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Scalar interactions, Gravity and Black Holes

Carlo Branchina

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**Scalar interactions, Gravity and Black
Holes**

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Alle mie nonne

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Introduction

Last century has witnessed a radical change of paradigms in our understanding of the laws of nature. The resolution of the long-standing problem of the black-body radiation known as the "ultraviolet catastrophe" by Planck in 1901 together with the subsequent explanation of the photoelectric effect by Einstein in 1905 opened the way to the concept of quanta of energy. This would eventually find a more profound understanding with the advent of the theory of Quantum Mechanics (QM), formalized, among others, by Heisenberg, Schrödinger and Dirac, through which we understand, by means of the widely accepted Born interpretation, the laws of Nature at distances much smaller than the human scales. On the other hand, the new, quantum effects are seen to be heavily suppressed at the macroscopic scales we usually experience, and the quantum description reduces to the classical, Newtonian one, in these conditions.

Besides the quantum, another revolution, which we may call the "relativistic revolution", started in 1905 when Einstein first developed the theory of Special Relativity (SR). The concepts of absolute time and immobile space on which rested the classical Newton's theory were abandoned to the idea of the more intricate spacetime, where those that appear to us as two different and separated components are actually intertwined and do not exist by themselves. The new, relativistic effects that emerge in this framework are of the greatest importance in the description of the motion of bodies with velocity v close to the speed of light. Nevertheless, for the much smaller values of v that we experience in everyday's life, those effect are seen to be suppressed, and the description given by classical mechanics turns out to be appropriate in this regime. In the SR framework, the spacetime only plays the role of a background in the description of the motion of bodies. In 1915, the relativistic paradigm was enlarged to account also for the dynamics of such spacetime with the theory of General Relativity (GR), where what we perceive as gravity is identified with the geometry of the space-time itself. The study of solutions to the theory's equations lead to several important results, among which the discovery of black holes (BHs), and the establishment of the Standard Model of Cosmology, that describes the history of the Universe from its first few instants of life.

The profound idea these two "revolutions" share, that historically, for obvious reasons, struggled to come to light, is that the fundamental laws of Nature do not always

(regardless of any condition) have to look as they commonly appear to us, but actually only need to reproduce the world as we see it in the particular conditions characteristic of our experience. In other words, the quantum and relativistic paradigms do not enter in conflict with the Newtonian description of Nature, but rather explain how the classical picture is embedded in something more fundamental and how it is re-obtained as a perfectly valid effective description in an appropriately defined limit.

Since this first example, the concept of effective descriptions has been at the core of our understanding of physics. Quantum Mechanics and Special Relativity themselves are only specific corners of more important structures. One of the greatest achievements of Quantum Mechanics, for example, is the explicit solution for the hydrogen atom with its quantized energy levels. The correspondence between the energy differences of such levels and the empirically observed Balmer's formula is usually argued to provide, already in the context of Quantum Mechanics, an explanation of the hydrogen atom emission spectrum in terms of the transition of the electron between the different levels. However, the quantum mechanical evolution by itself does not account for such transitions between stationary states. The solution of the puzzle is again the same: Quantum Mechanics is only one of the different layers towards the fundamental description of Nature, and as such it has a well-defined regime of validity. For the problem at hand, it should be noted that Quantum Mechanics is intrinsically non-relativistic, and it is for this reason that it can only describe processes with a fixed number of particles. After some not completely satisfactory attempts, the combination of Quantum Mechanics with the theory of Special Relativity eventually resulted in the foundation of the Quantum Field Theory (QFT) framework in the 30s. In the early days, QFT was only thought to be the quantum theory of fields, like the electromagnetic field, that were already known classically, in the same way as QM is the quantum theory of a point-like particle. It was only later that the QFT paradigm was realized to be an inevitable consequence of the QM and SR¹ merging [8], and quantum fields were understood to be the basic ingredients in this new description, through which particles are described as excitations, lumps of energy of these fields.

Historically, Quantum Electrodynamics (QED), the quantum theory that describes the electromagnetic interaction of a single fermionic particle, usually the electron, was the first developed QFT that found successful experimental verification. The most usual calculation technique that allows to make falsifiable theoretical predictions is perturbation theory, where physical amplitudes like the scattering of n particles are calculated through a series expansion in the parameter \hbar that controls the (hopefully smaller and smaller) quantum corrections over the classical result, encoded in the zeroth order term. The perturbative analysis rests on the assumption that an interacting theory can be treated as a small perturbation of a free (non-interacting) one. As the strength of an interaction is determined by the value of its coupling constants, the validity of the perturbative expansion is related (but not completely determined) to the smallness of such couplings. Of course, this does not always need to be the case, and many physical phenomena that can be described within the QFT framework indeed necessitate of a non-perturbative approach.

The appearance of divergent integrals at higher orders of the perturbative expansion seemed as an insurmountable problem until Feynman, Schwinger and Tomonaga

¹ Plus the cluster decomposition principle

developed in 1949, for QED, a method to extract physical contributions from these divergent integrals: the Renormalization. In the process of renormalization, each parameter is divided in an observable and an unobservable part, and the divergences are isolated in the unobservable one through renormalization conditions. A QFT where a finite number of renormalization conditions is sufficient to render all higher orders in the perturbative expansion finite and calculable, is said to be renormalizable. On the contrary, non-renormalizable theories necessitate of new renormalization conditions at each order of the expansion and the theory needs an infinite number of parameters where to isolate the divergences to be rendered finite. The common lore, in the first decades of QFT, was that only renormalizable QFTs are consistent and of physical relevance. This was the cause of two serious concerns: a quantum field theoretic description of gravity that reduces to GR in the classical (as opposed to quantum) limit would necessarily be non-renormalizable, as well as the theory of weak interactions proposed by Fermi in 1933.

The resolution of the second puzzle came when, in the context of QFT, Glashow, Weinberg and Salam landed in 1967 a unified theory of three (electromagnetic, strong and weak) of the four fundamental forces, the Standard Model (SM), that was later shown in 1971 by 't Hooft to be renormalizable. In the SM, the electromagnetic and weak interactions unify at the Fermi scale $\mu_F = 247$ GeV and both QED and the Fermi theory are verified to be obtained as the low-energy effective descriptions of the Standard Model. The non-renormalizable four-fermion contact interaction of the Fermi theory is understood to emerge as an effective description of a more fundamental interaction mediated by the massive gauge bosons W and Z when the energy scale of observation is sufficiently smaller than their masses. In a low-energy effective theory valid for scales up to a certain Λ , in fact, the particles (and more generally any degree of freedom) with masses $m > \Lambda$ do not explicitly appear, and their (suppressed) contribution is encoded in the values of certain low-energy coupling constants, that might often render the effective theory non-renormalizable.

A great boost in the process of understanding the whole concept of effective descriptions in the context of QFT came from the development of a more physical approach to renormalization, the Renormalization Group (RG), started by Stueckelberg and Petermann in 1953 and Gell-Mann and Low in 1954. The main idea of the Renormalization Group (RG), especially in the formulation that followed from the Gell-Mann and Low seminal work, is that a physical process at a certain energy scale is better expressed in terms of the parameters defined *at that* energy scale. The original example of Gell-Mann and Low is very intuitive and pedagogical in this sense [9]. They consider the Serber-Uelhing potential, i.e. the one-loop (next to leading order) quantum correction in the perturbative expansion to the Coulomb potential between two particles. At small distances $r \ll \hbar/mc$, it takes the form

$$V(r) = \frac{qq'}{4\pi r} \left\{ 1 + \frac{2\alpha}{3\pi} \left(\ln \left(\frac{\hbar}{mcr} \right) - \frac{5}{6} - \ln \gamma + \mathcal{O}(\alpha^2) \right) \right\},$$

where q and q' are the experimentally measured charges, α is the QED fine structure constant, r is the space separation between the two charges and γ is the Euler-Mascheroni constant $\gamma \sim 0.58$. When $r \rightarrow 0$, each particle feels the bare charge of the other one. On the contrary, at a finite distance, each charge feels the "bare charge" of the

other plus all the vacuum polarization effects of QED that screen it, i.e. the so-called "dressed charge". The charges q and q' are experimentally measured at a specific distance scale \tilde{r} (distance scale \sim energy scale $^{-1}$), and of course do not correspond to the bare charges. Apart from the divergence at $r = 0$, that is the well-known logarithmic divergence one would have extrapolating QED up to infinite energy, it is clear that a redefinition of the charges such that

$$q(r)q'(r) \equiv qq' \left\{ 1 + \frac{2\alpha}{3\pi} \left(\ln \left(\frac{\hbar}{mcr} \right) - \frac{5}{6} - \ln \gamma + \mathcal{O}(\alpha^2) \right) \right\}$$

greatly simplifies the expression of $V(r)$. The charges $q(r)$ and $q'(r)$ are those we referred to as relevant quantities *at* the distance scale r . As functions of r , they incorporate in their definition the way the quantum fluctuations effectively correct, or in jaergon "dress", the bare quantities as the distance/energy scale varies. The physical charge $q (= q(\tilde{r}))$ shows divergences only when it is expanded in terms of the bare charge $q_0 (= q(0))$. An expansion of the generic charge $q(r)$ in terms of the generic charge $q(r')$ only shows finite terms of the type $\ln(r'/r)$.

Since the seminal works of 1953 and 1954, the RG has been greatly developed and has found a large number of applications, especially in condensed matter physics and in the description of critical phenomena. The intimate connection between statistical physics and QFT in the Feynman's path integral formalism has then allowed for a cross-fertilizing exchange between the two. The deeper understanding of the Renormalization Group came with Kadanoff's idea of the blocking transformation for a spin system known as the Ising model. In a region close to the critical point, where a phase transition occurs, the spin system is completely dominated by the (thermal or quantum, depending on what kind of phase transition we are describing) fluctuations. The spins are correlated between each other and show a collective behaviour. Grouping the spins in blocks and rescaling the distances, we obtain a completely equivalent effective description of the system, not in terms of the original, fundamental, degrees of freedom, but in terms of new, derived parameters. Iterating this procedure, the subsequent transformations generate an evolution of the parameters with the length (or energy) scale. The latter then "run" (i.e. evolve) with the energy/distance scale, and this running encodes the way the fluctuations dress their bare values. In other words, a Quantum Field Theory is defined by a curve in parameter space, and each point on this curve determine how the theory looks at a specific energy scale.

This idea found a robust formulation in the works of Wilson in the 70s. Starting from the theory at a given high energy scale Λ , the Wilson RG aims at reaching the effective description of the theory at a lower scale $k < \Lambda$ by means of a stepwise evaluation of the fluctuations. The central idea is that both in QFT and in the theory of critical phenomena the difficulties arise when one tries to treat many degrees of freedom (and many scales) all at once (as it is done in the perturbative framework for e.g.). The iterative procedure thus aims at decouplings all these scales considering only a small subsets of them at each step. In the meanwhile, another approach to the Renormalization Group was developed, directly in the particle physics language, by Callan and Symanzik. This is built in the context of perturbative QFT, and the running encodes the evolution with energy of renormalized quantities. Still in the context of perturbation theory, yet another (much less common) version of the RG was developed to account

for the running of the bare quantities, where the word "bare" now refers to quantities obtained from regularized integrals encountered in the intermediate steps of renormalization. Unlike these two, the Wilson method is intrinsically non-perturbative: it reduces to the perturbative ones only under specific approximations.

Under these influences, the approach to QFT has radically changed. Quantum Field Theories², and in particular the Standard Model, are understood to be effective descriptions of something more fundamental. That the Standard Model cannot be the ultimate layer in the description of Nature is already obvious from the fact it does not describe gravity. Other experimental and observational difficulties come from the missing description of neutrino masses, dark matter, the naturalness/hierarchy problem, the strong CP problem, and many others. Even though these shortcomings are present, the tight experimental scrutiny under which it has been put has decreed a spectacular triumph for the SM predictions obtained within the framework of perturbation theory. Strictly speaking, the infinities appear in the perturbative expansion only if we allow the intermediate states in the calculation of the quantum corrections to carry any value of momentum (and thus of energy) up to infinity. If this were the case, it would mean that with the QFT paradigm we have found the ultimate layer in the description of Nature: quantum fields would be the most fundamental, microscopic and high energetic objects. Nevertheless, as alluded previously, at sufficiently low energies the QFT paradigm is unavoidable as an effective description. Whatever the complete/fundamental theory is, it must be well described by a QFT in a certain energy range, and the mentioned triumph of the Standard Model predictions suggest that this QFT is either the SM or a slight modification of it.

All these observations boil down to the concept of effective field theories (EFTs), quantum field theories with a built-in UV (energy) scale Λ , below which the QFT gives an appropriate description. Above it, new degrees of freedom take over, and the theory needs to be appropriately "UV completed". Whatever the completion is (an UV safe QFT or something built on a completely different paradigm) its imprint on the lower energy effective field theory is encoded in the value it provides for the parameters that define the latter at the high energy scale Λ . The value of the same parameters at lower scales is then obtained by means of the renormalization group methods mentioned above.

The new perspective gained on QFTs as low-energy effective descriptions of more general structures has also opened up a way to a field theoretic description of gravity, where some of its low-energy aspects can be studied. Although gravity is completely absent in the Standard Model, the reason why the predictions of the latter have been experimentally verified to a great amount of precision is that the gravitational coupling G_N is parametrically much smaller than the other couplings, so that gravity can successfully be ignored in a first approximation. Of course, the low-energy field theoretic description of the (unknown) fundamental theory should describe both the SM and gravity. The non-renormalizable character of the gravitational EFT is of no concern, as it is not a fundamental theory and the operators that make it non-renormalizable are suppressed by powers of the Newton coupling. In the most extreme case scenario, the EFT describing both the SM and gravity can be valid up the Planck scale $M_P \sim G_N^{-1/2} \sim 10^{19}$ GeV, and for phenomena whose typical scale is at accessible ener-

² Apart from some attempts at asymptotically safe theories in the UV

gies $E \lesssim 10^3 - 10^4$ GeV, the impact of these terms on physical quantities is enormously suppressed by powers of E/M_P .

The present thesis is divided in two main parts, both framed in the logic described above. The first topic is somehow related to the issue of the quantization of gravity and its low-energy EFT description. Although the fundamental theory of nature is still eluding us, some candidates have been proposed, among which String Theory (ST) is perhaps the most studied one. String Theories are defined in $D > 4$ dimensions, and should reduce, as any other candidate (see the discussion above), to a four-dimensional field theory including both the SM and gravity in the appropriate low-energy limit after the additional dimensions have been compactified, i.e. integrated out to be made effectively "invisible". In this respect, the Swampland program aims to single out, among the space of all self-consistent EFTs, those respecting known features of ST. Its philosophy was later extended to include more general properties of Quantum Gravity (QG). In other words, the Swampland program aims to turn around and go in the opposite direction to the common QFT approach, very successful in the past, that gravity has no influence on physics at low enough energies and can be ignored in such regimes, and aims at pointing out the additional constraints that its presence imposes on low-energy EFTs for them to enjoy a UV completion consistent with Quantum Gravity. The criteria are of conjectural nature and the Weak Gravity Conjecture (WGC), originally proposed for a $U(1)$ gauge theory, is one of the best established among them. Since its inception, several extensions have been discussed. The first part of the thesis is dedicated to a study of few such extensions, both in the case of theories with scalar fields and in the case of theories with an additional weakly coupled gauge boson, and is based on [1, 2, 3, 4]. The case with scalar fields was regarded with much interest, as the latter commonly appear in the compactification process of both string theories and field theories with more than four dimensions, and the guiding principle to define a WGC in this case had not been identified. The theories with additional, weakly coupled gauge bosons have been of interest for a while now as a possibility to realize a portal between the SM particles and dark matter.

The second part of this work is dedicated to the study of the so-called naturalness/hierarchy problem in the context of the Wilsonian approach to QFT briefly outlined above. Already in the old days of Renormalization, scalar masses were known to get severe divergent contributions from the quantum fluctuations. In modern perturbative language, scalar (squared) masses, and in particular the Higgs mass, are better said to get contributions that are quadratically proportional to the ultimate UV scale Λ . Physically, this means that scalar masses are extremely sensitive to the left-over of the UV completion whatever the latter is, which by itself might be a reason of concern. On the more practical side, moreover, this seems to be totally in contrast with the experimentally measured Higgs mass $m_H \sim 125$ GeV and with the fact that no sign of the new physics implied by the traditionally proposed solutions, as for instance supersymmetry (SUSY), has been detected so far. During the last years, several models and mechanisms have been proposed to cancel the UV power sensitivity and naturally generate a small Higgs mass. The second part of the thesis contains a thorough analysis of both the problem and some of these proposed solutions within the Wilsonian approach. A new possible way to think about the problem by taking in full consideration the Wilson's lesson will also emerge. This will be based on [5, 6]

Part I

The Weak Gravity Conjecture

CHAPTER 1

The Weak Gravity Conjecture in the Swampland program

1.1 The Swampland Program

Our current understanding of the physical world and its elementary building blocks is based on two widely appreciated theories (see the Introduction for a more thorough discussion). On the one hand, the Standard Model describes three of the four fundamental interactions, the electromagnetic, weak and strong interactions, in the framework of Quantum Field Theory. The understanding of the gravitational interaction relies instead on Einstein's theory of General Relativity. The modern approach to QFT based on the concept of Effective Field Theories has, at least in the energy regime where the physical description is given in terms of a QFT, partially circumvented the long-standing problem of the quantization of gravity. At sufficiently low energies, General Relativity, or even theories beyond it, can be treated as an EFT. The treatment is not fully quantistic, but rather semi-classical, and will be briefly described later.

Needless to say, this surely does not solve the issue of the quantization of gravity, and the merging of a gravitational EFT to the SM (or extensions of it) does not provide a fundamental theory¹, as discussed in the Introduction. A much studied proposal for a Theory of Everything (TOE) is String Theory, where the most fundamental objects of which all matter is made are tiny vibrating strings. The potential of String Theory to be a TOE comes from the finite size of the strings, as opposed to the lack of particles' shape in the field theoretic description. It was already recognized in classical electrodynamics, and in particular in the calculation of the Abraham-Lorentz force, that it is the approximation of particles with point-like objects that leads to problematic infinities. It was, however, extremely difficult to develop a model with finite-size elementary objects. The struggle with infinities in the old days of Renormalization in QFT is of the same nature, the divergences appearing only when the theory is extrapolated down to zero distances/infinite energy. It is indeed well-known that, even in QFT, non-local interactions generate UV finite contributions and do not suffer from these problems.

¹ Apart from some attempts to make such theories finite, like Asymptotically Safe Gravity.

Consistency conditions impose that the different known String Theories must live in $D = 10$ spacetime dimensions. The way to reconcile this with the fact that we only experience 4 dimensions is to consider the additional ones in the shape of a (sufficiently small) compact manifold. This can be understood with a very simple pedagogical example. Suppose we have a perfectly cylindrical rope and we attach both its ends somewhere to do some rope walking. The struggle of a human being to stay in equilibrium on the rope comes from the fact that, on its scales, the rope is effectively a one-dimensional system that can be well approximated by a line. If, on the other hand, an ant is put on the same rope, it can experience the circular dimension and will realize that the system is actually a two-dimensional cylinder. The process of effectively hiding the extra dimensions to low-energy (macroscopic) observers goes under the name of "compactification". The arguments discussed in the Introduction about the inevitability of the QFT description make it that, upon compactification, String Theory must reduce in a given low energy range to a four-dimensional EFT that contains both gravity and the Standard Model. More generally, any theory of Quantum Gravity necessarily have an energy range where it is well approximated by an EFT.

A top-down approach to derive a falsifiable field theory limit from string theory has proved to be extremely difficult. The huge number of string vacua, about which the low-energy EFTs are constructed, was originally thought to be too large to generate any constraint on them. A different, rather down-top approach, was developed when it was realized that this is not the case: not all self-consistent EFTs can actually be derived from string theory [10]. Although it still leaves a huge landscape of possible models, UV completion into a string theory was observed to be more constraining than only the requirement of self-consistency of the field theory. The natural idea that followed from this observation was to derive non-trivial bounds on EFTs from the requirement that they should be completed at high energies in a string theory. A low-energy observer in the EFT domain of validity could not derive those bounds from any principle. The effective theories that do not respect the constraints so derived are said to fall into the Swampland, while those that do respect them belong to the Landscape. From its inception, another approach to the Swampland has been developed, whose aim is to pin out general properties of Quantum Gravity that might distinguish between EFTs that can have a UV completion that respects such properties and EFTs that cannot. Proof of the criteria is beyond the scopes and the possibilities of the Swampland program, as this would require a knowledge of Quantum Gravity and of the derivation of its low-energy limit that would make the program itself useless. In the second approach, that is the one adopted in this thesis, the idea is rather to start from general properties of Quantum Gravity that we either know or reasonably think should be in place, and try to motivate how such properties should reflect in the EFT. In the more stringy approach, it is customary to gather evidence from string examples and promote features that are shared by those example to general principles. Of course, as we do not have control of the UV theory we are trying to shape, some arguments might be weaker than others and sometimes they might even turn out to be misleading. The various conjectures should better be seen as suggestions, or signposts, as discussed for example in [11], towards the theory of Quantum Gravity and/or the link between string theory and field theory. For a pedagogical introduction to the Swampland program see [11, 12, 13].

Among the physical systems that allow to shed some light on properties of quantum gravity even without a proper theory that describes it, black holes certainly play a central role. These objects are so dense that their behaviour is completely dictated by the gravitational field in a strong coupling regime, that corresponds to the one where quantum gravity should become of great importance. At large distances, however, they are well understood as extended solutions of GR. The semiclassical treatment of their interactions and their description as thermodynamic systems with a temperature and an entropy has allowed to describe some of their quantum aspects, such as the black hole evaporation, and provide a bridge between the classical and quantum regime. While the search for the complete theory progresses, physical processes related to black holes represent an incomparable window on the quantum properties of gravity, and indeed they frequently appear to justify or suggest Swampland criteria.

1.2 Gravity as an Effective Field Theory

Before moving to the central subject of this chapter, I will briefly present here how the EFT formalism is applied to gravity (for a thorough and pedagogical introduction to the subject see [14, 15, 16]). This provides the framework for the subsequent discussion on the Weak Gravity Conjecture.

The starting point of this presentation is the action

$$S = S_{EH} + S_m. \quad (1.2.1)$$

In this equation, S_{EH} is the well-known Einstein-Hilbert action for the gravitational field

$$S_{EH} = \int d^4x \sqrt{-g} \frac{R}{2\kappa^2}, \quad (1.2.2)$$

where $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar and $R_{\mu\nu}$ the Ricci tensor, g the determinant of the metric and κ is related to the Newton's constant G_N and the reduced Planck mass through $\kappa^2 = 8\pi G_N = 1/M_P^2$. S_m is the action for a matter field that, for simplicity, will be taken to be a real scalar field

$$S_m = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right). \quad (1.2.3)$$

That the variation of S with respect to the metric field $g_{\mu\nu}$ leads to the Einstein's equations

$$\begin{aligned} \delta S_{EH} &= \int d^4x \frac{\sqrt{-g}}{2\kappa^2} \left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R \right) \delta g^{\mu\nu} \\ \delta S_m &= \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} \\ \frac{\delta S}{\delta g^{\mu\nu}} &= 0 \iff R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R = \kappa^2 T_{\mu\nu}, \end{aligned} \quad (1.2.4)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the scalar field, follows from

$$\delta \det g = \det(g + \delta g) - \det g = e^{\text{Tr} \log(g + \delta g)} - e^{\text{Tr} \log g} = \text{Tr}(g^{-1} \delta g) = g^{\mu\nu} \delta g_{\mu\nu}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{\delta g}{\sqrt{-g}} = -\frac{1}{2}\frac{g^{\mu\nu}\delta g_{\mu\nu}}{\sqrt{-g}}, \quad (1.2.5)$$

where order higher than one in $\delta g_{\mu\nu}$ have been ignored.

In the weak-field regime, the gravitational field $g_{\mu\nu}$ is expanded about the Minkowski background as $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$. Imposing the orthogonality condition $g_{\mu\rho}g^{\rho\nu} = \delta_{\mu}^{\nu}$, the expansion for the inverse metric $g^{\mu\nu} = \eta^{\mu\nu} - 2\kappa h^{\mu\nu} + 4\kappa^2 h^{\mu\rho}h_{\rho}^{\nu} + \mathcal{O}(\kappa^3)$ is obtained. As for the usual gauge symmetries, the general coordinate invariance of General Relativity introduces a redundancy that is translated in the fact that the ansatz for the metric is not unique. A particularly convenient choice of gauge where to perform the calculations is the de-Donder gauge, where we impose

$$g^{\mu\nu}\Gamma_{\mu\nu}^{\rho} = 0. \quad (1.2.6)$$

At first order in $\kappa(h)$, this translates in the well-known condition $\partial_{\mu}h_{\nu}^{\mu} - 1/2\partial_{\nu}h_{\rho}^{\rho} = 0$. It is now a simple matter of calculation to expand the action S_{EH} in powers of κ . For the determinant of the metric, we write

$$\sqrt{-g} = e^{\frac{1}{2}\log(\det g)} = e^{\frac{1}{2}\text{Tr}(\log g)} \simeq e^{\frac{1}{2}\text{Tr}\left(2\kappa h - \frac{(2\kappa h)^2}{2} + \mathcal{O}(\kappa^3)\right)} \simeq 1 + \kappa h_{\mu}^{\mu} - \kappa^2 h^{\mu\rho}h_{\rho\mu} + \frac{\kappa^2}{2}(h_{\mu}^{\mu})^2 + \mathcal{O}(\kappa^3). \quad (1.2.7)$$

A direct expansion of the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}\left(\partial_{\rho}\Gamma_{\mu\nu}^{\rho} - \partial_{\nu}\Gamma_{\rho\mu}^{\rho} - \Gamma_{\mu\sigma}^{\rho}\Gamma_{\nu\rho}^{\sigma} + \Gamma_{\mu\nu}^{\rho}\Gamma_{\rho\sigma}^{\sigma}\right), \quad (1.2.8)$$

where Γ are the Christoffel symbols, is slightly more complicated. A great simplification, that turns out to be extremely useful if one wishes to go beyond second order in the product $\sqrt{-g}R$ and describe self-interactions of gravitons, comes after some simple algebraic steps are performed.

Using the equality $\log \det g = \text{Tr} \log g$, it is easy to verify that

$$\partial_{\nu}g = g g^{\mu\lambda}\partial_{\nu}g_{\mu\lambda}, \quad (1.2.9)$$

and $\Gamma_{\mu\nu}^{\mu}$ can then be written as

$$\Gamma_{\mu\nu}^{\mu} = \frac{1}{2}\partial_{\nu}\log g. \quad (1.2.10)$$

This means that $\Gamma_{\mu\nu}^{\mu}$ is a derivative and, in passing, we note that in turn this also shows that $\partial_{\rho}\Gamma_{\mu\nu}^{\mu}$ is symmetric in the indices ρ and ν , as it should be since a term of this type appears in the definition of the Ricci tensor (1.2.8). From the property $(\sqrt{-g})^{-1}\partial_{\nu}\sqrt{-g} = 1/2g^{-1}\partial_{\nu}g$ it is straightforwardly obtained that

$$\sqrt{-g}\Gamma_{\mu\nu}^{\mu} = \partial_{\nu}\sqrt{-g}. \quad (1.2.11)$$

From the definition of the Christoffels

$$\Gamma_{\mu\nu}^{\rho} = \frac{g^{\rho\sigma}}{2}\left(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\right), \quad (1.2.12)$$

it is immediate to realize that

$$\partial_{\sigma}g_{\mu\nu} = \Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma}. \quad (1.2.13)$$

Moreover, the condition $\partial_\sigma g_\nu^\alpha = 0$ leads to the relation

$$\partial_\sigma g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \partial_\sigma g_{\mu\nu}. \quad (1.2.14)$$

Combining (1.2.11), (1.2.13) and (1.2.14) leads to

$$\partial_\nu (\sqrt{-g} g^{\mu\nu}) = -\sqrt{-g} g^{\nu\rho} \Gamma_{\rho\nu}^\mu \quad (1.2.15)$$

$$\partial_\sigma (\sqrt{-g} g^{\mu\nu}) = \sqrt{-g} (g^{\mu\nu} \Gamma_{\alpha\sigma}^\alpha - g^{\nu\beta} \Gamma_{\beta\sigma}^\mu - g^{\mu\alpha} \Gamma_{\alpha\sigma}^\nu). \quad (1.2.16)$$

Combining now equations (1.2.11), (1.2.15) and (1.2.16), it is finally seen that up to integration by parts the action

$$\tilde{S}_{EH} = \int d^4x \frac{\sqrt{-g}}{2\kappa^2} g^{\mu\nu} (\Gamma_{\mu\sigma}^\rho \Gamma_{\nu\rho}^\sigma - \Gamma_{\mu\nu}^\rho \Gamma_{\rho\sigma}^\sigma) \quad (1.2.17)$$

is actually equivalent to the original one (1.2.2). Equation (1.2.17) turns out to be an extremely useful simplification when one wishes to expand the gravitational action in h , particularly if one wishes to go beyond second order.

Before doing so, we should first observe that, inserting the ansatz for the metric, the Einstein equation (1.2.4) at first order in h reads

$$\square \left(h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h^\rho{}_\rho \right) = -\kappa T_{\mu\nu} \quad (1.2.18)$$

that, in D generic dimensions², can be easily rewritten as

$$\square h_{\mu\nu} = -\kappa \left(T_{\mu\nu} + \frac{\eta_{\mu\nu}}{D-2} T^\rho{}_\rho \right). \quad (1.2.19)$$

The linearized equation (1.2.19) shows that $h_{\mu\nu}$ respects the usual massless Klein-Gordon equation with source given by the r.h.s. of the same equation. Beyond the linear approximation, the higher orders in h will themselves serve as a source to the gravitational field. This makes it clear that $h_{\mu\nu}$ is the field to be quantized. The quantization proceeds as usual with the other fields. The general solution of the linearized equation of motion in the absence of sources is given in terms of plane waves. Two helicities, $\lambda = \pm 2$, are taken for the graviton $h_{\mu\nu}$ to make the theory parity-symmetric and to match the two degrees of freedom described by GR. We indicate the graviton polarization tensor with $\epsilon_{\mu\nu}(\lambda)$, that can be taken as the product of two helicity-one polarization vectors: $\epsilon_{\mu\nu}(\lambda) = \epsilon_\mu(\lambda/2) \epsilon_\nu(\lambda/2)$. The field is then expressed in terms of its creation-annihilation operators as

$$h_{\mu\nu} = \sum_{\lambda=++,--} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \{ a(p, \lambda) \epsilon_{\mu\nu}(p, \lambda) e^{-ipx} + a^\dagger(p, \lambda) \epsilon_{\mu\nu}^*(p, \lambda) e^{ipx} \} \quad (1.2.20)$$

and is finally quantized promoting the coefficients a and a^\dagger to operators obeying the usual commutation relations

$$[a(p, \lambda), a^\dagger(p', \lambda')] = (2\pi)^3 \delta^3(p - p') \delta_{\lambda, \lambda'}. \quad (1.2.21)$$

² Up until now, the results have all been shown for the particular case of $D = 4$ dimensions for the sake of simplicity, but the dimension of the spacetime has never played any role and the results are valid for any value of D

To keep track of the necessary choice of gauge, it is usual to insert in the action the gauge-fixing term

$$\sqrt{-g} \mathcal{L}_{GF} = \sqrt{-g} \xi \partial_\mu \left(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) \partial^\rho \left(h_{\rho\nu} - \frac{1}{2} \eta_{\rho\nu} h \right), \quad (1.2.22)$$

where here and thereafter in the equations h stands for h^ρ_ρ . The de Donder gauge corresponds to $\xi = 1$. This can be easily verified by a "brute-force" calculation. Expanding the gravitational action to second order in de Donder gauge, the quadratic term

$$\frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} - \frac{1}{4} h^\rho_\rho \square h^\sigma_\sigma = \frac{1}{2} h_{\mu\nu} \mathcal{O}^{\mu\nu\rho\sigma} h_{\rho\sigma} \quad (1.2.23)$$

is obtained for h , where $\mathcal{O}^{\mu\nu\rho\sigma}$ is a differential operator defined as

$$\mathcal{O}^{\mu\nu\rho\sigma} = \frac{\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}}{2} \square. \quad (1.2.24)$$

In D dimensions, the inverse operator in Fourier space

$$D^{\mu\nu\rho\sigma}(p) = \frac{i}{p^2} \mathcal{P}^{\mu\nu\rho\sigma}, \quad \mathcal{P}^{\mu\nu\rho\sigma} \equiv \frac{\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}}{2} - \frac{\eta^{\mu\nu} \eta^{\rho\sigma}}{D-2} \quad (1.2.25)$$

then defines the graviton propagator. Note that, due to the cancellation between the prefactor $1/(2\kappa^2)$ and the κ^2 factor coming from the expansion to second-order, the quadratic term correctly is of order κ^0 . All the terms coming from the expansion at order n carry a factor κ^{n-2} . Accordingly, all terms beyond second order that arise from this expansion are self-interacting terms of the gravitational field with the same, universal, coupling constant given by κ . As κ is the inverse of the reduced Planck mass M_P , this also evidentiates the non-renormalizable character of the gravitational EFT discussed in the Introduction.

We report here, with in mind the future application we will make of it, the three-graviton interacting term:

$$\begin{aligned} (\sqrt{-g} \mathcal{L})_{h^3} = & \frac{\kappa}{2} (h \partial_\mu h \partial^\mu h - 2h \partial_\mu h^{\mu\nu} \partial_\nu h - 2h^\mu_\nu \partial_\mu h \partial^\nu h - 4h_{\mu\nu} \partial_\rho h^{\mu\nu} \partial^\rho h - h \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} \\ & + 4h_{\mu\nu} \partial^\nu h^{\mu\rho} \partial_\rho h + 4h_{\mu\nu} \partial_\rho h^{\mu\rho} \partial^\nu h + 2h \partial_\mu h_{\nu\rho} \partial^\nu h^{\mu\rho} + 4h_{\mu\nu} \partial_\rho h^{\mu\nu} \partial_\sigma h^{\rho\sigma} \\ & + 4h_{\mu\nu} \partial_\rho h^{\mu\sigma} \partial^\rho h^\nu_\sigma - 4h_{\mu\nu} \partial_\rho h^{\mu\sigma} \partial_\sigma h^\nu_\rho + 2h_{\mu\nu} \partial^\mu h_{\rho\sigma} \partial^\nu h^{\rho\sigma} \\ & - 8h_{\mu\nu} \partial^\nu h_{\rho\sigma} \partial^\sigma h^{\mu\rho}). \end{aligned} \quad (1.2.26)$$

The gravitational interaction of the scalar field is obtained expanding $\sqrt{-g}$ and $g^{\mu\nu}$ in (1.2.3). At first order in κ it is given by the coupling to the energy-momentum tensor

$$\sqrt{-g} \mathcal{L}_{\phi^2 h} = -\kappa h^{\mu\nu} \left(\partial_\mu \phi \partial_\nu \phi - \frac{\eta_{\mu\nu}}{2} (\partial\phi \cdot \partial\phi - m^2 \phi^2) \right) \quad (1.2.27)$$

and at second order by

$$\sqrt{-g} \mathcal{L}_{\phi^2 h^2} = 2\kappa^2 \left(h^{\mu\rho} h^\nu_\rho - \frac{1}{2} h^{\mu\nu} h \right) \partial_\mu \phi \partial_\nu \phi - \frac{\kappa^2}{2} \left(h^{\rho\sigma} h_{\rho\sigma} - \frac{1}{2} h^2 \right) (\partial\phi \cdot \partial\phi - m^2 \phi^2). \quad (1.2.28)$$

As for the self-interactions of the graviton, the coupling constant is again given by κ . Actually, the coupling of any field to gravity turns out to be held up by κ . This is a rather important point, as it encodes the Equivalence Principle on which GR is based: gravity couples in the same way to any field. Going further in the expansion, n -point interacting terms with $n > 4$, $n - 2$ gravitons and 2 scalars are obtained. We will not be concerned with this kind of terms here. Interaction vertices between gravitons and $n > 2$ scalar fields arise upon expansion of $\sqrt{-g}$ when higher powers of ϕ are included in the matter action.

1.3 The U(1) Weak Gravity Conjecture

One of the first and most studied Swampland criteria is the Weak Gravity Conjecture, originally proposed in [17]. Let us consider a 4-dimensional $U(1)$ gauge theory in an EFT of Quantum Gravity

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{2\kappa^2} - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right) + S_{\text{matter}}, \quad (1.3.1)$$

where g is the gauge coupling and S_{matter} is the action for some matter fields. The statement of the Weak Gravity Conjecture is that such a theory necessarily contains a state of charge q and mass m that respects the condition

$$gq \geq \frac{1}{\sqrt{2}} \kappa m. \quad (1.3.2)$$

By itself, this equation does not have a sharp meaning, as the parameters that appear in it are running parameters. To make it more precise, it is of the utmost importance to discuss the physical motivations behind it. Before doing so, we can already observe that the most natural assumption one could make is to consider the mass m as the pole mass of the particle, and consequently the gauge and gravitational couplings g and κ that appears in (1.3.2) should be taken to represent their value at the pole mass scale, $g \equiv g(m)$, $\kappa \equiv \kappa(m)$. Here and throughout the next sections, unless explicitly said differently, g and κ will always indicate the value of the couplings at the pole mass.

The first, obvious, consequence of the WGC is that for a state that respects (1.3.2) the repulsive Coulomb force between two of its copies is stronger than the corresponding attractive Newton's force. The factor $1/\sqrt{2}$ can be easily verified to be the correct one for this confrontation from a simple direct calculation (it will be derived in the case of D generic dimensions later). This is trivially verified in our own world for all the known charged elementary particles. This same feature can be stated in other, different ways. Equation (1.3.2) can be said to ensure that the theory contains at least one self-repulsive particle, or that it makes gravity subdominant in the scattering amplitudes related to this particle (in particular, in the 4 point amplitude between two initial and two final matter states, where the coefficients of the t and u pole are both given by the same combination constrained in (1.3.2)). All these statements here are redundant, but this will not necessarily be the case in more complicated set-ups. Expressing the consequences of (1.3.2) in all these seemingly equivalent ways is however useful to understand what could possibly go wrong if the conjecture were not to hold.

One possibility is the following. Suppose there is only one particle in the spectrum and that it does not respect the conjecture; its mass over charge ratio is $\kappa m/gq > \sqrt{2}$. In suitable units³

$$\tilde{m} = \frac{\kappa^2 m}{8\pi}, \quad \tilde{q}^2 = \frac{\kappa^2 q^2}{32\pi^2}, \quad (1.3.3)$$

we write this for simplicity $\tilde{m}/\tilde{q} > 1$. The net force between two copies of this particle is attractive, and the two will then form a (non relativistic) bound state of charge $\tilde{Q} = 2\tilde{q}$ and mass $\tilde{M} < 2\tilde{m}$ (that this classical expectation holds true at the quantum level will be shown in section 1.6.1). The bound state will thus have a mass over charge ratio smaller than the original elementary particle, $\tilde{M}/\tilde{Q} > 1$. For a generic state of mass M and charge Q decaying into a set of particles with masses m_i and charges q_i , the triangle inequality

$$\frac{M}{Q} \geq \sum_i \frac{m_i}{Q} = \frac{1}{Q} \sum_i \frac{m_i}{q_i} q_i \geq \left(\frac{m}{q}\right)_{\min} \quad (1.3.4)$$

shows that this can only happen if the mass over charge ratio of the decaying state is greater than that of at least one of the particles it decays into. In our case, the bound state formed by the self-attractive particle has a smaller mass to charge ratio than the particle itself, and it is then stable. Adding another particle, the latter will feel an attractive force from the bound state, that will grow further. The mass to charge ratio for the new bound state is again smaller than that of its constituents, ensuring its stability. Iterating this procedure, the composite state will grow further and further until the value $\tilde{M}/\tilde{Q} = 1$ is obtained, where the bound state might turn into an extremal Reissner-Nordström (RN) black hole (more generally into a state that respects a no-force condition), and the balance between attractive and repulsive force is reached. A violation of the WGC thus leads to towers of stable states culminating either in extremal RN black holes or states that respects a no force condition, which might reasonably be argued to be a feature to avoid.

As a special case of the example above, we have seen that violation of the WGC forbids extremal Reissner-Nordström black holes to decay. The Reissner-Nordström black holes are static solutions to the Einstein-Maxwell equations that take the form

$$ds^2 = - \left(1 - \frac{2\tilde{M}}{r} + \frac{\tilde{Q}^2}{r^2}\right) dt^2 + \left(1 - \frac{2\tilde{M}}{r} + \frac{\tilde{Q}^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (1.3.5)$$

These black holes have two horizons, corresponding to the roots of

$$P(r) \equiv r^2 - 2\tilde{M}r + \tilde{Q}^2, \quad (1.3.6)$$

when $\tilde{M} > |\tilde{Q}|$. In the following, we drop the tilde in the notation, as it will be evident when we work in geometrical units. If, starting from this condition, the mass is decreased, the outer horizon (the event horizon) shrinks. On the contrary, when the charge is increased, the inner horizon (the Cauchy surface) grows. Either way, when the mass over charge ratio decreases, the two horizons get closer and closer until they

³ These correspond to the geometrical units frequently used in the study of charged black holes, with the slight difference that in the latter case the gauge coupling is usually included in the definition of the charge: $\tilde{Q}^2 = \frac{\kappa^2 q^2 Q^2}{32\pi^2}$

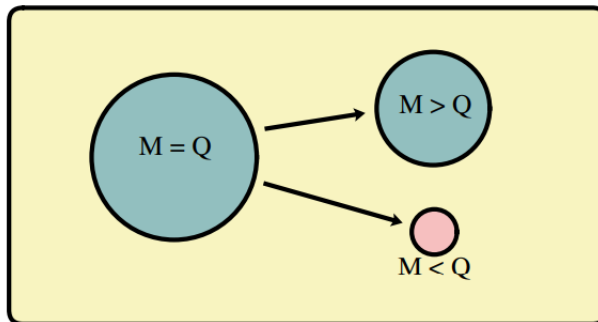


Figure 1.1 – Extremal black holes can only decay into a particle that respects the WGC and a non-extremal black hole.

meet when $M = |Q|$. In the limit $M \rightarrow |Q|^+$, the black holes are on the verge of violating the cosmic censorship hypothesis [18] and exposing a naked singularity, that would indeed be unveiled for $M < |Q|$. It is for this reason that the solutions with $M = |Q|$ are referred to as extremal black holes. Non-extremal black holes have a non-zero temperature

$$T = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{\left(M + \sqrt{M^2 - Q^2}\right)^2} \quad (1.3.7)$$

and can radiate their mass and charge away through Hawking evaporation, until they eventually thermalize at $T = 0$ ending up in an extremal one. The property of the extremal black holes to be endpoints of the Hawking evaporation makes it that they cannot radiate away through this mechanism. For the triangle inequality (1.3.4) extremal black holes can only decay if a particle with smaller mass to charge ratio exists, that is, if a particle that respects (1.3.2) exists. The WGC is thus seen to open up a possibility for extremal black holes to decay into non-extremal ones, and then continue their evaporation through the Hawking mechanism. This is schematically shown in Fig. 1.1, taken from [17], and Fig. 1.2. Another way to phrase it could be that the WGC requires that at least one particle exists in the parametric region complementary to the BH region of existence.

The central question on this side seems to be whether all black holes should be able to decay and/or whether they should be able to radiate away all their charge. In addition to the fact that, as a principle, it would be to the least singular that only extremal black holes are forbidden to decay and/or evaporate, a stronger argument may be the following. If they were not able to decay, this would mean that extremal black holes of any charge (and associated mass) could exist. These would be stable and consequently "trapped" in their state. The theory would then suffer of a possible infinite number of stable states called "remnants". In parameter space, the entire orange line of Fig. 1.2 would be populated by stable states.

Although none of the arguments presented above provides a proof of the conjecture, they certainly point out some particular features the UV theory would have to cope with was it not respected. Furthermore, in this simple set-up, all the arguments are related and seem redundant. As we will see in the following, this is rather a coinci-

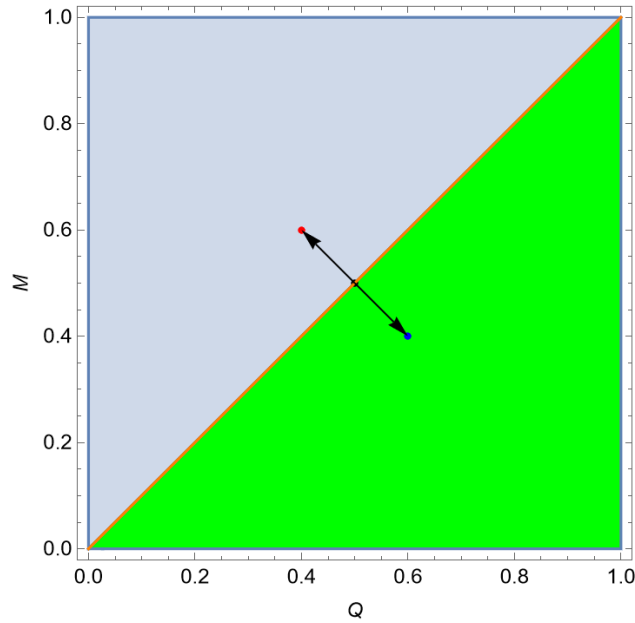


Figure 1.2 – The black hole region of existence is depicted in blue. Extremal black holes are found on the orange line. From the point of view of GR, naked singularities are found in the green region. The WGC requires that a particle (blue point) exists such that an extremal black hole (black point) can decay to a non extremal one (red point) by emission of this particle.

dence due to the simplicity of the theory, and it does not appear in more complicated set-ups. Concerning the question of the scale at which (1.3.2) should be required, the arguments provided here involve the pole mass of the particle and points in the direction of the expectation we started with. In fact, the results for the scattering amplitudes have been discussed only at the tree-level, and additionally classical forces arise from them in the NR limit. The bound state argument is used for non-relativistic bound states formed with the particles initially at rest, and the minimal requirement for the decay of the black hole is that it produces a particle at rest.

Summarizing, the WGC can be formulated as the consequence of any one of the following physical requirements:

- All black holes should be able to decay and evaporate
- Not all states in Quantum Gravity should form stable gravitationally bound states
- It exists at least one particle such that the long range force between two of its copies is repulsive
- Gravity is subdominant in scattering amplitudes.

Using these same arguments, the conjecture is readily generalized to the case of D generic dimensions as

$$gq \geq \sqrt{\frac{D-3}{D-2}} \kappa m, \tag{1.3.8}$$

where κ is now proportional to $1/M_{P_D}^{D-2}$ and M_{P_D} is the D -dimensional Planck mass. The Reissner-Nordström metric generalizes in D dimensions to the Reissner-Nordström-Tangherlini [19]. Re-inserting physical units to see the precise factors, it reads (see also [20] for a careful and more recent derivation)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + d\Omega_{D-2}^2 \quad (1.3.9)$$

with

$$f(r) = 1 - \frac{2\kappa^2}{(D-2)A_{D-2}} \frac{M}{r^{D-3}} + \frac{\kappa^2}{(D-2)(D-3)A_{D-2}^2} \frac{Q^2}{r^{2(D-3)}}, \quad (1.3.10)$$

where A_{D-2} is the area of the $D-2$ unit sphere. The function $f(r)$ has two zeroes, corresponding to two horizons, when

$$\left(\frac{\kappa^2 M}{(D-2)A_{D-2}} \right)^2 - \frac{\kappa^2}{(D-2)(D-3)A_{D-2}^2} Q^2 \geq 0 \Leftrightarrow \kappa^2 M^2 \geq \frac{D-2}{D-3} Q^2. \quad (1.3.11)$$

The same numerical factor can be easily derived from the comparison of the Coulomb and Newton force for a generic charged particle (we take a scalar for simplicity), which is obtained from the calculation of the two t -channel diagrams of Fig. 1.3, that give

$$\begin{aligned} \mathcal{M}^{(t)} = & g^2 q^2 \frac{(p_1 + p_3) \cdot (p_2 + p_4)}{t} \quad (1.3.12) \\ & - \frac{\kappa^2}{4} \left[\left(p_{1\mu} p_{3\nu} + p_{3\mu} p_{1\nu} - \eta_{\mu\nu} (p_1 \cdot p_3 - m_n^2) \right) \frac{\mathcal{P}^{\mu\nu\alpha\beta}}{t} \left(p_{2\alpha} p_{4\beta} + p_{4\alpha} p_{2\beta} - \eta_{\alpha\beta} (p_2 \cdot p_4 - m_n^2) \right) \right]. \end{aligned}$$

Evaluating $\mathcal{M}^{(t)}$ in the non-relativistic (NR) limit

$$\frac{s - 4m_n^2}{m_n^2} \rightarrow 0, \quad \frac{t}{m_n^2} \rightarrow 0, \quad \text{and} \quad \frac{u}{m_n^2} \rightarrow 0, \quad (1.3.13)$$

where s , t and u are the usual Mandelstam variables defined as

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2, \quad (1.3.14)$$

it reduces to

$$\mathcal{M}_{NR}^{(t)} = 4 \frac{m^2}{t} \left(g^2 q^2 - \frac{D-3}{D-2} \kappa^2 m^2 \right). \quad (1.3.15)$$

Matching this with the Born expansion and dividing by $4m^2$ to account for the QFT normalization as opposed to the classical one shows that it is the comparison between the terms inside the parenthesis in (1.3.15) that determines the long range character of the classical force felt by the particle. The same combination appears as a factor of the u -channel in the complete scattering amplitude, so the WGC bound is sufficient to ensure subdominance of gravity in the whole amplitude.

If, in $D = 4$ dimensions we suppose that the theory also contains magnetic monopoles and that the same reasoning that led to (1.3.2) apply to them, up to a numerical factor this implies

$$g_{\text{mag}} q_{\text{mag}} \gtrsim \kappa m_{\text{mag}} = \frac{m}{M_P}. \quad (1.3.16)$$

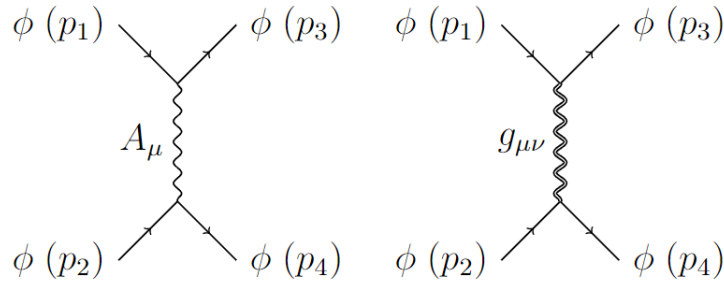


Figure 1.3 – Diagrams contributing to the long range force between two charged scalar particles denoted here by ϕ .

Use of the Dirac quantization property $g_{\text{mag}}q_{\text{mag}} = (gq)^{-1}$ and the expectation that the mass of the monopole is linearly dependent on the physical cut-off Λ of the EFT

$$m_{\text{mag}} \sim \frac{\Lambda}{g^2} \quad (1.3.17)$$

leads to the relation

$$\Lambda \lesssim gM_P. \quad (1.3.18)$$

For small, perturbative values of g , this inequality implies a breaking of the EFT description at a scale lower than M_P . It is customarily considered that gravity places an upper limit on the domain of validity of the EFT only at the scale where it becomes strongly coupled, that is M_P , while maybe other phenomena may predict a lower value for the cut-off. Equation (1.3.18) is meant to indicate that this expectation is too naive, and that gravity itself predicts a cut-off lower than M_P .

The derivation of the magnetic version of the WGC (1.3.18) is somewhat looser than that of the electric one (1.3.2). It is however widely appreciated for that it seems to give a quantitative description of what would go wrong if a gauge coupling was smoothly sent to 0. This has to do with an old idea that Quantum Gravity forbids the existence of global symmetries, and can again be understood in terms of black hole physics.

Suppose we create a black hole throwing particles with global charge against each other. It is well-known from the No-Hair theorem that black holes do not carry global charges, which physically means that their horizon is insensible to them. The black hole we have formed is a Schwarzschild one with

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \quad (1.3.19)$$

and a finite temperature

$$T = \frac{1}{8\pi M}, \quad (1.3.20)$$

through which it can emit Hawking radiation, but it cannot lose its global charge. Actually, the global charge of the black hole remains undefined, and in turn this means that there is an infinite uncertainty related to it for an observer placed outside the horizon. This infinite uncertainty corresponds to an infinite entropy, which is in contrast

with the determination of the black hole entropy through the Hawking-Bekenstein relation. Moreover, the evaporation would continue until the Quantum Gravity regime is reached at scales around M_P . Keeping an open mind, the black hole could lose some charge in the vicinity of the Planckian regime, where Quantum Gravity takes over, but this cannot be guaranteed to be sufficient to discharge it completely. As a result, the endpoint of the evaporation would be a stable remnant with any value of the global charge possible. In other words, the theory would suffer from the presence of an infinite number of remnants.

This argument seems very similar to the previous one invoked to justify equation (1.3.2), and the merit of the magnetic WGC (1.3.18) is that it actually links them. An interpretation of (1.3.18) follows naturally after noticing that in the limit $g \rightarrow 0$ a gauge symmetry becomes physically indistinguishable from a global one. One can get convinced of this in many ways. In the field theory language, it is immediate from (1.3.1) to realize that the gauge field dynamics is frozen in that limit or, reabsorbing the factor g in a redefinition of the gauge field, that the latter is completely decoupled from matter, which is the same. On the black hole side, the Reissner-Nordström black holes have been shown to have, for any value of the mass, a maximal value of the charge. Being the charge quantized semiclassically, this means that for a fixed value of the mass M , the number of distinct black holes, corresponding to the different charges it can assume, is

$$N \sim \frac{M}{g M_P}, \quad (1.3.21)$$

where we have reintroduced the Planck mass to make it more evident that N is a pure number. For the Schwarzschild black holes, on the contrary, no constraint arise and an infinite number of distinct black holes are found for any value of the mass, as argued above. In the limit $g \rightarrow 0$, the same is obtained for the RN solution.

The magnetic WGC predicts that as g is continuously sent to 0, the cut-off of the effective description decreases and tend to 0 at the same rate, so that this provides a physical mechanism that stops the gauge coupling from reaching a null value. In the stringy approach to the Swampland, a strong motivation against global symmetries come from the fact that they always are gauged in string theory.

In the next sections, we will discuss some extensions of the WGC to more complicated set-ups related to the works we wish to present [1, 2, 3, 4]. For each case, we will try to individuate, among the different motivations outlined in this section, the ones that might apply in the case of interest.

1.4 The Weak Gravity Conjecture with several $U(1)$

A straightforward extension of the Weak Gravity Conjecture to the case where several $U(1)$ gauge groups are present seems to be given by the requirement that at least one particle respecting (1.3.2) for each one of the groups is present. Such a generalization has actually been shown to be too naive in [21], where the correct way to generalize the conjecture was first presented. Consider a theory with N $U(1)$ gauge groups labeled by the subscript a , and particles denoted by i with masses m_i and charges $q_{i,a}$ with respect to the a -th gauge group. The groups are supposed to be unbroken, so that

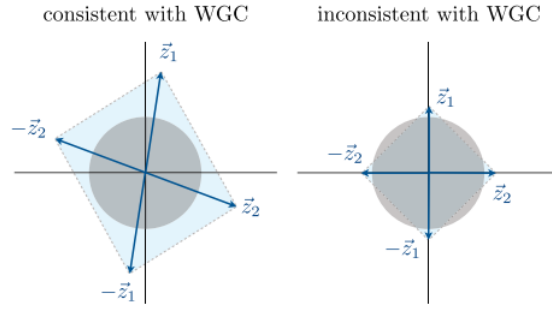


Figure 1.4 – The grey sphere is the unit sphere and denotes the region of existence of black holes. The blue quadrilateral is formed by the particles that are taken to respect the WGC. Any black hole in the theory can decay only if the blue quadrilateral contains the sphere.

in case some kinetic mixing is present this can be removed by a rotation of the gauge bosons. The proper generalization of the WGC must then necessarily be $SO(N)$ invariant, and we use a compact vector notation for the charges, $\vec{q}_i = (q_{i,1}, q_{i,2}, \dots, q_{i,N})$, and for the charge to mass ratios, $\vec{z}_i = \vec{q}_i/m_i$, of the particles. In this set-up, the simplest attempt would be to require that at least one state with $|\vec{z}_i| > 1$ exists. This is however too mild. The requirement $|\vec{z}_i| > 1$ might be fulfilled, for example, by a vector $\vec{z}_i = (z, 0, \dots, 0)$ with $z > 1$. Extremal black holes charged under another gauge group, as for example a black hole with charge $\vec{Q} = (0, \dots, 0, Q)$, are not allowed to decay with the mild requirement proposed above. The crucial observation is that, when several gauge factors are present, the existence condition for the associated charged black holes is

$$|\vec{Q}|^2 \leq M^2. \quad (1.4.1)$$

In a space with axis $\vec{z}_1, \vec{z}_2, \dots, \vec{z}_N$, this corresponds to the unit sphere, and the extremal black holes are those found on the boundary of such a sphere. Any state that is found either inside the sphere or on its boundary, must be able to decay. For this to happen, there must be in the spectrum a sufficient number of particles to generate a convex hull that encloses the black hole region, as shown for the case of 2 gauge groups in Fig. 1.4, taken from [21]. In the left panel, the black hole region is completely enclosed in the convex hull, while this is not the case in the right panel. There, the black holes found in the grey region outside the blue box cannot decay and they actually end up to form stable remnants.

More quantitatively, this can be understood as follows. Consider a black hole of mass M and charge \vec{Q} with charge to mass ratio \vec{Z} decaying into a collection of n particles, of which n_i with masses m_i and charges \vec{q}_i . For energy and charge conservation, we have

$$\vec{Q} = \sum_i n_i \vec{q}_i, \quad M > \sum_i n_i m_i. \quad (1.4.2)$$

The charge over mass ratio of the black hole can be written as

$$\vec{Z} = \frac{\sum_i n_i \vec{q}_i}{M} = \sum_i \left(\frac{n_i m_i}{M} \right) \vec{z}_i, \quad (1.4.3)$$

with

$$\sum_i \frac{n_i m_i}{M} < 1. \quad (1.4.4)$$

This means that the convex hull formed by the black hole, the unitary sphere, is a subunitary weighted average of the one formed by the set of particles we have taken it to decay into, which is nothing but a more rigorous way to say the same thing we graphically inferred.

For the case of two gauge groups, it was shown that the conjecture requires that two particles (with respective antiparticles) exist whose charge to mass ratios \vec{z}_1 and \vec{z}_2 verify the relation

$$(|\vec{z}_1|^2 - 1) (|\vec{z}_2|^2 - 1) > (1 + |\vec{z}_1 \cdot \vec{z}_2|)^2. \quad (1.4.5)$$

If we take the particles to have orthogonal charges of equal strength, $|\vec{z}_1| = |\vec{z}_2| = z$ and $\vec{z}_1 \cdot \vec{z}_2 = 0$, as in the right panel of Fig. 1.4, equation (1.4.5) implies $z > \sqrt{2}$, as one would impose measuring the distance from the origin to any one of the four edges.

Repeating the calculation the led to (1.3.15), it is readily seen that the condition one obtains imposing any one of the other requirements (repulsive long-range force, gravitational subdominance and that not all states form gravitationally bound states) coincide and agree with the one derived here specifically devised to allow for the decay of extremal black holes.

A phenomenologically interesting prediction of the WGC in the case of a theory with a $U(1)$ dark photon with tiny mixing with the SM photon, based on [2], will be discussed in chapter 2.

1.5 Dilatonic Weak Gravity Conjecture

Dilatonic gauge theories usually result from the compactification of higher dimensional theories of gravity. The circle compactification of a $D + 1$ dimensional to a D dimensional field theory with gravity is extensively derived in Appendix A. The higher dimensional gravitational field g_{MN} is effectively described, upon compactification on a circle, by the lower dimensional gravitational field $g_{\mu\nu}$, a scalar field ϕ and a gauge field A_μ . The scalar and gauge fields are effective descriptions of the additional degrees of freedom of the gravitational field in $D + 1$ dimensions, and can be schematically thought to be related to the g_{MN} components as: $\phi \sim g_{D+1,D+1}$, $A_\mu \sim g_{\mu,D+1}$. The scalar field ϕ is referred to as the dilaton or the radion, as it controls the size of the extra dimension, and the gauge field is usually referred to as the graviphoton. As can be seen in (A.1.27), the key feature that distinguishes dilatonic theories from usual gauge theories comes from their kinetic term, which is obtained from

$$-\frac{1}{4} e^{-2\gamma\kappa\phi} F^2, \quad (1.5.1)$$

where γ here is a generic dilatonic coupling constant (in particular it is unrelated to the specific one of Appendix A) and the VEV of ϕ determines the value of the gauge coupling.

Dilatonic black holes were first discovered and studied in [22, 23]. An analysis in a slightly different set-up to derive the corresponding extension of the WGC was first

presented in [24]. It is customary, to simplify numerical factors, to consider the action

$$S = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} (R - 2(\partial\phi)^2 - e^{-2\alpha\phi} F^2), \quad (1.5.2)$$

where dimensionless fields are used for simplicity. The dimensionfull physical fields obtained from them are

$$\tilde{\phi} = \sqrt{2} \frac{\phi}{\kappa}, \quad \tilde{A}_\mu = \sqrt{2} \frac{A_\mu}{\kappa}. \quad (1.5.3)$$

The solution to the equations of motion take the form

$$\left\{ \begin{aligned} ds^2 &= - \left[\left(1 - \frac{r_\pm}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \right] dt^2 + \left[\left(1 - \frac{r_\pm}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \right]^{-1} dr^2 \\ &\quad + r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} d\Omega_2^2, \\ e^{2\alpha\phi} &= e^{2\alpha\phi_0} \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \\ F &= \frac{1}{\sqrt{4\pi G}} \frac{Q e^{2\alpha\phi_0}}{r} dt \wedge dr, \end{aligned} \right. \quad (1.5.4)$$

where r_+ and r_- are two integration constants related to the mass and the charge of the black hole as measured at infinity (see for instance [25]) through

$$\left\{ \begin{aligned} 2M &= r_+ + \frac{1-\alpha^2}{1+\alpha^2} r_-, \\ Q^2 e^{2\alpha\phi_0} &= \frac{r_+ r_-}{1+\alpha^2}. \end{aligned} \right. \quad (1.5.5)$$

These relationships are obtained expanding the time component g_{00} of the metric for $r \gg r_+, r_-$ and individuating the coefficients of r^{-1} and r^{-2} of such an expansion. Those correspond, respectively, to the asymptotic flux of the gravitational and electromagnetic field produced at the black hole horizon. These black holes also carry a scalar charge D as a secondary charge, which can be again measured from the flux at infinity:

$$D = \frac{1}{4\pi} \lim_{r \rightarrow \infty} \int d^2\Sigma^\mu \nabla_\mu \phi, \quad \Leftrightarrow \quad \phi = \phi_0 - \frac{D}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (1.5.6)$$

It is easily checked that the Reissner-Nordström solution is recovered for $\alpha = 0$.

A peculiar feature of these solutions is that for any value of $\alpha \neq 0$ they have an extended singularity at the surface $r = r_-$, whose area vanishes. This means that this family of black holes only posses a single horizon, found at $r = r_+$, when $r_+ > r_-$. When the two are equal, the horizon is degenerate with the singularity. As a consequence, we define extremal states those obtained in the limit $r_+ \rightarrow r_-^+$. This type of extremality is rather different than the Reissner-Nordström one encountered in the previous sections. A dilatonic solution with $r_+ = r_-$ already shows a naked singularity, whereas a RN solution with $M^2 = Q^2$ has a singularity protected by one horizon. In the extremal limit, the black hole states here are on the verge of showing a naked singularity because the horizon and the singularity itself are getting closer and closer, as opposed to the RN case where extremal black holes are on the verge of showing a naked singularity because the outer and inner horizon becomes degenerate in the limit and cease to exist beyond it. This dilatonic type of extremality is conceptually closer to one we could "artificially" define for Schwarzschild black holes. From (1.3.19), we

see that those solutions possess a singularity at $r = 0$ protected by a single horizon when $r_h = 2M > 0$. Extremal states could then be defined as those on the verge of showing a naked singularity in the limit $M \rightarrow 0$. A more thorough discussion of these properties and the peculiar behaviour of dilatonic black holes will be presented in the chapter 5 dedicated to the work [3]. Note also that the singularity at $r = r_-$ is a space-like singularity for the dilatonic black holes, which means that the singularity is not avoidable for an observer that crosses the horizon. This is similar to the case of the Schwarzschild black holes, where the singularity at $r = 0$ is space-like. On the contrary, Reissner-Nordström black holes have a time-like singularity at $r = 0$, while the variable t is space-like in the region between the Cauchy surface r_C and the event horizon r_h . This means that an observer that enters into the black hole from outside through the event horizon is forced to reach the Cauchy surface, but once he crosses it he can freely move in the region $0 < r < r_C$, and decide to cross it again from the inside. As he does this, he is then forced to move towards the event horizon and exit the black hole. This can be repeated infinitely many times.

In terms of physical parameters, it can be observed that the horizon shrinks when the mass is decreased, and equivalently the singularity grows when the charge is increased. The condition for the singularity to be shielded by an horizon is simply

$$Q^2 e^{2\alpha\phi_0} < (1 + \alpha^2)M^2. \quad (1.5.7)$$

Requiring that all the black holes in this family can decay, meaning that also the black holes obtained as we reach the extremal limit do so, imposes the existence of at least one particle with mass m and charge q that satisfies a relationship opposite to (1.5.7). This can be seen as follows. Suppose an extremal (or quasi extremal) state decays into a particle of charge q and mass m . The new state will have a charge $Q' = Q - q$ and mass $M' \leq M - m$. Imposing that this new state is a subextremal black hole means

$$Q'^2 e^{2\alpha\phi_0} < (1 + \alpha^2)M'^2 \Rightarrow (2qQ - q^2) e^{2\alpha\phi_0} > (1 + \alpha^2) (2mM + m^2). \quad (1.5.8)$$

Expanding the above expression for q/Q , $m/M \ll 1$ and using the extremality condition for Q and M , the relation

$$qe^{\alpha\phi_0} > \sqrt{1 + \alpha^2} m \quad (1.5.9)$$

is obtained. In D generic dimensions and in physical units the bound takes the form

$$q^2 e^{\sqrt{2}\alpha\kappa\phi_0} > \left(\frac{D-3}{D-2} + \frac{\alpha^2}{2} \right) \kappa^2 m^2. \quad (1.5.10)$$

The cases $\alpha = 1$ and $\alpha = \sqrt{2(D-1)/(D-2)}$ are of particular interest as they correspond, respectively, to those that arise from string and Kaluza-Klein compactifications.

The constraint (1.5.10) can be further understood deriving it as a condition on the character of the long range force. We take the action

$$S = \int d^D x \sqrt{-g} \left(\frac{R}{2\kappa^2} - \frac{1}{4} e^{-\sqrt{2}\alpha\kappa\phi} F^2 + \frac{1}{2} (\partial\phi)^2 + D_\mu \varphi D^\mu \varphi^* - e^{\sqrt{2}\alpha\kappa\phi} m^2 \varphi^* \varphi \right), \quad (1.5.11)$$

with D_μ the usual covariant derivative and where, as suggested by the most common result of the circle compactification (see Appendix A), the field dependent exponential factors in front of the mass term and the gauge kinetic term (the inverse

in this case) have been identified. A more general argument for the identification of the field dependence of these two terms was given in [26]. Expanding the dilaton about a generic background value ϕ_0 , calculation of the $2 \rightarrow 2$ scattering amplitude $\varphi(p_1)\varphi(p_2) \rightarrow \varphi(p_3)\varphi(p_4)$ leads to

$$\begin{aligned} \mathcal{M} = & q^2 e^{\sqrt{2}\alpha\kappa\phi_0} \left(\frac{(p_1 + p_3) \cdot (p_2 + p_4)}{t} + \frac{(p_1 + p_4) \cdot (p_2 + p_3)}{u} \right) - 2\alpha^2 \kappa^2 \bar{m}^4 \left(\frac{1}{t} + \frac{1}{u} \right) \\ & - \frac{\kappa^2}{4} \left[\left(p_{1\mu} p_{3\nu} + p_{3\mu} p_{1\nu} - \eta_{\mu\nu} (p_1 \cdot p_3 - \bar{m}^2) \right) \frac{\mathcal{P}^{\mu\nu\alpha\beta}}{t} \left(p_{2\alpha} p_{4\beta} + p_{4\alpha} p_{2\beta} - \eta_{\alpha\beta} (p_2 \cdot p_4 - \bar{m}^2) \right) \right. \\ & \left. + (t, p_3, p_4) \leftrightarrow (u, p_4, p_3) \right], \end{aligned} \quad (1.5.12)$$

where we have defined $\bar{m}^2 \equiv e^{\sqrt{2}\alpha\kappa\phi_0} m^2$. In the NR limit, this reduces to

$$\mathcal{M}_{NR} = 4\bar{m}^2 \left[q^2 e^{\sqrt{2}\alpha\kappa\phi_0} - \kappa^2 \bar{m}^2 \left(\frac{\alpha^2}{2} + \frac{D-3}{D-2} \right) \right] \left(\frac{1}{t} + \frac{1}{u} \right). \quad (1.5.13)$$

The quantity inside the square bracket corresponds to the combination constrained in (1.5.10). The bound looks rather similar to the original one in (1.3.2). The factor in front of the charge is the gauge coupling induced by ϕ_0 , and the additional term accounts for the attractive scalar force mediated by the dilaton for states with three-point coupling to the scalar as in (1.5.11).

Once again, any one of the different motivations invoked for the $U(1)$ WGC lead to the same bound, at least assuming all states are coupled to the dilaton through an exponential factor in front of their mass that coincides with the inverse of the exponential in front of F^2 . A fundamental property of the dilatonic black holes is that they carry a secondary scalar charge, which ensured the presence of a term that could be related to a scalar force in their extremality bound. In the next section, we will overview some of the attempts towards a definition of a more generic WGC for theories with scalar fields, and dedicate the chapters 3 and 4, based respectively on [1] and [4], to the exposition of our contribution to the field. The chapter 5 will report our study of the generalization of the dilatonic WGC in (asymptotically) de Sitter and Anti-de Sitter spacetime from the corresponding black hole solutions [3].

1.6 WGC in the presence of scalar fields

The first approach to the problem of the definition of a Weak Gravity Conjecture with scalar fields beyond the dilatonic case was made in [27] in terms of massless scalars coupled to the WGC state in a four-dimensional theory. It was later generalized to the case of a D -dimensional theory in [26]. In terms of the additional scalar force the scalars mediate, this choice had the advantage that the latter could be considered on the same ground as the gauge and gravitational forces, making it rather simple to justify the constraint in light of the "no bound state" argument. However, unless some symmetry, as for e.g. supersymmetry, or some other yet unknown mechanism, is at work, interacting scalars cannot be kept massless against quantum corrections. Actually, the quadratic scalar operators are the less stable ones, as they suffer from the

well-known naturalness problem, which will be extensively discussed in the second part of the thesis. This means that the generic analysis in terms of long range forces, that we will develop in the following, leave open the question of how the scalar mediators are kept massless, or at least parametrically lighter than the WGC state. We will briefly come back to this point at the end of this section.

In light of this, and to gain some black hole intuition on the problem, the analysis of [27] starts with the study of extremal black holes in $\mathcal{N} = 2$ Supergravity. It was shown that a black hole solution to the action

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{2\kappa^2} + g_{ij} \partial_\mu \phi^i \partial^\mu \bar{\phi}^j + \mathcal{I}_{ij} \mathcal{F}_{\mu\nu}^I \mathcal{F}^{J,\mu\nu} + \mathcal{R}_{IJ} \mathcal{F}_{\mu\nu}^I (\star\mathcal{F})^{J,\mu\nu} \right), \quad (1.6.1)$$

where g_{ij} is the metric on the scalar field space, $\star\mathcal{F}$ is the Hodge dual of the electromagnetic tensor \mathcal{F} and \mathcal{I}_{ij} and \mathcal{R}_{ij} define the gauge kinetic terms and are related through electro-magnetic duality, have charges and masses that in the extremal limit are related, in geometrical units, through:

$$Q^2 = M^2 + 4g^{ij} \partial_i M \bar{\partial}_j M. \quad (1.6.2)$$

As commonly in black hole physics, the quantities are calculated at spacial infinity. These black hole solutions are BPS states, which means that they can only decay to other BPS states. As the relation (1.6.2) follows from general properties of BPS states and is dictated by the structure of $\mathcal{N} = 2$, the particles the extremal black holes can decay into need to satisfy the same kind of relation. The decay of extremal black holes, in this set-up, only enforces the presence of a BPS particle in the spectrum with $q^2 = m^2 + 4g^{ij} \partial_i m \bar{\partial}_j m$. Nevertheless, in light of the difficulty to keep scalar massless discussed above, this was recognized as a great underlying motivation to more general derivations. The possibility that an equation like (1.6.2) could be written outside the supergravity realm was also hinted in [27].

Consider now a simpler, and less constrained, set-up given by

$$S = \int d^Dx \sqrt{-g} \left(\frac{R}{2\kappa^2} - \frac{1}{4} F^2 + \frac{1}{2} (\partial\phi)^2 + |D\varphi|^2 - m^2(\phi) \varphi \varphi^* \right), \quad (1.6.3)$$

where ϕ is a (classically) massless scalar field, φ a charged matter field to be identified with the desired WGC state, whose interactions with the massless field have been included in a definition of a field-dependent "mass" term, which is nothing but the usual quadratic operator used in many contexts in QFT. The three-point coupling $2m\partial_\phi m|_{\phi=0}$ is straightforwardly obtained for the operator $\phi\varphi\varphi^*$ expanding $m^2(\phi)$ around the background value of ϕ (that for free massless scalars can only be $\phi = 0$)

$$m^2(\phi) = m^2(\phi) \Big|_{\phi=0} + 2m\partial_\phi m \Big|_{\phi=0} \phi + \mathcal{O}(\phi^2). \quad (1.6.4)$$

The calculation of the $\varphi\varphi \rightarrow \varphi\varphi$ scattering amplitude gives the same result as in (1.5.12) with the replacement $2\alpha^2\kappa^2\bar{m}^4 \leftrightarrow 4m^2(\partial_\phi m)^2$, where the quantity on the right hand side is understood to be evaluated at the background value of ϕ . The requirement that in a theory with gravity, gauge and scalar fields (as (1.6.3)) there is at least one state that respects the inequality (in physical units)

$$g^2 q^2 \geq \frac{D-3}{D-2} \kappa^2 m^2 + (\partial_\phi m)^2, \quad (1.6.5)$$

ensures that such a state feels an overall long-range repulsive force and does not form a bound state with itself. Gravitational interactions are also ensured to be subdominant with respect to both gauge ones and the combination of gauge and scalar ones, while nothing can be said for the comparison between gravity and scalar interactions so far. For the black hole argument to predict the same bound, it would first be necessary to actually demonstrate the property hinted in [27] that extremal black holes solutions to an Einstein-Maxwell-scalar theory carry the secondary scalar charge such that they respect (1.6.2), and then to prove a sort of conservation and/or linearity of the scalar charge such that the black hole can only decay if it gives away more charge than the combination of mass and scalar charge. The dilatonic black holes studied in the previous section do respect (1.6.2) in the extremal limit, and the bound obtained from them provides an example of a WGC of the form (1.6.5).

A peculiar feature of (1.6.5) is that, when $g \rightarrow 0$, it seems to require that both the mass of the WGC state and its coupling to the scalar field vanish, which was again interpreted as a possible manifestation of a quantum gravity obstruction to global symmetries. Another possibly interesting limit arises when there is a cancellation between the gauge and scalar forces, $g^2 q^2 \sim (\partial_\phi m)^2$. A "light" particle of mass $m^2 \sim (g^2 q^2 - (\partial_\phi m)^2)/\kappa^2$ should then be present in the spectrum, according to (1.6.5). These were claimed in [28] to be UV/IR mixing properties of quantum gravity, and it was observed that, would the WGC state be a scalar, the second observation could be of interest for the naturalness problem. Of course, this is a delicate claim, as already recognized in [28]. Equation (1.6.5) rests more than others on the no bound state and long-range force arguments, that are intrinsically classical in nature, and have nothing to do with quantum fluctuations⁴. Assuming the bound in (1.6.5) has been given a solid justification, a cancellation between the gauge and scalar forces means that, at the pole mass scale, the mass of the WGC state is small. In light of the discussion in chapters 7, 8 on the works [5, 6], it should be noted here that this is only a point-like information, that does not constrain in any way the running of the mass parameter. Even more, the requirement that the gauge and scalar forces cancel between each other seems to require itself an high degree of fine-tuning. As their difference has to be multiplied by the Planck mass, a low upper bound on the mass, say of the order of 10^2 GeV, can only be obtained if there is a sufficient cancellation between g^2 and $\mu^2 \equiv (\partial_\phi m)^2$

$$\sqrt{g^2 - \mu^2} \sim 10^{-17}, \quad (1.6.6)$$

which requires a tuning of g^2 and μ^2 up to the 34th digit. This is exactly the same degree of fine-tuning that is necessary in the framework of perturbative renormalization to obtain a renormalized Higgs mass $m_{H,R} \sim 125$ GeV when the physical UV cut-off is taken at the Planck scale.

Finally, with respect to the problem of keeping the scalar mediators massless discussed above, it is straightforward to recognize that, for a massive scalar mediator of mass m_ϕ , equation (1.6.5) gets a correction of order m_ϕ/m . If the scalar mediator is kept sufficiently lighter than the WGC state, the analysis based on the long range forces/bound states formation described above (classically) still holds to a good approximation.

⁴ It has quantum validity, as shown in section 1.6.1, but only in non-relativistic quantum mechanics, which is not at the core of the problem

Besides the observation of the particular limits described by the bound, [28] provided a stronger motivation for (1.6.5). Consider a five-dimensional gauge theory

$$S^{(5D)} = \int d^5x \sqrt{\hat{g}} \left(\frac{\hat{R}}{2\hat{\kappa}^2} - \frac{1}{4\hat{g}^2} \hat{F}^2 - \left| \hat{D}\hat{\varphi} \right|^2 - \hat{m}^2 |\hat{\varphi}|^2 \right), \quad (1.6.7)$$

where hatted quantities are understood to be five-dimensional. Upon compactification on a circle, the five-dimensional degrees of freedom of gravity distributes into four-dimensional gravity, the graviphoton and the radion/dilaton, while the degrees of freedom of the gauge field are distributed into a four dimensional gauge field and a scalar field. Keeping only the zero modes, the four dimensional action can be schematically written as

$$S^{(4D)} = \int d^4x \sqrt{-g} \left(\frac{R}{2\kappa^2} + \mathcal{I}_{IJ} \hat{F}_{\mu\nu}^I F^{J,\mu\nu} + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial h)^2 + |D\varphi|^2 - m^2(\phi, h) |\varphi|^2 \right), \quad (1.6.8)$$

where \mathcal{I} is the kinetic mixing matrix between the graviphoton and the "original" gauge field, ϕ is the radion/dilaton and h is the scalar produced by the component \hat{A}_5 of the gauge field along the circle. Assuming $\hat{\varphi}$ to be the $U(1)$ WGC state in five dimensions means that its mass and charge are related through

$$\hat{g}\hat{q} \geq \sqrt{\frac{2}{3}} \hat{\kappa}\hat{m}, \quad (1.6.9)$$

and that the overall long range force it feels interacting with another copy of itself is repulsive. This information is not lost in the compactification process. A complete calculation, that goes beyond that of [28], can be found in Chapter 4, where the work [4] will be presented. The result is that the way this information is encoded in the lower dimensional theory is through equation (1.6.5). In other words, generalizing to a generic number of dimensions and anticipating on [4], if one has a state in $D + 1$ that respects the original, $U(1)$, WGC, the zero mode obtained upon compactification automatically respects the scalar version of [27]. This tie between scalar fields and higher dimensional degrees of freedom of gravity and/or gauge fields gives a stronger motivation to (1.6.5). A violation of it would correspond to a violation of the $U(1)$ WGC in a possible higher dimensional embedding, and things can be reconciled only if the different arguments about extremal black hole decay and the formation of a tower of bound states can be successfully encompassed.

1.6.1 The Repulsive Force Conjecture

The study of the WGC in the presence of scalar fields was later pushed forward in [29], where a thorough analysis is carried out to give a more rigorous definition of both the forms of the conjecture with and without massless scalars. The two motivations most commonly used in the literature are there differentiated: the black hole decay, on the one side, and the bound state/self-repulsive character of long range forces, on the other. In full generality, black holes in a two derivative Einstein-Maxwell theory with massless scalars have an extremality bound that defines a region in the parameter

space spanned by the charge to mass ratio $\vec{z} = \vec{q}/m$ inside which sub-extremal black holes are found. The states that lie either on the boundary or outside this region are called super-extremal. Assuming that the charge is quantized means that the possible charges that a state can assume in a given theory form a lattice Γ_q . A rational direction in the charge \vec{q} space is defined, starting from the origin, as a direction where populated lattice points are found. The black-hole motivated Weak Gravity Conjecture can then be stated as the requirement that in every rational direction in charge space a superextremal state is found. Such a state can either be a particle state or a multiparticle bound state. A slightly different motivation than the one given in [17] is provided. The two-derivative Einstein-Maxwell-scalar action provides a gravitational effective field theory valid at large distances, where the curvature effects are sufficiently suppressed for the Ricci scalar to be dominant with respect to higher curvature (higher derivative) contributions,

$$R \gg R^2, R_{\mu\nu}R^{\mu\nu}, \dots, \quad (1.6.10)$$

so that the latter can be ignored. The black hole solutions found in General Relativity are approximate solutions of a more general theory that are valid in the low curvature limit. Accordingly, the curvature corrections will be of greater importance for lighter black holes and can be neglected for heavy ones. If the WGC is violated, these corrections make the light black holes subextremal in all their range of existence. In particular, this means that, for a fixed charge, the corresponding lightest black holes (the analog of the extremal black holes in GR) are strictly subextremal. For larger charges, the lightest black holes will still be subextremal but with smaller corrections, meaning that their charge to mass ratio grows, until asymptotically the GR domain is obtained and the lightest black holes are the extremal ones. In this excursion from small to large black holes, the charge to mass extremal ratio of the GR solution is asymptotically reached from below. Because of the triangle inequality (1.3.4), all these lightest black holes per fixed charge are stable, leading once again to the conclusion that a theory that violates the WGC would have to cope with the unfriendly prediction of an infinite number of stable black hole states.

To better distinguish between the two different motivations, the expression "Repulsive Force Conjecture" (RFC) was coined for the long-range forces/bound-state motivated WGC. We will adopt it in this section to make the difference clear. In the Einstein-Maxwell-scalar theory of interest, the classical force between two massive particles, denoted 1 and 2, in $d \geq 3$ spatial dimensions read

$$F_{12}(r) = \frac{\mathcal{G}^{ab}q_{1,a}q_{2,b}}{r^{d-2}} - \kappa^2 \frac{d-3}{d-2} \frac{m_1 m_2}{r^{d-2}} - \frac{g^{ij}\mu_{1,i}\mu_{2,j}}{r^{d-2}} + \dots, \quad (1.6.11)$$

where \mathcal{G}^{ab} is the gauge coupling matrix, g^{ij} the scalar coupling matrices, μ_i the "scalar charge", i.e. the coupling of the matter state to the massless scalar ϕ_i , and the dots indicate terms suppressed by higher powers of r . Those are of no interest to determine the long-range character of the force and only determine short range interactions, that in turn will be of relevance only to determine details of bound states. The coefficient of the overall long range force between two particles I and J falling off like $r^{-(d-2)}$ is denoted \mathcal{F}_{IJ}

$$\mathcal{F}_{IJ} \equiv \mathcal{G}^{ab}q_{1,a}q_{2,b} - \kappa^2 \frac{d-3}{d-2} m_1 m_2 - g^{ij}\mu_{1,i}\mu_{2,j}. \quad (1.6.12)$$

We say that two particles are mutually repulsive if $\mathcal{F}_{IJ} \geq 0$, and a single particle is self-repulsive if $\mathcal{F}_{II} \geq 0$. We define a multiparticle state to be weakly self-repulsive if its total mass, charge and scalar charge make it self-repulsive. Denoting n^I the number of its constituent particles I , this means that $\sum_{I,J} n^I n^J \mathcal{F}_{IJ} \geq 0$. A multiparticle state is instead strongly self-repulsive when any two of its constituents particles are mutually and self repulsive. In equations, this boils down to $\mathcal{F}_{IJ} \geq 0$ for all I, J . After some not-completely satisfactory attempts, the RFC is then stated as the requirement that for every rational direction in charge space there is a strongly self-repulsive multiparticle state. This is stronger than the conjecture in [27] and it was meant to encompass the problem, already acknowledged there, that the latter encounters in the case of multiple $U(1)$ gauge bosons. Namely, Palti's conjecture would require that for any massless photon a self-repulsive particle exists, but it might well be that such a particle owe its repulsive character to the charge under a different gauge boson to the one that it is "attached" to through this requirement. In other words, it might well be that a spectrum respecting equation (1.6.5) in the case of several gauge bosons reduces to a weaker constraint than the convex hull in the limit when there are no massless scalars. This is basically the same thing that was observed in [21], namely that the requirement that a particle with $|\vec{z}| > 1$ exists for every massless gauge boson is not sufficient.

The proposed RFC was motivated as follows. A violation of the conjecture means that there is at least one rational direction in charge space where no strongly self-repulsive multiparticle state is found. For any state in this direction, this means that there are at least either two mutually attractive constituents particles or one self-attractive particle. Suppose we pick one such state in this direction. When it is the first of the two possibilities which is realized, a new multiparticle state with the same charge and a smaller mass can be obtained substituting the two constituents by their bound state. When it is the second condition which is realized, combining two copies of this multiparticle state and replacing the two copies of the self-attractive particle with their bound state, a new state with twice the charge and less than twice the mass is obtained. In either cases, the newly formed state is in the same direction in charge space and with a larger charge to mass ratio, which means that it is stable against decay onto the original state. If the RFC is to be violated, the newly formed states in turn cannot be strongly self-repulsive. The procedure can then be repeated an infinite number of times, and sequentially each time the larger bound states that are formed are stable. This leads to the conclusion that an infinite tower of multiparticles states is generated in a theory that violates the RFC.

The WGC and the RFC as formulated above are equivalent only in the case with no massless scalars and a single gauge boson. When (still with no massless scalars) several gauge bosons are present, the WGC conjecture is determined by the convex hull condition, which amounts to require the presence of enough superextremal multiparticle states to enclose the black hole region. This is clearly a weaker statement than the RFC. In the case where massless scalars are present, if those enter the black hole solution in a way different than they enter into the combination of long range forces, this can actually make the WGC and the RFC differ even more, a situation which is depicted in Fig. 1.5, taken from [29]. The important take-home message is that the RFC might well be a different statement than the WGC and can be justified independently, which puts it on a firmer ground than it might have seemed before.

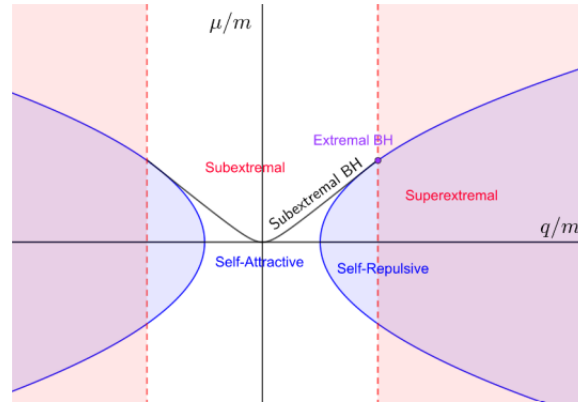


Figure 1.5 – Regions where of superextremality are depicted in red, while regions of self-repulsiveness are depicted in blue. Extremality here is only determined by $q/m > c$, with c a constant. Self-repulsiveness is determined by $\mu^2/m^2 \leq q^2/m^2 - 1$.

Finally, another crucial point for the RFC was pointed out. The question of whether two mutually attractive particles form a bound state had been investigated in the WGC literature only classically. The validity of the classical prediction that any two such particles do form a bound state relies on the knowledge that non-relativistic quantum effects should not determine the long range behaviour of the force between the particles. In [29] it was pointed out that this is specific to the case of $D = 4$ dimensions, and that attention should be paid when working in $D > 4$.

As the WGC bound is meant to be imposed at the scale of the particle's pole mass, it is actually natural to study the question quantum-mechanically. In D dimensions, with one time coordinate and $d \equiv D - 1$ spatial dimensions, the bound state solves the two-body time-independent Schrödinger equation

$$\left\{ -\frac{\hbar^2}{2\mu} \nabla_d^2 - \frac{\mathcal{K}}{r^{d-2}} \right\} \Psi(\vec{r}) = E\Psi(\vec{r}), \quad (1.6.13)$$

where ∇^2 is the d -dimensional Laplacian, $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass given in terms of the constituents masses, and \mathcal{K} is the coefficient of the long range force given by $\mathcal{K} = -\mathcal{F}_{12}/V_{d-1}$, with V_{d-1} the volume of the unit $d - 1$ sphere. For $\mathcal{K} > 0$, the classical force is attractive. The Laplacian can be expressed in spherical coordinates as

$$\nabla_d^2 = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{\nabla_{\Omega_{d-1}}^2}{r^2}, \quad (1.6.14)$$

where $\nabla_{\Omega_{d-1}}^2$ is the Laplacian on the $d - 1$ sphere. This can be written in the hyperspherical harmonics basis, where the radial and angular variables are separated as $\Psi(\vec{r}) = R(r)Y(\theta_1, \dots, \theta_{d-1})$ and the sphere Laplacian applied to R has eigenvalues $-l(l + d - 2)$, with l the angular momentum. The equation for the radial coordinate is then obtained

$$\frac{d^2 R}{dr^2} + \frac{d-1}{r} \frac{dR}{dr} + \frac{2\mu}{\hbar^2} \left\{ E + \frac{\mathcal{K}}{r^{d-2}} - \frac{\hbar^2 l(l + d - 2)}{2\mu r^2} \right\} = 0. \quad (1.6.15)$$

This can be further simplified defining $u(r) \equiv r^{(d-1)/2} R$, and the equation for u takes

the simple form

$$\frac{d^2 u}{dr^2} + \frac{2\mu}{\hbar^2} \{E - V_{\text{eff}}\} u = 0, \quad (1.6.16)$$

where the effective potential has been defined in the usual way as

$$V_{\text{eff}} = -\frac{\mathcal{K}}{r^{d-2}} + \frac{\hbar^2}{2\mu} \left(\frac{l(l+d-2)}{r^2} + \frac{(d-3)(d-1)}{4r^2} \right). \quad (1.6.17)$$

This is nothing but the generalization to the case of d generic space dimensions of the usual steps that lead to the well-known quantum-mechanical solution to the hydrogen atom (or any other three-dimensional system with central potential). In particular, these steps allow to write the radial equation as a one-dimensional Schrödinger equation, from which it is possible to read the total potential effectively felt by the particles as r varies. In usual, $d = 3$, quantum mechanics, it is a well-established result that a non vanishing angular momentum generates a repulsive centrifugal barrier $\sim r^{-2}$. The contribution from the "original" potential falls-off like r^{-1} and is dominant at large r , leading to the conclusion that the classical analysis of the sign of \mathcal{K} is sufficient to determine whether two particles form a bound state or not. Schematically, the way quantum mechanics change the classical picture is through the introduction of the centrifugal barrier that only determines the minimal width of the bound state. Another important property of the $d = 3$ result is that the centrifugal barrier is not present for vanishing angular momentum $l = 0$. When $d > 3$, things drastically change. In the passage from R to u an additional term proportional to $d - 3$ that falls-off like r^{-2} is generated. The contribution that in $d = 3$ is given by the angular momentum is now shifted, so that even in the case $l = 0$ a term falling off like r^{-2} is present. The fall-off of the classical potential proportional to \mathcal{K} is steeper the greater the value of d . In $d = 4$ space dimensions, the original potential and the additional r^{-2} terms are comparable, and for positive values of \mathcal{K} , that is, for an attractive classical force, a bound state can form only if

$$\mathcal{K} > \frac{3\hbar^2}{8\mu}, \quad (1.6.18)$$

where the right hand side corresponds to the r^{-2} contribution inside the parenthesis in (1.6.17) calculated for $l = 0$. For states with non-zero angular momentum the repulsive barrier is stronger, and the condition is then more severe. For $d > 4$, the behaviour is opposite to the $d = 3$ one. The "centrifugal" term (which is not centrifugal anymore) is actually the dominant term at large r and acts repulsively. Only small bound states can exist for positive values of \mathcal{K} , and the "centrifugal" force acts as a confining force for them: for distances larger than a certain maximal value, the bound state is ripped apart. The minimal size of these bound states depends on details of the short-range interactions.

Equations (1.6.16) and (1.6.17) show that the study of the sign of \mathcal{K} is sufficient to enquire about the formation of (non-relativistic) bound states both at the classical and at the quantum level only when $d = 3$, corresponding to the usual four space-time dimensions we experience. For larger values of d , the question is more subtle. Of course, $\mathcal{K} > 0$ still is a necessary condition to allow for a bound state, but clearly it is not a sufficient one. This should be kept in mind when bound state arguments are

used to motivate the kind of Weak Gravity Conjecture that in [29] was termed Repulsive Force Conjecture. Nevertheless, it should also be noted that higher-dimensional theories, that is theories with more than four-spacetime dimensions, should reduce at sufficiently low energies to a four dimensional effective field theory that includes gravity upon compactification. As a consequence, although a violation of the RFC in $D > 4$ spacetime dimensions does not guarantee the formation of an infinite tower of stable bound states (in D dimensions), it might still lead to such a problematic prediction in the effective four dimensional theory. Actually, it turns out that compactifications over tori lead to conservation of the attractive/repulsive character of the force. This was already briefly discussed in [29]. An example of a conservation of the overall force sign and character has been provided here in the discussion below equation (1.6.7), and further examples will be given in Chapter 4.

1.7 WGC, Scalars and the Distance Conjecture

Scalar fields have a special place in the Swampland program. They always appear in the process of compactification, as we have already discussed in the previous sections, and their vacuum expectation value (VEV) usually determines the strenght of some lower dimensional effective couplings and mass terms. Such scalars often appear massless at tree-level, and in sufficiently controlled set-ups (for example, with sufficient supersymmetry) can be maintained so. The massless scalars that appear after compactification are usually referred to as "moduli" field, and the way their vacuum value affects the effective theory has been studied with particular attention. An example of this behaviour is provided by the gauge coupling and the mass spectrum obtained from the circle compactification of a field theory combined with gravity (see for instance (A.2.9) and (A.2.12)). Schematically, the compactification of a field $\hat{\Phi}$ on a circle of radius L leads in the lower dimensional effective theory to a tower of states where the quantized momentum along the compact dimension n/L is seen as an effective mass term. In equations, we have:

$$m_{n(D)}^2 = m_{(D+1)}^2 + e^{\sqrt{2}\gamma\kappa\phi_0} \frac{n^2}{L^2}; \quad (1.7.1)$$

where ϕ_0 is the VEV of the dilaton ϕ , $m_{(D+1)}^2$ is a $D+1$ -dimensional mass term and γ the dilatonic coupling that, for simplicity, we leave unspecified and fix to be positive. The states that respect this mass formula are called the Kaluza-Klein (KK) modes. In string theory, they have to be supplemented with winding modes, that express the possibility that the string, as an extended object, wraps the compact dimension p times. After compactification, the generic mass term has the form

$$m_{n,p(D)}^2 = m_{(D+1)}^2 + e^{\sqrt{2}\gamma\kappa\phi_0} \frac{n^2}{L^2} + e^{-\sqrt{2}\gamma\kappa\phi_0} \frac{p^2 L^2}{l_s^4}, \quad (1.7.2)$$

where l_s is the string length. Consider now a typical configuration in the field theory domain with a VEV $\kappa\phi_0^{(1)} \ll 1$, such that both exponentials are $\mathcal{O}(1)$, and a variation of the VEV ϕ_0 from $\phi_0^{(1)}$ to $\phi_0^{(2)}$. Defining the variable $\delta\phi$ as their difference, $\delta\phi \equiv \phi_0^{(2)} -$

$\phi_0^{(1)}$, we can distinguish two behaviours. When $\delta\phi > 0$, the different levels of the KK tower are all magnified, while the levels of the winding tower are all lowered. The converse happens when $\delta\phi < 0$. Independently from the sign, for VEV variations of order greater than the Planck mass $\gamma\kappa|\delta\phi| \gtrsim 1$, either the KK tower or the winding tower show in the effective lower dimensional theory an infinite number of states that become light at an exponential rate.

That in both the limits $\kappa\delta\phi \gg 1$, $\kappa\delta\phi \ll -1$ there is (at least at tree-level) one tower whose mass scale exponentially decreases is a deep property of string theory, as opposed to the field theory case where it is true only in the second limit. This is related to the extended nature of the strings, that allows for the existence of winding states, and to the non-trivial interplay between KK and winding modes, that are related through the exchange

$$L^2 \leftrightarrow \frac{l_s^4}{L^2}. \quad (1.7.3)$$

The latter automatically enforces also the exchange $\phi_0 \rightarrow -\phi_0$, as the dilaton encodes the $g_{D+1,D+1}$ degree of freedom and controls the radius of the extra dimension. From a stringy perspective, then, the two regions of parameter space with $\kappa\phi_0 \gg 1$ and $\kappa\phi_0 \ll -1$ are considered not to admit a QFT description. Of course, this latter statement does not add anything new to our previous knowledge, since scalar VEVs greater than the Planck mass are surely not admitted in an EFT that has to be completed into a theory of Quantum Gravity, barrying the trials towards and asymptotically free QFT of gravity. The example aims to go beyond the plain and simple observation of the breakdown of the EFT and to point out *how* such a breakdown might be signaled when scalar field VEVs grow up to the Planck scale.

On the side of the stringy approach to the Swampland, these features have been elevated to the rank of a conjecture. This was first proposed in [30] in the following terms. Consider an EFT with gravity where a moduli space \mathcal{M} is parametrized by the VEV of a set of scalar fields ϕ^i with no potential. Starting from a point P and sweeping the possible values for the scalar VEVs towards a point P' which is an infinite distance⁵ far away from P , one encounters a tower of states that becomes light at an exponential rate as

$$m(P') \sim m(P)e^{-\gamma\kappa|\vec{\phi}_0(P')-\vec{\phi}_0(P)|}, \quad (1.7.4)$$

where $\vec{\phi}_0$ is the position vector in the moduli space $\vec{\phi}_0 \equiv (\phi_0^1, \dots, \phi_0^n)$.

