diag}_{i \in [n]} \left(\Sigma_i \tilde{Q}^\delta(D') \right)$$

admits a unique solution $\delta(D') \in \mathcal{D}_n$ and, under the hypotheses of Theorem G.1, one can estimate:

$$\|\mathbb{E}[Q] - \tilde{Q}^{\delta(D)}(D)\|_F \leq O(\sqrt{\log n}).$$

Remark G.3. *Note here that although there are no term $1/\sqrt{n}$ in the bound, this result still provides some good convergence bound for the estimation of linear forms like the Stieltjes transform or more generally any $\frac{1}{n} \text{Tr}(AQ)$ when $\|A\| \leq 1$ (because then $\frac{1}{n} \|A\|_F \leq O(1/\sqrt{n})$).*

Remark G.4. *Unlike in the result of Chapter F (for instance Theorem F.25), there is no resort to the high probability event \mathcal{A}_Q here. It is because, as we will see later in Lemma G.6, in the current setting all the drawings of the resolvent Q are bounded thanks to the hypothesis $\kappa^2 \kappa_D \leq 1 - \varepsilon$. In the sens, the parameter z of Chapter F is here always far from the spectrum of $\frac{1}{n}XX^T$.*

Remark G.5. *Let us give two examples of the matrices $D^{(i)}$ that one could encounter in practice:*

- *For all $i \in [n]$, $D_i = f(x_i)$ for $f : \mathbb{R}^p \rightarrow \mathbb{R}$, bounded, then, D_i just depends on x_i so one can merely take $D^{(i)} = D_{-i}$ for all $i \in [n]$.*
- *For the robust regression described by Equation G.1, as in [SLTC21], we can assume for simplicity³ $\|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty \leq O(1)$. If we choose $D = \text{Diag}(f'(x_i^T \beta))$, then it is convenient to assume $\frac{1}{n} \|f'\|_\infty \|X\|^2 \leq 1 - \varepsilon$ (which implies in particular $\frac{1}{n} \|X\|^2 \|D\| \leq 1 - \varepsilon$) so that β is well defined, being solution of a contractive fixed point equation. One can further introduce $\beta^{(i)} \in \mathbb{R}^p$, the unique solution to*

$$\beta^{(i)} = \frac{1}{n} \sum_{\substack{1 \leq j \leq n \\ j \neq i}} f(x_j^T \beta^{(i)}) x_j.$$

By construction, $\beta^{(i)}$ is independent of x_i and so is:

$$D^{(i)} \equiv \text{Diag} \left(f'(x_1^T \beta^{(i)}), \dots, f'(x_{i-1}^T \beta^{(i)}), 0, f'(x_{i+1}^T \beta^{(i)}), \dots, f'(x_n^T \beta^{(i)}) \right).$$

Besides $\|D_{-i} - D_{-i}^{(i)}\|_F \leq \|f''\|_\infty \|X_{-i}^T (\beta - \beta^{(i)})\|_F$. Now, the identities:

$$X_{-i}^T \beta = \frac{1}{n} X_{-i}^T X f(X^T \beta) \quad \text{and} \quad X_{-i}^T \beta^{(i)} = \frac{1}{n} X_{-i}^T X_{-i} f(X_{-i}^T \beta^{(i)})$$

(where f is applied entry-wise) imply:

$$\|X_{-i}^T (\beta - \beta^{(i)})\|_F \leq \frac{1}{n} \|f''\|_\infty \|X_{-i}\|^2 \|X_{-i}^T (\beta - \beta^{(i)})\|_F + \frac{1}{n} f(x_i^T \beta) X_{-i}^T x_i.$$

We can then deduce (since $\frac{1}{n} \|f''\|_\infty \|X_{-i}\|^2 \leq 1 - \varepsilon$ by hypothesis):

$$\|D_{-i} - D_{-i}^{(i)}\|_F \leq \|f''\|_\infty \|X_{-i}^T (\beta - \beta^{(i)})\|_F \leq \frac{\|f''\|_\infty}{n\varepsilon} f(x_i^T \beta) X_{-i}^T x_i \leq O(1).$$

³The bound $\|f\|_\infty \leq O(1)$ is not necessary to set the concentration of Q , but it avoids a lot of complications.

G.1 Concentration of Q

Lemma G.6. *Under the assumptions of Theorem G.1, $\|Q\| \leq \frac{1}{\varepsilon} \leq O(1)$.*

Then we can show a Lipschitz concentration of Q but with looser observable diameter than the one given by Theorem G.1 (as for $XDXT$, we get better concentration speed in the linear concentration framework).

Lemma G.7. *Under the hypotheses of Theorem G.1:*

$$\left(Q, \frac{1}{\sqrt{n}}QX\right) \propto \mathcal{E}_2 \quad \text{in } (\mathcal{M}_{p,n}, \|\cdot\|_F).$$

Proof. Let us just show the concentration of the resolvent, the tuple is treated the same way. If we note $\phi(X, D) = Q$ and we introduce $X' \in \mathcal{M}_{p,n}$ and $D' \in \mathcal{D}_n$, satisfying $\|X'\| \leq \kappa\sqrt{n}$ and $\|D'\| \leq \kappa_D$ as X, D , we can bound:

$$\begin{aligned} & \|\phi(X, D) - \phi(X', D)\|_F \\ &= \frac{1}{n} \|\phi(X, D)(X - X')DX^T\phi(X', D)\|_F + \frac{1}{n} \|\phi(X, D)X'D(X - X')^T\phi(X', D)\|_F \\ &\leq \frac{2\kappa\kappa_D}{\varepsilon^2\sqrt{n}} \|X - X'\|_F, \end{aligned}$$

thanks to the hypotheses and Lemma G.6 given above. The same way, we can bound:

$$\|\phi(X, D) - \phi(X, D')\|_F \leq \frac{\kappa^2}{\varepsilon^2} \|D - D'\|_F$$

Therefore, as a $O(1)$ -Lipschitz transformation of (X, D) , $Q \propto \mathcal{E}_2$. □

G.2 Control on the dependency on x_i

The dependence between Q and x_i prevent us from bounding straightforwardly $\|Qx_i\|$ with Lemma G.6 and the hypotheses on x_i . We can still disentangle this dependence thanks to the notations:

$$Q_{-i} = \left(I_p - \frac{1}{n}X_{-i}^TDX_{-i}^T\right)^{-1} \quad \text{and} \quad Q_{-i}^{(i)} = \left(I_p - \frac{1}{n}X_{-i}^TD^{(i)}X_{-i}^T\right)^{-1}.$$

We can indeed bound:

$$\|\mathbb{E}[Q_{-i}^{(i)}x_i]\| \leq \|\mathbb{E}[Q_{-i}^{(i)}]\mathbb{E}[x_i]\| \leq O(1), \quad (\text{G.2})$$

and we even have interesting concentration properties that will be important later:

Lemma G.8. *Under the assumptions of Theorem G.1:*

$$Q_{-i}^{(i)}x_i, \frac{1}{\sqrt{n}}X_{-i}^TQ_{-i}^{(i)}x_i \in O(1) \pm \mathcal{E}_2.$$

Proof. Considering $u \in \mathbb{R}^p$, deterministic such that $\|u\| \leq 1$, we can bound thanks to the independence between $Q_{-i}^{(i)}$ and x_i :

$$\left|u^TQ_{-i}^{(i)}x_i - \mathbb{E}[u^TQ_{-i}^{(i)}x_i]\right| \leq \left|u^TQ_{-i}^{(i)}(x_i - \mathbb{E}[x_i])\right| + \left|u^T(Q_{-i}^{(i)} - \mathbb{E}[Q_{-i}^{(i)}])\mathbb{E}[x_i]\right|.$$

Therefore, the concentrations $x_i \propto \mathcal{E}_2$ and $Q_{-i}^{(i)} \propto \mathcal{E}_2$ given in Lemma G.7 imply that there exist two constants $C, c > 0$ such that $\forall t > 0$ such that if we note \mathcal{A}_{-i} , the sigma algebra generated by X_{-i} (it is independent with x_i):

$$\begin{aligned} & \mathbb{P}\left(\left|u^TQ_{-i}^{(i)}x_i - \mathbb{E}[u^TQ_{-i}^{(i)}x_i]\right| \geq t\right) \\ &\leq \mathbb{E}\left[\mathbb{P}\left(\left|u^TQ_{-i}^{(i)}(x_i - \mathbb{E}[x_i])\right| \geq \frac{t}{2} \mid \mathcal{A}_{-i}\right)\right] + \mathbb{P}\left(\left|u^T(Q_{-i}^{(i)} - \mathbb{E}[Q_{-i}^{(i)}])\mathbb{E}[x_i]\right| \geq \frac{t}{2}\right) \\ &\leq \mathbb{E}\left[Ce^{(t/c\|Q_{-i}^{(i)}\|)^2}\right] + Ce^{(t/c\|\mathbb{E}[x_i]\|)^2} \leq C'e^{-t^2/c'}, \end{aligned}$$

for some constants $C', c' > 0$, thanks to the bounds $\|\mathbb{E}[x_i]\| \leq O(1)$ given in the assumptions and $\|Q_{-i}^{(i)}\| \leq O(1)$ given by Lemma G.6.

The linear concentration of $X_{-i}^T Q_{-i}^{(i)} x_i / \sqrt{n}$ is proven the same way since one can show as in Lemma G.7 that $(X, D) \mapsto X_{-i}^T Q_{-i}^{(i)} / \sqrt{n}$ is $O(1)$ -Lipschitz on $\{\|X\| \leq \kappa\sqrt{n}, \|D\| \leq \kappa_D\}$, and therefore, $X_{-i}^T Q_{-i}^{(i)} / \sqrt{n} \propto \mathcal{E}_2$. \square

Let us adapt the Schur identity to the presence of the diagonal matrix D_i :

$$Q = Q_{-i} - \frac{1}{n} \frac{D_i Q_{-i} x_i x_i^T Q_{-i}}{1 + D_i \Delta_i} \quad \text{and} \quad Q x_i = \frac{Q_{-i} x_i}{1 + D_i \Delta_i},$$

where we noted $\Delta_i \equiv \frac{1}{n} x_i^T Q_{-i} x_i$. The link between $Q_{-i} x_i$ and $Q_{-i}^{(i)} x_i$ is made thanks to:

Lemma G.9. *Under the hypotheses of Theorem G.1, for all $i \in [n]$:*

$$\|Q_{-i} x_i - Q_{-i}^{(i)} x_i\| \in O(\sqrt{\log n}) \pm \mathcal{E}_2(\sqrt{\log n}).$$

Let us first prove a Lemma of independent interest:

Lemma G.10. *Under the hypotheses of Theorem G.1, for all $i \in [n]$*

$$\left\| \frac{1}{\sqrt{n}} X_{-i}^T Q_{-i}^{(i)} x_i \right\|_{\infty} \in O(\sqrt{\log n}) \pm \mathcal{E}_2(\sqrt{\log n}).$$

Proof. The control on the variation is given by Lemma G.8 ($\|\cdot\|_{\infty} \leq \|\cdot\|_F$) and the bound on the expectation is a consequence of Proposition B.22 and the bound:

$$\frac{1}{\sqrt{n}} \left\| \mathbb{E} \left[X_{-i}^T Q_{-i}^{(i)} x_i \right] \right\|_{\infty} \leq \frac{1}{\sqrt{n}} \left\| \mathbb{E} \left[X_{-i}^T Q_{-i}^{(i)} \right] \mathbb{E} [x_i] \right\| \leq O(1). \quad (\text{G.3})$$

\square

Proof of Lemma G.9. Let us bound directly:

$$\begin{aligned} \left\| (Q_{-i} - Q_{-i}^{(i)}) x_i \right\| &\leq \left\| \frac{1}{n} Q_{-i} X_{-i} (D_{-i}^{(i)} - D_{-i}) X_{-i}^T Q_{-i}^{(i)} x_i \right\| \\ &\leq \frac{1}{n} \|Q_{-i} X_{-i}\| \|D_{-i}^{(i)} - D_{-i}\|_F \|X_{-i}^T Q_{-i}^{(i)} x_i\|_{\infty} \leq O\left(\frac{1}{\sqrt{n}} \|X_{-i}^T Q_{-i}^{(i)} x_i\|_{\infty}\right). \end{aligned}$$

We can then conclude thanks to Lemma G.10. \square

G.3 Proof of the concentration

Let us first provide a preliminary result that will allow us to set that $x_i^T Q A Q x_i$ behaves more or less like a $O(\sqrt{\log n})$ -Lipschitz observation of (X, D, Y) .

Lemma G.11. *Under the hypotheses of Theorem G.1, $\forall i \in [n]$, and for any deterministic matrices $U, V \in \mathcal{M}_p$ such that $\|U\|, \|V\| \leq 1$:*

$$\|V Q X\|_{\infty} \in O\left(\sqrt{\log n}\right) \pm \mathcal{E}_2\left(\sqrt{\log n}\right).$$

Be careful that the bound would not have been so tight for $\|Q X U\|_{\infty}$ given $U, V \in \mathcal{M}_n$. We just need a small lemma to be able to bound $1 + \Delta_i D_i$ from below, it is basically a rewriting of Lemma F.7 bounding Λ in Chapter F.

Lemma G.12. *Under the hypotheses of Theorem G.1, $\forall i \in [n]$:*

$$|\Delta_i| \leq \frac{\kappa^2}{\varepsilon} \quad \text{and} \quad \varepsilon \leq 1 + D_i \Delta_i \leq 1 + \frac{\kappa \kappa_D}{\varepsilon}$$

Proof. Let us simply bound for any $i \in [n]$: $\|\Delta_i\| = \frac{1}{n}|x_i^T Q x_i| \leq \frac{\kappa^2}{\varepsilon}$ thanks to Lemma G.6. We can then directly deduce the upper bound of $|1 + D_i \Delta_i|$.

For the lower bound, let us introduce again the matrix $\tilde{Q} = (I_n - D^{1/2} X^T X D^{1/2})^{-1}$, we can bound as in Lemma G.6 $\|\tilde{Q}\| \leq \frac{1}{\varepsilon}$, and we can show again that:

$$1 + D_i \Delta_i = 1 + \frac{D_i}{n} x_i^T Q x_i = \frac{1}{Q_i},$$

which allows us to bound $|1 + D_i \Delta_i| \geq \frac{1}{\|\tilde{Q}\|} \geq \varepsilon$. \square

Proof. Following the same identities and arguments presented in the proof of Lemma G.9, we can bound thanks to Lemma G.12

$$\begin{aligned} \|VQX\|_\infty &= \sup_{i \in [n]} \left\| \frac{VQ_{-i}^{(i)} x_i + \frac{1}{n} V(Q_{-i} - Q_{-i}^{(i)}) x_i}{1 + D_i \Delta_i} \right\|_\infty \\ &\leq O \left(\sup_{i \in [n]} \left(\|VQ_{-i}^{(i)} x_i\|_\infty, \frac{1}{\sqrt{n}} \|X_{-i}^T Q_{-i}^{(i)} x_i\|_\infty \right) \right). \end{aligned}$$

Introducing, as in Section B.4, (e_1, \dots, e_p) and (f_1, \dots, f_n) , respectively, the canonical basis of \mathbb{R}^p and \mathbb{R}^n we know from Lemma G.8 that for all $k \in [p]$ and $i, j \in [n]$:

$$e_k^T V Q_{-i}^{(i)} x_i \in O(1) \pm \mathcal{E}_2 \quad \text{and} \quad \frac{1}{\sqrt{n}} f_j^T X_{-i}^T Q_{-i}^{(i)} x_i \in O(1) \pm \mathcal{E}_2,$$

since $|\mathbb{E}[e_k^T V Q_{-i}^{(i)} x_i]| \leq \|\mathbb{E}[Q_{-i}^{(i)}]\| \|\mathbb{E}[x_i]\| \leq O(1)$ and similarly, $|\mathbb{E}[f_j^T X_{-i}^T Q_{-i}^{(i)} x_i]/\sqrt{n}| \leq O(1)$. Following the arguments displayed in Section B.4, there exist four constants K, C, c, c' (all $\leq O(1)$) such that we can bound:

$$\begin{aligned} \mathbb{P}(\|VQX\|_\infty \geq t) &\leq \mathbb{P} \left(\sup_{\substack{i, j \in [n] \\ k \in [p]}} e_k^T V Q_{-i}^{(i)} x_i + \frac{1}{\sqrt{n}} f_j^T X_{-i}^T Q_{-i}^{(i)} x_i \geq \frac{t}{K} \right) \\ &\leq \max \left(1, n^2 p C e^{-t^2/c} \right) \leq \max(e, C) e^{-K^2 t^2 / c' \log(n^2 p)}. \end{aligned}$$

We can then deduce the concentration of $\|VQX\|_\infty$ since $\log(n^2 p) \leq O(\log(n))$. \square

Let us now prove three results of progressive difficulty. Since $Q \propto \mathcal{E}_2$ and $x_i \propto \mathcal{E}_2$, one can follow the lines of Lemma F.40 to set:

Lemma G.13. *Under the hypotheses of Theorem G.1:*

$$\Delta_i \equiv \frac{1}{n} x_i^T Q_{-i} x_i \in \bar{\Delta}_i \pm \mathcal{E}_2 \left(\sqrt{\frac{1}{n}} \right) + \mathcal{E}_1 \left(\sqrt{\frac{1}{n}} \right),$$

where we noted $\bar{\Delta}_i \equiv \mathbb{E}[\frac{1}{n} x_i^T Q_{-i} x_i]$ (recall that $\forall i \in [n]: |\bar{\Delta}_i| \leq \frac{\kappa^2}{\varepsilon}$).

We let the reader refer to the proof of Lemma F.40 or more simply to next result to get justifications for this lemma.

Lemma G.14. *Under the hypotheses of Theorem G.1, given a deterministic matrix $A \in \mathcal{M}_{p,n}$:*

$$x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i \in O(\sqrt{n} \|A\|_F) \pm \mathcal{E}_2(\|A\|_F) + \mathcal{E}_1(\|A\|).$$

Proof. Let us bound:

$$\begin{aligned} &\left| x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i - \mathbb{E} \left[x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i \right] \right| \\ &\leq \left| x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i - \text{Tr} \left(\Sigma_i Q_{-i}^{(i)} A Q_{-i}^{(i)} \right) \right| + \left| \text{Tr} \left(\Sigma_i Q_{-i}^{(i)} A Q_{-i}^{(i)} \right) - \text{Tr} \left(\Sigma_i \mathbb{E} \left[Q_{-i}^{(i)} A Q_{-i}^{(i)} \right] \right) \right| \end{aligned}$$

One can then deduce the result from Lemma B.19 (setting that if $Z \in O(\sigma \pm \mathcal{E}_q(\sigma))$ and $0 \leq Y \leq Z$, then $Y \in O(\sigma) \pm \mathcal{E}_q(\sigma)$) applied to the concentrations:

- $x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i \mid \mathcal{A}_{-i} \in \text{Tr} \left(\Sigma_i Q_{-i}^{(i)} A Q_{-i}^{(i)} \right) \pm \mathcal{E}_2(\|A\|_F/\varepsilon^2) + \mathcal{E}_1(\|A\|/\varepsilon^2)$ thanks to Hanson-Wright inequality (Proposition B.41),
- $\left| \text{Tr} \left(\Sigma_i Q_{-i}^{(i)} A Q_{-i}^{(i)} \right) \right|$ is a $(\frac{1}{\varepsilon} \|\Sigma_i\| \|A\|_F)$ -Lipschitz transformation of $Q_{-i}^{(i)}$, and therefore, one can deduce from Lemma G.7 that $\text{Tr} \left(\Sigma_i Q_{-i}^{(i)} A Q_{-i}^{(i)} \right) \in \text{Tr} \left(\Sigma_i \mathbb{E} \left[Q_{-i}^{(i)} A Q_{-i}^{(i)} \right] \right) \pm \mathcal{E}_2(\|A\|_F)$.

Besides, one can bound:

$$\left| \mathbb{E} \left[x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i \right] \right| = \left| \text{Tr} \left(\Sigma_i \mathbb{E} \left[Q_{-i}^{(i)} A Q_{-i}^{(i)} \right] \right) \right| \leq O(\sqrt{p} \|A\|_F).$$

□

Lemma G.15. *Under the hypotheses of Theorem G.1, given a deterministic matrix $A \in \mathcal{M}_{p,n}$ such that $\|A\|_F \leq 1$:*

$$\frac{1}{n} \|X^T Q A Q X\|_d \in O \left(\frac{1}{\sqrt{n}} \right) \pm \mathcal{E}_1 \left(\sqrt{\frac{\log n}{n}} \right).$$

This Lemma in particular gives us the concentration of any diagonal term of the random matrix $\frac{1}{n} X^T Q A Q X$, i.e. of any $\frac{1}{n} x_i^T Q A Q x_i$, $i \in [n]$.

Proof. To prove the concentration, let us introduce again the decomposition $A = U^T \Lambda V$, with $U, V \in \mathcal{O}_p$ and $\Lambda \in \mathcal{D}_p$. We are going to bound the variation of $\frac{1}{n} \|X^T Q A Q X\|_d$ towards the variations of $\frac{1}{\sqrt{n}} V Q X \in \mathcal{E}_2$ (see Lemma G.7). Let us define the mapping $\phi : \mathcal{M}_{p,n}^2 \rightarrow \mathbb{R}$ satisfying for all $M, P \in \mathcal{M}_{p,n}$, $\phi(M, P) = \|M^T \Lambda P\|_d$ (with that definition, $\frac{1}{n} \|X^T Q A Q X\|_d = \phi(\frac{1}{\sqrt{n}} V Q X, \frac{1}{\sqrt{n}} U Q X)$). Given 4 variables M, P, M', P' satisfying $\|M\|, \|P\|, \|M'\|, \|P'\| \leq \frac{\kappa}{\varepsilon}$ we can bound as in the proof of Corollary B.43:

$$|\phi(M, P) - \phi(M', P)| \leq \|(M - M')^T \Lambda P\|_d \leq \|M - M'\|_F \|P\|_\infty \|\Lambda\|_F \leq \|M - M'\|_F \|P\|_\infty,$$

and the same way, $|\phi(M, P) - \phi(M, P')| \leq \|P - P'\|_F \|M\|_\infty$. We further invoke Lemma G.11 that provides the concentration:

$$\left(\frac{1}{\sqrt{n}} \|V Q X\|_\infty, \frac{1}{\sqrt{n}} \|U Q X\|_\infty \right) \in O \left(\sqrt{\frac{\log n}{n}} \right) \pm \mathcal{E}_2 \left(\sqrt{\frac{\log n}{n}} \right). \quad (\text{G.4})$$

We can then deduce from Theorem B.46 the concentration $\sqrt{\frac{n}{\log n}} \phi(\frac{1}{\sqrt{n}} V Q X, \frac{1}{\sqrt{n}} U Q X) \in \mathcal{E}_2 + \mathcal{E}_1 \in \mathcal{E}_1$, from which we deduce the concentration of $\|X^T Q A Q X\|_d$.

For the estimation, let us first express:

$$x_i^T Q A Q x_i = x_i^T Q_{-i} A Q_{-i} x_i (1 + D_i \Delta_i)$$

Thanks to Lemmas G.16, G.14 and the assumptions on the concentration of D_i , one can bound:

$$\left\| x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i (1 + D_i \Delta_i) \right\| \leq O(\sqrt{n}).$$

To be able to replace $Q_{-i}^{(i)}$ with Q_{-i} in the previous inequality, one can bound:

$$\begin{aligned} & \left| x_i^T Q A Q x_i - (1 + (1 + D_i \Delta_i)) \left(x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i \right) \right| \\ & \leq \left(1 + \frac{\kappa_D \kappa^2}{\varepsilon} \right) \left(\left| x_i^T (Q_{-i} - Q_{-i}^{(i)}) A (Q_{-i} - Q_{-i}^{(i)}) x_i \right| + \left| x_i^T (Q_{-i} - Q_{-i}^{(i)}) A Q_{-i}^{(i)} x_i \right| \right. \\ & \quad \left. + \left| x_i^T Q_{-i}^{(i)} A (Q_{-i} - Q_{-i}^{(i)}) x_i \right| \right) \end{aligned}$$

We can then invoke the concentrations:

- $\|x_i^T Q_{-i}^{(i)} A\|, \|A Q_{-i}^{(i)} x_i\| \in O(1) \pm \mathcal{E}_2$ thanks to Lemmas G.8 and B.26.
- $\|(Q_{-i} - Q_{-i}^{(i)}) x_i\|, \|x_i (Q_{-i} - Q_{-i}^{(i)})\| \in O(\sqrt{\log n}) \pm \mathcal{E}_2(\sqrt{\log n})$ thanks to Lemma G.9
- $\left| x_i^T Q_{-i}^{(i)} A Q_{-i}^{(i)} x_i \right| \in O(\sqrt{n}) \pm \mathcal{E}_1$ thanks to Lemma G.14.

to be able to finally bound $\mathbb{E}[\frac{1}{n} \|X^T Q A Q X\|_d] \leq O(\sqrt{n})$ thanks to (G.4) (with $U = V = I_p$).

□

Proof of Theorem G.1. Let us consider $A \in \mathcal{M}_{p,n}$ such that $\|A\|_F \leq 1$ and let us note $\phi(X, D) = \text{Tr}(AQ)$. We abusively work with X, D and independent copies X', D' satisfying $\|X\|, \|X'\| \leq \sqrt{n}\kappa$ and $\|D\|, \|D'\| \leq \kappa_D$ as if they were deterministic variables, and we note $Q'_X \equiv \phi(X', D)$, $Q'_D \equiv \phi(X, D')$. Let us bound the variations

$$|\phi(X, D) - \phi(X', D)| = \frac{1}{n} |\text{Tr}(AQ(X - X')DXQ'_X)| \leq \frac{\kappa\kappa_D}{\varepsilon^2\sqrt{n}} \|X - X'\|_F.$$

We can also bound as in the proof of Proposition B.48:

$$|\phi(X, D) - \phi(X, D')| \leq \frac{1}{n} \|XQ'_D A Q X\|_d \|D - D'\|_F.$$

The concentration $1/\sqrt{n\log n} \|XQ'_D A Q X\|_d \in O(1/\sqrt{\log n}) \pm \mathcal{E}_1$ is provided by Lemma G.15 (actually Lemma G.15 gives the concentration of $\|XQ A Q X\|_d$, but the proof remains the same if one replaces one of the Q with Q'_D , for a diagonal matrix D' , independent with D). To simplify the use of Theorem B.46, let us note $\tilde{\phi} \equiv \sqrt{\frac{n}{\log n}} \phi$. Then we have:

- $\left| \tilde{\phi}(X, D) - \tilde{\phi}(X', D) \right| \leq \Psi_1(X, X', D) \|X - X'\|_F$ with $\Psi_1(X, D, D') \propto \mathcal{E}_2 + \mathcal{E}_1$,
- $\left| \tilde{\phi}(X, D) - \tilde{\phi}(X, D') \right| \leq \Psi_2(X, D, D') \|D - D'\|_F$ with $\Psi_2(X, D, D') \propto \mathcal{E}_2 + \mathcal{E}_1$,

and we can add an other inequality with an imaginary variable to be able to apply Theorem B.46 with $m = 3$, $\sigma = \mu_1 = \mu_2 = \mu_3 = 1$ which gives us $\tilde{\phi}(X, D) \propto \mathcal{E}_2 + \mathcal{E}_1 + \mathcal{E}_{3/2}$ from which we can deduce the concentration of $\text{Tr}(AQ) = \sqrt{\log n/n} \tilde{\phi}(X, D)$. □

G.4 Proof of the estimation

Lemma G.16. *Under the hypotheses of Theorem G.1, given a deterministic matrix $A \in \mathcal{M}_{p,n}$ such that $\|A\|_F \leq 1$:*

$$x_i^T Q A Q x_i (1 + D_i \bar{\Delta}_i) \in O(\sqrt{n}) \pm \mathcal{E}_{2/3} \left(\sqrt{\log n} \right).$$

Proof. Let us first show the concentration of:

$$\phi(X, D) \equiv x_i^T Q A Q x_i (1 + D_i \Delta_i) = x_i^T Q A Q_{-i} x_i = x_i^T Q_{-i} A Q x_i.$$

Given $X, X' \in X(\Omega)$, we note $Q' \equiv (I_p - \frac{1}{n} X' D X'^T)^{-1}$ and $\Delta' \equiv \frac{1}{n} x_i^T Q' x_i$. One can bound:

$$\begin{aligned} |\phi(X, D) - \phi(X', D)| &\leq \left| x_i^T (Q - Q') A Q_{-i} x_i \right| + \left| x_i^T Q' A Q x_i D_i (\Delta_i - \Delta'_i) \right| + \left| x_i^T Q'_i A (Q - Q') x_i \right| \end{aligned}$$

1. One can then bound ($|x_i^T Q'_i A (Q - Q') x_i|$ is treated the same way):

$$\begin{aligned} &\left| x_i^T (Q - Q') A Q_{-i} x_i \right| \\ &= \frac{1}{n} \left| x_i^T Q (X - X') D X'^T Q' A Q_{-i} x_i \right| + \frac{1}{n} \left| x_i^T Q X' D (X - X')^T Q' A Q_{-i} x_i \right| \\ &\leq \frac{2\kappa^2 \kappa_D}{\varepsilon^2} \|A Q_{-i} x_i\| \|X - X'\|, \end{aligned}$$

2. besides:

$$\begin{aligned} |\Delta_i - \Delta'_i| &\leq \frac{1}{n^2} |x_i^T (Q - Q_{-i}) x_i| + \frac{2}{n} |x_i^T Q'_{-i} (x_i - x'_i)| \\ &\leq \left(\frac{2\kappa^3 \kappa_D}{\varepsilon^2 \sqrt{n}} + \frac{2\kappa}{\varepsilon \sqrt{n}} \right) \|X - X'\|, \end{aligned}$$

Therefore:

$$|\phi(X, D) - \phi(X', D)| \leq O \left(\frac{1}{\sqrt{n}} |x_i^T Q' A Q x_i| + \|A Q_{-i} x_i\| \right) \|X - X'\|.$$

