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Normalized solutions for Sobolev critical Schrödinger equations: existence, multiplicity and stability issues

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École Doctorale Carnot-Pasteur

THÈSE DE DOCTORAT

Présentée par:

Thanh Trung Le

en vue de l'obtention du grade de Docteur de l'Université de Bourgogne Franche-Comté

Spécialité: Mathématiques

**Normalized solutions for Sobolev critical Schrödinger equations:
existence, multiplicity and stability issues**

Solutions normalisées pour des équations de Schrödinger Sobolev
critique: existence, multiplicité et stabilité

Soutenue publiquement le 13 octobre 2022 devant le jury composé de:

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Résumé

Dans cette thèse, nous considérons deux types d'équations de Schrödinger non linéaires (NLS), à savoir une classe d'équations de Schrödinger non linéaire avec une non linéarité de type *mixed powers* sur \mathbb{R}^N et une classe d'équations non linéaires de Schrödinger-Poisson-Slater sur \mathbb{R}^3 . Ces deux types de NLS apparaissent dans divers modèles mathématiques et physiques et ont attiré beaucoup d'attention ces dernières années.

Du point de vue physique, puisque, en plus d'être une quantité conservée pour l'équation de l'évolution, la masse a souvent une signification physique claire; par exemple, elle représente l'alimentation électrique en optique non linéaire, ou le nombre total d'atomes dans la condensation de Bose-Einstein, etc., nous nous concentrons sur l'étude des solutions ayant une masse prescrite, à savoir les solutions normalisées. Des questions d'existence, de multiplicité et de stabilité de ces solutions sont examinées dans cette thèse. Nous nous occupons à la fois de cas sous-critiques de Sobolev et de cas critiques de Sobolev. Une attention particulière est accordée aux cas critiques de Sobolev dans lesquels de nombreux problèmes restent ouverts. Puisque les solutions normalisées sont obtenues comme points critiques, sous contrainte, d'une fonctionnelle, les principaux ingrédients de nos preuves sont variationnels.

La thèse se compose de quatre chapitres. Le Chapitre 1 est une introduction à cette thèse contenant une brève présentation des questions traitées et des résultats obtenus. Dans le Chapitre 2, nous étudions une classe d'équations non linéaires de Schrödinger sur \mathbb{R}^N avec une nonlinéarité mixte Sobolev critique. Dans une situation où la fonctionnelle associée est non bornée inférieurement sur la contrainte, nous prouvons l'existence de deux points critiques sur la contrainte, un minimiseur local, et un point selle se trouvant au niveau d'un col de montagne. Nous montrons également que les ondes stationnaires associées à l'ensemble des minimiseurs locaux sont orbitalement stables et que celles associées aux point selles situés au niveau du col sont fortement instables. La principale difficulté est la présence de l'exposant critique de Sobolev. En ce qui concerne les minimiseurs locaux, il n'est pas possible d'utiliser de manière standard le principe de concentration par compacité développé par P. L. Lions. Par ailleurs même en ayant obtenu la compacité de l'ensemble des suites minimisantes, l'existence globale du problème d'évolution associée reste à montrer pour établir la stabilité orbitale. En ce qui concerne le point selle, nous avons besoin d'une estimation stricte sur le niveau du col associé et celle-ci est obtenue en utilisant des fonctions de test.

Dans le Chapitre 3 nous étudions une classe d'équations de Schrödinger-Poisson-Slater sur \mathbb{R}^3 . Nous considérons plusieurs classes de paramètres, certains impliquant que la fonctionnelle sera non bornée inférieurement sur la contrainte. Dans le cas où la structure géométrique des fonctionnelles associées suggère l'existence de minima locaux ou globaux, nous développons un argument pour traiter simultanément les cas sous-critiques de Sobolev et

les cas critiques de Sobolev. Dans le cas où la structure géométrique des fonctionnelles associées suggère l'existence d'un point selle, nous avons besoin de deux arguments distincts pour traiter les cas Sobolev sous-critique et Sobolev critique. Enfin, au Chapitre 4, nous présentons quelques remarques finales sur les deux équations examinées dans cette thèse et aussi nous proposons quelques problèmes ouverts.

Mots clefs

Equations de Schrödinger, équations de Schrödinger-Poisson-Slater, exposant critique de Sobolev, masse prescrite, solution normalisée, multiplicité de solutions, état fondamental, stabilité orbitale, instabilité forte par blow-up, minimiseur local ou global, point selle au niveau du col, méthodes variationnelles, identité de type Pohozaev.

Abstract

In this thesis, we consider two types of nonlinear Schrödinger equations (NLS), namely a class of nonlinear Schrödinger equations with mixed power nonlinearities in \mathbb{R}^N and a class of Schrödinger-Poisson-Slater equations in \mathbb{R}^3 . These two types of NLS arise in various mathematical and physical models and have drawn wide attention in recent years.

From the physical point of view, since, in addition to being a conserved quantity for the evolution equation, the mass often has a clear physical meaning; for instance, it represents the power supply in nonlinear optics, or the total number of atoms in Bose-Einstein condensation, etc, we focus on studying solutions having prescribed mass, namely normalized solutions. The existence, multiplicity, and stability issues of such solutions are considered in this thesis. We deal with both Sobolev sub-critical and Sobolev critical cases. Particular attention is paid to Sobolev critical cases in which many open problems remain. Since normalized solutions are found as critical points of an associated functional on a constraint, the main ingredients of our proofs are variational methods.

The thesis consists of four chapters. Chapter 1 is an introduction to this thesis containing a brief presentation of issues treated and obtained results. In Chapter 2, we study Sobolev critical nonlinear Schrödinger equations with mixed power nonlinearities in \mathbb{R}^N . In a situation where the associated functional is unbounded from below on the constraint, we prove the existence of two constrained critical points, one local minimizer, and one saddle point lying at a mountain pass level. We also show that the standing waves associated with the local minimizer are orbitally stable and the associated standing waves corresponding with saddle points lying at mountain pass levels are strongly unstable. The main difficulty is the presence of the Sobolev critical exponent. Concerning the local minimizer, it is not possible to use in a standard way the compactness by concentration approach developed by P. L. Lions. Even having the compactness, the global existence in evolution is still unknown. For the existence of the saddle point, we need a strict upper estimate of the associated mountain pass level that we derive using testing functions.

In Chapter 3, we study Schrödinger-Poisson-Slater equations in \mathbb{R}^3 . We deal with some range of parameters under which the associated functional restricted on the constraint will sometimes be bounded, sometimes be unbounded. In the case where the geometric structure of the associated functional suggests the existence of local minima or global minima, we develop an argument to deal with both Sobolev sub-critical and Sobolev critical cases. In the case where the geometric structure of the associated functional suggests the existence of a saddle point, we need two different arguments to deal with Sobolev sub-critical and Sobolev critical cases. Finally, in Chapter 4, we present some concluding remarks about the two equations considered in this thesis and also we propose some open problems.

Keywords

Nonlinear Schrödinger equation, Schrödinger-Poisson-Slater equation, Sobolev critical exponent, prescribed mass, normalized solution, multiplicity of solutions, ground states, orbital stability, strong instability by blow-up, local or global minimizer, saddle point lying at mountain pass level, variational methods, Pohozaev type identity.

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Notations

We list below the notations that we use throughout the thesis:

- (1) \mathbb{R} , \mathbb{C} denote respectively the set of real numbers, complex numbers. \mathbb{R}_+ denotes the interval $(0, \infty)$.
- (2) \mathbb{R}^N denotes the N -dimensional real Euclidean space, and the typical point in \mathbb{R}^N is $x = (x_1, x_2, \dots, x_N)$.
- (3) ∇u denotes the gradient of a differentiable function u , namely $\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \dots, \partial_{x_N} u)$.
- (4) Δu denotes the Laplacian operator of a twice-differentiable function u , namely

$$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}.$$

- (5) 2^* denotes the critical exponent of the Sobolev embedding, namely

$$2^* = \frac{2N}{N-2} \text{ if } N \geq 3 \text{ and } 2^* = +\infty \text{ if } N = 1, 2.$$

- (6) 2_α and 2_α^* denote lower and upper critical exponents respectively that come from the Hardy-Littlewood-Sobolev inequality (see [63, Chapter 4]), namely

$$2_\alpha := \frac{N+\alpha}{N}, \quad 2_\alpha^* := \begin{cases} +\infty & \text{if } N = 1, 2, \\ \frac{N+\alpha}{N-2} & \text{if } N \geq 3 \end{cases} \quad \text{with } \alpha \in (0, N).$$

- (7) $L^\infty(\mathbb{R}^N)$ denotes the set of almost everywhere bounded functions defined in \mathbb{R}^N .
- (8) For any $1 \leq q < \infty$, we write $L^q(\mathbb{R}^N)$ as the usual Lebesgue space endowed with the norm

$$\|u\|_{L^q(\mathbb{R}^N)}^q := \int_{\mathbb{R}^N} |u|^q dx.$$

- (9) $H^1 := H^1(\mathbb{R}^N, \mathbb{R})$ denotes the usual Hilbert space of functions defined from \mathbb{R}^N into \mathbb{R} with the norm

$$\|u\|_{H^1(\mathbb{R}^N)} = \|\nabla u\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)},$$

and $H := H^1(\mathbb{R}^N, \mathbb{C})$ denotes the Hilbert space of functions defined from \mathbb{R}^N into \mathbb{C} .

(10) \mathcal{S} denotes the best constant in the Sobolev inequality, see (2.2.1).

(11) For convenience, we denote for $u \in H^1(\mathbb{R}^3)$ the following quantities

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy, \quad C(u) := \int_{\mathbb{R}^3} |u|^p dx.$$

(12) The notation $a \sim b$ means that $Cb \leq a \leq C'b$ for some $C, C' > 0$.

(13) The open ball in \mathbb{R}^3 with center at 0 and radius $R > 0$ is denoted by B_R .

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Chapter 1

Introduction

The thesis which collects some works obtained during my Ph.D is devoted to the study of normalized solutions for two types of nonlinear elliptic partial differential equations, namely a class of nonlinear Schrödinger equations in \mathbb{R}^N and a class of Schrödinger-Poisson-Slater equations in \mathbb{R}^3 . [Chapter 2](#) corresponds to the paper [\[46\]](#) in collaboration with Louis Jeanjean, Jacek Jendrej (Université Sorbonne Paris Nord) and Nicola Visciglia (Università Degli Studi di Pisa) and the paper [\[48\]](#) in collaboration with Louis Jeanjean. [Chapter 3](#) corresponds to the work with Louis Jeanjean in the paper [\[47\]](#).

1.1 Normalized solutions for a Sobolev critical nonlinear Schrödinger equation

In [Chapter 2](#), we study the existence and stability of standing waves of prescribed mass for the time-dependent nonlinear Schrödinger equations with mixed power nonlinearities

$$i\partial_t v + \Delta v + \mu v|v|^{q-2} + v|v|^{2^*-2} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (1.1.1)$$

where $N \geq 3$, $v : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $\mu > 0$, $2 < q < 2 + \frac{4}{N}$ and $2^* = \frac{2N}{N-2}$.

The nonlinear Schrödinger equation (NLS) with pure and mixed power nonlinearities has attracted much attention in the last decades. The local existence result for the pure power energy critical NLS has been established in [\[29\]](#). The corresponding global existence and scattering for defocusing quintic NLS in dimension $N = 3$ has been established in the papers [\[22, 33\]](#) respectively in the radial and non-radial case. We also quote the concentration-compactness/rigidity approach introduced in [\[54\]](#) in order to study global existence and scattering in the focusing energy critical NLS below the ground state. Concerning the case of NLS with mixed nonlinearities let us quote [\[1, 2, 30, 32, 57, 68, 69, 84\]](#).

We recall that standing waves to [\(1.1.1\)](#) are solutions of the form $v(t, x) = e^{-i\lambda t} u(x)$, $\lambda \in \mathbb{R}$. Then the function $u(x)$ satisfies the equation

$$-\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N. \quad (1.1.2)$$

When looking for solutions to [\(1.1.2\)](#) a possible choice is to consider $\lambda \in \mathbb{R}$ fixed and to search for solutions as critical points of the action functional

$$\mathcal{A}_{\lambda, \mu}(u) := \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\lambda}{2} \|u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

In this case one usually focus on the existence of minimal action solutions, namely of solutions minimizing $\mathcal{A}_{\lambda,\mu}$ among all non-trivial solutions. In that direction, we refer to [6] where, relying on the pioneering work of Brezis-Nirenberg [25], the existence of positive real solutions for equations of the type of (1.1.2) is addressed in a very general setting; to [1, 2] which concerns the case where $q > 2 + 4/N$ and $\mu > 0$; to [30, 68] where the fixed $\lambda \in \mathbb{R}$ problem is analyzed for $q = 2 + 4/N$ and $\mu < 0$; see also [57] and the reference therein.

Alternatively, we can search for solutions to (1.1.2) having a prescribed L^2 -norm. Defining on $H := H^1(\mathbb{R}^N, \mathbb{C})$ the Energy functional

$$F_\mu(u) := \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

it is standard to check that F_μ is of class C^1 and that a critical point of F_μ restricted to the (mass) constraint

$$S(c) := \{u \in H : \|u\|_{L^2(\mathbb{R}^N)}^2 = c\}$$

gives rise to a solution to (1.1.2), satisfying $\|u\|_{L^2(\mathbb{R}^N)}^2 = c$.

In this approach the parameter $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier. In particular, $\lambda \in \mathbb{R}$ does depend on the solution and is not a priori given. This approach, that we shall follow here, is relevant from the physical point of view, in particular, since the L^2 norm is a preserved quantity of the evolution and since the variational characterization of such solutions is often a strong help to analyze their orbital stability, see for example, [15, 28, 80, 81].

Let us define

$$m(c) := \inf_{u \in S(c)} F_\mu(u).$$

Depending on the range of parameters we shall consider $m(c)$ will be finite or not. If, following the introduction of the Compactness by Concentration Principle of P. L. Lions [64, 65], the search of normalized solutions corresponding to a global minimizer of a functional restricted to an L^2 norm constraint is now a classical topic, the search of critical points when the functional is unbounded from below on the constraint remained for a long time much less studied. In the frame of this thesis, namely for a functional corresponding to an autonomous equation lying on all the space \mathbb{R}^N , [45] was for a long time the sole contribution. This direction of research was likely brought to the attention of the community by the papers [8, 15] both published in 2013. Since then numerous contributions flourished within this topic and we just mention, among many possible choices, the works, [9, 11, 12, 14, 21, 31, 41, 50]. We also refer to [10, 70] for non-autonomous problems set on \mathbb{R}^N and to [73, 75, 76] for contributions when the underlying equation is set on a bounded domain of \mathbb{R}^N . In the above-mentioned papers, the involved nonlinearities were Sobolev subcritical. It was only in 2020 that was first treated in [81] a problem involving a Sobolev critical nonlinearity.

For future reference, we now recall some definitions:

Definition 1.1.1. *We say that $u_c \in S(c)$ is a ground state solution to (1.1.2) if it is a solution having minimal Energy among all the solutions which belong to $S(c)$. Namely, if*

$$F_\mu(u_c) = \inf \left\{ F_\mu(u), u \in S(c), \left(F_\mu|_{S(c)} \right)'(u) = 0 \right\}.$$

Definition 1.1.2. $Z \subset H$ is stable if: $Z \neq \emptyset$ and for any $v \in Z$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\varphi \in H$ satisfies $\|\varphi - v\|_H < \delta$ then $u_\varphi(t)$ is globally defined and $\inf_{z \in Z} \|u_\varphi(t) - z\|_H < \varepsilon$ for all $t \in \mathbb{R}$, where $u_\varphi(t)$ is the solution to (1.1.1) corresponding to the initial condition φ .

A standing wave $e^{-i\lambda t}u(x)$ is strongly unstable if for every $\varepsilon > 0$ there exist $\varphi \in H$ such that $\|u - \varphi\|_H < \varepsilon$, and $u_\varphi(t)$ blows-up in finite time.

It is well-known that the study of problems with mixed nonlinearities and the type of results one can expect, depend on the behavior of the nonlinearities at infinity, namely on the value of the various power exponents. In particular, this behavior determines whether the functional is bounded from below on $S(c)$. One speaks of a mass subcritical case if it is bounded from below on $S(c)$ for any $c > 0$, and of a mass supercritical case if the functional is unbounded from below on $S(c)$ for any $c > 0$. One also refers to a mass critical case when the boundedness from below does depend on the value $c > 0$. To be more precise, consider an equation of the form

$$i\partial_t v + \Delta v + \mu v|v|^{p_1-2} + v|v|^{p_2-2} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (1.1.3)$$

where it is assumed that $2 < p_1 \leq p_2 \leq 2^*$. The threshold exponent is the so-called L^2 -critical exponent

$$p_c = 2 + \frac{4}{N}.$$

A very complete analysis of the various cases that may happen for (1.1.3), depending on the values of (p_1, p_2) , has been provided recently in [81]. Let us just recall here some rough elements. If both p_1 and p_2 are strictly less than p_c then the associated Energy functional is bounded from below on $S(c)$ and to find a ground state (see Definition 1.1.1) one looks for a global minimum on $S(c)$. The problem then directly falls into the setting covered by the Compactness by Concentration Principle introduced by P.L. Lions [64, 65] which, for more complicated equations, in particular non autonomous ones, is still a very active field. Such solutions are expected to be orbitally stable, see Definition 1.1.2. If $p_c \leq p_1 \leq p_2 \leq 2^*$, then the Energy functional is unbounded from below on $S(c)$ but it is possible to show that a ground state exists. This ground state is characterized as a critical point of *mountain-pass type* and it lies at a strictly positive level of the Energy functional. Such ground states are expected to be strongly unstable, see Definition 1.1.2. We refer, for the link between the variational characterization of a solution and its instability, to the classical paper [20], and to [45, 56, 80, 81] for more recent developments.

In the case we consider here: $2 < p_1 < p_c < p_2 = 2^*$, the Energy functional is thus unbounded from below on $S(c)$ but, as we shall see, the presence of the lower order, mass subcritical term $-\mu\|u\|_{L^q(\mathbb{R}^N)}^q$ created, for sufficiently small values of $c > 0$, a geometry of local minima on $S(c)$. The presence of such geometry, in problems which are mass supercritical, had already been observed in several related situations. In [13, 14] for related scalar problems, in [40] in the case of a system or [73] for an evolution problem set on a bounded domain. Actually, Soave studied the equation (1.1.3) in [81] and derived, for any small $c > 0$ depending on $\mu > 0$, the existence of a ground state solution as a local minimizer to (1.1.3) in the case $2 < p_1 < p_c < p_2 = 2^*$. Soave also raised two open problems:

1. If a set $Z \subset H$ denotes all ground states to (1.1.3), is Z orbitally stable? In the subcritical Sobolev case ($p_2 < 2^*$), the set Z is orbitally stable, see [80], however, this was an open problem in the Sobolev critical case ($p_2 = 2^*$). As already explained in [81],

to obtain the orbital stability of the set Z , following the classical approach laid down in [28], two ingredients are essential. First, the relative compactness, up to translation, of all minimizing sequences for F_μ on $V(c)$. Secondly, the global existence of solutions to (1.1.1) for initial data close to Z . It does not seem possible to use [81, Theorem 1.1] as a starting point to prove the orbital stability of Z , since the existence of a ground state in [81, Theorem 1.1] is obtained through the study of one particular (locally) minimizing sequence which is radially symmetric and hence, the relative compactness of all minimizing sequences for F_μ on $V(c)$ is not guaranteed.

2. Since $F_\mu|_{S(c)}$ is unbounded from below, it could be natural to expect that there exists a second critical point lying at a mountain pass level. To obtain such solution, following the approach in [45], one constructs a special bounded Palais-Smale sequence and then proves this sequence strongly convergence. In the Sobolev critical case, the existence of a bounded Palais-Smale sequence can be proved, however, the convergence of such sequence is a very delicate problem.

The motivation for our study originated from the above two open problems. To solve the first open problem, for any fixed $\mu > 0$, we shall find an explicit value $c_0 = c_0(\mu) > 0$ such that, for any $c \in (0, c_0)$, there exists a set $V(c) \subset S(c)$ having the property that

$$m(c) := \inf_{u \in V(c)} F_\mu(u) < 0 < \inf_{u \in \partial V(c)} F_\mu(u). \quad (1.1.4)$$

The sets $V(c)$ and $\partial V(c)$ are given by

$$V(c) := \{u \in S(c) : \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 < \rho_0\}, \quad \partial V(c) := \{u \in S(c) : \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 = \rho_0\}$$

for a suitable $\rho_0 > 0$, depending only on $c_0 > 0$ but not on $c \in (0, c_0)$. We also introduce the set

$$\mathcal{M}_c := \{u \in V(c) : F_\mu(u) = m(c)\}.$$

Our first result is,

Theorem 1.1.3. *Let $N \geq 3$, $2 < q < 2 + \frac{4}{N}$. For any $\mu > 0$ there exists a $c_0 = c_0(\mu) > 0$ such that, for any $c \in (0, c_0)$, F_μ restricted to $S(c)$ has a ground state. This ground state is a (local) minimizer of F_μ in the set $V(c)$ and any ground state for F_μ on $S(c)$ is a local minimizer of F_μ on $V(c)$. In addition, if $(u_n) \subset V(c)$ is such that $F_\mu(u_n) \rightarrow m(c)$ then, up to translation, $u_n \rightarrow u \in \mathcal{M}_c$ in H .*

To obtain the relative compactness of all minimizing sequences, the fact that one minimizes only on a subset of $S(c)$, in contrast to a global minimization on all $S(c)$, increases the difficulty to rule out a possible *dichotomy*. Different strategies have been recently implemented to deal with this issue [13, 40, 80], all relying on a suitable choice of the set where the local minima is searched. In the presence of a Sobolev critical term an additional difficulty arises. In a Sobolev subcritical setting, if a sequence $(v_n) \subset S(c)$ is *vanishing* in the sense of P.L. Lions, see [65, Lemma I.1], one would immediately get

$$\liminf_{n \rightarrow \infty} F_\mu(v_n) = \liminf_{n \rightarrow \infty} \frac{1}{2} \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 \geq 0.$$

Thus the *vanishing* can directly be rule out knowing that $m(c) < 0$. Here [65, Lemma I.1] does not apply anymore; the term $\|v_n\|_{2^*}$ may not go to 0 if (v_n) is *vanishing*. Thus we need a better understanding of this possible loss of compactness and this leads to our definition of the set $V(c)$. This is presented in Section 2.3 where we prove the below theorem which will both imply the existence of a ground state (see Theorem 1.1.3) but also, as it may be expected, will be a crucial step to derive the orbital stability of the set \mathcal{M}_c .

Theorem 1.1.4. *For any $c \in (0, c_0)$, if $(u_n) \subset \{u \in H : \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 < \rho_0\}$ is such that $\|u_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow c$ and $F_\mu(u_n) \rightarrow m(c)$ then, up to translation, $u_n \xrightarrow{H} u \in \mathcal{M}_c$. In particular the set \mathcal{M}_c is compact in H , up to translation.*

We shall now focus on the (orbital) stability of the set \mathcal{M}_c . The fact that ground states are characterized as local minima suggests, despite the problem being mass supercritical, that the set \mathcal{M}_c could be orbitally stable. Actually, such orbital stability results have now been proved, on related problems (but always Sobolev subcritical) in several recent papers [13, 40, 80]. Along this line we now present the following result:

Theorem 1.1.5. *Let $N \geq 3$, $2 < q < 2 + \frac{4}{N}$, $\mu > 0$ and $c_0 = c_0(\mu) > 0$ be given in [Theorem 1.1.3](#). Then, for any $c \in (0, c_0)$, the set \mathcal{M}_c is compact, up to translation, and it is orbitally stable.*

As to the global existence of solutions to (1.1.1), it is also affected by the presence of the Sobolev critical exponent. In Sobolev subcritical cases, it is well known [27] that if, for an initial datum $\varphi \in H$, the maximum time of existence $T_\varphi^{max} > 0$ is finite then necessarily the corresponding solution v satisfies $\|\nabla v(t)\|_2 \rightarrow +\infty$ as $t \rightarrow T_\varphi^{max}$. Thus, a uniform a priori bound on $\|\nabla v(t)\|_2$ yields global existence. Note that, by conservation of the Mass and Energy, in view of (1.1.4), for an initial datum in $V(c) \cap \{u \in S(c) : F_\mu(u) < 0\}$, the evolution takes place in the (bounded) set $V(c)$. Thus, in a subcritical setting, the global existence would follow directly. However, in our case it is unknown if the previous blow-up alternative holds and hence, we cannot deduce global existence just since the evolution takes place in $V(c)$, see [27, Theorem 4.5.1] or [84, Proposition 3.2] for more details. To overcome this difficulty, building on the pioneering work of Cazenave-Weissler [29], see also [27, Section 4.5], we first derive an upper bound on the propagator $e^{it\Delta}$ which provides a kind of uniform local existence result, see [Section 2.4](#). Next, using the information that all minimizing sequences are, up to translation, compact and also specifically and crucially that \mathcal{M}_c is compact, up to translation, we manage to show that, for initial data sufficiently close to the set \mathcal{M}_c the global existence holds and this leads to the orbital stability of \mathcal{M}_c , proving [Theorem 1.1.5](#).

To solve the second open problem, we search for a solution lying at a mountain pass level. This type of solution has indeed been obtained recently on related problems, see, for example, [14, 31, 73]. In particular, on (1.1.3) the existence of such a mountain pass geometry had been observed in [80] in a Sobolev subcritical setting, namely when $p_2 < 2^*$, and a corresponding solution had been obtained. However, when one considers the Sobolev critical case $p_2 = 2^*$, an additional difficulty arises due to the fact that to prove the existence of such a solution one needs a precise upper estimate of the associated mountain pass level. Roughly speaking this upper estimate is crucial to guarantee that a Palais-Smale sequence at the mountain pass level does not carry a bubble which, by vanishing when passing to the weak limit, would prevent its strong convergence in $H^1(\mathbb{R}^N)$.

The need to obtain, in problem involving a Sobolev critical term, a sharp estimate on some minimax levels is known since the pioneering work of Brezis-Nirenberg [25] and the usual way to derive such strict upper bound is through the use of testing functions. It will also be the case here but we shall need, in this context, to overcome non-standard difficulties due to the fact that we search for a solution with a prescribed norm. In [81] such difficulties were first encountered and overcome but under the assumption that $p_c \leq p_1 \leq p_2 \leq 2^*$. In that case there is no solution at an energy level below the mountain pass level. In the problem we are considering, the need to respect L^2 -constraint, combined with the existence of a ground

state solution makes things more complex. Indeed, it appears necessary for proving the strict inequality that we need, see (1.1.8) in Proposition 1.1.9, to control precisely the interaction between standard truncated extremal Sobolev functions, as recalled in Lemma 2.7.1, with a suitable sequence of ground states for $m(c_n)$ with $c_n \rightarrow c$.

Let \mathcal{S} denote the best constant in the Sobolev inequality, see (2.2.1). We now state the following result.

Theorem 1.1.6. *Let $N \geq 4$, $2 < q < 2 + \frac{4}{N}$, $\mu > 0$ and $c_0 = c_0(\mu) > 0$ be given in Theorem 1.1.3. Then, for any $c \in (0, c_0)$, there exists a second solution $v_c \in S(c)$ which satisfies*

$$0 < F_\mu(v_c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N}.$$

In particular, $v_c \in S(c)$ is not a ground state.

Theorem 1.1.6 can be complemented in the following way.

Theorem 1.1.7. *Under the assumptions of Theorem 1.1.6 we have,*

(i) *For any fixed $\mu > 0$ and assuming that $c \in (0, c_0(\mu))$,*

$$\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \mathcal{S}^{\frac{N}{2}} \quad \text{and} \quad F_\mu(v_c) \rightarrow \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad \text{as} \quad c \rightarrow 0.$$

(ii) *For any fixed $c > 0$, $v_c \in S(c)$ exists for any $\mu > 0$ sufficiently small and*

$$\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \mathcal{S}^{\frac{N}{2}} \quad \text{and} \quad F_\mu(v_c) \rightarrow \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad \text{as} \quad \mu \rightarrow 0.$$

Let us now give some elements of the strategy of the proof of Theorem 1.1.6 which is presented in Section 2.5. We define

$$Q_\mu(u) := \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu \gamma_q \|u\|_{L^q(\mathbb{R}^N)}^q - \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

where

$$\gamma_q := \frac{N(q-2)}{2q}. \tag{1.1.5}$$

It is well known, see, for example, [45, Lemma 2.7], that all critical points of F_μ restricted to $S(c)$ and thus any solution to (1.1.2) satisfies $Q_\mu(u) = 0$. Introducing the set

$$\Lambda(c) := \{u \in S(c) : Q_\mu(u) = 0\}.$$

we shall show that it admits the decomposition into the disjoint union $\Lambda(c) = \Lambda^+(c) \cup \Lambda^-(c)$, where

$$\Lambda^+(c) := \{u \in \Lambda(c) : F_\mu(u) < 0\}, \quad \text{and} \quad \Lambda^-(c) := \{u \in \Lambda(c) : F_\mu(u) > 0\}. \tag{1.1.6}$$

The ground state $u_c \in S(c)$ obtained in Theorem 1.1.3, see also [81], lies on $\Lambda^+(c)$ and can be characterized by

$$F_\mu(u_c) = \inf_{u \in \Lambda^+(c)} F_\mu(u) = \inf_{u \in V(c)} F_\mu(u) = m(c).$$

The critical point $v_c \in S(c)$ obtained in [Theorem 1.1.6](#) will satisfy

$$F_\mu(v_c) = \inf_{u \in \Lambda^-(c)} F_\mu(u).$$

We denote by $H_r^1(\mathbb{R}^N)$ the subspace of functions in $H^1(\mathbb{R}^N)$ which are radially symmetric with respect to 0, and we define $S_r(c) := S(c) \cap H_r^1(\mathbb{R}^N)$. Accordingly, we also set $\Lambda_r^+(c) = \Lambda^+(c) \cap H_r^1(\mathbb{R}^N)$ and $\Lambda_r^-(c) = \Lambda^-(c) \cap H_r^1(\mathbb{R}^N)$.

Let

$$M^0(c) := \inf_{g \in \Gamma^0(c)} \max_{t \in [0, \infty)} F_\mu(g(t)) \quad (1.1.7)$$

where

$$\Gamma^0(c) := \{g \in C([0, \infty), S_r(c)) : g(0) \in \Lambda_r^+(c), \exists t_g \text{ s.t. } g(t) \in E_c \forall t \geq t_g\}$$

with

$$E_c := \{u \in S(c) : F_\mu(u) < 2m(c)\} \neq \emptyset.$$

The proof of [Theorem 1.1.6](#) will follow directly from the three below propositions.

Proposition 1.1.8. *Let $N \geq 3$. For any $c \in (0, c_0)$, there exists a Palais-Smale sequence $(u_n) \subset S_r(c)$ for F_μ restricted to $S(c)$ at level $M^0(c)$, with $Q_\mu(u_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Proposition 1.1.9. *Let $N \geq 3$. For any $c \in (0, c_0)$, if it holds that*

$$M^0(c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad (1.1.8)$$

then the Palais-Smale sequence obtained in [Proposition 1.1.8](#) is, up to subsequence, strongly convergent in $H_r^1(\mathbb{R}^N)$.

Proposition 1.1.10. *For any $c \in (0, c_0)$, if $N \geq 4$ it holds that*

$$M^0(c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N}.$$

If, as a consequence of Ekeland variational principle, the geometry of the mountain pass implies the existence of a Palais-Smale sequence (a PS sequence for short) at the mountain pass Energy level it is now a well-identified difficulty that such sequences may not be bounded. To obtain a bounded PS sequence one needs to explicit a sequence having additional properties. Hence, we first prove that the existence of a PS sequence u_n which satisfies $Q_\mu(u_n) \rightarrow 0$ as $n \rightarrow \infty$ in [Proposition 1.1.8](#). The condition that $Q_\mu(u_n) \rightarrow 0$ as $n \rightarrow \infty$, incorporated into the variational procedure the information that any solution must satisfy the Pohozaev type identity $Q_\mu(u) = 0$, see [\[45\]](#) in that direction. The convergence of such PS sequence is proved in [Proposition 1.1.9](#) under the assumption of the strict upper bound of mountain pass level $M^0(c)$, see [\(1.1.8\)](#). To establish [Proposition 1.1.9](#), we shall make use of arguments first presented in [\[81, Proposition 3.1\]](#). It is important to notice that the strong convergence is only obtained by working in $H_r^1(\mathbb{R}^N)$. [Proposition 1.1.10](#) is the heart of the proof of [Theorem 1.1.6](#) where the strict inequality [\(1.1.8\)](#) is proved by using test functions. We construct test functions that could be viewed as the sum of a truncated extremal function

of the Sobolev inequality on \mathbb{R}^N translated far away from the origin. This choice of testing functions are sufficient to prove our strict inequality when $N \geq 4$ but we miss it in the case $N = 3$.

Finally, we state the strongly unstable of the standing waves obtained in [Theorem 1.1.6](#). Actually this result is a direct consequence of the variational characterization of the solution obtained in [Theorem 1.1.6](#), combined with recent advances on the subject of instability by blow-up contained in [\[80, 81\]](#).

Theorem 1.1.11. *Under the assumptions of [Theorem 1.1.6](#) the associated standing wave $e^{-\lambda t} v_c(x)$ is strongly unstable.*

1.2 Normalized solutions for a Schrödinger-Poisson-Slater equation

In [Chapter 3](#), we consider the following Schrödinger-Poisson-Slater equations:

$$i\partial_t v + \Delta v + \gamma(|x|^{-1} * |v|^2)v + a|v|^{p-2}v = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^3, \quad (1.2.1)$$

where $v : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$, $\gamma \in \mathbb{R}$, $a \in \mathbb{R}$ and $p \in (\frac{10}{3}, 6]$. We look for standing wave solutions to [\(1.2.1\)](#), namely to solutions of the form $v(t, x) = e^{i\lambda t} u(x)$, $\lambda \in \mathbb{R}$. Then the function $u(x)$ satisfies the equation

$$-\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3. \quad (1.2.2)$$

Motivated by the fact that the L^2 -norm is a preserved quantity of the evolution we focus on the search of solutions to [\(1.2.2\)](#) with prescribed L^2 -norm. It is standard that for some prescribed $c > 0$, a solution of [\(1.2.2\)](#) with $\|u\|_{L^2(\mathbb{R}^3)}^2 = c$ can be obtained as a critical point of the *Energy* functional

$$F(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\gamma}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy - \frac{a}{p} \int_{\mathbb{R}^3} |u|^p dx$$

restricted to

$$S(c) := \{u \in H^1(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3)}^2 = c\}.$$

Then the parameter $\lambda \in \mathbb{R}$ in [\(1.2.2\)](#) appears as a Lagrange multiplier, it is an unknown of the problem. As we know, $F(u)$ is a well-defined and C^1 functional on $S(c)$ for any $p \in (2, 6]$ (see [\[77\]](#) for example).

Let us define

$$m(c) = \inf_{u \in S(c)} F(u). \quad (1.2.3)$$

Depending on the range of parameters we shall consider $m(c)$ will be finite or not. The case where $\gamma < 0$ and $a > 0$ in [\(1.2.2\)](#) has been the most studied so far. When $p \in (2, \frac{10}{3})$ it can be shown that $m(c) \in (-\infty, 0]$ for any $c > 0$ and it is also the case when $p = \frac{10}{3}$ and $c > 0$ is small. It is shown in [\[17\]](#) that minimizer exists if $p \in (2, 3)$ and $c > 0$ is small enough, see also [\[78\]](#) for the special case $p = \frac{8}{3}$. The case $p \in (3, \frac{10}{3})$ was considered in [\[16, 51\]](#), see also [\[55\]](#) for a closely related problem. In [\[51\]](#) the existence of a threshold value $c_0 > 0$ such that $m(c)$ has a minimizer if and only if $c \in [c_0, \infty)$ was established. It was also proved in [\[51\]](#) that a

minimizer does not exist for any $c > 0$ if $p = 3$ or $p = \frac{10}{3}$. We also refer to [26] for related results. When $p \in (\frac{10}{3}, 6]$ a scaling argument reveals that $m(c) = -\infty$ but nevertheless it was proved in [15] that, when $p \in (\frac{10}{3}, 6)$ there exists, for $c > 0$ small enough a critical point of F constrained to $S(c)$ at a strictly positive level. In this work we complement the result of [15] by showing that when $p = 6$ and for any $c > 0$ there does not exist positive solutions, see Theorem 1.2.5.

Even if some open problems remain when $\gamma < 0$ and $a > 0$, we shall mainly concentrate here on the others cases: $(\gamma < 0, a < 0)$, $(\gamma > 0, a > 0)$ and $(\gamma > 0, a < 0)$. We define, for short, the following quantities

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy, \quad C(u) := \int_{\mathbb{R}^3} |u|^p dx.$$

For $u \in S(c)$, we set $u^t(x) := t^{\frac{3}{2}} u(tx)$, $t > 0$, then

$$u^t \in S(c), \quad A(u^t) = t^2 A(u), \quad B(u^t) = tB(u), \quad C(u^t) = t^\sigma C(u),$$

where

$$2 < \sigma := \frac{3(p-2)}{2} \leq 6, \quad (1.2.4)$$

due to $p \in (\frac{10}{3}, 6]$. For $u \in S(c)$, we define the fiber map

$$t \in (0, \infty) \mapsto g_u(t) := F(u^t) = \frac{1}{2} t^2 A(u) - \frac{\gamma}{4} t B(u) - \frac{a}{p} t^\sigma C(u).$$

Hence, we have

$$g'_u(t) = tA(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} t^{\sigma-1} C(u) = \frac{1}{t} Q(u^t),$$

where

$$Q(u) = A(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} C(u).$$

Actually the condition $Q(u) = 0$ corresponds to a Pohozaev identity and the set

$$\Lambda(c) := \{u \in S(c) : Q(u) = 0\} = \{u \in S(c) : g'_u(1) = 0\}$$

appears as a natural constraint. Indeed, if $u \in S(c)$, then $t > 0$ is a critical point for g_u if and only if $u^t \in \Lambda(c)$. In particular, $u \in \Lambda(c)$ if and only if 1 is a critical point of g_u .

First we briefly consider the case $\gamma < 0, a < 0$. For any $u \in S(c)$, we have that $g'_u(t) > 0$ for all $t > 0$, hence the fiber map $g_u(t)$ is strictly increasing and so we can state the following non-existence result:

Theorem 1.2.1. *Assume that $\gamma < 0, a < 0$. Then $F(u)$ has no critical point on $S(c)$.*

Next, we consider the case $\gamma > 0, a > 0$. In this case, let

$$c_1 := \left(\frac{4}{\gamma K_H} \frac{\sigma - 2}{\sigma - 1} \right)^{\frac{3p-10}{4(p-3)}} \left(\frac{p}{a\sigma(\sigma-1)K_{GN}} \right)^{\frac{1}{2(p-3)}} > 0, \quad (1.2.5)$$

where σ is defined by (1.2.4) and K_H, K_{GN} are some suitable constants. We also introduce the decomposition of $\Lambda(c)$ into the disjoint union $\Lambda(c) = \Lambda^+(c) \cup \Lambda^0(c) \cup \Lambda^-(c)$, where

$$\begin{aligned}\Lambda^+(c) &:= \{u \in \Lambda(c) : g_u''(1) > 0\} = \{u \in S(c) : g_u'(1) = 0, g_u''(1) > 0\}, \\ \Lambda^0(c) &:= \{u \in \Lambda(c) : g_u''(1) = 0\} = \{u \in S(c) : g_u'(1) = 0, g_u''(1) = 0\}, \\ \Lambda^-(c) &:= \{u \in \Lambda(c) : g_u''(1) < 0\} = \{u \in S(c) : g_u'(1) = 0, g_u''(1) < 0\}.\end{aligned}$$

For any $c \in (0, c_1)$ we can prove that $\Lambda^0(c) = \emptyset$ and $\Lambda^+(c) \neq \emptyset$, $\Lambda^-(c) \neq \emptyset$. Since F is bounded from below on $\Lambda(c)$, we can define

$$\gamma^+(c) := \inf_{u \in \Lambda^+(c)} F(u) \quad \text{and} \quad \gamma^-(c) := \inf_{u \in \Lambda^-(c)} F(u). \quad (1.2.6)$$

Our first main result is

Theorem 1.2.2. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma > 0$, $a > 0$ and let $c_1 > 0$ be defined by (1.2.5). For any $c \in (0, c_1)$, there exist $u_c^+ \in \Lambda^+(c)$ such that $F(u_c^+) = \gamma^+(c)$ and $u_c^- \in \Lambda^-(c)$ such that $F(u_c^-) = \gamma^-(c)$. The functions u_c^+, u_c^- are bounded continuous positive Schwarz symmetric functions. In addition there exist $\lambda_c^+ > 0$ and $\lambda_c^- > 0$ such that (u_c^+, λ_c^+) and (u_c^-, λ_c^-) are solutions to (1.2.2).*

If the geometrical structure of F restricted to $S(c)$ is identical in the Sobolev subcritical case $p \in (\frac{10}{3}, 6)$ and in the Sobolev critical case $p = 6$, the proof that the levels $\gamma^+(c)$ and $\gamma^-(c)$ are indeed reached requires additional, more involved, arguments in the case $p = 6$. In particular, showing that $\gamma^-(c)$ is attained requires to check that the following inequality holds

$$\gamma^-(c) < \gamma^+(c) + \frac{1}{3\sqrt{aK_{GN}}}. \quad (1.2.7)$$

It is known since the pioneering work of Brezis-Nirenberg [25] that the way to derive such a strict upper bound is through the use of testing functions. In Chapter 2, considering the equation

$$-\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N, \quad (1.2.8)$$

with $N \geq 3$, $\mu > 0$, $2 < q < 2 + \frac{4}{N}$ and $2^* = \frac{2N}{N-2}$ we face the need to establish a similar inequality. We constructed test functions which could be viewed as the sum of a truncated extremal function of the Sobolev inequality on \mathbb{R}^N centered at the origin and of u_c^+ translated far away from the origin. This choice of testing functions was sufficient to prove our strict inequality when $N \geq 4$ but we missed it in the case $N = 3$. Very recently, in [89] the authors introduced an alternative choice of testing functions which allowed to treat, in a unified way, the case $N = 3$ and $N \geq 4$ for (1.2.8). The strategy in [89], recording of the one introduced by G. Tarantello in [85], is on the contrary, to located the extremal functions where the solution u_c^+ takes its greater values (the origin thus). The idea behind the proof is that the interaction decreases the value of the *Energy* with respect to the case where the supports would be disjoint. Since (1.2.2) is set on \mathbb{R}^3 , we believe in view of our experience on (1.2.8), more appropriate to follow the approach of [89] to check the inequality (1.2.7) for any $c \in (0, c_1)$.

The results of Theorem 1.2.2 are complemented in several directions. First, we show that the solution $u^+(c)$ obtained in Theorem 1.2.2 can be characterized as a local minima for F restricted to $S(c)$. We treat here the full range $p \in (\frac{10}{3}, 6]$ with a single proof. More precisely we show,

Theorem 1.2.3. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma > 0$, $a > 0$ and let $c \in (0, c_1)$. Then we have $\Lambda^+(c) \subset V(c)$ and*

$$\gamma^+(c) = \inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in V(c)} F(u)$$

where

$$V(c) := \{u \in S(c) | A(u) < k_1\}$$

for some $k_1 > 0$ independent of $c \in (0, c_1)$. In addition, any minimizing sequence for F on $V(c)$ is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$.