The conjecture has been extensively studied in various string set-ups to gather evidence that this might be a generic feature of string theory, according to the stringy Swampland philosophy. A refinement of the conjecture has been presented in [31, 32], where the meaning of "infinite distance" was sharpened in the sense described in the footnote, and it was proposed that the conjecture might also apply to scalars with potentials and their corresponding field space. It is also believed that the coupling γ should not be parametrically smaller than one, such that the exponentially decreasing behaviour of the masses truly occurs for distances of order the Planck mass.

Studies of the Distance Conjecture beyond string theory have not been performed and it is not evident that its prediction could be attributed to intrinsic properties of

⁵ Of course, the statement of an infinite distance by itself has no sharp meaning. Here, "infinite" means a distance greater than the Planck mass such that $\gamma\kappa\delta\phi > 1$.

quantum gravity. Nevertheless, for the scope of this thesis, our interest towards this conjecture comes from the possible links to the WGC that have been observed. The prediction that in a certain limit of scalar VEV field space a tower of states become lighter and lighter implying the breakdown of the EFT at a lower and lower scale, seems to fit very well with the picture given by the magnetic WGC. The upper limit on the cut-off scale imposed by the magnetic WGC is controlled by the gauge coupling. Equations (A.2.9) and (A.2.12) provide an example of the fact that the limits studied by the distance conjecture might correspond to a weak coupling limit, making the connection sharper. Possible links due to entropy bounds have been observed. Our main concern will be on the possibility that, if found, a scalar version of the WGC, i.e. a bound on the relative strength of scalar and gravitational interactions in a theory with no gauge fields, have some ties with the distance conjecture.

1.8 Towards a Scalar WGC

A first attempt towards a direct comparison between scalar interactions and gravity was already made in [27]. It was observed that, when several $U(1)$ gauge fields are present, a quantity that was considered to encode the interaction between two BPS $\mathcal{N} = 2$ states might vanish. This was interpreted as if in the absence of gauge interactions some sort of cancellation between gravitational contributions and scalar contributions is at place. The generalization into an equation in terms of more physical quantities, rather than one in terms of central and symplectic charges proper of $\mathcal{N} = 2$, and the individuation of a physical principle behind it was however hosted by some difficulties. Scalar mediated forces are attractive, and the long range force argument surely cannot be used. Moreover, it is not clear how and why self-attractive particles should be a problem in this context. Nevertheless, to mimick the result of $\mathcal{N} = 2$ and specify (1.6.5) to a theory with scalar mediators and no gauge fields, it was still proposed that a particle respecting the inequality

$$g^{ij} \partial_i m \partial_j m > \sqrt{\frac{D-3}{D-2}} \kappa^2 m^2, \quad (1.8.1)$$

with g^{ij} the metric on the scalar manifold and ∂_i the derivative with respect to the field ϕ_i , should exist. This was assumed to encode the idea that gravity is the weakest force, which is one of the many ways one can look at the original WGC (1.3.2). The question of where this equation might come from and why this is the right comparison to make was left as an open question. A property of (1.8.1) that was greatly appreciated is the following. Restricting to the simplest case of one scalar mediator and using geometrical units not to bother with the numerical coefficient and the coupling κ , it was observed that if (1.8.1) is considered as a differential equation on the "mass" term

$$|\partial_\phi m| > m \quad (1.8.2)$$

this would be solved by a function

$$m = e^{\pm\alpha\phi}, \quad |\alpha| > 1. \quad (1.8.3)$$

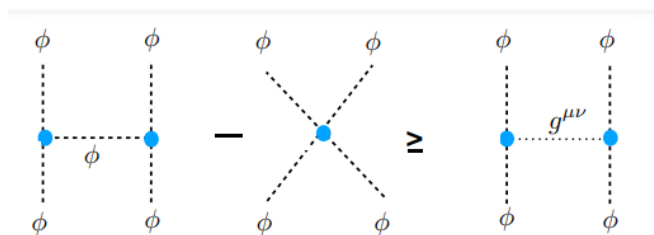


Figure 1.6 – The SSWGCG is supposed to bound the combination of the three and four point self interactions of the scalar field ϕ to make gravity the weakest force

Before discussing the result, an important comment is in order. In (1.6.5) and (1.8.1) the "mass function" $m(\phi)$ is defined to have a compact notation for both a proper mass term and interacting terms of the WGC state with the scalars (as, for e.g., we usually do in the calculation of the effective potential). In this compact notation, the actual mass and three-point coupling are the zeroth and first order term in an expansion around the background values of the scalar fields. In other words, the inequality described by those equations, strictly speaking, need only to be respected when the quantities m and $\partial_\phi m$ are the coefficients of the first two terms of a Taylor expansion around ϕ_0 and are calculated *at* ϕ_0 . It corresponds to the point-like requirement that, about the specific value ϕ_0 of the scalar field VEV, a certain relation exists between the couplings. It does not say anything about what happens for a different background value ϕ'_0 . Imposing the inequality as a differential one on the quadratic operator $m(\phi)$ is equivalent to asking that scalar interactions dominate with respect to gravitational interactions for any background value of ϕ , which is of course a much stronger statement.

Uplifting the constraint on the couplings to a differential inequality means that different VEVs of ϕ are being probed, and in turns this makes it possible to link the scalar WGC to the distance conjecture. The result (1.8.3) is such that for large $\alpha|\phi_0|$, in physical units $\alpha\kappa|\phi_0| \gtrsim 1$, each one of the two solutions for the mass function $m(\phi)$ decreases at an exponential rate in one of the two limits. This was taken as an indication that the states that saturate the scalar WGC might be the same ones that ensure the breakdown of the EFT when transplanckian values in the scalar field space are reached. They could even be the first states of a tower, thus signaling the usual types of structure encountered in theories with extra dimensions. The lower bound on α in (1.8.3) ensures that the exponential fall truly happens no further than the Planck scale.

Although these nice features point towards the idea that a relation between the differential constraint (1.8.2) and the distance conjecture exists, the result of the former seems to miss one key feature of the latter. The solution (1.8.3) can account for an exponential fall only in one direction of field space. There is no single function which is a solution to (1.8.2) and is characterized by an exponentially decreasing behaviour in both the cases $\kappa\phi > 1$ and $\kappa\phi < -1$. In [33], this was actually taken as an indication that the bound (1.8.1) might be incomplete. Building on the idea already contained in [17] that gravity has to be the weakest of all interactions, they proposed to postulate that the interactions of any scalar have to be stronger than gravity in the whole field space. In particular, they formulate the conjecture as a constraint on the self-interactions of a single scalar. Although conceptually not new and actually in agreement with the

original spirit of the WGC, this is different from the previous attempt towards a scalar WGC as it encompasses the identification and distinction of a scalar mediator and a WGC state. For a single scalar field ϕ with potential $V(\phi)$, the requirement is that for any value of ϕ , $V(\phi)$ respects the differential constraint

$$2(V''')^2 - V''V'''' \geq \kappa^2(V'')^2, \quad (1.8.4)$$

where prime symbols indicate ϕ -derivatives. This bound was named the "Strong Scalar Weak Gravity Conjecture" (SSWGC) and it was claimed to have a simple interpretation in terms of the $\phi\phi \rightarrow \phi\phi$ scattering amplitude as described by Fig. 1.6, taken from [33]. According to that, the first term should describe the attractive scalar mediated interaction that reduces to an effective contact interaction at large distances. The inclusion of a repulsive genuine contact term given by V'''' was shown to be crucial to make the conjecture consistent with simple axion potentials. It describes an UV contribution (a Dirac delta in r -space). For this reason, the bound (1.8.4) was considered to encapsulate some sort of intrinsic UV/IR mixing described by the interplay between the three and four point couplings. This was further motivated by the study of the extremal condition for which (1.8.4) is saturated. Writing the equation in terms of $m^2 \equiv V''$, the most general solution is obtained as

$$m^2(\phi) = \frac{1}{Ae^{\kappa\phi} + Be^{-\kappa\phi}}, \quad (1.8.5)$$

where A and B are two integration constants. The denominator of the solution is rather suggestive, as the combination of a positive and a negative exponential reminds of the string spectrum (1.7.2) with a built-in duality for the exchange

$$\phi \leftrightarrow -\phi, \quad A \leftrightarrow B. \quad (1.8.6)$$

The structure of (1.8.5) is such that in both the limits $\kappa\phi > 1$ and $\kappa\phi < -1$ it predicts an exponentially damped mass function, and expanding in the asymptotic regions we obtain

$$m^2_{\kappa\phi \gg 1} \rightarrow Ae^{-\kappa\phi}, \quad m^2_{\kappa\phi \ll -1} \rightarrow Be^{\kappa\phi}. \quad (1.8.7)$$

In [33] the authors discuss the possibility that these states are actually signaling the presence of a tower, of which they are the first states and that the extremal states not only might "know" about the existence of extra dimensions, but also about string theory and the winding states. For the potential, the solution (1.8.5) takes the form

$$V(\phi) = c_1 + c_2\phi + i \frac{\text{Li}_2\left(-\frac{i\sqrt{A}e^{\kappa\phi}}{\sqrt{B}}\right) - \text{Li}_2\left(\frac{i\sqrt{A}e^{\kappa\phi}}{\sqrt{B}}\right)}{2\sqrt{A}\sqrt{B}\kappa^2} \quad (1.8.8)$$

where c_1 and c_2 are two further integration constants and Li_2 is the dilogarithm function. For specific choices of the constants, the above potential show a peculiar behaviour with a minimum and a linear growth for large $\kappa\phi$.

Application of (1.8.4) to simple potentials reveals that, although it is respected to a high degree, some of the most common potentials violate it at least in a region of parameter space. For example, the quartic potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \quad (1.8.9)$$

with $m^2 > 0$ respects the bound only for $\phi^2 \gtrsim (2/3) m^2/\lambda$. If (1.8.4) truly translates the idea that gravity is weaker than scalar self-interactions, we should then conclude that for a scalar in a quartic potential gravity is subdominant only when the field is expanded around values of the background $|\phi_0| \gtrsim (2/3) m^2/\lambda$. With the quartic potential above, the VEV of ϕ is (classically) at $\phi = 0$. This is the vacuum about which we should expand our theory, and according to (1.8.4) we should conclude that gravity is the strongest force! This violation of the bound in the ϕ^4 theory was further analyzed in [34], where equation (1.8.4) was dismissed. A physical counterexample was given in terms of a dilute gas of scalar atoms, of which helium 4 atoms are a prime example, that is well described by a potential of the form (1.8.9) with $\lambda > 0$. As we will see more thoroughly in chapter 3, where the work [1] will be presented, the failure of (1.8.4) is mainly due to the fact that the constraint does not become trivial in the decoupling limit of gravity $\kappa \rightarrow 0$. It instead remains as a constraint on the derivatives of the potential, and accordingly it cannot be attributed (at least, not entirely) to properties of quantum gravity. Even forgetting about this, the derivation of the bound is not clear, and the identification of the different terms with the diagrams of Fig. 1.6 is not straightforward. In particular, the factor 2 in front of the third derivative of the potential is rather unjustified. It is however of the greatest importance, in the interpretation given for the extremal states. When it is replaced by a factor 1, for example, the extremal solution takes the form

$$m^2(\phi) = Ae^{-\frac{\kappa^2 \phi^2}{2} + B\phi} \quad (1.8.10)$$

while for a generic prefactor $k \neq 1$ it is

$$m^2(\phi) = \left(Ae^{\sqrt{k-1}\kappa\phi} + Be^{-\sqrt{k-1}\kappa\phi} \right)^{-\frac{1}{k-1}}. \quad (1.8.11)$$

The solution (1.8.10) trivially shows an exponentially decreasing behaviour for both $\kappa\phi > 1$ and $\kappa\phi < 1$ as long as the quadratic term in the exponential dominates with respect to the linear one. The duality $\phi \rightarrow -\phi$ is approximately realized only in the asymptotic regions of the ϕ space. The solution (1.8.11) is well-behaved only for $k > 1$, and all the features of the case $k = 2$ are present. For $k < 1$, the solution is imaginary and certainly cannot be considered. This shows that the conclusions reached in [33] about the extremal states can only be supported for $k \geq 1$. A way to actually derive a constraint from a physical principle will be presented in chapter 3, and the corresponding value of k will naturally emerge.

1.9 WGC in de Sitter Spacetime

1.9.1 Cosmological expansion and de Sitter spacetime

So far, our discussion has been limited to the special case of an asymptotically flat spacetime. The black hole solutions shown in the previous sections indeed share the property that the metric reduces to the Minkowski one in the $r \rightarrow \infty$ limit. Analogously, the calculation of scattering amplitudes and the related derivation of the classical force is performed in the usual S -matrix formalism of flat-space QFT, where the

asymptotic properties of Minkowski spacetime, that is, the existence of a spatial infinity, allows to define the $\langle in |$ and $| out \rangle$ states on which the LSZ formula is built. Although the weakness of the gravitational coupling and the experimental success of the Standard Model suggest that the flat space-time is an extremely good low-energy approximation to the real spacetime we live in, we have reasons not to expect that asymptotic flatness is realized at a more fundamental level.

Cosmological observations have led to the discovery that the Universe is expanding at an accelerating rate [35, 36]. In the Standard Model of Cosmology, based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) solution to Einstein's equation for a homogeneous and isotropic Universe, an accelerated expansion can only be accounted for if it is driven by an energy component with a negative pressure and an equation of state $w \equiv p/\rho < -1/3$, where p is its pressure and ρ its energy density. For instance, in the FLRW metric

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{\sqrt{1 - Kr^2}} + r^2 d\Omega_2^2 \right), \quad (1.9.1)$$

where $a(t)$ is the comoving factor and K a parameter that dictates the curvature of three dimensional constant time surfaces, ${}^3R = 6K/a^2$, the Einstein's equations reduce to the combination of the Friedmann equation

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} \rho - \frac{K}{a^2} \quad (1.9.2)$$

and an equation for the acceleration \ddot{a}

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6} (\rho + 3p). \quad (1.9.3)$$

From (1.9.3) it is rather evident that an accelerated expansion requires an equation of state $w < -1/3$. Among the different components, about 70% of the energy density of the Universe today consists of the energy component that is thought to be responsible for the accelerated expansion, that we refer to as dark energy. It might come either in the form of a positive cosmological constant Λ_c , with equation of state $w = -1$, or in the form of a slowly varying scalar field ϕ , usually called "quintessence", with equation of state

$$w = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \quad (1.9.4)$$

A quintessence dark energy should be well described, at least in a first approximation, by a positive cosmological constant, so that in the following we shall concentrate on this case. We will briefly come back to the issue of the quintessence in the chapter dedicated to the Scalar WGC we have proposed in [1].

A Universe with a positive cosmological constant is described by a de Sitter (dS) spacetime (for an introductory review on dS see [37, 38]). A generic D -dimensional de Sitter spacetime can be more simply visualized as an hyperboloid in a $D + 1$ -dimensional Minkowski spacetime satisfying the equation

$$-X_0^2 + X_1^2 + \dots + X_D^2 = l^2, \quad (1.9.5)$$

with l the dS radius. Calculation of the induced metric shows that the Ricci tensor is

$$R_{\mu\nu} = \frac{D-1}{l^2} g_{\mu\nu}, \quad (1.9.6)$$

from which it is straightforward that in dS spacetime the vacuum equation is

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R + g_{\mu\nu} \Lambda_c = 0 \quad (1.9.7)$$

with

$$\Lambda_c = \frac{(D-1)(D-2)}{2l^2}. \quad (1.9.8)$$

This shows that dS spacetime truly describes a Universe with a positive cosmological constant. In static coordinates,

$$\begin{cases} X_0 = \sqrt{1-r^2} \sinh t \\ X_i = r \sin \theta_1 \cdots \sin \theta_{i-1} \cos \theta_i, & i = 1, \dots, D-1 \\ X_d = \sqrt{1-r^2} \cosh t, \end{cases} \quad (1.9.9)$$

the metric reads

$$ds^2 = -(1-H^2 r^2) dt^2 + (1-H^2 r^2)^{-1} dr^2 + r^2 d\Omega_{D-2}^2, \quad (1.9.10)$$

where $H^2 = l^{-2}$ is the Hubble parameter. These coordinates have two remarkable features: $\partial/\partial t$ is clearly a Killing vector, meaning that a time evolution can be defined at least in a certain region of the spacetime, and they show what is usually called a cosmological horizon at $r^2 = 1/H^2$. It is a generic property of dS that no observer can access the entire spacetime, and this limitation is expressed here by the presence of the horizon. We define the accessible region where the t variable is time-like (and $\partial/\partial t$ points towards the future) the causal or static patch of dS.

The geometrical properties of dS spacetime make it a complicated task to even develop a formalism where to calculate scattering amplitudes. For a recently proposed approach that, based on bootstrap techniques, tries to encompass these problems, see [39, 40]. On the other hand, the study of black holes existence and extremality conditions is possible and more accessible. In the following, the Reissner-Nordström de Sitter black holes will be studied to derive the WGC constraint imposed by the black hole decay in dS spacetime.

1.9.2 Black holes in dS and the WGC

A four-dimensional Einstein-Maxwell theory in dS

$$S = \int d^4x \sqrt{-g} \left(\frac{R - 2\Lambda_c}{2\kappa^2} - \frac{1}{4g^2} F^2 \right) \quad (1.9.11)$$

admits a charged black hole solution of the form [41, 42, 43]

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad (1.9.12)$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - H^2 r^2. \quad (1.9.13)$$

The parameters that appear are defined in geometrical units (1.3.3) and the definition of the Hubble parameter $H^2 = \Lambda_c/3$ have also been used. When $Q = 0$, the Schwarzschild dS black holes are recovered. Finding the zeroes of the function (1.9.13) in that case shows that these solutions have two horizons, the cosmological horizon and an event horizon, when $M^2 < 1/(27H^2)$. When the equality is reached, the cosmological and event horizon are degenerate, and for $M^2 > 1/(27H^2)$ the metric shows a naked singularity at $r = 0$. In this case, we say that the whole causal patch of dS has been eaten by the black hole [44]. The near horizon limit when $M^2 \rightarrow 1/(27H^2)$ from below should be investigated by means of appropriate coordinates, that define what we call a (neutral in this case) Nariai black hole [45]. The first peculiar property of black holes in dS that we already encounter here is that the presence of a cosmological horizon at a finite distance puts an upper bound on the black hole mass. The study of the zeroes of (1.9.13) in the general case with $Q \neq 0$ is more complicated. We define for simplicity the fourth degree polynomial

$$P(r) \equiv -r^2 f(r) = H^2 r^4 - r^2 + 2Mr - Q^2. \quad (1.9.14)$$

The product of the four roots of P is $-Q^2 < 0$, so that P necessarily have two real roots, one positive and one negative. The other two roots can either be real and of the same sign, or complex conjugate. From this, we can already conclude that these solutions can only have either three or one horizon. The nature of the last two roots is determined by the sign of the determinant of the fourth degree polynomial

$$\Delta = 16H^6 \left(-27H^2 \frac{M^4}{H^4} + \left(\frac{1}{H^2} + 36Q^2 \right) \frac{M^2}{H^2} - Q^2 \left(\frac{1}{H^2} + 4Q^2 \right)^2 \right). \quad (1.9.15)$$

We have three cases:

- When $\Delta > 0$, three roots are real and positive and one is negative;
- When $\Delta = 0$ two of the three positive roots are degenerate, and one is negative. The equation $\Delta = 0$ is quadratic in the variable M^2/H^2 and it has two solutions;
- When $\Delta < 0$ two of the three roots above are complex conjugate, one is positive and one is negative.

The two roots of Δ are

$$\begin{aligned} M_-^2(H, Q^2) &= \frac{H}{54} \left[\frac{1}{H} \left(\frac{1}{H^2} + 36Q^2 \right) - \left(\frac{1}{H^2} - 12Q^2 \right)^{3/2} \right] \\ M_+^2(H, Q^2) &= \frac{H}{54} \left[\frac{1}{H} \left(\frac{1}{H^2} + 36Q^2 \right) + \left(\frac{1}{H^2} - 12Q^2 \right)^{3/2} \right]. \end{aligned} \quad (1.9.16)$$

The behaviour of the solution is shown in Fig. 1.7, taken from [46]. The two roots are signaled by the green and red line, respectively. In between the two, in the blue

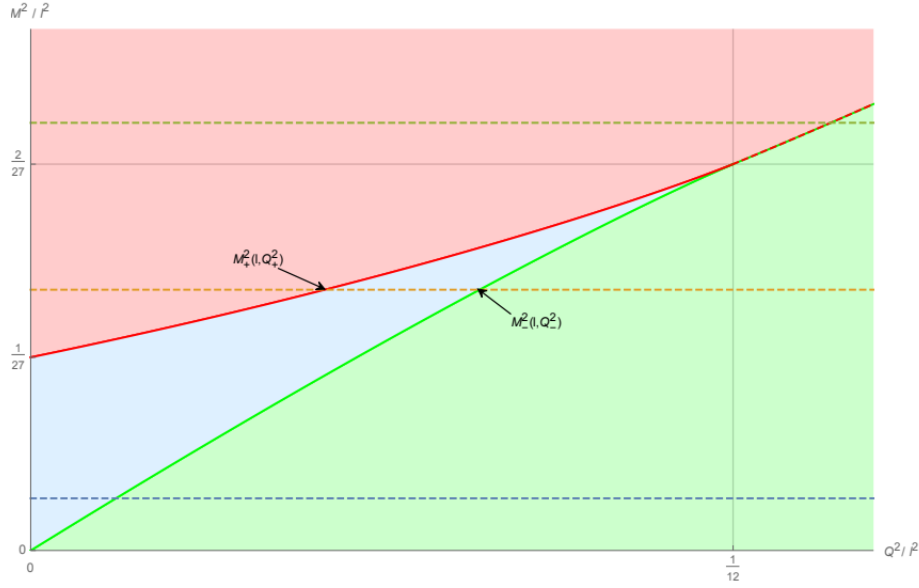


Figure 1.7 – Parameter space of the RN dS black hole solution. Black holes with three horizons are found in the blue region. The green region correspond to solutions with only a cosmological horizon, while the red region to solutions where the dS causal patch has been completely eaten by a giant black hole. Extremal black holes are found on the green curve, Nariai on the red one.

region, black hole solutions with three horizons are found. They possess a cosmological horizon, an event horizon and an inner Cauchy surface. As it happens for the Schwarzschild BHs, also the RN solution in dS is qualitatively similar to the corresponding Minkowski one, with the important addition of a maximal radius dictated by the dS length. Starting from a point in the blue region, we can understand the phase space of the black hole depicted in the figure following the evolution in two different directions: either growing the charge (horizontal line) or the mass (vertical line).

- Growing the charge: The event horizon shrinks while the inner horizon grows, until the two meet on the green curve. Those are the black holes we call extremal. Moving forward in the same direction, solutions with only the cosmological horizon are found in the green region.
- Growing the mass: The event horizon grows while the cosmological horizon shrinks, until the two meet on the red curve. Those are the charged Nariai black holes. In the red region, the event horizon would be bigger than the cosmological horizon: we interpret those solutions as if the black hole has eaten the whole dS causal patch.

The green and red curve meet at the point $(MH, QH) = (2/27, 1/12)$. The latter defines the maximal mass and charge for this family of black holes.

Another important thing to point out is that, for a given fixed charge Q , there are both a minimal and a maximal mass M , whereas in Minkowski only a lower bound on the mass is present. The green line is the equivalent of this Minkowski type of extremality, while the red line is conceptually new and completely related to the finite

size of dS. For a three-horizon black hole solution, we can individuate four regions of space. Indicating respectively the inner Cauchy surface, event horizon and cosmological horizon with r_c, r_h and r_{dS} we have

- $0 < r < r_c$: the t variable is time-like, and the electromagnetic energy density is the dominating one;
- $r_c < r < r_h$: the t variable is space-like, and it is now the gravitational energy density which is dominating;
- $r_h < r < r_{dS}$: the t variable is again time-like and $f(r)$ is dominated by the constant term
- $r_{dS} < r$: the t variable is space-like and it is the vacuum energy that dominates.

The two kind of extremality of the RNdS black holes suggest that there are two distinct decay processes to attention. One the one side, Nariai black holes should be able to decay, otherwise the red line would be infinitely populated by stable states, and they should do it without causing the complete disrapture of the dS patch. Extremal black holes should also be able to decay, otherwise the green line would be infinitely populated by stable states, without showing a naked singularity. The first mechanism has been studied in [47, 48]. It clearly is an intrinsically dS physical process, and as such it cannot be related to the flat space WGC. The analysis of the discharge of the Nariai black holes through Schwinger pair production lead to the conclusion that Nariai black holes can discharge themselves without "eating" all of the space if a particle of mass m and charge q such that

$$m^2 \gg gqHM_P \quad (1.9.17)$$

exists. Interestingly, the new dS process imposes a lower bound on the mass of the particle state that ensures the discharge. In the limit of a small cosmological constant $H \rightarrow 0$ the constraint disappear, which is consistent with the fact that Nariai black holes do not exist in Minkowski.

The second physical process, on the other hand, is conceptually the equivalent of the flat space one. Following [46], the decay of extremal RNdS black holes imposes the existence of at least one particle for which the electromagnetic energy density is strong enough to forbid the formation of horizons (apart from the cosmological one). In this way, when the extremal black hole decays emitting this particle, it loses more electromagnetic energy than gravitational one, so that it ends up in a subextremal black hole. This boils down to the requirement that at least one particle is found in the green region of Fig. 1.7. Indicating with upper case letters the charge and mass in geometrical units and with lower case letters the same quantities in physical units, quantitatively we have two distinct situations:

- For $Q^2 \leq (12H^2)^{-1} \Leftrightarrow g^2q^2 \leq \frac{8\pi^2}{3\kappa^2H^2}$ and $M^2 \leq 2/(27H^2) \Leftrightarrow m^2 \leq 2\frac{(8\pi)^2}{27\kappa^4H^2}$

we require the existence of a particle of mass m (M) and charge q (Q) such that

$$M^2 \leq M_-^2(H, q) \Leftrightarrow m^2 \leq \frac{(8\pi)^2}{\kappa^4} \frac{H}{54} \left[\frac{9\kappa^2}{8\pi^2H} g^2q^2 + \frac{1}{H^3} - \left(\frac{1}{H^2} - \frac{3\kappa^2}{8\pi^2} g^2q^2 \right)^{3/2} \right], \quad (1.9.18)$$

- For $Q \geq (12H^2)^{-1} \Leftrightarrow g^2 q^2 \leq \frac{8\pi^2}{3\kappa^2 H^2}$

we require the existence of a particle of mass m (M) and charge q (Q) such that

$$M^2 \leq \frac{3}{2}H^2 \left(Q^2 + \frac{5}{36} \frac{1}{H^2} \right)^2 \Leftrightarrow m \leq \frac{5}{12\sqrt{6}} \frac{8\pi}{\kappa^2 H} + \sqrt{\frac{3}{2}} H \frac{g^2 q^2}{4\pi}. \quad (1.9.19)$$

Expanding the results about the flat space limit, that is for $H \rightarrow 0$, the second of the two regions individuated above disappear, and expansion of (1.9.18) reads

$$M^2 < Q^2 - Q^4 H^2 + \mathcal{O}(H^4) \Leftrightarrow m^2 \leq 2 \frac{g^2 q^2}{\kappa^2} - \frac{g^4 q^4}{16\pi^2} H^2 + \mathcal{O}(H^4). \quad (1.9.20)$$

This correctly reproduces the flat-space WGC (1.3.2) when $H = 0$. As expected, the mechanism that allows for the decay of the extremal black holes that live on the green line of Fig. 1.7 is intimately related to the flat space one. If our aim is that of describing a dS Universe that can provide a good approximation to the one we live in, which is consistent with an extremely small cosmological constant of order $\Lambda_c \sim 10^{-122} M_P$ [49], this physical process certainly is of more practical interest than the discharge of the Nariai black holes. Equation (1.9.20) provides the correction to the WGC bounds (1.3.2) for a small non-vanishing vacuum energy/curvature of the Universe. It is rather comforting to see that the analysis in Minkowski space truly is a first order approximation to an analysis in a large dS space. Equation (1.9.17) certainly remains as a necessary consistency condition, and for all practical uses in the limit $H \rightarrow 0$ it basically reduces to the requirement that not all particle states are massless.

Chapter 5 will be dedicated to the presentation of [3], where the study of dS dilatonic black holes allowed us to define a dilatonic WGC in dS space. The Anti de Sitter (AdS) case for both the RN and the dilatonic black holes will also be presented, and confrontation with yet another mechanism proper of AdS is discussed in [3].

CHAPTER 2

WGC and U(1) mixing

2.1 Introduction

The extension of the WGC to the case of multiple gauge bosons [21] discussed in the previous chapter provided a generalization of (1.3.2). The same reasoning apply to magnetically charged black holes, which are protected by horizons as long as $|\vec{Q}_m|^2 \leq M^2$, with \vec{Q}_m the magnetic charge. Following the same route that brought to (1.3.18), the definition of a magnetic WGC for multiple gauge fields is rather straightforward. If, for simplicity, we follow the example we presented below (1.4.5) and consider the case of two orthogonal charges of the same strenght q_m , the requirement simply is that each of them satisfy $g_m q_m \geq \sqrt{2}m$ in geometrical units, i.e. $g_m q_m \geq \kappa m$ in physical units, from which it is then straightforward to derive the constraint

$$\Lambda \lesssim g_m^{-1} M_P. \quad (2.1.1)$$

The equations in the general case are certainly more intricated. It is of no use, for our scopes, to derive the precise form and coefficients of the generalization of (1.3.18) to the product of gauge fields. Certainly, as already discussed for the electric case, a weaker statement than that of the precise bound is the requirement that for each direction in charge space there is at least one state with superextremal charge to mass ratio. With

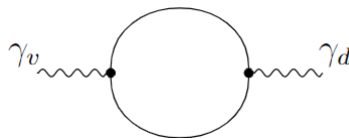


Figure 2.1 – One loop vacuum-polarization diagram generating a kinetic mixing between the visible photon γ_v and a dark photon γ_d .

the identifications that lead to (1.3.18), this translates into the requirement that the physical cutoff of the effective theory is

$$\Lambda \lesssim \min_{\{g\}} g M_P, \quad (2.1.2)$$

where $\{g\}$ is the set of electric couplings associated to the set of magnetic couplings $\{g_m\}$ through the Dirac quantization condition. As already stressed, the above bound, that we proposed for the first time in [2] in agreement with reasonings exposed in [50], misses of precision in the numerical factors, but encodes the qualitative prediction that the WGC for multiple $U(1)$ s imposes when it is applied to magnetic monopoles. Note that it describes a quantum gravity obstruction to the limit where any one of the gauge couplings in the theory vanishes, i.e. an obstruction to any one of the gauge symmetries to become global.

In [2], we have analyzed the consequences that this result have in theories where a so-called "dark photon" is present. Besides the energy content covered by dark energy, that we discussed in section 1.9.1 and amounts to approximately 70% of the total energy density of the Universe, the other $\sim 30\%$ is covered in large part ($\sim 25\%$) by dark matter. It is a hypothetical form of matter not yet discovered which should be composed by massive particles with little or no interactions with the Standard Model ones. The requirement that their interactions with the particles we have discovered so far (and in particular with the photon) is small is of course necessary to explain the difficulties in their detection. In this respect, an interesting realization comes in the following context. Suppose we add to the SM a single $U(1)$ gauge boson, that we call "dark photon", and a whole matter sector charged under this dark $U(1)$ with no interaction with the SM at all. Suppose that at least one state charged under the SM photon couples to this additional gauge boson. Then, labeling with a v and a d quantities related respectively to the photon and dark photon, vacuum polarization diagrams with the two different photons on each side generate a kinetic mixing ϵ_{vd} between the two gauge bosons at the one-loop level (see Fig. 2.1). This actually forms a portal through which all of the matter charged under the dark photon can interact with the SM photon, and thus the whole visible sector. For these interactions to be suppressed with respect to the SM ones, the kinetic mixing (and thus the dark gauge coupling, as we will discuss) needs to be sufficiently small. Quantitatively, of course, it depends on the specific model (for a thorough introduction on the subject see [51]).

At the time where [2] was written, the observation of an electronic recoil excess by the XENON1T collaboration [52] had driven considerable interest towards theories with a gauge kinetic mixing between the dark and visible photon of order $\epsilon_{vd} \sim 10^{-15} - 10^{-16}$. This was shown to provide a good fit of the data [53]. Although the work has been presentend in the wake of that observation, and numerical examples have been given according to the specific prediction $\epsilon_{vd} \sim 10^{-15} - 10^{-16}$ for concreteness, it should be regarded as independent of it. Its aim really is to point out a specific prediction of the Weak Gravity Conjecture in Beyond the Standard Model (BSM) models with a weakly coupled dark photon.

2.2 Discussion

The main prediction of [2] is that if the magnetic form of the WGC holds and a dark photon portal is realized in the Universe, a consequence of their combination is that the scale at which the model breaks down and new physics appears is parametrically smaller than both the Planck scale and the scale (1.3.18) obtained applying the $U(1)$ WGC to the visible photon. The one-loop induced gauge kinetic mixing is given as

$$\epsilon_{vd} = \frac{g_v g_d}{16\pi^2} \sum_i q_v^{(i)} q_d^{(i)} \ln \frac{m_i^2}{\mu^2}, \quad (2.2.1)$$

where m_i , $q_v^{(i)}$ and $q_d^{(i)}$ the mass and charges of the states coupled to both the visible and dark photon. With a gauge kinetic mixing ϵ_{vd} and the known value of the electromagnetic coupling, it was shown that the dark gauge coupling can be roughly estimated

$$g_d \sim (10^2 - 10^3) \epsilon_{vd}. \quad (2.2.2)$$

The Swampland completeness hypothesis was used to treat the logarithmic factor in (2.2.1) as an order $\mathcal{O}(1) - \mathcal{O}(10)$ number. This is in agreement with the Swampland philosophy and it goes in the direction of creating a network of the different conjectures to see what their combined predictions are. It is however not necessary. Although large logarithms are dangerous when performing precision calculations (for convergence of the perturbative series), this does not represent a problem for our order of magnitudes observation. Large logarithms would indeed require an even smaller coupling g_d and thus cut-off Λ . On the contrary, a sufficiently tuned cancellation between the logarithmic factors can be of greater concern as it would produce tiny values of ϵ_{vd} with no need for a strongly suppressed coupling constant. However, if a complete cancellation happens at the one-loop order, higher loops as well as gravitational loops would then become effectively the "leading" contributions to the kinetic mixing, and they would in turn generate a non-vanishing value of ϵ_{vd} . This was discussed in the text, and the results of [54] were used to argue that, at least for the numbers implied by XENON1T, the lower bound one can expect on ϵ_{vd} even in the case where the leading contribution is given by gravitational loops is still not sufficient to cancel our prediction (although it alleviates it). Of course, the complete cancellation that we are discussing is not simple to realize, and certainly it is not expected to happen. Nevertheless, it should be kept in mind as a possible way to make less stringent the prediction of a low coupling g_d and sequentially of a low cut-off Λ as a consequence of a tiny kinetic mixing ϵ_{vd} .

With all these caveats, the actual value of the predicted cut-off scale is strikingly small, of order $\mathcal{O}(100)$ TeV. This could be accessible in a not so far future, for example at the Future Circular Collider (FCC). This is great news, of course: the combination of the different pieces put here together might make a quantum gravity effect visible at low enough energies for us to reach them. A specific string realization of such a scenario is discussed in the paper, where the visible and dark sector are taken to live, respectively, on a D5 and D9 brane. Working out the relations between the gauge couplings and string scale M_s , it is in fact found for the latter that a kinetic mixing $\epsilon \sim 10^{-15}$ implies $M_s \sim \mathcal{O}(100) TeV$. Needless to say, this goes in the direction of the stringy approach to the Swampland.

Below, [2] is reported in full integrality.



U(1) mixing and the weak gravity conjecture

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Abstract Tiny values for gauge couplings of dark photons allow to suppress their kinetic mixing with ordinary photons. We point out that the weak gravity conjecture predicts consequently low ultraviolet cut-offs where new degrees of freedom might appear. In particular, a mixing angle of $\mathcal{O}(10^{-15})$, required in order to fit the excess reported by XENON1T, corresponds to new physics below $\mathcal{O}(100)$ TeV, thus accessible at a future circular collider. We show that possible realizations are provided by compactifications with six large extra dimensions and a string scale of order $\mathcal{O}(100)$ TeV.

This short note aims to investigate some possible relations between $U(1)$ mixing and the Weak Gravity Conjecture (WGC) [1, 2] (for a review see e.g. [3]). In particular, the case of tiny mixing has witnessed a recent surge of interest following the results announced by XENON1T [4]. The collaboration has reported an excess between 1 and 7 keV, close to the lower threshold of the experiment, with a peak around 2–3 keV. The significance of this excess could melt with a re-analysis of the signal. This could either follow from an accumulation of more data or from more thorough searches for evidences of contamination of the apparatus by some impurities as for the tritium hypothesis suggested by the collaboration in [4]. In the meanwhile, the possibility that it could be a signal of new physics does not seem excluded.

A possible fit of the data in terms of dark photons coupled to the Standard Model (SM) through a kinetic mixing portal [5–8] was analyzed in [9–11]. While solar emitted dark photons are not favored, a scenario where light dark photons with masses of 2–4 keV are absorbed by the xenon seems to correctly reproduce the excess, though with a reduced significance due to a look elsewhere effect. This can be achieved for a tiny visible-dark photon kinetic mixing parameter in the

range

$$\epsilon \simeq \mathcal{O}(10^{-16} - 10^{-15}), \quad (1)$$

which is in agreement with the upper bound limit given by XENON1T on ϵ , that also claims a 3σ significance for a 2.3 keV dark photon over the background. This was argued in [12] to lead to the correct result for the dark photon relic density. The dark photon in XENON1T can also appear as a vector portal for fermionic or bosonic dark matter, where, depending on the model, the mixing can take different values. In [13–15], mixing parameters of order $\epsilon \simeq \mathcal{O}(10^{-4})$, $\mathcal{O}(10^{-7})$ or $\mathcal{O}(10^{-10})$, with, respectively, an order $\mathcal{O}(\text{GeV})$ massive dark photon in the first two cases and a massless one in the last one have been advocated. We discuss below a possible origin of such mixing parameter, especially for challenging tiny values, where we find that the WGC allows to hope for an accompanying signal at collider experiments.

We focus on the sector of the low energy effective field theory describing the $U(1)$ gauge groups representations and interactions. One of the two, $U(1)_v$, is called visible as we have in mind hypercharge or electromagnetism. Another, $U(1)_d$, corresponds to an extra factor we call “dark” $U(1)$, having in mind a hidden sector. It is straightforward to generalize to cases with more abelian gauge groups. The associated gauge fields and gauge fields strengths are denoted as $A_{(v)}^\mu$, $F_{(v)}^{\mu\nu}$ and $A_{(d)}^\mu$, $F_{(d)}^{\mu\nu}$, respectively. The corresponding two-derivative Lagrangian reads:

$$\mathcal{L} \supset -\frac{1}{4} F_{(v)}^{\mu\nu} F_{(v)\mu\nu} - \frac{1}{4} F_{(d)}^{\mu\nu} F_{(d)\mu\nu} - \frac{\epsilon_{vd}}{2} F_{(v)}^{\mu\nu} F_{(d)\mu\nu} + g_v J_{(v)}^\mu A_{(v)\mu} + g_d J_{(d)}^\mu A_{(d)\mu}. \quad (2)$$

For massless visible and dark photons, this mixing in the two-derivative Lagrangian can be eliminated by performing the appropriate rotation. When the $U(1)_d$ gauge boson acquires a mass, through a Stueckelberg or Higgs mechanisms, the mixing has physical implications. The visible and

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dark photons couple in the new basis to the currents J_v^μ and J_d^μ through:

$$\mathcal{L} \supset \left[\frac{g_d}{\sqrt{1 - \epsilon_{vd}^2}} J_{(d)}^\mu - \frac{\epsilon_{vd} g_v}{\sqrt{1 - \epsilon_{vd}^2}} J_{(v)}^\mu \right] A_{(d)\mu} + g_v J_{(v)}^\mu A_{(v)\mu}, \tag{3}$$

thus implying that the visible matter is charged under the dark gauge symmetry with charge $\sim \epsilon_{vd} g_v$.

It is most natural to assume that the dark $U(1)$ mass and mixing vanish in the fundamental theory at the ultra-violet (UV) cut-off and are generated at lower energies. The mixing can be generated at one loop by states with masses m_i and charges $(q_v^{(i)}, q_d^{(i)})$ under $(U(1)_v, U(1)_d)$. It is then given by:

$$\epsilon_{vd} = \frac{g_v g_d}{16\pi^2} \sum_i q_v^{(i)} q_d^{(i)} \ln \frac{m_i^2}{\mu^2}, \tag{4}$$

where μ^2 is the renormalisation scale. In the case of the hyper-charge $U(1)_v \equiv U(1)_Y$ we have $g_v = g'$ and $q_v^{(i)} = Y^{(i)}$, while $g_v = g' \cos \theta_w$ and $q_v^{(i)} = q_{em}^{(i)}$ the electrical charge for $U(1)_v \equiv U(1)_{em}$.

In order to generate such a small mixing as the one required by XENON1T, we either require the dark photon coupling to be appropriately small, a cancellation in the one-loop logarithms, or appeal to higher order non-renormalisable operators. The cancellation can be partial, for instance between particles with (order one) charges $(q_v^{(i)}, q_d^{(i)})$ and $(q_v^{(j)}, q_d^{(j)} = -q_d^{(i)})$ and masses m_i and m_j with $m_j = m_i + \Delta m_{ij}$. For $\Delta m_{ij} \ll m_i$, we have an approximation:

$$\epsilon_{vd} \sim \frac{g_v g_d}{16\pi^2} \frac{\Delta m_{ij}}{m_i}. \tag{5}$$

For complete cancellation, this one loop contribution is replaced by higher loop ones. However, gravitational loops are expected to show up at some order and lead to a lower bound. It was shown in [16] that this is expected at six loop order giving rise to an $\epsilon_{vd} \gtrsim \mathcal{O}(10^{-13})$ for a *bona fide* four dimensional theory. We shall discuss below the first alternative of a tiny dark sector coupling.

We start by considering an abelian gauge symmetry $U(1)$ with gauge coupling g . The weak gravity conjecture requires the presence of at least one state with a mass:

$$m^2 \leq 2g^2 q^2 M_P^2, \tag{6}$$

where we use natural units $\hbar = c = 1$ and $M_P = \sqrt{8\pi G} \sim 2.4 \times 10^{18}$ GeV is the reduced Planck mass. Obviously this is satisfied by all the Standard Model states for the

hypercharge/electromagnetic gauge symmetry. This bound was generalised to the case of multiple $U(1)$ s by the replacement $q^2 \rightarrow \sum_i q_i^2$ in (6) [17]. In theories with supersymmetry, extra-dimensions can furnish a set of BPS states (Kaluza–Klein modes or solitonic objects as branes) that saturate this bound. Their masses will receive corrections after supersymmetry breaking but some should still satisfy the bound.

An explicit model is required for an exact computation of the mixing given by (4). Here, we would like to comment on possible states that will contribute to the sum. Following the Completeness Hypothesis of [18], all the sets of $U(1)$ charges are present in the theory. It was suggested in [19] that all the spots of the charge lattice are occupied by super-extremal states, i.e. satisfying the WGC. This lattice weak gravity conjecture was shown to be too strong; it does not hold in Kaluza–Klein theories within generic compactifications. It was subsequently replaced by the sub-Lattice Weak Gravity Conjecture [20]. This states instead that a super-extremal particle should exist only at every spot in a finite index sub-lattice of the full charge lattice. Later, causality and analyticity constraints for lower dimensional theories, obtained through dimensional reduction, has been used in [21] to propose the Tower Weak Gravity Conjecture: there is an infinite tower of states satisfying the bounds of the WGC. These states will contribute to generating a mixing between the $U(1)$ s. For two $U(1)$ s, the masses of the charged particles can be expressed as $m = c\sqrt{(g_v q_v)^2 + (g_d q_d)^2}$, where $c < 1$ is a state-dependent constant. For the following discussion, we use two integers i and j for the visible and dark charge of the particle respectively, as would be for quantized charges forming a lattice in charge space. Equation (4) then becomes in this scheme

$$\epsilon_{vd} = \frac{g_v g_d}{16\pi^2} \sum_{i,j} q_i q_j \ln \left(\frac{c^{(i,j)} [(g_v q_i)^2 + (g_d q_j)^2]}{\mu^2} \right). \tag{7}$$

Though the number of states is infinite, we include in the loop only states below the cut-off. If a particle with charge (q_i, q_j) is in the spectrum, there are also particles with charge $(q_i, -q_j)$, $(-q_i, q_j)$ and $(-q_i, -q_j)$ giving

$$\epsilon_{vd} \simeq \frac{g_v g_d}{16\pi^2} \sum_{i,j} q_i q_j \ln \left(\frac{c^{(i,j)} c^{(-i,-j)}}{c^{(-i,j)} c^{(i,-j)}} \right), \tag{8}$$

which as a result of the diverse cancellation between different contributions, could remain small (for typical sizes, see for example discussion in [8]).

The most relevant facet of the WGC for this work is the prediction of an ultraviolet cut-off scale for the effective field theory at $\Lambda_{UV} \lesssim g M_{Pl}$. This was dubbed as the magnetic weak gravity conjecture in [2] and, in the weak coupling limit $g \rightarrow 0$, it predicts the absence of global symmetries in quan-

tum gravity [18, 22, 23]. For electromagnetic or hypercharge gauge coupling, the cut-off scale set by the WGC remains close to the Planck scale.

We generalize here, as done by [24], this requirement to the case with multiple $U(1)$ gauge groups by requiring that none of the gauge symmetry factors should turn into a continuous global symmetry by taking the corresponding coupling to vanish. This implies that a tiny value of the dark photon gauge coupling, introduced to make the mixing tiny, require a UV cut-off at most of order $\Lambda_{UV} \lesssim g_d M_P$. This is sensibly lower than M_P and could have important consequences in phenomenology and cosmology.

Starting from (4), we identify the visible photon with the SM photon, i.e. $g_v = e \sim 0.3$, and the logarithm to be $\mathcal{O}(1 - 10)$, then

$$\begin{aligned} \epsilon_{vd} &\sim \frac{g_v g_d}{16\pi^2} \sim \mathcal{O}(10^{-3} - 10^{-2}) g_d \\ \Rightarrow g_d &\sim \mathcal{O}(10^2 - 10^3) \epsilon_{vd}. \end{aligned} \tag{9}$$

Per se, the WGC does not provide information on the new physics required at $\Lambda_{UV} \lesssim g_d M_{Pl}$. A simple possibility is that the $U(1)_d$ becomes part of some non-abelian gauge group $SU(2)_D$ with field strength $F_{(D)}^{\mu\nu}$ broken by a vacuum expectation value (v.e.v) of $\langle \Sigma \rangle = v \simeq \Lambda_{UV}/g_d$ of a field in the adjoint representation. One could then induce a contribution ϵ_{vd}^{NR} to the kinetic mixing through the effective non-renormalizable operator (see e.g. [25]):

$$\frac{c^{NR}}{M_P} Tr \left[\Sigma F_{(D)}^{\mu\nu} \right] F_{(v)\mu\nu} \Rightarrow \epsilon_{vd}^{NR} \simeq \frac{c^{NR}}{M_P} v, \tag{10}$$

where c^{NR} is a constant. For this contribution to remain sub-leading, we require:

$$c^{NR} v \lesssim \epsilon_{vd} M_P \Rightarrow c^{NR} v \lesssim 10^{-3} g_d M_P, \tag{11}$$

which for $\epsilon_{vd} \sim 10^{-15}$ gives $c^{NR} v \lesssim \text{TeV}$.

Kinetic mixing might also arise from D-terms in supersymmetric theories through effective operators [26, 27]:

$$\frac{D^2}{\Lambda_D^4} F_{(d)}^{\mu\nu} F_{(v)\mu\nu}, \tag{12}$$

that are expected to be very small. For example, they can be suppressed by the value of the ratio SM Higgs v.e.v over the scale Λ_D for hypercharge D -term or through powers of the dark sector coupling for the dark $U(1)$ D -term.

In the following we will use (9) to compute g_d from ϵ . A value of $\epsilon_{vd} \sim 10^{-15}$ as in (1) would require $g_d \sim \mathcal{O}(10^{-13} - 10^{-12})$. The WGC implies then that the theory has a UV cut-off:

$$\Lambda_{UV} \lesssim g_d M_P \sim \mathcal{O}(10^2 - 10^3) \text{TeV}. \tag{13}$$

Therefore, new physics must appear below energies of order $\mathcal{O}(100)$ TeV. Such physics could be accessible at future experiments at collider, such as the 100 TeV Future Circular Collider (FCC).

Following the SLP, such a scenario is consistent with quantum gravity only if it could arise from a string theory model. We will discuss now one possible venue for realizing this UV completion in a string theory. We do not attempt an explicit string model building which is beyond the scope of this work. We contemplate the possibility that a hierarchy $g_d \ll g_v$ is obtained through the suppression of g_d by the volume of the internal compactified space. More precisely, we consider a scenario where we start from ten-dimensional type IIB string theory compactified on a six-dimensional space of volume $V_6 \equiv (2\pi R)^6$. The four-dimensional reduced Planck mass M_P is related to the string scale mass M_s and string coupling g_s through (multiplied by 2 for type I strings):

$$M_P^2 = \frac{R^6 M_s^6}{2\pi g_s^2} M_s^2. \tag{14}$$

The visible $U(1)_v$ is taken to live on a D5-brane wrapping a small two-dimensional cycle of approximate string size of volume $(2\pi r)^2 \gtrsim 4\pi^2 M_s^{-2}$. Then, the visible coupling reads:

$$g_v^2 = \frac{2\pi g_s}{r^2 M_s^2} \simeq 2\pi g_s. \tag{15}$$

The dark $U(1)_d$ is instead chosen to live on a D9-brane wrapping the whole six-dimensional compact space and its gauge coupling is given by:

$$g_d^2 = \frac{2\pi g_s}{R^6 M_s^6}. \tag{16}$$

Then, we get:

$$\begin{aligned} \epsilon_{vd} &\sim \frac{g_v g_d}{16\pi^2} \sim \frac{g_s}{8\pi R^3 M_s^3} \\ \Rightarrow \epsilon_{vd} &\sim \frac{1}{\sqrt{128\pi^3}} \frac{M_s}{M_P} \sim 10^{-2} \frac{M_s}{M_P}, \end{aligned} \tag{17}$$

thus

$$\epsilon_{vd} \sim 10^{-15} \Rightarrow M_s \sim \mathcal{O}(100) \text{TeV}. \tag{18}$$

This is merely two orders of magnitude above the proposals of TeV strings for solving the hierarchy problem [28–37]. Note that our analysis is similar to the analysis performed in [27]. However there is a notable difference in that we impose that the $U(1)_d$ dark propagates in the whole large dimensions, thus six in this example, therefore we have considered D5–D9 branes instead of D3–D7, leading to different results, and in particular allowing smaller values of the mixing. The

Dp-D(p-4) set-up is enforced by supersymmetry, but, in the case of low string scale, we could have taken instead, without change in our results, a non-supersymmetric configuration of D3–D9 branes, our world being non-supersymmetric at least up to TeV energy scales. However, one should keep in mind that some of the non-supersymmetric configurations tend to fall in the Swampland [38].

The above scenario implies the appearance of large extra-dimensions at a scale of order:

$$\frac{1}{R} = \left(\frac{M_s}{\sqrt{8\pi} M_P} \frac{1}{\alpha_{YM}} \right)^{1/3} M_s, \quad (19)$$

where we have identified the tree-value of the SM gauge couplings as $\alpha_{YM} = g_s/2$. Taking an approximate value for $\alpha_{YM} \sim 1/25$, we get

$$\frac{1}{R} \sim \mathcal{O}(10) \text{ GeV}. \quad (20)$$

Though these values of the compactification energy scale might seem low, they are not experimentally excluded. Gauge bosons propagate in these extra dimensions, in addition to the gravitons. However, in contrast to the case in [28–30, 39–41], these are Kaluza–Klein excitations of the dark $U(1)$ with tiny couplings. It is the production of a huge number of them that will compensate the coupling strong suppression. They can be observed as missing energy at collider experiments in particular at a 100 TeV collider.

We can express the string mass scale and the compactification radius as a function of g_d and M_P :

$$M_s \sim \sqrt{g_s g_d} M_P \quad \text{and} \quad \frac{1}{R} \sim \frac{g_d^{3/4}}{(8\pi)^{1/6}} M_P \lesssim g_d M_P. \quad (21)$$

The most stringent bound on ϵ today is $\epsilon \sim \mathcal{O}(10^{-16})$ (see e.g. [42]), with a mass for the dark photon around the keV. For this case, one obtains for the string mass scale $M_s \sim 10^4$ GeV, and $\frac{1}{R} \sim 0.1 - 1$ GeV. Smaller values of ϵ cannot be obtained through our simple large extra-dimension setup as they will conflict with the current experimental limit on the string scale. For different values of the dark photon mass, the constraints on ϵ are weaker, and consequently the extra-dimension and string mass scales are set at higher energies. Taking the three values mentioned above we can have for $\epsilon \sim 10^{-10}$, $\epsilon \sim 10^{-7}$ and $\epsilon \sim 10^{-4}$, respectively, a string mass scale and an inverse compactification radius of order $M_s \sim 10^{10}$ GeV and $\frac{1}{R} \sim 10^7$ GeV, $M_s \sim 10^{13}$ GeV and $\frac{1}{R} \sim 10^{11}$ GeV, and finally $M_s \sim 10^{16}$ GeV and $\frac{1}{R} \sim 10^{15}$ GeV. The intermediate scale $\sim 10^{11}$ GeV has diverse motivations [36, 43]. It also corresponds to the energy where the SM quartic Higgs coupling vanish, thus a scale where new degrees of freedom might be expected. Though we cannot proceed to

the same string embedding as we have done above, for kinetic mixing as small as $\epsilon \sim 10^{-23}$ the WGC requires new physics around the scale $\Lambda \sim 1-10$ MeV, that could then in turn be constrained by the Big Bang Nucleosynthesis.

In our way to generate the large hierarchy between the two couplings g_v and g_d , we have assumed the existence of small cycle with a size of order of the string scale inside six large compact dimensions. This is not the case in the simplest toroidal compactifications and requires some warping. Thus, the KK excitations of the dark photon are not expected in general to exhibit the same spectrum as in the simplest case. However, assuming that the rough behaviour of the density of states goes with the energy as E^6/M_s^6 , a sizable value of the effective coupling between SM states and the dark photons is reached only at energies of the order of M_s .

In most phenomenological applications, the dark $U(1)$ is massive. The WGC in [2] concerns massless $U(1)$ gauge bosons. For instance, in the case of a massive $U(1)$, the charge is not conserved and there is no problem of remnants as charged black holes decay. However, one may argue that if the weak gravity states masses m_{WGC} are much bigger than the dark photon mass m_{γ_d} , the massive case is a (Higgs) phase of the same theory and remains in the landscape. Moreover, the comparison of gravity and gauge forces should be done at energies of order m_{WGC} and makes sense in the region $m_{\gamma_d} \ll m_{WGC}$. Finally, we have explicitly illustrated the WGC prediction for the UV cut-off of the theory by a type IIB string scenario that we do not expect to break down because of an infrared Higgsing of the $U(1)$. In fact, [24] have argued, through the explicit investigations of the properties of the WGC charge lattice, that the bounds used here on the mass, combination of charges ratios and ultraviolet cut-off of the theory remain true. A detailed discussion of the expected masses for the dark photon in different string settings is provided elsewhere [44].

To conclude, we would like to stress that the main aim of this work is not to add to the plethora of XENON1T analysis and interpretation, but to point out the amusing coincidence that the observation of kinetic mixing between ordinary and dark photon would suggest new physics at scales that should be probed by a future collider.

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CHAPTER 3

Revisiting the Scalar Weak Gravity Conjecture

3.1 Introduction

Since the extension of the Weak Gravity Conjecture to dilatonic theories proposed in [24], the interest towards the possibility that similar reasonings might impose more generally constraints on scalar couplings have raised. As discussed in the first Chapter, the history of the scalar versions of the WGC is somehow more complicated than that of the original one. In the case where gauge fields are still present, a first formulation was given in [27]. This was investigated further in [29] where the no bound state argument was suitably formalized and the conjecture was given a quite solid motivation. The subsequent extension to the case where no gauge fields are present is certainly the most delicate one. Among the different arguments that motivated the WGC (1.3.2), only the requirement that gravity is the weakest of all forces can be applied. Under this "slogan", two different constraints were proposed in [27] and [33]. The first one coincides with the requirement that in a theory with a massless mediator there is a matter state such that the long-range scalar force between two of its copies is stronger than gravity. In some sense, of course, this could be taken to translate the idea that gravity is weaker than scalar forces. It is not clear, however, why one should restrict to the analysis of long range forces only. When using the no bound state argument, there is no ambiguity in that: the overall long range force determines whether the particles are attracted to each other or not, and the role of short range interactions is only to determine details of the bound state those might form. The attractive character of scalar forces at large distances does not allow to apply such a reasoning in this case. Moreover, in full generality, keeping the scalar massless is possible only if some mechanisms (e.g. some symmetry) protects it, which is already a requirement that would make the conjecture non-universal. For a massive scalar mediator, the well-known position space Yukawa potential scales as

$$V_{\text{scalar}}(r) \sim \mu^2 \frac{e^{-m_\phi r}}{r}, \quad (3.1.1)$$

where m_ϕ is the mass of the scalar mediator, μ the scalar charge of the desired WGC particle and r the distance between the two interacting copies of that test particle. In this case, gravity is surely dominant at some very large distance $r \gg m_\phi^{-1}$, and what the constraint proposed in [27] truly says is that in a range of distances $r \lesssim m_\phi^{-1}$ scalar interactions are the dominant ones. For this to make sense, of course, one need that $m_\phi \ll m$, with m the mass of the WGC particle, otherwise the range of dominance of the scalar force is smaller than the Compton wavelength of the WGC state and the force arising from (3.1.1) certainly cannot be considered as a long range one.

The formulation of [33] was meant to bypass all these drawbacks. The long used slogan "gravity is the weakest force" was elevated to its literal meaning that gravity actually should present itself in the EFT as the weakest of *all* forces. If this applies in all cases, it should also be true for a single self-interacting scalar, and this was argued to be the way the problem of the Scalar WGC should be approached. Nevertheless, the bound proposed in [33] is not unambiguously defined, and its origin is unclear. The claimed diagrammatic interpretation do associate to each one of the terms a specific diagram, but the combination in which they appear and the coefficients in front of them is not justified and is actually rather puzzling. As seen in Section 1.8, the value of the coefficients is fundamental for the interpretation of the states that saturate the bound. To this respect, in both [27] and [33] the proposed bounds were elevated to a differential equation to scan different values of the scalar VEV about which we expand the theory. This is a necessary step for the interpretation of the extremal states they put forward. Actually, in [33] it was argued that the idea that "gravity is the weakest force" should hold true for any value of the VEV. Of course, a first limitation one could put to this claim is that it should hold true for any value of the VEV *in* the domain of validity of the EFT. Nevertheless, the proposed bounds do not really implement this idea. The example of the quartic potential, which has a whole region in field space around its minimum where the constraint is not satisfied, has been given in section 1.8. The violation provided by a physical system described by the quartic potential was also discussed on the basis of [34].

In [1], we provided a new formulation of the Scalar WGC where the sentence "gravity is the weakest force" is given a precise meaning. We show there what it truly means to compare the self-interactions of a scalar to its gravitational interactions, and the bound that truly translates this idea naturally emerges from that.

3.2 Discussion

The procedure followed in [1] is rather simple in its final development, but it clarifies many points. To give meaning to the say "gravity is the weakest force", one should first specify to which process and at what scales this applies. For example, one could say that, based on the experimental success of the Standard Model, and in particular of the perturbative decoupling argument for gravity, at the energies we have probed so far gravitational contributions should not be dominant in scattering amplitudes. For a self-interacting scalar ϕ , we individuate the physical process in the four-point scattering amplitude $\phi\phi \rightarrow \phi\phi$. Following the $U(1)$ WGC, the bound should be imposed at the scale of the WGC state, which in this formulation is the scalar itself. At such scales, the

theory is non-relativistic, and must be described accordingly. This is a central point, which is also in agreement with the no bound states argument used to justify various forms of the WGC where it applies. In the (strict) non-relativistic (NR) limit, the distinction between contact interactions and interactions mediated by a massive mediator does not exist anymore; the self-couplings of the scalar are mapped into a single non-relativistic four-point coupling. It is the application of the NR limit that fixes the combination of the original three and four point couplings μ and λ that appear in the constraint to be

$$\tilde{\lambda} = \lambda - \frac{5}{3} \frac{\mu^2}{m^2}. \quad (3.2.1)$$

The original four point coupling can be either attractive ($\lambda < 0$) or repulsive ($\lambda > 0$). In the former case, higher powers of ϕ in the potential are needed to stabilize it (classically). In the latter, the contact interaction contrasts the scalar mediated one, and the effective NR four point coupling can either be attractive or repulsive, depending on which one of the two dominates. Of course, an expansion about the limit would reveal that at a more fundamental¹ level the interaction with coupling μ/m is not exactly a contact one, but the mass of the self-interacting scalar make it such that outside its Compton wavelength the mediated interaction falls-off exponentially. What we call a contact interaction here more precisely is an interaction which appears contact on a sphere of radius the Compton wavelength. Smaller distances are related to energies outside the NR regime. Three-point couplings to massless particles, that in a non-relativistic theory we usually describe as coupling to "radiation" fields, on the other hand, can still be written [55]. For the case of the graviton, for example, this gives rise to both a long-range interaction and a contact one. The first one cannot be contrasted: in a theory of a self-interacting scalar, gravity will always dominate at distances sufficiently larger than the Compton wavelength of the scalar. The s-channel graviton mediated amplitude is the one that collapses to a point-like interaction in the strict NR limit. The main point in the definition of the Scalar WGC in [1] is that the comparison should be made between this component of the gravitationally mediated amplitude and the effective four-point NR coupling. Corrections to this bound can be organized in a series expansion about the NR limit. To this end, the sign of the effective four point coupling is of no interest: whether it is attractive or repulsive, it should be dominant with respect to the gravitational contributions. Note that it is this observation (and its mathematical translation) that make our bound trivial in the decoupling limit of gravity $\kappa \rightarrow 0$, unlike (1.8.4). When the overall short range force is repulsive, it forbids the formation of bound states of radius $r \lesssim r_c$, with r_c the Compton wavelength of the scalar. When it is attractive, smaller bound states can form, but the conjectured bound would assure that it is held up by scalar forces. In some loose sense, these arguments could be used to rely the Scalar WGC to the no gravitationally bound state argument, although it is not clear what would go wrong in such a theory if bound states held up by gravity with $r < r_c$ were able to form (see also [34] for discussions on this point).

Investigating the bound about different values of the VEV reveal some peculiarities. When $\lambda > 0$, so that the original contact interaction is repulsive, the competition between three and four point coupling is such that it can generate a zero of $\tilde{\lambda}$ as a function of the VEV ϕ_0 . This might be worrisome, as it defines a region of gravitational

¹ More fundamental here means quantum field theoretical as opposed to classical.

dominance. However, this regions turns out to be of width

$$\delta\phi_0 \sim \frac{m^2(\phi_0)}{M_P}, \quad (3.2.2)$$

where $m(\phi_0)$ is the mass of the scalar excitations around the VEV ϕ_0 . It is unclear, and would be actually interesting to investigate, whether such a region that is expected to be small in the EFT range of validity truly is accessible or not. One might expect, from the canonical commutation relations of the scalar

$$[\phi(t, \vec{x}), \Pi(t, \vec{y})] = i\delta^3(\vec{x} - \vec{y}), \quad (3.2.3)$$

with $\Pi(t, \vec{x}) = \partial_t\phi(t, \vec{x})$, that a formulation in a theory with a built-in cutoff scale $\Lambda \leq M_P$ could lead to a indetermination relation of the form

$$\delta\phi(t, \vec{x})\delta\Pi(t, \vec{y}) \geq i\hbar C m_\phi^2 |\vec{x} - \vec{y}|, \quad (3.2.4)$$

with C a generic constant and $|\vec{x} - \vec{y}| \geq M_P^{-1}$. This is absolutely speculative and there are no rigorous result in this direction.

In D generic dimensions, the bound proposed in [1] generalizes to

$$\left| \frac{5}{3} \frac{\mu^2}{m^2} - \lambda \right| \geq \frac{D-1}{D-2} \kappa^2 m^2. \quad (3.2.5)$$

The precise coefficient will be derived in [4], chapter 4. In [1] we did not care about the precise value of the coefficient in front of $\kappa^2 m^2$, as it was unimportant for the discussion, and it was indeed absorbed in a redefinition of the Planck scale. Nevertheless, this corrects the coefficients in (1.8.4).

The bound has been checked on several potentials of interest for phenomenological applications, especially for the case of quintessence. The result of this analysis is that it seems to be possible, with the most common quintessence potentials proposed in the literature, to realize a quintessence dark energy while keeping gravity the weakest force, at least in the sub-Planckian regime where ϕ is expanded about $\phi_0 \lesssim M_P$.

The application of the same reasoning that lead to (3.2.5) to the case of theories with moduli fields is straightforward. As already discussed in other cases, upon elevation of the constraint (3.2.5) to a differential inequality, the moduli dependence of the WGC states can be studied. In particular, for the case of a complex modulus Φ the extremal states have been shown to satisfy a differential equation (see 4.10 in the text)

$$\partial_\Phi \partial_{\bar{\Phi}} m^2 = 2\tilde{\kappa}^2 m^2, \quad (3.2.6)$$

where $\tilde{\kappa}$ is the conveniently redefined gravitational coupling in terms of the Planck mass \tilde{M}_P used in the text and m is the mass function of the matter state. The most general solution to this, of which only the part of interest was shown in [1], is

$$\begin{aligned} m_X^2(\Phi, \bar{\Phi}) = & A e^{\sqrt{2}\tilde{\kappa}(\Phi+\bar{\Phi})} + B e^{-\sqrt{2}\tilde{\kappa}(\Phi+\bar{\Phi})} + C e^{\sqrt{2}i\tilde{\kappa}(\Phi-\bar{\Phi})} + D e^{-\sqrt{2}i\tilde{\kappa}(\Phi-\bar{\Phi})} \\ & + \gamma_1 I_0\left(2\sqrt{2}\tilde{\kappa}|\Phi|\right) + \gamma_2 K_0\left(2\sqrt{2}\tilde{\kappa}|\Phi|\right), \end{aligned} \quad (3.2.7)$$

where $A, B, C, D, \gamma_1, \gamma_2$ are non-independent integration constants. The appearance of the Bessel functions can be better understood requiring that the mass have a $|\Phi|^2$ -dependent term $m_X^2(\Phi, \bar{\Phi}) = f(|\Phi|^2)$. Then

$$\partial_\Phi \partial_{\bar{\Phi}} f(|\Phi|^2) = \partial_\Phi (\partial_{|\Phi|^2} f(|\Phi|^2) \partial_{\bar{\Phi}} |\Phi|^2) = \partial_{|\Phi|^2} f(|\Phi|^2) + |\Phi|^2 \partial_{|\Phi|^2}^2 f(|\Phi|^2). \quad (3.2.8)$$

Simplifying the notation by putting $|\Phi|^2 = x$, the function should solve the equation

$$x f''(x) + f'(x) = 2\tilde{\kappa}^2 f(x), \quad (3.2.9)$$

whose solution is given by a combination of Bessel functions. Forgetting about the latter, which might encode interesting physical predictions but are not of direct interest here, we can use the following parametrization

$$\Phi = \frac{1}{\sqrt{2}}(\phi + i\chi), \quad e^{\sqrt{2}\frac{\Phi+\bar{\Phi}}{M_{Pl}}} = e^{2\tilde{\kappa}\phi}, \quad e^{\tilde{\kappa}\phi} = R_\phi, \quad \text{and} \quad e^{\tilde{\kappa}\chi} = R_\chi \quad (3.2.10)$$

to write

$$m_X^2(R) = \frac{m_-^2}{R_\phi^2} + m_+^2 R_\phi^2 + \frac{n_-^2}{R_\chi^2} + n_+^2 R_\chi^2. \quad (3.2.11)$$

This has the same form as the string states squared mass formula (1.7.2) for a compactification over a two-dimensional tori, with two KK and winding-like towers corresponding to the two scalar degrees of freedom. This shows that, among the different constraints that one can impose following our logic, there is at least one that is saturated by string-like states. Actually, this is more general, as our equation have the same form as (1.8.4) when the absolute value can be ignored with no need to change signs, and the solutions are of the type described in (1.8.11) with $k = 5/3$. In conclusion, our proposed bound, which has a stronger diagrammatic motivation and interpretation than those studied before, with coefficients completely determined by the physical requirements, also have the very welcome feature that string-like states saturate it. Note that this is slightly different than the similar property observed in [33], where the string-like spectrum appeared in the denominator of the extremal mass function (1.8.5). The relation of our bound to the distance conjecture automatically follows from the discussion above.



Revisiting the scalar weak gravity conjecture

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Abstract We revisit the scalar weak gravity conjecture and investigate the possibility to impose that scalar interactions dominate over gravitational ones. More precisely, we look for consequences of assuming that, for leading scalar interactions, the corresponding gravitational contribution is sub-dominant in the non-relativistic limit. For a single massive scalar particle, this leads us to compare four-point self-interactions in different type of potentials. For axion-like particles, we retrieve the result of the axion weak gravity conjecture: the decay constant f is bounded by the Planck mass, $f < M_{Pl}$. Similar bounds are obtained for exponential potentials. For quartic, power law and Starobinsky potentials, we exclude large trans-Planckian field excursions. We then discuss the case of moduli that determine the scalars masses. We retrieve the exponential dependence as requested by the Swampland distance conjecture. We also find extremal state masses with field dependence that reproduces both the Kaluza-Klein and winding modes behaviour. In particular cases, our constraints can be put in the form of the Refined de Sitter Conjecture.

1 Introduction

Among the *a priori* consistent low energy quantum field theories, it is believed that some cannot be embedded in a theory of quantum gravity. They form what is denoted as the *swampland* [1, 2] (see [3, 4] for a review). One of the selection criteria of consistent effective theories is provided by the Weak Gravity Conjecture (WGC) [5]. It claims that, in a theory with $U(1)$ gauge symmetry with coupling g , a state of charge q and mass m satisfying the inequality

$$gq \geq \frac{m}{M_{Pl}} \quad (1.1)$$

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must exist. Considering the charge over mass ratio, this condition can be obtained requiring that extremal black holes do decay entirely, leaving no remnants. It is furthermore consistent with black hole physics based arguments for non-existence of global symmetries in quantum gravity. This conjecture was claimed to be valid in any theory of quantum gravity and has been shown to hold in known examples in string theory.

There are two aspects of (1.1) that are useful to stress. First, in theories with $\mathcal{N} \geq 2$ supersymmetries, the central charge Z of the supersymmetry algebra is given by gq and is related to the mass of the BPS state through $|Z| = m$ (in Planck units). The relation (1.1) goes in opposite direction of the BPS condition. It can be therefore tempting to look for other forms of such conjectures by considering the extremal states identities and turning it to an (anti-BPS) inequality. This was stressed in [6].

The other useful aspect is the appearance of an ultraviolet scale $\Lambda \sim gqM_{Pl}$, controlled by the gauge coupling g , which sets the cut-off of the EFT. This was dubbed as the magnetic weak gravity conjecture in [5] and is clearly related to the non-existence of global symmetries in quantum gravity in the limit of weakly coupled gauge theories $g \rightarrow 0$.

Following the proposal of the WGC, another form was put forward as a Repulsive Force Conjecture (RFC) [6, 7]. This postulates the existence of a state within the $U(1)$ theory with the property that, taken far apart, two copies of such state feel a repulsive force between each other. This avoids gravitational bound states. It was accurately described in [7], where many of its consequences were exhibited.

Going beyond gauge fields and writing a similar conjecture for scalar fields, possibly complementary to swampland conjectures, is not straightforward. First, there is no such obvious argument on decay of black holes that can be used to induce the form of the conjecture. Second, to test in all generality different scalar conjectures in a quantum gravity theory is not easy. The scalar sector of the theory is very sensitive to the supersymmetry breaking. Implementing super-

symmetry breaking in a string theory and extracting the full corrections to the scalar potential of a single real field in flat space-time is a non trivial problem. Moreover, supersymmetric models involve complex scalars, and it is not evident how to disentangle all the facets of constraints applying on one real scalar. With the lack of non-supersymmetric string theory examples, one is lead to postulate some form of the scalar conjecture and evaluate it by investigating the consequences. The hope is that even this modest trial and error method will turn out to be useful and will allow us to shed some light on the landscape of the effective field theories coupled to gravity. This way of proceeding applies to the conjectures discussed below.

A Scalar Weak Gravity Conjecture (SWGCG) was investigated in [6] as a special case of the RFC. In the context of the RFC, the scalar field is massless and one is interested in the long range interactions it mediates. In an attempt to retrieve the Swampland Distance Conjecture mass formulae, it was proposed that:

$$g^{ij} \partial_i m \partial_j m \geq m^2 \quad (1.2)$$

where $\partial_i m \equiv \partial m / \partial \phi_i$ is the derivative of the mass term m with respect to the scalar field ϕ_i and g^{ij} is the appropriate metric on the space of fields. In a footnote of [6], it was also mentioned that, looking at different forms of the equalities satisfied by the central charge in $\mathcal{N} = 2$, another possible form of the conjecture could have been:

$$g^{ij} \partial_i \partial_j m^2 \geq g^{ij} \partial_i m \partial_j m + m^2. \quad (1.3)$$

The constraint (1.2) does not involve repulsive interactions and as such cannot be considered as a realization of the RFC. It seemed puzzling in the RFC set-up discussed in [6], as scalar mediated forces are attractive, and the possibility (1.3) was not pursued any further, with the exception of a few comments in [8]. It was somehow dismissed due to the lack of simple physical interpretation.

All these considerations led to the proposal of another form of the conjecture for scalar fields in [9]: the mass m of an interacting scalar field satisfies the bound [10]:

$$m^2 \frac{\partial^2}{\partial \phi^2} \left(\frac{1}{m^2} \right) \geq \frac{1}{M_{Pl}^2} \quad (1.4)$$

This was obtained by modifying by a factor 2 and an additional four-point contact interaction the inequality (1.2) expressed as derivatives of the scalar potential. This form of the conjecture was motivated by a set of implications [9–13], some of which might be of phenomenological importance. However, it raises some questions about its origin and the meaning of the corresponding inequality. As a consequence of the (1.4), for states with a mass depending on the scalar ϕ , the equality in (1.4) is reached for

$$m^2(\phi) = \frac{m_0^2}{Ae^{-\phi} + Be^{\phi}} \quad (1.5)$$

where A and B are integration constants. Through the identification $e^{-\phi} = R^2$, the result of (1.5) has been interpreted in [9] as an indication of the extended nature of the fundamental states.

Taken as such, the above proposals were dismissed in [14], because of inconsistent implications for simple scalar potentials, and it was instead suggested that scalar particles should be subject to constraints in such a way that they would not form bound states with size smaller than their Compton wavelength. No generic alternative formulation for these constraints on the scalar potential was proposed.

In this work, we will postulate that in the appropriate low energy limit, for the leading interaction, the gravitational contribution must be sub-leading. For particular scalar fields, we will propose an explicit set-up, based on the computation of four-point functions, for comparing the different interactions. The resulting inequalities will reproduce different forms of the Swampland conjectures, and, in a particular case, the inequality will be saturated for masses of the form (4.13):

$$m_X^2(\phi) = m_-^2 e^{-2\phi} + m_+^2 e^{2\phi}. \quad (1.6)$$

instead of (1.5).

The paper is organized as follows. In Sect. 2, we formulate the constraint of dominance of scalar interactions with respect to the gravitational ones for the case of a single massive scalar field self-interacting. We illustrate the constraint by the simplest example of a single real field with a cubic and quartic potential. A few other examples are studied in Sect. 3. Those include the quartic complex potential, the axion, the exponential and the Starobinsky potential. In the Sect. 4, we discuss an extension to moduli and massless scalars. Section 5 presents our conclusions.

2 Scalar vs gravity in the non-relativistic regime

The Weak Gravity Conjecture states that for any abelian gauge symmetry $U(1)$ there is at least one state with gauge self-interaction stronger than the gravitational one. Here, we will investigate a possible extension of the conjecture to the case of scalar fields.

We start with the case of a single self-interacting massive scalar field. We will postulate that for this scalar field the self-interaction is stronger than the gravitational one.

This assertion calls for a few immediate remarks. First, we need to specify at which scale the different interactions are computed and compared. This is chosen to be of order of the mass of the self-interacting particle. This is consistent with the fact that the Weak Gravity Conjecture makes statements about properties of effective field theories. At these energy

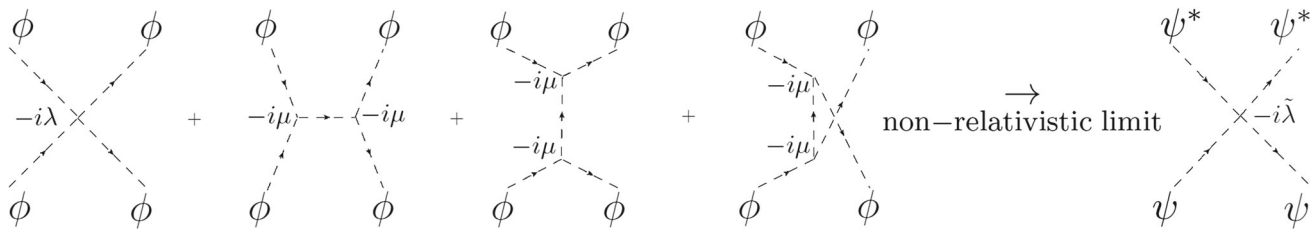


Fig. 1 The identification of the $2 \rightarrow 2$ scattering in the non relativistic theory coming from the corresponding scattering in the relativistic case

scales, the non-relativistic theory is a good approximation. This means, for example, that in scattering processes the particle number is conserved. We shall therefore investigate the strength of the interactions by computing the simplest scattering processes. Precisely, we will compare the four-point amplitude contribution of the scalar self-interaction versus the gravitational one.

We work in the non-relativistic limit and keep only the leading order in $1/c^2$. The gravitational forces are then expected to be well described by the Newtonian potential. Higher order corrections, as those given by the Einstein–Infeld–Hoffman Lagrangian, will be neglected. In practice, instead of dealing with the potential in coordinates space, we will work in the Fourier-transform space by computing the scattering amplitudes. The dominance of scalar self-interaction means in particular that all the higher dimensional non-renormalizable interactions suppressed by higher powers of the Planck mass should be subdominant and may be neglected. We will see below that this preeminence can happen to be violated in isolated regions of size $\frac{\Delta\phi^2}{m^2} \sim \frac{m^2}{M_{Pl}^2}$ where the interactions can switch nature between attractive and repulsive.

We restrict to four-dimensional Minkowski space-time and use from now on natural units $\hbar = c = 1$. We first investigate the simplest case of cubic and quartic potential and discuss other forms of scalar potentials in the next section.

We consider a real scalar ϕ with the potential:

$$V(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{\mu}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4. \tag{2.1}$$

In string theory, our fiducial quantum gravity theory, all the low energy parameters are field dependent. But we will consider here that the other scalar fields are fixed to their vacuum value and decouple from the dynamics of the low energy effective action under scrutiny. At energy scales $E \sim m_0$, the theory is non-relativistic and can be described by the corresponding limit. We study fluctuations around $\phi = 0$ and make the field redefinition:

$$\phi(x) = \frac{1}{\sqrt{2m_0}} \left(\psi(\mathbf{x}, t)e^{-im_0t} + \psi^*(\mathbf{x}, t)e^{im_0t} \right) \tag{2.2}$$

where the phase e^{-im_0t} is introduced to take into account the leading m_0 term in the non-relativistic limit expansion $E \simeq m_0 + \mathbf{p}^2/2m_0$ where \mathbf{p} is the particle three-dimensional momentum. The denominator $\sqrt{2m_0}$ comes from the different normalizations in relativistic and non-relativistic quantum mechanics.

The potential for the non-relativistic field ψ should be of the form

$$V_{eff}(\psi\psi^*) = m_0\psi\psi^* + \frac{\tilde{\lambda}}{16m_0^2}(\psi\psi^*)^2. \tag{2.3}$$

We now want to relate the single non-relativistic coupling $\tilde{\lambda}$ with the coefficients of the relativistic potential. We identify the low energy limit of the $2 \rightarrow 2$ scattering in the ϕ description with the corresponding scattering of four ψ states. This leads trivially to $\lambda = \tilde{\lambda}$ when $\mu = 0$ in (2.1). In the case where $\mu \neq 0$, we will have to take into account the contributions to the $2 \rightarrow 2$ scattering from the exchange of a virtual ϕ . We have in this case three diagrams, one for each channel, as shown in Fig. 1. We can compute the non relativistic limit of each one of them. This is obtained requiring $s - 4m_0^2 \ll m_0^2$, where $s = (p_1 + p_2)^2$ is the usual Mandelstam variable and p_1, p_2 the four-momenta of the initial states. We also have $t = -\frac{1}{2}(s - 4m_0^2)(1 - \cos(\theta))$ and $u = -\frac{1}{2}(s - 4m_0^2)(1 + \cos(\theta))$, θ being the angle between the in-going and out-going particles momenta in the center of mass frame. This basic computation yields the s-channel contribution as:

$$(-i\mu)^2 \frac{i}{s - m_0^2} = \frac{-i\mu^2}{3m_0^2} + \mathcal{O}\left(\frac{s - 4m_0^2}{m_0^2}\right), \tag{2.4}$$

and the t-channel as:

$$(-i\mu)^2 \frac{i}{t - m_0^2} = \frac{i\mu^2}{m_0^2} + \mathcal{O}\left(\frac{s - 4m_0^2}{m_0^2}\right). \tag{2.5}$$

Finally, the u-channel contribution is the same as the t-channel one. Summing up the three contributions we obtain $i\frac{5}{3}\frac{\mu^2}{m_0^2}$, so that the effective four-point self-interaction coupling in the non-relativistic limit is:

$$\tilde{\lambda} = \lambda - \frac{5}{3}\frac{\mu^2}{m_0^2}. \tag{2.6}$$

In computing the gravitational interaction, we have assumed $m_0^2 > 0$. Both attractive and repulsive forces can be obtained from the quartic self-interaction, through the choice of $\lambda < 0$ and $\lambda > 0$ respectively. On the other hand, the trilinear term always leads to an attractive force in a $2 \rightarrow 2$ states scattering. However, when $\lambda < 0$ the stability of the potential means that additional non renormalizable terms are important and should be taken into account. In the case of $\lambda > 0$, Eq. (2.6) shows the competition between the attractive and repulsive interactions in the non-relativistic limit. The resulting sign of $\tilde{\lambda}$ tells us about the attractive or repulsive nature of the effective interaction and, in the case where they are in competition, which one of the two terms dominate at energies $E \sim m_0$.

In the WGC the gauge and gravity forces have similar dependence in the distance between the scattering particles at leading order. There are two corrections, one from the evolution of gauge coupling with energy and the other from post-Newtonian effects. This is not the case for the scalar interaction. In the non-relativistic limit, the scalar potential is approximated by a delta distribution in space while the gravitational potential is Newtonian. A point-like interaction arises from integrating out massive mediators. In the infrared, at energies below the mass scale, the gravitational scattering exhibits a divergence coming from the t and u channels. Obviously, to compare a Newtonian potential at long distance with the strength of the scalar localised interaction is not very instructive. It is essential in the comparison to fix the energy scale, and naturally it is given by the mass of the scalar particle, and consider the gravitational scattering in the s -channel at $s \sim 4m_0^2$.

Requiring that gravity is the weakest force at low energy amounts then to impose:

$$|\tilde{\lambda}| = \left| \lambda - \frac{5\mu^2}{3m_0^2} \right| \geq \frac{m_0^2}{M_{Pl}^2}. \tag{2.7}$$

We have put an absolute value on the left hand side so that it holds independently of the sign of the self-interaction. Note also that, in the spirit of [5, 15, 16], the quantity $\sqrt{|\tilde{\lambda}|}M_{Pl}$, could be interpreted as an ultra-violet cut-off scale dictated by quantum gravity. In particular, this means that both the limits $\lambda \rightarrow 0$ and $\mu \rightarrow 0$ cannot be taken simultaneously. Cancellation of the two terms in $\tilde{\lambda}$, as we said, might encode the change of nature of the scalar interactions on a region of the phase space that need to be studied case by case.

Below, we will work in more generic field background values and potentials, therefore we will impose a stronger condition

$$4m_0^2 \left| \frac{\partial^4 V_{eff}}{\partial^2 \psi \partial^2 \psi^*} \right|_{\psi=0} \geq \frac{\tilde{c}}{M_{Pl}^2} \left| \frac{\partial^2 V_{eff}}{\partial \psi \partial \psi^*} \right|_{\psi=0}^2 \tag{2.8}$$

and take the order one constant \tilde{c} to be $\tilde{c} = 1$, which amounts to redefine the Planck mass to \tilde{M}_{Pl} . The r.h.s. of (2.8) represents the gravitational attractive interaction between the two particles only when we work at the minimum of the potential and the squared mass is positive defined.

We focus now on the simplest case $\mu = 0$ and investigate the relative strengths of self-interaction and gravitational one when ϕ sweeps the range of possible values. For this purpose we consider small perturbations $\delta\phi$, corresponding to the above ψ , around background values ϕ . We expand

$$\begin{aligned} V(\phi + \delta\phi) &= \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \\ &+ m_0^2\phi\delta\phi + \frac{\lambda}{3!}\phi^3\delta\phi \\ &+ \frac{1}{2}\left(m_0^2 + \frac{\lambda}{2}\phi^2\right)(\delta\phi)^2 \\ &+ \frac{\lambda}{3!}\phi(\delta\phi)^3 + \frac{\lambda}{4!}(\delta\phi)^4. \end{aligned} \tag{2.9}$$

From (2.9), we can immediately read the mass term, the cubic and the quartic couplings for $\delta\phi$ and the effective quartic coupling in the non-relativistic limit. Those are given by:

$$\begin{aligned} m_{\delta\phi}^2(\phi) &= m_0^2 + \frac{\lambda}{2}\phi^2, \quad \mu_{\delta\phi} = \lambda\phi, \quad \lambda_{\delta\phi} = \lambda \\ \tilde{\lambda} &= \lambda - \frac{5}{3}\frac{\lambda^2\phi^2}{m_0^2 + \lambda/2\phi^2}. \end{aligned} \tag{2.10}$$

We restrict to the case with $m_0^2, \lambda > 0$ to explicitly exhibit the competition between the attractive and repulsive terms. Requiring gravity to be the weakest force leads to

$$\left| \lambda - \frac{5}{3}\frac{\lambda^2\phi^2}{m_0^2 + \lambda/2\phi^2} \right| \geq \frac{1}{\tilde{M}_{Pl}^2} \left(m_0^2 + \frac{\lambda}{2}\phi^2 \right). \tag{2.11}$$

The term inside the absolute value of (2.11) vanishes for $\phi^2 = \frac{6}{7}\frac{m_0^2}{\lambda}$. The cubic term dominates above this turning point, a region where the interaction is attractive. The quartic one dominates instead below the turning point, making the scalar interaction repulsive.

We first investigate the $\phi^2 \leq \frac{6}{7}\frac{m_0^2}{\lambda}$ region where (2.11) reads

$$\phi^4 + \left(4\frac{m_0^2}{\lambda} + \frac{14}{3}\tilde{M}_{Pl}^2 \right) \phi^2 + 4\frac{m_0^4}{\lambda^2} - 4\tilde{M}_{Pl}^2\frac{m_0^2}{\lambda} \leq 0. \tag{2.12}$$

Assuming $\lambda \geq \frac{m_0^2}{\tilde{M}_{Pl}^2}$ and discarding the solutions with $\phi^2 < 0$, this is verified inside the region

$$0 \leq \phi^2 \leq -2\frac{m_0^2}{\lambda} - \frac{7}{3}\tilde{M}_{Pl}^2 + \frac{7}{3}\tilde{M}_{Pl}^2\sqrt{1 + \frac{120}{49}\frac{m_0^2}{\lambda\tilde{M}_{Pl}^2}}. \tag{2.13}$$

At the first order in $\frac{1}{\tilde{M}_{Pl}^2}$, this is obtained:

$$\phi^2 \lesssim \frac{6 m_0^2}{7 \lambda} - \frac{600 m_0^4}{343 \lambda^2} \frac{1}{\tilde{M}_{Pl}^2} \tag{2.14}$$

which exhibits a small region of order \tilde{M}_{Pl}^{-2} below the critical value where gravity is stronger than quartic scalar self-interaction.

For $\lambda \leq \frac{m_0^2}{\tilde{M}_{Pl}^2}$, the turning point happens at a scale $\phi^2 \sim \frac{m_0^2}{\lambda} \geq \tilde{M}_{Pl}^2$ and, as the inequality would not be solved for $\phi^2 \leq \frac{6 m_0^2}{7 \lambda}$, this would translate in gravity being stronger than scalar interactions all the way up to the Planck scale.

Let us now turn to the case $\phi^2 \geq \frac{6 m_0^2}{7 \lambda}$. There, the inequality translates into

$$\phi^4 + \left(4 \frac{m_0^2}{\lambda} - \frac{14}{3} \tilde{M}_{Pl}^2\right) \phi^2 + 4 \frac{m_0^4}{\lambda^2} + 4 \tilde{M}_{Pl}^2 \frac{m_0^2}{\lambda} \leq 0. \tag{2.15}$$

At leading order in $\frac{m_0^2}{\tilde{M}_{Pl}^2}$, the region where the inequality is verified is given by

$$\begin{aligned} \frac{6 m_0^2}{7 \lambda} + \frac{600 m_0^2}{343 \lambda^2} \frac{m_0^2}{\tilde{M}_{Pl}^2} &\lesssim \phi^2 \\ &\lesssim \frac{14}{3} \tilde{M}_{Pl}^2 - \frac{6 m_0^2}{7 \lambda} + \mathcal{O}(\tilde{M}_{Pl}^{-2}) \end{aligned} \tag{2.16}$$

In conclusion, up to the Planck scale, the gravity seems to dominate only around the special value $\phi^2 = \frac{6 m_0^2}{7 \lambda}$ in a symmetric interval of radius $\Delta \phi^2 \sim \frac{m_0^4}{\tilde{M}_{Pl}^2}$. It would be interesting to investigate, for explicit examples of quantum gravity, if the theory can be insensitive to such small field excursion regions, but this goes beyond the scope of this work.

3 Single scalar field potentials

In this section, we would like to investigate what the implications of requiring gravity to be weaker than the scalar field self-interactions in the non-relativistic limit are on different potentials. More precisely, we will consider *very slowly rolling fields*, having in mind possible cosmological applications. We impose the condition (2.8) and extract its implications for the involved scales and couplings.

3.1 The Mexican hat or Higgs-like quartic potential

We consider the quartic scalar potential

$$V(\phi, \bar{\phi}) = -m^2 \bar{\phi} \phi + \lambda (\bar{\phi} \phi)^2. \tag{3.1}$$

with $\lambda > 0$, insuring stability, and $m^2 > 0$.

It is convenient to use the parametrization $\phi(x) = \frac{1}{\sqrt{2}} \rho(x) e^{i\pi(x)}$. This potential develops a minimum at $\rho^2 = \frac{m^2}{\lambda}$. This theory has a global $U(1)$ symmetry,¹ which is spontaneously broken at the minimum, and $\pi(x)$ is the associated Goldstone boson. The final mass of $\pi(x)$ depends on details of the complete theory. It might be generated by the higher order terms breaking the global symmetry, as dictated for instance by the WGC. It could also be that the $U(1)$ symmetry is gauged. Then $\pi(x)$ gives rise to the longitudinal mode of the massive gauge boson. We will focus here only on the field $\rho(x)$ which plays in the latter case the role of the Higgs field.

We consider a small perturbation $\delta\rho(x)$ around a the background value $\rho(x)$. The expansion of the potential, up to $\mathcal{O}(\delta\rho^4)$, reads:

$$\begin{aligned} V(\rho + \delta\rho) \simeq & -\frac{1}{2} m^2 \rho^2 + \frac{\lambda}{4} \rho^4 + (\lambda \rho^3 - m^2 \rho) \delta\rho \\ & + \frac{1}{2} (3\lambda \rho^2 - m^2) \delta\rho^2 + \lambda \rho \delta\rho^3 + \frac{\lambda}{4} \delta\rho^4. \end{aligned} \tag{3.2}$$

The effective mass term, trilinear and quartic couplings of $\delta\rho(x)$ are then given by $m_{\delta\rho}^2 = 3\lambda \rho^2 - m^2$, $\mu_{\delta\rho} = 6\lambda \rho$, $\lambda_{\delta\rho} = 6\lambda$, respectively. The $\delta\rho(x)$ resulting quartic self-interaction $\tilde{\lambda}$ at low energies can now be computed to be

$$\tilde{\lambda} = 6\lambda - \frac{60\lambda^2 \rho^2}{3\lambda \rho^2 - m^2} = -6\lambda \frac{(m^2 + 7\lambda \rho^2)}{3\lambda \rho^2 - m^2}. \tag{3.3}$$

Vanishing self-interaction, i.e. a null value for $\tilde{\lambda}$, corresponds to $m^2 \lambda + 7\lambda^2 \rho^2 = 0$. This is obviously never satisfied here.

We discard the region $\rho^2 < \frac{m^2}{3\lambda}$ where the effective mass of $\delta\rho(x)$ is either tachyonic or vanishing, though we have checked that the inequality (2.8) is satisfied.

We will investigate the region $m_{\delta\rho}^2 > 0$, i.e. $\rho^2 > \frac{m^2}{3\lambda}$. We have:

$$9 \frac{\lambda^2}{\tilde{M}_{Pl}^2} \rho^4 - \left(6\lambda \frac{m^2}{\tilde{M}_{Pl}^2} + 42\lambda^2\right) \rho^2 + \frac{m^4}{\tilde{M}_{Pl}^2} - 6m^2 \lambda \leq 0. \tag{3.4}$$

Discarding the region $\rho^2 \in \left[0, \frac{m^2}{3\lambda}\right]$ as discussed above, the inequality is satisfied for:

$$\frac{m^2}{3\lambda} < \rho^2 \leq \frac{14}{3} \tilde{M}_{Pl}^2 + \frac{17}{21} \frac{m^2}{\lambda} + \mathcal{O}(\tilde{M}_{Pl}^{-2}) \tag{3.5}$$

It is worth mentioning that at the minimum, where $\rho^2 = \frac{m^2}{\lambda} \equiv v$, we get $\tilde{\lambda} = -24\lambda$, and the conjecture is then verified in the case:

$$\lambda \geq \frac{1}{12} \frac{m^2}{\tilde{M}_{Pl}^2} \sim 10^{-17} \Leftrightarrow v^2 \leq 12 \tilde{M}_{Pl}^2 \sim 10^{37} GeV^2, \tag{3.6}$$

¹ Quantum gravity requires that either the symmetry is gauged or broken. However, the latter might be sub-leading to the quartic self-interaction considered here.

where we have taken m to be the electroweak scale.

3.2 Axion-like potential

Let's consider the case of the axion potential:

$$V(\phi) = \mu^4 \left(1 - \cos\left(\frac{\phi}{f_a}\right) \right). \tag{3.7}$$

Expanding this potential around a fixed value ϕ_0 and excluding points where $\cos\left(\frac{\phi_0}{f_a}\right) = 0$ as the state becomes massless and our non-relativistic limit no more applies, we obtain up to fourth order in $\delta\phi$:

$$V(\phi) \simeq \mu^4 \left[1 - \cos\left(\frac{\phi_0}{f_a}\right) + \sin\left(\frac{\phi_0}{f_a}\right) \frac{\delta\phi}{f_a} + \frac{1}{2} \cos\left(\frac{\phi_0}{f_a}\right) \frac{(\delta\phi)^2}{f_a^2} - \frac{1}{3!} \sin\left(\frac{\phi_0}{f_a}\right) \frac{(\delta\phi)^3}{f_a^3} - \frac{1}{4!} \cos\left(\frac{\phi_0}{f_a}\right) \frac{(\delta\phi)^4}{f_a^4} \right], \tag{3.8}$$

from which we can read $\tilde{\lambda} = -\frac{1}{f_a^4} \left(\cos\left(\frac{\phi_0}{f_a}\right) + \frac{5}{3} \frac{\sin^2(\phi_0/f_a)}{\cos(\phi_0/f_a)} \right)$. Requiring gravity to be the weakest force leads to

$$\frac{1}{f_a^2} \left| \cos\left(\frac{\phi_0}{f_a}\right) + \frac{5}{3} \frac{\sin^2\left(\frac{\phi_0}{f_a}\right)}{\cos\left(\frac{\phi_0}{f_a}\right)} \right| \geq \frac{1}{\tilde{M}_{Pl}^2} \left| \cos\left(\frac{\phi_0}{f_a}\right) \right|, \tag{3.9}$$

which yields

$$\frac{1}{f_a^2} \left| 1 + \frac{5}{3} \tan^2\left(\frac{\phi_0}{f_a}\right) \right| \geq \frac{1}{\tilde{M}_{Pl}^2}. \tag{3.10}$$

We have expanded around a generic background value ϕ_0 thus (3.10) leads to:

$$f_a^2 \leq \tilde{M}_{Pl}^2 \tag{3.11}$$

We therefore retrieve the Axion Weak Gravity Conjecture, which requires an axion decay constant lower the Planck scale [5, 17–30]. Note that, in the r.h.s. of (3.9), we have taken the absolute value of the squared mass term. Here we see the inequality as taken on derivatives of the potential since the squared mass can be negative.