We are then left to bound the variation towards D which is slightly more tricky. Let us consider $D' \in \mathcal{D}_n$ and note $Q'' \equiv (I_p - \frac{1}{n} X D' X^T)^{-1}$ and $\Delta' \equiv \frac{1}{n} x_i^T Q'' x_i$. This time one should decompose followingly:

$$\begin{aligned} |\phi(X, D) - \phi(X, D')| &\leq |x_i^T (Q_i - Q'_i) A Q x_i| + |x_i^T Q'_i A Q x_i D_i (\Delta_i - \Delta'_i)| + |x_i^T Q'' A (Q_i - Q'_i) x_i| \end{aligned}$$

1. First let us take advantage of the independence between x_i and X_{-i} to bound as in the proof of Lemma G.9:

$$\begin{aligned} |x_i^T (Q_i - Q'_i) A Q x_i| &= \frac{1}{n} |x_i^T Q_i X_{-i} (D - D') X_{-i} Q'_i A Q x_i| \\ &\leq \frac{\kappa^2}{\varepsilon^2} \left\| \frac{1}{\sqrt{n}} x_i^T Q_i X_{-i} \right\|_{\infty} \|A Q_{-i} x_i\| \|D - D'\|_F \end{aligned}$$

2. Second, let us bound:

$$|\Delta_i - \Delta'_i| \leq \frac{1}{n^2} |x_i^T Q_{-i} X_{-i} (D - D') X_{-i} Q_{-i} x_i| \leq \frac{\kappa^4}{\varepsilon^2 \sqrt{n}} \|D - D'\|$$

That allows us to set that:

$$|\phi(X, D) - \phi(X, D')| \leq O \left(\frac{1}{\sqrt{n}} |x_i^T Q' A Q x_i| + \left\| \frac{1}{\sqrt{n}} x_i^T Q_i X_{-i} \right\|_{\infty} \|A Q_{-i} x_i\| \right) \|D - D'\|_F$$

We have then all the elements to apply Theorem B.46 with the concentrations:

- $\frac{1}{\sqrt{\log n}} \|A Q_{-i} x_i\| \in O(1) \pm \mathcal{E}_2(1)$ thanks to Lemmas G.8, G.9 and the fact that $\mathbb{E}[\|A Q_{-i}^{(i)} x_i\|] \leq \sqrt{\mathbb{E}[x_i^T Q_{-i}^{(i)} A A Q_{-i}^{(i)} x_i]} \leq \frac{1}{\varepsilon} \|\mathbb{E}[x_i x_i^T]\|^{1/2} \|A\|_F \leq O(1)$.
- $\frac{1}{\sqrt{n \log n}} |x_i^T Q' A Q x_i| \in O(1/\sqrt{\log n}) \pm \mathcal{E}_2$ thanks to Lemma G.15
- $\left\| \frac{1}{\sqrt{n \log n}} x_i^T Q_i X_{-i} \right\|_{\infty} \in O(1) \pm \mathcal{E}_2$ thanks to Lemma G.10.

and the parameters $\sigma = 1$ and $\mu = (1, 1, 1)$ to obtain:

$$x_i^T Q A Q x_i (1 + D_i \Delta_i) \in \mathcal{E}_2 \left(\sqrt{\log n} \right) + \mathcal{E}_1 \left(\sqrt{\log n} \right) + \mathcal{E}_{2/3} \left(\sqrt{\log n} \right).$$

To show the concentration of $x_i^T Q A Q x_i (1 + D_i \bar{\Delta}_i)$, note that:

$$|x_i^T Q A Q x_i (1 + D_i \bar{\Delta}_i) - x_i^T Q A Q x_i (1 + D_i \Delta_i)| \leq \kappa_D |x_i^T Q A Q x_i| |\Delta_i - \bar{\Delta}_i|,$$

Lemma A.7 allows us to set the concentration of the product between $x_i^T Q A Q x_i \in O(\sqrt{n}) \pm \mathcal{E}_1$ and $|\Delta_i - \bar{\Delta}_i| \in 0 \pm \mathcal{E}_2(1/\sqrt{n}) + \mathcal{E}_1(1/n)$ satisfying $|\Delta_i - \bar{\Delta}_i| \leq \frac{\kappa^2}{\varepsilon}$:

$$|x_i^T Q A Q x_i| |\Delta_i - \bar{\Delta}_i| \in O(1) \pm \mathcal{E}_2(1) + \mathcal{E}_1(1/\sqrt{n})$$

Lemma B.19 then allows us to conclude on the concentration of $x_i^T Q A Q x_i (1 + D_i \Delta_i)$. \square

The existence of the deterministic parameters $\delta \in \mathcal{D}_n$ such that:

$$\delta = \frac{1}{n} \text{Diag}_{i \in [n]} \text{Tr}(\Sigma_i \tilde{Q}^\delta(D))$$

$$\left(\text{recall that } \tilde{Q}^\delta(D) \equiv \left(I_p - \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\frac{D_i}{1 + D_i \delta_i} \right] \Sigma_i \right)^{-1} \right)$$

is a consequence of Theorem F.18. Note that Corollary F.24 giving a deterministic equivalent for $Q = (I_p - \frac{1}{n} X X^T)^{-1}$ imposes the columns of X to be independent but they can possibly be non identically distributed. It concerns in particular the case of matrices $(I_p - \frac{1}{n} X \tilde{D} X^T)^{-1}$ for deterministic diagonal matrices \tilde{D} as stated below. It is at the basis of the estimation of $\mathbb{E}[Q] = \mathbb{E}[(I_p - \frac{1}{n} X D X^T)^{-1}]$.

Proof of Theorem G.2. Let us introduce the resolvent $\bar{Q} \equiv (I_p - \frac{1}{n} X \bar{D} X^T)^{-1}$ where we defined

$$\bar{D} \equiv \frac{\mathbb{E} \left[\frac{D}{I_p + \Delta D} \right]}{I_p - \Delta \mathbb{E} \left[\frac{D}{I_p + \Delta D} \right]}.$$

As will be understood later, this elaborated definition is taken for \bar{D} to satisfy the following relation:

$$\frac{\bar{D}}{I_p + \bar{D} \Delta} = \mathbb{E} \left[\frac{D}{I_p + \Delta D} \right],$$

it implies in particular that $\tilde{Q}^\delta(D) = \tilde{Q}^\delta(\bar{D})$ for any $\delta \in \mathcal{D}_n$. Let us then consider a deterministic matrix $A \in \mathcal{M}_p$, such that $\|A\|_F \leq 1$ and bound:

$$\begin{aligned} & \left| \mathbb{E}[\text{Tr}(AQ)] - \mathbb{E}[\text{Tr}(A\bar{Q})] \right| \\ &= \frac{1}{n} \sum_{i=1}^n \left| \mathbb{E} \left[x_i^T Q A \bar{Q} x_i \bar{\Delta}_i^{-1} (\bar{\Delta}_i D_i + 1 - (\bar{\Delta}_i \bar{D}_i + 1)) \right] \right| \\ &= \frac{1}{n} \sum_{i=1}^n \left| \mathbb{E} \left[x_i^T Q A \bar{Q} x_i \bar{\Delta}_i^{-1} (\bar{\Delta}_i D_i + 1) (\bar{\Delta}_i \bar{D}_i + 1) \left(\frac{1}{\bar{\Delta}_i \bar{D}_i + 1} - \frac{1}{\bar{\Delta}_i D_i + 1} \right) \right] \right| \\ &= \frac{1}{n} \sum_{i=1}^n \left| \mathbb{E} \left[x_i^T Q A \bar{Q} x_i (\bar{\Delta}_i D_i + 1) (\bar{\Delta}_i \bar{D}_i + 1) \left(\frac{D_i}{\bar{\Delta}_i D_i + 1} - \mathbb{E} \left[\frac{D_i}{\bar{\Delta}_i D_i + 1} \right] \right) \right] \right| \\ &= \frac{1}{n} \sum_{i=1}^n \left| \mathbb{E} \left[\left(x_i^T Q A \bar{Q} x_i (\bar{\Delta}_i D_i + 1) - \mathbb{E} [x_i^T Q A \bar{Q} x_i (\bar{\Delta}_i D_i + 1)] \right) \frac{D_i (\bar{\Delta}_i \bar{D}_i + 1)}{\bar{\Delta}_i D_i + 1} \right] \right| \\ &= \kappa_D \left(\frac{\kappa_D \kappa^2}{\varepsilon} + 1 \right) \sup_{i \in [n]} \mathbb{E} \left[\left| x_i^T Q A \bar{Q} x_i (\bar{\Delta}_i D_i + 1) - \mathbb{E} [x_i^T Q A \bar{Q} x_i (\bar{\Delta}_i D_i + 1)] \right| \right] \\ &\leq O \left(\sqrt{\log n} \right) \end{aligned}$$

Thanks to Lemma G.16. Then Corollary F.24 allows us to state that the deterministic diagonal matrix $\delta(\bar{D}) \in \mathcal{D}_n$ solution to:

$$\delta = \frac{1}{n} \text{Tr} \left(\Sigma_i \tilde{Q}^\delta(\bar{D}) \right) \quad \left(= \frac{1}{n} \text{Tr} \left(\Sigma_i \tilde{Q}^\delta(D) \right) \right),$$

satisfies the estimation:

$$\left\| \mathbb{E}[Q] - \tilde{Q}^{\delta(D)}(D) \right\|_F \leq \left\| \mathbb{E}[Q] - \mathbb{E}[\bar{Q}] \right\|_F + \left\| \mathbb{E}[\bar{Q}] - \tilde{Q}^{\delta(D)}(D) \right\|_F \leq O \left(\sqrt{\log n} + \frac{1}{\sqrt{n}} \right) \leq O \left(\sqrt{\log n} \right).$$

□

The same way that Theorem G.1 can be linked to Proposition B.48 giving the linear concentration of XDX^T , the next proposition can be linked to Proposition B.45 giving the Lipschitz concentration of $XDX^T u$ for any deterministic $u \in \mathbb{R}^p$.

Proposition G.17. *In the setting of Theorem G.1, for any deterministic vector $u \in \mathbb{R}^p$ such that $\|u\| \leq O(1)$:*

$$Qu \propto \mathcal{E}_1 \left(\sqrt{\frac{\log n}{n}} \right).$$

Proof. With the same variables $X, X' \in \mathcal{M}_{p,n}$, $D, D' \in \mathcal{D}_n$ and with the same notations Q, Q'_X, Q'_D as in the proof of Theorem G.1, we bound:

$$\begin{aligned} \|Qu - Q'_X u\| &= \frac{1}{n} \|Q(X - X')DX^T Q'_X u\| + \frac{1}{n} \|QX'D(X - X')^T Q'_X u\| \\ &\leq \frac{\kappa\kappa_D}{\varepsilon^2\sqrt{n}} \|X - X'\|, \end{aligned}$$

Second:

$$\|Qu - Q'_D u\| = \frac{1}{n} \|Q'_D X(D - D')X^T Qu\| \leq \frac{\kappa}{\varepsilon\sqrt{n}} \|D - D'\|_F \|X^T Qu\|_\infty,$$

and we know from Lemma G.11 that $\|X^T Qu\|_\infty \in O(\sqrt{\log n}) + \mathcal{E}_2(\sqrt{\log n})$, which allows us to conclude with Theorem B.46 that:

$$\sqrt{\frac{n}{\log n}} Qu \propto \mathcal{E}_2 + \mathcal{E}_1,$$

but the \mathcal{E}_2 decay can here be removed since the \mathcal{E}_1 is looser. □

Part III

Applications to machine learning

We present here some selected applications of our PhD research. Despite the fact that they are all relying on the random matrix inferences we presented in the previous part, they are also quite diverse and will provide different insights into machine learning algorithms. The first one gives some elementary key of understanding of the choice of activation functions in neural networks. The second one gives some theoretical clue to apprehend heavy tailed distribution through the study of scatter matrix estimation. The third application gives theoretical ground to the leave-one-out technique that allows to predict the performances of wide range of algorithms. For that we provide a very general but not fully achieved study of empirical risk minimization that one could call more precisely ℓ_2 -regularized convex robust regression.

All those applications rely on quite similar hypotheses concerning a “raw” data matrix $X = (x_1, \dots, x_n)$, close to the hypotheses adopted for the study of the resolvent that we recall below. Depending on the application, we will give some modification of these assumptions to adapt to the object and settings under study, we will then keep the number and add a “bis” or “ter” to highlight the fact that it is not the initial assumption assumed in previous part.

In all the application, we will assume that the dimension of the data p is of the same order or lower than the number of data n . A classical normalization with n will ensure the convergence.

Assumption 1. $p \leq O(n)$.

The concentration hypothesis on X will not always be satisfied. In particular, for ELM the data is considered to be deterministic and we rather look at the randomness of the weights of the neural network. For the robust estimation of the scatter matrix, X will present some non concentrated component. Depending on the applications, we will therefore have some modifications of this assumption.

Assumption 2. $X \propto \mathcal{E}_2$

When X is random we will always assume the independence of the data since it is at the basis of our study of the resolvent. We however do not not assume that the data are identically distributed.

Assumption 3. X has independent columns $x_1, \dots, x_n \in \mathbb{R}^p$.

Recalling the notation:

$$\forall i \in [n] : \quad \mu_i \equiv \mathbb{E}[x_i] \quad \Sigma_i \equiv \mathbb{E}[x_i x_i^T] \quad \text{and} \quad C_i \equiv \Sigma_i - \mu_i \mu_i^T,$$

we know from Proposition B.20 that $\sup_{1 \leq i \leq n} \|C_i\| \leq O(1)$. We then present the last hypothesis that allows the convergence of the resolvent in the Frobenius norm. Actually in [LLC18], we could develop our study with the presence of a signal of norm of order $O(\sqrt{n})$ in the data, but in that case we also had to assume that the number of different distributions was bounded by a constant independent with n . We preferred here not to consider this case to simplify the message.

Assumption 4. $\sup_{i \in [n]} \|\mu_i\| \leq O(1)$.

Assumption 5. $\exists K > 0$ such that $\forall n \in \mathbb{N}, \forall i \in [n] : \Sigma_i \geq KI_p$.

Chapter H

Influence of activation function in ELM

Artificial neural networks, developed in the late fifties [Ros58] in an attempt to develop machines capable of brain-like behaviors, know today an unprecedented research interest, notably in its applications to computer vision and machine learning at large [KSH12, Sch15] where superhuman performances on specific tasks are now commonly achieved. Recent progress in neural network performances however find their source in the processing power of modern computers as well as in the availability of large datasets rather than in the development of new mathematics. In fact, for lack of appropriate tools to understand the theoretical behavior of the non-linear activations and deterministic data dependence underlying these networks, the discrepancy between mathematical and practical (heuristic) studies of neural networks has kept widening. A first salient problem in harnessing neural networks lies in their being completely designed upon a deterministic training dataset $X = [x_1, \dots, x_n] \in \mathcal{M}_{p,n}$, so that their resulting performances intricately depend first and foremost on X . Recent works have nonetheless established that, when smartly designed, mere randomly connected neural networks can achieve performances close to those reached by entirely data-driven network designs [RR07, SKC⁺11]. As a matter of fact, to handle gigantic databases, the computationally expensive learning phase (the so-called backpropagation of the error method) typical of deep neural network structures becomes impractical, while it was recently shown that smartly designed single-layer random networks (as studied presently) can already reach superhuman capabilities [CGBZ15] and beat expert knowledge in specific fields [JH04]. These various findings have opened the road to the study of neural networks by means of statistical and probabilistic tools [CHM⁺15, GSB15]. The second problem relates to the non-linear activation functions present at each neuron, which have long been known (as opposed to linear activations) to help design universal approximators for any input-output target map [HSW89].

In this work, we propose an original random matrix-based approach to understand the end-to-end regression performance of single-layer random artificial neural networks, sometimes referred to as extreme learning machines [HZS06, HZDZ12], when the number n and size p of the input dataset are large and scale proportionally with the number m of neurons in the network. Our approach has several interesting features both for theoretical and practical considerations. It is the first one of the few known attempts to move the random matrix realm away from matrices with independent or linearly dependent column entries. Notable exceptions are the line of works surrounding kernel random matrices [El 10, CB16] as well as large dimensional robust statistics models [CPS15, ZCS14]. Here, to alleviate the non-linear difficulty, we exploit concentration of measure arguments [Led05], thereby pushing further the original ideas of [El 09] established for simpler random matrix models. While we believe that more powerful, albeit more computational intensive, tools (such as an appropriate adaptation of the Gaussian tools advocated in [PŚ11]) cannot be avoided to handle advanced considerations in neural networks, we demonstrate here that the concentration of measure phenomenon allows one to fully characterize the main quantities at the heart of the single-layer regression problem at hand.

In terms of practical applications, our findings shed light on the already incompletely understood extreme learning machines which have proved extremely efficient in handling machine learning problems involving large to huge datasets [HZDZ12, CGBZ15] at a computationally affordable cost. But our objective is also to pave the path to the understanding of more involved neural network structures, featuring notably multiple layers and some steps of learning by means of backpropagation of the error.

Our main contribution is therefore twofold. By assuming a random m -neuron single-layer network

with Lipschitz continuous activation function $\sigma(\cdot)$, fulfilling a *ridge-regression* task from an input dataset $x_1, \dots, x_n \in \mathbb{R}^p$ to an output sequence $y_1, \dots, y_n \in \mathbb{R}^d$ reconstituted as s_1, \dots, s_n , we first provide a deterministic equivalent for the so-called *training* mean-square error $\frac{1}{n} \sum_{i=1}^n \|y_i - s_i\|^2$ in the limit where n, m, p grow at the same rate. In passing, as a result of independent mathematical interest, we obtain a deterministic equivalent for the empirical spectral measure of the Gram matrix

$$G = \frac{1}{n} \sigma(W^T X)^T \sigma(W^T X)$$

where $W \propto \mathcal{E}_2$ has independent entries (for instance $W_{i,j} \propto \mathcal{N}(0,1)$), $X = [x_1, \dots, x_n]$ is such that $\limsup_n \|XX^T\| < \infty$, and $\sigma(\cdot)$ is applied entry-wise.)

Our second result, for which we only provide heuristic arguments, unveils a deterministic approximation for the *testing* mean-square error $\frac{1}{\hat{n}} \sum_{i=1}^{\hat{n}} \|\hat{y}_i - \hat{s}_i\|^2$, which is the error achieved when new data $\hat{x}_1, \dots, \hat{x}_{\hat{n}}$ with corresponding outputs $\hat{y}_1, \dots, \hat{y}_{\hat{n}}$, are evaluated by the aforementioned regressor (designed solely from X and y_1, \dots, y_n) as $\hat{s}_1, \dots, \hat{s}_{\hat{n}}$. Simulations on real image datasets are provided that corroborate our results.

In practical terms, these findings provide new insights into the roles played by the activation function $\sigma(\cdot)$, the random distribution of the entries of W , and the ridge-regression parameter in the neural network performance. We notably exhibit and prove some peculiar behaviors, such as the impossibility for the network to carry out elementary Gaussian mixture classification tasks, when either the activation function or the random weights distribution are ill chosen.

Besides, for the practitioner, the theoretical formulas retrieved in this work allow for a fast offline tuning of the aforementioned hyperparameters of the neural network, notably when n is not too large compared to p . The graphical results provided in the course of the section were particularly obtained within a 100 to 500-fold gain in computation time between theory and simulations.

The remainder of the chapter is structured as follows: in Section H.1, we introduce the mathematical model of the system under investigation. Our main results are then described and discussed in Section H.2. Section H.3 discusses our main findings.

H.1 System Model

We consider a single layer random neural network defined as follows. Each input data $x \in \mathbb{R}^p$ is multiplied by the transpose matrix of $W = (w_1, \dots, w_n) \in \mathcal{M}_{p,m}$; a non-linear function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is then applied entry-wise to the vector $W^T x$. The output $s \in \mathbb{R}^d$ of the neural network is the inner product $\beta^T \sigma(W^T x)$ for some matrix $\beta \in \mathcal{M}_{n,d}$ to be designed, i.e.,

$$s = \beta^T \sigma(W^T x). \quad (\text{H.1})$$

The m neurons of the network are the virtual units operating the mapping $w_i^T x \mapsto \sigma(w_i^T x)$, for $1 \leq i \leq m$.

The neural network operates in two phases: a training phase where the regression matrix β is learned based on a known input-output dataset pair (X, Y) and a testing phase where, for β now fixed, the network operates on a new input dataset \hat{X} with corresponding unknown output \hat{Y} .

During the training phase, based on a set of known input $X = [x_1, \dots, x_n] \in \mathcal{M}_{p,n}$ and output $Y = [y_1, \dots, y_n] \in \mathcal{M}_{d,n}$ datasets, the matrix β is chosen so as to minimize the mean square error $\frac{1}{n} \sum_{i=1}^n \|s_i - y_i\|^2 + \gamma \|\beta\|_F^2$, where $s_i = \beta^T \sigma(W^T x_i)$ and $\gamma > 0$ is some regularization factor. This leads β to be explicitly given by the *ridge-regressor*

$$\beta = \frac{1}{n} Z \left(\frac{1}{n} Z^T Z + \gamma I_m \right)^{-1} Y^T$$

where we defined $Z \equiv \sigma(W^T X)$. We will also denote $Q \equiv \left(\frac{1}{n} Z^T Z + \gamma I_m \right)^{-1}$, the *resolvent* of $\frac{1}{n} Z^T Z$.

The main theoretical interest of this section is to provide preliminary results on the behavior of the random matrix Q which intervenes as a key quantity in the evaluation of the neural network performances. Notably, it easily appears that the mean-square error E_{train} on the training dataset X , given by

$$E_{\text{train}} = \frac{1}{n} \|Y^T - Z^T \beta\|_F^2 = \frac{\gamma^2}{n} \text{Tr} Y^T Y Q^2 \quad (\text{H.2})$$

is strongly related to Q . Under the growth rate assumptions on n, p, m taken below, it shall appear that the random variable E_{train} concentrates around its mean, letting then appear $\mathbb{E}[Q^2]$ as a key quantity in the asymptotic evaluation of E_{train} .

The testing phase of the neural network is more interesting in practice as it unveils the actual performance of neural networks. For a test dataset $\hat{X} \in \mathcal{M}_{p, \hat{n}}$ of length \hat{n} , with unknown output $\hat{Y} \in \mathcal{M}_{d, \hat{n}}$, the test mean-square error is defined by

$$E_{\text{test}} = \frac{1}{\hat{n}} \|\hat{Y}^T - \hat{Z}^T \beta\|_F^2$$

where $\hat{Z} = \sigma(W^T \hat{X})$ and β is the same as used in (H.2) (and thus only depends on (X, Y) and γ). One of the key questions in the analysis of such an elementary neural network lies in the determination of γ which minimizes E_{test} (and is thus said to have good *generalization* performance). Notably, small γ values are known to reduce E_{train} but to induce the popular *overfitting* issue which generally increases E_{test} , while large γ values engender both large values for E_{train} and E_{test} .

From a mathematical standpoint though, the study of E_{test} brings forward some technical difficulties that do not allow for as a simple treatment through the present concentration of measure methodology as the study of E_{train} . Nonetheless, the analysis of E_{train} allows at least for heuristic approaches to become available, which we shall exploit to propose an asymptotic deterministic approximation for E_{test} .

To study the large data and network behavior of Q and E_{train} , we shall make the following growth rate assumptions:

Assumption 1 bis (Growth Rate). $p \leq O(n)$, $m \leq O(n)$. In addition:

Besides, as explained, we assume that X is deterministic therefore the concentration hypotheses make the following assumption on W .

Assumption 2 bis. The matrix W has independent identically distributed columns w_1, \dots, w_m .

Assumption 3 bis (Concentrated W). $W \propto \mathcal{E}_2$.

For $a = \varphi(b) \in \mathbb{R}^\ell$, $\ell \geq 1$, with $b \sim \mathcal{N}(0, I_\ell)$, we shall subsequently denote $a \sim \mathcal{N}_\varphi(0, I_\ell)$. Assumption 3 bis concerns in particular any matrix W satisfying $\forall (i, j) \in [p] \times [m]$: $W_{ij} \sim \mathcal{N}(0, 1)$ or $W_{ij} \sim \mathcal{U}(-1, 1)$ (the uniform distribution on $[-1, 1]$).

Assumption 4 bis. $\sup_{i \in [m]} \|\mathbb{E}[w_i]\| \leq O(1)$, $\|X\| < O(1)$ and $\|Y\|_\infty < O(1)$.

Finally, we add the following regularity condition on the function σ .

Assumption 6 (Function σ). The function σ is Lipschitz continuous with parameter λ_σ independent of n .

Under Assumptions 1 bis-6, one can deduce that $\|\mathbb{E}[\sigma(w_i^T X)]\| \leq O(1)$ which allow us to apply the results on the resolvent as one will see below. This assumption holds for many of the activation functions traditionally considered in neural networks, such as sigmoid functions, the rectified linear unit $\sigma(t) = \max(t, 0)$, or the absolute value operator.

H.2 Main Results

2.1 Main technical results and training performance

A key quantity in the performance evaluation of the neural network under study is the matrix

$$\Sigma = \mathbb{E} [\sigma(w^T X)^T \sigma(w^T X)] \quad (\text{H.3})$$

of size $m \times m$ where w is any columns of W (they are identically distributed).

The deterministic equivalent of the resolvent $Q = (\frac{1}{n} Z^T Z + \gamma I_m)^{-1}$ is given thanks to Corollary F.24:

$$\left\| \mathbb{E}[Q] - \tilde{Q} \right\|_F \leq O\left(\frac{1}{\sqrt{n}}\right) \quad (\text{H.4})$$

where we defined as usual:

$$\tilde{Q} \equiv \left(\frac{m}{n} \frac{\Sigma}{1 + \delta} + \gamma I_m \right)^{-1}$$

and δ satisfies $\delta = \frac{1}{n} \text{Tr} \Sigma \tilde{Q}$.

For convenience, in the following, we shall denote

$$\tilde{\Sigma} = \frac{m}{n} \frac{\Sigma}{1 + \delta}. \quad (\text{H.5})$$

Estimation H.4 also provides the central step in the evaluation of E_{train} , for which not only $\mathbb{E}[Q]$ but also $\mathbb{E}[Q^2]$ needs be estimated. This last ingredient is provided by Proposition F.48 that we expressed for the case $m = 2$ below.

Proposition H.1 (Asymptotic equivalent for $\mathbb{E}[Q A Q]$). *Let Assumptions 1 bis-6 hold and $A \in \mathcal{M}_{m,m}$ be a symmetric non-negative definite matrix which is either Σ or a matrix with uniformly bounded operator norm (with respect to n). Then, for all $\varepsilon > 0$, there exists $c > 0$ such that*

$$\left\| \mathbb{E}[Q A Q] - \left(\tilde{Q} A \tilde{Q} + \frac{\frac{1}{n} \text{Tr}(\tilde{\Sigma} \tilde{Q} A \tilde{Q})}{1 - \frac{1}{n} \text{Tr} \tilde{\Sigma}^2 \tilde{Q}^2} \tilde{Q} \tilde{\Sigma} \tilde{Q} \right) \right\|_F \leq O\left(\frac{1}{\sqrt{n}}\right).$$

As an immediate consequence of Proposition H.1, we have the following result on the training mean-square error of single-layer random neural networks.

Proposition H.2 (Asymptotic training mean-square error). *Let Assumptions 1 bis-6 hold and \tilde{Q} , $\tilde{\Sigma}$ be defined as in Estimation H.4 and (H.5). Then, for all $\varepsilon > 0$,*

$$E_{\text{train}} \in \bar{E}_{\text{train}} \pm \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right),$$

where:

$$\begin{aligned} E_{\text{train}} &= \frac{1}{n} \|Y^T - Z^T \beta\|_F^2 = \frac{\gamma^2}{n} \text{Tr} Y^T Y Q^2 \\ \bar{E}_{\text{train}} &= \frac{\gamma^2}{n} \text{Tr} Y^T Y \tilde{Q} \left[\frac{\frac{1}{n} \text{Tr} \tilde{\Sigma} \tilde{Q}^2}{1 - \frac{1}{n} \text{Tr}(\tilde{\Sigma} \tilde{Q})^2} \tilde{\Sigma} + I_m \right] \tilde{Q}. \end{aligned}$$

Since \tilde{Q} and Σ share the same orthogonal eigenvector basis, it appears that E_{train} depends on the alignment between the right singular vectors of Y and the eigenvectors of Σ , with weighting coefficients

$$\left(\frac{\gamma}{\lambda_i + \gamma} \right)^2 \left(1 + \lambda_i \frac{\frac{1}{n} \sum_{j=1}^n \lambda_j (\lambda_j + \gamma)^{-2}}{1 - \frac{1}{n} \sum_{j=1}^n \lambda_j^2 (\lambda_j + \gamma)^{-2}} \right)$$

where we denoted $\lambda_i = \lambda_i(\tilde{\Sigma})$, $1 \leq i \leq n$, the eigenvalues of $\tilde{\Sigma}$ (which depend on γ through $\lambda_i(\tilde{\Sigma}) = \frac{m}{n(1+\delta)}\lambda_i(\Sigma)$). If $\liminf_n m/n > 1$, it is easily seen that $\delta \rightarrow 0$ as $\gamma \rightarrow 0$, in which case, $E_{\text{train}} \rightarrow 0$ almost surely. However, in the more interesting case in practice where $\limsup_n m/n < 1$, $\delta \rightarrow \infty$ as $\gamma \rightarrow 0$ and E_{train} consequently does not have a simple limit (see Section 3.3 for more discussion on this aspect).

Proposition H.2 is also reminiscent of applied random matrix works on empirical covariance matrix models, such as [BS07, KKHN09], then further emphasizing the strong connection between the non-linear matrix $\sigma(W^T X)$ and its linear counterpart $W\Sigma^{\frac{1}{2}}$.

