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THÈSE DE DOCTORAT
Discipline : Mathématiques

présentée par

Jérémy Pierard de Maujouy

**Géométrie de Cartan généralisée et
théories des champs**

**Field theory and
generalised Cartan geometry**

dirigée par Frédéric HÉLEIN

Soutenue le 13 Décembre 2022 devant le jury composé de :

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Résumé

Nous présentons la géométrie de Cartan généralisée qui est une structure géométrique localement isomorphe au fibré principal associé à une géométrie de Cartan sur une variété différentiable et proposons de formuler les théories de champs dans le cadre de la géométrie de Cartan généralisée.

Nous proposons une interprétation du modèle de Hélein-Vey pour la gravitation d'Einstein-Cartan comme une géométrie de Cartan généralisée résultant d'équations variationnelles (et vérifiant des équations de champs physiques) et abordons le modèle depuis une perspective géométrique originale. Nous expliquons notamment le “mécanisme d'annulation” crucial pour le modèle comme une propriété de certaines composantes des équations d'Euler-Lagrange. Cette approche nous permet d'élargir la théorie afin de coupler des spineurs de Dirac à l'espace-temps d'Hélein-Vey-Einstein-Cartan modélisé par une géométrie de Cartan généralisée. Les solutions satisfont les équations d'Einstein-Cartan-Dirac sur un espace-temps sous-jacent qui peut être une variété différentielle mais peut également présenter des singularités, par exemple de type orbifold.

Nous étudions également la question de savoir si une géométrie de Cartan généralisée correspond à une géométrie de Cartan au sens standard et par cela corrigeons une omission dans le traitement d'Hélein et Vey de leur modèle. Il s'agit de l'occasion de fournir un nouveau traitement à la question de l'intégration de l'action d'une algèbre de Lie en une action de groupe. Nous abordons également le problème de la structure différentielle de l'espace quotient et de savoir si l'application quotient définit un fibré principal. Nous définissons le “groupoïde de pseudo-action locale” qui est un groupoïde de Lie encodant toute la structure globale associée à une action d'algèbre de Lie sur une variété différentielle et établissons un résultat d'équivalence avec la construction de la complétion telle que définie par Kamber et Michor.

Mots-clés

Géométrie de Cartan généralisée, Gravitation d'Einstein-Cartan, Action d'algèbres de Lie

Generalised Cartan geometry and Field Theory

Abstract

We introduce generalised Cartan geometry as a geometrical structure which is locally isomorphic to the principal bundle associated with a Cartan geometry on a manifold and make the proposal to formulate general field theories in the framework of generalised Cartan geometry.

We give an interpretation of Hélein-Vey's model for Einstein-Cartan gravitation [HV16] as a dynamically generated generalised Cartan geometry (which furthermore satisfies physical field equations) and give a new geometrical perspective of the underlying mechanisms of the model. In particular, we explain the crucial "cancellation mechanism" as a property of suitable components of the Euler-Lagrange equation. This allows us to formulate an extension of the theory which couples Dirac spinors to the spacetime geometry modelled as a generalised Cartan geometry. The solutions satisfy Einstein-Cartan-Dirac equations on an underlying spacetime which can be a smooth manifold but may also present singularities, for example of orbifold type.

We study in which case a generalised Cartan geometry comes from a honest Cartan geometry and thereby fix an omission in the original treatment of Hélein and Vey. This is the occasion to give a new treatment to the question of integrating the action of a Lie algebra into a Lie group action. We also investigate when the quotient space is a smooth manifold and the quotient map defines a principal bundle. We define the so-called "local pseudo-action groupoid" which is a Lie groupoid encoding all the global structure associated to the action of a Lie algebra on a smooth manifold, and prove a result of equivalence with the completion construction such as defined by Kamber and Michor.

Keywords

Generalised Cartan geometry, Einstein-Cartan gravitation, Action of Lie algebra

Introduction (Français)

Contexte

La théorie de la Relativité Générale est une des avancées majeures du XXe siècle concernant notre compréhension de l'univers. Son influence est difficile à quantifier tant elle est omniprésente. Les conséquences les plus évidentes sont dans les domaines de l'astronomie et de la cosmologie : elle supprime la théorie de la gravitation de Newton qui était restée détrônée depuis plus de deux siècles. Elle prédit notamment l'existence et la formation de trous noirs, dont les conséquences sont aujourd'hui bien supportées par les observations.

La Relativité Générale a également eu, et a toujours, une influence considérable dans le domaine des mathématiques. Le domaine de la géométrie différentielle aurait-il connu un tel essor au cours du XXe siècle s'il n'offrait pas un cadre mathématique dans lequel s'inscrit naturellement la théorie ?

Il y a d'autres retombées plus concrètes, notamment dans le domaine de la technologie grand public : le système des GPS, aujourd'hui accessible en quelques clics à tout un chacun, est basé sur les prédictions de la Relativité Générale quant à l'écoulement relatif du temps.

Plus loin encore, la théorie de la Relativité Générale a de profondes implications philosophiques. L'absence d'une notion absolue de temps, déjà présente dans la théorie de la relativité restreinte, est une importante remise en cause de l'idée intuitive de temps.

Aujourd'hui, la théorie de la Relativité Générale fait pourtant presque partie de l'imaginaire collectif. Une image très souvent présentée est celle d'un espace bidimensionnel semblable à une nappe sur lequel un astre posé tel une boule de pétanque forme un creux. Le physicien Albert Einstein qui est la figure principale derrière la théorie, est un personnage connu du grand public. La notion d'espace-temps et le phénomène des trous noirs sont chose commune dans la littérature, le cinéma et le jeu de science-fiction.

Pourtant, la théorie telle qu'elle est comprise aujourd'hui est en conflit avec ce qui est généralement présenté comme l'autre avancée majeure de la physique du XXe siècle : la physique quantique. Pour cette raison, bien que la théorie soit forte de plus d'un siècle d'approfondissements et de confirmations, une partie de la recherche aujourd'hui s'intéresse à des approches alternatives à la théorie d'Einstein.

L'interaction gravitationnelle forme avec les interactions électromagnétique, forte et faible les quatre interactions fondamentales qui régissent notre compréhension actuelle du monde physique. Les trois dernières sont comprises dans comme des théories dites "de jauge", associées à des groupes de symétrie, et formalisées dans le cadre de la physique quantique. En dépit de la tension présente avec la physique quantique, il existe une approche de la théorie de la Relativité Générale qui la présente comme une théorie de jauge associée au groupe de Poincaré, groupe de symétrie de l'espace-temps plat et dépourvu de matière [Heh14]. Là où les autres groupes de jauge agissent sur des degrés de liberté "internes" aux particules, l'action du groupe de Poincaré ne peut cependant être découplée de l'espace-temps.

Dans [Tol78], Toller propose une approche à la théorie de la Relativité qui ne se formule non pas sur l'espace-temps, mais sur *l'espace des observateurs de l'espace-temps*, modélisé par un fibré des repères au dessus de l'espace-temps. Il s'agit d'un espace géométrique avec plus de dimensions que l'espace-temps. Les dimensions supérieures sont associées à l'action du groupe de Lorentz qui est un sous-groupe du groupe de Poincaré. À la même période, Regge et Ne'eman discutent de formulations de la gravitation et de la super-gravitation sur le groupe de Poincaré [NR78c]. Ces deux idées ne sont pas sans rappeler la proposition de Lurçat de formuler la théorie quantique des champs des spineurs sur le fibré des repères de l'espace de Minkowski, modélisé par le groupe de Poincaré, plutôt que sur l'espace de Minkowski [Lur64]. Il s'agit d'une tentative de donner au spin, caractéristique intrinsèque de la matière à l'échelle quantique, une modélisation de nature géométrique.

Cette approche fut reprise plus récemment par Hélein et Vey dans [HV16]. Ils formulent un modèle de théorie des champs défini sur un espace de dimension 10 tel que les solutions munissent cet espace de la structure de fibré des repères au dessus d'un espace-temps qui de plus vérifie les équations de champs de la Relativité Générale d'Einstein. La théorie développée dans cet article forme le point initial de la thèse présente.

Le modèle d'Hélein-Vey

Le modèle d'Hélein-Vey fut construit dans le formalisme multisymplectique et repose sur plusieurs faits remarquables. Le premier est qu'en ajoutant au Lagrangien d'Hilbert-Einstein d'autres termes similaires, il est possible d'obtenir des équations variationnelles dont les solutions fournissent à une variété différentielle une structure aussi forte que celle de fibré des repères (à plusieurs réserves près, dont nous discutons plus loin). Le second est que les contributions de ces termes dans les équations de champs disparaissent grâce à un subtil mécanisme qui permet de les rassembler dans un terme exact qui disparaît par intégration.

Le modèle est construit sur une variété de dimension 10, qui doit devenir un espace de repères pseudo-orthonormés au dessus d'un espace-temps de dimension 4. On distingue deux champs :

- Un corepère à valeur dans l'algèbre de Poincaré $\mathfrak{p} \simeq \mathfrak{so}_{1,3} \ltimes \mathbb{R}^{1,3}$. La composante $\mathbb{R}^{1,3}$ est un candidat pour une forme de soudure et la composante $\mathfrak{so}_{1,3}$ un candidat pour une forme de connexion.
- Une 8-forme à valeur dans \mathfrak{p}^* dont une composante est fixée (et reproduit le terme d'Einstein-Hilbert). Les composantes libres sont en un sens des variables conjuguées au corepère et peuvent s'interpréter comme de multiplicateurs de Lagrange.

On note $Q \rightarrow \mathcal{P}$ le fibré dont ces champs (rassemblés en un) sont des sections. Il y a sur Q une 1-forme canonique à valeur dans \mathfrak{p} que l'on note λ ainsi qu'une 8-forme canonique à valeur dans \mathfrak{p}^* , que l'on note p . Le modèle est décrit par la forme de Poincaré-Cartan (un intermédiaire entre un Lagrangien et un Hamiltonien) Θ présentée dans l'encadré 1.

Le premier objet de cette thèse est d'adapter le modèle d'Hélein-Vey afin de modéliser de la matière couplée à l'espace-temps : en l'occurrence des spineurs de Dirac. Il s'avère que le modèle d'Hélein-Vey est plus proche d'une théorie alternative à la Relativité Générale appelée *théorie d'Einstein-Cartan* [Tra06]. Alors qu'en Relativité Générale la géométrie de l'espace-temps est déterminée par une *métrique* qui joue un rôle de potentiel gravitationnel, la théorie d'Einstein-Cartan utilise à la fois une connexion linéaire sur l'espace tangent à l'espace-temps, qui modélise le champ gravitationnel, ainsi qu'une métrique compatible, qui détermine entre autres la structure causale de l'espace-temps. Il y a un degré de liberté supplémentaire : la *torsion* de la connexion considérée. Cette torsion est notamment couplée à la "densité de spin" de la matière présente, et interagit avec les spineurs de Dirac.

$e : T\mathcal{E} \xrightarrow{\sim} \mathcal{E} \times \mathbb{R}^{1,3}$ $\omega : T\mathcal{E} \rightarrow \mathfrak{so}_{1,3}$ $\mathcal{L}[e, \omega] = \eta^{bb'} e_{ab'}^{(2)} \wedge \left(d\omega + \frac{1}{2} [\omega \wedge \omega] \right)_b^a$	$\mathfrak{p} \simeq \mathfrak{so}_{1,3} \ltimes \mathbb{R}^{1,3}$ $\dim \mathcal{P} = \dim \mathfrak{p} = 10$ $Q \hookrightarrow \Lambda^8 T^* \mathcal{P} \otimes \mathfrak{p}^* \times_{\mathcal{P}} T^* \mathcal{P} \otimes \mathfrak{p}$ $p : Q \rightarrow \Lambda^8 T^* \mathcal{P} \otimes \mathfrak{p}^*$ $\lambda : Q \rightarrow T^* \mathcal{P} \otimes \mathfrak{p}$ $\Theta = p \wedge \left(d\lambda + \frac{1}{2} [\lambda \wedge \lambda] \right)$
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Le Lagrangien d'Einstein-Hilbert sur un espace-temps \mathcal{E} (dans le formalisme des tétrades)

Figure 1: La forme de Poincaré-Cartan de Hélein-Vey (suivant nos notations)

La géométrie de la Relativité Générale est convenablement modélisée dans le cadre de la *géométrie de Cartan* [Wis10]. Il s'agit d'un cadre général dans lequel une variété différentielle est équipée d'une structure permettant de la comparer infinitésimalement à une géométrie de référence qui est modélisable par un espace homogène G/H . Dans le cadre de la Relativité Générale, cette géométrie de référence est celle de l'espace de Minkowski, qui est le modèle pour un espace-temps absolument vide. La géométrie de Cartan fournit un cadre permettant naturellement de travailler avec de la torsion.

Ce travail mena à étudier en détail les mécanismes à l'œuvre dans [HV16]. Il est possible d'interpréter les termes supplémentaires du modèle d'Hélein-Vey comme des termes impliquant des multiplicateurs de Lagrange ajoutant aux équations variationnelles des équations de contrainte. Puisque les spineurs sur l'espace-temps se représentent naturellement comme des fonctions définies sur l'espace des repères et qui vérifient certaines équations d'invariance, cette perspective permet d'ajouter simplement à la théorie des champs modélisant des spineurs.

Par ailleurs, le mécanisme permettant d'isoler et annuler la contribution des multiplicateurs de Lagrange aux équations d'Einstein reste énigmatique dans l'article d'Hélein-Vey. Il est la conclusion d'une série de calculs conséquents peu éclairants sur la situation géométrique. En particulier ils comportent un changement de variable relativement délicat à effectuer mais indispensable afin d'obtenir un terme exact. Dans cette thèse nous proposons une approche plus géométrique qui clarifient l'argument. Le changement de variable en question, effectué sur les équations d'Euler-Lagrange, peut être interprété comme les équations d'Euler-Lagrange associées à des variations des champs d'une forme précise. Nous élucidons cette forme et expliquons les raisons pour lesquelles cela permet d'aboutir à des termes exacts.

Il s'avère cependant que la structure géométrique définie par les solutions du modèle d'Hélein-Vey ne définit un véritable fibré des repères que sous certaines hypothèses de nature topologiques globales. Il s'agit d'un défaut dans l'argument de [HV16] identifié pendant l'étude de l'article. La raison essentielle est la suivante : les équations variationnelles sont de nature purement locale, tandis que la structure d'un fibré de repères est en partie globale. En l'occurrence, il y a une action de groupe telle que l'espace des orbites soit une variété différentielle (l'espace-temps) et que l'espace initial définisse par dessus un fibré principal (localement trivial).

De par leur nature locale, les équations variationnelles ne peuvent fournir qu'une action

d'algèbre de Lie. Suivent alors les questions de l'intégration en une action de groupe de Lie puis de la construction de l'espace des orbites et de sa régularité. Seulement alors devient-il possible d'affirmer que l'on a bien un espace de fibrés au dessus de l'espace-temps.

Si cela semble de prime abord une limite au modèle, une étude attentive de la structure géométrique locale obtenue dans le modèle d'Hélein-Vey révèle que, dans le cas où l'on travaille bien avec un fibré des repères standard, elle est suffisante pour définir l'essentiel des éléments nécessaires à la formulation d'une théorie des champs sur l'espace-temps sous-jacent. En ce sens, une variété munie de cette structure géométrique locale devient un candidat approprié pour formuler une théorie des champs. Nous nommons une telle variété fibré des repères généralisé (*generalised frame bundle*), ce qui est une abréviation pour le terme plus précis *fibré des repères avec connexion généralisé*.

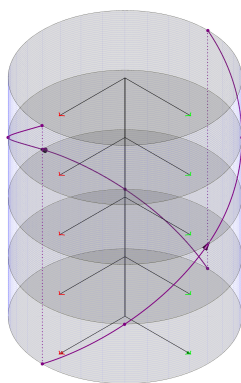


Figure 2: Un tore déplié vu comme un fibré des repères “tordu” (Section 7.1)

Un premier exemple simple de fibré des repères généralisé est celui du tore (plein) vu comme fibré des repères “tordu” au dessus du cône $\mathbb{B}^2/(x \sim -x)$, représenté dans la Figure 2. Le fibré des repères orthonormaux directs au dessus d'un disque \mathbb{B}^2 de dimension 2 est isomorphe au tore plein $\mathbb{B}^2 \times \text{SO}_2$ (de manière SO_2 -équivariante). Le tore “tordu” est construit en isolant une moitié du tore puis en recollant les extrémités *après une rotation de 180 degrés*. Les orbites sous SO_2 gardent la même longueur à l'exception de l'orbite centrale qui devient deux fois plus courte, et l'espace des orbites s'identifie ainsi au cône. Dans le tore tordu, la fibre au dessus du sommet du cône est isomorphe à $\text{PSO}_2 = \text{SO}_2 / (\mathbb{Z}_2)$ mais les autres fibres sont isomorphes à SO_2 .

Cet exemple s'adapte directement à d'autres groupes. Nous traiterons plus particulièrement d'un fibré des repères généralisé au dessus du cône quadridimensionnel $\mathbb{R}^4/(x \sim -x)$ avec une action de Spin_4 qui est libre et transitive au dessus de chaque point hormis au dessus de l'origine, qui a une fibre isomorphe à $\text{Spin}_3 \times \text{SO}_3$. La conséquence est que les spineurs sur le cône ont une composante (de chiralité spécifiée) qui s'annule nécessairement à l'origine.

Cette structure se généralise immédiatement au cadre de la géométrie de Cartan, auquel cas nous parlons de *fibré de Cartan généralisé* (*generalised Cartan bundle*) et de *géométrie de Cartan généralisée*. Le modèle local de la géométrie de Cartan généralisée est spécifié par une paire d'algèbres de Lie $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Nous utiliserons le terme $(\mathfrak{g}, \mathfrak{h})$ -géométrie de Cartan généralisée lorsqu'il

est souhaitable de spécifier le modèle.

Géométrie de Cartan généralisée

$$\begin{aligned}
 \mathfrak{h} &\hookrightarrow \mathfrak{g} \\
 \varpi^A &: T\mathcal{P} \xrightarrow{\sim} \mathcal{P} \times \mathfrak{g} \\
 \alpha^a &= T\mathcal{P} \xrightarrow{\sim} \mathcal{P} \times \mathfrak{g} \rightarrow \mathcal{P} \times \mathfrak{g}/\mathfrak{h} \\
 d\varpi^A + \frac{1}{2} [\varpi \wedge \varpi]^A &= \frac{1}{2} \Omega_{b,c}^A \alpha^a \wedge \alpha^b
 \end{aligned}$$

L'équation définissant une connexion de Cartan généralisée

Dans l'optique de mieux cerner la portée de cette généralisation, nous abordons la question de la reconstruction de l'action de groupe et de l'espace-temps sous-jacent pour les géométries de Cartan généralisées. Pour une $(\mathfrak{g}, \mathfrak{h})$ -géométrie de Cartan généralisée sur une variété P , il convient d'abord de sélectionner un groupe de Lie connexe H intégrant \mathfrak{h} . Il y a alors deux obstructions possibles à l'intégration de l'action de \mathfrak{h} en une action de H : la première est qu'il puisse y avoir ambiguïté sur l'action d'un élément de H en fonction du chemin utilisé dans H pour l'atteindre. La seconde est que les flots des éléments de \mathfrak{h} puissent ne pas être complets et n'être définis que jusqu'à certains temps.