3.3 Inverse power-law effective scalar potential

Another scalar potential is the inverse power-law one, frequently used in cosmological applications. It reads

$$V(\phi) = M^{4+p} \phi^{-p}, \tag{3.12}$$

where $p > 0$ is a constant and M sets the energy scale. In the general case, we expand the potential as a Taylor series

$$\begin{aligned} \frac{1}{M^{4+p}} V(\phi_0 + \delta\phi) &= \phi_0^{-p} - p\phi_0^{-p-1} \delta\phi \\ &+ \frac{p(p+1)}{2} \phi_0^{-p-2} (\delta\phi)^2 \\ &- \frac{p(p+1)(p+2)}{3!} \phi_0^{-p-3} (\delta\phi)^3 \\ &+ \frac{p(p+1)(p+2)(p+3)}{4!} \\ &\times \phi_0^{-p-4} (\delta\phi)^4. \end{aligned} \tag{3.13}$$

The effective quartic interaction in the non-relativistic limit is given by

$$\tilde{\lambda} = -\frac{p(p+1)(p+2)}{3} (2p+1) \phi_0^{-p-4}. \tag{3.14}$$

The gravitational interaction will thus be weaker than the scalar self-interaction in the non-relativistic limit if

$$\begin{aligned} \frac{p(p+1)(p+2)}{3} (2p+1) \left| \phi_0^{-p-4} \right| \\ \geq \frac{p(p+1)}{\tilde{M}_{Pl}^2} \left| \phi_0^{-p-2} \right|. \end{aligned} \tag{3.15}$$

which is satisfied for

$$\phi_0^2 \leq \frac{(p+2)(2p+1)}{3} \tilde{M}_{Pl}^2, \tag{3.16}$$

therefore forbidding large trans-Planckian excursions.

3.4 Exponential scalar potential

Another popular class of scalar potentials is represented by sums of exponential functions. We focus here on the simplest case

$$V(\phi) = \Lambda_0 e^{-\lambda\phi/f}. \tag{3.17}$$

The expansion around a background value ϕ_0 reads

$$\begin{aligned} V(\phi_0 + \delta\phi) &= \Lambda_0 e^{-\lambda\phi_0/f} \left[1 - \lambda \frac{\delta\phi}{f} \right. \\ &+ \frac{1}{2} \lambda^2 \left(\frac{\delta\phi}{f} \right)^2 - \frac{1}{3!} \lambda^3 \left(\frac{\delta\phi}{f} \right)^3 \\ &\left. + \frac{1}{4!} \lambda^4 \left(\frac{\delta\phi}{f} \right)^4 \right], \end{aligned} \tag{3.18}$$

and the self-interaction of the scalar field in the non-relativistic limit is encoded in the $\tilde{\lambda}$ quartic coupling

$$\tilde{\lambda} = \Lambda_0 e^{-\lambda\phi_0/f} \left(\frac{\lambda^4}{f^4} - \frac{5}{3} \frac{\lambda^4}{f^4} \right) = -\frac{2}{3} \frac{\lambda^4}{f^4} \Lambda_0 e^{-\lambda\phi_0/f}. \tag{3.19}$$

Application of our bound is straightforward and yields the following inequality

$$\frac{2}{3} \frac{\lambda^2}{f^2} \geq \frac{1}{\tilde{M}_{Pl}^2}, \tag{3.20}$$

The weak gravity regime under scrutiny is realized for scalars with an exponential potential as long as their scale does not exceed the Planck one, with

$$f^2 \leq \frac{2}{3} \lambda^2 \tilde{M}_{Pl}^2. \tag{3.21}$$

This bound still allows for a cosmological expansion (see e.g. [31]), but is in conflict with the requirement obtained in [32], as we will discuss below.

Let's consider the case of a double exponential potential

$$V(\phi) = \Lambda_1 e^{-\lambda_1 \phi/f} + \Lambda_2 e^{-\lambda_2 \phi/f}, \tag{3.22}$$

with the assumption $\lambda_1 \sim \lambda_2$. We develop each exponential as in (3.18) to get

$$\tilde{\lambda} = \Lambda_1 \frac{\lambda_1^4}{f^4} e^{-\lambda_1 \phi_0/f} + \Lambda_2 \frac{\lambda_2^4}{f^4} e^{-\lambda_2 \phi_0/f} - \frac{5}{3} \frac{1}{f^4} \frac{(\Lambda_1 \lambda_1^3 e^{-\lambda_1 \phi_0/f} + \Lambda_2 \lambda_2^3 e^{-\lambda_2 \phi_0/f})^2}{\Lambda_1 \lambda_1^2 e^{-\lambda_1 \phi_0/f} + \Lambda_2 \lambda_2^2 e^{-\lambda_2 \phi_0/f}}, \tag{3.23}$$

which can be rewritten as

$$\tilde{\lambda} = -\frac{1}{f^4} \frac{\frac{2}{3} \Lambda_1^2 \lambda_1^6 e^{-2\lambda_1 \phi_0/f} + \frac{2}{3} \Lambda_2^2 \lambda_2^6 e^{-2\lambda_2 \phi_0/f} + \Lambda_1 \Lambda_2 \lambda_1^2 \lambda_2^2 (\frac{10}{3} \lambda_1 \lambda_2 - \lambda_1^2 - \lambda_2^2) e^{-(\lambda_1 + \lambda_2) \phi_0/f}}{\Lambda_1 \lambda_1^2 e^{-\lambda_1 \phi_0/f} + \Lambda_2 \lambda_2^2 e^{-\lambda_2 \phi_0/f}}. \tag{3.24}$$

The analysis of this constraint on a double exponential is somehow quite involved, and not useful here to discuss in full generality. In the case where $\lambda_1^2 + \lambda_2^2 \leq \frac{10}{3} \lambda_1 \lambda_2$, all three terms in the numerator have the same sign. For $\Lambda_{1,2} > 0$ ($\Lambda_{1,2} < 0$) the scalar self-interaction is attractive (repulsive). The condition for gravity to be the weakest force reads

$$\frac{\tilde{M}_{Pl}^4}{\Lambda^4} \tilde{\lambda} = \frac{-\frac{256}{27} e^{-4\sqrt{2/3} \phi_0/\tilde{M}_{Pl}} + \frac{80}{27} e^{-3\sqrt{2/3} \phi_0/\tilde{M}_{Pl}} - \frac{16}{27} e^{-2\sqrt{2/3} \phi_0/\tilde{M}_{Pl}}}{2e^{-2\sqrt{2/3} \phi_0/\tilde{M}_{Pl}} - e^{-\sqrt{2/3} \phi_0/\tilde{M}_{Pl}}}. \tag{3.27}$$

$$\begin{aligned} I(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, f) &= \lambda_1^4 \Lambda_1^2 \left(\frac{2}{3} \frac{\lambda_1^2}{f^2} - \frac{1}{\tilde{M}_{Pl}^2} \right) \\ &\quad e^{-2\lambda_1 \phi_0/f} + \lambda_2^4 \Lambda_2^2 \left(\frac{2}{3} \frac{\lambda_2^2}{f^2} - \frac{1}{\tilde{M}_{Pl}^2} \right) e^{-2\lambda_2 \phi_0/f} \\ &\quad + \Lambda_1 \Lambda_2 \lambda_1^2 \lambda_2^2 \left(\frac{10/3 \lambda_1 \lambda_2 - \lambda_1^2 - \lambda_2^2}{f^2} - \frac{2}{\tilde{M}_{Pl}^2} \right) \\ &\quad e^{-(\lambda_1 + \lambda_2) \phi_0/f} \\ &\geq 0 \end{aligned} \tag{3.25}$$

It is verified for mass scales not exceeding the value $f^2 \sim \frac{2}{3} \lambda_{1,2}^2 \tilde{M}_{Pl}^2$.

3.5 Starobinsky potential

The power-law and the exponential potentials are frequently used in early Universe cosmology. We investigate here the implications of (2.8) for the Starobinsky's potential [33].

We consider the potential:

$$V(\phi) = \Lambda^4 \left(1 - e^{-\sqrt{2/3} \phi/\tilde{M}_{Pl}} \right)^2 \tag{3.26}$$

and expand it around a background field value ϕ_0 , and study

the leading order contribution to the quartic self-interaction perturbation $\delta\phi = \phi - \phi_0$. The non-relativistic regime quartic coupling $\tilde{\lambda}$ is given by:

The weakness of the gravitational interaction reads now

$$\left| -\frac{16}{9}e^{-2\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} + \frac{5}{9}e^{-\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} - \frac{1}{9} \right| \geq \left| e^{-2\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} - e^{-\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} + \frac{1}{4} \right|, \tag{3.28}$$

where we have put the absolute value on the r.h.s. to stress its positivity even if it is useless, being the square of a real quantity. Nevertheless, we still should study the sign and the strength of the l.h.s. of (3.28). The term inside the absolute value is always negative, meaning the scalar interaction is always attractive. So we can just drop the absolute values in Eq. (3.28). Simple algebra finally leads us to the conclusion that gravity is weaker than the scalar self-interaction if

$$\phi_0 \leq \sqrt{\frac{3}{2}} \ln \left(\frac{14}{\sqrt{51}-4} \right) \tilde{M}_{Pl} \sim 2\tilde{M}_{Pl}. \tag{3.29}$$

The coefficient in front of \tilde{M}_{Pl} in the above equation is of order 1. Slightly before reaching this scale, we would encounter tachyonic modes for $\phi \sim \sqrt{\frac{3}{2}} \ln(2)\tilde{M}_{Pl}$. In this Starobinsky’s model, self-interactions are strong enough to keep gravity the weakest force all the way up to the Planck scale.

3.6 Weak gravity and quintessence

One of the popular use of the above scalar potentials is for inducing cosmic acceleration, more precisely using ϕ as the quintessence field. We discuss here some direct implications of our constraints for such applications.

The late time cosmic acceleration may indeed be understood either in terms of a cosmological constant, in the context of the Λ CDM, or in terms of a dynamical scalar field, slowly rolling towards the minimum of its potential [34,35]. In the equation of state, the ratio pressure/energy density w is fixed to the value $w = -1$ in the first case, while it becomes a dynamical variable in the case of the quintessence [36]. The swampland criteria seem to be in favor of the latter scenario, that, with parameters tightened by the current observations, may fit into the program (see [32]). In this context, for the dark energy to take over the control of the expansion of the Universe at late times, the quintessence field needs to be very light, with mass of order the Hubble parameter as measured today $m \lesssim H_0 \sim 10^{-33} eV$. The corresponding potential is unknown and forms similar to those studied above have been considered (see for a review [31]). Requirements for the evolution equations of a scalar field ϕ to have a fixed point realizing the desired equation of state can be expressed as

$$\begin{cases} w_{eff} \equiv \frac{\rho_\phi + \rho_m}{P_\phi + P_m} = w_\phi > -\frac{1}{3}; \\ \Omega_\phi \equiv \frac{\rho_\phi}{3M_{Pl}^2 H^2} = 1, \end{cases} \tag{3.30}$$

where we denote with the subscript m the matter contribution, and ϕ for the quintessence one, and [37]:

$$\left(M_{Pl} \frac{V'(\phi)}{V(\phi)} \right)^2 \equiv \lambda^{*2} < 2. \tag{3.31}$$

Obviously, $w_\phi \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}/2 - V(\phi)}{\dot{\phi}/2 + V(\phi)}$, leads to different dynamics for the different potentials.

The axion potential gives the thawing solution, where the field and its corresponding equation of state are almost constant in the early cosmological era, with $w_\phi = -1$, and then starts to evolve after the mass drops below the Hubble parameter, leading to $w_\phi \geq -1$ [38,39]. The axion shift symmetry might allow to tame loop corrections. The condition (3.31) reads then²

$$\sin^2 \left(\frac{\phi}{f_a} \right) < 2 \frac{f_a^2}{M_{Pl}^2} \left(1 + \cos \left(\frac{\phi}{f_a} \right) \right)^2. \tag{3.32}$$

The requirement $f_a \leq M_{Pl}$ allows the axion-like fifth force to be stronger than gravity when ϕ gets sufficiently close to 0 for Eq. (3.32) to be realized. Observational constraints allow this model to be used for quintessence with $w_0 \in]-1, -0.7[$, w_0 being today’s value.

The power law potential gives rise to the tracking solution [40,41]. This allows for a cosmic evolution from the so-called scaling fixed point $(x, y) = \left(\sqrt{\frac{3}{2}} \frac{1+w_m}{\lambda}, \sqrt{\frac{3}{2}} \frac{1-w_m^2}{\lambda^2} \right)$, with $x = \frac{\dot{\phi}}{\sqrt{6}M_{Pl}H}$ and $y = \frac{\sqrt{V(\phi)}}{\sqrt{3}M_{Pl}H}$, where matter dominates, to the fixed point $(x, y) = (\lambda^*/\sqrt{6}, \sqrt{1-\lambda^{*2}/6})$, where the cosmic acceleration can be realized [37]. The behaviour of the equation of state is opposite to the previous case, as w slowly decreases with the evolution. Equation (3.31) gives

$$\phi^2 > \frac{1}{2} p^2 M_{Pl}^2, \tag{3.33}$$

Unless the p parameter is tuned to be very small, this calls for trans-Planckian values of the field, as we should have expected since the potential is monotonically decreasing to reach its asymptotic value $V = 0$ at infinity. Together with our constraint of weak gravity $\phi^2 \leq \frac{(p+2)(2p+1)}{3} M_{Pl}^2$, this leads to:

$$\frac{(p+2)(2p+1)}{3} > \frac{p^2}{2}, \tag{3.34}$$

which is valid for all positive powers. Of course, the applicability of the effective field theory treatment at trans-Planckian

² Note, that for the cosmological application, we have taken, as in [38], the potential to be $V(\phi) = \mu^4 \left(1 + \cos \left(\frac{\phi}{f_a} \right) \right)$. This corresponds to a shift of the minimum in (3.7) with no consequence for the analysis performed in Sect. 3.2.

scales is for the least questionable. Observations have led to constrain the tracker equation of state so tightly that the current accepted range of value for the exponent p is very restricted. Indeed, the upper bound on p was argued to be $p < 0.107$ in [42], or $p < 0.17$ in [43], so that positive integers should be excluded, making it difficult to realize power law potentials within the observational bounds in particle physics models.

The single exponential potential is popular as the cosmological evolution is there described by a closed system of equation [37, 44]. However, the fact that λ^* is constant in this case leads to strongly constrain this potential. It is realized again in the fixed point mentioned above but to be reached from the trivial fixed point $(x, y) = (0, 0)$ [31]. In particular, the transition from the more interesting scaling fixed point $(x, y) = \left(\sqrt{\frac{3}{2}} \frac{1+w_m}{\lambda}, \sqrt{\frac{3}{2}} \frac{1-w_m^2}{\lambda^2}\right)$ is forbidden. This can be circumvented by taking the case of a double exponential potential, as in Eq. (3.22). The solution which is realized in this case is a tracking one with constant Ω_ϕ [49].

The exponential potentials with decay constants respecting the upper bound discussed may well fit into the proposed inequality with

$$\lambda^2 \frac{M_{Pl}^2}{f^2} > \frac{3}{2}. \tag{3.35}$$

For the epoch of cosmic acceleration to be realized we need instead

$$\lambda^2 \frac{M_{Pl}^2}{f^2} < 2. \tag{3.36}$$

As we see, this seems to leave a window for both the weakness of gravity and the period of cosmic accelerated expansion to be realized through an exponential potential.

These type of potentials have also been constrained with current observations in the interest of other swampland conjectures, namely the de Sitter and the TCC conjectures [32, 45]. It was argued in [32] we should have for an exponential potential $\lambda^* = \lambda \frac{M_{Pl}}{f_a} \leq 0.6$. This was devised to be in agreement with the de Sitter conjecture with the constant c there appearing bounded to be $c \leq 0.6$. This bound is sensitive to uncertainties in the data as was investigated in e.g. [46, 47]. This seems to leave as the only viable conclusion that an exponential quintessence model can only lead to fifth force interactions weaker than gravity. However, [48] has hinted to the possibility that dark matter-dark energy coupling may relax constraints on λ .

A double exponential is usually devised to respect both constraints coming from big-bang nucleosynthesis and cosmic acceleration. As such, one exponent, λ_1 , is taken to give $\lambda_1 \frac{M_{Pl}}{f} \sim 1 - 10$, while the second is expected to take over at late times and respects the same bounds as those for the single exponent [32, 42, 49]. In this case, the weak gravity

may be realized in the early Universe as long as the double exponential is concerned, but at late time, one faces the same strong constraints as discussed above.

4 Multiple scalar and moduli fields

We consider now more complex situations with multiple scalar fields. The preeminence of the scalar interaction over the gravitational one has to be formulated in more general terms to account for these cases. In particular, we need to specify what are the processes we should consider to compare scalar and gravitational interactions.

In the case of multiple scalars, we assume that *in the appropriate low energy limit, for the leading interaction, the gravitational contribution must be sub-leading*. The focus on the leading scalar interaction can be seen as parallel to constraining the biggest ratio q/m in the WGC.

Let's illustrate the meaning of this statement. First consider the case of a massive scalar X , taken to be complex for simplicity. The leading interaction is given by the Yukawa coupling to another real scalar field ϕ and is described by :

$$\mathcal{L}_{int} = \mu\phi|X|^2 + \dots \tag{4.1}$$

where the dots stand for sub-leading higher order terms. We can write the potential as:

$$V(X, \phi) = m_X^2(\phi) |X|^2, \quad \mu = \partial_\phi m_X^2 \tag{4.2}$$

The preeminence of scalar interactions must be taken at the mass scale $\sim 2m_X$ and reads then:

$$|\partial_\phi m_X| \geq \frac{m_X}{\tilde{M}_{Pl}} \tag{4.3}$$

We can square the above three-point amplitudes on each side, $2X \rightarrow \phi$ on the left and $2X \rightarrow G$, on the right side, where G is the graviton. The comparison concerns then two $XX^* \rightarrow XX^*$ processes, at the energy scale m_X , one through scalar and the other through graviton exchange. This leads to the following potentials for X :

$$V_{scalar}(r) = -\frac{\mu^2}{4m_X^2 r}, \quad V_{grav}(r) = -\frac{m_X^2}{\tilde{M}_{Pl}^2 r} \tag{4.4}$$

Now, both scalar and gravitational interactions have similar dependence in the inter-particles distance and the comparison is straightforward:

$$\frac{\mu^2}{4m_X^2} \geq \frac{m_X^2}{\tilde{M}_{Pl}^2} \tag{4.5}$$

which can be written:

$$\partial_\phi m_X \partial_\phi m_X \geq \frac{m_X^2}{\tilde{M}_{Pl}^2} \tag{4.6}$$

In the extremal case saturating the above inequality, the solution is given by:

$$m_X^2(\phi) = m_0^2 e^{\pm 2\phi/\tilde{M}_{Pl}}. \tag{4.7}$$

This is the Swampland Distance Conjecture (SDC) [2, 45, 50–54]. The inequality (4.6) has been proposed by [6] in order to retrieve (4.7) and discussed by [6, 12, 55] with different motivations.

Let us now move forward and consider another case: a massless complex modulus field Φ , therefore with vanishing potential. We assume again that the theory contains at least one complex scalar field X such that the mass of X and its different couplings are functions of ϕ . For simplicity, we also assume that X has no tadpole and its vacuum expectation value vanishes, $\langle X \rangle = 0$. Under these assumptions, the scalar potential then takes the form:

$$V(X, \Phi) = m_X^2(\Phi)|X|^2 + \dots \quad m_X^2 = m_{X0}^2 + \lambda_\Phi |\Phi|^2 + \dots \tag{4.8}$$

where

$$\lambda_\Phi = \partial_\Phi \partial_{\bar{\Phi}} m_X^2(\Phi, \bar{\Phi}) \tag{4.9}$$

represents now the leading non-gravitational interaction of Φ . Here, m_{X0}^2 is a contribution to the squared mass independent of Φ , but depending on other fields while λ_Φ gives a scalar four-point interaction term of Φ and X obtained by expanding (4.8) in powers of Φ and $\bar{\Phi}$. The weakness of gravitational interaction becomes a statement comparing on one side the annihilation of two X states into two Φ state (and vice-versa) and on the other side the same channel through graviton exchange, both taken at the threshold energy scale $\sim 2m_X$.

As the modulus is massless, the gravitational interaction gets an enhancing factor of 2 compared to the massive case, analogous to the case of the gravitational deflection of light. In this case, the statement that the gravitational interaction is weaker reads:³

$$\partial_\Phi \partial_{\bar{\Phi}} m_X^2 \geq 2 \frac{m_X^2}{\tilde{M}_{Pl}^2} \tag{4.10}$$

If the state X has a self-quartic interaction, then we will also have to check a similar constraint on the self coupling $|\tilde{\lambda}_4| \tilde{M}_{Pl}^2 \geq m_X^2$.

³ For real fields, the inequality reads $g^{ij} \partial_i \partial_j m_X^2 \geq 2nm_X^2/\tilde{M}_{Pl}^2$ where g^{ij} and n are the metric in the space and the number of moduli fields. The dots in (4.8) include Φ^2 and $\bar{\Phi}^2$ as required to recover the case of real fields scattering and account for an extra factor of 2.

The extremal case corresponds to the case of equality in (4.10). It is solved for:⁴

$$m_X^2(\Phi, \bar{\Phi}) = m_-^2 e^{-\sqrt{2}\frac{\Phi+\bar{\Phi}}{\tilde{M}_{Pl}}} + m_+^2 e^{\sqrt{2}\frac{\Phi+\bar{\Phi}}{\tilde{M}_{Pl}}} \tag{4.11}$$

We can use the following parametrization:

$$\begin{aligned} \Phi &= \frac{1}{\sqrt{2}}(\phi + i\chi), \\ e^{\sqrt{2}\frac{\Phi+\bar{\Phi}}{\tilde{M}_{Pl}}} &= e^{2\frac{\phi}{\tilde{M}_{Pl}}}, \quad \text{and} \quad e^{\frac{\phi}{\tilde{M}_{Pl}}} = R \end{aligned} \tag{4.12}$$

then:

$$m_X^2(R) = \frac{m_-^2}{R^2} + m_+^2 R^2 \tag{4.13}$$

which is the well known formula for string states squared masses with the $\frac{m_-^2}{R^2}$ as the low energy Kaluza-Klein modes and $m_+^2 R^2$ the winding modes that are typical to extended objects, strings, winding around a compactified dimension. The (4.13) differs sensibly from (1.4) as it extremizes a different inequality.

Note that in the statement about the preeminence of the scalar interaction, the two fields Φ and X play a symmetric role.

Now, consider the case where the field ϕ is a modulus appearing only as a parameter in the couplings of the massive scalar X ($\langle X \rangle = 0$), through

$$V(X, \phi) = m_X^2(\phi)X^2 + \sum_{n \geq 4} \lambda_n(\phi)X^n \tag{4.14}$$

Then, the condition (4.3) can be written as:

$$\left. \frac{|\partial_\phi V(X, \phi)|}{V} \right|_{X=0} \geq \frac{\sqrt{c}}{M_{Pl}} \tag{4.15}$$

while the condition (4.10) reads now:

$$\left. \frac{|\partial_\phi \partial_{\bar{\phi}} V(X, \phi)|}{V} \right|_{X=0} \geq \frac{2\tilde{c}}{M_{Pl}^2} \tag{4.16}$$

where we note the similarity with the Refined de Sitter Conjectures [56–65] (in (4.16) when the second derivative is negative).

A popular way to look at the Weak Gravity Conjecture rests on the fact that the equality in (1.1) relates to the BPS states relation. In [6], it was suggested that the identity satisfied by the central charge in $\mathcal{N} = 2$ supersymmetry [66]

$$g^{i\bar{j}} D_i \bar{D}_{\bar{j}} |Z|^2 = g^{i\bar{j}} D_i Z \bar{D}_{\bar{j}} \bar{Z} + n|Z|^2 \tag{4.17}$$

⁴ Note that this is not the most general solution but we focus on reproducing the toroidal compactification dependence. Moreover, as the potential (4.8) and the Eq. (4.10) are symmetric under the exchange of the real and imaginary part, we choose to focus on the real part of the field only i.e. KK and winding excitation along one of the torus dimensions.

can be used to extract a bound on the mass m as in the BPS case $|Z| = m$:

$$g^{ij} \partial_i \partial_j m^2 \geq g^{ij} \partial_i m \partial_j m + nm^2 \tag{4.18}$$

with derivatives are with respect to scalar fields, and g_{ij} is the corresponding metric. Here, we would like to contemplate a different possibility. Following [66], the right hand side of (4.17) is identified with the scalar potential of the black hole solution, and it was shown that it implies that at the critical point the potential satisfies (in reduced Planck mass units):

$$\partial_i \partial_{\bar{j}} V \Big|_{critical} = 2G_{i\bar{j}} V_{critical} \tag{4.19}$$

We would like to contemplate here the possibility to extend this relation, beyond its derivation in the $\mathcal{N} = 2$ world, to

$$|\partial_i \partial_{\bar{j}} V| \geq cV \tag{4.20}$$

as given by (4.16). Along this line of thought, we note the similarity of (4.10), up to a factor 2 due to the masslessness of our field Φ , and the equation [66]:

$$\begin{aligned} \partial_i \partial_{\bar{j}} m(\Phi, \bar{\Phi}, p, q) \Big|_{critical} \\ = \frac{1}{2} G_{i\bar{j}}(\Phi, \bar{\Phi}) m(\Phi, \bar{\Phi}, p, q)_{critical} \end{aligned} \tag{4.21}$$

where $\Phi, \bar{\Phi}$ are moduli fields, p, q electrical and magnetic charges, m is the black hole mass and $G_{i\bar{j}}$ is the scalar metric on the moduli space.

Finally, let us comment that while supersymmetry was not explicitly invoked here, it might be required to insure the stability of some flat directions, therefore moduli fields, when radiative corrections are taken into account.

5 Conclusions

In contrast with the WGC, there is no obvious, no totally convincing road towards uncovering a law governing the scalar potential in quantum gravity. The main ideas have been reviewed in the introduction. Their variety can be considered as an evidence both for the difficulty and risks in writing such constraints and for the interest in investigating their implications.

We postulate that in the appropriate low energy limit, for the leading interaction, the gravitational contribution must be sub-leading. Such a statement is hollow if one does not specify which process is concerned and the energy scale at which the interaction strengths are compared. We provided answers for these questions for some cases and found that we retrieve some forms of the Swampland conjectures.

The constraint (2.8) differs from previous proposed inequalities. Indeed, [6,7] focused on massless scalars and their role in the formation of gravitational bound states.

Strictly speaking, the logic behind their inequalities would lead to (4.6) but with an opposite sign for the r.h.s. part. This is due to the fact that their arguments constrain repulsive interactions to be stronger than gravitational one, while the scalar mediated one is attractive. While the logic in this work differs, in the massless case (4.5) agrees with one of the proposals of [6], that was also discussed further in [9,12,55]. This is all but surprising as the different arguments were put such as one recovers the SDC, which corresponds to the ubiquitous Kaluza-Klein states present in String theory compactifications. Our analysis differs also in the fact that we have also considered self-interacting scalars but only focused on the case of neutral states.

The conjecture presented in [9] leads to an inequality that would constrain in qualitatively similar manner attractive self-interactions for a massive particle (non-tachyonic), but we were not able to recover their coefficients for the different contributions. Moreover, the field dependence of the extremal states squared mass (1.5) differs sensibly from our result (4.13).

The main playground for testing different conjectures about quantum gravity is string compactifications and their effective supergravity theories. While they represent an opportunity to put the conjecture on firm grounds (see [67] for a recent proposal), one should be able to disentangle what is due to generic quantum gravity from what is due to supersymmetry, other symmetries or just consistency of the precise string theory compactification. Here, we have kept the analysis on a very basic level which we believe is sufficient to stress the main points. We plan to test our constraints in string compactification models in the future.

We end by mentioning two immediate remarks. For the Standard Model Higgs scalar, it was found that the running quartic coupling vanishes at energies of order 10^{11} GeV [68], we should therefore contemplate this intermediate energy scale as an ultra-violet cut-off. Scalar interactions determine the behaviour of spherically symmetric cosmological clumps. The size and dynamics of these objects is different depending on the quartic self-interaction coupling λ . For the case of repulsive complex scalars, massive boson stars, with masses comparable to the fermionic ones, are allowed only when the relevant *relativistic* parameter $\lambda M_{pl}^2/m^2$ is big [69]. This is a prediction of the weak gravity conjecture discussed here.

Going through the implications of our weak gravity requirement we recovered, in the corresponding cases and forms, some of the Swampland program expectations: the Axion Weak Gravity Conjecture, the Swampland Distance Conjecture, the string Kaluza-Klein and winding modes mass formula and the Swampland de Sitter Conjecture. It would be interesting to investigate if a formulation from general principles of the preeminence of scalar interactions when com-

pared to gravitational ones can lead to a unified Swampland conjecture that rules them all.

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CHAPTER 4

Kaluza-Klein theories and the Scalar Weak Gravity Conjectures

4.1 Introduction: compactification and Pair Production WGC

As formulated in [1], with the precise dimension-dependent coefficients ignored, the Scalar WGC was only proposed as a criterion for a fixed number of dimensions, that one would naturally take to be four. An extension to the case of D generic dimensions necessitate of an inspection of the behaviour of the conjecture under dimensional reduction. If, upon compactification, the DOFs are mixed in such a way that, following the logic we proposed, the bounds in different dimensions cannot be related to each other, this would certainly raise some question about the validity of our proposal. It would mean that the information encoded in the inequality depends on the way we describe the theory, which would make it more difficult (not necessarily impossible) to consider it as if it was translating a generic property of low energy EFTs of Quantum Gravity.

In [4], the Kaluza Klein compactification of a $D + 1$ dimensional theory on a circle was used as a playground to test the behaviour of scalar operators under compactification. The interest towards compactified theories goes well beyond the idea to check the behaviour of our Scalar WGC. Compactification of gravity automatically generates a dilatonic theory. If, in addition, the higher dimensional theory contains gauge groups, the lower dimensional one is ensured to have several gauge groups and several scalars. Dimensionally reduced theories are then an extremely useful playground to test various formulations of the WGC with gauge and scalar fields. In particular, after our paper [1], another work appeared [56], whose aim was that of pointing out a way to reformulate the original WGC (1.3.2) following a logic that would be extremely easy to apply to the case of a scalar field theory. The proposed reformulation goes under the name of Pair Production Weak Gravity Conjecture.

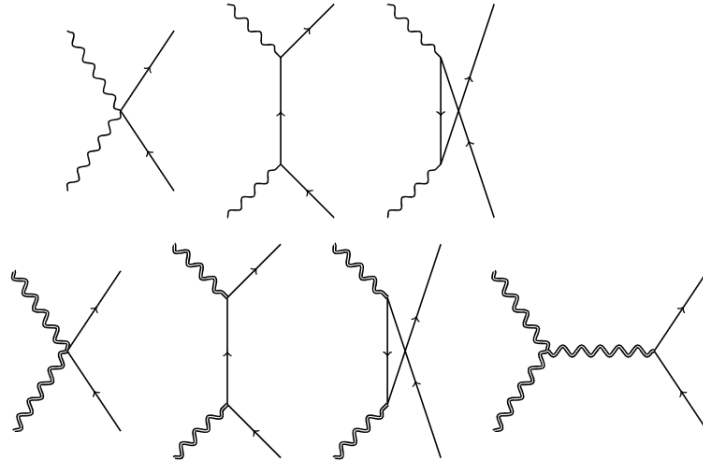


Figure 4.1 – The relevant diagrams for the pair production formulation of the WGC are depicted here. In the first line, the diagrams for the photon pair production are shown, while in the second line those for the graviton pair production. The first diagram of the first line is only present when the matter state is a scalar.

As we stressed in [1], the first thing that is needed to define a WGC is to give a precise meaning to the say "gravity is the weakest force". In [56], it was argued that this should in particular apply to the the pair production of two WGC states from massless particles. In fact, it was observed there, that if we consider either a bosonic or a fermionic particle coupled to a gauge group and gravity, the original WGC of [17] could be reformulated as the requirement that the cross-section for the production of a couple of matter states from photons ($\gamma\gamma \rightarrow \psi\bar{\psi}$) at threshold should be greater than the corresponding production from gravitons ($GG \rightarrow \psi\bar{\psi}$). In a more physical way, one could state this as the requirement that, when two massive particles are seen to be produced in a detector, we require the probability that the two have been produced from two gauge bosons to be greater than the probability that the two have been produced from two gravitons. For this statement not to be ambiguous, however, one should also require that gravitational contributions to $\gamma\gamma \rightarrow \psi\bar{\psi}$, that at tree-level appear in an s-channel diagram with the graviton as a mediator, are ignored. Although this might seem a strange requirement in an EFT that contains gravity together with the other fields and that should describe their interplay, it was postulated in [56] that the comparison should be restricted to purely gravitational and purely non-gravitational diagrams, as if one is comparing the result he would obtain in a plain gauge theory with the one he would obtain in a theory of gravity coupled to the matter state. The relevant diagrams considered in [56] are shown in Fig 4.1. The result for the gravitational diagrams was obtained in [56] from previous results on the gravitational Compton scattering [57, 58, 59] by means of cross-symmetry arguments. A careful derivation will be presented below with [4]. The final result is that, in the $D = 4$ dimensions case investigated in [56], the squared amplitude for the photon production reduces in the threshold limit

$$\frac{s - 4m_n^2}{m_n^2} \rightarrow 0, \quad \frac{t + m_n^2}{m_n^2} \rightarrow 0, \quad \frac{u + m_n^2}{m_n^2} \rightarrow 0, \quad (4.1.1)$$

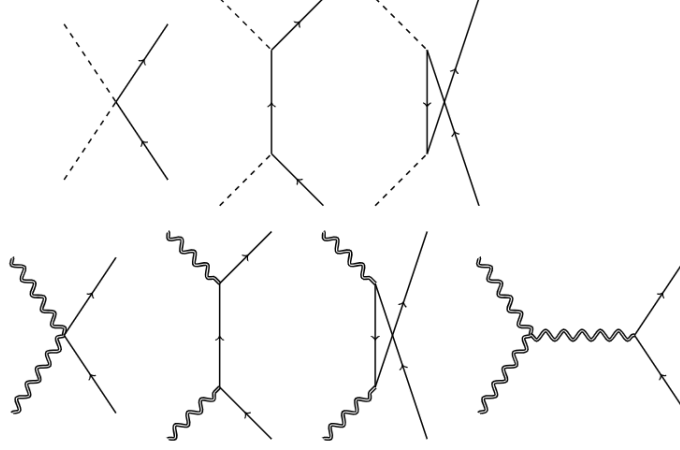


Figure 4.2 – The relevant diagrams for the pair production formulation of the Scalar WGC are depicted here. In the first line, the diagrams for the scalar pair production are shown, while in the second line those for the graviton pair production.

to

$$|\mathcal{M}_{\gamma\gamma}|^2 \rightarrow 4(gq)^4, \quad (4.1.2)$$

with g is the gauge coupling and q the charge of the state. The corresponding gravitational production is

$$|\mathcal{M}_{GG}|^2 \rightarrow \kappa^4 m^4 \quad (4.1.3)$$

with m the mass of the state. Requiring $|\mathcal{M}_{\gamma\gamma}|^2 \geq |\mathcal{M}_{GG}|^2$ then leads to $\sqrt{2}gq \geq \kappa m$, which is equivalent to (1.3.2).

The same logic can then be applied to other cases, as for example the case of a theory with a modulus field and a matter field, which was argued to provide a new criterion for a Scalar WGC. Consider a theory of the form

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + |\partial\varphi|^2 - m^2(\phi)|\varphi|^2, \quad (4.1.4)$$

where the interaction of the matter state φ with the massless scalar ϕ are encoded in the function $m(\phi)$. Requiring the purely scalar production to be more important than the purely gravitational one, which amounts to the comparison of the diagrams in Fig. 4.2, leads in the threshold limit to

$$\left| (\partial_\phi m^2)^2 - m^2 \partial_\phi^2 m^2 \right|_{\phi=\phi_0} \geq \kappa^2 m^4(\phi_0). \quad (4.1.5)$$

After promoting that to a differential equation, with all the consequences discussed in the previous chapters, this was considered to provide the correction to the bound the same authors proposed in [33]. This bound is however different from ours [1] in the value of the coefficients in front of the different terms. What we already see here is that, although in the previously discussed case of a gauge theory the proposed pair production criterion ($\gamma\gamma/GG \rightarrow \psi\bar{\psi}$) gave the same relative coefficient as the comparison of the gauge and gravitational long range forces ($\psi\psi \rightarrow \psi\psi$), pair production from a couple of moduli ($\phi\phi \rightarrow \varphi\bar{\varphi}$) does not lead to the same bound as the comparison of the short range scalar and gravitational interactions in $\varphi\varphi \rightarrow \varphi\varphi$. Another important

difference with both our proposal and that of [33] is that this criterion does not apply to the case where the matter state φ is the only state in the theory, besides gravity. The logic of constraining the production of pairs from massless particles cannot apply to self-interacting scalars, of course. If we were to blindly force it, a new scalar diagram would appear in the s-channel and we would fall back to our case [1]. The two constraints are conceptually different and lead to different bounds. Of course, if the idea that gravity has to be subdominant in the low energy EFT scattering amplitudes is taken to its full meaning, both bounds are of interest as they implement the idea in two different channels. Differently from what done in [56], the bound obtained from the pair production criterion (4.1.5) should not be applied to single scalar potentials. Doing that requires a change of the underlying criterion to that of [1], and the inclusion of the new s-channel diagram. A blind application of (4.1.5) with the identification $\partial_\phi^2 V(\phi) = m^2(\phi)$ has no physical meaning.

In [4] we study the question of the behaviour of the Scalar WGC under compactification and of the Pair Production re-formulation of WGC criteria in the context of a Kaluza Klein theory obtained from a circle compactification.

4.2 Discussion

After a brief recap of the compactification of gravity and scalar fields on a circle (see Appendix A for a more thorough derivation), various questions were studied. It was first verified that KK states of a higher dimensional massless scalar field saturate the dilatonic WGC. The relevant amplitudes for the pair production of KK states from massless states were then calculated. Several results of interest have been found:

- In $D=4$ dimensions, the gravitationally mediated s-channel diagram for the photon pair production vanishes in the threshold limit. This is true both for a plain $U(1)$ gauge theory and a dilatonic gauge theory. It is peculiar to the case $D = 4$ and does not apply to any other value of D .
- The gravitationally mediated s-channel diagram for the scalar pair production is non-vanishing for all D .
- Production from mixed initial states $G\gamma$ and $G\phi$ vanishes in the limit of interest
- For a plain $U(1)$ gauge theory, the four dimensional factorization between gravitational and gauge pair production that in [56] was inferred from cross-symmetry is successfully verified. More generally, we find that the coefficient of the factorization is independent of the dimension D , so that the ratio of the gravitational and photon pair production is the same for all values of D .

All these results boil down to the following conclusions. In 4 dimensions, the Pair Production WGC is verified from a direct calculation to reproduce the original one. The gravitationally mediated photon production vanishes, so that in this case there might be an ambiguity whether one should follow [56] or allow for such diagrams to be included. The generalization of the bound to the case where both gauge and scalar

are present was only sketched in [56] to be

$$\sum_{ij} |\mathcal{M}_{ij}|^2 \geq |\mathcal{M}_{GG}|^2, \quad (4.2.1)$$

where the indices i and j run over the non gravitational massless states (photons and moduli). Whatever the generalization is, we have found that the KK states have $\mathcal{M}_{\phi\phi} = \mathcal{M}_{\gamma\phi} = 0$ in the threshold limit, and $\mathcal{M}_{\gamma\gamma}$ is modified by the presence of the dilaton in such a way that the request $|\mathcal{M}_{\gamma\gamma}|^2 \geq |\mathcal{M}_{GG}|^2$ formally reproduces the dilatonic WGC. Using the relations between the charge and the mass, it is readily seen that the bound is actually saturated. As the KK states are those that saturate the dilatonic WGC, this gives a precise indication that the reformulation in terms of the pair production criterion should lead to an equivalent bound as the WGC also in the dilatonic case. Note that the gravitationally mediated scalar production does not vanish, so that the requirement to ignore them, that in the gauge theory might have seemed useless, is now of capital importance.

Unfortunately, however, all this can only be said when $D = 4$. Getting back to the case of a plain $U(1)$ gauge theory, our direct calculation finds that the factorization property that ensured the appearance of the good coefficient in the comparison of gauge and gravitational amplitudes in $D = 4$ dimensions, is actually independent of D . Whatever the value of D , the result for the amplitudes is

$$|\mathcal{M}_{\gamma\gamma}|^2 \rightarrow \frac{4}{D-2}(gq)^4, \quad |\mathcal{M}_{GG}|^2 \rightarrow \frac{1}{D-2}(\kappa m)^4 \quad \frac{|\mathcal{M}_{\gamma\gamma}|^2}{|\mathcal{M}_{GG}|^2} \rightarrow \frac{4(gq)^4}{(\kappa m)^4} \quad (4.2.2)$$

so that the requirement $|\mathcal{M}_{\gamma\gamma}|^2 \geq |\mathcal{M}_{GG}|^2$ does not reproduce (1.3.8). The mixed $G\gamma$ diagram vanish for any value of D , and addition of the gravitationally mediated photon s-channel does not change the conclusion, as it would amount to the requirement

$$\frac{(2(D-2)(gq)^2 + (D-4)\kappa^2 m^2)^2}{(D-2)^3} - \frac{(\kappa m)^4}{D-2} \geq 0 \iff (gq)^2 \geq \frac{(\kappa m)^2}{D-2}. \quad (4.2.3)$$

With these results at hand, we are forced to conclude that the proposed reformulation in terms of pair productions only works in the special case $D = 4$, where it reproduces the correct coefficients "by accident" rather than for a deep underlying physical principle that relates it to the original conjecture.

As for our Scalar WGC, it was easily checked that it is well behaved under compactification. Requiring that our bound (3.2.5) is satisfied by a state $\hat{\Phi}$ in $D + 1$ dimensions, it is automatically satisfied by its zero mode φ_0 in D dimensions if one carefully considers all the contact interactions that arise in the NR limit and requires that the four-point effective NR coupling is dominated by the self-interactions. In particular, the dilatonic coupling generates an s-channel contribution that is fundamental to prove the validity of the D dimensional bound. A particular feature we have found is that the dilaton mediated s-channel has a sign opposite to the graviton mediated s-channel. The constraint in D dimensions is recovered thanks to a cancellation between gravitational and dilatonic contributions, rather than a sum. This is the opposite of what happens with long range forces. It was also shown that the addition of a non minimal-coupling to

gravity $\hat{\Phi}^2 R$ introduces new terms in the Scalar WGC that are well-behaved under the compactification process. In D dimensions, (3.2.5) is modified to

$$\left| \frac{5}{3} \frac{\mu^2}{m^2} - \lambda \right| \geq \left(\frac{d-1}{d-2} + 4\xi \frac{d+1}{d-2} + 16\xi^2 \frac{d-1}{d-2} \right) \kappa^2 m^2, \quad (4.2.4)$$

where ξ is the coupling constant associated to the non-minimal interaction.

As was already discussed a few times above, the circle compactification of a gauge theory leads to a reshuffling of the degrees of freedom such that a higher dimensional gauge field decomposes into a lower dimensional gauge field and a bunch of scalars. In particular, for the $5D \rightarrow 4D$ S^1 compactification, the internal component \hat{A}_5 of a gauge field \hat{A}_M behaves in the effective 4-dimensional theory as a scalar field. This is explicitly shown in the paper [4]. The identification of internal components of gauge fields as lower dimensional scalars is a well-known property that, in full generality, is referred to as gauge-Higgs unification or Hosotani mechanism [60, 61]. In this respect, the D -dimensional one-loop effective potential for the Wilson line of \hat{A}_{D+1} 's zero mode was calculated in [62]. This result is part of a vast literature on the subject of radiative corrections in KK theories. Although the claim of a UV-insensitiveness of the effective potential in these models has generated a strong debate (see for example [63, 64, 65, 66]), this is of no concern for the result of [62], as it can be shown that even when the UV sensitivity appears the potential can be renormalized¹ [7], so that the result for the renormalized effective potential is still that of [62] (see equation 99 in the paper [4]). Application of our Scalar WGC bound to such an effective potential reveals that the gravitational subdominance can be realized for the $h_0 \equiv A_5^{(0\text{-mode})}$ self interactions only if the compactification radius L is

$$L^2 \gtrsim \frac{\hat{\kappa}^2}{\hat{g}^2}, \quad (4.2.5)$$

where $\hat{\kappa}$ and \hat{g} are the gravitational and gauge couplings in 5 dimensions. This has a two-fold consequence. For perturbative values of \hat{g} , $\hat{g} \sim \mathcal{O}(10^{-2} - 10^{-1})$, it tells us that the compactification radius should be bigger than the Planck length, which makes sense if any of the KK states should appear in the low-energy EFT. The other information it encodes is that, in the limit $\hat{g} \rightarrow 0$, h_0 self-interactions can only be made subdominant for an extremely large radius of compactification, which disrupts the picture of the compactified theory as the extra dimensions are now on the same ground as the 4 non-compact ones. This might be seen as another obstacle to the limit of vanishing gauge couplings.

Below, we present an almost completed version of [4]. The work is complete in its results, and will be soon sent to a journal after some refinements on its presentation.

¹ This is opposed to other similar cases [63, 64], discussed in the same period, where the possible UV sensitivity is not polynomial in the field.

Newton versus Coulomb for Kaluza-Klein modes

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Abstract

We consider a set of elementary compactifications of $D + 1$ to D spacetime dimensions on a circle: first for pure general relativity, then in the presence of a scalar field, first free then with a non minimal coupling to the Ricci scalar, and finally in the presence of gauge bosons. We compute the tree-level amplitudes in order to compare some gravitational and non-gravitational amplitudes. This allows us to recover the known constraints of the $U(1)$, dilatonic and scalar Weak Gravity Conjectures in some cases, and to show the interplay of the different interactions. We study the KK modes pair-production in different dimensions. We also discuss the contribution to some of these amplitudes of the non-minimal coupling in higher dimensions for scalar fields to the Ricci scalar.

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1 Introduction

Among the Swampland conjectures [1], one of the most popular and best tested is probably the Weak Gravity Conjecture (WGC). Its simplest formulation [2] considers the case of a D -dimensional $U(1)$ gauge theory, with a coupling constant g , and requires the existence of at least one state of mass m and charge q which satisfies:

$$gq \geq \sqrt{\frac{D-3}{D-2}} \kappa_D m, \quad (1.1)$$

where κ_D is defined as $\kappa_D^2 = 8\pi G_D = \frac{1}{M_{P,D}^{D-2}}$ with $M_{P,D}$ the reduced Planck mass in D dimensions. This inequality implies, among others, that in the non-relativistic limit, the Newton force is not stronger than the Coulomb force. The particular states for which the equality in (1.1) is satisfied are said to saturate the WGC. In this work we will be interested in a particular case of them.

The present work is dedicated to the study of two different generalizations of the WGC: one that arises when the gauge interaction is complemented by a dilaton interaction [3, 4], and another [5–7] that broadly requires the dominance of scalar interactions with respect to gravity in some scattering processes depending on the specific theory. We are interested in the modes that propagate in an extra dimension forming a tower of KK excitations [8–11]. We will explicitly show that these modes undergo gravitational and non-gravitational interactions of equal intensity, which allows us to use them as probes for the conjectured inequalities generalizing the one mentioned above. They will also be useful to investigate the behavior of the scalar WGC under compactification.

Obviously, the KK excitations considered here saturate the inequalities conjectured only at the classical level, to which our study will be limited, since both terms of these inequalities are in general corrected by quantum effects. However, one has in mind that extending the theory with enough supersymmetries, the KK modes can be BPS states which saturate them even at the quantum level.

The fact that KK modes saturate the inequalities of the various conjectures is a known property, but we will give a derivation of it here in a simple form that we have not found in the existing literature. Our derivation of the various inequalities will be based on amplitude calculations, not for example on the conditions for decay of extremal black holes, and some of the explicit expressions for the amplitudes needed to make the comparisons seem to be either missing or scattered and hard to find, so we hope that presenting them altogether here might be useful.

This work is organized as follows. Section 2 reviews the well-known reduction of KK from $D + 1$ to D dimensions of the Hilbert-Einstein action and a massless scalar. It allows us to introduce our notations, presents the Lagrangian expansion needed to extract the Feynmann rules for calculating amplitudes, and compute the numerical factor in the total derivative term, often misquoted in the literature, which will be useful in Section 5. The dilatonic WGC inequality is derived in Section 3, where we also calculate various KK pair production amplitudes. In Section 4, we consider adding a mass term for the scalar in $D + 1$ dimensions and we find our form of the scalar WGC. A non-minimal coupling to gravity is considered in section 5. The interactions due to the presence of higher dimensional gauge fields are discussed in section 6. Our conclusions are presented in section 7. Finally, some technical details about our calculations are gathered in appendices.

2 Expansion to Second Order in the Gravitational Field

We work with the signature $(+, -, \dots, -)$. The $D + 1$ dimensional quantities will be denoted with a hat. We use Latin and Greek letters for the $D+1$ and D -dimensional coordinates, respectively. We denote by x the D non-compact and by $z \equiv z + 2\pi L$ the compact coordinates. We recall the steps of the simple dimensional reduction of a free real massless scalar field $\hat{\Phi}$ coupled to General Relativity:

$$\mathcal{S}^{(D+1)} = \mathcal{S}_{EH}^{(D+1)} + \mathcal{S}_{\Phi,0}^{(D+1)}, \quad (2.1)$$

where

$$\mathcal{S}_{EH}^{(D+1)} = \frac{1}{2\hat{\kappa}^2} \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \hat{R}, \quad (2.2)$$

and

$$\mathcal{S}_{\Phi,0}^{(D+1)} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \frac{1}{2} \hat{g}^{MN} \partial_M \hat{\Phi} \partial_N \hat{\Phi} \quad (2.3)$$

The Ricci scalar \hat{R} is computed from the metric \hat{g}_{MN} . In the simplest compactification from $D+1$ to D dimensions it takes the form

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi}g_{\mu\nu} - e^{2\beta\phi}A_\mu A_\nu & e^{2\beta\phi}A_\mu \\ e^{2\beta\phi}A_\nu & -e^{2\beta\phi} \end{pmatrix} \quad (2.4)$$

with ϕ , A_μ and $g_{\mu\nu}$ D -dimensional fields independent of the z coordinate:

$$\mathcal{S}_{EH}^{(D+1)} = \frac{1}{2\hat{\kappa}^2} \int d^{D+1}x \sqrt{(-1)^{D-1}g} e^{((D-2)\alpha+\beta)\phi} \left\{ R - [2(1-D)\alpha - 2\beta]\square\phi - [(D-2)(1-D)\alpha^2 + 2\beta((2-D)\alpha - \beta)](\partial\phi)^2 - \frac{1}{4}e^{2(\beta-\alpha)\phi}F^2 \right\}. \quad (2.5)$$

where g is the determinant of the D -dimensional metric. A canonical D -dimensional Einstein-Hilbert action is obtained for

$$(D-2)\alpha + \beta = 0. \quad (2.6)$$

and the canonical dilaton kinetic term fixes the constant α to be:

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}. \quad (2.7)$$

Since all fields are independent of z , we can perform the integration over this coordinate to obtain, keeping only the zero modes,¹

$$\mathcal{S}_{0,0}^{(D)} = \frac{2\pi L}{2\hat{\kappa}^2} \int d^Dx \sqrt{(-1)^{D-1}g} \left[R + 2\alpha\square\phi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{2(1-D)\alpha\phi}F^2 \right]. \quad (2.8)$$

We define the D -dimensional constant κ in terms of the $(D+1)$ -dimensional $\hat{\kappa}$ as

$$\frac{1}{\kappa^2} = \frac{2\pi L}{\hat{\kappa}^2} \implies M_P^{D-2} = 2\pi L \hat{M}_P^{D-1} \quad (2.9)$$

In (2.4), the ϕ and A_μ fields are dimensionless. Dimensional fields, that we denote $\tilde{\phi}$ and \tilde{A}_μ , can be written as

$$\tilde{\phi} = \frac{\phi}{\sqrt{2\kappa}}; \quad \tilde{A}_\mu = \frac{A_\mu}{\sqrt{2\kappa}} \quad (2.10)$$

The action of the D -dimensional gauge and scalar fields, denoted as the graviphoton and the dilaton, respectively, reads:

$$\mathcal{S}_{0,0}^{(D)} = \int d^Dx \sqrt{(-1)^{D-1}g} \left[\frac{R}{2\kappa^2} + 2\alpha\square\tilde{\phi} + \frac{1}{2}(\partial\tilde{\phi})^2 - \frac{1}{4}e^{2\sqrt{2}(1-D)\alpha\kappa\tilde{\phi}}\tilde{F}^2 \right]. \quad (2.11)$$

¹The factor in front of the D'Alembertian operator, 2α , corrects the expression sometimes found in the literature, $(D-3)\alpha$. As long as only minimal coupling to gravity is considered, the difference is harmless.

In the following, with the exception of section 5, the second term in (2.11), being a total derivative, will be discarded and, for notational simplicity, we remove the tilde in our notation.

For simplicity, we restrict to the simplest case where the field $\hat{\Phi}$ is periodic and single-valued on the compact dimension

$$\hat{\Phi}(x, z + 2\pi L) = \hat{\Phi}(x, z), \quad \hat{\Phi}(x, z) = \frac{1}{\sqrt{2\pi L}} \sum_{n=-\infty}^{+\infty} \varphi_n(x) e^{\frac{inz}{L}}, \quad (2.12)$$

which leads to

$$\begin{aligned} \mathcal{S} = \int d^D x \sqrt{(-1)^{D-1} g} & \left\{ \frac{R}{2\kappa^2} + \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}}\kappa\phi} F^2 + \frac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 \right. \\ & + \sum_{n=1}^{\infty} \left(\partial_\mu \varphi_n \partial^\mu \varphi_n^* - \frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi} \varphi_n \varphi_n^* \right) \\ & \left. + \sum_{n=1}^{\infty} \left(i\sqrt{2}\kappa \frac{n}{L} A^\mu (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) + 2\kappa^2 \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* \right) \right\}, \quad (2.13) \end{aligned}$$

where we have chosen in (2.7) the positive root for α . The complex scalars φ_n form the Kaluza-Klein (KK) tower and appear minimally coupled to the graviphoton. Around a generic background value ϕ_0 for the dilaton, the gauge coupling g is given by

$$g^2 = e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}. \quad (2.14)$$

For each KK mode, the mass and charge read

$$gq_n = \sqrt{2}\kappa \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \quad m_n = \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}. \quad (2.15)$$

This shows that they are related through

$$(gq_n)^2 = 2\kappa^2 m_n^2, \quad (2.16)$$

saturating the dilatonic WGC condition. This is expected as all the interactions unify to descend from the unique gravitational interaction of a free scalar field in higher dimensions. Useful for the rest of the manuscript is to derive this result proceeding instead with the expansion of the metric (2.4) to second order:

$$\hat{g}_{MN} = \hat{\zeta}_{MN} + 2\hat{\kappa}\hat{h}_{MN} + 4\hat{\kappa}^2\hat{f}_{MN} + o(\hat{\kappa}^3) \quad (2.17)$$

where:

$$\hat{\zeta}_{MN} = \begin{pmatrix} e^{2\sqrt{2}\alpha\hat{\kappa}\phi_0}\eta_{\mu\nu} & 0 \\ 0 & -e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \end{pmatrix}. \quad (2.18)$$

is the background metric and $\hat{\kappa}^2\hat{f}_{MN} \ll \hat{\kappa}\hat{h}_{MN} \ll 1$, for all M, N . We write the perturbation as

$$\begin{cases} \hat{g}_{MN} = \hat{\zeta}_{MN} + 2\hat{\kappa}\hat{h}_{MN} + 4\hat{\kappa}^2\hat{f}_{MN} + \mathcal{O}(\hat{\kappa}^3) \\ \hat{g}^{MN} = \hat{\zeta}^{MN} + 2\hat{\kappa}\hat{t}^{MN} + 4\hat{\kappa}^2\hat{l}^{MN} + \mathcal{O}(\hat{\kappa}^3). \end{cases} \quad (2.19)$$

The relation $\hat{g}_{MP}\hat{g}^{PN} \equiv \delta_M^N$ reads

$$\begin{cases} \hat{t}^{MN} = -\hat{h}^{MN} \\ \hat{l}^{MN} + \hat{f}^{MN} = \hat{h}_P^M \hat{h}^{PN}, \end{cases} \quad (2.20)$$

where it is understood that the indices are raised and lowered with the background metric $\hat{\zeta}$, then

$$\begin{aligned} \sqrt{(-1)^D} \hat{g} \mathcal{L}_\Phi = & \sqrt{(-1)^D} \hat{\zeta} \left[\frac{1}{2} \partial_M \hat{\Phi} \partial^M \hat{\Phi} - \frac{\hat{\kappa}'}{2} \hat{h}^{MN} \left(\partial_M \hat{\Phi} \partial_N \hat{\Phi} - \frac{1}{2} \hat{\zeta}_{MN} \partial_P \hat{\Phi} \partial^P \hat{\Phi} \right) \right. \\ & + \frac{\hat{\kappa}'^2}{2} \left(\hat{l}^{MN} - \frac{1}{2} \hat{h}^{MN} \hat{h}_P^P \right) \partial_M \hat{\Phi} \partial_N \hat{\Phi} \\ & \left. + \frac{\hat{\kappa}'^2}{4} \left(\hat{f}_P^P - \frac{1}{2} \hat{h}_{MP} \hat{h}^{PM} + \frac{1}{4} (\hat{h}_P^P)^2 \right) \partial_M \hat{\Phi} \partial^M \hat{\Phi} \right]. \end{aligned} \quad (2.21)$$

where $\hat{\kappa}' \equiv 2\hat{\kappa}$. With:

$$\hat{h}^{MN} = \frac{1}{\sqrt{2\pi L}} \begin{pmatrix} e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} (\sqrt{2}\alpha\phi \eta^{\mu\nu} + h^{\mu\nu}) & -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \frac{A^\mu}{\sqrt{2}} \\ -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \frac{A^\nu}{\sqrt{2}} & -e^{-2\sqrt{2}\beta\hat{\kappa}\phi_0} \sqrt{2}\beta\phi \end{pmatrix}, \quad (2.22)$$

and using $\sqrt{(-1)^D} \hat{\zeta} = e^{\sqrt{2}(D\alpha+\beta)\hat{\kappa}\phi_0}$, this leads to the coupling between the leading order fluctuations \hat{h}^{MN} of the metric and the stress-energy-momentum of the scalar field $\hat{T}_{MN}^{\hat{\Phi}}$:

$$\begin{aligned} \mathcal{L}_{int}^{(1)} = & -\hat{\kappa} \hat{h}^{MN} \hat{T}_{MN}^{\hat{\Phi}} = -\hat{\kappa} h^{\mu\nu} T_{\mu\nu}^{(\varphi_0, \varphi_n)} \\ & - i\sqrt{2}\hat{\kappa} A^\mu \sum_{n=1}^{\infty} \frac{n}{L} (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) - 2\sqrt{\frac{D-1}{D-2}} \hat{\kappa} e^{2\sqrt{\frac{D-1}{D-2}}\hat{\kappa}\phi_0} \phi \sum_{n=1}^{\infty} \frac{n^2}{L^2} \varphi_n \varphi_n^*. \end{aligned} \quad (2.23)$$

Next, we identify \hat{f}_{MN} from the metric decomposition at second order:

$$\hat{f}_{MN} = \frac{1}{2\pi L} \begin{pmatrix} e^{2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha^2 \phi^2 \eta_{\mu\nu} + \sqrt{2}\alpha\phi h_{\mu\nu} + f_{\mu\nu} \right) & -\frac{1}{2} e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} A_\mu A_\nu & e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta\phi A_\mu \\ e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta\phi A_\nu & & -e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta^2 \phi^2 \end{pmatrix}. \quad (2.24)$$

With this result, \hat{l}^{MN} in (2.21) is given by

$$\hat{l}^{MN} = \hat{h}^{MP} \hat{h}_P^N - \hat{f}^{MN} \quad (2.25)$$

Using (2.24) and (2.22) one obtains

$$\hat{l}^{MN} = \frac{1}{2\pi L} \begin{pmatrix} e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha^2 \phi^2 \eta^{\mu\nu} + \sqrt{2}\alpha\phi h^{\mu\nu} + l^{\mu\nu} \right) & -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha\phi A^\mu + \frac{1}{\sqrt{2}} h^{\mu\rho} A_\rho \right) \\ -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha\phi A^\nu + \frac{1}{\sqrt{2}} h^\nu_\rho A^\rho \right) & -e^{-2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta^2 \phi^2 + e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \frac{1}{2} A_\rho A^\rho \end{pmatrix}. \quad (2.26)$$

We define $J_{\mu,n} = (\varphi_n \partial_\mu \varphi_n^* - \varphi_n^* \partial_\mu \varphi_n)$, then the second order interaction in the Lagrangian is given by

$$\mathcal{L}_{int}^{(2)} = \frac{1}{2} \partial_\mu \varphi_0 \partial_\nu \varphi_0 \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} + \frac{1}{2} (D^2 \alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2) \phi^2 \right) \right] \quad (2.27)$$

$$\begin{aligned}
& + \frac{1}{2}((D-2)\alpha + \beta)\phi h_\rho^\rho \left[\eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h_\rho^\rho h^{\mu\nu} \right] \\
& + \sum_{n=1}^{\infty} \partial_\mu \varphi_n \partial_\nu \varphi_n^* \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} + \frac{1}{2}(D^2\alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2) \phi^2 \right. \right. \\
& \quad \left. \left. + \frac{1}{2}((D-2)\alpha + \beta)\phi h_\rho^\rho \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h_\rho^\rho h^{\mu\nu} \right] \\
& - \sum_{n=1}^{\infty} \frac{n^2}{L^2} |\varphi_n|^2 \left[-A^2 e^{\sqrt{2}(D\alpha-\beta)\kappa\phi_0} \left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} + \left(\frac{1}{2}(D\alpha + \beta) - \beta \right) \phi h_\rho^\rho \right. \right. \\
& \quad \left. \left. + \frac{1}{2}(D^2\alpha^2 + 2D\alpha\beta + \beta^2 - 4D\alpha\beta - 4\beta^2 + 4\beta^2)\phi^2 \right) \right] \\
& - \sum_{n=1}^{\infty} i \frac{n}{L} h^{\rho\sigma} A_\rho J_{\sigma,n} + i \frac{n}{L} A^\rho J_{\rho,n} \left(-\frac{h_\sigma^\sigma}{2} - ((D-2)\alpha + \beta)\phi \right)
\end{aligned}$$

This expression simplifies using the relation between β and α (2.6). In particular, the coefficients of ϕ^2 and ϕh_ρ^ρ vanish. One obtains

$$\begin{aligned}
\mathcal{L}_{int}^{(2)} &= \frac{1}{2} \partial_\mu \varphi_0 \partial_\nu \varphi_0 \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2} h_\rho^\rho h^{\mu\nu} \right] \\
&+ \sum_{n=1}^{\infty} \partial_\mu \varphi_n \partial_\nu \varphi_n^* \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2} h_\rho^\rho h^{\mu\nu} \right] \\
&- \sum_{n=1}^{\infty} \frac{n^2}{L^2} |\varphi_n|^2 e^{2\sqrt{2}(D-1)\alpha\kappa\phi_0} \left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \\
&- \sum_{n=1}^{\infty} \frac{n^2}{L^2} |\varphi_n|^2 \left[-A^2 + e^{2\sqrt{2}(D-1)\alpha\kappa\phi_0} (2(D-1)^2\alpha^2\phi^2 + (D-1)\alpha\phi h_\rho^\rho) \right] \\
&+ i \frac{n}{L} A^\rho \frac{h_\sigma^\sigma}{2} J_{\rho,n} - i \frac{n}{L} h^{\rho\sigma} A_\rho J_{\sigma,n}
\end{aligned} \tag{2.28}$$

which shows how the gauge invariance of the graviphoton is recovered in this expansion at second order in $\hat{\kappa}$ and exhibits the minimal coupling of the graviphoton to the tower of scalars in $\hat{\kappa}$.

3 Scattering Amplitudes and Weak Gravity Conjectures

In this section, we will compute diverse $2 \rightarrow 2$ amplitudes in the simple model defined above and compare two sets to be identified, one denoted as *gravitational* and the other as *non-gravitational* mediated interactions.

We expand the dilaton around its background value ϕ_0 as $\phi_0 + \phi$ in the action (2.13) to obtain:

$$\mathcal{S}_f = \int d^D x \sqrt{(-1)^{D-1} g} \left\{ \frac{R}{2\kappa^2} + \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \sum_{m=0}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}}\kappa \right)^m \frac{\phi^m}{m!} F^2 \right\}$$

$$\begin{aligned}
& + \frac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 + \sum_{n=1}^{\infty} \partial_\mu \varphi_n \partial^\mu \varphi_n^* \\
& - \sum_{n=1}^{\infty} \left(\frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}} \kappa \phi_0} \sum_{m=0}^{\infty} \left(2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \varphi_n \varphi_n^* \right) \\
& + \sum_{n=1}^{\infty} \left(i\sqrt{2} \kappa \frac{n}{L} A^\mu (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) + 2\kappa^2 \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* \right) \Big\} \quad (3.1)
\end{aligned}$$

where diverse interactions can be identified. For instance:

- 3 and 4-point vertices for minimally-coupled scalars to graviphotons appear in the last line. We can identify the KK electric charges

$$gq_n = \sqrt{2} \kappa \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}} \kappa \phi_0}. \quad (3.2)$$

- In the third line, the m -th term ($m \neq 0$) in the sum gives a $(2+m)$ -point interaction with m dilatons and two KK scalars with coupling

$$-i \left(2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}} \kappa \phi_0}. \quad (3.3)$$

- The m -th term in the sum in front of F^2 in the first line gives a coupling of m dilatons with two gauge fields

$$-i \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m (p_1 \cdot p_2 \eta_{\mu\nu} - p_{1\nu} p_{2\mu}). \quad (3.4)$$

Expansion of the metric around flat space-time $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$ gives the usual minimal couplings to gravity for both the matter fields (φ_0, φ_n) and the massless mediators (ϕ, A_μ).

3.1 The Dilatonic WGC

Consider the tree-level $2 \rightarrow 2$ scattering^{2,3} $\varphi_n(p_1)\varphi_n(p_2) \rightarrow \varphi_n(p_3)\varphi_n(p_4)$:

$$\begin{aligned}
i\mathcal{M} = & ig^2 q_n^2 \left(\frac{(p_1 + p_3) \cdot (p_2 + p_4)}{t} + \frac{(p_1 + p_4) \cdot (p_2 + p_3)}{u} \right) - 4i \frac{D-1}{D-2} \kappa^2 m_n^4 \left(\frac{1}{t} + \frac{1}{u} \right) \\
& - \frac{\kappa^2}{4} \left[(p_{1\mu} p_{3\nu} + p_{3\mu} p_{1\nu} - \eta_{\mu\nu} (p_1 \cdot p_3 - m_n^2)) \frac{i\mathcal{P}^{\mu\nu\alpha\beta}}{t} (p_{2\alpha} p_{4\beta} + p_{4\alpha} p_{2\beta} - \eta_{\alpha\beta} (p_2 \cdot p_4 - m_n^2)) \right]
\end{aligned}$$

²We adopt here this simple notation where $\varphi_n(p)$, or $|\varphi_n(p)\rangle$ should not be viewed as the field operator acting on the vacuum but to represent a one-particle state of momentum p .

³Here and throughout, s, t and u will denote the Mandelstam variables.

$$\left. \begin{aligned} & + (t, p_3, p_4) \leftrightarrow (u, p_4, p_3) \end{aligned} \right] \quad (3.5)$$

where \mathcal{P} is the usual massless spin-2 projector

$$\mathcal{P}^{\alpha\beta\rho\sigma} = \frac{\eta^{\alpha\rho}\eta^{\beta\sigma} + \eta^{\alpha\sigma}\eta^{\beta\rho}}{2} - \frac{\eta^{\alpha\beta}\eta^{\rho\sigma}}{D-2} \quad (3.6)$$

and we have separated the contributions from the exchanges of the gauge boson, the dilaton and the graviton, respectively.

Taking the non-relativistic (NR) limit

$$\frac{s - 4m_n^2}{m_n^2} \rightarrow 0, \quad \frac{t}{m_n^2} \rightarrow 0, \quad \text{and} \quad \frac{u}{m_n^2} \rightarrow 0 \quad (3.7)$$

and expressing the charge in terms of the mass we obtain

$$i\mathcal{M} \rightarrow i\mathcal{M}_{NR} = 4im_n^2 \left[g^2 q_n^2 - \kappa^2 m_n^2 \left(\frac{D-1}{D-2} + \frac{D-3}{D-2} \right) \right] \left(\frac{1}{t} + \frac{1}{u} \right) = 0. \quad (3.8)$$

The relation between the charge and the mass (2.16) ensures the cancellation between the three forces. It is straightforward to generalize this to see that dominance of the gauge interaction requires that a state with charge q and mass m satisfying the relation

$$g^2 q^2 \geq \left(\frac{\alpha^2}{2} + \frac{D-3}{D-2} \right) \kappa^2 m^2, \quad (3.9)$$

where α is the dilatonic coupling of the form $e^{2\sqrt{2}\alpha\kappa\phi} F^2$, exists. We have therefore recovered in this explicit amplitude computation the Dilatonic Weak Gravity Conjecture that was derived in [3] (see also [4] for its generalization) from the study of the extremal Einstein-Maxwell-dilaton black hole solutions. In the absence of the massless dilaton field $\alpha = 0$, one trivially retrieves the original WGC condition

$$g^2 q^2 \geq \frac{D-3}{D-2} \kappa^2 m^2. \quad (3.10)$$

3.2 Amplitudes for Pair Production

Consider the production of a pair of matter states, here scalar KK states, of momenta p_3, p_4 from *massless* particles of momenta p_1, p_2 . We can split the production processes into two sets:

- Non-gravitational production: a pair of KK scalar modes $|\varphi_n, \varphi_n^*\rangle$ can arise from a pair of photons $\langle\gamma, \gamma|$, a pair of dilatons $\langle\phi, \phi|$, or a dilaton and a photon $\langle\phi, \gamma|$.
- Gravitational production: this includes the presence of a graviton G in initial states as $\langle G, G|$, $\langle G, \gamma|$ or $\langle G, \phi|$, but also gravitons as intermediate states in the production from $\langle\gamma, \gamma|$ or $\langle\phi, \phi|$. For later convenience, we further divide the gravitational production processes into purely gravitational (the $\langle G, G|$ production) and mixed (all the others).

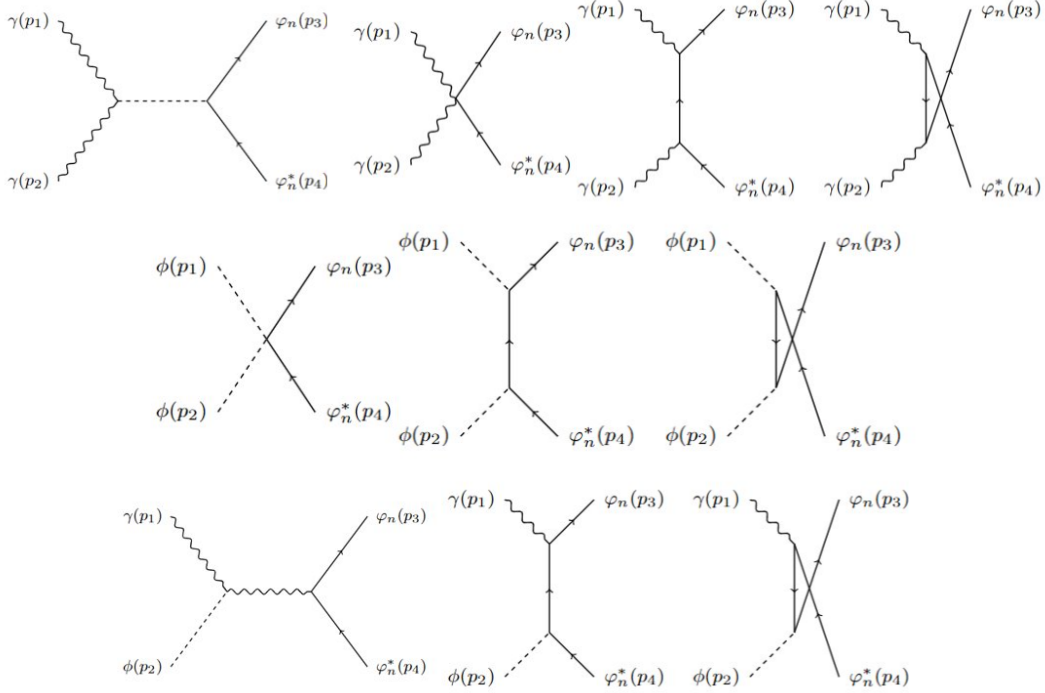


Figure 1: Feynman diagrams for the non-gravitational production of a pair of matter states φ_n, φ_n^* from two photons (first line), two dilatons (second line) and a dilaton and a photon (third line).

3.2.1 Non gravitational amplitudes

The production from photons $\gamma\gamma \rightarrow \varphi_n\varphi_n^*$ occurs through the coupling to the $U(1)$ gauge boson plus an s-channel term mediated by the dilaton, as depicted in the first line of figure 1. These give:

$$i\mathcal{M}_{\gamma\gamma} = ig^2 q_n^2 \epsilon_\mu(p_1)\epsilon_\nu(p_2) \left(\frac{(2p_3^\mu - p_1^\mu)(2p_4^\nu - p_2^\nu)}{t - m_n^2} + \frac{(2p_4^\mu - p_1^\mu)(2p_3^\nu - p_2^\nu)}{u - m_n^2} + 2\eta^{\mu\nu} \right) \quad (3.11)$$

$$- 2ig^2 q_n^2 \frac{D-1}{D-2} \epsilon_\mu(p_1)\epsilon_\nu(p_2) \frac{p_1 \cdot p_2 \eta^{\mu\nu} - p_1^\nu p_2^\mu}{s}.$$

We are interested in the threshold limit

$$\frac{s - 4m_n^2}{m_n^2} \rightarrow 0, \quad \frac{t + m_n^2}{m_n^2} \rightarrow 0, \quad \frac{u + m_n^2}{m_n^2} \rightarrow 0, \quad (3.12)$$

leading to

$$|\mathcal{M}_{\gamma\gamma}|^2 \xrightarrow{\text{Threshold}} \frac{4}{(D-2)^2} \left[(D-2) - \frac{3(D-1)^2}{4(D-2)} + \frac{D-1}{D-2} \right] g^4 q_n^4 = \left(\frac{D-3}{D-2} \right)^2 \frac{g^4 q_n^4}{D-2} \quad (3.13)$$

We note that, in a $U(1)$ gauge theory with no dilaton, the amplitude would be given by the first line of (3.11) only, that means in the threshold limit $4g^4 q_n^4 / (D-2)$ for a state of charge q .

The production from a dilation pair $\phi\phi \rightarrow \varphi_n\varphi_n^*$ (second line of figure 1) is immediately recognized to give a null result in the limit of interest:

$$i\mathcal{M}_{\phi\phi} = -4i\kappa^2 \frac{D-1}{D-2} m_n^4 \left(\frac{1}{t - m_n^2} + \frac{1}{u - m_n^2} \right) - 4i\kappa^2 \frac{D-1}{D-2} m_n^2 \xrightarrow{\text{Threshold}} 0. \quad (3.14)$$

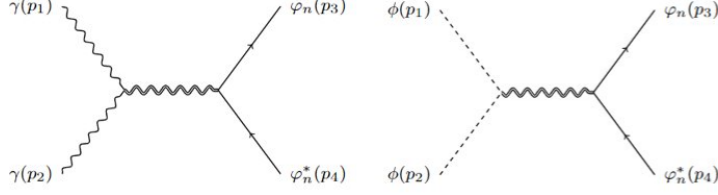


Figure 2: Feynman diagrams for pair production, gravitationally mediated, from photons and dilatons.

Finally, the production from the pair photon-dilaton $\phi\gamma \rightarrow \varphi_n\varphi_n^*$ receives contributions from the three s, t and u -channels (see the third line of figure 1)

$$i\mathcal{M}_{\gamma(p_1)\phi(p_2)} = \epsilon_\mu(p_1) \left\{ -2\sqrt{\frac{D-1}{D-2}} \kappa g q_n (p_1 \cdot (p_1 + p_2) g^{\mu\rho} - p_1^\rho (p_1 + p_2)^\mu) (p_3 - p_4)_\rho \frac{i}{s} \right. \\ \left. + 2i\sqrt{\frac{D-1}{D-2}} \kappa g q_n m_n^2 \left(\frac{(2p_3 - p_1)^\mu}{t - m_n^2} - \frac{(2p_4 - p_1)^\mu}{u - m_n^2} \right) \right\}, \quad (3.15)$$

and this is easily verified to give a null contribution in the threshold limit.

3.2.2 Mixed amplitudes

We consider now the “mixed gravitational” processes: we start by computing the graviton s -channel mediation for $\gamma\gamma$ and $\phi\phi$ initial states, then the amplitudes with initial states γG and ϕG . We present hereafter the results for the particular case $D = 4$. When it will be of interest, we will show the results for a generic number of dimensions D .

The additional contribution to the $\gamma\gamma$ and $\phi\phi$ productions described in figure 2 respectively read

$$i\mathcal{M}_{\gamma\gamma}^G = -\kappa^2 \left\{ (p_1 \cdot p_2)(\epsilon_{1\alpha}\epsilon_{2\beta} + \epsilon_{1\beta}\epsilon_{2\alpha}) + (p_{1\alpha}p_{2\beta} + p_{1\beta}p_{2\alpha})(\epsilon_1 \cdot \epsilon_2) - (\epsilon_{1\alpha}p_{2\beta} + p_{2\alpha}\epsilon_{1\beta})(p_1 \cdot \epsilon_2) \right. \\ \left. - (\epsilon_{2\alpha}p_{1\beta} + p_{1\alpha}\epsilon_{2\beta})(p_2 \cdot \epsilon_1) - \eta_{\alpha\beta}(p_1 \cdot p_2 \epsilon_1 \cdot \epsilon_2 - \epsilon_1 \cdot p_2 \epsilon_2 \cdot p_1) \right\} \frac{i\mathcal{P}^{\alpha\beta\rho\sigma}}{s} \\ \left\{ p_{3\rho}p_{4\sigma} + p_{3\sigma}p_{4\rho} - \eta_{\rho\sigma}(p_3 \cdot p_4 + m^2) \right\}, \quad (3.16)$$

and

$$i\mathcal{M}_{\phi\phi}^G = -\kappa^2 \left\{ p_{1\alpha}p_{2\beta} + p_{1\beta}p_{2\alpha} - \eta_{\alpha\beta}p_1 \cdot p_2 \right\} \frac{i\mathcal{P}^{\alpha\beta\rho\sigma}}{s} \left\{ p_{3\rho}p_{4\sigma} + p_{3\sigma}p_{4\rho} - \eta_{\rho\sigma}(p_3 \cdot p_4 + m^2) \right\}, \quad (3.17)$$

where $\epsilon_i = \epsilon(p_i)$. For the $\gamma\gamma \rightarrow \varphi_n\varphi_n^*$ amplitude, a (simpler) way to compute this is through projecting onto a specific basis for the polarizations ϵ (see Appendix B).

Working in the center of mass frame for the massive particles, we obtain the different components of the graviton mediated $\gamma\gamma \rightarrow \varphi_n\varphi_n^*$ as follows

$$i\mathcal{M}_{+,+}^G = i\mathcal{M}_{-,-}^G = -i\frac{\kappa^2}{s} [tu - m_n^4 + (m_n^2 - u)^2 + su] = 0$$

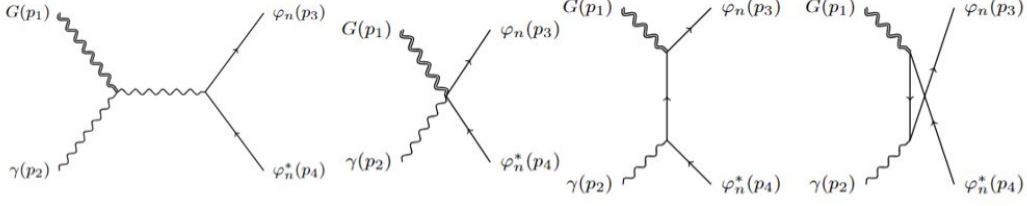


Figure 3: Feynman diagrams for the mixed pair production from a graviton and a photon.

$$i\mathcal{M}_{+,-}^G = i\mathcal{M}_{-,+}^G = i\frac{\kappa^2}{s} [tu - m_n^4], \quad (3.18)$$

where the \pm sign refers to the helicities of the incoming gauge bosons. In the threshold limit the graviton mediated contribution vanishes for both components.

In D dimensions, the whole $\mathcal{M}_{\gamma\gamma}$ amplitude reads

$$|\mathcal{M}_{\gamma\gamma}|^2 \xrightarrow{\text{Threshold}} \frac{(2(D-2)(gq)^2 + (D-4)\kappa^2 m^2)^2}{(D-2)^3} \quad (3.19)$$

for a generic $U(1)$ gauge theory (i.e. when the dilaton is put to zero) and

$$|\mathcal{M}_{\gamma\gamma}|^2 \xrightarrow{\text{Threshold}} \frac{((D-3)(gq_n)^2 + (D-4)\kappa^2 m_n^2)^2}{(D-2)^3} \quad (3.20)$$

in the dilatonic theory we are studying here. Both the results for the $U(1)$ and dilatonic theory ((3.13) and discussion below) are recovered in the limit $\kappa \rightarrow 0$. It is instructive to note, from these equations, that the vanishing of the graviton mediated contribution to the production from a photon pair is specific to the case of $D = 4$ dimensions, and in $D \neq 4$ dimensions mixed terms of the form $g^2 q^2 \times \kappa^2 m^2$ are generated.

For the $\phi\phi \rightarrow \varphi_n \varphi_n^*$ the amplitude reads

$$i\mathcal{M}_{\phi\phi}^G = -i\frac{\kappa^2}{s} [m_n^4 - ut - m_n^2 s]. \quad (3.21)$$

This results in a non vanishing contribution in the limit of interest such that

$$i\mathcal{M}_{\phi\phi}^G = i\kappa^2 m_n^2. \quad (3.22)$$

Concerning the mixed initial states, we have both $\gamma G \rightarrow \varphi_n \varphi_n^*$ (see figure 3) and $\phi G \rightarrow \varphi_n \varphi_n^*$ (see figure 4). Each of these two processes receive contributions from four diagrams.

Starting with the graviton-photon production, the amplitude $G(p_1)\gamma(p_2) \rightarrow \varphi_n \varphi_n^*$ takes the form

$$i\mathcal{M}_{G\gamma}^{\text{mix.}} = i\kappa g q_n \left(\frac{4(\epsilon_1 \cdot p_3)^2 \epsilon_2 \cdot p_4}{t - m_n^2} - \frac{4(\epsilon_1 \cdot p_4)^2 \epsilon_2 \cdot p_3}{u - m_n^2} + 2\epsilon_1 \cdot \epsilon_2 \epsilon_1 \cdot (p_3 - p_4) - \frac{(p_1 + p_2) \cdot p_2 (2\epsilon_1 \cdot \epsilon_2 \epsilon_1 \cdot (p_3 - p_4))}{s} \right) \quad (3.23)$$

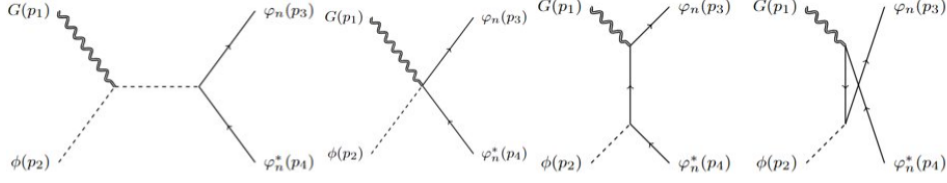


Figure 4: Feynman diagrams for the mixed pair production from a graviton and a dilaton.

and so for the different choices of graviton and photon helicities:

$$\begin{cases} i\mathcal{M}_{++,+}^{\text{mix.}} = -i\mathcal{M}_{--,-}^{\text{mix.}} = -i\kappa g q_n \sqrt{2 \frac{tu - m_n^4}{s}} \left(\frac{m_n^4 - tu}{(t - m_n^2)(u - m_n^2)} + 3 \right) \\ i\mathcal{M}_{+,-,+}^{\text{mix.}} = -i\mathcal{M}_{-,-,+}^{\text{mix.}} = i\kappa g q_n \sqrt{2 \frac{tu - m_n^4}{s}} \left(\frac{m_n^4 - tu}{(t - m_n^2)(u - m_n^2)} \right). \end{cases} \quad (3.24)$$

It is immediately verified that all these contributions vanish in the threshold limit where $t \rightarrow -m_n^2$ and $u \rightarrow -m_n^2$.

The same vanishing limit at threshold holds for the mixed graviton-dilaton production, where the amplitude is

$$i\mathcal{M}_{G\phi}^{\text{mix.}} = -2i\kappa\mu_n \left(\frac{(\epsilon_1 \cdot p_3)^2}{t - m_n^2} + \frac{(\epsilon_1 \cdot p_4)^2}{u - m_n^2} \right) \quad (3.25)$$

with $\mu_n = \sqrt{6}\kappa m_n^2$ the three-point $\phi\varphi_n\varphi_n^*$ $D = 4$ coupling, and finally

$$i\mathcal{M}_{++}^{\text{mix.}} = i\mathcal{M}_{--}^{\text{mix.}} = i\kappa\mu_n \frac{tu - m_n^4}{(t - m_n^2)(u - m_n^2)}. \quad (3.26)$$

From the explicit results presented in Appendix B, it is also immediate to realize that the mixed contributions vanish at threshold for all D .

3.2.3 Gravitational production amplitudes

Finally, we discuss the purely gravitational production. The starting point for the expression of the amplitude is rather long. It receives in fact contribution from the four diagrams of figure 5, each one with vertices determined from a two-derivative interacting term (some details about two-derivative interactions are discussed in Appendix A). We prefer to give here a more compact expression that is obtained after some algebra:

$$\begin{aligned} i\mathcal{M}_{GG} = \frac{\kappa^2}{2} \left(-\frac{8(p_3 \cdot \epsilon_1)^2 (p_4 \cdot \epsilon_2)^2}{t - m_n^2} - \frac{8(p_3 \cdot \epsilon_2)^2 (p_4 \cdot \epsilon_1)^2}{u - m_n^2} \right. \\ \left. - 2 \frac{(\epsilon_1 \cdot \epsilon_2)^2 (m_n^4 - tu - sm_n^2)}{s} - 4\epsilon_1 \cdot \epsilon_2 (p_3 \cdot \epsilon_2 p_4 \cdot \epsilon_1 + p_3 \cdot \epsilon_1 p_4 \cdot \epsilon_2) \right) \quad (3.27) \end{aligned}$$

The complete results for each one of the four diagrams contributing to the amplitude are presented in Appendix B, together with the description of the helicity method. Using now the specific basis for

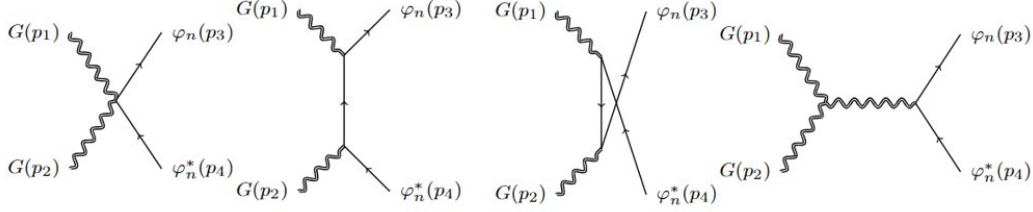


Figure 5: Feynman diagrams for the production of a pair of matter states from two gravitons.

$D = 4$ dimensions, we find

$$\begin{aligned}
i\mathcal{M}_{++,++} &= i\mathcal{M}_{--,--} = i\kappa^2 \left(\frac{(m_n^4 - tu) m^2}{(t - m_n^2)(u - m_n^2)} + m_n^2 \right) \\
i\mathcal{M}_{++,--} &= i\mathcal{M}_{--,++} = i\kappa^2 \frac{(m_n^4 - tu)^2}{s(t - m_n^2)(u - m_n^2)},
\end{aligned} \tag{3.28}$$

Comparing this result with the one obtained from the $\gamma\gamma$ production in the case with no dilaton, we verify the factorization

$$\begin{aligned}
\mathcal{M}_{++,++}^{(GG)} &= \frac{\kappa^2}{4(gq)^4} \frac{(t - m_n^2)(u - m_n^2)}{s} \mathcal{M}_{+,+}^{(\gamma\gamma)} \\
\mathcal{M}_{++,--}^{(GG)} &= \frac{\kappa^2}{4(gq)^4} \frac{(t - m_n^2)(u - m_n^2)}{s} \mathcal{M}_{+,-}^{(\gamma\gamma)}.
\end{aligned} \tag{3.29}$$

The corresponding factorization for the comparison between the gravitational Compton scattering $G\varphi \rightarrow G\varphi$ (with φ a generic scalar field) and the usual Compton scattering was found in [12, 13] (see also [14]).

From the above results, in the threshold limit we have

$$|\mathcal{M}_{GG}|^2 = \frac{1}{4} \left(|\mathcal{M}_{++,++}|^2 + |\mathcal{M}_{++,--}|^2 + |\mathcal{M}_{--,++}|^2 + |\mathcal{M}_{--,--}|^2 \right) \rightarrow \frac{\kappa^4 m_n^4}{2}. \tag{3.30}$$

Note that the result $|\mathcal{M}_{GG}|^2 \rightarrow \kappa^4 m^4/2$, and more generally the "purely gravitational" pair production, is independent from the presence of the dilaton. This is easily generalized to the case of generic D (see again Appendix B for details) and leads in the threshold limit to

$$|\mathcal{M}_{GG}|^2 \rightarrow \frac{1}{D-2} \kappa^4 m_n^4. \tag{3.31}$$

3.2.4 Gravitational vs gauge amplitudes

When the dilaton is put to zero, the requirement

$$|\mathcal{M}_{\gamma\gamma}|^2 \underset{\text{Threshold}}{\geq} |\mathcal{M}_{GG}|^2 \tag{3.32}$$

gives the original $U(1)$, $D = 4$ WGC bound $\sqrt{2}gq \geq \kappa m$.

Using cross-symmetry on the results of [12–14], the authors of [7] observed that (3.32) leads to the WGC relation and proposed (3.32) as a possible alternative formulation of the WGC. In [7], the graviton-mediated diagram was not taken into account in the γ amplitude. Our calculation shows that in the threshold limit, the contribution of this additional diagram disappears. Therefore, in the four-dimensional $U(1)$ gauge theory, we can safely compare, as in the (3.32), the $\gamma\gamma$ and GG productions without having to neglect any contribution.

Our calculation also shows that in $D = 4$ dimensions, the KK states saturate (3.32). In fact, we emphasize again that the gravitational amplitude \mathcal{M}_{GG} , here, does not care about the presence of the dilaton: whether the theory is a simple $U(1)$ or a dilatonic $U(1)$, the result for \mathcal{M}_{GG} is unchanged. On the other hand, the amplitude $\mathcal{M}_{\gamma\gamma}$ receives an additional contribution which changes the numerical coefficient in front of $g^4 q^4$ from 2 to $1/8$. Since the $\mathcal{M}_{\gamma\phi}$ and $\mathcal{M}_{\phi\phi}$ amplitudes both vanish in the threshold limit, the comparison of the pair production processes in this KK theory leads to

$$\frac{g^4 q_n^4}{8} \geq \frac{\kappa^4 m_n^4}{2} \implies gq \geq \sqrt{2}\kappa m \quad (3.33)$$

and (2.16) shows that KK states saturate it.

However, if, in the presence of the dilaton, we consider gravitationally mediated diagrams for $\gamma\gamma$ and $\phi\phi$ amplitudes, there is a non-vanishing contribution that comes from $\mathcal{M}_{\phi\phi}^G$ in (3.22), and this would clearly spoil the saturation observed for the KK states. The inclusion of the mixed production channels $G\gamma$ (3.23) and $G\phi$ (3.25) cannot restore the saturation property, since both do not contribute in the limit of interest. The dilatonic WGC will be recovered only if the contributions from graviton exchanges in $\gamma\gamma$ and $\phi\phi$ amplitudes are not included.

Note also that the pairwise production comparison does not reproduce the constraints of WGCs in more than 4 dimensions. The $\mathcal{M}_{\gamma\gamma}$ and \mathcal{M}_{GG} amplitudes lead, for any D , to compare $\sqrt{2}gq$ and κm . For the case of a simple theory $U(1)$, setting as quoted above the dilaton to zero in our calculations, the result for the production from a photon pair in D dimensions in the threshold limit is

$$|\mathcal{M}_{\gamma\gamma}|^2 = \frac{4}{D-2}(gq)^4. \quad (3.34)$$

In Appendix B we learn that the purely gravitational production of pairs gives, in the same limit of interest,

$$|\mathcal{M}_{GG}|^2 = \frac{1}{D-2}(\kappa m)^4. \quad (3.35)$$

By comparing (3.34) and (3.35), it is immediate to observe that requiring $|\mathcal{M}_{\gamma\gamma}|^2 \geq |\mathcal{M}_{GG}|^2$, one *does not reproduce* the WGC bound

$$gq \geq \sqrt{\frac{D-3}{D-2}}\kappa m. \quad (3.36)$$

Similarly, the comparison of purely gravitational pair production and purely non-gravitational pair production in the KK theory we consider here amounts to a comparison of the results

$$|\mathcal{M}_{\gamma\gamma}|^2 \rightarrow \frac{(D-3)^2}{(D-2)^3}g^4 q_n^4, \quad |\mathcal{M}_{\phi\phi}|^2 \rightarrow 0, \quad |\mathcal{M}_{GG}|^2 \rightarrow \frac{1}{D-2}\kappa^4 m_n^4. \quad (3.37)$$

Using (2.16), it is immediate to realize that the KK states saturate the (3.32) (or an equivalent generalization of it to include the $\mathcal{M}_{\phi\phi}$ contribution which disappears here) only for $D = 4$. The results of section 3.2.2 show that the addition of mixed contributions does not change this.

4 Massive and Self-interacting Scalars

We next consider the presence of mass and self-interacting terms in the higher dimensional scalar theory. The KK scalar modes are no more extremal states of the WGC, but this set-up will allow us to retrieve Scalar Weak Gravity Conjectures which are postulated to constrain the relative strength of the additional terms.

We will consider the simple extension of (2.1)

$$\mathcal{S}_{int} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \left[-\frac{1}{2} \hat{m}^2 \hat{\Phi}^2 + \frac{\hat{\mu}}{3!} \hat{\Phi}^3 - \frac{\hat{\lambda}}{4!} \hat{\Phi}^4 \right]. \quad (4.1)$$

Here, \hat{m} has mass dimension one, $\hat{\mu}$ has dimension $3 - \frac{D+1}{2}$ and $\hat{\lambda}$ has dimension $4 - (D+1)$. Using the ansatz (2.12), it is straightforward to see that the action takes the form

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_f + \mathcal{S}_{int} \\ &= \int d^D x \sqrt{(-1)^{D-1} g} \left\{ \frac{R}{2\kappa^2} + \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}}\kappa\phi} F^2 + \frac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 - \frac{1}{2} e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi}} \hat{m}^2 \varphi_0^2 \right. \\ &\quad + \sum_{n=1}^{\infty} \partial_\mu \varphi_n \partial^\mu \varphi_n^* - \sum_{n=1}^{\infty} \left(e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi} \frac{n^2}{L^2} + e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi}} \hat{m}^2 \right) \varphi_n \varphi_n^* \\ &\quad + \sum_{n=1}^{\infty} \left[i\sqrt{2}\kappa \frac{n}{L} A^\mu (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) + 2\kappa^2 \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* \right] \\ &\quad + e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi} \left[\frac{\mu}{3!} \varphi_0^3 - \frac{\lambda}{4!} \varphi_0^4 + \mu \varphi_0 \sum_{n=1}^{\infty} \varphi_n \varphi_n^* - \frac{\lambda}{2} \varphi_0^2 \sum_{n=1}^{\infty} \varphi_n \varphi_n^* \right. \\ &\quad - \frac{\lambda}{2} \varphi_0 \sum_{m,n=1}^{\infty} (\varphi_m \varphi_m \varphi_{n+m}^* + \varphi_m^* \varphi_n^* \varphi_{n+m}) + \frac{\mu}{2} \sum_{n,m=1}^{\infty} (\varphi_n \varphi_m \varphi_{n+m}^* + \varphi_n^* \varphi_m^* \varphi_{n+m}) \\ &\quad - \frac{\lambda}{3!} \sum_{m,n,p=1}^{\infty} (\varphi_m \varphi_n \varphi_p \varphi_{m+n+p}^* + \varphi_m^* \varphi_n^* \varphi_p^* \varphi_{m+n+p}) \\ &\quad \left. - \frac{\lambda}{2} \sum_{n=1}^{\infty} \varphi_n \varphi_n^* \sum_{m=1}^{\infty} \varphi_m \varphi_m^* - \frac{\lambda}{4} \sum_{\substack{m,n,p=1 \\ m \neq p, n \neq p, m+n > p}}^{\infty} \varphi_m \varphi_n \varphi_p^* \varphi_{n+m-p}^* \right\}, \quad (4.2) \end{aligned}$$

where we have kept the notation compact, but, in our perturbative analysis, the dilaton will again be expanded around a background value ϕ_0 as above. The couplings constants μ and λ are defined, from their higher dimensional counterpart, as

$$\mu = \frac{\hat{\mu}}{\sqrt{2\pi L}}, \quad \lambda = \frac{\hat{\lambda}}{2\pi L}. \quad (4.3)$$

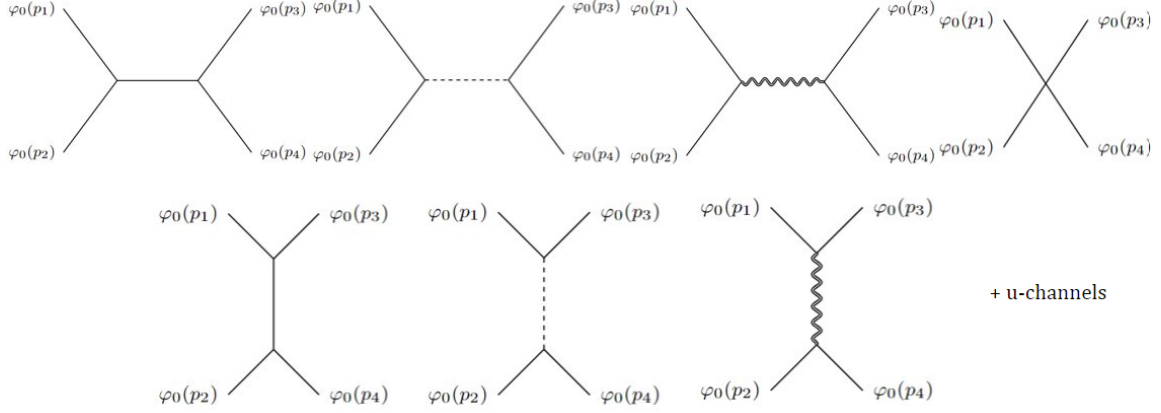


Figure 6: Feynman diagrams for the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ scattering when a potential for the higher dimensional scalar, “parent” of φ_0 , has been turned on.