2.2 Testing performance

As previously mentioned, harnessing the asymptotic testing performance E_{test} seems, to the best of the authors' knowledge, out of current reach with the sole concentration of measure arguments used for the proof of the previous main results. Nonetheless, if not fully effective, these arguments allow for an intuitive derivation of a deterministic equivalent for E_{test} , which is strongly supported by simulation results.

To introduce this result, let $\hat{X} = [\hat{x}_1, \dots, \hat{x}_{\hat{n}}] \in \mathcal{M}_{p, \hat{n}}$ be a set of input data with corresponding output $\hat{Y} = [\hat{y}_1, \dots, \hat{y}_{\hat{n}}] \in \mathcal{M}_{d, \hat{n}}$. We also define $\hat{Z} = \sigma(W^T \hat{X}) \in \mathcal{M}_{p, \hat{n}}$. We assume that \hat{X} and \hat{Y} satisfy the same growth rate conditions as X and Y in Assumption 1 bis. To introduce our claim, we need to extend the definition of Σ in (H.3) and $\tilde{\Sigma}$ in (H.5) to the following notations: for all pair of matrices (A, B) of appropriate dimensions,

$$\begin{aligned}\Sigma_{AB} &= \mathbb{E} [\sigma(w^T A)^T \sigma(w^T B)] \\ \tilde{\Sigma}_{AB} &= \frac{m}{n} \frac{\Sigma_{AB}}{1 + \delta}\end{aligned}$$

where $w \sim \mathcal{N}_\varphi(0, I_p)$. In particular, $\Sigma = \Sigma_{XX}$ and $\tilde{\Sigma} = \tilde{\Sigma}_{XX}$.

With these notations in place, we are in position to state our estimation of the test error.

Proposition H.3 (Deterministic equivalent for E_{test}). *Let Assumptions 1 bis-6 hold and \hat{X}, \hat{Y} satisfy the same conditions as X, Y in Assumption 1 bis. Then, for all $\varepsilon > 0$,*

$$E_{\text{test}} \in \bar{E}_{\text{test}} \pm \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right),$$

where:

$$\begin{aligned}E_{\text{test}} &= \frac{1}{\hat{n}} \left\| \hat{Y}^T - \hat{Z}^T \beta \right\|_F^2 \\ \bar{E}_{\text{test}} &= \frac{1}{\hat{n}} \left\| \hat{Y}^T - \tilde{\Sigma}_{X\hat{X}}^T \tilde{Q} \hat{Y}^T \right\|_F^2 \\ &\quad + \frac{\frac{1}{n} \text{Tr} Y^T Y \tilde{Q} \tilde{\Sigma} \tilde{Q}}{1 - \frac{1}{n} \text{Tr}(\tilde{\Sigma} \tilde{Q})^2} \left[\frac{1}{\hat{n}} \text{Tr} \tilde{\Sigma}_{\hat{X}\hat{X}} - \frac{1}{\hat{n}} \text{Tr}(I_m + \gamma \tilde{Q})(\tilde{\Sigma}_{X\hat{X}} \tilde{\Sigma}_{\hat{X}X} \tilde{Q}) \right].\end{aligned}$$

While not immediate at first sight, one can confirm (using notably the relation $\tilde{\Sigma} \tilde{Q} + \gamma \tilde{Q} = I_m$) that, for $(\hat{X}, \hat{Y}) = (X, Y)$, $\bar{E}_{\text{train}} = \bar{E}_{\text{test}}$, as expected.

In order to evaluate practically the results of Proposition H.2 and Proposition H.3, it is a first step to be capable of estimating the values of Σ_{AB} for various $\sigma(\cdot)$ activation functions of practical interest. Such results, which call for completely different mathematical tools (mostly based on integration tricks), are provided in the subsequent section.

2.3 Evaluation of Σ_{AB}

The evaluation of $\Sigma_{AB} = \mathbb{E}[\sigma(w^T A)^T \sigma(w^T B)]$ for arbitrary matrices A, B naturally boils down to the evaluation of its individual entries and thus to the calculus, for arbitrary vectors $a, b \in \mathbb{R}^p$ of

$$\mathbb{E}[\sigma(w^T a) \sigma(w^T b)] = (2\pi)^{-\frac{p}{2}} \int \sigma(\varphi(\tilde{w})^T a) \sigma(\varphi(\tilde{w})^T b) e^{-\frac{1}{2} \|\varphi(\tilde{w})\|^2} d\tilde{w}. \quad (\text{H.6})$$

$\sigma(t)$	Σ_{ab}
t	$a^T b$
$\max(t, 0)$	$\frac{1}{2\pi} \ a\ \ b\ \left(\angle(a, b) \operatorname{acos}(-\angle(a, b)) + \sqrt{1 - \angle(a, b)^2} \right)$
$ t $	$\frac{2}{\pi} \ a\ \ b\ \left(\angle(a, b) \operatorname{asin}(\angle(a, b)) + \sqrt{1 - \angle(a, b)^2} \right)$
$\operatorname{erf}(t)$	$\frac{2}{\pi} \operatorname{asin} \left(\frac{2a^T b}{\sqrt{(1+2\ a\ ^2)(1+2\ b\ ^2)}} \right)$
$1_{\{t>0\}}$	$\frac{1}{2} - \frac{1}{2\pi} \operatorname{acos}(\angle(a, b))$
$\operatorname{sign}(t)$	$\frac{2}{\pi} \operatorname{asin}(\angle(a, b))$
$\cos(t)$	$\exp(-\frac{1}{2}(\ a\ ^2 + \ b\ ^2)) \cosh(a^T b)$
$\sin(t)$	$\exp(-\frac{1}{2}(\ a\ ^2 + \ b\ ^2)) \sinh(a^T b)$.

Table H.1: Values of Σ_{ab} for $w \sim \mathcal{N}(0, I_p)$, $\angle(a, b) \equiv \frac{a^T b}{\|a\| \|b\|}$.

The evaluation of (H.6) can be obtained through various integration tricks for a wide family of mappings $\varphi(\cdot)$ and activation functions $\sigma(\cdot)$. The most popular activation functions in neural networks are sigmoid functions, such as $\sigma(t) = \operatorname{erf}(t) \equiv \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$, as well as the so-called rectified linear unit (ReLU) defined by $\sigma(t) = \max(t, 0)$ which has been recently popularized as a result of its robust behavior in deep neural networks. In physical artificial neural networks implemented using light projections, $\sigma(t) = |t|$ is the preferred choice. Note that all aforementioned functions are Lipschitz continuous and therefore in accordance with Assumption 6.

Despite their not abiding by the prescription of Assumptions 3 bis and 6, we believe that the following results could be extended to more general settings, as discussed in Section H.3. In particular, one of the key ingredients in the proof of all our results is that the vector $\sigma(w^T X)$ follows a concentration of measure phenomenon, induced by the Gaussianity of \tilde{w} (if $w = \varphi(\tilde{w})$), the Lipschitz character of σ and the norm boundedness of X . It is likely, although not necessarily simple to prove, that $\sigma(w^T X)$ may still concentrate under relaxed assumptions. This is likely the case for more generic vectors w than $\mathcal{N}_\varphi(0, I_p)$ as well as for a larger class of activation functions, such as polynomial or piece-wise Lipschitz continuous functions.

In anticipation of these likely generalizations, we provide in Table H.1 the values of Σ_{ab} for $w \sim \mathcal{N}(0, I_p)$ (i.e., for $\varphi(t) = t$) and for a set of functions $\sigma(\cdot)$ not necessarily satisfying Assumption 6. Denoting $\Sigma \equiv \Sigma(\sigma(t))$, it is interesting to remark that, since $\arccos(x) = -\arcsin(x) + \frac{\pi}{2}$, $\Sigma(\max(t, 0)) = \Sigma(\frac{1}{2}t) + \Sigma(\frac{1}{2}|t|)$. Also, $[\Sigma(\cos(t)) + \Sigma(\sin(t))]_{a,b} = \exp(-\frac{1}{2}\|a - b\|^2)$, a result reminiscent of [RR07].¹ Finally, note that $\Sigma(\operatorname{erf}(\kappa t)) \rightarrow \Sigma(\operatorname{sign}(t))$ as $\kappa \rightarrow \infty$, inducing that the extension by continuity of $\operatorname{erf}(\kappa t)$ to $\operatorname{sign}(t)$ propagates to their associated kernels.

In addition to these results for $w \sim \mathcal{N}(0, I_p)$, we also evaluated $\Sigma_{ab} = \mathbb{E}[\sigma(w^T a)\sigma(w^T b)]$ for $\sigma(t) = \zeta_2 t^2 + \zeta_1 t + \zeta_0$ and $w \in \mathbb{R}^p$ a vector of independent and identically distributed entries of zero mean and moments of order k equal to m_k (so $m_1 = 0$); w is not restricted here to satisfy $w \sim \mathcal{N}_\varphi(0, I_p)$. In this case, we find

$$\begin{aligned} \Sigma_{ab} &= \zeta_2^2 [m_2^2 (2(a^T b)^2 + \|a\|^2 \|b\|^2) + (m_4 - 3m_2^2)(a^2)^T (b^2)] + \zeta_1^2 m_2 a^T b \\ &\quad + \zeta_2 \zeta_1 m_3 [(a^2)^T b + a^T (b^2)] + \zeta_2 \zeta_0 m_2 [\|a\|^2 + \|b\|^2] + \zeta_0^2 \end{aligned} \quad (\text{H.7})$$

where we defined $(a^2) \equiv [a_1^2, \dots, a_p^2]^T$.

It is already interesting to remark that, while classical random matrix models exhibit a well-known universality property — in the sense that their limiting spectral distribution is independent of the moments (higher than two) of the entries of the involved random matrix, here W —, for $\sigma(\cdot)$ a polynomial of order two, Σ and thus μ_n strongly depend on $\mathbb{E}[W_{ij}^k]$ for $k = 3, 4$. We shall see in Section H.3 that this remark

¹It is in particular not difficult to prove, based on our framework, that, as $m/n \rightarrow \infty$, a random neural network composed of $m/2$ neurons with activation function $\sigma(t) = \cos(t)$ and $m/2$ neurons with activation function $\sigma(t) = \sin(t)$ implements a Gaussian difference kernel.

has troubling consequences. We will notably infer (and confirm via simulations) that the studied neural network may provably fail to fulfill a specific task if the W_{ij} are Bernoulli with zero mean and unit variance but succeed with possibly high performance if the W_{ij} are standard Gaussian (which is explained by the disappearance or not of the term $(a^T b)^2$ and $(a^2)^T (b^2)$ in (H.7) if $m_4 = m_2^2$).

H.3 Practical Outcomes

We discuss in this section the outcomes of our main results in terms of neural network application.

3.1 Simulation Results

We first provide in this section a simulation corroborating the findings of Proposition H.2 and suggesting the validity of Proposition H.3. To this end, we consider the task of classifying the popular MNIST image database [LCB98], composed of grayscale handwritten digits of size 28×28 , with a neural network composed of $m = 512$ units and standard Gaussian W . We represent here each image as a $p = 784$ -size vector; 1024 images of sevens and 1024 images of nines were extracted from the database and were evenly split in 512 training and test images, respectively. The database images were jointly centered and scaled so to fall close to the setting of Assumption 1 bis on X and \hat{X} (an admissible preprocessing intervention). The columns of the output values Y and \hat{Y} were taken as unidimensional ($d = 1$) with $Y_{1j}, \hat{Y}_{1j} \in \{-1, 1\}$ depending on the image class. Figure H.1 displays the simulated (averaged over 100 realizations of W) versus theoretical values of E_{train} and E_{test} for three choices of Lipschitz continuous functions $\sigma(\cdot)$, as a function of γ .

Note that a perfect match between theory and practice is observed, for both E_{train} and E_{test} , which is a strong indicator of both the validity of Proposition H.3 and the adequacy of Assumption 1 bis to the MNIST dataset.

We subsequently provide in Figure H.2 the comparison between theoretical formulas and practical simulations for a set of functions $\sigma(\cdot)$ which do not ascribe by Assumption 6, i.e., either discontinuous or non-Lipschitz maps. The closeness between both sets of curves is again remarkably good, although to a lesser extent than for the Lipschitz continuous functions of Figure H.1. Also, the achieved performances are generally worse than those observed in Figure H.1.

It should be noted that the performance estimates provided by Proposition H.2 and Proposition H.3 can be efficiently implemented at low computational cost in practice. Indeed, by diagonalizing Σ (which is a marginal cost independent of γ), \bar{E}_{train} can be computed for all γ through mere vector operations; similarly \bar{E}_{test} is obtained by the marginal cost of a basis change of $\Sigma_{\hat{X}X}$ and the matrix product $\Sigma_{X\hat{X}}\Sigma_{\hat{X}X}$, all remaining operations being accessible through vector operations. As a consequence, the simulation times needed to generate the aforementioned theoretical curves using the linked Python script were found to be 100 to 500 times faster than to generate the simulated network performances. Beyond their theoretical interest, the provided formulas thereby allow for an efficient offline tuning of the network hyperparameters, notably the choice of an appropriate value for the ridge-regression parameter γ .

3.2 The hidden kernel

Estimation H.4 and the subsequent theoretical findings importantly reveal that the neural network performances are directly related to the Gram matrix Σ , which acts as a deterministic kernel on the dataset X . This remark is in fact a well-known result found e.g., in [HZDZ12] where a link between the present single-layer networks and the so-called least-square support vector machine method is exhibited, or in [Wil98] where it is shown that, as $m \rightarrow \infty$ alone, the neural network behaves as a mere kernel operator. These remarks were in fact put at an advantage in [RR07], where computationally efficient kernel methods were proposed based on random single-layer neural networks.

As discussed previously, the formulas for \bar{E}_{train} and \bar{E}_{test} notably suggest that good performances are achieved if the dominant eigenvectors of Σ show a good alignment to Y (and similarly for $\Sigma_{X\hat{X}}$ and \hat{Y}). This naturally drives us to finding a priori simple regression tasks where ill-choices of Σ may annihilate the neural

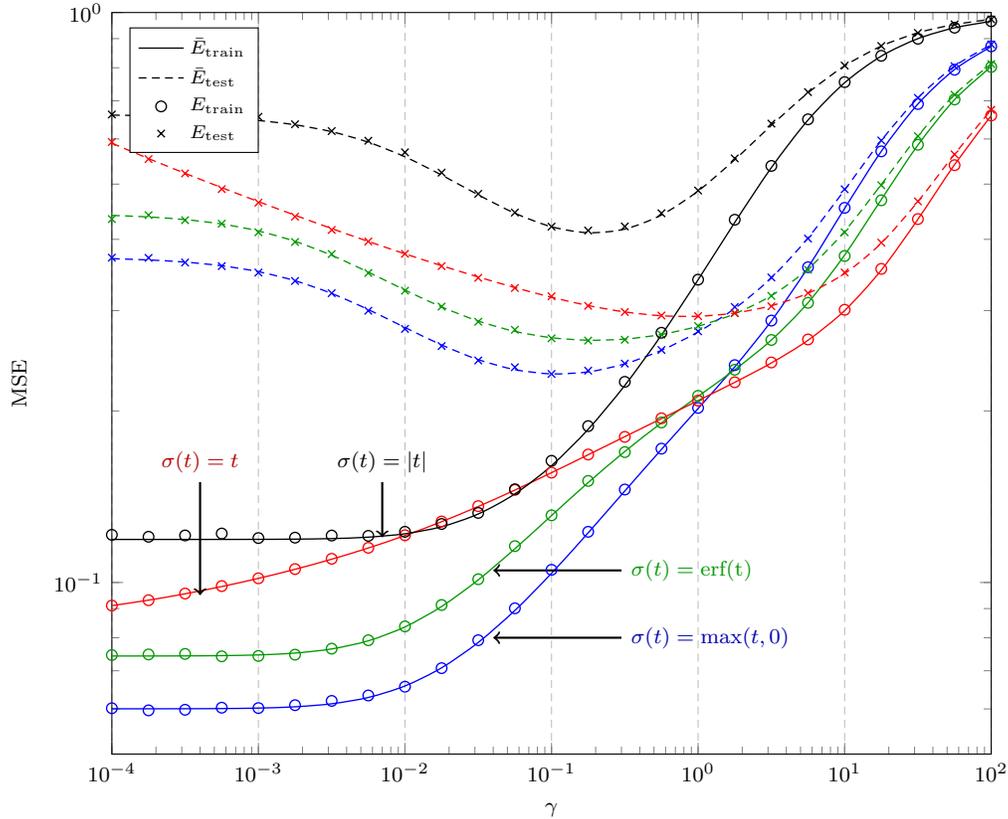


Figure H.1: Neural network performance for Lipschitz continuous $\sigma(\cdot)$, $W_{ij} \sim \mathcal{N}(0, 1)$, as a function of γ , for 2-class MNIST data (sevens, nines), $m = 512$, $n = \hat{n} = 1024$, $p = 784$.

network performance. Following recent works on the asymptotic performance analysis of kernel methods for Gaussian mixture models [CB16, LC17, MC17] and [CK16], we describe here such a task.

Let $x_1, \dots, x_{n/2} \sim \mathcal{N}(0, \frac{1}{p}C_1)$ and $x_{n/2+1}, \dots, x_T \sim \mathcal{N}(0, \frac{1}{p}C_2)$ where C_1 and C_2 are such that $\text{Tr} C_1 = \text{Tr} C_2$, $\|C_1\|, \|C_2\|$ are bounded, and $\text{Tr}(C_1 - C_2)^2 = O(p)$. Accordingly, $y_1, \dots, y_{n/2+1} = -1$ and $y_{n/2+1}, \dots, y_T = 1$. It is proved in the aforementioned articles that, under these conditions, it is theoretically possible, in the large p, T limit, to classify the data using a kernel least-square support vector machine (that is, with a training dataset) or with a kernel spectral clustering method (that is, in a completely unsupervised manner) with a non-trivial limiting error probability (i.e., neither zero nor one). This scenario has the interesting feature that $x_i^T x_j \rightarrow 0$ almost surely for all $i \neq j$ while $\|x_i\|^2 - \frac{1}{p} \text{Tr}(\frac{1}{2}C_1 + \frac{1}{2}C_2) \rightarrow 0$, almost surely, irrespective of the class of x_i , thereby allowing for a Taylor expansion of the non-linear kernels as early proposed in [El 10].

Transposed to our present setting, the aforementioned Taylor expansion allows for a consistent approximation $\tilde{\Sigma}$ of Σ by an *information-plus-noise* (spiked) random matrix model (see e.g., [LV10, BGN12]). In the present Gaussian mixture context, it is shown in [CB16] that data classification is (asymptotically at least) only possible if $\tilde{\Sigma}_{ij}$ explicitly contains the quadratic term $(x_i^T x_j)^2$ (or combinations of $(x_i^2)^T x_j$, $(x_j^2)^T x_i$, and $(x_i^2)^T (x_j^2)$). In particular, recalling Table H.1 and letting $a, b \sim \mathcal{N}(0, C_i)$ with $i = 1, 2$, it is easily seen that only $\max(t, 0)$, $|t|$, and $\cos(t)$ can realize the task. Indeed, we have the following Taylor expansions around

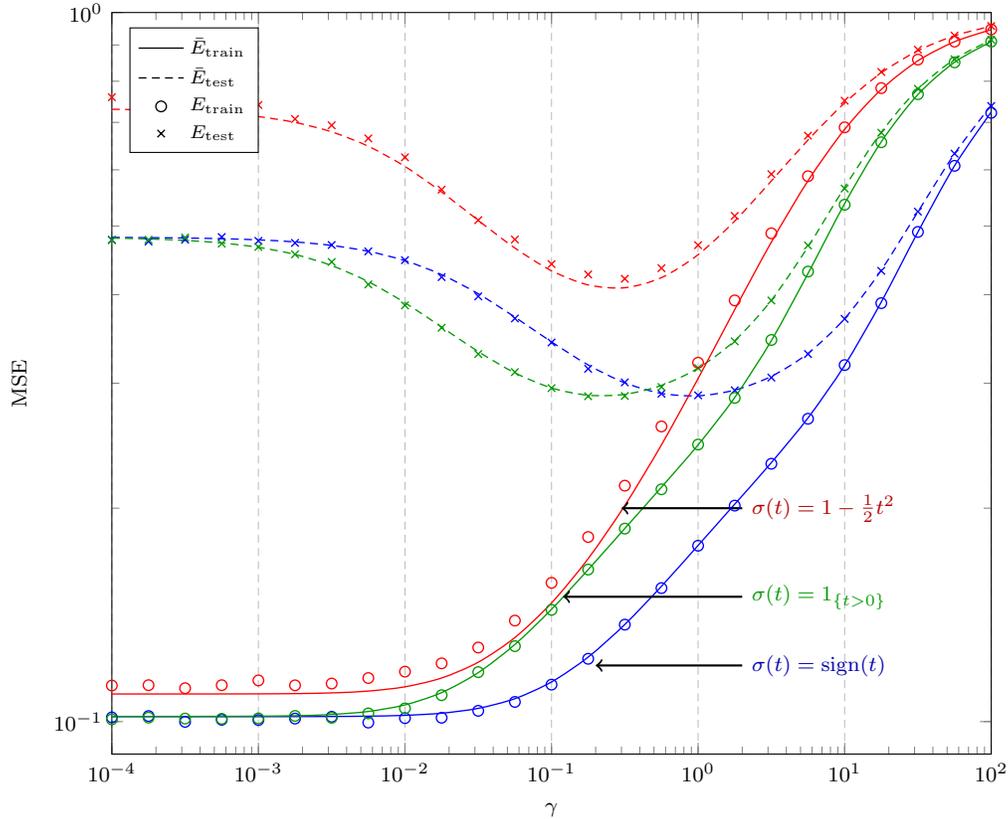


Figure H.2: Neural network performance for $\sigma(\cdot)$ either discontinuous or non Lipschitz, $W_{ij} \sim \mathcal{N}(0, 1)$, as a function of γ , for 2-class MNIST data (sevens, nines), $m = 512$, $n = \hat{n} = 1024$, $p = 784$.

$x = 0$:

$$\begin{aligned} \operatorname{asin}(x) &= x + O(x^3) \\ \operatorname{sinh}(x) &= x + O(x^3) \\ \operatorname{acos}(x) &= \frac{\pi}{2} - x + O(x^3) \\ \operatorname{cosh}(x) &= 1 + \frac{x^2}{2} + O(x^3) \\ x \operatorname{acos}(-x) + \sqrt{1-x^2} &= 1 + \frac{\pi x}{2} + \frac{x^2}{2} + O(x^3) \\ x \operatorname{asin}(x) + \sqrt{1-x^2} &= 1 + \frac{x^2}{2} + O(x^3) \end{aligned}$$

where only the last three functions exhibit a quadratic term.

More surprisingly maybe, recalling now Equation (H.7) which considers non-necessarily Gaussian W_{ij} with moments m_k of order k , a more refined analysis shows that the aforementioned Gaussian mixture classification task will fail if $m_3 = 0$ and $m_4 = m_2^2$, so for instance for $W_{ij} \in \{-1, 1\}$ Bernoulli with parameter $\frac{1}{2}$. The performance comparison of this scenario is shown in the top part of Figure H.3 for $\sigma(t) = -\frac{1}{2}t^2 + 1$ and $C_1 = \operatorname{Diag}(I_{p/2}, 4I_{p/2})$, $C_2 = \operatorname{Diag}(4I_{p/2}, I_{p/2})$, for $W_{ij} \sim \mathcal{N}(0, 1)$ and $W_{ij} \sim \operatorname{Bern}$ (that is, Bernoulli $\{(-1, \frac{1}{2}), (1, \frac{1}{2})\}$). The choice of $\sigma(t) = \zeta_2 t^2 + \zeta_1 t + \zeta_0$ with $\zeta_1 = 0$ is motivated by [CB16, CK16], where it is shown, in a somewhat different setting, that this choice is optimal for class recovery. Note that, while the test performances are overall rather weak in this setting, for $W_{ij} \sim \mathcal{N}(0, 1)$, E_{test} drops below one (the amplitude of the \hat{Y}_{ij}), thereby indicating that non-trivial classification is performed. This is not so for the

Bernoulli $W_{ij} \sim \text{Bern}$ case where E_{test} is systematically greater than $|\hat{Y}_{ij}|$. This is theoretically explained by the fact that, from Equation (H.7), Σ_{ij} contains structural information about the data classes through the term $2m_2^2(x_i^T x_j)^2 + (m_4 - 3m_2^2)(x_i^2)^T(x_j^2)$ which induces an information-plus-noise model for Σ as long as $2m_2^2 + (m_4 - 3m_2^2) \neq 0$, i.e., $m_4 \neq m_2^2$ (see [CB16] for details). This is visually seen in the bottom part of Figure H.3 where the Gaussian scenario presents an isolated eigenvalue for Σ with corresponding structured eigenvector, which is not the case of the Bernoulli scenario. To complete this discussion, it appears relevant in the present setting to choose W_{ij} in such a way that $m_4 - m_2^2$ is far from zero, thus suggesting the interest of heavy-tailed distributions. To confirm this prediction, Figure H.3 additionally displays the performance achieved and the spectrum of Σ observed for $W_{ij} \sim \text{Stud}$, that is, following a Student-t distribution with degree of freedom $\nu = 7$ normalized to unit variance (in this case $m_2 = 1$ and $m_4 = 5$). Figure H.3 confirms the large superiority of this choice over the Gaussian case (note nonetheless the slight inaccuracy of our theoretical formulas in this case, which is likely due to too small values of n, m, p to accommodate W_{ij} with higher order moments, an observation which is confirmed in simulations when letting ν be even smaller).

3.3 Limiting cases

We have suggested that Σ contains, in its dominant eigenmodes, all the usable information describing X . In the Gaussian mixture example above, it was notably shown that Σ may completely fail to contain this information, resulting in the impossibility to perform a classification task, even if one were to take infinitely many neurons in the network. For Σ containing useful information about X , it is intuitive to expect that both $\inf_\gamma \bar{E}_{\text{train}}$ and $\inf_\gamma \bar{E}_{\text{test}}$ become smaller as m/n and n/p become large. It is in fact easy to see that, if Σ is invertible (which is likely to occur in most cases if $\liminf_n n/p > 1$), then

$$\begin{aligned} \lim_{n \rightarrow \infty} \bar{E}_{\text{train}} &= 0 \\ \lim_{n \rightarrow \infty} \bar{E}_{\text{test}} &= \frac{1}{\hat{n}} \left\| \hat{Y}^T - \Sigma_{\hat{X}X} \Sigma^{-1} Y^T \right\|_F^2 \end{aligned}$$

and we fall back on the performance of a classical kernel regression. It is interesting in particular to note that, as the number of neurons m becomes large, the effect of γ flattens out. Therefore, a smart choice of γ is only relevant for small (and thus computationally more efficient) neuron layers. This observation is depicted in Figure H.4 where it is made clear that a growth of m reduces E_{train} to zero while E_{test} saturates to a non-zero limit which becomes increasingly irrespective of γ . Note additionally the interesting phenomenon occurring for $m \leq n$ where too small values of γ induce important performance losses, thereby suggesting a strong importance of proper choices of γ in this regime.

Of course, practical interest lies precisely in situations where n is not too large. We may thus subsequently assume that $\limsup_n m/n < 1$. In this case, as suggested by Figures H.1–H.2, the mean-square error performances achieved as $\gamma \rightarrow 0$ may predict the superiority of specific choices of $\sigma(\cdot)$ for optimally chosen γ . It is important for this study to differentiate between cases where $r \equiv \text{rank}(\Sigma)$ is smaller or greater than m . Indeed, observe that, with the spectral decomposition $\Sigma = U_r \Lambda_r U_r^T$ for $\Lambda_r \in \mathcal{D}_r$ diagonal and $U_r \in \mathcal{M}_{m,r}$,

$$\delta = \frac{1}{n} \text{Tr} \Sigma \left(\frac{m}{n} \frac{\Sigma}{1 + \delta} + \gamma I_m \right)^{-1} = \frac{1}{n} \text{Tr} \Lambda_r \left(\frac{m}{n} \frac{\Lambda_r}{1 + \delta} + \gamma I_r \right)^{-1}$$

which satisfies, as $\gamma \rightarrow 0$,

$$\begin{cases} \delta \rightarrow \frac{r}{m-r} & , r < m \\ \gamma \delta \rightarrow \Delta = \frac{1}{n} \text{Tr} \Sigma \left(\frac{m}{n} \frac{\Sigma}{\Delta} + I_m \right)^{-1} & , r \geq m. \end{cases}$$

A phase transition therefore exists whereby δ assumes a finite positive value in the small γ limit if $r/m < 1$, or scales like $1/\gamma$ otherwise.