La solution à la seconde obstruction est la construction d'une "globalisation" de P dans laquelle l'action de \mathfrak{h} devient complète. La première obstruction est capturée par la propriété d'"univalence", déjà présente dans [Pal57]. La résolution de l'obstruction est possible en passant à un quotient adapté de la variété considéré. Chacune de ces deux opérations est cependant susceptible de produire des singularités.

Nous construisons un "quotient univalent" de P qui est (sous certaines hypothèses) le quotient approprié sur lequel les éléments de H agissent de manière univoque. Sa construction est effectuée dans le cadre des *groupoïdes de Lie*, qui sont des structures géométriques permettant de traiter d'action possiblement "partielle" et "multivaluée" d'un groupe de Lie sur une variété.

Nous montrons que la succession du passage au quotient univalent puis de la globalisation coïncide avec la *complétion*, qui est une procédure proposée dans [KM04]. Nous proposons également une construction alternative à la complétion : considérer le quotient univalent muni d'une "action locale" du groupe H . Il y a équivalence en un sens précis (équivalence faible de groupoïdes de Lie) entre le quotient univalent muni de cette action locale et la complétion munie de l'action de H . En ce sens, travailler sur la complétion n'offre pas plus de possibilités que travailler avec l'action locale, et il convient de choisir la structure la plus adaptée à la situation.

Portée de notre travail

Le principal point d'ombre dans le modèle d'Hélein-Vey, et par conséquent dans notre travail, est que le mécanisme permettant d'annuler la contribution des multiplicateurs de Lagrange ne s'applique qu'à une partie des équations d'Euler-Lagrange. Par conséquent notre analyse ne porte que sur une partie des équations : le principe variationnel impose a priori des contraintes

plus fortes que les seules équations de champs (standard) que nous isolons. Il est cependant à noter que les équations restantes sont sujettes à de fortes symétries de jauge. Ces équations supplémentaires nécessitent un travail supplémentaire et font de fait l’objet de recherches actuelles¹ – il serait possible que les multiplicateurs de Lagrange aient une influence sur la géométrie de l’espace-temps.

Une autre “limite” à nos résultats porte sur l’intégration des géométries de Cartan généralisées. Afin de pouvoir reconstituer un espace-temps sous-jacent qui présente un minimum de régularité, il est souhaitable d’avoir une propriété de *propreté*. Il s’agit cependant d’une propriété de nature globale et qui ne peut être obtenue par des équations variationnelles locales : il s’agit en générale d’une propriété à établir ou supposer de manière *ad hoc*. La *propreté* peut être toutefois obtenue comme conséquence de la compacité du groupe de Lie considéré. Les constructions de quotient et de complétion peuvent par ailleurs aboutir à des espaces *non séparés*, qui sont potentiellement pathologiques du point de vue de la géométrie différentielle.

Contributions

Voici l’essentiel des contributions originales de la présente thèse :

- La proposition d’utiliser des géométries de Cartan généralisées en théorie des champs (Section 10.1). La géométrie de Cartan est généralement utilisée pour étudier la géométrie de la variété de base, mais peut également s’accommoder de degrés de liberté “internes”, à savoir une composante *non-effective* du groupe de structure. Cela peut s’agir d’un degré de liberté discret comme pour le cas de $\mathbb{R}^{1,3} \simeq (\text{Spin}_{1,3}^+ \ltimes \mathbb{R}^{1,3}) / \text{Spin}_{1,3}$ ou bien de degrés de liberté continus dans le cas d’une théorie de jauge.
- Une nouvelle interprétation du modèle d’Hélein-Vey s’appuyant sur la notion de géométrie de Cartan généralisée (Section 10.3) ainsi qu’une explication du mécanisme d’annulation de la contribution des multiplicateurs (Section 11.4). En particulier l’identification d’un terme exact est justifiée par l’utilisation de certaines composantes spécifiques des équations d’Euler-Lagrange plutôt qu’obtenue après un changement de variables.
- L’extension du modèle d’Hélein-Vey couplant la gravitation à des spineurs de Dirac (Chapitre 11). La construction de ce modèle illustre bien la méthode par laquelle nous proposons de formuler les théories des champs dans le cadre d’une géométrie de Cartan généralisée. Cette extension se fait de manière assez naturelle grâce à notre analyse du modèle d’Hélein-Vey et est plus simple dans la mesure où la géométrie de Cartan généralisée est déjà obtenue grâce au terme original du modèle d’Hélein-Vey.

Ces trois premières contributions sont présentées dans l’article [Pie22] disponible en prépublication à l’heure de la rédaction.

- Une étude détaillée de la question de l’intégration d’une algèbre de Lie en action de groupe de Lie reposant sur des méthodes de groupoïdes de Lie (Chapitre 8). Si la question n’est pas nouvelle [Pal57; KM04; Bla14], nous proposons quelques nouvelles constructions et fournissons un cadre général depuis lequel il est possible de comparer les différentes approches (Théorème 8.1.34). Ceci est l’objet d’un article en cours de préparation [Pie].

Structure de la thèse

Le travail qui suit est organisé en deux parties.

¹ *Travail en cours* par Emmanuel Vignes.

La première partie rassemble la présentation du cadre géométrique dans lequel se situe le travail ainsi que la définition des géométries de Cartan généralisées et la question de leur intégration. La seconde partie rassemble les considérations qui relèvent de la théorie des champs et en particulier la partie du travail portant sur le modèle de Hélein-Vey et son extension.

La thèse commence au Chapitre 1 par présentation des conventions et définitions générales utilisées tout au long du texte.

La partie *Geometry* se décompose ainsi :

2. Le Chapitre 2 présente sans démonstration des définitions et résultats fondamentaux sur les espaces des jets qui seront essentiellement utilisés dans les Chapitres 3 et 9.
3. Le Chapitre 3 présente les structures de **fibré principal** et de connexion principale, de fibré des repères et de la généralisation des ***G*-structures**, équipés d'une ***forme de soudure***. Une dernière section présente des notions élémentaires de **géométrie de Cartan** (d'ordre 1) ainsi que certaines des propriétés élémentaires, qui seront utiles dans l'ensemble du texte.

4. Le Chapitre 4 présente dans le détail la structure des **algèbres de Clifford** et des nombreux isomorphismes qui les relient, des groupes Spin^+ , Spin et Pin ainsi que de leurs représentations dites spinorielles. Dans une deuxième section, on présente les notions de **structures spin** sur une variété et l'**opérateur de Dirac** ainsi que ses propriétés élémentaires, *en présence de torsion*.

L'avant-dernière section discute des **structures pin**. Il y a en effet non-unicité du groupe Pin et l'approche covariante des structures spin, définies de manière indépendantes de la métrique, produit des groupes différents des groupes Pin construits à partir des algèbres de Clifford.

La dernière section rassemble des remarques portant sur la multitude de **conventions de signes** qui entourent les algèbres de Clifford et les spineurs.

5. Le Chapitre 5 établit quelques formules concernant la **géométrie de Riemann-Cartan**, c'est-à-dire de variétés munis d'une métrique de signature arbitraire ainsi que d'une connexion compatible mais pouvant présenter de la torsion. Nous nous intéressons aux relations entre **la courbure de Ricci et la torsion**, ainsi que l'influence de la **torsion** sur l'**opérateur de Dirac**. Ces résultats sont essentiellement utilisés dans la décomposition des équations de champs dans la Section 11.4.

6. Le Chapitre 6 rassemble trois sections essentiellement indépendantes. La première présente la notion de **groupoïde de Lie** en insistant sur les *groupoïdes d'action* comme modèle de référence, présente plusieurs exemples ainsi que des constructions qui seront nécessaires dans cette thèse, ou bien utiles pour mieux comprendre l'idée des groupoïdes de Lie.

La seconde section présente quelques constructions sur les **variétés feuilletées** qui seront essentielles dans le Chapitre 8, notamment le **groupoïde d'holonomie**.

La troisième section présente succinctement la notion d'**orbifold** qui est une généralisation de la notion de variété différentielle acceptant certaines classes de singularités de manière à pouvoir considérer certains **espaces quotients**. Les constructions du Chapitre 8 produiront parfois des orbifolds.

7. Le Chapitre 7 présente la notion de **géométrie de Cartan généralisée**, qui est une généralisation directe de la structure locale du fibré principal d'une géométrie de Cartan (présentée dans le Chapitre 3). Le chapitre commence avec une illustration simple de la notion avant de procéder à une définition générale. Quelques exemples sont présentés dans

une seconde section. Le chapitre se termine avec une section regroupant des formules qui seront utilisées à de nombreuses reprises dans les Sections 10.3 et 11.2.

8. Le Chapitre 8 traite de la question de l’**“intégration” d’une géométrie de Cartan généralisée** en une géométrie de Cartan standard. Une première section traite de l’**intégration de l’action d’une algèbre de Lie** dans le détail. Il y a deux approches, reposant toutes deux sur la construction d’un feuilletage sur la variété produit $P \times H$ avec l’algèbre de Lie \mathfrak{h} agissant sur P .

Dans la première on construit un groupoïde dit “de **pseudo-action locale**” qui décrit les différentes manières dont les éléments de H peuvent agir sur P . Il s’agit alors de comparer ce groupoïde à une construction standard appelée “groupoïde d’action”. Cette approche peut-être rapprochée des travaux [Pal57] mais repose sur un formalisme différent.

Dans la seconde approche on construit directement une “**complétion**” de P qui est naturellement équipée d’une action de H . Il s’agit de la construction de [KM04]. La question est alors d’étudier la relation entre P et cette complétion.

Enfin, ces deux approches sont mises en relations : elles sont “équivalentes” précisément lorsque l’action de \mathfrak{h} sur P vérifie la propriété d’**univalence**.

La seconde section aborde la question de savoir si l’action de H sur P définit un fibré principal au dessus de l’espace des orbites. La première sous-question est la suivante : est-ce que l’espace des orbites est régulier ? Le cas échéant, est-ce que P forme bien un fibré H -principal au dessus de l’espace des orbites ? La réponse repose crucialement sur une propriété de *propreté*, qui est une propriété globale de nature topologique sur l’action de groupe.

Les résultats des deux sections précédentes sont appliqués à la question de l’intégration d’une géométrie de Cartan généralisée dans la troisième section. Dans cette situation, une 1-forme de Cartan définit une action d’algèbre de Lie, que l’on souhaite intégrer comme action de groupe de Lie avant d’interpréter la 1-forme comme un connexion de Cartan sur un fibré principal au dessus de l’espace des orbites.

La partie *Field Theory* se décompose ainsi :

9. Le Chapitre 9 présente le formalisme géométrique de l’**approche covariante des théorie des champs**, essentiellement pour des théories dites du premier ordre.

Après une présentation du formalisme Lagrangien nous présentons la dérivation des équations d’Euler-Lagrange via la **forme de Poincaré-Cartan**. Sous cette formulation, il est tout à fait naturel d’arriver au **formalisme (pré)-multisymplectique**, dans lequel les équations variationnelles prennent la forme des équations de **Hamilton-de Donder-Weyl**.

Enfin, nous présentons l’idée générale des **multiplicateurs de Lagrange** qui permettent d’imposer des contraintes en tant qu’équations variationnelles additionnelles. Le formalisme des formes différentielles permet de considérer naturellement une classe de multiplicateurs “non-holonomes”, qui joueront un rôle essentiel dans les Chapitres 10 et 11. En particulier, il y a une description générale d’un mécanisme par lequel il est possible d’isoler des composantes des équations variationnelles dont les contributions des multiplicateurs peuvent être rassemblées dans une forme différentielle exacte.

Le chapitre se termine sur une présentation très succincte de deux approches géométriques alternatives (bien que non sans relation) à la théorie des champs.

10. Le Chapitre 10 présente notre approche du **modèle d'Hélein-Vey**. Dans la première section, nous esquissons une procédure générale par laquelle une classe large de théorie des champs peut-être relevée à l'espace des repères. La géométrie de Cartan fournit un formalisme adaptée pour traiter de manière unifiée les théories de jauge dites "internes" et les théories de l'espace-temps. En conséquence, ces théories peuvent être naturellement étendues au cadre de la géométrie de Cartan généralisée. La théorie d'Hélein-Vey va plus loin et propose d'obtenir cette même géométrie de Cartan généralisée comme produit des équations variationnelles.

La théorie des champs en question traitée dans [HV16] est la gravitation d'Einstein-Cartan. Après une brève introduction à cette théorie, nous décrivons le modèle d'Hélein-Vey qui se place sur une variété de dimension 10. Nous dérivons ensuite les équations variationnelles associées sur la variété de dimension 10.

La question d'obtenir des équations de champs sur l'espace-temps sous-jacent et notamment le problème de la contribution des multiplicateurs est traitée dans le chapitre suivant, dans le modèle qui comporte également des spineurs de Dirac.

11. Le Chapitre 11 présente notre extension du modèle d'Hélein-Vey qui couple la gravitation à des **spineurs de Dirac**. Après une brève présentation du Lagrangien de Dirac standard, nous construisons la formulation dans le cadre d'une géométrie de Cartan généralisée, ce qui permet de coupler ce modèle au modèle d'Hélein-Vey. Nous dérivons ensuite les équations variationnelles.

La dernière section présente notre approche du mécanisme par lequel les contributions des multiplicateurs de Lagrange sont isolables puis annulables. En considérant des composantes spécifiques des équations d'Euler-Lagrange, il est possible de rassembler ces contributions dans des termes exacts. Nous montrons alors que dans le cas où la géométrie de Cartan généralisée est bien une géométrie de Cartan standard, ces termes disparaissent par intégration dans lorsque le groupe de structure du fibré est *compact*, ce qui correspond par exemple au cas de la gravitation *Riemannienne*.

Il est alors possible de dériver les équations d'Einstein-Cartan-Dirac-Sciama-Kibble décrivant un espace-temps d'Einstein-Cartan dynamique et couplé à un champ de spineurs de Dirac. Le chapitre se termine sur une brève analyse de la forme des équations.

Introduction (English)

Background

The theory of General Relativity is one of the major breakthroughs of the XXth century in our understanding of the universe. The range of its consequences can hardly be overstated. Perhaps the most expectable implications belong to the domains of astronomy and cosmology: General Relativity replaces Newton's theory of universal gravitation which remained unchallenged for almost two centuries. Its predictions include the presence and the formation of black holes, the influence of which are nowadays well supported by observations.

In spite of all its successes, General Relativity faces a major challenge: its current formulation is not compatible with quantum physics, which is another major breakthrough of the XXth century. This is why alternative theories are still being investigated today.

Along with the so-called electromagnetic interaction, weak interaction and strong interaction, gravitational interaction belongs to the few fundamental laws which rule physical phenomena at the fundamental level, according to our current understanding. The former three belong to the so-called "gauge theories" which are associated to specific symmetry groups and are well formalised within quantum physics. Despite the existing conflict with quantum physics, there is an approach to General Relativity which aims to understand it as a gauge theory associated with the Poincaré group, symmetry group of the flat and empty Minkowski spacetime [Heh14]. However the action of the Poincaré group cannot be dissociated from spacetime, whereas the other gauge groups act on the "internal" degrees of freedom of particles.

One proposal is to approach the theory of Relativity not on spacetime but on the *space of observers of spacetime*, modelled by a frame bundle above the manifold of spacetime [Tol78]. It is a higher dimensional geometrical space with extra dimensions (compared to spacetime) on which the Lorentz group, a subgroup of the Poincaré group, acts. Roughly in the same period, there are discussions about formulating gravitation and its extension, super-gravitation, over the Poincaré "group manifold" [NR78c]. These two ideas may echo an earlier proposal of a formulation of the quantum theory of spinor fields on the frame bundle of the Minkowski space, modelled by the Poincaré group, instead of formulating it on the Minkowski space [Lur64]. The motivation behind this proposal is to give to spin, an intrinsic property of matter at quantum level, a geometrical modelling.

More recently, Hélein and Vey proposed a new model along similar lines [HV16]. They construct a field theory defined over a 10-dimensional space such that the solutions endow this space with the structure of a frame bundle over a spacetime which furthermore satisfies the field equations of Einstein's General Relativity. Their theory is the starting point of the present thesis.

The Hélein-Vey model

The Hélein-Vey model is constructed in the so-called multisymplectic formalism and relies on a few remarkable facts. First, using similar terms to the Lagrangian of Hilbert-Einstein, it is possible to obtain variational equations such that the solutions provide to the source manifold the strong structure of a frame bundle (with a few caveats which are discussed below). Second, the contributions of these terms in the field equations vanish by a subtle “cancellation mechanism” which allows gathering them into an exact term which vanishes upon integration.

The model is constructed over a 10-manifold which is expected to become a pseudo-orthonormal frame bundle above of 4-dimensional spacetime. There are two fields:

- A coframe with value in the Poincaré algebra $\mathfrak{p} \simeq \mathfrak{so}_{1,3} \ltimes \mathbb{R}^{1,3}$. The $\mathbb{R}^{1,3}$ component is expected to become a solder form and the $\mathfrak{so}_{1,3}$ component a connection form.
- A differential 8-form with value in the dual space \mathfrak{p}^* which has some component fixed (and which reproduces the Einstein-Hilbert term). The unrestricted components can be interpreted as conjugated variables to the coframe and play the role of Lagrange multipliers.

Let us denote $Q \rightarrow \mathcal{P}$ the bundle of which these fields, gathered into one, are a section. It is equipped with a canonical \mathfrak{p} -valued 1-form which we denote λ and a canonical \mathfrak{p}^* -valued 8-form which we denote p . The model is encoded in the Poincaré-Cartan form Θ (it is halfway between a Lagrangian and a Hamiltonian) which is introduced in Figure 3.

$e : T\mathcal{E} \xrightarrow{\sim} \mathcal{E} \times \mathbb{R}^{1,3}$ $\omega : T\mathcal{E} \rightarrow \mathfrak{so}_{1,3}$ $\mathcal{L}[e, \omega] = \eta^{bb'} e_{ab'}^{(2)} \wedge \left(d\omega + \frac{1}{2} [\omega \wedge \omega] \right)_b^a$	$\mathfrak{p} \simeq \mathfrak{so}_{1,3} \ltimes \mathbb{R}^{1,3}$ $\dim \mathcal{P} = \dim \mathfrak{p} = 10$ $Q \subseteq \Lambda^8 T^* \mathcal{P} \otimes \mathfrak{p}^* \times_{\mathcal{P}} T^* \mathcal{P} \otimes \mathfrak{p}$ $p : Q \rightarrow \Lambda^8 T^* \mathcal{P} \otimes \mathfrak{p}^*$ $\lambda : Q \rightarrow T^* \mathcal{P} \otimes \mathfrak{p}$ $\Theta = p \wedge \left(d\lambda + \frac{1}{2} [\lambda \wedge \lambda] \right)$
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The Einstein-Hilbert Lagrangian (in tetradic formalism) on a spacetime \mathcal{E}

Figure 3: Hélein-Vey’s Poincaré-Cartan form (in our notation)

The first aim of this thesis is to extend the Hélein-Vey model in order to model matter fields coupled to space-time: we handle the case of Dirac spinors. The Hélein-Vey model turns out to be closer to an alternative theory to General Relativity called *Einstein-Cartan theory* [Tra06]. While General Relativity determines the geometry of the manifold of spacetime by a *metric* which plays the role of the gravitational potential, Einstein-Cartan theory uses both a *metric* and tangent connection on spacetime, which models the gravitational field. This allows for one extra degree of freedom: the so-called *torsion* of the tangent connection. In particular, this torsion becomes coupled to the “spin density” of matter through the field equations, and interacts with Dirac spinors.