The tree-level masses for the zero mode φ_0 and the KK excitations are given by:

$$m_0^2 = e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \hat{m}^2, \quad m_n^2 = e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \frac{n^2}{L^2} + e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \hat{m}^2. \quad (4.4)$$

4.1 The Scalar Weak Gravity Conjecture

We start by computing the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ amplitude. The diagrams intervening in the scattering are presented in the figure 6. The non-relativistic limit of the tree-level amplitude reads

$$\begin{aligned} i\mathcal{M} = & i e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \left[e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0} \frac{5}{3} \frac{\mu^2}{m_0^2}} - \lambda \right] \\ & - \frac{i}{(D-1)(D-2)} \kappa^2 m_0^2 - 4 \frac{i}{(D-1)(D-2)} \kappa^2 m_0^4 \left(\frac{1}{t} + \frac{1}{u} \right) \\ & + i \frac{D-1}{D-2} \kappa^2 m_0^2 - 4i \frac{D-3}{D-2} \kappa^2 m_0^4 \left(\frac{1}{t} + \frac{1}{u} \right), \end{aligned} \quad (4.5)$$

where the different lines correspond to the contributions from the self-interaction, dilaton and graviton exchanges, respectively.

Following [6], we compare the contributions to the amplitude at the energy scale given by the (massive) external states at rest. In the non-relativistic limit, we can further split (4.5) into contributions from short and long range interactions. We can identify an effective contact interaction:

$$\begin{aligned} i\mathcal{M}_{CT}^{(D)} = & i e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \left(\frac{5}{3} \frac{\mu^2}{\hat{m}^2} - \lambda - \frac{1}{(D-1)(D-2)} \kappa^2 \hat{m}^2 + \frac{D-1}{D-2} \kappa^2 \hat{m}^2 \right) \\ = & i \frac{e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}}}{2\pi L} \left(\frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \hat{\lambda} + 2\pi L \frac{D}{D-1} \kappa^2 \hat{m}^2 \right). \end{aligned} \quad (4.6)$$

where in the first line we can identify the contributions from the scalar interaction for the first two terms, then from the dilaton and graviton, respectively. Using (2.9) and the $(D+1)$ -gravitational

coupling $\hat{\kappa} = \sqrt{2\pi L} \kappa$, the last term is recognized to be the gravitational s-channel contribution to the $\hat{\Phi}\hat{\Phi} \rightarrow \hat{\Phi}\hat{\Phi}$ scattering in $D + 1$ dimensions:

$$i\mathcal{M}_{CT}^{(D+1)} = i \frac{e^{\frac{2}{\sqrt{(D-1)(D-2)}} \kappa \phi_0}}{2\pi L} \left(\frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \hat{\lambda} + \frac{(D+1)-1}{(D+1)-2} \hat{\kappa}^2 \hat{m}^2 \right). \quad (4.7)$$

The above equation illustrates the fact that constraining the scalar interactions of the field $\hat{\Phi}$ to be dominant with respect to gravity in $D + 1$ dimensions is enough to ensure that the scalar interactions of the zero mode φ_0 are dominant with respect to the combination of gravitational and dilatonic contributions in D dimensions. In other words, the effective (tree-level) non-relativistic four-point function of the zero mode φ_0 that emerges in the reduced-dimensional theory is the same as the effective non-relativistic four-point coupling for the "parent" field $\hat{\Phi}$ in the higher-dimensional theory. Requiring that in such a contact term, the contributions of the $\hat{\Phi}$ self-interactions are the dominant ones in the $D + 1$ dimensions automatically ensures that the same property holds for the φ_0 self-interactions with respect to the set of interactions that appear in the D dimensional theory.

It is interesting to observe that the higher dimensional result is recovered here thanks to a cancellation, rather than an addition, between the graviton and dilaton mediated diagrams. This is dictated by the form of the D -dependent coefficient $\gamma_s(D) \equiv (D-1)/(D-2)$ appearing in front of the graviton-mediated amplitude in the s -channel which decreases with D : $\gamma_s(D+1) < \gamma_s(D)$. The dimension-dependent factor appearing in the t and u -channels, $\gamma_{t,u}(D) \equiv (D-3)/(D-2)$ vary in the opposite direction. In other words, the peculiar feature is that, for the contact terms, the spin-2 and spin-0 bosonic mediators give opposite contributions. This feature will also appear in the amplitudes computed with the non minimal coupling to gravity. As a consequence of particular interest in the case of a massive dilaton the higher dimensional sub-dominance of gravity does not imply that gravity by itself (i.e. without the dilaton) is subdominant in the lower dimensional theory too. This violation happens in the parametric region

$$\frac{D}{D-1} \hat{\kappa}^2 \hat{m}^2 \leq \left| \frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \hat{\lambda} \right| \leq \frac{D-1}{D-2} \hat{\kappa}^2 \hat{m}^2, \quad (4.8)$$

which is an interval of length $\hat{\kappa}^2 \hat{m}^2 / (D-1)(D-2)$ inversely proportional to the dimension D .

The amplitude $\varphi_n \varphi_n \rightarrow \varphi_n \varphi_n$ provides a generalization in the presence of self-interacting terms of the computation done in section 3.1. The scattering amplitude receives contributions from gauge bosons, dilatons, gravitons in the t and u -channels, φ_0 exchange, from the s -channel exchange of a φ_{2n} particle and from a 4-point contact term. These are the diagrams that are presented in figure 7 and lead to

$$i\mathcal{M} = -ie^{\frac{4}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \mu^2 \left(\frac{1}{s - m_{2n}^2} + \frac{1}{t - m_0^2} + \frac{1}{u - m_0^2} \right) - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \\ + i \left(\frac{1}{t} + \frac{1}{u} \right) \left(4g^2 q_n^2 m_n^2 - 4 \frac{D-3}{D-2} \frac{m_n^4}{M_P^{D-2}} - (\partial_\phi m_n^2)^2 \right) \quad (4.9)$$

with

$$\partial_\phi m_n^2 = \frac{1}{M_P^{(D-2)/2}} \left(\frac{2}{\sqrt{(D-1)(D-2)}} e^{\frac{2}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \hat{m}^2 + 2\sqrt{\frac{D-1}{D-2}} e^{2\sqrt{\frac{D-1}{D-2}} \kappa \phi_0} \frac{n^2}{L^2} \right). \quad (4.10)$$

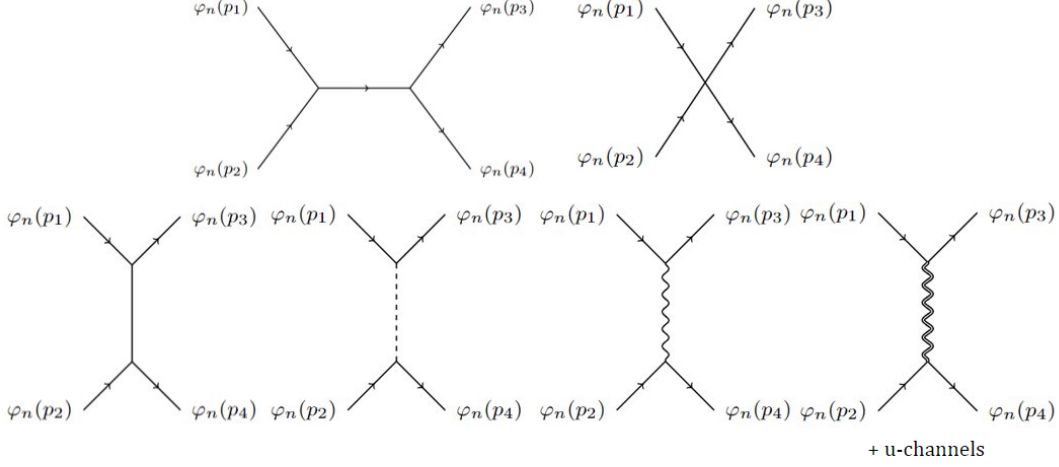


Figure 7: Feynman diagrams for the $\varphi_n \varphi_n \rightarrow \varphi_n \varphi_n$ scattering in the t -channel.

4.2 Massive dilatons

Let us consider for our illustrative discussion a simple potential for the dilaton in a polynomial expansion of the form

$$V(\phi) = \frac{1}{2} m_\phi^2 \phi^2 - \frac{\mu_\phi}{3!} \phi^3 + \frac{\lambda_\phi}{4!} \phi^4. \quad (4.11)$$

In the $\varphi_0 \varphi_0 \rightarrow \varphi_0 \varphi_0$ scattering amplitude (4.5), the addition of a dilaton mass gives in the non-relativistic limit

$$\begin{aligned} i\mathcal{M}(\varphi_0 \varphi_0 \rightarrow \varphi_0 \varphi_0) &= i e^{\frac{2}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \left[e^{\frac{2}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \frac{5}{3} \frac{\mu^2}{m_0^2} - \lambda \right] \\ &\quad - 4 \frac{i}{(D-1)(D-2)} \kappa^2 m_0^4 \frac{1}{s - m_\phi^2} - 4 \frac{i}{(D-1)(D-2)} \kappa^2 m_0^4 \left(\frac{1}{t - m_\phi^2} + \frac{1}{u - m_\phi^2} \right) \\ &\quad + i \frac{D-1}{D-2} \kappa^2 m_0^2 - 4i \frac{D-3}{D-2} \kappa^2 m_0^4 \left(\frac{1}{t} + \frac{1}{u} \right), \end{aligned} \quad (4.12)$$

where the limit still needs to be implemented in the dilaton propagators according to its mass. We can thus follow the evolution of \mathcal{M} with respect to m_ϕ to better expand it.

For the $\varphi_n \varphi_n \rightarrow \varphi_n \varphi_n$ case, the scattering amplitude with the massive dilaton reads

$$\begin{aligned} i\mathcal{M}(\varphi_n \varphi_n \rightarrow \varphi_n \varphi_n) &= -i e^{\frac{4}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \mu^2 \left(\frac{1}{s - m_{2n}^2} + \frac{1}{t - m_0^2} + \frac{1}{u - m_0^2} \right) - i \lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \\ &\quad - i (\partial_\phi m_n^2)^2 \left(\frac{1}{t - m_\phi^2} + \frac{1}{u - m_\phi^2} \right) + i \left(\frac{1}{t} + \frac{1}{u} \right) \left(4g^2 q_n^2 m_n^2 - 4 \frac{D-3}{D-2} \kappa^2 m_n^4 \right). \end{aligned} \quad (4.13)$$

Putting all the analysis for both the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ and $\varphi_n\varphi_n \rightarrow \varphi_n\varphi_n$ scattering amplitudes together, we give a brief overview of the results here.

When the mass m_ϕ of the dilaton is less than that of the zero mode, m_0 , its mass can be neglected to first order in an expansion, in powers of m_ϕ over the exchanged momentum, and requiring that the self-interactions of a scalar field dominate in $D + 1$ dimensions is sufficient to ensure that the same property is verified by its zero mode in D dimensions; a result that follows from the studies of the previous sections. As soon as the mass of the dilaton is comparable to that of the 0-mode, the massless dilaton approximation is no longer adequate and an appropriate discussion must be made for different denominators involving m_ϕ , m_0 , m_n and m_{2n} . The analysis can be done easily but it is cumbersome and not really illuminating. In short, there is no easy way to relate combinations appearing in D dimensions in this case with quantities already constrained, by assumption, in $D + 1$ dimensions.

5 $\hat{\Phi}^2 R$ interaction

Let us consider now the effect on the different D-dimensional amplitudes of the presence of a non-minimal coupling to gravity of the form

$$S_{(\xi)} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \frac{\xi}{2} \hat{\Phi}^2 \hat{R}, \quad (5.1)$$

with \hat{R} the Ricci scalar (see for example [15]). We assume here that $\langle \hat{\Phi} \rangle = 0$ as a non-vanishing vev would correspond to a redefinition of the Planck mass and a shift of the canonical fields. After compactification, one gets:

$$S_{(\xi)} = \int d^D x \sqrt{(-1)^{D-1} g} \left[\xi \left(R - \kappa^2 (\partial\phi)^2 - \frac{2\kappa}{\sqrt{(D-1)(D-2)}} \nabla_\mu \partial^\mu \phi - \frac{1}{2} e^{-2\sqrt{\frac{D-1}{D-2}} \kappa \phi} \kappa^2 F^2 \right) \times \left(\frac{\varphi_0^2}{2} + \sum_{n=1}^{\infty} \varphi_n \varphi_n^* \right) \right]. \quad (5.2)$$

This leads to new three-point couplings. First, using the linear expansion of the metric $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$, the R term gives the new coupling $\kappa(\partial_\mu \partial_\lambda h^{\mu\lambda} - \square h^\lambda_\lambda)$ ($\varphi_0^2 + 2 \sum_{n=1}^{\infty} \varphi_n \varphi_n^*$) of the graviton to the scalar matter fields. Then, the $\nabla_\mu \partial^\mu \phi$ term, that we discard in previous sections as it takes the form of a total derivative, gives an additional three-point vertex between the dilaton and the matter fields and can enter, for example, in the computation of the dilatonic force in the non-relativistic limit. At first order in κ , we can write $\kappa \nabla_\mu \partial^\mu \phi = \kappa \partial_\mu \partial^\mu \phi + \mathcal{O}(\kappa^2)$, the Christoffel symbols starting themselves at order κ .

The $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ amplitude resulting from the action (5.2), receives a contribution from the dilaton exchange (see Appendix A for some details on the Feynman rules for two-derivative vertices)

$$i\mathcal{M}_\phi = -i \frac{4}{(D-1)(D-2)} \xi^2 \kappa^2 (s+t+u) = -i \frac{16}{(D-1)(D-2)} \xi^2 \kappa^2 m_0^2. \quad (5.3)$$

and one from the graviton

$$i\mathcal{M}_G = 4i\frac{D-1}{D-2}\xi^2\kappa^2(s+t+u) = 16i\frac{D-1}{D-2}\xi^2\kappa^2m_0^2. \quad (5.4)$$

Their sum gives

$$i\mathcal{M}_{(non-minimal)} = i\mathcal{M}_\phi + i\mathcal{M}_G = 4i\frac{D}{D-1}\xi^2\kappa^2(s+t+u) = 16i\frac{D}{D-1}\xi^2\kappa^2m_0^2. \quad (5.5)$$

This matches the result one would obtain for the $\hat{\Phi}\hat{\Phi} \rightarrow \hat{\Phi}\hat{\Phi}$ scattering in $D+1$ dimensions.

At this point, we have computed tree-level four point amplitudes where both vertices arise either from minimal or non-minimal couplings to gravity in $D+1$ dimensions. In order to compute the total $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ amplitude we need to compute the contribution from “mixed” diagrams involving one minimal and one non-minimal vertices. This mixed gravitational diagrams give in the s -channel

$$i\mathcal{M}_{s\text{-channel}}^{G\text{-mix.}} = -2i\xi\frac{\kappa^2}{s}\left(2p_1 \cdot (p_1 + p_2)p_2 \cdot (p_1 + p_2) - (p_1 + p_2)^2(p_1 \cdot p_2 + m_0^2) + \frac{2}{D-2}(p_1 + p_2)^2(p_1 \cdot p_2) - \frac{D}{D-2}(p_1 + p_2)^2(p_1 \cdot p_2 + m_0^2)\right) \quad (5.6)$$

and in the t -channel

$$i\mathcal{M}_{t\text{-channel}}^{G\text{-mix.}} = 2i\xi\frac{\kappa^2}{t}\left(2p_1 \cdot (p_1 - p_3)p_3 \cdot (p_1 - p_3) - (p_1 - p_3)^2(p_1 \cdot p_3 - m_0^2) + \frac{2}{D-2}(p_1 - p_3)^2(p_1 \cdot p_3) - \frac{D}{D-2}(p_1 - p_3)^2(p_1 \cdot p_3 - m_0^2)\right), \quad (5.7)$$

while the u channel can be obtained through the replacements $t \leftrightarrow u$ and $p_3 \leftrightarrow p_4$. After some simple algebra, their sum reads

$$i\mathcal{M}_{s\text{-channel}}^{G\text{-mix.}} = i\xi\kappa^2\left(s + \frac{4m_0^2}{D-2}\right); i\mathcal{M}_{t\text{-channel}}^{G\text{-mix.}} = i\xi\kappa^2\left(t + \frac{4m_0^2}{D-2}\right); i\mathcal{M}_{u\text{-channel}}^{G\text{-mix.}} = i\xi\kappa^2\left(u + \frac{4m_0^2}{D-2}\right) \\ \implies i\mathcal{M}^{G\text{-mix.}} = i\xi\kappa^2\left(s + t + u + \frac{12}{D-2}m_0^2\right) = 4i\xi\kappa^2\frac{D+1}{D-2}m_0^2. \quad (5.8)$$

The computation of the similar mixed diagrams with dilaton exchange gives

$$i\mathcal{M}^{\phi\text{-mix.}} = -12i\xi\kappa^2\frac{m_0^2}{(D-1)(D-2)}, \quad (5.9)$$

where each channel contributes the same amount.

Summing up all the contributions, the final result for the amplitude is

$$i\mathcal{M}^{\text{mix.}} = 4i\xi\kappa^2\frac{D+2}{D-1}m_0^2, \quad (5.10)$$

as it is expected from the higher dimensional Lagrangian. Again, the higher dimensional gravitational contribution is obtained after a cancellation between the effective spin-2 and spin-0 mediators. From

the two results obtained above, we see that the direct non-minimal coupling to gravity (5.1) contributes with a constant term in the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ amplitude. If one takes the non-minimal coupling into account from the start and modifies the SWGC in D generic dimensions requiring

$$\left| \frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \lambda \right| \geq \left(\frac{D-1}{D-2} + 4\xi \frac{D+1}{D-2} + 16\xi^2 \frac{D-1}{D-2} \right) \hat{\kappa}^2 \hat{m}^2, \quad (5.11)$$

the same property will be respected by the zero mode φ_0 in $D-1$ dimensions with the replacement of hatted by unhatted quantities $\hat{\mu}^2, \dots \rightarrow \mu^2, \dots$.

In the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ scattering, the four point amplitudes appear as a sum of the three channels s, t, u whose coefficients add-up to a factor $s+t+u = 4m_0^2$. Therefore, the total amplitude does not increase with the exchanged momentum. This is not always the case as for example in the two examples of the $\varphi_n\varphi_n \rightarrow \varphi_n\varphi_n$ or $\varphi_n\varphi_n^* \rightarrow \varphi_n\varphi_n^*$ scattering amplitudes. The computation of the available channels, t and u in the first case, s and t in the second, proceeds as in the φ_0 case described above, but these contributions with two or one non minimal vertex do not close the sum $s+t+u$, as was the case in (5.4) and (5.8).

6 Higher dimensional gauge theory

So far, we have considered gravitational and scalar interactions in the higher dimensional theory. We will discuss now the case with gauge interactions. We consider a charged scalar $\hat{\Phi}$ of charge q and mass \hat{M} minimally coupled to a $U(1)$ gauge field \hat{B}_M with gauge coupling \hat{g} in $D+1$ dimensions

$$\mathcal{S}_{EH,\Phi,H}^{(D+1)} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \left\{ \frac{\hat{R}}{2\hat{\kappa}^2} + \hat{D}_M \hat{\Phi} \hat{D}^M \hat{\Phi}^* - \hat{M}^2 \hat{\Phi} \hat{\Phi}^* - \frac{1}{4} \hat{H}_{MN} \hat{H}^{MN} \right\}, \quad (6.1)$$

where \hat{H} is the field strength for the gauge field \hat{B} and \hat{D}_M the $D+1$ dimensional covariant derivative $\hat{D}_M \equiv \partial_M - i\hat{g}' q \hat{B}_M$, with \hat{g}' the gauge coupling. For simplicity, we choose the following periodicities for the fields

$$\begin{aligned} \hat{B}_M(x, z + 2\pi L) &= \hat{B}_M(x, z), & \hat{B}_M(x, z) &= \frac{1}{\sqrt{2\pi L}} \sum_{n=-\infty}^{+\infty} B_{(n)M}(x) e^{\frac{inz}{L}} \\ \hat{\Phi}(x, z + 2\pi L) &= e^{i2\pi q_\Phi} \hat{\Phi}(x, z), & \hat{\Phi}(x, z) &= \frac{1}{\sqrt{2\pi L}} \sum_{n=-\infty}^{+\infty} \varphi_n(x) e^{i(n+q_\Phi)\frac{z}{L}}, \end{aligned} \quad (6.2)$$

where q_Φ is a putative charge of $\hat{\Phi}$ under an internal symmetry. The compactification of the (kinetic term of the) gauge field gives the lagrangian

$$\begin{aligned} \mathcal{L}_H^{(D)} &= -e^{-2\alpha\phi} \left(\frac{H_0^2}{4} + \sum_{n=1}^{\infty} \frac{|H_{(n)}|^2}{2} \right) + e^{-2\beta\phi} \left(\frac{(\partial h_0)^2}{2} + \sum_{n=1}^{\infty} \left| \partial h_n - i \frac{n}{L} B_{(n)} \right|^2 \right) \\ &+ e^{-2\alpha\phi} A^\mu \left(-H_{(0)\mu\nu} \partial^\nu h_0 + \sum_{n=1}^{\infty} H_{(n)\mu\nu} \left(\partial^\nu h_n^* - i \frac{n}{L} B_{(n)}^{*\nu} \right) + H_{(n)\mu\nu}^* \left(\partial^\nu h_n - i \frac{n}{L} B_{(n)}^\nu \right) \right) \end{aligned}$$

$$\begin{aligned}
& + e^{-2\alpha\phi} \left[A^2 \left(\frac{(\partial h_0)^2}{2} + \sum_{n=1}^{\infty} \left| \partial h_n - i \frac{n}{L} B_{(n)} \right|^2 \right) \right. \\
& \quad \left. + A^\mu A^\nu \left(\partial_\mu h_0 \partial_\nu h_0 + 2 \sum_{n=1}^{\infty} \left(\partial_\mu h_n - i \frac{n}{L} B_{(n)\mu} \right) \left(\partial_\nu h_n - i \frac{n}{L} B_{(n)\nu} \right)^* \right) \right], \tag{6.3}
\end{aligned}$$

where $h_0 \equiv B_{(0)z}$ is a real scalar corresponding to the zero mode of the gauge field \hat{B}_M component along the compact dimension z and $h_n \equiv B_{(n)z}$ are the complex scalars forming the KK tower of the same field. From the above action, each field h_n is seen to generate a mass for the KK excitations $B_{(n)\mu}$ of the non-compact components of the gauge field, that are then complex massive vectors, and to behave as the Goldstones in the Higgs mechanism (or in a Stueckelberg mechanism). Note that the relations $B_{(-n)\mu} = B_{(n)\mu}^*$ and $h_{-n} = h_n^*$ are valid, although the same cannot be said for the Fourier modes of the complex field $\hat{\Phi}$.

The D -dimensional lagrangian obtained from the kinetic and mass term of the scalar field $\hat{\Phi}$ reads

$$\begin{aligned}
\mathcal{L}_\Phi^{(D)} = & \sum_{n=-\infty}^{+\infty} |D\varphi_n|^2 - \left(e^{2\alpha\phi} \hat{M}^2 + e^{-2(\beta-\alpha)\phi} \left[\frac{n+q\Phi}{L} - g'qh_0 \right]^2 \right) |\varphi_n|^2 \\
& + g'q \sum_{\substack{n,p=-\infty \\ n \neq 0}}^{+\infty} \left[i B_{(n)}^\mu (\varphi_p \partial_\mu \varphi_{n+p}^* - \partial_\mu \varphi_p \varphi_{n+p}^*) - 2g'q B_{(0)\mu} B_{(n)}^\mu \varphi_p \varphi_{n+p}^* \right. \\
& \quad \left. - g'q \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} B_{(n)\mu} B_{(m)}^\mu \varphi_p \varphi_{n+m+p}^* \right] \\
& + g'q A^\mu \left(\sum_{\substack{n,p=-\infty \\ n \neq 0}}^{+\infty} i h_n (\partial_\mu \varphi_p \varphi_{n+p}^* - \varphi_p \partial_\mu \varphi_{n+p}^*) - 2 \sum_{n,p=-\infty}^{+\infty} \frac{n+p+q\varphi}{L} B_{(n)\mu} \varphi_p \varphi_{n+p}^* \right. \\
& \quad \left. + 2g'qh_0 \sum_{n,p=-\infty}^{+\infty} B_{(n)\mu} \varphi_p \varphi_{n+p}^* + 2g'q \sum_{\substack{n,m,p=-\infty \\ m \neq 0}}^{+\infty} h_m B_{(n)\mu} \varphi_p \varphi_{n+p}^* \right) \\
& + \left(A^2 + e^{-2(\beta-\alpha)\phi} \right) \left(2g'q \sum_{\substack{n,p=-\infty \\ n \neq 0}}^{+\infty} \left[\frac{n+p+q\varphi}{L} - g'qh_0 \right] h_n \varphi_p \varphi_{n+p}^* \right. \\
& \quad \left. - g'^2 q^2 \sum_{\substack{n,m,p=-\infty \\ m \neq 0}}^{+\infty} h_n h_m \varphi_p \varphi_{n+m+p}^* \right), \tag{6.4}
\end{aligned}$$

where $g'q \equiv \hat{g}'q/\sqrt{2\pi L}$ and when acting on φ_n

$$D_\mu \equiv \partial_\mu - ig'q B_{(0)\mu} - ig \left[\left(\frac{n+q\Phi}{L} - g'qh_0 \right) \right] A_\mu, \tag{6.5}$$

from which one can read the charge under the graviphoton. The h_0 term in this expression is a manifestation of the Aharonov-Bohm effect for the Wilson line of B_z , $\oint_z B_z$.

Here we are interested in comparing the different gravitational and non-gravitational long range classical interactions, which can be obtained from the t -channel amplitudes. The t -channel contribution to the $\varphi_n(p_1)\varphi_n(p_2) \rightarrow \varphi_n(p_3)\varphi_n(p_4)$ scattering amplitude is

$$\begin{aligned}
i\mathcal{M}_n = & \frac{i}{t} \left(g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} + 2\kappa^2 \left(\frac{n+q_\Phi}{L} - g'q e^{-\sqrt{\frac{D-2}{D-1}}\kappa\phi_0} \bar{h}_0 \right)^2 e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \right) (p_1+p_3) \cdot (p_2+p_4) \\
& - \frac{i}{t} \left[4g'^2 q^2 \left(g'q\bar{h}_0 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} - \frac{n+q_\Phi}{L} e^{\frac{D}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \right)^2 \right. \\
& + \left(2\sqrt{\frac{D-1}{D-2}}\kappa \left(\frac{n+q_\Phi}{L} - g'q\bar{h}_0 e^{-\sqrt{\frac{D-2}{D-1}}\kappa\phi_0} \right)^2 e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \right. \\
& \left. \left. + \frac{2}{\sqrt{(D-1)(D-2)}\kappa\hat{M}^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \right)^2 \right], \tag{6.6}
\end{aligned}$$

where we have omitted writing the gravitational contribution, to avoid lengthy expressions, only to reinsert it in the next step when we perform the non-relativistic limit. The mass of the n th KK state can be read from the first line of the action in (6.4)

$$m_n^2 = \left(\frac{n+q_\Phi}{L} - g'q e^{-\sqrt{\frac{D-2}{D-1}}\kappa\phi_0} \bar{h}_0 \right)^2 e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} + \hat{M}^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \tag{6.7}$$

Let us first consider the simplest case where $q_\Phi = \bar{h}_0 = \hat{M} = 0$. In the non-relativistic limit, for $n \neq 0$, the coefficient of $\frac{1}{t}$ in the t -channel amplitude takes the form

$$\begin{aligned}
\mathcal{M}_n^{\text{t-pole}} = & \left(g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} + 2\kappa^2 m_n^2 \right) 4m_n^2 - 4g'^2 q^2 m_n^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \\
& - 4\frac{D-1}{D-2}\kappa^2 m_n^4 - 4\frac{D-3}{D-2}\kappa^2 m_n^4 \\
= & 0, \tag{6.8}
\end{aligned}$$

where m_n^2 in this case is simply $m_n^2 = e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} n^2/L^2$ and the gravitational scattering has been reinserted. The vanishing amplitude results from the (expected) two by two cancellation of interactions for the massive KK modes: namely gravitational vs dilatonic and D -dimensional gauge vs scalar from the $(D+1)$ -direction gauge field component. The $n = 0$ amplitude is different as the zero mode is massless with our specific choice. The non gravitational amplitude reads

$$i\mathcal{M}_0^{\text{relativistic}} = \frac{i}{t} g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} (p_1+p_3) \cdot (p_2+p_4). \tag{6.9}$$

Let us now consider the case $q_\Phi \neq 0$. The zero mode is massive

$$m_0^2 = e^{\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \frac{q_\Phi^2}{L^2}, \tag{6.10}$$

and the corresponding four-point amplitude is given by (again, we do not include here the gravitational contribution whose expression for generic exchanger momenta is long and not very illuminating)

$$\begin{aligned}
i\mathcal{M}_0 = & \frac{i}{t} \left(g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} + 2\kappa^2 \frac{q_\Phi^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}} \kappa \phi_0} \right) (p_1 + p_3) \cdot (p_2 + p_4) \\
& - \frac{i}{t} \left[4g'^2 q^2 \frac{q_\Phi^2}{L^2} e^{2\frac{D}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} + 4\frac{D-1}{D-2} \kappa^2 \frac{q_\Phi^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}} \kappa \phi_0} \right].
\end{aligned} \tag{6.11}$$

In the non-relativistic limit, the total amplitude obtained by adding the gravitational contribution to (6.11), cancels. The non-periodicity, which makes the zero mode massive, also generates couplings at h_0 and ϕ , whose exchanges cancel, respectively, the gauge and gravitational amplitudes of the zero mode. This is to be expected since integer values of q_Φ reshuffle the KK states; what was the zero mode becomes one of the massive modes for which we have seen that the total amplitude disappears. It is immediate to verify that the same is true for generic $n \neq 0$, \mathcal{M}_n remains null, and the same thing happens if one turns on \bar{h}_0 , as can be easily verified.

We can now study the general case. It is immediately verified that, after some algebra, in the non-relativistic limit the scattering amplitude (6.6) simplifies to

$$\begin{aligned}
i\mathcal{M}_{NR}^{(D)} = & 4ie^{\frac{4}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \hat{M}^2 \left(g'^2 q^2 - \frac{D-2}{D-1} \kappa^2 \hat{M}^2 \right) \\
= & 4ie^{\frac{4}{\sqrt{(D-1)(D-2)}} \kappa \phi_0} \frac{\hat{M}^2}{2\pi L} \left(\hat{g}'^2 q^2 - \frac{(D+1)-3}{(D+1)-2} \hat{\kappa}^2 \hat{M}^2 \right) \propto i\mathcal{M}_{NR}^{(D+1)}
\end{aligned} \tag{6.12}$$

where one recognizes in the combination inside the parenthesis the $D+1$ dimensional corresponding dependence. The q_Φ and \bar{h}_0 dependences cancel out to leave this simple expression only in terms of the higher dimensional mass and charge. We conclude that the requirement that the state in $D+1$ dimensions feels a repulsive long range force ensures that the KK modes in D dimensions also feel a repulsive long range force.

The mapping of the $D+1$ dimensional $U(1)$ WGC into the D dimensional form of the conjecture with gauge and scalar fields was discussed in [3] from the requirement of extremal black holes and black p-branes decays, leading to the establishment of the dilatonic WGC, and in [16] for the special case of a five to four dimensional circle compactification retaining only the zero modes. The analysis presented here generalizes, from the standpoint of scattering amplitudes, the connection between these different forms of the conjecture to the case with several gauge and scalar fields with reasonings involving the whole Kaluza-Klein tower.

6.1 Effective potential for h_0

Finally, we comment on the confrontation of the effective one-loop potential for the Wilson line with the scalar WGC of [6]. The potential is generated by the integration of the KK excitations⁴. In the

⁴We use here the results of the effective potentials investigated in details for example in [17] and at the one-loop level in a type I non-supersymmetric string model in [18].

case of a circle compactification from five to four dimensions, the potential takes the simple form

$$V_{\text{eff}}(h_0) = -\frac{3}{64\pi^6 L^4} \sum_{n=1}^{\infty} \frac{\cos(2\pi n g' q h_0 L)}{|n|^5} = -\frac{3 \left(\text{Li}_5 \left(e^{-2\pi i g' q h_0 L} \right) + \text{Li}_5 \left(e^{2\pi i g' q h_0 L} \right) \right)}{128\pi^6 L^4}, \quad (6.13)$$

where the symbols Li_n denote the usual Polylogarithm functions defined as

$$\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}. \quad (6.14)$$

For the Wilson line to satisfy the Scalar WGC inequality of [6] around a generic background value \bar{h}_0 (we indicate with η the excitations around it, $h_0 = \bar{h}_0 + \eta$), one then needs

$$L^2 \geq \frac{3\kappa^2}{2\pi^2 g'^2 q^2} \left[\frac{\text{Li}_3(e^{ix}) + \text{Li}_3(e^{-ix})}{\left| \frac{20}{9} \frac{(\text{Li}_2(e^{ix}) - \text{Li}_2(e^{-ix}))^2}{\text{Li}_3(e^{ix}) + \text{Li}_3(e^{-ix})} - \log(2 - 2\cos x) \right|} \right]. \quad (6.15)$$

where x is defined to be $x \equiv 2\pi g' q \bar{h}_0 L$, to be respected for $m_\eta^2 > 0$, while the inequality is trivially verified for $m_\eta^2 < 0$, but this case is of no interest. In the inequality (6.15), the factor inside the square parenthesis on the right hand side is periodic and reaches a maximal value around 0.6 – 0.7 in the regions of parameters where $m_\eta^2 > 0$. Taken to be approximately an order one, the gravitational sub-dominance is then realized around any background value \bar{h}_0 if ⁵

$$L^2 \geq \frac{3\hat{\kappa}^2}{2\pi^2 \hat{g}'^2 q^2} = \frac{3}{2\pi^2 g'^2 q^2} \frac{1}{M_P^2}, \quad (6.16)$$

which means that the compactification length cannot be parametrically smaller than the Planck's one as expected.

From (6.3) and (6.4), it is immediate to observe that the self-couplings induced by radiative corrections are not the only ones that can appear in the 4-point function $\eta\eta \rightarrow \eta\eta$. A first contribution may come from the kinetic term of h_0 , coupled to the dilaton as in (6.3). This gives a two derivative vertex that would then induce contributions to the four point function proportional to the scalar product of external momenta ($p_1 \cdot p_2 \times p_3 \cdot p_4$ in the s-channel, and so on). For the effective four point non relativistic coupling, this only accounts for a shift of the gravitational contribution, the second term in (6.16). In particular, the numerical coefficient 3/2 should be changed with 5 in (6.16) and all the subsequent inequalities.

7 Conclusions

An extra dimension for our space-time was originally introduced to unify gravity with electromagnetism: [8–11]. From the point of view of a lower dimensional observer, this unification makes the KK modes undergo attractive gravitational plus scalar interactions and repulsive electric interactions with the same intensity. This motivated the use of the KK states interactions in this work to extract

⁵Note that $\frac{\kappa^2}{g'^2} = \frac{\hat{\kappa}^2}{\hat{g}'^2}$, so we can express the bound either in terms of five- or four-dimensional quantities in the same form.

the form of the inequalities that appear when one is interested in comparing gravitational interactions to other types of interactions.

Taking into account the scalar interaction due to the presence of a dilaton, the calculation of four-point amplitudes allowed us to find the inequalities of the Dilatonic WGC. Our observations go further, with the extension of the construction to include interactions in the higher dimension, and we have shown how the Scalar WGC is found as well as the behavior of these conjectures under dimensional reduction. Meanwhile, we have also computed a number of scattering amplitudes for the pair production of KK states and have been able to compare the contributions of the different channels for spacetime dimensions $D \geq 4$.

A Lagrangians with derivative interactions

One subtlety that we wish to address here is related to the nature and the use of derivative interactions in perturbation theory. The perturbative expansion is an expansion of the exponential $e^{-i \int d^D x \mathcal{H}_I}$ in powers of \mathcal{H}_I , the interaction hamiltonian in the interaction picture. When the lagrangian presents derivative interactions, one should be careful to correctly construct \mathcal{H}_I before announcing the Feynman rules. Interactions containing more than one derivative of fields can generate new genuine additional Feynman rules [19]. The analog of this result was found, in the path integral formalism, in [20]. We illustrate this in two simple examples closely related to the cases studied.

A.1 Interactions with derivatives of a gauge field

We first present the case of the theory defined by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}} \kappa \phi} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu). \quad (\text{A.1})$$

We have singled out here only the part of interest to us to highlight the interaction between the dilaton ϕ and derivatives of the graviphoton A_μ . We will work in the usual radiation gauge $A_0 = 0$, $\vec{\nabla} \cdot \vec{A} = 0$. Computation of the canonical conjugate momenta give us

$$\begin{cases} \Pi_{A_0} = 0 \\ \Pi_{A_i} = - \left(1 + \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right) F^{0i} \\ \Pi_\phi = \partial_0 \phi. \end{cases} \quad (\text{A.2})$$

The fact that $\Pi_{A_0} = 0$ is, of course, what we should expect in a canonical formalism. The Heisenberg picture hamiltonian is obtained as

$$\begin{aligned} \mathcal{H} &= \Pi_{A_\mu} \partial_0 A_\mu + \Pi_\phi \partial_0 \phi - \mathcal{L} \\ &= -\frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \partial_0 \phi \partial_0 \phi + \frac{1}{2} \partial_i \phi \partial_i \phi - \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \left(F^{0i} F_{0i} - \frac{F_{\mu\nu} F^{\mu\nu}}{4} \right) \\ &= \frac{1}{2} \Pi_{A_i} \Pi_{A_i} + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \Pi_\phi \Pi_\phi + \frac{1}{2} \partial_i \phi \partial_i \phi + \frac{1}{4} \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} F^{\mu\nu} F_{\mu\nu} \end{aligned}$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} \left[\left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right]^2 F^{0i} F_{0i}. \quad (\text{A.3})$$

The transition to the interaction picture is done making the following replacements:

$$\begin{cases} \Pi_{A_i} \rightarrow -F^{0i} (= \Pi_{A_i, I}) \\ F_{ij} \rightarrow F_{ij} \\ F^{0i} \rightarrow F^{0i} \left(1 + \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right)^{-1} \\ \Pi_{\phi} \rightarrow \partial_0 \phi \\ \phi \rightarrow \phi \\ \partial_0 \phi \rightarrow \partial_0 \phi \end{cases} \quad (\text{A.4})$$

Some simple algebra finally get us to the interaction picture hamiltonian in the form

$$\begin{aligned} \mathcal{H} = & -\frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \partial_0 \phi \partial_0 \phi + \frac{1}{2} \partial_i \phi \partial_i \phi + \frac{1}{4} \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} F^{\mu\nu} F_{\mu\nu} \\ & - \frac{1}{2} \frac{\sum_{m=1}^{\infty} \left[\left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right]^2}{1 + \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!}} F^{0i} F_{0i}. \end{aligned} \quad (\text{A.5})$$

Careful construction of the interaction hamiltonian reveals the presence of an additional term to the naive expectation, to the extent that

$$\mathcal{H}_I = -\mathcal{L}_I - \frac{1}{2} \frac{\sum_{m=1}^{\infty} \left[\left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right]^2}{1 + \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!}} F^{0i} F_{0i}, \quad (\text{A.6})$$

with the new term sharing the same structure with the one found in the model of [19].

Combining this result with the two derivative propagator⁶

$$\langle \partial_{\mu} A_{\rho} \partial_{\nu} A_{\sigma} \rangle (q) = i\eta_{\rho\sigma} \frac{q_{\mu} q_{\nu}}{q^2 (+i\epsilon)} - i\eta_{\rho\sigma} \eta_{\mu 0} \eta_{\nu 0} \quad (\text{A.7})$$

we finally have the explicit form of the non standard Feynman rules we should consider in the minimally coupled (i.e. with $\xi = 0$) dimensionally reduced theory. The additional term consists in an infinite series in powers of $\kappa\phi$ starting at order 2 and defining a vertex with two gauge bosons. As such, it will not enter any of the computations we have performed, but certainly need to be considered, alongside with the propagator corrections, even at tree level, when looking at different physical processes, like $\phi\phi \rightarrow \gamma\gamma$ and $\phi\gamma \rightarrow \phi\gamma$ ones.

⁶Given here in the covariant gauge, to keep a simple notation.

A.2 Toy model for the two-derivative interaction of the non-minimal coupling

The second model we present here aims to capture the main properties of the new vertices brought in by the non-minimal coupling to gravity. We explicitly show, with the simplest toy model, that the different additional pieces due to such derivatives cancel each other, allowing the use of naive perturbation theory.

Let us take, for definiteness, the following lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\varphi)^2 + \frac{a}{2}\kappa(\partial^2\phi)\varphi^2 + \frac{b}{2}\kappa^2(\partial\phi)^2\varphi^2 = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\varphi)^2 - a\kappa(\partial\phi \cdot \partial\varphi)\varphi + \frac{b}{2}\kappa^2(\partial\phi)^2\varphi^2, \quad (\text{A.8})$$

where a and b are dimensionless constants. In keeping the parallel with the cases discussed in the text, one should think of ϕ as a massless mediator and φ the matter field. The addition of a mass term for φ does not change the computations.

The conjugate momenta are

$$\begin{cases} \Pi_\phi = \partial_0\phi - a\kappa\phi\partial_0\varphi + b\kappa^2\partial_0\phi\varphi^2 \\ \Pi_\varphi = \partial_0\varphi - a\kappa\varphi\partial_0\phi, \end{cases} \quad (\text{A.9})$$

and, inverting the relations, we obtain

$$\begin{cases} \partial_0\phi = \frac{\Pi_\phi + a\kappa\varphi\Pi_\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \\ \partial_0\varphi = \Pi_\varphi + a\kappa\varphi \frac{\Pi_\phi + a\kappa\varphi\Pi_\varphi}{1 + (b-a^2)\kappa^2\varphi^2}. \end{cases} \quad (\text{A.10})$$

Following the steps described above, the interaction picture hamiltonian is obtained:

$$\begin{aligned} \mathcal{H} &= \frac{\partial_0\phi(\partial_0\phi + a\kappa\varphi\partial_0\varphi)}{1 + (b-a^2)\kappa^2\varphi^2} + \partial_0\varphi \left(\partial_0\varphi + a\kappa\varphi \frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right) - \frac{1}{2} \left(\frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 \\ &\quad - \frac{1}{2} \left(\partial_0\varphi + a\kappa\varphi \frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 + \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}\partial_i\varphi\partial_i\varphi + a\kappa\varphi\partial_0\varphi \frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \\ &\quad + a^2\kappa^2\varphi^2 \left(\frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 - \frac{b}{2}\kappa^2\varphi^2 \left(\frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 - a\kappa\varphi\partial_i\phi\partial_i\varphi + \frac{b}{2}\kappa^2\varphi^2\partial_i\phi\partial_i\phi \end{aligned} \quad (\text{A.11})$$

Expanding to second order in κ , to match the usual contributions to the $\varphi\varphi \rightarrow \varphi\varphi$ or $\phi\phi \rightarrow \varphi\varphi$ amplitudes from (A.8), we get

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}(\partial_0\phi\partial_0\phi + \partial_i\phi\partial_i\phi) + \frac{1}{2}(\partial_0\varphi\partial_0\varphi + \partial_i\varphi\partial_i\varphi) + a\kappa\varphi(\partial_0\varphi\partial_0\phi - \partial_i\varphi\partial_i\phi) - \frac{b}{2}\kappa^2\varphi^2(\partial_0\phi\partial_0\phi - \partial_i\phi\partial_i\phi) \\ &\quad + \frac{a^2}{2}\kappa^2\varphi^2(\partial_0\phi\partial_0\phi + \partial_0\varphi\partial_0\varphi) + O(\kappa^3). \end{aligned} \quad (\text{A.12})$$

We recognize, in the first line, the sum $\mathcal{H}_{\text{free}} - \mathcal{L}_I$ that is usually found in perturbation theory with no derivative interactions. The operator in the second line, as well as all the higher orders that can

be derived from (A.11), are due to the derivative interactions in (A.8). Equation (A.12) shows that, at the level of the interaction picture hamiltonian, we get additional 4-point vertices with respect to the usual ones.

We now check the impact of such additional interactive terms through the explicit computation of the $\varphi(p_1)\varphi(p_2) \rightarrow \varphi(q_1)\varphi(q_2)$ scattering amplitude. Taking into account the corrections to the scalar propagator (analogous to (A.7)), the usual $(-\mathcal{L}_I)$ interactions give, in each one of the s, t and u channels

$$i\mathcal{M}_{(-\mathcal{L}_I)} = -ia^2\kappa^2 P_\mu P_\nu \left(\frac{P^\mu P^\nu}{P^2} - \eta_0^\mu \eta_0^\nu \right), \quad (\text{A.13})$$

where P is the appropriate momentum factor in each channel ($P = p_1 + p_2$, $P = p_1 - p_3$ and $P = p_1 - p_4$, respectively, in s, t and u). After some algebra, the four φ contact term in (A.12) accounts for a contribution

$$i\mathcal{M}_{\text{contact}} = -2ia^2\kappa^2(p_{1,0}^2 + p_{2,0}^2 + q_{1,0}^2 - q_{1,0}p_{1,0}), \quad (\text{A.14})$$

where the notation $p_{i,0}$ means the zero component of the momentum p_i .

Putting it all together one gets

$$i\mathcal{M} = -ia^2\kappa^2 \left\{ s+t+u - ((p_{1,0} + p_{2,0})^2 + (p_{1,0} - q_{1,0})^2 + (p_{1,0} - q_{2,0})^2) + 2(p_{1,0}^2 + p_{2,0}^2 + q_{1,0}^2 - q_{1,0}p_{1,0}) \right\}. \quad (\text{A.15})$$

Using momentum conservation one can show that, again after some algebra, the non covariant pieces cancel leaving the same result one would have guessed using the naive Feynman rules from the lagrangian (A.8) associating the appropriate momentum factor to each derivative:

$$i\mathcal{M} = -ia^2\kappa^2 (s + t + u). \quad (\text{A.16})$$

The type of vertices being the same, this same cancellation happens in the “pair production”-like amplitude $\phi\phi \rightarrow \varphi\varphi$.

This toy model explicitly shows the cancellation between different non covariant pieces arising in the computation of amplitudes with two derivative vertices and justifies, a posteriori, the use of naive perturbation theory we made in section 5.

B Helicity basis and Mandelstam variables

In the computation of the pair production diagrams, we need to deal with external states polarizations for massless helicity-1 and helicity-2 particles. This is of no concern when we compute the squared amplitude, as it is usually treated by means of the replacements $\sum_{\text{pol}} \epsilon_\mu(p)\epsilon_\nu^*(p) \rightarrow -g_{\mu\nu}$ for photon amplitudes and $\sum_{\text{pol}} \epsilon_{\mu\nu}(p)\epsilon_{\rho\sigma}^*(p) = \sum_{\text{pol}} \epsilon_\mu(p)\epsilon_\nu(p)\epsilon_\rho^*(p)\epsilon_\sigma^*(p) \rightarrow \mathcal{P}_{\mu\nu\rho\sigma}$ for graviton ones. If, on the other hand, we want to consider the amplitude more directly and not its square, we need to choose a basis for the polarizations and the momentum, and perform the calculations within this basis.

For the case of the pair production, the in-going states relevant here are either photons or gravitons, while the outgoing ones are massive particles. We perform here the computations in the center of momentum frame.

Starting from the $D = 4$ case, we write the momenta

$$p_1 = E_p(1, 0, 0, 1), \quad p_2 = E_p(1, 0, 0, -1), \quad p_3 = (E_p, p \sin \theta, 0, p \cos \theta), \quad p_4 = (E_p, -p \sin \theta, 0, -p \cos \theta) \quad (\text{B.1})$$

and the polarizations

$$\epsilon_1^\pm \equiv \epsilon(p_1)^\pm = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0), \quad \epsilon_2^\pm \equiv \epsilon(p_2)^\pm = \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0). \quad (\text{B.2})$$

The scalar products appearing in the amplitudes can now be explicitly performed in this particular basis and the results can then be rewritten in terms of the Mandelstam variables using the following relations:

$$p^2 = \frac{s - 4m^2}{4}, \quad \sin^2 \theta = \frac{(t - u)^2}{s(s - 4m^2)}, \quad \cos^2 \theta = \frac{4tu - 4m^2}{s(s - 4m^2)} \quad (\text{B.3})$$

At this point, we need to separate the contributions coming from different helicities. For definiteness, we refer now to the amplitude in (3.11), that we report here for the reader's convenience

$$i\mathcal{M}_{\gamma\gamma} = ig^2 q_n^2 \epsilon_\mu(p_1) \epsilon_\nu(p_2) \left(\frac{(2p_3^\mu - p_1^\mu)(2p_4^\nu - p_2^\nu)}{t - m_n^2} + \frac{(2p_4^\mu - p_1^\mu)(2p_3^\nu - p_2^\nu)}{u - m_n^2} + 2\eta^{\mu\nu} \right) - 2ig^2 q_n^2 \frac{D-1}{D-2} \epsilon_\mu(p_1) \epsilon_\nu(p_2) \frac{p_1 \cdot p_2 \eta^{\mu\nu} - p_1^\nu p_2^\mu}{s}.$$

A great simplification comes when we deal more directly with the amplitudes components. We can in fact use the property⁷ $\epsilon(p) \cdot p = 0$. With our choice of basis, we also have $\epsilon(p_1) \cdot p_2 = \epsilon(p_2) \cdot p_1 = 0$, so that, for the purposes of the calculation with the helicity method, we can use the following expression for the amplitude

$$\mathcal{M}_{\gamma\gamma} = 4g^2 q_n^2 \left\{ \frac{\epsilon(p_1) \cdot p_3 \epsilon(p_2) \cdot p_4}{t - m_n^2} + \frac{\epsilon(p_1) \cdot p_4 \epsilon(p_2) \cdot p_3}{u - m_n^2} + \frac{\epsilon(p_1) \cdot \epsilon(p_2)}{2} \left(1 - \frac{D-1}{D-2} \frac{p_1 \cdot p_2}{s} \right) \right\}. \quad (\text{B.4})$$

We denote with $\mathcal{M}_{\pm\pm}$ the different contributions, with the \pm referring to the helicities of the polarization. We have then

$$i\mathcal{M}_{++} = 2i(gq_n)^2 \left(\frac{m_n^2 s}{(t - m_n^2)(u - m_n^2)} - \gamma_d \frac{3}{4} \right), \quad i\mathcal{M}_{+-} = -2i(gq_n)^2 \frac{(m_n^4 - ut)}{(t - m_n^2)(u - m_n^2)}, \quad (\text{B.5})$$

where we have introduced a factor γ_d in front of the term arising from the dilaton such that we retrieve the result for our KK theory when $\gamma_d = 1$ and the usual result for a $U(1)$ gauge theory when $\gamma_d = 0$. To compute the total amplitude, we average over the in-going polarizations and obtain in the threshold limit

$$|\mathcal{M}_{\gamma\gamma}|^2 = \frac{1}{4} (2|\mathcal{M}_{++}|^2 + 2|\mathcal{M}_{+-}|^2) \rightarrow 2 \left(1 - \gamma_d \frac{3}{4} \right)^2 (gq_n)^4. \quad (\text{B.6})$$

⁷When using the usual shortcut $\sum_{\text{pol}} \epsilon_\mu(p) \epsilon_\nu(p) = -g_{\mu\nu}$ this simplification cannot be used.

When $\gamma_d = 0$, the overall numerical factor is 2, while for $\gamma_d = 1$, it is $1/8$, matching the results obtained in Section 3.2 for $D = 4$. It is immediate to realize that, in the threshold limit, only the $\epsilon(p_1) \cdot \epsilon(p_2)$ term contributes.

The same method outlined above can be used for any other number of dimensions D , where the gauge bosons have $D - 2$ independent helicity states. For instance, in the $D = 5$ case, the helicity basis can be taken as

$$\begin{aligned}\epsilon_1^1 &= \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0) & \epsilon_2^1 &= \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0) \\ \epsilon_1^2 &= \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0) & \epsilon_2^2 &= \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0) \\ \epsilon_1^3 &= (0, 0, 0, 1, 0) & \epsilon_2^3 &= (0, 0, 0, -1, 0).\end{aligned}\tag{B.7}$$

For any $D > 4$, the polarization basis can be chosen such that, for both p_1 and p_2 , the first two polarizations are the same as in $D = 4$, while the other polarizations are $\epsilon_1^i = (0, \dots, \underbrace{1}_{i+1}, \dots, 0)$ and $\epsilon_2^i = (0, \dots, \underbrace{-1}_{i+1}, \dots, 0)$. For an even number of dimensions, one may chose the basis in an equivalent way as an ensemble of two by two circular polarizations. In $D = 6$ dimensions, for instance, this would give

$$\begin{aligned}\epsilon_1^1 &= \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0, 0) & \epsilon_2^1 &= \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0, 0) \\ \epsilon_1^2 &= \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0, 0) & \epsilon_2^2 &= \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0, 0) \\ \epsilon_1^3 &= \frac{1}{\sqrt{2}}(0, 0, 0, -1, -i, 0) & \epsilon_2^3 &= \frac{1}{\sqrt{2}}(0, 0, 0, 1, -i, 0) \\ \epsilon_1^4 &= \frac{1}{\sqrt{2}}(0, 0, 0, 1, -i, 0) & \epsilon_2^4 &= \frac{1}{\sqrt{2}}(0, 0, 0, -1, -i, 0).\end{aligned}\tag{B.8}$$

Of course, the results are independent of the particular choice.

Whatever specific basis one choses, from (B.4) it follows that in the threshold limit, as already observed for the specific case $D = 4$, only the diagonal terms \mathcal{M}_{ii} are non zero, and they all give the same contribution

$$\mathcal{M}_{ii} \rightarrow 2(gq)^4 \left(1 - \frac{1}{2} \frac{D-1}{D-2}\right).\tag{B.9}$$

It is then straightforward to extract the value of the amplitude in the threshold limit for D generic dimensions as

$$|\mathcal{M}|^2 \rightarrow \frac{1}{(D-2)^2} (D-2) |\mathcal{M}_{ii}|^2 = \frac{4}{D-2} (gq)^4 \left(1 - \frac{1}{2} \frac{D-1}{D-2}\right)^2 = \left(\frac{D-3}{D-2}\right)^2 \frac{(gq_n)^4}{D-2}.\tag{B.10}$$

This result of course matches that shown in (3.13), that was obtained by means of the usual trick $\sum_{\text{pol}} \epsilon_\mu(p) \epsilon_\nu(p) = -g_{\mu\nu}$. Note also that when the dilaton is put to zero (i.e. when the second contribution in the parenthesis (B.10) is put to zero) we re-obtain the result

$$|\mathcal{M}_{\gamma\gamma}|^2 \rightarrow \frac{4}{D-2} (gq)^4.\tag{B.11}$$

The same procedure can now be used to extract the different components of the purely gravitational amplitude of section 3.2.3. The four diagrams contribute in the amount

$$\begin{aligned}
\mathcal{M}_{t\text{-pole}} &= -\frac{4\kappa^2(\epsilon_1 \cdot p_3)^2(\epsilon_2 \cdot p_4)^2}{t - m_n^2} \\
\mathcal{M}_{u\text{-pole}} &= -\frac{4\kappa^2(\epsilon_2 \cdot p_3)^2(\epsilon_1 \cdot p_4)^2}{u - m_n^2} \\
\mathcal{M}_{\text{seagull}} &= 2\kappa^2 \epsilon_1 \cdot \epsilon_2 (\epsilon_1 \cdot \epsilon_2 (p_3 \cdot p_4 + m_n^2) - 2\epsilon_2 \cdot p_3 \epsilon_1 \cdot p_4 - 2\epsilon_1 \cdot p_3 \epsilon_2 \cdot p_4)
\end{aligned} \tag{B.12}$$

and

$$\begin{aligned}
\mathcal{M}_{g\text{-pole}} &= \frac{2\epsilon_1 \cdot \epsilon_2}{D-2} \left\{ 2p_1 \cdot p_2 \left[(D-2)(\epsilon_{2,\lambda}\epsilon_{1,\tau} + \epsilon_{1,\lambda}\epsilon_{2,\tau}) - \epsilon_1 \cdot \epsilon_2 \eta_{\lambda\tau} \right] \right. \\
&\quad + p_1 \cdot p_2 \left[4\epsilon_1 \cdot \epsilon_2 \eta_{\lambda\tau} - 2(D-2)(\epsilon_{2,\lambda}\epsilon_{1,\tau} + \epsilon_{1,\lambda}\epsilon_{2,\tau}) \right] \\
&\quad + D\epsilon_1 \cdot \epsilon_2 (p_{1,\lambda}p_{1,\tau} + p_{2,\lambda}p_{2,\tau} + p_{1,\lambda}(p_1 + p_2)_\tau + p_{2,\lambda}(p_1 + p_2)_\tau) \\
&\quad + 2Dp_1 \cdot p_2 \epsilon_{2,\lambda}\epsilon_{1,\tau} + 2(D-2)p_1 \cdot p_2 \epsilon_{1,\lambda}\epsilon_{2,\tau} - 2p_1 \cdot p_2 \epsilon_1 \cdot \epsilon_2 \eta_{\lambda\tau} \\
&\quad + 2\epsilon_1 \cdot \epsilon_2 p_{2,\lambda}p_{1,\tau} + 2\epsilon_1 \cdot \epsilon_2 p_{1,\lambda}p_{2,\tau} - 2\epsilon_1 \cdot \epsilon_2 (p_1 + p_2)_\lambda p_{1,\tau} \\
&\quad - 2\epsilon_1 \cdot \epsilon_2 (p_1 + p_2)_\lambda p_{2,\tau} - 2\epsilon_1 \cdot \epsilon_2 p_{1,\lambda}(p_1 + p_2)_\tau - 2\epsilon_1 \cdot \epsilon_2 p_{2,\lambda}(p_1 + p_2)_\tau \\
&\quad \left. - 4p_2 \cdot (p_1 + p_2) \epsilon_{2,\lambda}\epsilon_{1,\tau} \right\} \left(p^{3,\lambda} p^{4,\tau} + p^{4,\lambda} p^{3,\tau} - g^{\lambda\tau} (p_3 \cdot p_4 + m_n^2) \right) \tag{B.13}
\end{aligned}$$

to give (3.27), reported here for simplicity

$$\begin{aligned}
i\mathcal{M}_{GG} &= \frac{\kappa^2}{2} \left(-\frac{8(p_3 \cdot \epsilon_1)^2(p_4 \cdot \epsilon_2)^2}{t - m_n^2} - \frac{8(p_3 \cdot \epsilon_2)^2(p_4 \cdot \epsilon_1)^2}{u - m_n^2} \right. \\
&\quad \left. - 2\frac{(\epsilon_1 \cdot \epsilon_2)^2 (m_n^4 - tu - sm_n^2)}{s} - 4\epsilon_1 \cdot \epsilon_2 (p_3 \cdot \epsilon_2 p_4 \cdot \epsilon_1 + p_3 \cdot \epsilon_1 p_4 \cdot \epsilon_2) \right)
\end{aligned}$$

As in the previous case, it is again easily verified that in the threshold limit only the diagonal \mathcal{M}_{ii} terms are non-vanishing and that they all give the same result. In terms of the above amplitude, such non-vanishing contribution is given by the $(\epsilon_1 \cdot \epsilon_2)^2$ term that results in

$$\mathcal{M}_{ii} \rightarrow \kappa^2 m_n^2. \tag{B.14}$$

It is now straightforward to obtain, from these considerations, the result for the squared amplitude in D generic dimensions:

$$|\mathcal{M}|^2 \rightarrow \frac{1}{(D-2)^2} (D-2) |\mathcal{M}_{ii}|^2 = \frac{\kappa^4 m_n^4}{D-2}, \tag{B.15}$$

which is the result quoted in the text (3.35).

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CHAPTER 5

Dilatonic WGC in (Anti-)de Sitter space

5.1 Introduction

The work [3] started as a continuation of [46] when we asked ourselves how the extremal condition of dS Reissner Nordström black holes is modified in the presence of a dilaton. This is when we realized that a study of dilatonic black holes in asymptotically dS space had never been performed. Similarly, a comprehensive study was missing also in the AdS case.

Dilatonic black hole solutions in asymptotically (A)dS space-time were derived for the first time in [67], and later re-obtained with different methods in [68, 69]. They are presented in full details below where we report [3], and it is of no use to repeat their presentation here. Some features of the AdS case were studied in [70, 71]. In the latter, a particular mechanism proper of AdS spacetime (that was previously presented for the RN AdS black holes [72, 73]) was discussed to allow for the decay of extremal black holes. As for the case of the dS WGC proposed in [47, 48], this is a physical property of the non-asymptotically flat space that has no Minkowskian counterpart. This is not in contrast with our findings, that are intended to find the corrections to the flat space constraints that comes for small values of the (absolute value of the) cosmological constant $|\Lambda_c|$. It rather provides an additional mechanism for black hole decay complementary to the one available in flat space.

The study of dilatonic (A)dS black holes will be presented from scratch in the following, where we report [3] in its full integrality.

5.2 Discussion

The core of the paper is dedicated to the analysis of the conditions for the existence of horizons in (A)dS dilatonic black hole solutions. They are found at the coordinates

r where the time-component of the metric

$$g_{00}(r) = - \left[\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \right] \quad (5.2.1)$$

vanishes. Similarly to what happened in the flat case (1.5.4), the surface $r = r_-$ indicates the extended singularity of these black holes.

In both asymptotically dS and AdS, we have found that the dilatonic coupling plays a special role in determining the properties of the black hole. Specifically, the value $\alpha_c^2 \equiv 1/3$ defines a turning point and we can distinguish three different behaviours.

- For $\alpha > \alpha_c$, the dS black holes have been found to have two horizons, a cosmological and an event horizon, while the AdS only have an event horizon. The singularity at $r = r_-$ is space-like. In both cases the extremal limit is obtained as the event horizon and the extended singularity merge. In terms of the mass and the charge (in the units defined in the paper), this happens for $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$, which is the same extremality condition found in flat space. From the point of view of the charge to mass ratio of the extremal solutions, there seem to be here a total insensitiveness to both the presence and the sign of the cosmological constant. In the limit $\alpha \rightarrow \infty$, the Schwarzschild (A)dS solution is recovered.
- For $\alpha = \alpha_c$ the number of horizons of the black holes is unchanged. The extremal solution is again reached in the limit where the event horizon coincides with the singularity, but now this translates in a relationship between the charge and the mass which is different from the flat-space one. In particular the region of existence becomes larger (smaller) in the (A)dS case. The correction to the flat space extremality condition due to the cosmological constant can be organized in a series expansion in the Hubble parameter H : $Q^2 e^{2\alpha_c\phi_0} = 4/3M^2 \pm 4^3/3^4 M^4 H^2 + \mathcal{O}(M^6 H^4)$, where the upper (lower) sign refers to (A)dS.
- For $0 < \alpha < \alpha_c$, the dS black holes still have their two (cosmological and event) horizons. The AdS black holes, on the contrary, drastically change with respect to the case $\alpha \geq \alpha_c$ and now have two horizons, an event horizon and a Cauchy surface. This is the only case where $\lim_{r \rightarrow r_-} g_{00} < 0$ and it has a two-fold consequence. First, the singularity of AdS black holes changes its nature and becomes a time-like one. This means in particular that the singularity can be avoided (as, for example, the $r = 0$ singularity of the Reissner Nordström black holes). The second consequence is that the extremality now is not anymore of the dilatonic type (event horizon = singularity) but of Reissner-Nordström type (event horizon = Cauchy surface). Interestingly, this does not change the fact that the relation between the charge and the mass of the extremal black holes can still be organized in a series in powers of H around the flat space condition (see eq. (5.7) of the paper). On the other hand, the extremality for the dS black holes still seems to be of dilatonic type. As the event horizon and the singularity get closer and closer, however, something interesting happens: the metric becomes complex before they can ever meet, making it impossible to define extremal dilatonic dS solutions in the parametric region $0 < \alpha < \alpha_c$.

To better understand these properties we attempted at a first thermodynamic analysis¹. In asymptotically flat space, this was first carried out in [75], where it was found that the Hawking temperature of extremal black holes diverges for $\alpha > 1$, is finite and equal $1/(8\pi M)$ (the same value as for Schwarzschild black holes) for $\alpha = 1$ (this is true for all black holes, not only extremal ones) and vanishes for $\alpha < 1$, while the Hawking-Beckenstein entropy vanishes for any value $\alpha \neq 0$. The interpretation we give of this behaviour is that it is a reflection of the interpolation of the dilatonic black holes between a Schwarzschild ($\alpha \rightarrow \infty$) and a Reissner-Nordström solution ($\alpha = 0$) with a turning point at $\alpha = 1$. In fact, as mentioned in section 1.5, we may define a formal extremal limit for Schwarzschild black holes as the one where $r_h \rightarrow r_s$, where r_h is the horizon and r_s the singularity placed at $r = 0$. The limit $r_h \rightarrow r_s$ corresponds to $M \rightarrow 0$ and, consequently, $T \rightarrow \infty$. In particular, the fact that for $\alpha = 1$, that corresponds to the result of a string compactification, no trivial endpoints of Hawking evaporation ($T = 0$) could be found was indicated in [23] as a puzzling property.

In (A)dS, we have found that the temperature of the extremal states diverges for both $\alpha > 1$ and $\alpha_c < \alpha < 1$, is finite and equal to

$$T = \frac{1}{4\pi} \left(\frac{1}{2M} \mp 2MH^2 \right), \quad (5.2.2)$$

for $\alpha = 1$, with the upper (lower) sign referring to (A)dS, and vanishes for $\alpha = \alpha_c$. For $0 < \alpha < \alpha_c$, T vanishes in the AdS case and cannot be defined in the dS case. The divergence for $\alpha > 1$ is partially and implicitly dictated by the cosmological constant, and is slightly different from the flat space one, while the $\alpha_c < \alpha < 1$ divergence is completely driven by the cosmological constant. Together with (5.2.2), this indicates that although for $\alpha > \alpha_c$ it does not appear in the charge-mass relation, extremal black holes do know about the presence of the cosmological constant. Trying to replicate our interpretation of the asymptotically flat case, we would like to conclude that the behaviour of the temperature is a reflection of the interpolation between Schwarzschild and RN solutions with turning point at² $\alpha = \alpha_c$. In this picture, the appearance of a Cauchy surface in AdS for $\alpha < \alpha_c$ could be understood as the only way the black hole has to approach the RN behaviour avoiding a divergence of the extremal temperature. Inspection of the temperature, that we report here for simplicity

$$T = \frac{1}{4\pi} \left[\frac{r_+}{r_h} \left(1 - \frac{r_-}{r_h^2} \right)^{\frac{1-\alpha^2}{1+\alpha^2}} + \frac{1-\alpha^2}{1+\alpha^2} \left(1 - \frac{r_+}{r_h} \right) \left(1 - \frac{r_-}{r_h} \right)^{-\frac{2\alpha^2}{1+\alpha^2}} \frac{r_-}{r_h^2} \mp 2H^2 r_h \left(1 - \frac{r_-}{r_h} \right)^{\frac{2\alpha^2}{1+\alpha^2}} \mp 2 \frac{\alpha^2}{1+\alpha^2} H^2 r_- \left(1 - \frac{r_-}{r_h} \right)^{-\frac{1-\alpha^2}{1+\alpha^2}} \right], \quad (5.2.3)$$

indeed shows that the only way to make sure that in the whole range $\alpha < \alpha_c$ no singularity of the temperature arises in the extremal limit is that the latter does not imply

¹ Here we limit ourselves to an analysis for black holes much smaller than the size of dS, i.e. close to the extremal limit and with small values of the cosmological constant. More intricate features of larger black holes in dS are described for e.g. in [74].

² Or maybe two turning points at $\alpha = 1$ and $\alpha = \alpha_c$. The first one would be a generalization of the flat space one, while the second is an intimate property of (A)dS.

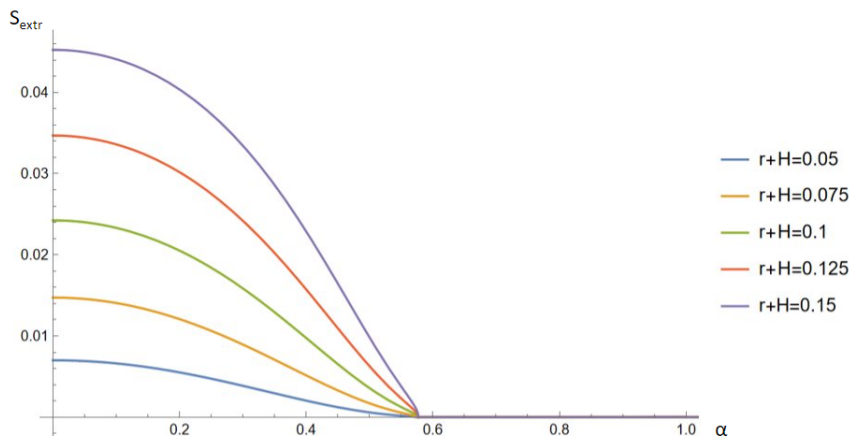


Figure 5.1 – Extremal entropy of AdS extremal black holes as a function of α for different choices of the constant r_+ .

a coincidence between the horizon and the singularity $r_h \rightarrow r_-$. This also enforces a non-vanishing value of the Hawking-Beckenstein entropy of extremal (A)dS states, which has no flat-space counterpart. From the equation

$$S = \pi r_h^2 \left(1 - \frac{r_-}{r_h} \right)^{\frac{2\alpha^2}{1+\alpha^2}}, \quad (5.2.4)$$

it is clear that this is only possible in the region of parameter space where $r_h \neq r_-$ in the extremal limit.

The entropy of extremal AdS black holes can be evaluated in the following way (this was left out of the paper for future investigations in this direction). Their horizon is located at r_{0-} (eq. 4.13 of the paper), which is a function of r_+ and r_- . The two constants r_+ and r_- are related at extremality by the equation $F(r_{0-}) = 0$ (eq. 4.14). The latter cannot be solved to obtain a function $r_-(r_+)$ in full generality, but it is possible to pursue the computation in the small H limit, that is the one we are interested in. Using the expansion

$$r_- = r_+ + cr_+^{\gamma+1}H^\gamma + o(r_+^{\gamma+1}H^\gamma) \quad (5.2.5)$$

found for the $0 < \alpha < \alpha_c$ extremal AdS black holes in eq. 5.5 with $\gamma = (1 + \alpha^2)/(1 - \alpha^2)$ (eq. 5.6), we need to determine c from

$$\frac{1 + \alpha^2}{2(1 - \alpha^2)}c + \left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)}c \right)^{\frac{3\alpha^2 - 1}{1 + \alpha^2}} = 0. \quad (5.2.6)$$

To obtain an analytic solution, we should again revert to an approximation. The most important parametric region for the α -dependence of the extremal black holes' entropy is in the neighborhood of $\alpha = \alpha_c$, where a possible discontinuous behaviour have to be analyzed. Accordingly, we first expand the second term in (5.2.6) about $\alpha = \alpha_c$ as

$$\left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)}c \right)^{\frac{3\alpha^2 - 1}{1 + \alpha^2}} = e^{\frac{3\alpha^2 - 1}{1 + \alpha^2} \ln \left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)}c \right)} \quad (5.2.7)$$

$$\simeq 1 + \frac{3\alpha^2 - 1}{1 + \alpha^2} \ln \left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)} c \right) + \frac{1}{2} \left(\frac{3\alpha^2 - 1}{1 + \alpha^2} \right)^2 \ln^2 \left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)} c \right) + \mathcal{O}((3\alpha^2 - 1)^3)$$

and then the logarithms using

$$\begin{aligned} \frac{3\alpha^2 - 1}{1 + \alpha^2} \ln \left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)} c \right) &\simeq 1 + \frac{3\alpha^2 - 1}{1 + \alpha^2} \left[\frac{3\alpha^2 - 1}{2(1 - \alpha^2)} c - 1 - \frac{1}{2} \left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)} c \right)^2 \right] \\ &+ \frac{1}{2} \left(\frac{3\alpha^2 - 1}{1 + \alpha^2} \right)^2 \left(\frac{3\alpha^2 - 1}{2(1 - \alpha^2)} c \right)^2 + \mathcal{O}((3\alpha^2 - 1)^3). \end{aligned} \quad (5.2.8)$$

Injecting the results of the expansions in (5.2.6), a quadratic equation for c is obtained. Only one of the two solutions to the quadratic equation for c gives a tiny correction to the solution one obtains at first order; the other one is unphysical. Keeping only this solution for c , we obtain the (rather lengthy, it is of no use to show it here) expression of $r_-(r_+)$. Besides α , we now have only one parameter left, r_+ , that allows to move on the curve that defines the extremality. The entropy is fully determined and its evolution with α is shown in Fig. 5.1 for different choices of r_+H . Numerical simulations show that for the values chosen in the plot the second order determination of c pursued here gives a very good approximation to the exact solution (5.2.6) even in the region $\alpha \ll \alpha_c$. The behaviour of the entropy is extremely interesting and has a strong resemblance to that of an order parameter in a second-order phase transition. Again, this seems to give an indication that something peculiar is happening at $\alpha = \alpha_c$, which is probably better captured by the thermal properties of the black holes. A further investigation of this would require to build a suitable framework (maybe a suitable application of the theory of critical phenomena) for this case. This would have driven the paper in a completely new direction at the risk of generating too much confusion. It was then left for future work.

Due to the form of g_{00} (5.2.1), the dS black holes cannot develop a Cauchy surface. In the description we are trying to build, the obstruction to extremality that appears for $0 < \alpha < \alpha_c$ could be due to the fact that, as the value $\alpha = 0$ is approached, the solution cannot reproduce the properties of the RN dS black holes and develops an instability that makes the extremal region inaccessible.

There are two things that might cast some doubts on our interpretation in (A)dS. First, we should understand how the case $\alpha = 1$ fits in the picture. It is the only point, in the whole range $\alpha_c < \alpha < \infty$, where the extremal temperature does not diverge. Note however that the finiteness of the temperature comes from the vanishing of the exponent $|(1 - \alpha^2)/(1 + \alpha^2)|$ of the term $1 - r_-/r_h$ (the one that causes the divergence) for $\alpha \rightarrow 1$: it is not a discontinuity in α of such exponent. On the contrary, the vanishing of the temperature for $\alpha \rightarrow \alpha_c$ is discontinuous in α . It is caused by a factorization in the metric that makes the prefactor in front of the divergence null. This happens abruptly for $\alpha = \alpha_c$ because extremal states do not have $r_+ = r_-$ anymore. Even more, there is another peculiarity that makes the case $\alpha = 1$ particularly interesting. In asymptotically dS, extremal black holes have $D = M$, where D is the scalar charge, and the singularity is located at $r_s = 2D$. The condition $4D^2H^2 < 1$, that appeared in the intermediate steps of the analysis in the paper (see the discussion between eq. 4.3 and eq. 4.4), physically corresponds to the requirement that the singularity is smaller than the Hubble horizon of dS, $r_H = H^{-1}$. An extremal state of size the Hubble horizon is

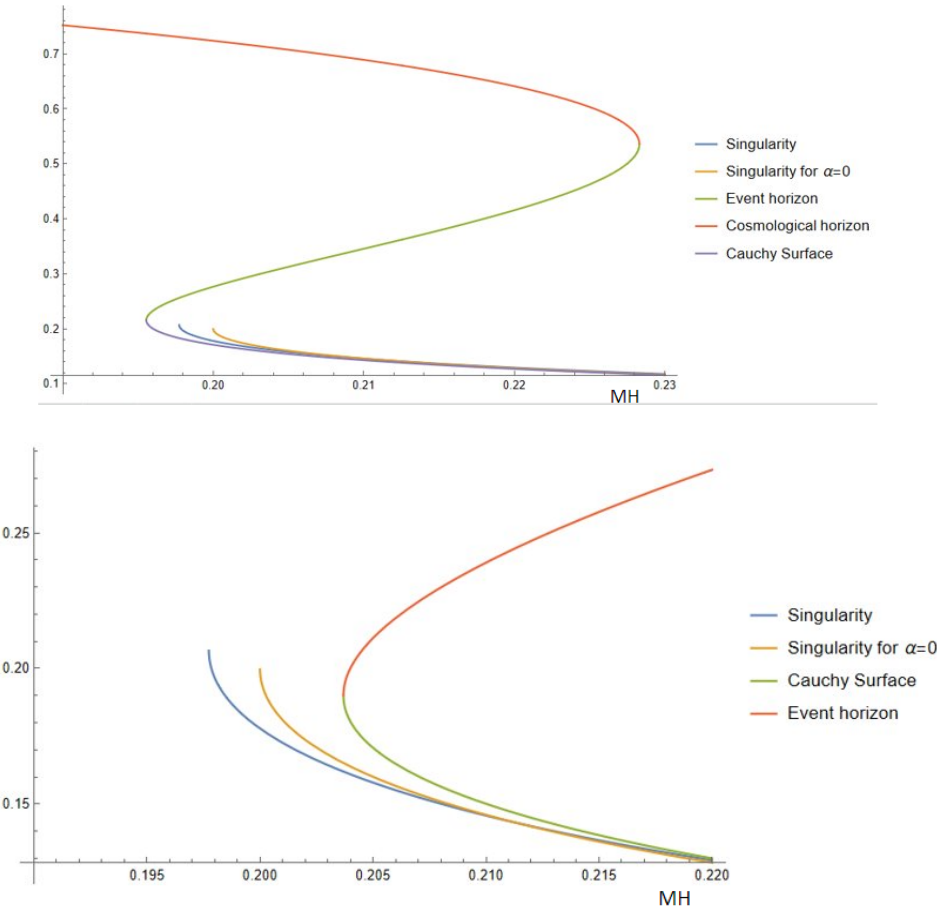


Figure 5.2 – The picture shows the relation between the horizons of the RN dS and AdS (respectively first and second panel) and the singularity of the corresponding dilatonic black holes as a function of MH . The other parameters here are fixed through the choice $QH = 0.2$ and $\alpha = 0.15$.

actually found to define the maximal charge and mass that a dilatonic dS black hole with $\alpha = 1$ can have, and is represented in the Fig. 1 of the paper by the point where the yellow, green and red curves meet. These solutions have $4M^2H^2 = 1$ which means that they are trivial endpoints of the Hawking evaporation. This is a completely new feature which is proper of de Sitter.

The second matter is related to the role of the singularity r_- . The singular surface $r = r_-$ can be written in terms of the charge and mass using equation 2.10 of the paper

$$r_- = \frac{(1 + \alpha^2)Q^2 e^{2\alpha\phi_0}}{M + \sqrt{M^2 - (1 - \alpha^2)Q^2 e^{2\alpha\phi_0}}}. \quad (5.2.9)$$

Independently from the cosmological constant, in the limit $\alpha \rightarrow 0$ we have $r_- \rightarrow M - \sqrt{M^2 - Q^2}$, where the right hand side is the expression of the Cauchy surface of flat space RN black holes, as can be easily checked from (1.3.5). For $\Lambda_c = 0$, it might seem natural from that to identify the singularity of the dilatonic black holes as a sort of generalization of the Cauchy surface of the Reissner Nordström ones. The question of the instability of Cauchy surfaces and the prediction of their disappearance under perturbations of Reissner-Nordström or Kerr solutions has been studied from different perspectives, and this might seem to go in that direction. For the RN (A)dS black holes the expression of the Cauchy surface is more cumbersome, and does not coincide with the limiting expression for r_- . Graphical representations do not show any definite relation between the Cauchy surface and r_- that might point towards an identification of the singularity as a generalization of the Cauchy surface, as depicted in Fig. 5.2. The interpretation given above seems, at least naively, to go in an opposite direction to that, pointing towards an identification of role of the extended singularity for dilatonic black holes with the role of the point singularity for RN ones. A comprehensive understanding of the behaviour of these solutions might come only if all these peculiarities can be put together in a single picture.

An insight on the occurrence of the complex region for the metric comes from the point-particle reduction ("skeletonization") of the black hole (see [76, 77]), whose virtue is that it allows to go beyond their region of existence in parameter space. The discussed dS obstruction can then be understood in terms of a flat-space QFT. The occurrence of a complex metric is indeed a common property of flat and (A)dS spacetimes, but appears inside the domain of existence of the BHs only in dS. The study has shown that in the locus of parameter space where the metric becomes complex, the force between a black hole and a Kaluza-Klein state vanishes and the probability of emission of a pair of dilaton from the black hole diverges. The skeletonization framework might lead to a deeper understanding of the problem at hand, that was left for future work. Intuitively, the resemblance to the case of the overcharged hydrogen atom, with the ground state energy

$$E_{1s} = mc^2 \sqrt{1 - Z\alpha^2} \quad (5.2.10)$$

becoming complex for $Z > 137$ (see eq. 1.1.27 in [78]), is striking once we interpret g_{00} as a potential energy. The emergence of complex potentials in field theory is related to an instability of the system [79], and, if the analogy is successfully verified, the point where the metric becomes complex could be understood as the composite state being on the verge of becoming overcharged and spitting out charged particles to remain

stable. This would give a physical mechanism for the observed obstruction when it happens for charges smaller than the extremal one, and a suitable point particle theory of the black hole might help uncover it. This would also allow to define a dilatonic dS WGC for $0 < \alpha < \alpha_{cr}$, which is the only region where we could not do it. For all the other cases, the dilatonic (A)dS WGC conjecture has been formulated as the requirement that a state in the region complementary to that of the black hole (but not eating the whole dS patch) exists. If the curve where the metric becomes complex is truly impassable in the sense that it cannot admit addition of charge, for all that matters for the definition of a WGC this is equivalent to the usual extremality. The WGC constraint would simply require that a state allowing a black hole on the verge of becoming overcharged exists, which boils down to the requirement that a state with

$$q^2 e^{2\alpha\phi_0} \geq \frac{m^2}{1 - \alpha^2}, \quad (5.2.11)$$

in geometrical units, exists. The bound would be (explicitly) independent of H , but implicitly related to a specific property of dS, with no Minkowskian counterpart.

The paper [3] is integrally reported below.

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Dilatonic (Anti-)de Sitter black holes and Weak Gravity Conjecture

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ABSTRACT: Einstein-Maxwell-dilaton theory with non-trivial dilaton potential is known to admit asymptotically flat and (Anti-)de Sitter charged black hole solutions. We investigate the conditions for the presence of horizons as function of the parameters mass M , charge Q and dilaton coupling strength α . We observe that there is a value of α which separate two regions, one where the black hole is Reissner-Nordström-like from a region where it is Schwarzschild-like. We find that for de Sitter and small non-vanishing α , the extremal case is not reached by the solution. We also discuss the attractive or repulsive nature of the leading long distance interaction between two such black holes, or a test particle and one black hole, from a world-line effective field theory point of view. Finally, we discuss possible modifications of the Weak Gravity Conjecture in the presence of both a dilatonic coupling and a cosmological constant.

KEYWORDS: Beyond Standard Model, Black Holes

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1 Introduction

While global symmetries seem fine in Quantum Field Theory (QFT), their existence is a no-go for any Quantum Theory of Gravity: global charges are not conserved when falling in black holes (see [1]). This illustrates the fact that not all model builder's ingredients are allowed in Nature: some consistent QFTs will never be derived from an ultraviolet (UV) theory that includes quantum gravity, such as String Theory. They fall in the Swampland, contrary to those that form the Landscape (see [2, 3]). Maybe the best-tested condition that discriminates between the two sets of theories is the Weak Gravity Conjecture [4]. This requires, for an abelian U(1) gauge symmetry, the presence of at least one state carrying a charge Q bigger than its mass M , measured in Planck length units: $M^2 < Q^2$. Arguments based on black hole (BH) physics have allowed to extend this conjecture to either Einstein-Maxwell-dilaton theory in flat space-time [5] or to de Sitter space-time [6]. It was the aim of this work to put the two together: Einstein-Maxwell-dilaton theory with (Anti-) de Sitter ((A)dS) backgrounds.

The present work began then when we asked ourselves what happens to the de Sitter Weak Gravity Conjecture in the case of a dilatonic gauge coupling. In [6], the loci of black hole horizons were interpreted as the result of the competition between a repulsive electromagnetic energy density on one side, and gravity on the other, with attractive and repulsive contributions from the black hole mass and the cosmological constant, respectively. It was suggested that the parameter region where the electromagnetic contribution dominates defines simultaneously the absence of a black hole solution and the WGC conditions. It was comforting to see that in the limit of vanishing cosmological constant one recovers the flat space-time result, in contrast with previous attempts [7, 8] that provide complementary criteria for the consistency of the theory and therefore different proposals for a WGC. We were also interested in the dilaton as an example of scalar field that allows to probe the Scalar WGC [9–18].

An extension of the Reissner-Nordström de Sitter black hole solution [19–22] to the case of Einstein-Maxwell-dilaton theory was constructed in [23–25]. This is a generalisation of the well known flat space-time solution of Gibbons-Maeda [26] and Garfinkle-Horowitz-Strominger [27]. For the de Sitter background, we were not able to find in the literature a discussion on the conditions for the existence of horizons with a dilatonic coupling $\alpha \neq 0$. Some aspects of the asymptotically Anti-de Sitter metric were discussed for the case of AdS₅ in [24] and for AdS₄ in some limits by [28]. Some properties of these solutions, as the photon spheres, were considered in [29]. It is the main goal of this work to provide the missing comprehensive analysis. In the different cases, the WGC states will then be considered to be contained within the parametric regions complementary to those where black holes exist.

The paper is organized as follows. Section 2 presents the black hole solution and some formulae generic to all values of α . A very brief review of the asymptotically flat space-time is given in section 3 for completeness and comparison. The horizons of charged dilatonic de Sitter black hole are described in section 4. The Anti-de Sitter case is studied in section 5. Some thermodynamic quantities are computed in section 6. The issue of attractive and repulsive forces, in the case of asymptotically flat space-time, are analyzed in section 7. For the convenience of the readers, our results are summarized in section 8 with our conclusions.

2 Einstein-Maxwell-dilaton black holes

The Reissner-Nordström black holes are parametrized by their charge \tilde{Q} and mass \tilde{M} . It is useful to define the analog of such quantities in the so-called geometrized units:

$$M = \frac{\kappa^2 \tilde{M}}{8\pi}, \quad Q^2 = \frac{\kappa^2 \tilde{Q}^2}{32\pi^2} \quad \Rightarrow \quad \frac{M^2}{Q^2} = \frac{\kappa^2 \tilde{M}^2}{2 \tilde{Q}^2}, \quad (2.1)$$

with $\kappa^2 = 1/M_P^2 = 8\pi G \equiv 8\pi$ and G Newton's constant. The absence of a naked singularity requires $Q \leq M$.

In the following we consider the extension provided by the Einstein-Maxwell-dilaton action

$$\mathcal{S} = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} \left(R - 2(\partial\phi)^2 - e^{-2\alpha\phi} F^2 - V(\phi) \right), \quad (2.2)$$

where $F_{\mu\nu}$ is the field strength tensor related to the massless gauge field A_μ . For $V(\phi) = 0$, the values $\alpha = 1$ and $\alpha = \sqrt{3}$ are those obtained from string theory and Kaluza-Klein compactifications, respectively. Note that ϕ and A_μ here are dimensionless. The dimensionful physical fields are

$$\tilde{\phi} = \sqrt{2}M_P \phi; \quad \tilde{A}_\mu = \sqrt{2}M_P A_\mu. \quad (2.3)$$

In the following, for notation simplicity, we will use ϕ for both the dimensionful and the dimensionless fields.

A static, spherically symmetric solution to the Einstein's equations was given in [23] for $V(\phi)$ of the form

$$V(\phi) = \frac{2}{3} \frac{\Lambda}{(1 + \alpha^2)^2} \left[\alpha^2(3\alpha^2 - 1)e^{-2\frac{\phi - \phi_0}{\alpha}} + (3 - \alpha^2)e^{2\alpha(\phi - \phi_0)} + 8\alpha^2 e^{\alpha(\phi - \phi_0) - \frac{\phi - \phi_0}{\alpha}} \right], \quad (2.4)$$

where Λ is the cosmological constant and ϕ_0 the asymptotic value of $\phi(r)$ for $r \rightarrow \infty$.

Contemplating the form of the action in (2.2), we can identify the gauge couplings $g = e^{\alpha\phi}$ and its asymptotic value $g_0 = e^{\alpha\phi_0}$. Then (2.4) can be written as:

$$V(\phi) = \frac{2}{3} \frac{\Lambda}{(1 + \alpha^2)^2} \left[\alpha^2(3\alpha^2 - 1) \left(\frac{g}{g_0}\right)^{-2/\alpha^2} + (3 - \alpha^2) \left(\frac{g}{g_0}\right)^2 + 8\alpha^2 \left(\frac{g}{g_0}\right)^{1-1/\alpha^2} \right]. \quad (2.5)$$

From the sign of the exponent of the coupling in each term, at least for some rational values of α , we can associate the first term to non-perturbative contributions, the second to perturbative while the third is non-perturbative for $\alpha < 1$, perturbative correction for $\alpha > 1$ and a tree-level contribution for $\alpha = 1$. In the latter case, the potential takes the simple form:

$$V(\phi) = \frac{1}{3} \Lambda \left[\frac{g_0^2}{g^2} + \frac{g^2}{g_0^2} + 4 \right]. \quad (2.6)$$

where one could associate the first, second and third terms to D -brane fluxes, one-loop effect and tree-level cosmological constant contributions, respectively. One can go further and try to imagine different realizations of such kind of potential in models with flux compactifications. One could start with a gauge theory living on a brane wrapping a cycle of volume V in the internal which has a gauge coupling g that go as $V^{-1/2}$ and assumes that the dilaton ϕ measures this volume, $V = e^{-2\alpha\phi}$ in string length units. The potential can be written as

$$V(\phi) = \frac{2}{3} \frac{\Lambda}{(1 + \alpha^2)^2} \left[\alpha^2(3\alpha^2 - 1) \left(\frac{V}{V_0}\right)^{1/\alpha^2} + (3 - \alpha^2) \frac{V_0}{V} + 8\alpha^2 \left(\frac{V}{V_0}\right)^{(\alpha^2 - 1)/2\alpha^2} \right]. \quad (2.7)$$

One can then identify the first and the third terms as resulting from fluxes inside cycles that have smaller or bigger volumes that measure $1/\alpha^2$ and $(1 - \alpha^2)/2\alpha^2$ of the volume V .

It is straightforward to see that for the (A)dS case the potential has a global (maximum) minimum as long as $1/\sqrt{3} \leq \alpha \leq \sqrt{3}$. It has only a local minimum, a local maximum, and it is unbounded from below for $0 < \alpha < 1/\sqrt{3}$ and $\alpha > \sqrt{3}$. For the AdS case, $\Lambda < 0$,

embeddings of the five-dimensional version of this potential and associated black hole solutions in supergravity, or in string theory, have been briefly discussed in [24]. It was pointed out in [24] that these correspond to what is known as superstars or giant gravitons, and for some peculiar values of α , as $0, 1/\sqrt{3}, 2/\sqrt{3}$ they are consistent solutions of truncated $N = 2$ supergravity.

For the dS case, $\Lambda > 0$, the situation is more complicate. No asymptotically dS space-time has been constructed from string theory. Present attempts rest mainly on vacua from supergravity equations of motion with possible presence of non-perturbative contributions from branes. It is not clear if further investigations including all quantum corrections will allow to construct such solutions. Strictly speaking the conjectures forbid stable solution, thus one could still consider the possibility of long lived vacua for very small values of the cosmological constant as the expected life-time is of order $H^{-1} \log H$ (see, for example, [30]). This is an important issue that goes beyond the scope of this paper but should be kept in mind of the reader.

The black hole metric solution of the equations of motion reads

$$\left\{ \begin{array}{l} ds^2 = - \left[\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \right] dt^2 \\ \quad + \left[\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \right]^{-1} dr^2 \\ \quad + r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} d\Omega_2^2, \\ e^{2\alpha\phi} = e^{2\alpha\phi_0} \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \\ F = \frac{1}{\sqrt{4\pi G}} \frac{Q e^{2\alpha\phi_0}}{r} dt \wedge dr. \end{array} \right. \quad (2.8)$$

Here H^2 is the Hubble parameter $H^2 = |\Lambda|/3$. When $\Lambda = 0$, this reproduces the asymptotically flat black hole solutions of [26, 27]. Otherwise, the solution is either an asymptotically dS (upper sign) or AdS (lower sign) space-time. The relation between the integration constants r_+ , r_- and the mass and charge is

$$\left\{ \begin{array}{l} 2M = r_+ + \frac{1 - \alpha^2}{1 + \alpha^2} r_-, \\ Q^2 e^{2\alpha\phi_0} = \frac{r_+ r_-}{1 + \alpha^2}, \\ D = \frac{\alpha}{1 + \alpha^2} r_-, \end{array} \right. \quad (2.9)$$

where D is the scalar charge of the black hole defined as the integral over a two sphere at infinity, $D = \frac{1}{4\pi} \lim_{r \rightarrow \infty} \int d^2 \Sigma^\mu \nabla_\mu \phi$, or, equivalently, through the expansion $\phi = \phi_0 - \frac{D}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$ at large r . This family of solutions have only two independent parameters (in addition to the constant asymptotic value ϕ_0): r_+, r_- or Q, M . Inverting the relations, we

obtain r_+ , r_- from Q and M as

$$\begin{cases} r_+ = M \pm \sqrt{M^2 - (1 - \alpha^2)Q^2 e^{2\alpha\phi_0}} \\ r_- = \frac{(1 + \alpha^2)Q^2 e^{2\alpha\phi_0}}{M \pm \sqrt{M^2 - (1 - \alpha^2)Q^2 e^{2\alpha\phi_0}}}, \end{cases} \quad (2.10)$$

and D

$$D = \alpha \frac{Q^2 e^{2\alpha\phi_0}}{M \pm \sqrt{M^2 - (1 - \alpha^2)Q^2 e^{2\alpha\phi_0}}}, \quad (2.11)$$

as long as $(1 - \alpha^2)Q^2 e^{2\alpha\phi_0} > M^2$. In order to map the uncharged $Q = 0$ case to the Schwarzschild metric, we choose this limit to correspond exclusively to $r_- = 0$. Contrary to the choice $r_+ = 0$, this allows the metric (2.8) to truly take the desired form. Thus, we will consider in the following only solutions with a ‘+’ sign in (2.10) and (2.11).

As a consequence of the above relations between (M, Q) and the integration constants (r_+, r_-) , some peculiarities arise.

- When $\alpha \geq 1$, probing the (r_+, r_-) plane allows to sweep the entire (M, Q) one. The region $r_+ < [(\alpha^2 - 1)/(\alpha^2 + 1)] r_-$ defines negative masses $M < 0$ and is unphysical. A bijection is then defined between the $r_+ \geq [(\alpha^2 - 1)/(\alpha^2 + 1)] r_-$ portion of the (r_+, r_-) plane and the whole (M, Q) one.
- When $0 < \alpha < 1$, for $M^2 < (1 - \alpha^2)Q^2 e^{2\alpha\phi_0}$ both the constants r_+, r_- and the metric become complex. A part of the (M, Q) plane is inaccessible to the solution, a manifestation of the fact that

$$M^2 - (1 - \alpha^2)Q^2 e^{2\alpha\phi_0} = \left(\frac{r_+}{2} - \frac{1 - \alpha^2}{1 + \alpha^2} \frac{r_-}{2} \right)^2$$

is always positive in the parametric coordinates system (r_+, r_-) . There, writing $r_- = r_+ \tan \theta$, the charge-to-mass ratio

$$\frac{Q^2 e^{2\alpha\phi_0}}{M^2} = \frac{4}{1 + \alpha^2} \frac{\tan \theta}{\left(1 + \frac{1 - \alpha^2}{1 + \alpha^2} \tan \theta\right)^2}$$

monotonically increases from 0 to $1/(1 - \alpha^2)$ for $\theta \in \left[0, \arctan \frac{1 + \alpha^2}{1 - \alpha^2}\right]$, reaches its maximal value and then monotonically decreases to 0 for $\theta \in \left[\arctan \frac{1 + \alpha^2}{1 - \alpha^2}, \frac{\pi}{2}\right]$. In this second copy of the $(1 - \alpha^2)Q^2 e^{2\alpha\phi_0} < M^2$ parametric region, Q vanishes for $r_+ = 0$ and, following the discussion below (2.11), we discard it. The bijection is now defined between the $r_+ \geq [(1 - \alpha^2)/(1 + \alpha^2)] r_-$ and the $M^2 \geq (1 - \alpha^2)Q^2 e^{2\alpha\phi_0}$ portions of the planes.

Combining them, these observations show that $r_+ < |(1 - \alpha^2)/(1 + \alpha^2)| r_-$ defines a non physical region for all $\alpha \neq 0$. The Reissner-Nordström solution ($\alpha = 0$) does not suffer from the same issue, as will be discussed in the next section.

3 Asymptotically flat black holes: $\Lambda = 0$

For $\alpha = 0$, $g_{00} = -\left(1 - \frac{r_+ + r_-}{r} + \frac{r_+ r_-}{r^2}\right) = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$, we recover the Reissner-Nordström solution. The r_+ and r_- constants do not enter separately into the metric anymore but only through the combinations $r_+ + r_-$ and $r_+ r_-$. It is thanks to this property that Reissner-Nordström solutions (either flat or asymptotically (A)dS) do not suffer from the complex valued region discussed above.

When $\alpha \neq 0$, r_- is the location of a singular surface while r_+ is the only event horizon of the black hole. The condition for the singularity to be shielded by the horizon is simply $r_+ > r_-$, that is:

$$Q^2 e^{2\alpha\phi_0} < (1 + \alpha^2) M^2 \tag{3.1}$$

In this case of asymptotically flat black holes, the complex valued region is beyond the reach of the black hole solution.

4 Dilatonic de Sitter black holes: $\Lambda > 0$

When $\alpha = 0$, the dilaton decouples and we recover the Reissner-Nordström-de Sitter solution studied in [6]. For $\alpha \neq 0$, one needs to distinguish between several cases, corresponding to different behaviours of g_{00} . Here, r_+ does not determine the location of the horizon anymore, while r_- still indicates the coordinate of a singular surface.