As a consequence, if $r < m$, as $\gamma \rightarrow 0$, $\tilde{\Sigma} \rightarrow \frac{m}{n}(1 - \frac{r}{n})\Sigma$ and $\tilde{Q} \sim \frac{T}{n-r} U_r \Lambda_r^{-1} U_r^T + \frac{1}{\gamma} V_r V_r^T$, where $V_r \in \mathcal{M}_{m,(m-r)}$ is any matrix such that $[U_r \ V_r]$ is orthogonal, so that $\tilde{\Sigma} \tilde{Q} \rightarrow U_r U_r^T$ and $\tilde{\Sigma} \tilde{Q}^2 \rightarrow U_r \Lambda_r^{-1} U_r^T$; and thus, $\bar{E}_{\text{train}} \rightarrow \frac{1}{n} \text{Tr} Y V_r V_r^T Y^T = \frac{1}{n} \|Y V_r\|_F^2$, which states that the residual training error corresponds to the energy of Y not captured by the space spanned by Σ . Since E_{train} is an increasing function of γ , so is

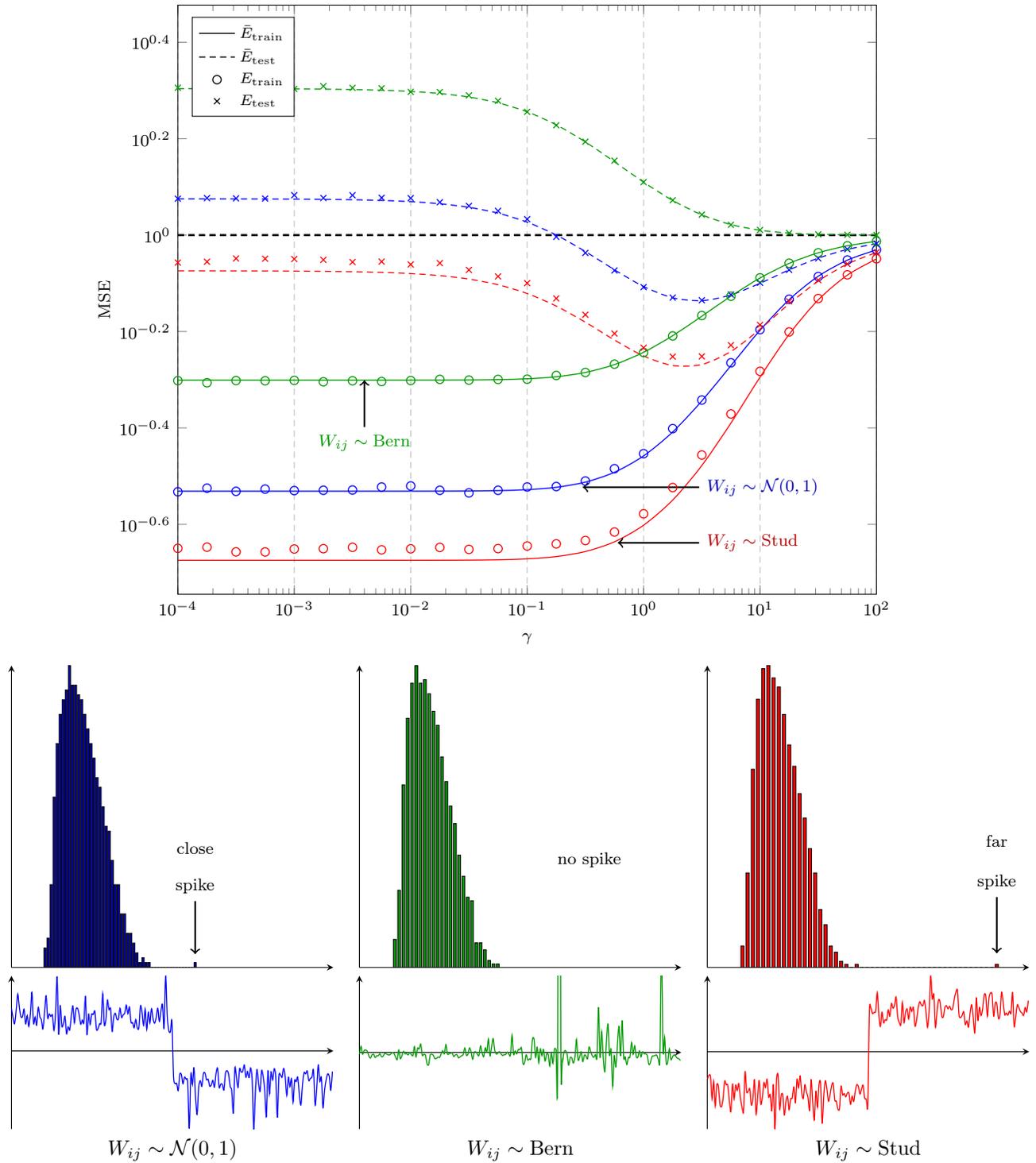


Figure H.3: (Top) Neural network performance for $\sigma(t) = -\frac{1}{2}t^2 + 1$, with different W_{ij} , for a 2-class Gaussian mixture model (see details in text), $n = 512$, $T = \hat{n} = 1024$, $p = 256$. (Bottom) Spectra and second eigenvector of Σ for different W_{ij} (first eigenvalues are of order n and not shown; associated eigenvectors are provably non informative).

\bar{E}_{train} (at least for all large n) and thus $\frac{1}{n} \|YV_r\|_F^2$ corresponds to the lowest achievable asymptotic training error.

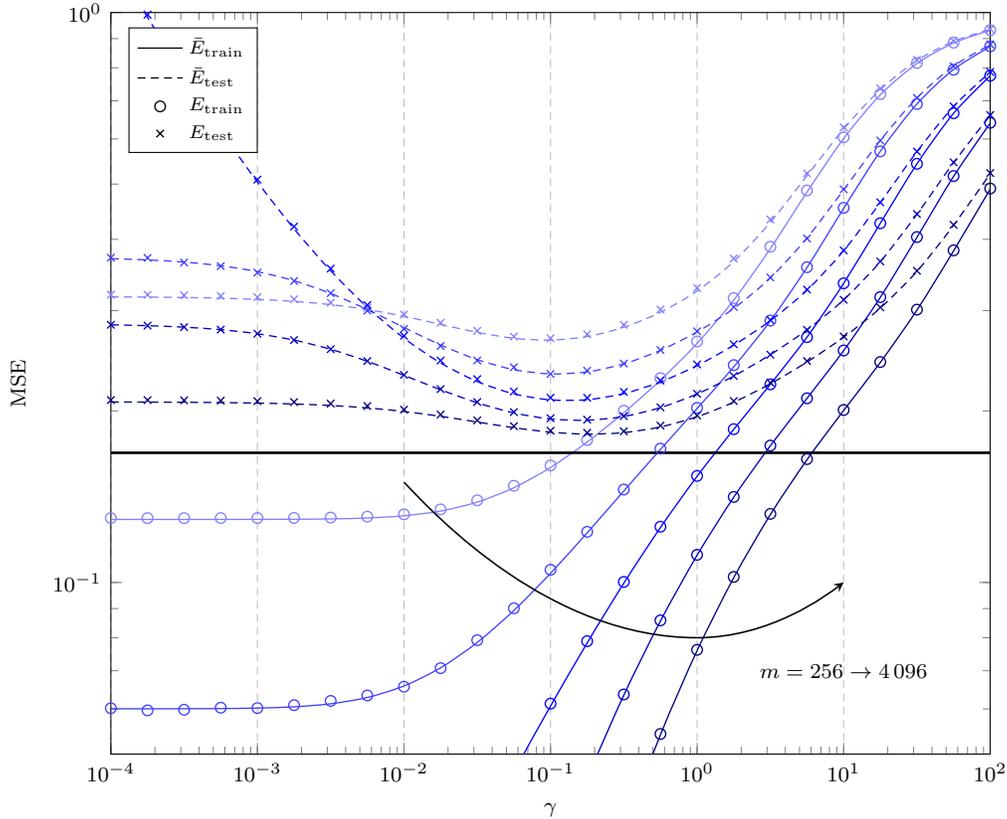


Figure H.4: Neural network performance for growing m (256, 512, 1024, 2048, 4096) as a function of γ , $\sigma(t) = \max(t, 0)$; 2-class MNIST data (sevens, nines), $n = \hat{n} = 1024$, $p = 784$. Limiting ($m = \infty$) \bar{E}_{test} shown in thick black line.

If instead $r > m$ (which is the most likely outcome in practice), as $\gamma \rightarrow 0$, $\tilde{Q} \sim \frac{1}{\gamma} \left(\frac{m}{n} \frac{\Sigma}{\Delta} + I_m \right)^{-1}$ and thus

$$\bar{E}_{\text{train}} \xrightarrow{\gamma \rightarrow 0} \frac{1}{n} \text{Tr} Y Q_{\Delta} \left[\frac{\frac{1}{n} \text{Tr} \tilde{\Sigma}_{\Delta} Q_{\Delta}^2}{1 - \frac{1}{n} \text{Tr} (\tilde{\Sigma}_{\Delta} Q_{\Delta})^2} \tilde{\Sigma}_{\Delta} + I_m \right] Q_{\Delta} Y^T$$

where $\tilde{\Sigma}_{\Delta} = \frac{m}{n} \frac{\Sigma}{\Delta}$ and $Q_{\Delta} = \left(\frac{m}{n} \frac{\Sigma}{\Delta} + I_m \right)^{-1}$.

These results suggest that neural networks should be designed both in a way that reduces the rank of Σ while maintaining a strong alignment between the dominant eigenvectors of Σ and the output matrix Y .

Interestingly, if X is assumed as above to be extracted from a Gaussian mixture and that $Y \in \mathcal{M}_{1,T}$ is a classification vector with $Y_{1j} \in \{-1, 1\}$, then the tools proposed in [CB16] (related to spike random matrix analysis) allow for an explicit evaluation of the aforementioned limits as n, m, p grow large. This analysis is however cumbersome and outside of the scope of the present work.

3.4 Derivation of Σ_{ab}

Gaussian w

In this section, we evaluate the terms Σ_{ab} provided in Table H.1. The proof for the term corresponding to $\sigma(t) = \text{erf}(t)$ can already be found in [Wil98, Section 3.1] and is not recalled here. For the other functions $\sigma(\cdot)$, we follow a similar approach as in [Wil98], as detailed next.

The evaluation of Σ_{ab} for $w \sim \mathcal{N}(0, I_p)$ requires to estimate

$$\mathcal{I} \equiv (2\pi)^{-\frac{p}{2}} \int_{\mathbb{R}^p} \sigma(w^T a) \sigma(w^T b) e^{-\frac{1}{2}\|w\|^2} dw.$$

Assume that a and b are not linearly dependent. It is convenient to observe that this integral can be reduced to a two-dimensional integration by considering the basis e_1, \dots, e_p defined (for instance) by

$$e_1 = \frac{a}{\|a\|}, \quad e_2 = \frac{\frac{b}{\|b\|} - \frac{a^T b}{\|a\|\|b\|} \frac{a}{\|a\|}}{\sqrt{1 - \frac{(a^T b)^2}{\|a\|^2\|b\|^2}}}$$

and e_3, \dots, e_p any completion of the basis. By letting $w = \tilde{w}_1 e_1 + \dots + \tilde{w}_p e_p$ and $a = \tilde{a}_1 e_1$ ($\tilde{a}_1 = \|a\|$), $b = \tilde{b}_1 e_1 + \tilde{b}_2 e_2$ (where $\tilde{b}_1 = \frac{a^T b}{\|a\|}$ and $\tilde{b}_2 = \|b\| \sqrt{1 - \frac{(a^T b)^2}{\|a\|^2\|b\|^2}}$), this reduces \mathcal{I} to

$$\mathcal{I} = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(\tilde{w}_1 \tilde{a}_1) \sigma(\tilde{w}_1 \tilde{b}_1 + \tilde{w}_2 \tilde{b}_2) e^{-\frac{1}{2}(\tilde{w}_1^2 + \tilde{w}_2^2)} d\tilde{w}_1 d\tilde{w}_2.$$

Letting $\tilde{w} = [\tilde{w}_1, \tilde{w}_2]^T$, $\tilde{a} = [\tilde{a}_1, 0]^T$ and $\tilde{b} = [\tilde{b}_1, \tilde{b}_2]^T$, this is conveniently written as the two-dimensional integral

$$\mathcal{I} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sigma(\tilde{w}^T \tilde{a}) \sigma(\tilde{w}^T \tilde{b}) e^{-\frac{1}{2}\|\tilde{w}\|^2} d\tilde{w}.$$

The case where a and b would be linearly dependent can then be obtained by continuity arguments.

The function $\sigma(t) = \max(t, 0)$. For this function, we have

$$\mathcal{I} = \frac{1}{2\pi} \int_{\min(\tilde{w}^T \tilde{a}, \tilde{w}^T \tilde{b}) \geq 0} \tilde{w}^T \tilde{a} \cdot \tilde{w}^T \tilde{b} \cdot e^{-\frac{1}{2}\|\tilde{w}\|^2} d\tilde{w}.$$

Since $\tilde{a} = \tilde{a}_1 e_1$, a simple geometric representation lets us observe that

$$\left\{ \tilde{w} \mid \min(\tilde{w}^T \tilde{a}, \tilde{w}^T \tilde{b}) \geq 0 \right\} = \left\{ r \cos(\theta) e_1 + r \sin(\theta) e_2 \mid r \geq 0, \theta \in [\theta_0 - \frac{\pi}{2}, \frac{\pi}{2}] \right\}$$

where we defined $\theta_0 \equiv \arccos\left(\frac{\tilde{b}_1}{\|\tilde{b}\|}\right) = -\arcsin\left(\frac{\tilde{b}_1}{\|\tilde{b}\|}\right) + \frac{\pi}{2}$. We may thus operate a polar coordinate change of variable (with inverse Jacobian determinant equal to r) to obtain

$$\begin{aligned} \mathcal{I} &= \frac{1}{2\pi} \int_{\theta_0 - \frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^+} (r \cos(\theta) \tilde{a}_1) (r \cos(\theta) \tilde{b}_1 + r \sin(\theta) \tilde{b}_2) r e^{-\frac{1}{2}r^2} d\theta dr \\ &= \tilde{a}_1 \frac{1}{2\pi} \int_{\theta_0 - \frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) (\cos(\theta) \tilde{b}_1 + \sin(\theta) \tilde{b}_2) d\theta \int_{\mathbb{R}^+} r^3 e^{-\frac{1}{2}r^2} dr. \end{aligned}$$

With two integration by parts, we have that $\int_{\mathbb{R}^+} r^3 e^{-\frac{1}{2}r^2} dr = 2$. Classical trigonometric formulas also provide

$$\begin{aligned} \int_{\theta_0 - \frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)^2 d\theta &= \frac{1}{2} (\pi - \theta_0) + \frac{1}{2} \sin(2\theta_0) \\ &= \frac{1}{2} \left(\pi - \arccos\left(\frac{\tilde{b}_1}{\|\tilde{b}\|}\right) + \frac{\tilde{b}_1}{\|\tilde{b}\|} \frac{\tilde{b}_2}{\|\tilde{b}\|} \right) \\ \int_{\theta_0 - \frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) \sin(\theta) d\theta &= \frac{1}{2} \sin^2(\theta_0) = \frac{1}{2} \left(\frac{\tilde{b}_2}{\|\tilde{b}\|} \right)^2 \end{aligned}$$

where we used in particular $\sin(2 \arccos(x)) = 2x\sqrt{1-x^2}$. Altogether, this is after simplification and replacement of \tilde{a}_1, \tilde{b}_1 and \tilde{b}_2 ,

$$\mathcal{I} = \frac{1}{2\pi} \|a\| \|b\| \left(\sqrt{1 - \angle(a, b)^2} + \angle(a, b) \arccos(-\angle(a, b)) \right).$$

It is worth noticing that this may be more compactly written as

$$\mathcal{I} = \frac{1}{2\pi} \|a\| \|b\| \int_{-1}^{\angle(a, b)} \arccos(-x) dx.$$

which is minimum for $\angle(a, b) \rightarrow -1$ (since $\arccos(-x) \geq 0$ on $[-1, 1]$) and takes there the limiting value zero. Hence $\mathcal{I} > 0$ for a and b not linearly dependent.

For a and b linearly dependent, we simply have $\mathcal{I} = 0$ for $\angle(a, b) = -1$ and $\mathcal{I} = \frac{1}{2} \|a\| \|b\|$ for $\angle(a, b) = 1$.

The function $\sigma(t) = |t|$ Since $|t| = \max(t, 0) + \max(-t, 0)$, we have

$$\begin{aligned} |w^T a| \cdot |w^T b| &= \max(w^T a, 0) \max(w^T b, 0) + \max(w^T(-a), 0) \max(w^T(-b), 0) \\ &\quad + \max(w^T(-a), 0) \max(w^T b, 0) + \max(w^T a, 0) \max(w^T(-b), 0). \end{aligned}$$

Hence, reusing the results above, we have here

$$\mathcal{I} = \frac{\|a\| \|b\|}{2\pi} \left(4\sqrt{1 - \angle(a, b)^2} + 2\angle(a, b) \arccos(-\angle(a, b)) - 2\angle(a, b) \arccos(\angle(a, b)) \right).$$

Using the identity $\arccos(-x) - \arccos(x) = 2 \arcsin(x)$ provides the expected result.

The function $\sigma(t) = 1_{t \geq 0}$. With the same notations as in the case $\sigma(t) = \max(t, 0)$, we have to evaluate

$$\mathcal{I} = \frac{1}{2\pi} \int_{\min(\tilde{w}^T \tilde{a}, \tilde{w}^T \tilde{b}) \geq 0} e^{-\frac{1}{2} \|\tilde{w}\|^2} d\tilde{w}.$$

After a polar coordinate change of variable, this is

$$\mathcal{I} = \frac{1}{2\pi} \int_{\theta_0 - \frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{\mathbb{R}^+} r e^{-\frac{1}{2} r^2} dr = \frac{1}{2} - \frac{\theta_0}{2\pi}$$

from which the result unfolds.

The function $\sigma(t) = \text{sign}(t)$. Here it suffices to note that $\text{sign}(t) = 1_{t \geq 0} - 1_{-t \geq 0}$ so that

$$\begin{aligned} \sigma(w^T a) \sigma(w^T b) &= 1_{w^T a \geq 0} 1_{w^T b \geq 0} + 1_{w^T(-a) \geq 0} 1_{w^T(-b) \geq 0} \\ &\quad - 1_{w^T(-a) \geq 0} 1_{w^T b \geq 0} - 1_{w^T a \geq 0} 1_{w^T(-b) \geq 0} \end{aligned}$$

and to apply the result of the previous section, with either (a, b) , $(-a, b)$, $(a, -b)$ or $(-a, -b)$. Since $\arccos(-x) = -\arccos(x) + \pi$, we conclude that

$$\mathcal{I} = (2\pi)^{-\frac{p}{2}} \int_{\mathbb{R}^p} \text{sign}(w^T a) \text{sign}(w^T b) e^{-\frac{1}{2} \|w\|^2} dw = 1 - \frac{2\theta_0}{\pi}.$$

The functions $\sigma(t) = \cos(t)$ and $\sigma(t) = \sin(t)$. Let us first consider $\sigma(t) = \cos(t)$. We have here to evaluate

$$\begin{aligned} \mathcal{I} &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \cos(\tilde{w}^T \tilde{a}) \cos(\tilde{w}^T \tilde{b}) e^{-\frac{1}{2} \|\tilde{w}\|^2} d\tilde{w} \\ &= \frac{1}{8\pi} \int_{\mathbb{R}^2} \left(e^{i\tilde{w}^T \tilde{a}} + e^{-i\tilde{w}^T \tilde{a}} \right) \left(e^{i\tilde{w}^T \tilde{b}} + e^{-i\tilde{w}^T \tilde{b}} \right) e^{-\frac{1}{2} \|\tilde{w}\|^2} d\tilde{w} \end{aligned}$$

which boils down to evaluating, for $d \in \{\tilde{a} + \tilde{b}, \tilde{a} - \tilde{b}, -\tilde{a} + \tilde{b}, -\tilde{a} - \tilde{b}\}$, the integral

$$e^{-\frac{1}{2}\|d\|^2} \int_{\mathbb{R}^2} e^{-\frac{1}{2}\|\tilde{w}-\imath d\|^2} d\tilde{w} = (2\pi)e^{-\frac{1}{2}\|d\|^2}.$$

Altogether, we find

$$\mathcal{I} = \frac{1}{2} \left(e^{-\frac{1}{2}\|a+b\|^2} + e^{-\frac{1}{2}\|a-b\|^2} \right) = e^{-\frac{1}{2}(\|a\|+\|b\|)^2} \cosh(a^T b).$$

For $\sigma(t) = \sin(t)$, it suffices to appropriately adapt the signs in the expression of \mathcal{I} (using the relation $\sin(t) = \frac{1}{2i}(e^t + e^{-t})$) to obtain in the end

$$\mathcal{I} = \frac{1}{2} \left(e^{-\frac{1}{2}\|a+b\|^2} + e^{-\frac{1}{2}\|a-b\|^2} \right) = e^{-\frac{1}{2}(\|a\|+\|b\|)^2} \sinh(a^T b)$$

as desired.

3.5 Polynomial $\sigma(\cdot)$ and generic w

In this section, we prove Equation H.7 for $\sigma(t) = \zeta_2 t^2 + \zeta_1 t + \zeta_0$ and $w \in \mathbb{R}^p$ a random vector with independent and identically distributed entries of zero mean and moment of order k equal to m_k . The result is based on standard combinatorics. We are to evaluate

$$\Sigma_{ab} = \mathbb{E} \left[(\zeta_2(w^T a)^2 + \zeta_1 w^T a + \zeta_0) (\zeta_2(w^T b)^2 + \zeta_1 w^T b + \zeta_0) \right].$$

After development, it appears that one needs only assess, for say vectors $c, d \in \mathbb{R}^p$ that take values in $\{a, b\}$, the moments

$$\begin{aligned} \mathbb{E}[(w^T c)^2 (w^T d)^2] &= \sum_{i_1 i_2 j_1 j_2} c_{i_1} c_{i_2} d_{j_1} d_{j_2} \mathbb{E}[w_{i_1} w_{i_2} w_{j_1} w_{j_2}] \\ &= \sum_{i_1} m_4 c_{i_1}^2 d_{i_1}^2 + \sum_{i_1 \neq j_1} m_2^2 c_{i_1}^2 d_{j_1}^2 + 2 \sum_{i_1 \neq i_2} m_2^2 c_{i_1} d_{i_1} c_{i_2} d_{i_2} \\ &= \sum_{i_1} m_4 c_{i_1}^2 d_{i_1}^2 + \left(\sum_{i_1 j_1} - \sum_{i_1 = j_1} \right) m_2^2 c_{i_1}^2 d_{j_1}^2 \\ &\quad + 2 \left(\sum_{i_1 i_2} - \sum_{i_1 \neq i_2} \right) m_2^2 c_{i_1} d_{i_1} c_{i_2} d_{i_2} \\ &= m_4 (c^2)^T (d^2) + m_2^2 (\|c\|^2 \|d\|^2 - (c^2)^T (d^2)) \\ &\quad + 2m_2^2 ((c^T d)^2 - (c^2)^T (d^2)) \\ &= (m_4 - 3m_2^2) (c^2)^T (d^2) + m_2^2 (\|c\|^2 \|d\|^2 + 2(c^T d)^2) \\ \mathbb{E}[(w^T c)^2 (w^T d)] &= \sum_{i_1 i_2 j} c_{i_1} c_{i_2} d_j \mathbb{E}[w_{i_1} w_{i_2} w_j] = \sum_{i_1} m_3 c_{i_1}^2 d_{i_1} = m_3 (c^2)^T d \\ \mathbb{E}[(w^T c)^2] &= \sum_{i_1 i_2} c_{i_1} c_{i_2} \mathbb{E}[w_{i_1} w_{i_2}] = m_2 \|c\|^2 \end{aligned}$$

where we recall the definition $(a^2) = [a_1^2, \dots, a_p^2]^T$. Gathering all the terms for appropriate selections of c, d leads to (H.7).

Chapter I

Robust Estimation

Robust estimators of covariance (or scatter) are necessary ersatz for the classical sample covariance matrix when the dataset $X = (x_1, \dots, x_n)$ present some diverging statistical properties, such as unbounded second moments of the x_i 's. We study here the M-estimator of scatter $\hat{\Sigma}$ initially introduced in [Hub64] defined as the solution (if it exists) to the following fixed point equation:

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n u \left(\frac{1}{n} x_i^T (\hat{\Sigma} + \gamma I_p)^{-1} x_i \right) x_i x_i^T, \quad (\text{I.1})$$

where $\gamma > 0$ is a regularization parameter and $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a mapping that tends to zero at $+\infty$, and whose object is to control outlying data. The literature in this domain has so far divided the study of $\hat{\Sigma}$ into (i) a first exploration of conditions for its existence and uniqueness as a *deterministic* solution to (I.1) (e.g., [Hub64, Mar76, Tyl87]) and (ii) an independent analysis of its statistical properties when seen as a random object (in the large n regime [CP08] or in the large n, p regime [CM14, ZCS14]).

In this chapter, we claim that the study of the conditions of existence (i) and statistical behavior (ii) of $\hat{\Sigma}$ can be conveniently carried out jointly. Specifically, by means of a flexible framework based on concentration of measure theory and semi-metric d_s stability argument, we simultaneously explore the existence and large dimensional (n, p large) spectral properties of $\hat{\Sigma}$. Our findings may be summarized as the following three main contributions to robust statistics and more generally to large dimensional statistics.

First, the proposed concentration of measure framework has the advantage of relaxing the assumptions of independence in the entries of x_i made in previous works [CM14, ZCS14], thereby allowing for possibly complex and quite realistic data models. In detail, our data model decomposes x_i as $x_i = \sqrt{\tau_i} z_i + m$ where the z_1, \dots, z_n are independent random vectors satisfying a concentration of measure hypothesis (in particular, the z_i 's could arise from a very generic generative model, .e.g, $z_i = h(\tilde{z}_i)$ for $\tilde{z}_i \sim \mathcal{N}(0, I_q)$ and $h : \mathbb{R}^q \rightarrow \mathbb{R}^p$ a 1-Lipschitz mapping), m is a deterministic vector (a signal or information common to all data) and τ_i are arbitrary (possibly large) deterministic values.¹ This setting naturally arises in many engineering applications, such as in antenna array processing (radar, brain signal processing, etc.) where the τ_i 's model noise impulsiveness and m is an informative signal to be detected by the experimenter [OPP⁺11], or in statistical finance where the x_i 's model asset returns with high volatility and m is the market leading direction [YCM14].

Second, as compared to previous works in the field [Mar76, KT91, OT14, OPP20, CM14], our frameworks allows for the relaxation of some of the classically posed constraints on the mapping u made. Specifically, u is here only required to be 1-Lipschitz with respect to the “stable semi-metric”, which is equivalent to assuming that $t \mapsto tu(t)$ is non-decreasing and that $t \mapsto u(t)/t$ is non-increasing. The semi-metric naturally arises when studying the *resolvent* $(\hat{\Sigma} + \gamma I_p)^{-1}$ of $\hat{\Sigma}$, which is at the core of our large p, n analysis of $\hat{\Sigma}$, using modern tools from random matrix theory. To establish concentration properties in the large dimensional regime on $\hat{\Sigma}$, under our framework, the function u is nonetheless further requested to be such that $t \rightarrow tu(t)$ is strictly smaller than 1 ($\hat{\Sigma}$ is however still defined without this condition). Yet, and most importantly, u needs not be a non-increasing function, as demanded by most works in the field.

¹We may alternatively assume the τ_i random independent of $Z = (z_1, \dots, z_n)$.

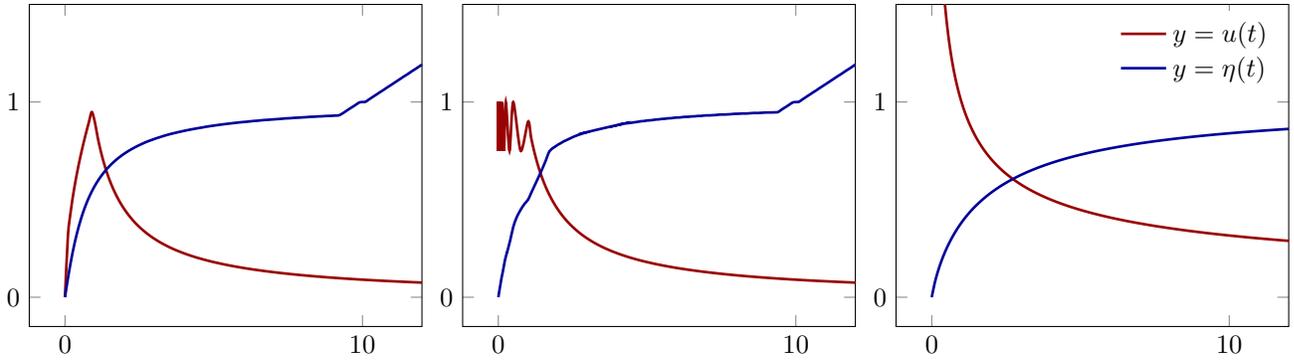


Figure I.1: Three stable mappings u and their associated η mappings. **(Left)** $u : t \mapsto \max(\sqrt{t}, \frac{1}{t})$, **(Center)** mapping introduced in Remark E.12 having no limit in 0, **(Right)** $u : t \mapsto \frac{1}{\sqrt{t}}$, and $\eta : t \mapsto \sqrt{\sqrt{\frac{x^2}{4} + 1} - \frac{x}{2}}$; this last choice does not satisfy our hypotheses because u is not bounded and $\lim_{t \rightarrow \infty} tu(t) = \infty \geq 1$ (then $t \mapsto \frac{\eta(t)}{t}$ is not bounded from below).

Third, the ‘‘Lipschitz and stable semi-metric’’ properties of the model are consistently articulated so as to propagate the concentration properties from Z to the robust scatter matrix $\hat{\Sigma}$. The core technical result allowing for this articulation is Proposition I.10. This combined framework provides the rate of convergence of the Stieltjes transform of the spectral distribution of $\hat{\Sigma}$ to its large n, p limit along with conditions guaranteeing the possibility to recover the signal m from the asymptotic statistical properties of $\hat{\Sigma}$.

I.1 Main Results

We place ourselves under the random matrix regime where p , the size of data $x_1, \dots, x_n \in \mathbb{R}^p$ is of the same order as n , the number of data – for practical use, imagine that $10^{-2} \leq \frac{p}{n} \leq 10^2$. The convergence results will be expressed as functions of the quasi asymptotic quantities p and n that are thought of as tending to infinity (in practice our results are extremely accurate already for $p, n \geq 100$). We will then work with the notations $a_{n,p} \leq O(b_{n,p})$ or $a_{n,p} \geq O(b_{n,p})$ to signify that there exists a constant K independent of p and n such that $a_{n,p} \leq Kb_{n,p}$ or $a_{n,p} \geq Kb_{n,p}$, respectively, and to simplify the notation, most of the time, the indices n, p will be omitted. In particular we have $O(n) \leq p \leq O(n)$. Our hypotheses concern four central objects:

- $Z = (z_1, \dots, z_n) \in \mathcal{M}_{p,n}$ satisfies the concentration of measure phenomenon; all the random vectors z_1, \dots, z_n are independent and $\sup_{1 \leq i \leq n} \|\mathbb{E}[z_i]\| \leq O(1)$;
- $\tau = \text{Diag}(\tau_1, \dots, \tau_n) \in \mathcal{D}_n^+$ satisfy $\forall i \in [n], \tau_i > 0$ and $\frac{1}{n} \sum_{i=1}^n \tau_i \leq O(1)$;
- $m \in \mathbb{R}^p$ and $\|m\| \leq O(1)$;
- $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is bounded, $t \mapsto tu(t)$ is non-decreasing, $t \mapsto \frac{u(t)}{t}$ is non-increasing and $\forall t > 0: tu(t) < 1$.