As we shall see $\gamma^+(c) < \gamma^-(c)$ and combined with the property that any critical point lies in $\Lambda(c)$ it implies that the solution u_c^+ obtained in [Theorem 1.2.2](#) is a ground state. Following [\[14\]](#) a ground state is defined as a solution $v \in S(c)$ to [\(1.2.2\)](#) which has minimal Energy among all the solutions which belong to $S(c)$. Namely, if

$$F(v) = \inf \left\{ F(u), u \in S(c), \left(F|_{S(c)} \right)'(u) = 0 \right\}.$$

Let us now denote

$$\mathcal{M}_c := \{u \in V(c) : F(u) = \gamma^+(c)\}.$$

Hence, \mathcal{M}_c is the set of all ground states. The property that any minimizing sequence for F restricted to $V(c)$ is, up to translation, strongly converging is known to be a key ingredient to show that the set \mathcal{M}_c is orbitally stable. If $p \in (\frac{10}{3}, 6)$ the orbital stability of \mathcal{M}_c indeed follows directly from [Theorem 1.2.3](#) by the classical arguments of [\[28\]](#). In the case $p = 6$ the situation is more delicate as the existence of a uniform $H^1(\mathbb{R}^3)$ bound on the solution of [\(1.2.1\)](#) during its lifespan is not sufficient to guarantee that blow-up may not occurs. We refer to [\[27\]](#) for more details. We do not prove anything in that direction but strongly believe that the set \mathcal{M}_c is orbitally stable. Actually, such a result has been obtained on the equation [\(1.2.8\)](#) in [Chapter 2](#).

We also discuss the behavior of the associated Lagrange multipliers in [Chapter 3](#) and show that if the behavior of λ_c^+ is essentially the same for the cases $p \in (\frac{10}{3}, 6)$ and $p = 6$, there is a distinct behavior for λ_c^- . Besides, we also establish the property that the map $c \mapsto \gamma^-(c)$ is strictly decreasing.

Next, we consider the case $\gamma > 0$, $a < 0$. Recalling the definition of $m(c)$ given in [\(1.2.3\)](#) we shall show that $-\infty < m(c) < 0$ and then we prove the following result.

Theorem 1.2.4. *Let $p \in (\frac{10}{3}, 6]$, $\gamma > 0$ and $a < 0$. For any $c > 0$, the infimum $m(c)$ is achieved and any minimizing sequence for [\(1.2.3\)](#) is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$ to a solution of [\(1.2.2\)](#). In addition, the associated Lagrange multiplier is positive.*

Even if the proof of [Theorem 1.2.4](#) follows the lines of the proof of [Theorem 1.2.3](#), the change of sign in front of the power term requires some adaptations. Here again the orbital stability of the set of minimizers should follow directly from the classical arguments of [\[28\]](#) if $p \in (\frac{10}{3}, 6)$ and it should also be the case when $p = 6$ by adapting the arguments of [Chapter 2](#). Note that the behavior of the associated Lagrange multipliers is also studied.

In the last part of [Chapter 3](#) we consider the case $\gamma < 0$, $a > 0$ and $p = 6$. Based on Liouville-type result which are proved for exterior domain of \mathbb{R}^N by [\[7, Theorem 2.1\]](#), we obtain the following result:

Theorem 1.2.5. *Let $p = 6$, $\gamma < 0$ and $a > 0$. For any $c > 0$, we have that*

- (i) *If $u \in H^1(\mathbb{R}^3)$ is a non-trivial solution to (1.2.2) then the associated Lagrange multiplier λ is negative and*

$$F(u) > \frac{1}{3\sqrt{aK_{GN}}}.$$

- (ii) *Equation (1.2.2) has no positive solution in $H^1(\mathbb{R}^3)$.*

Summarizing, the known results concerning the existence of solutions to (1.2.2) are collected in the following table:

	Sobolev subcritical case $\frac{10}{3} < p < 6$	Sobolev critical case $p = 6$
$\gamma < 0, a < 0$	No solution for any $c > 0$	
$\gamma < 0, a > 0$	One solution for c small enough: a saddle point, see [15]	No positive solution for any $c > 0$
$\gamma > 0, a < 0$	One solution for any $c > 0$: a global minimizer	
$\gamma > 0, a > 0$	Two solutions for c small enough: one local minimizer, one saddle point	

Table 1.1: The existence of solutions to (1.2.2)

Chapter 2

Multiple normalized solutions for the Sobolev critical Schrödinger equation

This chapter is precisely the results in the papers [46] and [48] combined. For the convenience of reading, this chapter is presented self-contained with the rest of the thesis.

2.1 Introduction

In this chapter, we study the existence and orbital stability of ground state standing waves of prescribed mass for the nonlinear Schrödinger equation with mixed power nonlinearities

$$i\partial_t v + \Delta v + \mu v|v|^{q-2} + v|v|^{2^*-2} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (2.1.1)$$

where $N \geq 3$, $v : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $\mu > 0$, $2 < q < 2 + \frac{4}{N}$ and $2^* = \frac{2N}{N-2}$.

We recall that standing waves to (2.1.1) are solutions of the form $v(t, x) = e^{-i\lambda t} u(x)$, $\lambda \in \mathbb{R}$. Then the function $u(x)$ satisfies the equation

$$-\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N. \quad (2.1.2)$$

Motivated by the fact that the L^2 -norm is a preserved quantity of the evolution we focus on the search of solutions to (2.1.2) having a prescribed L^2 -norm. Defining on $H := H^1(\mathbb{R}^N, \mathbb{C})$ the Energy functional

$$F_\mu(u) := \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

it is standard to check that F_μ is of class C^1 and that a critical point of F_μ restricted to the (mass) constraint

$$S(c) := \{u \in H : \|u\|_{L^2(\mathbb{R}^N)}^2 = c\}$$

gives rise to a solution to (2.1.2), satisfying $\|u\|_{L^2(\mathbb{R}^N)}^2 = c$. In this approach the parameter $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier. In particular, $\lambda \in \mathbb{R}$ does depend on the solution and is not a priori given.

We shall focus on the existence of ground state solutions.

Definition 2.1.1. We say that $u_c \in S(c)$ is a ground state solution to (2.1.2) if it is a solution having minimal Energy among all the solutions which belong to $S(c)$. Namely, if

$$F_\mu(u_c) = \inf \left\{ F_\mu(u), u \in S(c), \left(F_\mu|_{S(c)} \right)'(u) = 0 \right\}.$$

Note that this definition keeps a meaning even in situations where the Energy F_μ is unbounded from below on $S(c)$. Implicit in [45], this definition was formally introduced, on a related model, in [14] and is now becoming standard.

It is well-known that the study of problems with mixed nonlinearities and the type of results one can expect, depend on the behavior of the nonlinearities at infinity, namely on the value of the various power exponents. In particular, this behavior determines whether the functional is bounded from below on $S(c)$. One speaks of a mass subcritical case if it is bounded from below on $S(c)$ for any $c > 0$, and of a mass supercritical case if the functional is unbounded from below on $S(c)$ for any $c > 0$. One also refers to a mass critical case when the boundedness from below does depend on the value $c > 0$. To be more precise, consider an equation of the form

$$i\partial_t v + \Delta v + \mu v|v|^{p_1-2} + v|v|^{p_2-2} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (2.1.3)$$

where it is assumed that $2 < p_1 \leq p_2 \leq 2^*$. The threshold exponent is the so-called L^2 -critical exponent

$$p_c = 2 + \frac{4}{N}.$$

A very complete analysis of the various cases that may happen for (2.1.3), depending on the values of (p_1, p_2) , has been provided recently in [80, 81]. Let us just recall here some rough elements. If both p_1 and p_2 are strictly less than p_c then the associated Energy functional is bounded from below on $S(c)$ and to find a ground state one looks for a global minimum on $S(c)$. The problem then directly falls into the setting covered by the Compactness by Concentration Principle introduced by P.L. Lions [64, 65] which, for more complicated equations, in particular non autonomous ones, is still a very active field. Such solutions are expected to be orbitally stable, see Definition 2.1.5 below. If $p_c \leq p_1 \leq p_2 \leq 2^*$, then the Energy functional is unbounded from below on $S(c)$ but it is possible to show that a ground state exists. This ground state is characterized as a critical point of *mountain-pass type* and it lies at a strictly positive level of the Energy functional. Such ground states are expected to be orbitally unstable. We refer, for the link between the variational characterization of a solution and its instability, to the classical paper [20], and to [45, 56, 80, 81] for more recent developments.

In the case we consider here : $2 < p_1 < p_c < p_2 = 2^*$, the Energy functional is thus unbounded from below on $S(c)$ but, as we shall see, the presence of the lower order, mass subcritical term $-\mu \|u\|_{L^q(\mathbb{R}^N)}^q$ created, for sufficiently small values of $c > 0$, a geometry of local minima on $S(c)$. The presence of such geometry, in problems which are mass supercritical, had already been observed in several related situations. In [13, 14] for related scalar problems, in [40] in the case of a system or [73] for an evolution problem set on a bounded domain. Actually, it was already observed on (2.1.1) in [81].

Precisely, for any fixed $\mu > 0$, we shall find an explicit value $c_0 = c_0(\mu) > 0$ such that, for any $c \in (0, c_0)$, there exists a set $V(c) \subset S(c)$ having the property that

$$m(c) := \inf_{u \in V(c)} F_\mu(u) < 0 < \inf_{u \in \partial V(c)} F_\mu(u). \quad (2.1.4)$$

The sets $V(c)$ and $\partial V(c)$ are given by

$$V(c) := \{u \in S(c) : \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 < \rho_0\}, \quad \partial V(c) := \{u \in S(c) : \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 = \rho_0\}$$

for a suitable $\rho_0 > 0$, depending only on $c_0 > 0$ but not on $c \in (0, c_0)$. We also introduce the set

$$\mathcal{M}_c := \{u \in V(c) : F_\mu(u) = m(c)\}.$$

Our first result is,

Theorem 2.1.2. *Let $N \geq 3$, $2 < q < 2 + \frac{4}{N}$. For any $\mu > 0$ there exists a $c_0 = c_0(\mu) > 0$ such that, for any $c \in (0, c_0)$, F_μ restricted to $S(c)$ has a ground state. This ground state is a (local) minimizer of F_μ in the set $V(c)$ and any ground state for F_μ on $S(c)$ is a local minimizer of F_μ on $V(c)$. In addition, if $(u_n) \subset V(c)$ is such that $F_\mu(u_n) \rightarrow m(c)$ then, up to translation, $u_n \rightarrow u \in \mathcal{M}_c$ in $H^1(\mathbb{R}^N, \mathbb{C})$.*

Remark 2.1.3. *The value of $c_0 = c_0(\mu) > 0$ is explicit and is given in (2.3.1)-(2.3.2). In particular $c_0 > 0$ can be taken arbitrary large by taking $\mu > 0$ small enough.*

Remark 2.1.4.

- (i) *If $u \in S(c)$ is a ground state then the associated Lagrange multiplier $\lambda \in \mathbb{R}$ in (2.1.2) satisfies $\lambda < 0$, see Lemma 2.2.1.*
- (ii) *There exists a ground state which is a real valued, positive, radially symmetric decreasing function, see Lemma 2.2.2.*
- (iv) *The map $c \mapsto m(c)$ is non-increasing and that $m(c) \rightarrow 0$ as $c \rightarrow 0$, see Lemma 2.3.5.*
- (iv) *More globally, under the assumption of Theorem 2.1.2 it can be proved that, for any $c \in (0, c_0)$, \mathcal{M}_c has the following structure:*

$$\mathcal{M}_c = \{e^{i\theta} u, \text{ for some } \theta \in \mathbb{R}, u \in \tilde{\mathcal{M}}_c, u > 0\},$$

where

$$\tilde{\mathcal{M}}_c = \{u \in S(c) \cap H^1(\mathbb{R}^N, \mathbb{R}), F_\mu(u) = m(c)\}.$$

Indeed, this description directly follows from the convergence, up to translation, of the minimizing sequences of F_μ restricted to $V(c)$, applying the argument of [42, Section 3]. We leave the details to the interested reader.

We shall now focus on the (orbital) stability of the set \mathcal{M}_c . Following the terminology of [28], see also [42], we give the following definition.

Definition 2.1.5. *$Z \subset H$ is stable if: $Z \neq \emptyset$ and for any $v \in Z$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\varphi \in H$ satisfies $\|\varphi - v\|_H < \delta$ then $u_\varphi(t)$ is globally defined and $\inf_{z \in Z} \|u_\varphi(t) - z\|_H < \varepsilon$ for all $t \in \mathbb{R}$, where $u_\varphi(t)$ is the solution to (2.1.1) corresponding to the initial condition φ .*

Notice that the orbital stability of the set Z implies the global existence of solutions to (2.1.1) for initial datum φ close enough to the set Z . We underline that this fact is non trivial due to the critical exponent that appears in (2.1.1), even if the H norm of the solution is uniformly bounded on the lifespan of the solution.

The fact that ground states are characterized as local minima suggests, despite the problem being mass supercritical, that the set \mathcal{M}_c could be orbitally stable. Actually, such orbital stability results have now been proved, on related problems (but always Sobolev subcritical) in several recent papers [13, 40, 80]. Along this line we now present the main result of this chapter.

Theorem 2.1.6. *Let $N \geq 3$, $2 < q < 2 + \frac{4}{N}$, $\mu > 0$ and $c_0 = c_0(\mu) > 0$ be given in Theorem 2.1.2. Then, for any $c \in (0, c_0)$, the set \mathcal{M}_c is compact, up to translation, and it is orbitally stable.*

In [81], Soave studied equation (2.1.1) and derived, for any small $c > 0$ depending on $\mu > 0$, an existence result which is very similar to the one contained in Theorem 2.1.2, see [81, Theorem 1.1]. However, it does not seem possible to use [81, Theorem 1.1] as a starting point to prove Theorem 2.1.6. The existence of a ground state in [81, Theorem 1.1] is obtained through the study of one particular (locally) minimizing sequence which is radially symmetric. As already explained in [81], to obtain the orbital stability of the set \mathcal{M}_c , following the classical approach laid down in [28], two ingredients are essential. First, the relative compactness, up to translation, of all minimizing sequences for F_μ on $V(c)$, as guaranteed by our Theorem 2.1.2. Secondly, the global existence of solutions to (2.1.1) for initial data close to \mathcal{M}_c .

To obtain the relative compactness of all minimizing sequences, the fact that one minimizes only on a subset of $S(c)$, in contrast to a global minimization on all $S(c)$, increases the difficulty to rule out a possible *dichotomy*. Different strategies have been recently implemented to deal with this issue [13, 40, 80], all relying on a suitable choice of the set where the local minima is searched. In the presence of a Sobolev critical term an additional difficulty arises. In a Sobolev subcritical setting, if a sequence $(v_n) \subset S(c)$ is *vanishing* then applying [65, Lemma I.1] one would immediately get

$$\liminf_{n \rightarrow \infty} F_\mu(v_n) = \liminf_{n \rightarrow \infty} \frac{1}{2} \|\nabla v_n\|_2^2 \geq 0.$$

Thus the *vanishing* can directly be rule out knowing that $m(c) < 0$. Here [65, Lemma I.1] does not apply anymore; the term $\|v_n\|_{2^*}$ may not go to 0 if (v_n) is *vanishing*. Thus we need a better understanding of this possible loss of compactness and this leads to our definition of the set $V(c)$. As to the global existence of solutions to (2.1.1), it is also affected by the presence of the Sobolev critical exponent. In Sobolev subcritical cases, it is well known [27] that if, for an initial datum $\varphi \in H$, the maximum time of existence $T_\varphi^{max} > 0$ is finite then necessarily the corresponding solution v satisfies $\|\nabla v(t)\|_2 \rightarrow +\infty$ as $t \rightarrow T_\varphi^{max}$. Thus, a uniform a priori bound on $\|\nabla v(t)\|_2$ yields global existence. Note that, by conservation of the Mass and Energy, in view of (2.1.4), for an initial datum in $V(c) \cap \{u \in S(c) : F_\mu(u) < 0\}$, the evolution takes place in the (bounded) set $V(c)$. Thus, in a subcritical setting, the global existence would follow directly. However, in our case it is unknown if the previous blow-up alternative holds and hence, we cannot deduce global existence just since the evolution takes place in $V(c)$, see [27, Theorem 4.5.1] or [84, Proposition 3.2] for more details. To overcome this difficulty, building on the pioneering work of Cazenave-Weissler [29], see also [27, Section 4.5], we first derive an upper bound on the propagator $e^{it\Delta}$ which provides a kind of uniform local existence result, see Proposition 2.4.3. Next, using the information that all minimizing sequences are, up to translation, compact and also specifically and crucially that \mathcal{M}_c is compact, up to translation, we manage to show that, for initial data sufficiently close to the set \mathcal{M}_c the global existence holds and this leads to the orbital stability of \mathcal{M}_c , proving Theorem 2.1.6.

Remark 2.1.7. *We point out that, in order to prove [Theorem 2.1.6](#), we have only established the global existence of solutions for initial data close to \mathcal{M}_c . We believe it would be interesting to inquire if the global existence holds away from \mathcal{M}_c , typically for any initial data in $V(c) \cap \{u \in S(c) : F_\mu(u) < 0\}$. If so, investigating the long time behavior of these solutions would be worth to. Our guess is that these solutions evolve toward the sum of an element of \mathcal{M}_c and a part which scatter. However, so far nothing is known in that direction.*

Now, we focus on the second solution to [\(2.1.1\)](#). We observe that the structure of local minima, for a functional which is unbounded from below, suggests the possibility to search for a solution lying at a mountain pass level. This type of solution has indeed been obtained recently on related problems, see, for example, [\[14, 31, 73\]](#). In particular, on [\(2.1.3\)](#) the existence of such a mountain pass geometry had been observed in [\[80\]](#) in a Sobolev subcritical setting, namely when $p_2 < 2^*$, and a corresponding solution had been obtained. However, when one considers the Sobolev critical case $p_2 = 2^*$, an additional difficulty arises due to the fact that to prove the existence of such a solution one needs a precise upper estimate of the associated mountain pass level. Roughly speaking this upper estimate is crucial to guarantee that a Palais-Smale sequence at the mountain pass level does not carry a bubble which, by vanishing when passing to the weak limit, would prevent its strong convergence in $H^1(\mathbb{R}^N)$.

The need to obtain, in problem involving a Sobolev critical term, a sharp estimate on some minimax levels is known since the pioneering work of Brezis-Nirenberg [\[25\]](#) and the usual way to derive such strict upper bound is through the use of testing functions. It will also be the case here but we shall need, in this context, to overcome non-standard difficulties due to the fact that we search for a solution with a prescribed norm. In [\[81\]](#) such difficulties were first encountered and overcome but under the assumption that $p_c \leq p_1 \leq p_2 \leq 2^*$. In that case there is no solution at an energy level below the mountain pass level. In the problem we are considering, the need to respect L^2 -constraint, combined with the existence of a ground state solution makes things more complex. Indeed, it appears necessary for proving the strict inequality that we need, see [\(2.1.8\)](#) in [Proposition 2.1.13](#), to control precisely the interaction between standard truncated extremal Sobolev functions, as recalled in [Lemma 2.7.1](#), with a suitable sequence of ground states for $m(c_n)$ with $c_n \rightarrow c$, see the proof of [Proposition 2.1.18](#) for more details. Actually, the existence of a second solution to [\(2.1.2\)](#) was proposed in [\[81\]](#) as an open problem.

From here until the end of the thesis, \mathcal{S} denotes the best constant in the Sobolev inequality, see [\(2.2.1\)](#). We now state the main result.

Theorem 2.1.8. *Let $N \geq 4$, $2 < q < 2 + \frac{4}{N}$, $\mu > 0$ and $c_0 = c_0(\mu) > 0$ be given in [Theorem 2.1.2](#). Then, for any $c \in (0, c_0)$, there exists a second solution $v_c \in S(c)$ which satisfies*

$$0 < F_\mu(v_c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N}.$$

In particular, $v_c \in S(c)$ is not a ground state.

[Theorem 2.1.8](#) can be complemented in the following way.

Theorem 2.1.9. *Under the assumptions of [Theorem 2.1.8](#) we have,*

(i) For any fixed $\mu > 0$ and assuming that $c \in (0, c_0(\mu))$,

$$\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \mathcal{S}^{\frac{N}{2}} \quad \text{and} \quad F_\mu(v_c) \rightarrow \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad \text{as} \quad c \rightarrow 0.$$

(ii) For any fixed $c > 0$, $v_c \in S(c)$ exists for any $\mu > 0$ sufficiently small and

$$\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \mathcal{S}^{\frac{N}{2}} \quad \text{and} \quad F_\mu(v_c) \rightarrow \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad \text{as} \quad \mu \rightarrow 0.$$

Remark 2.1.10. *Theorem 2.1.9 (ii) can be set in parallel with [81, Theorem 1.4 2)]. Note that a particular emphasis is given in [81] on the behavior of the solutions as $\mu \rightarrow 0$, in the spirit of the so-called Brezis-Nirenberg problem. In that direction, but for a fixed $\lambda \in \mathbb{R}$ problem, we also refer to [32].*

Theorem 2.1.11. *Under the assumptions of Theorem 2.1.8 the associated standing wave $e^{-\lambda t} v_c(x)$ is strongly unstable.*

We do not claim any originality in Theorem 2.1.11. Actually this result is a direct consequence of the variational characterization of the solution obtained in Theorem 2.1.8, combined with recent advances on the subject of instability by blow-up contained in [80, 81].

Let us now give some elements of the strategy of the proof of Theorem 2.1.8. We define

$$Q_\mu(u) := \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu \gamma_q \|u\|_{L^q(\mathbb{R}^N)}^q - \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

where

$$\gamma_q := \frac{N(q-2)}{2q}. \quad (2.1.5)$$

It is well known that all critical points of F_μ restricted to $S(c)$ and thus any solution to (2.1.2) satisfies $Q_\mu(u) = 0$, see Lemma 2.2.1. Introducing the set

$$\Lambda(c) := \{u \in S(c) : Q_\mu(u) = 0\}.$$

we shall show, see Lemma 2.2.3, that it admits the decomposition into the disjoint union $\Lambda(c) = \Lambda^+(c) \cup \Lambda^-(c)$, where

$$\Lambda^+(c) := \{u \in \Lambda(c) : F_\mu(u) < 0\}, \quad \text{and} \quad \Lambda^-(c) := \{u \in \Lambda(c) : F_\mu(u) > 0\}. \quad (2.1.6)$$

The ground state $u_c \in S(c)$ obtained in Theorem 2.1.2, see also [81], lies on $\Lambda^+(c)$ and can be characterized by

$$F_\mu(u_c) = \inf_{u \in \Lambda^+(c)} F_\mu(u) = \inf_{u \in V(c)} F_\mu(u) = m(c).$$

The critical point $v_c \in S(c)$ obtained in Theorem 2.1.8 will satisfy, see Remark 2.5.4,

$$F_\mu(v_c) = \inf_{u \in \Lambda^-(c)} F_\mu(u).$$

The proof of Theorem 2.1.8 will follow directly from the three propositions below.

We denote by $H_r^1(\mathbb{R}^N)$ the subspace of functions in $H^1(\mathbb{R}^N)$ which are radially symmetric with respect to 0, and we define $S_r(c) := S(c) \cap H_r^1(\mathbb{R}^N)$. Accordingly, we also set $\Lambda_r^+(c) = \Lambda^+(c) \cap H_r^1(\mathbb{R}^N)$ and $\Lambda_r^-(c) = \Lambda^-(c) \cap H_r^1(\mathbb{R}^N)$.

Let

$$M^0(c) := \inf_{g \in \Gamma^0(c)} \max_{t \in [0, \infty)} F_\mu(g(t)) \quad (2.1.7)$$

where

$$\Gamma^0(c) := \{g \in C([0, \infty), S_r(c)) : g(0) \in \Lambda_r^+(c), \exists t_g \text{ s.t. } g(t) \in E_c \forall t \geq t_g\}$$

with

$$E_c := \{u \in S(c) : F_\mu(u) < 2m(c)\} \neq \emptyset.$$

Proposition 2.1.12. *Let $N \geq 3$. For any $c \in (0, c_0)$, there exists a Palais-Smale sequence $(u_n) \subset S_r(c)$ for F_μ restricted to $S(c)$ at level $M^0(c)$, with $Q_\mu(u_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Proposition 2.1.13. *Let $N \geq 3$. For any $c \in (0, c_0)$, if it holds that*

$$M^0(c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad (2.1.8)$$

then the Palais-Smale sequence obtained in Proposition 2.1.12 is, up to subsequence, strongly convergent in $H_r^1(\mathbb{R}^N)$.

Proposition 2.1.14. *For any $c \in (0, c_0)$, if $N \geq 4$ it holds that*

$$M^0(c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N}.$$

Remark 2.1.15. *If, as a consequence of Ekeland variational principle, the geometry of the mountain pass implies the existence of a Palais-Smale sequence (a PS sequence for short) at the mountain pass Energy level it is now a well-identified difficulty that such sequences may not be bounded. To obtain a bounded PS sequence one needs to explicit a sequence having additional properties. The condition that $Q_\mu(u_n) \rightarrow 0$ as $n \rightarrow \infty$, incorporated into the variational procedure the information that any solution must satisfy the Pohozaev type identity $Q_\mu(u) = 0$, see [45] in that direction.*

Remark 2.1.16. *To establish Proposition 2.1.13, we shall make use of arguments first presented in [81, Proposition 3.1]. It is important to notice that the strong convergence is only obtained by working in $H_r^1(\mathbb{R}^N)$. Indeed the strong convergence in $L^q(\mathbb{R}^N)$ of any weakly converging sequence in $H_r^1(\mathbb{R}^N)$ is crucially used.*

The proof of Proposition 2.1.14, which is the heart of the proof of Theorem 2.1.8, can be divided into two parts whose proofs require different types of arguments. Let

$$M(c) := \inf_{h \in \Gamma(c)} \max_{t \in [0, \infty)} F_\mu(h(t))$$

where

$$\Gamma(c) := \{h \in C([0, \infty), S(c)) : h(0) \in V(c) \cap \{u : F_\mu(u) < 0\}, \exists t_h \text{ s.t. } h(t) \in E_c \forall t \geq t_h\}.$$

Proposition 2.1.17. *Let $N \geq 3$. For any $c \in (0, c_0)$, it holds that*

$$M^0(c) \leq M(c).$$

Proposition 2.1.18. *For any $c \in (0, c_0)$, if $N \geq 4$ we have that*

$$M(c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N}. \quad (2.1.9)$$

Even if the conclusion of [Proposition 2.1.17](#) may somehow been expected, the proof of this result is rather involved. Due to the fact that the symmetric rearrangement map is not continuous from $H_+^1(\mathbb{R}^N)$, the subspace of non negative functions in $H^1(\mathbb{R}^N)$, to $H_+^1(\mathbb{R}^N)$ if $N \geq 2$, see [\[3, 4\]](#), it is not possible to replace a given path (of non negative functions which is not restrictive) by a path which would be a Schwarz rearrangement (elements by elements) of the initial path, see [\[80, Remark 5.2\]](#) for a discussion in that direction. Actually, if the strict inequality of [\(2.1.4\)](#) guarantees that the functional has a mountain pass geometry, it is not a sufficient information to prove that $M^0(c) \leq M(c)$. A better understanding of the geometry of the functional F_μ is required and, for this purpose, we introduce a set $W(c)$, directly connected with the decomposition $\Lambda(c) = \Lambda^+(c) \cup \Lambda^-(c)$ and study its relation with $V(c)$, see [Lemma 2.2.4](#).

Note that we need to prove that $M^0(c) \leq M(c)$ because, on one hand the compactness of the Palais-Smale sequence at the mountain pass level can only be obtained by working in $H_r^1(\mathbb{R}^N)$, on the other hand to show the strict inequality in [Proposition 2.1.18](#) we need to work with testing functions, testing paths actually, which are not radial. The idea of using non-radial test functions to estimate a mountain pass level defined on a radial space seems to be new.

Remark 2.1.19. *It is only in [Proposition 2.1.18](#) that appears the need to restrict ourselves to $N \geq 4$ in [Theorem 2.1.8](#), [Theorem 2.1.9](#) and [Theorem 2.1.11](#). The strict inequality in [Proposition 2.1.18](#) is proved by using test functions. We construct test functions that could be viewed as the sum of a truncated extremal function of the Sobolev inequality on \mathbb{R}^N translated far away from the origin. This choice of testing functions are sufficient to prove our strict inequality when $N \geq 4$ but we miss it in the case $N = 3$.*

[Chapter 2](#) is organized as follows. [Section 2.2](#) is devoted to some preliminaries. In particular we clarify the structure of the set $\Lambda(c)$ and introduce our set $W(c)$ which will prove essential in the proof of [Proposition 2.1.17](#). We shall study the existence of ground state solutions and prove [Theorem 2.1.2](#) in [Section 2.3](#). The orbital stability of the ground state solutions corresponded [Theorem 2.1.6](#) will be proved in [Section 2.4](#). In [Section 2.5](#), the proofs of the existence of standing waves lying at mountain pass levels are given. More specifically, the proof of [Proposition 2.1.12](#), [Proposition 2.1.13](#) and [Proposition 2.1.14](#) are given in [Subsection 2.5.1](#), [Subsection 2.5.2](#) and [Subsection 2.5.3](#) respectively and finally [Subsection 2.5.4](#) presents the proof of [Theorem 2.1.6](#) and additional properties. The strong instability of such standing waves are studied in [Section 2.6](#).

2.2 Preliminary results

We shall make use of the following classical inequalities : For any $N \geq 3$ there exists an optimal constant $\mathcal{S} > 0$ depending only on N , such that

$$\mathcal{S}\|f\|_2^2 \leq \|\nabla f\|_2^2, \quad \forall f \in H^1(\mathbb{R}^N), \quad (\text{Sobolev inequality}) \quad (2.2.1)$$

see [23, Theorem 9.9]. If $N \geq 2$ and $p \in [2, \frac{2N}{N-2})$ then

$$\|f\|_p \leq C_{N,p} \|\nabla f\|_2^\beta \|f\|_2^{(1-\beta)}, \quad \text{with } \beta = N \left(\frac{1}{2} - \frac{1}{p} \right) \quad (\text{Gagliardo-Nirenberg inequality}), \quad (2.2.2)$$

for all $f \in H^1(\mathbb{R}^N)$, see [72].

Now, let $u \in S(c)$ be arbitrary but fixed. For $s \in \mathbb{R}_+$ we set

$$u_s(x) := s^{\frac{N}{2}} u(sx).$$

Clearly $u_s \in S(c)$ for any $s \in \mathbb{R}_+$. We define on \mathbb{R}_+ the fiber map,

$$\psi_u(s) := F_\mu(u_s) = \frac{s^2}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} s^{q\gamma_q} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{s^{2^*}}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}, \quad (2.2.3)$$

where γ_q is given in (2.1.5). Note that $\gamma_q \in (0, 1)$ and $q\gamma_q \in (0, 2)$. We also have

$$\psi'_u(s) = s \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu \gamma_q s^{q\gamma_q-1} \|u\|_{L^q(\mathbb{R}^N)}^q - s^{2^*-1} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = \frac{1}{s} Q_\mu(u_s), \quad (2.2.4)$$

where

$$Q_\mu(u) = \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu \gamma_q \|u\|_{L^q(\mathbb{R}^N)}^q - \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

Lemma 2.2.1. *Let $N \geq 3$. If $(u, \lambda) \in H^1(\mathbb{R}^N) \setminus \{0\} \times \mathbb{R}$ is a solution to*

$$-\Delta u - \mu|u|^{q-2}u - |u|^{2^*-2}u = \lambda u, \quad (2.2.5)$$

then $Q_\mu(u) = 0$ and $\lambda < 0$.

Proof. The fact that any solution to (2.2.5) satisfies $Q_\mu(u) = 0$ is a direct consequence of the Pohozaev identity see, for example, [45, Lemma 2.7]. Now we deduce from (2.2.5) that

$$\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu \|u\|_{L^q(\mathbb{R}^N)}^q - \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = \lambda \|u\|_{L^2(\mathbb{R}^N)}^2. \quad (2.2.6)$$

Combining (2.2.6) with $Q_\mu(u) = 0$ we obtain that

$$\lambda \|u\|_{L^2(\mathbb{R}^N)}^2 = -\mu(1 - \gamma_q) \|u\|_{L^q(\mathbb{R}^N)}^q$$

which proves the lemma since $\gamma_q \in (0, 1)$. □

Lemma 2.2.2. *Let $N \geq 3$. For any $\mu > 0$ and any $c \in (0, c_0)$, if $m(c)$ is reached then*

(i) *$m(c)$ is also reached by a positive, radially symmetric non-increasing function, denoted u_c that satisfies, for a $\lambda_c \in \mathbb{R}$,*

$$-\Delta u_c - \mu|u_c|^{q-2}u_c - |u_c|^{2^*-2}u_c = \lambda_c u_c \quad \text{in } \mathbb{R}^N. \quad (2.2.7)$$

(ii) *any ground state is contained in $V(c)$.*

Proof. (i) Assuming that $m(c)$ is reached at u_0 that satisfies

$$-\Delta u_0 - \mu |u_0|^{q-2} u_0 - |u_0|^{2^*-2} u_0 = \lambda_0 u_0 \quad \text{in } \mathbb{R}^N,$$

for a $\lambda_0 \in \mathbb{R}$. Now, let u_c be the Schwarz rearrangement of $|u_0|$. Hence u_c is a positive, radially symmetric non-increasing function. We also have that

$$\|u_c\|_{L^2(\mathbb{R}^N)}^2 = \|u_0\|_{L^2(\mathbb{R}^N)}^2 = c, \quad \|\nabla u_c\|_{L^2(\mathbb{R}^N)}^2 \leq \|\nabla u_0\|_{L^2(\mathbb{R}^N)}^2 < \rho_0 \quad \text{and} \quad F_\mu(u_c) \leq F_\mu(u_0).$$

This implies that $u_c \in V(c)$ and hence $F_\mu(u_c) = F_\mu(u_0)$. Thus, $m(c)$ is reached by u_c that satisfies (2.2.7) for a $\lambda_c \in \mathbb{R}$.

(ii) By Lemma 2.2.1, all critical points u of F_μ restricted to $S(c)$ satisfies

$$Q_\mu(u) = \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu \gamma_q \|u\|_{L^q(\mathbb{R}^N)}^q - \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = 0.$$

By (2.2.4), for any $v \in S(c)$ and any $s \in (0, \infty)$,

$$\psi'_v(s) = \frac{1}{s} Q_\mu(v_s). \quad (2.2.8)$$

We observe that any $u \in S(c)$ can be written as $u = v_s$ with $v \in S(c)$, $\|\nabla v\|_2 = 1$ and $s \in (0, \infty)$.

We deduce from (2.2.8) that if $w \in S(c)$ is a ground state there exists a $v \in S(c)$, $\|\nabla v\|_2^2 = 1$ and a $s_0 \in (0, \infty)$ such that $w = v_{s_0}$, $F_\mu(w) = \psi_v(s_0)$ and $\psi'_v(s_0) = 0$. Namely, $s_0 \in (0, \infty)$ is a zero of the function ψ'_v .

Now, since $\psi_v(s) \rightarrow 0^-$, $\|\nabla v_s\|_2 \rightarrow 0$, as $s \rightarrow 0$ and $\psi_v(s) = F_\mu(v_s) \geq 0$ when $v_s \in \partial V(c) = \{u \in S(c) : \|\nabla u\|_2^2 = \rho_0\}$, necessarily ψ'_v has a first zero $s_1 > 0$ corresponding to a local minima. In particular, $v_{s_1} \in V(c)$ and $F(v_{s_1}) = \psi_v(s_1) < 0$. Also, from $\psi_v(s_1) < 0$, $\psi_v(s) \geq 0$ when $v_s \in \partial V(c)$ and $\psi_v(s) \rightarrow -\infty$ as $s \rightarrow \infty$, ψ_v has a second zero $s_2 > s_1$ corresponding to a local maxima of ψ_v . Since v_{s_2} satisfies $F(v_{s_2}) = \psi_v(s_2) \geq 0$, we have that $m(c) \leq F(v_{s_1}) < F(v_{s_2})$. In particular, since $m(c)$ is reached, v_{s_2} cannot be a ground state.

To conclude the proof of (ii) it then just suffices to show that ψ'_v has at most two zeros, since this will imply $s_0 = s_1$ and $w = v_{s_0} = v_{s_1} \in V(c)$. However, this is equivalent to showing that the function

$$s \mapsto \frac{\psi'_v(s)}{s}$$

has at most two zeros. We have

$$\theta(s) := \frac{\psi'_v(s)}{s} = \|\nabla v_s\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu N(q-2)}{2q} s^{\alpha_0} \|v_s\|_{L^q(\mathbb{R}^N)}^q - s^{\alpha_2} \|v_s\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

and

$$\theta'(s) = -\alpha_0 \frac{\mu N(q-2)}{2q} s^{\alpha_0-1} \|v_s\|_{L^q(\mathbb{R}^N)}^q - \alpha_2 s^{\alpha_2-1} \|v_s\|_{L^{2^*}(\mathbb{R}^N)}^{2^*},$$

where

$$\alpha_0 := \frac{N(q-2)}{2} - 2 < 0, \quad \text{and} \quad \alpha_2 := \frac{4}{N-2} > 0.$$

Since $\alpha_0 < 0$ and $\alpha_2 > 0$, the equation $\theta'(s) = 0$ has a unique solution, and $\theta(s)$ has indeed at most two zeros. \square

Lemma 2.2.3. *Let $N \geq 3$ and $c \in (0, c_0)$. For every $u \in S(c)$, the function ψ_u has exactly two critical points s_u^+ and s_u^- with $0 < s_u^+ < s_u^-$. Moreover:*

(i) s_u^+ is a local minimum point for ψ_u , $F_\mu(u_{s_u^+}) < 0$ and $u_{s_u^+} \in V(c)$.

(ii) s_u^- is a global maximum point for ψ_u , $\psi'_u(s) < 0$, for all $s > s_u^+$ and

$$F_\mu(u_{s_u^-}) \geq \inf_{u \in \partial V(c)} F_\mu(u) > 0.$$

(iii) $\psi''_u(s_u^-) < 0$ and the map $u \in S(c) \mapsto s_u^+ \in \mathbb{R}$ is of class C^1 .

Proof. Let $u \in S(c)$ be arbitrary. Since $\psi_u(s) \rightarrow 0^-$, $\|\nabla u_s\|_{L^2(\mathbb{R}^N)} \rightarrow 0$, as $s \rightarrow 0$ and $\psi_u(s) = F_\mu(u_s) > 0$ when $u_s \in \partial V(c) = \{v \in S(c) : \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 = \rho_0\}$, necessarily ψ'_u has a first zero $s_u^+ > 0$ corresponding to a local minima. In particular, $u_{s_u^+} \in V(c)$ and $F(u_{s_u^+}) = \psi_u(s_u^+) < 0$. Now, from $\psi_u(s_u^+) < 0$, $\psi_u(s) > 0$ when $u_s \in \partial V(c)$ and $\psi_u(s) \rightarrow -\infty$ as $s \rightarrow \infty$, ψ'_u has a second zero $s_u^- > s_u^+$ corresponding to a local maxima of ψ_u with $F_\mu(u_{s_u^-}) \geq \inf_{u \in \partial V(c)} F_\mu(u) > 0$.

To conclude the proofs of (i) and (ii), it just suffices to show that ψ'_u has at most two zeros. However, this is equivalent to showing that the function

$$s \mapsto \frac{\psi'_u(s)}{s}$$

has at most two zeros. We have

$$\theta(s) := \frac{\psi'_u(s)}{s} = \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu\gamma_q s^{q\gamma_q - 2} \|u\|_{L^q(\mathbb{R}^N)}^q - s^{2^* - 2} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

and

$$\theta'(s) = -\mu(q\gamma_q - 2)\gamma_q s^{q\gamma_q - 3} \|u\|_{L^q(\mathbb{R}^N)}^q - (2^* - 2)s^{2^* - 3} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

Since $q\gamma_q - 2 < 0$ and $2^* - 2 > 0$, the equation $\theta'(s) = 0$ has a unique solution and hence $\theta(s)$ has indeed at most two zero points.

To establish (iii) let us first show that $\psi''_u(s_u^+) < 0$. In this aim, first note that in view of (i) and (ii), $\psi''_u(s)$ has a zero $s_u^0 \in (s_u^-, s_u^+)$. Now, by direct calculations

$$\psi''_u(s) = \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \mu\gamma_q(q\gamma_q - 1)s^{q\gamma_q - 2} \|u\|_{L^q(\mathbb{R}^N)}^q - (2^* - 1)s^{2^* - 2} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

We distinguish two cases. If $q\gamma_q - 1 \leq 0$ then $\psi''_u(s)$ has at most one zero and we are done. If $q\gamma_q - 1 > 0$, then, knowing that $\psi''_u(s)$ has a zero we deduce that $\psi''_u(s)$ has exactly two zeros that we denote by $s_u^1 < s_u^2$. To conclude it suffices to show that $s_u^0 = s_u^2$ since this would imply that s_u^+ cannot be a zero of $\psi''_u(s)$. To show this, we assume by contradiction that $s_u^0 = s_u^1$. Then, since $\psi''_u(s) < 0$ for $s \in (0, s_u^1)$ and recording that $\psi'_u(s) < 0$ for $s > 0$ small we deduce that $\psi'_u(s) < 0$ for $s \in (0, s_u^0)$. This contradicts the fact that $s_u^+ < s_u^0$ satisfies $\psi'(s_u^+) = 0$. At this point we have proved that $\psi''_u(s_u^+) < 0$. Now (iii) follows from a direct application of the Implicit Function Theorem to the C^1 function $\Psi : \mathbb{R} \times S(c) \mapsto \mathbb{R}$ defined by $\Psi(s, u) = \psi'_u(s)$, taking into account that $\Psi(s_u^+, u) = 0$ and $\partial_s \Psi(s_u^+, u) = \psi''_u(s_u^+) < 0$. \square

In view of Lemma 2.2.3, the set $\Lambda(c) := \{u \in S(c) : Q_\mu(u) = 0\}$ admits the decomposition into the disjoint union $\Lambda(c) = \Lambda^+(c) \cup \Lambda^-(c)$, see (2.1.6) for the definitions of $\Lambda^+(c)$ and $\Lambda^-(c)$.

Lemma 2.2.4. *Let $N \geq 3$. Introducing, for any $c \in (0, c_0)$, the set*

$$W(c) := \{u \in S(c) : s_u^- > 1\}$$

it holds that

(i) $\Lambda^+(c) \subset W(c)$.

(ii) $\partial W(c) = \Lambda^-(c)$ and $\inf_{u \in \partial W(c)} F_\mu(u) > 0$.

(iii) $V(c) \cap \{u : F_\mu(u) < 0\} \subset W(c)$.

(iv) If $m(c)$ is reached then $\inf_{u \in W(c)} F_\mu(u)$ is reached and $\inf_{u \in W(c)} F_\mu(u) = m(c)$.

Proof. Points (i) and (ii) are direct consequence of Lemma 2.2.3 and of the definition of $W(c)$. To prove (iii) we assume by contradiction that there exists a $v \in V(c) \cap \{u : F_\mu(u) < 0\}$ with $v \notin W(c)$. Since $v \notin W(c)$, then by Lemma 2.2.3 (ii) we know that $\psi'_v(s) < 0$ for all $s \geq 1$. Thus, for all $s \geq 1$,

$$F_\mu(v_s) = \psi_v(s) \leq \psi_v(1) = F_\mu(v) < 0. \quad (2.2.9)$$

But, since $v \in V(c)$ there exists a $s_0 > 1$ such that $v_{s_0} \in \partial V(c)$. Recording that $F_\mu(u) > 0$ for any $u \in \partial V(c)$ we get a contradiction with (2.2.9). This proves (iii). Now, still in view of Lemma 2.2.3 and the definition of $W(c)$ we have that

$$\inf_{u \in W(c)} F_\mu(u) = \inf_{u \in \Lambda^+(c)} F_\mu(u). \quad (2.2.10)$$

Also, we know from Lemma 2.2.2(ii) that $\Lambda^+(c) \subset V(c)$ and since any minimizer for F_μ on $V(c)$ must belong to $\Lambda^+(c)$ it follows that

$$m(c) = \inf_{u \in V(c)} F_\mu(u) = \inf_{u \in \Lambda^+(c)} F_\mu(u). \quad (2.2.11)$$

Gathering (2.2.10) and (2.2.11) and recording that $m(c)$ is reached the conclusion follows. \square

2.3 Existence of ground state solutions

Now, letting

$$\alpha_0 := \frac{N(q-2)}{2} - 2, \quad \alpha_1 := \frac{2N - q(N-2)}{2}, \quad \alpha_2 := \frac{4}{N-2},$$

we consider the function $f(c, \rho)$ defined on $(0, \infty) \times (0, \infty)$ by

$$f(c, \rho) = \frac{1}{2} - \frac{\mu}{q} C_{N,q}^q \rho^{\frac{\alpha_0}{2}} c^{\frac{\alpha_1}{2}} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \rho^{\frac{\alpha_2}{2}},$$

and, for each $c \in (0, \infty)$, its restriction $g_c(\rho)$ defined on $(0, \infty)$ by $\rho \mapsto g_c(\rho) := f(c, \rho)$.