A suitable framework to understand the geometry of General Relativity is the so-called *Cartan geometry* [Wis10]. It is a general framework in which a differentiable manifold is equipped with a structure relating it locally to a reference geometry which is modelled by a homogeneous space G/H . Indeed General Relativity can be understood as a “curved” version of Special Relativity which takes place in the Minkowski space: at every point of spacetime there exists a referential in free fall in which physics² should behave as in the Minkowski space.

This work led us to study the inner workings of the model of Hélein-Vey. We interpret the extra terms in the model as Lagrange multiplier terms imposing constraints as extra variational equations. Since spinor fields on spacetime can be naturally modelled as functions defined on the space of frames and satisfying certain invariance properties, this perspective allows a relatively straightforward addition of spinor fields to the model.

Moreover, the “cancellation mechanism” of the Lagrange multipliers remains somewhat obscure in [HV16]. It is obtained through lengthy and involved calculations which do not shed much light on the underlying geometry. In particular, they involve a rather delicate change of coordinates which is indispensable to obtain an exact term. In this thesis we propose a more geometrical approach which gives a more satisfying explication of the argument. The change of variable, operated on the Euler-Lagrange equations, can be understood as selecting the Euler-Lagrange equations corresponding to very specific variations of the fields. We investigate the form of these specific variations and explain how they lead to exact terms.

However it turns out that the geometrical structure obtained from the solutions of the Hélein-Vey model does not always define a frame bundle: some topological hypotheses of global nature are required. This is a defect in [HV16] we identified when studying the article. It comes down to the following fact: variational equations are of local nature while the structure of a frame bundle is partly a global structure. More precisely, there is a group action such that the orbit space is a differentiable manifold (spacetime) and the starting space defines a (locally trivial) principal bundle above it.

Due to their local character, variational equations can only provide a *Lie algebra* action. This brings the following questions: can the action be integrated into a group action? Is the orbit space a smooth manifold? Only then can be formulated the question of whether we obtain a frame bundle above spacetime.

A close look at the local geometrical structure obtained in the Hélein-Vey model reveals that, at least in the case of a standard frame bundle, it is sufficient to define the important elements required in order to formulate a field theory over the underlying spacetime. As a consequence, what seemed like a limit to the model can be taken as a motivation to formulate field theories in a new geometrical framework. We call a manifold with such a local structure a generalised frame bundle, which is an abbreviation for the more exact term of *generalised frame bundle with connection*.

A first simple example of generalised frame bundle is that of a (bulk) torus seen as a “twisted” frame bundle above the cone $\mathbb{B}^2/(x \sim -x)$. This is represented in Figure 4. The bundle of direct orthonormal frames above a 2-dimensional disc \mathbb{B}^2 is isomorphic to the bulk torus $\mathbb{B}^2 \times \text{SO}_2$ (in a SO_2 -equivariant manner). The “twisted” torus is obtained by isolating a half of the torus (cutting along a horizontal plane) then gluing the two ends *after a 180 degree twist*. The orbits under SO_2 keep the same length at the exception of the central orbit which becomes half as long, and the orbit space becomes a cone. In the twisted torus the fibre above the “tip” of the cone (the singular point) is isomorphic to $\text{PSO}_2 = \text{SO}_2/(\mathbb{Z}_2)$ while the other fibres are isomorphic to SO_2 .

It is straightforward to adapt this example to other groups. We will be especially interested in a generalised fibre bundle above the 4-dimensional cone $\mathbb{R}^4/(x \sim -x)$ that has an action of Spin_4 which is free and transitive above each point but above the origin, which has a fibre isomorphic

²This is an overly simplified statement of the *principle of equivalence*.

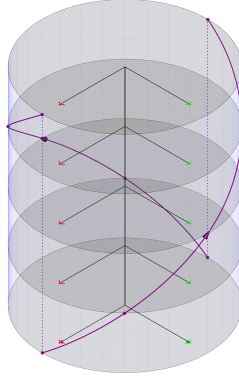


Figure 4: An unfolded (bulk) torus as a “twisted” frame bundle (Section 7.1)

to $\text{Spin}_3 \times \text{SO}_3$. As a consequence, spinors on the cone have a component (of specified chirality) which necessarily vanishes at the origin.

This structure can be immediately generalised to the framework of Cartan geometry, in which case we will speak of *generalised Cartan bundle* and *generalised Cartan geometry*. The local model for generalised Cartan geometry is specified by a pair of Lie algebra $\mathfrak{h} \hookrightarrow \mathfrak{g}$. We will use the term generalised $(\mathfrak{g}, \mathfrak{h})$ -Cartan geometry when we want to specify the model.

Generalised Cartan geometry

$$\begin{aligned}
 & \mathfrak{h} \hookrightarrow \mathfrak{g} \\
 & \varpi^A : T\mathcal{P} \xrightarrow{\sim} \mathcal{P} \times \mathfrak{g} \\
 & \alpha^a = T\mathcal{P} \xrightarrow{\varpi} \mathcal{P} \times \mathfrak{g} \rightarrow \mathcal{P} \times \mathfrak{g}/\mathfrak{h} \\
 & \quad \quad \quad \xrightarrow{\sim} \\
 & d\varpi^A + \frac{1}{2} [\varpi \wedge \varpi]^A = \frac{1}{2} \Omega_{b,c}^A \alpha^a \wedge \alpha^b
 \end{aligned}$$

The equation defining a generalised Cartan geometry

In order to better grasp the scope of this generalisation, we study the question of building a group action and an underlying spacetime for generalised Cartan geometry. Given a generalised $(\mathfrak{g}, \mathfrak{h})$ -Cartan geometry on a manifold P , one must first choose a connected Lie group H which integrates \mathfrak{h} . There are then two kinds of obstruction to the integration of the action of \mathfrak{h} into an action of H : the first one is that there may be ambiguity on how an element of H acts, depending

on the path used in H to reach it. The second one is that the flows of the vector fields associated to elements of \mathfrak{h} may not be complete and only be defined up to a certain time.

The first obstruction is captured by the property of “univalence”, already formulated in [Pal57]. It can be resolved by constructing a suitable quotient of the considered manifold. The solution to the second obstruction is the construction of a “globalisation” of P in which the action of \mathfrak{h} becomes complete. Both operations may give rise to singularities.

We introduce a “univalent quotient” of P which is, assuming suitable hypotheses, the appropriate quotient on which the elements of H act without ambiguity. It is constructed using *Lie groupoids*, which are geometrical structures convenient for working with possibly *partially defined* and *multivalued* actions of a Lie group on a manifold. Even in the case H is simply-connected, if the action of \mathfrak{h} is non-complete, the univalent quotient can be smaller than P .

We show that taking the univalent quotient, followed by globalisation, coincides with *completion*, a procedure proposed in [KM04]. We also propose an alternative construction: working directly with the univalent quotient endowed with a “local group action” of H . There is equivalence, in a precise sense (weak equivalence of Lie groupoids), between the univalent quotient equipped with the local action and the completion equipped with the group action of H . Accordingly, working with the completion does not offer new possibilities compared to working with the local action, and we recommend choosing the structure which is the most adequate to the problem at hand.

Scope of our work

The main dissatisfaction point in the Hélein-Vey model, hence in our work as well, is that we are able to cancel the contribution of the Lagrange multipliers *only in part of the Euler-Lagrange equations*. As a result, our analysis only applies to part of the equations: the variational principle imposes *a priori* stronger constraints than the sole (standard) field equations we extract. However the remaining equations have a strong gauge symmetry. The analysis of these equations requires further work and is the subject of current research³ – the fields we call Lagrange multipliers might turn out to have an influence on spacetime geometry.

Another “limit” of our results concerns the integration of generalised Cartan geometries. In order to reconstruct an underlying spacetime which presents reasonable regularity, it is desirable to have a property of *properness*. It is however a global property which cannot be obtained by local variational equations: it is a property that has to be established in the case at hand, or assumed in an *ad-hoc* manner. Properness can however be obtained as a consequence of the compactness of the considered Lie group. Moreover the quotient construction and completion construction may result in *non-Hausdorff* spaces, which are potentially pathological from a differentiable manifold perspective.

Original contributions

Let us present the main contributions of the present thesis:

- A proposal to use generalised Cartan geometries in field theory (Section 10.1). Cartan geometry is usually used in order to study the geometry of the underlying manifold but it can also integrate “internal” degrees of freedom, namely *non-effective* parts of the structure group. It may be a discrete degree of freedom as in the case of $\mathbb{R}^{1,3} \simeq (\text{Spin}_{1,3}^+ \ltimes \mathbb{R}^{1,3}) / \text{Spin}_{1,3}$ or continuous degrees of freedom as in the case of a gauge theory.

³Work in progress by E. Vignes.

- An original interpretation of the Hélein-Vey model relying on the notion of generalised Cartan geometry (Section 10.3) and an explanation of the cancelling mechanism applying to the contribution of the Lagrange multipliers (Section 11.4). Notably, the identification of an exact term is justified by the extraction of specific components of the Euler-Lagrange equations rather than obtained through of change of variables.
- An extension of the Hélein-Vey model coupling gravitation to Dirac spinors (Chapter 11). Our construction of this model provides an illustration of the method by which we propose to formulate field theories in the framework of generalised Cartan geometry. This extension is done in a rather straightforward manner thanks to our analysis of the Hélein-Vey model and is simpler since the generalised Cartan geometry is already obtained with the original term of the Hélein-Vey model.

These three contributions are presented in the article [Pie22] available in prepublication at press time.

- A detailed study of the integration of a Lie algebra action into a Lie group action relying on Lie groupoid methods (Chapter 8). Although the question is hardly new [Pal57; KM04; Bla14], we propose new constructions and provide a general framework in which it is possible to compare the different approaches (Theorem 8.1.34). This is the subject of an article in preparation [Pie].

Outline of the thesis

This thesis is organised in two parts.

In the first part, we present the geometrical background necessary for our work, introduce generalised Cartan geometry and handle the question of their integration.

In the second part, we introduce the general background required for field theory then present our work on the Hélein-Vey model as well as its extension with Dirac spinors.

We start by introducing in Chapter 1 general definitions and conventions which will be used throughout the thesis.

The *Geometry* Part is decomposed as follows:

2. Chapter 2 presents without demonstration definitions and basic results on jet spaces which will be mainly used in Chapters 3 and 9.
3. Chapter 3 presents the structures of **principal bundle** and principal connection, frame bundles and their generalisation as ***G*-structures**, equipped with a ***solder form***. In a last section we present the basic notions in (first order) **Cartan geometry** as well as some elementary properties which we will be useful in the following sections.
4. Chapter 4 presents in detail the structure of **Clifford algebras** and the many isomorphisms between them, the Spin^+ , Spin and Pin groups as well as their so-called spinor representations. In a second section we present the notions of **spin structure** on a manifold and the **Dirac operator** as well as its basic properties *in presence of torsion*.

The third section discusses **pin structures**. There is indeed non-unicity of the Pin group and the covariant approach to spin structures, which defines them independently of any metric, results in groups which differ from the Pin groups constructed from Clifford algebras.

The last section gathers remarks pertaining to the various **sign conventions** around the Clifford algebras and the spinors.

5. Chapter 5 establishes a few formulas in **Riemann-Cartan geometry**, namely manifolds endowed with a metric of arbitrary (non-degenerate) signature and a compatible connection which is allowed to present torsion. Our interest is the relations between **Ricci curvature and torsion**, as well as the effect of **torsion** on the **Dirac operator**. These results will be mainly used in the decomposition of field equations done in Section 11.4.

6. Chapter 6 is made of three essentially independent sections. The first one introduces the notion of **Lie groupoids** with an emphasis on *action groupoids* as a reference model, provides several examples as well as constructions which will be useful for our work, or helpful in order to get a better grasp on Lie groupoids.

In a second section we present a few constructions on **foliated manifolds** which will be central in Chapter 8, especially the *holonomy groupoid*.

The third section is a succinct presentation of the notion of **orbifold** which is a generalisation of differentiable manifolds which allows for some singularities in order to handle certain **quotient spaces**. Constructions of Chapter 8 may result in orbifolds.

7. Chapter 7 introduces the notion of **generalised Cartan geometry**, which is a direct generalisation of the local structure of the principal bundle of a Cartan geometry (introduced in Chapter 3). The chapter starts with a simple illustration of the notion before moving to a general definition. In a second section we present a few examples. The chapter ends with a section gathering formulas which will be used in Sections 10.3 and 11.2.

8. Chapter 8 handles the question of the **“integration” of a generalised Cartan geometry** into a standard Cartan geometry. A first section deals in detail with the **integration of a Lie algebra action**. There are two approaches, both relying on the construction of a foliation on the product manifold $P \times H$ with the Lie algebra \mathfrak{h} acting on P .

In the first one we build a Lie groupoid of **“local pseudo-action”** which describes the different ways elements of H can act on P . The question is then to compare this groupoid to the standard construction called **“action groupoid”**. This approach can be likened to the work in [Pal57] however it relies on a different formalism.

The second approach consists in directly constructing a **“completion”** of P which is naturally endowed with an action of H . This construction can be found in [KM04]. The question is then to study the relation between P and the completion.

Finally, we relate these two approaches: they are **“equivalent”** precisely when the action of \mathfrak{h} on P satisfies the property of **univalence**.

The second section handles the problem of whether the action of H on P defines a principal bundle above the orbit space. First, we need to understand under what hypotheses the orbit space is regular. If that is the case, then one can ask whether P defines an H -principal bundle. The answer relies in an essential way on the property of **properness**, which is a global property of topological character on the group action.

The results of both sections are then applied to the integration of a generalised Cartan geometry in a third section. In this situation, a so-called **Cartan 1-form** defines a Lie algebra action, which we want to integrate into a group action then interpret the 1-form as Cartan connection on a principal bundle over the orbit space.

The *Field Theory* Part is decomposed as follows:

9. Chapter 9 introduces the geometrical formalism for **the covariant approach to field theory**, essentially for the so-called **first order theories**.

After introducing the Lagrangian formalism, we present the derivation of the Euler-Lagrange equations using the *Poincaré-Cartan form*. In this formulation, the **(pre-)multisymplectic formalism** appears naturally and the variational equations take the form of *Hamilton-de Donder-Weyl* equations.

Finally, we present the general idea of **Lagrange multipliers** which are a way to impose constraints as additional variational equations. The formalism of differential forms allows considering a class a “non-holonomic” multipliers which will have an essential role in Chapters 10 and 11. Notably, we give a general description of the cancelling mechanism according to which it is possible to isolate in the variational equations components in which the contributions of the multipliers can be gathered in an exact differential form.

The chapter closes on a very succinct presentation of two alternative (although not unrelated) geometrical approaches to field theory.

10. Chapter 10 presents our approach to the **Hélein-Vey model**. In a first section we sketch a general procedure according to which it is possible to lift a large class of gauge field theory to the frame bundle. Cartan geometry provides an appropriate formalism to handle in a unified manner the usual “internal” gauge theories and theories of spacetime. Consequently, these theories can naturally be extended to the framework of generalised Cartan geometry. The Hélein-Vey theory goes further and proposes to obtain this generalised Cartan geometry as a result of the variational equations.

The field theory which is handled in [HV16] is the Einstein-Cartan theory of gravitation. After a brief introduction to this theory, we introduce the Hélein-Vey model which takes place on a 10-manifold. We derive the corresponding variational equations on the 10-manifold.

The derivation of field equations on the underlying spacetime and in particular the handling of the contribution of the multipliers is postponed to the following chapter, in a model which also contains Dirac spinors.

11. Chapter 11 introduces our extension to the Hélein-Vey model which couples gravitation to **Dirac spinors**. After a short presentation of the standard Dirac Lagrangian, we construct a formulation in the framework of a generalised Cartan geometry, which allows coupling this model to the Hélein-Vey model. We then proceed to derive the variational equations.

The last section presents our approach to the cancellation mechanism. By considering specific components of the Euler-Lagrange equation, it is possible to gather the contributions of the Lagrange multipliers in exact terms. We then show that in the case the generalised Cartan geometry is a honest Cartan geometry, these term vanish by integration in the case the structure group is *compact*, which corresponds to the case of *Riemannian* gravitation.

It is then possible to derive the Einstein-Cartan-Dirac-Sciama-Kibble equations describing a dynamical Einstein-Cartan spacetime coupled to a Dirac spinor field. The chapter closes with a brief analysis of the form of the field equations.

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Chapter 1

Conventions and definitions

Contents

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1.1 Spaces and manifolds

When not specified, we shall assume smooth manifolds to be Hausdorff paracompact.

Definition 1.1.1 (Completely regular topological space). A *completely regular* topological space is a topological space X which has all points closed and such that for any closed subset C and $x \in X \setminus C$ there exists a continuous real function such that $f(C) = \{0\}$ and $f(x) = 1$.

Note that completely regular spaces are sometimes defined without the requirement that the points are closed. The definition we give here is the one used in [Pal61]. With this definition completely regular spaces are necessarily Hausdorff.

Example 1.1.2. All metrisable topological spaces are completely regular, in particular all manifolds.

Since we will be dealing with quotients of manifolds, it will be convenient to consider more general smooth structures than that of smooth manifolds. They will be defined by the (local) data of smooth maps from the manifold to \mathbb{R} .

Definition 1.1.3 (Smooth structure). A *smooth structure* on a topological space X is the data for each open subset \mathcal{U} of a subset of real maps, called “smooth maps”:

$$\mathcal{C}_X^\infty(\mathcal{U}) \subset \mathcal{C}(X, \mathbb{R})$$

which satisfies the following properties:

1. For \mathcal{V} an open subset of \mathcal{U} , smooth maps on \mathcal{U} restrict to \mathcal{V} to smooth maps:

$$\forall u \in \mathcal{C}_X^\infty(\mathcal{U}), u|_{\mathcal{V}} \in \mathcal{C}_X^\infty(\mathcal{V})$$

2. Given a family of open subsets $(\mathcal{U}_i)_{i \in I}$ and $u \in \mathcal{C}(\bigcup_I \mathcal{U}_i, \mathbb{R})$,

$$u \in \mathcal{C}_X^\infty \left(\bigcup_I \mathcal{U}_i \right) \Leftrightarrow \forall i \in I, u|_{\mathcal{U}_i} \in \mathcal{C}_X^\infty(\mathcal{U}_i)$$

3. For any open subset \mathcal{U} , any integer k and any family $(u_j)_{1 \leq j \leq k}$ of smooth functions over \mathcal{U} , for all \mathcal{C}^∞ maps $g : \mathbb{R}^k \rightarrow \mathbb{R}$, the composite $g \circ (u_1, \dots, u_k)$ belongs to $\mathcal{C}_X^\infty(\mathcal{U})$.