4.1 $\alpha = 1$

The $\alpha = 1$ case allows for explicit expressions of the black hole horizons. The metric can be written as

$$ds^2 = -\left(1 - \frac{2M}{r} - H^2 r(r - 2D)\right) dt^2 + \left(1 - \frac{2M}{r} - H^2 r(r - 2D)\right)^{-1} dr^2 + r(r - 2D) d\Omega_2^2. \tag{4.1}$$

D is related to M and Q through

$$D = \frac{Q^2 e^{2\phi_0}}{2M} \tag{4.2}$$

and $r = r_- = 2D$ is a singular surface. The horizons correspond to the loci of the roots of the polynomial of degree 3 in r :

$$P(r) = H^2 r^3 - 2DH^2 r^2 - r + 2M \tag{4.3}$$

Their explicit expression is not very illuminating. We find more instructive, in particular for discussing below $\alpha \neq 1$, to provide a description of the behaviour of the roots as functions of M and D .

First, note that $P(r) \xrightarrow{r \rightarrow +\infty} +\infty$, and can have two extrema $R_- < R_+$ given by the roots of $P'(r) = 3H^2 r^2 - 4DH^2 r - 1$. As $R_- R_+ = -1$, $R_- < 0$ while $R_+ = \frac{2}{3}D + \frac{1}{6}\sqrt{16D^2 + \frac{12}{H^2}} > 0$. We are interested only in solutions of $P(r) = 0$ in the region $r > 2D$

outside the singularity. Therefore, we will discuss the signs of $P(2D)$ and, when $R_+ > 2D$, $P(R_+)$.

The case of $R_+ < 2D$, i.e. $D^2H^2 > \frac{1}{4}$, corresponds to $r_-^2H^2 > 1$, which means that the radius of the singular surface is greater than the Hubble's one. No black hole solutions can arise there: when $P(2D) < 0$ one root is present, otherwise the polynomial is always positive for all $r > 2D$.

We restrict from now on to $R_+ > 2D$. When $P(2D) = 2(M - D) \leq 0$, P only has one root. If $P(2D) > 0$, there can be 0,1 or 2 roots, depending on the sign of P at the minimum R_+ .

Studying this case, i.e. $M > D$, we have

$$P(R_+) = -\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M - \sqrt{4D^2 + \frac{3}{H^2}} \left(\frac{8}{27}D^2H^2 + \frac{2}{9} \right). \quad (4.4)$$

If $-\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M$ is negative, the sign of $P(R_+)$ is fixed to be negative, and there are two zeros for P . In order to further investigate the sign of $P(R_+)$, it is helpful to consider the function

$$\begin{cases} U(D) \equiv -\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M \\ D < D_1 \rightarrow U(D) > 0 \\ D = D_1 \rightarrow U(D) = 0 \\ D > D_1 \rightarrow U(D) < 0 \end{cases} \quad (4.5)$$

We have $U(0) = 2M > 0$ and U is decreasing with D . There is so one solution D_1 such that $U(D_1) = 0$. For $D > D_1$, U is negative and P has two zeros. The region $D < D_1$, where $U(D)$ is positive, needs further investigation.

It is easier for the rest of the computation, with $D < D_1$, to reformulate the zeros of $P(R_+)$ as the zeros of a simpler function:

$$\begin{aligned} P(R_+) = 0 &\Leftrightarrow \left(-\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M \right)^2 = \left(4D^2 + \frac{3}{H^2} \right) \left(\frac{8}{27}D^2H^2 + \frac{2}{9} \right)^2 \\ &\Leftrightarrow -\left(\frac{4}{3} \right)^3 H^2 M Q(D) = 0, \end{aligned}$$

where Q is a function of D defined as

$$Q(D) = D^3 + \frac{1}{16H^2M}D^2 + \frac{9}{8H^2}D - \frac{27M}{16H^2} + \frac{1}{16H^4M}, \quad (4.6)$$

so that $P(R_+) < 0$ when $Q(D) > 0$. Q is an increasing function for positive D . The sign of $Q(0) = -\frac{27M}{16H^2} + \frac{1}{16H^4M}$ discriminates between two cases. If $Q(0) > 0$, $Q(D)$ is positive for all positive D . If $Q(0) < 0$, there is one D_0 such that $Q(D_0) = 0$:

$$\begin{cases} D < D_0 \rightarrow Q(D) < 0 \Rightarrow P(R_+) > 0 \Rightarrow P(r) \neq 0, \forall r \in \mathbb{R}^+ \\ D = D_0 \rightarrow Q(D) = 0 \\ D > D_0 \rightarrow Q(D) > 0 \Rightarrow P(R_+) < 0. \end{cases} \quad (4.7)$$

Imposing the necessary condition $D^2H^2 < \frac{1}{4}$, we shall now group all cases. There are three possibilities corresponding to 0, 1 or 2 roots.

- For P to have 2 roots, the first condition to be satisfied is $P(2D) > 0$, i.e. $D < M$. If $D > D_1$, P has two roots and there is no need for further investigations. If $D < D_1$, P is also assured to have two roots when $M^2H^2 < \frac{1}{27}$. On the contrary, when $M^2H^2 \geq \frac{1}{27}$, P has two roots when the additional condition $Q(D) > 0 \Leftrightarrow D > D_0$ is met.
- There are two scenarios where P has one root. The first is realized when $P(2D) \leq 0$, corresponding to $D \geq M$. As $P(0) > 0$, this happens when one of the two roots above is behind the singularity. The second scenario is met when $P(2D) \geq 0$ and $P(R_+) = 0$, corresponding to $M \geq D$ and $D = D_0$ with $D < D_1$. This latter case is found when the two horizons discussed in the previous point coincide.
- Finally, the case where P does not have roots corresponds to $D < M$, $D < D_1$, $D < D_0$ and $M^2H^2 > \frac{1}{27}$.

All the cases listed above depend on the values of D_0 and D_1 . Those are given in terms of M as roots of the polynomials Q and U . More compact expressions, that we present below, can be given using the variables $Y = DH$ and $X = MH$. This gives:

$$\begin{aligned}
 Y_0 = & -\frac{1}{48X} + \frac{1}{48} \left(-\frac{1}{X^3} - \frac{2^4 \cdot 3^3 \cdot 5}{X} + 2^7 \cdot 3^6 X \right. \\
 & \left. + \frac{48\sqrt{3}}{X^2} \sqrt{1 + 2^2 \cdot 3^4 X^2 + 2^4 \cdot 3^7 X^4 + 2^6 \cdot 3^9 X^6} \right)^{\frac{1}{3}} \\
 & - \frac{16}{3} \left(\frac{27}{8} - \frac{1}{256X^2} \right) \left(-\frac{1}{X^3} - \frac{2^4 \cdot 3^3 \cdot 5}{X} + 2^7 \cdot 3^6 X \right. \\
 & \left. + \frac{48\sqrt{3}}{X^2} \sqrt{1 + 2^2 \cdot 3^4 X^2 + 2^4 \cdot 3^7 X^4 + 2^6 \cdot 3^9 X^6} \right)^{-\frac{1}{3}} \\
 Y_1 = & \frac{3}{2\sqrt{6}} \left(-3\sqrt{6}X + \sqrt{1 + 54X^2} \right)^{-\frac{1}{3}} - \frac{3}{2\sqrt{6}} \left(-3\sqrt{6}X + \sqrt{1 + 54X^2} \right)^{\frac{1}{3}} \quad (4.8)
 \end{aligned}$$

Actually, the value of Y_0 presented just above is complex for $X < \frac{1}{12\sqrt{6}}$. In that range of parameters, of the three roots of $Q(D)$, it is another one which is real, corresponding to Y_0 with an absolute value taken on the factors elevated to the $\pm\frac{1}{3}$ power and on the factor $\frac{27}{8} - \frac{1}{256X^2}$. However, as we are only interested in $D > 0$ and D_0 is positive only for $X > \frac{1}{\sqrt{27}}$, the expression for Y_0 given above is real in the whole range of interest for D and the absolute values are of no use.

The different cases for the black hole horizons are represented graphically in figure 1. Instead of D , we used the electric charge Q (actually, Q really is Qe^{ϕ_0}) to define x -axis. The green curve represents $D = D_0$, while the yellow one is $M = D$. The function D_1 , represented by the dashed blue curve, is below $M = D$ for $D^2H^2 < \frac{1}{4}$ so that, according to our previous findings, it plays no role in the separation of the different regimes. In the region between the green and the yellow curves, black hole solutions with two horizons

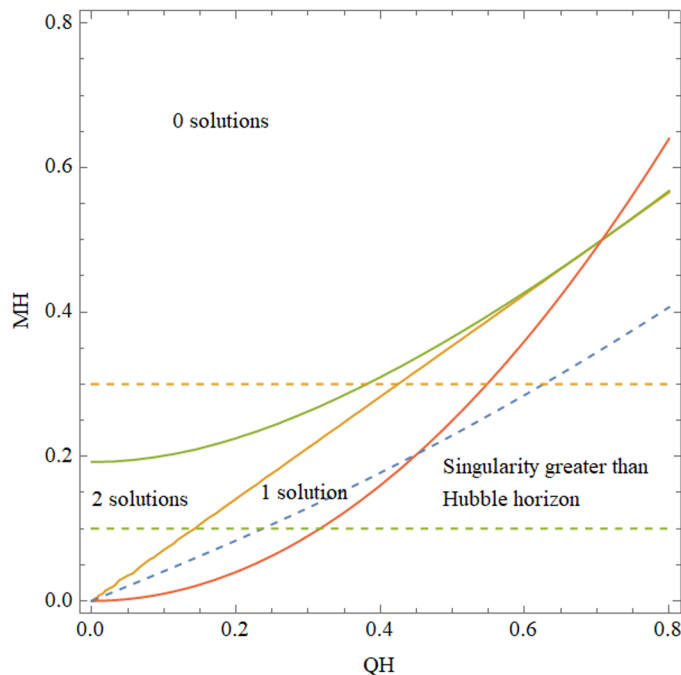


Figure 1. Number of horizons of the $\alpha = 1$ de Sitter black hole as a function of MH and QH . The green curve represents HD_0 , the yellow one the limit $\sqrt{2}MH = Qe^{\alpha\phi_0}H$, and the red one $D^2H^2 = \frac{1}{4}$. Dotted lines are for intermediate steps and discussions in the text.

are found. For $Q = 0$, the discriminant between solutions with two and zero horizons is $M = \frac{1}{\sqrt{27}H}$, as it should. Solutions describing a naked singularity with a cosmological horizon are found below the yellow curve. Finally, the red curve is defined by $2DH = 1$. On its right, the radius of the singularity is greater than the Hubble's.

To illustrate the solution, we now follow two straight horizontal lines, like the green and yellow dashed ones, in figure 1, with $MH < \frac{1}{\sqrt{27}}$ in one case and $\frac{1}{\sqrt{27}} < MH \leq \frac{1}{2}$ in the other.

- $MH < \frac{1}{\sqrt{27}}$. At $Q = 0$ there are two horizons: the event and the cosmological horizon. As Q grows, the radius of the cosmological horizon and of the singularity increase while that of the event horizon decreases until the value $Qe^{\phi_0} = \sqrt{2}M$ is reached. Here, the event horizon coincides with the singularity. For $Qe^{\phi_0} > \sqrt{2}M$, only the cosmological horizon surrounds $r_- = \frac{Q^2e^{2\alpha\phi_0}}{M}$. The radius of the singular surface increases with Q until it meets the Hubble radius when $Q^2e^{2\alpha\phi_0} = \frac{M}{H}$.
- $\frac{1}{\sqrt{27}} < MH \leq \frac{1}{2}$. With $MH < \frac{1}{2}$, at $Q = 0$ no horizons are present. This remains true until the condition $\frac{Q^2e^{2\alpha\phi_0}}{2M} = D_0$ is met: at this point, one horizon appears. Here, two roots of g_{00} coincide, meaning that the event and cosmological horizons have the same size. As Q further grows the two horizons disentangle, the radius of the event horizon shrinks, while that of the cosmological horizon increases. From now on, the analysis is the same as in the previous point: when the condition $Qe^{\alpha\phi_0} = \sqrt{2}M$ is reached, the singularity merge with the event horizon, and for greater charges

the solutions only show a cosmological horizon. When $MH = \frac{1}{2}$, the region with two horizons disappears. At the point $MH = \frac{1}{2}$, $Qe^{\alpha\phi_0}H = \sqrt{2}MH = \frac{1}{\sqrt{2}}$, the green, yellow and red curves meet. Here, both the two roots of g_{00} coincide with the singularity that coincides, in turn, with the Hubble horizon. Put simply, the locations of the singularity, the event, the cosmological and the Hubble horizons all coincide. In terms of the previously defined quantities, this corresponds to the case where $P(2D) = P(R_+) = 0$ with $R_+ = 2D$. This point defines the maximal mass and charge for which a black hole solution exists. Larger charges allow for the presence of a cosmological horizon, with the singularity bigger than the Hubble surface.

For $MH > \frac{1}{2}$, no black hole solution is possible: the singularity is either naked, when $Qe^{\alpha\phi_0} < \sqrt{2}M$, or shielded by a cosmological horizon when $Qe^{\alpha\phi_0} > \sqrt{2}M$, with the latter coinciding with the singularity when the equality is verified. The condition for the singularity to be bigger than the Hubble horizon is now met before this last one.

If we follow the arguments of [6] to infer the WGC condition from the absence of event horizons shielding the singularity, the WGC would require the existence of a state with mass m and charge q , in geometrized units, satisfying $qe^{\phi_0} > \sqrt{2}m$. This corresponds to the dilatonic WGC bound in asymptotically flat space-time, as discussed above. Thus, for $\alpha = 1$, the dilatonic WGC seems to be insensitive to the presence of a cosmological constant.

4.2 $\alpha > 1$

We first study the $\alpha \rightarrow \infty$ limit, where one should recover the Schwarzschild-de Sitter solution, and then look at the generic $\alpha > 1$ case.

In the $\alpha \rightarrow \infty$ limit the metric reads

$$ds^2 = - \left[\frac{1 - \frac{r_+}{r}}{1 - \frac{r_-}{r}} - H^2 r^2 \left(1 - \frac{r_-}{r} \right)^2 \right] dt^2 + \left[\frac{1 - \frac{r_+}{r}}{1 - \frac{r_-}{r}} - H^2 r^2 \left(1 - \frac{r_-}{r} \right)^2 \right]^{-1} dr^2 + r^2 \left(1 - \frac{r_-}{r} \right)^2 d\Omega_2^2. \quad (4.9)$$

To study the horizons of the above metric, we need to find the roots of the polynomial

$$G(r) \equiv H^2(r - r_-)^3 - (r - r_+) = 0. \quad (4.10)$$

G has two extrema: a minimum at $r = r_- + \frac{1}{\sqrt{3}H}$ and a maximum at $r = r_- - \frac{1}{\sqrt{3}H}$. The latter is inside the singular surface. The knowledge of the values on the singular surface, $G(r_-)$, and at its minimum, $G(r_- + \frac{1}{\sqrt{3}H})$, allows to find the number of roots of G . We have

$$\begin{cases} G(r_-) = r_+ - r_- \\ G\left(r_- + \frac{1}{\sqrt{3}H}\right) = r_+ - r_- - \frac{2}{\sqrt{27}H}. \end{cases} \quad (4.11)$$

For $0 < r_+ - r_- < \frac{2}{\sqrt{27}H}$, the singularity is protected by two horizons: the event and the cosmological horizons. When $r_+ - r_- = \frac{2}{\sqrt{27}H}$, the two horizons merge. Above, neither the

event nor the cosmological horizon are present. At $r_+ = r_-$, the event horizon coincides with the singularity. Using (2.10) one obtains, for $\alpha \rightarrow \infty$, $r_+ - r_- = 2M$. Thus, we discard the $r_+ < r_-$ region as corresponding to negative masses. In the $\alpha \rightarrow \infty$ limit of (2.8) the Schwarzschild-de Sitter solution is thus recovered: the discriminant between a naked and a shielded singularity is the sign of $M - \frac{1}{\sqrt{27}H}$.

Now, we consider the general metric (2.8) and define a new function F that vanishes for the same values of r than g_{00} :

$$F(r) \equiv r - r_+ - H^2 r^3 \left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2 - 1}{\alpha^2 + 1}}. \quad (4.12)$$

To investigate the solutions of $F(r) = 0$, we divide F into the sum of two contributions: $A(r) \equiv r - r_+$, and $B(r) \equiv H^2 r^3 \left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2 - 1}{\alpha^2 + 1}}$.

The intersection points of the two curves defined by A and B give the zeros of F . We carry the analysis in two regions of the parameter space:

- For $r_+ \leq r_-$, $A(r_-) \geq 0$ and the two curves always cross in one point. Accordingly there is, in this case, only one zero, corresponding to the cosmological horizon.
- For $r_+ > r_-$, $A(r_-) < 0$ and there are either two, one or zero solutions depending on the location of the point r_0 where $B'(r_0) = 1$. $B(r_0) \leq A(r_0)$ corresponds to the case where the function has two zeros, coalescing into one when the equality is satisfied. $B(r_0) > A(r_0)$ will determine the horizon-less regime where the dS space-time causal patch has been completely eaten by the black hole.

We consider, from now on, $r_+ > r_-$. To discriminate between the different regimes we just described, we proceed in the following way.

First, we observe that the limit for the two zeros to collapse into one is obtained where $A(r)$ and $B(r)$ are tangent, thus $F(r_0) = 0$ and $F'(r_0) = 0$ ($B'(r_0) = 1$). Consider¹ $r_{0\pm}$ two functions of r_{\pm} given by:

$$r_{0\pm} = \frac{(3 - \alpha^2)r_- + 3(1 + \alpha^2)r_+}{4(1 + \alpha^2)} \pm \sqrt{\left(\frac{(3 - \alpha^2)r_- + 3(1 + \alpha^2)r_+}{4(1 + \alpha^2)}\right)^2 - 2\frac{r_+ r_-}{1 + \alpha^2}}. \quad (4.13)$$

When

$$\begin{cases} F(r_{0\pm}) = 0 \\ F'(r_{0\pm}) = 0 \end{cases} \quad (4.14)$$

the event and cosmological horizons coincide. As $F(r) \rightarrow -\infty$ for $r \rightarrow \infty$, starting with $F(r_-) < 0$, if $F(r)$ takes a positive value at some coordinate value this ensures that it crosses twice the abscissa axis thus allowing the existence of two horizons. Therefore, when $r_{0\pm}$ are both greater than r_- , $r_+ > r_-$, the black hole solution exists in the parameter region of (r_+, r_-) where $F(r_{0+}) > 0$ and $F(r_{0-}) < 0$.

¹The solutions of the system are always two as the equation $F(r) = 0$, with the prior $F'(r) = 0$, reduces to a quadratic equation for r .

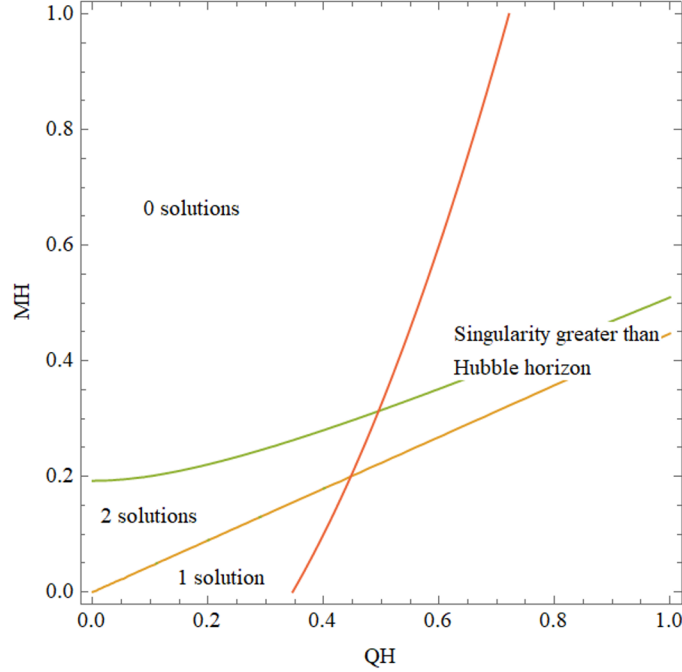


Figure 2. Number of horizons of the $\alpha > 1$ de Sitter black hole as a function of MH and QH . The green curve represents $F(r_{0+}) = 0$, the yellow one the limit $Q^2 = (1 + \alpha^2)M^2$, and the red one $r_-H = 1$. We chose for the illustration $\alpha = 2$ and $\phi_0 = 0$.

Next, note that for $\alpha > 1$, $r_{0-} < r_-$ does not intervene. Using $F(r_-) < 0$, the region where two horizons are present is defined by $F(r_{0+}) > 0$. In terms of M and Q , $F(r_{0+}) > 0$ translates to

$$\begin{aligned}
 & \left((1 - 2\alpha^2)M + (\alpha^2 - 2)\sqrt{M^2 - (1 - \alpha^2)Q^2e^{2\alpha\phi_0}} + \sqrt{P(M, Q, \alpha, \phi_0)} \right) \\
 & - H^2 \left((10 + 9\alpha^2)M - (4 + 3\alpha^2)\sqrt{M^2 - (1 - \alpha^2)Q^2e^{2\alpha\phi_0}} + \sqrt{P(M, Q, \alpha, \phi_0)} \right)^{\frac{3\alpha^2 - 1}{1 + \alpha^2}} \\
 & \times \left((4 + 3\alpha^2)M - \sqrt{M^2 - (1 - \alpha^2)Q^2e^{2\alpha\phi_0}} + \sqrt{P(M, Q, \alpha, \phi_0)} \right)^{\frac{4}{1 + \alpha^2}} > 0,
 \end{aligned} \tag{4.15}$$

where $P(M, Q, \alpha, \phi_0)$ is defined by

$$\begin{aligned}
 P(M, Q, \alpha, \phi_0) = & (17 + 24\alpha^2 + 9\alpha^4)M^2 - (9 + 15\alpha^2 + 8\alpha^4)Q^2e^{2\alpha\phi_0} \\
 & - (8 + 6\alpha^2)M\sqrt{M^2 - (1 - \alpha^2)Q^2e^{2\alpha\phi_0}}.
 \end{aligned} \tag{4.16}$$

Figure 2 presents the results of the discussion above. In analogy to the case $\alpha = 1$, we have added a further constraint for the singularity to be inside the Hubble horizon, $r_- < \frac{1}{H}$. When the black hole charge vanishes, the equation $F(r_{0+}) = 0$ reduces to $M = \frac{1}{\sqrt{27}H}$. Consider in this figure a point in the region corresponding to a black hole with two horizons and vary the charge or the mass:

- Increasing the mass, the event horizon reaches the cosmological one for the black hole mass M such that $F(r_{0+}) = 0$. Beyond this value, the singularity is naked.
- Increasing the charge, instead, we encounter at some point the line $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$, where the event horizon and the singular surface r_- coalesce. Continuing to increase the charge, the electromagnetic energy density becomes strong enough to prevent the formation of an event horizon. The WGC states are expected to have mass and charge in this region of parameters.

The WGC would then require the existence of a state with $q^2 e^{2\alpha\phi_0} > (1 + \alpha^2)m^2$, as in the case $\alpha = 1$. Although the presence of a cosmological constant changes the form of $g_{00}(r)$, in the $\alpha > 1$ case the weak gravity bound would take the same form as in asymptotically flat space-time.

4.3 $\alpha < 1$

For $\alpha < 1$, both the terms in g_{00} given by:

$$g_{00}(r) = - \left[\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} - H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \right]$$

vanish when $r \rightarrow r_-$. For $\alpha^2 = \frac{1}{3}$, the r_- dependence factorizes and g_{00} can be written as

$$g_{00}(r) \Big|_{\alpha^2=\frac{1}{3}} = - \left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}} \left(1 - \frac{r_+}{r} - H^2 r^2\right), \quad (4.17)$$

where the second factor takes the form of the g_{00} of a Schwarzschild-de Sitter metric with mass $M^* \equiv \frac{r_+}{2}$. The relative importance of the two terms in g_{00} depends on whether $\alpha^2 < \frac{1}{3}$ or $\alpha^2 > \frac{1}{3}$. A priori, we may expect a dilatonic-like black hole behaviour for $\frac{1}{\sqrt{3}} < \alpha < 1$, similar to the $\alpha > 1$, while a different, de Sitter-like black hole, behaviour for $\alpha < \frac{1}{\sqrt{3}}$. We thus split the $\alpha < 1$ analysis in three parts: $\frac{1}{\sqrt{3}} < \alpha < 1$, $\alpha = \frac{1}{\sqrt{3}}$ and $\alpha < \frac{1}{\sqrt{3}}$.

4.3.1 $\frac{1}{\sqrt{3}} < \alpha < 1$

Factorizing the $(1 - \frac{r_-}{r})$ term, we study the zeros of F defined in (4.12), with $3\alpha^2 - 1 > 0$. The only difference with the $\alpha > 1$ case comes from the convergence of the first term in g_{00} when $r \rightarrow r_-$. The convergence is to 0^+ when $r_+ > r_-$ and to 0^- when $r_+ < r_-$. Concretely, this does not affect the zeros of F , and thus the results obtained in the case $\alpha \geq 1$. In the (Q, M) plane, the boundaries of the region allowing black holes is still given by $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$ and $F(r_{0+}) = 0$. Note that the most involved part of the analysis in the case $\alpha > 1$ was for the situation $r_+ > r_-$ and used $r_{0-} < r_-$. While in the region $\alpha \in]\frac{1}{\sqrt{3}}, 1[$, r_{0-} can become greater than r_- , this only happens when $r_+ < r_-$ and therefore does not modify that analysis. Black hole arguments would again indicate for the WGC the existence of a particle satisfying $q^2 e^{2\alpha\phi_0} > (1 + \alpha^2)m^2$. As long as the second term in g_{00} (*de Sitter-like*) is sub-dominant, the transition between black holes and naked singularities (with a cosmological horizon) seems to happen in the same parametric region as in asymptotically flat space-time.

4.3.2 $\alpha = \frac{1}{\sqrt{3}}$

The $\alpha = \frac{1}{\sqrt{3}}$ case allows for explicit expressions of the horizons and can be studied in full details. The second factor in (4.17) can be seen as the time component of a Schwarzschild-de Sitter metric with an effective mass $M^* \equiv r_+/2$. This factor has two zeros for $r_+ < \frac{2}{\sqrt{27}H}$, degenerate for $r_+ = \frac{2}{\sqrt{27}H}$, and none for $r_+ > \frac{2}{\sqrt{27}H}$. The roots of the polynomial $P(r) \equiv r - r_+ - H^2 r^3$ are:

$$\left\{ \begin{array}{l} r_c = \frac{1}{H} \left(\frac{\left(\frac{2}{3}\right)^{1/3}}{\left(-9r_+H + \sqrt{3}\sqrt{-4 + 27r_+^2 H^2}\right)^{1/3}} + \frac{\left(-9r_+H + \sqrt{3}\sqrt{-4 + 27r_+^2 H^2}\right)^{1/3}}{2^{1/3}3^{2/3}} \right) \\ r_h = -\frac{1}{H} \left(\frac{\left(\frac{2}{3}\right)^{1/3} e^{-i\pi/3}}{\left(-9r_+H + \sqrt{3}\sqrt{-4 + 27r_+^2 H^2}\right)^{1/3}} + \frac{\left(-9r_+H + \sqrt{3}\sqrt{-4 + 27r_+^2 H^2}\right)^{1/3} e^{i\pi/3}}{2^{1/3}3^{2/3}} \right) \\ r_{--} = -\frac{1}{H} \left(\frac{\left(\frac{2}{3}\right)^{1/3} e^{i\pi/3}}{\left(-9r_+H + \sqrt{3}\sqrt{-4 + 27r_+^2 H^2}\right)^{1/3}} + \frac{\left(-9r_+H + \sqrt{3}\sqrt{-4 + 27r_+^2 H^2}\right)^{1/3} e^{-i\pi/3}}{2^{1/3}3^{2/3}} \right), \end{array} \right. \quad (4.18)$$

where r_c , r_h are the cosmological and the event horizons, respectively, and r_{--} is negative thus of no physical interest. In fact, one can see from the coefficients of P that the product of the roots is $-\frac{r_+}{H^2} < 0$ and their sum is null. So, there are either two real positive roots and a negative one (corresponding to the case where we have two horizons) or two complex conjugate and a negative root (corresponding to the case where no horizon is present). The transition between these two regimes happens when the horizons coincide, $r_c = r_h$, i.e. for $r_+ = \frac{2}{\sqrt{27}H}$. We also require that these roots are located outside the singular surface at r_- . In order to study the behaviour of the roots of g_{00} , we consider the equations:

$$\left\{ \begin{array}{l} r_c - r_- = 0 \\ r_h - r_- = 0 \\ r_+ = \frac{2}{\sqrt{27}H}. \end{array} \right. \quad (4.19)$$

We can see in figure 3 (left panel) the different regimes for g_{00} in the (r_+H, r_-H) plane. It is instructive to understand the (r_+H, r_-H) diagram before moving to the physical parameters M and Q . The red curve is $r_+ = \frac{2}{\sqrt{27}H}$, while the blue and yellow ones represent

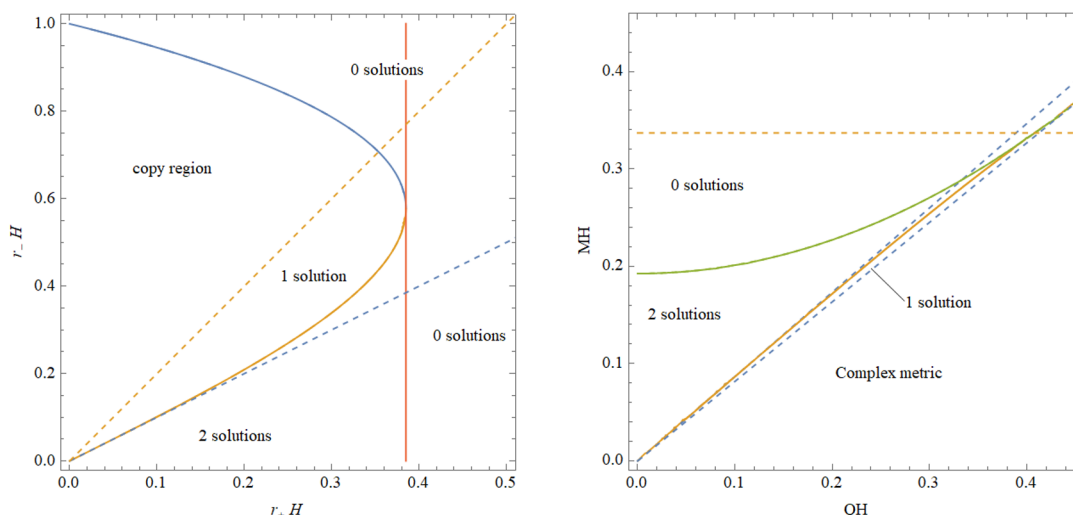


Figure 3. Number of horizons for $\alpha^2 = \frac{1}{3}$ in $(r_+ H, r_- H)$ (left) and (QH, MH) coordinates (right). *Left:* the region below the blue curve corresponds to $r_c > r_-$ and the region below the yellow one to $r_h > r_-$. The intersection of the two regions is populated by black hole solutions. On its right, the dS space-time causal patch is completely eaten. The blue dotted curve is $r_+ = r_-$, the yellow dotted one is $r_+ = r_-/2$. The portion of plane above the latter does not have counterpart in (QH, MH) . *Right:* green and yellow curves translate, respectively, the blue and yellow ones from left panel. Blue dotted curves are $Q^2 = (4/3)M^2$ and $Q^2 = (3/2)M^2$. The union of the region between the yellow curve and the lower dotted blue one from $(0, 0)$ to $(\frac{1}{\sqrt{6}}, \frac{7}{12\sqrt{3}})$ and the one between the green and the same dotted line from $(\frac{1}{\sqrt{6}}, \frac{7}{12\sqrt{3}})$ to $(\frac{\sqrt{3}}{4}, \frac{1}{2\sqrt{2}})$ give the region with a singularity surrounded only by a cosmological horizon. The yellow dotted line represents the maximal black hole mass: $M_{\max} = \frac{7}{12\sqrt{3}H}$.

$r_c - r_-$ and $r_h - r_-$, respectively. They intersect each other at the point $(r_+, r_-) = (\frac{2}{\sqrt{27}H}, \frac{1}{\sqrt{3}H})$. As long as $r_+ < \frac{2}{\sqrt{27}H}$, $r_c > r_-$ is realized below the blue curve and $r_h > r_-$ below the yellow one. As a consistency check of the method used for generic α , one can verify here, thanks to the explicit expressions of the horizons, the equivalence of the condition $r_c \geq r_-$, $r_+ \leq \frac{2}{\sqrt{27}H}$ with $F(r_{0+}) \geq 0$ and of $r_h \geq r_-$, $r_+ \leq \frac{2}{\sqrt{27}H}$ with $F(r_{0-}) \leq 0$. The different regimes are related in the following way:

- The transition between a black hole solution and a naked singularity with no cosmological horizon happens for $r_+ = \frac{2}{\sqrt{27}H}$ when $r_- \leq \frac{1}{\sqrt{3}H}$.
- A transition from the black hole to a naked singularity with a cosmological horizon can only happen in the combined interval $r_+ \in [0, \frac{2}{\sqrt{27}H}]$, $r_- \in [0, \frac{1}{\sqrt{3}H}]$ when we cross the yellow curve representing $r_h = r_-$. We find here a new bound compared to the $r_+ = r_-$ (blue dashed one) present for $\alpha^2 > \frac{1}{3}$.
- The regions defining a naked singularity with or without a cosmological horizon meet on the blue curve when $r_- > \frac{1}{\sqrt{3}H}$.

The region of existence of the black hole is larger than what it would have been if it was bounded by $r_+ = r_-$ ($Q^2 e^{2\alpha\phi_0} = \frac{4}{3}M^2 (= (1 + \alpha^2)M^2)$). The line $r_+ = r_-/2$, correspond-

ing to $Q^2 = (3/2)M^2 (= M^2/(1 - \alpha^2))$, is also shown in the diagram (yellow dashed one). It entirely lies in a region where the singularity is naked. Along this dotted line, the separation between the regions where the singularity is surrounded or not by a cosmological horizon is given by its intersection with the blue curve. The problematic region of a complex valued metric limiting the definition of the coordinates does not intervene in the black hole region. It can only be reached after the singularity has been exposed. Note that each value of (QH, MH) corresponds to two choices of the coordinates (r_+, r_-) , one of which is above the line $r_+ = r_-/2$ and one below it. However, only this last region leads to black hole solutions.

A similar analysis can be carried in terms of the mass and charge parameters (M, Q) , using the equations $r_+ = M + \sqrt{M^2 - \frac{2}{3}Q^2 e^{2\alpha\phi_0}}$, $r_- = \frac{4}{3}Q^2 / (M + \sqrt{M^2 - \frac{2}{3}Q^2 e^{2\alpha\phi_0}})$. It is convenient to use $\hat{M} \equiv MH$ and $\hat{Q} \equiv e^{\alpha\phi_0}QH$, and the conditions (4.19) then take the following form

$$r_c - r_- = 0 \Leftrightarrow \frac{\left(\frac{2}{3}\right)^{1/3}}{\left(-9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{-4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3} + \frac{\left(-9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{-4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3} 3^{2/3}}} = \frac{4\hat{Q}^2}{3\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}, \quad (4.20)$$

$$r_h - r_- = 0 \Leftrightarrow \frac{\left(\frac{2}{3}\right)^{1/3} e^{-i\pi/3}}{\left(-9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{-4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3} + \frac{e^{i\pi/3}\left(-9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{-4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3} 3^{2/3}}} = -\frac{4\hat{Q}^2}{3\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}, \quad (4.21)$$

and

$$r_+ = \frac{2}{\sqrt{27}H} \Leftrightarrow \hat{M} = \frac{1}{\sqrt{27}} + \frac{\sqrt{3}\hat{Q}^2}{2}. \quad (4.22)$$

It is possible to identify the *triple point* $(r_+, r_-) = \left(\frac{2}{\sqrt{27}H}, \frac{1}{\sqrt{3}H}\right)$ where the regions with two, one and zero solutions meet with $(\hat{Q}, \hat{M}) = \left(\frac{1}{\sqrt{6}}, \frac{7}{12\sqrt{3}}\right)$. We now have all the elements to understand the phase diagram in the (\hat{Q}, \hat{M}) plane as displayed in figure 3 (*right* panel):

- Restricting to masses below the *triple point*, thus $\hat{M} < \frac{7}{12\sqrt{3}}$, the upper bound on the mass allowing the two-horizons solution is given by $r_+ = \frac{2}{\sqrt{27}H}$, as shown in the (r_+H, r_-H) plane. It is represented by the green curve, $\hat{M} = \frac{1}{\sqrt{27}} + \frac{\sqrt{3}\hat{Q}^2}{2}$, from $(0, \frac{1}{\sqrt{27}})$ up to the triple point. Above it, the bound from (4.20) is stronger and the green curve now corresponds to $r_c = r_-$.
- The lower bound on this black hole region is given by (4.21) and it is represented in yellow in figure 3. It corresponds to the limit where the event horizon coincides with the singularity. Note that the lower bound yellow curve starts at $(0,0)$ and crosses the upper bound green curve at the *triple point*. After that, the yellow curve runs above the green one and does not bound any physical region.

The graphical representation of the two bounds reveals that for masses $M > \frac{7}{12\sqrt{3}H}$ the event horizon cannot form: this is the maximal mass above which asymptotically de Sitter black hole solutions are no more possible (yellow dashed line). Accordingly, this point corresponds to a maximal charge $Q_{\max} = \frac{1}{\sqrt{6}H}$.

- Singularities with only a cosmological horizon are found in the domain given by the union of: (1) the region between the yellow and the lower dashed blue curve from the origin up to the *triple point*, (2) the region between the green and the same dashed blue curve, now above it. On this dashed curve $Q^2 e^{2\alpha\phi_0} = (3/2)M^2$ marks the limit of definition of the metric.

The $(r_c = r_-)$ green curve crosses the blue one, corresponding to $Q^2 e^{2\alpha\phi_0} = (3/2)M^2$ in the point $(Qe^{\alpha\phi_0}, M) = (\frac{\sqrt{3}}{4H}, \frac{1}{2\sqrt{2}H})$. This is a point of maximal charge and mass. Above it, the green curve delimiting singularities with and without cosmological horizon cannot be drawn: either it is not defined, or it lies inside the inaccessible region (complex metric).

To confirm that (4.21) can be seen as a WGC bound, one can look at its behaviour when $H \rightarrow 0$. To look at this limit, let us rewrite r_h , in the region where it is real, as

$$r_h = \frac{2}{\sqrt{3}H} \sin\left(\frac{\theta}{3}\right), \tag{4.23}$$

where the angle θ is defined by $\sin(\theta) = (3\sqrt{3}/2)r_+H$ and $\cos(\theta) = \sqrt{1 - (27/4)r_+^2H^2}$. In the limit $H \ll 1/r_+$, one obtains

$$r_h = r_+ + H^2 r_+^3 + \mathcal{O}(r_+^3 H^4). \tag{4.24}$$

Looking at $r_h - r_- = 0$, replacing r_+ and r_- by their definition in function of M and Q (2.10), one can write the expansion of Q in powers of H as

$$Q^2 e^{2\alpha\phi_0} = \frac{4}{3}M^2 + \frac{4^3}{3^4}M^4 H^2 + \mathcal{O}(M^6 H^4). \tag{4.25}$$

In the limit $H \rightarrow 0$, the bound given by (4.21) reduces to (3.1).

In conclusion, for $\alpha = 1/\sqrt{3}$, the study of horizons of these dilatonic black holes would rather suggest (4.21) as a WGC bound than (3.1).

4.3.3 $\alpha < \frac{1}{\sqrt{3}}$

As shown above, when $\alpha = 1/\sqrt{3}$, $r_+ = r_-$ is no longer the black hole extremality condition, as it was for all cases with $\alpha > 1/\sqrt{3}$. In the following, we will see that this remains true for $\alpha < 1/\sqrt{3}$.

The first difference one can observe with respect to previous cases is the change in the behaviour of the derivative of $g_{00}(r)$ in a neighborhood of the singularity. Leading terms are given by

$$\begin{cases} \partial_r(g_{00}) \underset{r \rightarrow r_-}{\sim} -\frac{1-\alpha^2}{1+\alpha^2} \frac{r_-}{r^2} \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{-2\frac{\alpha^2}{1+\alpha^2}} & \text{for } \alpha > \frac{1}{\sqrt{3}} \\ \partial_r(g_{00}) \underset{r \rightarrow r_-}{\sim} 2\frac{\alpha^2}{1+\alpha^2} H^2 r_- \left(1 - \frac{r_-}{r}\right)^{-\frac{1-\alpha^2}{1+\alpha^2}} & \text{for } \alpha < \frac{1}{\sqrt{3}}. \end{cases} \quad (4.26)$$

When $\alpha < 1/\sqrt{3}$, the sign of the derivative at the vicinity of the singular surface $r = r_-$ is independent of r_+ and is always positive, with $\lim_{r \rightarrow r_-^+} g_{00}(r) = 0^+$: g_{00} always reaches 0 from above. This, combined with the asymptotic value $g_{00} \rightarrow +\infty$ when $r \rightarrow \infty$, implies that the metric exhibits a horizon only if the parameters in g_{00} are such that the function is decreasing in an interval to reach a negative minimum. In this situation, there are two horizons, coincident when the minimum of g_{00} is 0. This leads to a first conclusion:

- In the parametric (QH, MH) space, a singularity surrounded only by a cosmological horizon can only appear on a curve, rather than in a portion of the plane as happened for all cases with $\alpha \geq \frac{1}{\sqrt{3}}$.

Remember that, in contrast with the asymptotically flat case, here both $r_+ > r_-$ and $r_+ \leq r_-$ are now allowed. The method used above to investigate the limits between regions with different behaviours of the horizons was valid only for $r_+ > r_-$ but can now be extended to all the situations. We proceed thus by using the function F defined in (4.12), and its decomposition into A and B . For $\alpha^2 < 1/3$, $B \rightarrow +\infty$ when $r \rightarrow r_-$. In a neighborhood of r_- we always have $B(r) > A(r)$.

We notice that the condition $F(r_{0-}) < 0$ is not very illuminating in this case of $\alpha^2 < 1/3$. Indeed, for $r_+ > r_-$, $F(r_{0-})$ is always negative while for $r_+ \leq r_-$, expanding (4.13), we have $r_{0-} \leq r_-$, i.e. r_{0-} lies inside the singular surface and $F(r_{0-}) < 0$ should not be considered.

One should recall here that, for any $\alpha \neq 0$, the solution is plagued by the appearance of the complex valued metric for $Q^2 e^{2\alpha\phi_0} (1 - \alpha^2) > M^2$. Sweeping the whole $r_+, r_- \in \mathbb{R}^+$ parametric space, we have no access to that region. All we can say here is that constraints from $F(r_{0-})$ do not appear in the region where the metric is defined real. For $\alpha = 0$, the metric is real valued in the whole (Q, M) plane: this allows us to verify that $F(r_{0-}) = 0$ matches the condition for the existence of naked singularities with a cosmological horizon in the RNdS metric found in [6].

Therefore:

- The condition for the existence of black hole solutions is given by $F(r_{0+}) > 0$. When $F(r_{0+}) = 0$, the event and cosmological horizons coincide. $F(r_{0+}) < 0$ defines naked singularities with no cosmological horizon.

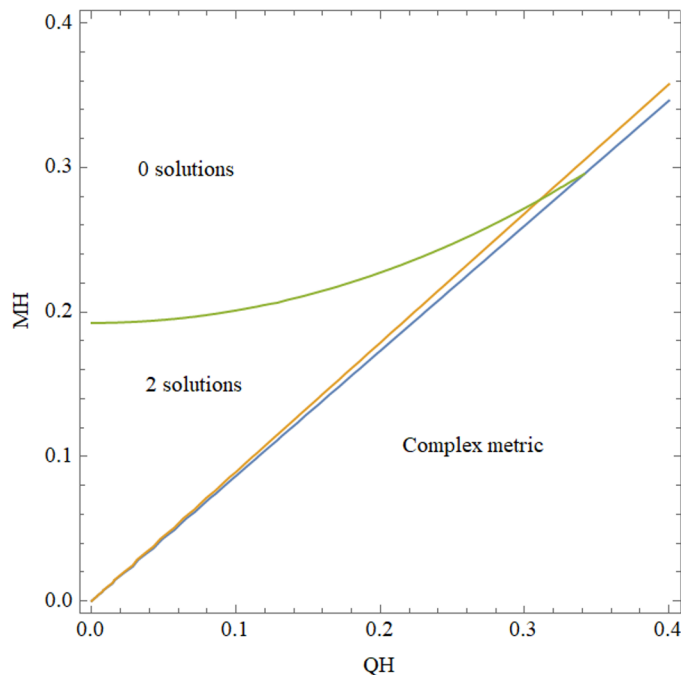


Figure 4. Number of horizons for $\alpha < 1/\sqrt{3}$, here illustrated by the value $\alpha = 1/2$. The green curve represents $F(r_{0+}) = 0$ and gives an upper bound on the mass. The yellow line represents $Q^2 = (1 + \alpha^2)M^2$, that does not play anymore the same role for $\alpha < 1/\sqrt{3}$. The blue one is $Q^2 = M^2/(1 - \alpha^2)$. In the region between the green and the blue curves, cosmological and event horizons are present. Below the blue one, r_+ , r_- and the metric become complex valued.

There is a maximal mass, above which there is no black hole solution. This mass corresponds to the point where the curve defined by $F(r_{0+}) = 0$ crosses the line defined by $Q^2 e^{2\alpha\phi_0} (1 - \alpha^2) = M^2$. It is given by

$$M_{\max} = \frac{1}{2\sqrt{2}H} \left(\frac{1 - 3\alpha^2}{2(1 - \alpha^2)} \right)^{\frac{1-3\alpha^2}{2(1+\alpha^2)}} \quad (4.27)$$

The behaviour of the horizons for the asymptotically de Sitter metric for $\alpha < 1/\sqrt{3}$ is described in figure 4 where we have taken, for an explicit illustrative example, $\alpha = 1/2$.

In figure 4, the region of the (QH, MH) plane with two horizons shows an upper bound represented by the green curve, $F(r_{0+}) = 0$. On the green curve, the event and cosmological horizons coincide. The lower bound is given by the blue curve, where $Q^2 e^{2\alpha\phi_0} (1 - \alpha^2) = M^2$ and the metric is on the verge of becoming complex. Plots of g_{00} reveal that, on this line, the event horizon and the singularity are still far apart. Approaching this line from above (the black hole solution region), we see that the event horizon and the singularity get closer but never touch. Expected solutions with only the cosmological horizon seem to be hidden inside the inaccessible region. The maximal mass for the black hole, corresponding to the crossing point of the green and blue curves are given by $M_{\max}H = 2^{-\frac{8}{5}} 3^{\frac{1}{10}} \simeq 0.3$.

We can compare with the RN-dS black hole solution studied in [6, 22], corresponding here to $\alpha = 0$. In that case the (QH, MH) plane shows a central region with three horizons surrounded by two regions with one horizon. One of them is attained, in parametric space,

after the event horizon has reached the cosmological one, and is interpreted as a dS space-time causal patch eaten by the black hole. The other, related in [6] to dS-WGC states, is beyond the locus of the coincidence of the inner and event horizons. For $0 < \alpha < 1/\sqrt{3}$, there are two horizons in a central region and zero horizons outside. Strictly speaking, $r = r_-$ is a zero of g_{00} for all $\alpha < 1$ but the inner horizon is traded for a singularity. The instability of Cauchy horizons may provide an additional motivation towards their identification. However, we have observed that for $\alpha \rightarrow 0$, while the two cosmological and event horizons tend to their corresponding surfaces in RN-dS, numerically the singularity r_- seems not to coincide exactly with the inner horizon but lies slightly above: it is no more a singularity neither an horizon. In fact, in this limit, the expression of $r_- = r_-(Q, M)$, given in (2.10), takes the same form as the RN black hole inner horizon and becomes trivially this surface when $H \rightarrow 0$.

Finally, note that for $\alpha = 0$ the parametric equations $F(r_{0\pm}) = 0$ reproduce the relations separating the regions with different horizons in [6]

$$\begin{cases} F(r_{0-}) = 0 \Leftrightarrow_{\alpha=0} M_-^2 = \frac{1}{54l} [l(l^2 + 36Q^2) - (l^2 - 12Q^2)^{\frac{3}{2}}] \\ F(r_{0+}) = 0 \Leftrightarrow_{\alpha=0} M_+^2 = \frac{1}{54l} [l(l^2 + 36Q^2) + (l^2 - 12Q^2)^{\frac{3}{2}}] \end{cases} \quad (4.28)$$

with $l = \frac{1}{H}$. It is M_-^2 , and thus $F(r_{0-})$, that marks the transition between black holes and naked singularities with cosmological horizon. However, the solution of $F(r_{0-}) = 0$ can not be used for $0 < \alpha < 1/\sqrt{3}$, as the metric is complex in that region. Note that in all previous literature, because the asymptotically flat metric always shows a naked singularity before turning complex, this region was simply ignored.

5 Dilatonic Anti-de Sitter black holes

Changing the sign of the H^2 terms in the metric (2.8) allows to obtain a particular class of dilatonic asymptotically AdS black hole solutions. For completeness, we will investigate the phase space exhibiting the behaviour of the horizons as one varies α , M and Q using the same method as for the de Sitter case.

We start by briefly recalling the Reissner Nordström AdS case as it will correspond to the $\alpha \rightarrow 0$ limit. The time component of the metric is $g_{00}(r) = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} + H^2 r^2\right)$. Its roots are given by those of the polynomial $G(r) \equiv H^2 r^4 + r^2 - 2Mr + Q^2$. They are two (degenerate in the extremal case) real positive roots as long as

$$M^2 \geq \frac{1}{54} \left(36Q^2 - \frac{1}{H^2} + \frac{(1 + 12H^2Q^2)^{\frac{3}{2}}}{H^2} \right). \quad (5.1)$$

The presence of horizons can be inspected through the study of the zeros of the function:

$$F_{\text{AdS}}(r) \equiv r - r_+ + H^2 r^3 \left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2 - 1}{1 + \alpha^2}} = -r \left(1 - \frac{r_-}{r}\right)^{\frac{1 - \alpha^2}{1 + \alpha^2}} g_{00}(r). \quad (5.2)$$

This turns out to be much simpler to study than the corresponding dS function F . It is indeed straightforward to see that whenever $r_+ < r_-$, $F_{\text{AdS}} > 0$ for all $r \in [r_-, \infty[$ and so

the function cannot have zeros. It will prove useful to split F_{AdS} into the sum of the two contributions $A_{\text{AdS}}(r) \equiv r - r_+$, a straight line, and $B_{\text{AdS}}(r) \equiv -H^2 r^3 \left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2 - 1}{1 + \alpha^2}}$ which is always negative. In this way, the problem is again recast in terms of the intersection points of A_{AdS} and B_{AdS} . We split the discussion into three parts depending on the value of α .

$\alpha^2 > \frac{1}{3}$: when approaching the singularity, B_{AdS} goes to 0^- . As a consequence, the curves defined by A_{AdS} and B_{AdS} have either one intersection point when $r_+ \geq r_-$ ($A_{\text{AdS}}(r_-) \leq 0$), or no intersections at all when $r_+ < r_-$ ($A_{\text{AdS}}(r_-) > 0$). We conclude that the discriminant between the black hole regime and the naked singularity is given by $r_+ = r_-$ i.e. $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$.

$\alpha^2 = \frac{1}{3}$: B_{AdS} does not depend on r_- : this is again related to a factorization in the metric as we have seen in the dS case. As such, F_{AdS} now corresponds to the Schwarzschild-AdS polynomial $F_{\text{AdS}}(r) = r - r_+ + H^2 r^3$, whose roots are given by

$$\left\{ \begin{array}{l} r_h = \frac{1}{H} \left(-\frac{\left(\frac{2}{3}\right)^{1/3}}{\left(9r_+H + \sqrt{3}\sqrt{4 + 27r_+^2 H^2}\right)^{1/3}} + \frac{\left(9r_+H + \sqrt{3}\sqrt{4 + 27r_+^2 H^2}\right)^{1/3}}{2^{1/3}3^{2/3}} \right) \\ r_{--} = \frac{1}{H} \left(\frac{\left(\frac{2}{3}\right)^{1/3} e^{-i\pi/3}}{\left(9r_+H + \sqrt{3}\sqrt{4 + 27r_+^2 H^2}\right)^{1/3}} - \frac{\left(9r_+H + \sqrt{3}\sqrt{4 + 27r_+^2 H^2}\right)^{1/3} e^{i\pi/3}}{2^{1/3}3^{2/3}} \right) \\ r_{--}^* = \frac{1}{H} \left(-\frac{\left(\frac{2}{3}\right)^{1/3} e^{i\pi/3}}{\left(9r_+H + \sqrt{3}\sqrt{4 + 27r_+^2 H^2}\right)^{1/3}} + \frac{\left(9r_+H + \sqrt{3}\sqrt{4 + 27r_+^2 H^2}\right)^{1/3} e^{-i\pi/3}}{2^{1/3}3^{2/3}} \right), \end{array} \right. \quad (5.3)$$

where r_h is the horizon and r_{--} and r_{--}^* are two complex conjugate (non-physical) roots. Accordingly, the condition for the singularity to be shielded by the horizon is just $r_h > r_-$, which reads:

$$\begin{aligned} & \frac{\left(9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3}3^{2/3}} \\ & - \frac{\left(\frac{2}{3}\right)^{1/3}}{\left(9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}} \\ & > \frac{4\hat{Q}^2}{3\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}. \end{aligned} \quad (5.4)$$

In the (\hat{Q}, \hat{M}) space this gives a lower bound on the mass which is a little higher than the asymptotically flat case: $\hat{Q}^2 > (4/3)\hat{M}^2$, as shown in figure 5 (left panel). The $H \rightarrow 0$ limit of (5.4) is again given by (4.25) with the change of sign in front of the H^2 term.

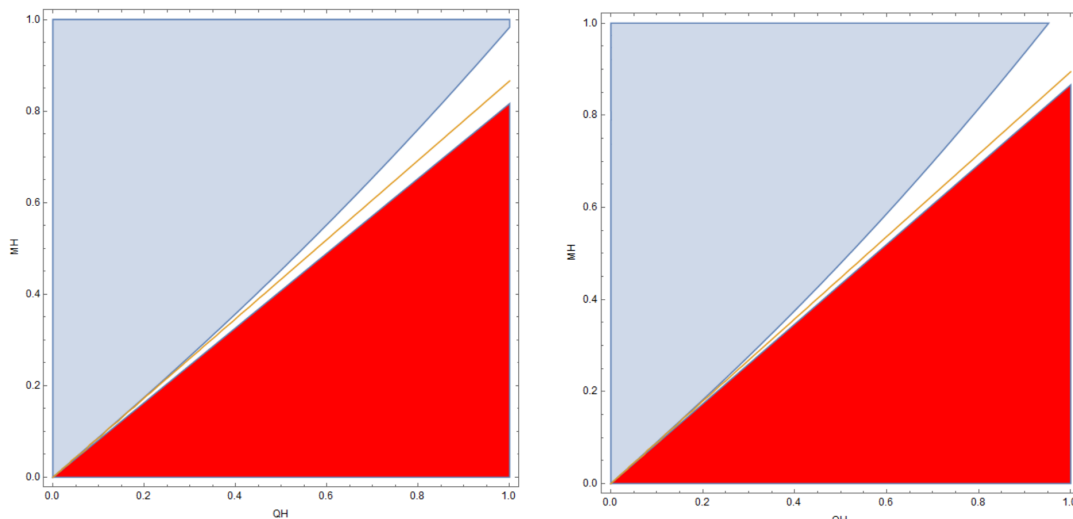


Figure 5. The dilatonic AdS black hole case. The *left panel* describes the phase diagram for the $\alpha^2 = 1/3$, the *right panel* shows the explicit example of $\alpha^2 = 1/4$ to illustrate the situation for $\alpha^2 < 1/3$. In both cases, the blue region corresponds to black hole solutions, the yellow line is the flat-space discriminant between shielded and naked singularities (playing no role here but shown for comparison) and the red region is the inaccessible region where the metric becomes complex.

$\alpha^2 < \frac{1}{3}$: this is again the most intricate parametric region. Here, B_{AdS} diverges to $-\infty$ when $r \rightarrow r_-$. For $r_+ \leq r_-$, since A_{AdS} is positive for all $r \geq r_-$, and so is the difference $A_{\text{AdS}} - B_{\text{AdS}}$, no horizon can ever be present.

On the other hand, when $r_+ > r_-$ we have $A_{\text{AdS}}(r) < 0$ for all $r \in [r_-, r_+[$, so the combination $A_{\text{AdS}} - B_{\text{AdS}}$ could result to be negative there. As B_{AdS} is a concave function of r , two roots will be present when A_{AdS} and B_{AdS} intersect, collapsing to one when they are tangent, and zero otherwise. The region of parameters allowing the presence of two horizons, the black hole solution region, is obtained as in the dS case by solving the combined equations $F_{\text{AdS}} = 0$ and $F'_{\text{AdS}} = 0$. The solutions to this system are the same $r_{0\pm}$ found in (4.13).² In the (r_+H, r_-H) plane, $F_{\text{AdS}}(r_{0+})$ is always null or positive, leading to no constraint in practice. As a consequence:

- The condition for the singularity to be shielded can be simply expressed as $F_{\text{AdS}}(r_{0-}) \leq 0$, with the equality being satisfied by extremal solutions with coincident horizons.

This leads to a lower bound on the mass, that lies above the flat-space one ($Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$), as illustrated in the example of figure 5 (*right panel*).

For $\alpha^2 = 1/3$, the lower bound on the mass coincides with the one obtained above by simply requiring $r_h - r_- \geq 0$, the set of curves are continuously connected. The presence of black holes with two horizons is a new characteristic that was not present for $\alpha \geq 1/\sqrt{3}$.

As in the dS case, we verify again the equivalence between the limit $\lim_{\alpha \rightarrow 0} F_{\text{AdS}}(r_{0-}) \leq 0$ and (5.1).

²Writing $F'_{\text{AdS}}(r)$ with the prior $F_{\text{AdS}}(r) = 0$, one sees that it is independent of the sign in front of the $H^2 r^3$ term and gives the same equations as in the dS case, $F'(r) = 0$ with the prior $F(r) = 0$.

Note that the singularity at $r = r_-$ changes its nature: from a space-like one, as it happens when it is behind the single $\alpha^2 > 1/3$ horizon, to a time-like one. This is dictated by the derivative of g_{00} that diverges now to $-\infty$ for $r \rightarrow r_-^+$. The $\alpha^2 < 1/3$ AdS black holes are the only ones where the singularity can be avoided: $r = r_-$ is not in the future light-cone of all the observers that crosses the horizons, g_{00} becomes time-like again before reaching it, as already observed by [28]. Of all the setups we studied, this is the only case where $\lim_{r \rightarrow r_-^+} g_{00} < 0$ in a black hole parametric region.

This is similar to what happens in the Reissner Nordstrom AdS metric for the $r = 0$ singularity. Varying α , starting with $\alpha^2 > 1/3$, we encounter at $\alpha^2 = 1/3$ a transition from Schwarzschild-AdS like black holes, with only one horizon and a space-like singularity, to Reissner-Nordstrom AdS like ones, with two horizons and a time-like singularity. For all $\alpha \neq 0$, the singularity at $r = r_-$ resembles here the singularity at $r = 0$ of $\alpha = 0$. Note that we do not encounter the issue of a complex metric, in contrast with the dS case, as the naked singularity bound is reached for values $M^2 > (1 - \alpha^2)Q^2$.

It can be interesting to look at the first correction to the flat space-time condition $r_- = r_+$ for small H . For $H = 0$, $F(r_{0-}) = 0$ reduces to $r_{0-} = r_+$ which is equivalent to $r_- = r_+$.

In order to find the first term in the expansion in H , we set

$$r_- = r_+ + cr_+^{\gamma+1}H^\gamma + o(r_+^{\gamma+1}H^\gamma), \tag{5.5}$$

where the constants c and γ have to be fixed.

From (5.5), it is possible to express r_{0-} as $r_{0-} = r_+ + \frac{(1+\alpha^2)}{2(1-\alpha^2)}cr_+^{\gamma+1}H^\gamma + o(r_+^{\gamma+1}H^\gamma)$. Requiring $F(r_{0-}) = 0$ at first order gives:

$$\gamma = \frac{1 + \alpha^2}{1 - \alpha^2} \quad \text{and} \quad \frac{1 + \alpha^2}{2(1 - \alpha^2)}c + \left[\frac{3\alpha^2 - 1}{2(1 - \alpha^2)}c \right]^{\frac{3\alpha^2 - 1}{1 + \alpha^2}} = 0. \tag{5.6}$$

For $\alpha < 1/\sqrt{3}$, c is single valued, negative, with the limits $c \rightarrow -2$ when $\alpha \rightarrow 0$ and $c \rightarrow -1$ for $\alpha \rightarrow 1/\sqrt{3}$. Plugging the relation between (r_+, r_-) and (Q, M) given by (2.10) in (5.5), we can look for the corresponding relation $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2 + bM^{2+\delta}H^\delta$, that defines the boundary of the black hole region. It is possible to determine the constants δ and b : $\delta = \gamma$ and $b = \alpha^2(1 + \alpha^2)^{\frac{2}{1-\alpha^2}}c$. Thus, the constraint $F(r_{0-}) = 0$ can be expanded for $H \rightarrow 0$ as

$$Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2 + \alpha^2(1 + \alpha^2)^{\frac{2}{1-\alpha^2}}cM^{\frac{3-\alpha^2}{1-\alpha^2}}H^{\frac{1+\alpha^2}{1-\alpha^2}} + o\left(M^{\frac{3-\alpha^2}{1-\alpha^2}}H^{\frac{1+\alpha^2}{1-\alpha^2}}\right). \tag{5.7}$$

We see that for $\alpha = 1/\sqrt{3}$, it reduces to

$$Q^2 e^{2\alpha\phi_0} = \frac{4}{3}M^2 - \frac{4^3}{3^4}M^4H^2 + o(M^4H^2). \tag{5.8}$$

It is the same equation as for the dS case, with a difference of sign. For $\alpha \rightarrow 0$, the power of H tends to 1, but the coefficient in front vanishes. This is coherent with [22], since there is no linear term in the expansion for small H .

6 Thermodynamics

The analysis of the existence of horizons shows that above and below the value $\alpha = 1/\sqrt{3}$ both the dS and AdS black holes have different properties. We have seen these differences in the behaviours of g_{00} and its first derivative $\partial_r g_{00}$. In this section, we shall analyse them using thermodynamic quantities of the black holes.³

The Hawking temperature of the black holes is given by

$$T = -\frac{\partial_r g_{00}}{4\pi} \Big|_{r=r_h} = \frac{1}{4\pi} \left[\frac{r_+}{r_h} \left(1 - \frac{r_-}{r_h^2}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} + \frac{1-\alpha^2}{1+\alpha^2} \left(1 - \frac{r_+}{r_h}\right) \left(1 - \frac{r_-}{r_h}\right)^{-\frac{2\alpha^2}{1+\alpha^2}} \frac{r_-}{r_h^2} \mp 2H^2 r_h \left(1 - \frac{r_-}{r_h}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \mp 2\frac{\alpha^2}{1+\alpha^2} H^2 r_- \left(1 - \frac{r_-}{r_h}\right)^{-\frac{1-\alpha^2}{1+\alpha^2}} \right], \quad (6.1)$$

where r_h is the radial coordinate of the event horizon. The upper (lower) sign refers to the dS (AdS) solution. The above expression could be simplified using the relation between r_+, r_- and r_h given by $g_{00}(r_h) = 0$, but it is more useful for our purposes to keep it in this form. Note that the second term in (6.1) does not vanish identically as now $r_h \neq r_+$ in general. We write the Hawking-Beckenstein black hole entropy proportional to the horizon area as

$$S = \pi r_h^2 \left(1 - \frac{r_-}{r_h}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \quad (6.2)$$

Note that this is the same as in asymptotically flat space and vanishes when $r_h \rightarrow r_-$ for any $\alpha \neq 0$.

For $\alpha > 1/\sqrt{3}$ extremal solutions are defined to have $r_h \rightarrow r_- = r_+$. The temperature diverges in the limit $r_h \rightarrow r_-$ when $\alpha \neq 1$ and we distinguish two cases.

- For $\alpha > 1$, the first and the second term of (6.1) lead the divergence. The temperature goes as

$$T \Big|_{r_h \rightarrow r_-} \sim \frac{1}{4\pi r_h} \left(1 + \frac{1-\alpha^2}{1+\alpha^2}\right) \left(1 - \frac{r_-}{r_h}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}. \quad (6.3)$$

- For $1/\sqrt{3} < \alpha < 1$, the divergence is lead by the last term ($H \neq 0$)

$$T \Big|_{r_h \rightarrow r_-} \sim \frac{1}{2\pi} \frac{\alpha^2}{1+\alpha^2} H^2 r_- \left(1 - \frac{r_-}{r_h}\right)^{-\frac{1-\alpha^2}{1+\alpha^2}} \quad (6.4)$$

For $\alpha = 1$, the temperature is finite and reads

$$T = \frac{1}{4\pi} \left(\frac{1}{2M} \mp 2MH^2 \right) \quad (6.5)$$

In this case, with the extremality condition $D = M$ and the necessary requirement for the singularity to be smaller than the Hubble radius in the dS case, $4D^2 H^2 \leq 1$, the

³For the physics of thermodynamic quantities in de Sitter space-time, we refer the reader to [31].

expression (6.5) is always positive (or null). Actually, this points out a rather interesting property: the extremal solution with the singularity of the same size as the Hubble horizon (denoted by the point where the green, yellow and red curve of figure 1 meet) has a null temperature. It is thus a trivial endpoint of Hawking evaporation. This is not the case in the asymptotically flat metric where $T = 1/8\pi M$ and one questioned whether the extremal solutions are endpoints of Hawking evaporation or not [32]. Note that the finiteness of T comes from the vanishing of the exponent $|(1 - \alpha^2)/(1 + \alpha^2)|$ in the continuous limit $\alpha \rightarrow 1$.

The divergence of the temperature was discussed in [32] for the asymptotically flat $\alpha > 1$ case. The divergence for $1/\sqrt{3} < \alpha < 1$ is new and entirely due to the presence of a non-vanishing cosmological constant. For the presence of horizons, the $1/\sqrt{3} < \alpha < 1$ and $\alpha > 1$ black holes share the same properties and are not much sensitive to the value of H when they approach extremal solutions. The temperature, however, has a different form in the two cases and shows a dependence on H .

When $0 < \alpha \leq 1/\sqrt{3}$, extremal solutions no longer have $r_+ = r_-$. We distinguish two cases.

- For $\alpha = 1/\sqrt{3}$, the extremality condition $r_h = r_-$ is reached when $F(r_{0-}) = 0$ (defined in (4.12) and (4.13)) or $F_{\text{AdS}}(r_{0-}) = 0$ (defined in (5.2)). The temperature of such black holes is

$$T_{r_h \rightarrow r_-} \sim \frac{1}{8\pi} \frac{r_-}{r_h^2} \left(1 - \frac{r_+}{r_h} \mp H^2 r_h^2\right) \left(1 - \frac{r_-}{r_h}\right)^{-\frac{1}{2}} = 0, \tag{6.6}$$

thus vanishes for the extremal solution as the factor inside the first parenthesis corresponds to $g_{00}(r_h)$ and is identically null. If one were to blindly take the limit $\alpha \rightarrow 1/\sqrt{3}$ of (6.4), the temperature of extremal solutions would seem to diverge with an exponent 1/2. This shows a discontinuity in the α -dependence of such exponent that can be traced back to the factorization in the metric and the loss of the extremality condition $r_+ = r_-$.

- When $0 < \alpha < 1/\sqrt{3}$, we have seen in both the dS and AdS cases that $r_h = r_-$ is never reached for different reasons.

In the dS case, extremal solutions could not be defined within the domain of a real valued metric.

In the AdS case, the black holes on the verge of exposing a naked singularity have the event horizon coincident with their inner horizon. As these do not coincide with the singularity, the temperature does not diverge anymore but vanishes as $\partial_r g_{00}(r_h) = 0$.

A simple interpretation of the behaviour of the temperature is as follows. The $\alpha \rightarrow \infty$ and $\alpha \rightarrow 0$ limits of such black holes are given by Schwarzschild and Reissner-Nordström black holes, respectively, with temperatures given by

$$T_{Sc} = \frac{1}{8\pi M}, \quad T_{RN} = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{\left(M + \sqrt{M^2 - Q^2}\right)^2} \tag{6.7}$$

in asymptotically flat space. In that case, the dilatonic black holes have a temperature

$$T = \frac{1}{4\pi r_+} \left(1 - \frac{r_-}{r_+}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}, \quad (6.8)$$

diverging for $\alpha > 1$, finite for $\alpha = 1$, and vanishing for $0 < \alpha < 1$ extremal solutions. We retrieve a vanishing temperature for small α , here $\alpha = 0$, as the extremality condition reads now $M = Q$. For large values of α , the extremality condition obtained by identification of the horizon $r_h (= r_+)$ with the singularity r_- , would formally correspond in the Schwarzschild case to put the horizon at the origin i.e. formally take the limit M tends to 0 in the black hole solution which in turn leads to a divergent temperature.

This remains true also in asymptotically (A)dS space-time. For large α , we consider the temperature of the Schwarzschild (A)dS black hole

$$T_{Sc} = \frac{1}{4\pi} \frac{1 \mp 3H^2 r_h^2}{r_h}, \quad (6.9)$$

where r_h is the radius of the event horizon.⁴ The extremality corresponds again to a formal $r_h \rightarrow 0$ limit, obtained when $M \rightarrow 0$ in (6.9). This leads to a divergent temperature. For small α , we consider instead the Reissner-Nordström (A)dS black holes, and T has a similar form [22]

$$T_{RN} = \frac{1}{4\pi} \frac{1 - \frac{Q^2}{r_h^2} \mp 3H^2 r_h^2}{r_h}, \quad (6.10)$$

where r_h indicates again the radius of the event horizon but defined by a different metric. The region of validity for the dS and AdS solutions are given in (4.28) and (5.1), respectively. Extremality is obtained then by taking the lower mass bound for which the temperature vanishes.

In this picture, the transition between diverging and vanishing temperatures of extremal dilatonic black holes could be then seen as the thermal footprint of a transition from a Schwarzschild-like behaviour to a Reissner-Nordström like one. Such transition happens for $\alpha = 1$ in asymptotically flat space, where the temperature is equal to $T = \frac{1}{8\pi M}$, and for $\alpha = 1/\sqrt{3}$ in asymptotically (A)dS space, where $T = 0$.

For the AdS black holes, one can observe a peculiar behaviour of the entropy formula applied to the extremal solution. It trivially vanishes above $\alpha = 1/\sqrt{3}$ as one has a naked singularity. It is finite for $\alpha < 1/\sqrt{3}$, increasing as $\alpha \rightarrow 0$: there, the extremal condition corresponds to the coincidence of two (non-singular) horizons.

7 Test particles in charged dilatonic black hole metric

The weak gravity conjectures, for abelian gauge symmetries, dilatonic or scalar interactions, have been formulated as constraints on the non-relativistic and, often but not always,

⁴Note, that in the dS case, the singularity is shielded as long as $M^2 H^2 < 1/27$, while in the AdS case it is for any mass $M > 0$. We do not discuss issues related to thermal equilibrium of observers in the region between the event and cosmological horizons in asymptotically dS space [33]. We restrict to the small H limit where the two are far away.

large distance interactions between charged states. These states can be elementary in the theory, but also solitonic as D-branes. Accordingly, we wish to study non-relativistic, large distance, leading interactions between the charged black holes. The latter, separated by very large distances and interacting through gravitons, gauge bosons and scalar fields with large wavelengths compared to their typical size, i.e. their horizon radius, look like point particles. One challenge for the point-like particle description of the interactions is to identify here the expression of the scalar coupling and associated scalar charge of these states.

It appears instructive to first consider the simpler case of a test particle submitted to the forces generated by a black hole. Also, taking in our computations the limit $H = 0$, allows to compare with the available results of explicit amplitude computation.

7.1 Large distance action of the dilatonic black holes on a test particle

In our effective theory description, the scalar charge of a point-like particle with respect to the dilaton ϕ appears encoded in the field-dependent mass $m(\phi)$. This in turn will be translated into a three-point coupling in a field theory context, as we shall discuss later. The action for the motion of a test particle of mass $m(\phi)$ and charge q in the black hole geometry is given by

$$S_m = \int d\tau \left(-m(\phi) \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + \sqrt{4\pi G} g q A_\mu \dot{x}^\mu \right), \quad (7.1)$$

where x^μ represent the particle's coordinates and the dot indicates a derivative with respect to the proper time τ . The last term is the coupling to the abelian gauge field A_μ , with a gauge coupling constant g . The mass m and the charge q are in geometrized units for consistency. The geodesic equations are

$$-m(\phi) \left(\ddot{x}^\mu + \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho \right) + \sqrt{4\pi G} g q F_{\rho}^\mu \dot{x}^\rho - \frac{dm(\phi)}{d\phi} \left(\dot{x}^\mu \dot{x}^\rho \partial_\rho \phi - \dot{x}^\rho \dot{x}_\rho \partial^\mu \phi \right) = 0, \quad (7.2)$$

where the Γ s denote the Christoffel symbols and $F_{\mu\rho}$ is the gauge field strength. Here, we are mainly interested in the last term, interpreted as a scalar force.

To study the motion of the test particle in the space-time defined by the metric (2.8), we first rewrite the Lagrangian as

$$\mathcal{L} = -m(\phi) \sqrt{f(r) \dot{t}^2 - \frac{\dot{r}^2}{f(r)} - r^2 g(r) \dot{\theta}^2 - r^2 g(r) \sin^2 \theta \dot{\phi}^2} - \frac{e^{2\alpha\phi_0} q Q}{r} \dot{t}, \quad (7.3)$$

where the gauge field was chosen as $A = \left(-\frac{gQ}{\sqrt{4\pi G} r}, 0, 0, 0 \right)$ and the gauge coupling is now $g = e^{\alpha\phi_0}$. In (7.3), we have introduced two functions f and g :

$$\begin{cases} f(r) \equiv \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \\ g(r) \equiv \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \end{cases} \quad (7.4)$$

where the \mp signs depend on whether we consider an asymptotically dS or AdS space-time, respectively.

We shall start by taking $H = 0$, corresponding to the asymptotically flat space-time solution, and discuss the $H \neq 0$ case in a second moment.

The spherical symmetry allows us to restrict the analysis to the equatorial plane $\theta = \frac{\pi}{2}$. The two Killing vectors ∂_t and ∂_φ correspond to two constant conserved quantities E and L , proportional to the energy and angular momentum as measured at infinity, respectively. They are given by:

$$\begin{cases} E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} = m(\phi)f(r)\dot{t} + \frac{e^{2\alpha\phi_0}qQ}{r} \\ L = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m(\phi)r^2g(r)\dot{\varphi}, \end{cases} \quad (7.5)$$

In (7.5) we have used the normalization $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ whose explicit form reads

$$-f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2g(r)\dot{\varphi}^2 = -1, \quad (7.6)$$

or, using (7.5),

$$-\frac{1}{m^2(\phi)f(r)}\left(E - \frac{e^{2\alpha\phi_0}qQ}{r}\right)^2 + \frac{\dot{r}^2}{f(r)} + \frac{L^2}{m^2(\phi)r^2g(r)} = -1. \quad (7.7)$$

Restricting to radial paths and null angular momentum L , this gives

$$\left(\frac{dr}{d\tau}\right)^2 = -f(r) + \frac{1}{m^2(\phi)}\left(E - \frac{e^{2\alpha\phi_0}qQ}{r}\right)^2. \quad (7.8)$$

After putting the equation in the form $\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) = 0$, one can read the forces from the $1/r$ coefficient in $V_{\text{eff}}(r)$, the Newtonian approximation being recovered at large distances ($r \gg r_-$). In this limit, using (2.8), the leading order of f and m are

$$f(r) = 1 - \frac{1}{r}\left(r_+ + \frac{1-\alpha^2}{1+\alpha^2}r_-\right) + \mathcal{O}\left(\frac{1}{r^2}\right) \quad (7.9)$$

$$m^2(\phi) = m^2\left(\phi_0 - \frac{\alpha}{1+\alpha^2}\frac{r_-}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) = m^2(\phi_0) - \frac{dm^2}{d\phi}\Bigg|_{\phi_0} \frac{\alpha}{1+\alpha^2}\frac{r_-}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (7.10)$$

Together with the identification in (2.9), this gives

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{\frac{E^2}{m_0^2} - 1}{2} + \frac{M}{r} + \frac{E^2}{m_0^2}\frac{m'_0 D}{r} - \frac{E}{m_0}\frac{e^{2\alpha\phi_0}\frac{q}{m_0}Q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad (7.11)$$

where the $'$ stands for the derivative with respects to ϕ and the subscripts 0 denote quantities evaluated at $\phi = \phi_0$. In (7.11), M , Q and D are the mass, the charge and scalar charge of the black hole expressed in geometrized units. Note that D is a secondary charge related to M and Q in (2.11). The E/m_0 factors should be seen as relativistic corrections,

intrinsically present in the GR framework, and important for $v/c \sim 1$. At leading order, the non-relativistic potential V_{pp} felt by a point-particle takes then the form

$$V_{pp}(r) \equiv m_0 V_{\text{eff}}(r) = -\frac{m_0 M + m'_0 D - e^{2\alpha\phi_0} q Q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (7.12)$$

We have therefore shown that the forces felt by a test particle match the result for the non-relativistic limit of the $2 \rightarrow 2$ scattering amplitude between this test particle and a state with the gauge charge, scalar charge and mass of the black hole.

Before turning to the question of how to extend this picture to the case of scattering of two point-like dilatonic black holes, we give here the final expression for the second order term in the expansion of $V_{\text{eff}}(r)$

$$V_{\text{eff}}^{(2)}(r) = -\frac{1}{2r^2} \left[\frac{e^{4\alpha\phi_0} q^2}{m_0^2} Q^2 - (1 - \alpha^2) (e^{2\alpha\phi_0} Q^2 - D^2) - \frac{1}{2} \frac{E^2}{m_0^2} D^2 \frac{m''_0{}^2}{m_0^2} + \frac{E^2}{m_0^2} \frac{1 + \alpha^2}{\alpha} D^2 \frac{m'_0}{m} - 4 \frac{E}{m_0} e^{2\alpha\phi_0} \frac{q}{m_0} Q D \frac{m'_0}{m_0} + 4 \frac{E^2}{m_0^2} D^2 \left(\frac{m'_0}{m_0}\right)^2 \right], \quad (7.13)$$

This generalizes the $1/r^2$ term one finds in the Reissner-Nordström case:

$$V_{\text{eff}}^{(2)}(r) = -\frac{1}{2r^2} \left(\frac{q^2}{m^2} Q^2 - Q^2 \right) \quad (7.14)$$

that is recovered in the limit $\alpha \rightarrow 0$ (thus $D \rightarrow 0$), giving a non vanishing contribution even for purely radial motion. Actually, in the dilatonic case, the expansion of the various terms formally gives contributions to the different orders in $1/r$.

For generic paths with $L \neq 0$, the $1/r^2$ term gets contributions both from (7.13) and from the angular momentum term

$$\frac{1}{m^2} \frac{L^2}{r^2} \frac{f(r)}{g(r)} \quad \text{with} \quad \frac{f(r)}{g(r)} = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1-3\alpha^2}{1+\alpha^2}} \quad (7.15)$$

The explicit expression of the angular momentum-related potential term is

$$\begin{aligned} \frac{1}{m^2} \frac{L^2}{r^2} \frac{f}{g} &= \frac{L^2}{m_0^2 r^2} \left[1 + \frac{2D \frac{m'_0}{m_0} - 2M + 2\alpha D}{r} \right. \\ &\quad + \frac{1}{r^2} \left[\frac{1 + \alpha^2}{\alpha} \frac{m'_0}{m_0} D - \frac{m_0'' D}{m_0^2} \frac{D}{2} + 2 \left(\frac{m'_0}{m_0}\right)^2 + (1 - 3\alpha^2) e^{2\alpha\phi_0} Q^2 \right. \\ &\quad \left. \left. + 4(1 + \alpha^2) \frac{m'_0}{m} D^2 - 2(1 + \alpha^2)(1 - 3\alpha^2) D^2 - 2 \frac{m'_0}{m_0} D M - 2 \frac{1 - \alpha^2}{\alpha} \frac{m'_0}{m_0} D \right] \right. \\ &\quad \left. + \mathcal{O}\left(\frac{1}{r^3}\right) \right]. \quad (7.16) \end{aligned}$$

In the limit $\alpha \rightarrow 0$ this reduces to

$$\frac{L^2}{m^2 r^2} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \quad (7.17)$$

An important difference for $\alpha \neq 0$, for the overall potential sourced by both V_{eff} and the angular momentum, is that the sub-leading contributions are parts of a formally infinite expansion in powers of r_-/r , where $r = r_-$ is the location of the singularity. As we approach r_- , higher orders become important and the whole expansion needs to be taken into account. Contrary to Schwarzschild or Reissner-Nordström solutions, there is no fixed-order dominant term whose sign determines whether the overall effective potential is attractive or repulsive around the singularity.

In the presence of a non-zero cosmological constant, the computation leading to (7.6) and (7.7) is unchanged but with the additional $\mp H^2 r^2 (1 - \frac{r_-}{r})^{\frac{2\alpha^2}{1+\alpha^2}}$ in $f(r)$. At large $r \gg r_-$, this can be expanded to give

$$\mp H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \underset{r \gg r_-}{\sim} \mp \left(H^2 r^2 - 2\alpha D H^2 r + (\alpha^2 - 1) D^2 H^2 + \frac{2}{3} \frac{\alpha^2 - 1}{\alpha} \frac{D^3 H^2}{r} + \frac{1}{6} \frac{(\alpha^2 - 1)(\alpha^2 + 3)}{\alpha^2} \frac{D^4 H^2}{r^2} + \mathcal{O}\left(\frac{r_-^5}{r^5}\right) \right). \quad (7.18)$$

The result is an additional contribution to the $1/r$ potential, trivially vanishing for $H \rightarrow 0$, $\alpha \rightarrow 0$ and $\alpha = 1$. Finally, the additional contribution to the angular momentum terms, given by $(L^2/m^2(\phi)r^2)(f(r)/g(r))$ leads to $\mp L^2 H^2/m_0^2$, at leading order.

7.2 Forces between two point-like states with the black holes charges

We will investigate now the interaction between two point-like states both describing the black hole type solutions. These states will be characterized by their mass, their charge and their coupling to the dilaton ϕ . We will pursue this description in region of parameters of the solution even beyond the extremal solution limit, therefore point-like states not corresponding to black holes anymore, in an attempt to get some indication of what happens where the metric becomes complex.

The question of how to associate the parameters of a dilatonic black hole to a particle state was addressed in [34, 35] for the case of an asymptotically flat space-time solution.

The black hole parameters (say its ADM mass, gauge charge and scalar charge) are defined at infinity. As such, for a point particle to effectively describe this black hole, its charge q , mass $m(\phi)$ and first derivative $m'(\phi)$ observed at infinity must satisfy the conditions

$$\begin{cases} m(\phi_0) = M = \frac{1}{2} \left(r_+ + \frac{1 - \alpha^2}{1 + \alpha^2} r_- \right) \\ q = Q = \sqrt{\frac{r_+ r_-}{1 + \alpha^2}} e^{-\alpha \phi_0} \\ m'(\phi)|_{\phi_0} = D = \frac{\alpha}{1 + \alpha^2} r_-, \end{cases} \quad (7.19)$$

where ϕ_0 is the asymptotic value of ϕ at infinity.

In order to obtain an explicit expression for the scalar charge/scalar coupling of the point-like black hole approximation, we express the relation (2.11) as:

$$D = \frac{\alpha}{1 - \alpha^2} \left(M - \sqrt{M^2 - (1 - \alpha^2) Q^2 e^{2\alpha \phi_0}} \right) \quad (7.20)$$

and consider that the point-like state lives in a region where $\phi_0 \simeq \phi$, generated by the other (distant) black hole, and therefore has a coupling to the dilaton given by:

$$\frac{dm}{d\phi} = \frac{\alpha}{1-\alpha^2} \left(m(\phi) - \sqrt{m^2(\phi) - (1-\alpha^2)q^2 e^{2\alpha\phi}} \right). \quad (7.21)$$

As was shown in [34, 35], the useful parameters to describe the scalar interactions are

$$\gamma(\phi) \equiv \frac{d}{d\phi} \ln m(\phi) \quad \text{and} \quad \beta(\phi) \equiv \frac{d\gamma(\phi)}{d\phi}, \quad (7.22)$$

and the mass $m(\phi)$ can be expanded around a background value $\bar{\phi}$ as

$$m(\phi) = m(\bar{\phi}) \left(1 + \gamma(\bar{\phi})(\phi - \bar{\phi}) + \frac{1}{2} \left(\gamma^2(\bar{\phi}) + \beta(\bar{\phi}) \right) (\phi - \bar{\phi})^2 + \mathcal{O} \left((\phi - \bar{\phi})^3 \right) \right). \quad (7.23)$$

Using (7.21) one obtains

$$\begin{cases} \gamma(\phi) = \frac{\alpha}{1-\alpha^2} \left(1 - \sqrt{1 - (1-\alpha^2) \frac{q^2}{m^2(\phi)} e^{2\alpha\phi}} \right) \\ \beta(\phi) = \frac{\alpha^2}{1-\alpha^2} \frac{q^2 e^{2\alpha\phi}}{m^2(\phi)} \left(1 - \frac{\alpha^2}{\sqrt{1 - (1-\alpha^2) \frac{q^2}{m^2(\phi)} e^{2\alpha\phi}}} \right). \end{cases} \quad (7.24)$$

With these formulae at hand, we can now extend the analysis of one black hole and a test particle case to the present case with two black holes.

For $\alpha = 1$, it is easy to see that an explicit solution is $m(\phi) = \sqrt{\mu^2 + q^2 e^{2\phi}}/2$, where μ is an integration constant. It is useful for the discussion below to recall that geometrized units have been used so far and that the ϕ field here is dimensionless (see (2.3)). In terms of physical quantities, this translates into

$$m(\phi) = \sqrt{\mu^2 + M_P^2 q^2 e^{\sqrt{2} \frac{\phi}{M_P}}}, \quad (7.25)$$

where, although we have used the same notation for simplicity, the quantities should now be understood to be the physical ones.

The tree-level t -channel contribution to the $2 \rightarrow 2$ scattering amplitude of 2 such states with same charge q and mass $m(\phi)$ reads:⁵

$$\begin{aligned} \mathcal{A} &= \frac{4m^2}{t} \left(q^2 e^{\sqrt{2} \frac{\phi}{M_P}} - (\partial_\phi m)^2 - \frac{1}{2} \frac{m^2}{M_P^2} \right) \Big|_{\bar{\phi}} \\ &= \frac{4m^2}{t} \left(q^2 e^{\sqrt{2} \frac{\phi}{M_P}} - \frac{1}{2} \frac{M_P^2}{m^2} q^4 e^{2\sqrt{2} \frac{\phi}{M_P}} - \frac{1}{2} \frac{m^2}{M_P^2} \right) \Big|_{\bar{\phi}}, \end{aligned} \quad (7.26)$$

⁵The two states being the same, they have the same asymptotic value of ϕ thus $\bar{\phi} = \phi_0$. We however keep the bar notation here.

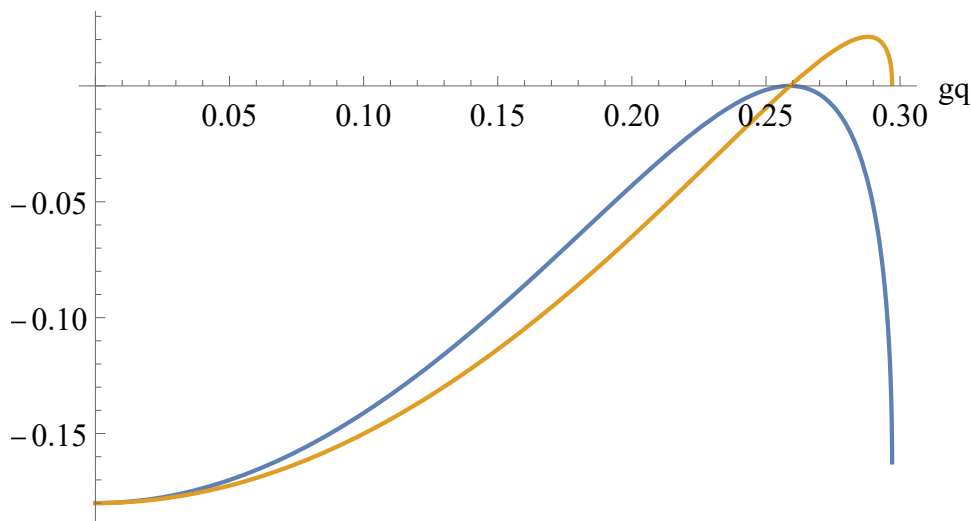


Figure 6. Comparison of the leading overall forces for the scattering of two point-like states approximating far-away black holes in blue, and one Kaluza-Klein-like state and one similar to the black hole in yellow. Here, $\alpha = 0.7$, and $m = 0.3$ in Planck units.

where the bar from now on indicates quantities evaluated at the background value $\bar{\phi}$. The amplitude can then be put in the simple form

$$\mathcal{A} = -2 \frac{M_P^2}{t} \left(\frac{\bar{m}^2}{M_P^2} - q^2 e^{\sqrt{2} \frac{\bar{\phi}}{M_P}} \right)^2 = -\frac{2\mu^4}{M_P^2 t} \quad (7.27)$$

from which it is straightforward to observe that it vanishes for $q^2 e^{\sqrt{2} \frac{\bar{\phi}}{M_P}} = \bar{m}^2 / M_P^2$ ($\mu = 0$) and is always negative otherwise. The no-force condition is readily seen to correspond, once we revert again to geometrized units, to the black hole extremality

$$q^2 e^{2\bar{\phi}} = 2\bar{m}^2. \quad (7.28)$$

As such, for $\alpha = 1$, the leading classical force between two particles with charge q and mass (7.25) giving an effective description of a pair of the same black holes is always attractive and vanishes only for extremal states.

Turning now to generic values of α , $\gamma(\phi)$ can be rewritten in terms of physical quantities as

$$\gamma(\phi) = \frac{\alpha}{1 - \alpha^2} \left(1 - \sqrt{1 - 2M_P^2(1 - \alpha^2) \frac{q^2}{m^2(\phi)} e^{\sqrt{2}\alpha \frac{\phi}{M_P}}} \right). \quad (7.29)$$

The tree-level contribution to the force between two states is given by the coefficient of the t -channel pole is

$$\begin{aligned} \mathcal{A}^{\text{t-pole}} &= 4e^{\sqrt{2}\alpha \frac{\bar{\phi}}{M_P}} q^2 \bar{m}^2 - 2 \frac{\bar{m}^4}{M_P^2} \\ &\quad - 2 \frac{\alpha^2}{(1 - \alpha^2)^2} \frac{\bar{m}^4}{M_P^2} \left(1 - \sqrt{1 - 2M_P^2(1 - \alpha^2) \frac{q^2}{m_b^2} e^{\sqrt{2}\alpha \frac{\phi}{M_P}}} \right)^2 \end{aligned} \quad (7.30)$$

The resulting behaviour of \mathcal{A} is represented by the blue curve in figure 6. We observe again that the overall force between two particles is always attractive, even beyond the point $q^2 e^{\sqrt{2}\alpha \frac{\bar{\phi}}{M_P}} = \frac{1+\alpha^2}{2} \frac{\bar{m}^2}{M_P^2}$ corresponding to extremality, which is again found to be the only point where the force vanishes. Contrary to the Reissner-Nordstrom case, the dilatonic coupling does not allow repulsive forces beyond extremality: increasing q at fixed m , the scalar force grows at least as strong as the gauge one. As observed in [34] and later shown in [36], in the asymptotic $\phi \rightarrow \infty$ limit, the solution to (7.21) takes the form $\sqrt{(1+\alpha^2)/2} (m(\phi)/M_P) = q e^{\frac{\alpha}{\sqrt{2}} \frac{\bar{\phi}}{M_P}}$. We note here that it coincides with the extremal relation.

We now discuss the force felt by a particle of charge q and Kaluza-Klein-like mass $m(\phi) = m_A e^{\alpha\phi}$ (in geometrized units). Our result for the effective potential for the motion of such a particle in the asymptotically flat background metric, when applied for this case, agrees with [37]. We will address here the attractive or repulsive nature of the leading interaction, the $1/r$ contribution, for the case of our point-like states.

The effective potential takes the form (7.12), with $m' = \alpha m$. If we choose the two, particle and black hole, states such that they share the same mass ($m(\bar{\phi}) = M$) and charge ($q = Q$), we find that, for $\alpha < 1$, the overall force, proportional to $M^2 + \alpha M D - e^{2\alpha\bar{\phi}} Q^2$, as shown by the yellow curve in figure 6, vanishes at two points:

- the point defining the extremality condition

$$M^2 = \frac{e^{2\alpha\bar{\phi}} Q^2}{1 + \alpha^2} \tag{7.31}$$

- and in the point

$$M^2 = (1 - \alpha^2) e^{2\alpha\bar{\phi}} Q^2 \tag{7.32}$$

where the metric is on the verge of becoming complex.

This remains valid as long as $m/q = M/Q$.

Denoting the point-like state approximating a black hole as S , we now turn to the computation of the amplitude $S \rightarrow S\phi\phi$ for the emission of a pair of dilatons due to the couplings in (7.23). It takes the form

$$\mathcal{A}(S \rightarrow S\phi\phi) = -2i \frac{\bar{m}^4}{M_P^2} \bar{\gamma}^2 \left(\frac{1}{t - \bar{m}^2} + \frac{1}{u - \bar{m}^2} \right) - i \frac{\bar{m}^2}{M_P^2} (2\bar{\gamma}^2 + \bar{\beta}), \tag{7.33}$$

where $\bar{\beta}$ is now written in physical units as the other quantities. At threshold, $t, u = -\bar{m}^2$, and this simplifies to

$$\mathcal{A}(S \rightarrow S\phi\phi) = -i \frac{\bar{m}^2}{M_P^2} \bar{\beta} = -2i \frac{\alpha^2}{1 - \alpha^2} q^2 e^{\sqrt{2}\alpha \frac{\bar{\phi}}{M_P}} \left(1 - \frac{\alpha^2}{\sqrt{1 - (1 - \alpha^2) 2M_P^2 \frac{q^2}{\bar{m}^2} e^{\sqrt{2}\alpha \frac{\bar{\phi}}{M_P}}}} \right). \tag{7.34}$$

For any finite value $\alpha \geq 1$, the amplitude is finite as well and reduces to $\mathcal{A} = -2iq^2 e^{\sqrt{2}\alpha \frac{\phi}{M_P}}$ for $\alpha = 1$. However, for $\alpha < 1$ the amplitude diverges at

$$q^2 e^{\sqrt{2}\alpha \frac{\phi}{M_P}} = \frac{1}{2(1 - \alpha^2)} \frac{\bar{m}^2}{M_P^2}, \tag{7.35}$$

corresponding to the largest charge before the metric becomes complex. It is also easily verified that the amplitude (7.34) vanishes when the no force (extremality) condition (7.31) is met.

8 Summary and conclusions

We give here an overview of the results obtained above for the existence of horizons in the black hole solution (2.8). We also attempt to infer from them new bounds for the Dilatonic Weak Gravity Conjecture (DWGC). We will keep separate the discussions about asymptotically flat, AdS and dS space-time. For AdS, this extends previous partial results for the dilatonic black hole solution given in [24, 28]. Also, the authors of [28] focused on the region $r_+ \gg r_-$. Whereas, our analysis of the horizons in the dS solution is to our knowledge new. Finally, for the WGC in these cases, we follow [6], where the dS-WGC bound was conjectured to be set by the boundary, in the (QH, MH) plane, between the black hole solution region exhibiting both an event and a cosmological horizon and the naked singularity region with only a cosmological horizon. The limit where the cosmological and event horizons coincide corresponds to Nariai black hole cases. The dilaton profile is then constant $\phi = \phi_N$. Given a generic potential $V(\phi)$, a necessary condition for the Nariai black hole to exist is given by the bound [49, 50]

$$\left| \frac{V'}{V}(\phi_N) \right| \leq 2\alpha. \tag{8.1}$$

This inequality was used in [50] to derive a generic bound on the potential, without a prior condition on ϕ . For the class of potentials $V(\phi)$ given by (2.4), it is easy to find generic ranges of dilaton values where the condition (8.1) is not verified. However, in the different cases we discuss here, the Nariai black hole solutions are present for specific values of the dilaton field:

$$e^{2\alpha\phi_N} = e^{2\alpha\phi_0} \left(1 - \frac{r_-}{r_c} \right)^{\frac{2\alpha^2}{1+\alpha^2}}. \tag{8.2}$$

and, for such values of ϕ_N , the bound (8.1) is always satisfied.

We are particularly interested in the behaviour of the black hole solutions as a function of the dilaton coupling α . We notice that the $\alpha \rightarrow \infty$ and $\alpha \rightarrow 0$ limits of such black holes are approximated by Schwarzschild and Reissner-Nordström black holes, and we find a value of $\alpha = 1, 1/\sqrt{3}$ which separates between the two behaviours in flat and (A)dS backgrounds, respectively.

The flat space-time BH and DWGC: we have retrieved the well-known result [26, 27]

$$Q^2 e^{2\alpha\phi_0} \leq (1 + \alpha^2) M^2 \tag{8.3}$$

As formulated by [5], we assume that the WGC corresponds to the opposite inequality.

The asymptotically AdS BH and AdS-DWGC:

- For $\alpha > 1/\sqrt{3}$: the black hole solutions exhibit only one (event) horizon. It is located outside the singular surface as long as

$$Q^2 e^{2\alpha\phi_0} < (1 + \alpha^2)M^2 \tag{8.4}$$

and the two surfaces coincide when the inequality turns to equality. The DWGC condition is the same as in the asymptotically flat-space one.

- When $\alpha = 1/\sqrt{3}$: the black hole solutions still possess only one horizon. The coincidence of that horizon with the singularity is no more obtained for $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$ but for a smaller charge now, saturating the inequality

$$\begin{aligned} & \frac{\left(9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3} 3^{2/3}} \\ & - \frac{\left(\frac{2}{3}\right)^{1/3}}{\left(9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}} \\ & \geq \frac{4\hat{Q}^2}{3\left(M + \sqrt{M^2 - \frac{2}{3}\hat{Q}^2}\right)}. \end{aligned} \tag{8.5}$$

The expansion for small H (large $L = \frac{1}{H}$, with L the AdS length scale) gives

$$Q^2 e^{(2/\sqrt{3})\phi_0} \leq \frac{4}{3}M^2 - \frac{4^3}{3^4}M^4 H^2 + o(H^2). \tag{8.6}$$

which reproduces the flat space-time case for $H \rightarrow 0$.