For future reference, note that for any $N \geq 3$, $\alpha_0 \in (-2, 0)$, $\alpha_1 \in \left[\frac{4}{N}, 2\right)$ and $\alpha_2 \in (0, 4]$.

Lemma 2.3.1. *For each $c > 0$, the function $g_c(\rho)$ has a unique global maximum and the maximum value satisfies*

$$\begin{cases} \max_{\rho>0} g_c(\rho) > 0 & \text{if } c < c_0, \\ \max_{\rho>0} g_c(\rho) = 0 & \text{if } c = c_0, \\ \max_{\rho>0} g_c(\rho) < 0 & \text{if } c > c_0, \end{cases}$$

where

$$c_0 := \left(\frac{1}{2K} \right)^{\frac{N}{2}} > 0, \quad (2.3.1)$$

with

$$K := \frac{\mu}{q} C_{N,q}^q \left[-\frac{\alpha_0 \mu C_{N,q}^q 2^* S^{\frac{2^*}{2}}}{\alpha_2 q} \right]^{\frac{\alpha_0}{a_2 - \alpha_0}} + \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \left[-\frac{\alpha_0 \mu C_{N,q}^q 2^* S^{\frac{2^*}{2}}}{\alpha_2 q} \right]^{\frac{\alpha_2}{a_2 - \alpha_0}} > 0. \quad (2.3.2)$$

Proof. By definition of $g_c(\rho)$, we have that

$$g'_c(\rho) = -\frac{\alpha_0 \mu}{2} \frac{C_{N,q}^q}{q} \rho^{\frac{\alpha_0}{2} - 1} c^{\frac{\alpha_1}{2}} - \frac{\alpha_2}{2} \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \rho^{\frac{\alpha_2}{2} - 1}.$$

Hence, the equation $g'_c(\rho) = 0$ has a unique solution given by

$$\rho_c = \left[-\frac{\alpha_0 \mu C_{N,q}^q 2^* S^{\frac{2^*}{2}}}{\alpha_2 q} \right]^{\frac{2}{a_2 - \alpha_0}} c^{\frac{\alpha_1}{a_2 - \alpha_0}}. \quad (2.3.3)$$

Taking into account that $g_c(\rho) \rightarrow -\infty$ as $\rho \rightarrow 0$ and $g_c(\rho) \rightarrow -\infty$ as $\rho \rightarrow \infty$, we obtain that ρ_c is the unique global maximum point of $g_c(\rho)$ and the maximum value is

$$\begin{aligned} \max_{\rho>0} g_c(\rho) &= \frac{1}{2} - \frac{\mu}{q} C_{N,q}^q \left[-\frac{\alpha_0 \mu C_{N,q}^q 2^* S^{\frac{2^*}{2}}}{\alpha_2 q} \right]^{\frac{\alpha_0}{a_2 - \alpha_0}} c^{\frac{\alpha_0 \alpha_1}{2(a_2 - \alpha_0)}} c^{\frac{\alpha_1}{2}} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \left[-\frac{\alpha_0 \mu C_{N,q}^q 2^* S^{\frac{2^*}{2}}}{\alpha_2 q} \right]^{\frac{\alpha_2}{a_2 - \alpha_0}} c^{\frac{\alpha_1 \alpha_2}{2(a_2 - \alpha_0)}} \\ &= \frac{1}{2} - \frac{\mu}{q} C_{N,q}^q \left[-\frac{\alpha_0 \mu C_{N,q}^q 2^* S^{\frac{2^*}{2}}}{\alpha_2 q} \right]^{\frac{\alpha_0}{a_2 - \alpha_0}} c^{\frac{\alpha_1 \alpha_2}{2(a_2 - \alpha_0)}} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \left[-\frac{\alpha_0 \mu C_{N,q}^q 2^* S^{\frac{2^*}{2}}}{\alpha_2 q} \right]^{\frac{\alpha_2}{a_2 - \alpha_0}} c^{\frac{\alpha_1 \alpha_2}{2(a_2 - \alpha_0)}} \\ &= \frac{1}{2} - K c^{\frac{2}{N}}. \end{aligned}$$

By the definition of c_0 , we have that $\max_{\rho>0} g_{c_0}(\rho) = 0$, and hence the lemma follows. \square

Lemma 2.3.2. *Let $(c_1, \rho_1) \in (0, \infty) \times (0, \infty)$ be such that $f(c_1, \rho_1) \geq 0$. Then for any $c_2 \in (0, c_1]$, we have that*

$$f(c_2, \rho_2) \geq 0 \quad \text{if } \rho_2 \in \left[\frac{c_2}{c_1} \rho_1, \rho_1 \right].$$

Proof. Since $c \rightarrow f(\cdot, \rho)$ is a non-increasing function we clearly have that

$$f(c_2, \rho_1) \geq f(c_1, \rho_1) \geq 0. \quad (2.3.4)$$

Now taking into account that $\alpha_0 + \alpha_1 = q - 2 > 0$ we have, by direct calculations, that

$$f\left(c_2, \frac{c_2}{c_1} \rho_1\right) \geq f(c_1, \rho_1) \geq 0. \quad (2.3.5)$$

We observe that if $g_{c_2}(\rho') \geq 0$ and $g_{c_2}(\rho'') \geq 0$ then

$$f(c_2, \rho) = g_{c_2}(\rho) \geq 0 \quad \text{for any } \rho \in [\rho', \rho'']. \quad (2.3.6)$$

Indeed, if $g_{c_2}(\rho) < 0$ for some $\rho \in (\rho', \rho'')$ then there exists a local minimum point on (ρ_1, ρ_2) and this contradicts the fact that the function $g_{c_2}(\rho)$ has a unique critical point which has to coincide necessarily with its unique global maximum (see [Lemma 2.3.1](#)). By (2.3.4), (2.3.5), we can choose $\rho' = (c_2/c_1)\rho_1$ and $\rho'' = \rho_1$, and (2.3.6) implies the lemma. \square

Lemma 2.3.3. *For any $u \in S(c)$, we have that*

$$F_\mu(u) \geq \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 f(c, \|\nabla u\|_{L^2(\mathbb{R}^N)}^2).$$

Proof. Applying the Gagliardo-Nirenberg inequality (2.2.2) and the Sobolev inequality (2.2.1) we obtain that, for any $u \in S(c)$,

$$\begin{aligned} F_\mu(u) &= \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ &\geq \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} C_{N,q}^q \|\nabla u\|_2^{\alpha_0+2} \|u\|_2^{\alpha_1} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \|\nabla u\|_{L^2(\mathbb{R}^N)}^{2^*} \\ &= \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \left[\frac{1}{2} - \frac{\mu}{q} C_{N,q}^q \|\nabla u\|_2^{\alpha_0} \|u\|_2^{\alpha_1} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \|\nabla u\|_{L^2(\mathbb{R}^N)}^{\alpha_2} \right] \\ &= \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 f(\|u\|_2^2, \|\nabla u\|_{L^2(\mathbb{R}^N)}^2). \end{aligned}$$

The lemma is proved. \square

Now let $c_0 > 0$ be given by (2.3.1) and $\rho_0 := \rho_{c_0} > 0$ being determined by (2.3.3). Note that by [Lemma 2.3.1](#) and [Lemma 2.3.2](#), we have that $f(c_0, \rho_0) = 0$ and $f(c, \rho_0) > 0$ for all $c \in (0, c_0)$. We define

$$B_{\rho_0} := \{u \in H : \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 < \rho_0\} \quad \text{and} \quad V(c) := S(c) \cap B_{\rho_0}.$$

We shall now consider the following local minimization problem: for any $c \in (0, c_0)$,

$$m(c) := \inf_{u \in V(c)} F_\mu(u) \quad (2.3.7)$$

and also consider the set

$$\mathcal{M}_c := \{u \in V(c) : F_\mu(u) = m(c)\}. \quad (2.3.8)$$

The main aim of this section is the following result.

Theorem 2.3.4. For any $c \in (0, c_0)$, if $(u_n) \subset B_{\rho_0}$ is such that $\|u_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow c$ and $F_\mu(u_n) \rightarrow m(c)$ then, up to translation, $u_n \xrightarrow{H} u \in \mathcal{M}_c$. In particular the set \mathcal{M}_c is compact in H , up to translation.

[Theorem 2.3.4](#) will both imply the existence of a ground state but also, as it may be expected, will be a crucial step to derive the orbital stability of the set \mathcal{M}_c .

In order to prove [Theorem 2.3.4](#) we collect some properties of $m(c)$ defined in [\(2.3.7\)](#).

Lemma 2.3.5. It holds that

(i) For any $c \in (0, c_0)$,

$$m(c) = \inf_{u \in V(c)} F_\mu(u) < 0 < \inf_{u \in \partial V(c)} F_\mu(u). \quad (2.3.9)$$

(ii) $c \in (0, c_0) \mapsto m(c)$ is a continuous mapping.

(iii) For any $c \in (0, c_0)$, we have for all $\alpha \in (0, c)$: $m(c) \leq m(\alpha) + m(c - \alpha)$ and if $m(\alpha)$ or $m(c - \alpha)$ is reached then the inequality is strict.

(iv) For any $c \in (0, c_0)$, there exists a $d = d(c) > 0$ such that $m(c - \alpha) \leq m(c) + d\alpha$ for any $\alpha \in (0, c)$.

Proof. (i) For any $u \in \partial V(c)$ we have $\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 = \rho_0$. Thus, using [Lemma 2.3.3](#), we get

$$F_\mu(u) \geq \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 f(\|u\|_2^2, \|\nabla u\|_{L^2(\mathbb{R}^N)}^2) = \rho_0 f(c, \rho_0) > 0.$$

Now let $u \in S(c)$ be arbitrary but fixed. Recall from [Section 2.2](#) that for any $s \in (0, \infty)$, we have $u_s \in S(c)$ and

$$\psi_u(s) = F_\mu(u_s) = \frac{s^2}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} s^{q\gamma_q} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{s^{2^*}}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

Taking into account that $q\gamma_q < 2$ and $2^* > 2$ we see that $\psi_u(s) \rightarrow 0^-$, as $s \rightarrow 0$. Therefore, there exists $s_0 > 0$ small enough such that $\|\nabla(u_{s_0})\|_{L^2(\mathbb{R}^N)}^2 = s_0^2 \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 < \rho_0$ and $F_\mu(u_{s_0}) = \psi_u(s_0) < 0$. This implies that $m(c) < 0$.

(ii) Let $c \in (0, c_0)$ be arbitrary and $(c_n) \subset (0, c_0)$ be such that $c_n \rightarrow c$. From the definition of $m(c_n)$ and since $m(c_n) < 0$, see (i), for any $\varepsilon > 0$ sufficiently small, there exists $u_n \in V(c_n)$ such that

$$F_\mu(u_n) \leq m(c_n) + \varepsilon \quad \text{and} \quad F_\mu(u_n) < 0. \quad (2.3.10)$$

We set $y_n := \sqrt{\frac{c}{c_n}} u_n$ and hence $y_n \in S(c)$. We have that $y_n \in V(c)$. Indeed, if $c_n \geq c$, then

$$\|\nabla y_n\|_{L^2(\mathbb{R}^N)}^2 = \frac{c}{c_n} \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 \leq \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 < \rho_0.$$

If $c_n < c$, by [Lemma 2.3.2](#), we have $f(c_n, \rho) \geq 0$ for any $\rho \in \left[\frac{c_n}{c} \rho_0, \rho_0\right]$. Hence, we deduce from [Lemma 2.3.3](#) and [\(2.3.10\)](#) that $f(c_n, \|\nabla u_n\|_2^2) < 0$, thus $\|\nabla u_n\|_2^2 < \frac{c_n}{c} \rho_0$ and

$$\|\nabla y_n\|_{L^2(\mathbb{R}^N)}^2 = \frac{c}{c_n} \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 < \frac{c}{c_n} \frac{c_n}{c} \rho_0 = \rho_0.$$

Since $y_n \in V(c)$ we can write

$$m(c) \leq F_\mu(y_n) = F_\mu(u_n) + [F_\mu(y_n) - F_\mu(u_n)]$$

where

$$F_\mu(y_n) - F_\mu(u_n) = -\frac{1}{2}\left(\frac{c}{c_n} - 1\right)\|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q}\left[\left(\frac{c}{c_n}\right)^{\frac{q}{2}} - 1\right]\|u_n\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*}\left[\left(\frac{c}{c_n}\right)^{\frac{2^*}{2}} - 1\right]\|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

Since $\|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 < \rho_0$, also $\|u_n\|_{L^q(\mathbb{R}^N)}^q$ and $\|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$ are uniformly bounded. Thus, we have that as $n \rightarrow \infty$,

$$m(c) \leq F_\mu(y_n) = F_\mu(u_n) + o_n(1). \quad (2.3.11)$$

Combining (2.3.10) and (2.3.11), we get

$$m(c) \leq m(c_n) + \varepsilon + o_n(1).$$

Now, let $u \in V(c)$ be such that

$$F_\mu(u) \leq m(c) + \varepsilon \quad \text{and} \quad F_\mu(u) < 0.$$

Set $u_n := \sqrt{\frac{c_n}{c}}u$ and hence $u_n \in S(c_n)$. Clearly, $\|\nabla u\|_2^2 < \rho_0$ and $c_n \rightarrow c$ imply $\|\nabla u_n\|_2^2 < \rho_0$ for n large enough, so that $u_n \in V(c_n)$. Also, $F_\mu(u_n) \rightarrow F_\mu(u)$. We thus have

$$m(c_n) \leq F_\mu(u_n) = F_\mu(u) + [F_\mu(u_n) - F_\mu(u)] \leq m(c) + \varepsilon + o_n(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $m(c_n) \rightarrow m(c)$. The point (ii) follows.

(iii) Note that, fixed $\alpha \in (0, c)$, it is sufficient to prove that the following holds

$$\forall \theta \in \left(1, \frac{c}{\alpha}\right] : m(\theta\alpha) \leq \theta m(\alpha) \quad (2.3.12)$$

and that, if $m(\alpha)$ is reached, the inequality is strict. Indeed, if (2.3.12) holds then we have

$$m(c) = \frac{c-\alpha}{c}m(c) + \frac{\alpha}{c}m(c) = \frac{c-\alpha}{c}m\left(\frac{c}{c-\alpha}(c-\alpha)\right) + \frac{\alpha}{c}m\left(\frac{c}{\alpha}\alpha\right) \leq m(c-\alpha) + m(\alpha),$$

with a strict inequality if $m(\alpha)$ is reached. To prove that (2.3.12) holds, note that in view of (i), for any $\varepsilon > 0$ sufficiently small, there exists a $u \in V(\alpha)$ such that

$$F_\mu(u) \leq m(\alpha) + \varepsilon \quad \text{and} \quad F_\mu(u) < 0. \quad (2.3.13)$$

In view of Lemma 2.3.2, $f(\alpha, \rho) \geq 0$ for any $\rho \in \left[\frac{\alpha}{c}\rho_0, \rho_0\right]$. Hence, we can deduce from Lemma 2.3.3 and (2.3.13) that

$$\|\nabla u\|_2^2 < \frac{\alpha}{c}\rho_0. \quad (2.3.14)$$

Consider now $v = \sqrt{\theta}u$. We first note that $\|v\|_2^2 = \theta\|u\|_2^2 = \theta\alpha$ and also, because of (2.3.14), $\|\nabla v\|_2^2 = \theta\|\nabla u\|_2^2 < \rho_0$. Thus $v \in V(\theta\alpha)$ and we can write

$$\begin{aligned} m(\theta\alpha) &\leq F_\mu(v) = \frac{1}{2}\theta\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q}\theta^{\frac{q}{2}}\|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*}\theta^{\frac{2^*}{2}}\|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ &< \frac{1}{2}\theta\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q}\theta\|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*}\theta\|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ &= \theta\left(\frac{1}{2}\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q}\|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*}\|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\right) = \theta F_\mu(u) \leq \theta(m(\alpha) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that $m(\theta\alpha) \leq \theta m(\alpha)$. If $m(\alpha)$ is reached then we can let $\varepsilon = 0$ in (2.3.13) and thus the strict inequality follows.

(iv) Let $u \in V(c)$ be a minimizer of $m(c)$ and set $y_\alpha := \sqrt{\frac{c-\alpha}{c}} \cdot u$. Since $y_\alpha \in V(c-\alpha)$ we have

$$m(c-\alpha) \leq F_\mu(y_\alpha) = m(c) + [F_\mu(y_\alpha) - F_\mu(u)]$$

with

$$F_\mu(y_\alpha) - F_\mu(u) = -\frac{\alpha}{2c} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \left[\left(\frac{c-\alpha}{c} \right)^q - 1 \right] \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \left[\left(\frac{c-\alpha}{c} \right)^{2^*} - 1 \right] \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

The result now follows from the observation that $\alpha \mapsto F_\mu(y_\alpha) - F_\mu(u)$ is of class C^1 on $(0, c)$. \square

Lemma 2.3.6. *Let $(v_n) \subset B_{\rho_0}$ be such that $\|v_n\|_{L^q(\mathbb{R}^N)} \rightarrow 0$. Then there exists a $\beta_0 > 0$ such that*

$$F_\mu(v_n) \geq \beta_0 \|\nabla v_n\|_2^2 + o_n(1).$$

Proof. Indeed, using the Sobolev inequality (2.2.1), we obtain that

$$\begin{aligned} F_\mu(v_n) &= \frac{1}{2} \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2^*} \|v_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + o_n(1) \geq \frac{1}{2} \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^{2^*} + o_n(1) \\ &= \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 \left[\frac{1}{2} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^{\frac{\alpha_2}{2}} \right] + o_n(1) \geq \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 \left[\frac{1}{2} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \rho_0^{\frac{\alpha_2}{2}} \right] + o_n(1). \end{aligned}$$

Now, since $f(c_0, \rho_0) = 0$, we have that

$$\beta_0 := \left[\frac{1}{2} - \frac{1}{2^*} \frac{1}{S^{\frac{2^*}{2}}} \rho_0^{\frac{\alpha_2}{2}} \right] = \frac{\mu}{q} C_{N,q}^q \rho_0^{\frac{\alpha_0}{2}} c_0^{\frac{\alpha_1}{2}} > 0.$$

\square

Lemma 2.3.7. *For any $c \in (0, c_0)$, let $(u_n) \subset B_{\rho_0}$ be such that $\|u_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow c$ and $F_\mu(u_n) \rightarrow m(c)$. Then, there exist a $\beta_1 > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that*

$$\int_{B(y_n, R)} |u_n|^2 dx \geq \beta_1 > 0, \quad \text{for some } R > 0. \quad (2.3.15)$$

Proof. We assume by contradiction that (2.3.15) does not hold. By $(u_n) \subset B_{\rho_0}$ and $\|u_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow c$, the sequence (u_n) is bounded in H . From [65, Lemma I.1] and since $2 < q < 2^*$, we deduce that $\|u_n\|_{L^q(\mathbb{R}^N)} \rightarrow 0$, as $n \rightarrow \infty$. At this point, Lemma 2.3.6 implies that $F_\mu(u_n) \geq o_n(1)$. This contradicts the fact that $m(c) < 0$ and the lemma follows. \square

Proof of Theorem 2.3.4. We know from Lemma 2.3.7 and Rellich compactness theorem that there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$u_n(x - y_n) \rightharpoonup u_c \neq 0 \quad \text{in } H.$$

Our aim is to prove that $w_n(x) := u_n(x - y_n) - u_c(x) \rightarrow 0$ in H . Clearly

$$\begin{aligned} \|u_n\|_{L^2(\mathbb{R}^N)}^2 &= \|u_n(x - y_n)\|_{L^2(\mathbb{R}^N)}^2 = \|u_n(x - y_n) - u_c(x)\|_{L^2(\mathbb{R}^N)}^2 + \|u_c\|_{L^2(\mathbb{R}^N)}^2 + o_n(1) \\ &= \|w_n\|_{L^2(\mathbb{R}^N)}^2 + \|u_c\|_{L^2(\mathbb{R}^N)}^2 + o_n(1). \end{aligned}$$

Thus, we have

$$\|w_n\|_{L^2(\mathbb{R}^N)}^2 = \|u_n\|_{L^2(\mathbb{R}^N)}^2 - \|u_c\|_{L^2(\mathbb{R}^N)}^2 + o_n(1) = c - \|u_c\|_{L^2(\mathbb{R}^N)}^2 + o_n(1). \quad (2.3.16)$$

By a similar argument,

$$\|\nabla w_n\|_{L^2(\mathbb{R}^N)}^2 = \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 - \|\nabla u_c\|_{L^2(\mathbb{R}^N)}^2 + o_n(1). \quad (2.3.17)$$

More generally it is direct to show, using the Brezis-Lieb lemma [24, Theorem 1], that the other terms in F_μ also enjoy a the splitting property, and thus we have

$$F_\mu(w_n) + F_\mu(u_c) = F_\mu(u_n(x - y_n)) + o_n(1),$$

and, by the translational invariance, we obtain

$$F_\mu(u_n) = F_\mu(u_n(x - y_n)) = F_\mu(w_n) + F_\mu(u_c) + o_n(1). \quad (2.3.18)$$

Now, we claim that

$$\|w_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow 0. \quad (2.3.19)$$

In order to prove this, let us denote $c_1 := \|u_c\|_{L^2(\mathbb{R}^N)}^2 > 0$. By (2.3.16), if we show that $c_1 = c$ then the claim follows. We assume by contradiction that $c_1 < c$. In view of (2.3.16) and (2.3.17), for n large enough, we have $\|w_n\|_{L^2(\mathbb{R}^N)}^2 \leq c$ and $\|\nabla w_n\|_{L^2(\mathbb{R}^N)}^2 \leq \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 < \rho_0$. Hence, we obtain that $w_n \in V(\|w_n\|_{L^2(\mathbb{R}^N)}^2)$ and $F_\mu(w_n) \geq m(\|w_n\|_{L^2(\mathbb{R}^N)}^2)$. Recoding that $F_\mu(u_n) \rightarrow m(c)$, in view of (2.3.18), we have

$$m(c) = F_\mu(w_n) + F_\mu(u_c) + o_n(1) \geq m(\|w_n\|_{L^2(\mathbb{R}^N)}^2) + F_\mu(u_c) + o_n(1).$$

Since the map $c \mapsto m(c)$ is continuous (see Lemma 2.3.5(ii)) and in view of (2.3.16), we deduce that

$$m(c) \geq m(c - c_1) + F_\mu(u_c). \quad (2.3.20)$$

We also have that $u_c \in V(c_1)$ by the weak limit. This implies that $F_\mu(u_c) \geq m(c_1)$. If $F_\mu(u_c) > m(c_1)$, then it follows from (2.3.20) and Lemma 2.3.5(iii) that

$$m(c) > m(c - c_1) + m(c_1) \geq m(c - c_1 + c_1) = m(c),$$

which is impossible. Hence, we have $F_\mu(u_c) = m(c_1)$, namely u_c is a local minimizer on $V(c_1)$. So, using Lemma 2.3.5(iii) with the strict inequality, we deduce from (2.3.20) that

$$m(c) \geq m(c - c_1) + F_\mu(u_c) = m(c - c_1) + m(c_1) > m(c - c_1 + c_1) = m(c),$$

which is impossible. Thus, the claim (2.3.19) follows and from (2.3.16) we deduce that $\|u_c\|_{L^2(\mathbb{R}^N)}^2 = c$.

Let us now show that $\|\nabla w_n\|_2^2 \rightarrow 0$. This will prove that $w_n \rightarrow 0$ in H and completes the proof. In this aim first observe that in view of (2.3.17) and since $u_c \neq 0$, we have $\|\nabla w_n\|_{L^2(\mathbb{R}^N)}^2 \leq \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 < \rho_0$, for n large enough. Hence $(w_n) \subset B_{\rho_0}$ and in particular it is bounded in H . Then by using the Gagliardo-Nirenberg inequality (2.2.2), and by recalling $\|w_n\|_2^2 \rightarrow 0$ we also have $\|w_n\|_q^q \rightarrow 0$. Thus Lemma 2.3.6 implies that

$$F_\mu(w_n) \geq \beta_0 \|\nabla w_n\|_{L^2(\mathbb{R}^N)}^2 + o_n(1) \text{ where } \beta_0 > 0. \quad (2.3.21)$$

Now we remember that

$$F_\mu(u_n) = F_\mu(u_c) + F_\mu(w_n) + o_n(1) \rightarrow m(c).$$

Since $u_c \in V(c)$ by weak limit, we have that $F_\mu(u_c) \geq m(c)$ and hence $F_\mu(w_n) \leq o_n(1)$. In view of (2.3.21), we then conclude that $\|\nabla w_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow 0$. \square

We end this section with,

Proof of Theorem 2.1.2. The fact that if $(u_n) \subset V(c)$ is such that $F_\mu(u_n) \rightarrow m(c)$ then, up to translation, $u_n \rightarrow u \in \mathcal{M}_c$ in H follows from Theorem 2.3.4. In particular, it insures the existence of a minimizer for F_μ on $V(c)$. The fact that this minimizer is a ground state and that any ground state for F_μ on $S(c)$ belongs to $V(c)$ was proved in Lemma 2.2.2. \square

2.4 Orbital stability of the ground state solutions

Now, we focus on the local existence of solutions to the following Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u + \mu u|u|^{q-2} + u|u|^{2^*-2} = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad N \geq 3 \\ u(0, x) = \varphi(x) \in H. \end{cases} \quad (2.4.1)$$

Denoting $g : \mathbb{C} \rightarrow \mathbb{C}$ by $g(u) := \mu u|u|^{q-2} + u|u|^{2^*-2}$, (2.4.1) reads as

$$i\partial_t u + \Delta u + g(u) = 0.$$

Next we give the notion of integral equation associated with (2.4.1). In order to do that first we give another definition.

Definition 2.4.1. *If $N \geq 3$ the pair (p, r) is said to be (Schrödinger) admissible if*

$$\frac{2}{p} + \frac{N}{r} = \frac{N}{2}, \quad p, r \in [2, \infty].$$

We shall work with two particular admissible pairs (see Lemma 2.4.5):

$$(p_1, r_1) := \left(\frac{4q}{(q-2)(N-2)}, \frac{Nq}{q+N-2} \right),$$

and

$$(p_2, r_2) := \left(\frac{4 \times 2^*}{(2^* - 2)(N - 2)}, \frac{N \times 2^*}{2^* + N - 2} \right).$$

Along with those couples we introduce the spaces $Y_T := Y_{p_1, r_1, T} \cap Y_{p_2, r_2, T}$ and $X_T := X_{p_1, r_1, T} \cap X_{p_2, r_2, T}$ equipped with the following norms:

$$\|w\|_{Y_T} = \|w\|_{Y_{p_1, r_1, T}} + \|w\|_{Y_{p_2, r_2, T}}, \quad \text{and} \quad \|w\|_{X_T} = \|w\|_{X_{p_1, r_1, T}} + \|w\|_{X_{p_2, r_2, T}}. \quad (2.4.2)$$

where for a generic function $w(t, x)$ defined on the time-space strip $[0, T) \times \mathbb{R}^N$ we have defined:

$$\|w(t, x)\|_{Y_{p, r, T}} = \left(\int_0^T \|w(t, \cdot)\|_{L^r(\mathbb{R}^N)}^p dt \right)^{\frac{1}{p}} \quad \text{and} \quad \|w(t, x)\|_{X_{p, r, T}} = \left(\int_0^T \|w(t, \cdot)\|_{W^{1, r}(\mathbb{R}^N)}^p dt \right)^{\frac{1}{p}}.$$

Now, we give a definition of an integral solution to (2.4.1) on the time interval (for more detail we refer to [27, Chapter 2]):

Definition 2.4.2. *Let $T > 0$. We say that $u(t, x)$ is an integral solution of the Cauchy problem (2.4.1) on the time interval $[0, T]$ if:*

1. $u \in \mathcal{C}([0, T], H) \cap X_T$;
2. for all $t \in (0, T)$ it holds $u(t) = e^{it\Delta}\varphi - i \int_0^t e^{i(t-s)\Delta}g(u(s))ds$.

The first main result of this section is the following local existence result. We do not claim a real originality here, related versions already exist in the literature, see for example [54, Theorem 2.5]. However, we believe convenient to the reader to provide a version specifically adapted to our problem and to give a proof of this result as self-contained as possible.

Proposition 2.4.3. *There exists $\gamma_0 > 0$ such that if $\varphi \in H$ and $T \in (0, 1]$ satisfy*

$$\|e^{it\Delta}\varphi\|_{X_T} \leq \gamma_0,$$

then there exists a unique integral solution $u(t, x)$ to (2.4.1) on the time interval $[0, T]$. Moreover $u(t, x) \in X_{p, r, T}$ for every admissible couple (p, r) and satisfies the following conservation laws:

$$F_\mu(u(t)) = F_\mu(\varphi), \quad \|u(t)\|_{L^2(\mathbb{R}^N)} = \|\varphi\|_{L^2(\mathbb{R}^N)}, \quad \text{for all } t \in [0, T]. \quad (2.4.3)$$

In order to prove Proposition 2.4.3 we need some preliminary results.

Let us recall Strichartz's estimates that will be useful in the sequel (see for example [27, Theorem 2.3.3 and Remark 2.3.8] and [53] for the endpoint estimates).

Proposition 2.4.4. *Let $N \geq 3$ then for every admissible pairs (p, r) and (\tilde{p}, \tilde{r}) , there exists a constant $C > 0$ such that for every $T > 0$, the following properties hold:*

- (i) For every $\varphi \in L^2(\mathbb{R}^N)$, the function $t \mapsto e^{it\Delta}\varphi$ belongs to $Y_{p, r, T} \cap \mathcal{C}([0, T], L^2(\mathbb{R}^N))$ and

$$\|e^{it\Delta}\varphi\|_{Y_{p, r, T}} \leq C\|\varphi\|_2.$$

(ii) Let $F \in Y_{\tilde{p}, \tilde{r}, T}$, where we use a prime to denote conjugate indices. Then the function

$$t \mapsto \Phi_F(t) := \int_0^t e^{i(t-s)\Delta} F(s) ds$$

belongs to $Y_{p,r,T} \cap \mathcal{C}([0, T], L^2(\mathbb{R}^N))$ and

$$\|\Phi_F\|_{Y_{p,r,T}} \leq C \|F\|_{Y_{\tilde{p}, \tilde{r}, T}}.$$

(iii) For every $\varphi \in H$, the function $t \mapsto e^{it\Delta} \varphi$ belongs to $X_{p,r,T} \cap \mathcal{C}([0, T], H)$ and

$$\|e^{it\Delta} \varphi\|_{X_{p,r,T}} \leq C \|\varphi\|_H.$$

The following result will be useful in the sequel.

Lemma 2.4.5. *Let $N \geq 3$ and $2 < \alpha \leq 2^*$ be given. Then the couple (p, r) defined as follows*

$$p := \frac{4\alpha}{(\alpha - 2)(N - 2)} \quad \text{and} \quad r := \frac{N\alpha}{\alpha + N - 2}$$

is admissible. Moreover for every admissible couple (\tilde{p}, \tilde{r}) there exists a constant $C > 0$ such that for every $T > 0$ the following inequalities hold:

$$\left\| \int_0^t e^{i(t-s)\Delta} [\nabla g_\alpha(u(s))] ds \right\|_{Y_{\tilde{p}, \tilde{r}, T}} \leq CT^\mu \|\nabla u\|_{Y_{p,r,T}}^{\alpha-1}, \quad (2.4.4)$$

$$\left\| \int_0^t e^{i(t-s)\Delta} [g_\alpha(u(s)) - g_\alpha(v(s))] ds \right\|_{Y_{\tilde{p}, \tilde{r}, T}} \leq CT^\mu (\|\nabla u\|_{Y_{p,r,T}}^{\alpha-2} + \|\nabla v\|_{Y_{p,r,T}}^{\alpha-2}) \|u - v\|_{Y_{p,r,T}}, \quad (2.4.5)$$

where $g_\alpha(u) := |u|^{\alpha-2} u$ and $\mu := \frac{(N-2)(2^* - \alpha)}{4} \geq 0$.

Proof. By direct calculations, one can check that

$$\frac{2}{p} + \frac{N}{r} = \frac{N}{2} \quad \text{and} \quad p, r \geq 2.$$

Hence, (p, r) is an admissible pair. Also it is easy to check that there exists a $C > 0$ such that :

$$|g'_\alpha(u)| \leq C|u|^{\alpha-2}, \quad (2.4.6)$$

$$|g_\alpha(u) - g_\alpha(v)| \leq C|u - v|(|u|^{\alpha-2} + |v|^{\alpha-2}). \quad (2.4.7)$$

Combining (2.4.6) and the Chain Rule, gives

$$|\nabla g_\alpha(u)| = |g'_\alpha(u) \nabla u| \leq C |\nabla u| |u|^{\alpha-2}.$$

Using Hölder's inequality, we obtain that

$$\|\nabla g_\alpha(u)\|_{L^{r'}(\mathbb{R}^N)} \leq C \|\nabla u\| |u|^{\alpha-2} \|u\|_{L^{r'}(\mathbb{R}^N)}^{\alpha-2} \leq C \|\nabla u\|_{L^{r'}(\mathbb{R}^N)} \|u\|_{L^{r'}(\mathbb{R}^N)}^{\alpha-1},$$

where we also used the Sobolev embedding of $W^{1,r}(\mathbb{R}^N)$ into $L^{r^*}(\mathbb{R}^N)$ with $r^* := \frac{Nr}{N-r}$, see [23, Theorem 9.9]. Hence, using Hölder's inequality,

$$\begin{aligned} \|\nabla g_\alpha(u)\|_{Y_{p',r',T}} &= \left(\int_0^T \|\nabla g_\alpha(u)\|_{L^{p'}(\mathbb{R}^N)}^{p'} dt \right)^{\frac{1}{p'}} \leq C \left(\int_0^T \|\nabla u\|_r^{(\alpha-1)p'} dt \right)^{\frac{1}{p'}} \\ &\leq CT^{(\alpha-1)\left(\frac{1}{(\alpha-1)p'} - \frac{1}{p}\right)} \left(\int_0^T \|\nabla u\|_r^p dt \right)^{\frac{\alpha-1}{p}} = CT^\mu \|\nabla u\|_{Y_{p,r,T}}^{\alpha-1}. \end{aligned}$$

At this point (2.4.4) follows by applying Proposition 2.4.4 (ii). To establish (2.4.5) note that by (2.4.7) and the Hölder's inequality, we have

$$\|g_\alpha(u) - g_\alpha(v)\|_{L^{r'}(\mathbb{R}^N)} \leq C \| |u - v| (|u|^{\alpha-2} + |v|^{\alpha-2}) \|_{L^{r'}(\mathbb{R}^N)} \leq C \|u - v\|_{L^r(\mathbb{R}^N)} \| |u| + |v| \|_{L^{r^*}(\mathbb{R}^N)}^{\alpha-2}.$$

Hence, we can deduce that

$$\begin{aligned} \|g_\alpha(u) - g_\alpha(v)\|_{Y_{p',r',T}} &= \left(\int_0^T \|g_\alpha(u) - g_\alpha(v)\|_{L^{r'}(\mathbb{R}^N)}^{p'} dt \right)^{\frac{1}{p'}} \\ &\leq C \left(\int_0^T \|u - v\|_{L^r(\mathbb{R}^N)}^{p'} \| |u| + |v| \|_{L^{r^*}(\mathbb{R}^N)}^{(\alpha-2)p'} dt \right)^{\frac{1}{p'}} \\ &\leq C \left(\int_0^T \|u - v\|_{L^r(\mathbb{R}^N)}^p dt \right)^{\frac{1}{p}} \left(\int_0^T \| |u| + |v| \|_{L^{r^*}(\mathbb{R}^N)}^{\frac{(\alpha-2)pp'}{p-p'}} dt \right)^{\frac{p-p'}{pp'}} \\ &\leq CT^\mu \|u - v\|_{Y_{p,r,T}} \left(\int_0^T \| |u| + |v| \|_{L^{r^*}(\mathbb{R}^N)}^p dt \right)^{\frac{\alpha-2}{p}} \\ &= CT^\mu \| |u| + |v| \|_{Y_{p,r^*,T}}^{\alpha-2} \|u - v\|_{Y_{p,r,T}} \\ &\leq CT^\mu \left(\|u\|_{Y_{p,r^*,T}} + \|v\|_{Y_{p,r^*,T}} \right)^{\alpha-2} \|u - v\|_{Y_{p,r,T}}. \end{aligned}$$

The inequality (2.4.5) follows by applying the previous Sobolev embedding and Proposition 2.4.4 (ii). \square

In order to prove Proposition 2.4.3 we shall need two lemmas from Functional Analysis.

Lemma 2.4.6. *For all $1 < p, r < \infty$, $X_{p,r,T}$ is a separable reflexive Banach space.*

Proof. This is a direct consequence of Phillips' theorem, see [35, Chapter IV, Corollary 2]. \square

Lemma 2.4.7. *For all $R, T > 0$ the metric space $(B_{R,T}, d)$, where*

$$B_{R,T} := \{u \in X_T : \|u\|_{X_T} \leq R\},$$

and

$$d(u, v) := \|u - v\|_{Y_T}$$

is complete.

Proof. Let (u_n) be a Cauchy sequence. Since Y_T is a Banach space, there exists $u \in Y_T$ such that

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{Y_T} = 0.$$

It remains to show that $u \in B_{R,T}$.

By taking a subsequence, we can assume that $l_1 := \lim_{n \rightarrow \infty} \|u_n\|_{X_{p_1, r_1, T}}$ and $l_2 := \lim_{n \rightarrow \infty} \|u_n\|_{X_{p_2, r_2, T}}$ exist. By [Lemma 2.4.6](#), there exists a subsequence of (u_n) which converges weakly in $X_{p_1, r_1, T}$. In particular, this sequence converges in the sense of distributions and hence the limit equals u . Thus,

$$\|u\|_{X_{p_1, r_1, T}} \leq l_1.$$

Similarly,

$$\|u\|_{X_{p_2, r_2, T}} \leq l_2.$$

Taking the sum, we get $\|u\|_{X_T} \leq l_1 + l_2 \leq R$. \square

Proof of Proposition 2.4.3. Step 1. Existence and uniqueness in $B_{2\gamma_0, T}$ for γ_0 small enough. For any $u \in X_T$ and $t \in [0, T]$, we define

$$\Phi(u)(t) := e^{it\Delta} \varphi + i \int_0^t e^{i(t-s)\Delta} g(u(s)) ds. \quad (2.4.8)$$

We claim that, if $\gamma_0 > 0$ is small enough, then Φ defines a contraction on the metric space $(B_{2\gamma_0, T}, d)$ (see [Lemma 2.4.7](#)).

Let $u \in B_{2\gamma_0, T}$ and consider any admissible pair (\tilde{p}, \tilde{r}) . Let $T \in (0, 1]$ and apply [Lemma 2.4.5](#). We deduce from (2.4.4) and (2.4.8) that

$$\|\nabla \Phi(u) - e^{it\Delta} \nabla \varphi\|_{Y_{\tilde{p}, \tilde{r}, T}} \leq C \|\nabla u\|_{Y_{p_1, r_1, T}}^{q-1} + C \|\nabla u\|_{Y_{p_2, r_2, T}}^{2^*-1} \leq C 2^q \gamma_0^{q-1}, \quad \forall u \in B_{2\gamma_0, T}.$$

Similarly, we deduce from (2.4.5) (applied with $v = 0$) that

$$\|\Phi(u) - e^{it\Delta} \varphi\|_{Y_{\tilde{p}, \tilde{r}, T}} \leq C \|\nabla u\|_{Y_{p_1, r_1, T}}^{q-2} \|u\|_{Y_{p_1, r_1, T}} + C \|\nabla u\|_{Y_{p_2, r_2, T}}^{2^*-2} \|u\|_{Y_{p_2, r_2, T}} \leq C 2^q \gamma_0^{q-1}, \quad \forall u \in B_{2\gamma_0, T}.$$

In particular if we choose $(\tilde{p}, \tilde{r}) = (p_1, r_1)$ and $(\tilde{p}, \tilde{r}) = (p_2, r_2)$ then

$$\|\Phi(u)\|_{X_T} \leq \gamma_0 + C 2^q \gamma_0^{q-1}$$

and hence if $\gamma_0 > 0$ is small enough in such a way that $C 2^{q+2} \gamma_0^{q-1} \leq \gamma_0$, then $B_{2\gamma_0, T}$ is an invariant set of Φ .

Now, let $u, v \in B_{2\gamma_0, T}$. By (2.4.5), we have for every admissible pair (\tilde{p}, \tilde{r})

$$\begin{aligned} & \|\Phi(u) - \Phi(v)\|_{Y_{\tilde{p}, \tilde{r}, T}} \\ & \leq C \left(\|\nabla u\|_{Y_{p_1, r_1, T}}^{q-2} + \|\nabla v\|_{Y_{p_1, r_1, T}}^{q-2} \right) \|u - v\|_{Y_{p_1, r_1, T}} + C \left(\|\nabla u\|_{Y_{p_2, r_2, T}}^{2^*-2} + \|\nabla v\|_{Y_{p_2, r_2, T}}^{2^*-2} \right) \|u - v\|_{Y_{p_2, r_2, T}} \\ & \leq C 2^q \gamma_0^{q-2} (\|u - v\|_{Y_{p_1, r_1, T}} + \|u - v\|_{Y_{p_2, r_2, T}}), \quad \forall u, v \in B_{2\gamma_0, T}. \end{aligned}$$

In particular if we choose $(\tilde{p}, \tilde{r}) = (p_1, r_1)$ and $(\tilde{p}, \tilde{r}) = (p_2, r_2)$ then

$$\|\Phi(u) - \Phi(v)\|_{Y_T} \leq C 2^{q+1} \gamma_0^{q-2} \|u - v\|_{Y_T}$$

and if we choose $\gamma_0 > 0$ small enough in such a way that $C2^{q+1}\gamma_0^{q-2} < \frac{1}{2}$ then Φ is a contraction on $(B_{2\gamma_0, T}, d)$. In particular Φ has one unique fixed point in this space. The property $u \in C([0, T], H)$ and $u \in X_{p, r, T}$ for every admissible couple (p, r) is straightforward and follows by Strichartz estimates.

Step 2. Uniqueness in X_T . Assume $u_1(t, x)$ and $u_2(t, x)$ are two fixed points of Φ in the space X_T . We define $T_0 = \sup\{\bar{T} \in [0, T] \mid \sup_i \|u_i(t, x)\|_{X_{\bar{T}}} \leq 2\gamma_0\}$. It is easy to show that $T_0 \in (0, \bar{T})$ and arguing as in step 1 the operator Φ is a contraction on $(B_{2\gamma_0, T_0}, d)$. Hence by uniqueness of the fixed point in this space necessarily $u_1(t, x) = u_2(t, x)$ in X_{T_0} . Moreover since $u_i(t, x) \in \mathcal{C}([0, T_0]; H)$ we have $u_1(T_0, x) = u_2(T_0, x) = \psi(x)$. Hence at time T_0 the solutions coincide and starting from T_0 (that we can also identify with $T_0 = 0$ by using the translation invariance w.r.t. to time of the equation), we can apply again the step 1 in the ball $(B_{2\gamma_0, \bar{T}}, d)$ with initial condition $\psi(x)$, where $\bar{T} > 0$ is such that $\|e^{it\Delta}\psi\|_{X_{\bar{T}}} \leq \gamma_0$. Again by uniqueness of the fixed point of Φ in the space $(B_{2\gamma_0, \bar{T}}, d)$ we deduce that $u_1(t, x) = u_2(t, x)$ in $X_{T_0+\bar{T}}$, hence contradicting the definition of T_0 unless $T_0 = T$.