Those conditions are sufficient to retrieve part of the statistical properties of Z and of the signal m from the data matrix

$$X = Z\sqrt{\tau} + m\mathbf{1}^T$$

through the robust scatter matrix $\hat{\Sigma}$ defined in Equation (I.1). The standard sample covariance matrix $\frac{1}{n}XX^T$ instead inefficiently estimates some of these statistics due to the presence of possibly large (outlying) τ_i 's (although $\frac{1}{n} \sum_{i=1}^n \tau_i \leq O(1)$, it is allowed for some τ_i 's to be of order $\tau_i \geq O(n)$). The robust scatter matrix controls this outlying behavior by mitigating the impact of the high energy data x_i with the tapering action of the mapping u induced by the hypothesis $tu(t) < 1$ (see Figure I.2).

Introducing the diagonal matrix $\hat{\Delta}$ solution to the fixed point equation:

$$\hat{\Delta} = \frac{1}{n} x_i^T \left(\frac{1}{n} X^T u(\hat{\Delta}) X + \gamma I_p \right)^{-1} x_i,$$

(with $u(\cdot)$ operating entry-wise on the diagonal elements of $\hat{\Delta}$) the robust scatter matrix is simply $\hat{\Sigma} = \frac{1}{n} X u(\hat{\Delta}) X^T$, and the tapering action is revealed by low values of $u(\hat{\Delta})_{ii}$ when τ_i is large. As shown on the central display of Figure I.2, compared to $\frac{1}{n} X X^T$, $\hat{\Sigma} = \frac{1}{n} X u(\hat{\Delta}) X^T$ has a cleaner spectral behavior which lets appear the signal induced by m as an isolated eigenvalue-eigenvector pair. This eigenvector can then be exploited to estimate m (this is a classical random matrix inference problem that will not be studied here).

This chapter precisely shows that the spectral distribution of $\hat{\Sigma}$ is asymptotical equivalent to the spectral distribution of $\frac{1}{n} Z^T U Z$ where U is a deterministic diagonal matrix satisfying $\|U\| \leq O(1)$. Interestingly, the definition of U merely depends on the second order moments of z_1, \dots, z_n which we denote, $\forall i \in [n]$, $\Sigma_i \equiv \mathbb{E}[z_i z_i^T]$, on the vector $\tau \in \mathbb{R}^n$ of the τ_i 's, on the function u , but not on the signal m . The definition of U relies on the introduction of a function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ derived from u and defined as the solution to

$$\forall t \in \mathbb{R}_+ : \eta(t) = \frac{t}{1 + tu(\eta(t))}$$

and on the diagonal matrix $\Lambda_z : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$. For any $z \in \mathbb{R}_+$ and $\Delta \in \mathcal{D}_n^+$, $\Lambda_z(\Delta)$ is defined as the unique solution to the n equations:

$$\forall i \in [n], \Lambda_z(\Delta)_i = \frac{1}{n} \text{Tr} \left(\Sigma_i \left(\frac{1}{n} \sum_{j=1}^n \frac{\Sigma_j \Delta_j}{1 + \Delta_j \Lambda_z(\Delta)_j} + z I_p \right)^{-1} \right).$$

Introducing the resolvent $R : z \mapsto (z I_p + \hat{\Sigma})^{-1}$, we have the concentration:

Theorem I.1. *For any $z \geq O(1)$, and any deterministic matrix $A \in \mathcal{M}_p$ such that $\|A\|_* \leq O(1)$ there exist two constants $C, c > 0$ ($C, c \sim O(1)$) such that, for any $0 < \varepsilon \leq 1$,*

$$\mathbb{P} \left(\left| \text{Tr}(AR(z)) - \frac{1}{p} \text{Tr} \left(A \left(\frac{1}{n} \sum_{i=1}^n \frac{U_i \Sigma_i}{1 + \Lambda_z(U)_i U_i} + z I_p \right)^{-1} \right) \right| \geq \varepsilon \right) \leq C e^{-c n \varepsilon^2}$$

where $U = \text{diag}(U_1, \dots, U_n) \in \mathcal{D}_n^+$ satisfies $U \leq O(1)$ and is the unique solution to the equation:

$$U = \tau \cdot u \circ \eta(\tau \Lambda_\gamma(U)) \tag{I.2}$$

(the mappings u and η are applied entry-wise on the diagonal terms on \mathcal{D}_n^+).

The proof of the existence and uniqueness of U is based on contractiveness arguments for the semi-metric d_s introduced in Part II (see Theorem E.10), therefore, in practice, a precise estimate of U is merely obtained from successive iteration of (I.2).

Employing this theorem with $A = \frac{1}{p} I_p$ (then $\|A\|_* = 1$), classical random matrix theory inferences allow us to estimate the spectral distribution of the robust scatter matrix from the estimation of its Stieltjes transform $m(z) = \frac{1}{p} \text{Tr}(R(-z))$. This is confirmed in Figure I.2 which depicts the eigenvalue distribution of the sample covariance of the data matrix X : (i) deprived of the influence of τ (i.e., for $\tau = I_n$), (ii) corrected with the robust scatter matrix (i.e., it is here the sample covariance matrix of the equivalent data $X u(\hat{\Delta})^{1/2}$), and (iii) without any modification on X . For the two first spectral distributions, we displayed their estimation with the Stieltjes transform as per Theorem I.1.

We additionally can provide guarantees on the alignment of the eigen vector v_{\max} associated to the highest eigen value with the signal m , as expressed on Figure I.2. Indeed if we take $A = m m^T$ (then $\|A\|_* = \|m\|^2 \leq O(1)$), for any path γ containing the highest eigen value of $\hat{\Sigma}$ but no other value from the bulk we have the identity:

$$\frac{1}{2i\pi} \oint_{\gamma} m^T R(-z) m dz = (v_{\max}^T m)^2.$$

Estimating $m^T R(z) m$ therefore leads to estimating the projection $(v_{\max}^T m)^2$.

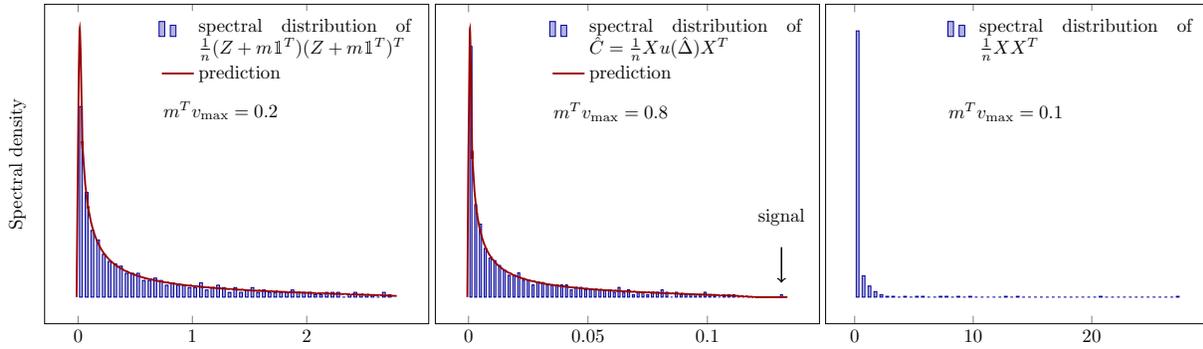


Figure I.2: Spectral distributions of the matrices $\frac{1}{n}(Z + m\mathbb{1}^T)(Z + m\mathbb{1}^T)^T$, $\hat{\Sigma}$ and $\frac{1}{n}XX^T$ against their large dimensional prediction; $p = 500$, $n = 400$ (null eigenvalues removed), $u : t \mapsto \min(t, \frac{1}{1+5t})$, the variables τ_1, \dots, τ_n are drawn independently from a Student distribution with 1 degree of freedom, $m = \mathbb{1}/\sqrt{p} \in \mathbb{R}^p$; $Z = \sin(W)$ for $W \sim \mathcal{N}(0, AA^T)$ where $A \in \mathcal{M}_p$ is a fixed matrix whose entries are drawn from the Gaussian distribution with zero mean and unit variance ($Z \propto \mathcal{E}_2$ by construction). The population covariance and mean of Z are computed with a set of p^2 independent realizations of Z . The values of the projections of the signal m against the eigenvector v_{\max} associated to the largest eigenvalue reveals that, with the robust scatter approach, the diverging action of τ in the model can be turned into an advantage to infer the signal m from the data. The choice of the mapping u is not optimized, our goal here is just to show that non monotonic functions are suited to robust statistics as long as they satisfy our assumptions.

I.2 Preliminaries for the study of the resolvent

Let \mathcal{S}_p be the set of symmetric matrices of size p and \mathcal{S}_p^+ the set of symmetric nonnegative matrices. Given $S, T \in \mathcal{S}_p$, we denote $S \leq T$ iff $T - S \in \mathcal{S}_p^+$. We will extensively work with the set $(\mathcal{S}_p^+)^n$ which will be denoted for simplicity \mathcal{S}_p^n . Given $S \in \mathcal{S}_p^n$, we finally let $S_1, \dots, S_n \in \mathcal{S}_p^+$ be its n components.

Given two sequences of scalars $a_{n,p}, b_{n,p}$, the notation $a_{n,p} \sim O(b_{n,p})$ means that $a_{n,p} \leq O(b_{n,p})$ and $a_{n,p} \geq O(b_{n,p})$. We extend those characterizations to diagonal matrices: given $\Delta \in \mathcal{D}_n^+$, $\Delta \leq O(1)$ indicates that $\|\Delta\| \leq O(1)$ while $\Delta \geq O(1)$ means that $\|\frac{1}{\Delta}\| \leq O(1)$ and $\Delta \sim O(1)$ means that $O(1) \leq \Delta \leq O(1)$.

The different assumptions leading to the main results are presented progressively so that the reader easily understands their importance and direct implications. A full recollection of all these assumptions is provided at the beginning of our manuscript.

2.1 The resolvent behind robust statistics and its contracting properties

Given $\gamma > 0$ and $S \in \mathcal{S}_p^n$, we introduce the resolvent function at the core of our study :

$$\begin{aligned} \mathcal{R}_\gamma : \mathcal{S}_p^n \times \mathcal{D}_n^+ &\longrightarrow \mathcal{M}_p \\ (S, \Delta) &\longmapsto \left(\frac{1}{n} \sum_{i=1}^n \Delta_i S_i + \gamma I_p \right)^{-1}. \end{aligned}$$

Given a dataset $X = (x_1, \dots, x_n) \in \mathcal{M}_{p,n}$, if we note $X \cdot X^T = (x_i x_i^T)_{1 \leq i \leq n} \in \mathcal{S}_p^n$, the robust estimation of the scatter matrix then reads (if well defined):

$$\hat{\Sigma} = \frac{1}{n} X u(\hat{\Delta}) X^T \quad \text{with} \quad \hat{\Delta} = \text{Diag} \left(\frac{1}{n} x_i^T \mathcal{R}_\gamma(X \cdot X^T, u(\hat{\Delta})) x_i \right)_{1 \leq i \leq n}. \quad (\text{I.3})$$

In the following, we will denote for simplicity $\mathcal{R}_\gamma^X \equiv \mathcal{R}_\gamma(X \cdot X^T, u(\hat{\Delta}))$. To understand the behavior (structural, spectral, statistical) of $\hat{\Sigma}$, one needs first to understand the behavior of the resolvent $\mathcal{R}_\gamma(S, \Delta)$ for general $S \in \mathcal{S}_p^n$ and $\Delta \in \mathcal{D}_n^+$. We document in this subsection its contracting properties.

As the scalar γ will rarely change in the remainder, it will be sometimes omitted for readability.

Lemma I.2. Given $\gamma > 0$, $S \in \mathcal{S}_p^n$, $M \in \mathcal{M}_{p,n}$ and $\Delta \in \mathcal{D}_n^+$:

$$\|R_\gamma(S, \Delta)\| \leq \frac{1}{\gamma}; \quad \left\| \frac{1}{\sqrt{n}} R_\gamma(M \cdot M^T, \Delta) M \Delta^{\frac{1}{2}} \right\| \leq \frac{1}{\sqrt{\gamma}}; \quad \left\| \frac{1}{n} R_\gamma(S, \Delta) \sum_{l=1}^k \Delta_l S_l \right\| \leq 1.$$

Proof. We can bound in the space of symmetric matrices:

$$\frac{1}{n} \sum_{i=1}^n \Delta_i S_i + \gamma I_p \geq \gamma I_n,$$

thus $R_\gamma(S, \Delta) \leq \frac{1}{\gamma} I_n$. Besides, we can write:

$$R_\gamma(S, \Delta) \frac{1}{n} \sum_{i=1}^n \Delta_i S_i = I_p - \gamma R_\gamma(S, \Delta) \leq I_p.$$

Noting $S_i = m_i m_i^T$ and $S = (S_1, \dots, S_n) = M \cdot M^T$, we can then deduce that:

$$\frac{1}{\sqrt{n}} R_\gamma(M \cdot M^T, \Delta) M \Delta^{\frac{1}{2}} = \left(R_\gamma(S, \Delta) \frac{1}{n} \sum_{i=1}^n \Delta_i S_i R_\gamma(S, \Delta) \right)^{1/2},$$

which provides the second bound. \square

Given $M \in \mathcal{M}_{p,n}$, and $S \in \mathcal{S}_p^n$, further define the mapping $I_\gamma : \mathcal{S}_p^n \times \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$,

$$I(S, \Delta) = \text{Diag} \left(\frac{1}{n} \text{Tr}(S_i R_\gamma(S, \Delta)) \right)_{1 \leq i \leq n}.$$

With the notation $I_\gamma^X(\Delta) \equiv I(X \cdot X^T, \Delta)$, the fixed point $\hat{\Delta}$ defined in (I.3) is simply $\hat{\Delta} = I_\gamma^X(u(\hat{\Delta}))$. To prove the existence and uniqueness of $\hat{\Delta}$ we exploit the Banach fixed-point theorem to find contracting properties on the mapping $\Delta \mapsto I_\gamma^X(u(\Delta))$ for which $\hat{\Delta}$ is a fixed point. As we see in the following lemma, the contractive character does not appear relatively to the spectral norm on \mathcal{D}_n^+ but relatively to the stable semi-metric presented in Part II, Chapter E.

Lemma I.3. Given $S \in \mathcal{S}_p^n$ and $\Delta, \Delta' \in \mathcal{D}_n^+$, we have (the index γ being omitted)

$$d_s(I(S, \Delta), I(S, \Delta')) < \max(\phi_\gamma^S(\Delta), \phi_\gamma^S(\Delta')) d_s(\Delta, \Delta'),$$

where $\phi_\gamma^S(\Delta) \equiv \|1 - \gamma R_\gamma(S, \Delta)\|$

Proof. Given $a \in \{1, \dots, k\}$, we can bound thanks to Cauchy Shwarz inequality:

$$\begin{aligned} |I(S, \Delta)_a - I(S, \Delta')_a| &= \left| \frac{1}{n} \text{Tr}(S_a (R_\gamma(S, \Delta') - R_\gamma(S, \Delta))) \right| \\ &= \left| \frac{1}{n} \sum_{b=1}^k \text{Tr}(S_a R_\gamma(S, \Delta') S_b (\Delta'_b - \Delta_b) R_\gamma(S, \Delta)) \right| \\ &\leq \frac{1}{n} \sqrt{\sum_{b=1}^k \text{Tr} \left(S_a R_\gamma(S, \Delta) \frac{S_b |\Delta'_b - \Delta_b|}{\sqrt{\Delta_b \Delta'_b}} \Delta_b R_\gamma(S, \Delta) \right)} \\ &\quad \cdot \sqrt{\sum_{b=1}^k \text{Tr} \left(S_a R_\gamma(S, \Delta') \frac{S_b |\Delta'_b - \Delta_b|}{\sqrt{\Delta_b \Delta'_b}} \Delta'_b R_\gamma(S, \Delta') \right)} \\ &\leq \left\| \frac{\Delta' - \Delta}{\sqrt{\Delta \Delta'}} \right\| \sqrt{\frac{1}{n} \text{Tr}(S_a R_\gamma(S, \Delta) (1 - \gamma R_\gamma(S, \Delta)))} \\ &\quad \cdot \sqrt{\frac{1}{n} \text{Tr}(S_a R_\gamma(S, \Delta') (1 - \gamma R_\gamma(S, \Delta')))} \\ &\leq \sqrt{\phi_\gamma^S(\Delta) \phi_\gamma^S(\Delta')} \left\| \frac{\Delta' - \Delta}{\sqrt{\Delta \Delta'}} \right\| \sqrt{I(S, \Delta)_a I(S, \Delta')_a} \end{aligned}$$

□

If one sees the term $\left\| \frac{\Delta - \Delta'}{\sqrt{\Delta \Delta'}} \right\|$ as a distance between Δ and Δ' , then Lemma I.3 sets the 1-Lipschitz character of $I(S, \cdot) : \Delta \mapsto I(S, \Delta)$, which is a fundamental property in what follows. We present in the next subsection a precise description of such functions that will be called *stable mappings*.

2.2 Fixed Point theorem applied to resolvent equations

To employ Theorem E.10 to the fixed point equation satisfied by $\hat{\Delta}$, a first step is to look at $\Delta \mapsto I_\gamma(S, \Delta) = \text{Diag}(\text{Tr}(S_i R(S, \Delta)))_{1 \leq i \leq n}$ that we know to be stable from Lemma I.3. The control on the Lipschitz parameter required by Theorem E.10 is issued from the following preliminary Lemma.

Lemma I.4. *Given $S \in \mathcal{S}_p^n$ and a function $f : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$ bounded by $f_0 \in \mathcal{D}_n^+$,*

$$\forall \Delta \in \mathcal{D}_n^+ : \frac{I_p}{f_0 \|S\| + \gamma} \leq R(S, f(\Delta)) \leq \frac{I_p}{\gamma} \quad \text{where} \quad \|S\| \equiv \frac{1}{n} \left\| \sum_{i=1}^n S_a \right\|.$$

Combined with Lemma I.3, this result allows us to build a family of contracting stable mappings with the composition $\tilde{I}(S, \cdot) \circ f$ when $f \in \mathcal{S}(\mathcal{D}_n^+)$ is bounded from above (recall that any f in the stable class $\mathcal{S}(\mathcal{D}_n^+)$ defined in Definition 22 satisfies $\forall \Delta, \Delta' \in \mathcal{D}_n^+, d_s(f(\Delta), f(\Delta')) \leq d_s(\Delta, \Delta')$). We thus obtain the following corollary to Theorem E.10.

Corollary I.5. *Given $f \in \mathcal{S}(\mathcal{D}_n^+)$, and a family of non-negative and non-zero symmetric matrices $S = (S_1, \dots, S_k) \in \mathcal{S}_p^n$, noting λ_f the Lipschitz parameters of f for the semi-metric d_s , if we assume that f is bounded from above or that $\lambda_f < 1$, then the fixed point equation*

$$\Delta = I^S(f(\Delta))$$

admits a unique solution in \mathcal{D}_n^+ .

Remark I.6. *To give a counterexample when f is nor contractive for the semi-metric, nor bounded from above, let us consider the mapping $f : t \mapsto \frac{1}{t}$ and the sequence of matrix $S = (I_p, \dots, I_p)$, then we have the equivalence:*

$$\Delta = I^S(f(\Delta)) \quad \iff \quad \forall i \in [n], \gamma \Delta_i + 1 = \frac{p}{n}$$

which cannot be satisfied for any $\Delta \in \mathcal{D}_n^+$ when $p < n$ (when $p \geq n$, the existence and uniqueness of the solution can be shown using the contractivity of $I^S \circ f$ for the spectral norm – but this is another problem).

Proof. We can first deduce from Lemma I.3 that:

$$d_s(I^S(f(\Delta)), I^S(f(\Delta'))) < \lambda_f \max(\phi^S(f(\Delta)), \phi^S(f(\Delta'))) d_s(\Delta, \Delta')$$

When $\lambda_f = 1$ but $\forall \Delta \in \mathcal{D}_n^+, f(\Delta) \leq f_0 < \infty$, we can still deduce the contractivity of $I^S \circ f$ thanks to Lemma I.4 that allows us to bound:

$$\max(\phi^S(f(\Delta)), \phi^S(f(\Delta'))) \leq \sup_{\Delta \in \mathcal{D}_n^+} \|1 - \gamma R^S(f(\Delta))\| \leq \frac{1}{1 + \frac{\gamma}{f_0 \|S\|}} < 1,$$

and conclude thanks to Theorem E.10. □

We will thus suppose from here on that u is a bounded² stable function, to be able to use Corollary I.5 and set the existence and uniqueness of $\hat{\Delta}$ and $\hat{\Sigma}$ as defined in (I.3).

Assumption 7. *$u \in \mathcal{S}(\mathbb{R}_+)$ and there exists $u^\infty > 0$ such that $\forall t \in \mathbb{R}_+, u(t) \leq u^\infty$.*

²We will see later that we need also to assume that $u_\times : t \mapsto tu(t)$ is bounded which implies that u behaves on the infinity as a mapping $t \mapsto \frac{\alpha}{t}$ which is not contractive for the semi-metric d_s . Still, to be able to apply Corollary I.5, we need to assume that u is bounded.

Proposition I.7. *For $X \in \mathcal{M}_{p,n}$, there exists a unique diagonal matrix $\hat{\Delta} \in \mathcal{D}_n^+$ such that*

$$\hat{\Delta} = I^X \left(u(\hat{\Delta}) \right).$$

Now that $\hat{\Delta}$, and therefore $\hat{\Sigma}$ are perfectly defined, let us introduce additional assumptions to be able to infer concentration properties on $\hat{\Delta}$.

2.3 Assumptions on Z and deterministic equivalent of the resolvent

Having proved the existence and uniqueness of $\hat{\Sigma}$, we now introduce statistical conditions on X to study $\hat{\Sigma}$ in the large dimensional $n, p \rightarrow \infty$ limit. We work under Assumptions 1 that imposes $p \leq O(n)$, then since X was some heavy tail behavior, one needs to make the concentration hypothesis on the n p -dimensional random vector $Z = (z_1, \dots, z_n) \in \mathcal{M}_{p,n}$. One therefore assume:

Assumption 1 ter. $Z \propto \mathcal{E}_2$.

Assumption 2 ter. *The random vectors z_1, \dots, z_n are all independent.*

We denote their means $\mu_i \equiv \mathbb{E}[z_i] \in \mathbb{R}^p$, their second order statistic matrix (or non-centered covariance matrix) $\Sigma_i \equiv \mathbb{E}[z_i z_i^T]$ and their covariance matrices $C_i = \Sigma_i - \mu_i \mu_i^T \in \mathcal{M}_p$. With this new notation, we still assume Assumptions 4 and 5 that impose upper bound on μ_1, \dots, μ_n and lower bound on $\Sigma_1, \dots, \Sigma_n$.

Given $\Delta \in \mathcal{D}_n^+$ we will extensively use the notation $R_\gamma^Z(\Delta) = R(Z \cdot Z^T, \Delta) = (\frac{1}{n} Z \Delta Z^T + \gamma I_p)^{-1}$. The study here is more simple than in Part II because we are just interested in the specific case where $z < 0$ and for that reason, we further look at $R \equiv R_z^Z(\Delta)$ for $z > 0$ and even $z \geq O(1)$ (for concentration issues).

We recall from Chapter F and more precisely Remark F.4 that if $|z| \geq O(1)$, then R is a $O(\frac{\|\Delta\|^{1/2}}{\sqrt{n}})$ -Lipschitz transformation of Z and, therefore, we can deduce that³:

$$R \propto \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right) \tag{I.4}$$

We express the deterministic equivalent thanks to the diagonal matrix $\Lambda^\Sigma(\Delta) \in \mathcal{D}_n^+$ defined, as the unique solution to:

$$\Lambda^\Sigma(\Delta) = I^\Sigma \left(\frac{\Delta}{I_n + \Delta \Lambda^\Sigma(\Delta)} \right);$$

its existence and uniqueness is proven thanks to Corollary I.5 (here $f : \Delta \rightarrow \frac{\Delta}{I_n + \Delta \Lambda^\Sigma(\Delta)}$ is bounded by $\|\Delta\|$). This equation allows us to compute $\Lambda^\Sigma(\Delta)$ iteratively via the standard fixed-point algorithm. The deterministic equivalent $\tilde{R}_z^\Sigma(\Delta)$ of $R_z^Z(\Delta)$ is then defined as follows:

$$\tilde{R}_z^\Sigma(\Delta) \equiv R_z \left(\Sigma, \frac{\Delta}{I_n + \Delta \Lambda^\Sigma} \right) = \left(\frac{1}{n} \sum_{i=1}^n \frac{\Delta_i \Sigma_i}{1 + \Delta_i \Lambda_i^\Sigma} + z I_p \right)^{-1}.$$

Then one can show thanks to Theorem F.25 that for any $\Delta \in \mathcal{D}_n^+$ and $A \in \mathcal{M}_p$ be deterministic matrices such that $\|\Delta\| \leq O(1)$ and $\|A\|_F \leq O(1)$:

$$\text{Tr}(A R_z^Z(\Delta)) \in \text{Tr} \left(A \tilde{R}_z^\Sigma(\Delta) \right) \pm \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right) \tag{I.5}$$

This theorem will later allow us to estimate the Stieltjes transform of the spectral distribution of $\hat{\Sigma}$. Note that here again, since $\frac{1}{n} Z \Delta Z^T + \gamma I_p \geq \gamma I_p$, the resolvent is always invertible and therefore one does not need to remove the low probability event \mathcal{A}_Q introduced in Chapter F (here the parameter z of Chapter F is always at a distance bigger than a constant to the spectrum). For this purpose, we need the next corollary to predict the asymptotic behavior of $\hat{\Delta}$ defined in (I.3). Recall that $I^X : \Delta \mapsto I(X \cdot X^T, \Delta)$. Then the following holds (it is a combination of Lemmas F.6, F.8 and F.23).

³Here, no need to restrain the study to \mathcal{A}_Q since z is far from the spectrum.

Corollary I.8. For all $\Delta \in \mathcal{D}_n^+$ with $\|\Delta\| \leq O(1)$,

$$I^Z(\Delta) \in \frac{\Lambda^\Sigma(\Delta)}{I_n + \Delta \Lambda^\Sigma(\Delta)} \pm \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right) \quad \text{in } (\mathcal{D}_n, \|\cdot\|_F)$$

Remark I.9. It is possible to extend the results of (I.5) and Corollary I.8 to the broader case where each mean μ_i ($i \in [n]$) can be decomposed as the sum of a particular component $\dot{\mu}_i$ of low energy (i.e. with a low norm) and a bigger component proportional to a general signal s of high energy as follows:

$$\mu_i = \dot{\mu}_i + t_i s, \tag{I.6}$$

where $t_1, \dots, t_n > 0$ are n scalars satisfying $\frac{1}{n} \sum_{i=1}^n t_i \geq 1$ and $\sup_{i \leq 1 \leq n} t_i \leq O(1)$. The concentration results are then almost the same with a speed $O(\sqrt{\log n/n})$ replacing $O(1/\sqrt{n})$ and the concentration in (I.5) is then only true for $A \in \mathcal{M}_p$ satisfying $\|A\|_* \leq O(1)$ (where $\|A\|_* = \text{Tr}((AA^T)^{1/2})$).

I.3 Estimation of the robust scatter matrix

3.1 Setting and strategy of the proof

Having set up the necessary tools and preliminary results, we now concentrate on our target objective. Let $x_i = \sqrt{\tau_i} z_i + m$, $1 \leq i \leq n$, where τ_i is a deterministic positive variable, $m \in \mathbb{R}^p$ is a deterministic vector, and z_1, \dots, z_n are the random vectors presented in the previous section. For $X = (x_1, \dots, x_n) \in \mathcal{M}_{p,n}$, we write $X = Z\tau^{\frac{1}{2}} + m\mathbb{1}^T$ where $\tau \equiv \text{Diag}(\tau_i)_{1 \leq i \leq n} \in \mathcal{D}_n^+$ and $\mathbb{1} \equiv (1, \dots, 1) \in \mathbb{R}^n$. The basic idea to estimate $\hat{\Delta}$, as a solution to the fixed-point equation $\hat{\Delta} = I^X(u(\hat{\Delta}))$, consists in retrieving a deterministic equivalent also solution to a (now deterministic) fixed-point equation. For this, we use the following central result concerning the stability of the semi-metric d_s to perturbations (it is quite similar to Proposition E.23 that was concerning solutions lying in $\mathcal{D}(\mathbb{H})$).