We will call a topological space equipped with a smooth structure a *smooth space*.

Remark. In the literature it is found under the names of “differential space” or “Mostow space” [Mes16; Bat+17].

In this language, a smooth manifold is defined as follows:

Definition 1.1.4 (Smooth manifold). A smooth manifold M of dimension $n \in \mathbb{N}$ is a topological space equipped with a smooth structure which is locally Euclidean in the following sense:

- M with its smooth structure is locally isomorphic to \mathbb{R}^n .

and which satisfies the following two global topological requirements:

- M is Hausdorff.
- M is paracompact.

A smooth space which satisfies the first requirement but not the global ones will be called a *non-Hausdorff manifold*¹ (without mention of paracompactness).

Notice that according to this language, a manifold is not a space with a global structure (like an atlas) but a space with a global structure *which satisfies a local property*. In particular, this allows talking about a space being a manifold *in the neighbourhood of one point*, or on an open subset. In this way, it will be possible to handle singular spaces, like orbifolds, which require to omit some subset in order to obtain a manifold.

There is a corresponding notion of smooth map which is actually very practical: a map between smooth spaces is close if its “components” are smooth:

Definition 1.1.5. Let X and Y be smooth spaces and $f : X \rightarrow Y$ a continuous map. The map f is called *smooth* if

$$f^* \mathcal{C}_Y^\infty \subseteq \mathcal{C}_X^\infty$$

namely if for all open subset $\mathcal{U} \subset Y$,

$$\forall u \in \mathcal{C}_Y^\infty(\mathcal{U}), u \circ f \in \mathcal{C}_X^\infty(f^{-1}(\mathcal{U}))$$

This formalism indeed extends the theory of smooth manifolds according to the following theorem:

Theorem 1.1.6 ([Mos79]). *Let M be and N be two smooth manifolds. Then a smooth map $M \rightarrow N$ is exactly a \mathcal{C}^∞ map for the \mathcal{C}^∞ -manifold structures.*

Definition 1.1.7 (Fibre bundle). Let P be and M be two smooth manifolds. By *fibre bundle over M* with fibre F we will mean a smooth manifold P with a smooth map $P \xrightarrow{p} M$ such that there exists a basis of open subsets (\mathcal{U}) of M such that over each \mathcal{U} the fibre bundle is diffeomorphic to a cartesian product:

¹It is sometimes called *premanifold* [Bou07].

$$\begin{array}{ccc}
 p^{-1}(\mathcal{U}) & \xrightarrow{\sim} & F \times \mathcal{U} \\
 & \searrow & \swarrow p_2 \\
 & \mathcal{U} &
 \end{array}$$

$p|_{p^{-1}(\mathcal{U})}$

On a fibre bundle $P \xrightarrow{p} M$ the *vertical (tangent) bundle* is the vector bundle on P which is composed at each point of the tangent vectors to the fibre:

$$VP := \ker(dp) \subset TP$$

Vectors (and vector fields) belonging to VP are called *vertical*, while differential forms that have vanishing contraction with any vertical vector are called *horizontal* (also *semi-basic*).

Definition 1.1.8 (Fibred product of manifolds). Let X, Y and Z be three manifolds with two maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$. Define the following subset of $X \times Y$:

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

It may be written more accurately as $X \times_{(f,g)} Y$.

When it is a submanifold of $X \times Y$, it is called the *fibre product of X and Y over Z* . It is part of the following commutative diagram:

$$\begin{array}{ccc}
 X \times_Z Y & \longrightarrow & Y \\
 \downarrow & & \downarrow g \\
 X & \xrightarrow{f} & Z
 \end{array}$$

The diagram is sometimes interpreted asymmetrically and the fibre product with its map to X is called the *pullback of Y to/over X (by/under f)*. In this case, the map $X \times_Z Y$ is sometimes called the pullback of g under f .

Theorem 1.1.9 ([Mic08]). *The fibre product always exists when f and g are transversal, namely*

$$\forall (x, y) \in X \times_Z Y, \quad df(T_x X) + dg(T_y Y) = T_{f(x)} Z$$

In this case, $\dim X \times_Z Y = \dim X + \dim Y - \dim Z$. In particular, it exists when either f or g is a submersion.

1.2 Lie groups and algebras actions

In this thesis *Lie algebra* will mean finite dimensional real Lie algebra. Let us recall that for a Lie group G , its Lie algebra \mathfrak{g} can be identified as the tangent space at identity $T_e G$ provided with the infinitesimal adjoint action bracket:

$$[\xi_1, \xi_2] = \text{ad}_{\xi_1}(\xi_2)$$

It is naturally isomorphic to the Lie algebra of *left*-invariant vector fields on G (hence corresponding to a *right* action on G). On the other hand *right*-invariant vector fields on G form a Lie algebra which is anti-isomorphic to \mathfrak{g} in the following sense: writing L_ξ for the right-invariant vector field such that $L_\xi(e) = \xi$, the following holds:

$$[L_{\xi_1}, L_{\xi_2}] = L_{-[\xi_1, \xi_2]}$$

Definition 1.2.1 (G -space, G -manifold). Given a Lie group G , a G -space is a topological space² M with a continuous (left) action of G : a continuous map

$$\rho : G \times M \rightarrow M \text{ such that } \rho(g_1 g_2, x) = \rho(g_1, \rho(g_2, x))$$

A G -manifold is a smooth manifold with a smooth action of G .

Definition 1.2.2 (\mathfrak{g} -manifold). Denoting by \mathfrak{g} a Lie algebra, a right \mathfrak{g} -manifold is a (smooth) manifold with an action of \mathfrak{g} by smooth vector fields:

$$\xi \in \mathfrak{g} \rightarrow \bar{\xi} \in \Gamma(TX) \text{ such that } [\bar{\xi}_1, \bar{\xi}_2] = \overline{[\xi_1, \xi_2]}$$

The vector fields $\bar{\xi}$ are called *fundamental vector fields* on the \mathfrak{g} -manifold.

Remark. A *right* smooth action of a Lie group differentiates to a right action of the associated Lie algebra.

Similarly, a smooth group action on the left differentiates to a representation of the associated Lie algebra which reverses the bracket.

Definition 1.2.3 (Principal G -bundle). Given a Lie group G , a principal G -bundle is a fibre bundle $P \xrightarrow{p} M$ with G acting freely on P such that the fibres are the orbits under G . We will use the usual convention that principal G -bundles have a *right* group action.

Definition 1.2.4 (Equivariant bundle). An *equivariant bundle* on a G -space M is a fibre bundle $P \xrightarrow{p} M$ endowed with an action of G which lifts the action on M :

$$\forall (g, x) \in G \times P, \quad p(g \cdot x) = g \cdot p(x)$$

namely the following diagram commutes:

$$\begin{array}{ccc} G \times P & \longrightarrow & G \\ \downarrow \text{id}_G \times p & & \downarrow p \\ G \times M & \longrightarrow & M \end{array}$$

An *equivariant section* of an equivariant bundle is a section which is invariant under the action of G :

$$\phi : M \rightarrow P, \forall x \in M, \quad \begin{cases} p(\phi(x)) = x \\ \phi(g \cdot x) = g \cdot \phi(x) \end{cases}$$

Namely, the following diagram commutes:

$$\begin{array}{ccc} G \times P & \longrightarrow & P \\ \text{id}_G \times \phi \uparrow & & \uparrow \phi \\ G \times M & \longrightarrow & M \end{array}$$

We will mainly be interested about equivariant sections of bundles of the form $\Lambda^\bullet T^*M \otimes V$ with V a linear representation of G .

Let M be a G -space. At each point $x \in M$ there is an *isotropy group*

$$G_x := \{g \in G \mid g \cdot x = x\}$$

²In particular we do not require in the definition complete regularity as [Pal61] for example.

and an *orbital map*

$$g \in G \mapsto g \cdot x \in M$$

which factors through a continuous (or smooth) injection $G/G_x \rightarrow M$ onto the orbit of x .

Definition 1.2.5 (Properties of actions of Lie groups and algebras). The action ρ of a group G (resp. a Lie algebra \mathfrak{g}) on a manifold M , is said:

- *effective* (also *faithful*) if no non-trivial element acts trivially:

$$\begin{aligned} \forall g \in G, \quad \rho(g) = \text{id}_M &\implies g = e \\ \forall \xi \in \mathfrak{g}, \quad \rho(\xi) = 0_{TM} &\implies \xi = 0 \end{aligned}$$

- *free* if the isotropy group (resp. algebra) at each point is trivial:

$$\begin{aligned} \forall x \in M, g \in G, \quad g \cdot x = x &\implies g = e \\ \forall x \in M, \forall \xi \in \mathfrak{g}, \quad \bar{\xi}|_x = 0 &\implies \xi = 0 \end{aligned}$$

- *transitive* if the orbital maps are surjective (resp. the vectors representing the Lie algebra span the whole tangent space at each point ³):

$$\begin{aligned} \forall x \in M, y \in M, \exists g \in G, \quad g \cdot x = y \\ \forall x \in M, X \in TM, \exists \xi \in \mathfrak{g}, \quad \xi \cdot x = X \end{aligned}$$

Furthermore, the action of a Lie algebra on a manifold is said *complete* when the flow of every fundamental vectors fields is complete. We also say that the \mathfrak{g} -manifold is complete. These definitions allow introducing the following notion:

Definition 1.2.6 (G -torsor). A G -torsor is a free transitive G -space.

The following notion will be useful when we will discuss the orbits of a group action.

Definition 1.2.7 (Type of an orbit). Let M be a G -space. If x and y are two points of M belonging to the same orbit, their isotropy groups are conjugate.

The *orbit type* (also called isotropy type⁴) of an orbit of G is the conjugacy class of its isotropy groups. We write it with brackets $[O]$ and use the term “orbit of type G/K ” for a (closed) subgroup $K \subset G$.

Note that orbits with different types may be homeomorphic, for example different presentations of the circle group as quotients of the real line \mathbb{R} . On the other hand, orbits with the same type may not be homeomorphic (see however 8.2.6).

Given two vector representations V_1 and V_2 of G , we will write

$$V_1 \rightarrow_G V_2$$

for a G -equivariant map and

$$V_1 \simeq_G V_2$$

to state their equivalence as representations of G . Similarly, for two vector representations V_1 and V_2 of \mathfrak{g} , we will write

$$V_1 \rightarrow_{\mathfrak{g}} V_2$$

³This is sometimes called *locally transitive* or *infinitesimally transitive*.

⁴In [Bre72] it is made a difference between *orbit type* and *isotropy type*. We are using here the isotropy type.

for a \mathfrak{g} -equivariant map and

$$V_1 \simeq_{\mathfrak{g}} V_2$$

when they are equivalent as representations of \mathfrak{g} .

1.3 Superscript and subscript conventions

Abstract indices and summations

Let E be a finite dimensional vector space. We write E^* for the dual vector space.

We will use the following convention: *subscripts* for bases $(e_i)_{i \in I}$ of E and *superscripts* for bases $(\epsilon^i)_{i \in I}$ of the dual space E^* . This convention requires a prior choice of which space is the base space and which is the dual. We write $(e^*i)_{i \in I}$ for the dual basis to the basis $(e_i)_{i \in I}$.

A vector x belonging to E decomposes into the basis as follows

$$x = \sum_{i \in I} x^i e_i$$

Similarly, a covector α belonging to E^* decomposes into the dual basis as follows:

$$\alpha = \sum_{i \in I} \alpha_i e^*i$$

In this way, coordinates in a basis use the index opposite to that of the basis. Indeed, this is justified by the following:

$$x^i = e^*i(x), \quad \alpha_i = \alpha(e_i)$$

If x and α are expressed in coordinates in dual bases, their contraction is readily computed as follows:

$$\alpha(x) = \sum_{i \in I} \alpha_i x^i$$

This formula holds independently of the basis-dual basis pair used.

This leads to the so-called “abstract index notation”: a vector x can be identified to the system of its coordinates (x^i) . In the abstract index notation, rather than working in a specific basis, superscript and subscript are used with respect to a generic basis and are simply an indication of whether the object belongs to E or E^* ⁵.

For a further formalisation, a vector x can be seen as a basis-dependent system of coordinates (x^i) which transforms contravariantly under change of basis. Correspondingly, a covector α can be seen as a basis-dependent system of coordinates (α_i) which transforms covariantly under change of basis. This perspective ties in with the frame bundle approach to tensors on manifolds that we will present in Section 3.2. Since contraction is a “geometrical” operation, namely it is independent of the basis, the construction

$$\sum_{i \in I} \alpha_i x^i$$

is well defined even without specification of a basis – as long as the indices are understood as referring to dual bases.

⁵Our abstract indices differ superficially from the conventions in [Wal84]: while x^i denotes for them the vector x , it will denote for us the system of coordinates of x^i which depends on a non-fixed basis

Since contractions are omnipresent when manipulating various tensors, we will use *Einstein's summation convention* which implies an implicit summation when an identical index is repeated above and below:

$$\alpha_i x^i := \sum_{i \in I} \alpha_i x^i$$

The convention does not hold when an index is repeated at the same position, since this does not result in a basis-independent construction. In fact it is best to avoid such repetitions. When summing expressions with a repeated index at the same position (which requires specifying a working basis) the summation shall be explicitly written. The summation convention can also be used with elements of the generic basis/dual basis in the following way:

$$\begin{aligned} x &= x^i e_i \\ \alpha &= \alpha_j e^{*j} \end{aligned}$$

A tensor $T \in E^{\otimes r} \otimes E^{*s}$ decomposes as

$$T = T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s} e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r} \otimes e^{*j_1} \otimes e^{*j_2} \otimes \dots \otimes e^{*j_s}$$

and can be referred to using its system of coordinates $(T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s})$ so that contractions can be very conveniently written using the summation convention:

$$x^{j_1} T^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$$

According to these conventions, the Kronecker symbol δ_j^i represents the identity $\text{Id}_E \in E \otimes E^*$.

Antisymmetric tensors

We will be working with tensors with antisymmetric components. Since they can be represented as elements of a tensor product or an exterior product, we record here our conventions.

We will be using the standard embedding

$$\begin{aligned} \Lambda^k E &\hookrightarrow T^k E \\ u_1 \wedge \dots \wedge u_k &\mapsto \sum_{\sigma \in S_k} \text{sgn}(\sigma) u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(k)} \end{aligned}$$

so that for example given a basis (e_i) of E there is the corresponding “wedge” basis of $\Lambda^2 E$:

$$(e_i \wedge e_j)_{i < j} \equiv (e_i \otimes e_j - e_j \otimes e_i)_{i < j}$$

An element $A \in \Lambda^k E$ can then be decomposed over the tensor basis of $T^k E$ and over the wedge basis of Λ^k . The convention is that the components $A^{i_1 i_2 \dots i_k}$ correspond to the tensor basis, namely:

$$A = A^{i_1 i_2 \dots i_k} e_{i_1} \otimes \dots \otimes e_{i_k} \in \Lambda^k E \hookrightarrow T^k E$$

Now the Einstein summation convention consistently sums over *all possible indices*, which prevents naturally using $\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}\}_{i_1 < i_2 < \dots < i_k}$ as a basis in implicit sums. It is therefore used as a redundant family of generators $\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}\}_{i_1, i_2, \dots, i_k \in I}$ with suitable normalisation factors. Accordingly, $A \in \Lambda^k E$ will be written as follows:

$$A = \frac{1}{k!} A^{i_1 i_2 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k}$$

Metrics

Assume E is provided with a non-degenerate symmetric bilinear form η . We will use the following notion throughout the text

Definition 1.3.1 (Pseudonormed vectors and pseudo-orthonormal bases). Let e be a vector. It is *pseudonormed* if $\langle e|e \rangle = \pm 1$.

A basis of pseudonormed vectors which are orthogonal between each other is called a *pseudo-orthonormal basis*.

The bilinear form η defines an isomorphism

$$\eta : \begin{cases} E & \xrightarrow{\sim} E^* \\ x & \mapsto \eta(x, \cdot) \end{cases}$$

We will also use a transposition notation ${}^t x$ for the image of x under η . There is an inverse isomorphism η^{-1} which also defines a non-degenerate bilinear form on E^* .

As an element of $E^* \otimes E^*$, the bilinear form will be represented as η_{ij} . Similarly, the inverse bilinear form can be represented as an element of $E \otimes E$ and will be represented as η^{ij} without -1 superscript, as is standard practice. The inverse relation between the two can be expressed as follows:

$$\begin{aligned} \eta^{ij} \eta_{jk} &= \delta_k^i \\ \eta_{ij} \eta^{jk} &= \delta_i^k \end{aligned}$$

The bilinear form is often used to “raise and lower” indices as follows: given a vector $x = (x^i)$, the image under η is written as

$$x_i = \eta_{ij} x^j$$

The inverse form can be used to lower indices as follows:

$$\alpha^i = \eta^{ij} \alpha_j$$

The inverse relation between η and η^{-1} implies that composing raising and lowering of the same index is consistent: lowering then raising an index results in no change and identically for raising then lowering. The convention for the order of the indices will be to keep the relative order of upper indices and lower indices, and place upper indices before lower indices. For example given a tensor $T^a{}_{bc}$ the corresponding totally covariant and totally contravariant tensors are

$$\begin{aligned} T_{dbc} &:= g_{da} T^a{}_{bc} \\ T^{ade} &:= g^{db} g^{ec} T^a{}_{bc} \end{aligned}$$

given a metric g on the space in which the indices a, b, c live.

Let us specify a basis (e_i) of E . Then the dual basis and the transposed basis satisfy the following relation:

$${}^t e_i = \eta_{ij} e^{*j}$$

which can be reversed as follows:

$$e^{*j} = \eta^{ji} ({}^t e_i)$$

Note however that using the transposition notation breaks our convention of using superscripts for bases of the dual vector space and thus shall be done with care.

These conventions will mainly be used on elements and sections of vector bundles throughout the text.

1.4 Conventions for dual forms

Let E be a n -dimensional vector space of finite dimension n over \mathbb{R} . Provide it with a volume element $\text{vol} \in \Lambda^n E$. The volume element defines an isomorphism $\Lambda^n E \simeq \mathbb{R}$. In most cases, we will actually use a space (bundle) of *linear forms* as E . This section will make use of the Einstein summation convention for repeated indices (as stated in Section 1.3).

We define here the notation for interior products between p -covectors and q -vectors. We will write $\wedge_{i:1 \rightarrow q} \alpha^i$ for the wedge product $\alpha^1 \wedge \alpha^2 \cdots \wedge \alpha^q$. Let $\alpha \in \Lambda^q E$ and $(X_j) \in (E^*)^p$. We will use the following convention :

$$\boxed{(X_1 \wedge X_2 \wedge \cdots \wedge X_p) \lrcorner \alpha = i_{X_1 \wedge X_2 \wedge \cdots \wedge X_p} \alpha = i_{X_p} i_{X_{p-1}} \cdots i_{X_1} \alpha} \quad (1.1)$$

The volume element provides isomorphisms $\Lambda^p E^* \rightarrow \Lambda^{n-p} E$ under the following contraction

$$X \in \Lambda^p E^* \mapsto X \lrcorner \text{vol} \quad (1.2)$$

In the case E is provided with an inner product, precomposing the isomorphisms by the induced inner product on the exterior powers $\Lambda^p E \rightarrow \Lambda^p E^*$ gives the *Hodge duality operator* \star .