Step 3. Conservation laws. The proof of (2.4.3) is rather classical. In particular it follows by Proposition 1 and Proposition 2 in [74]. Another possibility is to follow the proof of Propositions 5.3 and 5.4 in [38], that can be repeated *mutatis mutandis* in the context of (2.4.1). The minor modification compared with [38] is that we use the end-point Strichartz estimate in order to treat the Sobolev critical nonlinearity. \square

We shall prove that the set \mathcal{M}_c defined in (2.3.8) is orbitally stable. In particular a non-trivial point concerns the fact that the local solutions, whose existence has been established in Proposition 2.4.3, can be extended to global solutions provided that the initial datum is close to \mathcal{M}_c . The main difficulty is related to the criticality of the nonlinearity in (2.4.1).

To simplify the next statement we denote by $u_\varphi(t)$ the integral solution associated with (2.4.1) and we denote by T_φ^{max} its maximal time of existence.

Theorem 2.4.8. *Let $v \in \mathcal{M}_c$. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that:*

$$\forall \varphi \in H \text{ s.t. } \|\varphi - v\|_H < \delta \implies \sup_{t \in [0, T_\varphi^{max}]} \text{dist}_{H^1}(u_\varphi(t), \mathcal{M}_c) < \varepsilon. \quad (2.4.9)$$

In particular we have

$$u_\varphi(t) = m_c(t) + r(t), \quad \forall t \in [0, T_\varphi^{max}], \text{ where } m_c(t) \in \mathcal{M}_c, \|r(t)\|_H < \varepsilon. \quad (2.4.10)$$

Proof. Suppose the theorem is false. Then there exists $(\delta_n) \subset \mathbb{R}^+$ a decreasing sequence converging to 0 and $(\varphi_n) \subset H$ satisfying

$$\|\varphi_n - v\|_H < \delta_n$$

and

$$\sup_{t \in [0, T_{\varphi_n}^{max}]} \text{dist}_{H^1}(u_{\varphi_n}(t), \mathcal{M}_c) > \varepsilon_0,$$

for some $\varepsilon_0 > 0$. We observe that $\|\varphi_n\|_2^2 \rightarrow c$ and, by continuity of F_μ , $F_\mu(\varphi_n) \rightarrow m(c)$. By conservation laws, for $n \in \mathbb{N}$ large enough, u_{φ_n} will remain inside of B_{ρ_0} for all $t \in [0, T_{\varphi_n}^{max}]$. Indeed, if for some time $\bar{t} > 0$ $\|\nabla u_{\varphi_n}(\bar{t})\|_2^2 = \rho_0$ then, in view of Lemma 2.3.5 (i) we have that $F_\mu(u_{\varphi_n}(\bar{t})) \geq 0$ in contradiction with $m(c) < 0$. Now let $t_n > 0$ be the first time such that $\text{dist}_{H^1}(u_{\varphi_n}(t_n), \mathcal{M}_c) = \varepsilon_0$ and set $u_n := u_{\varphi_n}(t_n)$. By conservation laws, $(u_n) \subset B_{\rho_0}$ satisfies

$\|u_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow c$ and $F_\mu(u_n) \rightarrow m(c)$ and thus, in view of [Theorem 2.3.4](#), it converges, up to translation, to an element of \mathcal{M}_c . Since \mathcal{M}_c is invariant under translation this contradicts the equality $\text{dist}_{H^1}(u_n, \mathcal{M}_c) = \varepsilon_0 > 0$. \square

The rest of this section is devoted to show that $T_\varphi^{\max} = \infty$ and it will conclude the proof of [Theorem 2.1.6](#).

Proposition 2.4.9. *Let $\mathcal{K} \subset H \setminus \{0\}$ be compact up to translation and assume that (p, r) is an admissible pair with $p \neq \infty$. Then, for every $\gamma > 0$ there exists $\varepsilon = \varepsilon(\gamma) > 0$ and $T = T(\gamma) > 0$ such that*

$$\sup_{\{\varphi \in H | \text{dist}_{H^1}(\varphi, \mathcal{K}) < \varepsilon\}} \|e^{it\Delta} \varphi\|_{X_{p,r,T}} < \gamma.$$

Proof. We first claim, for every $\gamma > 0$, the existence of a $T > 0$ such that

$$\sup_{\varphi \in \mathcal{K}} \|e^{it\Delta} \varphi\|_{X_{p,r,T}} < \frac{\gamma}{2}. \quad (2.4.11)$$

If it is not true then there exists sequences $(\varphi_n) \subset \mathcal{K}$ and $(T_n) \subset \mathbb{R}^+$ such that $T_n \rightarrow 0$ and

$$\|e^{it\Delta} \varphi_n\|_{X_{p,r,T_n}} \geq \bar{\gamma} \quad (2.4.12)$$

for a suitable $\bar{\gamma} > 0$. Since \mathcal{K} is compact up to translation, passing to a subsequence, there exists a sequence $(x_n) \subset \mathbb{R}^N$ such that

$$\tilde{\varphi}_n(\cdot) := \varphi_n(\cdot - x_n) \xrightarrow{H} \varphi(\cdot)$$

for a $\varphi \in H$. By continuity (induced by Strichartz's estimates) we have, for every $\tilde{T} > 0$,

$$\|e^{it\Delta} \tilde{\varphi}_n\|_{X_{p,r,\tilde{T}}} \rightarrow \|e^{it\Delta} \varphi\|_{X_{p,r,\tilde{T}}}. \quad (2.4.13)$$

Also, recording the translation invariance of Strichartz's estimates we get from [\(2.4.12\)](#) that

$$\|e^{it\Delta} \tilde{\varphi}_n\|_{X_{p,r,T_n}} = \|e^{it\Delta} \varphi_n\|_{X_{p,r,T_n}} \geq \bar{\gamma}. \quad (2.4.14)$$

Now, by [Proposition 2.4.4](#) (iii), we have $e^{it\Delta} \varphi \in X_{p,r,1}$, namely the function

$$[0, 1] \ni t \rightarrow g(t) := \|e^{it\Delta} \varphi\|_{W^{1,r}(\mathbb{R}^N)}^p$$

belongs to $L^1([0, 1])$. Then by applying the Dominated Convergence Theorem we get

$$\|\chi_{[0,\tilde{T}]}(t)g(t)\|_{L^1([0,1])} \rightarrow 0 \quad \text{as } \tilde{T} \rightarrow 0,$$

namely $\|e^{it\Delta} \varphi\|_{X_{p,r,\tilde{T}}}^p \rightarrow 0$ as $\tilde{T} \rightarrow 0$. Hence, we can choose $\tilde{T} > 0$ such that

$$\|e^{it\Delta} \varphi\|_{X_{p,r,\tilde{T}}} < \bar{\gamma}. \quad (2.4.15)$$

At this point gathering [\(2.4.13\)](#)- [\(2.4.15\)](#) we get a contradiction and the claim holds. Now, fix a $T > 0$ such that [\(2.4.11\)](#) holds. By [Proposition 2.4.4](#) (iii), we have

$$\|e^{it\Delta} \eta\|_{X_{p,r,T}} \leq C\|\eta\|_H, \quad \forall \eta \in H.$$

Thus, assuming that $\|\eta\|_H < \frac{\gamma}{2C} := \varepsilon$, we obtain that

$$\|e^{it\Delta}\eta\|_{X_{p,r,T}} < \frac{\gamma}{2}.$$

Summarizing, we get that, for all $\varphi \in \mathcal{K}$ and all $\eta \in H$ such that $\|\eta\|_H < \varepsilon$,

$$\|e^{it\Delta}(\varphi + \eta)\|_{X_{p,r,T}} \leq \|e^{it\Delta}\varphi\|_{X_{p,r,T}} + \|e^{it\Delta}\eta\|_{X_{p,r,T}} < \gamma.$$

This implies the proposition. \square

Proposition 2.4.10. *Let $\mathcal{K} \subset H \setminus \{0\}$ be compact up to translation. Then, for every $\gamma > 0$ there exists $\varepsilon = \varepsilon(\gamma) > 0$ and $T = T(\gamma) > 0$ such that*

$$\sup_{\{\varphi \in H | \text{dist}_{H^1}(\varphi, \mathcal{K}) < \varepsilon\}} \|e^{it\Delta}\varphi\|_{X_T} < \gamma.$$

Proof. We apply [Proposition 2.4.9](#) twice with the admissible pairs (p_1, r_1) and (p_2, r_2) . Then, the proposition follows from the definition of the norm X_T given in [\(2.4.2\)](#). \square

Theorem 2.4.11. *Let $\mathcal{K} \subset H \setminus \{0\}$ be compact up to translation. Then there exist $\varepsilon_0 > 0$ and $T_0 > 0$ such that the Cauchy problem [\(2.4.1\)](#), where φ satisfies $\text{dist}_{H^1}(\varphi, \mathcal{K}) < \varepsilon_0$, has a unique solution on the time interval $[0, T_0]$ in the sense of [Definition 2.4.2](#).*

Proof. We apply [Proposition 2.4.10](#) where $\gamma = \gamma_0$ is given in [Proposition 2.4.3](#). Then [Proposition 2.4.3](#) guarantees that the theorem holds for $\varepsilon_0 = \varepsilon(\gamma_0) > 0$ and $T_0 = \min\{T(\gamma_0), 1\} > 0$. \square

Theorem 2.4.12. *Let \mathcal{M}_c be defined in [\(2.3.8\)](#). Then there exists a $\delta_0 > 0$ such that, if $\varphi \in H$ satisfies $\text{dist}_{H^1}(\varphi, \mathcal{M}_c) < \delta_0$ the corresponding solution to [\(2.4.1\)](#) satisfies $T_\varphi^{\max} = \infty$.*

Proof. We make use of [Theorem 2.4.11](#) where we choose $\mathcal{K} = \mathcal{M}_c$. By [Theorem 2.4.8](#), we can choose a $\delta_0 > 0$ such that [\(2.4.9\)](#) and [\(2.4.10\)](#) holds for $\varepsilon = \varepsilon_0$ where $\varepsilon_0 > 0$ is given in [Theorem 2.4.11](#). Then [Theorem 2.4.8](#) guarantees that the solution $u_\varphi(t)$ where $\text{dist}_{H^1}(\varphi, \mathcal{M}_c) < \delta_0$ satisfies $\text{dist}_{H^1}(u_\varphi(t), \mathcal{M}_c) < \varepsilon_0$ up to the maximum time of existence $T_\varphi^{\max} \geq T_0$. Since, at any time in $(0, T_\varphi^{\max})$ we can apply again [Theorem 2.4.11](#) that guarantees an uniform additional time of existence $T_0 > 0$, this contradicts the definition of T_φ^{\max} if $T_\varphi^{\max} < \infty$. \square

At this point we can give,

Proof of [Theorem 2.1.6](#). The fact that \mathcal{M}_c is compact, up to translation, was established in [Theorem 2.3.4](#). The orbital stability of \mathcal{M}_c , in the sense of [Definition 2.1.5](#) follows from [Theorem 2.4.8](#) and [Theorem 2.4.12](#). \square

2.5 Existence of standing waves lying at mountain pass levels

2.5.1 The proof of [Proposition 2.1.12](#)

We follow the strategy introduced in [\[45\]](#) and consider the functional $\tilde{F}_\mu : \mathbb{R}_+ \times H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\tilde{F}_\mu(s, u) := F_\mu(u_s) = \psi_u(s) = \frac{s^2}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} s^{q\gamma_q} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{s^{2^*}}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

Note that

$$\partial_s \tilde{F}_\mu(s, u) = \psi'_u(s) = \frac{1}{s} Q_\mu(u_s) \quad (2.5.1)$$

and, for any $v \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} \partial_u \tilde{F}_\mu(s, u)(v) &= s^2 \int_{\mathbb{R}^N} \nabla u \nabla v dx - \mu s^{q\gamma_q} \int_{\mathbb{R}^N} |u|^{q-2} u v dx - s^{2^*} \int_{\mathbb{R}^N} |u|^{2^*-2} u v dx \\ &= F'_\mu(u_s)(v_s). \end{aligned} \quad (2.5.2)$$

We recall that the tangent space at a point $u \in S(c)$ is defined as

$$T_u S(c) = \{v \in H^1(\mathbb{R}^N) : \langle u, v \rangle_{L^2(\mathbb{R}^N)} = 0\},$$

and that, for any $u \in S(c)$ and any $v \in T_u S(c)$,

$$\langle F'_{\mu|_{S(c)}}(u), v \rangle = \langle F'_\mu(u), v \rangle. \quad (2.5.3)$$

Lemma 2.5.1. *Let $N \geq 3$. For $u \in S(c)$ and $s > 0$, the map*

$$T_u S(c) \rightarrow T_{u_s} S(c), \quad \phi \mapsto \phi_s$$

is a linear isomorphism with inverse

$$T_{u_s} S(c) \rightarrow T_u S(c), \quad \psi \mapsto \psi_\perp.$$

Proof. We follow the approach in [11, Lemma 3.6]. For $\phi \in T_u S(c)$ and for $t > 0$, we have

$$\int_{\mathbb{R}^N} u_t(x) \phi_t(x) dx = \int_{\mathbb{R}^N} t^N u(tx) \phi(tx) dx = \int_{\mathbb{R}^N} u(y) \phi(y) dy = 0.$$

As a consequence, $\phi_t \in T_{u_t} S(c)$ and the map is well defined. Clearly it is linear. Taking into account that, for every $t, s > 0$ and $w \in H^1(\mathbb{R}^N)$,

$$w_{ts} = (ts)^{\frac{N}{2}} w(tsx) = (w_t)_s,$$

we obtain that the map is linear isomorphism. \square

Definition 2.5.2. *Given $c > 0$, we say that \tilde{F}_μ has a mountain pass geometry on $\mathbb{R}_+ \times S_r(c)$ at level $\tilde{M}(c)$ if*

$$\tilde{M}(c) := \inf_{\tilde{h} \in \tilde{\Gamma}(c)} \max_{t \in [0, \infty)} \tilde{F}_\mu(\tilde{h}(t)) > \max\{\tilde{F}_\mu(\tilde{h}(0)), \tilde{F}_\mu(\tilde{h}(t_{\tilde{h}}))\}$$

where

$$\tilde{\Gamma}(c) := \{\tilde{h} \in C([0, \infty), \mathbb{R}_+ \times S_r(c)) : \tilde{h}(0) \in (1, \Lambda_r^+(c)), \exists t_{\tilde{h}} > 0 \text{ s.t. } \tilde{h}(t) \in (1, E_c) \forall t \geq t_{\tilde{h}}\}.$$

Recording that the definition of $M^0(c)$ is given in (2.1.7), we have,

Lemma 2.5.3. *Let $N \geq 3$. For any $c \in (0, c_0)$, \tilde{F}_μ has a mountain pass geometry at the level $\tilde{M}(c)$. Moreover, $M^0(c) = \tilde{M}(c)$.*

Proof. Let $h \in \Gamma^0(r)$, since $\tilde{h}(t) = (1, h(t)) \in \tilde{\Gamma}(c)$ and $\tilde{F}_\mu(\tilde{h}(t)) = F_\mu(h(t))$ for all $t \in \mathbb{R}_+$, we have that $M^0(c) \geq \tilde{M}(c)$. Next, we shall prove that $\tilde{M}(c) \geq M^0(c)$. For all $\tilde{h}(t) = (s(t), v(t)) \in \tilde{\Gamma}(c)$, we have $s(0) = 1$, $v(0) \in \Lambda_r^+(c)$ and there exists a $t_{\tilde{h}} > 0$ such that $s(t) = 1$, $v(t) \in E_c$ for all $t \geq t_{\tilde{h}}$. Setting $h(t) = v(t)_{s(t)}$, we have that h is continuous from $[0, \infty)$ into $S_r(c)$ and

$$h(0) = v(0)_{s(0)} = v(0) \in \Lambda_r^+(c), \quad h(t) = v(t)_{s(t)} = v(t) \in E_c \quad \forall t \geq t_{\tilde{h}}.$$

Hence, $h \in \Gamma^0(r)$ and $\tilde{F}_\mu(\tilde{h}(t)) = F_\mu(v(t)_{s(t)}) = F_\mu(h(t))$. Thus, $\tilde{M}(c) \geq M^0(c)$ and finally $M^0(c) = \tilde{M}(c)$.

Now we claim that

$$M^0(c) > 0 \quad \text{for any } c \in (0, c_0). \quad (2.5.4)$$

Indeed, let $g \in \Gamma^0(c)$ be arbitrary. Since $g(0) \in \Lambda_r^+(c)$ in particular $g(0) \in V(c)$. Now for $t > 0$ large, since $F_\mu(g(t)) < 2m(c)$, necessarily in view of (2.3.9), $g(t) \notin V(c)$. By continuity of g there exists a $t_0 > 0$ such that $g(t_0) \in \partial V(c)$ and using again (2.3.9) we conclude.

At this point observing that

$$\max\{\tilde{F}_\mu(\tilde{h}(0)), \tilde{F}_\mu(\tilde{h}(t_{\tilde{h}}))\} = \max\{F_\mu(h(0)), F_\mu(h(t_{\tilde{h}}))\} < 0$$

it follows that \tilde{F}_μ has a mountain pass geometry at level $\tilde{M}(c)$ for all $0 < c < c_0$. \square

Proof of Proposition 2.1.12. Following [37, Section 5], we set

1. $\mathcal{F} = \{\tilde{h}([0, \infty)) : \tilde{h} \in \tilde{\Gamma}(c)\}$.
2. $B = (1, \Lambda_r^+(c)) \cup (1, E_c)$.
3. $F = \{(s, u) \in \mathbb{R}_+ \times S_r(c) : \tilde{F}_\mu(s, u) \geq \tilde{M}(c)\}$.

Since \tilde{F}_μ has a mountain pass geometry at level $\tilde{M}(c)$ (see Lemma 2.5.3) and by the definition of the superlevel set F , we obtain $F \setminus B = F$ and

$$\sup_{(1, u) \in B} \tilde{F}_\mu(s, u) \leq \tilde{M}(c) \leq \inf_{(s, u) \in F} \tilde{F}_\mu(s, u). \quad (2.5.5)$$

For any $A \in \mathcal{F}$, there exists a $h_0 \in \tilde{\Gamma}(c)$ such that $A = h_0([0, \infty))$ and

$$\tilde{M}(c) = \inf_{\tilde{h} \in \tilde{\Gamma}(c)} \max_{t \in [0, \infty)} \tilde{F}_\mu(\tilde{h}(t)) \leq \max_{t \in [0, \infty)} \tilde{F}_\mu(h_0(t)).$$

Hence, there exists a $t_0 \in [0, \infty)$ such that $\tilde{M}(c) \leq \tilde{F}_\mu(h_0(t_0))$. This means that $h_0(t_0) \in F$ and consequently,

$$A \cap F \setminus B \neq \emptyset, \quad \forall A \in \mathcal{F}. \quad (2.5.6)$$

Now, for all $(s, u) \in \mathbb{R}_+ \times S_r(c)$, we have

$$\tilde{F}_\mu(s, u) = F_\mu(u_s) = F_\mu(|u|_s) = \tilde{F}_\mu(1, |u|_s).$$

Hence, for any minimizing sequence $(z_n = (\alpha_n, \beta_n)) \subset \tilde{\Gamma}(c)$ for $\tilde{M}(c)$, we have that the sequence $(y_n = (1, |\beta_n|_{\alpha_n}))$ is also a minimizing sequence for $\tilde{M}(c)$.

Using the terminology in [37, Section 5], it means that \mathcal{F} is a homotopy stable family of compact subset of $\mathbb{R} \times S_r(c)$ with extended closed boundary B and the superlevel set F is a dual set for \mathcal{F} . By (2.5.5) and (2.5.6), we can apply [37, Theorem 5.2] with the minimizing sequence $\{y_n = (1, |\beta_n|_{\alpha_n})\}$. This implies that there exists a Palais-Smale sequence $(s_n, w_n) \subset \mathbb{R}_+ \times S_r(c)$ for \tilde{F}_μ restricted to $\mathbb{R}_+ \times S_r(c)$ at level $\tilde{M}(c)$, that is, as $n \rightarrow \infty$,

$$\partial_s \tilde{F}_\mu(s_n, w_n) \rightarrow 0, \quad (2.5.7)$$

and

$$\|\partial_u \tilde{F}_\mu(s_n, w_n)\|_{(T_{w_n} S(r))^*} \rightarrow 0, \quad (2.5.8)$$

with the additional property that

$$|s_n - 1| + \|w_n - |\beta_n|_{\alpha_n}([0, \infty))\|_{H^1(\mathbb{R}^N)} \rightarrow 0. \quad (2.5.9)$$

By (2.5.1), (2.5.7) and since (s_n) is bounded due to (2.5.9), we obtain $Q_\mu((w_n)_{s_n}) \rightarrow 0$ as $n \rightarrow \infty$. Also, by (2.5.2), the condition (2.5.8) implies that

$$F'_\mu((w_n)_{s_n})(\phi)_{s_n} \rightarrow 0, \quad (2.5.10)$$

as $n \rightarrow \infty$, for every $\phi \in T_{w_n} S_r(c)$. Let then $u_n := (w_n)_{s_n}$. By (2.5.3), (2.5.10) and Lemma 2.5.1, we obtain that $(u_n) \subset S_r(c)$ is a Palais-Smale sequence for F_μ restricted to $S_r(c)$ at level $M^0(c)$, with $Q_\mu(u_n) \rightarrow 0$. Since the problem is invariant under rotations, $(u_n) \subset S_r(c)$ is also the Palais-Smale sequence for F_μ restricted to $S(c)$ at level $M^0(c)$, with $Q_\mu(u_n) \rightarrow 0$. \square

2.5.2 The proof of Proposition 2.1.13

Now, we give the

Proof of Proposition 2.1.13. Let $(u_n) \subset H_r^1(\mathbb{R}^N)$ be given by Proposition 2.1.12. To show its convergence we proceed in three steps.

Step 1: $(u_n) \subset H_r^1(\mathbb{R}^N)$ is bounded.

Since $Q_\mu(u_n) \rightarrow 0$, we have, using the Gagliardo-Nirenberg inequality (2.2.2),

$$\begin{aligned} F_\mu(u_n) &= \frac{1}{N} \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \left(1 - \frac{q\gamma_q}{2^*}\right) \|u_n\|_{L^q(\mathbb{R}^N)}^q + o_n(1) \\ &\geq \frac{1}{N} \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} C_{N,q}^q \left(1 - \frac{q\gamma_q}{2^*}\right) c^{(1-\gamma_q)q} \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^{q\gamma_q} + o_n(1), \end{aligned}$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. Since $F_\mu(u_n) \rightarrow M^0(c) < \infty$ and $q\gamma_q < 2$ the conclusion follows.

Step 2: $(u_n) \subset H_r^1(\mathbb{R}^N)$ has a non-trivial weak limit.

Since $(u_n) \subset H_r^1(\mathbb{R}^N)$ is a bounded sequence, by the compact embedding of $H_r^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$, there exists a $u \in H_r^1(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^N)$, $u_n \rightarrow u$ strongly in $L^q(\mathbb{R}^N)$ and a.e in \mathbb{R}^N .

Let us assume now, by contradiction, that u is trivial. Then, $\|u_n\|_{L^q(\mathbb{R}^N)} \rightarrow 0$ and since $Q_\mu(u_n) \rightarrow 0$, using the Sobolev embedding, see (2.2.1), we deduce that

$$\mathcal{S} \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 \leq \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + o_n(1). \quad (2.5.11)$$

We distinguish the two cases

$$\text{either (i) } \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \rightarrow 0 \quad \text{or (ii) } \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \rightarrow \ell > 0.$$

If (i) holds then, in view of (2.5.11), we also have that $\|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow 0$ which implies that $F_\mu(u_n) \rightarrow 0$ contradicting the fact that $M^0(c) > 0$, see (2.5.4). If (ii) holds we deduce from (2.5.11) that

$$\|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \geq \mathcal{S}^{\frac{N}{2}} + o_n(1)$$

and thus, recording that $Q_\mu(u_n) \rightarrow 0$ and $\|u_n\|_{L^q(\mathbb{R}^N)} \rightarrow 0$, it follows that

$$\|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 = \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + o_n(1) \geq \mathcal{S}^{\frac{N}{2}} + o_n(1). \quad (2.5.12)$$

From (2.5.12) we deduce that

$$F_\mu(u_n) = \frac{1}{N} \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 + o_n(1) \geq \frac{1}{N} \mathcal{S}^{\frac{N}{2}} + o_n(1).$$

But, since $m(c) < 0$, necessarily $M^0(c) < \frac{\mathcal{S}^{\frac{N}{2}}}{N}$ and we also have a contradiction.

Step 3: $(u_n) \subset H_r^1(\mathbb{R}^N)$ strongly converges.

Since (u_n) is bounded, following [19, Lemma 3], we know that

$$\left(F_\mu|_{\mathcal{S}(c)}\right)'(u_n) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^N) \iff F'_\mu(u_n) - \frac{1}{c} \langle F'_\mu(u_n), u_n \rangle u_n \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

Thus, for any $w \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} o_n(1) &= \left\langle F'_\mu(u_n) - \frac{1}{c} \langle F'_\mu(u_n), u_n \rangle u_n, w \right\rangle \\ &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla w - \mu |u_n|^{q-2} u_n w - |u_n|^{2^*-2} u_n w - \lambda_n u_n w \, dx, \end{aligned} \quad (2.5.13)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$ and

$$c \lambda_n = \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 - \mu \|u_n\|_{L^q(\mathbb{R}^N)}^q - \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + o_n(1). \quad (2.5.14)$$

In particular $(\lambda_n) \subset \mathbb{R}$ is bounded and, up to a subsequence, $\lambda_n \rightarrow \lambda \in \mathbb{R}$. Now, passing to the limit in (2.5.13) by weak convergence, we obtain that

$$-\Delta u - \mu |u|^{q-2} u - |u|^{2^*-2} u = \lambda u. \quad (2.5.15)$$

Thus in view of Lemma 2.2.1, $Q_\mu(u) = 0$ and $\lambda < 0$.

Let $(v_n) \subset H_r^1(\mathbb{R}^N)$ be such that $v_n = u_n - u$. We have that $v_n \rightharpoonup 0$ weakly in $H_r^1(\mathbb{R}^N)$, $v_n \rightarrow 0$ strongly in $L^q(\mathbb{R}^N)$ and a.e. in \mathbb{R}^N . Thus

$$\|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 = \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 + o_n(1)$$

and also, by the Brezis-Lieb Lemma [24],

$$\|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + \|v_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + o_n(1). \quad (2.5.16)$$

In particular,

$$F_\mu(u_n) = F_\mu(u) + F_\mu(v_n) + o_n(1). \quad (2.5.17)$$

and

$$Q_\mu(u_n) = Q_\mu(u) + Q_\mu(v_n) + o_n(1). \quad (2.5.18)$$

Here again we distinguish the two cases

$$\text{either (i) } \|v_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \rightarrow 0 \quad \text{or (ii) } \|v_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \rightarrow \ell > 0.$$

Assuming that (ii) holds, and since $Q_\mu(u) = 0$, we deduce from (2.5.18) that

$$\|v_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = \|\nabla v_n\|_{L^2(\mathbb{R}^N)}^2 + o_n(1).$$

Then, reasoning as in Step 2, it follows that

$$F_\mu(v_n) \geq \frac{\mathcal{S}^{\frac{N}{2}}}{N} + o_n(1)$$

which leads, in view of (2.5.17), to

$$F_\mu(u_n) \geq F_\mu(u) + \frac{\mathcal{S}^{\frac{N}{2}}}{N} + o_n(1).$$

At this point, recording from Remark 2.1.4 that $c \mapsto m(c)$ is non increasing, using $Q_\mu(u) = 0$ and since, by property of the weak limit, $\|u\|_{L^2(\mathbb{R}^N)}^2 \leq c$, we get that

$$F_\mu(u) \geq m\left(\|u\|_{L^2(\mathbb{R}^N)}^2\right) \geq m(c).$$

Thus, $F_\mu(u_n) \rightarrow M^0(c)$ satisfies

$$F_\mu(u_n) \geq m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N} + o_n(1)$$

which contradicts our assumption on $M^0(c)$.

It remains to show that if (i) holds then $(u_n) \subset H_r^1(\mathbb{R}^N)$ converges strongly. Since (i) holds, we get from (2.5.16) that $\|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \rightarrow \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$. Choosing $w = u_n$ in (2.5.13) we deduce since u is solution to (2.5.15) that

$$\begin{aligned} & \|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 - \lambda_n \|u_n\|_{L^2(\mathbb{R}^N)}^2 - \mu \|u_n\|_{L^q(\mathbb{R}^N)}^q - \|u_n\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ & \rightarrow \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \lambda \|u\|_{L^2(\mathbb{R}^N)}^2 - \mu \|u\|_{L^q(\mathbb{R}^N)}^q - \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = 0. \end{aligned}$$

Therefore, taking into account that $\|u_n\|_{L^q(\mathbb{R}^N)}^q \rightarrow \|u\|_{L^q(\mathbb{R}^N)}^q$ due to $(u_n) \subset H_r^1(\mathbb{R}^N)$ and since $\lambda_n \rightarrow \lambda$, we obtain that

$$\|\nabla u_n\|_{L^2(\mathbb{R}^N)}^2 - \lambda \|u_n\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \lambda \|u\|_{L^2(\mathbb{R}^N)}^2.$$

By $\lambda < 0$ (since u is non trivial), see Lemma 2.2.1 we conclude that $u_n \rightarrow u$ strongly in $H_r^1(\mathbb{R}^N)$. At this point the proposition is proved. \square

2.5.3 The proof of Proposition 2.1.14

As announced in the Introduction to show that Proposition 2.1.14 holds we shall rely on Proposition 2.1.17 and Proposition 2.1.18.

Proof of Proposition 2.1.17. We shall proceed into three steps.

Step 1: For any $c \in (0, c_0)$, it holds that

$$M(c) \geq \inf_{u \in \Lambda^-(c)} F_\mu(u).$$

Let $h \in \Gamma(c)$. We have $h(0) \in V(c) \cap \{u : F_\mu(u) < 0\}$ and thus, in view of Lemma 2.2.4(iii), $h(0) \in W(c)$ or equivalently $s_{h(0)}^+ > 1$. Since $h \in \Gamma(c)$, we also have that for t large enough,

$$F_\mu(h(t)) \leq 2m(c) < m(c).$$

Thus, from Lemma 2.2.4(iv), we get that $h(t) \notin W(c)$ for t large enough or equivalently that $s_{h(t)}^+ < 1$ for such $t > 0$. By the continuity of h and of $u \mapsto s_u^+$, see Lemma 2.2.3(iii), we deduce that there exists a $t_0 > 0$ such that $s_{h(t_0)}^+ = 1$, namely such that $h(t_0) \in \partial W(c)$. Thus we have that

$$M(c) \geq \inf_{u \in \partial W(c)} F_\mu(u) = \inf_{u \in \Lambda^-(c)} F_\mu(u)$$

due to Lemma 2.2.4(ii).

Step 2: For any $c \in (0, c_0)$, it holds that

$$\inf_{u \in \Lambda^-(c)} F_\mu(u) \geq \inf_{u \in \Lambda_r^-(c)} F_\mu(u).$$

For any $u \in \Lambda^-(c)$, let v be the Schwarz rearrangement of $|u|$. We claim that $\psi_v(s) \leq \psi_u(s)$ for all $s \geq 0$. Indeed, we have

$$\begin{aligned} \psi_v(s) &= \frac{s^2}{2} \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} s^{\frac{N(q-2)}{2}} \|v\|_{L^q(\mathbb{R}^N)}^q - \frac{s^{2^*}}{2^*} \|v\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\ &\leq \frac{s^2}{2} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} s^{\frac{N(q-2)}{2}} \|u\|_{L^q(\mathbb{R}^N)}^q - \frac{s^{2^*}}{2^*} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = \psi_u(s). \end{aligned}$$

Recording, see Lemma 2.2.3, that s_u^- is the unique global maximum point for ψ_u , we deduce from the above claim that

$$\psi_u(s_u^-) \geq \psi_u(s_v^+) \geq \psi_v(s_v^+).$$

Since $u \in \Lambda^-(c)$, we have that $s_u^- = 1$ and hence

$$F_\mu(u) = \psi_u(1) = \psi_u(s_u^-) \geq \psi_v(s_v^+) = F_\mu(v_{s_v^+}).$$

Recording that $v_{s_v^+} \in \Lambda_r^-(c)$, we deduce that

$$\inf_{u \in \Lambda^-(c)} F_\mu(u) \geq \inf_{u \in \Lambda_r^-(c)} F_\mu(u).$$

Step 3: For any $c \in (0, c_0)$, it holds that

$$\inf_{u \in \Lambda_r^-(c)} F_\mu(u) \geq M^0(c).$$

Let $u \in \Lambda_r^-(c)$ and $s_1 > 0$ be such that $u_{s_1} \in E_c$. Let us consider the map

$$g_u : t \in [0, \infty) \mapsto u_{(1-t)s_u^+ + ts_1} \in S_r(c).$$

We have that $g_u \in C([0, \infty), S_r(c))$ and

$$g_u(0) = u_{s_u^+} \in \Lambda_r^+(c) \quad \text{and} \quad g_u(1) = u_{s_1} \in E_c.$$

Hence, we get $g_u \in \Gamma^0(c)$ and

$$F_\mu(u) = \max_{s>0} F_\mu(u_s) \geq \max_{t \in [0, \infty)} F_\mu(g_u(t)) \geq \inf_{g \in \Gamma^0(c)} \max_{t \in [0, \infty)} F_\mu(g(t)) = M^0(c).$$

Finally, from **Steps 1, 2 and 3** we deduce that [Proposition 2.1.17](#) holds. \square

Remark 2.5.4. Trivially, since $\Gamma^0(c) \subset \Gamma(c)$, one has $M(c) \leq M^0(c)$. Thus, from [Proposition 2.1.17](#) we deduce that

$$M^0(c) = M(c) = \inf_{u \in \Lambda^-(c)} F_\mu(u).$$

In the rest of this subsection, we shall prove [Proposition 2.1.18](#). Firstly, we need some lemmas.

Lemma 2.5.5. Let $N \geq 3$. For any $c \in (0, c_0)$, the following property holds

$$M(c) \leq \inf_{h \in \mathcal{G}(c)} \max_{t \in [0, \infty)} F_\mu(h(t))$$

where

$$\mathcal{G}(c) := \left\{ h \in C([0, \infty), \cup_{d \in [\frac{c}{2}, c]} S(d)) : h(0) \in \mathcal{M}_d \text{ for some } d \in \left[\frac{c}{2}, c \right], \exists t_0 = t_0(h) \text{ s.t. } h(t) \in E_c \forall t \geq t_0 \right\}.$$

Proof. Let any $h \in \mathcal{G}(c)$. We define the function

$$t \mapsto \theta(t) := \sqrt{\frac{\|h(t)\|_{L^2(\mathbb{R}^N)}^2}{c}}.$$

Note that θ is the continuous function from $[0, \infty)$ into \mathbb{R} and $\theta(t) \leq 1$ for all t . Now we set

$$g(t)(x) := \theta(t)^{\frac{N}{2}-1} h(t)(\theta(t)x). \tag{2.5.19}$$

By direct computations, we obtain that

$$\begin{aligned} \|g(t)\|_{L^2(\mathbb{R}^N)}^2 &= \frac{1}{[\theta(t)]^2} \|h(t)\|_{L^2(\mathbb{R}^N)}^2 = c, & \|\nabla g(t)\|_{L^2(\mathbb{R}^N)}^2 &= \|\nabla h(t)\|_{L^2(\mathbb{R}^N)}^2, \\ \|g(t)\|_{L^q(\mathbb{R}^N)}^q &= [\theta(t)]^{(\frac{N}{2}-1)q-N} \|h(t)\|_{L^q(\mathbb{R}^N)}^q, & \|g(t)\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} &= \|h(t)\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}. \end{aligned}$$

Hence,

$$\begin{aligned}
 F_\mu(g(t)) &= \frac{1}{2} \|\nabla g(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \|g(t)\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \|g(t)\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\
 &= \frac{1}{2} \|\nabla h(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} [\theta(t)]^{(\frac{N}{2}-1)q-N} \|h(t)\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \|h(t)\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \\
 &\leq \frac{1}{2} \|\nabla h(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \|h(t)\|_{L^q(\mathbb{R}^N)}^q - \frac{1}{2^*} \|h(t)\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = F_\mu(h(t))
 \end{aligned}$$

due to $\theta(t) \leq 1$ for all t and $(\frac{N}{2}-1)q-N < 0$. Noting that $F_\mu(g(0)) \leq F_\mu(h(0)) < 0$ and that $\|\nabla g(0)\|_{L^2(\mathbb{R}^N)}^2 = \|\nabla h(0)\|_{L^2(\mathbb{R}^N)}^2 < \rho_0$ we deduce that $g(0) \in V(c) \cap \{u : F_\mu(u) < 0\}$ and hence that $g \in \Gamma(c)$. At this point the lemma is proved. \square

Let u_ε be an extremal function for the Sobolev inequality in \mathbb{R}^N defined by

$$u_\varepsilon(x) := \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{N-2}{2}}}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^N. \quad (2.5.20)$$

Let $\xi \in C_0^\infty(\mathbb{R}^N)$ be a radially non-increasing cut-off function with $\xi \equiv 1$ in B_1 , $\xi \equiv 0$ in $\mathbb{R}^N \setminus B_2$. Setting $U_\varepsilon(x) = \xi(x)u_\varepsilon(x)$ we shall prove the following lemma.

Lemma 2.5.6. *Let $N \geq 3$ and $u \in H^1(\mathbb{R}^N)$ be a nonnegative function. For every $\varepsilon > 0$ and every $t > 0$ we have*

$$F_\mu(u + tU_\varepsilon) \leq F_\mu(u) + t \int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla U_\varepsilon(x) dx + \frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.$$

Proof. We have, for any $\varepsilon > 0$ and any $t > 0$,

$$\|\nabla(u + tU_\varepsilon)\|_{L^2(\mathbb{R}^N)}^2 = \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + 2t \int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla U_\varepsilon(x) dx + t^2 \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2.$$

Also, since both $u \in H^1(\mathbb{R}^N)$ and U_ε are non negative,

$$\|u + tU_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \geq \|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + t^{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

and

$$\|u + tU_\varepsilon\|_{L^q(\mathbb{R}^N)}^q \geq \|u\|_{L^q(\mathbb{R}^N)}^q + t^q \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q.$$

Therefore, we obtain that

$$\begin{aligned}
 F_\mu(u + tU_\varepsilon) &\leq \frac{1}{2} \left[\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + 2t \int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla U_\varepsilon(x) dx + t^2 \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 \right] \\
 &\quad - \frac{\mu}{q} \left[\|u\|_{L^q(\mathbb{R}^N)}^q + t^q \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q \right] - \frac{1}{2^*} \left[\|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} + t^{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right] \\
 &= F_\mu(u) + t \int_{\mathbb{R}^N} \nabla u(x) \cdot \nabla U_\varepsilon(x) dx + \frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}.
 \end{aligned}$$

\square

From now we fix a sequence $(\varepsilon_n) \subset \mathbb{R}_+$ such that $\varepsilon_n \rightarrow 0$.

Lemma 2.5.7. *Let $N \geq 3$. There exists $0 < t_0 < t_1 < \infty$ such that, for any sequence $(u_n) \subset H^1(\mathbb{R}^N)$ satisfying*

$$\int_{\mathbb{R}^N} \nabla u_n(x) \cdot \nabla U_{\varepsilon_n}(x) dx \leq 1, \quad \forall n \in \mathbb{N}, \quad (2.5.21)$$

setting

$$I_n(t) := t \int_{\mathbb{R}^N} \nabla u_n(x) \cdot \nabla U_{\varepsilon_n}(x) dx + \frac{t^2}{2} \|\nabla U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_{\varepsilon_n}\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_{\varepsilon_n}\|_{L^{2^*}(\mathbb{R}^N)}^{2^*},$$

we have, for any $n \in \mathbb{N}$ large enough,

- (i) if $I_n(t) \geq \frac{1}{2N} \mathcal{S}^{\frac{N}{2}}$ then necessarily $t \geq t_0$,
- (ii) $I_n(t) \leq 2m(c)$ for any $t \geq t_1$.

Proof. Observe that

$$I_n(t) \leq t \int_{\mathbb{R}^N} \nabla u_n(x) \cdot \nabla U_{\varepsilon_n}(x) dx + \frac{t^2}{2} \|\nabla U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2.$$

We have that $\|\nabla U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \mathcal{S}^{\frac{N}{2}} > 0$, see [Lemma 2.7.1](#). Thus in view of (2.5.21), if $t \rightarrow 0$ then $I_n(t) < \frac{1}{2N} \mathcal{S}^{\frac{N}{2}}$. Hence, there exists $t_0 > 0$ such that if $I_n(t) \geq \frac{1}{2N} \mathcal{S}^{\frac{N}{2}}$ then necessarily $t \geq t_0$ and point (i) holds. We also have

$$I(t) \leq t \int_{\mathbb{R}^N} \nabla u_n(x) \cdot \nabla U_{\varepsilon_n}(x) dx + \frac{t^2}{2} \|\nabla U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 - \frac{t^{2^*}}{2^*} \|U_{\varepsilon_n}\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}$$

with $\|\nabla U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \mathcal{S}^{\frac{N}{2}} > 0$ and $\|U_{\varepsilon_n}\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \rightarrow \mathcal{S}^{\frac{N}{2}} > 0$, see [Lemma 2.7.1](#). Thus, in view of (2.5.21), there exists a $t_1 > 0$ such that $I_n(t) \leq 2m(c)$, for all $t \geq t_1$, if $n \in \mathbb{N}$ is large enough. Thus point (ii) also holds. \square

We define by \mathcal{M}_c^0 the set of elements of \mathcal{M}_c which have the properties guarantee by [Lemma 2.2.2](#).

Lemma 2.5.8. *Let $N \geq 3$, $c \in (0, c_0)$ and $u_c \in \mathcal{M}_c^0$. For any $\varepsilon > 0$ there exists a $y_\varepsilon \in \mathbb{R}^N$ such that*

$$2 \int_{\mathbb{R}^N} u_c(x - y_\varepsilon) U_\varepsilon(x) dx \leq t_1 \|U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 \quad (2.5.22)$$

where $t_1 > 0$ is provided by [Lemma 2.5.7](#) and

$$\int_{\mathbb{R}^N} \nabla u_c(x - y_\varepsilon) \cdot \nabla U_\varepsilon(x) dx \leq \|U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2. \quad (2.5.23)$$

Proof. Since $u_c \in \mathcal{M}_c^0$ is a radial, non-increasing function, we know from [18, Radial Lemma A.IV] that

$$|u_c(z)| \leq C(N) |z|^{-\frac{N}{2}} \sqrt{c}, \quad \forall |z| \geq 1, \quad (2.5.24)$$

and thus,

$$\int_{\mathbb{R}^N} u_c(x-y)U_\varepsilon(x)dx \leq C(N)\sqrt{c} \int_{\mathbb{R}^N} |x-y|^{-\frac{N}{2}} U_\varepsilon(x)dx. \quad (2.5.25)$$

Using that the function U_ε is compactly supported in B_2 , we have that, for $|y|$ large enough,

$$\int_{\mathbb{R}^N} |x-y|^{-\frac{N}{2}} U_\varepsilon(x)dx \leq \int_{B_2} \left|\frac{y}{2}\right|^{-\frac{N}{2}} U_\varepsilon(x)dx.$$

At this point, we deduce from (2.5.25) that, for $|y|$ large enough,

$$\int_{\mathbb{R}^N} u_c(x-y)U_\varepsilon(x)dx \leq C(N)\sqrt{c} \left|\frac{y}{2}\right|^{-\frac{N}{2}} \int_{\mathbb{R}^N} U_\varepsilon(x)dx.$$

Using Hölder's inequality, we obtain that $\|U_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq |B_2|^{\frac{1}{2}}\|U_\varepsilon\|_{L^2(\mathbb{R}^N)}$ and (2.5.22) follows. Now, since for any $y \in \mathbb{R}^N$, $u_c(\cdot - y) \geq 0$ is solution to the equation

$$-\Delta u - \mu|u|^{q-2}u - |u|^{2^*-2}u = \lambda_c u \quad \text{in } \mathbb{R}^N, \quad (2.5.26)$$

for some $\lambda_c < 0$, we have that

$$\int_{\mathbb{R}^N} \nabla u_c(x-y) \cdot \nabla U_\varepsilon(x)dx \leq \mu \int_{\mathbb{R}^N} |u_c(x-y)|^{q-1} U_\varepsilon(x)dx + \int_{\mathbb{R}^N} |u_c(x-y)|^{2^*-1} U_\varepsilon(x)dx. \quad (2.5.27)$$

Since, both $|u_c(z)|^{q-1} \leq |u_c(z)|$ and $|u_c(z)|^{2^*-1} \leq |u_c(z)|$ for $|z|$ large, see (2.5.24), reasoning as in the proof of (2.5.22) we readily check that (2.5.23) also holds. \square

Now we define the sequence $(c_n) \in \left[\frac{c}{2}, c\right)$ as follows

$$c_n := c - 2t_1^2 \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 \quad (2.5.28)$$

where $t_1 > 0$ is given in Lemma 2.5.7. Clearly, $c_n \rightarrow c$ as $n \rightarrow \infty$. Now, for each $n \in \mathbb{N}$, we fix a $u_{c_n} \in \mathcal{M}_{c_n}^0$.