- For $0 < \alpha < 1/\sqrt{3}$: the black holes have both an inner and an outer horizon. The extremality condition is now obtained when the two horizons coincide and is expressed as

$$F_{\text{AdS}}(r_{0-}) = 0 \tag{8.7}$$

where F_{AdS} is defined by (5.2) and r_{0-} by (4.13). Here the two coincident horizons are located outside the singularity. The expansion for small H gives

$$Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2 + \alpha^2(1 + \alpha^2)\frac{2}{1-\alpha^2} cM^{\frac{3-\alpha^2}{1-\alpha^2}} H^{\frac{1+\alpha^2}{1-\alpha^2}} + o\left(H^{\frac{1+\alpha^2}{1-\alpha^2}}\right), \tag{8.8}$$

where c is defined in (5.6). From this expansion one can see that the condition tends to the flat space one in the limit $H \rightarrow 0$. Black hole states solve $F_{\text{AdS}}(r_{0-}) < 0$.

- For $\alpha = 0$: this is the well studied case of charged AdS without dilaton (see for example [38–48]).

A WGC bound can be identified as the requirement of the presence of a state with a charge Q and a mass M that verifies an inequality opposite to the ones above respected by black holes with a horizon.

In [51–53], the classical decay of (near)-extremal solutions through the charged superradiance mechanism was used to obtain a WGC bound in asymptotically AdS space-time with a dilaton. The conjecture requires the existence of a state with mass m and charge q solving

$$q \geq \frac{\Delta}{\mu}, \quad \text{with} \quad \Delta = \frac{3H}{2} + \sqrt{\frac{9H^2}{4} + m^2}, \quad \mu = \frac{Q}{r_+}. \quad (8.9)$$

where Δ is the minimum frequency of a scalar perturbation in AdS and μ the difference between the component A_t of the gauge field at infinity and on the horizon for extremal solutions. The condition for the onset of superradiance was obtained considering the leading order in the horizon radius for small black hole and the equation $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$ to define extremal states. With these assumptions, eq. (8.9) reads

$$q \geq \Delta \sqrt{1 + \alpha^2}. \quad (8.10)$$

Here, for $\alpha \leq 1/\sqrt{3}$, we found that the extremality condition receives corrections. Accordingly, the bound from (8.9) becomes

$$q \geq \frac{r_{0-}}{\sqrt{r_+ r_-}} \Delta \sqrt{1 + \alpha^2}, \quad (8.11)$$

where r_+ and r_- are related by the condition $F_{\text{AdS}}(r_{0-}) = 0$, $r_{0-} = r_{0-}(r_+, r_-)$. Again, F_{AdS} is defined by (5.2) and r_{0-} in (4.13). For $r_+ \geq r_-$, and thus for extremal solutions, $r_{0-} < \sqrt{r_+ r_-}$, so that the bound (8.11) is weaker than (8.10). Using (4.13) and the expansion of the extremality condition for small H (5.5), this gives at leading order:

$$q \geq \Delta \sqrt{1 + \alpha^2} \left(1 + \frac{\alpha^2}{1 - \alpha^2} c r_+^\gamma H^\gamma + o(r_+^\gamma H^\gamma) \right), \quad (8.12)$$

where $\gamma = (1 + \alpha^2)/(1 - \alpha^2)$ and c is a constant solution of the equation given in (5.6). Note that the expression of the minimum frequency in AdS might also receive corrections which we expect to be sub-leading (for RN-AdS, we have $\omega = H\Delta + o(r_h H^2)$).

The asymptotically dS-BH and dS-DWGC:

- For $\alpha > 1/\sqrt{3}$, the condition for the event horizon to coincide with the singularity is given by $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$. It is again the same as in both asymptotically flat and AdS space.
- When $\alpha = 1/\sqrt{3}$, the extremal solution solutions solve

$$\frac{\left(\frac{2}{3}\right)^{1/3} e^{-i\pi/3}}{\left(-9 \left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{-4 + 27 \left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}} \quad (8.13)$$

$$\begin{aligned}
 & + \frac{e^{i\pi/3} \left(-9 \left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2} \right) + \sqrt{3} \sqrt{-4 + 27 \left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2} \right)} \right)^{1/3}}{2^{1/3} 3^{2/3}} \\
 & = - \frac{4\hat{Q}^2}{3 \left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2} \right)}.
 \end{aligned}$$

This condition can be expanded for small H as

$$Q^2 e^{(2/\sqrt{3})\phi_0} = \frac{4}{3}M^2 + \frac{4^3}{3^4}M^4H^2 + \mathcal{O}(M^6H^4), \tag{8.14}$$

showing that it allows for slightly greater charges than the corresponding flat space limit. It goes to the flat space-time condition for $H \rightarrow 0$.

- For $0 < \alpha < 1/\sqrt{3}$:

For a real valued metric, one has always either two horizons or, trivially for huge masses, a naked singularity. We have found no region of parameters with a (real) metric exhibiting only a naked singularity with a cosmological horizon.

In both limits $\alpha \rightarrow 0$ and $\alpha \rightarrow 1/\sqrt{3}$ limits, we retrieve the $\alpha = 0$ and $\alpha = 1/\sqrt{3}$ results, respectively, where a transition from a black hole to a naked singularity with cosmological horizon happens for $F(r_{0-}) = 0$ (defined by (4.12) and (4.13)).

We discuss further this case below.

- For $\alpha = 0$

The phases of the dS-RN black hole metric are described in details in [6, 22]. The condition for the existence of the black hole is

$$Q^2 \leq M^2 + M^4H^2 + \mathcal{O}(M^6H^4) \tag{8.15}$$

with $M^2H^2 \leq \frac{2}{27}$.

The appearance of a complex valued metric leaves the lower bound on the black hole mass for the $0 < \alpha < 1/\sqrt{3}$ asymptotically dS solutions undefined. Because previous literature focused on the asymptotically flat metric which always shows a naked singularity before reaching the complex valued metric region, it was not investigated. Here, however, the condition for the metric becoming complex,

$$Q^2 e^{2\alpha\phi_0} > \frac{M^2}{1 - \alpha^2} \tag{8.16}$$

can be reached within the domain of the black hole solution region and represents a new bound.

To investigate the nature of this bound, we studied, in the asymptotically flat space, the behaviour of the interaction between point-like particles approximating at long distances the charged dilatonic black holes. The main difficulty is then to express the coupling of the

dilaton to the point-like particles as a function of the charge and mass which is non-trivial for the generic solution. A first approximation was to adopt the relation between scalar and gauge charges and mass as it appears for the black hole solution at infinity. We found then that the self interaction between two such states is always attractive, even in the super-extremal case, and is null only for extremal solutions with $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2)M^2$. Increasing further the charge to mass ratio, the production of dilaton pairs seems to diverge for $\alpha < 1$ when we reach $Q^2 e^{2\alpha\phi_0} = M^2/(1 - \alpha^2)$ above which the metric becomes complex. Considering instead a particle with a mass $m(\phi) = m_A e^{\alpha\phi}$, interacting with a black hole of same mass and charge at large distance, we found that for $0 < \alpha < 1$ the leading order forces between these two states cancel at the extremality but also at the point where the metric is on the verge of becoming complex. We might consider that (8.16) represents a new dilatonic de Sitter WGC bound for this domain of dilaton couplings ($0 < \alpha < 1/\sqrt{3}$), but further investigation is needed to confirm or infirm this. Also, it will be interesting to consider the effect of higher derivative corrections to the black hole solutions.

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Part II

Wilson RG and the Naturalness/Hierarchy problem

CHAPTER 6

The Wilson approach and the Renormalization Group

6.1 Introduction

Quantum Field Theory naturally resulted as a necessary framework for the combination of Quantum Mechanics with the theory of Special Relativity. Predictions within this framework are most usually made by means of the perturbative method, through which physical amplitudes are calculated in a series in the small parameter \hbar . The appearance of divergences at higher orders of the perturbative expansion has been the source of the greatest difficulties until the advent of renormalization. In the process of renormalizing a theory, physical quantities are divided into an observable and an unobservable part. Divergences are then absorbed in the latter to extract physical predictions from divergent integrals.

In its original formulation, renormalization provides a technique to make falsifiable predictions that has given spectacular results when compared to experiments, but with little or no physical interpretation. A more profound understanding of its meaning came with the renormalization group. As mentioned in the Introduction, this was initiated by Stueckelberg and Petermann [80], who realized that the many ways of extracting sensible predictions from divergent integrals (in jaergon, the many renormalization prescriptions), could all be related between themselves by transformations forming a semi-group, to which they gave the name of "Renormalization Group". This is the usual (textbook) way Renormalization Group equations are still presented today: as equations that express independence of the final results from the renormalization prescription. The Gell-Mann and Low example of the Serber-Uehling potential [9] discussed in the introduction gives a different, more physical, perspective on the Renormalization Group as a description of the way the theory evolves with the observational scale. As revealed by Wilson himself in his Nobel lecture [81], this was the main inspiration for his later works, and indeed it contains in a nutshell the main idea of what would later become the Wilsonian Renormalization Group, that was developed in [82, 83, 84].

The crucial observation, in Wilson work, is that the difficulties encountered in Quantum Field Theory are nothing but a specific example of something more general. Many systems in Nature share the property that although their laws are governed by underlying microscopic dynamics, at sufficiently large scales the effects due to the microscopic constituents average out and the averaged quantities satisfy classical macroscopic equations. The classical mechanics theory of rigid bodies, hydrodynamics and thermodynamics, applied to certain systems, are prime examples of this. On the other hand, critical phenomena and quantum field theories (as well as fluid dynamics in the turbulence regime) involve (infinitely) many degrees of freedom and scales that contribute all at once. For this reason, they do not belong to this class.

The RG approach developed by Wilson provides a framework to deal with this kind of problems. The idea is that, in systems with infinitely many scales, the "averaging out" that in the first class of models is automatically done should now be performed one step at a time. This is immediately understood in position space: quantum/thermal fluctuations should first be calculated on the smallest scale we can reach, and then, one step at a time, on larger and larger scales. In the next section, I illustrate the method adopting the convenient formulation given in [85].

6.2 RG Equations: Scalar theory in the unbroken phase

Consider the theory of a single scalar field ϕ in Euclidean space (the Wilson method has been extended to theories with fermions in [86, 87], while the case of gauge bosons is more delicate and has been investigated in [88, 89, 90]). Ultraviolet divergences only appear when QFTs are extrapolated up to infinite energy/zero distance, and are a manifestation that the theory is not a fundamental one, but is obtained after the very high frequency fluctuations of a more fundamental theory have been averaged out. Suppose then that, as a result of this operation, we know the theory at the highest energy/smallest distance scale where it is still valid. Let us call this energy scale Λ , and call the action defined at this scale $S_\Lambda[\phi]$. The field $\phi(x)$ contains all the Fourier modes from 0 up to Λ

$$\phi(x) = \sum_{|q|=0}^{\Lambda} \phi_q e^{iq \cdot x}. \quad (6.2.1)$$

The parameters (coupling constants, masses, ...) that appear in the action S_Λ of course depend on the scale Λ . In the same sense pointed out by Gell-Mann and Low in [9], to describe physical processes at the typical scales of the experiments we perform, we need to know what the form of the theory and the value of its parameters are at those scales. In other words, given S_Λ , we should solve the general problem of how to determine the action S_k at a lower scale $k < \Lambda$. We can approach this problem dividing the field $\phi(x)$ in a background field $\phi_0(x)$ and a fluctuation $\eta(x)$ such that ϕ_0 contains all the Fourier modes from 0 to k and η the Fourier modes from k to Λ . In the following, we drop the subscript 0 on ϕ as it will be evident that it is a background field. The action at the scale k would then be obtained integrating out the fluctuations η from

$$e^{-\frac{1}{\hbar} S_k[\phi]} = \int [D\eta] e^{-\frac{1}{\hbar} S_\Lambda[\phi+\eta]} = \int [\Pi_{|q| \in]k, \Lambda}] d\eta_q] e^{-\frac{1}{\hbar} S_\Lambda[\phi+\eta]}. \quad (6.2.2)$$

The problem with this equation is that the exact evaluation of the quantum fluctuation integral is (as it is well-known) rather complicated and we have to resort to some approximation. If we compute the right hand side by means of a loop expansion in powers of \hbar , the right hand side of (6.2.2) typically develops power like terms and logarithms, that will appear respectively as $\Lambda^n - (\Lambda - k)^n$ and $\log(\Lambda/k)$. For a sufficiently large hierarchy between Λ and k , these contributions are large and put at risk the validity of the expansion, which was our concern in the first place. Nevertheless, this trial suggests what could be a better way to attack the problem, that is precisely what Wilson did.

Suppose we have successfully defined the theory at a scale $k < \Lambda$, and we want to define it at a still lower scale $p < k$. To truly disentangle the scales, we should reach p from k recursively by averaging out the fluctuations in infinitesimal momentum shells one at a time. In equations, we first define the action at a scale $k - \delta k$ infinitesimally smaller than k through

$$e^{-\frac{1}{\hbar}S_{k-\delta k}[\phi]} = \int [D\eta] e^{-\frac{1}{\hbar}S_k[\phi+\eta]}, \quad (6.2.3)$$

where ϕ contains all the Fourier modes from 0 up to $k - \delta k$ and η only a few modes in the infinitesimal shell $]k - \delta k, k]$. Note that we have temporarily reinserted the parameter \hbar to keep track of the expansions we will make. Once $S_{k-\delta k}$ is known, we calculate $S_{k-2\delta k}$ from it, and so on until we reach p . Evaluation of $S_{k-\delta k}$ from S_k in a loop expansion, the power and logarithmic terms that were dangerous in the previous case now appear as $k^n - (k - \delta k)^n \sim n \delta k$, $\log(k/(k - \delta k)) \sim \delta k/k$, which are verified to be small parameters of order δk . It was pointed out in [85] (see [91] for a careful derivation) that the one-loop approximation obtained expanding the action in the right hand side of (6.2.3) up to order $\mathcal{O}(\eta^2)$ is "exact", in the sense that higher order terms generate corrections of order $(\delta k)^2$ and vanish in the limit $\delta k \rightarrow 0$. In equations, we expand

$$S_k[\phi + \eta] = S_k[\phi] + \frac{\hbar}{2} \int d^D x d^D y \frac{\delta^2 S_k[\phi + \eta]}{\delta \eta(x) \delta \eta(y)} \Bigg|_{\eta(x)=0} \eta(x) \eta(y) + \mathcal{O}(\eta^3) \quad (6.2.4)$$

with

$$\frac{\delta S_k[\phi + \eta]}{\delta \eta(x)} \Bigg|_{\eta(x)=0} = 0 \quad (6.2.5)$$

This is the great advantage of the Wilson method: the stepwise evaluation of the quantum fluctuations in small but finite shells of width δk allows to find a new small parameter, namely $\delta k/k$. The expansion (6.2.4) in the evaluation of (6.2.3) then rests on the smallness of this parameter that, in the limit $\delta k \rightarrow 0$ makes the expansion exact. According to that, in the following, we drop the \hbar and come back to usual notations, as the small parameter that governs the expansion is now $\delta k/k$.

Before proceeding with the derivation of the Wegner-Houghton equation, we observe that (as it is clear from (6.2.4) and (6.2.5)), the latter is obtained assuming that the saddle point of the blocking transformation defined in (6.2.3) and (6.2.4) is the trivial one, namely $\eta = 0$, although this was noted much later [92]. We will come back to this observation and discuss what happens when non-trivial saddle points are found later.

From (6.2.3) and (6.2.4) we obtain in the limit $\delta k \rightarrow 0$ an integro-differential equation, the Wegner-Houghton equation, for the action S_k . It describes the evolution of the action with the scale k , and S_Λ serves as a boundary condition. The resulting equation is too complicated to seek for a general solution, and we should revert to some approximation to make it more manageable. To this end, we first write the action in a derivative expansion

$$S_k[\phi] = \int d^D x \frac{Z_k(\phi)}{2} \partial_\mu \phi \partial^\mu \phi + U_k(\phi) + Y_k(\phi) (\partial_\mu \phi \partial^\mu \phi)^2 + \dots, \quad (6.2.6)$$

where the dots indicate higher derivative terms. The evolution equation for S_k then becomes a system of coupled evolution equations for Z_k, U_k, Y_k, \dots . For our purposes here, it will be sufficient to consider the lowest order approximation to the derivative expansion, the local potential approximation (LPA), that amounts to set $Z_k = 1, Y_k, \dots = 0$.

Consistently with the LPA, we take ϕ as a homogeneous background field. Inserting (6.2.4) in (6.2.3) we obtain

$$e^{-S_{k-\delta k}[\phi]} = e^{-S_k[\phi]} \int [D'\eta] e^{-\frac{1}{2} \int d^D x \int d^D y \frac{\delta^2 S_k[\phi+\eta]}{\delta \eta(x) \delta \eta(y)} \Big|_{\eta(x)=0} \eta(x) \eta(y)}, \quad (6.2.7)$$

where the symbol D' indicates that the field η that we are integrating out only contains modes in the shell $[k - \delta k, k]$. The integral on the right hand side is Gaussian and it is readily solved by standard techniques to give $\det(\delta^2 S_k / \delta \eta^2)^{-1/2}$. Inserting now the form of the action as dictated by the LPA and evaluating the determinant as usual, we obtain an evolution equation for U_k

$$U_{k-\delta k}(\phi) = U_k(\phi) + \frac{1}{2} \int' \frac{d^D p}{(2\pi)^D} \log(p^2 + U_k''(\phi)) (1 + \mathcal{O}(\delta k)) \quad (6.2.8)$$

where the symbol \int' means that that the integration is performed only in the shell $k - \delta k < p \leq k$ and $U_k''(\phi)$ indicates the second derivative of U_k with respect to ϕ . In the limit $\delta k \rightarrow 0$ the differential equation

$$k \frac{\partial}{\partial k} U_k(\phi) = -\frac{N_D}{2} k^D \log(k^2 + U_k''(\phi)) \quad (6.2.9)$$

is obtained, where N_D is the result of the angular integration in D dimensions, $N_D = 2/((4\pi)^{D/2} \Gamma(D/2))$. This is a non-perturbative evolution equation for the potential $U_k(\phi)$. The potential defined at the scale Λ serves as a boundary condition to the flow of U_k from $k = \Lambda$ down to $k = 0$, where the usual effective potential obtained from the Legendre transform of the functional $W = -\log Z$ (with Z the partition function), is found.

Before discussing this result further, we should see how it translates when written in terms of the usual coupling constants. Suppose the potential is expressed in a polynomial expansion in the field, and, for simplicity, that there is a discrete Z_2 symmetry $\phi \rightarrow -\phi$. From the explicit expression

$$U_k(\phi) = \Omega_k + \frac{1}{2} m_k^2 \phi^2 + \frac{\lambda_k}{4!} \phi^4 + \sum_{n \geq 3} \frac{\lambda_k^{(2n)}}{n!} \phi^n \quad (6.2.10)$$

one obtains a set of infinitely many coupled differential equations for the couplings

$$k \frac{\partial \Omega_k}{\partial k} \equiv \beta_{\Omega_k} = -\frac{N_D k^D}{2} \log \left(\frac{k^2 + m_k^2}{k^2} \right) \quad (6.2.11)$$

$$k \frac{\partial m_k^2}{\partial k} \equiv \beta_{m_k^2} = -\frac{N_D k^D}{2} \frac{\lambda_k}{k^2 + m_k^2} \quad (6.2.12)$$

$$k \frac{\partial \lambda_k}{\partial k} \equiv \beta_{\lambda_k} = -\frac{N_D k^D}{2} \left(\frac{\lambda_k^{(6)}}{k^2 + m_k^2} - \frac{3\lambda_k^2}{(k^2 + m_k^2)^2} \right) \quad (6.2.13)$$

$$k \frac{\partial \lambda_k^{(6)}}{\partial k} \equiv \beta_{\lambda_k^{(6)}} = -\frac{N_D k^D}{2} \left(\frac{\lambda_k^{(8)}}{k^2 + m_k^2} - \frac{15\lambda_k \lambda_k^{(6)}}{(k^2 + m_k^2)^2} + \frac{30\lambda_k^3}{(k^2 + m_k^2)^3} \right) \quad (6.2.14)$$

...

where β_i are the so-called beta functions that govern the running of the couplings. It is customary to subtract, from (6.2.9) the field-independent quantity $\log(k^2)$ or $\log(k^2 + U_k''(0))$, which only changes the factor inside the logarithm of the equation for the cosmological constant Ω_k and is nothing but the usual normalization of the fluctuation determinant. The equations (6.2.11)-(6.2.14) are the Wilsonian RG equations that describe the flow of the couplings in the LPA. The Wilsonian RG equations are a system of infinitely many coupled differential equations where all the couplings (either implicitly or explicitly) are intertwined and influence each other.

A rather important point, that is evident from (6.2.11)-(6.2.14), is that all the couplings $\lambda^{(2n)}$ contribute to the evolution of U_k , there is no a priori distinction between renormalizable and non-renormalizable couplings. In the RG framework, the classification of the operators is made in terms of the analysis of the relevant, marginal and irrelevant directions in the space of the coupling constants, that is not universal but crucially depends on the region of parameter space where one works. More precisely, the analysis is performed in the vicinity of a fixed point of the RG transformation.

Fixed point are defined as follows. Starting with the dimensionful coupling constants introduced in (6.2.10), let us consider the dimensionless couplings $g^{(2)}(k) \equiv k^{-2} m_k^2$, $g^{(4)}(k) \equiv k^{D-4} \lambda_k$, $g^{(6)}(k) \equiv k^{2(D-3)} \lambda_k^{(6)}$, ..., and truncate the parameter space to n couplings only (in the following we use the vectorial notation $\vec{g} \equiv (g^{(2)}, g^{(4)}, \dots, g^{(2n)}) = g^{(i)} \vec{e}_i$, with \vec{e}_i canonical basis). If we indicate with R the RG transformation, a fixed point in the parameter space is a point where

$$R(\vec{g}_*) = \vec{g}_*. \quad (6.2.15)$$

In the vicinity of a fixed point the RG equations can (often but not always) be linearized¹. For a small deviation with respect to \vec{g}_* , the eigenvalues of the linearization matrix (the $\tilde{\beta}$ are the equivalent for the dimensionless couplings of the β functions defined in (6.2.11)-(6.2.14))

$$R_{ij}^{(l)} = \frac{\partial \tilde{\beta}^i}{\partial g^{(j)}}, \quad (6.2.16)$$

¹ Marginal couplings around the Gaussian fixed point are an example of a case where the linearization cannot be performed as the linear term vanishes and the leading contribution is quadratic.

determine how the couplings behave in the vicinity of the fixed point. Actually, around the fixed point \vec{g}_* the RG equations can be written as

$$k \frac{d}{dk} (\vec{g}(k) - \vec{g}_*) = R^{(l)} \Big|_{\vec{g}=\vec{g}_*} \cdot (\vec{g}(k) - \vec{g}_*). \quad (6.2.17)$$

Let us indicate the matrix $R^{(l)} \Big|_{\vec{g}=\vec{g}_*}$ with M , its eigenvectors and eigenvalues with \vec{u}_i and $y^{(i)}$ respectively (there is no sum on the index i below):

$$M \vec{u}_i = y^{(i)} \vec{u}_i, \quad (6.2.18)$$

Expanding $\vec{g}(k) - \vec{g}_*$ in terms of the basis given by the eigenvectors of M

$$\vec{g}(k) - \vec{g}_* = v^{(i)}(k) \vec{u}_i \quad (6.2.19)$$

the system (6.2.17) becomes (again, no sum)

$$k \frac{d}{dk} v^{(i)}(k) = y^{(i)} v^{(i)}(k) \quad (6.2.20)$$

whose solution is ($i = 1, 2, \dots, n$)

$$v^{(i)}(k) = v^{(i)}(k_0) \left(\frac{k}{k_0} \right)^{y^{(i)}}, \quad (6.2.21)$$

where k_0 is an arbitrary value of k . The quantities $v^{(i)}(k)$ are the scaling parameters, that allow to define relevant and irrelevant directions around the fixed point \vec{g}_* .

Following the flow towards the IR ($k \rightarrow 0$) (that is how the coarse graining is defined), the direction given by the eigenvector \vec{u}_i is said to be (IR) relevant if the eigenvalue $y^{(i)}$ is negative. In this case, in fact, while moving towards the IR, $v^{(i)}(k)$ increases. Those eigendirections with $y^{(i)}$ positive instead are called (IR) irrelevant, as $v^{(i)}(k)$ decreases for $k \rightarrow 0$.

We now observe that if S is the matrix that diagonalizes M , i.e. $S^{-1}MS = Y = \text{diag}(y^{(1)}, y^{(2)}, \dots, y^{(n)})$, it is also $\vec{u}_i = S_{ij} \vec{e}_i$, and the original couplings $g^{(i)}(k)$ are obtained as a linear combination of the $v^{(i)}(k)$

$$g^{(i)}(k) = g_*^{(i)} + S_{ij} v^{(i)}(k). \quad (6.2.22)$$

Therefore, the IR ($k \rightarrow 0$) flow of the $g^{(i)}(k)$ in the vicinity of the fixed point \vec{g}_* is dominated by the directions corresponding to eigenvectors of M with negative eigenvalues. For the coupling $g_k^{(i)}$ not to explode while moving towards the IR, the $v^{(i)}(k)$ corresponding to such directions need to be fine-tuned. In other words, the IR flow of the couplings $g_k^{(i)}$ is sensible to the precise value of the $v^{(i)}(k)$ along the directions with negative eigenvalues. On the contrary, for the eigendirections characterized by positive eigenvalues the system is not really sensible to the precise value of the $v^{(i)}(k)$, as their contribution is more and more suppressed as k decreases. The analysis presented so far can be easily reverted if we consider the flow in the opposite, UV, direction. Following the UV flow, in fact, the role of positive and negative eigenvalues is opposite to what we have seen above. Eigendirections with negative eigenvalues are UV irrelevant, while those with positive eigenvalues are UV relevant.

There is a nice and simple geometrical interpretation of what we have discussed above. Given a fixed point \vec{g}_* , in the parameter space $(g^{(1)}, g^{(2)}, \dots, g^{(n)})$ we can define an IR critical surface Σ_{IR} as the surface such that for any "initial condition" at $k = k_0$ the flow towards the IR brings the point $(g^{(1)}(k), g^{(2)}(k), \dots, g^{(n)}(k))$ to collapse on \vec{g}_* for $k \rightarrow 0$. Similarly we can define a UV critical surface Σ_{UV} as the surface such that for any "initial condition" at $k = k_0$ the flow towards the UV takes the point to collapse on \vec{g}_* for $k \rightarrow \infty$. The analysis presented above is referred to the tangent planes (determined by the eigendirections of M) to the above defined surfaces at \vec{g}_* . It is interesting to note that, if for a given theory we have, say, two fixed points, $\vec{g}_*^{(1)}$ and $\vec{g}_*^{(2)}$, and for instance $\Sigma_{IR}^{(1)}$ is a IR critical surface for $\vec{g}_*^{(1)}$, while $\Sigma_{UV}^{(2)}$ is a UV critical surface for $\vec{g}_*^{(2)}$, if the intersection of these two surfaces is still a surface $\Sigma = \Sigma_{IR}^{(1)} \cap \Sigma_{UV}^{(2)}$, for any initial condition on Σ , the flow will collapse on $\vec{g}_*^{(1)}$ for $k \rightarrow 0$ and on $\vec{g}_*^{(2)}$ for $k \rightarrow \infty$.

We also have to observe that if M has eigenvectors with null eigenvalues, these are the so called marginal directions. To uncover the behaviour of the flow along these eigendirections, we necessarily have to go beyond the linear approximation. These directions will be then classified as marginal (weakly) IR relevant or irrelevant depending on whether they (slowly) tend to grow or decrease as k evolves down to the IR (similarly for the flow towards the UV). We finally note that, when the fixed point is the Gaussian one, i.e. $\vec{g}_* = \vec{0}$, the classification presented above coincides with the usual perturbative classification, and actually gives a stronger motivation for it (see, for e.g., [93]), but the RG method also captures other regimes where couplings enjoy different, non-perturbative, behaviours. This will be discussed further in Chapter 8, where [6] will be presented. In Appendix B I present a careful derivation of the toy example of [93], that clarifies, and gives a specific example, of how the question of perturbative renormalizability emerges in the larger context of the Renormalization Group.

From now on, we specify to the case $D = 4$, for the sake of clarity. Expanding the equations in the UV regime $k^2 \gg m_k^2$, we obtain

$$k \frac{\partial}{\partial k} \Omega_k = -\frac{k^2 m_k^2}{16\pi^2} + \frac{m_k^4}{32\pi^2} \quad (6.2.23)$$

$$k \frac{\partial}{\partial k} m_k^2 = -\frac{k^2 \lambda_k}{16\pi^2} + \frac{\lambda_k m_k^2}{16\pi^2} \quad (6.2.24)$$

$$k \frac{\partial}{\partial k} \lambda_k = \frac{3\lambda_k^2 + m_k^2 \lambda_k^{(6)}}{16\pi^2} - \frac{\lambda_k^{(6)} k^2}{16\pi^2} \quad (6.2.25)$$

$$k \frac{\partial}{\partial k} \lambda_k^{(6)} = \frac{15\lambda_k \lambda_k^{(6)} + m_k^2 \lambda_k^{(8)}}{16\pi^2} - \frac{\lambda_k^{(8)} k^2}{16\pi^2} \quad (6.2.26)$$

...

that, upon truncation² to ϕ^4 by setting $\lambda_k^{(6)} = \lambda_k^{(8)} = \dots = 0$, match the usual bare RG equations obtained in perturbation theory (see [5], where it is also shown how to derive the perturbative Callan-Symanzik equations [94, 95] from the Wilsonian equations). We see here how the bare RG equations, that are obtained in the perturbative framework requiring independence of bare quantities from the technical (as opposed

² This is what is usually done in the perturbation theory relying on the observation that the coupling $\lambda^{(6)}$ has mass dimension $[\lambda^{(6)}] = [\text{mass}]^{-2}$, so that at scales $p \ll \Lambda$ it will be suppressed by powers of p^2/Λ^2 .

to physical) cut-off used to regularize divergent integrals, provide a specific approximation of the complete equations. This will be studied further in [5], Chapter 7.

6.3 Cruising down towards the IR: broken vs unbroken phases

A comparison of the sets of equations (6.2.11)-(6.2.14) and (6.2.23)-(6.2.26) reveal that, as the ultimate IR scale $k = 0$ is approached, the Wilsonian equations present some features that are missed even by the bare perturbative ones. Depending on the sign of m_k^2 , we should analyze two different cases.

When $m_k^2 > 0$, the term inside the logarithm in (6.2.11) and the denominators appearing in (6.2.12)-(6.2.14) are always positive. If we generically indicate m_k^2 and λ_k with $m_k^2 \equiv \lambda_k^{(2)}$ and $\lambda_k \equiv \lambda_k^{(4)}$, it is straightforward to observe that all the beta functions in the right hand side of (6.2.11)-(6.2.14) are given by combinations of the coupling constants that all appear in the ratio

$$\frac{\lambda_k^{(n)}}{k^2 + \lambda_k^{(2)}}. \quad (6.3.1)$$

In the UV regime, that we have appropriately defined to be the one where $k^2 \gg \lambda_k^{(2)}$, this is expanded as

$$\frac{\lambda_k^{(n)}}{k^2 + \lambda_k^{(2)}} \sim \frac{\lambda_k^{(n)}}{k^2} \left[1 - \frac{\lambda_k^{(2)}}{k^2} + \mathcal{O} \left(\frac{(\lambda_k^{(2)})^2}{k^4} \right) \right] \quad (6.3.2)$$

and the equations coincide with (6.2.23)-(6.2.26). Going down with the scale k , a scale \bar{k} where $m_{\bar{k}}^2 = \bar{k}^2$ can be found. This represents a turning point between the UV scaling and an IR scaling. The latter is obtained below the scale \bar{k} where the combinations (6.3.1) are expanded as

$$\frac{\lambda_k^{(n)}}{k^2 + \lambda_k^{(2)}} \sim \frac{\lambda_k^{(n)}}{\lambda_k^{(2)}} \left[1 - \frac{k^2}{\lambda_k^{(2)}} + \mathcal{O} \left(\frac{k^4}{(\lambda_k^{(2)})^2} \right) \right]. \quad (6.3.3)$$

In this new IR regime, for all practical reasons the evolution freezes (the same can be easily verified deriving the equations for the couplings from [86, 87] for a theory that include fermions). It is in fact easy to convince that the factor k^4 (or k^D in a generic number of dimensions) common to all the beta functions in (6.2.11)-(6.2.14) is not sufficient to make the quantities have a significant evolution from \bar{k} down to $k = 0$ with respect to the one above \bar{k} . This is a feature that is completely absent in the bare (and certainly also in the renormalized) perturbative RG equations, where in a large region below \bar{k} the running shows a freezing which is only a bit milder than the actual one. Before reaching $k = 0$, however, it completely deviates from this behaviour. In the RG language developed here, where (6.2.23)-(6.2.26) are obtained expanding the original

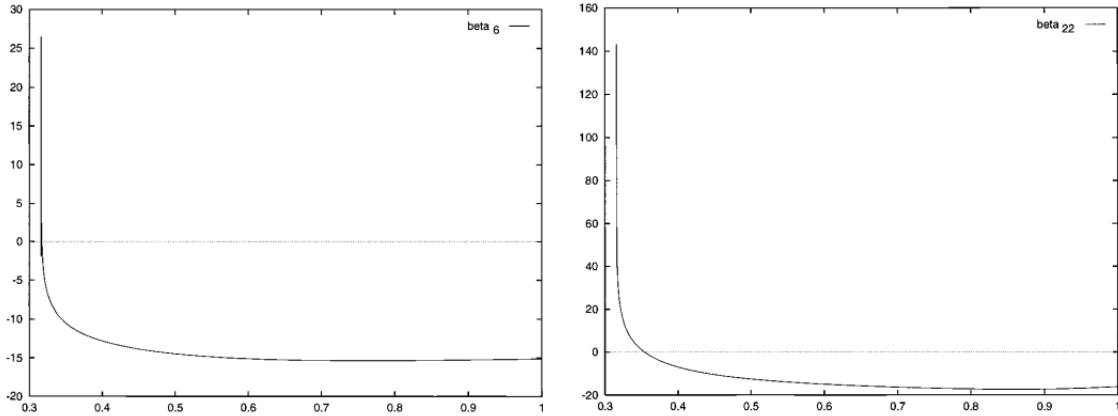


Figure 6.1 – These pictures show the evolution of $\log \left| \partial \tilde{\beta}_n(\tilde{k}) / \partial \tilde{\lambda}_\Lambda^{(6)} \right|$ for the truncation $2N = 22$, where the quantities with a tilde are dimensionless and defined as $k = \Lambda \tilde{k}$, $m_k^2 = k^2 \tilde{m}_k^2$, $\lambda_k^{(n)} = k^{4-n}$ and the beta functions accordingly. The *left panel* shows the evolution of $\log \left| \partial \tilde{\beta}_6(\tilde{k}) / \partial \tilde{\lambda}_\Lambda^{(6)} \right|$, while the *right panel* the evolution of $\log \left| \partial \tilde{\beta}_{22}(\tilde{k}) / \partial \tilde{\lambda}_\Lambda^{(6)} \right|$.

equations for $k^2 \gg m_k^2$, it is rather obvious why this happens: the perturbative equations cannot be trusted anymore. The IR freezing does not change the character of the operators, the non-renormalizable couplings remain irrelevant and do not experience a new phase.

A more interesting phenomenon happens when $m_k^2 < 0$. In this case a critical scale k_{cr} can be found in the flow of m_k^2 where the inverse propagator

$$D(k) = k^2 + m_k^2 \quad (6.3.4)$$

vanishes. When $k_{\text{cr}} = 0$, this is the equivalent of the Coleman-Weinberg mechanism [96], where a seemingly massless theory at $\phi = 0$ actually undergoes spontaneous symmetry breaking and develops a new minimum at $\phi \neq 0$, that serves as a new (homogeneous) vacuum about which the excitations are massive. On the contrary, a non-vanishing critical scale $k_{\text{cr}} \neq 0$ indicates an instability of the system with respect to fluctuations of non-vanishing momentum. In fact, the inverse propagator $D(k)$ measures the restoring force of fluctuations as it is a combination of their kinetic energy and potential energy/inertia. The physical meaning of all this is that the homogeneous vacuum is unstable against fluctuations [91, 92] and develops the so-called spinodal instability, that favors a phase separation and thus the formation of a non-homogeneous vacuum (see [97, 98]). Of course, the equations (6.2.11)-(6.2.14) cannot be trusted anymore, and one should go back to square one to derive an equation for this case. Before doing so, however, it is interesting to study how k_{cr} is reached from above.

This question was studied in [91]. The ansatz (6.2.10) for a polynomial expansion of the potential was taken and the flow was numerically solved truncating it at a large value N of n . This allowed to keep track of the non-renormalizable couplings and the way their effects pile up as the scale of the instability is reached. It was found that in the UV region $k^2 \gg k_{\text{cr}}^2$, the couplings follow their usual perturbative behaviour with no surprise. In particular, the non-renormalizable couplings are effectively suppressed

and irrelevant in the flow. As the scale k_{cr} is approached, however, for a sufficiently large N ($2N \geq 22$), all the coupling constants explode and develop singularities. As the system is of infinitely many coupled equations, as soon as one coupling starts to grow indefinitely, all the other couplings follow the same behaviour. If the system was truncated for $N < 22$, the divergences would seem to be tamed and, after some peaked oscillations, the couplings would relax to 0 as $k \rightarrow k_{\text{cr}}$. This is actually an illusion, and it shows that, unlike the UV regime, the region in the vicinity of k_{cr} requires that we consider as many non-renormalizable operators as possible to get the correct predictions. This was discussed to be due to the fact that if not enough operator are piled up, their collective effect cannot make $\lambda^{(4)}$ transition from positive to negative values, and sequentially all the other couplings experience strong oscillation but eventually collapse to 0. The quantity

$$\frac{\partial \beta_n(k)}{\partial \lambda_\Lambda^{(6)}}, \quad (6.3.5)$$

where β_n is the beta function of the n th coupling constant, was calculated to show the violation of universality, according to which changes in the boundary values of irrelevant couplings can be re-absorbed into the running of relevant couplings so that (for $m > 4$ in $D = 4$)

$$\frac{\partial \beta_n(k)}{\partial \lambda_\Lambda^{(m)}} = 0. \quad (6.3.6)$$

This gives a quantitative prediction of how much the usual perturbative arguments are overcome, and it is depicted in Fig. 6.1, taken from [91]. Although the Wegner-Houghton equation (and a fortiori (6.2.11)-(6.2.14)) certainly cannot enter the spinodal instability region and should be stopped before the flow gets too close to k_{cr} , the presence of these new scaling laws can be seen as a precursor, an indicator of the instability. The phase separation is certainly not described, but in some sense it is anticipated.

To describe the flow of the Wilsonian action in the instability region, it is necessary to get back to equation (6.2.3). In the expansion of the action $S_k[\phi + \eta]$ (6.2.4), it was tacitly assumed that the path integral is saturated by the configuration with a trivial saddle point $\eta_0(x) = 0$. One should now seek a non-homogeneous configuration that describes the spinodal phase. As it was realized in [92], this amounts to take a non-trivial saddle point $\eta_0(x) \neq 0$, so that the background is not the homogeneous field ϕ anymore. Unlike the case with $\eta_0 = 0$, when $\eta_0 \neq 0$ the zeroth order term $S_k[\phi + \eta_0]$ in the expansion (6.2.4) provides a tree-level evolution for the Wilsonian action when inserted in (6.2.3). This is now the leading contribution, the one-loop being order \hbar suppressed with respect to this one. At leading order, one can follow the tree-level renormalization that reads

$$e^{-\frac{1}{\hbar} S_{k-\delta k}[\phi]} = \int [D\eta] e^{-\frac{1}{\hbar} S_k[\phi+\eta]} = e^{-\frac{1}{\hbar} S_k[\phi+\eta_0]}, \quad (6.3.7)$$

where η_0 is the configuration that saturates the path integral, i.e. minimizes the action $S_k[\phi + \eta]$. Taking the quantization volume to be a box of length L , in the LPA in D generic dimensions $S_{k-\delta k}[\phi] = L^D U_{k-\delta k}[\phi]$, and the equation can be rewritten in the following way

$$L^D U_{k-\delta k}[\phi] = \min_{\{\eta\}} \int d^D x \frac{1}{2} \partial_\mu \eta \partial^\mu \eta + U_k[\phi + \eta] + \mathcal{O}(\delta k) \quad (6.3.8)$$

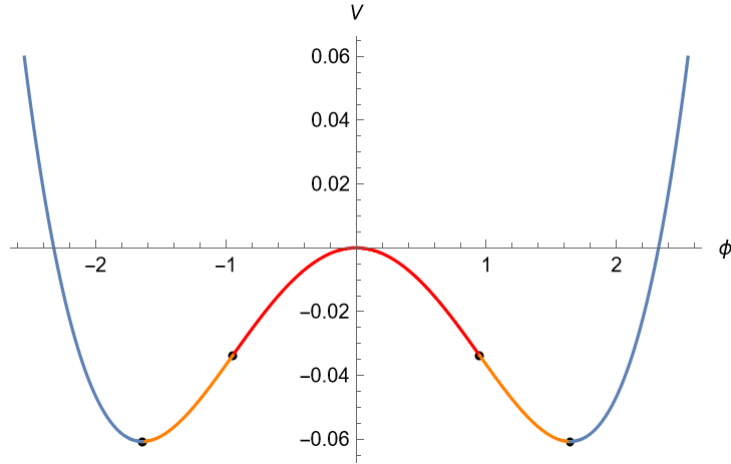


Figure 6.2 – Unstable (red), metastable (orange) and stable (blue) regions for a symmetry breaking potential $V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$ with $m^2 > 0, \lambda > 0$. The black dots show the point where different behaviours connect and correspond to the zeros of the second and first derivative of V . The values $m = 0.3$ and $\lambda = 0.2$ have been chosen for the graphical representation.

The fluctuation η contains modes in the shell $[k - \delta k, k]$ and the saddle point has the generic form

$$\eta_0(x) = \sum_{p \in [k - \delta k, k]} \psi_p e^{ip \cdot x} + \psi_p^* e^{-ip \cdot x}. \quad (6.3.9)$$

A particularly interesting ansatz for the saddle point was considered in [92]. Motivated by arguments of energy-entropy balance, the authors consider a saddle point configuration given by a single plane wave

$$\eta_0(x) = \bar{\eta}_k e^{ik \cdot x} + \bar{\eta}_k^* e^{-ik \cdot x} = 2\rho_k \cos(k \cdot x + \alpha_k). \quad (6.3.10)$$

Before any renormalization, a single plane wave fluctuation about a trivial saddle point ($\rho_k = 0$) in a potential

$$V(\phi) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad (6.3.11)$$

with $m^2 > 0$ is an unstable mode about a homogeneous vacuum ϕ_0 if it has a negative restoring force $k^2 + V_k''(\phi)|_{\phi=\phi_0} = k^2 + m^2 + (\lambda/2)\phi_0^2 < 0$. Schematically, it means that if a small fluctuation is given around ϕ_0 , not only the latter does not relax back to ϕ_0 , it also happens that its amplitude grows to larger values of ϕ until it finds a minimum where it can sit. The instability region extends from the origin $\phi = 0$ to the point $\phi^2 = 2m^2/\lambda$ where $V''(\phi)$ vanishes. This is the region where the fluctuation is more sensible to the concave part of the potential and is depicted in red in Fig. 6.2. For vacuum values $2m^2/\lambda < \phi_0^2 < 6m^2/\lambda$, the fluctuations are now more sensible to the positive curvature about the minimum in $\phi^2 = 6m^2/\lambda$. Nevertheless, they still are attracted to larger values of $|\phi_0|$, and this defines a metastable region, depicted in orange in Fig. 6.2.

In agreement with the result that above the critical scale non-renormalizable operators really are irrelevant, suppose now the parameters have been runned from Λ all

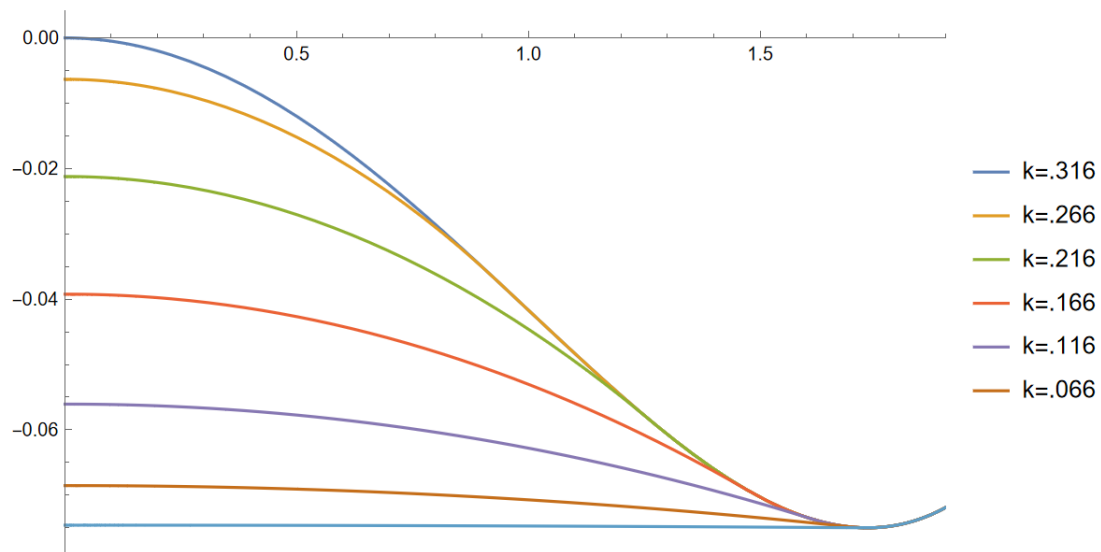


Figure 6.3 – Tree-level evolution of the symmetry breaking potential in the spinodal region. The critical scale has been chosen to be $k_{\text{cr}} = 0.316$. Several curves are shown, until the last one that realizes the Maxwell construction at $k = 0$.

the way down to (approximately) $k = k_{\text{cr}}$ in the ϕ^4 truncation. At $k = k_{\text{cr}}$ the potential is of the form

$$U_{k_{\text{cr}}}(\phi) = \frac{m_{k_{\text{cr}}}^2}{2}\phi^2 + \frac{\lambda_{k_{\text{cr}}}}{4!}\phi^4 \quad (6.3.12)$$

with $m_{k_{\text{cr}}}^2 = -k_{\text{cr}}^2$. The scale k_{cr} signals the first value obtained starting from Λ where at least one homogeneous background is unstable, i.e. the background with $\phi = 0$. It is only a matter of manipulations of the integral to get convinced that with the single plane wave ansatz, equation (6.3.8) becomes

$$U_{k-\delta k}(\phi) = \min_{\{\rho\}} \left[k^2 \rho_k^2 + \frac{1}{2} \int_{-1}^1 du U_k(\phi + 2\rho_k \cos(\pi u)) \right]. \quad (6.3.13)$$

This is a non-perturbative evolution equation that allows to follow the flow of the potential inside the spinodal region. It is found that as soon as $k < k_{\text{cr}}$, $\rho_k \neq 0$ in the region $0 < \phi^2 < 6(k_{\text{cr}}^2 - k^2)/\lambda$. Specifically, the saddle point has an amplitude

$$2\rho_k(\phi) = 6 \frac{k_{\text{cr}}^2 - k^2}{g} - \phi. \quad (6.3.14)$$

Following the evolution down to $k = 0$, the main result is that the instability region grows until it eventually incorporates the whole range from the origin to the minimum. For larger values of ϕ , there is no tree-level evolution of the potential, which is in agreement with the fact that non-trivial saddle points should only appear for unstable modes. In the instability region, the potential takes the form

$$U_k(\phi) = -\frac{1}{2}k^2\phi^2 - \frac{3}{2\lambda}(m^2 - k^2)^2. \quad (6.3.15)$$

From k_{cr}^2 down to $k = 0$, the mass term m_k^2 follows k^2 . In terms of the dimensionless quantities $\tilde{m}_k^2 = m_k^2/k^2$ and $\tilde{\lambda}_k = \lambda_k$, one way to say this is that in its tree-level evolution the potential has reached a fixed point $(\tilde{m}, \tilde{\lambda}) = (-1, 0)$. Even more, the potential (6.3.15) realizes the Maxwell construction at $k = 0$, as it can be seen in Fig. 6.3. This result is actually of the greatest importance. The effective potential as defined from the Legendre transform of the functional W , is by definition a convex function of the field. The usual way the calculation of the effective potential is performed in the symmetry broken phase, is to assume that there is some underlying phenomenon realizing the Maxwell construction that allows to define the effective potential even though it does not appear as a convex function of the field. What we see here is that the RG evolution (6.3.13) describes a mechanism through which the construction is automatically performed by the creation of a new vacuum configuration as the ultimate IR scale $k = 0$ is reached ($\rho_k \neq 0$) that make the restoring force $D(k) \geq 0$ in the whole range of ϕ . Finally, it was also shown in [92] that the difference $U_k - U_{k-\delta k}$ is order $\mathcal{O}(\delta k)$ also for the tree-level renormalization, so that the blocking transformation converges as $\delta k \rightarrow 0$.

The two cases we have discussed show two typical behaviours predicted by the Wilsonian RG for the evolution of the couplings down to $k = 0$. In the next sections, where the Wilson approach will be applied to the Higgs boson to see if and how it can shed some light on the naturalness/hierarchy problem, we will be mainly interested with the phenomenon of the freezing, as no spinodal instability appears in the Higgs case.

Dimensional Regularization, Wilsonian RG and the Naturalness/Hierarchy problem

7.1 Introduction: the Naturalness/Hierarchy problem

In its original form, Renormalization was conceived as a way to systematically extract sensible results from the divergent integrals that appear beyond leading order in the perturbative expansion. In this framework, scalar masses in four dimensions were recognized to require a rather severe cancellation as they receive quadratically divergent contributions from the coupling to any degree of freedom, whether it is the scalar itself or another scalar, a fermion or a gauge boson. In the hard cut-off regularization, the one loop correction to the two-point function in a ϕ^4 theory¹

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!} \phi^4, \quad (7.1.1)$$

with $m_0^2 = m^2 + \delta m^2$ and $\lambda_0 = \lambda + \delta \lambda$, is

$$\Sigma = \frac{\lambda \Lambda^2}{32\pi^2} + \frac{\lambda m^2 \log\left(\frac{m^2}{\Lambda^2}\right)}{32\pi^2} + \delta m^2 + O\left(\frac{m^4}{\Lambda^2}\right). \quad (7.1.2)$$

Schematically, the renormalized mass m^2 is made finite in the limit $\Lambda \rightarrow \infty$ reabsorbing first the quadratic (as well as the logarithmic) divergence in the counterterm δm^2 . Specifically, δm^2 can be fixed imposing

$$\delta m^2 = -\frac{\lambda \Lambda^2}{32\pi^2} + \frac{\lambda m^2}{32\pi^2} \log \frac{\Lambda^2}{\mu^2}, \quad (7.1.3)$$

where μ is an arbitrary subtraction scale defined by the renormalization conditions.

¹ To keep the notation the simplest, we do not bother inserting any field strength renormalization, as it would turn out to be trivial at the one-loop order.

In a nutshell, this is historically the essence of what we now call the naturalness problem of the scalar masses. In the language of the perturbative renormalization, the quantum corrections to the mass of scalars are quadratically divergent. Inserting in the intermediate steps a technical (as opposed to physical) ultimate UV scale Λ , the corrections are proportional to Λ^2 . In principle, still in the philosophy of the old approach to renormalization, one could proceed in the following way: the contribution could be first canceled by an appropriate choice of the counterterm δm^2 , and then the cut-off is sent to infinity. In this way, one ends up with a finite, "small" prediction for the quantum correction to the mass.

This is all not very satisfactory from the point of view of a physical understanding of what happens, and of course it is not the way we think about QFT and Renormalization today. As we have extensively discussed, QFTs are effective theories valid up to some ultimate *physical* scale Λ . Note that, although we use the same notation, this is completely different from the cut-off that appeared in the previous equations. The latter was merely a technical device introduced to regularize the integral and extract a finite part from it, with no meaning after all. Leaving the RG approach aside for the moment, if we give Λ the meaning of a physical scale where the integration *has* to be stopped not to go outside the regime of validity of the QFT, there is no divergence anymore, but rather a quadratic sensitivity to the ultimate UV scale. Still, this is a cause of concern. In general, in QFT, we expect physical quantities not to be much smaller than their quantum corrections. In other words, for a quantity A that receives quantum corrections δA , we expect $|A| \gtrsim |\delta A|$. This can be traced back to a more general principle commonly accepted in physics, according to which dimensionless numbers that appear in physical laws are not expected to be neither extremely large nor extremely small, and goes under the name of naturalness principle. For the problem at hand, this means that scalar masses are expected to be

$$m^2 \sim \Lambda^2. \tag{7.1.4}$$

A more rigorous definition of a principle of naturalness is due to 't Hooft [99]. It states that it is technically natural for an operator \mathcal{O} to be small if in the limit $\mathcal{O} \rightarrow 0$ the symmetry of the theory is enhanced. In this sense, fermion masses are protected from large quantum corrections because in the limit $m \rightarrow 0$ the theory enjoys an additional chiral symmetry. The latter acts in a custodial manner, ensuring that fermion masses are only multiplicatively renormalized (i.e. the quantum corrections are proportional to the mass itself). Scalar masses do not lead to an enhanced symmetry in the $m \rightarrow 0$ limit², so that nothing protects them from getting too large contributions.

By itself, this is already a rather disturbing property, as it means that scalar masses have a severe sensitivity to the ultimate scale of the EFT, and so sequentially to the details of the physics determined by the leftover of the higher energy degrees of freedom. On the more practical side, in the Standard Model the only scalar that appears is the Higgs boson, whose measured (pole) mass is $m_H \sim 125$ GeV [100, 101]. Still in the context of the hard cut-off regularization (or, equivalently for what concerns the expressions, calculating loop integrals up to the physical cut-off Λ), the quantum

² Apart from the possibility that if the mass of a scalar is the only dimensionfull parameter in a theory, in the limit $m \rightarrow 0$ the latter enjoys a classical scale symmetry. However, this is on a different ground and can be protective only in very specific and constrained cases. We will come back to this point.

correction to the Higgs mass reads

$$\delta m_H^2 = \frac{\alpha}{2} \Lambda^2 + \frac{\gamma}{2} m_H^2 \log \frac{\Lambda^2}{m_H^2} + \mathcal{O} \left(\frac{m_H^2}{\Lambda^2} \right) \quad (7.1.5)$$

where α and γ are combinations of the SM coupling constants given, at one loop, by (see for e.g. [102])

$$16\pi^2 \alpha = 12y_t^2 - 12\lambda - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2 \quad (7.1.6)$$

$$16\pi^2 \gamma = 6y_t^2 + 12\lambda - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2. \quad (7.1.7)$$

More specifically, y_t is the top Yukawa coupling, λ is the quartic coupling of the Higgs and g_1 and g_2 the $SU(2)_L$ and $U(1)_Y$ gauge couplings. Unfortunately, with the experimentally measured value of the couplings the combination that appears in α is known not to vanish. The condition $\alpha = 0$, either in the SM or in slight modifications of it, is known as the Veltman condition [103]. The way it appears in this framework, it seems only a combination of constants (that in the perturbative RG are substituted with their running quantities through a procedure known as a "one-loop improvement"). It will be better discussed later in terms of the Wilsonian RG and running couplings.

An EFT valid up to the Planck scale $M_P \sim 1.2 \cdot 10^{19}$ GeV, where gravity becomes strongly coupled, requires a cancellation between the one-loop correction and the counterterm at the level of the 31st digit to reproduce the experimental result. This is a highly non trivial fine-tuning: two numbers of order 10^{35} should cancel each other to leave a number of order $\sim 10^4$. This is what generally goes under the name of the electroweak naturalness/hierarchy problem. The word naturalness refers to the fact that scalar masses, and so the Higgs mass, have corrections quadratically proportional to the ultimate UV scale, while the name hierarchy refers to the fact that this expectation gets worse the higher the scale where the Standard Model cease to be valid. If it is valid all the way up to the Planck scale, the hierarchy problem is equivalent to the question of why the gravitational coupling is so much smaller than the gauge ones, or, in other words, why is there such a difference in energy between the Fermi and the Planck scale. Historically, the problem was formulated in terms of quadratic divergences rather than quadratic sensitivity to a physical scale. Quite often, the expression "quadratic divergences" is still used. Of course, in an effective quantum field theory with a built-in range of validity, it makes no sense to talk about divergences, as the theory cannot be taken to be valid at all energies. This is usually³ done out of tradition, but can sometimes be misleading.

Supersymmetry (SUSY) [104, 105], but also composite Higgs models [106], represent the traditional approach to the problem based on symmetry arguments. For e.g., supersymmetry requires that all particles come with (super)partners of opposite statistics (fermions for bosons, bosons for fermions) so that the theory is made up by the combination of sectors with an equal number of fermionic and bosonic degrees of freedom with same mass and couplings. The different signs of the loop corrections

³ Unless when frameworks of UV complete QFTs are claimed, where, for the very meaning of what an UV complete QFT is, the integration is made in the whole range of momenta and it makes sense to talk about the presence or the absence of divergences.

brought by bosons and fermions (see (7.1.6)) allow to cancel all the contributions. Unfortunately, no sign of the physics implied by supersymmetry (or composite models) has been found experimentally. It is certainly not realized at low energies, and one of the most difficult task is the description of a mechanism that allows to break SUSY at low energies but recover it at energies higher than those we have probed so far. In that case, the theory would be the SM from the IR up to the scale of SUSY breaking, and a supersymmetric extension of it from that scale on. Eventually, the latter would either result to be the complete framework or would find itself a completion into an even more fundamental theory. However, if the scale of SUSY breaking Λ_{SUSY} is far from the electroweak scale μ_F , $\Lambda_{\text{SUSY}} \gg \mu_F$, this reintroduces the problem, as the theory below Λ_{SUSY} would show a quadratic sensitivity to Λ_{SUSY} (see the introduction of [5]). The same can be said for the scale of compositeness.

Due to the difficulties encountered by these traditional approaches, many other ways to look at the problem and to formulate it have been proposed, either using other regularization methods, the perturbative RG, proposing mechanisms to lower the cut-off scale and many others. Some of these, that will be of interest to us, are reviewed in [5], that is reported in full integrality at the end of this chapter. Particular attention will be drawn towards claims about the possibility that power-like contributions predicted by Wilsonian calculations (identified with perturbative hard cut-off calculations where Λ is a physical scale not to be sent to infinity) are unphysical and to be discarded (see [107, 108, 109, 110, 111] and [5] for other references), and that the physically meaningful results for the evaluation of the quantum fluctuations are better captured by the dimensional regularization method (DR) [112, 113].

In certain of these approaches it is assumed that dimensional regularization might be the *correct* way to perform calculations in effective field theories and that, for some still unknown reasons, it might behave like the true, physical cut-off. Contrary to common textbook statements, where dimensional regularization is presented as a useful technique to obtain the renormalized theory but with no physical interpretation [114], the authors that follow this line of reasoning propose the idea that DR is endowed with special physical properties that have to be unveiled in order to understand what are the intrinsic features that make DR the correct way to perform calculations. The consequence of this would be that, as DR explicitly shows only (finite) logarithmic and no power-like contributions for the calculation over the whole range of momenta $|k| \in [0, \infty[$, the naturalness problem would simply evaporate, or at least it would find a different formulation: it would only be related to the presence of heavy particles coupled to the Higgs (see in addition also [115, 116] and again [5] for other references), whereas the Wilsonian scale Λ has a much wider significance. In this respect, it should be noted however that, although the results of DR do not show quadratic divergences⁴, they have poles in even dimensions. Already in [103] it was empirically observed that the diagrams that in four dimensions receive quadratic divergences in the hard cut-off regularization can be individuated in DR as those that have poles for all even dimensions $D \geq 2$. Diagrams that in the hard cut-off regularization only receive logarithmically divergent contributions have poles for even dimensions $D \geq 4$. Since then, the common lore is that the quadratic divergences of the hard cut-off regularization can be

⁴ Here the word divergence is purposely used, as calculations in DR are done with no limitation on the momentum.

traced back in DR through poles in $D = 2$ dimensions. Our results [5] will clarify this point as the relation between quadratic divergences and $D = 2$ poles will rigorously emerge from the calculation.

In the scenario reported above, the use of DR makes it that classically scale-invariant models would only suffer of a soft (logarithmic) breaking of the symmetry at the quantum level, which is a very welcome feature as it allows to generate masses through the Coleman-Weinberg mechanism [96]. Many of the works following the direction outlined in the previous paragraph actually are based on classically conformal extensions of the SM. This was even pushed forward to the very definition of a "conformal invariant regularization", that consists in a modification of DR where the 't Hooft scale is replaced by field-dependent quantities, with the hope that scale invariance might be preserved even at the quantum level [117, 118].

Another approach that will be carefully investigated is one based on the use of perturbative Callan-Symanzik RG equations to connect the UV and IR values of the Higgs mass, of which we will say more later.

The rise of interest towards these ideas, well documented in the paper, was one of the main motivations for the work [5]. We provided a careful analysis of the properties of DR to see how perturbative calculations in this scheme relate to the Wilsonian approach developed in the previous chapter. This allowed us then to answer the question whether DR truly is endowed with special physical properties or not. We also derived the link between the Callan-Symanzik and the Wilsonian RG equations, to see whether the former can really be applied to solve the naturalness/hierarchy problem. The analysis of [5] was later pushed forward in [6], chapter 8, where a fully Wilsonian perspective on the problem is adopted and a new way to look at it naturally emerges.

7.2 Discussion

The first part of the paper [5] is dedicated to the calculation of a well known quantity, the one-loop effective potential of the ϕ^4 theory in D generic dimensions (the same is done for the propagator in the Appendix B). A long and tedious route was purposely taken, with the goal of showing how the DR formal rules are derived from the generic loop integral. This is achieved by a comparison of the results obtained in DR and in the hard cut-off regularization. The latter was shown to provide, when Λ is taken as a physical scale not to be trespassed, a specific approximation of the Wilsonian potential U_k at $k = 0$ in the LPA.

The common way DR is presented is the following:

- the loop integral is formally extended to complex values of the dimension D ;
- It is observed that the integral is convergent for $\text{Re}(D) < 2$ (in terms of physical dimensions only $D = 1$). The result for $\text{Re}(D) < 2$ is then obtained in terms of a beta function. This is easily translated in the products of $\Gamma(z)$ functions, which have a pole at $z = 0$, and is proportional to $\Gamma(1 - D/2)$;
- Replacing the $\Gamma(z)$ functions with their analytic extension, that have poles at $z = 0, -1, -2, \dots$, and forgetting about their origin, the result is formally extended to generic complex values of D , except those where the poles appear: $D = 2, 4, \dots$

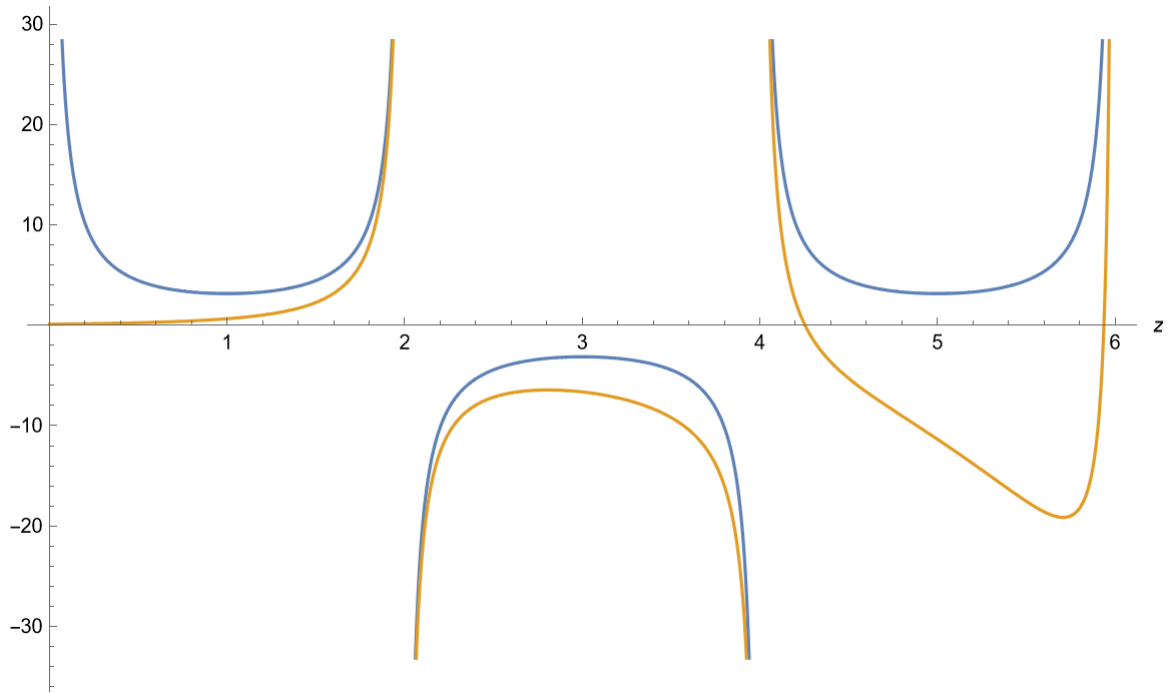


Figure 7.1 – The figure shows the analytic extension of the complete (blue) and incomplete (yellow) beta functions $\bar{B}(1 - \frac{z}{2}, \frac{z}{2})$ and $\bar{B}_i(1 - \frac{z}{2}, \frac{z}{2}; u)$ as a function of z for $u = 0.1$. As the poles are approached, the figure shows that the two functions have the same pole structure.

- The formal expression obtained in this way is expanded around $D = 4$. The pole is canceled by an appropriate choice of the counterterms. The expression one is left with coincides with the result for renormalized quantities obtained in the framework of the hard cut-off regularization.

Two convenient features, opposed to what happens in a hard cut-off regularization (and a fortiori in a Wilsonian calculation) are observed: (i) no power-like term appear: the renormalization of the mass is only multiplicative. In turn, this means that classically scale invariant theories only suffer from a soft logarithmic breaking due to the appearance of a mass scale μ through which the mass dimension of the four-point coupling λ in $D \neq 4$ dimensions is considered; (ii) higher powers ϕ^6, ϕ^8, \dots are not generated: application of DR to a (four dimensional) theory with only dimension 4 operators never generates operator of dimension different than 4.

All of this of course cannot be taken automatically, and the various extensions and formal steps need to find a proper justification. This is why we perform the full calculation of the D -dimensional effective potential the way we do. To make the manipulations justifiable, the integral has to be made convergent, and the momentum integration has to be stopped at Λ . In turn, this makes it possible to see how DR disposes of the cut-off Λ , that, we stress again, in the Wilson approach is a *physical* scale. For any value of the positive integer D , the result is given by a surface term plus the difference of a complete and an incomplete beta function. Individually, *both* the complete and incomplete beta functions have poles at $D = 2, 4, 6, \dots$. Of course, as the expression we started with is finite, the poles of one function cancel with those of the other function.

The cut-off only appears in the surface term and in the incomplete beta function, that can be written as (see equation 32 of the paper)

$$\bar{B}_i\left(1 - \frac{z}{2}, \frac{z}{2}; u\right) = \frac{2}{2-z} \left(\frac{M^2}{\Lambda^2}\right)^{\frac{2-z}{2}} - \frac{2}{4-z} \left(\frac{M^2}{\Lambda^2}\right)^{\frac{4-z}{2}} + \frac{2}{6-z} \left(\frac{M^2}{\Lambda^2}\right)^{\frac{6-z}{2}} + \dots \quad (7.2.1)$$

where $u = M^2/(M^2 + \Lambda^2)$ and M^2 is the usual (quadratic) fluctuation operator made up of the mass and the quartic coupling $M^2(\phi) = m^2 + \lambda/2 \phi^2$ (equation 8 in [5]). Note that an ab initio logarithmically divergent integral would lead to a term $\bar{B}_i(2 - z/2, z/2; u)$ and the first term would be missing in the expansion. This equation shows the structure of the poles of the incomplete beta function. As the final result is free of poles, this also enforces the structure of the poles of the complete beta function, a property which was successfully verified (see also Fig. 7.1). Altogether, the result is of course that of the hard cut-off regularization: the poles disappear and one is left with the usual Λ -dependent terms. The result of DR is obtained if one throws away the surface term and the incomplete beta function. The effect of this is to hide the Λ dependence and introduce spurious poles in the result. Formally, one should introduce an intermediate step in the renormalization of the theory, splitting the counterterms in two parts

$$\delta\Omega = \delta\Omega_1 + \delta\Omega_2, \quad \delta m^2 = \delta m_1^2 + \delta m_2^2, \quad \delta\lambda = \delta\lambda_1 + \delta\lambda_2, \quad (7.2.2)$$

and reabsorb the divergences in Λ and the poles of only the incomplete beta function in a first set of counterterms ($\delta\Omega_2, \delta m_2^2, \delta\lambda_2$). The resulting expression has no cut-off dependence and shows poles for even values of D . Of course, this is an artifact of the cancellation performed with the set of counterterms indexed "2". It has nothing to do with special physical properties of DR. In the figure 1 of the paper, this is pictorially represented. The formal result of DR is represented by bubble 2, which is obtained from the full result of bubble 1 via the partial cancellation performed through ($\delta\Omega_2, \delta m_2^2, \delta\lambda_2$). When one uses the formal rules of DR, this cancellation is hidden: one arrives directly at bubble 2 without even seeing that there is a bubble 1 behind it.

At this point, it seems rather natural to ask how this procedure can result in the same renormalized potential as the one obtained with a full calculation. This was explained in [5] thanks to the following two observations (we specify to the case $D = 4$):

- In the expansion of the incomplete beta function, the pole term $2/\epsilon$ and the $\log \Lambda^2/M^2$ come from the expansion of the same term (see equations 32, 39 and 41);
- In the expansion of the complete beta function, in turn, the pole term $2/\epsilon$ and the $\log M^2/\mu^2$ come from the expansion of the same term (see equations 38 and 40).

As the polar terms need to cancel between each other, this means that the $\log \Lambda^2/M^2$ and the $\log M^2/\mu^2$ must have the same coefficient and opposite sign. This proves that dimensional regularization necessarily gives the same result as the hard cut-off calculation once the potential (or a generic n-point function) is renormalized through (among the other cancellations) the subtraction of a term $\log \Lambda^2/\mu^2$ to $\log \Lambda^2/M^2$.

From the higher Wilsonian standpoint of the running action, the one loop effective potential V_{1l} calculated with the hard cut-off is an approximation to the running potential U_k found at $k = 0$ in the LPA with trivial saddle point where the whole domain of

integration $[0, \Lambda]$ contributes at once (so that it also relies on the validity of the perturbative expansion as the small parameter $\delta k/k$ is lost). The bare perturbative RG equation can be obtained from V_{1l} requiring independence on the cut-off, i.e. $\Lambda(d/d\Lambda)V_{1l} = 0$. This is commonly done with the result in equation 16 (or 43) of [5], where V_{1l} has been expanded for $M^2 \ll \Lambda^2$. The expansion is actually a fundamental step in the renormalization: the renormalization conditions are imposed on n-point functions, and the expansion is thus necessary to obtain a systematic expression organized in powers of ϕ . Depending on how many operators are kept, a certain truncation of the set of equations (6.2.23)-(6.2.26) is obtained. As a side note, we observe that if, on the contrary, the cut-off independence is required before the result of V_{1l} is expanded for $M^2/\Lambda^2 \ll 1$, the system (6.2.11)-(6.2.14) is re-obtained. This is less common, and surely outside the framework of perturbation theory. In fact, it is easy to convince that the improvement of the one-loop result obtained differentiating Λ in this precise manner reproduces the Wilsonian derivation in the LPA with trivial saddle and the classical potential as UV boundary. This short-cut was used for example in [119] to (re)-derive Wilsonian RG equations with fermions.

The Callan-Symanzik RG equations are obtained requiring independence of the renormalized potential from the 't Hooft scale (or subtraction scale) μ . In particular, this leads for the ϕ^4 theory to the set of equations 55-57 in the paper. The running equation for the mass is multiplicative

$$\mu \frac{dm^2}{d\mu} = \frac{\lambda}{16\pi^2} m^2. \quad (7.2.3)$$

The question of how to relate such an equation to the Wilsonian one was answered in the paper. Starting from the UV expansion obtained in the LPA (6.2.24), it was shown that (7.2.3) is obtained if at each scale μ one subtracts a critical mass

$$m_{\mu, \text{cr}}^2 \equiv \frac{\lambda_\mu}{16\pi^2} \mu \delta\mu \quad (7.2.4)$$

from the running Wilsonian mass m_μ^2 . It is the combination $\tilde{m}_\mu^2 \equiv m_\mu^2 - m_{\mu, \text{cr}}^2$, the subtracted mass, that runs with (7.2.3). In the derivation of this result, attention should be paid to the fact that the subtraction needs to be done at each step of the blocking transformation, and then one has to start over again. An explicit example of this will be given in [6] for both the ϕ^4 theory and the Standard Model.

Building on the correspondence between quantum field theories and critical phenomena, the physical picture that emerge is clear: the quadratic term in (6.2.12) indicates the position of a critical surface in parameter space, on which the mass parameter is multiplicatively renormalized. This is the same that is done in the Ginzburg-Landau theory when the equivalent of the mass term (that is, the coefficient of the quadratic operator) is proportional to the deviation⁵ from the critical temperature $T - T_c$. The Callan-Symanzik equation describes the evolution of the deviations with respect to the

⁵ The parameter that measures the deviation from the critical surface is only multiplicatively renormalized and behaves as in a scaling regime. The word critical is used below in light of the connection between correlation lengths and the masses in the correspondence between the theory of critical phenomena and QFT. The limit $m^2 \ll \Lambda^2$ corresponds to $\xi \gg a$ with a the lattice spacing/inter-atomic distance.

critical surface. In this context, the fine-tuning is more generically seen as the subtraction of the critical mass, which forces the system in the critical region. The naturalness problem then finds a precise formulation in this RG framework. Somehow, we know (experimentally for the Higgs) that the physics described by QFT (at least at low energies) happens in a critical regime where physical masses are $m^2 \ll \Lambda^2$. What we do not know, and is the essence of the problem, is what physical mechanism drives the system towards this critical regime. Until this is understood, we need to force our system by hand to stay within the critical region.

The take-home message of the analysis is twofold. Concerning dimensional regularization, it is recognized as an extremely useful technical tool with a very precise physical meaning. It realizes the two steps (integration of quantum fluctuations and tuning towards the critical surface) that characterize a renormalized Wilsonian EFT in the limit of no decoupling (shell $[0, \Lambda]$) in one go. In this sense, we agree with [120] that the results of DR are physical and not only technical, but unfortunately cannot support the idea that DR might possess special physical properties. Even more, our calculation finds that when its results (or the results of any other regularization scheme) are used to obtain the Callan-Symanzik equations, it is fundamental to recognize that the resulting equation for the mass parameter encodes the evolution of the subtracted mass, i.e. the quantity that measures the deviations with respect to the critical surface. This role of the term appearing in the perturbative RG equation was recognized in other works [121, 122, 123] (see [5] for more literature), with whom we agree on this point. In light of our results, what we can not agree on (and this is one of the main point on which we wanted to give a new perspective in [5] and [6]) is the idea that only deviations from the critical surface are of importance and that other terms should be discarded. The Renormalization Group, and in particular the formulation due to Wilson, has a far greater domain of validity (see for instance [124] and references therein), and certainly need not to be forced to the critical regime. This will be further discussed and clarified in [6].

Callan-Symanzik equations are used in yet another approach, where the presence of the quadratic term is recognized, although not in a completely satisfactory way from the Wilsonian standpoint (i.e. not in terms of the sliding scale k). To clarify this point, let us refer again for a moment to the ϕ^4 theory. If one does not cancel the Λ^2 dependence in (7.1.2) and only defines the counterterm (7.1.3) to cancel the $\log \Lambda$, so that no fine-tuning has been performed, the perturbative result for the mass at the one-loop level is

$$m_{1l}^2 = m^2 + \frac{\lambda}{32\pi^2} \left(\Lambda^2 + m^2 \left(\log \frac{m^2}{\mu^2} \right) \right). \quad (7.2.5)$$

Combining it with the way the Callan-Symanzik equations are usually derived, i.e. requiring independence of the final result on μ , to free the running mass $m(\mu)$ from the quadratic contribution $\sim \Lambda^2$ it is only necessary to cancel it once. The cancellation of the constant would then generate a small mass at all scales: the renormalized RG equation

$$\mu \frac{dm^2}{d\mu} = \gamma m^2 \quad (7.2.6)$$

with small, perturbative values for γ (besides the example of the ϕ^4 theory, either the SM one (7.1.6) or a BSM one) ensures that if $m_\mu^2 \ll \Lambda$ is valid at one scale, then it is

valid at all scales in the QFT regime. This is the approach proposed in [115, 125, 126] (see [5] for other references and followers of these ideas). Either relying on a quantum gravity miracle outside the QFT realm that would ensure $m^2(\Lambda) \ll \Lambda^2$ ([115]), or searching in QFT a particular extension of the SM that realizes the Veltman condition at a scale $\Lambda \lesssim M_P$ (in the SM at two-loop it happens at $\sim 10^{23}$ GeV), the crucial point is that if a mechanism realizes a cancellation of the Λ^2 term at a scale \bar{k} , then the latter naturally disappears at all other scales and the naturalness/hierarchy problem is solved. It is clear that our results go in a direction opposite to this: in the Wilsonian RG the "quadratic divergence" is only an approximation to a quadratic running that appears when we take the macroscopic shell $[0, \Lambda]$. As now said several times, the Callan-Symanzik equation by construction incorporates even more than the usually acknowledged fine-tuning, as they realize the subtraction of the quadratic term at *each* scale μ .

In conclusion, the consequence of all this is that all the different approaches that rely either on the use of dimensional regularization and/or the use of the Callan-Symanzik equations unfortunately cannot provide a solution to the naturalness/hierarchy problem. DR and the renormalized perturbative RG equations are specifically devised to go directly to renormalized results/the critical region by hand without even showing it: the fine-tuning or subtraction is intrinsic in their formulation. Unfortunately we need to conclude that, by construction, they cannot shed any light on the problem.

Dimensional regularization, Wilsonian RG, and the naturalness and hierarchy problem

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While it is usually stated that dimensional regularization (DR) has no direct physical interpretation, consensus has recently grown on the idea that it might be endowed with special physical properties that would provide the mechanism that solves the naturalness and hierarchy problem. Comparing direct Wilsonian calculations with the corresponding DR ones, we find that DR indeed has a well-defined physical meaning, and we point out its limitations. In particular, our results show that DR cannot provide the solution to the naturalness and hierarchy problem. The absence of too large corrections to the Higgs boson mass is due to a secretly realized fine-tuning, rather than special physical properties of DR. We also investigate these issues within the Wilsonian RG framework, and by comparison with the usual perturbative RG analysis, we show that several popular proposals for the resolution of the problem, commonly considered as physical mechanisms free of fine-tuning, again secretly implement the tuning.

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I. INTRODUCTION

The Standard Model (SM) of particle physics is a very successful theory, and the discovery of the Higgs boson [1,2] is one of the most important findings of the last years. However, it is not a complete theory, and the search for physics beyond the Standard Model (BSM) is one of the strongest driving forces of present experimental and theoretical physics. Several fundamental unsolved questions (dark matter, matter-antimatter asymmetry, neutrino masses, the flavor and the strong CP problem, the problem with the unification of the gauge couplings) urge us to find the way beyond the SM. Among them, the naturalness and hierarchy problems.

Any quantum field theory (QFT) that contains scalar fields is confronted with the naturalness problem. It is formulated in different (equivalent) ways, the essential point being that the quantum corrections to the mass of a scalar field are typically proportional to the “highest mass scale” of the theory. When

this scale is too large, we have to resort to an “unnatural” tuning of the mass parameter, a “fine-tuning”.

One way to formulate the problem is as follows (see, for instance, [3]). If the higher energy model assumed to embed the SM contains a field of large mass M coupled to the Higgs field $H(x)$, its mass m_H receives corrections proportional to M ($\gg m_H$). As an example, we can consider GUT models, that contain scalar fields ϕ with masses $M \gg m_H$. These fields are coupled to H through terms of the kind $\lambda_\phi H^2 \phi^2$, so that m_H receives corrections as (μ is the subtraction or ’t Hooft scale)

$$\Delta m_H^2 \sim \lambda_\phi M^2 \ln \frac{M^2}{\mu^2}. \quad (1)$$

Similarly, in a supersymmetric extension of the SM, where SUSY is broken by a large stop mass $\tilde{m}_t \gg m_H$, the Higgs mass receives a large correction (y_t is the top Yukawa coupling),

$$\Delta m_H^2 \sim y_t \tilde{m}_t^2 \ln \frac{\tilde{m}_t^2}{\mu^2}. \quad (2)$$

The same problem can be formulated in a Wilsonian effective field theory (WEFT) framework [4], where a QFT is defined with a built-in cutoff Λ , the scale above which the theory loses its validity and has to be replaced with a (not better specified) higher energy theory. In this framework,

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the “highest mass scale” of the SM is Λ , and Δm_H^2 takes the form,

$$\Delta m_H^2 \sim \alpha \Lambda^2, \quad (3)$$

where α is a combination of coupling constants.

Despite their apparent difference, Eqs. (1)–(2) on the one side, and (3) on the other one, have the same physical content. In fact, the SM is an effective “low-energy” theory valid up to a certain “new physics” scale,¹ irrespective of being such a scale an ultraviolet (UV) cutoff Λ (that, in particular, could be the Planck mass M_P), or the mass of a heavy GUT scalar or of a heavy stop.

From the low-energy perspective, i.e., from the perspective of the SM alone, we do not know its UV completion. The Higgs mass m_H^2 receives a “quadratic correction” [in the form (1) or (2) or (3)], as a generic “leftover” of the higher energy theory that embeds the SM. The reason is that in the low-energy theory (the SM in this case), there is no symmetry that protects m_H^2 from getting such a large contribution.

Referring to the above example of a SUSY embedding, it is often stated that, even though (due to supersymmetry) there are no “quadratic divergences”, still there is a large correction to m_H^2 [see (2)], concluding that the naturalness problem is not related to the occurrence of quadratic divergences, but rather to the presence of high mass scales in the theory [3,5]. However, from the viewpoint of the lower energy theory (the SM), m_H^2 *does get* a “quadratically divergent” correction, that is \tilde{m}_t^2 , the scale above which the SM is no longer valid. This is the physical meaning of “quadratically divergent” correction. Below \tilde{m}_t the theory is the SM, not its supersymmetric extension. Therefore, we should not refer to the absence of quadratic divergences in the SUSY theory, but rather to the presence of quadratic divergences in the SM.

If we generically indicate with Λ the scale above which the SM has to be replaced by a higher energy theory (the “highest mass scale” of the SM), the radiative corrections to m_H^2 are proportional to Λ^2 . We stress that Λ is not a cutoff to be sent to infinity, but rather a physical scale above which the physics cannot be described in terms of the low-energy SM degrees of freedom (d.o.f.). The “quadratic correction” to m_H^2 is considered as “unnatural” in the sense that it is too large: $\Delta m_H^2 \sim \Lambda^2$. Therefore, the relevant question is whether there exists a “physical mechanism” (a symmetry, a dynamical mechanism, ...) that could free m_H^2 from such a large correction, allowing to get

$$m_H^2 \ll \Lambda^2. \quad (4)$$

In the absence of such a mechanism, we are led to resort to an *ad hoc* “fine-tuning” of the mass.

A great progress in our understanding of (renormalization in) QFTs comes from the Wilson’s lesson, that relies on the deep connection between QFTs and statistical physics.² In the Wilsonian framework, first the fluctuation modes are eliminated within a tiny shell, then a tuning of the parameters towards the critical region is realized. Iterating this procedure, the renormalized theory is finally obtained. For an interesting implementation of these ideas in the Hamiltonian framework, see [6].

This puts the renormalization of a QFT on a totally different perspective than a mere affair of “cancellation of divergences”. However, it gives no clue on the *physical mechanism* that drives the system toward the “critical regime” $m_H^2 \ll \Lambda^2$. In this respect, there is a profound difference between critical phenomena and QFTs. While the mechanism that drives the statistical system towards the critical regime is well known (in the case of a ferromagnet, this is the tuning of the temperature towards T_c), for QFTs, we do not know what triggers the system towards the critical regime, i.e., towards the renormalized theory.

The above considerations indicate what the “physical” way of posing the naturalness problem should be. The mass m_H^2 at the scale Λ is “naturally” expected to be $m_H^2(\Lambda) \sim \Lambda^2$. Which physical mechanism drives the SM toward the critical regime $m_H^2(v) \ll \Lambda^2$ (v is the Fermi scale)?

In other words, which physical mechanism introduces such an unnatural hierarchy among physical scales?

Traditional approaches, as supersymmetry and/or composite models, have to cope with the unfriendly constraints that come from the LHC results: the compositeness scale or the SUSY breaking scale should be just around the corner, in the TeV regime, but no sign of new physics has been observed so far. This leads several authors to speculate that the SM could be valid up to some very high energy scale, the Planck scale M_P or so. If this is the case, alternative approaches to the naturalness problem have to be envisaged.

In this respect, some authors consider (classical) conformal extensions of the SM, where the quantum fluctuations break the conformal symmetry only softly. By taking models with no intrinsic mass scale, i.e., containing only operators of dimension four, and calculating the quantum fluctuations using dimensional regularization (DR), only a logarithmic breaking of the conformal symmetry is

²From the theory of critical phenomena, we know that the critical regime is reached when the correlation length ξ among statistical fluctuations becomes much larger than the interatomic distance a . For a ferromagnet, this happens when the temperature T approaches the critical temperature T_c , and for T close to T_c , we have $\xi \sim |T - T_c|^{-\nu}$, where $\nu(> 0)$ is the appropriate critical exponent. QFTs and critical phenomena are connected through the correspondence $m_H \rightarrow \frac{1}{\xi}$, $\Lambda \rightarrow \frac{1}{a}$, and the requirement (4) in QFT corresponds to the tuning of the statistical system toward the critical regime.

¹Barring the possibility that the SM is the theory of everything.

realized, and small masses appear through a Coleman-Weinberg mechanism [7–28].³

The central assumption of these approaches is that DR plays a special role in defining the renormalized theory (see, for instance, [7]), grasping an element of truth that is missed by the Wilson’s theory. The two methods are regarded as physically different (see, for instance, [19]): the Wilsonian one needs a “fine-tuning”, while DR seems to be dispensed of it. Even more, the Wilsonian paradigm is downgraded to a “computational technique” that *improperly* insists in giving a physical meaning to the elimination of momentum shells in the construction of the effective action, and the naturalness problem is viewed as a problem of the “effective theory ideology” [19].

In view of the enormous success that these ideas have gained in the last years [7–28,31,32], it is of the greatest importance to investigate on these issues. Is it really possible that DR encodes *physical* properties that makes it the correct tool to calculate the quantum fluctuations in QFT, while the Wilsonian strategy produces unphysical terms?

If this would turn out to be the case, such a finding would represent a breakthrough, and the physical mechanism that makes $m_H^2 \ll \Lambda^2$ would be uncovered. In this respect, it is worth to stress that the more common lore is that, although DR is a powerful technique to calculate radiative corrections, it has no direct physical interpretation.

One of the main goals of the present work is to perform a thorough analysis of the Wilsonian and DR methods for calculations in QFT. From this analysis, the *physical meaning* and the *limits* of DR will clearly emerge. In our opinion, this represents a relevant progress in our understanding of renormalization, that allows us to make a correct use of DR. Notable recent examples of physical effects that cannot be captured by DR calculations are in [33,34], where it is shown that an effective field theory can be derived from string theory only if a Wilsonian perspective is adopted. In this framework, the decoupling of states above and below the physical cutoff scale can be derived (contrary to what happens when calculations are performed in DR), and, despite largely diffused Swampland arguments, this allows for a positive value for the cosmological constant at cosmological scales, even if a negative value is found at the string/Kaluza-Klein (KK) scale. An older example is given by theories with spontaneous symmetry breaking (SSB), where the RG flow of the coupling constants deviates significantly from the perturbative one when the infrared (IR) region is approached [35], and eventually, the RG equations get totally modified [36].

³Conformal extensions of the SM have also been advocated elsewhere [29,30], in a somehow different perspective: the couplings should not run with the scale (vanishing β functions), and the model should have enough constraints so that all the parameters should be fixed (predicted). This very ambitious program, however, has not yet found realistic implementations.

With regard to the naturalness and hierarchy problem, another largely considered possibility consists in assuming that the UV completion of the SM provides for $m_H^2(\Lambda)$ a value $m_H^2(\Lambda) \ll \Lambda^2$. In this scenario, once the problem of the “large correction” is fixed in the UV, the RG equation for the running $m_H^2(\mu)$ provides the “small” measured value for the physical mass $m_H^2(v)$, which turns out to be of the same order of $m_H^2(\Lambda)$; then it seems that the naturalness and hierarchy problem can be solved this way. From this standpoint, the “explanation” for the smallness of $m_H^2(v)$ is pushed towards the unknown realm of the Standard Model UV completion (quantum gravity, string theory, ...): the higher energy theory should operate the “miracle” of leaving us with a value of $m_H^2(\Lambda)$ much lower than the naturally expected one, $m_H^2(\Lambda) \sim \Lambda^2$.

Somehow complementary to the previous one, another scenario considers that at the scale Λ (as naturally expected) $m_H^2(\Lambda) \sim \Lambda^2$. A much lower value of m_H^2 is obtained at the Fermi scale, through a mechanism presented as “self-organized criticality”, where the critical regime $m_H^2(v) \ll m_H^2(\Lambda)$ should be reached without resorting to any fine-tuning.

In the present work, we carefully investigate both these scenarios and show that our previous results on the physical meaning of the renormalization procedures are crucial to ascertain the viability of these proposed mechanisms. Although it is not immediately apparent, we will see that they both hide a fine-tuning that makes them unfit to solve the naturalness and hierarchy problem.

The rest of the paper is organized as follows. To pave the way to our analysis, in Sec. II, we briefly review the main steps that lead to the renormalized one-loop effective potential of a single component scalar field theory in $d = 4$ dimensions. We use dimensional regularization, momentum cutoff, and Wilsonian flow equations, and compare the results. In Sec. III, we calculate the effective potential of a scalar theory in d dimensions with the help of a momentum cutoff, establish the connection between WEFT strategy and DR calculations, and show how DR hides the necessary fine-tuning. In Sec. IV, we consider the Wilsonian RG flows, showing again how the usual perturbative RG equations hide the fine-tuning. In Sec. V, we apply the results of the previous sections to the SM, and compare our results with previous literature. Section VI is for the conclusions.

II. DIMENSIONAL REGULARIZATION AND WEFT

In the last years, there has been growing consensus on the idea that DR might play a special role in defining QFTs and that the DR and the WEFT approaches are physically different, with the latter missing some “truth” that is instead encoded in DR [7–28,31,32]. From this perspective, WEFT is viewed as a calculation procedure that *improperly insists* in defining the renormalized theory through the successive

elimination of modes and the naturalness problem as an *artifact* of this approach [19].

Clearly, if one would find that DR is really endowed with special properties that make it the *correct physical way* to define QFTs, while the WEFT paradigm has to be discarded, this would be an earthquake for our understanding of QFTs, and the naturalness and hierarchy problem would simply evaporate.

The present section is devoted to a thorough investigation of these issues. For our analysis, a great simplification comes from the observation that, in order to investigate upon these questions, there is no need to consider the full SM. As we will see, the essence of the problem is entirely captured by considering the simpler ϕ^4 theory. Moreover, it will be sufficient to restrict ourselves to the lowest order of approximation, focusing in particular on the one-loop effective potential $V(\phi)$. We stress that, when calculated with the help of a momentum cutoff, the one-loop potential provides an implementation (and an approximation) of the WEFT strategy (see Sec. II C).

For a single component d -dimensional ϕ^4 theory, the action is

$$S[\phi] = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \Omega_0 + \frac{1}{2} m_0^2 \phi^2 + \frac{\mu^{4-d} \lambda_0}{4!} \phi^4 \right), \quad (5)$$

where μ is a mass scale introduced to keep λ_0 dimensionless,

$$\Omega_0 = \Omega + \delta\Omega, \quad m_0^2 = m^2 + \delta m^2, \quad \lambda_0 = \lambda + \delta\lambda, \quad (6)$$

are the bare parameters, $\delta\Omega$, δm^2 , and $\delta\lambda$ the counterterms, Ω , m^2 , and λ the renormalized parameters. The one-loop effective potential is

$$V_{1l}(\phi) = \Omega_0 + \frac{m_0^2}{2} \phi^2 + \frac{\mu^{4-d} \lambda_0}{4!} \phi^4 + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln \left(1 + \frac{m^2 + \frac{1}{2} \mu^{4-d} \lambda \phi^2}{k^2} \right). \quad (7)$$

The integral in (7) converges only for $d < 2$, that for integer values of d means only for $d = 1$, while for $d \geq 2$ it is UV divergent. Integrating over the angular variables and defining

$$M^2(\phi) \equiv m^2 + \frac{1}{2} \mu^{4-d} \lambda \phi^2, \quad (8)$$

for the one-loop correction to the potential $\delta V(\phi)$, we have

$$\begin{aligned} \delta V &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln \left(1 + \frac{M^2(\phi)}{k^2} \right) \\ &= \frac{1}{2(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \int_0^\infty dk^2 (k^2)^{\frac{d}{2}-1} \ln \left(1 + \frac{M^2(\phi)}{k^2} \right). \end{aligned} \quad (9)$$

We now briefly review the steps that lead to the renormalized one-loop potential, using first DR and then a momentum cutoff Λ to calculate (9) (where the latter, as stressed above, realizes an approximation of the WEFT strategy). We reassure the experts, familiar with the few lines of Secs. II A and II B, that they are reported here only as functional to our analysis.

A. Effective potential in 4D. Dimensional regularization

The first observation that leads to the DR strategy consists in noting that the right-hand side of (9) can be extended to complex d but converges only for $\text{Re}(d) < 2$. We have

$$\begin{aligned} &\int_0^\infty dk^2 (k^2)^{d/2-1} \ln \left(1 + \frac{M^2(\phi)}{k^2} \right) \\ &= \frac{2}{d} \mu^d \left(\frac{M^2(\phi)}{\mu^2} \right)^{\frac{d}{2}} B \left(1 - \frac{d}{2}, \frac{d}{2} \right) \\ &= \frac{2}{d} \mu^d \left(\frac{M^2(\phi)}{\mu^2} \right)^{\frac{d}{2}} \Gamma \left(1 - \frac{d}{2} \right) \Gamma \left(\frac{d}{2} \right), \end{aligned} \quad (10)$$

where $B(\alpha, \beta)$ and $\Gamma(z)$ are the special beta and gamma functions. Together with some of their properties, they are given in Appendix A.

The second observation is that, if we replace $\Gamma(z)$ with its analytic extension $\bar{\Gamma}(z)$ (see Appendix A), the second line of (10) can be extended to generic complex values of d . The function $\bar{\Gamma}(z)$ is obtained with the help of the Weierstrass representation for $\Gamma(z)^{-1}$ [see (A5)], from which we see that $\bar{\Gamma}(z)$ has poles for $z = 0, -1, -2, \dots$

Inserting the last member of (10) (with $\Gamma(z)$ replaced by $\bar{\Gamma}(z)$) in the right-hand side of (9), and using the relation $\bar{\Gamma}(z+1) = z\bar{\Gamma}(z)$, the DR rules for calculating δV are as follows. First we replace (for any complex $d \neq 2, 4, 6, \dots$)

$$\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln \left(1 + \frac{M^2(\phi)}{k^2} \right) \rightarrow -\frac{\mu^d}{2(4\pi)^{\frac{d}{2}}} \left(\frac{M^2(\phi)}{\mu^2} \right)^{\frac{d}{2}} \bar{\Gamma} \left(-\frac{d}{2} \right). \quad (11)$$

Successively, we expand the right hand side of (11) around $d = 4$ ($\epsilon \equiv 4 - d$),

$$\begin{aligned} &\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln \left(1 + \frac{M^2(\phi)}{k^2} \right) \\ &\equiv \frac{\mu^{-\epsilon} [M^2(\phi)]^2}{64\pi^2} \left(-\frac{2}{\epsilon} + \gamma + \ln \frac{M^2(\phi)}{4\pi\mu^2} - \frac{3}{2} \right) + \mathcal{O}(\epsilon). \end{aligned} \quad (12)$$

Then we cancel the pole in ϵ with the help of the counterterms $\delta\Omega$, δm^2 , $\delta\lambda$ in (6), that in the $\overline{\text{MS}}$ scheme are ($\bar{\epsilon} \equiv \epsilon(1 + \frac{\epsilon}{2} \ln \frac{\epsilon'}{4\pi})$)

$$\delta\Omega = \frac{m^4}{32\pi^2\bar{\epsilon}}\mu^{-\epsilon}, \quad \delta m^2 = \frac{\lambda m^2}{16\pi^2\bar{\epsilon}}, \quad \delta\lambda = \frac{3\lambda^2}{16\pi^2\bar{\epsilon}}. \quad (13)$$

Finally, we take the limit $\epsilon \rightarrow 0$, and the renormalized one-loop potential reads (for $\Omega = 0$)

$$V_{1l}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{1}{64\pi^2} \left(m^2 + \frac{\lambda}{2}\phi^2 \right)^2 \left[\ln \left(\frac{m^2 + \frac{\lambda}{2}\phi^2}{\mu^2} \right) - \frac{3}{2} \right]. \quad (14)$$

B. Effective potential in 4D. Momentum cutoff

If we calculate the loop integral in (7) for $d = 4$ with a sharp momentum cutoff, the one-loop correction to the potential is

$$\delta V(\phi) = \frac{1}{64\pi^2} \left[\Lambda^4 \ln \left(1 + \frac{M^2(\phi)}{\Lambda^2} \right) + \Lambda^2 M^2(\phi) - [M^2(\phi)]^2 \ln \left(\frac{\Lambda^2 + M^2(\phi)}{M^2(\phi)} \right) \right]. \quad (15)$$

Taking $\frac{\phi^2}{\Lambda^2}, \frac{m^2}{\Lambda^2} \ll 1$, and expanding the right-hand side of (15) in powers of $\frac{M^2}{\Lambda^2}$,

$$V_{1l}(\phi) = \Omega_0 + \frac{m_0^2}{2}\phi^2 + \frac{\lambda_0}{4!}\phi^4 + \frac{\Lambda^2 M^2}{32\pi^2} - \frac{(M^2)^2}{64\pi^2} \left(\ln \frac{\Lambda^2}{M^2} + \frac{1}{2} \right) + O \left(\frac{\phi^6}{\Lambda^2} \right). \quad (16)$$

Inserting (6) in (16), with

$$\begin{aligned} \delta\Omega &= -\frac{\Lambda^2 m^2}{32\pi^2} + \frac{m^4}{64\pi^2} \left[\ln \left(\frac{\Lambda^2}{\mu^2} \right) - 1 \right]; \\ \delta m^2 &= -\frac{\lambda\Lambda^2}{32\pi^2} + \frac{\lambda m^2}{32\pi^2} \left[\ln \left(\frac{\Lambda^2}{\mu^2} \right) - 1 \right]; \\ \delta\lambda &= \frac{3\lambda^2}{32\pi^2} \left[\ln \left(\frac{\Lambda^2}{\mu^2} \right) - 1 \right], \end{aligned} \quad (17)$$

and neglecting the cutoff suppressed terms $\frac{\phi^6}{\Lambda^2}, \frac{\phi^8}{\Lambda^4}, \dots$, for the renormalized one-loop potential, we find the same result obtained with DR, i.e., Eq. (14).