Let $(e^i)_{1 \leq i \leq n}$ be a *direct* basis of E and (u_i) its dual basis. Let I be a sequence of p indices in $\llbracket 1, n \rrbracket$. Define

$$e^I = \bigwedge_{i:1 \rightarrow p} e^{I_i}$$

and dually

$$u_I = \bigwedge_{i:1 \rightarrow p} u_i$$

We will explicitly use the map $\Lambda^p E^* \rightarrow \Lambda^{n-p} E$ and adopt the notation

$$e_I^{(n-p)} := u_I \lrcorner \text{vol} \quad (1.3)$$

In components, vol is represented by the *Levi-Civita symbol* commonly written $\varepsilon_{i_1 \dots i_n}$, interpreted as a *completely antisymmetric* rank n tensor *in a basis of determinant 1*. For example, the duality between E^* and $\Lambda^{n-1} E$ is expressed, for $A = A^i u_i \in E^*$ as follows :

$$(A^i u_i) \lrcorner \text{vol} = A^i (u_i \lrcorner \text{vol}) = A^i e_i^{(n-1)} \quad (1.4)$$

$$(A \lrcorner \text{vol})_{i_1 \dots i_{n-1}} = A^{i_0} \varepsilon_{i_0 i_1 \dots i_{n-1}} \quad (1.5)$$

which generalizes to the case of p -forms on E in a straightforward manner.

From now on we assume that I does not contain repeated indices of the basis so that u_I and e^I are nonzero. The conventions have been chosen such that for two increasing multi-indices I and J ,

$$u_I \lrcorner e^J = \delta_I^J \quad (1.6)$$

Define I^c as the set of indices absent from I , identified with the corresponding increasing sequence. We will write $\epsilon(I)$ for the sign of the permutation which has I as its 1st p values and then I^c (the

permutation is a shuffle in the case I follows the increasing order) so that

$$e^I \wedge e^{I^c} = \epsilon(I) \text{vol} \quad (1.7)$$

We can then express the contraction in the following way

$$e_I^{(n-p)} = u_{I \lrcorner} \text{vol} = u_{I \lrcorner} \epsilon(I) e^I \wedge e^{I^c} = (u_{I \lrcorner} e^I) \epsilon(I) \wedge e^{I^c} = \epsilon(I) e^{I^c} \quad (1.8)$$

Note that this formula can act as a definition in the case (e^i) form a general family of vectors (not assumed to be a basis). Combining (1.7,1.8) we obtain the following formula stating that $e_I^{(n-p)}$ represent a (twisted) 1-form on $\Lambda^p E$: for $f_J e^J \in \Lambda^p E$

$$\boxed{(f_J e^J) \wedge e_I^{(n-p)} = f_I \text{vol}} \quad (1.9)$$

which justifies placing I as a subscript in $e_I^{(n-p)}$.

Part I
Geometry

Chapter 2

Jet Spaces

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In this section, we present the general structure of jet spaces. They are elementary constructions of differential geometry, which will be useful for considering Cartan geometry and which are also crucially involved in the formalism of field theory in physics. General references are [KMS93; GMS09; KL08].

2.1 Jets of a smooth map

The idea of jets of a smooth map is the geometric (covariant) realisation of the idea of polynomial expansion of a smooth map.

Definition 2.1.1 (Contact to the k th order). Let X and Y be two smooth manifolds and $f, g : X \rightarrow Y$ two smooth maps. Let x be a point in X . Let k be an integer. Then f and g are said to *have contact to order k at x* if $f(x) = g(x)$ and in every chart around x and around $f(x)$, f and g have the same partial derivatives of order 1 to k . Contact to the k -th order holds as soon as the partial derivatives match in one given couple of charts.

We will call equivalence classes *contact elements of order k* and write them $j_x^k(f)$.

Remark (Omission of contact order). Let X and Y be two smooth manifolds and $f, g : X \rightarrow Y$ two smooth maps. Let k be an integer and x a point in X . If f and g have contact to order k at x , then for every integer $l \leq k$ they have contact to order l .

Theorem 2.1.2 (Composition of contact elements). *Let X, Y and Z be three smooth manifolds and $f : X \rightarrow Y, g : Y \rightarrow Z$ two smooth maps. Let k be an integer. Let x be a point of X .*

Then $j_x^k(g \circ f)$ only depends on $j_x^k(f)$ and $j_{f(x)}^k(g)$.

2.2 Jet bundles

The notion of contact element allows realising higher order differentials of a smooth map as equivalence classes, whereas they cannot be naturally realised as tensors. These equivalence classes are naturally organised into smooth manifolds.

We will be especially interested in the contact elements of sections of a fibre bundle.

Definition 2.2.1. Let $Y \xrightarrow{\pi} X$ be a fibre bundle and k an integer. The *jet bundle of order k of π* , also called the *k -th prolongation of π* , is the reunion for all x in X of the space of contact elements to the order k of local sections of π in a neighbourhood of x . It has a natural structure of smooth manifold which smoothly projects to Y (by taking the image of x by representatives of the contact class). It is written

$$\mathcal{J}^k(\pi)$$

or sometimes more implicitly $\mathcal{J}^k(Y)$.

The jet spaces of increasing order form a sequence of bundles above X :

$$\cdots \rightarrow \mathcal{J}^3(\pi) \xrightarrow{\pi_2^3} \mathcal{J}^2(\pi) \xrightarrow{\pi_1^2} \mathcal{J}^1(\pi) \xrightarrow{\pi_0^1} \mathcal{J}^0(\pi) \simeq Y \xrightarrow{\pi} X$$

We name π_k^l the fibration map $\mathcal{J}^l(\pi) \rightarrow \mathcal{J}^k(\pi)$ for $l \geq k$ and π^l the fibration map $\mathcal{J}^l(\pi) \rightarrow X$.

In the case of a trivial bundle $F \times X \rightarrow X$, the jet space contains exactly all contact elements of maps from X to F . It is called the *jet space of order k from X to F* and we write it $\mathcal{J}^k(X, F)$.

Example 2.2.2. • The 0th order jet space is naturally identified to the total space of the fibre bundle

$$\mathcal{J}^0(\pi) \simeq Y$$

- Vectors on a manifold are often constructed as contact elements from a line. Consequently, there is a natural diffeomorphism

$$\mathcal{J}^1(\mathbb{R}, X) \simeq \mathbb{R} \times TX$$

above $\mathbb{R} \times X$.

- Linear forms on a manifold are usually constructed as the dual to the tangent bundle. They can also be constructed as contact elements of maps to a line. There is a natural diffeomorphism as follows:

$$\mathcal{J}^1(X, \mathbb{R}) \simeq T^*X \times \mathbb{R}$$

above $X \times \mathbb{R}$.

Example 2.2.3 (Connections on a fibre bundle). Jet bundles of sections provide a natural frame to study the geometry of Ehresmann connections.

Let $Y \xrightarrow{\pi} X$ be a fibre bundle¹.

An *Ehresmann connection* on π is the data of a smooth tangent distribution HY on Y , called the *horizontal distribution*, such that

$$TY = VY \oplus HY$$

In particular, HY is identified through π to π^*TX :

$$d\pi : HY \xrightarrow{\sim} \pi^*TX$$

¹The construction holds more generally for a surjective submersion.

As a consequence, for every y in Y , HY defines a section of the projection

$$TY \rightarrow \pi^*TX$$

Furthermore, for each such linear section, there exists for every point $y \in Y$ a local section σ of $Y \xrightarrow{\pi} X$ defined on a neighbourhood of $\pi(y)$ such that

$$\begin{aligned} \sigma(\pi(y)) &= y \\ d\sigma(T_{\pi(y)}X) &= HY \end{aligned}$$

This means that an Ehresmann connection associates to each point y of Y a 1-jet of section of $Y \xrightarrow{\pi} X$ sending $\pi(y)$ to y . Namely, it defines a section of $\mathcal{J}^1(\pi) \rightarrow Y$.

Conversely, a section of $\mathcal{J}^1(\pi) \rightarrow Y$ defines at each point of Y a tangent subspace supplementary to VY which is the tangent space to the graphs of representatives of the contact element. There is in fact a natural isomorphism

$$\mathcal{J}^1(\pi) \times_Y TY \xrightarrow{\sim} \mathcal{J}^1(\pi) \times_Y (\pi^*TX \oplus VY)$$

In conclusion, there is a natural bijection between Ehresmann connections of a fibre bundle $Y \xrightarrow{\pi} X$ and sections of the 1st order jet bundle

$$\mathcal{J}^1(\pi) \rightarrow Y$$

Theorem 2.2.4. *Let $Y \xrightarrow{\pi} X$ be a fibre bundle and k an integer. Then $\mathcal{J}^{k+1}(\pi) \rightarrow \mathcal{J}^k(\pi)$ is an affine bundle modelled on the pullback bundle of the bundle of homogeneous degree $k + 1$ polynomial maps from TX to the bundle of vertical tangent vectors on Y :*

$$\text{Sym}^{k+1} T^*X \otimes VY$$

We give a more detailed construction for the first order jet bundles.

Theorem 2.2.5 (Space of first order jets). *Let $Y \xrightarrow{\pi} X$ be a fibre bundle. Then the space of first order jets of sections of π is the submanifold of $\mathcal{J}^1(X, Y) \simeq \pi_X^*T^*X \otimes_{X \times Y} \pi_Y^*TY$ which satisfies the following affine equation:*

$$d\pi \circ \sigma = \pi^* \text{id}_{TX} \quad \text{with } \sigma \in \pi_X^*T^*X \otimes_{X \times Y} \pi_Y^*TY$$

with $\otimes_{X \times T}$ standing for the tensor product of vector bundles above $X \times Y$.

The structure of affine bundle under the vector bundle $\pi^*T^*X \otimes VY$ can be explained by the pullback construction in the following diagram:

$$\begin{array}{ccccc} & & \mathcal{J}^1(Y) & \xrightarrow{\quad} & 1 \\ & & \downarrow & \lrcorner & \downarrow \pi^*(\text{id}_{TX}) \\ \pi^*(T^*X) \otimes VY & \longrightarrow & \pi^*(T^*X) \otimes TY & \longrightarrow & \pi^*(T^*X \otimes TX) \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

If X is of dimension n and Y is a bundle with fibre dimension d then $\mathcal{J}^1(\pi)$ has fibre dimension nd over Y .

Remark. Let $Y \rightarrow X$ be a fibre bundle equipped with an Ehresmann connection. It can be represented as a section $Y \rightarrow \mathcal{J}^1(\pi)$. In particular, it induces an affine bundle isomorphism

$$\pi^*(T^*X) \otimes VY \xrightarrow{\sim} \mathcal{J}^1(\pi)$$

fibred above Y .

Definition 2.2.6 (Prolongation of sections to order k). Let $Y \xrightarrow{\pi} X$ be a fibre bundle and k an integer.

Let $s : X \rightarrow Y$ be a smooth section of π . Then s admits a natural smooth lift through $\mathcal{J}^k(\pi) \rightarrow Y$ which is called the *prolongation of s to order k* or the k -jet of s and is defined as follows:

$$j^k(s) : x \in X \mapsto j_x^k(s) \in \mathcal{J}^k(\pi)$$

The following diagram commutes :

$$\begin{array}{ccc} & \mathcal{J}^k(\pi) & \\ j^k(s) \nearrow & & \searrow \pi_0^k \\ X & \xrightarrow{s} & Y \end{array}$$

This construction is crucial and allows imposing *differential conditions* on a map by geometrically requiring that its prolongations take value in a specific subset of the jet spaces.

Example 2.2.7 (Differential of a map). Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a smooth map. Then the 1-jet of f can be described as follows:

$$x \in \mathbb{R}^m \mapsto (x, f(x), df(x)) \in \mathcal{J}^1(\mathbb{R}^m \rightarrow \mathbb{R}^n) \simeq \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m*} \otimes \mathbb{R}^n$$

Jet spaces give the geometrical framework to work with constructions depending not only on the value of maps but on their jet to a certain order.

Example 2.2.8 (Pullback of differential forms). Let $Y \xrightarrow{\pi} X$ be a fibre bundle. Then for any smooth section $X \xrightarrow{s} Y$ and any differential form $\alpha \in \Omega^\bullet(Y)$, the pullback of α through s depends at each point of X of s at the first order. More precisely, there exists a (graded) map

$$\begin{aligned} \mathcal{J}^1(\pi) \times_Y \Lambda^\bullet T^*Y &\rightarrow \pi^* \Lambda^\bullet T^*X \\ (j^1 s, \gamma) &\mapsto s^* \gamma \end{aligned}$$

which is fibred over Y and straightforwardly constructed as

$$\begin{array}{c} \mathcal{J}^1(\pi) \times_Y \Lambda^\bullet T^*Y \\ \downarrow \\ \text{Hom}(\pi^* T X, T Y) \otimes_Y \Lambda^\bullet T^*Y \\ \downarrow \\ \text{Hom}(\Lambda^\bullet \pi^* T X, \Lambda^\bullet T Y) \otimes (\Lambda^\bullet T Y)^* \\ \downarrow \\ \Lambda^\bullet \pi^* T^* X \simeq \pi^* \Lambda^\bullet T^* X \end{array}$$

Then given a smooth section $X \xrightarrow{s} Y$ and a differential form $\alpha \in \Omega^\bullet(Y)$, the pullback $s^*\alpha$ can be constructed as follows:

$$X \xrightarrow{j^1 s} \mathcal{J}^1(\pi) \xrightarrow{(\text{id}, \alpha \circ \pi_Y^1)} \mathcal{J}^1(\pi) \times_Y \Lambda^\bullet T^* Y \rightarrow \Lambda^\bullet T^* X$$

Theorem 2.2.9 (Functoriality). *Let $Y_1 \xrightarrow{\pi_1} X_1$ and $Y_2 \xrightarrow{\pi_2} X_2$ be two fibre bundles. Let (f, ϕ) be a fibre morphism, there is a commutative diagram as follows:*

$$\begin{array}{ccc} Y_1 & \xrightarrow{\phi} & Y_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

Assume furthermore that f is a local diffeomorphism. Then for each integer k there is an induced map $\mathcal{J}^k(\phi) : \mathcal{J}^k(\pi_1) \rightarrow \mathcal{J}^k(\pi_2)$ such that for any contact element $j_x^k \sigma$ of order k of a section of π_1 , the following equation holds (and the right-hand term is well defined):

$$j^k \phi \circ j_x^k \sigma = \mathcal{J}^k(\phi)(j_x^k \sigma) \circ j_x^k f \quad (2.1)$$

Remark. Since f is locally invertible, its contact elements are as well so that Equation (2.1) can be turned into a *definition* of the induced map.

In particular, any fibration automorphism lifts to automorphisms of the jet bundles. Similarly to the case of general jet spaces, to an infinitesimal bundle automorphism of π is associated an induced map on the jet bundle of section at every order. The case of first order jet spaces is described in Example 2.2.10

Local coordinates on the first jet bundle

We write F a fibre of the locally trivial bundle $Y \xrightarrow{\pi} X$. Let (x^i) be a local coordinate system on an open subset $\mathcal{U} \subset X$ and $y : Y \rightarrow F$ local fibre coordinates on Y defining a local trivialisation:

$$\begin{array}{ccc} Y|_{\mathcal{U}} & \xrightarrow[\sim]{(\pi, y)} & \mathcal{U} \times F \\ & \searrow \pi & \swarrow \pi_{\mathcal{U}} \\ & \mathcal{U} & \end{array}$$

The tangent bundle has the following local trivialisation

$$\begin{array}{ccc} TY|_{\mathcal{U}} & \xrightarrow[\sim]{(D\pi, Dy)} & T\mathcal{U} \times TF \\ Dy \downarrow & & \downarrow \text{id} \times \pi_{TF} \\ \pi^* T\mathcal{U} & \xrightarrow{\sim} & T\mathcal{U} \times F \end{array}$$

and the tensor product with $\pi^*(T^*\mathcal{U})$ gives

$$\begin{array}{ccc} \pi^*(T^*\mathcal{U}) \otimes TY|_{\mathcal{U}} & \xrightarrow[\sim]{\text{id}_{T^*\mathcal{U}} \otimes (D\pi \oplus Dy)} & T^*\mathcal{U} \otimes (T\mathcal{U} \oplus TF) \\ \text{id}_{T^*\mathcal{U}} \otimes D\pi \downarrow & & \downarrow \\ \pi^*(T^*\mathcal{U} \otimes T\mathcal{U}) & \xrightarrow{\sim} & (T^*\mathcal{U} \otimes T\mathcal{U}) \times F \end{array}$$

The jet bundle corresponds to the inverse image of the section $\pi^* \text{id}_{T\mathcal{U}} : Y \rightarrow \pi^*(T^*\mathcal{U} \otimes T\mathcal{U})$ hence we obtain

$$\begin{array}{ccc} \mathcal{J}^1 Y|_{\mathcal{U}} & \xrightarrow[\text{id} \otimes D_y]{\sim} & T^*\mathcal{U} \otimes T\mathcal{U} \\ \downarrow & & \downarrow \\ Y|_{\mathcal{U}} & \xrightarrow[\text{(\pi, y)}]{\sim} & \mathcal{U} \times F \end{array}$$

The local trivialisation is given by base space coordinates (x^i) , fibre coordinates (y^a) and 1st order coordinates (v_i^a) corresponding to the fibre of $T^*\mathcal{U} \otimes T\mathcal{U} \rightarrow \mathcal{U} \times F$. Notice that the bundle trivialisation gives sections of $\mathcal{J}^1 Y \rightarrow Y$ and according to the remark following Theorem 2.2.5 this identifies the affine bundle with the underlying vector bundle $\pi^*(T^*U) \otimes VY \rightarrow Y$.

Of course, it is possible to be completely explicit by using local coordinates on F which allows identifying the jets with jets between Euclidean spaces. They are easily described using Taylor polynomials.

Example 2.2.10 (1st order prolongation of a projectable vector field [Olv86]). Let $Y \xrightarrow{\pi} X$ be a fibre bundle and $u \in \Gamma(TY)$ an infinitesimal bundle automorphism, namely a projectable vector field.

Let us use local fibration coordinates (x^i, y^a, v_i^a) on $\mathcal{J}^1(\pi)$. Then if

$$u = u^i \partial_{x^i} + u^a \partial_{y^a}$$

(with u^i independent of y^a) the first order prolongation takes the form

$$j^1 u = u^i \partial_{x^i} + u^a \partial_{y^a} + (\partial_{x^i} u^a + v_i^B \partial_{y^B} u^a) \partial_{v_i^a}$$

2.3 Contact structure of first order jet bundles

Beside their fibration over the bundle total space and the tower of affine bundles, jet bundles have more structure which is a manifestation of their construction as spaces of contact elements. We will focus in this section on the spaces of first order jets for simplicity and because it will be the main case of interest for us.

Definition 2.3.1. Let $Y \xrightarrow{\pi} X$ be a fibre bundle. Let $\phi : X \rightarrow \mathcal{J}^1(\pi)$ be a section of the first order jet bundle. It is called *holonomic* (or *integrable*) if there exists a smooth section $s : X \rightarrow Y$ such that $\phi = j^1 s$.