Lemma 2.5.9. *Under the setting introduced above, for any $n \in \mathbb{N}$ large enough, there exists a $y_n \in \mathbb{R}^N$ such that*

$$c_n \leq \|u_{c_n}(\cdot - y_n) + tU_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 \leq c, \quad \forall t \in [0, t_1], \quad (2.5.29)$$

and

$$\int_{\mathbb{R}^N} \nabla u_{c_n}(x - y_n) \cdot \nabla U_{\varepsilon_n}(x)dx \leq \max\{1, \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2\}. \quad (2.5.30)$$

Proof. For any $n \in \mathbb{N}$, according to Lemma 2.5.8, we can choose a $y_n \in \mathbb{R}^N$ such that

$$2 \int_{\mathbb{R}^N} u_{c_n}(x - y_n)U_{\varepsilon_n}(x)dx \leq t_1 \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2.$$

Now, for $t \in [0, t_1]$, we have,

$$\begin{aligned} \|u_{c_n}(\cdot - y_n) + tU_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 &= \|u_{c_n}(\cdot - y_n)\|_{L^2(\mathbb{R}^N)}^2 + 2t \int_{\mathbb{R}^N} u_{c_n}(x - y_n)U_{\varepsilon_n}(x)dx + t^2 \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq c_n + 2t_1 \int_{\mathbb{R}^N} u_{c_n}(x - y_n)U_{\varepsilon_n}(x)dx + t_1^2 \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq c_n + 2t_1^2 \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 = c \end{aligned}$$

where for the last inequality we have used the definition of c_n given in (2.5.28). The first inequality in (2.5.29) is obvious by the positivity of u_{c_n} and U_{ε_n} . Finally note that (2.5.30) directly holds in view of (2.5.23) and since $\|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 \rightarrow 0$ as $n \rightarrow \infty$. \square

Lemma 2.5.10. *Under the setting introduced above, we define,*

$$\gamma_n(t) = \begin{cases} u_{c_n}(\cdot - y_n) + tU_{\varepsilon_n} & \text{if } t \in [0, t_1], \\ \gamma_n(t_1) & \text{if } t \geq t_1. \end{cases}$$

Then $\gamma_n \in \mathcal{G}(c)$, for any $n \in \mathbb{N}$ large enough. In addition

$$M(c) \leq \max_{t \in [0, \infty)} F_\mu(\gamma_n(t)) \leq \max \left\{ \max_{t \in [t_0, t_1]} F_\mu(\gamma_n(t)), m(c) + \frac{2}{3N} \mathcal{S}^{\frac{N}{2}} \right\}. \quad (2.5.31)$$

Proof. Let $n \in \mathbb{N}$ be arbitrary fixed and large. To show that $\gamma_n \in \mathcal{G}(c)$ we first observe that $\gamma_n \in C([0, \infty), H^1(\mathbb{R}^N))$. Also, by (2.5.29), we get that

$$\gamma_n(t) \subset \bigcup_{d \in [\frac{\varepsilon}{2}, c]} S(d)$$

since $c_n \rightarrow c$. Now, note that by Lemma 2.5.6

$$\begin{aligned} F_\mu(\gamma_n(t)) &\leq m(c_n) + t \int_{\mathbb{R}^N} \nabla u_{c_n}(\cdot - y_n) \cdot \nabla U_{\varepsilon_n}(x) dx \\ &\quad + \frac{t^2}{2} \|\nabla U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_{\varepsilon_n}\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_{\varepsilon_n}\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}. \end{aligned} \quad (2.5.32)$$

In view of (2.5.30) we can apply Lemma 2.5.7 to deduce that, for $t \geq t_1$

$$F_\mu(\gamma_n(t)) \leq m(c_n) + 2m(c) \leq 2m(c). \quad (2.5.33)$$

We conclude that $\gamma_n \in \mathcal{G}(c)$ for any $n \in \mathbb{N}$ large enough. In particular the first inequality in (2.5.31) holds because of Lemma 2.5.5. Now, considering again (2.5.32) and recording that (2.5.30) apply, we deduce from Lemma 2.5.7 that, if $t \in [0, t_0]$,

$$F_\mu(\gamma_n(t)) \leq m(c_n) + \frac{1}{2N} \mathcal{S}^{\frac{N}{2}} \leq m(c) + \frac{2}{3N} \mathcal{S}^{\frac{N}{2}} \quad (2.5.34)$$

since $c_n \rightarrow c$. Gathering (2.5.33) and (2.5.34) we see that the second inequality in (2.5.31) holds. \square

Proof of Proposition 2.1.18. We assume the setting above. In view of Lemma 2.5.10 to show that

$$M(c) < m(c) + \frac{1}{N} \mathcal{S}^{\frac{N}{2}}$$

it suffices to show that

$$\max_{t \in [t_0, t_1]} F_\mu(u_{c_n}(\cdot - y_n) + tU_{\varepsilon_n}) < m(c) + \frac{1}{N} \mathcal{S}^{\frac{N}{2}}.$$

From [Lemma 2.3.5\(iv\)](#), [Lemma 2.5.6](#), [Lemma 2.5.9](#) and the definition of c_n given in (2.5.28) we can write

$$\begin{aligned}
 & \max_{t \in [t_0, t_1]} F_\mu(u_{c_n}(\cdot - y_n) + tU_{\varepsilon_n}) \\
 & \leq m(c_n) + \max_{t \in [t_0, t_1]} \left[t \|U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 + \frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right] \\
 & \leq m(c) + d(c - c_n) + t_1 \|U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 + \max_{t \in [t_0, t_1]} \left[\frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right] \\
 & \leq m(c) + 2t_1^2 d \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 + t_1 \|U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 \\
 & \quad + \max_{t \in [t_0, t_1]} \left[\frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right].
 \end{aligned}$$

To complete the proof it suffices to show, for $n \in \mathbb{N}$ sufficiently large,

$$J_n := (2t_1^2 d + t_1) \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 + \max_{t \in [t_0, t_1]} \left[\frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right] < \frac{1}{N} \mathcal{S}^{\frac{N}{2}}.$$

In this aim first note that

$$J_n \leq (2t_1^2 d + t_1) \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 + \max_{t > 0} \left[\frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right] - \frac{\mu t_0^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q.$$

But it holds, in view of the estimates of [Lemma 2.7.1\(i\)](#), that

$$\max_{t > 0} \left[\frac{t^2}{2} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{t^{2^*}}{2^*} \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \right] = \frac{1}{N} \mathcal{S}^{\frac{N}{2}} + O(\varepsilon_n^{N-2}).$$

Summarizing, the proof of [Proposition 2.1.18](#) will be completed if we manage to show that, for $n \in \mathbb{N}$ large enough,

$$(2t_1^2 d + t_1) \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t_0^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q + O(\varepsilon_n^{N-2}) < 0.$$

At this point we distinguish two cases.

Case 1 $N \geq 5$: By [Lemma 2.7.1](#) we have that, for some $K_1 > 0, K_2 > 0$, as $n \rightarrow \infty$,

$$\|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 = K_1 \varepsilon_n^2 + o(\varepsilon_n^{N-2}) \quad \text{and} \quad \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = K_2 \varepsilon_n^{N - \frac{(N-2)q}{2}} + o(\varepsilon_n^{N - \frac{(N-2)q}{2}}).$$

Thus, for some constants, $\tilde{K}_1 > 0, \tilde{K}_2 > 0$, for $n \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
 (2t_1^2 d + t_1) \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t_0^q}{q} \|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q + O(\varepsilon_n^{N-2}) &= \tilde{K}_1 \varepsilon_n^2 - \tilde{K}_2 \varepsilon_n^{N - \frac{(N-2)q}{2}} + O(\varepsilon_n^{N-2}) \\
 &\leq -\frac{\tilde{K}_2}{2} \varepsilon_n^{N - \frac{(N-2)q}{2}} < 0
 \end{aligned}$$

since $N - \frac{(N-2)q}{2} < \min\{N-2, 2\} \iff q > 2$.

Case 2 $N = 4$: By [Lemma 2.7.1](#) we have that as $n \rightarrow \infty$, for some $K_4 > 0$,

$$\|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 = \omega \varepsilon_n^2 |\log \varepsilon_n| + O(\varepsilon_n^2) \quad \text{and} \quad \|U_{\varepsilon_n}\|_{L^q(\mathbb{R}^N)}^q = K_4 \varepsilon_n^{4-q} + o(\varepsilon_n^{4-q}).$$

Thus, for some constants, $\tilde{K}_3 > 0$, $\tilde{K}_4 > 0$, for $n \in \mathbb{N}$ sufficiently large,

$$(2t_1^2 d + t_1) \|U_{\varepsilon_n}\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu t_0^q}{q} \|U_{\varepsilon_n}\|_{L^q(\mathbb{R}^N)}^q + O(\varepsilon_n^2) = \tilde{K}_3 \varepsilon_n^2 |\log \varepsilon_n| - \tilde{K}_4 \varepsilon_n^{4-q} + O(\varepsilon_n^2) \leq -\frac{\tilde{K}_4}{2} \varepsilon_n^{4-q} < 0$$

since $4 - q < 2$. In view of Cases 1 and 2 we deduce that the conclusion of [Proposition 2.1.18](#) holds if $N \geq 4$. \square

Remark 2.5.11. *Our analysis of the interaction between a solution characterized as a local minima and a suitable family of truncated extremal functions for the Sobolev inequality reminds us of the approach developed by G. Tarantello in [85]. However, in [85], the extremal functions are located in a set where the local minima solution takes its greater values. The idea being to prove, through delicate estimates, that this interaction does decrease the mountain pass value of the associated functional with respect to the case where the two supports would be fully disjoint. Here, on the contrary, our construction aims at separating sufficiently the regions where the functions concentrate and to show that the remaining interaction (remember our functions $u_c \in S(c)$ lie on all \mathbb{R}^N) can be assumed sufficiently small.*

2.5.4 The proofs of [Theorem 2.1.8](#) and [Theorem 2.1.9](#)

Proof of [Theorem 2.1.8](#). The proof follows directly combining [Proposition 2.1.12](#), [Proposition 2.1.13](#) and [Proposition 2.1.14](#). \square

Proof of [Theorem 2.1.9](#). For $\mu > 0$ fixed, let us prove that

$$F_\mu(v_c) \rightarrow \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad \text{as } c \rightarrow 0. \quad (2.5.35)$$

First, using that $Q_\mu(v_c) = 0$, we can write, using the Gagliardo-Nirenberg inequality [\(2.2.2\)](#)

$$\begin{aligned} F_\mu(v_c) &= \frac{1}{N} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \left(1 - \frac{q\gamma_q}{2^*}\right) \|u_n\|_{L^q(\mathbb{R}^N)}^q \\ &\geq \frac{1}{N} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} C_{N,q}^q \left(1 - \frac{q\gamma_q}{2^*}\right) c^{(1-\gamma_q)q} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^{q\gamma_q}. \end{aligned} \quad (2.5.36)$$

Since

$$F_\mu(v_c) = M^0(c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N} \leq \frac{\mathcal{S}^{\frac{N}{2}}}{N} \quad (2.5.37)$$

and $q\gamma_q < 2$ we deduce that $(v_c) \subset H_r^1(\mathbb{R}^N)$ is uniformly bounded with respect to $c \in (0, c_0)$. Thus, still using the Gagliardo-Nirenberg inequality, we deduce that

$$\|v_c\|_{L^q(\mathbb{R}^N)}^q \leq C_{N,q} \|v_c\|_{L^2(\mathbb{R}^N)}^{q(1-\gamma_q)} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^{q\gamma_q} \rightarrow 0 \quad \text{as } c \rightarrow 0. \quad (2.5.38)$$

Hence, recording that $Q_\mu(v_c) = 0$, we have that

$$\ell := \lim_{c \rightarrow 0} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 = \lim_{c \rightarrow 0} \|v_c\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \leq \frac{1}{\mathcal{S}^{\frac{2^*}{2}}} \lim_{c \rightarrow 0} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^{2^*} = \frac{1}{\mathcal{S}^{\frac{2^*}{2}}} \ell^{\frac{2^*}{2}}.$$

Therefore, either $\ell = 0$ or $\ell \geq \mathcal{S}^{\frac{N}{2}}$. We claim that $\ell = 0$ is impossible. Indeed, since $v_c \in \Lambda^-(c)$, we have that

$$\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 - \mu\gamma_q \|v_c\|_{L^q(\mathbb{R}^N)}^q - \|v_c\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = 0 \quad (2.5.39)$$

and

$$\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 - \mu\gamma_q(q\gamma_q - 1)\|v_c\|_{L^q(\mathbb{R}^N)}^q - (2^* - 1)\|v_c\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \leq 0. \quad (2.5.40)$$

Combining (2.5.39), (2.5.40) and using the Sobolev inequality, we get

$$\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{2^* - q\gamma_q}{2 - q\gamma_q} \|v_c\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \leq \frac{2^* - q\gamma_q}{2 - q\gamma_q} \frac{1}{\mathcal{S}^{\frac{2^*}{2}}} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^{2^*}$$

proving the claim.

At this point, in view of (2.5.38), using that $Q_\mu(v_c) = 0$, we have

$$M^0(c) = F_\mu(v_c) = \frac{1}{N} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{q} \left(1 - \frac{q\gamma_q}{2^*}\right) \|v_c\|_{L^q(\mathbb{R}^N)}^q = \frac{1}{N} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 + o_c(1) \geq \frac{\mathcal{S}^{\frac{N}{2}}}{N} + o_c(1),$$

where $o_c(1) \rightarrow 0$ as $c \rightarrow 0$. Taking into account that $m(c) \rightarrow 0$ as $c \rightarrow 0$, see Remark 2.1.4, and that

$$M^0(c) < m(c) + \frac{\mathcal{S}^{\frac{N}{2}}}{N}$$

we obtain (2.5.35). Clearly also the above proof shows that $\|\nabla v_c\|_{L^2(\mathbb{R}^N)}^2 \rightarrow \mathcal{S}^{\frac{N}{2}}$. This proves (i).

To show that (ii) holds we start to observe that, since $c_0(\mu) \rightarrow \infty$ as $\mu \rightarrow 0$, see Remark 2.1.3, $v_c \in S(c)$ exists for any $\mu \rightarrow 0$ sufficiently small. Now, (2.5.36)- (2.5.37) imply that $(v_c) \subset H_r^1(\mathbb{R}^N)$ is uniformly bounded as $\mu \rightarrow 0$ and thus, using the Gagliardo-Nirenberg inequality, we have that

$$\mu \|v_c\|_{L^q(\mathbb{R}^N)}^q \leq \mu C_{N,q} \|v_c\|_{L^2(\mathbb{R}^N)}^{q(1-\gamma_q)} \|\nabla v_c\|_{L^2(\mathbb{R}^N)}^{q\gamma_q} \rightarrow 0 \quad \text{as } \mu \rightarrow 0.$$

From here the rest of the proof is identical to the one of (i). \square

2.6 Strong instability of the standing waves lying at mountain pass levels

To prove the strong instability of the standing waves, we use the recent advances on the subject of instability by blow-up contained in [80, 81]. Now, we recall [81, Theorem 1.6] which in our notation reads as

Theorem 2.6.1. *Under the assumptions of Theorem 2.1.8, let $u \in S(c)$ be such that $F_\mu(u) < \inf_{\Lambda^+(c)} F_\mu$. Then, if $s_u^+ < 1$ and $|x|u \in L^2(\mathbb{R}^N, \mathbb{C})$, the solution ϕ of (2.1.1) with initial datum u blows-up in finite time.*

Proof of Theorem 2.1.11. Since, by Remark 2.5.4

$$F_\mu(v_c) = \inf_{u \in \Lambda^-(c)} F_\mu(u)$$

and $\psi_{v_c}(s)$ has a unique global maximum at $s^* = 1$, see Lemma 2.2.3, we have that $v_t := (v_c)_t$ satisfies

$$F_\mu(v_t) < \inf_{u \in \Lambda^-(c)} F_\mu(u)$$

for any $t > 1$. Clearly $s_{v_t}^+ = \frac{1}{t} < 1$ for any $t > 1$ and also $v_t \rightarrow v_c$ as $t \rightarrow 1^+$. Now, since $\lambda_c < 0$, by a classical decay argument, we have that v_c and thus v_t satisfies $|x|v_t \in L^2(\mathbb{R}^N, \mathbb{C})$. At this point, applying Theorem 2.6.1, we deduce that $v_c \in S(c)$ gives rise to an unstable standing wave. See for more details [81, Theorem 1.3] or [80, Theorem 1.4]. \square

Remark 2.6.2. *Theorem 2.6.1, see also [80, Theorem 1.13], is remarkable because it permits to detect a finite time blow-up occurs just by considering the 1-variable function ψ_u . We refer, for earlier results on the link between the variational characterization of a solution and its instability, to the classical paper [20], and to [56] for more recent developments.*

2.7 Appendix

Let $N \geq 3$, u_ε be the extremal functions for the Sobolev inequality in \mathbb{R}^N defined in (2.5.20) and $\xi \in C_0^\infty(\mathbb{R}^N)$ be a radially non-increasing cut-off function with $\xi \equiv 1$ in B_1 , $\xi \equiv 0$ in $\mathbb{R}^N \setminus B_2$.

Lemma 2.7.1. *Setting $U_\varepsilon := \xi u_\varepsilon$ and denoting by ω the area of the unit sphere in \mathbb{R}^N , we have, for $N \geq 3$,*

(i)

$$\|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 = S^{\frac{N}{2}} + O(\varepsilon^{N-2}) \quad \text{and} \quad \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}^2 = S^{\frac{N}{2}} + O(\varepsilon^N),$$

(ii) *For some positive constant $K > 0$,*

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = \begin{cases} K\varepsilon^{N - \frac{(N-2)}{2}q} + o(\varepsilon^{N - \frac{(N-2)}{2}q}) & \text{if } \frac{N}{N-2} < q < 2^*, \\ \omega\varepsilon^{\frac{N}{2}}|\log \varepsilon| + O(\varepsilon^{\frac{N}{2}}) & \text{if } q = \frac{N}{N-2}, \\ \omega\left(\int_0^2 \frac{\xi^q(r)}{r^{(N-2)q - (N-1)}} dr\right)\varepsilon^{\frac{N-2}{2}q} + o(\varepsilon^{\frac{N-2}{2}q}) & \text{if } 1 \leq q < \frac{N}{N-2}. \end{cases}$$

Proof. The point (i) is standard. See, for example, [83, pages 163-164]. We shall thus concentrate on point (ii). We have

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \xi^q(x) \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-2}{2}q} dx.$$

Passing to radial coordinates, we get

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = \omega \varepsilon^{\frac{N-2}{2}q} \int_0^2 \frac{\xi^q(r) r^{N-1}}{(\varepsilon^2 + r^2)^{\frac{N-2}{2}q}} dr \quad (2.7.1)$$

that can be decomposed as

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = \omega \varepsilon^{\frac{N-2}{2}q} \int_0^2 \frac{(\xi^q(r)-1)r^{N-1}}{(\varepsilon^2+r^2)^{\frac{N-2}{2}q}} dr + \omega \varepsilon^{\frac{N-2}{2}q} \int_0^2 \frac{r^{N-1}}{(\varepsilon^2+r^2)^{\frac{N-2}{2}q}} dr := I_1(\varepsilon) + I_2(\varepsilon). \quad (2.7.2)$$

Since $\xi(r) \equiv 1$ on $[0, 1]$, the integral in $I_1(\varepsilon)$ is converging and thus $I_1(\varepsilon) = O(\varepsilon^{\frac{N-2}{2}q})$. Now, by making a change of variable, we rewrite $I_2(\varepsilon)$ as

$$I_2(\varepsilon) = \omega \varepsilon^{N-\frac{N-2}{2}q} \int_0^{\frac{2}{\varepsilon}} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}q}} dr.$$

The integral in $I_2(\varepsilon)$ is converging, as $\varepsilon \rightarrow 0$, to a finite value if and only if $q > \frac{N}{N-2}$. Thus, when $q > \frac{N}{N-2}$, we have that, for some constant $K > 0$,

$$I_2(\varepsilon) = K \varepsilon^{N-\frac{N-2}{2}q} + o(\varepsilon^{N-\frac{N-2}{2}q})$$

and recording that $I_1(\varepsilon) = O(\varepsilon^{\frac{N-2}{2}q})$, we have

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = I_1(\varepsilon) + I_2(\varepsilon) = K \varepsilon^{N-\frac{N-2}{2}q} + o(\varepsilon^{N-\frac{N-2}{2}q}).$$

This proves point (ii) for $\frac{N}{N-2} < q < 2^*$. Now, assuming that $q = \frac{N}{N-2}$, and proceeding as in (2.7.2), we get that

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = \omega \varepsilon^{\frac{N}{2}} \int_0^2 \frac{(\xi^q(r)-1)r^{N-1}}{(\varepsilon^2+r^2)^{\frac{N}{2}}} dr + \omega \varepsilon^{\frac{N}{2}} \int_0^2 \frac{r^{N-1}}{(\varepsilon^2+r^2)^{\frac{N}{2}}} dr := I_1(\varepsilon) + I_2(\varepsilon) \quad (2.7.3)$$

with $I_1(\varepsilon) = O(\varepsilon^{\frac{N}{2}})$. Also,

$$I_2(\varepsilon) = \omega \varepsilon^{\frac{N}{2}} \int_0^{\frac{2}{\varepsilon}} \frac{r^{N-1}}{(1+r^2)^{\frac{N}{2}}} dr = \omega \varepsilon^{\frac{N}{2}} (|\log \varepsilon| + O(1)).$$

Summarizing, we obtain for $q = \frac{N}{N-2}$ that

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = \omega \varepsilon^{\frac{N}{2}} |\log \varepsilon| + O(\varepsilon^{\frac{N}{2}}).$$

It remains to study the case $1 \leq q < \frac{N}{N-2}$. Under this assumption, we observe that, for all $r > 0$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\xi^q(r)r^{N-1}}{(\varepsilon^2+r^2)^{\frac{N-2}{2}q}} = \frac{\xi^q(r)r^{N-1}}{r^{(N-2)q}} = \frac{\xi^q(r)}{r^{(N-2)q-(N-1)}}$$

and also that, for all $\varepsilon > 0$, for some constant $D > 0$

$$\left| \frac{\xi^q(r)r^{N-1}}{(\varepsilon^2+r^2)^{\frac{N-2}{2}q}} \right| \leq \frac{D}{r^{(N-2)q-(N-1)}} \in L^1([0, 2]).$$

Thus, from Lebesgue's theorem, we deduce from (2.7.1) that

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^N)}^q = \omega \left(\int_0^2 \frac{\xi^q(r)}{r^{(N-2)q-(N-1)}} dr \right) \varepsilon^{\frac{N-2}{2}q} + o(\varepsilon^{\frac{N-2}{2}q}).$$

This ends the proof of the lemma. \square

Chapter 3

Multiple normalized solutions for the Schrödinger-Poisson-Slater equation

This chapter is precisely the results in the paper [47]. For the convenience of reading, this chapter is presented self-contained with the rest of the thesis.

3.1 Introduction

In this chapter, we consider the following Schrödinger-Poisson-Slater equation:

$$i\partial_t v + \Delta v + \gamma(|x|^{-1} * |v|^2)v + a|v|^{p-2}v = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^3, \quad (3.1.1)$$

where $v : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$, $\gamma \in \mathbb{R}$, $a \in \mathbb{R}$ and $p \in (\frac{10}{3}, 6]$. We look for standing wave solutions to (3.1.1), namely to solutions of the form $v(t, x) = e^{i\lambda t} u(x)$, $\lambda \in \mathbb{R}$. Then the function $u(x)$ satisfies the equation

$$-\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3. \quad (3.1.2)$$

Motivated by the fact that the L^2 -norm is a preserved quantity of the evolution we focus on the search of solutions to (3.1.2) with prescribed L^2 -norm. It is standard that for some prescribed $c > 0$, a solution of (3.1.2) with $\|u\|_{L^2(\mathbb{R}^3)}^2 = c$ can be obtained as a critical point of the *Energy* functional

$$F(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\gamma}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy - \frac{a}{p} \int_{\mathbb{R}^3} |u|^p dx$$

restricted to

$$S(c) := \{u \in H^1(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3)}^2 = c\}.$$

Then the parameter $\lambda \in \mathbb{R}$ in (3.1.2) appears as a Lagrange multiplier, it is an unknown of the problem. As we know, $F(u)$ is a well-defined and C^1 functional on $S(c)$ for any $p \in (2, 6]$ (see [77] for example).

Let us define

$$m(c) = \inf_{u \in S(c)} F(u). \quad (3.1.3)$$

Depending on the range of parameters we shall consider $m(c)$ will be finite or not. The case where $\gamma < 0$ and $a > 0$ in (3.1.2) has been the most studied so far. When $p \in (2, \frac{10}{3})$ it can be shown that $m(c) \in (-\infty, 0]$ for any $c > 0$ and it is also the case when $p = \frac{10}{3}$ and $c > 0$ is small. It is shown in [17] that minimizer exists if $p \in (2, 3)$ and $c > 0$ is small enough, see also [78] for the special case $p = \frac{8}{3}$. The case $p \in (3, \frac{10}{3})$ was considered in [16, 51], see also [55] for a closely related problem. In [51] the existence of a threshold value $c_0 > 0$ such that $m(c)$ has a minimizer if and only if $c \in [c_0, \infty)$ was established. It was also proved in [51] that a minimizer does not exist for any $c > 0$ if $p = 3$ or $p = \frac{10}{3}$. We also refer to [26] for related results. When $p \in (\frac{10}{3}, 6)$ a scaling argument reveals that $m(c) = -\infty$ but nevertheless it was proved in [15] that, when $p \in (\frac{10}{3}, 6)$ there exists, for $c > 0$ small enough a critical point of F constrained to $S(c)$ at a strictly positive level. In this work we complement the result of [15] by showing that when $p = 6$ and for any $c > 0$ there does not exist positive solutions, see Theorem 3.1.9.

Even if some open problems remain when $\gamma < 0$ and $a > 0$, we shall mainly concentrate here on the others cases: $(\gamma < 0, a < 0)$, $(\gamma > 0, a > 0)$ and $(\gamma > 0, a < 0)$. We define, for short, the following quantities

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy, \quad C(u) := \int_{\mathbb{R}^3} |u|^p dx.$$

For $u \in S(c)$, we set $u^t(x) := t^{\frac{3}{2}} u(tx)$, $t > 0$, then

$$u^t \in S(c), \quad A(u^t) = t^2 A(u), \quad B(u^t) = tB(u), \quad C(u^t) = t^\sigma C(u),$$

where

$$2 < \sigma := \frac{3(p-2)}{2} \leq 6, \tag{3.1.4}$$

due to $p \in (\frac{10}{3}, 6]$. For $u \in S(c)$, we define the fiber map

$$t \in (0, \infty) \mapsto g_u(t) := F(u^t) = \frac{1}{2} t^2 A(u) - \frac{\gamma}{4} t B(u) - \frac{a}{p} t^\sigma C(u).$$

Hence, we have

$$g'_u(t) = tA(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} t^{\sigma-1} C(u) = \frac{1}{t} Q(u^t),$$

where

$$Q(u) = A(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} C(u).$$

Actually the condition $Q(u) = 0$ corresponds to a Pohozaev identity and the set

$$\Lambda(c) := \{u \in S(c) : Q(u) = 0\} = \{u \in S(c) : g'_u(1) = 0\}$$

appears as a natural constraint. Indeed, if $u \in S(c)$, then $t > 0$ is a critical point for g_u if and only if $u^t \in \Lambda(c)$. In particular, $u \in \Lambda(c)$ if and only if 1 is a critical point of g_u .

First we briefly consider the case $\gamma < 0, a < 0$. For any $u \in S(c)$, we have that $g'_u(t) > 0$ for all $t > 0$, hence the fiber map $g_u(t)$ is strictly increasing and so we can state the following non-existence result:

Theorem 3.1.1. *Assume that $\gamma < 0, a < 0$. Then $F(u)$ has no critical point on $S(c)$.*

Next, we consider the case $\gamma > 0, a > 0$. In this case, let

$$c_1 := \left(\frac{4}{\gamma K_H} \frac{\sigma - 2}{\sigma - 1} \right)^{\frac{3p-10}{4(p-3)}} \left(\frac{p}{a\sigma(\sigma-1)K_{GN}} \right)^{\frac{1}{2(p-3)}} > 0, \quad (3.1.5)$$

where σ is defined by (3.1.4) and K_H, K_{GN} are defined in Lemma 3.2.1. We also introduce the decomposition of $\Lambda(c)$ into the disjoint union $\Lambda(c) = \Lambda^+(c) \cup \Lambda^0(c) \cup \Lambda^-(c)$, where

$$\begin{aligned} \Lambda^+(c) &:= \{u \in \Lambda(c) : g_u''(1) > 0\} = \{u \in S(c) : g_u'(1) = 0, g_u''(1) > 0\}, \\ \Lambda^0(c) &:= \{u \in \Lambda(c) : g_u''(1) = 0\} = \{u \in S(c) : g_u'(1) = 0, g_u''(1) = 0\}, \\ \Lambda^-(c) &:= \{u \in \Lambda(c) : g_u''(1) < 0\} = \{u \in S(c) : g_u'(1) = 0, g_u''(1) < 0\}. \end{aligned}$$

By Lemma 3.3.3 and Lemma 3.3.4, for any $c \in (0, c_1)$ we have that $\Lambda^0(c) = \emptyset$ and $\Lambda^+(c) \neq \emptyset$, $\Lambda^-(c) \neq \emptyset$. Since F is bounded from below on $\Lambda(c)$ due to Lemma 3.3.1, we can define

$$\gamma^+(c) := \inf_{u \in \Lambda^+(c)} F(u) \quad \text{and} \quad \gamma^-(c) := \inf_{u \in \Lambda^-(c)} F(u). \quad (3.1.6)$$

Our first main result is

Theorem 3.1.2. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma > 0, a > 0$ and let $c_1 > 0$ be defined by (3.1.5). For any $c \in (0, c_1)$, there exist $u_c^+ \in \Lambda^+(c)$ such that $F(u_c^+) = \gamma^+(c)$ and $u_c^- \in \Lambda^-(c)$ such that $F(u_c^-) = \gamma^-(c)$. The functions u_c^+, u_c^- are bounded continuous positive Schwarz symmetric functions. In addition there exist $\lambda_c^+ > 0$ and $\lambda_c^- > 0$ such that (u_c^+, λ_c^+) and (u_c^-, λ_c^-) are solutions to (3.1.2).*

Remark 3.1.3. *In Theorem 3.1.2, borrowing an approach first introduced in [31], an effort is made to optimize the limit value $c_1 > 0$. As a consequence, compared to the works [80, 81] and our work in Chapter 2, we do not benefit from the property that $\gamma^-(c) \geq 0 = \sup_{u \in \Lambda^+(c)} F(u)$. Such property is a help to show the convergence of the Palais-Smale sequences in these works. Also, the fact that we may have $\gamma^-(c) < 0$ makes somehow more involved to prove that the level $\gamma^-(c)$ is reached by a radially symmetric function, a Schwartz function actually, see Lemma 3.3.6. It is not clear to us if $c_1 > 0$ is optimal. Nevertheless, we conjecture that there exists a $c_0 \geq c_1 > 0$ such that one solution exists when $c = c_0$ and that, at least positive solutions, do not exist when $c > c_0$.*

Remark 3.1.4. *As we shall see $\gamma^+(c) < \gamma^-(c)$ and combined with the property that any critical point lies in $\Lambda(c)$ it implies that the solution u_c^+ obtained in Theorem 3.1.2 is a ground state. Following [14] a ground state is defined as a solution $v \in S(c)$ to (3.1.2) which has minimal Energy among all the solutions which belong to $S(c)$. Namely, if*

$$F(v) = \inf \left\{ F(u), u \in S(c), \left(F|_{S(c)} \right)'(u) = 0 \right\}.$$

If the geometrical structure of F restricted to $S(c)$ is identical in the Sobolev subcritical case $p \in (\frac{10}{3}, 6)$ and in the Sobolev critical case $p = 6$, the proof that the levels $\gamma^+(c)$ and $\gamma^-(c)$ are indeed reached requires additional, more involved, arguments in the case $p = 6$. In particular, showing that $\gamma^-(c)$ is attained requires to check that the following inequality holds

$$\gamma^-(c) < \gamma^+(c) + \frac{1}{3\sqrt{a}K_{GN}}. \quad (3.1.7)$$

It is known since the pioneering work of Brezis-Nirenberg [25] that the way to derive such a strict upper bound is through the use of testing functions. In Chapter 2, considering the equation

$$-\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N, \quad (3.1.8)$$

with $N \geq 3$, $\mu > 0$, $2 < q < 2 + \frac{4}{N}$ and $2^* = \frac{2N}{N-2}$ we face the need to establish a similar inequality. We constructed test functions which could be viewed as the sum of a truncated extremal function of the Sobolev inequality on \mathbb{R}^N centered at the origin and of u_c^+ translated far away from the origin. This choice of testing functions was sufficient to prove our strict inequality when $N \geq 4$ but we missed it in the case $N = 3$. Very recently, in [89] the authors introduced an alternative choice of testing functions which allowed to treat, in a unified way, the case $N = 3$ and $N \geq 4$ for (3.1.8). The strategy in [89], recording of the one introduced by G. Tarantello in [85], is on the contrary, to located the extremal functions where the solution u_c^+ takes its greater values (the origin thus). The idea behind the proof is that the interaction decreases the value of the *Energy* with respect to the case where the supports would be disjoint. In this chapter, where (3.1.2) is set on \mathbb{R}^3 , we believe in view of our experience on (3.1.8), more appropriate to follow the approach of [89] to check the inequality (3.1.7) for any $c \in (0, c_1)$.

The results of Theorem 3.1.2 are complemented in several directions. First, we show that the solution $u^+(c)$ obtained in Theorem 3.1.2 can be characterized as a local minima for F restricted to $S(c)$. We treat here the full range $p \in (\frac{10}{3}, 6]$ with a single proof. More precisely we show,

Theorem 3.1.5. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma > 0$, $a > 0$ and let $c \in (0, c_1)$. Then we have $\Lambda^+(c) \subset V(c)$ and*

$$\gamma^+(c) = \inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in V(c)} F(u)$$

where

$$V(c) := \{u \in S(c) | A(u) < k_1\}$$

for some $k_1 > 0$ independent of $c \in (0, c_1)$ (see (3.3.45) for the definition of $k_1 > 0$). In addition, any minimizing sequence for F on $V(c)$ is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$.

Remark 3.1.6. *The proof of Lemma 3.3.20 which is a key step to established Theorem 3.1.5, reveals some additional properties of the set $V(c)$. Indeed, we have that $V(c) \subset S(c) \setminus \Lambda^-(c)$ and thus $V(c)$ is separating the sets $\Lambda^+(c)$ and $\Lambda^-(c)$. Also, for any $0 < c, \tilde{c} < c_1$, we have that $A(u) < k_1 < A(v)$ for all $u \in \Lambda^+(c), v \in \Lambda^-(\tilde{c})$, see (3.3.46) and (3.3.49).*

Remark 3.1.7. *To prove that the minimizing sequences for F on $V(c)$ are, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$ we follow an approach due to [43] that has already been used several times, see, for example, [39, 67]. The first step in this approach is to show that the sequences do not vanish. When $p = 6$, we rely for this, in an essential way, on the fact that $c_1 > 0$ is sufficiently small, see Lemma 3.3.22. This fact is also used to end the proof. Finally, note that since we allow the possibility that $\inf_{u \in \partial V(c)} F(u) < 0$ where $\partial V(c) := \{u \in S(c) | A(u) = k_1\}$ we must check that the minimizers do ly in $V(c)$.*

Let us now denote

$$\mathcal{M}_c := \{u \in V(c) : F(u) = \gamma^+(c)\}.$$

In view of [Remark 3.1.4](#), \mathcal{M}_c is the set of all ground states. The property that any minimizing sequence for F restricted to $V(c)$ is, up to translation, strongly converging is known to be a key ingredient to show that the set \mathcal{M}_c is orbitally stable. If $p \in (\frac{10}{3}, 6)$ the orbital stability of \mathcal{M}_c indeed follows directly from [Theorem 3.1.5](#) by the classical arguments of [\[28\]](#). In the case $p = 6$ the situation is more delicate as the existence of a uniform $H^1(\mathbb{R}^3)$ bound on the solution of [\(3.1.1\)](#) during its lifespan is not sufficient to guarantee that blow-up may not occur. We refer to [\[27\]](#) for more details. We do not prove anything in that direction but strongly believe that the set \mathcal{M}_c is orbitally stable. Actually, such a result has been obtained on the equation [\(3.1.8\)](#) in [Chapter 2](#).

We also discuss the behavior of the associated Lagrange multipliers and show that if the behavior of λ_c^+ is essentially the same for the cases $p \in (\frac{10}{3}, 6)$ and $p = 6$, see [Lemma 3.3.24](#), there is a distinct behavior for λ_c^- , see [Lemmas 3.3.25](#) and [3.3.26](#). Finally, in [Lemma 3.3.27](#), we establish the property that the map $c \mapsto \gamma^-(c)$ is strictly decreasing.

Next, we consider the case $\gamma > 0$, $a < 0$. Recalling the definition of $m(c)$ given in [\(3.1.3\)](#) we show in [Lemma 3.4.1](#), that $-\infty < m(c) < 0$ and then we prove the following result.

Theorem 3.1.8. *Let $p \in (\frac{10}{3}, 6]$, $\gamma > 0$ and $a < 0$. For any $c > 0$, the infimum $m(c)$ is achieved and any minimizing sequence for [\(3.1.3\)](#) is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$ to a solution of [\(3.1.2\)](#). In addition, the associated Lagrange multiplier is positive.*

Even if the proof of [Theorem 3.1.8](#) follows the lines of the proof of [Theorem 3.1.5](#), the change of sign in front of the power term requires some adaptations, see [Lemma 3.4.2](#) and [Lemma 3.4.4](#). Here again the orbital stability of the set of minimizers should follow directly from the classical arguments of [\[28\]](#) if $p \in (\frac{10}{3}, 6)$ and it should also be the case when $p = 6$ by adapting our arguments in [Chapter 2](#). Note that we also study the behavior of the associated Lagrange multipliers in [Lemma 3.4.5](#).

In the last part of the chapter we consider the case $\gamma < 0$, $a > 0$ and $p = 6$.

Theorem 3.1.9. *Let $p = 6$, $\gamma < 0$ and $a > 0$. For any $c > 0$, we have that*

- (i) *If $u \in H^1(\mathbb{R}^3)$ is a non-trivial solution to [\(3.1.2\)](#) then the associated Lagrange multiplier λ is negative and*

$$F(u) > \frac{1}{3\sqrt{aK_{GN}}}.$$

- (ii) *Equation [\(3.1.2\)](#) has no positive solution in $H^1(\mathbb{R}^3)$.*

Remark 3.1.10. *In [\[81, Theorem 1.2\]](#), considering the equation*

$$-\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N, \quad (3.1.9)$$

with $N \geq 3$, $2 < q < 2^$ and $\mu < 0$, it was proved that [\(3.1.9\)](#) has no positive solution $u \in H^1(\mathbb{R}^N)$ if $N = 3, 4$ or if $N \geq 5$ under the additional assumption $u \in L^p(\mathbb{R}^N)$ for some $p \in (0, \frac{N}{N-2}]$. In [Remark 3.5.2](#), partly using arguments used in the proof of [Theorem 3.1.9](#), we improve [\[81, Theorem 1.2\]](#) showing that [\(3.1.9\)](#) has no positive solution in $H^1(\mathbb{R}^N)$ for all $N \geq 3$ and no non-trivial radial solution for $N \geq 3$ and $q > 2 + \frac{2}{N-1}$.*

Remark 3.1.11. We propose as an open problem to investigate if there are radial solutions under the assumptions of [Theorem 3.1.9](#). See [Remark 3.5.3](#) in that direction.

This chapter is organized as follows. In [Section 3.2](#) we present some preliminary results. [Section 3.3](#) is devoted to the treatment of the case $\gamma > 0$, $a > 0$ and $p \in (\frac{10}{3}, 6]$. In [Subsection 3.3.1](#) we make explicit the geometrical structure of F on $S(c)$ and show the existence of a bounded Palais-Smale sequence $(u_n^+) \subset \Lambda^+(c)$ at the level $\gamma^+(c)$ and of a bounded Palais-Smale sequence $(u_n^-) \subset \Lambda^-(c)$ at the level $\gamma^-(c)$. In [Subsection 3.3.2](#) we give the proof of [Theorem 3.1.2](#) in the Sobolev subcritical case. [Subsection 3.3.3](#) is devoted to the proof of [Theorem 3.1.2](#) in the critical case. In [Subsection 3.3.4](#) we prove the convergence of all minimizing sequences associated to $\gamma^+(c)$, namely [Theorem 3.1.5](#). The behavior of the Lagrange multipliers and the property of the map $c \mapsto \gamma^-(c)$ are studied in [Subsection 3.3.5](#). In [Section 3.4](#) we treat the case $\gamma > 0$, $a < 0$ and $p \in (\frac{10}{3}, 6]$ and we prove [Theorem 3.1.8](#). Finally, in [Section 3.5](#), we consider the case $\gamma < 0$, $a > 0$ and $p = 6$, and prove [Theorem 3.1.9](#)

3.2 Preliminary results

In this section we present various preliminary results. When it is not specified they are assumed to hold for $\gamma \in \mathbb{R}$, $a \in \mathbb{R}$, $p \in (\frac{10}{3}, 6]$ and any $c > 0$. Firstly, we present the definitions of $\Lambda(c)$, $\Lambda^+(c)$, $\Lambda^0(c)$, $\Lambda^-(c)$ via $A(u)$, $B(u)$ and $C(u)$:

$$\begin{aligned}\Lambda(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u) \right\}, \\ \Lambda^+(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), A(u) > \frac{a\sigma(\sigma-1)}{p}C(u) \right\}, \\ \Lambda^0(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), A(u) = \frac{a\sigma(\sigma-1)}{p}C(u) \right\}, \\ \Lambda^-(c) &= \left\{ u \in S(c) : A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), A(u) < \frac{a\sigma(\sigma-1)}{p}C(u) \right\}.\end{aligned}$$

Lemma 3.2.1. Let $u \in S(c)$, there exists

- (i) a constant $K_H > 0$ such that $B(u) \leq K_H \sqrt{A(u)}c^{\frac{3}{2}}$.
- (ii) a constant $K_{GN} > 0$ such that $C(u) \leq K_{GN}[A(u)]^{\frac{\sigma}{2}}c^{\frac{6-p}{4}}$.