C. Wilsonian RG flow and one-loop effective potential

As mentioned above, the one-loop effective potential calculated with a momentum cutoff provides an approximation to the potential obtained within the WEFT framework. To elucidate this point, let us consider the Wilsonian

effective action $S_k[\phi]$, where k is the running scale. Given the bare (tree-level) action $S_\Lambda[\Phi]$ [where $\Phi(x) = \sum_{0 < |p| < \Lambda} \varphi_p e^{ipx}$], $S_k[\phi]$ is obtained decomposing $\Phi(x) = \phi(x) + \phi'(x)$ [with $\phi(x) = \sum_{0 < |p| < k} \varphi_p e^{ipx}$ and $\phi'(x) = \sum_{k < |p| < \Lambda} \varphi_p e^{ipx}$], and integrating out the modes φ_p in the range $k < p < \Lambda$,

$$e^{-S_k[\phi]} \equiv \int D[\phi'] e^{-S_\Lambda[\phi + \phi']}. \quad (18)$$

The effective action is $\Gamma[\phi] = S_{k=0}[\phi]$, while the bare action is $S_\Lambda[\phi] = S_{k=\Lambda}[\phi]$.

At the infinitesimally lower scale $k - \delta k$, the Wilsonian action $S_{k-\delta k}[\phi]$ is obtained through an equation similar to (18),

$$e^{-S_{k-\delta k}[\phi]} = \int D[\phi'] e^{-S_k[\phi + \phi']}, \quad (19)$$

where ϕ' contains only modes in the infinitesimal shell $k - \delta k < p < k$.

Let us consider the gradient expansion for $S_k[\phi]$,

$$S_k[\phi] = \int d^d x \left(U_k(\phi) + \frac{Z_k(\phi)}{2} \partial_\mu \phi \partial_\mu \phi + Y_k(\phi) (\partial_\mu \phi \partial_\mu \phi)^2 + W_k(\phi) (\phi \partial_\mu \partial_\mu \phi)^2 + \dots \right), \quad (20)$$

and restrict ourselves to the local potential approximation (LPA), that amounts to

$$Z_k(\phi) = 1, \quad Y_k(\phi) = W_k(\phi) = \dots = 0. \quad (21)$$

By taking as background field $\phi(x)$, the homogeneous configuration,

$$\phi(x) = \phi_0, \quad (22)$$

and performing in (19) the integration over ϕ' under the assumption that the saddle point ϕ'_{sp} is trivial,⁴ i.e., $\phi'_{\text{sp}} = 0$, we get ($U_k''(\Phi) \equiv \frac{\partial^2 U_k(\Phi)}{\partial \Phi^2}$),

$$U_{k-\delta k}(\phi_0) = U_k(\phi_0) + \frac{1}{2} \int' \frac{d^d p}{(2\pi)^d} \ln \left(\frac{p^2 + U_k''(\phi_0)}{p^2} \right), \quad (23)$$

where the prime indicates that the integration is performed within the shell $[k - \delta k, k]$, and we have subtracted a field independent term. In the limit $\delta k \rightarrow 0$, we finally have

⁴The modifications to Eq. (23) that arise when a nontrivial saddle point $\phi'_{\text{sp}} \neq 0$ is present are discussed in [36].

$$k \frac{\partial}{\partial k} U_k(\phi_0) = -\frac{k^d}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \ln \left(\frac{k^2 + U_k''(\phi_0)}{k^2} \right), \quad (24)$$

that is the RG flow equation for the Wilsonian potential $U_k(\phi_0)$ in the LPA.

This is an intrinsically nonperturbative equation for $U_k(\phi)$, that implements the WEFT strategy. However, its nonperturbative nature becomes evident only for sufficiently small values of k , the IR regime (that will be better specified in Sec. IV). On the contrary, for sufficiently large values of k (UV regime), it reproduces the perturbative results.

As mentioned above, for $k = 0$ the Wilsonian potential $U_k(\phi)$ is the effective potential $V_{\text{eff}}(\phi)$, while for $k = \Lambda$ it is the bare (tree-level) potential $U_\Lambda(\phi)$. We now show under which approximation the perturbative one-loop effective potential $V_{1l}(\phi)$ is obtained from (24). Taking for $U_\Lambda(\phi)$,

$$U_\Lambda(\phi) = \Omega_0 + \frac{m_0^2}{2} \phi^2 + \frac{\mu^{4-d} \lambda_0}{4!} \phi^4, \quad (25)$$

and approximating $U_k(\phi)$ in the right-hand side of (24) with $U_\Lambda(\phi)$ (i.e., freezing $U_k(\phi)$ to its boundary value at $k = \Lambda$), we can integrate both sides of (24) in the whole momentum range $[0, \Lambda]$ (indicated by the upper case Λ) in the integral below) and get

$$V_{1l}(\phi) = \Omega_0 + \frac{m_0^2}{2} \phi^2 + \frac{\mu^{4-d} \lambda_0}{4!} \phi^4 + \frac{1}{2} \int^{(\Lambda)} \frac{d^d k}{(2\pi)^d} \ln \left(1 + \frac{m_0^2 + \frac{1}{2} \mu^{4-d} \lambda_0 \phi^2}{k^2} \right), \quad (26)$$

that is nothing but the one-loop effective potential (7) once we replace the bare values m_0^2 and λ_0 in the above integral with the corresponding renormalized values, which is coherent with the fact that the one-loop correction is $O(\hbar)$.

Equation (26) shows that the one-loop potential calculated with the hard cutoff Λ provides a specific implementation (and approximation) of the WEFT strategy. A smoothed equivalent implementation of WEFT is obtained by means of the proper-time regularization, and in Appendix B, we give an example of that. From now on, we will refer to the WEFT strategy having in mind one-loop calculations of the kind considered in this section.

D. DR versus WEFT

Let us compare now the DR and WEFT approaches for the calculation of the one-loop effective potential. Apart from the elementary observation that the two procedures give the same result (once the counterterms are appropriately chosen), we would like to make a couple of other comments, relevant to our subsequent analysis.

From the results briefly reviewed above, it seems that DR intrinsically avoids

- (i) the appearance of quadratic divergences, so that there is no need to fine-tune the scalar mass (the same holds true for the cosmological constant);
- (ii) the appearance of higher powers ϕ^6, ϕ^8, \dots , with coupling constants of inverse mass power dimensions, that on the contrary are present in WEFT [see (16)].

Moreover, when m^2 vanishes, at the quantum level the theory (that is clearly scale invariant at the classical level) shows only a soft (logarithmic) breaking of scale invariance, and the scalar mass is generated through the Coleman-Weinberg mechanism [7–9, 11–15, 18–20, 22–28].

Accordingly, some authors speculate that DR might grasp an element of truth that is missed by the WEFT scheme. Even more, the physical essence of the WEFT approach is questioned. The whole idea of including in the theory the quantum fluctuations via iterative integrations over infinitesimal momentum shells is considered as misleading, and the naturalness/hierarchy problem is regarded as an artifact of “the effective field theory ideology” [19].

We will show that, contrary to these expectations, DR does not encode any special physical principle, but is a *specific way* of implementing the WEFT paradigm that can be applied only when the perturbative expansion is valid. In particular, we will see that DR realizes the fine-tuning of the mass parameter, although it does it in a *hidden way*. To this end, in the next section we turn our attention to the calculation of the one-loop effective potential in d dimensions by means of a momentum cutoff.

III. EFFECTIVE POTENTIAL IN D DIMENSIONS

As a first step of our analysis, we calculate the loop integral in (7), that is the one-loop correction $\delta V(\phi)$ for the d -dimensional theory (we should not forget that d is a positive integer), by introducing a cutoff Λ ($u \equiv \frac{M^2}{M^2 + \Lambda^2}$),

$$\delta V(\phi) = \frac{1}{2} \int^{(\Lambda)} \frac{d^d k}{(2\pi)^d} \ln \left(1 + \frac{M^2}{k^2} \right) \equiv \delta V_1(\phi) + \delta V_2(\phi), \quad (27)$$

where we defined

$$\delta V_1(\phi) = \frac{\mu^d}{d(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \left(\frac{M^2}{\mu^2} \right)^{\frac{d}{2}} \int_u^1 dt (1-t)^{\frac{d}{2}-1} t^{-\frac{d}{2}}, \quad (28)$$

$$\delta V_2(\phi) = \frac{\mu^d}{d(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \left(\frac{\Lambda}{\mu} \right)^d \ln \left(1 + \frac{M^2}{\Lambda^2} \right). \quad (29)$$

As Λ is finite, both $\delta V_1(\phi)$ and $\delta V_2(\phi)$ are finite for any integer d (more generally, this holds true for any complex value of d , with $\text{Re } d > 0$).

For our scopes, we note in particular that the integral in (28) is finite. Moreover, it is not difficult to show that, for any value of the dimension d , we can write

$$\int_u^1 dt (1-t)^{\frac{d}{2}-1} t^{-\frac{d}{2}} = \lim_{z \rightarrow d} \left[\bar{B} \left(1 - \frac{z}{2}, \frac{z}{2} \right) - \bar{B}_i \left(1 - \frac{z}{2}, \frac{z}{2}; u \right) \right], \quad (30)$$

where $\bar{B}(\alpha, \beta)$ and $\bar{B}_i(\alpha, \beta; x)$ are the analytic extensions of the complete and incomplete beta functions $B(\alpha, \beta)$ and $B_i(\alpha, \beta; x)$, and are defined for $x \in \mathbb{R}$ and for generic complex values of α and β , excluding $\alpha, \beta = 0, -1, -2, \dots$ (the functions B , B_i , \bar{B} , \bar{B}_i , with the corresponding existence domains and some of their properties, are given in Appendix A).

From Appendix A, we also know that

$$\bar{B} \left(1 - \frac{z}{2}, \frac{z}{2} \right) = \bar{\Gamma} \left(1 - \frac{z}{2} \right) \bar{\Gamma} \left(\frac{z}{2} \right), \quad (31)$$

which shows that $\bar{B}(1 - \frac{z}{2}, \frac{z}{2})$ has poles in $z = 2, 4, 6, \dots$ [see (A5)]. Moreover, expanding $\bar{B}_i(1 - \frac{z}{2}, \frac{z}{2}, u)$ in powers of $M^2/\Lambda^2 \ll 1$ (remember that $u = \frac{M^2}{M^2 + \Lambda^2}$), we have

$$\begin{aligned} \bar{B}_i \left(1 - \frac{z}{2}, \frac{z}{2}, u \right) &= \frac{2}{2-z} \left(\frac{M^2}{\Lambda^2} \right)^{\frac{2-z}{2}} - \frac{2}{4-z} \left(\frac{M^2}{\Lambda^2} \right)^{\frac{4-z}{2}} \\ &+ \frac{2}{6-z} \left(\frac{M^2}{\Lambda^2} \right)^{\frac{6-z}{2}} + \dots \end{aligned} \quad (32)$$

As we noted above, the right-hand side of (30) is finite for any integer d , so the same must be true for the left-hand side of this equation. Therefore, as (31) and (32) show that both $\bar{B}(1 - \frac{z}{2}, \frac{z}{2})$ and $\bar{B}_i(1 - \frac{z}{2}, \frac{z}{2}, u)$ have poles for $z = 2, 4, 6, \dots$, when we consider in (30) the limit towards one of these values of z , the pole developed in (31) must cancel the one coming from (32). It can be easily shown that this is actually the case. For definiteness, in what follows, we consider in (30) only the case $d = 4$, i.e., the theory defined in $d = 4$ dimensions, and show explicitly such a cancellation. However, the calculations and considerations developed below can be easily extended to any of the values $d = 2, 4, 6, \dots$

From (28) and (30), we have

$$\begin{aligned} \delta V_1(\phi) &= \frac{\mu^d}{d(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \left(\frac{M^2}{\mu^2} \right)^{\frac{d}{2}} \int_u^1 dt t^{-\frac{d}{2}} (1-t)^{\frac{d}{2}-1} \\ &\equiv \lim_{z \rightarrow d} [A_1(z) - A_2(z)], \end{aligned} \quad (33)$$

where we defined

$$A_1(z) \equiv F(z) \cdot \bar{B} \left(1 - \frac{z}{2}, \frac{z}{2} \right) \quad (34)$$

$$A_2(z) \equiv F(z) \cdot \bar{B}_i \left(1 - \frac{z}{2}, \frac{z}{2}; u \right) \quad (35)$$

$$\text{with } F(z) \equiv \frac{\mu^z}{z(4\pi)^{\frac{z}{2}} \Gamma(\frac{z}{2})} \left(\frac{M^2}{\mu^2} \right)^{\frac{z}{2}}. \quad (36)$$

With the help of (31) and (32), we can expand F , \bar{B} and \bar{B}_i around $z = 4$. More specifically, we write $z = 4 - \epsilon$ and expand these functions around $\epsilon = 0$, thus getting

$$\begin{aligned} F(4 - \epsilon) &= \frac{\mu^{-\epsilon}}{64\pi^2} [M^2(\phi)]^2 \left[1 + \left(-\gamma_E + \log(4\pi) \right. \right. \\ &\left. \left. - \log \frac{M^2(\phi)}{\mu^2} + \frac{3}{2} \right) \frac{\epsilon}{2} \right] + \mathcal{O}(\epsilon^2) \end{aligned} \quad (37)$$

$$\bar{B} \left(-1 + \frac{\epsilon}{2}, 2 - \frac{\epsilon}{2} \right) = -\frac{2}{\epsilon} + \mathcal{O}(\epsilon) \quad (38)$$

$$\bar{B}_i \left(-1 + \frac{\epsilon}{2}, 2 - \frac{\epsilon}{2}, \frac{M^2}{M^2 + \Lambda^2} \right) = -\frac{2}{\epsilon} - \frac{\Lambda^2}{M^2} + \log \frac{\Lambda^2}{M^2} + \mathcal{O}(\epsilon). \quad (39)$$

Then, using (37)–(39), we can write $A_1(4 - \epsilon)$ and $A_2(4 - \epsilon)$ as

$$A_1(4 - \epsilon) = \frac{\mu^{-\epsilon} [M^2(\phi)]^2}{64\pi^2} \left(-\frac{2}{\epsilon} + \gamma + \ln \frac{M^2(\phi)}{4\pi\mu^2} - \frac{3}{2} \right) + \mathcal{O}(\epsilon) \quad (40)$$

$$\begin{aligned} A_2(4 - \epsilon) &= -\frac{\mu^{-\epsilon}}{64\pi^2} [M^2(\phi)]^2 \left(\frac{\Lambda^2}{M^2(\phi)} - \log \frac{\Lambda^2}{M^2(\phi)} \right) \\ &+ \frac{\mu^{-\epsilon} [M^2(\phi)]^2}{64\pi^2} \left(-\frac{2}{\epsilon} + \gamma + \ln \frac{M^2(\phi)}{4\pi\mu^2} - \frac{3}{2} \right) \\ &+ \mathcal{O}(\epsilon) + \mathcal{O} \left(\frac{M^2}{\Lambda^2} \right). \end{aligned} \quad (41)$$

Let us make now two observations that are crucial to our analysis. From (40) and (41), we see that in the difference $A_1(4 - \epsilon) - A_2(4 - \epsilon)$, that is nothing but Eq. (33) for $z = 4 - \epsilon$, the polar terms $\frac{1}{\epsilon}$ cancel each other (as expected), and the limit $\epsilon \rightarrow 0$ can be safely and easily taken. For $d = 4$, we have

$$\begin{aligned} \delta V_1(\phi)|_{d=4} &= \lim_{\epsilon \rightarrow 0} [A_1(4 - \epsilon) - A_2(4 - \epsilon)] \\ &= -\frac{1}{64\pi^2} [M^2(\phi)]^2 \left(\frac{\Lambda^2}{M^2(\phi)} - \log \frac{\Lambda^2}{M^2(\phi)} \right) \\ &+ \mathcal{O} \left(\frac{1}{\Lambda^2} \right). \end{aligned} \quad (42)$$

Taking now the surface term $\delta V_2(\phi)$ in (29) for the case $d = 4$, performing the expansion in $M^2/\Lambda^2 \ll 1$, and combining the result with $\delta V_1(\phi)$ above, we finally get

$$\begin{aligned} \delta V(\phi) &= \delta V_1 + \delta V_2 \\ &= \frac{\Lambda^2 M^2(\phi)}{32\pi^2} - \frac{[M^2(\phi)]^2}{64\pi^2} \left(\ln \frac{\Lambda^2}{M^2(\phi)} + \frac{1}{2} \right) + \mathcal{O}\left(\frac{\phi^6}{\Lambda^2}\right), \end{aligned} \quad (43)$$

that is nothing but (16), i.e., the result obtained directly for the four-dimensional theory when the loop integral is calculated with a cutoff. The quadratic and logarithmic divergences in (43) are then canceled with the help of the counterterms (17), and this finally gives the renormalized potential (14).

Our second observation is that (keeping aside for a moment the $\epsilon \rightarrow 0$ limit) if we conveniently write the one-loop correction to the potential as

$$\begin{aligned} \delta V(\phi) &= \delta V_1(\phi) + \delta V_2(\phi) \\ &= A_1(4 - \epsilon) + [\delta V_2(\phi) - A_2(4 - \epsilon)], \end{aligned} \quad (44)$$

and neglect the term in the square bracket of the last member (with no justification for the moment; we will comment on this point below), we have

$$\begin{aligned} \delta V(\phi) &= A_1(4 - \epsilon) \\ &= \frac{\mu^{-\epsilon} [M^2(\phi)]^2}{64\pi^2} \left(-\frac{2}{\epsilon} + \gamma + \ln \frac{M^2(\phi)}{4\pi\mu^2} - \frac{3}{2} \right) \\ &\quad + \mathcal{O}(\epsilon), \end{aligned} \quad (45)$$

that is nothing but the DR result (12) for δV . Taking the counterterms (13), for the renormalized potential $V_{1l}(\phi)$, we again obtain (14).

Referring to (27), we proceed with our analysis by noting that δV in $d = 4$ dimensions can be calculated in one of the following three equivalent ways:

- taking $d = 4$, and then calculating the integral;
- calculating the integral for generic d , and then replacing $d = 4$;
- calculating separately $A_1(4 - \epsilon)$ and $A_2(4 - \epsilon)$, expanding each of them around $\epsilon = 0$, considering the difference $A_1 - A_2$, and finally taking the limit $\epsilon \rightarrow 0$.

The procedure (c) is the one that we used in this section and is certainly more intricate, and definitely much longer and cumbersome than (a) and/or (b). However, for the purposes of our analysis, that is to *uncover the physical meaning* of DR, we need to refer to this one.

We have just seen that if we neglect A_2 and δV_2 in (44), we are left with the DR result. But what could justify the neglect of A_2 and δV_2 in (44)? To answer this question, we begin by noting that, when we use the procedure (c), we can write

$$\begin{aligned} V_{1l}(\phi) &= \lim_{\epsilon \rightarrow 0} \left[\Omega + \delta\Omega + \frac{1}{2}(m^2 + \delta m^2)\phi^2 + \frac{\mu^\epsilon}{4!}(\lambda + \delta\lambda)\phi^4 \right. \\ &\quad \left. + A_1(4 - \epsilon) - A_2(4 - \epsilon) + \delta V_2(\phi) \right], \end{aligned} \quad (46)$$

where $\delta V_2(\phi)$ is given by (29) (with d replaced by $4 - \epsilon$), while $A_1(4 - \epsilon)$ and $A_2(4 - \epsilon)$ are given by (40) and (41), respectively. As in the difference $A_1(4 - \epsilon) - A_2(4 - \epsilon)$, the polar terms in ϵ disappear, in (46) we are left with the original WEFT result, and the divergences in Λ are canceled by the counterterms $\delta\Omega$, δm^2 and $\delta\lambda$ given in (17).

However, we now follow a different pattern, that *naturally* leads to the DR recipes and allows to find the *physical meaning* of DR. As we will see, this represents an advancement in our understanding of renormalization that allows us to avoid misinterpretations and misuses of DR.

Going back to the splitting (44) for δV , and defining $\Delta V_2(\phi)$ as given below,

$$\begin{aligned} \delta V(\phi) &= A_1(4 - \epsilon) + [\delta V_2(\phi) - A_2(4 - \epsilon)] \\ &\equiv A_1(4 - \epsilon) + \Delta V_2(\phi), \end{aligned} \quad (47)$$

our objective is to realize the cancellation of the divergences separately in $A_1(4 - \epsilon)$ and in $\Delta V_2(\phi)$, starting with $\Delta V_2(\phi)$. Note that, while in $A_1(4 - \epsilon)$ only divergences for $\epsilon \rightarrow 0$ appear, $\Delta V_2(\phi)$ contains divergences for $\epsilon \rightarrow 0$ as well as for $\Lambda \rightarrow \infty$.

In order to realize such a separate cancellation, we begin by making the splitting

$$\begin{aligned} \delta\Omega &= \delta\Omega_1 + \delta\Omega_2, & \delta m^2 &= \delta m_1^2 + \delta m_2^2, \\ \delta\lambda &= \delta\lambda_1 + \delta\lambda_2. \end{aligned} \quad (48)$$

Choosing

$$\delta\Omega_2 = -\frac{\Lambda^2 m^2}{32\pi^2} + \frac{m^4}{64\pi^2} \left[\ln \left(\frac{\Lambda^2}{\mu^2} \right) - 1 \right] - \frac{m^4}{32\pi^2 \bar{\epsilon}} \mu^{-\epsilon} \quad (49)$$

$$\delta m_2^2 = -\frac{\lambda \Lambda^2}{32\pi^2} + \frac{\lambda m^2}{32\pi^2} \left[\ln \left(\frac{\Lambda^2}{\mu^2} \right) - 1 \right] - \frac{\lambda m^2}{16\pi^2 \bar{\epsilon}} \quad (50)$$

$$\delta\lambda_2 = \frac{3\lambda^2}{32\pi^2} \left[\ln \left(\frac{\Lambda^2}{\mu^2} \right) - 1 \right] - \frac{3\lambda^2}{16\pi^2 \bar{\epsilon}}, \quad (51)$$

and inserting (49), (50), and (51) in (46), we have

$$\begin{aligned} V_{1l}(\phi) &= \lim_{\epsilon \rightarrow 0} \left[\Omega + \delta\Omega_1 + \frac{m^2 + \delta m_1^2}{2} \phi^2 + \frac{\lambda + \delta\lambda_1}{4!} \phi^4 \right. \\ &\quad \left. + A_1(4 - \epsilon) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) + \mathcal{O}(\epsilon) \right]. \end{aligned} \quad (52)$$

Apart from the harmless $\mathcal{O}\left(\frac{1}{\Lambda^2}\right)$ and $\mathcal{O}(\epsilon)$ terms, (52) is the one-loop potential that we would have obtained

following directly the DR “rules”. In fact, the final form (14) for the renormalized potential is obtained from (52) once $\delta\Omega_1$, δm_1^2 , and $\delta\lambda_1$ are chosen according to the $\overline{\text{MS}}$ counterterms in (13).

This is the result that allows to uncover the *physical content* of DR and will enable us to answer one of the questions that motivated this work, namely whether or not DR is endowed with special properties that make it the *correct physical way* to define QFTs [19], thus, helping in solving the naturalness problem.

We have just shown that (52), which in a DR setting is obtained from the well-known recipes, actually comes from the introduction of an intermediate step in the process of obtaining the renormalized potential, that in the Wilsonian language (theory of critical phenomena) corresponds to the tuning toward the critical regime (critical surface). As already stressed (see Sec. II C), the calculation of the one-loop effective potential with a momentum cutoff Λ is a specific implementation of the Wilson’s strategy in the perturbative regime.

The rules of DR are nothing but a “short cut” that allows to derive for $V_{1l}(\phi)$ directly the right-hand side of (52). We stress that, following the alternative (and longer) path (c), we have learnt that we are never dispensed from the *necessity* of subtracting the quadratically divergent contribution to the mass of the scalar particle. Such a subtraction is realized through the counterterm δm_2^2 in (51). When we adopt the short cut that takes (52) as the starting point for the calculation of the one-loop effective potential, this cancellation is *hidden*, and it seems we are dispensed of it.

The above remarks are illustrated in Fig. 1. Let us start by considering the “bare potential”, represented by bubble ①, and defined by $V(\phi) = \Omega_0 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0\mu^{4-d}}{4!}\phi^4$. To obtain the one-loop correction $\delta V = \delta V_1 + \delta V_2$, we *have to* sum (integration in $d^d k$) over the momenta of the intermediate virtual states, see (27), (28), and (29). The explicit calculation of this sum, arrested to the maximal value $|k| = \Lambda$, allows us to determine the counterterms $\delta\Omega$, δm^2 , and $\delta\lambda$, so that the one-loop “renormalized potential” $V_{1l}(\phi) = V(\phi) + \delta V(\phi)$ of bubble ③ is obtained. In Fig. 1, this calculation is represented by the line that connects ① with ③.

The same result is obtained by following a different, but *totally equivalent*, pattern. We can go to bubble ③, the “renormalized potential”, passing first from bubble ②.

The divergences in Λ are *anceled* in step ① \rightarrow ② [see (49)–(51)]. This is the *hidden fine-tuning* that leads to the “DR bare potential”. The DR counterterms $\delta\Omega_1$, δm_1^2 , and $\delta\lambda_1$ are determined in step ② \rightarrow ③. The important lesson is that the cancellation of the quadratic divergences is *secretly* realized when, through the DR recipes, we directly access the “DR bare potential” of bubble ②.

The above results show that DR is a smart calculation technique that, when the conditions for the perturbative

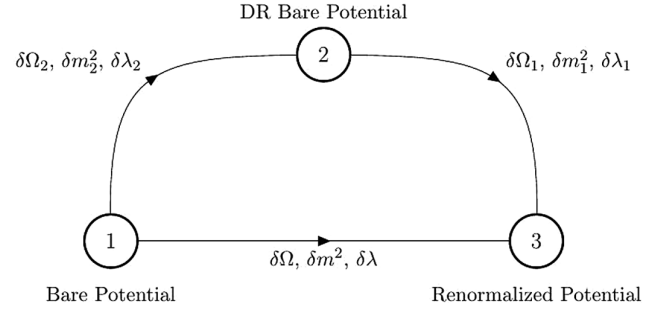


FIG. 1. Bubble “1” represents the bare potential (25), bubble “3” the renormalized potential (14). The line connecting “1” with “3” represents the calculation of the one-loop potential in $d = 4$ dimensions with a UV cutoff, and the determination of the counterterms (17). Bubble “2” represents the bare potential (52) in DR language. The line connecting “1” with “2” represents the calculation of the one-loop potential performed in d -dimensions with a UV cutoff, and the determination of the counterterms (49)–(51). The line connecting “2” with “3” represents the determination of the $\overline{\text{MS}}$ counterterms (13).

expansion are satisfied, implements both steps of the *physical* Wilsonian EFT calculation (the integration over the momentum modes, and the tuning towards the critical surface) at once.⁵ In other words, DR is an efficient technique that takes us *directly* to renormalized quantities, and, as such, is a very welcome tool. At the same time, our results show that DR is not endowed with any special physical property, despite claims to the contrary [7,19] that received a large follow-up [8–18,20–28,31,32].

The above results can also be obtained by means of a formal short cut that again shows how the unwanted terms are secretly canceled.

Let us start with (27) for $\delta V = \delta V_1 + \delta V_2$, where δV_1 and δV_2 are given in (28) and (29), respectively. Focusing on (28), and relaxing the physical requirement that d is an integer, we consider the integral in this equation for complex values of d , with $0 < \text{Re } d < 2$. Under this hypothesis, we can operate the (mathematically legitimate) splitting

$$\int_u^1 dt (1-t)^{\frac{d}{2}-1} t^{-\frac{d}{2}} = \int_0^1 dt (1-t)^{\frac{d}{2}-1} t^{-\frac{d}{2}} - \int_0^u dt (1-t)^{\frac{d}{2}-1} t^{-\frac{d}{2}}, \quad (53)$$

and note that (remember that $u = \frac{M^2}{M^2 + \Lambda^2}$) in (53) we can safely take the limit $\Lambda \rightarrow \infty$. Under this limit, the second term in the right-hand side of (53) vanishes, and we are left with the first term only, that is nothing but the beta function

⁵In Appendix B, we consider a different but equivalent implementation of the WEFt strategy, using a proper-time regularization, and apply it to the two-point vertex function rather than to the full effective potential.

$B(\alpha, \beta)$ (see Appendix A) of arguments $\alpha = 1 - \frac{d}{2}$ and $\beta = \frac{d}{2}$. Moreover, going to (29) for δV_2 , we see that for $0 < \text{Re } d < 2$, we have $\lim_{\Lambda \rightarrow \infty} \delta V_2 = 0$. Therefore, under these conditions, $\delta V = \delta V_1$. Replacing $B(1 - \frac{d}{2}, \frac{d}{2})$ with its analytic extension \bar{B} and pretending that we can extend the above results (obtained after sending $\Lambda \rightarrow \infty$!) to generic values of d , from (28), we immediately get the DR result.

The bottom line of the above derivation is that, operating with $0 < \text{Re } d < 2$, we can safely send the cutoff Λ to infinity. Therefore, when we extend to higher dimensions the results obtained this way, we may get the wrong impression that there is no need for a physical cut in the sum over the loop momenta.

Finally, to better understand the physical meaning of DR, and the reason why it gives the correct result for the perturbatively renormalized quantities, let us further compare the detailed calculations for the one-loop potential within the Wilsonian and DR frameworks.

Let us begin by considering $A_2(4 - \epsilon)$ in (41), and note that the terms $\log \Lambda^2/M^2$ (first line) and $2/\epsilon$ (second line) both come from the expansion around $\epsilon = 0$ of the same term in (32), namely the one with the pole in $d = 4$. This is why they are multiplied by the same factor. On the other hand, $A_1(4 - \epsilon)$ in (40) is given by the product of $F(4 - \epsilon)$ times $\bar{B}(-1 + \frac{\epsilon}{2}, 2 - \frac{\epsilon}{2})$ [see (34)]. A simple inspection of (37) and (38) shows that the terms $2/\epsilon$ and $\log M^2/\mu^2$ in $A_1(4 - \epsilon)$ have the same coefficient. Moreover, we already noted that the $2/\epsilon$ polar terms in $A_1(4 - \epsilon)$ and $A_2(4 - \epsilon)$ *must* have the same coefficient, otherwise there would be no cancellation of these “spurious singularities”. Therefore, $\log M^2/\mu^2$ in A_1 , and $\log M^2/\Lambda^2$ in A_2 *must* have the same coefficient.

This latter observation proves that the renormalized potential obtained from the DR rules *must* be the same as the renormalized potential derived from the physical Wilsonian calculation in the perturbative regime. In fact, the relevant part of the one-loop correction to the potential calculated with DR is the $\log M^2/\mu^2$ mentioned above, and contained in A_1 . At the same time, the coefficient of the divergent term $\log \Lambda^2/M^2$, obtained when performing the Wilsonian calculation, is *the same* as the coefficient of the similar term in A_2 [see (16) and (43)]. Therefore, the WEFT calculation needs counterterms [see (17)] that add up to a $\log \Lambda^2/\mu^2$ with *exactly the same coefficient*.

This simple chain of observations shows why, under the condition of the perturbative regime, the DR formal rules provide for the effective potential (and more generally, for any physical quantity) *exactly the same result* that is obtained when the direct WEFT physical calculation is performed. We have actually shown that the DR rules are obtained in the WEFT framework and are far from being in contrast with it. DR is a technique able to give the physically correct perturbative results, although the deep physical reason for that is very much hidden in the procedure.

In this respect, it is worth to compare our results with the analysis presented in [37] (see also [38]), whose main point is that the results obtained within the DR framework are “generic” (as opposed to specific) and then “physical”. As already stressed several times, our results show that DR is a realization of the Wilsonian calculation, that implements the cancellation of the quadratic contributions in a smart way, thus successfully providing the physical results at the Fermi scale. In [37], the same results are obtained through a technically different approach, namely through the introduction of higher derivative terms, thus showing the generality of the DR results.

Before ending this section, we would like to note that the results of our analysis are of particular interest when studying BSM models with classical scale invariance, where the use of dimensional regularization seems to suggest that scale invariance (apart from a welcome logarithmic violation) can be preserved also at the quantum level [7–28]. To better illustrate our point, let us refer in particular to [7,19]. By considering the possibility that the fundamental theory of nature does not possess any mass or length scale, in [19], only dimension four operators are kept, more precisely SM operators with dimensionless couplings, with the Higgs field nonminimally coupled to gravity. Similarly, in [7], a conformally extended version of the Standard Model is considered, with right-chiral neutrinos and a minimally enlarged scalar sector. In both cases, it seems that with these almost scale invariant models the naturalness and hierarchy problem is absent.

However, the reason why we only see a logarithmic violation of scale invariance is entirely due to the fact that the quantum corrections are calculated with DR. This is why the particle masses, generated through the Coleman-Weinberg mechanism, seem to not exhibit strong UV sensitivity. It is obvious that, if we consider a theory that contains only dimension four operators and at the same time, compute the radiative corrections with DR, operators of dimension two can never be generated, so that we get the impression that no fine-tuning is needed. However, as we have shown in the present section, DR contains a “hidden fine-tuning”, and the fact that dimension two operators do not appear when calculating radiative corrections is simply due to that. In this respect, we note that in the original Coleman and Weinberg calculation the effective potential is obtained introducing a momentum cutoff for the loop integrals, and the renormalized $m^2 = 0$ mass is obtained only after performing the fine-tuning [39].

IV. PERTUBATIVE, WILSONIAN, AND SUBTRACTED RG

Let us consider the Callan-Symanzik equation for the renormalized potential of Eq. (14), obtained by requiring independence of $V_{1l}(\phi)$ from μ ,

$$\mu \frac{d}{d\mu} V_{1l}(\phi) = \left(\mu \frac{\partial}{\partial \mu} + \beta_\Omega \frac{\partial}{\partial \Omega} + m^2 \gamma_m \frac{\partial}{\partial m^2} + \beta_\lambda \frac{\partial}{\partial \lambda} \right) V_{1l}(\phi) = 0. \quad (54)$$

Inserting (14) in (54), the one-loop RG functions read

$$\beta_\Omega = \mu \frac{d\Omega}{d\mu} = \frac{m^4}{32\pi^2} \quad (55)$$

$$\gamma_m = \frac{1}{m^2} \left(\mu \frac{dm^2}{d\mu} \right) = \frac{\lambda}{16\pi^2} \quad (56)$$

$$\beta_\lambda = \mu \frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}. \quad (57)$$

Below we will see that, once the subtraction that leads to the tuning towards the critical surface is performed, these RG functions coincide with the corresponding Wilsonian ones in the UV region (perturbative regime). We note that they are obtained in the perturbative regime and can be derived either using DR or the momentum cutoff calculation.

The flow of the bare parameters is obtained in a similar way, requiring independence of the bare effective potential in (16) from Λ . From an equation analogous to (54), we obtain

$$\Lambda \frac{d}{d\Lambda} \Omega_0 = -\frac{m_0^2 \Lambda^2}{16\pi^2} + \frac{m_0^4}{32\pi^2} \quad (58)$$

$$\Lambda \frac{d}{d\Lambda} m_0^2 = -\frac{\lambda_0 \Lambda^2}{16\pi^2} + \frac{\lambda_0 m_0^2}{16\pi^2} \quad (59)$$

$$\Lambda \frac{d}{d\Lambda} \lambda_0 = \frac{3\lambda_0^2}{16\pi^2}, \quad (60)$$

that, as we show below, are nothing but the Wilsonian RG equations for the running parameters in the UV regime.

To ascertain this point, let us go back to Eq. (24) for the Wilsonian potential $U_k(\phi)$, that for the reader's convenience, we write here for $d = 4$,

$$k \frac{\partial}{\partial k} U_k(\phi) = -\frac{k^4}{16\pi^2} \ln \left(\frac{k^2 + U_k''(\phi)}{k^2} \right). \quad (61)$$

Inserting in (61), the expansion,

$$U_k(\phi) = \Omega_k + \frac{1}{2} m_k^2 \phi^2 + \frac{1}{4!} \lambda_k \phi^4 + \frac{1}{6!} \lambda_k^{(6)} \phi^6 + \frac{1}{8!} \lambda_k^{(8)} \phi^8 + \dots, \quad (62)$$

where the Wilsonian RG parameters are [the upper label (i) denotes the i th derivative with respect to ϕ],

$$\begin{aligned} \Omega_k &= U_k(0), & m_k^2 &= U_k^{(2)}(0), & \lambda_k &= U_k^{(4)}(0), \\ \lambda_k^{(6)} &= U_k^{(6)}(0), & \lambda_k^{(8)} &= U_k^{(8)}(0), & \dots & \end{aligned} \quad (63)$$

for $\Omega_k, m_k^2, \lambda_k, \dots$ we easily get

$$k \frac{\partial \Omega_k}{\partial k} = -\frac{k^4}{16\pi^2} \log \left(\frac{k^2 + m_k^2}{k^2} \right) \quad (64)$$

$$k \frac{\partial m_k^2}{\partial k} = -\frac{k^4}{16\pi^2} \frac{\lambda_k}{k^2 + m_k^2} \quad (65)$$

$$k \frac{\partial \lambda_k}{\partial k} = -\frac{k^4}{16\pi^2} \left(\frac{\lambda_k^{(6)}}{k^2 + m_k^2} - \frac{3\lambda_k^2}{(k^2 + m_k^2)^2} \right) \quad (66)$$

$$\begin{aligned} k \frac{\partial \lambda_k^{(6)}}{\partial k} &= -\frac{k^4}{16\pi^2} \times \left(\frac{\lambda_k^{(8)}}{k^2 + m_k^2} - \frac{15\lambda_k \lambda_k^{(6)}}{(k^2 + m_k^2)^2} + \frac{30\lambda_k^3}{(k^2 + m_k^2)^3} \right) \\ &\dots \end{aligned} \quad (67)$$

These are the Wilsonian renormalization group equations in the framework of the local potential approximation. They form a set of infinitely many coupled differential equations and govern the nonperturbative flow of the theory parameters. If $k^2 + m_k^2 > 0$ in the whole range $[0, \Lambda]$, this flow essentially coincides with the perturbative one (see below). Similar results can be obtained for theories with scalars and fermions [40,41].

If, on the contrary, there exists a critical value k_{cr} where $k_{\text{cr}}^2 + m_{k_{\text{cr}}}^2 = 0$, that is the case when the theory manifests SSB, the nonperturbative nature of these equations becomes manifest when the region $k_{\text{cr}}^2 + m_{k_{\text{cr}}}^2 \gtrsim 0$ is approached. In this regime, the flow of the coupling constants deviates significantly from the perturbative one [35]. For values of $k < k_{\text{cr}}$, that is within the spinodal instability region, the flow equation (61) no longer holds, and has to be replaced with a new RG equation, that realizes the Maxwell construction for the SSB potential [36].

Limiting ourselves to the case when $k^2 + m_k^2 > 0$ and retaining for the potential $U_k(\phi)$ in (62) only terms up to the quartic coupling λ_k , this set of equations is truncated to Eqs. (64)–(66) only, where in the latter the term with $\lambda_k^{(6)}$ is missing.

Under the condition $k^2 \gg m_k^2$, i.e., in the UV regime, expanding these three equations in m_k^2/k^2 , we easily get

$$k \frac{\partial}{\partial k} \Omega_k = -\frac{k^2 m_k^2}{16\pi^2} + \frac{m_k^4}{32\pi^2} \quad (68)$$

$$k \frac{\partial}{\partial k} m_k^2 = -\frac{k^2 \lambda_k}{16\pi^2} + \frac{\lambda_k m_k^2}{16\pi^2} \quad (69)$$

$$k \frac{\partial}{\partial k} \lambda_k = \frac{3\lambda_k^2}{16\pi^2}, \quad (70)$$

that coincide with (58), (59), and (60) (once we replace k with Λ), that is what we wanted to show.

To understand the relation between the renormalized flow [Eqs. (55)–(57)] and the Wilsonian one [Eqs. (58)–(60), or equivalently (68)–(70)], we have to introduce first two “critical” parameters. Let us start with the mass. From the finite difference version of (59) [or equivalently (69)], we have

$$m_0^2(\Lambda - \delta\Lambda) = m_0^2(\Lambda) + \frac{\delta\Lambda \lambda_0(\Lambda)}{\Lambda 16\pi^2} \Lambda^2 - \frac{\delta\Lambda \lambda_0(\Lambda) m_0^2(\Lambda)}{\Lambda 16\pi^2} + \mathcal{O}\left(\frac{\delta\Lambda^2}{\Lambda^2}\right). \quad (71)$$

We now define the subtracted mass parameter $\tilde{m}^2(\Lambda - \delta\Lambda)$ at the scale $\Lambda - \delta\Lambda$ through the equation,

$$\tilde{m}^2(\Lambda - \delta\Lambda) \equiv m_0^2(\Lambda - \delta\Lambda) - m_{\text{cr}}^2(\Lambda), \quad (72)$$

where

$$m_{\text{cr}}^2(\Lambda) \equiv \frac{\lambda_0(\Lambda)}{16\pi^2} \Lambda \delta\Lambda, \quad (73)$$

is the *critical mass* that comes from the integration in the momentum shell $[\Lambda - \delta\Lambda, \Lambda]$ and vanishes in the $\delta\Lambda \rightarrow 0$ limit, so that we have the boundary,

$$\tilde{m}^2(\Lambda) = m_0^2(\Lambda). \quad (74)$$

With the help of (72) and (74), Eq. (71) can be written (in differential form) as

$$\frac{1}{\tilde{m}^2} \left(\Lambda \frac{d}{d\Lambda} \tilde{m}^2 \right) = \frac{\lambda_0}{16\pi^2}. \quad (75)$$

Comparing (75) with (56), we see that the perturbative flow of the renormalized mass m^2 is nothing but the flow of \tilde{m}^2 . The right-hand side of (75) is precisely the perturbative γ_m that appears in (56).

Similarly, by considering the finite difference version of (58) [or equivalently (68)], and defining the subtracted vacuum energy $\tilde{\Omega}$ through the equation,

$$\tilde{\Omega}(\Lambda - \delta\Lambda) \equiv \Omega_0(\Lambda - \delta\Lambda) - \Omega_{\text{cr}}(\Lambda), \quad (76)$$

where

$$\Omega_{\text{cr}}(\Lambda) \equiv \frac{\tilde{m}^2(\Lambda)}{16\pi^2} \Lambda \delta\Lambda, \quad (77)$$

is the *critical vacuum energy*, for $\tilde{\Omega}$ we obtain the flow equation,

$$\Lambda \frac{d}{d\Lambda} \tilde{\Omega} = \frac{\tilde{m}^4}{32\pi^2}. \quad (78)$$

As before, comparing (78) with (55), we see that the perturbative flow of the renormalized vacuum energy Ω coincides with the flow of $\tilde{\Omega}$, and the right-hand side of (78) is nothing but the perturbative β_Ω of (55).

For the dimensionless coupling λ there is obviously no subtraction to operate and in fact, comparing (60) with (57), we immediately see that the perturbative flow equation for the renormalized coupling λ coincides with the UV flow of λ_0 .

For the purposes of our analysis, it is important to stress that the perturbative flow equations of the positive mass dimension parameters m^2 and Ω , that can be obtained by using either DR or a momentum cutoff, are nothing but the RG equations of the *fine-tuned* parameters $\tilde{\Omega}$ and \tilde{m}^2 in the UV regime, i.e., the (UV) flow of the Wilsonian parameters subtracted of their critical values.⁶ This corresponds to the tuning towards the critical surface. We have then shown that the renormalized RG equations (55)–(57) contain the fine-tuning.

In the next section, we apply to the Standard Model the results and considerations developed here.

V. STANDARD MODEL. PERTURBATIVE AND WILSONIAN RG

The fact that at LHC no new particles have been observed allows to speculate that the SM could be valid all the way up to the Planck scale M_P , or to another high energy scale as for instance Λ_{GUT} , or even a trans-Planckian scale. From now on, we indicate this ultimate UV scale with Λ . For energies above this scale, we can imagine different scenarios: the SM could be replaced by a different theory, outside the QFT paradigm (string theory, loop quantum gravity, ...), or it might even be that, merging with quantum gravity, it could be extrapolated up to infinitely large energies [44].

Starting with the appropriate boundary conditions at Λ , the RG flows that connect Λ to the Fermi scale μ_F should provide the measured values of the coupling constants and of the particle masses as RG outputs at the scale μ_F . Let us concentrate on the running of the Higgs boson mass $m_H^2(\mu)$. As mentioned in the Introduction, a boundary condition typically regarded as a possible solution to the naturalness and hierarchy problem [3,45] is the so-called “miracle,”

$$m_H^2(\Lambda) \ll \Lambda^2, \quad (79)$$

that could come as a left-over of the Standard Model UV completion.

One specific implementation of (79) is obtained imposing the Veltman condition, i.e., the vanishing of the

⁶Other attempts toward a comparison between DR and Wilsonian flows are in [42,43].

quadratic divergences [46]. In the pure SM, such a condition is verified at $\Lambda \sim 10^{23}$ GeV (when the couplings are run with two-loop RG functions) [47,48]. If we would like to implement the Veltman condition at the Planck scale, $\Lambda = M_P$, we should consider extensions of the SM, as, for instance, in [49]. For the purposes of our analysis, however, it is totally irrelevant whether we consider the SM or a modified version of it. For this reason, in the following we concentrate on the SM.

Let us consider the perturbative RG flows, and restrict ourselves to the two-loop approximation for the RG functions [see, for instance, [50,51]]

$$\mu \frac{d}{d\mu} \lambda_i = \beta_{\lambda_i}, \quad (80)$$

$$\mu \frac{d}{d\mu} m_H^2 = m_H^2 \gamma_m, \quad (81)$$

where λ_i ($i = 1, \dots, 5$) stands for the SM quartic coupling λ , the top Yukawa coupling y_t and the three gauge couplings g_i .

When γ_m takes on perturbative values, i.e., $\gamma_m \ll 1$ (which is the case in the SM), and the RG equation for $m_H^2(\mu)$ is given by (81), we certainly have $m_H^2(\Lambda) \sim m_H^2(\mu_F)$. For instance, choosing $\mu_F = m_t$ and taking for $m_H(m_t)$ the value $m_H = 125.7$ GeV, if we take for Λ the scale where the Veltman condition is satisfied, Eq. (81) imposes the boundary $m_H(\Lambda) = 129.87$ GeV. More generally, similar results are obtained whenever the UV condition (79) is satisfied. Therefore, when the (79) miracle is realized, and the RG mass flow is governed by (81), it seems that the naturalness and hierarchy problem is solved.

However, we note that the miracle (79) can effectively protect m_H^2 from large quantum corrections only if the SM really provides the multiplicative renormalization encoded in (81). This latter condition is necessary to obtain $m_H^2(m_t) \sim m_H^2(\Lambda)$, i.e., the absence of hierarchy. From the previous section, we know that what runs in (81) is not the original Wilsonian mass $m_H^2(\mu)$ but rather the subtracted (i.e., renormalized) Higgs mass $\tilde{m}_H^2(\mu)$, where the fine-tuning of the quadratic divergence is already performed [see (75)]. Therefore, we cannot couple Eqs. (79) and (81) and pretend that the result $m_H^2(m_t) \sim m_H^2(\Lambda)$ provides a solution to the naturalness and hierarchy problem. In fact, whatever boundary $m_H^2(\Lambda)$ we use [including the boundary (79)], if we do not subtract the critical value of the mass, nothing can protect m_H^2 from getting a ‘‘quadratically divergent’’ ($\sim \Lambda^2$) contribution. As stressed in the previous section, such a subtraction is nothing but the *fine-tuning* and is necessary to switch from bare to renormalized mass.

It is worth to stress again that the SM is an EFT, where the physical UV cutoff Λ plays the role of a distinguished

scale, above which its UV completion has to be considered. But physics below Λ is governed by the SM, so the only consistent way of getting physical quantities is through effective quantum field theory calculations. Therefore, the appearance of quadratically divergent (i.e., proportional to Λ^2) contributions to the mass cannot be avoided: to reach the Fermi scale value $m_H \sim 125.7$ GeV, a fine-tuning *must* be operated.

Moreover we note that, as correctly pointed out in [33,34], to derive from string theory (or from any other UV completion) an effective field theory, it is necessary to take a Wilsonian perspective. In these recent papers, it was shown that the decoupling of states above and below the KK scale (i.e., the cutoff scale for the ‘‘low energy’’ effective theory) in a type IIB string theory can be demonstrated only by means of a bona fide Wilsonian calculation. Applying that to the well-known problem concerning the sign of the cosmological constant (CC), the author shows that at cosmological scales a positive value for CC can be obtained even if a negative value is found at the KK scale.

Other attempts to solve the naturalness and hierarchy problem, either within the SM, or in the framework of some SM extension, are based on the RG equations for the coupling constants of interest (the quartic Higgs coupling and/or additional couplings when SM extension are considered), with boundary conditions again fixed in the UV [49,51–57]. Running the quartic coupling λ down to the Fermi scale, the Higgs mass m_H is determined through the usual relation between λ and the vacuum expectation value (vev) of the Higgs field.⁷ It might seem that in this manner the problem of quadratic divergences is avoided. However, we should not forget that the vev gets a radiative correction from tadpole diagrams that, if not canceled, bring $\sim \Lambda^2$ contributions to the mass, and we have the same fine-tuning problem considered above. Moreover, although in some of these works [49,51,52,55,57] a sort of (softly broken) conformal invariance is apparently implemented, all these approaches still contain a hidden fine-tuning. The physical quantities are either obtained within a DR calculation or by means of subtracted RG equations.

Differently from the hidden fine-tuning of the previously discussed approaches, in [44,59–66] the presence of quadratic divergences is properly acknowledged, and it is correctly pointed out that they locate the critical surface in the couplings space. Moreover, the authors stress that the quantities we are interested in are the deviations of the physical parameters from their critical values (not the bare ones), that are nothing but the renormalized parameters. While this observation is correct, it does not give any indication on the physical mechanism that triggers the

⁷According to the chosen renormalization conditions, this relation is either the tree level or the radiatively corrected one [58].

approach to the critical surface. Although the authors refer to their approach as to a “self-organized criticality” phenomenon, they explicitly perform the subtraction of the quadratically divergent term in the mass parameter.⁸ Without such a subtraction, the system would never be driven towards the critical surface. In other words, they perform the usual fine-tuning of the theory. In a true self-organized critical phenomenon, a dynamical mechanism drives the physical system towards the critical surface, and no subtraction has ever to be performed.

The observation that the bare couplings and the critical surface are not universal quantities, and that “usually in quantum field theories they are not of much interest,” is certainly not a justification for performing the subtraction by hand. The essence of the naturalness problem consists in searching for the physical mechanism responsible for the suppression of the large radiative corrections to the mass. In [44,59–66], it is simply shown that, once the subtraction is performed by hand, with a large γ_m (~ 2), the theory can accommodate a large hierarchy between the Fermi and the Planck scale. Moreover, the circumstance that different choices of the cutoff scheme can give different values for the coefficient of the quadratic divergences [66] is not a problem in itself. If we think, for instance, of a supersymmetric theory, the cancellation of the quadratic divergences is related to the simultaneous presence of bosonic and fermionic superpartners, and occurs whatever specific cutoff scheme is chosen.

Finally, in a couple of recent papers [68,69], it is suggested that there might exist formulations of QFT that *ab initio* do not exhibit divergences. The authors claim that BPHZ is the most famous of these approaches. However, it is well known that the “R operation” in the BPHZ method corresponds to the renormalization operated via a specific choice of counterterms. In the original work of Bogoliubov and Parasiuk [70], this is implemented by employing a variation of the Hadamard regularization of singular integrals [71–74] for the subtraction of divergent contributions: it is precisely this operation that is abbreviated as R operation.

The authors also claim that another notable example of finite formulation of QFT, based on the Callan-Symanzik equations [75,76], is presented in [77,78]. However, as clearly explained in Callan’s lectures [78], the renormalized finite results are obtained in two steps. First, a modification of the functions in the loop integrals is obtained through a subtraction “à la Pauli-Villars”, thus, getting finite results. Successively, the renormalized quantities are obtained with the help of the Callan-Symanzik equations. Therefore, both BPHZ and the method explained in [78] are nothing else

⁸Similar observations are also done elsewhere; see, for instance, [67]. However, as already noted, forcing the system close to the critical surface is tantamount to subtract the quadratic terms and certainly does not explain what drives the system towards that regime.

than implementations of the subtraction of divergences, totally equivalent to the usual renormalization procedure, and as such cannot provide any *ab initio* finite formulation of QFT.

VI. SUMMARY, CONCLUSIONS, AND OUTLOOK

This work is focused on the evergreen subject of renormalization, with particular reference to the SM and BSM theories, and more generally, of theories containing scalar fields. We begin with a thorough analysis of dimensional regularization, usually considered as a useful calculation technique deprived of any direct physical interpretation. The analysis is done comparing the derivation of the one-loop effective potential in DR with a direct calculation performed in the framework of the Wilsonian effective field theory approach.

We have shown that DR implements at once both steps of the physical Wilsonian EFT calculation (the integration over the quantum fluctuation modes and the tuning towards the critical surface), provided that the conditions for the validity of the perturbative expansion are fulfilled. As such, it is a practical and welcome tool. In Sec. III, we have shown in detail that the DR results are nothing but an intermediate step in the Wilsonian derivation of renormalized physical quantities, where a *hidden* fine-tuning, that secretly realizes the tuning towards the critical region, is automatically encoded.

These findings enabled us to answer one of the physical questions that motivated the present study, a subject that has driven lot of recent research work [7–28,31,32], namely whether or not DR is endowed with special physical properties that make it the correct way to define QFTs. Our results definitely show that this is not the case. The physical mechanism that realizes the tuning toward the renormalized theory is not encoded in unknown physical properties of DR. As a consequence, DR cannot be of help in solving the naturalness and hierarchy problem.

In particular, we have shown that BSM models based on (classically) scale-invariant extensions of the Standard Model, where (apart from weak violations) the scale invariance is kept also at the quantum level through the use of dimensional regularization, do not provide a solution to the naturalness and hierarchy problem, as hoped by their proponents. In fact, when the classical lagrangian contains only operators of dimension four, and dimensional regularization is used, no terms with dimension different than four can ever be generated. In particular, this is the case for operators of dimension two, and we could get the impression that no fine-tuning is needed, and that the naturalness and hierarchy problem could be solved this way. As we have shown, however, DR contains a “hidden fine-tuning”, that invalidates such a conclusion.

We also analyzed the naturalness and hierarchy problem in the renormalization group framework. According to recent literature, if the UV completion of the SM provides

the boundary $m_H^2(\Lambda) \ll \Lambda^2$, the problem would disappear, as the perturbative anomalous dimension $\gamma_m (\ll 1)$ allows to get $m_H^2(\mu_F) \sim m_H^2(\Lambda)$ [3,45]. In another scenario, where the same RG equation (81) for $m_H^2(\mu)$ is used, it seems that the hierarchy $m_H^2(\mu_F) \ll m_H^2(\Lambda) \sim \Lambda^2$ can be accommodated, as the physical value $m_H^2(\mu_F)$ is obtained assuming that the UV completion of the SM provides for γ_m a nonperturbative value $\gamma_m \gtrsim 2$ [44,59–66]. However, as shown in detail in Sec. V, the RG equation used for $m_H^2(\mu)$ already contains the fine-tuning, and in both these scenarios, the suggested solution to the naturalness and hierarchy problem comes from this tuning. Therefore, they cannot solve the problem.

Again in the RG framework, the same question has been attacked in a different way, that might seem to circumvent the fine-tuning problem. Within the SM, or in one of its extensions, the RG equations for the couplings are considered with boundary conditions fixed in the UV [49,51–57]. Running the Higgs quartic coupling λ down to the Fermi scale, m_H is determined through the usual relation between λ and the vev of the Higgs field. However, we have shown that this does not solve the naturalness and hierarchy problem, as the vev gets radiative corrections from tadpole diagrams. If not canceled (fine-tuning), they bring quadratic divergent contributions to m_H^2 , again leaving the problem unsolved.

All these shortcomings are related in a way or another to the use of DR, of which we have shown the range of validity and limitations, providing examples where the direct physical Wilsonian calculations are needed, while DR gives incorrect results [33–36].

We also analyzed other recent attempts to solve the naturalness and hierarchy problem, that aim at a finite formulation of quantum field theories [68,69]. We showed that these calculations actually implement the usual subtraction of divergences and therefore, do not shade any light on the problem.

Before ending this section, we would like to discuss some possible continuations of our work. The methods employed in the present paper can be extended to implement the Wilsonian approach to gauge theories and quantum gravity, where some interesting attempts have already been made [79–81]. The same holds true for implementing conformal and/or scale invariance at the quantum level, where the present state of art is too poor. We plan to come to these issues in further studies.

At the same time, extending the methods of the present work, we plan to pursue our investigation on the physical mechanism that provides the tuning of the Higgs boson mass towards its experimental value.

APPENDIX A: SPECIAL FUNCTIONS

The special functions $B(\alpha, \beta)$ and $\Gamma(z)$ of complex arguments α, β , and z are defined by

$$B(\alpha, \beta) = \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} \quad \text{Re } \alpha, \text{Re } \beta > 0 \quad (\text{A1})$$

$$\Gamma(z) = \int_0^\infty d\tau \tau^{z-1} e^{-\tau} \quad \text{Re } z > 0, \quad (\text{A2})$$

where the conditions $\text{Re } \alpha > 0$, $\text{Re } \beta > 0$ and $\text{Re } z > 0$ guarantee the convergence of the integrals in (A1) and (A2). The functions B and Γ satisfy the relations,

$$\Gamma(z+1) = z\Gamma(z) \quad (\text{A3})$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (\text{A4})$$

and the inverse of $\Gamma(z)$ can be given with the help of the Weierstrass representation (γ_E is the Euler-Mascheroni constant),

$$\Gamma(z)^{-1} = z e^{\gamma_E z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \quad \text{Re } z > 0. \quad (\text{A5})$$

The right-hand side of (A5) has zeros for $z = 0, -1, -2, \dots$ (but these values are excluded by the condition $\text{Re } z > 0$) and is convergent for any z . The analytic extension $\bar{\Gamma}(z)$ of $\Gamma(z)$ is given by the inverse of the right-hand side of (A5), and then it is defined for generic values of z , with the exception of the zeros of (A5). The property (A3) holds also for $\bar{\Gamma}(z)$.

The analytic extension $\bar{B}(\alpha, \beta)$ of $B(\alpha, \beta)$ is obtained with the help of (A4), once the replacement $\Gamma \rightarrow \bar{\Gamma}$ is made. Due to the properties of the function $\bar{\Gamma}(z)$ defined above, the function $\bar{B}(\alpha, \beta)$ is defined for generic complex values of α and β , excluding $\alpha, \beta = 0, -1, -2, \dots$

Another special function used in the text is the incomplete beta function $B_i(\alpha, \beta; x)$ defined as

$$B_i(\alpha, \beta; x) \equiv \int_0^x dy y^{\alpha-1} (1-y)^{\beta-1} \quad \text{Re } \alpha, \text{Re } \beta > 0, x \in \mathbb{R}. \quad (\text{A6})$$

It is not difficult to show that, when $|x| < 1$, $\text{Re } \alpha > 0$, and $\text{Re } \beta > 0$, the function $B_i(\alpha, \beta; x)$ satisfies the relation,

$$B_i(\alpha, \beta; x) = \frac{x^\alpha}{\alpha} {}_2F_1(\alpha, 1-\beta; \alpha+1; x), \quad (\text{A7})$$

where ${}_2F_1(a, b; c; x)$ is the hypergeometric function,

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (\text{A8})$$

and $(y)_n$ are the Pochhammer symbols,

$$(y)_n \equiv \frac{\Gamma(y+n)}{\Gamma(y)} = y(y+1) \cdots (y+n-1). \quad (\text{A9})$$

From (A8) and (A9), we see that the right-hand side of (A7) is defined for any value of β and for generic values of α , [not only for $\text{Re } \alpha > 0$, and $\text{Re } \beta > 0$ as in (A6)], with the exception of the values $\alpha = 0, -1, -2, \dots$. With the help of (A7), the analytic extension $\tilde{B}_i(\alpha, \beta; x)$ of $B_i(\alpha, \beta; x)$ is then defined in this larger domain.

APPENDIX B: PROPAGATOR IN D -DIMENSIONS. PROPER TIME

In this appendix, we show that results similar to those obtained in Sec. III, where a hard momentum cutoff is used, can be obtained when we consider a proper-time regularization for the loop integrals. To this end, rather than resorting to the example of the full effective potential as in Sec. III, we consider only the one-loop two-point vertex function $\Gamma^{(2)}(0)$ in d dimensions for zero external momenta,

$$\Gamma^{(2)}(0) = m^2 + \delta m^2 + \frac{\lambda}{2} \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2}. \quad (\text{B1})$$

Within the proper-time regularization, the loop integral in (B1) is

$$\begin{aligned} \frac{1}{m^2} \int \frac{d^d k}{(2\pi)^d} \int_{m^2/\Lambda^2}^{\infty} d\tau e^{-\tau(k^2/m^2+1)} \\ = \frac{(m^2)^{d/2-1}}{(4\pi)^{d/2}} \int_{m^2/\Lambda^2}^{\infty} d\tau \tau^{-d/2} e^{-\tau}. \end{aligned} \quad (\text{B2})$$

Although d in (B2) is a positive integer, the integral in the right-hand side of this equation is convergent for any complex value of d . From the definition of the lower incomplete Gamma function $\Gamma_i(z; u)$ (with $z \in \mathbb{C}$ and $u \in \mathbb{R}$),

$$\Gamma_i(z; u) \equiv \int_u^{\infty} d\tau \tau^{z-1} e^{-\tau}, \quad (\text{B3})$$

we have

$$\int_{m^2/\Lambda^2}^{\infty} d\tau \tau^{-d/2} e^{-\tau} = \Gamma_i\left(1 - \frac{d}{2}; \frac{m^2}{\Lambda^2}\right). \quad (\text{B4})$$

Moreover, for $\text{Re } d < 2$, we have

$$\Gamma_i\left(1 - \frac{d}{2}; \frac{m^2}{\Lambda^2}\right) = \Gamma\left(1 - \frac{d}{2}\right) - \gamma\left(1 - \frac{d}{2}; \frac{m^2}{\Lambda^2}\right), \quad (\text{B5})$$

where $\gamma(z, u)$ is the upper incomplete Gamma function (with $\text{Re } z > 0$ and $u \in \mathbb{R}$),

$$\gamma(z; u) \equiv \int_0^u d\tau \tau^{z-1} e^{-\tau}. \quad (\text{B6})$$

This latter function satisfies the relation,

$$\gamma(z, u) = \frac{u^z}{z} {}_1F_1(z, z+1; -u), \quad (\text{B7})$$

where ${}_1F_1(a, b; u)$ is the hypergeometric function,

$${}_1F_1(a, b; u) = \sum_{n=0}^{\infty} \frac{(a)_n u^n}{(b)_n n!}, \quad (\text{B8})$$

and $(x)_n$ are the Pochhammer symbols defined in (A9).

The hypergeometric function ${}_1F_1(a, b; u)$ is defined for any complex value of u (infinite convergence radius), for any complex value of a , and for generic complex values of b , excluding $b = 0, -1, -2, \dots$. The analytic extension $\bar{\gamma}(z; u)$ of $\gamma(z; u)$ is defined through the right-hand side of (B7),

$$\bar{\gamma}(z, u) = \frac{u^z}{z} {}_1F_1(z, z+1; -u) \quad \text{for } z \neq 0, -1, -2, \dots, \quad (\text{B9})$$

and it is not difficult to see that, for any integer positive value of d ,

$$\begin{aligned} \int_{m^2/\Lambda^2}^{\infty} d\tau \tau^{-d/2} e^{-\tau} &= \Gamma_i\left(1 - \frac{d}{2}; \frac{m^2}{\Lambda^2}\right) \\ &= \lim_{z \rightarrow d} \left[\bar{\Gamma}\left(1 - \frac{z}{2}\right) - \bar{\gamma}\left(1 - \frac{z}{2}; \frac{m^2}{\Lambda^2}\right) \right]. \end{aligned} \quad (\text{B10})$$

The reader might find convenient at this point to compare the above equation with the analogous Eq. (30) of Sec. III. We already know that $\bar{\Gamma}(1 - \frac{z}{2})$ in (B10) has simple poles in $z = 2, 4, 6, \dots$. Moreover, expanding $\bar{\gamma}$ in powers of $\frac{m^2}{\Lambda^2} \ll 1$ with the help of (B8) and (B9), we have

$$\begin{aligned} \bar{\gamma}\left(1 - \frac{z}{2}, \frac{m^2}{\Lambda^2}\right) &= \frac{2}{2-z} \left(\frac{m^2}{\Lambda^2}\right)^{\frac{2-z}{2}} - \frac{2}{4-z} \left(\frac{m^2}{\Lambda^2}\right)^{\frac{4-z}{2}} \\ &\quad + \frac{1}{6-z} \left(\frac{m^2}{\Lambda^2}\right)^{\frac{6-z}{2}} + \dots \end{aligned} \quad (\text{B11})$$

that shows that $\bar{\gamma}(1 - \frac{z}{2}, \frac{m^2}{\Lambda^2})$ has simple poles in $z = 2, 4, 6, \dots$ as $\bar{\Gamma}(1 - \frac{z}{2})$.

By considering for the two-point function $\Gamma^{(2)}(0)$ in (B1) a proper-time regularization, we have found a result in all similar to the one obtained in Sec. III for the effective potential, where we used a hard momentum cutoff.

Following the same approach of Sec. III, from the proper-time regularization (that is a way of implementing the Wilsonian strategy with a smooth cutoff), we can again derive the DR rules.

Specifically, writing $z = 4 - \epsilon$ and expanding around $\epsilon = 0$, from (B2) and (B10), we have

$$\left[\frac{\lambda}{2} \mu^{4-d} \frac{(m^2)^{d/2-1}}{(4\pi)^{d/2}} \int_{m^2/\Lambda^2}^{\infty} d\tau \tau^{-d/2} e^{-\tau} \right]_{d=4} = \lim_{\epsilon \rightarrow 0} [C_1(4 - \epsilon) - C_2(4 - \epsilon)], \quad (\text{B12})$$

where

$$C_1(\epsilon) = \frac{\lambda m^2}{32\pi^2} \left(-\frac{2}{\epsilon} + \gamma_E - \ln(4\pi) \right) + \frac{\lambda m^2}{32\pi^2} \left(\ln \frac{m^2}{\mu^2} - 1 \right) + \mathcal{O}(\epsilon), \quad (\text{B13})$$

$$C_2(\epsilon) = \frac{\lambda m^2}{32\pi^2} \left(-\frac{2}{\epsilon} - \ln(4\pi) \right) - \frac{\lambda m^2}{32\pi^2} \ln \frac{m^2}{\mu^2} - \frac{\lambda \Lambda^2}{32\pi^2} + \frac{\lambda m^2}{32\pi^2} \ln \frac{\Lambda^2}{m^2} + \mathcal{O}(\epsilon) + \mathcal{O}(m^2/\Lambda^2). \quad (\text{B14})$$

The similarity of Eqs. (B13) and (B14) with Eqs. (40) and (41) of Sec. III is evident, and the way to obtain the DR rules following the same path illustrated in Fig. 1 is immediately clear. The fine-tuning for the mass parameter, as explained in detail in Sec. III, is hidden in the step ① \rightarrow ②, and again we see that DR is a way of implementing the Wilsonian calculation, incorporating the fine-tuning of the mass parameter. Naturally, again with reference to Fig. 1, if we follow the path ① \rightarrow ③, we obtain

$$\left[\frac{\lambda}{2} \mu^{4-d} \frac{(m^2)^{d/2-1}}{(4\pi)^{d/2}} \int_{m^2/\Lambda^2}^{\infty} d\tau \tau^{-d/2} e^{-\tau} \right]_{d=4} = \frac{\lambda \Lambda^2}{32\pi^2} - \frac{\lambda m^2}{32\pi^2} \ln \frac{\Lambda^2}{m^2} + \frac{\lambda m^2}{32\pi^2} (\gamma_E - 1), \quad (\text{B15})$$

and the fine-tuning has to be implemented in the usual manner.

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Physical Tuning and Naturalness

8.1 Introduction

The letter [6] continues and pushes forward the analysis started in [5]. In particular, it contains a further examination of the naturalness/hierarchy problem in the context of the Wilsonian RG and a proposal for a radically different (and fully Wilsonian) way to look at it.

In the previous work, it was shown that perturbative RG equations (the Callan-Symanzik equations) arise from the Wilsonian equations only if one considers the running of quantities subtracted of their critical value. This step is necessary to obtain an evolution equation which is multiplicative at each step of the blocking transformation. This is very well known and commonly performed in the study of critical phenomena, where the interest is mainly towards the critical regime and the flow equations are linearized about critical points to study how the system behaves in the scaling region around them. If we take the example of a phase transition driven by statistical fluctuations, one can physically understand this operation as a tuning of an external parameter, the temperature T , towards its critical value T_c . In the case of QFT, however, there is no physical interpretation for the operation that one does when forcing the system in the critical region. This was recognized to be the essence of the naturalness/hierarchy problem from the higher standpoint of the RG, which, unlike technical schemes, is intimately related to the very concept of evaluating and piling up the effects of quantum fluctuations and how, at different energy/distance scales, those are incorporated in the value of coupling constants.

In [5], we limited ourselves to give a precise statement of the problem and to observe that, in light of all this, some interesting patterns towards a solution proposed in the literature and based on the use of the Callan-Symanzik equations cannot work. We did not come up with a suggestion for how to ("Wilsonianly") escape the problem. For the radical change of viewpoint this requires, we thought it would be better to leave it to a specifically dedicated work [6], that is reported at the end of this chapter.

8.2 Discussion

The key result of [6] through which we managed to get a strong physical insight on the Wilsonian running of the Higgs mass and substantiate our proposal with simple analytical expressions of unambiguous interpretation is contained in equations 11 and 20 of the paper, respectively an approximate solution for the running mass in the ϕ^4 theory and the running Higgs mass in the SM.

The pedagogical example of the ϕ^4 theory allowed us to introduce and have analytical control over the approximation that is made to obtain the two equations. The full solution for, respectively, the two running parameters $\lambda(\mu)$ and $m(\mu)$ is given in equations 8 and 9. The first one is the well-known evolution equation of the four-point coupling, that shows the Landau pole at the scale $\mu_{\text{LP}} = e^{16\pi^2/3\lambda\Lambda}\Lambda$, which, for perturbative values of λ_Λ , is well beyond the reach of the EFT $\mu_{\text{LP}} \gg \Lambda$. The Fermi scale (individuated here with the top mass m_t) value $\lambda(m_t) = 0.1272$ is obtained from $\lambda(M_P) = 0.1402$ at the Planck scale. It is the extremely slow, logarithmic running of $\lambda(\mu)$ that ensures that the approximation for $m(\mu)$ obtained freezing λ to the constant value $\lambda \equiv \lambda(M_P)$ is a good one. Graphical representation and/or evaluation of both equations 9 and 11 show how good the agreement is. Equation 11 is of much simpler lecture and interpretation than equation 9. The running of $m^2(\mu)$, that we report below

$$m^2(\mu) = \left(\frac{\mu}{\Lambda}\right)^{\frac{\lambda}{16\pi^2}} \left(m^2(\Lambda) + \frac{\lambda\Lambda^2}{32\pi^2 - \lambda}\right) - \frac{\lambda\mu^2}{32\pi^2 - \lambda}, \quad (8.2.1)$$

is simply given by the sum of two terms. With $\lambda = 0.1402$, the first one is extremely slow and goes with $\sim \mu^{8.88 \cdot 10^{-4}}$; the second one is quadratic in the scale. As discussed in the text, it is the interplay between these two contributions that determine how the mass evolve from the UV scale down to the IR. Depending on the value of the constant inside the parenthesis, a scale $\bar{\mu}$ is found below which the running is, for all practical reasons, solely determined by the first term and becomes extremely slow.

The same analysis was then performed for the SM, and the equivalent of (8.2.1)

$$m_H^2(\mu) = \left(\frac{\mu}{\Lambda}\right)^\gamma \left(m_H^2(\Lambda) - \frac{\alpha\Lambda^2}{16\pi^2(2-\gamma)}\right) + \frac{\alpha\mu^2}{16\pi^2(2-\gamma)}, \quad (8.2.2)$$

was found as an approximate solution to equation 17 (γ and α are those defined in (7.1.6) and are frozen to their value at Λ). The comparison with the numerical solution for the complete equation again confirmed the validity of the approximation to a very high degree. A "one-loop improvement" of (8.2.2) where, similarly to what is done in standard perturbative calculations, we promote α and γ to running parameters replacing them with their solution $\alpha(\mu)$ and $\gamma(\mu)$, determines an agreement which is so good that the approximate and full solutions become practically indistinguishable.

The upside of equations (8.2.1) and (8.2.2) is that they show cristal clear several properties of the running of scalar masses. In particular, as discussed in the paper, they elucidate

- What is the fine-tuning and what is its role;
- What is the sense of the Callan-Symanzik (CS) equations and what is their limitation;

- What is the quantity that runs with the CS equations and is only multiplicatively renormalized
- What it means physically to extend the CS equations outside their realm of validity.

Each one of these questions finds a definite answer when a fully fledged Wilsonian approach is adopted.

The usual fine-tuning one encounters in perturbative calculations is reproduced in the term inside the parenthesis in (8.2.1) and (8.2.2). Its physical meaning is to determine the UV boundary value of the mass $m(\Lambda)$ (here we collectively indicate both the mass (8.2.1) and (8.2.2) with $m(\mu)$). In turn, depending on how precise the cancellation with Λ^2 is, this also determines the scale $\bar{\mu}$ and the value $m(\bar{\mu})$ at which the running of the mass practically freezes. The freezing scale is reached through a running which is quadratic with μ . In particular, taking for example $\Lambda = M_P$,

$$\frac{\alpha M_P^2}{16\pi^2(2-\gamma)} \sim 3.02 \cdot 10^{35} \text{GeV}^2, \quad (8.2.3)$$

and the experimentally measured value $m_H \sim 125.7 \text{ GeV}$ at m_t is reached if the cancellation

$$m_H^2(M_P) - \frac{\alpha M_P^2}{16\pi^2(2-\gamma)} \quad (8.2.4)$$

leaves a residual $\sim (125.4)^2$, i.e. if $m_H^2(M_P)$ cancel (8.2.3) up to the 31st digit. With such a cancellation, the running is quadratic in almost all the domain $[m_t, M_P]$.

It is immediate to observe from (8.2.2) that the variable

$$\tilde{m}_H^2(\mu) = m_H^2(\mu) - \frac{\alpha\mu^2}{16\pi^2(2-\gamma)} \quad (8.2.5)$$

solves the usual Callan-Symanzik equation

$$\mu \frac{d}{d\mu} m^2(\mu) = \gamma m^2(\mu). \quad (8.2.6)$$

This gives a specific example of the subtracted mass already discussed in [5]. This quantity, which measures the deviation of the mass from the critical surface found at $\sim \mu^2$ as μ varies, is the one which is multiplicatively renormalized at all scales and that follows a slow evolution. As already stressed several times, the perturbative RG equation (8.2.6) usually considered in the literature emerges only after the system has been forced in the critical region, and it follows the evolution of the deviations with respect to it, not of the full mass. In this respect, it should be noted that what we call fine-tuning (8.2.4) is nothing more and nothing less than the subtraction (8.2.5) performed at the scale Λ . Although hidden, it is obviously contained in (8.2.6).

As we pointed out in the paper [6], under certain assumptions (8.2.6) is nevertheless extremely useful even to follow the evolution of the Wilsonian mass. The freezing region is found below the scale $\bar{\mu}$, where

$$\frac{\alpha\mu^2}{16\pi^2(2-\gamma)} \ll m_H^2(\mu). \quad (8.2.7)$$

To very good accuracy, the subtracted and unsubtracted masses $\tilde{m}_H(\mu)$ and $m_H(\mu)$ coincide in this region. This means that if one fixes $\tilde{m}(\mu)$ at a scale μ^* in the freezing region to the same value the full mass take there after it has been run from Λ to μ^* , the running of $\tilde{m}_H(\mu)$ will coincide with that of $m_H(\mu)$ (approximately) up to $\bar{\mu}$. This is in particular true if one fixes both the values $m_H(\mu_F)$ and $\tilde{m}_H(\mu_F)$ to the experimentally measured value of the Higgs mass: the running encoded in the Callan-Symanzik equation (8.2.6) approximately coincides with the Wilsonian one around the Fermi scale (observe, for example, that with the values taken in equations (8.2.3) and (8.2.4), the presence or absence of the quadratic term μ^2 only determines a shift of the Higgs mass $m_H(m_t)$ from 125.7 to 125.4 GeV). To discriminate between the two, it is necessary to access energy regions that lie outside the freezing regime, where $\alpha\mu^2/16\pi^2(2-\gamma)$ is now comparable with $m_H^2(\mu)$ and the deviation to the critical surface does not coincide with the value of the mass itself anymore. We briefly commented in the paper about the possibility that the running of the Higgs mass could be experimentally investigated (in a way similar to what is done with the running of the bottom quark mass [127]) in the (hopefully) not too far future. As an illustrative example, in figure 2 of the paper the flow of the Wilsonian and the subtracted masses both realizing $m_H(m_t) = 125.7$ GeV and $\tilde{m}_H(m_t) = 125.7$ GeV are shown. Running up to energies $\mu = 200$ GeV, a difference of only ~ 0.8 GeV is generated between the two.

In light of the coincidence in the freezing region, the quantity $m(\mu)$ that appears in the perturbative RG equation (8.2.6) can be identified with the physical mass for $\mu < \bar{\mu}$. This equation can be used in the whole range of μ only if one interprets $m(\mu)$ as a "mathematical construct" that measures the deviation from the critical surface, which is ensured to be small at all energies by the tiny values of the anomalous mass dimension γ .

Our analysis suggests the following type of reasoning. Ignoring for a moment the (almost) constant contribution given by the first term in (8.2.2), in each shell $[\mu - \delta\mu, \mu]$ the size of the quantum fluctuations is $\alpha\mu\delta\mu/16\pi^2(2-\gamma)$. It is thus technically natural to expect that at any scale the mass is proportional to the combination $\alpha\mu^2/16\pi^2(2-\gamma)$. The first term, that we have ignored so far, is generated when, at the ultimate UV scale Λ , a small deviation

$$\frac{m_H^2(\Lambda) - \frac{\alpha\Lambda^2}{16\pi^2(2-\gamma)}}{m_H^2(\Lambda)} \ll 1, \quad (8.2.8)$$

from this expectation appears. The way this deviation enters into the running is through an extremely slow power-like running, so that it can be treated approximately as a constant for finite values of μ . At sufficiently low energies it becomes comparable with the "original" size of the quantum fluctuations $\propto \mu\delta\mu$. It thus causes a relaxation of the physical mass to an approximately constant value (for a related discussion of the freezing in the context of the Landau theory see [81]).

In this respect, one should not forget that, above Λ , the theory is not the SM (or any non-asymptotically free modifications of it), but is given in terms of new degrees of freedom. The UV completion of the SM is the one that determine the values of the parameters at the scale Λ . We, as low energy observers, only have access to the values experimentally measured. Once we agree that experiments require a Higgs mass $m_H(\mu_F) \sim 125$ GeV and that the physical way to include at a scale μ the quantum fluctuations of higher energy modes $p > \mu$ is through the Wilsonian successive elimination,

then it is up to the UV completion to generate the boundary value $m_H(\Lambda)$, and in particular the small deviation (8.2.8). In [6], we propose that this line of thought should be used to define a sort of experimentally IR driven Swampland approach. Through experiments, we (low energy observers) only know the values the parameters take at low energies with a certain precision. Schematically, for a generic quantity g_i we have at a scale μ_{low} $g_i(\mu_{\text{low}}) = \tilde{g}_i \pm \delta\tilde{g}_i$, with $\delta\tilde{g}_i$ the experimental uncertainty. Using the RG methods to follow the evolution of all the g_i in the domain of validity of the EFT, this defines a region of parameter space at the scale Λ . The couplings with UV boundary that fall in this region generate, at low energies, values compatible with the experimental observations. The Swampland program discussed in the first chapters of this thesis assume as a starting point properties of Quantum Gravity and/or String Theory, and then, if concretely possible, constrains the space of all possible EFTs to those that can enjoy a UV completion while respecting such features. Our idea is somehow complementary to this and basically requires an exchange of the roles. We propose to take as a starting point the results of experiments $\{g_i(\mu_{\text{low}})\} = \{\tilde{g}_i \pm \delta\tilde{g}_i\}$ to evolve them through exact RG equations (the ones discussed in this thesis work, or equivalently other implementations where the integration over the shell is performed in a smoother way [128]) in the region of validity of the EFT up to the ultimate UV scale, where $\{g_i(\Lambda)\} = \{\bar{g}_i \pm \delta\bar{g}_i\}$. A generic UV completion (or more simply a new, not necessarily fundamental, layer above the EFT domain) can generate the set of couplings observed at the energies we experimentally access if and only if it predicts values at the matching scale Λ that are found inside the region defined by $\{\bar{g}_i \pm \delta\bar{g}_i\}$. These UV completions would fall in our proposed "IR Landscape", while the others would have to be dismissed and fall in the "IR Swampland". This line of reasoning is not dissimilar to that in [129], where it was argued that the field theory limit of String Theory can only be derived if a Wilsonian perspective is adopted.

Before concluding, a comment on the observation at the end of [6] is in order. The sensitivity to UV boundary conditions, that in perturbation theory we call fine-tuning and here we have discussed as the deviation from criticality at Λ , is not necessarily something to be scared of. The renormalization group is not only useful and predictive for subtracted quantities in the scaling regime, but is rather global and contains much more than just the evolution around fixed points. In this respect, a notable example is given by the ϕ^4 theory in $D = 3$ dimensions, that describes the free energy

$$F[\phi] = \int d^3x \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m_k^2\phi^2 + \frac{\lambda_k}{4!}\phi^4 \quad (8.2.9)$$

of the Ginzburg-Landau theory for the ferromagnetic transition. The Wilsonian RG equations are obtained from (6.2.12) and (6.2.13)

$$k \frac{dm_k^2}{dk} = -\frac{k^3 \lambda_k}{4\pi^2 (k^2 + m_k^2)}, \quad k \frac{d\lambda_k}{dk} = \frac{3k^3 \lambda_k^2}{4\pi^2 (k^2 + m_k^2)^2}, \quad (8.2.10)$$

and moving to the dimensionless parameters $\tilde{m}_k^2 \equiv k^{-2}m_k^2$ and $\tilde{\lambda}_k \equiv k^{-1}\lambda_k$ (with $t \equiv \log k/k_0$ the RG time and k_0 a reference scale):

$$\frac{d\tilde{m}_k^2}{dt} = -2\tilde{m}_k^2 - \frac{\tilde{\lambda}_k}{4\pi^2(1 + \tilde{m}_k^2)} \quad (8.2.11)$$

$$\frac{d\tilde{\lambda}_k}{dt} = -\tilde{\lambda}_k + \frac{3\tilde{\lambda}_k^2}{4\pi^2(1+\tilde{m}_k^2)^2}. \quad (8.2.12)$$

This system of equations have two fixed points: the Gaussian $G \equiv (0, 0)$ and the Wilson-Fisher [130] $WF \equiv (\tilde{m} = -1/7, \tilde{\lambda} = 48\pi^2/49)$ respectively. The linearization matrix reads

$$\begin{pmatrix} \partial_{\tilde{m}_k^2} \beta_{\tilde{m}_k^2} & \partial_{\tilde{\lambda}_k} \beta_{\tilde{m}_k^2} \\ \partial_{\tilde{m}_k^2} \beta_{\tilde{\lambda}_k} & \partial_{\tilde{\lambda}_k} \beta_{\tilde{\lambda}_k} \end{pmatrix} = \begin{pmatrix} -2 + \frac{\tilde{\lambda}_k}{4\pi^2(1+\tilde{m}_k^2)^2} & -\frac{1}{4\pi^2(1+\tilde{m}_k^2)} \\ -\frac{3\tilde{\lambda}_k^2}{2\pi^2(1+\tilde{m}_k^2)^3} & -1 + \frac{3\tilde{\lambda}_k}{2\pi^2(1+\tilde{m}_k^2)^2} \end{pmatrix}. \quad (8.2.13)$$

Around G , its eigenvalues are -2 and -1 : they both are negative and they define two UV attractive directions in parameter space, given by the associated eigenvectors. The eigenvalues around the Wilson-Fisher fixed point are $1/3(-\sqrt{22}-1)$ and $1/3(\sqrt{22}-1)$. The first is negative and defines a UV attractive (IR repulsive) direction, while the second is positive and defines a UV repulsive (IR attractive) one. The resulting behaviour is depicted in figure 3 of the paper. Starting from boundary conditions in the vicinity of the Gaussian fixed point, three trajectories are shown. The blue and the red ones are obtained running in a four order of magnitude range for k the full Wilsonian equations, while the green one is obtained running in the same range the RG equations obtained after linearization around G :

$$\frac{d}{dt} \tilde{m}_k^2 = -2\tilde{m}_k^2 - \frac{1}{4\pi^2} \tilde{\lambda}_k \quad (8.2.14)$$

$$\frac{d}{dt} \tilde{\lambda}_k = -\tilde{\lambda}_k. \quad (8.2.15)$$

Comparing the blue and the red curves we see that, depending on the value of the parameters at the UV scale, the flow can have very different shapes. It can be attracted to the WF fixed point (blue curve) and experience a regime of slow-down where it evolves according to the scaling laws around it until the IR repulsive direction moves it away, or it can evolve as if it does not feel it (red curve). This example is taken here to argue that if we, as low energy observers, sit on an equivalent of the Wilson Fisher fixed point (or any other point in the much more complicated parameter space of the Theory of Everything that contains all the parameters one would ever need to describe the physics at any scale) it is not necessarily a strange feature that from our viewpoint there is a strong sensitivity to the UV boundary (the value where the Higgs mass freezes as determined by the combination (8.2.4) in our case). On the contrary, the very existence of the ferromagnetic transition tells us that the tuning is physical and necessary: to enter in the scaling region of the WF fixed point a fine tuning in the UV is *necessary*. In the case of the Higgs, this translates in the following. Since we observe a large hierarchy $m_H^2(\mu_F) \ll \Lambda^2$, whatever the UV completion is it *must* predict the good value $m_H^2(\Lambda)$ (see equation (8.2.8)) for the running to end up at the Fermi scale μ_F in the region where the parameters match those experimentally measured. This is the same thing an observer sitting on the vicinity of the Wilson-Fisher fixed point would say: if, at the scales proper of the observer, the flow is driven close to the WF fixed point, then *necessarily* the UV boundary condition is sufficiently fine-tuned for this to happen (blue curve vs red curve in the figure 3 of [6]).

This puts of course a severe constraint on the UV completion, and our hope is that developments in this direction might help to shed some light on the physics that lie beyond the Standard Model and more generally beyond the EFT paradigm.

Physical Tuning and Naturalness

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We present a radically new proposal for the solution of the naturalness/hierarchy problem, where the fine-tuning of the Higgs mass finds its physical explanation and the well-known multiplicative renormalization of the usual perturbative approach emerges as an IR property of the non-perturbative running of the mass.

The Standard Model (SM) of particle physics is an effective theory, i.e. a quantum field theory valid up to a certain scale (M_P , M_{GUT} , ...), above which it has to be replaced by its ultraviolet (UV) completion. This scale, that we generically indicate with Λ , is the *physical cut-off* of the theory: the SM effective Lagrangian $\mathcal{L}_{SM}^{(\Lambda)}$, allows to describe processes at momenta $p \lesssim \Lambda$.

Due to unsuppressed quantum fluctuations, the square of the Higgs boson mass m_H^2 receives contributions proportional to Λ^2 . In this respect, we stress that $m_H^2 \sim \Lambda^2$ indicates a “quadratic sensitivity” of m_H^2 to the ultimate scale of the theory, not a “quadratic divergence”. Moreover, this value of m_H^2 is nothing but the square of the running $m_H^2(\mu)$ at the scale $\mu = \Lambda$. If Λ is too large, $m_H^2(\Lambda)$ is “unnaturally” large, and this poses a problem of “hierarchy” with the Fermi scale μ_F , where $m_H(\mu_F) \sim 125$ GeV. Several attempts have been made towards the “solution” of this naturalness/hierarchy (NH) problem. Here we focus on three of them, as they will help to introduce our proposal.

1. A popular approach is based on the assumption that the UV completion of the SM could provide the condition

$$m_H^2(\Lambda) \ll \Lambda^2 \quad (1)$$

at the scale Λ [1–3]. Sometimes (1) is viewed as a “quantum gravity miracle” [1], that could result from a conspiracy among the SM couplings at the scale Λ . This is for instance the case of the so called Veltman condition¹ [4]. In such a scenario: (i) the naturalness problem is solved from physics “outside” the SM realm, as (1) is considered a left-over of its UV completion (or extensions of it); (ii) the hierarchy problem is solved “inside” the SM, by considering the perturbative renormalization group (RG) equation for $m_H^2(\mu)$ ($\gamma \ll 1$ is the perturbative anomalous mass dimension)

$$\mu \frac{d}{d\mu} m_H^2(\mu) = \gamma m_H^2(\mu). \quad (2)$$

From (1) and (2), in fact, $m_H^2(\mu_F)$ and $m_H^2(\Lambda)$ turn out to be of the same order: there is no problem of hierarchy.

2. A somehow complementary approach consists in considering again Eq.(2) for $m_H^2(\mu)$, but assuming this time that gravity could provide a non-perturbative value for γ (~ 2). In this case, the large hierarchy between the Fermi scale μ_F and the UV scale Λ can be accommodated [5–12]: the NH problem would then disappear.

3. Finally, some authors suggest that dimensional regularization (DR) could be endowed with special physical properties that make it the correct “physical” way to calculate the radiative corrections in quantum field theory (QFT). Again, if no new heavy particles are coupled to the Higgs boson, the NH problem would seem to be absent from the beginning [13–36].

As shown in a recent paper [37], however, none of these approaches provides a solution to the problem. The reason is that any effective field theory (EFT), including the SM, is *necessarily* defined and interpreted in a Wilsonian framework. The meaning of this statement is twofold: (i) the parameters (masses and couplings) $g_i(\Lambda)$ in the effective Lagrangian $\mathcal{L}_{SM}^{(\Lambda)}$ result from integrating out the higher energy modes $k > \Lambda$ related to the UV completion of the SM; (ii) the same parameters $g_i(\mu)$ at a lower scale $\mu < \Lambda$ result from integrating out the modes of the fields that appear in $\mathcal{L}_{SM}^{(\Lambda)}$ in the range $[\mu, \Lambda]$.

It is shown in [37] that DR provides a specific implementation of the Wilsonian strategy, where the fine-tuning is *automatically* encoded in the calculations, although in a *hidden* manner. As a consequence, DR cannot provide a solution to the problem. Moreover, it is shown that (2) is obtained when the “critical value” $m_{cr}^2(\mu)$ is subtracted to $m^2(\mu)$. In other words, $m_H^2(\mu)$ in (2) is not the Wilsonian mass $m^2(\mu)$. It is rather: $m_H^2(\mu) \equiv m^2(\mu) - m_{cr}^2(\mu)$. Equation (2) then incorporates the fine-tuning, and cannot be invoked to solve the NH problem.

In this *Letter* we make a radically different proposal, rooted in simple and (in our opinion) indisputable “facts”: (i) the SM is an EFT valid up to an ultimate UV scale Λ ; (ii) the Wilsonian integration of modes is

¹ If not directly to the SM itself, it can be applied to some of its extensions [3].

the only *physically consistent* way of including the quantum fluctuations in an EFT.

To introduce our proposal, we begin by considering the Wilsonian RG equations for the scalar ϕ^4 theory, whose (Euclidean) Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m_\Lambda^2 \phi^2 + \frac{\lambda_\Lambda}{4!} \phi^4, \quad (3)$$

where $m_\Lambda^2 \equiv m^2(\Lambda)$ and $\lambda_\Lambda \equiv \lambda(\Lambda)$ are the mass and coupling constant at the (physical) scale Λ . By considering the corresponding Wilsonian action within the so called ‘‘Local Potential approximation’’, $S_k[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + U_k(\phi) \right)$, and truncating the potential to the first two terms, $U_k(\phi) = \frac{1}{2} m_k^2 \phi^2 + \frac{\lambda_k}{4!} \phi^4$, the RG equations for m_k^2 and λ_k are (see [37, 38])

$$k \frac{dm_k^2}{dk} = -\frac{k^4}{16\pi^2} \frac{\lambda_k}{k^2 + m_k^2}, \quad (4)$$

$$k \frac{d\lambda_k}{dk} = \frac{k^4}{16\pi^2} \frac{3\lambda_k^2}{(k^2 + m_k^2)^2}. \quad (5)$$

When the UV boundaries for (4)-(5) are such that the condition $m_k^2 \ll k^2$ is satisfied in the whole range of integration, this system is well approximated by

$$k \frac{dm_k^2}{dk} = -\frac{\lambda_k}{16\pi^2} k^2 + \frac{\lambda_k}{16\pi^2} m_k^2 \quad (6)$$

$$k \frac{d\lambda_k}{dk} = \frac{3\lambda_k^2}{16\pi^2}. \quad (7)$$

Taking for instance ‘‘SM-like’’ IR boundaries, $m(\mu_F) = 125.7$ GeV and $\lambda(\mu_F) = 0.1272$, and solving both systems numerically, we find that the solutions to (4)-(5) and (6)-(7) coincide with great accuracy.

The flow equations (6)-(7) can be solved analytically. The solution to (7) (decoupled from (6)) is the well-known one-loop-improved running quartic coupling

$$\lambda(\mu) = \frac{\lambda_\Lambda}{1 - \frac{3}{16\pi^2} \lambda_\Lambda \log\left(\frac{\mu}{\Lambda}\right)}. \quad (8)$$

Then, inserting (8) in (6), this latter equation can also be solved analytically, and we find

$$\begin{aligned} m^2(\mu) &= \frac{1}{3 \cdot 2^{2/3} \left(3\lambda_\Lambda \log\left(\frac{\mu}{\Lambda}\right) - 16\pi^2 \right)} \left(2^{2/3} \Lambda^2 e^{\frac{32\pi^2}{3\lambda_\Lambda}} \times \right. \\ &\left. \left(16\pi^2 - 3\lambda_\Lambda \log\left(\frac{\mu}{\Lambda}\right) \right) E_{\frac{2}{3}} \left(\frac{32\pi^2}{3\lambda_\Lambda} - 2 \log\left(\frac{\mu}{\Lambda}\right) \right) \right. \\ &\left. + 4\lambda_\Lambda \sqrt[3]{-\frac{1}{\lambda_\Lambda}} \left(\Lambda^2 e^{\frac{32\pi^2}{3\lambda_\Lambda}} E_{\frac{2}{3}} \left(\frac{32\pi^2}{3\lambda_\Lambda} \right) + 3m_\Lambda^2 \right) \right. \\ &\left. \times \left(3\pi \log\left(\frac{\mu}{\Lambda}\right) - \frac{16\pi^3}{\lambda_\Lambda} \right)^{2/3} \right) \end{aligned} \quad (9)$$

where $E_{\frac{2}{3}}(x)$ is the generalized exponential integral function $E_p(x)$ with $p = \frac{2}{3}$.

The *non-perturbative* evolution equation that we have just found is important for our analysis. First of all we note that, expanding (9) for $\lambda_\Lambda \ll 1$ (and $\mu^2 \ll \Lambda^2$), we obtain the well-known perturbative result (for notational simplicity, from now on we replace $\lambda_\Lambda \rightarrow \lambda$)

$$m_\mu^2 = m_\Lambda^2 + \frac{\lambda}{32\pi^2} \left(\Lambda^2 - m_\Lambda^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) \right). \quad (10)$$

More important for our scopes, however, is to note that the flow equation (9) has a very interesting *non-perturbative* approximation, that can be obtained replacing λ_k with λ in the right hand side of (6),

$$m_\mu^2 = \left(\frac{\mu}{\Lambda} \right)^{\frac{\lambda}{16\pi^2}} \left(m_\Lambda^2 + \frac{\lambda \Lambda^2}{32\pi^2 - \lambda} \right) - \frac{\lambda \mu^2}{32\pi^2 - \lambda}. \quad (11)$$

Using for instance the same boundary values considered before (see below (7)), we easily check that (11) provides a very good approximation to the flow governed by (4).

Eq. (11) is a crucial result of the present work (below it will be extended to the SM) and contains several important lessons. First it shows how the fine-tuning usually realized in perturbative QFT operates in the Wilsonian framework: it simply fixes the boundary at the UV scale $\mu = \Lambda$ for the running of the mass m_μ^2 . This is encoded in the parenthesis in the right hand side of (11), where m_Λ^2 and $\frac{\lambda \Lambda^2}{32\pi^2 - \lambda}$ need to be *enormously fine-tuned* if at the IR scale μ_{low} we want $m_{\mu_{low}} \sim \mathcal{O}(100)$ GeV.