The structure we will now introduce allows identifying when a section of the jet bundle is holonomic.

Let $Y \xrightarrow{\pi} X$ be a fibre bundle. Let us use a generic local fibration coordinates systems (x^i, y^a) with x^i coordinates on X and y^a fibre coordinates on Y .

We can construct the following 1-forms on $\mathcal{J}^1(\pi)$:

$$\theta^a = dy^a - v_i^a dx^i$$

The family (θ^a) depends on the coordinate system used. They can however be gathered in a uniquely defined VY -valued form

$$\mathcal{J}^1(\pi) \times_Y TY \xrightarrow{\sim} \mathcal{J}^1(\pi) \times_Y (\pi^* TX \oplus VY)$$

mentioned in Example 2.2.3. The VY -valued form on $\mathcal{J}^1(\pi)$ is then the composition

$$T\mathcal{J}^1(\pi) \xrightarrow{d\pi_0^1} \mathcal{J}^1(\pi) \times_Y TY \xrightarrow{\sim} \mathcal{J}^1(\pi) \times_Y (\pi^*TX \oplus VY) \rightarrow (\pi_0^1)^*VY$$

The forms θ^a are called *contact 1-forms* and can be used to identify first order prolongations of maps from X to Y as according to the following theorem.

Theorem 2.3.2.

1. The differential forms α on $\mathcal{J}^1(X, Y)$ such that for all $f : X \rightarrow Y$,

$$(j^1f)^*\alpha = 0$$

form a differential ideal. It is called the contact ideal and its elements are called contact forms.

2. The contact ideal is (locally) algebraically generated by the forms θ^a and the forms $d\theta^a$ in any coordinate system.
3. A section $X \rightarrow \mathcal{J}^1(X, Y)$ is holonomic if and only if it pulls back every contact form to 0, or equivalently every contact 1-form.

The differentials of the contact 1-forms are straightforward to compute:

$$d\theta^a = -dv_i^a \wedge dx^i$$

Remark. The notion of “contact structure” of jet spaces is different from the notion of contact structure in the so-called “contact geometry” in which a contact structure is the data of a maximally non-integrable codimension one tangent distribution. They are however not unrelated, as the following example shows.

Example 2.3.3 (Jet space of maps to the line). Let X be a smooth manifold. As stated earlier, the first order jet space of maps from X to \mathbb{R} is identified to the following space:

$$T^*X \times \mathbb{R}$$

The line \mathbb{R} is equipped with a tautological coordinate $y : \mathbb{R} \rightarrow \mathbb{R}$. Consequently, $T\mathbb{R}$ is trivialised and the corresponding contact form can be taken with value in \mathbb{R} :

$$\theta = dy - p_i dq^i$$

in which we changed the coordinates (x^i, v_i^a) on X to a more familiar notation (q^i, p_i) . This is the standard form of (usual) contact forms in a Darboux coordinate system. If we restrict to $T^*X \times \{0\} \simeq T^*X$, we obtain a submanifold on which $dp_i \wedge dq^i$ defines an exact symplectic structure.

Example 2.3.4 (Jet space of maps from the line). Let Y be a smooth manifold. As stated earlier, the first order jet space of maps from \mathbb{R} to Y is identified to the following space:

$$\mathbb{R} \times TY$$

The line \mathbb{R} is equipped with a tautological coordinate $t : \mathbb{R} \rightarrow \mathbb{R}$. Consequently, $T^*\mathbb{R}$ is trivialised and the corresponding contact forms take the following form:

$$\theta^a = dy^a - v^a dt$$

If we have an arbitrary path $\mathbb{R} \rightarrow TY$ then it is the tangent path to a path $\mathbb{R} \rightarrow Y$ if and only if θ^a vanish on it. This is the “contact condition” or “holonomy condition”.

As mentioned earlier, contact 1-forms on $\mathcal{J}^1(X, Y)$ are all components of a geometrically defined $\pi_Y^1{}^*TY$ -valued 1-form $T\mathcal{J}^1(X, Y) \rightarrow \pi_Y^1{}^*TY$. As a consequence, its exterior powers define a graded algebra map

$$\Lambda^\bullet \pi_Y^1{}^*T^*Y \mapsto \Lambda^\bullet T^*\mathcal{J}^1(X, Y)$$

which takes value in the contact forms.

Remark. When the fibre bundle $Y \rightarrow X$ is trivial, or more generally is equipped with an Ehresmann connection, there is a consistently defined projection

$$TY \rightarrow VY$$

which can be used in place of $\partial_{y^a} \otimes \partial y^a$ and an isomorphism

$$\mathcal{J}^1(\pi) \xrightarrow{\sim} \pi^*T^*X \otimes VY$$

which can be used in place of $\partial_{y^a} \otimes v_i^a dx^i$ in the construction of the VY -valued contact form.

In particular, the connection 1-form $TY \rightarrow VY$ can be identified with the pullback of the VY -valued contact form by the section $Y \rightarrow \mathcal{J}^1(\pi)$.

The contact structure has an incarnation dual to the contact ideal, known as the Cartan distribution:

Theorem 2.3.5 (Cartan distribution, [KL08; KV98]). *Let $Y \xrightarrow{\pi} X$ be a fibre bundle and k an integer. The tangent planes to all holonomic sections of $\mathcal{J}^k(\pi)$ span a distribution called the Cartan distribution.*

The Cartan distribution is non-integrable and its annihilator is the space of contact 1-forms of $\mathcal{J}^k(\pi)$.

Under the hypotheses of Theorem 2.2.9, the induced map between jet spaces sends prolongations of sections to prolongations of sections and therefore preserves the contact structure and the Cartan distribution.

2.4 Anholonomic jets

Jet spaces can support non holonomic jets. The question of whether a given jet section is holonomic or not can be treated using the contact structure or equivalently the Cartan distribution. This gives a mean to construct a map satisfying given differential constraints.

In some situations, the differential data we have on a map we try to construct is not a higher order jet, but the jet of a jet (or a higher order iteration). Thus we have to check iteratively holonomy constraint in order to ascertain whether there exists a map with the corresponding jet data.

Definition 2.4.1 ([Ehr54]). Let X and Y be two smooth manifolds. For an integer k , the anholonomic space of jets of order k from X to Y is written $\tilde{\mathcal{J}}^k(X, Y)$ and is inductively constructed as follows:

$$\begin{aligned} \tilde{\mathcal{J}}^0(X, Y) &= X \times Y \\ \tilde{\mathcal{J}}^{k+1}(X, Y) &= \mathcal{J}^1(\tilde{\mathcal{J}}^k(X, Y) \rightarrow X) = \underbrace{\mathcal{J}^1 \cdots \mathcal{J}^1}_{k \text{ times}} \pi \end{aligned}$$

There is a natural inclusion

$$\mathcal{J}^k(X, Y) \hookrightarrow \tilde{\mathcal{J}}^k(X, Y)$$

Nomenclature brings the following important distinction: a *jet* in an anholonomic jet space is called holonomic if it belongs to the subspace of holonomic jets. In contrast, a *section* of the holonomic jet space is called holonomic if it is the prolongation of a smooth map. Similarly, a section of the anholonomic jet space is called holonomic if it takes value in the holonomic jet space and is holonomic as such.

2.5 Jets of diffeomorphisms

As mentioned earlier, jet spaces can be used to construct smooth maps satisfying differential conditions. In particular, differential equations correspond to subsets of jet spaces. Another subset which will interest us is that of *jets of local diffeomorphisms*. Namely, these are contact elements which have an “inverse contact element” in the sense that both left and right composition will give the contact element of identity at the considered points.

Definition 2.5.1. Let X and Y be two smooth manifolds and k be an integer.

The *manifold of invertible jets of order k* or *manifold of jets of diffeomorphisms of order k* is the submanifold of $\mathcal{J}^k(X, Y)$ of contact elements of maps which are a local diffeomorphism at the considered point. We write it

$$\mathcal{J}_{\text{diff}}^k(X, Y)$$

It is empty if X and Y are manifolds of different dimensions and an open and dense submanifold of $\mathcal{J}^k(X, Y)$ if X and Y have same dimension.

The inverse function theorem implies that a contact element of $\mathcal{J}^k(X, Y)$ for $k \geq 1$ belongs to $\mathcal{J}_{\text{diff}}^k(X, Y)$ if and only if the reduction at order 1 belongs to $\mathcal{J}_{\text{diff}}^1(X, Y)$. When X and Y are manifolds of the same dimension, $\mathcal{J}_{\text{diff}}^1(X, Y)$ is an open and dense submanifold of $\mathcal{J}^1(X, Y)$ and as a consequence the same holds at every order $k \geq 1$.

Assume that X and Y are manifolds of the same dimension. Let x and y be two points respectively in X and Y . Then the submanifold of invertible contact elements above $(x, y) \in X \times Y$ is isomorphic to the submanifold of linear isomorphisms

$$T_x X \xrightarrow{\sim} T_y Y$$

It has in particular a natural right action of $\text{GL}(T_x X)$ and a natural left action of $\text{GL}(T_y Y)$. They are both transitive and free thus make $\text{Iso}(T_x X, T_y Y)$ into a $\text{GL}(T_x X)$ -torsor and a $\text{GL}(T_y Y)$ -torsor respectively.

Let us give an illustration, albeit degenerate, of how this space can be used to build maps.

Example 2.5.2 (Graphs of symplectomorphisms). Let (X, ω_X) and (Y, ω_Y) be two symplectic manifolds of dimension $2n$.

It is well known that a map $f : X \rightarrow Y$ is a symplectomorphism ($f^* \omega_Y = \omega_X$) if and only if the difference symplectic form $\pi_Y^* \omega_Y - \pi_X^* \omega_X$ on $X \times Y$ vanishes on the graph of f . This implies in particular that f has to be a local diffeomorphism. This problem has a geometrical generalisation: Lagrangian submanifolds of $X \times Y$ for $\pi_Y^* \omega_Y - \pi_X^* \omega_X$ generalise graphs of symplectomorphisms. Around points on which they project transversally to X , they locally define a diffeomorphism from X to Y which preserves the symplectic form.

As explained in Example 2.2.8, the pullback of ω_Y is a well defined section of $(\pi_X^1)^* T^* X$ over $\mathcal{J}^1(X, Y)$ hence *a fortiori* over $\mathcal{J}_{\text{diff}}^1(X, Y)$. We write it $\pi^{1*} \omega_Y$. Thus the geometric question of

finding Lagrangian submanifolds of $X \times Y$ amounts here to finding submanifolds of $\mathcal{J}_{\text{diff}}^1(X, Y)$ of dimension $2n$ on which $\pi^1^* \omega_Y - \pi_X^1^* \omega_X$ and the contact ideal vanish.

The point is that $\Delta_X \omega := \pi^1^* \omega_Y - \pi_X^1^* \omega_X$ is now realised as a section of $(\pi_X^1)^* T^* X$. As a consequence, given a submanifold $L \subset \mathcal{J}^1(X, Y)$ with a point $l \in L$ at which L is transverse to $\pi_X^1 T^* X$ and such that $\Delta_X \omega$ vanishes on $T_l L$, necessarily $\Delta_X \omega|_l = 0$. Therefore the vanishing of $\Delta_X \omega$ on submanifolds of $\mathcal{J}_{\text{diff}}^1(X, Y)$ is equivalent to their inclusion in the locus of zeroes of $\Delta_X \omega$, at least at points at which they are transversal to π_X^1 .

Since contact elements of $\mathcal{J}_{\text{diff}}^k(X, Y)$ are invertible, there is bijection

$$\mathcal{J}_{\text{diff}}^k(X, Y) \rightarrow \mathcal{J}_{\text{diff}}^k(Y, X)$$

It can even be proved to be a diffeomorphism:

Theorem 2.5.3. *Let X and Y be two differentiable n -manifolds and k an integer.*

Then inversion of contact elements define a diffeomorphism

$$\mathcal{J}_{\text{diff}}^k(X, Y) \xrightarrow{\sim} \mathcal{J}_{\text{diff}}^k(Y, X)$$

Remark. Contact forms can be defined for jet spaces of arbitrary order and inversion preserves these contact structures. Indeed, given two manifolds X and Y , a holonomic section of $\mathcal{J}^k(X, Y)$ corresponds to a local diffeomorphism $f : X \rightarrow Y$, for which a local inverse g has a prolongation $j^k g$ to $\mathcal{J}^k(Y, X)$ such that the following diagram commutes (on the subsets on which it is well defined):

$$\begin{array}{ccc} \mathcal{J}_{\text{diff}}^k(X, Y) & \xrightarrow{\sim} & \mathcal{J}_{\text{diff}}^k(Y, X) \\ j^k f \uparrow & & j^k g \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

We present another familiar example, which will be a relevant model when considering Cartan geometry.

Example 2.5.4 (Frame bundles). Let X be a smooth n -manifold. Consider the space of diffeomorphism jets $\mathcal{J}_{\text{diff}}^1(\mathbb{R}^n, X)$.

The subspace of contact elements starting from $0 \in \mathbb{R}^n$ is the following subset of $\mathcal{J}_{\text{diff}}^1(\mathbb{R}^n, X)$:

$$\{(x, u) \in X \times T_0^* \mathbb{R}^n \otimes TX \mid u \text{ is a linear isomorphism } T_0 \mathbb{R}^n \xrightarrow{\sim} T_x X\}$$

In other words, this is isomorphic to the bundle of linear frames of TX .

There is a natural trivialisation $T_0 \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$, and the bundle of linear frames has a right action of $\text{GL}(T_0 \mathbb{R}^n) \simeq \text{GL}_n$ which gives it a structure of principal GL_n -bundle. According to the embedding

$$\mathcal{J}_{\text{diff}}^1(\mathbb{R}^n, X) \xrightarrow{\sim} \mathcal{J}_{\text{diff}}^1(X, \mathbb{R}^n) \hookrightarrow T^* X \otimes T \mathbb{R}^n$$

the frame bundle is equipped with a $T_0 \mathbb{R}^n \simeq \mathbb{R}^n$ -valued canonical form α , called the *solder form* of the frame bundle.

Since the whole tangent bundle of \mathbb{R}^n is naturally trivialised, α exists over the whole jet space. Using coordinates (z^i) on \mathbb{R}^n , the \mathbb{R}^n -valued contact form on $\mathcal{J}_{\text{diff}}(X, \mathbb{R}^n)$ takes the following form:

$$dz^i \otimes \frac{\partial}{\partial z^i} - \alpha \in \Omega^1(\mathcal{J}_{\text{diff}}(X, \mathbb{R}^n), \mathbb{R}^n)$$

2.6 Germs of smooth maps

We give a brief presentation of the more general notion of *germs* of maps. In our framework, germs is a purely topological notion and does not rely on the local structure of manifolds. Let X and Y be two topological spaces.

Definition 2.6.1. Let x be a point of X . Two maps f and g respectively defined on a neighbourhood \mathcal{U}_f and \mathcal{U}_g of x with value in Y have the same germ at x if there exists a neighbourhood \mathcal{U} of x which is included in $\mathcal{U}_f \cap \mathcal{U}_g$ and such that

$$f|_{\mathcal{U}} = g|_{\mathcal{U}}$$

In other words, the germ at x is a formalisation of the behaviour of a map in the neighbourhood of x .

Theorem 2.6.2. *The space of germs of maps from X to Y has natural projections to X and Y . It can furthermore be naturally equipped with a topology, called the étale topology, such that these projections are continuous.*

Similarly to jets, germs can be composed.

Assume now that X and Y are differentiable manifolds. We can consider germs of smooth maps. Then jets are a function of germs:

Theorem 2.6.3. *Let x be a point of X and f and g two smooth maps from X to Y . If f and g have the same germ at x , then they have contact to every order $k \in \mathbb{N}$.*

This defines a continuous map from the spaces of germs to the spaces of jets which is compatible with composition.

In particular, this makes superfluous the distinction between k -contact elements of globally defined maps and locally defined maps. Indeed, given a map f defined on a neighbourhood \mathcal{U} of x , using a smaller neighbourhood \mathcal{V} diffeomorphic to a ball, it is possible to build a local map which coincides with f on \mathcal{V} but is constant outside a neighbourhood of x , and can therefore be extended to the whole of X .

Chapter 3

Frame bundles and Cartan Geometry

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This chapter gathers definitions and notations on frame bundles, G -structures, connections and Cartan geometry we will be using in later chapters.

3.1 Principal bundles and connections

We start with a quick presentation of the main constructions on principal bundles which will be relevant for us. We only give a succinct account of the structure, a detailed treatment can be found in [KN96]. Let M be a smooth manifold and G a Lie group.

Principal bundles

Definition 3.1.1 (G -principal bundles). A G -principal bundle over M is a locally trivial fibre bundle $P \rightarrow M$ equipped with a right action of G such that one of the following equivalent properties is satisfied:

- G acts freely on P and transitively on the fibres above M .

- The submersion $P \rightarrow M$ identifies M with the quotient space P/G .

The Lie group G is called the *structure group* of the principal bundle.

Local trivialisations identify P with $M \times G$ but with G as a right G -space: fibres do not have in general a consistently defined group structure.

There is a natural notion of morphism of G -principal bundle, but we will be interested in relating principal bundles with *different structure groups*. The corresponding notion of morphism is as follows:

Definition 3.1.2 (Reduction of structure group). Let K be a Lie group equipped with a morphism $K \xrightarrow{\phi} G$. Let $P_G \rightarrow M$ be a G -principal bundle. A *reduction of structure group to K (along $K \xrightarrow{\phi} G$)* of P_G is the data of an K -principal bundle $P_K \rightarrow M$ equipped with a $(K \rightarrow G)$ -intertwining bundle map $f : P_K \rightarrow P_G$, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
 P_K \times K & \xrightarrow{f \times \phi} & P_G \times G \\
 \downarrow (p,h) \mapsto p \cdot h & & \downarrow (p,h) \mapsto p \cdot h \\
 P_K & \xrightarrow{f} & P_G \\
 & \searrow & \swarrow \\
 & M &
 \end{array}$$

Remark. The name “reduction” comes from the case in which K is an embedded subgroup of G . When $K \rightarrow G$ is surjective (thus a quotient map), they are often called *lifting of structure group*. Often the morphism $K \rightarrow G$ will be left implicit when there is no risk of confusion.

Example 3.1.3 (Orthonormal frames). Let $E \rightarrow M$ be a vector bundle of rank k over M . The linear frames of each one of its fibres can be gathered into a smooth fibre bundle above M . The group GL_k acts *from the right* on the linear frames of each fibre of E , in a free and transitive fashion. As a consequence, the bundle of linear frames of E has the structure of a GL_k -principal bundle over M . We write it $\mathrm{GL}(E)$.

If E is equipped with a positive inner product, this defines a notion of orthonormal frames. The orthonormal frames of E form a sub-fibre bundle of $\mathrm{GL}(E)$. Similarly to $\mathrm{GL}(E)$, the orthonormal frames of E form an O_k -principal bundle over M . Furthermore, the embedding into $\mathrm{GL}(E)$ is O_k equivariant: the inner product induces a reduction of structure group of $\mathrm{GL}(E)$ to O_k .