Proof. We first recall the Hardy-Littlewood-Sobolev inequality (see [63, Chapter 4]):

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \right| \leq C(N, \lambda, p, q) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}, \quad (3.2.1)$$

where $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, $p, q > 1$, $0 < \lambda < N$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{N} = 2.$$

Let us also recall the Gagliardo-Nirenberg inequality (see [72]) and the Sobolev inequality (see [23, Theorem 9.9]) in the unified form: if $N \geq 3$ and $p \in [2, \frac{2N}{N-2}]$ then

$$\|f\|_{L^p(\mathbb{R}^N)} \leq C(N, p) \|\nabla f\|_{L^2(\mathbb{R}^N)}^\beta \|f\|_{L^2(\mathbb{R}^N)}^{(1-\beta)}, \quad \text{with } \beta = N \left(\frac{1}{2} - \frac{1}{p} \right).$$

Applying the Hardy-Littlewood-Sobolev inequality we obtain

$$B(u) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy \leq K_1 \|u\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \quad (3.2.2)$$

and thus using the Gagliardo-Nirenberg inequality, we get

$$B(u) \leq K_1 \|u\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \leq K_1 K_2 \|\nabla u\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)}^3 = K_H \sqrt{A(u)} c^{\frac{3}{2}}.$$

Finally, applying the Sobolev, Gagliardo-Nirenberg inequality, we have

$$C(u) = \|u\|_{L^p(\mathbb{R}^3)}^p \leq K_{GN} \|\nabla u\|_{L^2(\mathbb{R}^3)}^\sigma \|u\|_{L^2(\mathbb{R}^3)}^{\frac{6-p}{2}} = K_{GN} [A(u)]^{\frac{\sigma}{2}} c^{\frac{6-p}{4}}.$$

□

Lemma 3.2.2. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma \in \mathbb{R}$ and $a \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^3)$ is a weak solution to*

$$-\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0, \quad (3.2.3)$$

then $Q(u) = 0$. Moreover, if $u \neq 0$ then we have

$$(i) \quad \lambda > 0 \text{ if } \gamma > 0 \text{ and } p \in (\frac{10}{3}, 6],$$

$$(ii) \quad \lambda < 0 \text{ if } \gamma < 0 \text{ and } p = 6.$$

Proof. Our proof is inspired by [15, Lemma 4.2]. The following Pohozaev type identity holds for $u \in H^1(\mathbb{R}^3)$ weak solution of (3.2.3) ([34], also see [77, Theorem 2.2]),

$$\frac{1}{2}A(u) + \frac{3\lambda}{2}D(u) - \frac{5\gamma}{4}B(u) - \frac{3a}{p}C(u) = 0, \quad \text{where } D(u) = \|u\|_{L^2(\mathbb{R}^3)}^2. \quad (3.2.4)$$

By multiplying (3.2.3) by u and integrating, we derive a second identity

$$A(u) + \lambda_c D(u) - \gamma B(u) - aC(u) = 0. \quad (3.2.5)$$

Combining (3.2.4) and (3.2.5), we get

$$A(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}C(u) = 0.$$

This means that $Q(u) = 0$. Using (3.2.4) and (3.2.5) again, we obtain

$$2(6-p)A(u) + (5p-12)\gamma B(u) = 2(3p-6)\lambda D(u). \quad (3.2.6)$$

If $\gamma > 0$ and $p \in (\frac{10}{3}, 6]$, we have

$$2(6-p) \geq 0, \quad (5p-12)\gamma > 0, \quad 2(3p-6) > 0.$$

Hence, $\lambda > 0$. If $\gamma < 0$ and $p = 6$, we have

$$2(6-p) = 0, \quad (5p-12)\gamma = 18\gamma < 0, \quad 2(3p-6) = 24 > 0.$$

This implies that $\lambda < 0$.

□

Lemma 3.2.3. *Let $p \in (\frac{10}{3}, 6]$. Assume that $\gamma \in \mathbb{R}$ and $a \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^3)$ is a weak solution to*

$$-\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0, \quad (3.2.7)$$

then $u \in L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3)$. Moreover, in case $\gamma > 0$, $a > 0$ we have that if $u \not\equiv 0$ and $u \geq 0$ then $u > 0$.

Proof. Applying [62, Theorem 2.1], we get that $u \in W_{loc}^{2,r}(\mathbb{R}^3)$ for every $r > 1$ and hence $u \in C(\mathbb{R}^3)$. Since $u \in H^1(\mathbb{R}^3)$, the Sobolev embedding (see [23, Corollary 9.10]) implies that $|u|^2 \in L^q(\mathbb{R}^3)$ for every $q \in [1, 3]$. Now, setting $K := |x|^{-1}$, we write $K := K_1 + K_2$ where $K_1 := K$ on $B(0, 1)$, $K_1 := 0$ on $\mathbb{R}^3 \setminus B(0, 1)$ and $K_2 := K - K_1$. Clearly $K_1 \in L^2(\mathbb{R}^3)$ and $K_2 \in L^4(\mathbb{R}^3)$. Applying [63, Lemma 2.20] with $K_1 \in L^2(\mathbb{R}^3)$, $|u|^2 \in L^2(\mathbb{R}^3)$ and with $K_2 \in L^4(\mathbb{R}^3)$, $|u|^2 \in L^{\frac{4}{3}}(\mathbb{R}^3)$, we obtain that $K_1 * |u|^2$ and $K_2 * |u|^2$ are continuous. Also

$$\lim_{|x| \rightarrow \infty} (K_1 * |u|^2)(x) = 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} (K_2 * |u|^2)(x) = 0.$$

Hence, we get that $K * |u|^2$ is continuous and

$$\lim_{|x| \rightarrow \infty} (K * |u|^2)(x) = 0. \quad (3.2.8)$$

Therefore, $K * |u|^2$ is bounded. At this point, we deduce from [81, Proposition B.1] that $u \in L^\infty(\mathbb{R}^3)$.

Now, if we assume that $\gamma > 0$, $a > 0$, $u \not\equiv 0$, $u \geq 0$, setting $v := -u \leq 0$ we get

$$-\Delta v + \lambda v = \gamma(|x|^{-1} * |v|^2)v + a|v|^{p-2}v \leq 0.$$

By Lemma 3.2.2, we have that $\lambda > 0$. We assume that there exists $x_0 \in \mathbb{R}^3$ such that $v(x_0) = 0$. For all $R > |x_0|$, we have that $v \in W^{2,r}(B_R)$ for every $r > 1$, $Lv := -\Delta v + \lambda v \leq 0$ in B_R with $\lambda > 0$ and $M := \max_{x \in B_R} v = 0$. At this point, applying [87, Theorem 3.27], in the particular case where $\Gamma = \emptyset$, we obtain that $v \equiv 0$ in B_R , and hence $u \equiv 0$ in B_R . The value $R > 0$ being arbitrarily large, this contradicts our assumption that $u \not\equiv 0$ and we conclude that $u > 0$. \square

Following [19], we recall that, for any $c > 0$, $S(c)$ is a submanifold codimension 1 of $H^1(\mathbb{R}^3)$ and the tangent space at a point $u \in S(c)$ is defined as

$$T_u S(c) = \{\varphi \in H^1(\mathbb{R}^3) : \langle u, \varphi \rangle_{L^2(\mathbb{R}^3)} = 0\}.$$

The restriction $F|_{S(c)} : S(c) \rightarrow \mathbb{R}$ is a C^1 functional on $S(c)$ and for any $u \in S(c)$ and any $v \in T_u S(c)$, we have

$$\langle F'|_{S(c)}(u), \varphi \rangle = \langle F'(u), \varphi \rangle.$$

We use the notation $\|dF|_{S(c)}\|_*$ to indicate the norm in the cotangent space $T_u S(c)'$, i.e the dual norm induced by the norm of $T_u S(c)$, i.e

$$\|dF|_{S(c)}(u)\|_* := \sup_{\|\varphi\| \leq 1, \varphi \in T_u S(c)} |dF(u)[\varphi]|. \quad (3.2.9)$$

We recall the following result, see Lemma 2.5.1,

Lemma 3.2.4. For $u \in S(c)$ and $t > 0$, the map

$$T_u S(c) \rightarrow T_{u^t} S(c), \quad \psi \mapsto \psi^t$$

is a linear isomorphism with inverse

$$T_{u^t} S(c) \rightarrow T_u S(c), \quad \phi \mapsto \phi^{\frac{1}{t}}.$$

Next, we recall a result concerning the convergence of the term B , see [77, Lemma 2.1],

Lemma 3.2.5. Let (u_n) be a sequence satisfying $u_n \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^3)$. Then we have

$$B(u_n) \rightarrow B(u).$$

3.3 The case $\gamma > 0$, $a > 0$ and $p \in (\frac{10}{3}, 6]$

3.3.1 The geometrical structure and the existence of bounded Palais-Smale sequences for $p \in (\frac{10}{3}, 6]$

In this subsection, we follow the approach first introduced in [31]. We shall always assume that $\gamma > 0$, $a > 0$ and $p \in (\frac{10}{3}, 6]$.

Lemma 3.3.1. For any $c \in (0, \infty)$, F restricted to $\Lambda(c)$ is coercive on $H^1(\mathbb{R}^3)$, namely when $(u_n) \subset H^1(\mathbb{R}^3)$ satisfies $\|u_n\| \rightarrow +\infty$ then $F(u_n) \rightarrow +\infty$. In particular F restricted to $\Lambda(c)$ is bounded from below.

Proof. Let $u \in \Lambda(c)$. Taking into account that

$$\frac{a}{p}C(u) = \frac{1}{\sigma}A(u) - \frac{\gamma}{4\sigma}B(u),$$

and using Lemma 3.2.1(i), we obtain

$$\begin{aligned} F(u) &= \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{1}{\sigma}A(u) + \frac{\gamma}{4\sigma}B(u) \\ &= \frac{\sigma-2}{2\sigma}A(u) - \frac{\gamma(\sigma-1)}{4}B(u) \geq \frac{\sigma-2}{2\sigma}A(u) - \frac{\gamma(\sigma-1)}{4}K_H\sqrt{A(u)}c^{\frac{3}{2}}. \end{aligned} \quad (3.3.1)$$

This concludes the proof. \square

For any $u \in S(c)$, we recall that

$$\begin{aligned} g_u(t) &= F(u^t) = \frac{1}{2}t^2A(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}t^\sigma C(u), \\ g'_u(t) &= tA(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}t^{\sigma-1}C(u) = \frac{1}{t}Q(u^t), \\ g''_u(t) &= A(u) - \frac{a\sigma(\sigma-1)}{p}t^{\sigma-2}C(u). \end{aligned}$$

For any $u \in S(c)$, we set

$$t_u^\star := \left(\frac{pA(u)}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}}.$$

This implies that t_u^\star is the unique solution of equation $g''_u(t) = 0$. So, we have

$$g''_u(t_u^\star) = 0, \quad g''_u(t) > 0 \text{ if } 0 < t < t_u^\star, \quad g''_u(t) < 0 \text{ if } t > t_u^\star. \quad (3.3.2)$$

Lemma 3.3.2. For any $c \in (0, c_1)$ and any $u \in S(c)$, we have $g'_u(t_u^\star) > 0$.

Proof. Let $u \in S(c)$ be arbitrary. By the definition of t_u^\star and by $g''_u(t_u^\star) = 0$, we have

$$\begin{aligned} g'_u(t_u^\star) &= t_u^\star A(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} (t_u^\star)^{\sigma-1} C(u) = t_u^\star A(u) - \frac{\gamma}{4} B(u) - \frac{1}{\sigma-1} t_u^\star A(u) \\ &= \frac{\sigma-2}{\sigma-1} t_u^\star A(u) - \frac{\gamma}{4} B(u) = \frac{\sigma-2}{\sigma-1} \left(\frac{pA(u)}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}} A(u) - \frac{\gamma}{4} B(u) \\ &= \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{pA(u)}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}} \sqrt{A(u)} - \frac{\gamma}{4} \frac{B(u)}{\sqrt{A(u)}} \right] \\ &= \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{p[A(u)]^{\frac{\sigma}{2}}}{a\sigma(\sigma-1)C(u)} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4} \frac{B(u)}{\sqrt{A(u)}} \right]. \end{aligned}$$

Applying [Lemma 3.2.1](#), we obtain

$$\begin{aligned} g'_u(t_u^\star) &\geq \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{p[A(u)]^{\frac{\sigma}{2}}}{a\sigma(\sigma-1)K_{GN}[A(u)]^{\frac{\sigma}{2}} c^{\frac{6-p}{4}}} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4} \frac{K_H \sqrt{A(u)} c^{\frac{3}{2}}}{\sqrt{A(u)}} \right] \\ &= \sqrt{A(u)} \left[\frac{\sigma-2}{\sigma-1} \left(\frac{p}{a\sigma(\sigma-1)K_{GN} c^{\frac{6-p}{4}}} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4} K_H c^{\frac{3}{2}} \right]. \end{aligned}$$

By direct computations, we now have

$$\frac{\sigma-2}{\sigma-1} \left(\frac{p}{a\sigma(\sigma-1)K_{GN} c^{\frac{6-p}{4}}} \right)^{\frac{1}{\sigma-2}} - \frac{\gamma}{4} K_H c^{\frac{3}{2}} > 0 \iff c < c_1.$$

Thus, we obtain that if $0 < c < c_1$ then $g'_u(t_u^\star) > 0$. \square

Lemma 3.3.3. For any $c \in (0, c_1)$, it holds that $\Lambda^0(c) = \emptyset$.

Proof. We assume that there exists $u \in \Lambda^0(c)$. Since $g''_u(1) = 0$ and t_u^\star is the unique solution of equation $g''_u(t) = 0$, we have $t_u^\star = 1$. So, we have $g'_u(t_u^\star) = g'_u(1) = 0$. This contradicts with $g'_u(t_u^\star) > 0$ in [Lemma 3.3.2](#). Thus, we obtain $\Lambda^0(c) = \emptyset$. \square

Lemma 3.3.4. For any $c \in (0, c_1)$ and any $u \in S(c)$, there exists

- (i) a unique $s_u^+ \in (0, t_u^\star)$ such that s_u^+ is a unique local minimum point for g_u and $u^{s_u^+} \in \Lambda^+(c)$.
- (ii) a unique $s_u^- \in (t_u^\star, \infty)$ such that s_u^- is a unique local maximum point for g_u and $u^{s_u^-} \in \Lambda^-(c)$.

Moreover, the maps $u \in S(c) \mapsto s_u^+ \in \mathbb{R}$ and $u \in S(c) \mapsto s_u^- \in \mathbb{R}$ are of class C^1 .

Proof. Taking into account that

$$g'_u(t) = tA(u) - \frac{\gamma}{4} B(u) - \frac{a\sigma}{p} t^{\sigma-1} C(u),$$

we have $g'_u(t) \rightarrow -\frac{\gamma}{4} B(u) < 0$ as $t \rightarrow 0$ and $g'_u(t) \rightarrow -\infty$ as $t \rightarrow +\infty$ due to $\sigma - 1 > 1$. By [Lemma 3.3.2](#), we have $g'_u(t_u^\star) > 0$. Therefore, the equation $g'_u(t) = 0$ has at least two solutions

s_u^+ and s_u^- with $0 < s_u^+ < t_u^* < s_u^-$. By (3.3.2), we have $g_u''(t) > 0$ for all $0 < t < t_u^*$. Hence, $g_u'(t)$ is strictly increasing function on $(0, t_u^*)$ and consequently $s_u^+ \in (0, t_u^*)$ is the unique local minimum point for g_u and $u^{s_u^+} \in \Lambda^+(c)$ due to $g_u''(s_u^+) = g_u''(s_u^+) > 0$. By the same argument, we obtain that $s_u^- \in (t_u^*, \infty)$ is a unique local maximum point for g_u and $u^{s_u^-} \in \Lambda^-(c)$.

In order to prove that $u \mapsto s_u^-$ are of class C^1 , we follow the argument in [80, Lemma 5.3]. It is a direct application of the Implicit Function Theorem on C^1 -function $\varphi(t, u) = g_u'(t)$. Taking into account that $\varphi(s_u^-, u) = g_u'(s_u^-) = 0$, $\partial_t \varphi(s_u^-, u) = g_u''(s_u^-) < 0$ and $\Lambda^0(c) = \emptyset$, we obtain $u \mapsto s_u^-$ is of class C^1 . The same argument proves that $u \mapsto s_u^+$ is of class C^1 . \square

Lemma 3.3.5. *For any $c \in (0, c_1)$, it holds that*

- (i) $F(u) < 0$ for all $u \in \Lambda^+(c)$,
- (ii) there exists $\alpha := \alpha(c) > 0$ such that $A(u) \geq \alpha$ for all $u \in \Lambda^-(c)$.

Proof. Let $u \in \Lambda^+(c)$, taking into account that

$$A(u) = \frac{\gamma}{4}B(u) + \frac{a\sigma}{p}C(u), \quad A(u) > \frac{a\sigma(\sigma-1)}{p}C(u),$$

we obtain

$$\begin{aligned} F(u) &= \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}C(u) + \frac{a(\sigma-1)}{p}C(u) \\ &< \frac{1}{2}A(u) - A(u) + \frac{1}{\sigma}A(u) = \frac{2-\sigma}{2\sigma}A(u). \end{aligned}$$

Since $\sigma > 2$, we have $F(u) < 0$. The point (i) is proved.

Let $u \in \Lambda^-(c)$, taking into account that

$$A(u) < \frac{a\sigma(\sigma-1)}{p}C(u),$$

and using Lemma 3.2.1, we obtain that

$$A(u) < \frac{a\sigma(\sigma-1)}{p}K_{GN}c^{\frac{6-p}{4}}[A(u)]^{\frac{\sigma}{2}}.$$

Since $\sigma > 2$, the point (ii) follows. \square

We define

$$S_r(c) := S(c) \cap H_r^1(\mathbb{R}^3), \quad \Lambda_r(c) := \Lambda(c) \cap H_r^1(\mathbb{R}^3), \quad \Lambda_r^\pm(c) := \Lambda^\pm(c) \cap H_r^1(\mathbb{R}^3).$$

Here $\Lambda^\pm(c)$ denotes either $\Lambda^+(c)$ or $\Lambda^-(c)$.

Lemma 3.3.6. *For any $c \in (0, c_1)$ it holds that*

$$\inf_{u \in \Lambda_r^\pm(c)} F(u) = \inf_{u \in \Lambda^\pm(c)} F(u).$$

Also, if $\inf_{u \in \Lambda^\pm(c)} F(u)$ is reached, it is reached by a Schwarz symmetric function.

Proof. Since $\Lambda_r^\pm(c) \subset \Lambda^\pm(c)$, we directly have

$$\inf_{u \in \Lambda_r^\pm(c)} F(u) \geq \inf_{u \in \Lambda^\pm(c)} F(u). \quad (3.3.3)$$

Therefore, it suffices to prove that

$$\inf_{u \in \Lambda_r^\pm(c)} F(u) \leq \inf_{u \in \Lambda^\pm(c)} F(u). \quad (3.3.4)$$

In this aim we start to note that

$$\inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in S(c)} \min_{0 < t \leq s_u^+} F(u^t) \quad \text{and} \quad \inf_{u \in \Lambda^-(c)} F(u) = \inf_{u \in S(c)} \max_{s_u^+ < t \leq s_u^-} F(u^t). \quad (3.3.5)$$

Now let $u \in S(c)$ and $v \in S_r(c)$ be the Schwarz rearrangement of $|u|$. Taking into account that $A(v) \leq A(u)$, $C(v) = C(u)$, and by the Riesz's rearrangement inequality (see [63, Section 3.7]), $B(v) \geq B(u)$, we have for all $t > 0$,

$$F(v^t) = \frac{1}{2}t^2 A(v) - \frac{\gamma}{4}tB(v) - \frac{a}{p}t^\sigma C(v) \leq \frac{1}{2}t^2 A(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}t^\sigma C(u) = F(u^t). \quad (3.3.6)$$

Observe that, for any $w \in S(c)$,

$$g'_w(t) = tA(w) - \frac{\gamma}{4}B(w) - \frac{a\sigma}{p}t^{\sigma-1}C(w) \quad \text{and} \quad g''_w(t) = A(w) - \frac{a\sigma(\sigma-1)}{p}t^{\sigma-2}C(w).$$

Thus we have

$$g'_v(0) \leq g'_u(0) < 0 \quad \text{and} \quad g''_v(t) \leq g''_u(t), \quad \forall t > 0.$$

This implies that $0 < s_u^+ \leq s_v^+ < s_v^- \leq s_u^-$. Hence, we deduce from (3.3.6) that

$$\min_{0 < t \leq s_v^+} F(v^t) \leq \min_{0 < t \leq s_u^+} F(u^t) \quad \text{and} \quad \max_{s_v^+ < t \leq s_v^-} F(v^t) \leq \max_{s_u^+ < t \leq s_u^-} F(u^t).$$

In view of (3.3.5), the inequality (3.3.4) holds. Now if $u_0 \in \Lambda^+(c)$ is such that $F(u_0) = \inf_{u \in \Lambda^+(c)} F(u)$ we see that v , the Schwarz rearrangement of $|u_0|$, belongs to $\Lambda_r^+(c)$. Indeed, if either $A(v) < A(u_0)$ or $B(v) > B(u_0)$ then $F(v^t) < F(u_0^t)$. Hence, in view of the above arguments, we get

$$\inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in S(c)} \min_{0 < t \leq s_u^+} F(u^t) \leq \min_{0 < t \leq s_v^+} F(v^t) < \min_{0 < t \leq s_{u_0}^+} F(u_0^t) = \inf_{u \in \Lambda^+(c)} F(u)$$

a contradiction. Thus $A(v) = A(u_0)$, $B(v) = B(u_0)$ and $C(v) = C(u_0)$ from which we deduce that $v \in \Lambda_r^+(c)$ and $F(v) = F(u_0)$. The case of $u_0 \in \Lambda^-(c)$ such that $F(u_0) = \inf_{u \in \Lambda^-(c)} F(u)$ is treated similarly. This ends the proof of the lemma. \square

Recalling that $\gamma^+(c)$ and $\gamma^-(c)$ are defined in (3.1.6) we have

Lemma 3.3.7. *For any $c \in (0, c_1)$, there exists a bounded Palais-Smale sequence $(u_n^+) \subset \Lambda_r^+(c)$ for F restricted to $S(c)$ at level $\gamma^+(c)$ and a bounded Palais-Smale sequence $(u_n^-) \subset \Lambda_r^-(c)$ for F restricted to $S(c)$ at level $\gamma^-(c)$.*

In order to prove [Lemma 3.3.7](#) we define the functions

$$\begin{aligned} I^+ : S(c) &\rightarrow \mathbb{R}, & I^+(u) &= F(u^{s_u^+}), \\ I^- : S(c) &\rightarrow \mathbb{R}, & I^-(u) &= F(u^{s_u^-}). \end{aligned}$$

Note that since the maps $u \mapsto s_u^+$ and $u \mapsto s_u^-$ are of class \mathbb{C}^1 , see [Lemma 3.3.4](#), the functionals I^+ and I^- are of class \mathbb{C}^1 .

Lemma 3.3.8. *For any $c \in (0, c_1)$, we have that $dI^+(u)[\psi] = dF(u^{s_u^+})[\psi^{s_u^+}]$ and $dI^-(u)[\psi] = dF(u^{s_u^-})[\psi^{s_u^-}]$ for any $u \in S(c)$, $\psi \in T_u S(c)$.*

Proof. We first give the proof for I^+ . Let $\psi \in T_u S(c)$, then $\psi = h'(0)$ where $h : (-\epsilon, \epsilon) \mapsto S(c)$ is a \mathbb{C}^1 -curve with $h(0) = u$. We consider the incremental quotient

$$\frac{I^+(h(t)) - I^+(h(0))}{t} = \frac{F(h(t)^{s_t}) - F(h(0)^{s_0})}{t}, \quad (3.3.7)$$

where $s_t := s_{h(t)}^+$, and hence $s_0 = s_u^+$. Recalling from [Lemma 3.3.4](#) that s_0 is a strict local minimum of $s \mapsto F(u^s)$ and $u \mapsto s_0$ is continuous, we get

$$\begin{aligned} &F(h(t)^{s_t}) - F(h(0)^{s_0}) \geq F(h(t)^{s_t}) - F(h(0)^{s_t}) \\ &= \frac{s_t^2}{2} \left[A(h(t)) - A(h(0)) \right] - \frac{\gamma s_t}{4} \left[B(h(t)) - B(h(0)) \right] - \frac{a s_t^\sigma}{p} \left[C(h(t)) - C(h(0)) \right] \\ &= s_t^2 \int_{\mathbb{R}^3} \nabla h(\tau_1 t) \cdot \nabla h'(\tau_1 t) dx - \gamma s_t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|h(\tau_2 t)(x)|^2 h(\tau_2 t)(y) h'(\tau_2 t)(y)}{|x-y|} dx dy \\ &\quad - a s_t^\sigma \int_{\mathbb{R}^3} |h(\tau_3 t)|^{p-2} h(\tau_3 t) h'(\tau_3 t) dx, \end{aligned}$$

for some $\tau_1, \tau_2, \tau_3 \in (0, 1)$. Analogously

$$\begin{aligned} &F(h(t)^{s_t}) - F(h(0)^{s_0}) \leq F(h(t)^{s_0}) - F(h(0)^{s_0}) = s_0^2 \int_{\mathbb{R}^3} \nabla h(\tau_4 t) \cdot \nabla h'(\tau_4 t) dx \\ &\quad - \gamma s_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|h(\tau_5 t)(x)|^2 h(\tau_5 t)(y) h'(\tau_5 t)(y)}{|x-y|} dx dy - a s_0^\sigma \int_{\mathbb{R}^3} |h(\tau_6 t)|^{p-2} h(\tau_6 t) h'(\tau_6 t) dx, \end{aligned}$$

for some $\tau_4, \tau_5, \tau_6 \in (0, 1)$. Now, from (3.3.7) we deduce that

$$\begin{aligned} &\lim_{t \rightarrow 0} \frac{I^+(h(t)) - I^+(h(0))}{t} \\ &= (s_u^+)^2 \int_{\mathbb{R}^3} \nabla u \nabla \psi dx - \gamma (s_u^+) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 u(y) \psi(y)}{|x-y|} dx dy - a (s_u^+)^\sigma \int_{\mathbb{R}^3} |u|^{p-2} u \psi dx \\ &= \int_{\mathbb{R}^3} \nabla u^{s_u^+} \nabla \psi^{s_u^+} dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u^{s_u^+}(x)|^2 u^{s_u^+}(y) \psi^{s_u^+}(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u^{s_u^+}|^{p-2} u^{s_u^+} \psi^{s_u^+} dx \\ &= dF(u^{s_u^+})[\psi^{s_u^+}], \end{aligned}$$

for any $u \in S(c)$, $\psi \in T_u S(c)$. The proof for I^- is similar. \square

Let \mathcal{G} be the set of all singletons belonging to $S_r(c)$. It is clearly a homotopy stable family of compact subsets of $S_r(c)$ with closed boundary (an empty boundary actually) in the sense of [\[37, Definition 3.1\]](#). In view of [Lemma 3.3.6](#) we have that

$$e_{\mathcal{G}}^+ := \inf_{A \in \mathcal{G}} \max_{u \in A} I^+(u) = \inf_{u \in S_r(c)} I^+(u) = \inf_{u \in \Lambda_r^+(c)} F(u) = \inf_{u \in \Lambda^+(c)} F(u) = \gamma^+(c).$$

$$e_{\mathcal{G}}^- := \inf_{A \in \mathcal{G}} \max_{u \in A} I^-(u) = \inf_{u \in S_r(c)} I^-(u) = \inf_{u \in \Lambda_r^-(c)} F(u) = \inf_{u \in \Lambda^-(c)} F(u) = \gamma^-(c).$$

Lemma 3.3.9. *For any $c \in (0, c_1)$, there exists a Palais-Smale sequence $(u_n^+) \subset \Lambda^+(c)$ for F restricted to $S_r(c)$ at level $e_{\mathcal{G}}^+$ and a Palais-Smale sequence $(u_n^-) \subset \Lambda^-(c)$ for F restricted to $S(c)$ at level $e_{\mathcal{G}}^-$.*

Proof. We first treat the case of $e_{\mathcal{G}}^+$. Let $(D_n) \subset \mathcal{G}$ be such that

$$\max_{u \in D_n} I^+(u) < e_{\mathcal{G}}^+ + \frac{1}{n},$$

and consider the homotopy

$$\eta : [0, 1] \times S(c) \mapsto S(c), \quad \eta(t, u) = u^{1-t+ts_u^+}.$$

From the definition of \mathcal{G} , we have

$$E_n := \eta(\{1\} \times D_n) = \{u^{s_u^+} : u \in D_n\} \in \mathcal{G}.$$

Lemma 3.3.4 implies that $E_n \subset \Lambda^+(c)$ for all $n \in \mathbb{N}$. Let $v \in E_n$, i.e. $v = u^{s_u^+}$ for some $u \in D_n$, and hence $I^+(v) = I^+(u)$. So, we have

$$\max_{v \in E_n} I^+(v) = \max_{u \in D_n} I^+(u).$$

Therefore, E_n is another minimizing sequence for $e_{\mathcal{G}}^+$. Applying [37, Theorem 3.2], in the particular case where the boundary $B = \emptyset$, there exists a Palais-Smale sequence (y_n) for I^+ on $S(c)$ at level $e_{\mathcal{G}}^+$ such that

$$\text{dist}_{H^1(\mathbb{R}^3)}(y_n, E_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.3.8)$$

Now writing $s_n := s_{y_n}^+$ we set $u_n^+ := y_n^{s_n} \in \Lambda^+(c)$. We claim that there exists a constant $C > 0$ such that

$$\frac{1}{C} \leq s_n^2 \leq C \quad (3.3.9)$$

for $n \in \mathbb{N}$ large enough. Indeed, notice first that

$$s_n^2 = \frac{A(u_n^+)}{A(y_n)}.$$

By $F(u_n^+) = I^+(y_n) \rightarrow e_{\mathcal{G}}^+ = \gamma^+(c) < 0$ we deduce from (3.3.1) that there exists $M > 0$ such that

$$\frac{1}{M} \leq A(u_n^+) \leq M. \quad (3.3.10)$$

On the other hand, since $E_n \in \Lambda^+(c)$ is a minimizing sequence for e_G^+ and F is $H^1(\mathbb{R}^3)$ coercive on $\Lambda^+(c)$, we obtain that E_n is uniformly bounded in $H^1(\mathbb{R}^3)$ and thus from (3.3.8), it implies that $\sup_n A(y_n) < \infty$. Also, since E_n is compact for every $n \in \mathbb{N}$, there exist a $v_n \in E_n$ such that $\|v_n - y_n\|_{H^1(\mathbb{R}^3)} \rightarrow 0$ as $n \rightarrow \infty$ due to (3.3.8). Using Lemma 3.3.1 again, we have, for a $\delta > 0$,

$$A(y_n) \geq A(v_n) - A(v_n - y_n) \geq \frac{\delta}{2}.$$

This proves the claim (3.3.9). From (3.2.9), and by Lemma 3.2.4, Lemma 3.3.8, we have

$$\begin{aligned} \|dF|_{S(c)}(u_n^+)\|_* &= \sup_{\|\psi\| \leq 1, \psi \in T_u S(c)} \left| dF(u_n^+)[\psi] \right| \\ &= \sup_{\|\psi\| \leq 1, \psi \in T_u S(c)} \left| dF(u_n^+) \left[\left(\psi^{\frac{1}{s_n}} \right)^{s_n} \right] \right| = \sup_{\|\psi\| \leq 1, \psi \in T_u S(c)} \left| dI^+(y_n) \left[\psi^{\frac{1}{s_n}} \right] \right|. \end{aligned}$$

This implies that $(u_n^+) \subset \Lambda^+(c)$ is a Palais-Smale sequence for F restricted to $S(c)$ at level e_G^+ since (y_n) is a Palais-Smale sequence for I^+ at level e_G^+ and $\|\psi^{\frac{1}{s_n}}\| \leq C_1 \|\psi\| \leq C_1$ due to (3.3.9). For the case of e_G^- the proof is identical except that we use Lemma 3.3.5(ii) along with (3.3.1) to conclude that there exists a $M > 0$ such that (3.3.10) holds for $A(u_n^-)$ replacing $A(u_n^+)$. \square

Proof of Lemma 3.3.7. Applying Lemma 3.3.9, we deduce that there exists a Palais-Smale sequence $(u_n^+) \subset \Lambda_r^+(c)$ for F restricted to $S(c)$ at level $e_G^+ = \gamma^+(c)$ and a Palais-Smale sequence $(u_n^-) \subset \Lambda_r^-(c)$ for F restricted to $S(c)$ at level $e_G^- = \gamma^-(c)$. In both cases the boundedness of these sequences follows from Lemma 3.3.1. \square

3.3.2 The compactness of our Palais-Smale sequences in the Sobolev subcritical case $p \in (\frac{10}{3}, 6)$

Lemma 3.3.10. *Let $p \in (\frac{10}{3}, 6)$. For any $c \in (0, c_1)$, if either $(u_n) \subset \Lambda^+(c)$ is a minimizing sequence for $\gamma^+(c)$ or $(u_n) \subset \Lambda^-(c)$ is a minimizing sequence for $\gamma^-(c)$, it weakly converges, up to translation, to a non-trivial limit.*

Proof. Since F restricted to $\Lambda(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see Lemma 3.3.1), (u_n) is bounded. Hence, up to translation, $u_n \rightharpoonup u_c$ weakly in $H^1(\mathbb{R}^3)$. Let us argue by contradiction assuming that $u_c = 0$, this means that (u_n) is vanishing. By [65, Lemma I.1], we have, for $2 < q < 6$,

$$\|u_n\|_{L^q(\mathbb{R}^3)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This implies that

$$C(u_n) \rightarrow 0, \quad \text{and} \quad B(u_n) \leq K_1 \|u_n\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \rightarrow 0,$$

due to (3.2.2). Since $(u_n) \subset \Lambda(c)$, we have $Q(u_n) = 0$, and hence

$$A(u_n) = \frac{\gamma}{4} B(u_n) + \frac{a\sigma}{p} C(u_n) \rightarrow 0. \quad (3.3.11)$$

If we assume that $(u_n) \subset \Lambda^-(c)$ we recall that by Lemma 3.3.5, there exists $\alpha > 0$ such that

$$A(u_n) \geq \alpha > 0, \quad \forall n \in \mathbb{N},$$

contradicting (3.3.11). If we assume that $(u_n) \subset \Lambda^+(c)$ then since

$$F(u_n) = \frac{1}{2}A(u_n) - \frac{\gamma}{4}B(u_n) - \frac{a}{p}C(u_n) \rightarrow 0$$

we reach a contradiction with the fact that

$$F(u_n) \rightarrow \gamma^+(c) = \inf_{u \in \Lambda^+(c)} F(u) < 0.$$

The lemma is proved. \square

Lemma 3.3.11. *Let $p \in (\frac{10}{3}, 6)$. Assume that a bounded Palais-Smale sequence $(u_n) \subset \Lambda_r(c)$ for F restricted to $S(c)$ is weakly convergent, up to translation, to the nonzero function u_c . Then, up to translation, $u_n \rightarrow u_c \in \Lambda_r(c)$ strongly in $H_r^1(\mathbb{R}^3)$. In particular u_c is a radial solution to (3.1.2) for some $\lambda_c > 0$ and $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.*

Proof. Since the embedding $H_r^1(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$ is compact for $q \in (2, 6)$, see [82] and, up to translation, $u_n \rightharpoonup u_c$ weakly in $H_r^1(\mathbb{R}^3)$, we have, up to translation, $u_n \rightarrow u_c$ strongly in $L^q(\mathbb{R}^3)$ for $q \in (2, 6)$ and a.e in \mathbb{R}^3 .

Since $(u_n) \subset H^1(\mathbb{R}^3)$ is bounded, following [19, Lemma 3], we know that

$$F'_{|S(c)}(u_n) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3) \iff F'(u_n) - \frac{1}{c}\langle F'(u_n), u_n \rangle u_n \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3).$$

Thus, for any $w \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} o_n(1) &= \left\langle F'(u_n) - \frac{1}{c}\langle F'(u_n), u_n \rangle u_n, w \right\rangle = \int_{\mathbb{R}^3} \nabla u_n \nabla w dx + \lambda_n \int_{\mathbb{R}^3} u_n w dx \\ &\quad - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 u_n(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_n|^{p-2} u_n w dx, \end{aligned} \quad (3.3.12)$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$ and

$$\lambda_n = \frac{-1}{c} [A(u_n) - \gamma B(u_n) - a C(u_n)] = \frac{1}{c} \left[\frac{3\gamma}{4} B(u_n) + a \left(1 - \frac{\sigma}{p} \right) C(u_n) \right],$$

due to $Q(u_n) = 0$. Since $u_n \in H_r^1(\mathbb{R}^3)$, we have $C(u_n) \rightarrow C(u_c)$ and $B(u_n) \rightarrow B(u_c)$ (see Lemma 3.2.5). Hence, we obtain that

$$\lambda_n \rightarrow \lambda_c = \frac{1}{c} \left[\frac{3\gamma}{4} B(u_c) + a \left(1 - \frac{\sigma}{p} \right) C(u_c) \right].$$

Now, using [92, Lemma 2.2], the equation (3.3.12) leads to

$$\int_{\mathbb{R}^3} \nabla u_c \nabla w dx + \lambda_c \int_{\mathbb{R}^3} u_c w dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c(x)|^2 u_c(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_c|^{p-2} u_c w dx = 0 \quad (3.3.13)$$

due to the weak convergence in $H_r^1(\mathbb{R}^3)$ and $\lambda_n \rightarrow \lambda_c \in \mathbb{R}$. This implies that (u_c, λ_c) satisfies

$$-\Delta u_c + \lambda_c u_c - \gamma(|x|^{-1} * |u_c|^2) u_c - a|u_c|^{p-2} u_c = 0 \quad \text{in } H^{-1}(\mathbb{R}^3).$$

By the assumption $u_c \neq 0$ and by [Lemma 3.2.2](#), we obtain that $Q(u_c) = 0$ and $\lambda_c > 0$.

Now choosing $w = u_n$ in [\(3.3.12\)](#) and choosing $w = u_c$ in [\(3.3.13\)](#), we obtain that

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \lambda_n \int_{\mathbb{R}^3} |u_n|^2 dx - \gamma B(u_n) - aC(u_n) \\ & \rightarrow \int_{\mathbb{R}^3} |\nabla u_c|^2 dx + \lambda_c \int_{\mathbb{R}^3} |u_c|^2 dx - \gamma B(u_c) - aC(u_c). \end{aligned}$$

We can deduce from $B(u_n) \rightarrow B(u_c)$, $C(u_n) \rightarrow C(u_c)$ and $\lambda_n \rightarrow \lambda_c$ that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \lambda_c \int_{\mathbb{R}^3} |u_n|^2 dx \rightarrow \int_{\mathbb{R}^3} |\nabla u_c|^2 dx + \lambda_c \int_{\mathbb{R}^3} |u_c|^2 dx.$$

Since $\lambda_c > 0$, we conclude that $u_n \rightarrow u_c$ strongly in $H_r^1(\mathbb{R}^3)$. The lemma is proved. \square

Proof of [Theorem 3.1.2](#) in the subcritical case $p \in (\frac{10}{3}, 6)$. We give the proof for $\gamma^+(c)$, the treatment for $\gamma^-(c)$ is identically. For any $c \in (0, c_1)$, by [Lemma 3.3.7](#), there exists a bounded Palais-Smale sequence $(u_n^+) \subset \Lambda_r^+(c)$ for F restricted to $S(c)$ at level $\gamma^+(c)$. From [Lemma 3.3.10](#) and [Lemma 3.3.11](#), we deduce that $u_n^+ \rightarrow u_c^+ \in \Lambda_r(c)$ strongly in $H_r^1(\mathbb{R}^3)$ and that there exists $\lambda_c^+ > 0$ such that (u_c^+, λ_c^+) is a solution to [\(3.1.2\)](#). Since $\Lambda^0(c) = \emptyset$ (see [Lemma 3.3.3](#)), we conclude that $u_c^+ \in \Lambda_r^+(c)$. From [Lemma 3.3.6](#) we can thus assume that u_c^+ is a Schwarz symmetric function. Hence, u_c^+ is non-negative. At this point, we can deduce from [Lemma 3.2.3](#) that u_c^+ is a bounded continuous positive function. \square

3.3.3 The compactness of our Palais-Smale sequences in the Sobolev critical case $p = 6$

Our next lemma is directly inspired from [\[81, Proposition 3.1\]](#).

Lemma 3.3.12. *Let $c \in (0, c_1)$ and $(u_n) \subset \Lambda_r^+(c)$ or $(u_n) \subset \Lambda_r^-(c)$ be a Palais-Smale sequence for F restricted to $S(c)$ at level $m \in \mathbb{R}$ which is weakly convergent, up to subsequence, to the function u_c . If $(u_n) \subset \Lambda_r^+(c)$ we assume that $m \neq 0$ and if $(u_n) \subset \Lambda_r^-(c)$ we assume that*

$$m < \frac{1}{3\sqrt{aK_{GN}}}. \quad (3.3.14)$$

Then $u_c \neq 0$ and we have the following alternative:

(i) either

$$F(u_c) \leq m - \frac{1}{3\sqrt{aK_{GN}}}, \quad (3.3.15)$$

(ii) or

$$u_n \rightarrow u_c \text{ strongly in } H_r^1(\mathbb{R}^3). \quad (3.3.16)$$

Proof. Since $u_n \rightharpoonup u_c$ weakly in $H_r^1(\mathbb{R}^3)$, we have, up to subsequence, $u_n \rightarrow u_c$ strongly in $L^q(\mathbb{R}^3)$ for $q \in (2, 6)$ and a.e in \mathbb{R}^3 .

Let us first show that $u_c \neq 0$. We argue by contradiction assuming that $u_c = 0$, this means that (u_n) is vanishing. By [65, Lemma I.1], we have, for $2 < q < 6$,

$$\|u_n\|_{L^q(\mathbb{R}^3)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This implies from (3.2.2) that

$$B(u_n) \leq K_1 \|u_n\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \rightarrow 0.$$

Since $(u_n) \subset \Lambda(c)$, we have

$$A(u_n) = aC(u_n) + o_n(1).$$

Passing to the limit as $n \rightarrow \infty$, up to subsequence we infer that

$$\lim_{n \rightarrow \infty} A(u_n) = \lim_{n \rightarrow \infty} aC(u_n) := \ell \geq 0.$$

Using Lemma 3.2.1(ii), we have

$$\ell = \lim_{n \rightarrow \infty} aC(u_n) \leq \lim_{n \rightarrow \infty} aK_{GN}[A(u_n)]^3 = aK_{GN}\ell^3.$$

Therefore, either $\ell = 0$ or $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. If $(u_n) \subset \Lambda^+(c)$, we have $A(u_n) > 5aC(u_n)$, and then $\ell = 0$. This implies that $F(u_n) \rightarrow 0$ and this contradicts the assumption that $m \neq 0$. Also, if $(u_n) \subset \Lambda^-(c)$, Lemma 3.3.5(ii) ensure that $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. Hence, we have

$$\begin{aligned} m + o_n(1) = F(u_n) &= \frac{\sigma - 2}{2\sigma} A(u_n) - \frac{\gamma(\sigma - 1)}{4\sigma} B(u_n) \\ &= \frac{1}{3} A(u_n) + o_n(1) = \frac{1}{3} \ell + o_n(1) \geq \frac{1}{3\sqrt{aK_{GN}}} + o_n(1), \end{aligned}$$

which contradicts our assumption (3.3.14). Thus, we have that $u_c \neq 0$.