There is another important lesson in (11). By simple inspection, we see that the combination

$$m_{\mu,r}^2 \equiv m_\mu^2 + \frac{\lambda \mu^2}{32\pi^2 - \lambda} \quad (12)$$

obeys the RG equation

$$\mu \frac{d}{d\mu} m_{\mu,r}^2 = \gamma m_{\mu,r}^2 \quad (13)$$

where $\gamma = \frac{\lambda}{16\pi^2}$ is the mass anomalous dimension for the ϕ^4 theory at one-loop order. Eq. (13) coincides with the well-known one-loop improved flow equation for the renormalized running mass. Therefore, $m_r^2(\mu)$ defined in (12) is the renormalized running mass. At the same time we note that

$$m_{\mu,cr}^2 \equiv -\frac{\lambda \mu^2}{32\pi^2 - \lambda} \quad (14)$$

is the ‘‘critical mass’’ defined at each value of the running scale μ , and that the subtraction in (13) drives the RG flow close to the critical surface of the Gaussian fixed point. The simple integration of (13) gives ($\mu_0 > \mu$)

$$m_{\mu,r}^2 = \left(\frac{\mu}{\mu_0} \right)^{\frac{\lambda}{16\pi^2}} m_{\mu_0,r}^2. \quad (15)$$

For the purposes of our analysis, it is important to note that we derived equation (13) in the Wilsonian framework, namely from the RG flow (11), whereas usually it

is derived in the context of “technical schemes”, as dimensional, heat kernel, or zeta function regularization. In this respect, we stress that, when the quantum fluctuations are calculated in the framework of a “technical scheme”, we only have access to (13) (and then to its solution (15)), but we are blind to the fact that the renormalized running mass is obtained only after operating at each scale μ the subtraction in (12). When, on the contrary, the quantum fluctuations are calculated within the Wilsonian “physical scheme”, we clearly see how the renormalized mass emerges.

There is a third important lesson contained in (11), that is related to the following question. Should we identify the physical running mass with (11) or with (15), i.e. with the original Wilsonian mass m_μ^2 , or with the subtracted mass $m_{\mu,r}^2$? In QFT the running mass is usually identified with (15). On the other hand, according to the definition of Wilsonian action, the running couplings $g_i(\mu)$ at the scale μ result from the integration over the quantum fluctuations in the range $[\mu, \Lambda]$, and are the effective couplings at this scale. This is true, in particular, for the mass. Therefore, it is the original Wilsonian mass m_μ^2 , not the subtracted $m_{\mu,r}^2$ that has to be identified with the physical mass at the scale μ .

Being this the case, how can we justify the traditional (textbook) approach to QFT, where it is $m_{\mu,r}^2$ that is identified with the physical running mass at the scale μ ?

The answer to this question comes from the comparison between our result (11) and (the textbook) equation (15). As long as we confine ourselves to sufficiently low values of μ (IR regime), the flow governed by (15) practically coincides with the flow (11). This overlap region is defined by the condition

$$\frac{\lambda\mu^2}{32\pi^2 - \lambda} \ll \left(\frac{\mu}{\Lambda}\right)^{\frac{\lambda}{16\pi^2}} \left(m_\Lambda^2 + \frac{\lambda\Lambda^2}{32\pi^2 - \lambda}\right), \quad (16)$$

where (we stress again) the term inside the parenthesis contains the fine-tuning necessary to obtain the IR (measured) value of the physical mass. Moreover, our equation (11) allows to find the energy range to which Eq. (15) is limited. Clearly, if we are interested in energy scales μ above the region determined by (16), we must go back to the original flow (11), that has a much wider range of validity.

We are now ready to move to the SM, and to present our proposal (a similar scenario holds if extended versions of the SM are considered). Following steps similar to those that led to (6), for the Higgs mass we get the Wilsonian RG equation

$$\mu \frac{d}{d\mu} m_H^2 = \alpha(\mu) \mu^2 + \gamma(\mu) m_H^2, \quad (17)$$

where $\alpha(\mu)$ is a combination of SM running couplings (gauge, Yukawa, scalar), that at one-loop level reads (for our purposes it is sufficient to restrict ourselves to the

one-loop order)

$$16\pi^2\alpha(\mu) = 12y_t^2 - 12\lambda - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2, \quad (18)$$

and $\gamma(\mu)$ is the mass anomalous dimension

$$16\pi^2\gamma(\mu) = 6y_t^2 + 12\lambda - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2. \quad (19)$$

Considering (as for the scalar theory above) constant values for the couplings, from (17) we obtain

$$m_H^2(\mu) = \left(\frac{\mu}{\Lambda}\right)^\gamma \left(m_H^2(\Lambda) - \frac{\alpha\Lambda^2}{2-\gamma}\right) + \frac{\alpha\mu^2}{2-\gamma}, \quad (20)$$

that, as it is easy to check, provides a very good approximation to the flow governed by (17). An improvement to (20) is obtained if α and γ outside the parenthesis are replaced with $\alpha(\mu)$ and $\gamma(\mu)$ (the term in parenthesis is an integration constant, where $\alpha = \alpha(\Lambda)$ and $\gamma = \gamma(\Lambda)$). In Fig. 1 both the solution to (17) and its analytical approximation (20) (with the improvement mentioned above) are plotted. They are practically indistinguishable.

Eq. (20) is one of the most important result of the present work, and deserves several comments. Before doing that, however, it is worth to derive few other related results. Let us define (as in (14)) the critical mass,

$$m_{H,cr}^2(\mu) \equiv \frac{\alpha\mu^2}{2-\gamma}, \quad (21)$$

and (as in (12)) the subtracted mass

$$m_{H,r}^2(\mu) \equiv m_H^2(\mu) - m_{H,cr}^2(\mu). \quad (22)$$

From (20) we derive the equation

$$\mu \frac{d}{d\mu} m_{H,r}^2(\mu) = \gamma m_{H,r}^2(\mu), \quad (23)$$

that once solved gives ($\mu_0 > \mu$)

$$m_{H,r}^2(\mu) = \left(\frac{\mu}{\mu_0}\right)^\gamma m_{H,r}^2(\mu_0). \quad (24)$$

Eq. (23) coincides with the well-known (textbook) one-loop improved RG equation for the renormalized running mass, and is nothing but Eq. (2). We then conclude that $m_{H,r}^2(\mu)$ defined in (22) is the usual renormalized running Higgs mass. However, going back to (20), we observe that (as explained in the section devoted to the scalar theory) it is the Wilsonian mass parameter $m_H^2(\mu)$ that has to be identified with the running Higgs mass. In connection with that, let us consider now the two following things.

(i) If we require that $m_H^2(\mu)$ at the Fermi scale μ_F is the measured $m_{H,exp}^2 \sim (125.7)^2 \text{ GeV}^2$, from (20) we see that $m_H^2(\Lambda)$ needs to be enormously fine-tuned.

(ii) Turning to the RG flow (24) for $m_{H,r}^2(\mu)$, and this time we require that it is $m_{H,r}^2(\mu_F)$ that takes the experimental value $\sim (125.7)^2 \text{ GeV}^2$, we see that the two flows

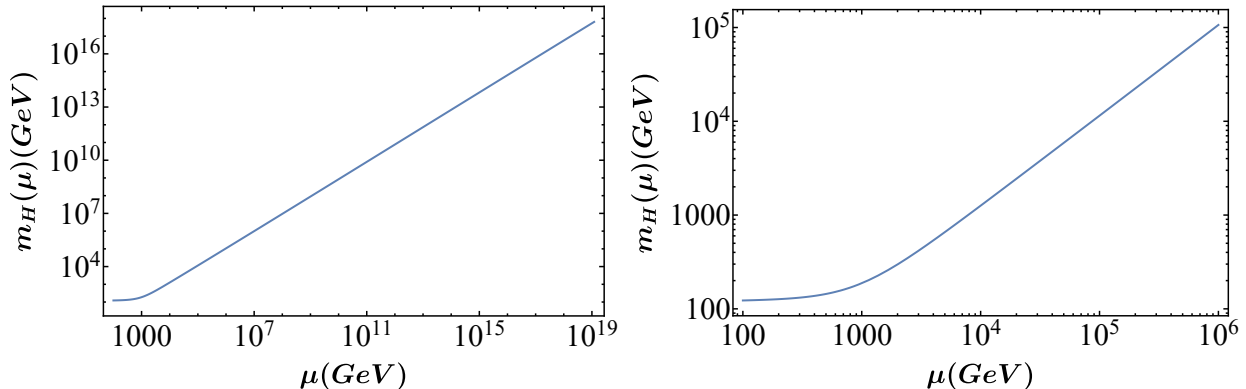


FIG. 1. *Left panel:* Log-log plot of $m_H(\mu)$, with UV boundary $m_H(M_P) \sim 6.347 \cdot 10^{17}$ GeV, see Eq. (17). The latter is coupled to the RG equations for the SM couplings, λ , y_t , g_1 , g_2 and g_3 , solved numerically using one-loop beta functions, and IR (experimental) boundary values: $\lambda(m_t) = 0.1272$, $y_t(m_t) = 0.9369$, $g_1(m_t) = 0.3587$, $g_2(m_t) = 0.6483$, $g_3(m_t) = 1.1671$ (m_t is the top quark mass). Eq. (20) (analytical approximation to the solution of (17)) is also plotted, but the two curves are indistinguishable. *Right panel:* Zoom in the region $10^2 - 10^6$ GeV of the running shown in the left panel. The “elbow” around $\mu \sim 10^3$ GeV signals that the IR flow is entering the region where $m_H(\mu)$ is very well approximated by $m_{H,r}(\mu)$ (Eq. (24)).

$m_H^2(\mu)$ and $m_{H,r}^2(\mu)$ coincide for all the values of μ that satisfy the condition

$$\frac{\alpha \mu^2}{2 - \gamma} \ll \left(\frac{\mu}{\Lambda}\right)^\gamma \left(m_H^2(\Lambda) - \frac{\alpha \Lambda^2}{2 - \gamma}\right). \quad (25)$$

These results contain crucial physical lessons. First of all we learn that the “fine-tuning” of $m_H^2(\Lambda)$, that in the traditional approach to QFT is *formally* realized through the introduction of counterterms, has a profound *physical* meaning. It provides the boundary at the UV scale Λ for the RG flow of the running mass $m_H^2(\mu)$. A very large value of m_H^2 at the UV scale Λ is *physically necessary* and welcome, not an unwanted result to get rid of.

Moreover, from Eq. (20) and from Fig. 1, we see that through a *quadratic running* that lasts for most of the $m_H^2(\mu)$ flow towards the IR, this finely tuned value of $m_H^2(\Lambda)$ allows to reach the experimental value of the Higgs mass at the Fermi scale. What is crucial to realize is that, proceeding towards the IR, the initial “quadratic running” $m_H^2(\mu) \sim \mu^2$ sooner or later gives the way to a lower energy running, where the “multiplicative renormalization” (see (24)) emerges. In schemes as DR we only have access to (24), but from a truly physical perspective the latter is an “emergent property” of the running, that rises when the flow approaches the IR.

This is a great change in the usual paradigm. In the physically unavoidable top-down Wilsonian approach, a large hierarchy between the UV and the IR values of m_H^2 , together with the fine-tuning of $m_H^2(\Lambda)$, are *physically mandatory* and the typical multiplicative renormalization (24) emerges as an IR property of the complete physical running (20).

Moreover Eq. (20) shows the limitations of (23), or equivalently of its solution (24): they can be used only at sufficiently low energies (where (25) is satisfied), that

in most of the cases are the only experimentally reachable energies. In the right panel of Fig. 1, a zoom of the $m_H(\mu)$ running is shown. The presence of an “elbow” near $\mu \sim 10^3$ GeV and the “freezing” of the $m_H(\mu)$ flow at lower scales signal that the first term in the right hand side of (20) takes over the second one. This realizes the “transition” from the additive to the multiplicative renormalization of the mass, that is the transition from (20) to (24). If one were to extend (24) outside its realm of validity, the experimental value $m_H = 125.7$ GeV at the Fermi scale would be reached starting with the UV boundary $m_H(M_P) \sim 132.4$ GeV. This is connected with one of the popular (but, as shown above, incorrect) approaches to the NH problem (see Eq. (1) and the related discussion).

The fine-tuning manifests itself through the term $(m_H^2(\Lambda) - \alpha/(2-\gamma)\Lambda^2)$ in (20), and the choice of this combination in the UV determines the measured (IR) value of m_H^2 . Therefore, taking into account the experimental uncertainties, we conclude that there exists a region of “tiny size” in the SM parameter space from which very large UV boundary values of m_H^2 give rise, through the RG flow, to the measured (within errors) value of the Higgs mass. Such a region can only be inherited from the ultimate UV completion of the SM (or of the yet unknown BSM), namely the Theory of Everything. In String Theory, for instance, an enormous variety of theories/vacua has to be considered, and the conditions for the existence of such a region are certainly met. Moreover, in connection with the quadratic dependence of the Higgs mass in the UV, we note that in the string framework the common expectation is that the Higgs mass at the string scale M_S is $m_H^2(M_S) \sim M_S^2$ [39].

Going back to the running (24) (multiplicative renormalization), we observe that it can be obtained within

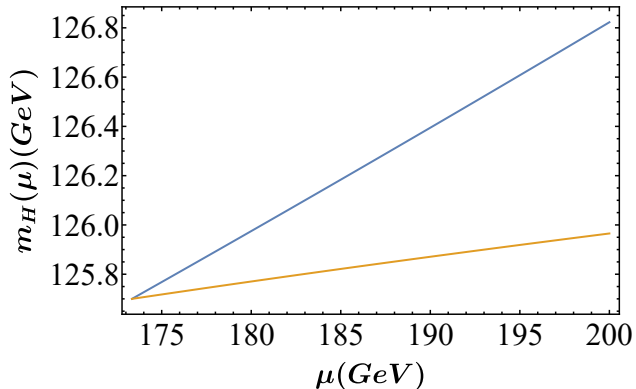


FIG. 2. This figure shows a focus of the $m_H(\mu)$ flow of Fig. 1, Eq. (20), in the IR region between m_t (where $m_H(m_t) \sim 125.7$ GeV) and 200 GeV (blue line). The yellow curve is the flow given by (24) again with $m_H(m_t) \sim 125.7$ GeV. Future experiments should allow to evidenciate the difference between these two flows.

different schemes (DR, heat kernel, ...), and that no physical content can ever be related to the choice of a specific scheme. However our analysis has shown that this behavior is related *only* to the IR sector of the flow. The whole UV \rightarrow IR running is given by the Wilsonian flow (20), while (24) is confined to the IR regime alone.

In light of these findings, an interesting question arises, that might be subject to experimental investigation in the (hopefully not too far) future. Although no one has observed up to now the running of the Higgs mass, we can consider physical processes that should allow to test the $m_H^2(\mu)$ flow (much in the same spirit of what is done with the running bottom quark mass [40]): think, for instance, of ongoing work on precision measurements of the trilinear coupling [41]. If future experiments will be able to enter the energy regime where the complete flow (20) and the approximate IR flow (24) start to be significantly different, and experimentally distinguishable, it should become possible to discriminate between these two alternatives (see Fig. 2).

In this respect, we observe that the connection between quantum field theory and statistical physics is usually done by establishing a one-to-one correspondence between the request $\xi \gg a$ in the theory of critical phenomena (a is the lattice spacing, ξ the correlation length) and the request $m^2 \ll \Lambda^2$ in the QFT framework (m is the particle mass, Λ the ultimate UV scale of the theory). When phrased in RG language, this corresponds to the tuning towards the “critical surface”, achieved through the subtraction of the “critical mass”: $m_{ren}^2(\mu) = m^2(\mu) - m_{cr}^2(\mu)$. However, we have seen that $m_{ren}^2(\mu)$ only captures the *final part* of the running of the physical mass. Actually, the RG flow is physically meaningful even far from the critical surface and from fixed points. This is in fact what happens to our flow (20),

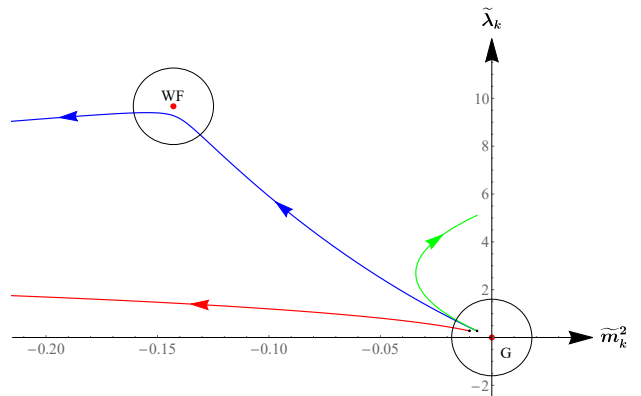


FIG. 3. RG flows (Eqs. (26) and (27)) in the parameter space $(\tilde{m}_k^2, \tilde{\lambda}_k)$ of a ϕ^4 theory in $d = 3$ dimensions. The blue and red flows emanate from the UV region close to the Gaussian fixed point G (different boundary values). The green line is obtained linearizing (26) and (27) around G, with the same boundary as the blue one.

that approaches the critical surface of the gaussian fixed point in the IR, giving eventually rise to the flow (24).

These points can be well illustrated if we go for a moment to $d = 3$ dimensions, and consider the Ginzburg-Landau free energy (used to describe the ferromagnetic transition), $F[\phi] = \int d^3x \left(\frac{1}{2}(\vec{\nabla}\phi)^2 + V_k(\phi) \right)$, with potential $V_k(\phi) = \frac{1}{2}m_k^2\phi^2 + \frac{\lambda_k}{4!}\phi^4$. The RG equations for the dimensionless couplings $\tilde{m}_k^2 \equiv k^{-2}m_k^2$ and $\tilde{\lambda}_k \equiv k^{-1}\lambda_k$ ($t \equiv \log k/k_0$, with k_0 a reference scale) are

$$\frac{d\tilde{m}_k^2}{dt} = -2\tilde{m}_k^2 - \frac{\tilde{\lambda}_k}{4\pi^2(1 + \tilde{m}_k^2)} \quad (26)$$

$$\frac{d\tilde{\lambda}_k}{dt} = -\tilde{\lambda}_k + \frac{3\tilde{\lambda}_k^2}{4\pi^2(1 + \tilde{m}_k^2)^2}. \quad (27)$$

It is immediate to see that these equations have a Gaussian and a Wilson-Fisher fixed point, G and WF in Fig. 3, and that G is an IR repulsive fixed point.

Fig. 3 conveys two messages. (a) - Let us consider the UV \rightarrow IR flow given by the blue line. In the region around G where (26) and (27) can be linearized, this flow is well approximated by the “subtracted flow” (green line), the analog of (24) in this case. Beyond this region, however, the green flow deviates from the true physical flow (blue line). The very existence of the ferromagnetic transition shows that the green flow *cannot* be the true one. (b) - The blue and red flows have slightly different UV boundaries. Thanks to the fine-tuning operated in the UV, the blue flow is driven towards WF (i.e. towards the ferromagnetic transition). This example clearly shows that, if (as it is certainly the case) the IR physics is dictated by WF, the fine-tuning in the UV is *physical* and *unavoidable*.

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APPENDIX A

Dimensional Reduction of massless scalar field on a circle

In this appendix, I will present in full details the circle compactification of a $D + 1$ -dimensional theory into D -dimensional one. This was referred to a number of times in the text, especially in chapter 4.

Notation

We will work in $D + 1$ dimensions, with the signature $(-, +, \dots, +)$ to simplify comparison with the existing literature. All the quantities defined in $D + 1$ dimensions will be denoted with a hat. For example, \hat{R} indicates the $D + 1$ -dimensional Ricci scalar. We use latin letters for the $D + 1$ dimensional indices, as M, N , greek ones μ, ν for uncompactified dimensions and z for the compactified one. In the same spirit, A, B will be used to define the local frame in $D + 1$ dimensions, a, b for the uncompactified ones and \tilde{z} for the compactified one. We use, in general dimension d , $\kappa_d^2 = 8\pi G_d = M_{P,d}^{-(d-2)}$, where G_d is the d -dimensional Newton constant and $M_{P,d}$ the d -dimensional reduced Planck mass.

A.1 Gravitational action

We discuss in this first section the compactification of pure $(D + 1)$ -dimensional gravity to D dimensions on a circle. I will present two equivalent methods to perform the calculations.

Direct computation in the Einstein frame

We start with the gravitational action in $D + 1$ dimension,

$$\mathcal{S} = \frac{1}{2\hat{\kappa}^2} \int d^{D+1}x \sqrt{-\hat{g}} \hat{R}, \quad (\text{A.1.1})$$

where the constant $\hat{\kappa}$ is understood to be the $D + 1$ -dimensional one. We assume here and for the following the so called cylinder condition

$$\partial_z \hat{g} = 0, \quad (\text{A.1.2})$$

that corresponds to consider only the zero modes of the higher dimensional gravitational field. We write the metric in the form

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{pmatrix} \quad (\text{A.1.3})$$

with ϕ , A_μ and $g_{\mu\nu}$ D -dimensional (dimensionless) fields independent of the z coordinate, as dictated by (A.1.2). The inverse metric is then

$$\hat{g}^{MN} = \begin{pmatrix} e^{-2\alpha\phi} g^{\mu\nu} & -e^{-2\alpha\phi} A_\mu \\ -e^{-2\alpha\phi} A_\nu & e^{-2\beta\phi} + e^{-2\alpha\phi} A_\rho A^\rho \end{pmatrix}. \quad (\text{A.1.4})$$

To describe the physics observed by a D -dimensional observer, we should express the action completely in terms of the D -dimensional fields. We will use the Cartan formalism to perform the dimensional reduction. We define the vielbeins \hat{e}_M^A with coordinates

$$\hat{e}_\mu^a = e^{\alpha\phi} e_\mu^a, \quad \hat{e}_z^a = 0, \quad \hat{e}_\mu^{\tilde{z}} = e^{\beta\phi} A_\mu \quad \text{and} \quad \hat{e}_z^{\tilde{z}} = e^{\beta\phi}, \quad (\text{A.1.5})$$

leading to the forms

$$\hat{e}^a = e^{\alpha\phi} e^a \quad \text{and} \quad \hat{e}^{\tilde{z}} = e^{\beta\phi} (A_\mu dx^\mu + dz).$$

To find the spin connection, we use the first Cartan structure equation

$$d\hat{e}^A + \hat{\omega}^A_B \wedge \hat{e}^B = 0. \quad (\text{A.1.6})$$

Plugging (A.1.5) in it and using the antisymmetry of the spin connection we find

$$\hat{\omega}^{ab} = \omega^{ab} + \alpha e^{-\alpha\phi} (\partial^b \phi \hat{e}^a - \partial^a \phi \hat{e}^b) + \frac{1}{2} e^{(\beta-2\alpha)\phi} F^{ba} \hat{e}^{\tilde{z}} \quad (\text{A.1.7})$$

$$\hat{\omega}^{a\tilde{z}} = \frac{1}{2} e^{(\beta-2\alpha)\phi} F_b^a \hat{e}^b - \beta e^{-\alpha\phi} \partial^a \phi \hat{e}^{\tilde{z}}, \quad (\text{A.1.8})$$

where the two-form F as usual is $F = dA$. The second Cartan equation gives us the Ricci form in terms of the spin connection as

$$\hat{R}_N^M = d\hat{\omega}_N^M + \hat{\omega}_K^M \wedge \hat{\omega}_N^K \quad (\text{A.1.9})$$

The two form components expressed in the usual frame are the Riemann tensor components, so that one can deduce the Ricci scalar from them. We have for the $M = a$, $N = z$ component

$$\hat{R}_{\tilde{z}}^a = d\hat{\omega}_{\tilde{z}}^a + \hat{\omega}_d^a \wedge \hat{\omega}_{\tilde{z}}^d. \quad (\text{A.1.10})$$

Finally we obtain

$$\hat{R}_{\tilde{z}}^a = e^{-2\alpha\phi} [(\alpha\beta - \beta^2) \partial_c \phi \partial^a \phi - \beta \partial_c \partial^a \phi] \hat{e}^c \wedge \hat{e}^{\tilde{z}} - \frac{1}{4} e^{2(\beta-2\alpha)\phi} F_d^a F_b^d \hat{e}^b \wedge \hat{e}^{\tilde{z}} - \beta e^{-\alpha\phi} \partial^d \phi \omega_d^a \wedge \hat{e}^{\tilde{z}} \quad (\text{A.1.11})$$

$$\begin{aligned}
 & -\alpha\beta e^{-2\alpha\phi}(\partial_b\phi\partial^b\phi\hat{e}^a\wedge\hat{e}^{\bar{z}}-\partial^a\phi\partial_d\phi\hat{e}^d\wedge\hat{e}^{\bar{z}}) \\
 & +\frac{1}{2}e^{(\beta-3\alpha)\phi}[(\beta-2\alpha)\partial_c\phi F_b^a+\partial_c F_b^a]\hat{e}^c\wedge\hat{e}^b-\frac{1}{2}e^{(\beta-2\alpha)\phi}F_b^a\omega_d^b\wedge\hat{e}^d \\
 & -\frac{1}{2}e^{(\beta-3\alpha)\phi}(\alpha F_b^a\partial_d\phi+\beta\partial^a\phi F_{bd})\hat{e}^b\wedge\hat{e}^d \\
 & +\frac{1}{2}e^{(\beta-2\alpha)\phi}F_b^d\omega_d^a\wedge\hat{e}^b+\frac{\alpha}{2}e^{(\beta-3\alpha)\phi}(\partial_d\phi F_b^d\hat{e}^a\wedge\hat{e}^b-\partial^a\phi F_b^d\hat{e}_d\wedge\hat{e}^b).
 \end{aligned}$$

From this, one can then extract the $\hat{R}^a_{\bar{z}c\bar{z}}$ and $\hat{R}^a_{\bar{z}bc}$ components of the Riemann tensor:

$$\hat{R}^a_{\bar{z}c\bar{z}}=e^{-2\alpha\phi}[(2\alpha\beta-\beta^2)\partial_c\phi\partial^a\phi-\beta\partial_c\partial^a\phi-\alpha\beta\partial_d\phi\partial^d\phi\delta_c^a]-\frac{1}{4}e^{2(\beta-2\alpha)\phi}F_d^aF_c^d-\beta e^{-2\alpha\phi}\partial^d\phi\omega_{dc}^a \quad (\text{A.1.12})$$

$$\begin{aligned}
 \hat{R}^a_{\bar{z}bc} &= \frac{1}{2}e^{(\beta-3\alpha)\phi}[(\beta-\alpha)\partial_c\phi F_b^a+\partial_c F_b^a-\beta\partial^a\phi F_{bc}+\alpha\partial_d\phi F_c^d\delta_b^a-\alpha\partial^a\phi F_{cb}-F_k^a\omega_{cb}^k+F_c^d\omega_{db}^a \\
 & \quad -\{b\leftrightarrow c\}] \quad (\text{A.1.13})
 \end{aligned}$$

We thus obtain two of the components of the Ricci tensor in the local frame, given as

$$\hat{R}_{\bar{z}\bar{z}}=\hat{R}^a_{\bar{z}a\bar{z}}=\beta e^{-2\alpha\phi}\{[(2-D)\alpha-\beta](\partial\phi)^2-\square\phi\}+\frac{1}{4}e^{2(\beta-2\alpha)\phi}F_{ad}F^{ad} \quad (\text{A.1.14})$$

and

$$\hat{R}_{\bar{z}c}=\hat{R}^a_{\bar{z}ac}=-\frac{1}{2}e^{(\beta-3\alpha)\phi}\{\nabla_a F_c^a+[3\beta-(4-D)\alpha]\partial_a\phi F_c^a\} \quad (\text{A.1.15})$$

with $(\partial\phi)^2=\partial_d\phi\partial^d\phi$, $\square\phi=\nabla_d\nabla^d\phi=\nabla_d(\partial^d\phi)=\partial_d\partial^d\phi+\omega_{ad}^d\partial^a\phi$ and $\nabla_a F_c^a=\partial_a F_c^a-\omega_{da}^a F_c^d+\omega_{ca}^k F_k^a$.

To find the missing components of the Ricci tensor we again apply the Cartan's second equation to

$$\hat{R}_b^a=d\hat{\omega}_b^a+\hat{\omega}_d^a\wedge\hat{\omega}_b^d+\hat{\omega}_{\bar{z}}^a\wedge\hat{\omega}_{\bar{z}b}^{\bar{z}}, \quad (\text{A.1.16})$$

giving

$$\begin{aligned}
 \hat{R}_b^a &= \frac{1}{2}\left\{-\partial_c[e^{(\beta-2\alpha)\phi}F_b^a]e^{-\alpha\phi}\hat{e}^c\wedge\hat{e}^{\bar{z}}-\beta e^{(\beta-3\alpha)\phi}F_b^a\partial_d\phi\hat{e}^d\wedge\hat{e}^{\bar{z}}-e^{(\beta-2\alpha)\phi}F_b^d\omega_d^a\wedge\hat{e}^{\bar{z}}\right. \\
 & \quad -\alpha e^{(\beta-3\alpha)\phi}F_b^d\partial_d\phi\hat{e}^a\wedge\hat{e}^{\bar{z}}+\alpha e^{(\beta-3\alpha)\phi}F_b^d\partial^a\phi\hat{e}_d\wedge\hat{e}^{\bar{z}}+e^{(\beta-2\alpha)\phi}F_d^a\omega_b^d\wedge\hat{e}^{\bar{z}} \\
 & \quad +\alpha e^{(\beta-3\alpha)\phi}F_d^a\partial_b\phi\hat{e}^d\wedge\hat{e}^{\bar{z}}-\alpha e^{(\beta-3\alpha)\phi}F_d^a\partial^d\phi\hat{e}_b\wedge\hat{e}^{\bar{z}}-\beta e^{(\beta-3\alpha)\phi}F_l^a\partial_b\phi\hat{e}^l\wedge\hat{e}^{\bar{z}} \\
 & \quad \left.+\beta e^{(\beta-3\alpha)\phi}\partial^a\phi F_{bk}\hat{e}^k\wedge\hat{e}^{\bar{z}}\right\} \\
 & +d\omega_b^a+\alpha e^{-\alpha\phi}\partial_c(e^{-\alpha\phi}\partial_b\phi)\hat{e}^c\wedge\hat{e}^a-\alpha^2 e^{-2\alpha\phi}\partial_b\phi\partial_d\phi\hat{e}^a\wedge\hat{e}^d-\alpha e^{-\alpha\phi}\partial_c(e^{-\alpha\phi}\partial^a\phi)\hat{e}^c\wedge\hat{e}_b \\
 & +\alpha e^{-\alpha\phi}\partial^a\phi\omega_{bd}\wedge\hat{e}^d+\alpha^2 e^{-2\alpha\phi}\partial^a\phi\partial_d\phi\hat{e}_b\wedge\hat{e}^d-\frac{1}{4}e^{2(\beta-2\alpha)\phi}F_b^aF_{kl}\hat{e}^k\wedge\hat{e}^l+\omega_d^a\wedge\omega_b^d \\
 & -\alpha e^{-\alpha\phi}\partial^d\phi\omega_d^a\wedge\hat{e}_b-\alpha e^{-\alpha\phi}\partial_d\phi\omega_b^d\wedge\hat{e}^a+\alpha e^{-\alpha\phi}\partial^a\phi\omega_b^d\wedge\hat{e}_d+\alpha^2 e^{-2\alpha\phi}\partial_d\phi\partial_b\phi\hat{e}^a\wedge\hat{e}^d \\
 & -\alpha^2 e^{-2\alpha\phi}(\partial\phi)^2\hat{e}^a\wedge\hat{e}_b+\alpha^2 e^{-2\alpha\phi}\partial^a\phi\partial^d\phi\hat{e}_d\wedge\hat{e}_b-\frac{1}{4}e^{2(\beta-2\alpha)\phi}F_l^aF_{bk}\hat{e}^l\wedge\hat{e}^k, \quad (\text{A.1.17})
 \end{aligned}$$

where the basis elements with a lower index are used as a shortcut for the contraction with local Minkowski metric, as in $\hat{e}_b = \eta_{bc}\hat{e}^c$. With the same reasoning as before one can extract now the $\hat{R}_{bc\bar{z}}$ and \hat{R}_{bcd}^a components of the form and use that to compute the \hat{R}_{bd} component of the Ricci tensor as

$$\begin{aligned}\hat{R}_{bd} &= \hat{R}_{b\bar{z}d}^{\bar{z}} + \hat{R}_{bad}^a \\ &= e^{-2\alpha\phi} \left\{ R_{bd} + \nabla_d(\partial_b\phi) [(2-D)\alpha - \beta] - \alpha\Box\phi\eta_{bd} + \partial_d\phi\partial_b\phi [\alpha^2(D-2) + 2\alpha\beta - \beta^2] \right. \\ &\quad \left. + (\partial\phi)^2\eta_{bd} [(2-D)\alpha^2 - \alpha\beta] \right\} - \frac{1}{2}e^{2(\beta-2\alpha)\phi}F_b^aF_{ad},\end{aligned}\quad (\text{A.1.18})$$

where R_{bd} is the D -dimensional Ricci tensor. We eventually get \hat{R} :

$$\begin{aligned}\hat{R} &= \eta^{bd}\hat{R}_{bd} + \hat{R}_{\bar{z}\bar{z}} \\ &= e^{-2\alpha\phi} \left\{ R + [2(1-D)\alpha - 2\beta]\Box\phi + [(D-2)(1-D)\alpha^2 + 2\beta(2-D)\alpha - 2\beta^2] (\partial\phi)^2 \right\} \\ &\quad - \frac{1}{4}e^{2(\beta-2\alpha)\phi}F_{ab}F^{ab}.\end{aligned}\quad (\text{A.1.19})$$

Finally, the determinant of the metric \hat{g} in $D+1$ dimensions is given by

$$\hat{g} = e^{(2D\alpha-2\beta)\phi}g, \quad (\text{A.1.20})$$

where g is the determinant of the D -dimensional metric, and thus plugging (A.1.20) and (A.1.19) in (A.1.1) we obtain

$$\begin{aligned}\mathcal{S} &= \frac{1}{2\hat{\kappa}^2} \int d^{D+1}x e^{(D\alpha+\beta)\phi} \sqrt{-g} \left\{ e^{-2\alpha\phi} \left[R + [2(1-D)\alpha - 2\beta]\Box\phi \right. \right. \\ &\quad \left. \left. + [(D-2)(1-D)\alpha^2 + 2\beta(2-D)\alpha - 2\beta^2] (\partial\phi)^2 \right] - \frac{1}{4}e^{2(\beta-2\alpha)\phi}F^2 \right\}.\end{aligned}\quad (\text{A.1.21})$$

We can now proceed to fix the constants. In order to get rid of the exponential in front of the D -dimensional Ricci scalar, that will allow us to get a canonical term for D -dimensional gravity, we require

$$(D-2)\alpha + \beta = 0. \quad (\text{A.1.22})$$

Accordingly, the scalar kinetic term reads $(D-2)(1-D)\alpha^2(\partial\phi)^2$. For it to be canonical we thus fix the constant α to the value

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}. \quad (\text{A.1.23})$$

Since all fields are independent of z , we can perform the integration over the extra dimension, that we take as compact in the shape of a circle of radius L . Of course, if the additional dimension needs to be hidden for low energy-observers, it should be defined on distances that could only be resolved with energies higher than those experimentally accessible. In that case, we should thus expect the radius L to be small. We can then rewrite the action

$$\mathcal{S} = \frac{2\pi L}{2\hat{\kappa}^2} \int d^Dx \sqrt{-g} \left(R - 2\alpha\Box\phi - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{2(1-D)\alpha\phi}F^2 \right). \quad (\text{A.1.24})$$

The negative sign in front of the kinetic term for the scalar field is the correct one as we are working with the $(-, +, \dots, +)$ signature. The coefficient in front of R now reads $2\pi L/(2\hat{\kappa}^2)$. We thus define the D -dimensional constant κ in terms of the $(D+1)$ -dimensional $\hat{\kappa}$ as

$$\frac{1}{\kappa^2} = \frac{2\pi L}{\hat{\kappa}^2} \implies M_{Pl}^{D-2} = 2\pi L \hat{M}_{Pl}^{D-1} \quad (\text{A.1.25})$$

Besides, from (A.1.3) we see that the ϕ and A_μ fields that we have used so far are dimensionless. The physical dimension-full fields, that we call $\tilde{\phi}$ and \tilde{A}_μ , are obtained once we reabsorb the $1/2\kappa^2$ pre-factor in front of their kinetic terms as

$$\tilde{\phi} = \frac{\phi}{\sqrt{2\kappa}} = \frac{M_{Pl}^{\frac{D-2}{2}}}{\sqrt{2}} \phi; \quad \tilde{A}_\mu = \frac{A_\mu}{\sqrt{2\kappa}} = \frac{M_{Pl}^{\frac{D-2}{2}}}{\sqrt{2}} A_\mu. \quad (\text{A.1.26})$$

In terms of D -dimensional quantities, the action of $(D+1)$ -dimensional pure gravity with one compact dimension in the shape of a circle reads:

$$\mathcal{S} = \int d^D x \sqrt{-g} \left(\frac{R}{2\kappa^2} - 2\alpha \square \tilde{\phi} - \frac{1}{2} (\partial \tilde{\phi})^2 - \frac{1}{4} e^{2\sqrt{2}(1-D)\alpha\kappa\tilde{\phi}} \tilde{F}^2 \right). \quad (\text{A.1.27})$$

The second term is a total derivative and, as such, we will not display it in the following. It was of importance only when studying the behaviour under compactification of the non minimal coupling to gravity of a scalar in [4]. The gauge field that we obtain under compactification is usually called graviphoton, and the scalar is called the radion and/or the dilaton. These names are due to the fact that the scalar field vev is fixes the length of the compactified dimension. In the rest of the work, we will remove the tilde and denote physical fields without it.

In the remainder of this section, we discuss the second method alluded above.

Computation via Weyl rescaling

In the calculation described above, we used a particular form of the metric that brought us all the way down to the action in (A.1.27) for an Einstein-Maxwell-scalar theory in the Einstein frame. Of course, one could ask whether this ad-hoc form of the metric could have been guessed from first principles and what happens if one uses a different metric. Here we try to fill this gap by showing how the same computation can be done using the Weyl transformations that relate different, physically equivalent, frames.

As a starter, we should compute the variation of the Ricci tensor under a Weyl rescaling of the metric $g \rightarrow \bar{g} e^{-2\omega(x)}$ in d generic dimensions.

The vielbeins transforms accordingly as $e^a = e^{-\omega(x)} \bar{e}^a$. Using the above mentioned Cartan's equations we can derive the following transformation rules for the spin connection and the Ricci form

$$\begin{cases} \omega_b^a = \bar{\omega}_b^a - \partial_b \omega \bar{e}^a + \partial^a \omega \bar{e}_b \\ R_b^a = \frac{1}{2} \bar{R}_b^a + \partial_c \partial^a \omega \bar{e}^c \wedge \bar{e}_b + \partial^d \omega \bar{\omega}_d^a \wedge \bar{e}_b + \partial^a \omega \partial^d \omega \bar{e}_d \wedge \bar{e}_b - \frac{1}{2} \partial_d \omega \partial^d \omega \bar{e}^a \wedge \bar{e}_b - \{^a \leftrightarrow b\}. \end{cases} \quad (\text{A.1.28})$$

In the same spirit as before, we can now extract from the components of the form the Ricci tensor

$$R_{bd} = R_{bad}^a = e^{2\omega} \left\{ \bar{R}_{bd} + (d-2)\nabla_b \nabla_d \omega + \nabla_a \nabla^a \omega \eta_{bd} + (d-2)\partial_b \omega \partial_d \omega - (d-2)\partial_a \omega \partial^a \omega \eta_{bd} \right\} \quad (\text{A.1.29})$$

and the Ricci scalar

$$R = e^{2\omega} \left(\bar{R} + 2(d-1)\square\omega + (d-2)(1-d)(\partial\omega)^2 \right). \quad (\text{A.1.30})$$

With this result in mind, we can now take the $D + 1$ -dimensional metric of the most generic form as

$$\hat{g}_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}. \quad (\text{A.1.31})$$

The corresponding vielbeins read

$$\hat{e}^a = \tilde{e}^a \text{ and } \hat{e}^{\tilde{z}} = A + dz, \text{ with } A = A_\mu dx^\mu. \quad (\text{A.1.32})$$

The spin connection and the associated Ricci form are

$$\hat{\omega}_{\tilde{z}a}^{\tilde{z}} = \frac{1}{2} F_{ab} \tilde{e}^b \text{ and } \hat{\omega}_b^a = \tilde{\omega}_b^a + \frac{1}{2} F_b^a \hat{e}^{\tilde{z}} \quad (\text{A.1.33})$$

$$\hat{R}_{\tilde{z}}^a = \frac{1}{2} \left(\partial_c F_b^a \tilde{e}^c \wedge \tilde{e}^b - F_b^a \tilde{\omega}_d^b \wedge \tilde{e}^d + F_l^c \tilde{\omega}_c^a \wedge \tilde{e}^l + \frac{1}{2} F_c^a F_l^c \tilde{e}^{\tilde{z}} \wedge \tilde{e}^l \right) \quad (\text{A.1.34})$$

$$\hat{R}_b^a = \tilde{R}_b^a + \frac{1}{2} \partial_c F_b^a \tilde{e}^c \wedge \hat{e}^{\tilde{z}} + \frac{1}{2} F_d^a \hat{e}^{\tilde{z}} \wedge \tilde{\omega}_b^d + \frac{1}{2} F_b^d \tilde{\omega}_d^a \wedge \hat{e}^{\tilde{z}} + \frac{1}{4} F_b^a F_{cd} \tilde{e}^c \wedge \tilde{e}^d - \frac{1}{4} F_l^a F_{db} \tilde{e}^l \wedge \tilde{e}^d \quad (\text{A.1.35})$$

The Ricci scalar is then

$$\hat{R} = \tilde{R} - \frac{1}{4} F^2. \quad (\text{A.1.36})$$

With the metric in (A.1.31) the determinant \hat{g} is $\hat{g} = \tilde{g}$.

We now perform a first Weyl rescaling of the $D + 1$ -dimensional metric from \hat{g} to $\hat{g}' = e^{2\omega(x)} \hat{g}$, where we take the function $\omega(x)$ to depend only on the non compact coordinates to preserve the cylinder condition (A.1.2). Using the relation $\sqrt{-\hat{g}} = e^{-(D+1)\omega(x)} \sqrt{-\tilde{g}}$ and (A.1.30) for $d = D + 1$ we obtain

$$\sqrt{-\hat{g}} \hat{R} = e^{(1-D)\omega} \sqrt{(-1)^D \tilde{g}'} \left(\tilde{R} - \frac{1}{4} \tilde{F}^2 + 2D \tilde{\square} \omega - D(D-1)(\tilde{\partial}\omega)^2 \right), \quad (\text{A.1.37})$$

where D -dimensional quantities have been explicitly displayed with the symbol on top. As we see, the D -dimensional gravitational term is not in the Einstein frame. We thus perform another Weyl rescaling to get rid of the exponential, this time on the lower dimensional metric. Writing the transformation as $\tilde{g} \rightarrow e^{-2\omega'(x)} g$ we get

$$\begin{aligned} \sqrt{-\tilde{g}} \hat{R} &= e^{(1-D)\omega + (2-D)\omega'} \sqrt{-g} \left(R + 2(D-1)\square\omega' + (1-D)(D-2)(\partial\omega')^2 \right) \\ &+ e^{(1-D)\omega - D\omega'} \sqrt{-g} \left(-\frac{1}{4} F^2 + 2D \tilde{\square} \omega - D(D-1)(\tilde{\partial}\omega)^2 \right) \\ &= \sqrt{-g} \left\{ e^{(1-D)\omega + (2-D)\omega'} \left(R + 2(D-1)\square\omega' + (1-D)(D-2)(\partial\omega')^2 \right) \right. \end{aligned} \quad (\text{A.1.38})$$

$$\left. +2D\Box\omega - D(D-1)(\partial\omega)^2 - \frac{1}{4}e^{-2\omega'}F^2 \right\},$$

where in the last line we have written all the operators with respect to the new metric g (or vielbein basis). Asking that the exponential in front of the Ricci scalar vanishes is equivalent to the requirement

$$(1-D)\omega(x) + (2-D)\omega'(x) = 0 \longrightarrow \omega(x) = -\frac{2-D}{1-D}\omega'(x). \quad (\text{A.1.39})$$

We see that if we want to end up in an Einstein frame, the functions defining the two different rescalings we have performed need to be proportional to one another. If, according to that, we now define $\omega(x) \equiv \beta\phi(x)$ and $\omega'(x) = (\alpha - \beta)\phi(x)$, we see that the action defined by (A.1.38) takes the form (A.1.21), thus justifying a posteriori the computation with the metric (A.1.3).

A.2 Free higher dimensional scalar field

We now move forward and start adding some particle content in our $D+1$ -dimensional theory in the form of a free real massless scalar field, so that the action reads

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_{\Phi}, \quad (\text{A.2.1})$$

where \mathcal{S}_{EH} is defined in (A.1.1) and

$$\mathcal{S}_{\Phi} = - \int d^{D+1}x \sqrt{-\hat{g}} \frac{1}{2} \hat{g}^{MN} \partial_M \hat{\Phi} \partial_N \hat{\Phi} \quad (\text{A.2.2})$$

Here we take the real scalar field Φ to be single valued at any point in space-time. Calling x the D non compact coordinates and z the compact one, this physical requirement fixes the periodicity along the compact dimension:

$$\hat{\Phi}(x, z + 2\pi L) = \hat{\Phi}(x, z). \quad (\text{A.2.3})$$

This in turn allows us to write the field Φ in the Fourier basis as

$$\hat{\Phi}(x, z) = \frac{1}{\sqrt{2\pi L}} \sum_{n=-\infty}^{+\infty} \varphi_n(x) e^{\frac{inz}{L}}, \quad (\text{A.2.4})$$

where the Fourier components only depends on the x -coordinate, thus realizing the periodic z -dependence through the basis elements.

Plugging the explicit form of the metric given in (A.1.4) together with this decomposition in the action (A.2.2) we find

$$\begin{aligned} \mathcal{S}_{\Phi} = -\frac{1}{2} \int d^{D+1}x \sqrt{-g} \sum_{n,m=-\infty}^{+\infty} \frac{e^{i\frac{(n+m)z}{L}}}{2\pi L} \left(g^{\mu\nu} \partial_{\mu} \varphi_n \partial_{\nu} \varphi_m - 2i \frac{\sqrt{2}m}{LM_{Pl}^{\frac{D-2}{2}}} A^{\mu} \partial_{\mu} \varphi_n \varphi_m \right. \\ \left. - \frac{2}{M_{Pl}^{D-2}} \frac{nm}{L^2} A_{\mu} A^{\mu} \varphi_n \varphi_m - \frac{nm}{L^2} e^{-2\sqrt{2}(\beta-\alpha)\frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_m \right) \end{aligned} \quad (\text{A.2.5})$$

The integration over the compactified dimension will give a term in $2\pi L \delta_{n,-m}$ that we can then use to perform the sum over one of the two variables, say m , and the action is thus written

$$\mathcal{S}_\Phi = -\frac{1}{2} \int d^D x \sqrt{-g} \sum_{n=-\infty}^{+\infty} \left(g^{\mu\nu} \partial_\mu \varphi_n \partial_\nu \varphi_{-n} + 2i \frac{\sqrt{2}n}{LM_{Pl}^{\frac{D-2}{2}}} A^\mu \partial_\mu \varphi_n \varphi_{-n} + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_{-n} + \frac{n^2}{L^2} e^{-2\sqrt{2}(\beta-\alpha) \frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_{-n} \right). \quad (\text{A.2.6})$$

The reality condition $\Phi(x, z) = \Phi^*(x, z)$ straightforwardly implies $\varphi_{-n} = \varphi_n^*$. For the 0-mode φ_0 this in turn implies that such mode is real.

To rewrite the action (A.2.6) in a more canonical and elegant way, we should first observe that all the bilinears $\varphi_n \varphi_{-n} = \varphi_n \varphi_n^*$ are symmetric under the transformation $n \rightarrow -n$. This straightforwardly applies to all but one term, giving

$$\begin{aligned} & \sum_{n=-\infty}^{+\infty} \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \varphi_n \partial_\nu \varphi_{-n} + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_{-n} + \frac{n^2}{L^2} e^{-2\sqrt{2}(\beta-\alpha) \frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_{-n} \right) = \\ & = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_0 \partial_\nu \varphi_0 + \sum_{n=1}^{\infty} \left(g^{\mu\nu} \partial_\mu \varphi_n \partial_\nu \varphi_n^* + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* + \frac{n^2}{L^2} e^{-2\sqrt{2}(\beta-\alpha) \frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_n^* \right). \end{aligned} \quad (\text{A.2.7})$$

For the remaining term, we see that under $n \rightarrow -n$ it transforms as $n A^\mu \partial_\mu \varphi_n \varphi_n^* \rightarrow -n A^\mu \partial_\mu \varphi_n^* \varphi_n$. Grouping the $(n, -n)$ terms two by two we thus get

$$\sum_{n=-\infty}^{+\infty} i \frac{\sqrt{2}n}{LM_{Pl}^{\frac{D-2}{2}}} A^\mu \partial_\mu \varphi_n \varphi_{-n} = \sum_{n=1}^{\infty} i \frac{\sqrt{2}n}{LM_{Pl}^{\frac{D-2}{2}}} A^\mu (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*), \quad (\text{A.2.8})$$

which together with the $A_\mu A^\mu$ term previously shown forms the well-known minimal coupling dictated by the gauge principle for a complex scalar field. The action (A.2.1) finally reads

$$\begin{aligned} \mathcal{S} = \int d^D x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} F^2 - \frac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 \right. \\ \left. - \sum_{n=1}^{\infty} \left(\partial_\mu \varphi_n \partial^\mu \varphi_n^* + \frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_n^* \right) \right. \\ \left. - \sum_{n=1}^{\infty} \left(i \frac{\sqrt{2}n}{LM_{Pl}^{\frac{D-2}{2}}} A^\mu (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* \right) \right\}, \end{aligned} \quad (\text{A.2.10})$$

where we have chosen the positive root for α (A.1.23).

Some comments are now in order. The first thing we observe is that a $D+1$ -dimensional

system with gravity and a real massless scalar field is described, in terms of D -dimensional quantities accessible to a low energy observer, as a rather involved system. Besides the presence of the graviphoton and the dilaton that we already encountered in the compactification of pure gravity, the higher dimensional real massless scalar compactifies down to a lower dimensional real massless scalar and an infinite tower of massive complex scalars which are the Kaluza-Klein states (KK-states) minimally coupled to the D -dimensional gauge field (the graviphoton). The ϕ -dependent mass of such fields in turn determines a coupling to the dilaton.

The field φ_0 represents a mode with no momentum along the compact dimension. It does not feel the presence of the additional dimension as given by the components $\hat{g}_{\mu z}$ and \hat{g}_{zz} , that we have written in terms of A_μ and ϕ . The φ_n 's, on the other hand, describe modes with non vanishing momentum along the compact dimension, and thus couple to A_μ and ϕ . The periodicity condition required in (A.2.3) implies the quantization of such momentum $p_z \sim n/L$, $n \in \mathbb{N}$. When the additional dimension is integrated out, this is effectively described as a mass term.

Coming back to (A.1.27), we see that the gauge coupling g is

$$g^2 = e^{2\sqrt{\frac{D-1}{D-2}} \frac{\phi}{M_{Pl}^{(D-2)/2}}}. \quad (\text{A.2.11})$$

For each KK mode, we can now give the explicit expression of both the mass and the charge. They read

$$gq_n = \frac{\sqrt{2}n}{LM_{Pl}^{(D-2)/2}} e^{\sqrt{\frac{D-1}{D-2}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} \quad \text{and} \quad m_n = \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}} \frac{\phi}{M_{Pl}^{(D-2)/2}}}. \quad (\text{A.2.12})$$

This shows that, in any space-time dimension D , the mass and the charge of the KK modes are related to each other as

$$(gq_n)^2 = 2 \frac{m_n^2}{M_{Pl}^{D-2}}. \quad (\text{A.2.13})$$

The charge to mass ratio of the KK-states is then equal to the bound (1.5.10) with $\alpha = \sqrt{3}$ in 4-dimensions.

A.3 Compactification via the higher dimensional gravitational vertices

In the previous (sub)sections we obtained the effective description of a $D + 1$ -dimensional theory of gravity and a massless scalar in terms D -dimensional quantities. In particular, fixing the dependence on the coordinate of the extra dimension and integrating it out, the (whole) original theory is rewritten in terms of D -dimensional fields. All the interactive terms for the KK modes of the scalar field displayed in (A.2.9) are of gravitational origin and come from the gravitational vertices of the higher dimensional scalar. We should in principle be able to derive all the lower dimensional vertices involving scalars from the corresponding $(D + 1)$ -dimensional gravitational ones. We can immediately see from (A.2.9) that, in the process, we will need to go up to second

order in κ . This is a more direct (and, to my knowledge, not much discussed) approach in the field theoretic language. I show how it works in the following.

The first thing one needs to do is to verify the perturbativity of the metric (A.1.3). The starting point is again the metric (A.1.3), that we rewrite here for the sake of clarity:

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{pmatrix}.$$

We should now expand this metric around a generic background, and then, using the conventions in (A.1.1), verify whether it is possible or not to write it as a formal development of the form

$$\hat{g}_{MN} = \hat{\zeta}_{MN} + 2\hat{\kappa}\hat{h}_{MN} + 4\hat{\kappa}^2\hat{f}_{MN} + \mathcal{O}(\hat{\kappa}^3), \quad (\text{A.3.1})$$

with $\hat{\kappa}^2\hat{f}_{MN} \ll \hat{\kappa}\hat{h}_{MN} \ll 1 \forall M, N$. Usually the perturbative expansion for gravity is set up around a Minkowski background. This need not be true in all cases as the dilaton ϕ may take a vev, and $\beta \neq \alpha$, so that in principle $\hat{\zeta}$ is not even proportional to $\hat{\eta}$. The lower dimensional metric is undetermined, and as such we can arbitrarily set up its perturbative expansion to be around the Minkowski background as $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} + 4\kappa^2 f_{\mu\nu} + \mathcal{O}(\kappa^3)$, where η is the Minkowski metric, as usual. Using now (A.1.25) and inserting the physical fields (A.1.26) we can write:

$$\begin{aligned} \hat{g}_{MN} &= \begin{pmatrix} e^{2\sqrt{2}\alpha\kappa\phi} (\eta_{\mu\nu} + 2\kappa h_{\mu\nu} + 4\kappa^2 f_{\mu\nu} + \dots) + 2\kappa^2 e^{2\sqrt{2}\beta\kappa\phi} A_\mu A_\nu & \sqrt{2}\kappa e^{2\sqrt{2}\beta\kappa\phi} A_\mu \\ \sqrt{2}\kappa e^{2\sqrt{2}\beta\kappa\phi} A_\nu & e^{2\sqrt{2}\beta\kappa\phi} \end{pmatrix} \\ &= \begin{pmatrix} e^{2\sqrt{2}\kappa\alpha\phi_0} \eta_{\mu\nu} & 0 \\ 0 & e^{2\sqrt{2}\beta\kappa\phi_0} \end{pmatrix} + \frac{2\hat{\kappa}}{\sqrt{2\pi L}} \begin{pmatrix} e^{2\sqrt{2}\alpha\kappa\phi_0} (\sqrt{2}\alpha\phi \eta_{\mu\nu} + h_{\mu\nu}) & e^{2\sqrt{2}\beta\kappa\phi_0} \frac{A_\mu}{\sqrt{2}} \\ e^{2\sqrt{2}\beta\kappa\phi_0} \frac{A_\nu}{\sqrt{2}} & e^{2\sqrt{2}\beta\kappa\phi_0} \sqrt{2}\beta\phi \end{pmatrix} \\ &+ \frac{4\hat{\kappa}^2}{2\pi L} \begin{pmatrix} e^{2\sqrt{2}\alpha\kappa\phi_0} \left(\alpha^2 \phi^2 \eta_{\mu\nu} + \sqrt{2}\alpha\phi h_{\mu\nu} + f_{\mu\nu} \right) + e^{2\sqrt{2}\beta\kappa\phi_0} \frac{A_\mu A_\nu}{2} & e^{2\sqrt{2}\beta\kappa\phi_0} \beta\phi A_\mu \\ e^{2\sqrt{2}\beta\kappa\phi_0} \beta\phi A_\nu & e^{2\sqrt{2}\beta\kappa\phi_0} \beta^2 \phi^2 \end{pmatrix} \\ &+ \mathcal{O}\left(\frac{\hat{\kappa}^3}{(2\pi L)^{3/2}}\right), \end{aligned} \quad (\text{A.3.2})$$

where we have expanded the dilaton around a generic background value as $\phi_0 + \phi$. We immediately see that the background is different from Minkowski except for the case $\phi_0 = 0$ since we have

$$\hat{\zeta}_{MN} = \begin{pmatrix} e^{2\sqrt{2}\alpha\kappa\phi_0} \eta_{\mu\nu} & 0 \\ 0 & e^{2\sqrt{2}\beta\kappa\phi_0} \end{pmatrix}. \quad (\text{A.3.3})$$

Nevertheless, we can say that, as long as the fields do not develop trans-Planckian values, our metric is perturbative in the sense of a Planck suppressed series around a background, even if the background is not flat.

We can now proceed to our declared goal of computing the lower dimensional vertices from the higher dimensional ones. In D generic dimensions, the perturbative expansion for the metric and inverse metric reads (greek indices and quantities with no hat are used here as this is general and unrelated to our compactification)

$$\begin{cases} g_{\mu\nu} = \zeta_{\mu\nu} + 2\kappa h_{\mu\nu} + 4\kappa^2 f_{\mu\nu} + \mathcal{O}(\kappa^3) \\ g^{\mu\nu} = \zeta^{\mu\nu} + 2\kappa t^{\mu\nu} + 4\kappa^2 l^{\mu\nu} + \mathcal{O}(\kappa^3). \end{cases} \quad (\text{A.3.4})$$

The relation $g_{\mu\rho}g^{\rho\nu} \equiv \delta_\mu^\nu$ is then expressed as

$$\begin{aligned} & (\zeta_{\mu\rho} + 2\kappa h_{\mu\rho} + 4\kappa^2 f_{\mu\rho}) (\zeta^{\rho\nu} + 2\kappa t^{\rho\nu} + 4\kappa^2 l^{\rho\nu}) = \\ & \delta_\mu^\nu + 2\kappa (\zeta^{\rho\nu} h_{\mu\rho} + \zeta_{\mu\rho} t^{\rho\nu}) + 4\kappa^2 (\zeta^{\rho\nu} f_{\mu\rho} + \zeta_{\mu\rho} l^{\rho\nu} + h_{\mu\rho} t^{\rho\nu}) + \mathcal{O}(\kappa^3) \equiv \delta_\mu^\nu, \end{aligned} \quad (\text{A.3.5})$$

and thus

$$\begin{cases} t^{\mu\nu} = -h^{\mu\nu} \\ l^{\mu\nu} + f^{\mu\nu} = h_\rho^\mu h^{\rho\nu}, \end{cases} \quad (\text{A.3.6})$$

where it is understood that the indices are raised and lowered with the background metric ζ . We find here the first and second order equations relating the perturbation of the metric and its inverse. It is common to find, in the literature, the particular choice $f_{\mu\nu} = 0, l^{\mu\nu} = h_\rho^\mu h^{\rho\nu}$ (with a minkowskian background). This is perfectly fine when one works with generic, non specified perturbations. In our case, however, as we already have imposed the specific form of the metric, we cannot use such convention but need to develop the gravitational formalism with the specific h, t, f, l dictated by our metric.

To go further, we now need to express the determinant of the metric. Defining $\kappa' \equiv 2\kappa$ and using g as the matrix representation of the metric $\sqrt{-\det(g)} = \exp\left(\frac{1}{2}\text{tr} \log(g)\right) = \exp\left\{\frac{1}{2}\text{tr} \left[\zeta + \kappa' \zeta^{-1} h + \kappa'^2 \zeta^{-1} f - \frac{(\zeta^{-1} \kappa' h + \kappa'^2 \zeta^{-1} f)^2}{2} \right] + \mathcal{O}(\kappa'^3) \right\}$ we have

$$\begin{aligned} & \sqrt{-\det(g)} = \\ & \sqrt{-\det(\zeta)} \left(1 + \frac{\kappa'}{2} \text{tr}(\zeta^{-1} h) + \frac{\kappa'^2}{2} \text{tr}(\zeta^{-1} f) - \frac{\kappa'^2}{4} \text{tr}((\zeta^{-1} h)^2) + \frac{\kappa'^2}{8} (\text{tr}(\zeta^{-1} h))^2 + \mathcal{O}(\kappa'^3) \right). \end{aligned} \quad (\text{A.3.7})$$

Let us now apply all of this to the $(D+1)$ -dimensional action \mathcal{S}_Φ (A.2.2), recovering the usual notation g for the determinant. We obtain

$$\begin{aligned} & \sqrt{-\hat{g}} \mathcal{L}_\Phi = -\sqrt{-\hat{g}} \frac{1}{2} \hat{g}^{MN} \partial_M \hat{\Phi} \partial_N \hat{\Phi} \implies \\ & -\sqrt{-\hat{\zeta}} \left(1 + \frac{\hat{\kappa}'}{2} \hat{h}_M^M + \frac{\hat{\kappa}'^2}{2} \hat{f}_M^M - \frac{\hat{\kappa}'^2}{4} \hat{h}^{MN} \hat{h}_{MN} + \frac{\hat{\kappa}'^2}{8} (\hat{h}_M^M)^2 \right) \left(\hat{\zeta}^{MN} - \hat{\kappa}' \hat{h}^{MN} + \hat{\kappa}'^2 \hat{l}^{MN} \right) \frac{1}{2} \partial_M \hat{\Phi} \partial_N \hat{\Phi} \\ & = -\sqrt{-\hat{\zeta}} \left[\frac{1}{2} \partial_M \hat{\Phi} \partial^M \hat{\Phi} + \frac{\hat{\kappa}'}{2} \hat{h}^{MN} \left(\partial_M \hat{\Phi} \partial_N \hat{\Phi} - \frac{1}{2} \hat{\eta}_{MN} \partial_P \hat{\Phi} \partial^P \hat{\Phi} \right) \right. \\ & \left. - \frac{\hat{\kappa}'^2}{2} \left(\hat{l}^{MN} - \frac{1}{2} \hat{h}^{MN} \hat{h}_P^P \right) \partial_M \hat{\Phi} \partial_N \hat{\Phi} - \frac{\hat{\kappa}'^2}{4} \left(\hat{f}_P^P - \frac{1}{2} \hat{h}_{MP} \hat{h}^{PM} + \frac{1}{4} (\hat{h}_P^P)^2 \right) \partial_M \hat{\Phi} \partial^M \hat{\Phi} \right]. \end{aligned} \quad (\text{A.3.8})$$

As we can see, at first order we recover the three point interaction

$$-\sqrt{-\hat{\zeta}} \frac{\hat{\kappa}'}{2} \hat{h}^{MN} T_{MN}^{\hat{\Phi}}$$

(the different sign in the definition of the stress-energy tensor with respect to the one usually displayed come from the signature), while at second order we now have a more generic structure due to the non-vanishing \hat{f}_{MN} . Of course, the above expansion

is rather independent of the particular system we are studying or the dimension. As we can see from (A.3.2), raising the indices with ζ to stay at lowest order we have in our case

$$\hat{h}^{MN} = \frac{1}{\sqrt{2\pi L}} \begin{pmatrix} e^{-2\sqrt{2}\alpha\kappa\phi_0} (\sqrt{2}\alpha\phi\eta^{\mu\nu} + h^{\mu\nu}) & e^{-2\sqrt{2}\alpha\kappa\phi_0} \frac{A^\mu}{\sqrt{2}} \\ e^{-2\sqrt{2}\alpha\kappa\phi_0} \frac{A^\nu}{\sqrt{2}} & e^{-2\sqrt{2}\beta\kappa\phi_0} \sqrt{2}\beta\phi \end{pmatrix}, \quad (\text{A.3.9})$$

and we can thus write, using $\sqrt{-\hat{\zeta}} = e^{\sqrt{2}(D\alpha+\beta)\kappa\phi_0}$,

$$\begin{aligned} -\sqrt{2\pi L}\sqrt{-\hat{\zeta}}\hat{h}^{MN}\hat{T}_{MN}^{\hat{\Phi}} &= e^{\sqrt{2}((D-2)\alpha+\beta)\kappa\phi_0} \left(\sqrt{2}\alpha\phi\eta^{\mu\nu} + h^{\mu\nu} \right) \left(\partial_\mu\hat{\Phi}\partial_\nu\hat{\Phi} - \frac{1}{2}\eta_{\mu\nu}\partial_P\hat{\Phi}\partial^P\hat{\Phi} \right) \\ &\quad + \sqrt{2}e^{\sqrt{2}((D-2)\alpha+\beta)\kappa\phi_0} A^\mu\partial_\mu\hat{\Phi}\partial_z\hat{\Phi} \\ &\quad + \sqrt{2}e^{\sqrt{2}(D\alpha-\beta)\kappa\phi_0} \beta\phi \left(\partial_z\hat{\Phi}\partial_z\hat{\Phi} - \frac{1}{2}\partial_P\hat{\Phi}\partial^P\hat{\Phi} \right). \end{aligned} \quad (\text{A.3.10})$$

Inserting now the Fourier decomposition (A.2.4) and performing the integration over the z variable we get, for the D -dimensional action,

$$\begin{aligned} &e^{\sqrt{2}((D-2)\alpha+\beta)\kappa\phi_0} \left(\sqrt{2}\alpha\phi\eta^{\mu\nu} + h^{\mu\nu} \right) \left[\partial_\mu\varphi_0\partial_\nu\varphi_0 + \sum_{n=1}^{\infty} \partial_\mu\varphi_n\partial_\nu\varphi_n^* + \partial_\nu\varphi_n\partial_\mu\varphi_n^* \right. \\ &\quad \left. - \eta_{\mu\nu} \left(\frac{1}{2}\partial_\rho\varphi_0\partial^\rho\varphi_0 + \sum_{n=1}^{\infty} \partial_\mu\varphi_n\partial^\mu\varphi_n^* + \frac{n^2}{L^2}\varphi_n\varphi_n^* \right) \right] - e^{\sqrt{2}((D-2)\alpha+\beta)\kappa\phi_0} \sqrt{2}A^\mu \sum_{n=1}^{\infty} i\frac{n}{L} (\partial_\mu\varphi_n\varphi_n^* - \varphi_n\partial_\mu\varphi_n^*) \\ &\quad + e^{\sqrt{2}(D\alpha-\beta)\kappa\phi_0} \sqrt{2}\beta\phi \left(\sum_{n=1}^{\infty} \frac{n^2}{L^2}\varphi_n\varphi_n^* - \frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0 - \sum_{n=1}^{\infty} \partial_\mu\varphi_n\partial^\mu\varphi_n^* \right) \end{aligned} \quad (\text{A.3.11})$$

The second term in the first line is $(-)$ the stress-energy tensor for φ_0 and φ_n . We have thus the D -dimensional gravitational interaction plus a bunch of other interactions. More specifically, this simplifies to

$$\begin{aligned} &-\sqrt{2\pi L}\sqrt{-\hat{\zeta}}\hat{h}^{MN}\hat{T}_{MN}^{\hat{\Phi}} = \quad (\text{A.3.12}) \\ &\quad -h^{\mu\nu}T_{\mu\nu}^{(\varphi_0,\varphi_n)} - i\sqrt{2}A^\mu \sum_{n=1}^{\infty} \frac{n}{L} (\partial_\mu\varphi_n\varphi_n^* - \varphi_n\partial_\mu\varphi_n^*) + \frac{(2-D)\alpha-\beta}{\sqrt{2}}\phi\partial_\mu\varphi_0\partial^\mu\varphi_0 \\ &\quad + \sqrt{2}[(2-D)\alpha-\beta]\phi \sum_{n=1}^{\infty} \partial_\mu\varphi_n\partial^\mu\varphi_n^* - \sqrt{2}(\beta-D\alpha)e^{\sqrt{2}(D\alpha-\beta)\kappa\phi_0}\phi \sum_{n=1}^{\infty} \frac{n^2}{L^2}\varphi_n\varphi_n^* \\ &= -h^{\mu\nu}T_{\mu\nu}^{(\varphi_0,\varphi_n)} - i\sqrt{2}A^\mu \sum_{n=1}^{\infty} \frac{n}{L} (\partial_\mu\varphi_n\varphi_n^* - \varphi_n\partial_\mu\varphi_n^*) - 2\sqrt{\frac{D-1}{D-2}}e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}\phi \sum_{n=1}^{\infty} \frac{n^2}{L^2}\varphi_n\varphi_n^*, \end{aligned}$$

where we have used $\beta = -(D-2)\alpha$ and $\alpha = (2(D-1)(D-2))^{-1/2}$. This can be written in a more compact form using again (A.1.25)

$$-\hat{\kappa}\hat{h}^{MN}\hat{T}_{MN}^{\hat{\Phi}} = -\kappa h^{\mu\nu}T_{\mu\nu}^{(\varphi_0,\varphi_n)} - i\sqrt{2}\kappa A^\mu \sum_{n=1}^{\infty} \frac{n}{L} (\partial_\mu\varphi_n\varphi_n^* - \varphi_n\partial_\mu\varphi_n^*)$$

$$-2\sqrt{\frac{D-1}{D-2}}\kappa e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}\phi\sum_{n=1}^{\infty}\frac{n^2}{L^2}\varphi_n\varphi_n^*. \quad (\text{A.3.13})$$

The above result exactly matches what one obtains at order κ from (A.2.9).

If one were to stop at first order, he would miss the gauge invariance of the theory with respect to the graviphoton. Put in different words, as the gauge invariance of the D -dimensional theory is a realization of the general coordinate invariance of the $D+1$ -dimensional one along the compact direction and we know we should recover it, consistency of the theory forces us to go to second order in κ . To proceed further, we identify \hat{f}_{MN} from (A.3.2):

$$\hat{f}_{MN} = \frac{1}{2\pi L} \begin{pmatrix} e^{2\sqrt{2}\alpha\kappa\phi_0} \left(\alpha^2\phi^2\eta_{\mu\nu} + \sqrt{2}\alpha\phi h_{\mu\nu} + f_{\mu\nu} \right) + e^{2\sqrt{2}\beta\kappa\phi_0} \frac{A_\mu A_\nu}{2} & e^{2\sqrt{2}\beta\kappa\phi_0} \beta\phi A_\mu \\ e^{2\sqrt{2}\beta\kappa\phi_0} \beta\phi A_\nu & e^{2\sqrt{2}\beta\kappa\phi_0} \beta^2\phi^2 \end{pmatrix}. \quad (\text{A.3.14})$$

With these results, $\hat{l}^{MN} = \hat{h}^{MP}\hat{h}^N_P - \hat{f}^{MN}$ is obtained as

$$\hat{l}^{MN} = \frac{1}{2\pi L} \begin{pmatrix} e^{-2\sqrt{2}\alpha\kappa\phi_0} \left(\alpha^2\phi^2\eta_{\mu\nu} + \sqrt{2}\alpha\phi h_{\mu\nu} + l^{\mu\nu} \right) & e^{-2\sqrt{2}\alpha\kappa\phi_0} \left(\alpha\phi A_\mu + \frac{1}{\sqrt{2}}h^{\mu\rho}A_\rho \right) \\ e^{-2\sqrt{2}\alpha\kappa\phi_0} \left(\alpha\phi A^\nu + \frac{1}{\sqrt{2}}h^\nu_\rho A^\rho \right) & e^{-2\sqrt{2}\beta\kappa\phi_0} \beta^2\phi^2 + e^{-2\sqrt{2}\alpha\kappa\phi_0} \frac{1}{2}A_\rho A^\rho \end{pmatrix}. \quad (\text{A.3.15})$$

As suggested by (A.3.8), we need to construct, with these ingredients, the following quantity

$$\begin{aligned} \hat{l}^{MN} - \frac{1}{2}\hat{h}^P_P\hat{h}^{MN} &= \frac{1}{2\pi L} \begin{pmatrix} e^{-2\sqrt{2}\alpha\kappa\phi_0} \left(\alpha^2\phi^2\eta^{\mu\nu} + \sqrt{2}\alpha\phi h^{\mu\nu} + l^{\mu\nu} \right) & e^{-2\sqrt{2}\alpha\kappa\phi_0} \alpha\phi A^\mu + \frac{1}{\sqrt{2}}h^{\mu\rho}A_\rho \\ e^{-2\sqrt{2}\alpha\kappa\phi_0} \alpha\phi A^\nu + \frac{1}{\sqrt{2}}h^\nu_\rho A^\rho & e^{-2\sqrt{2}\beta\kappa\phi_0} \beta^2\phi^2 + e^{-2\sqrt{2}\alpha\kappa\phi_0} \frac{1}{2}A_\rho A^\rho \end{pmatrix} \\ &\quad - \frac{\frac{D}{\sqrt{2}}\alpha\phi + \frac{h^P_P}{2} + \frac{\beta\phi}{\sqrt{2}}\beta\phi}{2\pi L} \begin{pmatrix} e^{-2\sqrt{2}\alpha\kappa\phi_0} \left(\sqrt{2}\alpha\phi\eta^{\mu\nu} + h^{\mu\nu} \right) & e^{-2\sqrt{2}\alpha\kappa\phi_0} \frac{A^\mu}{\sqrt{2}} \\ e^{-2\sqrt{2}\alpha\kappa\phi_0} \frac{A^\nu}{\sqrt{2}} & e^{-2\sqrt{2}\beta\kappa\phi_0} \sqrt{2}\beta\phi \end{pmatrix} \end{aligned} \quad (\text{A.3.16})$$

and we get

$$\frac{e^{-2\sqrt{2}\alpha\kappa\phi_0}}{2\pi L} \begin{pmatrix} -\alpha^2\phi^2\eta^{\mu\nu} - \frac{\alpha}{\sqrt{2}}\phi h^\rho_\rho\eta^{\mu\nu} + l^{\mu\nu} - \frac{h^\rho_\rho h^{\mu\nu}}{2} & \frac{1}{\sqrt{2}} \left(h^{\mu\rho}A_\rho - \frac{h^\rho_\rho}{2}A^\mu \right) \\ \frac{1}{\sqrt{2}} \left(h^\nu_\rho A^\rho - \frac{h^\rho_\rho}{2}A^\nu \right) & e^{-2\sqrt{2}(\beta-\alpha)\kappa\phi_0} \alpha(D(D-2)\alpha\phi^2 + \sqrt{2}h^\rho_\rho\phi) + \frac{A_\rho A^\rho}{2} \end{pmatrix}. \quad (\text{A.3.17})$$

For the second term of the second order expression of (A.3.8), we have

$$\begin{aligned} \hat{f}^P_P - \frac{1}{2}\hat{h}_{MP}\hat{h}^{PM} + \frac{1}{4}(\hat{h}^P_P)^2 &= D\alpha^2\phi^2 + \sqrt{2}\alpha\phi h^\mu_\mu + f^\mu_\mu + \frac{1}{2}e^{2\sqrt{2}(\beta-\alpha)\kappa\phi_0}A^2 + \beta^2\phi^2 \\ &\quad - \frac{1}{2} \left((2D\alpha^2 + \beta^2)\phi^2 + 2\sqrt{2}\alpha\phi h^\mu_\mu + h^{\mu\nu}h_{\mu\nu} + e^{2\sqrt{2}(\beta-\alpha)\kappa\phi_0}A^2 \right) \\ &\quad + \frac{1}{4} \left(2(D\alpha + \beta)^2\phi^2 + (h^\mu_\mu)^2 + 2\sqrt{2}(D\alpha + \beta)\phi h^\mu_\mu \right) \end{aligned} \quad (\text{A.3.18})$$

$$= f_\mu^\mu - \frac{1}{2} h^{\mu\nu} h_{\mu\nu} + \frac{1}{4} (h_\mu^\mu)^2 + \frac{1}{2} (D\alpha + \beta)^2 \phi^2 + \frac{1}{\sqrt{2}} (D\alpha + \beta) \phi h_\mu^\mu.$$

Summing up the two second order contributions with the Φ derivative in (A.3.8), one obtains

$$\begin{aligned} & \frac{1}{2\pi L} \left\{ e^{-2\sqrt{2}\alpha\kappa\phi_0} \left(-\alpha^2 \phi^2 \eta^{\mu\nu} - \frac{\alpha}{\sqrt{2}} \phi h_\rho^\rho \eta^{\mu\nu} + l^{\mu\nu} - \frac{h_\rho^\rho h^{\mu\nu}}{2} \right) \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi \right. \\ & + \sqrt{2} e^{-2\sqrt{2}\alpha\kappa\phi_0} \left(h^{\mu\rho} A_\rho - \frac{1}{2} h_\rho^\rho A^\mu \right) \frac{1}{2} \partial_\mu \Phi \partial_z \Phi \\ & + \left(e^{-2\sqrt{2}\beta\kappa\phi_0} (D(D-2)\alpha^2 \phi^2 + \sqrt{2}\alpha h_\rho^\rho \phi) + \frac{1}{2} e^{-2\sqrt{2}\alpha\kappa\phi_0} A_\rho A^\rho \right) \frac{1}{2} (\partial_z \Phi)^2 \left. \right\} \\ & + \frac{1}{2} \left(f_\mu^\mu - \frac{1}{2} h^{\mu\nu} h_{\mu\nu} + \frac{1}{4} (h_\mu^\mu)^2 + \frac{1}{2} (D\alpha + \beta)^2 \phi^2 + \frac{1}{\sqrt{2}} (D\alpha + \beta) \phi h_\mu^\mu \right) \times \\ & \left(e^{-2\sqrt{2}\alpha\kappa\phi_0} \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + e^{-2\sqrt{2}\beta\kappa\phi_0} \frac{1}{2} (\partial_z \Phi)^2 \right) \end{aligned} \quad (\text{A.3.19})$$

We now add to this the contribution given by the determinant. Using (A.2.4), integrating over the z dimension and defining $J_{\mu,n} = (\varphi_n \partial_\mu \varphi_n^* - \varphi_n^* \partial_\mu \varphi_n)$ the second order interaction is obtained

$$\begin{aligned} & \frac{1}{2} \partial_\mu \varphi_0 \partial_\nu \varphi_0 \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} + \frac{1}{2} (D^2 \alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2) \phi^2 \right. \right. \\ & \left. \left. + \frac{1}{2} ((D-2)\alpha + \beta) \phi h_\rho^\rho \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2} h_\rho^\rho h^{\mu\nu} \right] \\ & + \sum_{n=1}^{\infty} \partial_\mu \varphi_n \partial_\nu \varphi_n^* \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} + \frac{1}{2} (D^2 \alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2) \phi^2 \right. \right. \\ & \left. \left. + \frac{1}{2} ((D-2)\alpha + \beta) \phi h_\rho^\rho \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2} h_\rho^\rho h^{\mu\nu} \right] \quad (\text{A.3.20}) \\ & - \sum_{n=1}^{\infty} \frac{n^2}{L^2} |\varphi_n|^2 \left[A^2 + e^{\sqrt{2}(D\alpha-\beta)\kappa\phi_0} \left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} + \left(\frac{1}{2} (D\alpha + \beta) - \beta \right) \phi h_\rho^\rho \right. \right. \\ & \left. \left. + \frac{1}{2} (D^2 \alpha^2 + 2D\alpha\beta + \beta^2 - 4D\alpha\beta - 4\beta^2 + 4\beta^2) \phi^2 \right) \right] \\ & + \sum_{n=1}^{\infty} i \frac{n}{L} h^{\rho\sigma} A_\rho J_{\sigma,n} + i \frac{n}{L} A^\rho J_{\rho,n} \left(-\frac{h_\sigma^\sigma}{2} - ((D-2)\alpha + \beta) \phi \right) \end{aligned}$$

We can simplify the expression with the relation between β and α that we have already used to eliminate the exponential factor in front of the interaction term. In particular, this makes the terms in ϕ^2 and ϕh_ρ^ρ disappear, and one obtains

$$\begin{aligned} & \frac{1}{2} \partial_\mu \varphi_0 \partial_\nu \varphi_0 \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2} h_\rho^\rho h^{\mu\nu} \right] \quad (\text{A.3.21}) \\ & + \sum_{n=1}^{\infty} \partial_\mu \varphi_n \partial_\nu \varphi_n^* \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma} h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2} h_\rho^\rho h^{\mu\nu} \right] \end{aligned}$$

$$\begin{aligned}
 & - \sum_{n=1}^{\infty} \frac{n^2}{L^2} |\varphi_n|^2 e^{2\sqrt{2}(D-1)\alpha\kappa\phi_0} \left(\frac{f_\rho^\rho}{2} - \frac{h_{\rho\sigma}^{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \\
 & - \sum_{n=1}^{\infty} \frac{n^2}{L^2} |\varphi_n|^2 \left(A^2 + e^{2\sqrt{2}(D-1)\alpha\kappa\phi_0} (2(D-1)^2\alpha^2\phi^2 + (D-1)\alpha\phi h_\rho^\rho) \right) - i\frac{n}{L} A^\rho \frac{h_\sigma^\sigma}{2} J_{\rho,n} + i\frac{n}{L} h^{\rho\sigma} A_\rho J_{\sigma,n}.
 \end{aligned}$$

This calls for some remarks. The term which multiplies the $\frac{1}{2}\partial_\mu\varphi_0\partial_\nu\varphi_0$ is exactly the term obtained from the perturbation of the metric at second order. It corresponds then to the interaction term between two gravitons and two massless scalars. The same term is present for the φ_n field with an additional contribution from the mass term. We also see several terms of interaction between the A_μ fields, the graviton and the φ_n . There is no interaction between the gauge field and the φ_0 since the latter is not charged under the gauge symmetry. The interaction term between two gauge fields and two scalars fields was, at first order, the term missing in order to achieve the gauge invariance of the interaction. We might have expected to find an exponential factor in front of it. However, we should recall that in our previous derivation this exponential term appeared in front of the kinetic term of the A_μ , we should not search for it here. However, the expected exponential term is present in front of the interaction term between two dilatons ϕ and the mass term of φ_n . This computation of the interaction in the compactified theory gives the same result as the one we did in the section A.1.

APPENDIX B

Renormalizability in the language of the RG

In this appendix we study in full details the toy renormalization group equation proposed by Polchinski in [93]. This gives an account of how the question of perturbative renormalizability is seen in the larger sense of the RG.

These calculations were performed during a period of stay at the GGI in Florence with the invaluable help of other students, whom I thank in the appropriate section of this thesis.

B.1 Couplings and dimensions

Given a set of couplings g_α , corresponding to operators of dimension d_α in an effective field theory, we define their dimensionless version (in D dimensions) as

$$\tilde{\lambda}_k^{(\alpha)} = k^{d_\alpha - D} \lambda_k^{(\alpha)}, \quad (\text{B.1.1})$$

where k is the running scale.

We define the (dimensionless) β -functions associated to such couplings as

$$\beta_\alpha(\{\tilde{\lambda}_k\}) = k^{d_\alpha - D} \left(k \frac{d\tilde{\lambda}_k^{(\alpha)}}{dk} \right) \equiv e^{(d_\alpha - D)t} \left(\frac{d\tilde{\lambda}_t^{(\alpha)}}{dt} \right), \quad (\text{B.1.2})$$

having introduced the RG time $t \equiv \log(k/\Lambda)$, with Λ the cut-off. The scale dependence of the dimensionless couplings is given by the renormalization group equations

$$\frac{d\tilde{\lambda}_k^{(\alpha)}}{dt} = (d_\alpha - D) \lambda_t^{(\alpha)} + \beta_\alpha(\{\tilde{\lambda}_t\}). \quad (\text{B.1.3})$$

B.2 Two coupling theory

We consider now a theory in $D = 4$ dimensions with only two couplings (one relevant and one irrelevant): λ_k^4 , which is dimensionless, and λ_k^6 , which has dimension

of (mass)⁻². They correspond, respectively, to operators of dimension $d_4 = 4$ and $d_6 = 6$ in the Lagrangian. From B.1.3 we can obtain the evolution equations for the dimensionless couplings $\tilde{\lambda}_k^4 = \lambda_k^4$ and $\tilde{\lambda}_k^6 = k^{-2} \lambda_k^6$,

$$\begin{cases} \frac{d\tilde{\lambda}^4}{dt} = \beta_4(\tilde{\lambda}_4, \tilde{\lambda}_6) \\ \frac{d\tilde{\lambda}^6}{dt} = 2\tilde{\lambda}_6 + \beta_6(\tilde{\lambda}_4, \tilde{\lambda}_6) \end{cases}, \quad (\text{B.2.1})$$

where β -functions depend on the scale through the couplings and we have simplified the notation defining $\tilde{\lambda}_i \equiv \tilde{\lambda}_k^{(i)}$. Note that here we keep the notation generic, and do not specify to the β functions, and thus the RG equations, obtained in the Wilson approach. For a generic β function of a dimensionless coupling in D dimensions it is immediate to find the following identity

$$\frac{d\beta_i}{dt} = \sum_j \beta_{i,j} [(d_j - D) \lambda_j + \beta_j], \quad (\text{B.2.2})$$

where we have used the short notation

$$\beta_{i,j} \equiv \frac{\partial \beta_i}{\partial \tilde{\lambda}_j}. \quad (\text{B.2.3})$$

Take a particular solution of B.2.1, $(\bar{\lambda}_4, \bar{\lambda}_6)$, and consider small deviations $\epsilon_i = \tilde{\lambda}_i - \bar{\lambda}_i$. Then, expanding B.2.1 around $(\bar{\lambda}_4, \bar{\lambda}_6)$, one obtains at first order in ϵ_i

$$\begin{cases} d_t \epsilon_4 = \epsilon_4 \bar{\beta}_{4,4} + \epsilon_6 \bar{\beta}_{4,6} \\ d_t \epsilon_6 = 2 \epsilon_6 + \epsilon_4 \bar{\beta}_{6,4} + \epsilon_6 \bar{\beta}_{6,6} \end{cases}, \quad (\text{B.2.4})$$

where $d_t \equiv d/dt$ and $\bar{\beta}_{i,j} = \beta_{i,j}|_{(\bar{\lambda}_4, \bar{\lambda}_6)}$. The particular solution $(\bar{\lambda}_4, \bar{\lambda}_6)$ has been obtained from the flow equations after a certain RG time starting with a definite boundary condition. Different boundaries lead to different trajectories. Our goal is to study, for a fixed value $\lambda_4 \equiv \bar{\lambda}_4$, how much these are separated in the λ_6 direction.

To order ϵ , the vertical (λ_6) distance between the point $(\bar{\lambda}_4, \bar{\lambda}_6)$ and the corresponding point lying on a nearby trajectory in the $\lambda_4 - \lambda_6$ plane can be estimated as

$$\xi_6 \equiv \epsilon_6 - \epsilon_4 \left. \frac{d\lambda_6}{d\lambda_4} \right|_{\bar{\lambda}} = \epsilon_6 - \epsilon_4 \frac{d_t \bar{\lambda}_6}{d_t \bar{\lambda}_4} \equiv \epsilon_6 - \epsilon_4 \sigma. \quad (\text{B.2.5})$$

The slope σ in B.2.5 can be rewritten in a different form using B.2.1 evaluated at $(\bar{\lambda}_4, \bar{\lambda}_6)$, namely

$$\sigma \equiv \frac{d_t \bar{\lambda}_6}{d_t \bar{\lambda}_4} = \frac{2 \bar{\lambda}_6}{\bar{\beta}_4} + \frac{\bar{\beta}_6}{\bar{\beta}_4}, \quad (\text{B.2.6})$$

where $\bar{\beta}_i \equiv \beta_i(\bar{\lambda}_4, \bar{\lambda}_6)$. In order to study the scale dependence of the separation ξ_6 , we explicitly compute the quantity $\star \equiv d_t \xi_6 - 2 \xi_6$. Making use of Eqs. B.2.4 and Eqs. B.2.1 evaluated in $\bar{\lambda}$, one obtains

$$\star = d_t \epsilon_6 - (d_t \epsilon_4) \sigma - \epsilon_4 (d_t \sigma) - 2 \epsilon_6 + 2 \epsilon_4 \sigma \quad (\text{B.2.7})$$

$$= \epsilon_4 \bar{\beta}_{6,4} + \epsilon_6 \bar{\beta}_{6,6} - \epsilon_4 \bar{\beta}_{4,4} \sigma - \epsilon_6 \bar{\beta}_{4,6} \sigma - \epsilon_4 (d_t \sigma) + 2 \epsilon_4 \sigma \quad (\text{B.2.8})$$

and

$$d_t \sigma = d_t \left(\frac{2 \bar{\lambda}_6}{\bar{\beta}_4} + \frac{\bar{\beta}_6}{\bar{\beta}_4} \right) = \frac{2 (d_t \bar{\lambda}_6)}{\bar{\beta}_4} - 2 \frac{\bar{\lambda}_6}{\bar{\beta}_4} \frac{d_t \bar{\beta}_4}{\bar{\beta}_4} + \frac{d_t \bar{\beta}_6}{\bar{\beta}_4} - \frac{\bar{\beta}_6}{\bar{\beta}_4} \frac{d_t \bar{\beta}_4}{\bar{\beta}_4} \quad (\text{B.2.9})$$

$$= \frac{2 (d_t \bar{\lambda}_6)}{(d_t \bar{\lambda}_4)} - \frac{2 \bar{\lambda}_6}{\bar{\beta}_4} L_4 + \frac{\bar{\beta}_6}{\bar{\beta}_4} L_6 - \frac{\bar{\beta}_6}{\bar{\beta}_4} L_4 \quad (\text{B.2.10})$$

$$= 2 \sigma - \sigma L_4 + \frac{\bar{\beta}_6}{\bar{\beta}_4} L_6, \quad (\text{B.2.11})$$

where we have defined

$$L_i \equiv d_t \log(\bar{\beta}_i) = \frac{d_t \bar{\beta}_i}{\bar{\beta}_i} \quad (\text{B.2.12})$$

and used the definition of the slope σ in Eq. B.2.6. Therefore,

$$\begin{aligned} \star &= \epsilon_4 \left\{ \bar{\beta}_{6,4} - \bar{\beta}_{4,4} \sigma - 2 \sigma + \sigma L_4 - \frac{\bar{\beta}_6}{\bar{\beta}_4} L_6 + 2 \sigma \right\} + \\ &\quad \epsilon_6 \left\{ \bar{\beta}_{6,6} - \bar{\beta}_{4,6} \sigma \right\}. \end{aligned} \quad (\text{B.2.13})$$

Note that, using B.2.2 one obtains

$$L_i = \frac{1}{\bar{\beta}_i} \sum_j \bar{\beta}_{i,j} [\bar{\beta}_j + (d_j - 4) \bar{\lambda}_j] \quad (\text{B.2.14})$$

and hence expliciting L_6 we get

$$\begin{aligned} \star &= \epsilon_4 \left\{ \bar{\beta}_{6,4} - \bar{\beta}_{4,4} \sigma + \sigma L_4 - \bar{\beta}_{6,4} - \frac{\bar{\beta}_6}{\bar{\beta}_4} \bar{\beta}_{6,6} - \frac{2 \bar{\lambda}_6}{\bar{\beta}_4} \bar{\beta}_{6,6} \right\} + \\ &\quad \epsilon_6 \left\{ \bar{\beta}_{6,6} - \bar{\beta}_{4,6} \sigma \right\} \\ &= \epsilon_4 \sigma \left\{ L_4 - \bar{\beta}_{4,4} \right\} - \epsilon_4 \sigma \bar{\beta}_{6,6} + \epsilon_6 \left\{ \bar{\beta}_{6,6} - \sigma \bar{\beta}_{4,6} \right\} \\ &= \epsilon_6 \left\{ L_4 - \bar{\beta}_{4,4} \right\} - \xi_6 \left\{ L_4 - \bar{\beta}_{4,4} \right\} + \bar{\beta}_{6,6} [\epsilon_6 - \epsilon_4 \sigma] - \epsilon_6 \sigma \bar{\beta}_{4,6} \\ &= \epsilon_6 \left\{ L_4 - \bar{\beta}_{4,4} - \sigma \bar{\beta}_{4,6} \right\} + \xi_6 \left\{ \bar{\beta}_{4,4} + \bar{\beta}_{6,6} - L_4 \right\} \\ &= \epsilon_6 \left\{ \bar{\beta}_{4,4} + \frac{\bar{\beta}_6}{\bar{\beta}_4} \bar{\beta}_{4,6} + \frac{2 \bar{\lambda}_6}{\bar{\beta}_4} \bar{\beta}_{4,6} - \bar{\beta}_{4,4} - \sigma \bar{\beta}_{4,6} \right\} + \xi_6 \left\{ \bar{\beta}_{4,4} + \bar{\beta}_{6,6} - L_4 \right\} \\ &= \left\{ \bar{\beta}_{4,4} + \bar{\beta}_{6,6} - L_4 \right\} \xi_6. \end{aligned} \quad (\text{B.2.15})$$

We finally obtain

$$d_t \xi_6 = 2 \xi_6 + (\bar{\beta}_{4,4} + \bar{\beta}_{6,6} - d_t \log(\bar{\beta}_4)) \xi_6. \quad (\text{B.2.16})$$

The differential equation can be rewritten as

$$d_t \xi_6 = \Phi \xi_6, \quad \Phi = 2 + \bar{\beta}_{4,4} + \bar{\beta}_{6,6} - d_t \log(\bar{\beta}_4), \quad (\text{B.2.17})$$

whose solution is given by

$$\log \xi_6 \Big|_k^\Lambda = \int_k^\Lambda \frac{d\Lambda'}{\Lambda'} \Phi(\Lambda'), \quad (\text{B.2.18})$$

$$\xi_6(k) = \xi_6(\Lambda) \exp \left\{ \int_{\Lambda}^k \frac{d\mu}{\mu} (2 + \bar{\beta}_{4,4} + \bar{\beta}_{6,6} - \mu d_{\mu} \log(\bar{\beta}_4)) \right\}, \quad (\text{B.2.19})$$

$$\xi_6(k) = \xi_6(\Lambda) \left(\frac{k}{\Lambda} \right)^2 \left(\frac{\bar{\beta}_4(\Lambda)}{\bar{\beta}_4(k)} \right) \exp \left\{ \int_{\Lambda}^k \frac{d\mu}{\mu} (\bar{\beta}_{4,4} + \bar{\beta}_{6,6}) \right\}. \quad (\text{B.2.20})$$

We can therefore conclude that, as long as the integrand remains small and the coupling $\bar{\lambda}_4$ runs sufficiently slowly, the parameter ξ_6 is suppressed in the IR limit $\kappa^2 \ll \Lambda^2$ by inverse powers of the cut-off $(k/\Lambda)^2$. It follows that the coupling λ_6 is an irrelevant parameter. Indeed, as the boundary conditions at Λ are varied in a two-dimensional $\{\lambda_4, \lambda_6\}$ space, there is an enormous focusing on the IR such that the theory lies very close to a one-dimensional subspace. Whatever the boundary condition, at a fixed value of λ_4 the value of λ_6 is known for all the trajectories to accuracy $(k/\Lambda)^2$. This is how the usual arguments about renormalizability and the suppression of non-renormalizable couplings in the IR find a deeper explanation in the language of the Renormalization Group.

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Résumé: Quelle que soit la théorie microscopique fondamentale qui décrit l'Univers, à énergies suffisamment basses celle-ci doit être bien décrite par une théorie effective des champs, c'est à dire une théorie quantique des champs valable jusqu'à une certaine énergie maximale. C'est ici que de nouveaux degrés de libertés doivent se manifester et prendre le relais. Dans ce cadre, même en absence d'une théorie quantique de la gravité, il est possible d'en étudier certaines propriétés de faibles énergies. A cet égard, à partir d'aspects connus de la gravité quantique, le "Swampland" a pour but celui d'individuer comment ceux-ci se traduisent en contraintes pour la théorie des champs qui en donne une description effective à faibles énergies. Dans cette thèse, nous allons étudier une de ces contraintes connue comme la "Weak Gravity Conjecture", qui, dans sa forme d'origine, non seulement traduit l'idée que la gravité soit plus faible que les interactions électromagnétiques aux énergies dont nous faisons expérience, mais permet aussi que tous les trous noirs de Reissner-Nordström se désintègrent. Je montrerais comment il est possible d'étendre ceci à d'autres situations. En particulier, nous allons voir selon quelle logique il est possible de contraindre les interactions scalaires pour qu'elles soient dominantes par rapport à la gravité, et quelles contraintes posent la désintégration de trous noirs ayant une charge scalaire. Une prédiction de la conjecture pour les théories avec des bosons de jauge faiblement couplés au photon sera aussi discutée. Dans le domaine de validité de la théorie effective, la forme que celle-ci prend à une certaine énergie est reliée à celle qu'elle a à une autre énergie à travers le flux décrit par le groupe de renormalisation. Dans une deuxième partie de la thèse, je montrerais que la formulation du groupe de renormalisation donnée par Wilson peut nous apporter un point de vue différent sur le problème de "naturalness", notamment liée à la forte sensibilité des masses scalaires aux détails de la théorie d'hautes énergies qui complète la théorie effective. Une approche complémentaire au Swampland basée sur les résultats expérimentaux ressortira naturellement.

Abstract: Whatever the fundamental, microscopic, theory that describes the Universe is, at sufficiently low energies it should be well described by an effective field theory, i.e. a quantum field theory valid up to a certain maximal energy. It is at this energy scale that new degrees of freedom should manifest and take over. In this framework, even in the absence of a fundamental quantum theory of gravity, it is possible to study some of its low energy properties. Starting from known aspects of quantum gravity, the Swampland program aims to describe how those are translated into constraints on the quantum field theory that gives an effective description at low energies. In this thesis, we will study one such constraint, known as the Weak Gravity Conjecture, that, in its original formulation, not only translates the idea that gravity should be weaker than the electromagnetic interactions at the energy scale we usually experience, but also allows for the decay of all Reissner-Nordström black holes. I will show how to extend the conjecture to other setups. In particular, we will see what rationale can be used to constrain scalar interactions to be dominant with respect to gravity, and what kind of constraint does the decay of black holes with scalar charges impose. A prediction of the conjecture for a theory with gauge bosons weakly coupled to the Standard Model photon will be discussed. In the domain of validity of the effective field theory, the value of its parameters at a certain energy scale is related to the value at another scale through the evolution described by the Renormalization Group. In the second part of the thesis, I will show that the Wilson formulation of the Renormalization Group can provide a new way to look at the naturalness/hierarchy problem, related to the sensitivity of scalar masses to the details of the high energy theory that completes the Standard Model. A complementary approach to the Swampland based on the results of experiments will naturally emerge.