Proposition I.10. Let $f, f' : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$ be two mappings admitting a fixed point:

$$\Delta = f(\Delta) \quad \text{and} \quad \Delta' = f'(\Delta').$$

Further assume that $\Delta' \sim O(1)$ (i.e. that $\Delta' \geq O(1)$ and $\Delta' \leq O(1)$), that $\lambda \in (0, 1)$, the Lipschitz parameter of f for the semi-metric d_s satisfies $\lambda \geq O(1)$ and that:

$$\|f(\Delta') - f'(\Delta')\| \leq o(1).$$

Then, there exists a constant $K \leq O(1)$ such that

$$\|\Delta - \Delta'\| \leq K \|f(\Delta') - f'(\Delta')\|.$$

Proof. We want to apply Proposition E.23 with $f^s = f$, $\tilde{\Gamma}^s = \Delta$ and $\Gamma^s = \Delta' = f'(\Delta')$, we thus need to bound:

$$d_s(\Delta', f(\Delta')) = \left| \frac{|\Delta' - f(\Delta')|}{\sqrt{\Delta' f(\Delta')}} \right|$$

But we know that $|\Delta'| \geq O(1)$ and we can bound:

$$|f(\Delta')| \geq |f'(\Delta')| - |f(\Delta') - f'(\Delta')| \geq |\Delta'| - o(1) \geq O(1),$$

therefore $d_s(\Delta', f(\Delta')) \leq o(1)$ and we can apply Proposition E.23 to obtain:

$$\left\| \frac{\Delta - \Delta'}{\sqrt{\Delta \Delta'}} \right\| \leq K' \left\| \frac{f(\Delta') - \Delta'}{\sqrt{\Delta' \Delta}} \right\|.$$

Thus, since $O(1) \leq \|\Delta'\| \leq O(1)$, we have the bound:

$$\left\| \frac{\Delta - \Delta'}{\sqrt{\Delta}} \right\| \leq K'' \left\| \frac{f(\Delta') - \Delta'}{\sqrt{\Delta}} \right\| \quad (\text{I.7})$$

for some constant $K'' > 0$. We are left to bound from below and above $\|\Delta\|$ to recover the result of the theorem from (I.7). Considering the index i_0 such that $\Delta_{i_0} = \min(\Delta_i)_{1 \leq i \leq n}$, we have:

$$|\Delta_{i_0} - \Delta'_{i_0}| \leq K'' \sqrt{\Delta_{i_0}} \left\| \frac{f(\Delta') - \Delta'}{\sqrt{\Delta_{i_0}}} \right\| \leq o(1),$$

so that $\Delta_{i_0} \geq \Delta'_{i_0} - o(1) \geq O(1)$. On the other hand, one can bound again from (I.7):

$$\|\sqrt{\Delta}\| \leq \left\| \frac{\Delta'}{\sqrt{\Delta}} \right\| + K'' \left\| \frac{f(\Delta') - \Delta'}{\sqrt{\Delta}} \right\| \leq O(1).$$

As a consequence, $\Delta \sim O(1)$, and we can conclude from (I.7). \square

Proposition I.10 can be employed when Δ is random and Δ' is a deterministic equivalent (yet to be defined). If we let $f = I^X \circ u(\cdot)$ (and thus $\Delta = \hat{\Delta}$), it is not possible to state that $\Delta \sim O(1)$ since $I^X \circ u(\cdot) = \text{Diag}(\frac{1}{n} x_i^T R^X \circ u(\cdot) x_i)_{1 \leq i \leq n}$ scales with τ which might be unbounded. For this reason, in place of $\hat{\Delta}$, we will consider $\hat{D} \equiv \frac{\hat{\Delta}}{\tau}$ where:

$$\tau \equiv \text{Diag}(\max(\tau, 1)) \geq I_n.$$

We similarly denote $\bar{\tau} \equiv \text{Diag}(\min(\tau, 1)) \leq I_n$, note that $\tau = \bar{\tau}\tau$.

3.2 Definition of \tilde{D} , the deterministic equivalent of \hat{D}

The matrix $\hat{D} \equiv \frac{\hat{\Delta}}{\tau}$ satisfies the fixed point equation

$$\hat{D} = I^{\bar{Z}}(u^\tau(\hat{D})), \quad \text{where} \quad \bar{z}_i \equiv \frac{x_i}{\sqrt{\tau_i}} = \sqrt{\bar{\tau}_i} z_i + \frac{m}{\sqrt{\tau_i}} \quad \text{and} \quad u^\tau : \Delta \mapsto \tau u(\tau \Delta).$$

We will note from now on $\bar{m}_i = \mathbb{E}[\bar{z}_i]$ and $\bar{\Sigma}_i = \mathbb{E}[\bar{z}_i \bar{z}_i^T]$. In order to apply Corollary I.8 with the hypothesis described in Remark I.9, we will need a bound on the energy of the signal and on the τ_i 's.

Assumption 8. $\|m\| = O(1)$.

We still cannot apply Corollary I.8 since $\|u^\tau(\hat{D})\|$ is possibly unbounded. Still, let us assume for the moment that $\|u^\tau(\hat{D})\|$ is indeed bounded: then, following our strategy, we are led to introducing a deterministic diagonal matrix \tilde{D} ideally approaching \hat{D} and satisfying

$$\tilde{D} = \frac{\Lambda^{\bar{\Sigma}}(u^\tau(\tilde{D}))}{I_n + u^\tau(\tilde{D})\Lambda^{\bar{\Sigma}}(u^\tau(\tilde{D}))}, \quad (\text{I.8})$$

(where we recall $\Lambda^{\bar{\Sigma}}(u^\tau(\tilde{D})) = \check{I}(\bar{\Sigma}, \frac{u^\tau(\tilde{D})}{I_n + u^\tau(\tilde{D})\Lambda^{\bar{\Sigma}}(u^\tau(\tilde{D}))})$). Before proving the validity of the estimate \tilde{D} of \hat{D} , let us justify the validity of its definition (i.e., the existence and uniqueness of \tilde{D}). To this end, we first introduce a stable auxiliary mapping $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Proposition I.11. *Let $x \in \mathbb{R}_+$. Then the equation*

$$\eta = \frac{1}{\frac{1}{x} + u(\eta)}, \eta \in \mathbb{R}_+.$$

admits a unique solution that we denote $\eta(x)$. The mapping $\eta : \mathbb{R} \rightarrow \mathbb{R}$ is stable.

Proof. Let us show that the mapping $f : \eta \mapsto x/(1 + xu(\eta))$ is contracting for the stable semi-metric:

$$\begin{aligned} d_s(f(\eta), f(\eta')) &= d_s\left(\frac{1}{f(\eta)}, \frac{1}{f(\eta')}\right) = \frac{|u(\eta) - u(\eta')|}{\sqrt{\left(\frac{1}{x} + u(\eta)\right)\left(\frac{1}{x} + u(\eta')\right)}} \\ &\leq \sqrt{\frac{u(\eta)u(\eta')}{\left(\frac{1}{x} + u(\eta)\right)\left(\frac{1}{x} + u(\eta')\right)}} d_s(u(\eta), u(\eta')) \\ &\leq \frac{1}{1 + \frac{1}{u^\infty x}} d_s(\eta, \eta') \leq u^\infty d_s(\eta, \eta') \end{aligned}$$

The Theorem E.10 then allow us to conclude on the existence and uniqueness of $\eta(x)$. To prove the stability of η , we are going to use the characterization with the monotonicity of the functions $\eta_/ : x \mapsto \frac{\eta(x)}{x}$ and $\eta_\times \mapsto x\eta(x)$ presented in Property E.6. Let us consider $x, y \in \mathbb{R}_+$ such that $x \leq y$; if $\eta(x) \leq \eta(y)$, then $\eta_\times(x) \leq \eta_\times(y)$. Besides, since in addition $u_/$ is non-decreasing,

$$\eta_/(x) = \frac{1}{1 + xu(\eta(x))} \geq \frac{1}{1 + \frac{yu(y)u(\eta(x))}{\eta(x)}} \geq \frac{1}{1 + yu(\eta(y))} = \eta_/(y).$$

Similarly, if $\eta(x) \geq \eta(y)$, then $\eta_/(x) \geq \eta_/(y)$ and

$$\eta_\times(x) = \frac{1}{\frac{1}{x^2} + \frac{u(\eta(x))}{x}} \leq \frac{1}{\frac{1}{y^2} + \frac{\eta(x)u(\eta(y))}{\eta(y)}} \leq \frac{1}{\frac{1}{y^2} + \frac{u(\eta(y))}{y}} = \eta_\times(y).$$

We see that in both cases $\eta_/(x) \geq \eta_/(y)$ and $\eta_\times(x) \leq \eta_\times(y)$. Therefore, thanks to Property E.6, $\eta \in \mathcal{S}(\mathbb{R}_+)$. \square

The first equation of (I.8) can be rewritten $\tilde{D} = \eta_\tau(\Lambda^{\tilde{\Sigma}}(u^\tau(\tilde{D})))$, with $\eta_\tau : x \mapsto \frac{\eta(\tau x)}{\tau}$. To define \tilde{D} properly, we thus need to show that $\Lambda^{\tilde{\Sigma}}$ is stable (with the aim of employing Theorem E.10 again).

Proposition I.12. *For any $S \in \mathcal{S}_p^n$, the mapping $\Lambda^S : \mathcal{D}_n^+ \rightarrow \mathcal{D}_n^+$ is stable and satisfies $\forall \Delta, \Delta' \in \mathcal{D}_n^+$:*

$$d_s(\Lambda^S(\Delta), \Lambda^S(\Delta')) \leq \max(\phi_\gamma^S(\Delta), \phi_\gamma^S(\Delta')) d_s(\Delta, \Delta')$$

(with the notation provided in Lemma I.3.)

Proof. Given $S \in \mathcal{S}_p^n$ and $\Delta, \Delta' \in \mathcal{D}_n^+$, there exists $i_0 \in [n]$ such that, if we note $\lambda \equiv \max(\phi_\gamma^S(\Delta), \phi_\gamma^S(\Delta'))$, we can bound thanks to Lemma I.3:

$$\begin{aligned} d_s(\Lambda^S(\Delta), \Lambda^S(\Delta')) &= d_s\left(\tilde{I}\left(S, \frac{\Delta}{I_n + \Delta \Lambda^S(\Delta)}\right), \tilde{I}\left(S, \frac{\Delta'}{I_n + \Delta' \Lambda^S(\Delta')}\right)\right) \\ &\leq \lambda d_s\left(\frac{\Delta}{I_n + \Delta \Lambda^S(\Delta)}, \frac{\Delta'}{I_n + \Delta' \Lambda^S(\Delta')}\right) \\ &= \lambda d_s\left(\frac{I_n}{\Delta} + \Lambda^S(\Delta), \frac{I_n}{\Delta'} + \Lambda^S(\Delta')\right) \\ &= \lambda \frac{\left|\frac{1}{\Delta_{i_0}} + \Lambda^S(\Delta)_{i_0} - \frac{1}{\Delta'_{i_0}} + \Lambda^S(\Delta')_{i_0}\right|}{\sqrt{\left(\frac{1}{\Delta_{i_0}} + \Lambda^S(\Delta)_{i_0}\right)\left(\frac{1}{\Delta'_{i_0}} + \Lambda^S(\Delta')_{i_0}\right)}} \\ &\leq \lambda \max\left(\frac{\left|\frac{1}{\Delta_{i_0}} - \frac{1}{\Delta'_{i_0}}\right|}{\sqrt{\frac{1}{\Delta_{i_0}} \frac{1}{\Delta'_{i_0}}}}, \frac{|\Lambda^S(\Delta)_{i_0} - \Lambda^S(\Delta')_{i_0}|}{\sqrt{\Lambda^S(\Delta)_{i_0} \Lambda^S(\Delta')_{i_0}}}\right) \\ &\leq \lambda \max(d_s(\Delta, \Delta'), d_s(\Lambda^S(\Delta), \Lambda^S(\Delta'))) \end{aligned}$$

Thanks to Lemma I.3, the stability rules given in Property E.5, and the extra tools given by Lemma E.4 (already used to prove Property E.5). As a conclusion,

$$d_s(\Lambda^S(\Delta), \Lambda^S(\Delta')) \leq \lambda \max(d_s(\Delta, \Delta'), d_s(\Lambda^S(\Delta), \Lambda^S(\Delta'))),$$

which directly implies that $d_s(\Lambda^S(\Delta), \Lambda^S(\Delta')) \leq \lambda d_s(\Delta, \Delta')$. \square

We are thus now allowed to define \tilde{D} .

Proposition I.13. *There exists a unique diagonal matrix $\tilde{D} \in \mathcal{D}_n^+$ satisfying (I.8).*

Proof. We already know from Proposition I.12 that $D \mapsto \Lambda^{\tilde{\Sigma}}(u^\tau(D))$ is contractive for the semi metric d_s (we can indeed show as in the proof of Corollary I.5 that $\sup_{\Delta \in \mathcal{D}_n^+} \Phi_\gamma^{\tilde{\Sigma}}(u^\tau(\Delta)) < 1$ thanks to Lemma I.4 and since $u^\tau \leq \|\mathcal{I}\|u^\infty$ – be careful here that, possibly $\|\mathcal{I}\| \geq O(n)$, but that is not the question here). The same is true for $\eta_\tau(\Lambda^{\tilde{\Sigma}}(u^\tau(\tilde{D})))$ since η is stable; the existence and uniqueness of \tilde{D} thus unfold from Theorem E.10. \square

3.3 Concentration of \hat{D} around \tilde{D}

In order to establish the concentration of \hat{D} , we need an assumption on η to be able to bound $\tilde{D} = \eta_\tau(\Lambda^{\tilde{\Sigma}}(u^\tau(\tilde{D})))$. This assumption is expressed through a condition on u , justified by the following lemma that we already made visible on Figure I.1.

Lemma I.14. *The mapping η_τ is bounded from below iff, $\forall t \in \mathbb{R}_+$, $u_\times(t) = tu(t) < 1$.*

Proof. If there exists $\alpha > 0$ (and $\alpha < 1$) such that $\forall x \in \mathbb{R}_+$, $\frac{\eta(x)}{x} \geq \alpha$, then

$$\frac{\eta(x)}{x} + (1 - \alpha) \geq 1 \quad \text{and therefore:} \quad \frac{1}{\frac{1}{x} + u(\eta(x))} = \eta(x) \geq \frac{1}{\frac{1}{x} + \frac{1-\alpha}{\eta(x)}}$$

which implies that $u(\eta(x))\eta(x) \leq 1 - \alpha$. But since η is not bounded (otherwise $\lim_{t \rightarrow \infty} \frac{\eta(t)}{t} = 0 < \alpha$), there exists a sequence $(x_n)_{n \geq 0} \in \mathbb{R}_+^S$ such that $\eta(x_n) \rightarrow \infty$. Thus (u_\times being non-decreasing), $\forall t > 0$, $u_\times(t) \leq \lim_{n \rightarrow \infty} u(\eta(x_n))\eta(x_n) \leq 1 - \alpha$. Conversely, if $\forall t > 0$, $u_\times(t) < 1$, $\forall x \in \mathbb{R}_+$:

$$\frac{\eta(x)}{x} \geq \frac{1}{1 + u_\times^\infty \frac{x}{\eta(x)}} \quad \text{thus} \quad \frac{\eta(x)}{x} \geq 1 - u_\times^\infty > 0.$$

\square

Assumption 9. $u_\times^\infty < 1$, where $u_\times^\infty = \lim_{t \rightarrow \infty} tu(t)$.

We complete this extra assumption with two rather “loose” assumptions on τ .

Assumption 10. $\frac{1}{n} \sum_{i=1}^n \tau_i \leq O(1)$.

These assumptions imply the following important control.

Lemma I.15. $\tilde{D} \sim O(1)$.

Proof. We already know from our assumptions that $O(1) \leq \frac{1}{n} \text{Tr}(\Sigma_i) + \frac{1}{n} \frac{m^T m}{\tau} = \frac{1}{n} \text{Tr} \bar{\Sigma}_i \leq O(1)$ and we can then bound:

$$\Lambda^{\tilde{\Sigma}}(u^\tau(\tilde{D}))_i \geq \frac{O(1)}{\gamma + \frac{1}{n} \|\sum \Sigma_i \tau_i u(\tau_i \tilde{D})\|} \geq \frac{O(1)}{\gamma + \sup_{i \in [n]} \|\Sigma_i\| \frac{1}{n} \sum_{i=1}^n \tau_i u^\infty} \geq O(1).$$

Therefore, we can conclude thanks to Assumption 9 and Lemma I.14:

$$\tilde{D} = \frac{1}{\tau_i} \eta \left(\tau_i \Lambda^{\tilde{\Sigma}}(u^\tau(\tilde{D}))_i \right) \geq \eta_\tau^\infty \Lambda^{\tilde{\Sigma}}(u^\tau(\tilde{D}))_i \geq O(1).$$

\square

This control allows us to establish the concentration of \hat{D} :

Proposition I.16. *There exist two constants $C, c > 0$ ($C, c \sim O(1)$) such that:*

$$\forall \varepsilon \in (0, 1] : \quad \mathbb{P} \left(\left\| \hat{D} - \tilde{D} \right\| \geq \varepsilon \right) \leq C e^{-c n \varepsilon^2 / \log n}.$$

Proof. Let us check the hypotheses of Proposition I.10. Let us first bound the Lipschitz parameter (for the stable semi-metric) λ of $I^{\bar{Z}} \circ u^\tau$ around \tilde{D} defined as:

$$\forall \Delta \in \mathcal{D}_n^+ : \quad \left\| \frac{I^{\bar{Z}}(u^\tau(\Delta)) - I^{\bar{Z}}(u^\tau(\tilde{D}))}{\sqrt{I^{\bar{Z}}(u^\tau(\Delta))I^{\bar{Z}}(u^\tau(\tilde{D}))}} \right\| < \lambda \left\| \frac{\Delta - \tilde{D}}{\sqrt{\Delta \tilde{D}}} \right\|.$$

An inequality similar as in Lemma I.3 gives us:

$$\lambda \leq \sqrt{\|1 - \gamma \mathbf{R}^{\bar{Z}}(u^\tau(\tilde{D}))\|} \leq 1 - \frac{\gamma}{\gamma + \frac{1}{n} \|u^\tau(\tilde{D})\| \| \bar{Z} \bar{Z}^T \|}$$

(thanks to Lemma I.4).

First, the concentration $\bar{Z} \propto \mathcal{E}_2$ implies the concentration of its norm $\|Z\| \in \mathbb{E}[\|Z\|] \pm \mathcal{E}_2$ satisfying $K \equiv \mathbb{E}[\|Z\|]/\sqrt{n} \leq O(1)$. There exist then two supplementary constants $C, c > 0$ such that for all $n \in \mathbb{N}$, $\forall t > 0$:

$$\mathbb{P} \left(\frac{\|\bar{Z}\|}{\sqrt{n}} \geq t + K \right) \leq C e^{-n t^2},$$

and with, say, $t = K$, we see that with probability larger than $1 - C e^{-c K^2 n}$ (for some constants $C, c > 0$), $\|\bar{Z}\| \leq 2K\sqrt{n}$. There exists then a constant $K' > 0$, such that under this highly probable event $1 - \lambda \geq K'$.

Second, we know that $\bar{Z} = Z \bar{\tau}^{1/2} \sim \mathcal{E}_2$ and $u^\tau(\tilde{D}) \leq \frac{u_\infty}{\tilde{D}} \leq O(1)$ from Proposition I.15, therefore, $\bar{Z} u^\tau(\tilde{D}) \propto \mathcal{E}_2$ and we can then employ Corollary I.8 to state that $I^{\bar{Z}}(u^\tau(\tilde{D})) \in \tilde{D} \pm \mathcal{E}_2(1/\sqrt{n})$. Thus there exist two constants $C, c > 0$ such that

$$\forall t > 0 : \quad \pi_t \equiv \mathbb{P} \left(\left\| I^{\bar{Z}}(u^\tau(\tilde{D})) - \tilde{D} \right\| \geq t \right) \leq \sum_{i \in \mathbb{N}} \mathbb{P} \left(\left| I^{\bar{Z}}(u^\tau(\tilde{D}))_i - \tilde{D}_i \right| \geq t \right) \leq n C e^{-c n t^2}.$$

Now since $\pi_t \leq 1$, we can choose two constants $c', C' > 0$ depending on c, C but independent with n such that $\pi_t \leq C' e^{-c' n t^2 / \log n}$.

In this last inequality, we can take t small enough ($t = \frac{K'^2}{4\|1/\tilde{D}\|}$) such that on an event of probability larger than $1 - C' e^{-c' n / \log n}$ ($c' > 0$), we have:

$$1 - \lambda - \sqrt{\left\| \frac{I^{\bar{Z}}(u^\tau(\tilde{D})) - \tilde{D}}{\tilde{D}} \right\|} \geq \frac{K'}{2}.$$

Let us introduce the event:

$$\mathcal{A} \equiv \left\{ \|\bar{Z}\| \leq 2K\sqrt{n} \quad \text{and} \quad \left\| I^{\bar{Z}}(u^\tau(\tilde{D})) - \tilde{D} \right\| \geq \frac{K'^2}{4\|1/\tilde{D}\|} \right\}.$$

It satisfies $\mathbb{P}(\mathcal{A}^c) \leq C'' e^{-c'' n}$, for some constants $C'', c'' > 0$.

Applying Proposition I.10, we know that there exists a constant $K \leq O(1)$ such that $\forall n \in \mathbb{N}$:

$$\begin{aligned} \forall t > 0 : \quad \mathbb{P} \left(\left\| \hat{D} - \tilde{D} \right\| \geq t \right) &\leq \mathbb{P} \left(\left\| \hat{D} - \tilde{D} \right\| \geq t, \mathcal{A} \right) + C' e^{-c' n} \\ &\leq \pi_{t/K} + C'' e^{-c'' n} \leq C' e^{-c' n t^2 / \log n} + C'' e^{-c'' n}. \end{aligned}$$

We thus retrieve the result of the proposition bounding the value of t and choosing C and c appropriately. \square

It is even possible to give a deterministic equivalent of \hat{D} independent of the signal m .

Proposition I.17. *The fixed-point equation $D = \eta_\tau \circ \Lambda^{\bar{\tau}\Sigma} \circ u^\tau(D)$ admits a unique solution, denoted $\tilde{D}_{-m} \in \mathcal{D}_n^+$, and which satisfies $\|\tilde{D} - \tilde{D}_{-m}\| \leq O\left(\frac{1}{\sqrt{n}}\right)$.*

Proof. Let us first recall the notations $\mu_i = \mathbb{E}[z_i]$, $\Sigma_i = \mathbb{E}[z_i z_i^T] = C_i + \mu_i \mu_i^T$ and:

$$\bar{\Sigma}_i = \bar{\tau}_i C_i + \left(\sqrt{\bar{\tau}_i} \mu_i + \frac{m}{\sqrt{\bar{\tau}_i}} \right) \left(\sqrt{\bar{\tau}_i} \mu_i + \frac{m}{\sqrt{\bar{\tau}_i}} \right)^T = \bar{\tau}_i C_i + \bar{m}_i \bar{m}_i^T.$$

The existence and uniqueness of \tilde{D}_{-m} are justified for the same reasons as for \tilde{D} (just take $m = 0$). We want to employ again Proposition I.10, with the deterministic mappings:

$$f = \eta_\tau \circ \Lambda^{\bar{\tau}\Sigma} \circ u^\tau \quad \text{and} \quad f' = \eta_\tau \circ \Lambda^{\bar{\Sigma}} \circ u^\tau,$$

and with $\Delta = \tilde{D}_{-m}$ and $\Delta' = \tilde{D}$. We note that $\tilde{D} \sim O(1)$ and the Lipschitz parameter λ of f for the semi-metric satisfies a similar inequality as in the proof of Proposition I.16:

$$1 - \lambda \geq \frac{\gamma}{\gamma + u_\infty^\times \|\frac{1}{\tilde{D}}\| \sup \|\Sigma_i\|} \geq O(1).$$

We then need to bound the spectral norm $\|\eta_\tau \circ \Lambda^{\bar{\tau}\Sigma} \circ u^\tau(\tilde{D}) - \eta_\tau \circ \Lambda^{\bar{\Sigma}} \circ u^\tau(\tilde{D})\|$. Note that η is 1-Lipschitz for the absolute value because, for any $x, y \in \mathbb{R}_+$, the stability of η implies:

$$\frac{|\eta(x) - \eta(y)|}{|x - y|} \leq \sqrt{\frac{\eta(x)\eta(y)}{xy}} = \sqrt{\frac{1}{(1 + xu(\eta(x)))(1 + yu(\eta(y)))}} \leq 1.$$

Thus η_τ is also 1-Lipschitz. We are then left to bounding the distance (in spectral norm) between $\Lambda^{\bar{\tau}\Sigma} \circ u^\tau(\tilde{D})$ and $\Lambda^{\bar{\Sigma}} \circ u^\tau(\tilde{D})$, and we are naturally led to employing a second time Proposition I.10 since those two values are both fixed points of stable mappings:

$$\Lambda^{\bar{\Sigma}}(u^\tau(\tilde{D})) = \tilde{I}_{u^\tau(\tilde{D})}^{\bar{\Sigma}}(\Lambda^{\bar{\Sigma}}(u^\tau(\tilde{D}))) \quad \text{and} \quad \Lambda^{\bar{\tau}\Sigma}(u^\tau(\tilde{D})) = \tilde{I}_{u^\tau(\tilde{D})}^{\bar{\tau}\Sigma}(\Lambda^{\bar{\tau}\Sigma}(u^\tau(\tilde{D})))$$

where, for any $S \in \mathcal{S}_p^n$ and $\Delta \in \mathcal{D}_n^+$, $\tilde{I}_\Delta^S : \Lambda \mapsto I\left(S, \frac{\Delta}{I_n + \Delta \Lambda}\right)$. Once again, the first hypothesis is satisfied, $\Lambda^{\bar{\Sigma}}(u^\tau(\tilde{D})) \sim O(1)$ and λ' , the Lipschitz parameter of $\tilde{I}_\Delta^{\bar{\Sigma}}$ satisfies $1 - \lambda' \geq O(1)$. Noting for simplicity $\Delta \equiv u^\tau(\tilde{D})$, $\Lambda \equiv \Lambda^{\bar{\tau}\Sigma}(\Delta)$ and $\tilde{R}^S = \tilde{R}^S(S, \frac{\Delta}{I_n + \Delta \Lambda})$ (for $S = \bar{\Sigma}$ or $S = C$), we are left to bounding, for any $i \in [n]$,

$$\begin{aligned} \left| \tilde{I}_\Delta^{\bar{\tau}\Sigma}(\Lambda)_i - \tilde{I}_\Delta^{\bar{\Sigma}}(\Lambda)_i \right| &\leq \frac{1}{n \bar{\tau}_i} m^T \tilde{R}^{\bar{\tau}C} m + \frac{2}{n} \sqrt{\frac{\bar{\tau}_i}{\bar{\tau}_i}} m_i^T \tilde{R}^{\bar{\tau}\Sigma} m \\ &\quad + \left| \frac{1}{n} \text{Tr} \left(\Sigma_i \tilde{R}^{\bar{\tau}\Sigma} \left(\frac{1}{n} \sum_{j=1}^n \sqrt{\frac{\bar{\tau}_j}{\bar{\tau}_j}} (m_j^T m + m m_j^T) + \frac{1}{\bar{\tau}_j} m m^T \right) \tilde{R}^{\bar{\Sigma}} \right) \right| \\ &\leq O\left(\frac{1}{n} + \frac{1}{n} m^T \tilde{R}^{\bar{\Sigma}} \Sigma_i \tilde{R}^{\bar{\tau}\Sigma} m + \frac{1}{n} \sup_{j \in [n]} m^T \tilde{R}^{\bar{\Sigma}} \Sigma_i \tilde{R}^{\bar{\tau}\Sigma} m_j \right) \leq O\left(\frac{1}{n} \right) \end{aligned}$$

since $\frac{1}{\bar{\tau}_j}, \sqrt{\frac{\bar{\tau}_j}{\bar{\tau}_j}} \leq 1 \forall j \in [n]$. Applying twice Proposition I.10, we retrieve the result of the proposition. \square

Propositions I.16 and I.17 allow us to set the following result, proved similarly as Lemma B.18:

Corollary I.18. *There exist two constants $C, c > 0$ ($C, c \sim O(1)$) such that:*

$$\forall \varepsilon \in (0, 1] : \quad \mathbb{P} \left(\left\| \hat{D} - \tilde{D}_{-m} \right\| \geq \varepsilon \right) \leq C e^{-c n \varepsilon^2 / \log n}.$$

We now have all the elements to prove Theorem I.1.

Proof of Theorem I.1. Let us set $U \equiv \bar{\tau}u^\tau(\tilde{D}_{-m})$. We already know that $\|U\| \leq \sup_{i \in [n]} \frac{\bar{\tau}_i u_\infty^\infty}{[\tilde{D}_{-m}]_i} \leq O(1)$, which allows us to set, on the one hand, that for any deterministic matrix $A \in \mathcal{M}_p$ satisfying $\|A\|_* \leq O(1)$:

$$\mathrm{Tr}(A R_z^Z(U)) \in \mathrm{Tr}\left(A \tilde{R}_z^\Sigma(U)\right) \pm \mathcal{E}_2\left(\frac{1}{\sqrt{n}}\right),$$

thanks to (I.5) (it is true for all $A \in \mathcal{M}_p$ such that $\|A\|_F \leq O(1)$, so in particular for the matrices A satisfying $\|A\|_* \leq O(1)$).

On the second hand, placing ourselves in the overwhelming event where $\|Z\| \leq K\sqrt{n}$ as in the proof of Proposition I.16, we can bound thanks to Propositions I.16 and I.17:

$$\begin{aligned} |\mathrm{Tr}(A R_z^Z(U)) - \mathrm{Tr}(A R_z)| &\leq \frac{1}{n} \left\| R_z^Z(U) Z \left(\bar{\tau}u^\tau(\hat{\Delta}) - \bar{\tau}u^\tau(\tilde{\Delta}_{-m}) \right) Z^T R_z \right\| \\ &\leq O\left(\sup_{i \in [n]} \frac{\bar{\tau}_i u_\infty^\infty \|\hat{\Delta} - \tilde{\Delta}_{-m}\|}{\hat{D}_i[\tilde{\Delta}_{-m}]_i} \right) \end{aligned}$$

we can then conclude thanks to Corollary I.18. □

Chapter J

Robust convex regression

In modern statistical learning approaches, many optimization problems can be formulated in terms of convex minimization problem or fixed point equations (soft-max classification, M-estimators, low density classifiers, soft sm, logistic regression, optimal transport, adaboost, stochastic gradient descent...). As such, a theoretical characterization of the behavior of such objects becomes crucial in order to assess the performances of the underlying algorithms. This chapter has two main objectives: first it provides qualitative inferences concerning the statistical concentration of the parameter vector learned during the resolution of the problem and second, some quantitative inferences on a restricted range of regularized empirical risk minimization problems (see [ML19]) of objective function $\frac{1}{n} \sum_{i=1}^n h_i(x_i^T Y) + \frac{1}{2} \lambda \|Y\|^2$. Thanks to a convexity hypothesis on the mappings h_1, \dots, h_n , one can express these minimizing problem as a fixed point equation through differentiation

$$Y = \frac{1}{n} \sum_{i=1}^n f_i(x_i^T Y) x_i, \tag{J.1}$$

where we noted $f_i = h'_i$.