Now, since $(u_n) \subset H^1(\mathbb{R}^3)$ is bounded, following [19, Lemma 3], we know that

$$F'_{|S(c)}(u_n) \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3) \iff F'(u_n) - \frac{1}{c} \langle F'(u_n), u_n \rangle u_n \rightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3).$$

Thus, for any $w \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} o_n(1) = \left\langle F'(u_n) - \frac{1}{c} \langle F'(u_n), u_n \rangle u_n, w \right\rangle &= \int_{\mathbb{R}^3} \nabla u_n \nabla w dx + \lambda_n \int_{\mathbb{R}^3} u_n w dx \\ &\quad - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(x)|^2 u_n(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_n|^{p-2} u_n w dx, \end{aligned} \tag{3.3.17}$$

where $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$ and

$$\lambda_n = \frac{-1}{c} [A(u_n) - \gamma B(u_n) - aC(u_n)] = \frac{3\gamma}{4c} B(u_n),$$

due to $Q(u_n) = 0$. By $B(u_n) \rightarrow B(u_c)$ (see Lemma 3.2.5), we obtain that

$$\lambda_n \rightarrow \lambda_c = \frac{3\gamma}{4c} B(u_c). \tag{3.3.18}$$

Now, using [92, Lemma 2.2], the equation (3.3.17) leads to

$$\int_{\mathbb{R}^3} \nabla u_c \nabla w dx + \lambda_c \int_{\mathbb{R}^3} u_c w dx - \gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c(x)|^2 u_c(y) w(y)}{|x-y|} dx dy - a \int_{\mathbb{R}^3} |u_c|^{p-2} u_c w dx = 0 \quad (3.3.19)$$

due to the weak convergence in $H_r^1(\mathbb{R}^3)$ and $\lambda_n \rightarrow \lambda_c \in \mathbb{R}$. This implies that (u_c, λ_c) satisfies

$$-\Delta u_c + \lambda_c u_c - \gamma(|x|^{-1} * |u_c|^2) u_c - a|u_c|^{p-2} u_c = 0 \quad \text{in } H^{-1}(\mathbb{R}^3).$$

By Lemma 3.2.2, we obtain that $Q(u_c) = 0$ and $\lambda_c > 0$.

Let $v_n := u_n - u_c \rightarrow 0$ in $H_r^1(\mathbb{R}^3)$. It is direct to show, using the Brezis-Lieb lemma [24, Theorem 1], that

$$A(u_n) = A(u_c) + A(v_n) + o_n(1), \quad C(u_n) = C(u_c) + C(v_n) + o_n(1). \quad (3.3.20)$$

By $B(u_n) \rightarrow B(u_c)$ (see Lemma 3.2.5) and by $Q(u_n) = 0$, we have

$$A(u_c) + A(v_n) - \frac{\gamma}{4} B(u_c) - a[C(u_c) - C(v_n)] = o_n(1).$$

Taking into account that $Q(u_c) = 0$, we get $A(v_n) = aC(v_n) + o_n(1)$. Passing to the limit as $n \rightarrow \infty$, up to subsequence we infer that

$$\lim_{n \rightarrow \infty} A(v_n) = \lim_{n \rightarrow \infty} aC(v_n) := k \geq 0.$$

Using Lemma 3.2.1(ii), we have

$$k = \lim_{n \rightarrow \infty} aC(v_n) \leq \lim_{n \rightarrow \infty} aK_{GN}[A(v_n)]^3 = aK_{GN}k^3.$$

Therefore, either $k = 0$ or $k \geq (aK_{GN})^{-\frac{1}{2}}$.

If $k \geq (aK_{GN})^{-\frac{1}{2}}$, then by (3.3.20) and by $B(u_n) \rightarrow B(u_c)$, we have

$$\begin{aligned} m = \lim_{n \rightarrow \infty} F(u_n) &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} A(u_n) - \frac{\gamma}{4} B(u_n) - \frac{a}{6} C(u_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} A(u_c) + \frac{1}{2} A(v_n) - \frac{\gamma}{4} B(u_c) - \frac{a}{6} C(u_c) - \frac{a}{6} C(v_n) \right] \\ &= F(u_c) + \frac{1}{3} k \geq F(u_c) + \frac{1}{3} (aK_{GN})^{-\frac{1}{2}}. \end{aligned}$$

This implies that alternative (i) holds.

If instead $k = 0$, then by (3.3.20), we have $A(u_n) \rightarrow A(u_c)$ and $C(u_n) \rightarrow C(u_c)$. Choosing $w = u_n$ in (3.3.17) and $w = u_c$ in (3.3.19), we obtain that

$$A(u_n) + \lambda_n \|u_n\|_{L^2(\mathbb{R}^3)}^2 - \gamma B(u_n) - aC(u_n) \rightarrow A(u_c) + \lambda_c \|u_c\|_{L^2(\mathbb{R}^3)}^2 - \gamma B(u_c) - aC(u_c).$$

This implies that $\|u_n\|_{L^2(\mathbb{R}^3)} \rightarrow \|u_c\|_{L^2(\mathbb{R}^3)}$. Thus, we conclude that $u_n \rightarrow u_c$ strongly in $H_r^1(\mathbb{R}^3)$. \square

Proof of Theorem 3.1.2 in the critical case $p = 6$ for $\gamma^+(c)$. Since $\gamma^+(c) < 0$, the fact that it is reached is a direct consequence of Lemma 3.3.7, Lemma 3.3.12 and of the property, which is established in Lemma 3.3.21(iii) to come, that the map $c \mapsto \gamma^+(c)$ is non-increasing. The rest of the proof is identical to the one in the case $p \in (\frac{10}{3}, 6)$. \square

In the rest of this subsection, we shall prove [Theorem 3.1.2](#) in the critical case $p = 6$ for $\gamma^-(c)$.

Lemma 3.3.13. *Let $c \in (0, c_1)$. If*

$$\gamma^-(c) < \gamma^+(c) + \frac{1}{3\sqrt{aK_{GN}}} \quad (3.3.21)$$

then there exists a $u_c \in \Lambda_r^-(c)$ with $F(u_c) = \gamma^-(c)$ which is a radial solution to [\(3.1.2\)](#) for some $\lambda_c > 0$ with $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.

Proof. By [Lemma 3.3.7](#) there exists a Palais-Smale sequence $(u_n) \subset \Lambda^-(c)$ for F restricted to $S(c)$ at the level $\gamma^-(c)$. If [\(3.3.21\)](#) holds then necessarily [\(3.3.14\)](#), with $m = \gamma^-(c)$ holds, and [\(3.3.15\)](#) cannot hold. We deduce from [Lemma 3.3.12](#) that $u_n \rightarrow u_c$ strongly in $H^1(\mathbb{R}^N)$ and the conclusions follow. \square

Now we shall show that

Lemma 3.3.14. *For any $c \in (0, c_1)$, we have that*

$$\gamma^-(c) < \gamma^+(c) + \frac{1}{3\sqrt{aK_{GN}}}.$$

As already indicated our proof is inspired by [[89](#), Lemma 3.1]. Let u_ε be an extremal function for the Sobolev inequality in \mathbb{R}^3 defined by

$$u_\varepsilon(x) := \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{N-2}{2}}}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^3. \quad (3.3.22)$$

Let $\xi \in C_0^\infty(\mathbb{R}^N)$ be a radial non-increasing cut-off function with $\xi \equiv 1$ in B_1 , $\xi \equiv 0$ in $\mathbb{R}^N \setminus B_2$. Setting $U_\varepsilon(x) = \xi(x)u_\varepsilon(x)$ we recall the following result, see [Lemma 2.7.1](#).

Lemma 3.3.15. *Denoting ω the area of the unit sphere in \mathbb{R}^3 , we have*

(i)

$$\|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 = \sqrt{\frac{1}{K_{GN}}} + O(\varepsilon) \quad \text{and} \quad \|U_\varepsilon\|_{L^{2^*}(\mathbb{R}^3)}^2 = \sqrt{\frac{1}{K_{GN}}} + O(\varepsilon^3).$$

(ii) *For some positive constant $K > 0$,*

$$\|U_\varepsilon\|_{L^q(\mathbb{R}^3)}^q = \begin{cases} K\varepsilon^{3-\frac{q}{2}} + o(\varepsilon^{3-\frac{q}{2}}) & \text{if } q \in (3, 6), \\ \omega\varepsilon^{\frac{3}{2}}|\log \varepsilon| + O(\varepsilon^{\frac{3}{2}}) & \text{if } q = 3, \\ \omega\left(\int_0^2 \frac{\xi^q(r)}{r^{q-2}} dr\right)\varepsilon^{\frac{q}{2}} + o(\varepsilon^{\frac{q}{2}}) & \text{if } q \in [1, 3). \end{cases}$$

In the rest of the subsection we assume that $c \in (0, c_1)$ is arbitrary but fixed. Let u_c^+ be as provided by [Theorem 3.1.2](#). We recall that $u_c^+ \in \Lambda^+(c)$ satisfies $F(u_c^+) = \gamma^+(c)$ and is a bounded continuous positive Schwarz symmetric function.

Lemma 3.3.16. *For any $1 \leq p, q < \infty$, it holds that*

$$\int_{\mathbb{R}^3} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx \sim \int_{\mathbb{R}^3} |U_\varepsilon(x)|^q dx.$$

Proof. On one hand, since u_c^+ is bounded, we have that

$$\int_{\mathbb{R}^3} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx \leq \|u_c^+\|_{L^\infty(\mathbb{R}^3)}^p \int_{\mathbb{R}^3} |U_\varepsilon(x)|^q dx.$$

On the other hand, since $u_c^+ > 0$ on \mathbb{R}^3 is continuous and the function U_ε is compactly supported in B_2 , we have that

$$\begin{aligned} \int_{\mathbb{R}^3} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx &= \int_{B_2} |u_c^+(x)|^p |U_\varepsilon(x)|^q dx \geq \min_{x \in B_2} |u_c^+(x)|^p \int_{B_2} |U_\varepsilon(x)|^q dx \\ &= \min_{x \in B_2} |u_c^+(x)|^p \int_{\mathbb{R}^3} |U_\varepsilon(x)|^q dx. \end{aligned}$$

The lemma is proved. \square

For any $\varepsilon > 0$ and any $t > 0$, we have

$$A(u_c^+ + tU_\varepsilon) = \|\nabla(u_c^+ + tU_\varepsilon)\|_{L^2(\mathbb{R}^3)}^2 = A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + A(tU_\varepsilon) \quad (3.3.23)$$

and

$$\|u_c^+ + tU_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 = c + 2 \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx + \|tU_\varepsilon\|_{L^2(\mathbb{R}^3)}^2. \quad (3.3.24)$$

Using that, for all $a, b \geq 0$, $(a+b)^6 \geq a^6 + 6a^5b + 6ab^5 + b^6$, and that both $u_c^+ \in H^1(\mathbb{R}^N)$ and U_ε are non negative, we readily derive that

$$\begin{aligned} C(u_c^+ + tU_\varepsilon) &= \|u_c^+ + tU_\varepsilon\|_{L^6(\mathbb{R}^3)}^6 \\ &\geq C(u_c^+) + C(tU_\varepsilon) + 6 \int_{\mathbb{R}^3} (u_c^+(x))^5 (tU_\varepsilon(x)) dx + 6 \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x))^5 dx. \end{aligned} \quad (3.3.25)$$

Also, still using that $u_c^+ \in H^1(\mathbb{R}^N)$ and U_ε are non negative, we get by direct calculations that

$$\begin{aligned} B(u_c^+ + tU_\varepsilon) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x) + tU_\varepsilon(x)|^2 |u_c^+(y) + tU_\varepsilon(y)|^2}{|x-y|} dx dy \\ &\geq B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y) (tU_\varepsilon(y))}{|x-y|} dx dy. \end{aligned} \quad (3.3.26)$$

Finally, since u_c^+ is solution of the following equation

$$-\Delta u + \lambda_c^+ u - \gamma(|x|^{-1} * |u|^2)u - a|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^3$$

for a $\lambda_c^+ > 0$, we have that

$$\begin{aligned} -\lambda_c^+ \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x)) dx &= \int_{\mathbb{R}^3} \nabla u_c^+(x) \nabla(tU_\varepsilon(x)) dx \\ -\gamma \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y) (tU_\varepsilon(y))}{|x-y|} dx dy &- a \int_{\mathbb{R}^3} (u_c^+(x))^5 (tU_\varepsilon(x)) dx. \end{aligned} \quad (3.3.27)$$

Now, we define for $t > 0$, $w_{\varepsilon,t} = u_c^+ + tU_\varepsilon$ and $\bar{w}_{\varepsilon,t}(x) = \sqrt{\theta} w_{\varepsilon,t}(\theta x)$ with $\theta^2 = \frac{1}{c} \|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2$. The proof of [Lemma 3.3.14](#) will follow directly from the three lemmas below.

Lemma 3.3.17. *It holds that*

$$\gamma^-(c) \leq \sup_{t \geq 0} F(\bar{w}_{\varepsilon,t})$$

for $\varepsilon > 0$ sufficiently small.

Lemma 3.3.18. *There exist a $\varepsilon_0 > 0$ and $0 < t_0 < t_1 < \infty$ such that*

$$F(\bar{w}_{\varepsilon,t}) < \gamma^+(c) + \frac{1}{6\sqrt{aK_{GN}}}$$

for $t \notin [t_0, t_1]$ and any $\varepsilon \in (0, \varepsilon_0]$.

Lemma 3.3.19. *It holds that*

$$\max_{t \in [t_0, t_1]} F(\bar{w}_{\varepsilon,t}) < \gamma^+(c) + \frac{1}{3\sqrt{aK_{GN}}},$$

for any $\varepsilon \in (0, \varepsilon_0]$ where ε_0 and t_0, t_1 are provided by [Lemma 3.3.18](#).

Proof of Lemma 3.3.17. By direct calculation we get

$$A(\bar{w}_{\varepsilon,t}) = A(w_{\varepsilon,t}), \quad C(\bar{w}_{\varepsilon,t}) = C(w_{\varepsilon,t}), \quad (3.3.28)$$

and

$$\|\bar{w}_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2 = \theta^{-2} \|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2, \quad B(\bar{w}_{\varepsilon,t}) = \theta^{-3} B(w_{\varepsilon,t}). \quad (3.3.29)$$

Since $\theta^2 = \frac{1}{c} \|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2$, we have that $\bar{w}_{\varepsilon,t} \in S(c)$. By [Lemma 3.3.4](#) there exists $s_{\varepsilon,t}^- > 0$ such that $(\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-} \in \Lambda^-(c)$. We claim that $s_{\varepsilon,t}^- \rightarrow 0$ as $t \rightarrow +\infty$ uniformly for $\varepsilon > 0$ sufficiently small. Indeed, we have

$$A((\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-}) = \frac{\gamma}{4} B((\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-}) + aC((\bar{w}_{\varepsilon,t})^{s_{\varepsilon,t}^-})$$

or equivalently

$$(s_{\varepsilon,t}^-)A(\bar{w}_{\varepsilon,t}) = \frac{\gamma}{4} B(\bar{w}_{\varepsilon,t}) + a(s_{\varepsilon,t}^-)^5 C(\bar{w}_{\varepsilon,t}).$$

This implies that

$$A(\bar{w}_{\varepsilon,t}) \geq a(s_{\varepsilon,t}^-)^4 C(\bar{w}_{\varepsilon,t}). \quad (3.3.30)$$

In view of (3.3.23), (3.3.28), [Lemma 3.3.15\(i\)](#) and using Hölder's inequality, we have

$$\begin{aligned} A(\bar{w}_{\varepsilon,t}) &= A(w_{\varepsilon,t}) = A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + A(tU_\varepsilon) \\ &\leq A(u_c^+) + 2t \|\nabla u_c^+\|_{L^2(\mathbb{R}^3)} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^3)} + t^2 A(U_\varepsilon) \\ &\rightarrow A(u_c^+) + 2J\sqrt{A(u_c^+)t} + Jt^2 \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (3.3.31)$$

In view of (3.3.25), (3.3.28) and [Lemma 3.3.15\(i\)](#), we also have

$$C(\bar{w}_{\varepsilon,t}) = C(w_{\varepsilon,t}) \geq C(tU_\varepsilon) = t^6 C(U_\varepsilon) \rightarrow Lt^6 \text{ as } \varepsilon \rightarrow 0. \quad (3.3.32)$$

Combining (3.3.30)-(3.3.32), we obtain that, for $\varepsilon > 0$ sufficiently small

$$A(u_c^+) + J\sqrt{A(u_c^+)t} + Jt^2 \geq a(s_{\varepsilon,t}^-)^4 Lt^6,$$

which implies the claim. Since $\bar{w}_{\varepsilon,0} = w_{\varepsilon,0} = u_c^+$ and $u_c^+ \in \Lambda^+(c)$ we obtain, see [Lemma 3.3.4](#), that $s_{\varepsilon,0}^- > 1$. Still by [Lemma 3.3.4](#), the map $t \mapsto s_{\varepsilon,t}^-$ is continuous which implies that there exists $t_\varepsilon > 0$ such that $s_{\varepsilon,t_\varepsilon}^- = 1$. It follows that $\bar{w}_{\varepsilon,t_\varepsilon} \in \Lambda^-(c)$ and thus

$$\sup_{t \geq 0} F(\bar{w}_{\varepsilon,t}) \geq F(\bar{w}_{\varepsilon,t_\varepsilon}) \geq \gamma^-(c).$$

The lemma is proved. \square

Proof of Lemma 3.3.18. In view of (3.3.28) and (3.3.29), we have that

$$F(\bar{w}_{\varepsilon,t}) = \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4}\theta^{-3}B(w_{\varepsilon,t}) - \frac{a}{6}C(w_{\varepsilon,t}).$$

Hence, by (3.3.23), (3.3.25) and (3.3.26), we get that

$$\begin{aligned} F(\bar{w}_{\varepsilon,t}) &\leq \frac{1}{2} \left[A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + A(tU_\varepsilon) \right] - \frac{\gamma}{4}\theta^{-3}B(u_c^+) - \frac{a}{6} \left[C(u_c^+) + C(tU_\varepsilon) \right] \\ &= F(u_c^+) + \frac{\gamma}{4}(1 - \theta^{-3})B(u_c^+) + \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla(tU_\varepsilon(x)) dx + \frac{1}{2}A(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \\ &\leq F(u_c^+) + \frac{\gamma}{4}(1 - \theta^{-3})B(u_c^+) + t \|\nabla u_c^+\|_{L^2(\mathbb{R}^3)} \|\nabla U_\varepsilon\|_{L^2(\mathbb{R}^3)} + \frac{1}{2}t^2A(U_\varepsilon) - \frac{a}{6}t^6C(U_\varepsilon) := I(t). \end{aligned}$$

By [Lemma 3.3.15\(i\)](#), we have that, uniformly for $\varepsilon > 0$ small, $I(t) \rightarrow -\infty$ as $t \rightarrow +\infty$ and $I(t) \rightarrow F(u_c^+)$ as $t \rightarrow 0$ due to $\theta \rightarrow 1$. Hence, there exists $\varepsilon_0 > 0$ and $0 < t_0 < t_1 < \infty$ such that

$$F(\bar{w}_{\varepsilon,t}) < \gamma^+(c) + \frac{1}{6\sqrt{aK_{GN}}}$$

for $t \notin [t_0, t_1]$ and $\varepsilon \in (0, \varepsilon_0]$. The lemma is proved. \square

Proof of Lemma 3.3.19. We assume throughout the proof that $t \in [t_0, t_1]$. By using (3.3.26), we can write,

$$\begin{aligned} F(\bar{w}_{\varepsilon,t}) &= \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4}\theta^{-3}B(w_{\varepsilon,t}) - \frac{a}{6}C(w_{\varepsilon,t}) \\ &\leq \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4}\theta^{-3} \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \right] - \frac{a}{6}C(w_{\varepsilon,t}) \\ &= I_1 + I_2, \end{aligned} \tag{3.3.33}$$

where

$$I_1 := \frac{1}{2}A(w_{\varepsilon,t}) - \frac{\gamma}{4} \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \right] - \frac{a}{6}C(w_{\varepsilon,t}),$$

and

$$I_2 := \frac{\gamma}{4}(1 - \theta^{-3}) \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy \right]. \tag{3.3.34}$$

In view of (3.3.23), (3.3.25) and using crucially (3.3.27), we have

$$\begin{aligned}
 I_1 &\leq \frac{1}{2} \left[A(u_c^+) + 2 \int_{\mathbb{R}^3} \nabla u_c^+(x) \cdot \nabla (tU_\varepsilon(x)) dx + A(tU_\varepsilon) \right] \\
 &\quad - \frac{\gamma}{4} \left[B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y) (tU_\varepsilon(y))}{|x-y|} dx dy \right] \\
 &\quad - \frac{a}{6} \left[C(u_c^+) + C(tU_\varepsilon) + 6 \int_{\mathbb{R}^3} (u_c^+(x))^5 (tU_\varepsilon(x)) dx + 6 \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x))^5 dx \right] \\
 &= F(u_c^+) + F(tU_\varepsilon) - \lambda_c^+ \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) dx - a \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x))^5 dx.
 \end{aligned} \tag{3.3.35}$$

Now, we shall evaluate I_2 . By (3.3.24) and Lemma 3.3.15(ii), we get that

$$\begin{aligned}
 \theta^2 &= \frac{\|w_{\varepsilon,t}\|_{L^2(\mathbb{R}^3)}^2}{c} = 1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) dx + \frac{t^2}{c} \|U_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \\
 &= 1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) dx + \frac{t^2}{c} \left[\omega \left(\int_0^2 \xi(r) dr \right) \varepsilon + O(\varepsilon^2) \right] \\
 &= 1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) dx + O(\varepsilon).
 \end{aligned} \tag{3.3.36}$$

Note that, by Lemma 3.3.15(ii) and Lemma 3.3.16,

$$\int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) dx \sim \|\nabla U_\varepsilon\|_{L^1(\mathbb{R}^3)} = O(\varepsilon^{\frac{1}{2}}). \tag{3.3.37}$$

Observing that the Taylor expansion of $(1+x)^{-\frac{3}{2}}$ around $x=0$ is given by

$$(1+x)^{-\frac{3}{2}} = 1 - \frac{3}{2}x + O(x^2),$$

we get, in view of (3.3.36) and (3.3.37), that

$$\begin{aligned}
 1 - \theta^{-3} &= 1 - (\theta^2)^{-\frac{3}{2}} = 1 - \left[1 + \frac{2}{c} \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) + O(\varepsilon) \right]^{-\frac{3}{2}} \\
 &= 1 - \left[1 - \frac{3}{c} \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) + O(\varepsilon) \right] = \frac{3}{c} \int_{\mathbb{R}^3} u_c^+(x) (tU_\varepsilon(x)) + O(\varepsilon).
 \end{aligned} \tag{3.3.38}$$

Concerning the term $B(tU_\varepsilon)$, in view of (3.2.2) and Lemma 3.3.15(ii), we have

$$\begin{aligned}
 B(tU_\varepsilon) &= t^4 B(U_\varepsilon) \leq t^4 K_1 \|U_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \\
 &= t^4 K_1 \left(\|U_\varepsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)} \right)^{\frac{10}{6}} = t^4 K_1 \left(K_2 \varepsilon^{\frac{6}{5}} + o(\varepsilon^{\frac{6}{5}}) \right)^{\frac{10}{6}} = O(\varepsilon^2).
 \end{aligned} \tag{3.3.39}$$

Also, we can deduce from the Hardy-Littlewood-Sobolev inequality (3.2.1) and Lemma 3.3.16, Lemma 3.3.15(ii) that

$$\begin{aligned}
 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y) (tU_\varepsilon(y))}{|x-y|} dx dy &\leq K_2 \|u_c^+\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^2 \|u_c^+ U_\varepsilon\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \\
 &\leq K_3 \|U_\varepsilon\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} = O(\varepsilon^{\frac{3}{5}}).
 \end{aligned} \tag{3.3.40}$$

From (3.3.39) and (3.3.40) we deduce that

$$B(u_c^+) + B(tU_\varepsilon) + 4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_c^+(x)|^2 u_c^+(y)(tU_\varepsilon(y))}{|x-y|} dx dy = B(u_c^+) + O(\varepsilon^{\frac{3}{5}}). \quad (3.3.41)$$

Taking into account, see (3.3.18), that

$$c\lambda_c^+ = \frac{3\gamma}{4}B(u_c^+)$$

we obtain, combining (3.3.34), (3.3.37), (3.3.38) and (3.3.41), the following evaluation of I_2

$$I_2 \leq \frac{3\gamma}{4c}B(u_c^+) \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))dx + O(\varepsilon) = \lambda_c^+ \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))dx + O(\varepsilon). \quad (3.3.42)$$

At this point, in view of (3.3.33), (3.3.35) and (3.3.42) we deduce that

$$\begin{aligned} F(\bar{w}_{\varepsilon,t}) &\leq F(u_c^+) + F(tU_\varepsilon) - a \int_{\mathbb{R}^3} u_c^+(x)(tU_\varepsilon(x))^5 dx + O(\varepsilon) \\ &\leq F(u_c^+) + F(tU_\varepsilon) - at_0^5 \int_{\mathbb{R}^3} u_c^+(x)(U_\varepsilon(x))^5 dx + O(\varepsilon). \end{aligned} \quad (3.3.43)$$

In view of Lemma 3.3.15(i), a direct calculation shows that

$$\begin{aligned} \max_{t \in [t_0, t_1]} F(tU_\varepsilon) &= \max_{t \in [t_0, t_1]} \left[\frac{1}{2}A(tU_\varepsilon) - \frac{\gamma}{4}B(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \right] \\ &\leq \max_{t \in [t_0, t_1]} \left[\frac{1}{2}A(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \right] \\ &\leq \max_{t > 0} \left[\frac{1}{2}A(tU_\varepsilon) - \frac{a}{6}C(tU_\varepsilon) \right] = \frac{1}{3\sqrt{aK_{GN}}} + O(\varepsilon). \end{aligned} \quad (3.3.44)$$

In view of (3.3.43) and (3.3.44), by Lemma 3.3.15(ii) and Lemma 3.3.16, we conclude by observing that

$$-at_0^5 \int_{\mathbb{R}^3} u_c^+(x)(U_\varepsilon(x))^5 dx \sim -\|U_\varepsilon(x)\|_{L^5(\mathbb{R}^3)}^5 = -K\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}}).$$

□

Proof of Theorem 3.1.2 in the critical case $p = 6$ for $\gamma^-(c)$. We conclude that $\gamma^-(c)$ is reached by combining Lemma 3.3.7, Lemma 3.3.13 and Lemma 3.3.14. The rest of the proof is identical to the one in the case $p \in (\frac{10}{3}, 6)$. □

3.3.4 The compactness of any minimizing sequence associated to $\gamma^+(c)$ for $p \in (\frac{10}{3}, 6]$

In this subsection we give the proof of Theorem 3.1.5. For short we introduce the following notations,

$$M := \frac{p}{a\sigma(\sigma-1)K_{GN}}, \quad N := \frac{4(\sigma-2)}{\gamma(\sigma-1)K_H}, \quad k_0 := N^{-2}, \quad \text{and} \quad k_1 := k_0 c_1^3. \quad (3.3.45)$$

Note that

$$c_1 = N^{\frac{3p-10}{4(p-3)}} M^{\frac{1}{2(p-3)}}.$$

Lemma 3.3.20. *Let $p \in (\frac{10}{3}, 6]$ and $c \in (0, c_1)$.*

(i) *If $u \in \Lambda^+(c)$ then we have*

$$A(u) < k_0 c^3. \quad (3.3.46)$$

(ii) $\Lambda^+(c) \subset V(c)$ and

$$\gamma^+(c) = \inf_{u \in \Lambda^+(c)} F(u) = \inf_{u \in V(c)} F(u).$$

(iii) *If u_c is a minimizer for the minimization problem*

$$\inf_{u \in V(c)} F(u)$$

then $u_c \in V(c)$ and $\gamma^+(c)$ is reached.

Proof. i) Since $u \in \Lambda^+(c)$,

$$A(u) = \frac{\gamma}{4} B(u) + \frac{a\sigma}{p} C(u) \quad \text{and} \quad A(u) > \frac{a\sigma(\sigma-1)}{p} C(u).$$

Using Lemma 3.2.1(i), we have

$$\frac{\sigma-2}{\sigma-1} A(u) < \frac{\gamma}{4} B(u) \leq \frac{\gamma}{4} K_H \sqrt{A(u)} c^{\frac{3}{2}},$$

which implies that

$$A(u) < \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3 = N^{-2} c^3 = k_0 c^3 < k_0 c_1^3 = k_1. \quad (3.3.47)$$

Hence, the point (i) holds.

ii) By (3.3.47), we obtain that $\Lambda^+(c) \subset V(c)$ and hence

$$\inf_{u \in \Lambda^+(c)} F(u) \geq \inf_{u \in V(c)} F(u).$$

To prove the point (ii), it is sufficient to show that

$$\inf_{u \in \Lambda^+(c)} F(u) \leq \inf_{u \in V(c)} F(u). \quad (3.3.48)$$

Firstly, we claim that $\Lambda^-(c) \cap \overline{V(c)} = \emptyset$. Indeed, let $v \in \Lambda^-(c)$. Taking into account that

$$A(v) < \frac{a\sigma(\sigma-1)}{p} C(v),$$

and using Lemma 3.2.1(ii), we obtain that

$$A(v) < \frac{a\sigma(\sigma-1)}{p} K_{GN} c^{\frac{6-p}{4}} [A(v)]^{\frac{\sigma}{2}} = M^{-1} c^{\frac{6-p}{4}} [A(v)]^{\frac{\sigma}{2}}.$$

This implies that

$$A(v) > M^{\frac{4}{3p-10}} c^{-\frac{6-p}{3p-10}}.$$

By direct computations, we can check that

$$N^{-2}c_1^3 = M^{\frac{4}{3p-10}} c_1^{-\frac{6-p}{3p-10}},$$

which implies that for all $0 < c < c_1$,

$$k_1 = N^{-2}c_1^3 = M^{\frac{4}{3p-10}} c_1^{-\frac{6-p}{3p-10}} < M^{\frac{4}{3p-10}} c^{-\frac{6-p}{3p-10}} < A(v). \quad (3.3.49)$$

Therefore, the claim holds. Next, let $u \in S(c)$. Since the mapping $t \mapsto A(u^t)$ is continuous increasing, there exists a unique $t_u^1 > 0$ such that $A(u^{t_u^1}) = k_1$. By [Lemma 3.3.4](#) and [\(3.3.47\)](#), [\(3.3.49\)](#), we have

$$A(u^{s_u^+}) < A(u^{t_u^1}) < A(u^{s_u^-}),$$

which implies that

$$s_u^+ < t_u^1 < s_u^-.$$

Since $g_u'(t) > 0$ for all $t \in (s_u^+, s_u^-)$, we get that $g_u'(t) > 0$ for all $t \in (s_u^+, t_u^1]$ and hence

$$F(u^{s_u^+}) = g_u(s_u^+) < g_u(t) = F(u^t) \quad \forall t \in (s_u^+, t_u^1]. \quad (3.3.50)$$

Since s_u^+ is the unique local minimum point for g_u on $(0, s_u^-)$, we have that $F(u^{s_u^+}) \leq F(u^t)$ for all $t \in (0, t_u^1]$. Therefore, we obtain that

$$F(u^{s_u^+}) = \min\{F(u^t) | 0 < t \leq t_u^1\} = \min\{F(u^t) | t \in \mathbb{R}, A(u^t) \leq k_1\}.$$

In particular, if $u \in \overline{V(c)}$ we have

$$F(u^{s_u^+}) = \min\{F(u^t) | t \in \mathbb{R}, A(u^t) \leq k_1\} = \min\{F(u) | u \in \overline{V(c)}\} \leq F(u).$$

This implies [\(3.3.48\)](#) and the point [\(ii\)](#) is proved.

[\(iii\)](#) If we assume that $u_c \in \partial V(c)$, namely $A(u_c) = k_1$ and

$$F(u_c) = \min\{F(u) | u \in \overline{V(c)}\} = \min\{F(u^t) | t \in \mathbb{R}, A(u^t) \leq k_1\},$$

and we have a contradiction with [\(3.3.50\)](#). Thus, we have $u_c \in V(c)$. Now, since the minimizer u_c lies in the open (with respect to $S(c)$) set $V(c)$, we deduce from [Lemma 3.2.2](#) that $u_c \in \Lambda(c)$. By $\Lambda^-(c) \cap \overline{V(c)} = \emptyset$ and $\Lambda^0(c) = \emptyset$, we conclude that $u_c \in \Lambda^+(c)$ and thus $\gamma^+(c)$ is reached. \square

Lemma 3.3.21. *It holds that*

(i) $\gamma^+(c) < 0$, $\forall c \in (0, c_1)$.

(ii) $c \in (0, c_1) \mapsto \gamma^+(c)$ is a continuous mapping.

(iii) Let $c \in (0, c_1)$, for all $\alpha \in (0, c)$, we have $\gamma^+(c) \leq \gamma^+(\alpha) + \gamma^+(c - \alpha)$ and if $\gamma^+(\alpha)$ or $\gamma^+(c - \alpha)$ is reached then the inequality is strict.

Proof. Point (i) follows from Lemma 3.3.5. To prove (ii), let $c \in (0, c_1)$ be arbitrary and $(c_n) \subset (0, c_1)$ be such that $c_n \rightarrow c$. From the definition of $\gamma^+(c_n)$, for any $\varepsilon > 0$, there exists $u_n \in \Lambda^+(c_n)$ such that

$$F(u_n) \leq \gamma^+(c_n) + \varepsilon. \quad (3.3.51)$$

By (3.3.46), we have $A(u_n) < k_0 c_n^3$. We set $y_n := \sqrt{\frac{c}{c_n}} \cdot u_n$. Hence, we have $y_n \in S(c)$ and

$$A(y_n) = \frac{c}{c_n} A(u_n) < \frac{c}{c_n} k_0 c_n^3 = k_0 c_n^2 c < k_0 c_1^3 = k_1.$$

This implies that $y_n \in \overline{V(c)}$. Taking into account that $\frac{c}{c_n} \rightarrow 1$, we have

$$\gamma^+(c) \leq F(y_n) = F(u_n) + o_n(1). \quad (3.3.52)$$

Combining (3.3.51) and (3.3.52), we get

$$\gamma^+(c) \leq \gamma^+(c_n) + \varepsilon + o_n(1).$$

Reversing the argument we obtain similarly that

$$\gamma^+(c_n) \leq \gamma^+(c) + \varepsilon + o_n(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $\gamma^+(c_n) \rightarrow \gamma^+(c)$. The point (ii) follows.

iii) Note that, fixed $\alpha \in (0, c)$, it is sufficient to prove that the following holds

$$\forall \theta \in \left(1, \frac{c}{\alpha}\right]: \gamma^+(\theta\alpha) \leq \theta\gamma^+(\alpha) \quad (3.3.53)$$

and that, if $\gamma^+(\alpha)$ is reached, the inequality is strict. Indeed, if (3.3.53) holds then it follows directly that

$$\gamma^+(c) = \frac{c-\alpha}{c} \gamma^+(c) + \frac{\alpha}{c} \gamma^+(c) = \frac{c-\alpha}{c} \gamma^+\left(\frac{c}{c-\alpha}(c-\alpha)\right) + \frac{\alpha}{c} \gamma^+\left(\frac{c}{\alpha}\alpha\right) \leq \gamma^+(c-\alpha) + \gamma^+(\alpha)$$

with a strict inequality if $\gamma^+(\alpha)$ is reached. To prove that (3.3.53) holds, note that for any $\varepsilon > 0$ sufficiently small, there exist $u \in \Lambda^+(\alpha)$ such that

$$F(u) \leq \gamma^+(\alpha) + \varepsilon. \quad (3.3.54)$$

By (3.3.46), we have $A(u) < k_0 \alpha^3$. Consider now $v := \sqrt{\theta}u$, we have

$$\|v\|_{L^2(\mathbb{R}^3)} = \theta \|u\|_{L^2(\mathbb{R}^3)}, \quad A(v) = \theta A(u), \quad B(v) = \theta^2 B(u), \quad C(v) = \theta^3 C(u).$$

Therefore, we obtain that $v \in S(\theta\alpha)$ and

$$A(v) = \theta A(u) < k_0 \theta \alpha^3 < k_0 (\theta\alpha)^3 \leq k_0 c^3 < k_1.$$

Hence, $v \in \overline{V(\theta\alpha)}$ and we can write

$$\begin{aligned} \gamma^+(\theta\alpha) &\leq F(v) = \frac{1}{2}A(v) - \frac{\gamma}{4}B(v) - \frac{a}{p}C(v) = \frac{1}{2}\theta A(u) - \frac{\gamma}{4}\theta^2 B(u) - \frac{a}{p}\theta^3 C(u) \\ &< \frac{1}{2}\theta A(u) - \frac{\gamma}{4}\theta B(u) - \frac{a}{p}\theta C(u) = \theta F(u) \leq \theta(\gamma^+(\alpha) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that $\gamma^+(\theta\alpha) \leq \theta\gamma^+(\alpha)$. If $\gamma^+(\alpha)$ is reached then we can let $\varepsilon = 0$ in (3.3.54) and thus the strict inequality follows. \square

Lemma 3.3.22. *Let $(v_n) \subset H^1(\mathbb{R}^3)$ be such that $B(v_n) \rightarrow 0$ and $A(v_n) \leq k_1$. Then there exists a $b > 0$ such that*

$$F(v_n) \geq bA(v_n) + o_n(1). \quad (3.3.55)$$

Proof. Indeed, using $B(v_n) \rightarrow 0$ and Lemma 3.2.1(ii), we have

$$F(v_n) = \frac{1}{2}A(v_n) - \frac{a}{p}C(v_n) + o_n(1) \geq \frac{1}{2}A(v_n) - \frac{a}{p}K_{GN}c^{\frac{6-p}{4}}[A(v_n)]^{\frac{\sigma}{2}} + o_n(1) = bA(v_n) + o_n(1),$$

where

$$b := \frac{1}{2} - \limsup_{n \rightarrow \infty} \frac{a}{p}K_{GN}c^{\frac{6-p}{4}}[A(v_n)]^{\frac{\sigma}{2}-1} \geq \frac{1}{2} - \frac{a}{p}K_{GN}c_1^{\frac{6-p}{4}}k_1^{\frac{\sigma}{2}-1} = \frac{1}{2} - \frac{1}{\sigma(\sigma-1)}.$$

Hence, $b > 0$ due to $\sigma > 2$. The lemma is proved. \square

Lemma 3.3.23. *For any $c \in (0, c_1)$, any minimizing sequence (u_n) for F on $\overline{V(c)}$ is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$. In addition all minimizers lie in $V(c)$. In particular $\gamma^+(c)$ is reached.*

Proof. Since $(u_n) \subset \overline{V(c)}$, it is bounded in $H^1(\mathbb{R}^3)$. Also, from $\gamma^+(c) < 0$ we deduce from Lemma 3.3.22 that there exists a $\beta_0 > 0$ and a sequence $(y_n) \subset \mathbb{R}^3$ such that

$$\int_{B(y_n, R)} |u_n|^2 dx \geq \beta_0 > 0, \quad \text{for some } R > 0.$$

This implies that

$$u_n(x - y_n) \rightharpoonup u_c \neq 0 \quad \text{in } H^1(\mathbb{R}^3), \quad \text{for some } u_c \in H^1(\mathbb{R}^3).$$

Our aim is to prove that $w_n(x) := u_n(x - y_n) - u_c(x) \rightarrow 0$ in $H^1(\mathbb{R}^3)$. Clearly

$$\begin{aligned} \|u_n\|_{L^2(\mathbb{R}^3)}^2 &= \|u_n(x - y_n)\|_{L^2(\mathbb{R}^3)}^2 = \|u_n(x - y_n) - u_c(x)\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) \\ &= \|w_n\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1). \end{aligned}$$

Thus, we have

$$\|w_n\|_{L^2(\mathbb{R}^3)}^2 = \|u_n\|_{L^2(\mathbb{R}^3)}^2 - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) = c - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1). \quad (3.3.56)$$

By the similar argument,

$$A(w_n) = A(u_n) - A(u_c) + o_n(1). \quad (3.3.57)$$

More generally it is direct to show, using the Brezis-Lieb lemma [24, Theorem 1] for terms A and C , and using [92, Lemma 2.2] or [17, Proposition 3.1] for term B , that any term in F also enjoy a the splitting property, and thus we have

$$F(u_n - u_c) + F(u_c) = F(u_n) + o_n(1),$$

and by the translational invariance, we obtain

$$F(u_n) = F(u_n(x - y_n)) = F(u_n(x - y_n) - u_c(x)) + F(u_c) + o_n(1) = F(w_n) + F(u_c) + o_n(1). \quad (3.3.58)$$

Now, we claim that

$$\|w_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.3.59)$$

In order to prove this, let us denote $\tilde{c} := \|u_c\|_{L^2(\mathbb{R}^3)}^2 > 0$. By (3.3.56), if we show that $\tilde{c} = c$ then the claim follows. We assume by contradiction that $\tilde{c} < c$. In view of (3.3.56) and (3.3.57), for n large enough, we have $\|w_n\|_{L^2(\mathbb{R}^3)}^2 \leq c$ and $A(w_n) \leq A(u_n) \leq k_1$. Hence, we obtain that $w_n \in \overline{V(\|w_n\|_{L^2(\mathbb{R}^3)}^2)}$ and $F(w_n) \geq \gamma^+(\|w_n\|_{L^2(\mathbb{R}^3)}^2)$. Recording that $F(u_n) \rightarrow \gamma^+(c)$, in view of (3.3.58), we have

$$\gamma^+(c) = F(w_n) + F(u_c) \geq \gamma^+(\|w_n\|_{L^2(\mathbb{R}^3)}^2) + F(u_c).$$

Since the map $c \mapsto \gamma^+(c)$ is continuous (see Lemma 3.3.21(ii)) and in view of (3.3.56), we deduce that

$$\gamma^+(c) \geq \gamma^+(c - \tilde{c}) + F(u_c). \quad (3.3.60)$$

We also have that $u_c \in \overline{V(\tilde{c})}$ by the weak limit. This implies that $F(u_c) \geq \gamma^+(\tilde{c})$. If $F(u_c) > \gamma^+(\tilde{c})$, then it follows from (3.3.60) and Lemma 3.3.21(iii) that

$$\gamma^+(c) > \gamma^+(c - \tilde{c}) + \gamma^+(\tilde{c}) \geq \gamma^+(c - \tilde{c} + \tilde{c}) = \gamma^+(c),$$

which is impossible. Hence, we have $F(u_c) = \gamma^+(\tilde{c})$, namely u_c is local minimizer on $\overline{V(\tilde{c})}$. So, we can use Lemma 3.3.21(iii) with the strict inequality and we deduce from (3.3.60) that

$$\gamma^+(c) \geq \gamma^+(c - \tilde{c}) + F(u_c) = \gamma^+(c - \tilde{c}) + \gamma^+(\tilde{c}) > \gamma^+(c - \tilde{c} + \tilde{c}) = \gamma^+(c),$$

which is impossible. Thus, the claim follows and $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.