The converse holds as well: indeed specifying even a single orthonormal frame on each fibre of E is enough to specify the associated inner product.

Associated bundles

Let $P \rightarrow M$ be a G -principal bundle and X a *left* G -manifold.

Definition 3.1.4 (Associated bundle). The space

$$P[X] := P \times_G X = P \times X / ((pg, x) \sim (p, gx))$$

is a smooth manifold which forms a fibre bundle over M with fibres diffeomorphic to X . It is called *the associated bundle to P of fibre X* .

When X is a vector space and the action of G is linear, $P[X]$ has the structure of a vector bundle above M .

Remark. The definition involves X not as a smooth manifold but as a G -manifold: the action of G on X is part of the data.

The fibres of an associated bundle of fibre X can be identified with X , but this identification is only defined up to elements of G . Depending on the structure preserved by G the associated bundle can have different kind of structures, for example being itself a principal bundle over a Lie group, or a vector bundle.

Example 3.1.5. Let $E \rightarrow M$ be a vector bundle of dimension k over M . Recall that the frame bundle $\text{GL}(E)$ is a GL_k -principal bundle above M . Since GL_k naturally acts on \mathbb{R}^k , there exists an associated bundle $\text{GL}(E)[\mathbb{R}^k]$. We now prove that it is naturally isomorphic to E .

Since there is a natural embedding $\text{GL}(E) \hookrightarrow \text{Hom}(\mathbb{R}^k, E)$, there is a natural bundle map

$$\text{GL}(E) \times \mathbb{R}^k \rightarrow E$$

The inverse image of any set vector $u \in E_m$ is given by a frame $p \in \text{GL}(E)_m$ and the components $u|_p \in \mathbb{R}^k$ of u in p . Since for any g in GL_k , $u|_{pg} = g^{-1} \cdot u|_p$, one concludes that the inverse image of u is exactly an equivalence class for the equivalence relation

$$\forall g \in \text{GL}_k, \quad (pg, x) \sim (p, g \cdot x)$$

Namely, the bundle map factorises to a bundle isomorphism

$$\text{GL}(E)[\mathbb{R}^k] \xrightarrow{\sim} E$$

which can furthermore be proven to be linear.

Example 3.1.6 (Reduction of structure group). Let K be a closed subgroup of G and $P_K \hookrightarrow P_G = P$ a reduction of structure group to K . Then the action map

$$\begin{aligned} P_K \times G &\rightarrow P_G \\ (p, g) &\mapsto pg \end{aligned}$$

factors through a bundle isomorphism

$$\begin{aligned} P_K \times_K G &\xrightarrow{\sim} P_G \\ (p, g) &\mapsto pg \end{aligned}$$

Namely, a reduction of structure group exhibits P_G as an associated K -principal bundle.

Conversely, given a K -principal bundle $P_K \rightarrow M$, the G -manifold $P_K \times_K G$ always form a G -principal bundle over M which is naturally equipped with the reduction of structure group to K :

$$P_K \hookrightarrow P_K \times_K G$$

In summary, there is an equivalence between reductions of structure group of P_G to K and G -principal bundle isomorphisms from P_G to a G -principal bundle associated to an K -principal bundle.

Theorem 3.1.7 (Trivialisation of associated bundles). *We call π the fibration map $P \rightarrow M$. There is a natural trivialisation of fibre bundle above P :*

$$\pi^* P[X] \simeq P \times_M P[X] \xrightarrow{\sim} P \times X$$

which is G -intertwining in the sense that the following diagram commutes:

$$\begin{array}{ccc} \pi^*P[X] \times G & \longrightarrow & P \times X \times G \\ \downarrow \sim & & \downarrow (p,x,g) \mapsto (pg, g^{-1}x) \\ \pi^*P[X] & \longrightarrow & P \times X \end{array}$$

with $\pi^*P[X] \times G \rightarrow \pi^*P[X]$ the natural identification between $P[X]_{\pi(p)}$ and $P[X]_{\pi(pg)}$ for all $p \in P$.

Corollary 3.1.8 (Sections of an associated bundle, [KMS93], Section III.11.14). Let $\Gamma(P, X)^G$ denote the subset of G -equivariant maps $P \rightarrow X$.

The following map

$$\begin{array}{ccc} \Gamma(M, P[X]) & \rightarrow & \Gamma(P, X)^G \\ \sigma & \mapsto & \pi^*\sigma \end{array}$$

is a bijection.

Assume now that X is a linear representation V of G . The above map is a $\mathcal{C}^\infty(M)$ -linear isomorphism and extends to a graded $\Omega^\bullet(M)$ -module isomorphism between $P[V]$ -valued differential forms on M and V -valued differential forms on P which are

- *horizontal*.
- G -equivariant.

Such forms on P are called *basic* or *tensorial*; we write their space $\Omega_{\text{hor}}^\bullet(P, V)^G$:

$$\Omega^\bullet(M, P[V]) \simeq \Omega_{\text{hor}}^\bullet(P, V)^G$$

Given a reduction of structure group $P_K \hookrightarrow P_G$ and a G -manifold X , there is a bijection

$$\begin{array}{ccc} \Gamma(P_G, X)^G & \leftrightarrow & \Gamma(P_K, X)^K \\ A & \mapsto & A|_{P_K} \end{array}$$

which induces a bijection between the sections of the associated bundles. For a vector space V which has a group action of G , there is similarly a bijection between V -valued basic forms:

$$\begin{array}{ccc} \Omega^\bullet(P_G, V)^G & \leftrightarrow & \Omega^\bullet(P_K, V)^K \\ \alpha & \mapsto & \alpha|_{P_K} \end{array}$$

Theorem 3.1.9. *Let K be a closed subgroup of G . Then $P \rightarrow P/K$ has a structure of K -principal bundle. There is a natural associated bundle isomorphism*

$$P[G/K] \xrightarrow{\sim} P/K$$

There is a bijection between reductions of structure group of P to K and the set of global sections of P/K :

$$\begin{array}{ccc} \Gamma(M, P/K) & \xrightarrow{\sim} & \{\text{Reductions of structure group of } P \text{ to } K\} \\ \sigma & \mapsto & (\sigma^*P \rightarrow P/K) \end{array}$$

Example 3.1.10 (Inner products). Let E be a k -dimensional vector bundle over M . It is associated to $\mathrm{GL}(E)$. Since $\mathrm{GL}_k / \mathrm{O}_k$ is naturally identified to the set of inner products on \mathbb{R}^k , the associated bundle $\mathrm{GL}(E)[\mathrm{GL}_k / \mathrm{O}_k]$ is identified to the bundle of inner products on E . This gives a new perspective on the correspondence between inner products on E , namely sections of $\mathrm{GL}(E)[\mathrm{GL}_k / \mathrm{O}_k]$, and reductions of structure group of $\mathrm{GL}(E)$ to O_k .

Principal connections

The action of G on P induces an action of the associated Lie algebra \mathfrak{g} by differentiation of the 1-parameter subgroups of diffeomorphisms. For $\xi \in \mathfrak{g}$ we write $\bar{\xi}$ the corresponding vector field on P . Note that the vector fields $\bar{\xi}$ are vertical (they project to 0 in TM) and they span all the vertical directions.

Definition 3.1.11 (Principal Ehresmann connection on a principal bundle). Recall the notion of *Ehresmann connection* on P (Example 2.2.3) : it is the data of a smooth tangent distribution HP which is at every point of P a supplementary distribution to the vertical distribution VP^1 :

$$TP = VP \oplus HP$$

called the *horizontal distribution*.

An Ehresmann connection is called *principal* if it is invariant under the right action of G on P .

Hence an Ehresmann connection defines projections of a vector on P to horizontal and vertical components.

There is a natural notion of pullback of an Ehresmann connection to a pullback fibre bundle:

Lemma 3.1.12. Let $E \rightarrow M$ be a fibre bundle equipped with an Ehresmann connection $HE \subset TE$ and $f : X \rightarrow M$ a smooth map. Then the pullback bundle $f^*E \rightarrow X$ admits a unique Ehresmann connection $f^*HE \subset Tf^*E$ which can be constructed as follows: writing $\phi : f^*E \rightarrow E$ the natural bundle map,

$$f^*HE = \{u \in Tf^*E \mid D\phi(u) \in HE\}$$

Proof. Uniqueness follows once we know the above set defines an Ehresmann connection. Let us call d the fibre dimension of E . The essential argument is that ϕ is transverse to the distribution HE :

$$\mathrm{im} D\phi + \phi^*HE = \phi^*TE$$

Therefore f^*HE is at every point of f^*E of codimension d in Tf^*E . As a consequence, f^*HE is a vector subbundle of Tf^*E of corank d . Since $D\phi$ sends Vf^*E to VE , $f^*HE \cap Vf^*E = 0$ which allows concluding that f^*HE is supplementary to Vf^*E . \square

Principal connections are usually handled through the means of a 1-form, as we now introduce:

Definition 3.1.13 (Principal connections on a principal bundle). A *G-principal connection 1-form* on P is given by a \mathfrak{g} -valued 1-form ω on P such that:

- It is normalised for the action of \mathfrak{g} :

$$\forall \xi \in \mathfrak{g}, \quad \omega(\bar{\xi}) = \xi$$

¹Some authors ([KMS93; Mic08]) require an extra *completude* property related to parallel transport which we introduce below.

- It is G -equivariant as a \mathfrak{g} -valued 1-form:

$$R_g^* \omega = \text{Ad}_g^{-1} \omega$$

The adjective “ G -principal” will most often be omitted.

Theorem 3.1.14. *There is a bijection between principal Ehresmann connections and principal connection 1-forms on P :*

$$\omega \mapsto HP = \ker(\omega)$$

We say that a connection 1-form on P defines a connection on P .

A principal connection ω allows defining derivatives of sections of associated vector bundles on M .

Lemma 3.1.15. Let V be a linear representation of G and ω a connection 1-form on P . Let $\psi \in \Omega_{\text{hor}}^\bullet(P, V)$, in particular we do not assume it is equivariant.

Then ψ is \mathfrak{g} -equivariant if and only if the V -valued form

$$d\psi + \omega \cdot \psi$$

is horizontal, with $\omega \cdot \psi$ the V -valued form recursively defined such that $i_X(\omega \cdot \psi) = \omega(X) \cdot \psi - \omega \cdot i_X \psi$. In this case, $d\psi + \omega \cdot \psi$ is \mathfrak{g} -equivariant, and is G -equivariant when ψ is.

Since part of this lemma is not standard and will be crucial in the following sections, we provide a proof

Proof. Equivariance of ψ under \mathfrak{g} is equivalent to the following: for all ξ in \mathfrak{g} ,

$$\mathcal{L}_{\bar{\xi}} \psi + \xi \cdot \psi = 0$$

We simply need to reexpress each term, using the horizontality of ψ :

$$\begin{aligned} \mathcal{L}_{\bar{\xi}} \psi &= (i_{\bar{\xi}} d + d i_{\bar{\xi}}) \psi \\ &= i_{\bar{\xi}} d\psi + 0 \end{aligned}$$

and

$$\xi \cdot \psi = \omega(\bar{\xi}) \cdot \psi = i_{\bar{\xi}}(\omega \cdot \psi)$$

Therefore we obtain the following equation:

$$\mathcal{L}_{\bar{\xi}} \psi + \xi \cdot \psi = i_{\bar{\xi}}(d\psi + \omega \cdot \psi) \tag{3.1}$$

The form ψ is \mathfrak{g} -equivariant if and only if the right hand side of (3.1) vanishes for every ξ in \mathfrak{g} , which is equivalent to asserting that $d\psi + \omega \cdot \psi$ has vanishing contraction with every vertical vector field. Namely, it is a horizontal V -valued form.

Since ω is G -equivariant and a fortiori \mathfrak{g} -equivariant and d commutes with Lie derivatives and pullbacks, equivariance properties of ψ are naturally transferred to both $d\psi$ and $\omega \cdot \psi$ and therefore to $d\psi + \omega \cdot \psi$. \square

We obtain a differential operator of degree 1 which preserves basic forms on P :

Definition 3.1.16 (Covariant derivative). Let V be a linear representation of G . Assume P is equipped with a principal connection with connection 1-form ω .

Let Ψ be a section of $P[V]$ and ψ the associated element of $\Gamma(P, V)^G$. The *covariant derivative* of ψ is defined as follows:

$$d^\omega \psi := d\psi + \omega \cdot \psi \in \Omega^1(P, V)$$

The 1-form $d^\omega \psi$ is basic thus is associated to a $P[V]$ -valued 1-form on M . We write it

$$\nabla^\omega \Psi$$

and call it the covariant derivative of Ψ . The superscript ω is usually omitted when there is no risk of confusion.

This construction extends to V -valued basic forms: let ψ be in $\Omega_{\text{hor}}^k(P, V)^G$. We define the following product

$$\begin{array}{ccccc} \Omega^1(P, \mathfrak{g}) \otimes \Omega^k(P, V) & \longrightarrow & \Omega^{1+k}(P, \mathfrak{g} \otimes V) & \longrightarrow & \Omega^{k+1}(P, V) \\ \omega \otimes \psi & \longmapsto & \omega \wedge \psi & \longmapsto & \omega \cdot \psi \end{array}$$

It is used in the following definition of the *covariant exterior derivative* of ψ :

$$d^\omega \psi = d\psi + \omega \cdot \psi \tag{3.2}$$

It can be shown to correspond to the precomposition of $d\psi$ with the projection

$$\Lambda^{k+1}TP \rightarrow \Lambda^{k+1}HP \subset \Lambda^{k+1}TP$$

The $(k+1)$ -form $d^\omega \psi$ is then a horizontal V -valued $(k+1)$ -form which can be proved to be equivariant: as an element of $\Omega_{\text{hor}}^{k+1}(P, V)^G$ it is associated to a $P \times_G V$ -valued $(k+1)$ -form on M . In this way a connection allows to define derivatives of sections of vector bundles on M .

If Ψ is the associated $P[V]$ -valued k -form on M , we write the associated $P[V]$ -valued $(k+1)$ -form as follows:

$$d^\nabla \Psi$$

Theorem 3.1.17 (Induced principal connection, [KN96]). *Let K be a closed subgroup of G and $P_K \hookrightarrow P_G = P$ a reduction of structure group to P . Then there is a natural bijection between K -principal connections on P_K and G -principal connections on P_G such that $HP_G|_{P_K} \subset TP_K$:*

$$\begin{aligned} \{K\text{-principal connection on } P_K\} &\leftrightarrow \{G\text{-principal connection on } P_G \text{ tangent to } P_K\} \\ HP_K &\mapsto HP_G|_{P_K} = dR_g(HP_K|_p) \quad \text{for } p \in P_K \\ HP_G|_{P_K} &\leftarrow HP_K \\ \omega|_{P_K} &\leftarrow \omega \end{aligned}$$

If the connection on P_K is given by a 1-form ω_K then the corresponding connection 1-form on P_G is the factorisation to $P_G \simeq P_K \times_K G$ of the following 1-form:

$$\text{Ad}_g^{-1} \omega_K + \omega_G^{MC} \in \Omega^1(P_K \times G, \mathfrak{g})$$

with ω_G^{MC} the right Maurer-Cartan form on G (defined below in Example 3.3.5).

There is a direct correspondence between covariant derivatives on vector bundles and principal

connections on their frame bundles. The bridge can be given by the following notion of *linear connection* on a vector bundle:

Definition 3.1.18 (Linear connections). Let $E \rightarrow M$ be a vector bundle.

An Ehresmann connection HE on E is called a *linear connection* if it is stable by addition and by dilation. Let us define the fibrewise addition map $\text{Add} : E \times_M E \rightarrow E$ and for all $\lambda \in \mathbb{R}$, the multiplication map $\text{Dil}_\lambda : E \rightarrow E$. The Ehresmann connection has to satisfy the following²:

$$\begin{aligned} T \text{Add}(HE, HE) &\subseteq HE \\ \forall \lambda \in \mathbb{R}, T \text{Dil}_\lambda(HE) &\subseteq HE \end{aligned}$$

Theorem 3.1.19 ([Mic08; Pau]). Let $E \rightarrow M$ be a vector bundle with a linear frame bundle $P = \text{GL}(E) \rightarrow M$. There are natural bijections as follows:

$$\begin{array}{ccc} \{\text{GL}_n\text{-principal connections on } P\} & & HP \\ \updownarrow & & \downarrow \\ \{\text{Linear connections on } E\} & \cong & HE = HP \times_{\text{GL}_n} E \\ \updownarrow & & \downarrow \\ \{\text{Covariant derivatives on } E\} & & \nabla^{HE} \end{array}$$

Curvature of a principal connection

The obstruction to the integrability of the horizontal distribution HP is contained in the *curvature 2-form* associated to the connection, defined as follows:

$$\Omega := d\omega + \frac{1}{2}[\omega \wedge \omega] \in \Omega_{\text{hor}}^2(P, \mathfrak{g})^G \quad (3.3)$$

with

$$[\beta \wedge \gamma](X, Y) := [\beta(X), \gamma(Y)] - [\beta(Y), \gamma(X)]$$

The curvature 2-form vanishes if and only if HP is integrable, in which case its leaves project to M by surjective local diffeomorphisms. A connection with vanishing curvature is said to be *flat* [Mic08].

The curvature 2-form is also involved in the square of the covariant exterior derivative operator through the following *Ricci identity*:

$$\forall \Psi \in \Omega_{\text{hor}}^k(P, V)^G \quad d^\omega d^\omega \psi = \Omega \cdot \psi \quad (3.4)$$

using a similar notation to $\omega \cdot \psi$ for the product

$$\Omega^2(P, \mathfrak{g}) \otimes \Omega^k(P, V) \rightarrow \Omega^{k+2}(P, V)$$

It furthermore satisfies the *Bianchi identity*

$$d^\omega \Omega = 0 \quad (3.5)$$

²In fact any of the condition implies the other one, for similar reasons that additive continuous maps between vector spaces and 1-homogeneous maps between vector spaces which are differentiable at 0 are linear, respectively.

Parallel transport

Let $P \rightarrow M$ be a G -principal bundle equipped with a principal connection. We explained how it defines a covariant derivatives on sections of associated vector bundles. The corresponding integral structure, which does not need to consider vector bundle or even associated bundle, is called *parallel transport*.

Definition 3.1.20. Let $E \rightarrow M$ be a fibre bundle equipped with an Ehresmann connection. Let $c : I \rightarrow M$ be a smooth curve starting from m . The fibre bundle E can be “restricted to c ” by constructing the pullback fibre bundle $E_c := c^*E \rightarrow I$. The pullback of the Ehresmann connection to E_c defines an Ehresmann connection HE_c over $E_c \xrightarrow{\pi} I$ which is effectively a smooth distribution of rank 1.

The *parallel transport* associated to c is the flow of the horizontal vector field $X_c \in \Gamma(E_c, HE_c)$ uniquely defined by the two following conditions:

$$\begin{aligned} X_c &\in \Gamma(E_c, HE_c) \\ d\pi(X_c) &= \frac{\partial}{\partial t} \in \pi^*TI \end{aligned}$$

with $\frac{\partial}{\partial t}$ the positively oriented normalised vector field on I .