In the following, we stick to ℓ^2 -norm regularization although other kind of regularizations can be found in the literature [CM22]. We however believe in possible generalization of our approach as it was done very recently for the Lasso problem in [TSS⁺22]. The class of so called “ridge regularized robust regression estimators” ([BBEKY13, Kar13]) contains in particular softmax classifiers that were already studied in [MLC19b, SLTC21]. There are been continuous work made around this range of problems with different kind of techniques, approximate message passing in [SC19, DM16], convex Gaussian minmax Theorem in [TOS20, DKT22], Meanfield Theory in [MKUZ20] or Replica Method in [MKL⁺20, SZ22]. As a naive comparison to all these works we wish to emphasize the simplicity of the fixed point equation providing the first statistics of Y given here in Claim J.2. But the more important contribution is probably a rigorous justification of the leave-one-out technique introduced in [Kar13, BBEKY13], this contribution relies on Theorems G.1 and G.2 setting the concentration and the estimation of the resolvent $(I_p - \frac{1}{n} XDX^T)^{-1}$ when D is random and depends on X .

J.1 Main results

1.1 Theoretical results

The concentration of Y is proven thanks to Corollary C.9 one is then led to try and estimate its first statistics. For that purpose, one first needs to disentangle the contribution of each x_i in¹ $Y = \frac{1}{n} \sum_{i=1}^n f_i(x_i^T Y) x_i$. This is done thanks to the well known “leave-one-out” method that aims at expressing a formula to link Y with a

¹To give a arguably valid justification, we mention that if one can write $x_i Y$ as a functional $\xi_i(x_i Y_{-i})$ for Y_{-i} independent of i , then one can resort to Stein identities to estimate $f(\xi_i(x_i Y_{-i}) x_i$.

random vector Y_{-i} deprived of the contribution of x_i and defined followingly:

$$Y_{-i} = \frac{1}{n} \sum_{\substack{j=0 \\ j \neq i}}^n f_j(x_j^T Y_{-i}) x_j \quad (\text{J.2})$$

Our first result provides the link between Y and Y_{-i} , it is a combination of Propositions J.12 and J.9.

Proposition J.1. *Under classical Assumptions 1-4, we consider a two differentiable convex mappings $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in [n]$, such that $\|f_i\|_\infty, \|f_i'\|_\infty, \|f_i^{(2)}\|_\infty \leq \infty$, then for any $\delta \in \mathcal{D}_n$, let us note $\tilde{Q}^\delta \equiv (I_p - \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\frac{f_i'(x_i^T Y)}{1 - \delta_i f_i'(x_i^T Y)} \right] \Sigma_i)^{-1}$ where $\Sigma_i \equiv \mathbb{E}[x_i x_i^T]$. The equation $\delta = \frac{1}{n} \text{Diag}_{i \in [n]}(\text{Tr}(\Sigma_i \tilde{Q}^\delta))$ has a unique solution that we note Δ , and we have the concentration:*

$$\forall t > 0 : \mathbb{P}(|x_i^T Y - x_i^T Y_{-i} - \Delta_i f_i(x_i^T Y)| \geq t) \leq C e^{-cnt^2/\log n} + C e^{-cn},$$

for some constants $C, c > 0$ independent of n .

This proposition leads us to introduce the mapping ξ_i defined for any $z \in \mathbb{R}$ with the fixed point equation (see Proposition J.12)

$$\xi_i(z) = f_i(z + \Delta_i \xi_i(z)), \quad \xi_i(z) \in \mathbb{R}, \quad (\text{J.3})$$

then one can show (Proposition J.12 again) that $f_i(x_i^T Y) \in \xi_i(x_i^T Y_{-i}) \pm \mathcal{E}_1\left(\frac{1}{\sqrt{n}}\right)$. Finally, noting that $\xi_i'(t) = \frac{f_i'(\zeta_i(t))}{1 - \Delta_i f_i'(\zeta_i(t))}$ (where $\zeta_i(t)$ satisfies $\zeta_i(t) = t + \Delta_i f_i(\zeta_i(t))$), one is lead to deduce the following result that could not be demonstrated (although it was validated with practical examples as one can see in next subsection).

Claim J.2. *Considering sequence of Gaussian vectors $X_n = (x_1, \dots, x_n) \in \mathcal{M}_{p_n, n}$ and convex twice differentiable mappings $h_i : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the hypotheses of of Proposition J.1 if we note $m_i \equiv \mathbb{E}[x_i]$, $\Sigma_i = \mathbb{E}[x_i x_i^T]$, for all $n \in \mathbb{N}$, the unique vector m_Y satisfying:*

- $m_Y = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(\mathcal{N}(m_i, v_i))] \tilde{Q} m_i$
- $\Sigma_Y = m_Y m_Y^T + \tilde{Q} \tilde{S} \tilde{Q}$
- $\tilde{Q} = (I_p - \tilde{C})^{-1}$
- $\tilde{C} = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i'(\mathcal{N}(m_i, v_i))] C_i$
- $\tilde{S} = \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}[\xi_i^2(\mathcal{N}(m_i, v_i))] \Sigma_i$
- $\xi_i : \mathbb{R} \rightarrow \mathbb{R}$ is defined for any $z \in \mathbb{R}$ as the unique solution to $\xi_i(z) = f_i\left(z + \frac{1}{n} \text{Tr}(C_i \tilde{Q}) \xi_i(z)\right)$
- $m_i = m_i^T m_Y$
- $v_i = \frac{1}{n} \text{Tr}(\Sigma_Y \Sigma_i) - m_i^2$

Besides, if we note $\tilde{\Sigma} = \sum_{i=1}^n \xi_i(m_i^T m_Y)^2 \Sigma_i$, the solution Y to the fixed point equation (J.1) satisfies the estimation:

$$\|\mathbb{E}[Y] - m_Y\| \leq O\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad \|\mathbb{E}[Y Y^T] - \Sigma_Y\|_* \leq O\left(\frac{1}{\sqrt{n}}\right)$$

The estimation of the solutions to Claim J.2 are computed reducing the step of convergence at each iteration. The convergence is the ensured for the same reasons that transform (J.1) into the contractive equation (since $I_p \leq d^2(\Psi(X)) \leq (1 + \frac{1}{n} \sup_{i \in [n]} \|f_i'\|_\infty) I_p$):

$$Y = Y - \frac{Y - \frac{1}{n} \sum_{i=1}^n f(x_i^T Y) x_i}{1 + \frac{1}{n} \sup_{i \in [n]} \|f_i'\|_\infty \|X X^T\|}$$

We were not able to prove this claim that however appears right on all the different practical examples we studied. We give from Subsection 2.5 and in the next subsections some reachable inferences in the Gaussian case, however this approach has two main issues:

- Although we prove that some estimators satisfy those equations with vanishing error (when $n \rightarrow \infty$), we could not obtain the existence and uniqueness of the solution of the displayed fixed point equation,
- The fixed point equation in question relies on the computation of integrals $\mathbb{E}[\xi_i(z)]$ for $z \sim \mathcal{N}(m_z, v_z)$ and has therefore a very slow rate of convergence.

Those two reasons make us think that the Gaussian approach is not the good one and that one has to find under the general assumption of concentrated vectors a supplementary equation (giving a link between $\mathbb{E}[\xi_i[x_i^T Y_{-i}]$ and $\mathbb{E}[Y]$ for instance) that will allow us to prove Claim J.2. We give in next subsection some illustration of the validity of Claim J.2.

1.2 Application to the logistic regression

To illustrate our theoretical results in a simple way we present the example of a supervised classification method called the “the logistic regression” already studied by [MLC19a]. Considering a deterministic vector $m \in \mathbb{R}^p$ and a positive symmetric matrix $C \in \mathcal{M}_p$, we suppose we are given n Gaussian random vectors z_1, \dots, z_n , each one following the law $\mathcal{N}(y_i m, C)$ where $y_1, \dots, y_n \in \{-1, 1\}$ are the “labels” of z_i that determine the classes of our data. We assume for simplicity that the classes are balanced. Our classification problem aims at deducing from the *training set* z_1, \dots, z_n and the labels y_1, \dots, y_n a statistical characterization of our two classes that will allow us to classify a new coming data x , independent with the training set and following one or the other law. To introduce the problem, let us express the probability conditioned on a new data x , and knowing that z_i is Gaussian, that $y_i = y$, for a given $y \in \{0, 1\}$:

$$\begin{aligned} \mathbb{P}(y_i = y \mid z_i) &= \frac{\mathbb{P}(y_i = y)\mathbb{P}(z_i \mid y_i = y)}{\mathbb{P}(y_i = y)\mathbb{P}(z_i \mid y_i = y) + \mathbb{P}(y_i = -y)\mathbb{P}(z_i \mid y_i = -)} \\ &= \frac{e^{-(z_i - ym)^T C^{-1}(z_i - ym)/2}}{e^{-(z_i - ym)^T C^{-1}(z_i - ym)/2} + e^{-(z_i + ym)^T C^{-1}(z_i + ym)/2}} \\ &= \frac{1}{1 + e^{-yz_i^T C^{-1}m}} = \sigma(yz_i^T \beta^*), \end{aligned}$$

noting $\sigma : t \mapsto 1/(1 + e^{-t})$ and $\beta^* \equiv C^{-1}m$. The goal of the logistic regression is to try and estimate β to be able to classify the data depending on the highest value of $\mathbb{P}(y_i = 1 \mid z_i)$ and $\mathbb{P}(y_i = -1 \mid z_i)$. For that purpose, we solve a regularized maximum likelihood problem:

$$\min_{\beta \in \mathbb{R}^p} \frac{1}{p} \sum_{i=1}^n \rho(\beta^T x_i) + \frac{\lambda}{2} \|\beta\|^2$$

where $\rho(t) = \log(1 + e^{-t})$, $x_i = y_i z_i$ and $\lambda > 0$ is the regularizing parameter. Differentiating this minimizing problem, we obtain:

$$\beta = \frac{1}{\lambda n} \sum_{i=1}^n f(x_i^T \beta) x_i,$$

where $f : t \mapsto \frac{1}{1+e^t}$. If one chooses λ sufficiently big, the assumptions of Corollary C.9 are all satisfied, our results thus allow us to set the concentration $\beta \propto \mathcal{E}_2(1/\sqrt{n})$ and estimate its first statistics.

The system of equations provided in Conjecture J.2 can be solved by successive iteration, we can then deduce the performances of the algorithm from the statistics of Y . We depicted on Figure J.1 some of these predictions for a minimization problem slightly different where $\beta \in \mathcal{M}_{p,2}$, $y_i = (1, 0)$ if x_i in class \mathcal{C}_1 , $y = (0, 1)$ if x_i in class \mathcal{C}_2 and $f(x_i^T \beta) = \text{Softmax}(x_i^T \beta) \in \mathbb{R}^2$ for all $i \in [n]$. We displayed on on Figure J.2 the distribution of the random vectors $x^T \beta \in \mathbb{R}^2$, for different test data $x \in \mathbb{R}^p$ independent with $\beta \in \mathcal{M}_{p,2}$ and belonging to \mathcal{C}_1 or \mathcal{C}_2 (x and β are both randomly drawn). The index of the biggest entry of $x^T \beta$ provides the regressed class for this classifier. One can see that when n gets big, the distance between the two entries of $x^T \beta$ increases, one has indeed more data available to provide an accurate vector β .

The next step of the study is now to have a wider understanding of the formulas provided in Claim J.2 in order to provide fast classifier’s parameter optimization without a full coverage of the parameter space.

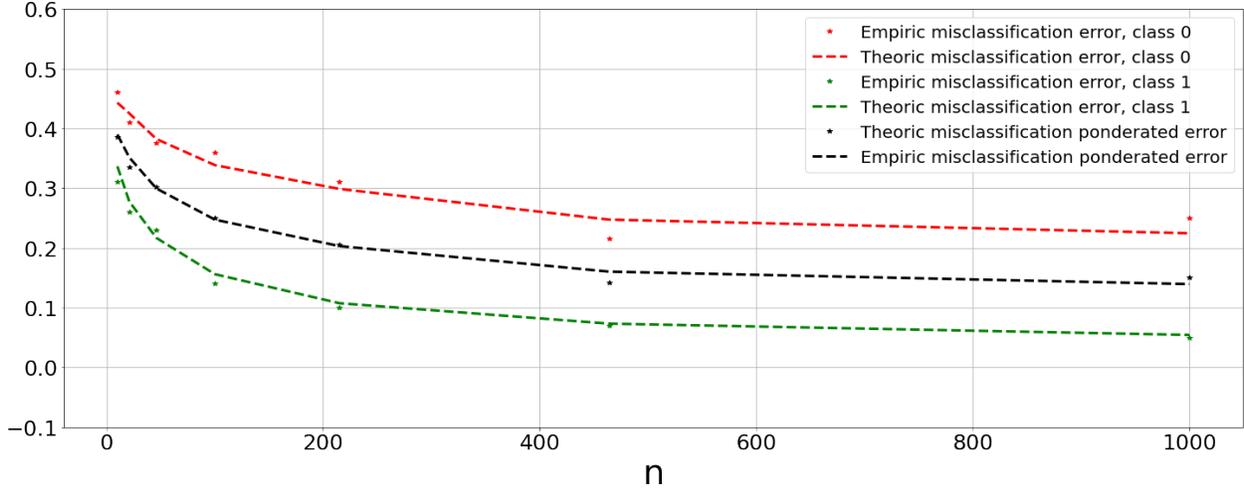


Figure J.1: Misclassification rate for two class of Gaussian random vectors with $p = 500$, $C_1 = C_2 = I_p$, $m_1 = (2, 0, 0, \dots)$, $m_2 = (0, 2, 0, \dots)$, with proportion $(\#C_1/n, \#C_2/n) = (\frac{1}{4}, \frac{3}{4})$ for different values for n . Mean taken on 200 drawings for the estimation of the empiric misclassification error.

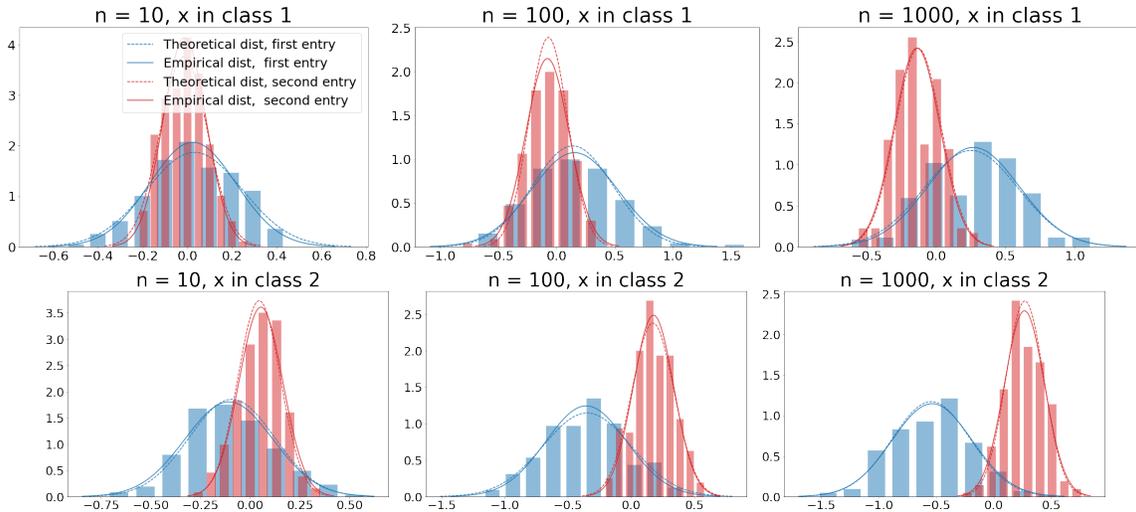


Figure J.2: Distribution of $x^T \beta$ with x , test data following the distribution of C_1 or C_2 , independent with β . Same setting as Figure J.1.

J.2 Fixed point equation depending on independent data x_1, \dots, x_n

We will keep Assumptions 1-5 concerning the definition of our regime and the concentration, independence and bounding hypotheses on X ; we further add the following assumptions on f , for reasons that will be clear later:

Assumption 11. *The mappings $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in [n]$, are twice differentiable and $\sup_{i \in [n]} \|f_i\|_\infty, \sup_{i \in [n]} \|f'_i\|_\infty, \sup_{i \in [n]} \|h_i^{(3)}\|_\infty \leq O(1)$.*

2.1 Leave-one-out approach

To estimate the expectation and covariance of Y , one needs to disentangle the influence of each data x_i on Y with the study of the random vector Y_{-i} defined with the fixed point equation (J.2) and that is independent with x_i by construction. To link Y with Y_{-i} we create a ‘‘bridge’’ defined by a parameter $t \in [0, 1]$, through a mapping $\Psi_{-i}^t : \mathcal{M}_{p,n} \rightarrow \mathcal{L}(E)$, defined for any $A \in \mathcal{M}_{p,n}$ and any $y \in E$ with:

$$\Psi_{-i}^t(A)(y) \equiv \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}} f_i(a_j^T y) a_j + t f_i(a_i^T y) a_i.$$

Then, noting for any $t \in [0, 1]$ $y_{-i}^A(t)$, the unique solution (when it exists) to $y_{-i}^A(t) = \Psi_{-i}^t(A)(y_{-i}^A(t))$, we see that:

$$Y_{-i} = y_{-i}^X(0) \quad \text{and} \quad Y = y_{-i}^X(1)$$

The next lemma sets the differentiability of the mapping $y_{-i}^A : [0, 1] \rightarrow E$.

Lemma J.3. *Under the differentiability assumption on h (Assumption 11) the mapping y_{-i}^A is differentiable and we have:*

$$y_{-i}^{A'}(t) = \frac{\chi_A'(t)}{n} \left(I_p + \frac{1}{n} A_{-i} D_A(t) A_{-i}^T \right)^{-1} a_i,$$

where $\chi_A : t \mapsto t f_i(a_i^T y_{-i}^A(t))$ and $D_A : t \mapsto \text{Diag}(-f'_i(x_i^T y_{-i}^A(t)))$.

Proof. Starting from the fixed point equation satisfied by Y in (J.1), we apply the inverse function theorem to the \mathcal{C}^1 bijective mapping:

$$\begin{aligned} \Theta : \mathbb{R} \times \mathbb{R}^p &\longrightarrow \mathbb{R} \times \mathbb{R}^p \\ (t, y) &\longmapsto \left(t, y - \frac{1}{n} \sum_{\substack{j=0 \\ j \neq i}} f_i(a_j^T y) a_j - \frac{t}{n} f_i(a_i^T y) a_i \right). \end{aligned}$$

It is indeed possible since $\mathbb{R} \times \mathbb{R}^p$ is a Banach space, $d\Theta$ is clearly bounded ($\Psi(A)$ is Lipschitz), and $\forall (t, y) \times (s, h) \in (\mathbb{R} \times \mathbb{R}^p)^2$:

$$\begin{aligned} d\Theta_{|(t,y)} \cdot (s, h) &= \left(s, h - \frac{s}{n} f_i(a_i^T y) a_i - \frac{t}{n} f'_i(a_i^T y) a_i^T h a_i - A_{-i} \text{Diag}(f'_i(a_i^T y))_{i \in [n]} A_{-i} h \right) \\ &= \left(s, h - \frac{s}{n} f_i(a_i^T y) a_i - A_{-i}(t) \text{Diag}(f'_i(a_i^T y))_{i \in [n]} A(t) h \right), \end{aligned}$$

if we note $A_{-i}(t) = (a_1, \dots, a_{i-1}, t a_i, a_{i+1}, \dots, a_n) \in \mathcal{M}_{p,n}$. Besides, $\forall t \in \mathbb{R}$, $f'_i(t) \leq 0$, thus $Q_{-i}^A(t) \equiv (I_p + A_{-i}(t) D_A(t) A)^{-1}$ is well defined and $d\Theta_{|(t,y)}$ is invertible with:

$$d\Theta_{|(t,y)}^{-1} = \begin{pmatrix} 1 & 0 \\ Q_{-i}^A(t) f_i(a_i^T y) a_i & Q_{-i}^A(t) \end{pmatrix}.$$

Therefore, Θ^{-1} is also C^1 and, we can differentiate $y_{-i}^A(t) = \Theta^{-1}(t, 0)$ to obtain the identity:

$$\begin{aligned} y_{-i}^{A'}(t) &= -A_{-i} D_A(t) A_{-i}^T y_{-i}^{A'}(t) \\ &\quad + \frac{1}{n} f_i(a_i^T y_{-i}^A(t)) a_i + \frac{t}{n} f'_i(a_i^T y_{-i}^A(t)) a_i^T y_{-i}^{A'}(t) a_i, \end{aligned} \quad (\text{J.4})$$

from which we retrieve directly the result of the Lemma. \square

2.2 Concentration of $t \mapsto y_{-i}^X(t)$

Introducing, for any $t \in [0, 1]$, and any $i \in [n]$, the diagonal matrix $D_{-i}(t) \in \mathcal{D}_n$ defined with

$$D_{-i}(t)_i = -t f'_i(x_i^T y_{-i}^X(t)) \quad \text{and} \quad \forall j \in [n], j \neq i : D_{-i}(t)_j = -f'_i(x_j^T y_{-i}^X(t))$$

the random matrix $d_2 \Psi_{-i}^t(A)|_y$ writes more simply $\frac{1}{n} X D_{-i}(t) X$ and we have the identity:

$$y_{-i}^X(t) = \frac{1}{n} f_i(x_i^T y_{-i}^X(t)) Q_{-i}(t) x_i, \quad (\text{J.5})$$

where, $Q_{-i}(t) \equiv Q(D_{-i}(t)) \equiv (I_p + \frac{1}{n} X D_{-i}(t) X^T)^{-1}$. Note in particular that:

$$\forall t \in [0, 1] : \|Q_{-i}(t)\| \leq 1 \quad (\text{J.6})$$

When $t = 0$, the random matrix $Q_{-i}(0)$, that we note Q_{-i} is then independent with x_i like $D_{-i} \equiv D_{-i}(0)$, since then $[D_{-i}]_i = 0$.

We are going to express the concentration of mappings defined on $[0, 1]$ and having value on vectorial spaces $(\mathbb{R}^p, \mathcal{M}_{p,n}$ or \mathcal{D}_n). Those concentrations are expressed with the infinite norm toward the parameter $t \in [0, 1]$, for simplicity, we will superscript the vectorial norms with ∞ to designate those norms. For instance:

$$\|y_{-i}^X(\cdot)\|^\infty = \sup_{t \in [0,1]} \|y_{-i}^X(t)\|.$$

We will also need to express norms on product of vectorial spaces. Given two normed vectorial spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, we note $\|\cdot\|_E + \|\cdot\|_F$ the norm defined on $E \times F$ as:

$$\forall (x, y) \in E \times F : (\|\cdot\|_E + \|\cdot\|_F)(x, y) = \|x\|_E + \|y\|_F.$$

We will now introduce a family of event indexed by the norm of $\|X\|$. Given $u > K$, we note:

$$\mathcal{A}_u \equiv \{\|X\| \leq u\sqrt{n}\}$$

It is possible to bound $\|X\|$ thanks to the bound

$$\|\mathbb{E}[X]\| \leq \sqrt{n} \sup_{1 \leq i \leq n} \|\mathbb{E}[x_i]\| \leq O(\sqrt{n})$$

Proposition B.22 allows us to set $\mathbb{E}[\|X\|] \leq \|\mathbb{E}[X]\| + O(\sqrt{n}) \leq O(\sqrt{n})$ and therefore $\|X\| \in O(\sqrt{n}) \pm \mathcal{E}_2$. In other words

$$\forall u \geq \nu, \mathbb{P}(A_u^c) \leq C e^{-cnu^2}, \quad \text{where } \nu \equiv \frac{2\mathbb{E}[\|X\|]}{\sqrt{n}} \leq O(1),$$

and $C, c > 0$ are two constants.

Proposition J.4. *For all $u \geq \nu$, and $i \in [n]$:*

$$\left(\frac{1}{u^2 \sqrt{n}} X, y_{-i}^X(\cdot) \right) \mid \mathcal{A}_u \propto \mathcal{E}_2 \left(\frac{u^2}{\sqrt{n}} \right) \quad \text{in } \left(\mathcal{M}_{p,n} \times (\mathbb{R}^p)^{[0,1]}, \|\cdot\|_F^\infty + \|\cdot\|^\infty \right),$$

Proof. Given a (sequence of) parameter $u \geq K$, we first know from Lemma B.4 and Remark B.5 that $(X \mid \mathcal{A}_u) \propto \mathcal{E}_2$. We are going to employ Corollary C.9 with the mapping $\Psi_{-i} : \mathcal{M}_{p,n} \rightarrow \mathcal{F}((\mathbb{R}^p)^{[0,1]})$ that is defined for any $A, y, t \in \mathcal{M}_{p,n} \times \mathbb{R}^p \times [0, 1]$ as:

$$\Psi_{-i}^t(A)(y) = \Psi(A)(y) - \frac{1-t}{n} h(a_i^T y), \quad (\text{J.7})$$

where we recall that $\Psi(A)(y) = \frac{1}{n} \sum_{i=1}^n h(a_i^T y)$. With this choice (recalling that $f = -\frac{1}{2}h'$), for all $A \in \mathcal{M}_{p,n}$, $y \in \mathbb{R}^p$:

$$d(\Psi_{-i}^t(A))|_y = \frac{1}{n} A_{-i} f_i(A_{-i}^T y) + \frac{t}{n} f_i(a_i^T y) a_i.$$

To apply Corollary C.9, we merely choose $y_0 : t \mapsto 0$, then for any $A \in X(\mathcal{A}_u)$:

$$\left\| d(\Psi_{-i}(A))|_{y_0(\cdot)} \right\|^\infty = \frac{1}{n} |f_i(0)| \left\| A_{-i} \mathbb{1} + \frac{t}{n} a_i \right\| \leq \|f\|_\infty u$$

We can easily bound for any $A \in X(\mathcal{A}_u)$, $y \in (\mathbb{R}^p)^{[0,1]}$:

$$I_p \leq d^2(\Psi_{-i}(A))|_y = -\frac{2}{n} A_{-i} \text{Diag}(f'_i(a_j^T y))_{j \in [n]} A_{-i} - \frac{2t}{n} f'_i(a_i^T y) a_i a_i^T + I_p \leq \|f'\|_\infty u^2$$

(recall that $\forall t \in \mathbb{R}$, $f'_i(t) \leq 0$).

Besides for any $y \in \mathcal{B}(y_0, u \|f\|_\infty)$ and any $A, B \in X(\mathcal{A}_u) \subset \mathcal{M}_{p,n}$:

$$\begin{aligned} \|d(\Psi(A))|_y - d(\Psi(B))|_y\|^\infty &\leq \frac{1}{n} \|(A - B) f_i(A^T y)\|^\infty + \frac{1}{n} \|B (f_i(A^T y) - f_i(B^T y))\|^\infty \\ &\leq \frac{1}{n} (|f_i(0)| + \|A\| \|y\|^\infty \|f'\|_\infty) \|A - B\| + \frac{1}{n} \|B\| \|A - B\| \|y\|^\infty \|f'\|_\infty \\ &\leq O\left(\frac{u^2}{\sqrt{n}}\right) \|A - B\|; \end{aligned}$$

and the same way, for any $a, b \in x_i(\mathcal{A}_u) \subset \mathbb{R}^p$ we show that:

$$\frac{1}{n} \|f_i(a^T y) a - f_i(b^T y) b\|^\infty \leq O\left(\frac{u^2}{\sqrt{n}}\right) \|a - b\|,$$

Therefore, noting $E \equiv (\mathcal{R}^p)^{[0,1]}$, we deduce from (J.7) that for all $u > K$, $A \mapsto d(\Psi_{-i}(A))|_y$ is $O\left(\frac{u^2}{\sqrt{n}}\right)$ -Lipschitz from $(X(\mathcal{A}_u), \|\cdot\|_F)$ to $(\mathcal{F}(E, E), \|\cdot\|_{\mathcal{B}(y_0, u)})$. We can then deduce from Corollary C.9 the result of the proposition. \square

Corollary J.5. $Y(\cdot) \propto \mathcal{E}_2(n^{-1/2}) + \mathcal{E}_{\frac{3}{2}}(n^{-3/2})$.

Proof. Noting that Y is a 1-Lipschitz transformation of $(\frac{1}{u^2 \sqrt{n}} X, y_{-i}^X(\cdot))$, let us bound thanks to Proposition J.4 for any $u \geq \nu$ and any independent copy of Y, Y' :

$$\mathbb{P}(|f_i(Y) - f_i(Y')| \geq t \mid \mathcal{A}_u) \leq C e^{-cnt^2/u^4} \quad \text{and} \quad \mathbb{P}(\mathcal{A}_u^c) \leq C e^{-cnu^2}.$$

One can then apply Lemma A.28 with $\sigma = 1/\sqrt{n}$, $m = 2$ and² $\eta = \eta_{(\mathcal{M}_{p,n}, \|\cdot\|)} = O(1/n)$ to obtain the result of the corollary. \square

Let us now prove the concentration of $y_{-i}^{X'}$ that we will integrate in next subsection.

Proposition J.6. Under \mathcal{A}_ν , $\forall i \in [n]$, $y_{-i}^{X'}(\cdot) \in \mathcal{E}_2\left(\frac{1}{\sqrt{n}}\right)$ in $((\mathbb{R}^p)^{[0,1]}, \|\cdot\|^\infty)$.