Let us now show that $A(w_n) \rightarrow 0$. This will complete the proof of the lemma. In this aim first observe that since (w_n) is a bounded sequence in $H^1(\mathbb{R}^N)$ we have, using Lemma 3.2.1(i), not only that $\|w_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0$ but also that $B(w_n) \rightarrow 0$. Now we remember that

$$F(u_n) = F(u_c) + F(w_n) + o_n(1) \rightarrow \gamma^+(c). \quad (3.3.61)$$

Since $u_c \in \overline{V(c)}$ by weak convergence property, we have, by Lemma 3.3.20(ii), that $F(u_c) \geq \gamma^+(c)$. Thus from (3.3.61) we deduce, on one hand, that necessarily $F(w_n) \leq o(1)$. On the other hand, since $A(w_n) \leq A(u_n) \leq k_1$, Lemma 3.3.22 implies that $F(w_n) \geq bA(w_n) + o_n(1)$ for some $b > 0$. Hence, we conclude $A(w_n) \rightarrow 0$ and thus that $u_n \rightarrow u_c \in \overline{V(c)}$ strongly in $H^1(\mathbb{R}^3)$. Finally, by Lemma 3.3.20(iii), we have $u_c \in V(c)$ and $\gamma^+(c)$ is reached. The lemma is proved. \square

3.3.5 Asymptotic behavior of the Lagrange multipliers and the monotonicity of the map $c \mapsto \gamma^-(c)$

Lemma 3.3.24. *Let $p \in (\frac{10}{3}, 6]$. There exist two constants $K_1 > 0$ and $K_2 > 0$ such that for any $c \in (0, c_1)$, if λ_c^+ is the Lagrange parameter associated to a solution u_c^+ lying at the level $\gamma^+(c)$ then we have*

$$|\gamma^+(c)| \leq K_1 c^3 \quad \text{and} \quad \lambda_c^+ \leq K_2 c^2.$$

Proof. By Lemma 3.3.20(i), we have

$$A(u_c^+) < N^{-2}c^3 = \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3.$$

Hence, we can deduce from Lemma 3.2.1(i) that

$$B(u_c^+) \leq K_H \sqrt{A(u_c^+)} c^{\frac{3}{2}} < \frac{\gamma(\sigma-1)K_H^2}{4(\sigma-2)} c^3.$$

Therefore, we have

$$\begin{aligned} |\gamma^+(c)| = |F(u_c^+)| &= \left| \frac{\sigma-2}{2\sigma} A(u_c^+) - \frac{\gamma(\sigma-1)}{4\sigma} B(u_c^+) \right| \leq \frac{\sigma-2}{2\sigma} A(u_c^+) + \frac{\gamma(\sigma-1)}{4\sigma} B(u_c^+) \\ &< \frac{\sigma-2}{2\sigma} \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3 + \frac{\gamma(\sigma-1)}{4\sigma} \frac{\gamma(\sigma-1)K_H^2}{4(\sigma-2)} c^3 \\ &= \frac{3\gamma^2(\sigma-1)^2 K_H^2}{32\sigma(\sigma-2)} c^3 := K_1 c^3. \end{aligned}$$

We deduce from (3.2.6) that

$$\begin{aligned} 2(3p-6)c\lambda_c^+ &= 2(6-p)A(u_c^+) + (5p-12)\gamma B(u_c^+) \\ &< 2(6-p) \left[\frac{\gamma(\sigma-1)K_H}{4(\sigma-2)} \right]^2 c^3 + (5p-12)\gamma \frac{\gamma(\sigma-1)K_H^2}{4(\sigma-2)} c^3. \end{aligned}$$

This implies that there exists a constant $K_2 > 0$ such that $\lambda_c^+ \leq K_2 c^2$. The lemma is proved. \square

Lemma 3.3.25. *Let $p \in (\frac{10}{3}, 6)$. There exist two constants $K_1 > 0$ and $K_2 > 0$ such that is λ_c^- denotes the Lagrange parameter associated to a solution u_c^- lying at the level $\gamma^-(c)$,*

$$|\gamma^-(c)| > K_1 c^{-\frac{6-p}{3p-10}} \quad \text{and} \quad \lambda_c^- > K_2 c^{-\frac{2p-4}{3p-10}}.$$

Proof. By $u_c^- \in \Lambda^-(c)$, we have

$$A(u_c^-) < \frac{a\sigma(\sigma-1)}{p} C(u_c^-).$$

Using Lemma 3.2.1(ii), we obtain that

$$A(u_c^-) < \frac{a\sigma(\sigma-1)}{p} K_{GN} c^{\frac{6-p}{4}} [A(u_c^-)]^{\frac{\sigma}{2}},$$

which implies that

$$A(u_c^-) > \left[\frac{p}{a\sigma(\sigma-1)K_{GN}} \right]^{\frac{2}{\sigma-2}} c^{-\frac{6-p}{3p-10}}.$$

We have that

$$\begin{aligned} |\gamma^-(c)| &= |F(u_c^-)| = \left| -\frac{1}{2}A(u_c^-) + \frac{a(\sigma-1)}{p}C(u_c^-) \right| \\ &> \frac{\sigma-2}{2\sigma}A(u_c^-) > \frac{\sigma-2}{2\sigma} \left[\frac{p}{a\sigma(\sigma-1)K_{GN}} \right]^{\frac{\sigma-2}{2}} c^{-\frac{6-p}{3p-10}} := K_1 c^{-\frac{6-p}{3p-10}}. \end{aligned}$$

We deduce from (3.2.6) that

$$\begin{aligned} \lambda_c^- &= \frac{1}{c} \frac{1}{2(3p-6)} [2(6-p)A(u_c^-) + (5p-12)\gamma B(u_c^-)] \\ &> \frac{1}{c} \frac{6-p}{3p-6} A(u_c^-) > \frac{1}{c} \frac{6-p}{3p-6} \left[\frac{p}{a\sigma(\sigma-1)K_{GN}} \right]^{\frac{\sigma-2}{2}} c^{-\frac{6-p}{3p-10}} := K_2 c^{-\frac{2p-4}{3p-10}}. \end{aligned}$$

The lemma is proved. \square

Lemma 3.3.26. *Let $p = 6$. There exists a constant $K_1 > 0$ such that if λ_c^- denote the Lagrange parameter associated to a solution u_c^- lying at the level $\gamma^-(c)$ then we have*

$$\gamma^-(c) \rightarrow \frac{1}{3\sqrt{aK_{GN}}} \quad \text{as } c \rightarrow 0 \quad \text{and} \quad \lambda_c^- \leq K_1 c^{\frac{1}{2}}.$$

Proof. Since $F(u)$ restricted to $\Lambda(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see Lemma 3.3.1) we have that $A(u_c^-)$ remains bounded. We deduce from (3.2.6) and Lemma 3.2.1(i) that

$$\lambda_c^- = \frac{1}{c} \frac{3\gamma}{4} B(u_c^-) \leq \frac{1}{c} \frac{3\gamma}{4} K_H \sqrt{A(u_c^-)} c^{\frac{3}{2}} := K_1 c^{\frac{1}{2}}.$$

We have that $B(u_c^-) \rightarrow 0$ as $c \rightarrow 0$ due to $B(u_c^-) \leq K_H \sqrt{A(u_c^-)} c^{\frac{3}{2}}$. Since $Q(u_c^-) = 0$, we have

$$A(u_c^-) = aC(u_c^-) + o_c(1),$$

where $o_c(1) \rightarrow 0$ as $c \rightarrow 0$. Passing to the limit as $c \rightarrow 0$, up to subsequence we infer that

$$\lim_{c \rightarrow 0} A(u_c^-) = \lim_{c \rightarrow 0} aC(u_c^-) := \ell \geq 0.$$

Using Lemma 3.2.1(ii), we have

$$\ell = \lim_{c \rightarrow 0} aC(u_c^-) \leq \lim_{c \rightarrow 0} aK_{GN}[A(u_c^-)]^3 = aK_{GN}\ell^3.$$

Therefore, either $\ell = 0$ or $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. Using Lemma 3.3.5(ii), we ensure that $\ell \geq (aK_{GN})^{-\frac{1}{2}}$. Hence, we have

$$\begin{aligned} \gamma^-(c) + o_c(1) &= F(u_c^-) = \frac{\sigma-2}{2\sigma}A(u_c^-) - \frac{\gamma(\sigma-1)}{4\sigma}B(u_c^-) = \frac{1}{3}A(u_c^-) + o_c(1) \\ &= \frac{1}{3}\ell + o_c(1) \geq \frac{1}{3\sqrt{aK_{GN}}} + o_c(1), \end{aligned}$$

which implies that

$$\gamma^-(c) \geq \frac{1}{3\sqrt{aK_{GN}}} \quad \text{as } c \rightarrow 0.$$

Recording Lemma 3.3.14, the lemma is proved. \square

Lemma 3.3.27. *When $p \in (\frac{10}{3}, 6]$, the function $c \mapsto \gamma^-(c)$ is strictly decreasing on $(0, c_1)$.*

Proof. Let $0 < c_2 < c_3 < c_1$. Since $\gamma^-(c_2)$ is reached, there exists $u \in S(c_2)$ such that $F(u) = \gamma^-(c_2)$. We define $v \in S(c_3)$ by $v(x) = \sqrt{\theta}u(\theta x)$ where $\theta = \sqrt{\frac{c_2}{c_3}} < 1$. By direct calculations we have

$$A(v) = A(u), \quad B(v) = \theta^{-3}B(u) \quad \text{and} \quad C(v) = \theta^{\frac{p}{2}-3}C(u). \quad (3.3.62)$$

Now observe that, since $\theta < 1$, for all $t > 0$,

$$\begin{aligned} F(v^t) &= \frac{1}{2}t^2A(v) - \frac{\gamma}{4}tB(v) - \frac{a}{p}t^\sigma C(v) \\ &= \frac{1}{2}t^2A(u) - \frac{\gamma}{4}t\theta^{-3}B(u) - \frac{a}{p}t^\sigma\theta^{\frac{p}{2}-3}C(u) < F(u^t). \end{aligned} \quad (3.3.63)$$

By (3.3.46) and (3.3.49), we have that

$$A(u^{s_u^+}) < k_1 < A(v^{s_v^-}),$$

and thus $s_u^+ < s_v^-$ due to $A(v) = A(u)$. Hence, we can deduce from (3.3.63) that

$$F(v^{s_v^-}) < \max_{s_u^+ < t} F(u^t) = F(u) = \gamma^-(c_2).$$

This implies that $\gamma^-(c_3) < \gamma^-(c_2)$ and hence, the lemma is proved. \square

3.4 The case $\gamma > 0$, $a < 0$ and $p \in (\frac{10}{3}, 6]$

Throughout this section, we assume that $\gamma > 0$, $a < 0$ and $p \in (\frac{10}{3}, 6]$.

Lemma 3.4.1. *F restricted to $S(c)$ is coercive on $H^1(\mathbb{R}^3)$ and bounded from below.*

Proof. Let $u \in S(c)$. Using Lemma 3.2.1(i), we obtain

$$F(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) \geq \frac{1}{2}A(u) - \frac{\gamma}{4}K_H\sqrt{A(u)}c^{\frac{3}{2}} - \frac{a}{p}C(u).$$

Since $\gamma > 0$, $a < 0$, this concludes the proof. \square

In what follows, we collect some basic properties of $m(c)$ defined in (3.1.3).

Lemma 3.4.2. *It holds that*

- (i) $m(c) < 0$, $\forall c > 0$.
- (ii) $c \mapsto m(c)$ is a continuous mapping.
- (iii) For any $c_2 > c_1 > 0$, we have $c_1 m(c_2) \leq c_2 m(c_1)$. If $m(c_1)$ is reached then the inequality is strict.
- (iv) For any $c_2, c_1 > 0$, we have $m(c_1 + c_2) \leq m(c_1) + m(c_2)$. If $m(c_1)$ or $m(c_2)$ is reached then the inequality is strict.

Proof. i) For any $u \in S(c)$, we recall that $u^t \in S(c)$ and

$$g_u(t) = F(u^t) = \frac{1}{2}t^2 A(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}t^\sigma C(u) \quad \text{and also} \quad g'_u(t) = tA(u) - \frac{\gamma}{4}B(u) - \frac{a\sigma}{p}t^{\sigma-1}C(u).$$

We observe that $g_u(t) \rightarrow 0$ and $g'_u(t) \rightarrow -\frac{\gamma}{4}B(u) < 0$ as $t \rightarrow 0$. Therefore, there exists $t_0 > 0$ such that $F(u^{t_0}) = g_u(t_0) < 0$. Thus, we have $m(c) < 0$.

ii) We assume that $c_n \rightarrow c$. From the definition of $m(c_n)$, for any $\varepsilon > 0$, there exists $u_n \in S(c_n)$ such that

$$F(u_n) \leq m(c_n) + \varepsilon. \quad (3.4.1)$$

We set $y_n := \sqrt{\frac{c}{c_n}} \cdot u_n$. Taking into account that $y_n \in S(c)$ and $\frac{c}{c_n} \rightarrow 1$, we have

$$m(c) \leq F(y_n) = F(u_n) + o_n(1). \quad (3.4.2)$$

Combining (3.4.1) and (3.4.2), we get

$$m(c) \leq m(c_n) + \varepsilon + o_n(1).$$

Reversing the argument we obtain similarly that

$$m(c_n) \leq m(c) + \varepsilon + o_n(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $m(c_n) \rightarrow m(c)$. The point (ii) follows.

iii) Let $t := \frac{c_2}{c_1} > 1$. For any $\varepsilon > 0$, there exist $u \in S(c_1)$ such that

$$F(u) \leq m(c_1) + \varepsilon. \quad (3.4.3)$$

Let $v := u(t^{-\frac{1}{3}}x)$. Then we have $\|v\|_{L^2(\mathbb{R}^3)}^2 = t\|u\|_{L^2(\mathbb{R}^3)}^2 = c_2$, hence $v \in S(c_2)$. Moreover, we have

$$A(v) = t^{\frac{1}{3}}A(u), \quad B(v) = t^{\frac{5}{3}}B(u), \quad C(v) = tC(u).$$

Therefore, we have

$$\begin{aligned} m(c_2) \leq F(v) &= \frac{1}{2}A(v) - \frac{\gamma}{4}B(v) - \frac{a}{p}C(v) = \frac{1}{2}t^{\frac{1}{3}}A(u) - \frac{\gamma}{4}t^{\frac{5}{3}}B(u) - \frac{a}{p}tC(u) \\ &< \frac{1}{2}tA(u) - \frac{\gamma}{4}tB(u) - \frac{a}{p}tC(u) = t \left(\frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{p}C(u) \right) \\ &= tF(u) \leq t(m(c_1) + \varepsilon) = \frac{c_2}{c_1}m(c_1) + \frac{c_2}{c_1}\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $c_1 m(c_2) \leq c_2 m(c_1)$. If $m(c_1)$ is reached then we can let $\varepsilon = 0$ in (3.4.3) and thus the strict inequality follows.

iv) Assume first that $0 < c_1 \leq c_2$. Then, by (iii), we have that

$$m(c_1 + c_2) \leq \frac{c_1 + c_2}{c_2}m(c_2) = m(c_2) + \frac{c_1}{c_2}m(c_2) \leq m(c_2) + \frac{c_1}{c_2} \frac{c_2}{c_1}m(c_1) = m(c_1) + m(c_2).$$

If $m(c_1)$ or $m(c_2)$ is reached, then we can use the strict inequality in (iii) and thus the strict inequality follows. The case $0 < c_2 < c_1$ can be treated reversing the role of c_1 and c_2 . \square

Lemma 3.4.3. *Let $(u_n) \subset S(c)$ be any minimizing sequence for $m(c)$. Then, there exist a $\beta_0 > 0$ and a sequence $(y_n) \in \mathbb{R}^3$ such that*

$$\int_{B(y_n, R)} |u_n|^2 dx \geq \beta_0 > 0, \quad \text{for some } R > 0. \quad (3.4.4)$$

Proof. Since F restricted to $S(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see Lemma 3.4.1), the sequence (u_n) is bounded. Now, we assume that (3.4.4) does not hold. By [65, Lemma I.1], we have, for $q \in (2, 6)$, $\|u_n\|_{L^q(\mathbb{R}^3)} \rightarrow 0$, as $n \rightarrow \infty$. This implies that

$$B(u_n) \leq K_1 \|u_n\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 \rightarrow 0,$$

due to (3.2.2). Hence, we obtain

$$F(u_n) = \frac{1}{2}A(u_n) - \frac{\gamma}{4}B(u_n) - \frac{a}{p}C(u_n) \rightarrow \frac{1}{2}A(u_n) - \frac{a}{p}C(u_n) \geq 0,$$

due to $a < 0$. This contradicts $F(u_n) \rightarrow m(c) < 0$, see Lemma 3.4.2(i). \square

Lemma 3.4.4. *Any minimizing sequence $(u_n) \subset S(c)$ for $m(c)$ is, up to translation, strongly convergent in $H^1(\mathbb{R}^3)$.*

Proof. Since F restricted to $S(c)$ is coercive on $H^1(\mathbb{R}^3)$ (see Lemma 3.4.1), the sequence (u_n) is bounded in $H^1(\mathbb{R}^3)$. We deduce from the weak convergence in $H^1(\mathbb{R}^3)$, the local compactness in $L^2(\mathbb{R}^3)$ and Lemma 3.4.3 that

$$u_n(x - y_n) \rightharpoonup u_c \neq 0 \quad \text{in } H^1(\mathbb{R}^3).$$

Our aim is to prove that $w_n(x) := u_n(x - y_n) - u_c(x) \rightarrow 0$ in $H^1(\mathbb{R}^3)$. Now, it is direct to show, using the Brezis-Lieb lemma [24, Theorem 1] for terms A and C , and using [92, Lemma 2.2] or [17, Proposition 3.1] for term B , that any term in F also enjoy a the splitting property, and thus we have

$$F(u_n - u_c) + F(u_c) = F(u_n) + o_n(1),$$

and by the translational invariance, we obtain

$$F(u_n) = F(u_n(x - y_n)) = F(u_n(x - y_n) - u_c(x)) + F(u_c) + o_n(1) = F(w_n) + F(u_c) + o_n(1), \quad (3.4.5)$$

and

$$\begin{aligned} \|u_n\|_{L^2(\mathbb{R}^3)}^2 &= \|u_n(x - y_n)\|_{L^2(\mathbb{R}^3)}^2 = \|u_n(x - y_n) - u_c(x)\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) \\ &= \|w_n\|_{L^2(\mathbb{R}^3)}^2 + \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1). \end{aligned}$$

Thus, we have

$$\|w_n\|_{L^2(\mathbb{R}^3)}^2 = \|u_n\|_{L^2(\mathbb{R}^3)}^2 - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1) = c - \|u_c\|_{L^2(\mathbb{R}^3)}^2 + o_n(1). \quad (3.4.6)$$

We claim that

$$\|w_n\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.4.7)$$

In order to prove this, let us denote $c_1 := \|u_c\|_{L^2(\mathbb{R}^3)}^2 > 0$. By (3.4.6), if we show that $c_1 = c$ then the claim follows. We assume by contradiction that $c_1 < c$. Recoding that $F(u_n) \rightarrow m(c)$, in view of (3.4.5), we have

$$m(c) = F(w_n) + F(u_c) \geq m\left(\|w_n\|_{L^2(\mathbb{R}^3)}^2\right) + F(u_c).$$

Since the map $c \mapsto m(c)$ is continuous (see Lemma 3.4.2(ii) and (3.4.6)), we deduce that

$$m(c) \geq m(c - c_1) + F(u_c). \quad (3.4.8)$$

If $F(u_c) > m(c_1)$, then it follows from Lemma 3.4.2(iv) that

$$m(c) > m(c - c_1) + m(c_1) \geq m(c - c_1 + c_1) = m(c),$$

which is impossible. Hence, we have $F(u_c) = m(c_1)$, namely u_c is global minimizer with respect to c_1 . So, we can using Lemma 3.4.2(iv) with the strict inequality and we deduce from (3.4.8) that

$$m(c) \geq m(c - c_1) + F(u_c) = m(c - c_1) + m(c_1) > m(c - c_1 + c_1) = m(c),$$

which is impossible. Thus, the claim follows and $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$.

At this point, since w_n is a bounded sequence in $H^1(\mathbb{R}^3)$ and by Lemma 3.2.1(i), we have

$$B(w_n) \leq K_H \sqrt{A(w_n)} \|w_n\|_{L^2(\mathbb{R}^3)}^3 \rightarrow 0.$$

Thus, we obtain that

$$\liminf_{n \rightarrow \infty} F(w_n) = \liminf_{n \rightarrow \infty} \left[\frac{1}{2} A(w_n) - \frac{a}{p} C(w_n) \right] \geq 0. \quad (3.4.9)$$

On the other hand, since $\|u_c\|_{L^2(\mathbb{R}^3)}^2 = c$, we deduce from (3.4.5) that

$$F(u_n) = F(w_n) + F(u_c) + o_n(1) \geq F(w_n) + m(c) + o_n(1),$$

and by $F(u_n) \rightarrow m(c)$, we have that

$$\limsup_{n \rightarrow \infty} F(w_n) \leq 0. \quad (3.4.10)$$

Combining (3.4.9) and (3.4.10), we obtain that $F(w_n) \rightarrow 0$. Hence, by (3.4.9) and $a < 0$, we have $A(w_n) \rightarrow 0$ and $C(w_n) \rightarrow 0$. Thus, we get $w_n \rightarrow 0$ in $H^1(\mathbb{R}^3)$. The lemma is completed. \square

Proof of Theorem 3.1.8. The proof follows directly from Lemma 3.4.4 for the convergence of the minimizing sequence and from Lemma 3.2.2 for the sign of the Lagrange parameter. \square

Lemma 3.4.5. *There exist three constants $K_1, K_2, K_3 > 0$ such that if λ_c denote the Lagrange parameter associated to a solution u_c lying at the level $m(c)$ then we have*

$$|m(c)| \leq K_1 c^3 + K_2 c^{2p-3} \quad \text{and} \quad \lambda_c \leq K_3 c^2.$$

Proof. By the fact that $m(c) < 0$ and by using [Lemma 3.2.1\(i\)](#), we get that

$$\begin{aligned} 0 > m(c) = F(u_c) &= \frac{1}{2}A(u_c) - \frac{\gamma}{4}B(u_c) - \frac{a}{p}C(u_c) \\ &\geq \frac{1}{2}A(u_c) - \frac{\gamma}{4}K_H\sqrt{A(u_c)}c^{\frac{3}{2}} - \frac{a}{p}C(u_c) \geq \frac{1}{2}A(u_c) - \frac{\gamma}{4}K_H\sqrt{A(u_c)}c^{\frac{3}{2}}, \end{aligned}$$

due to our assumption $\gamma > 0$ and $a < 0$. This implies that

$$\sqrt{A(u_c)} < \frac{\gamma K_H}{2}c^{\frac{3}{2}}.$$

Therefore, using again [Lemma 3.2.1](#), we obtain that

$$\begin{aligned} |m(c)| = |F(u_c)| &= \left| \frac{1}{2}A(u_c) - \frac{\gamma}{4}B(u_c) - \frac{a}{p}C(u_c) \right| \leq \frac{1}{2}A(u_c) + \frac{\gamma}{4}B(u_c) - \frac{a}{p}C(u_c) \\ &\leq \frac{1}{2}A(u_c) + \frac{\gamma K_H}{4}\sqrt{A(u_c)}c^{\frac{3}{2}} - \frac{aK_{GN}}{p}[A(u_c)]^{\frac{\sigma}{2}}c^{\frac{6-p}{4}} \\ &\leq \frac{\gamma^2 K_H^2}{8}c^3 - \frac{aK_{GN}}{p} \left[\frac{\gamma K_H}{2} \right]^\sigma c^{\frac{3\sigma}{2}}c^{\frac{6-p}{4}} := K_1c^3 + K_2c^{2p-3}. \end{aligned}$$

We deduce from [\(3.2.6\)](#) that

$$\begin{aligned} \lambda_c &= \frac{6-p}{3p-6} \frac{1}{c}A(u_c) + \frac{\gamma(5p-12)}{2(3p-6)} \frac{1}{c}B(u_c) \leq \frac{6-p}{3p-6} \frac{1}{c}A(u_c) + \frac{\gamma(5p-12)K_H}{2(3p-6)} \frac{1}{c}\sqrt{A(u_c)}c^{\frac{3}{2}} \\ &\leq \frac{6-p}{3p-6} \frac{1}{c} \frac{\gamma^2 K_H^2}{4}c^3 + \frac{\gamma(5p-12)K_H}{2(3p-6)} \frac{1}{c} \frac{\gamma K_H}{2}c^{\frac{3}{2}}c^{\frac{3}{2}} := K_3c^2. \end{aligned}$$

Thus, the lemma is proved. \square

3.5 The case $\gamma < 0$, $a > 0$ and $p = 6$

Throughout this section, we assume that $\gamma < 0$, $a > 0$ and $p = 6$. To prove the non-existence of the positive solution to [\(3.1.2\)](#), we first recall a Liouville-type result, see [\[7, Theorem 2.1\]](#),

Proposition 3.5.1. *Assume that $N \geq 3$ and the nonlinearity $f : (0, \infty) \mapsto (0, \infty)$ is continuous and satisfies*

$$\liminf_{s \rightarrow 0} s^{-\frac{N}{N-2}} f(s) > 0.$$

Then the differential inequality $-\Delta u \geq f(u)$ has no positive solution in any exterior domain of \mathbb{R}^N .

Proof of Theorem 3.1.9. Let $u \in H^1(\mathbb{R}^3)$ be a non-trivial solution to [\(3.1.2\)](#). By [Lemma 3.2.2](#), we have $\lambda < 0$ and $Q(u) = 0$. Hence,

$$aC(u) = A(u) - \frac{\gamma}{4}B(u) > A(u)$$

and using [Lemma 3.2.1\(ii\)](#), we obtain that

$$A(u) < aC(u) \leq aK_{GN}[A(u)]^3.$$

This implies that

$$A(u) > \sqrt{\frac{1}{aK_{GN}}}.$$

Using again $Q(u) = 0$ we have that

$$F(u) = \frac{1}{2}A(u) - \frac{\gamma}{4}B(u) - \frac{a}{6}C(u) = \frac{5a}{6}C(u) - \frac{1}{2}A(u) > \frac{1}{3}A(u) > \frac{1}{3\sqrt{aK_{GN}}},$$

proving point (i). To prove point (ii), we assume by contradiction that there exists a positive solution $u \in H^1(\mathbb{R}^3)$ to (3.1.2). Then, by point (i), the associated Lagrange multiplier λ is strictly negative. In view of (3.2.8), there exists $R_0 > 0$ large enough such that

$$(|x|^{-1} * |u|^2)(x) \leq -\frac{\lambda}{2\gamma} \quad \text{for } |x| > R_0.$$

Therefore, we get that

$$\begin{aligned} -\Delta u(x) &= \left(-\lambda + \gamma(|x|^{-1} * |u|^2)(x) + a|u(x)|^4\right)u(x) \\ &\geq \left(-\lambda + \gamma(|x|^{-1} * |u|^2)(x)\right)u(x) \geq -\frac{\lambda}{2}u(x) \quad \text{for } |x| > R_0. \end{aligned}$$

By applying Proposition 3.5.1 with $f(s) = -\frac{\lambda}{2}s$, we obtain a contradiction, and thus point (ii) holds. \square

Remark 3.5.2. In [81, Theorem 1.2], the author considers the equation

$$-\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 \quad \text{in } \mathbb{R}^N, \quad (3.5.1)$$

with $N \geq 3$, $2 < q < 2^*$ and $\mu < 0$. If $u \in H^1(\mathbb{R}^N)$ is a non-trivial solution to (3.5.1) then by [81, Theorem 1.2], the associated Lagrange multiplier λ is positive and following the arguments in [81, Proof of Theorem 1.2], one obtains that

$$-\Delta u \geq \frac{\lambda}{2}u \quad \text{for } |x| > R_1,$$

with $R_1 > 0$ large enough. Hence, by applying Proposition 3.5.1, we see that (3.5.1) has no positive solution $u \in H^1(\mathbb{R}^N)$ for all $N \geq 3$, improving slightly the conclusions of [81, Theorem 1.2]. Actually, borrowing an observation from [18], the non-existence results of [81, Theorem 1.2] can be further extended by showing that (3.5.1) has no non-trivial radial solutions in $H^1(\mathbb{R}^N)$ when $N \geq 3$ and $q > 2 + \frac{2}{N-1}$. Indeed, if $u \in H^1(\mathbb{R}^N)$ is a radial function by [18, Radial Lemma A.II], there exist constants $C > 0$ and $R_2 > 0$ such that

$$|u(x)| \leq C|x|^{-\frac{N-1}{2}} \quad \text{for } |x| > R_2.$$

Setting $V(x) = -\mu|u(x)|^{q-2} - |u(x)|^{2^*-2}$, we obtain that any radial solution $u \in H^1(\mathbb{R}^N)$ satisfies

$$-\Delta u(x) + V(x)u(x) = \lambda u(x), \quad (3.5.2)$$

where, since $q > 2 + \frac{2}{N-1}$,

$$\lim_{|x| \rightarrow \infty} |x| |V(x)| \leq \lim_{|x| \rightarrow \infty} \left[-\mu C |x|^{-\frac{(N-1)(q-2)}{2}+1} + C |x|^{-\frac{(N-1)(2^*-2)}{2}+1} \right] = 0.$$

Then (3.5.2) has no solution in view of Kato's result [52, page 404], also see [79] which states that Schrödinger operator $H = -\Delta + p(x)$ has no positive eigenvalue with an L^2 -eigenfunction if $p(x) = o(|x|^{-1})$.

Remark 3.5.3. One may wonder if a non-existence result for radial solutions also holds for (3.1.2) under the assumptions of Theorem 3.1.9. The difficulty one faces is that, for any $u \in H^1(\mathbb{R}^N)$, $(|x|^{-1} * |u|^2)(x) \geq C|x|^{-1}$ for $|x| > R$ for some $C, R > 0$ (see [15] or [71, Appendix A.4]). Thus, the result of Kato used in Remark 3.5.2 cannot be directly applied and the non-existence of radial solutions to (3.1.2) when $\gamma < 0$, $a > 0$ and $p = 6$ is an open problem.

Chapter 4

Concluding remarks and some open problems

In this chapter, we present some concluding remarks about the two equations considered in this thesis and also we propose some open problems.

4.1 On the Sobolev critical nonlinear Schrödinger equation

Concerning the Sobolev critical nonlinear Schrödinger equation with mixed power nonlinearities

$$\begin{cases} -\Delta u - \lambda u - \mu|u|^{q-2}u - |u|^{2^*-2}u = 0 & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)}^2 = c, \end{cases} \quad (4.1.1)$$

where $N \geq 3$, $\mu \in \mathbb{R}$, $2 < q < 2^*$, we present below some concluding remarks and also we propose open problems.

- (i) Let us begin with a remark about the existence of the second solution of mountain pass type in the case $2 < q < 2 + 4/N$. We proved in [Chapter 2](#) the existence of a second solution lying at mountain pass level for $\mu > 0$ small enough and $N \geq 4$. Moreover, it is not a ground state and the associated standing wave is strongly unstable. A key step in the proof of the existence of such a solution is that one needs a precise upper estimate of the associated mountain pass level. The need to obtain, in a problem involving a Sobolev critical term, a sharp estimate on some minimax levels is known since the pioneering work of Brezis-Nirenberg [\[25\]](#) and the usual way to derive such strict upper bound is through the use of testing functions. We constructed test functions that could be viewed as the sum of a truncated extremal function of the Sobolev inequality on \mathbb{R}^N translated far away from the origin. This choice of testing functions was sufficient to prove our strict inequality when $N \geq 4$ but we missed it in the case $N = 3$. Note that our approach proved nevertheless adequate to deal with the energy critical half-wave equation that was studied in [\[67\]](#). Very recently, in [\[89\]](#) the authors introduced an alternative choice of testing functions which allowed to treat, in a unified way, the case $N = 3$ and $N \geq 4$ for [\(4.1.1\)](#). The strategy in [\[89\]](#), recording of the one introduced by G. Tarantello in [\[85\]](#), is on the contrary, to locate the extremal functions where the ground state solution takes its greater values (the origin thus). The idea behind the

proof is that the interaction decreases the value of the *Energy* with respect to the case where the supports would be disjoint. The approach in [89] was applied in some different equations, see for example [59, 66] and the Schrödinger-Poisson-Slater equation studied in Chapter 3. However, in [60] our approach is more useful in some cases.

- (ii) In the case where $2 < q < 2 + 4/N$, it was proved in [81] that for and $\mu > 0$ small, (4.1.1) has a ground state as a local minimizer, however the stability of the associated standing wave is unknown. In Chapter 2, we develop a new argument to prove the existence of ground state for $\mu > 0$ small, which is useful to prove that the associated standing wave is orbital stable. To prove the stability, we have only established the global existence of solutions for initial data *close* to \mathcal{M}_c , the set containing all ground states. We believe it would be interesting to inquire if the global existence holds *away* from \mathcal{M}_c . If so, investigating the long time behavior of these solutions would be worth to. Our guess is that these solutions evolve toward the sum of an element of \mathcal{M}_c and a part which scatter. However, so far nothing is known in that direction, see Remark 2.1.7 for more detail.
- (iii) The above existence results hold for $\mu > 0$ small enough (equivalent $c > 0$ small enough), what happens if μ is large? If $2 < q < 2 + 4/N$, it was proved that (4.1.1) has no ground state for μ large enough in [88], where the authors suggested a new method using the relations between fixed-frequency solutions and normalized solutions. In [5], the authors established the existence of many solutions of mountain pass type for μ large and $2 < q < 2 + 4/N$ by using a minimax theorem found in [49] and truncation argument made in [36]. It was proved in [89] that if $q = 2 + 4/N$ then (4.1.1) has no ground state for μ large enough. In the case where $2 + 4/N < q < 2^*$, (4.1.1) has a ground state for all $\mu > 0$, see [89] and [58] with different proof, in particular, different choices of test functions in proving the upper estimate of the associated mountain pass level. Does (4.1.1) have many solutions of mountain pass type for $2 + 4/N \leq q < 2^*$? Since in this case, there exists a ground state for any $\mu > 0$, a second solution if exists must have an energy level greater than the ground state. Thus, it is difficult to prove the strict upper bound of the energy level of the second solution, and we conjecture the negative answer to the above question. However, so far nothing is known in that direction.
- (iv) In the case where $\mu < 0$, it was proved in [81] that for $2 < q < 2^*$, (4.1.1) has no positive solution $u \in H^1(\mathbb{R}^N)$ if $N = 3, 4$ or if $N \geq 5$ under the additional assumption $u \in L^p(\mathbb{R}^N)$ for some $p \in (0, \frac{N}{N-2}]$. We improve the above result showing that (4.1.1) has no positive solution in $H^1(\mathbb{R}^N)$ for all $N \geq 3$ and no non-trivial radial solution for $N \geq 3$ and $q > 2 + \frac{2}{N-1}$, see Remark 3.5.2 for more detail. If $2 < q < 2 + \frac{2}{N-1}$, the non-existence of radial solutions is an open problem.

Now, let us mention a non-autonomous equation on all the space \mathbb{R}^N . Recently, in [10] and [70], the following equation was studied

$$\begin{cases} -\Delta u + V(x)u - \lambda u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)}^2 = c, \end{cases} \quad (4.1.2)$$

where V is a fixed potential, $f(u) = |u|^{p-2}u$ with $2 + 4/N < p < 2^*$. Besides, the general non-linearity of the mass sub-critical case was studied in [44]. Under some assumptions on the

potential V and the mass constraint $c > 0$, some existence results of solutions to (4.1.2) are obtained. The main difficulty is the existence of the potential term $V(x)$. One loses the information on the geometric structure of the associated functional since the scaling argument can not apply to $V(x)$. The Pohozaev identity is complex and not really useful. The compactness issue is complex and becomes more complex if one assumes that V is not radial. To overcome these difficulties, the authors of the above-mentioned papers put some *strong assumptions* on V which help them can apply the technical arguments of autonomous problems. There is no really new idea to treat the potential term $V(x)$. Moreover, to our knowledge, the case where $f(u)$ is critical growth has not been considered yet. This is the reason that we are now interested in the non-autonomous problem, more specific, we consider (4.1.2) with mixed power nonlinearities $f(u) = \mu|u|^{q-2}q + |u|^{p-2}p$ with $2 < q < p \leq 2^*$. We expect that our experience in mixed power nonlinearities can be useful to treat this problem.

4.2 On the Schrödinger-Poisson-Slater equation

Concerning the Schrödinger-Poisson-Slater equation

$$\begin{cases} -\Delta u + \lambda u - \gamma(|x|^{-1} * |u|^2)u = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \|u\|_{L^2(\mathbb{R}^3)}^2 = c, \end{cases} \quad (4.2.1)$$

with $\gamma \in \mathbb{R}$, $a \in \mathbb{R}$, $p \in (\frac{10}{3}, 6]$ and $f(u) := a|u|^{p-2}u$, we propose the following open questions:

- (i) We proved in [Chapter 3](#) that there exists $c_1 > 0$ such that (4.2.1) has two solutions for any $0 < c < c_1$ in the case $\gamma > 0$ and $a > 0$. However, we don't know if the value $c_1 > 0$ is optimal and what happens if $c \geq c_1$. We conjecture that there exists a $c_0 \geq c_1 > 0$ such that one solution exists when $c = c_0$ and that, at least positive solutions, do not exist when $c > c_0$. Nevertheless, so far nothing is known in that direction, see [Remark 3.1.3](#) for more detail.
- (ii) In the case $p = 6$, $\gamma < 0$ and $a > 0$, it was proved that (4.2.1) has no positive solution in $H^1(\mathbb{R}^3)$ for any $c > 0$. We also expect the non-existence of non-trivial radial solutions. The difficulty one faces is that, for any $u \in H^1(\mathbb{R}^N)$, $(|x|^{-1} * |u|^2)(x) \geq C|x|^{-1}$ for $|x| > R$ for some $C, R > 0$. Thus, the result of Kato [[52](#)] cannot be directly applied and the non-existence of radial solutions is an open problem, see [Remark 3.5.3](#) for more detail.
- (iii) Let us mention the mixed nonlinear case $f(u) = a|u|^{q-2}u + \mu|u|^{p-2}u$ with $2 < q < p \leq 2^*$. The main difficult is that there are three nonlinear terms $(|x|^{-1} * |u|^2)u$, $|u|^{q-2}u$ and $|u|^{p-2}u$. Firstly, the geometric structure of the associated functional is more complex. Secondly, in the frame of this the thesis, namely for two nonlinear terms, we usually use the properties of the Pohozaev manifold keep only one nonlinear term, which helps us in some estimates. Thus, it seems difficult to apply in the mixed nonlinear case and the study of this case is an open problem.

Now, we introduce a general version of (4.2.1) which is usually called the Choquard equation

$$\begin{cases} -\Delta u + \lambda u - \gamma(I_\alpha * |u|^p)|u|^{p-2}u - \mu|u|^{q-2}u = 0 & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)}^2 = c, \end{cases} \quad (4.2.2)$$

where $N \geq 1$, $\alpha \in (0, N)$, $I_\alpha = \frac{C}{|x|^{N-\alpha}}$ with $C > 0$, $\gamma, \mu \in \mathbb{R}$, $2_\alpha \leq p \leq 2_\alpha^*$ and $2 < q \leq 2^*$ with notations: 2_α and 2_α^* are lower and upper critical exponents respectively that come from the Hardy-Littlewood-Sobolev inequality (see [63, Chapter 4]), namely

$$2_\alpha := \frac{N + \alpha}{N}, \quad 2_\alpha^* := \begin{cases} +\infty & \text{if } N = 1, 2, \\ \frac{N + \alpha}{N - 2} & \text{if } N \geq 3, \end{cases}$$

and L^2 -critical exponents $\bar{p} := \frac{N + \alpha + 2}{N}$, $\bar{q} := 2 + \frac{4}{N}$. Recently, (4.2.2) was studied in [59–61, 90, 91] for some range of parameters; and several the nonexistence, existence, multiple and stability results are obtained. We now present below some open problems.

In [91], the authors considered (4.2.2) with $N \geq 2$, $2_\alpha < p < 2_\alpha^*$ and $2 < q < 2^*$. Depending on the range of parameters γ , μ , p and q , the corresponding functionals are bounded or unbounded. Hence, many cases were considered in the paper and there are also some open cases proposed in [91, Remark 1.9].

Let us now focus on the two special cases: the lower and upper critical cases. The lower critical case $p = 2_\alpha$ was studied in [90] with the assumptions $\gamma > 0$ and $\mu > 0$. It was proved that (4.2.2) has a ground state if $N \geq 2$, $2 < q < \bar{q}$ and $\mu > 0$ large; if $N \geq 2$, $\bar{q} < q < 2^*$, $c > 0$ small and $\mu > 0$ large; and if $N \geq 3$, $q = 2^*$ $c > 0$ small, $\gamma > 0$ large and $\mu > 0$ large. In the case $N \geq 2$, $q = \bar{q}$, there is no solution if $\mu > 0$ is small. The authors also proposed some open problems in [90, Remark 1.7] concerning the stability issue of these solutions and the existence of solutions if $\mu > 0$ small. Recently, the authors in [61] treated the case $2 < q < \bar{q}$ where the existence of a ground state holds for any $\mu > 0$.

The upper critical case $p = 2_\alpha^*$ was studied in [59, 60] with the assumptions $\gamma = 1$, $\mu > 0$ and $N \geq 3$. It was proved in [60] that (4.2.2) has two solutions: one ground state and one mountain pass type if $2 < q < \bar{q}$ and $\mu > 0$ small; and the stability issue was also considered. In [59], the author proved the existence of a ground state if $q = \bar{q}$ and $\mu > 0$ small; and if $\bar{q} < q < 2^*$. In the case $q = \bar{q}$ and μ large, there is no ground state. If μ is large and $2 < q < \bar{q}$, the existence of solutions to (4.2.2) is unknown.

We observe that if $p = 2_\alpha$ or $p = 2_\alpha^*$ then the case $\gamma > 0$ and $\mu > 0$ in (4.2.2) has been the most studied so far. Other cases where $\gamma < 0$ or $\mu < 0$ are open problems. Especially, the Sobolev critical case $q = 2^*$ only studied in [90] with assumptions $N \geq 3$, $p = 2_\alpha$, $c > 0$ small, $\gamma > 0$ large and $\mu > 0$ large. So, it remains many open problems in the Sobolev critical case $q = 2^*$. We expect that our technical arguments in Chapter 3 which were applied only partially in [90] can be used to deal with these problems.

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Résumé

Dans cette thèse, nous considérons deux types d'équations de Schrödinger non linéaires (NLS), à savoir une classe d'équations de Schrödinger non linéaire avec une non linéarité de type *mixed powers* sur \mathbb{R}^N et une classe d'équations non linéaires de Schrödinger-Poisson-Slater sur \mathbb{R}^3 . Ces deux types de NLS apparaissent dans divers modèles mathématiques et physiques et ont attiré beaucoup d'attention ces dernières années.

Du point de vue physique, puisque, en plus d'être une quantité conservée pour l'équation de l'évolution, la masse a souvent une signification physique claire; par exemple, elle représente l'alimentation électrique en optique non linéaire, ou le nombre total d'atomes dans la condensation de Bose-Einstein, etc., nous nous concentrons sur l'étude des solutions ayant une masse prescrite, à savoir les solutions normalisées. Des questions d'existence, de multiplicité et de stabilité de ces solutions sont examinées dans cette thèse. Nous nous occupons à la fois de cas sous-critiques de Sobolev et de cas critiques de Sobolev. Une attention particulière est accordée aux cas critiques de Sobolev dans lesquels de nombreux problèmes restent ouverts. Puisque les solutions normalisées sont obtenues comme points critiques, sous contrainte, d'une fonctionnelle, les principaux ingrédients de nos preuves sont variationnels.

Mots clefs: Equations de Schrödinger, équations de Schrödinger-Poisson-Slater, exposant critique de Sobolev, masse prescrite, solution normalisée, multiplicité de solutions, état fondamental, stabilité orbitale, instabilité forte par blow-up, minimiseur local ou global, point selle au niveau du col, méthodes variationnelles, identité de type Pohozaev.

Abstract

In this thesis, we consider two types of nonlinear Schrödinger equations (NLS), namely a class of nonlinear Schrödinger equations with mixed power nonlinearities in \mathbb{R}^N and a class of Schrödinger-Poisson-Slater equations in \mathbb{R}^3 . These two types of NLS arise in various mathematical and physical models and have drawn wide attention in recent years.

From the physical point of view, since, in addition to being a conserved quantity for the evolution equation, the mass often has a clear physical meaning; for instance, it represents the power supply in nonlinear optics, or the total number of atoms in Bose-Einstein condensation, etc, we focus on studying solutions having prescribed mass, namely normalized solutions. The existence, multiplicity, and stability issues of such solutions are considered in this thesis. We deal with both Sobolev sub-critical and Sobolev critical cases. Particular attention is paid to Sobolev critical cases in which many open problems remain. Since normalized solutions are found as critical points of an associated functional on a constraint, the main ingredients of our proofs are variational methods.

Keywords: Nonlinear Schrödinger equation, Schrödinger-Poisson-Slater equation, Sobolev critical exponent, prescribed mass, normalized solution, multiplicity of solutions, ground states, orbital stability, strong instability by blow-up, local or global minimizer, saddle point lying at mountain pass level, variational methods, Pohozaev type identity.

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