One could propose as in Corollary J.5 a sharper control on the tail of $y_{-i}^X(\cdot)$ but the expression becomes complicated, so we prefer not to present it.

Proof. One could once again employ Corollary C.9, but it is straightforward here to employ the explicit formula given by (J.5):

$$y_{-i}^{X'}(t) = \frac{1}{n} f_i(x_i^T y_{-i}^X(t)) Q_{-i}(t) x_i,$$

that allows us to state that $y_{-i}^{X'}(\cdot)$ is concentrated as a $O(1/\sqrt{n})$ -Lipschitz transformation under \mathcal{A}_ν of the concentrated vector $(X, \sqrt{n} Y_{-i}(\cdot), D_{-i}(\cdot))$ (recall indeed that $\|Q_{-i}(t)\|, \|f_i\|_\infty, \|X\|/\sqrt{n}, \|D_{-i}\|^\infty \leq O(1)$). \square

²See Example B.21 to get more precision

2.3 Integration of $\frac{\partial y_{-i}^X(t)}{\partial t}$

Now that the concentration of the objects $y_{-i}^X(t)$ and $y_{-i}^X(t)'$ are well understood, we are able to integrate the formula provided by Lemma J.3 to express the link between Y and Y_{-i} . We just give some preliminary results to control the matrix $Q_{-i}(\cdot)$. In a first time, let us study $Q_{-i} \equiv Q_{-i}(0) = Q_{-i}(D_{-i})$ which is independent with x_i .

Proposition J.7. *Under \mathcal{A}_ν , $\forall i \in [n]$:*

$$\|Q_{-i}(\cdot)x_i - Q_{-i}x_i\| \in O\left(\sqrt{\log(n)}\right) \pm \mathcal{E}_2 \quad \text{in } (\mathbb{R}^{[0,1]}, \|\cdot\|_\infty)$$

This proposition is a consequence of the following lemma.

Lemma J.8. *Under \mathcal{A}_ν , $\sup_{t \in [0,1]} \|D_{-i} - D_{-i}(t)\|_F \leq O(1)$*

Proof. From identity:

$$X^T y_{-i}^X(t) = \frac{1}{n} X^T X_{-i} f(X^T y_{-i}^X(t)) + \frac{t}{n} X^T x_i f_i(x_i^T y_{-i}^X(t)),$$

(where given a vector $z \in \mathbb{R}^n$, $f(z) = (f_i(z_i))_{i \in [n]} \in \mathbb{R}^n$). We can deduce that:

$$\|X^T y_{-i}^X(t) - X^T Y_{-i}\| \leq \sup_{i \in [n]} \|f'_i\|_\infty \left\| \frac{1}{n} X^T X_{-i} \right\| \|X^T y_{-i}^X(t) - X^T Y_{-i}\| + \frac{t}{n} X^T x_i f_i(x_i^T y_{-i}^X(t)),$$

therefore $\|X^T y_{-i}^X(t) - X^T Y_{-i}\| \leq \frac{1}{n\varepsilon} X^T x_i f_i(x_i^T y_{-i}^X(t))$. Recalling the identity $D_{-i}(t) \equiv \text{Diag}(f'_i(X^T y_{-i}^X(t)))$, we can bound under \mathcal{A}_ν (recall that $\|X\|, \|x_i\| \leq O(\sqrt{n})$ and $\|f\|_\infty \leq O(1)$):

$$\|D_{-i} - D_{-i}(t)\|_F \leq \sup_{i \in [n]} \frac{\|f''_i\|_\infty}{\varepsilon} \frac{t}{n} \|X^T x_i f_i(x_i^T y_{-i}^X(t))\| \leq O(1) \quad (\text{J.8})$$

□

Proof of Proposition J.7. Under \mathcal{A}_ν and for any $t \in [0, 1]$, let us bound:

$$\begin{aligned} \|(Q_{-i}(t) - Q_{-i})x_i\| &\leq \frac{1}{n} \|Q_{-i}(t)X_{-i}(D_{-i} - D(t))X_{-i}^T Q_{-i}x_i\| \\ &\leq O\left(\frac{1}{\sqrt{n}}\right) \|X_{-i}^T Q_{-i}x_i\|_\infty, \end{aligned}$$

thanks to Lemma J.8 and since $\|X\| \leq O(\sqrt{n})$ and $\|Q_{-i}(t)\| \leq 1$.

Now, it is possible to show (see [LC22a, Lemma D.3]) that $X_{-i}^T Q_{-i}x_i \in \mathcal{E}_2(\sqrt{n})$. Besides, one can bound thanks to the independence between x_i and Q_{-i} (\mathcal{A}_ν being overwhelming, it preserves in a sense the independence relations):

$$\begin{aligned} \|\mathbb{E}[X_{-i}^T Q_{-i}x_i \mid \mathcal{A}_\nu]\|_\infty &\leq \|\mathbb{E}[X_{-i}^T Q_{-i} \mathbb{E}[x_i]]\|_\infty + O\left(\frac{1}{n}\right) \\ &\leq \mathbb{E}[\|X_{-i}\|] \|\mathbb{E}[x_i]\| \leq O(\sqrt{n}), \end{aligned}$$

derived from Assumption 4 and (J.6). Thus, Proposition B.22 allows us to bound:

$$\mathbb{E}[\|X_{-i}^T Q_{-i}x_i\|_\infty \mid \mathcal{A}_\nu] \leq O(\sqrt{\log(n)}),$$

from which we deduce the result of the proposition.

□

We have now all the elements to prove:

Proposition J.9. 7 6.8 Under \mathcal{A}_ν , $\forall i \in [n]$:

$$\left\| Y - Y_{-i} - \frac{1}{n} f_i(x_i^T Y) Q_{-i} x_i \right\| \in O\left(\frac{\sqrt{\log n}}{n}\right) \pm \mathcal{E}_2\left(\frac{1}{n}\right)$$

Proof. Setting $\chi(t) \equiv t f_i(x_i^T y_{-i}^X(t)) \in \mathbb{R}$, let us integrate between 0 and t the identity $y_{-i}^{X'}(t) = \chi'(t) \frac{1}{n} Q_{-i}(t) x_i$:

$$y_{-i}^X(t) - Y_{-i} = \frac{1}{n} f_i(x_i^T Y) Q_{-i} x_i + \frac{1}{n} \int_0^t \chi'(u) (Q_{-i}(u) - Q_{-i}(0)) x_i du. \quad (\text{J.9})$$

Now, $\chi'(u) = f_i(x_i^T y_{-i}^X(u)) + t f_i'(x_i^T y_{-i}^X(u)) x_i^T y_{-i}^{X'}(u)$, and we know from Proposition J.4 and Proposition J.6 that under \mathcal{A}_ν :

- $f_i(x_i^T y_{-i}^X(\cdot)) \in O(1) \pm \mathcal{E}_2$ and $\|f_i(x_i^T y_{-i}^X(\cdot))\|_\infty \leq O(1)$
- $f_i'(x_i^T y_{-i}^X(\cdot)) \in O(1) \pm \mathcal{E}_2$ and $\|f_i'(x_i^T y_{-i}^X(\cdot))\|_\infty \leq O(1)$
- $x_i^T y_{-i}^{X'}(\cdot) \in O(1) \pm \mathcal{E}_2$ and $\|x_i^T y_{-i}^{X'}(\cdot)\|_\infty \leq O(1)$

therefore $\chi'(\cdot) \in O(1) \pm \mathcal{E}_2$ in $(\mathbb{R}^{[0,1]}, \|\cdot\|_\infty)$, $\|\chi'\|_\infty \leq O(1)$ and one can bound:

$$\begin{aligned} & \|\chi'(\cdot)(Q_{-i}(u) - Q_{-i}(0)) x_i\| \\ & \leq |\chi'(\cdot)| \|(Q_{-i}(\cdot) - Q_{-i}(0)) x_i\| \in O(\sqrt{\log n}) \pm \mathcal{E}_2, \end{aligned}$$

in $(\mathbb{R}^{[0,1]}, \|\cdot\|_\infty)$. Since the integration between 0 and t is 1-Lipschitz for the infinite norm on $[0, t]$, we have the concentration:

$$\left\| \frac{1}{n} \int_0^t \chi'(u) (Q_{-i}(u) - Q_{-i}(0)) x_i du \right\| \in O(\sqrt{\log n}) \pm \mathcal{E}_2.$$

□

Note first that Y and Y_{-i} have comparable first statistics.

Corollary J.10.

$$\begin{cases} \|\mathbb{E}_{\mathcal{A}_\nu} [Y] - \mathbb{E}_{\mathcal{A}_\nu} [Y_{-i}]\| \leq O\left(\sqrt{\frac{\log n}{n}}\right) \\ \|\mathbb{E}_{\mathcal{A}_\nu} [Y Y^T] - \mathbb{E}_{\mathcal{A}_\nu} [Y_{-i} Y_{-i}^T]\|_* \leq O\left(\sqrt{\frac{\log n}{n}}\right), \end{cases} \quad (\text{J.10})$$

where we recall that $\|\cdot\|_*$ is the nuclear norm satisfying for any $A \in \mathcal{M}_{p,n}$, $\|A\|_* = \text{Tr}(\sqrt{AA^T})$.

Proof. It is just a consequence of Proposition J.9 and the bounds:

$$\begin{aligned} & \left\| \mathbb{E}_{\mathcal{A}_\nu} \left[\frac{1}{n} f_i(x_i^T Y) Q_{-i} x_i \right] \right\| \leq \frac{1}{n} \mathbb{E}_{\mathcal{A}_\nu} [|f_i(x_i^T Y)| \|x_i\| \|Q_{-i}\|] \leq O\left(\frac{1}{\sqrt{n}}\right) \\ & \left\| \mathbb{E}_{\mathcal{A}_\nu} \left[\frac{1}{n^2} f_i(x_i^T Y)^2 Q_{-i} x_i x_i^T Q_{-i} \right] \right\|_* \leq \frac{1}{n^2} \mathbb{E}_{\mathcal{A}_\nu} [|f_i(x_i^T Y)|^2 \|x_i\|^2 \|Q_{-i}\|^2] \leq O\left(\frac{1}{n}\right) \\ & \left\| \mathbb{E}_{\mathcal{A}_\nu} \left[\frac{1}{n} f_i(x_i^T Y)^2 Q_{-i} x_i Y_{-i}^T \right] \right\|_* \leq O\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

□

One can then wonder why we set a result as complex as Proposition J.9 if it was simply to obtain these simple relations between the first statistics of Y and Y_{-i} . It is because the behaviors of Y and Y_{-i} diverge when one looks at projections on x_i .

The observable diameter of order $O(\sqrt{\log n}/n)$ in Proposition J.9 allows us to keep good concentration bounds when Y is multiplied on the left by x_j^T , $j \in [n]$ (indeed, under \mathcal{A}_ν , $\|x_j\| \leq O(\sqrt{n})$). This time, the term $\frac{1}{n}f_i(x_i^T Y)x_j^T Q_{-i}x_i$ can be of order $O(1)$ in particular when $j = i$. For all $j \in [n]$, $\frac{1}{n}f_i(x_i^T Y)x_j Q_{-i}x_i \in \mathcal{E}_2(1/\sqrt{n})$ (as a $1/\sqrt{n}$ transformation of $(X, \sqrt{ny_{-i}^X(\cdot)})$) thus if $j \neq i$:

$$\mathbb{E}_{\mathcal{A}_\nu} \left[\frac{1}{n}f_i(x_i^T Y)x_j Q_{-i}x_i \right] \leq O\left(\frac{1}{\sqrt{n}}\right)$$

but when $j = i$ this quantity can be of order $O(1)$, therefore we obtain the concentrations:

Corollary J.11.

$$\begin{cases} x_j^T Y - x_j^T Y_{-i} \in O\left(\sqrt{\frac{\log n}{n}}\right) \pm \mathcal{E}_1\left(\frac{1}{\sqrt{n}}\right) & \text{when } j \neq i \\ x_i^T Y - x_i^T Y_{-i} - \frac{1}{n}x_i^T Q_{-i}x_i f_i(x_i^T Y) \in O\left(\sqrt{\frac{\log n}{n}}\right) \pm \mathcal{E}_1\left(\frac{1}{\sqrt{n}}\right). \end{cases} \quad (\text{J.11})$$

2.4 Implicit relation between $x_i^T Y$ and $x_i^T Y_{-i}$

The linear concentration (J.11) interests us particularly because it allows us to replace in the identity

$$Y = \frac{1}{n} \sum_{i=1}^n f_i(x_i^T Y)x_i,$$

the quantity $f_i(x_i^T Y)x_i$ with a quantity $f_i(\zeta_i(x_i^T Y_{-i}))x_i$ (for a given mapping $\zeta_i : \mathbb{R} \rightarrow \mathbb{R}$) that is more easy to manage thanks to the independence between x_i and Y_{-i} . The random variable is estimated thanks to a fixed point equation already introduced, in the case of Wishart matrices, in [SB95, PP09, LC22c]. Introducing the notation

$$\tilde{Q}^{\Lambda(\tilde{D})}(\tilde{D}) = \left(I_p + \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\frac{\tilde{D}_i \Sigma_i}{1 + \Lambda_i(\tilde{D})\tilde{D}_i} \right] \right)^{-1},$$

where, for all $\Gamma \in \mathcal{D}_n$, we note $\Lambda(\Gamma)$ the unique solution to:

$$\forall i \in [n] : \Lambda_i(\Gamma) = \frac{1}{n} \text{Tr} \left(\Sigma_i \tilde{Q}^{\Lambda(\Gamma)}(\Gamma) \right)$$

(see Theorem F.18 for existence and uniqueness).

We can then employ Theorems G.1 and G.2 to get:

$$Q \in Q^{\Lambda(D)}(D) \pm \mathcal{E}_1(\log(n)) + \mathcal{E}_{\frac{1}{2}}(1/\sqrt{n}) \quad \text{in } (\mathcal{M}_p, \|\cdot\|_F),$$

For simplicity, we will note from now on:

$$\Delta \equiv \Lambda(D),$$

(where the expectation in the definition of $\tilde{Q}^{\Lambda(D)}(D)$ is taken on \mathcal{A}_ν). One can then deduce in particular from this theorem that for any $i \in [n]$:

$$\frac{1}{n}x_i^T Q_{-i}x_i \mid \mathcal{A}_\nu \in \Delta_i \pm \mathcal{E}_1(\log(n)) + \mathcal{E}_{\frac{1}{2}}(1/\sqrt{n}) \quad (\text{J.12})$$

It sounds then natural to introduce the equation

$$z = x_i^T Y_{-i} + \Delta_i f_i(z), \quad z \in \mathbb{R} \quad (\text{J.13})$$

whose solution is close to $x_i^T Y$ as stated by next proposition.

Proposition J.12. *Given $i \in [n]$, and $x \in \mathbb{R}$, the equation:*

$$z = x + \Delta_i f_i(z), \quad z \in \mathbb{R}, \quad (\text{J.14})$$

admits a unique solution that we note $\zeta_i(x)$, one then has the approximation:

$$x_i^T Y \in \zeta_i(x_i^T Y_{-i}) \pm \mathcal{E}_1 \left(\frac{1}{\sqrt{n}} \right)$$

To prove this proposition, we are going to employ the following lemma which is just an adaptation of the result of Lemma C.7 concerning contractive mappings to the case of convex mappings.

Lemma J.13. *Given a twice differentiable convex mapping $f : \mathbb{R}^p \rightarrow \mathbb{R}$ such that for all $y \in \mathbb{R}^p$, $d^2 f|_y \geq \kappa$ and a vector $y_0 \in \mathbb{R}^p$ such that $\|df|_{y_0}\| \leq \tau$, we know that the fixed point $y^* = \min_{y \in \mathbb{R}^p} f_i(y)$ also satisfies:*

$$\|y^* - y_0\| \leq \frac{\tau}{\kappa}$$

Proof. One just has to introduce $K \equiv \sup_{y \in \mathcal{B}(y_0, \frac{\tau}{\kappa})} \|d^2 f|_y\|$, then the mapping $\phi : y \mapsto y - \frac{1}{K} df|_y$ is $(1 - \frac{K}{\kappa})$ -Lipschitz on $\mathcal{B}(y_0, \frac{\tau}{\kappa})$ and one can conclude with Lemma C.7. \square

Proof of Proposition J.12. As the solution of the minimizing convex problem ($\Delta_i > 0$):

$$\text{Minimize: } \phi(z) = \|z\|^2 + \Delta_i h(z) - x_i^T Y_{-i} z,$$

$\zeta(x_i^T Y_{-i} z)$ is well defined and unique. Now, we can bound under \mathcal{A}_ν :

$$\|d\phi|_{x_i^T Y}\| = \|x_i^T Y + \Delta_i f'_i(x_i^T Y) - x_i^T Y_{-i}\| \in O \left(\sqrt{\frac{\log n}{n}} \right) \pm \mathcal{E}_2 \left(\frac{1}{\sqrt{n}} \right)$$

thanks to Corollary J.11. We can therefore employ Lemma J.13 to deduce the result of the proposition. \square

We end this subsection with a little result that will allow us to differentiate ζ_i and:

$$\xi_i : t \mapsto f_i(\zeta_i(t))$$

Lemma J.14. *Given $i \in [n]$, the mapping ζ_i is differentiable and we have the identity:*

$$\xi'_i(t) = \frac{f'_i(\zeta_i(t))}{1 - \Delta_i f'_i(\zeta_i(t))}.$$

Proof. Considering $z, t \in \mathbb{R}$, let us express:

$$\zeta_i(z+t) - \zeta_i(z) = t + \Delta_i (f_i(\zeta_i(z+t)) - f_i(\zeta_i(z)))$$

thus $|\zeta_i(z+t) - \zeta_i(z)| \leq \frac{t}{1 - \Delta_i \|f'\|_\infty}$ (note that it implies that ζ_i is continuous). Let us bound:

$$\begin{aligned} & |f_i(\zeta_i(z+t)) - f_i(\zeta_i(z)) - f'_i(\zeta_i(z))(\zeta_i(z+t) - \zeta_i(z))| \\ & \leq \|f''\|_\infty |\zeta_i(z+t) - \zeta_i(z)|^2 \leq \frac{t^2 \|f''\|_\infty}{(1 - \Delta_i \|f'\|_\infty)^2} \end{aligned}$$

Dividing the upper identity by t we can bound:

$$\left| \frac{1}{t} (\zeta_i(z+t) - \zeta_i(z)) - 1 - \frac{\Delta_i}{t} f'_i(\zeta_i(z))(\zeta_i(z+t) - \zeta_i(z)) \right| \leq \frac{t \|f''\|_\infty \Delta_i}{(1 - \Delta_i \|f'\|_\infty)^2}.$$

We can then let t tend to 0 to conclude that ζ_i is differentiable and we obtain the identity:

$$\zeta'_i(t) = 1 + \Delta_i f'_i(\zeta_i(t)) \zeta'_i(t).$$

and one easily deduce from this formulation of ζ'_i the expression of ξ'_i \square

The formulation of ξ'_i leads us to introducing the notation

$$\tilde{Q} \equiv (I_p - \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi'_i(x_i^T Y_{-i}) \Sigma_i])^{-1}$$

that satisfies the estimation:

Corollary J.15. $\left\| \tilde{Q} - \tilde{Q}^\Delta(\mathbb{E}[D]) \right\|_F \leq O\left(\frac{\log n}{\sqrt{n}}\right)$

The proof can be done with the same diagonal matrix Δ but we think it would be more interesting to wait for the final definition of Δ to show the stronger result.

2.5 Expression of the mean and covariance of Y .

Introducing the notation $\forall i \in [n]$, $\xi_i = f \circ \zeta_i$, next Proposition gives us first estimations of the deterministic objects:

$$m_Y \equiv \mathbb{E}[Y] \qquad \text{and} \qquad \Sigma_Y \equiv \mathbb{E}[YY^T],$$

Proposition J.16. *Noting $\check{Y} = \frac{1}{n} \sum_{i=1}^n \xi_i(x_i^T Y_{-i}) x_i$, we can approximate:*

$$\|Y - \check{Y}\| \in O\left(\frac{\log n}{\sqrt{n}}\right) + \mathcal{E}_1\left(\frac{\log n}{\sqrt{n}}\right)$$

and we can estimate $\|m_Y - \mathbb{E}[\check{Y}]\|$, $\|\Sigma_Y - \mathbb{E}[\check{Y}\check{Y}^T]\|_* \leq O\left(\frac{\log n}{\sqrt{n}}\right)$.

Proof. Let us bound:

$$\|Y - \check{Y}\| \leq \left\| \frac{1}{n} \sum_{i=1}^n (f_i(x_i^T Y) - f_i(\zeta_i(x_i^T Y_{-i}))) x_i \right\| \leq O\left(\sup_{1 \leq i \leq n} |x_i^T Y - \zeta_i(x_i^T Y_{-i})|\right)$$

Besides, we know from Propositions J.12 that $(|x_i^T Y - \zeta_i(x_i^T Y_{-i})|)_{1 \leq i \leq n} \in \mathcal{E}_1\left(\frac{\log n}{\sqrt{n}}\right)$ in $(\mathbb{R}^n, \|\cdot\|_\infty)$, thus Proposition B.22 implies that:

$$\sup_{1 \leq i \leq n} |x_i^T Y - \zeta_i(x_i^T Y_{-i})| \in O\left(\frac{\log n}{\sqrt{n}}\right) + \mathcal{E}_1\left(\frac{\log n}{\sqrt{n}}\right),$$

from which we deduce the first result of the proposition. The estimation of the expectation and the non-centered covariance of Y is a direct consequence, indeed, for the covariance, note that for any deterministic matrix $A \in \mathcal{M}_p$ such that $\|A\| \leq 1$:

$$\begin{aligned} |\text{Tr}(A(\Sigma_Y - \mathbb{E}[\check{Y}\check{Y}^T]))| &\leq |\mathbb{E}[Y^T A Y - \check{Y}^T A \check{Y}]| = |\mathbb{E}[(Y - \check{Y})^T A (Y + \check{Y})]| \\ &\leq O(\|A\| \mathbb{E}[\|Y - \check{Y}\|]) \leq O\left(\frac{\log n}{\sqrt{n}}\right) \end{aligned}$$

□

2.6 Computation of the estimation of the mean and covariance of Y when X is Gaussian

The estimation given by Proposition J.16 becomes particularly interesting when X is Gaussian because in that case, the random variable $z_i \equiv x_i^T Y_{-i}$ is also Gaussian (when all the random vectors x_j are fixed, for $j \neq i$) and admits the statistics:

$$\mathbb{E}_i[z_i] = m_i^T Y_{-i} \qquad \text{and} \qquad \mathbb{E}_i[z_i^2] = Y_{-i}^T \Sigma_i Y_{-i}$$

(where we recall that $m_i \equiv \mathbb{E}[x_i]$ and $\Sigma_i \equiv \mathbb{E}[x_i x_i^T]$). The estimation of quantities of the form $\mathbb{E}_j[\xi(z_j) u^T x_j]$ will then be done in two steps that will be justified by Lemmas J.18 and J.19 below:

1. “separate” with Stein-like identities, the “functional part” $\xi(z_j)$ from the “vectorial part” $u^T x_j$ in $\mathbb{E}_j [\xi(z_j) u^T x_j]$,
2. show that the randomness brought by Y_{-i} in z_i can be neglected so that it can be approximated by a Gaussian random variable $\tilde{z}_i \sim \mathcal{N}(\mu_i, \nu_i)$ with:

$$\mu_i \equiv m_i^T m_Y \quad \text{and} \quad \nu_i \equiv \text{Tr}(\Sigma_Y \Sigma_i) - \mu_i^2, \quad i \in [n]. \quad (\text{J.15})$$

Proposition J.17. *Introducing the quantities:*

$$\tilde{m}_Y^{(1)} = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(\tilde{z}_i)] \tilde{Q} m_i \quad \text{and} \quad \tilde{C}_Y^{(1)} = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(\tilde{z}_i)^2] \tilde{Q} \Sigma_i \tilde{Q}$$

we have the estimations $\|m_Y - \tilde{m}_Y^{(1)}\|, \|C_Y - \tilde{C}_Y^{(1)}\|_* \leq O\left(\frac{\log n}{\sqrt{n}}\right)$.

The estimation merely rely on two lemmas. The first one is a derivation of the Stein identity:

Lemma J.18. *Given a Gaussian vector $x \sim \mathcal{N}(\mu, C)$ for $\mu \in \mathbb{R}^p$ and $C \in \mathcal{M}_p$ positive symmetric, two deterministic vectors $w, u \in \mathbb{R}^p$, and a deterministic matrix $A \in \mathcal{M}_{p,n}$, we have the identities:*

$$\begin{aligned} \mathbb{E}[f_i(w^T x) u^T x] &= \mathbb{E}[f_i(w^T x)] u^T \mu + \mathbb{E}[f'_i(w^T x)] u^T C w \\ \mathbb{E}[f_i(w^T x) x^T A x] &= \mathbb{E}[f_i(w^T x)] \text{Tr}(A(\mu \mu^T + C)) + \mathbb{E}[f'_i(w^T x)] w^T C (A + A^T) \mu \\ &\quad + \mathbb{E}[f''(w^T x)] w^T C A C w \end{aligned}$$

The second lemma allows us to integrate over Y , it will be proven after the proof of the proposition.

Lemma J.19. *Given two (sequences of) random variables $\mu \in \mathbb{R}$ and $\nu \in \mathbb{R}$ such that $0 < \tilde{\nu} \leq O(1)$ and two (sequences of) deterministic variable $\tilde{\mu}, \tilde{\nu} \in \mathbb{R}$ satisfying:*

$$\mu \in \tilde{\mu} \pm \mathcal{E}_2\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad \nu \in \tilde{\nu} \pm \mathcal{E}_2\left(\frac{1}{\sqrt{n}}\right),$$

if we consider a differentiable mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\|f'\|_\infty \leq O(1)$, then for any Gaussian random variable $z \sim \mathcal{N}(\mu, \nu)$, independent with ν and μ :

$$\mathbb{E}_z[f_i(z)] \in \mathbb{E}[f_i(\tilde{z})] \pm \mathcal{E}_2\left(\frac{1}{\sqrt{n}}\right)$$

where \mathbb{E}_z is the expectation taken only on the variation of z and $\tilde{z} \sim \mathcal{N}(\tilde{\mu}, \tilde{\nu})$.

Proof of Lemma J.19. Let us introduce a Gaussian random variable $y \sim \mathcal{N}(0, 1)$, independent with μ and ν . We can express:

$$\mathbb{E}_z[f_i(z)] \in \mathbb{E}[f_i(\tilde{z})] = \mathbb{E}_y[f_i(\mu + \sqrt{\nu}y)] \equiv \phi(\mu, \nu)$$

The mapping $y \mapsto f_i(\mu + \sqrt{\nu}y)e^{-y^2/2}$ is bounded, we can thus differentiate ϕ and we can bound:

$$\frac{\partial \phi}{\partial \nu} = \mathbb{E}_y[\sqrt{\nu} f'_i(\mu + \sqrt{\nu}y)] \leq O(1) \quad \frac{\partial \phi}{\partial \mu} = \mathbb{E}_y[f'_i(\mu + \sqrt{\nu}y)] \leq O(1)$$

Therefore as $O(1)$ -Lipschitz transformations of μ, ν under $\mathcal{A}_{\mu, \nu}$, we obtain the concentration $\phi(\mu, \nu) \in \phi(\tilde{\mu}, \tilde{\nu}) \pm \mathcal{E}_2(1/\sqrt{n})$ (one can invert the expectation and the function in concentration inequalities of random variables), which is exactly the result of the proposition. \square

Proof of Proposition J.17. Thanks to Lemma J.18, we can express for any $u \in \mathbb{R}^p$ symmetric, such that $\|u\| \leq O(1)$ and noting $\mathbb{E}_{-i} = \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \mathbb{E}_j$:

$$\begin{aligned} u^T m_Y &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{-i} [\mathbb{E}_i [\xi_i(z_i) u^T x_i]] + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(z_i)] u^T m_i + \mathbb{E}_{-i} [\mathbb{E}_i [\xi'_i(z_i)] u^T C_i Y_{-i}] + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(z_i)] u^T m_i + \mathbb{E}[\xi'_i(z_i)] u^T C_i m_Y + O\left(\frac{\log n}{\sqrt{n}}\right) \end{aligned}$$

thanks to Lemma J.19 and Corollary J.10. We can then deduce, replacing u by $u^T \tilde{Q}$:

$$u^T m_Y = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(z_i)] u^T \tilde{Q} m_i + O\left(\frac{\log n}{\sqrt{n}}\right).$$

To estimate the covariance, we can once again deduce from Proposition J.9 that for any $A \in \mathcal{M}_p$ such that $\|A\| \leq 1$:

$$\begin{aligned} \text{Tr}(A \Sigma_Y) &= \mathbb{E}[Y^T A Y] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{-i} [\mathbb{E}_i [\xi_i(z_i) Y^T A x_i]] + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{-i} [\mathbb{E}_i [\xi_i(z_i) Y_{-i}^T A x_i]] + \frac{1}{n} \mathbb{E}_{-i} [\mathbb{E}_i [\xi_i(z_i)^2 x_i^T Q A x_i]] + O\left(\frac{\log n}{\sqrt{n}}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(z_i)] \text{Tr}(m_Y A m_i) + \mathbb{E}[\xi'_i(z_i)] \text{Tr}(\Sigma_Y A C_i) \\ &\quad + \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(z_i)^2 \text{Tr}(\Sigma_i Q A)] + O\left(\frac{\log n}{\sqrt{n}}\right) \end{aligned}$$

note that we could remove several terms from the formula of Lemma J.19, since $\|Y_{-i}\| \leq 1$. Replacing A by $A \tilde{Q}$, and employing Theorems G.1 and G.2 we then obtain:

$$\text{Tr}(A \Sigma_Y) = m_Y^T A m_Y + \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\xi_i(z_i)^2 \text{Tr}(\Sigma_i \tilde{Q} A \tilde{Q})] + O\left(\frac{\log n}{\sqrt{n}}\right)$$

□

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