As we defined it, parallel transport is merely the solution of a geometrically constructed differential equation on a fibre bundle. As such, it may not exist globally. However in the G -principal case, equivariance gives enough homogeneity to the differential equation to ensure that global solutions always exist.

Theorem 3.1.21 (Completeness of principal connections, [Mic08]). *If $P \rightarrow M$ is a G -principal bundle equipped with a principal connection, then parallel transport is always globally defined. Namely, any curve in M has a globally defined parallel transport starting from any point in the fibre above the initial point of the curve.*

Furthermore, parallel transport commutes with the action of G on P .

Hence given a smooth path in M from m_1 to m_2 parallel transport gives an element PT_c in $\text{Hom}_G(P_{m_1}, P_{m_2})$. The concatenation of two paths is easily proved to give the composition of the corresponding parallel transports. Parallel transport along loops is called *holonomy*. Holonomies of any fibre of P form a group, called the *holonomy group* of the connection at the corresponding point of M .

Parallel transport acts as well on associated bundles. Let X be a G -manifold and consider the associated bundle $P[X]$. Given a smooth path c from m_1 to m_2 there is a mapping

$$P_{m_1} \xrightarrow[\text{PT}_c]{\sim} P_{m_2}$$

which commutes with the action on G . Therefore we can construct a map

$$P_{m_1} \times X \xrightarrow[\text{PT}_c \times \text{id}_X]{\sim} P_{m_2} \times X$$

which is G -equivariant for the action $(p, x) \cdot g = (p \cdot g, g^{-1} \cdot x)$. This induces a map between the quotients

$$P[X]_{m_1} = P_{m_1} \times_G X \xrightarrow{\sim} P_{m_2} \times_G X = P[X]_{m_2}$$

Thus we associated to the principal connection on P a parallel transport on the associated bundle $P[X]$. This associated parallel transport can in fact be associated with an Ehresmann connection on the associated bundle [KMS93].

We conclude this section on principal bundles with a characterisation of flat connections.

Theorem 3.1.22 (Flat connections and parallel transport). *The parallel transport of a smooth path is invariant under homotopy (with fixed endpoints) of the path if and only if the connection is flat.*

3.2 Frame bundles

We now turn our attention to frame bundles, which are principal bundles with more structure which ties them to the local geometry of the base manifold. Let M be a differentiable n -manifold.

3.2.1 The linear frame bundle of a manifold

At each point $m \in M$ the tangent space $T_m M$ is a n -dimensional vector space and has a n^2 -dimensional manifold of linear frames. They can be gathered into a fibre bundle over M which is called the (*linear*) *frame bundle of M* and which we write $\text{GL}(M)$.

Since the set of frames of $T_m M$ has a natural right action of GL_n which is both transitive and effective, $\text{GL}(M)$ has the structure of a GL_n -principal bundle over M (Example 3.1.3). Furthermore it is provided with a natural \mathbb{R}^n -valued 1-form which we now construct.

The canonical solder form

Let $\pi : \text{GL}(M) \rightarrow M$ be the fibration map. According to the examples of the previous section, we know that $\text{GL}(M)[\mathbb{R}^n] \simeq TM$. In particular, there is a natural “tautological” isomorphism of vector bundles

$$u : \pi^* TM \xrightarrow{\sim} \text{GL}(M) \times \mathbb{R}^n$$

Furthermore u is equivariant under the right action of GL_n : writing R_g for the right action of $g \in \text{GL}_n$ on $\text{GL}(M)$, the following diagram commutes

$$\begin{array}{ccc} \pi^* TM & \xrightarrow{u_{\mathbb{R}^n}} & \mathbb{R}^n \\ \downarrow R_{g*} & & \downarrow g^{-1} \\ \pi^* TM & \xrightarrow{u_{\mathbb{R}^n}} & \mathbb{R}^n \end{array}$$

The differential of π can be represented as a map

$$d\pi : T\text{GL}(M) \rightarrow \pi^* TM$$

so that composition gives a linear map

$$T\text{GL}(M) \xrightarrow{d\pi} \pi^* TM \xrightarrow{u_{\mathbb{R}^n}} \mathbb{R}^n$$

θ

The fibration map π is invariant under the action of GL_n :

$$\pi \circ R_g = \pi$$

thus for all $p \in \text{GL}(M)$, the follows holds:

$$d\pi \circ dR_g = R_{g*} d\pi$$

with $R_{g_*} : \pi^*TM \xrightarrow{\sim} R_{g_*}\pi^*TM$ the natural identification. Finally

$$\theta \circ dR_g|_p = u \circ d\pi \circ dR_g = u \circ R_{g_*}d\pi = g \cdot u \circ d\pi = g \cdot \theta$$

The linear map θ can be understood as a (horizontal) \mathbb{R} -valued 1-form on $GL(M)$ and is called the *canonical solder form* of $GL(M)$. As we just proved, it is equivariant under the action of GL_n .

Vectors, tensor fields and differential forms

Since TM is the associated bundle to $GL(M)$ with fibre \mathbb{R}^n , there is a natural $\mathcal{C}^\infty(M)$ -linear isomorphism

$$\Gamma(TM) \xrightarrow{\sim} \Gamma(GL(M), \mathbb{R}^n)^{GL_n}$$

according to 3.1.8. The same goes for tensor bundles $TM^{\otimes r} \otimes T^{*\otimes s}$ which are associated to $\mathbb{R}^{n \otimes r} \otimes \mathbb{R}^{n^* \otimes s}$

Differential forms on M can be represented on $GL(M)$ in two different ways, corresponding to whether they are considered as \mathbb{R} -valued differential forms on M or sections of $\Lambda^\bullet T^*M$. Correspondingly, there are graded $\mathcal{C}^\infty(M)$ -algebra isomorphisms

$$\Omega_{\text{hor}}(GL(M))^{GL_n} \simeq \Omega^\bullet(M) \simeq \Gamma(GL(M), \Lambda^\bullet \mathbb{R}^{n*})^{GL_n}$$

Due to this multiplicity of representations of differential forms on the frame bundle, manipulation on the frame bundle sometimes require a lot of care, as will be the case in Chapter 5.

As was illustrated in examples in the previous section, geometric structures can be defined on M by specifying a restricted class of linear bundles. The corresponding structures, called G -structures, are presented in the next section.

3.2.2 G -structures

When M has an orientation, it is possible to define a notion of “direct” frames. Conversely, given the class of direct frames, it is possible to identify a corresponding orientation. The same is true in pseudo-Riemannian geometry: it is equivalent to specify a metric or to specify the class of (pseudo)-orthonormal frames (with a specified ordered signature). The notion of G -structure sets the focus on the class of frames structured by the group relating all the frames of the specified class. More concretely:

Definition 3.2.1 (G -structure). Let M be an n -manifold. Let G be a Lie group equipped with an action on \mathbb{R}^n :

$$G \rightarrow GL_n$$

A G -structure on M is the data of a G -principal bundle $P \xrightarrow{\pi_P} M$ which has a G -equivariant bundle map to $GL(M)$, namely, the following diagram commutes:

$$\begin{array}{ccc} P \times G & \longrightarrow & GL(M) \times GL_n \\ \downarrow & & \downarrow \\ P & \longrightarrow & GL(M) \\ & \searrow & \swarrow \\ & M & \end{array}$$

Namely, it is a reduction of structure group of the linear frame bundle along $G \rightarrow \mathrm{GL}_n$.

Although the usual denomination is “ G -structure”, the action of G on \mathbb{R}^n is also an essential part of the defining data. As is standard, the action will often be left implicit. Note that the action does not have to be faithful, which will be especially relevant in Section 3.2.3.

Example 3.2.2. $\mathrm{GL}(M)$ defines a canonical GL_n -structure on M , which is part of the differential manifold structure.

Solder forms

The data of an equivariant map $P \rightarrow \mathrm{GL}(M)$ can be encoded in a \mathbb{R}^n -valued 1-form α on P which satisfies the following two requirements:

- The kernel is the vertical tangent bundle

$$VP := \ker(d\pi_P : TP \rightarrow \pi_P^*TM) \subset TP$$

- It is equivariant under the action of G on \mathbb{R}^n .

Such a 1-form is called a *solder form* (sometimes *soldering form*).

Stating that the kernel is VP is equivalent to asserting that α factors to an injective bundle mapping

$$\pi_P^*TM \hookrightarrow P \times \mathbb{R}^n$$

which is then necessarily an isomorphism for dimension reasons. Equivariance of the 1-form is equivalent to the equivariance of this mapping, so that such a 1-form indeed gives a mapping

$$P \rightarrow \mathrm{GL}(M)$$

It is then easily shown that the solder form on P is the pull back of θ by the obtained map $P \rightarrow \mathrm{GL}(M)$, and conversely the pullback of θ by an equivariant bundle map is always a solder form on P .

Furthermore, α inducing an equivariant bundle isomorphism

$$\pi_P^*TM \xrightarrow{\sim} P \times \mathbb{R}^n$$

implies that it further factors to a vector bundle isomorphism over M :

$$TM \simeq \pi_P^*TM/G \xrightarrow{\sim} P \times_G \mathbb{R}^n \simeq P[\mathbb{R}^n]$$

From this perspective, a solder form on P is associated to a $P[\mathbb{R}^n]$ -valued 1-form on M which is “non-degenerate”. Indeed this gives another characterisation of G -structures: they are G -principal bundles equipped with a vector bundle isomorphism

$$TM \simeq P[\mathbb{R}^n]$$

(G, V) -structures

For a basis-independent formulation of G -structures, it is possible to work with an n -dimensional representation V of G and to define a (G, V) -structure on M as a G -principal bundle with a map to the bundle of “ V -valued frames”, namely the bundle of isomorphisms between TM and $V \times M$:

Definition 3.2.3. Let G be a Lie group acting on a real vector space V of dimension n . A (G, V) -structure on a n -manifold M is a reduction of the structure group of the frame bundle of M to G acting on V . Namely, it is given by a G -principal bundle P and a soldering form which is equivalently given by any of the following structures:

- An isomorphism between the tangent bundle of M and the associated vector bundle:

$$\alpha : TM \xrightarrow{\sim} P[V]$$

- A G -equivariant V -valued horizontal 1-form $\alpha \in \Omega_{\text{hor}}^1(P; V)^G$ which is nondegenerate on horizontal directions.
- A G -equivariant bundle morphism from P to the V -indexed frames of M

$$P \rightarrow \text{Iso}(V, TM)$$

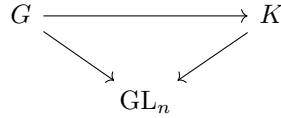
When G acts effectively on V , the mapping $P \rightarrow \text{Iso}(V, TM)$ is an embedding.

Change of structure group

Very similarly to associated principal bundles, it is possible to change the structure group of a G -structure along a group morphism which is compatible with the action on \mathbb{R}^n .

Theorem 3.2.4. Let G be a Lie group acting on \mathbb{R}^n and M a differentiable manifold equipped with a G -structure $P \rightarrow \text{GL}(M) \rightarrow M$.

Let K be another Lie group acting on \mathbb{R}^n equipped with a morphism from G compatible with the action on \mathbb{R}^n : namely, the following diagram commutes:



Then the associated K -principal bundle

$$P[K] = P \times_G K$$

is naturally equipped with an K -equivariant bundle map to $\text{GL}(M)$ hence it defines an K -structure over M .

For example, for an embedding $G \hookrightarrow K$, this corresponds to forgetting part of the G -structure. Conversely, it is possible to construct further reductions of the structure group with sections of quotient bundles of P .

Associated tensor fields

Tensor fields and tensor-valued differential forms can be defined using the principal bundle of a G -structure: by the same construction as for $\text{GL}(M)$, G -equivariant \mathbb{R}^n -valued forms can be identified with G -invariant $\pi_P^* TM$ -valued forms. The same goes for the tensor bundles.

Furthemore, given any finite dimensional linear representation V of G it is possible to consider G -equivariant V -valued fields (and differential forms). They are in natural bijection with section

of the *associated bundle* $P[V]$ on M which is a vector bundle over M with typical fibre V , as stated in Section 3.1

Similarly, there is an isomorphism

$$\Omega_{\text{hor}}^k(P, V)^G \simeq \Omega^k(M, P[V])$$

Curvature and torsion of a connection on a G -structure

A connection on a G -structure is simply a G -principal connection on the underlying G -principal bundle P . As such it can be defined by a connection 1-form ω . The connection has a curvature 2-form $\Omega \in \Omega_{\text{hor}}^2(P, \mathfrak{g})^G$. It is therefore associated to an element of $\Omega^2(M, P[\mathfrak{g}])$.

The action $\text{ad}_{\mathfrak{g}} : \mathfrak{g} \rightarrow \text{End}(\mathbb{R}^n)$ is G -equivariant and as such defines a G -invariant element of $\text{Hom}(\mathfrak{g}, \text{End}(\mathbb{R}^n))$. Therefore there is a corresponding element

$$\begin{aligned} P[\text{ad}_{\mathfrak{g}}] &\in \Gamma(M, P[\text{Hom}(\mathfrak{g}, \text{End}(\mathbb{R}^n))]) \simeq \Gamma(M, \text{Hom}(P[\mathfrak{g}], \text{End}(P[\mathbb{R}^n]))) \\ &\simeq \Gamma(M, \text{Hom}(P[\mathfrak{g}], \text{End}(TM))) \end{aligned}$$

The conclusion is that there is a natural bundle representation $P[\mathfrak{g}] \rightarrow \text{End}(TM)$ and therefore a $\text{End}(TM)$ -valued 2-form “curvature endomorphism”.

Furthermore, the G -structure equips P with a solder form α . Its covariant exterior derivative is called the *torsion* of ω :

$$\Theta := d^\omega \alpha \in \Omega_{\text{hor}}^2(P, \mathbb{R}^n) \quad (3.6)$$

It is associated to an element $T \in \Omega^2(M, TM)$ also called the torsion of the connection. The Ricci identity (3.4) applied to the soldering form is often called the *algebraic Ricci identity*:

$$d^\omega \Theta = \Omega \wedge \alpha \in \Omega_{\text{hor}}^3(P, \mathbb{R}^n) \quad (3.7)$$

3.2.3 Pulling back principal bundles

Let M be a manifold with a G -structure given by a principal bundle $P \rightarrow M$ equipped with a solder form α . Assume furthermore that there is another principal bundle $Q \rightarrow M$ with a structure group K . We are looking for a way to characterise the structure of the principal bundle Q as a structure on P .

The principal bundle Q can be pulled back through π_P to an K -principal bundle π_P^*Q on P . The fact that the principal bundle on P is a pullback bundle can be characterised by the structure of *G -equivariant K -principal bundle*: the action of G on P lifts to morphisms of principal bundle of π_P^*Q (which are identified with the identity on the spaces $Q_{\pi_P(p)}$). Namely, the following diagram commutes for every g in G :

$$\begin{array}{ccc} \pi_P^*Q & \xrightarrow{(p,q) \mapsto (p \cdot g, q)} & \pi_P^*Q \\ \downarrow & & \downarrow \\ P & \xrightarrow{\cdot g} & P \end{array}$$

The structure of π_P^*Q is clear when it is viewed as a fibred product of bundles:

$$\begin{array}{ccc}
 P \times_M Q \simeq \pi_P^*Q & \xrightarrow[\phi_Q]{\substack{G\text{-principal} \\ K\text{-equivariant}}} & Q \\
 \downarrow \phi_P & \lrcorner & \downarrow \text{K-principal} \\
 P & \xrightarrow[\text{G-principal}]{\pi_P} & M
 \end{array}$$

In particular the symmetry between the roles of P and Q is made manifest. The space $P \times_M Q$ has a structure of K -principal bundle over P , of G -principal bundle over Q and of $G \times K$ -principal bundle over M . In particular, the actions of G and K commute which implies that the bundle fibration over Q is K -equivariant.

Such structures have been explored in physics [KNR87] and generalised in mathematical studies [LLL21].

Principal connections

Assume now that Q is equipped with a principal connection with a connection form $A \in \Omega^1(Q, \mathfrak{k})^K$. The connection pulls back under the projection $P \times_M Q \xrightarrow{\phi_Q} Q$ into an K -principal connection on $P \times_M Q$ which we write ϕ_Q^*A . Since ϕ_Q is invariant under precomposition by the action of elements of G :

$$\phi_Q \circ R_g = \phi_Q$$

the same goes for ϕ_Q^*A :

$$R_g^* \phi_Q^*A = \phi_Q^*A$$

Thus the pullback connection on $P \times_M Q$ is G -invariant, or G -equivariant for the trivial adjoint action of G on $\mathfrak{k} \subset \mathfrak{g} \times \mathfrak{k}$. Similarly, a principal connection on P can be pulled back to $P \times_M Q$ to a Q -invariant connection.

Thus a principal connection on P and a principal connection on Q can be pulled back to $P \times_M Q$ and summed into a $\mathfrak{g} \oplus \mathfrak{k}$ -valued 1-form which is $G \times K$ -equivariant and normalised on $\mathfrak{g} \oplus \mathfrak{k}$: this is exactly a $G \times K$ -principal connection on $P \times_M Q$. Conversely, given a $G \times K$ -principal connection on $P \times_M Q$, its respective \mathfrak{g} and \mathfrak{k} components are G -equivariant and K -invariant, respectively K -equivariant and G -equivariant, and induce principal connections on P and Q . This is summed up in the following theorem

Theorem 3.2.5. *Let $P \rightarrow M$ be a G -principal bundle and $Q \rightarrow M$ a K -principal bundle. There is a natural bijection between $G \times K$ -principal connections on $P \times_M Q$ and couples of principal connections on P and Q .*

If P is equipped with a solder form α , it can also be pulled back to $P \times_M Q$ into a horizontal 1-form which is equivariant under G and invariant under K . Considering the trivial action of K on \mathbb{R}^n , $\phi_P^*\alpha$ is equivariant under $G \times K$. Furthermore its kernel is made of the vectors of $P \times_M Q$ mapped to vertical vectors on P : this is exactly the vertical vectors of $P \times_M Q \rightarrow M$. Thus we conclude:

Theorem 3.2.6. *If $P \rightarrow M$ with a solder form α defines a $(G \rightarrow \text{GL}_n)$ -structure and $Q \xrightarrow{\pi_Q} M$ is an K -principal bundle, then $\phi_Q^*\alpha$ defines a solder form on $P \times_M Q \rightarrow M$ for the representation $G \xrightarrow{p_G} K \rightarrow G \rightarrow \text{GL}_n$.*

More generally, given a representation V of G or K , it can be extended to a representation of $G \times K$ by a trivial action of the other factor. An equivariant V -valued field on P or Q can then be pulled back to $P \times_M Q$ to an equivariant V -valued field. Conversely, when K (resp. G) acts trivially on V , every $G \times K$ -equivariant field on $P \times_M Q$ factors to a G -equivariant field on P (resp. a K -equivariant field on Q).

3.3 Cartan geometry

In this section we discuss Cartan geometry which gives a general geometrical framework to understand the specifics of connections on G -structures and allows for broad generalisation. In this framework, a G -structure is interpreted as “contact configurations” with an affine model space and a connection gives a notion of “rolling without sliding” of the affine model space along paths. The general notion for suitable model spaces, not necessarily affine, is given by *Klein geometry*.

3.3.1 Klein geometries

We start with the definition of Klein geometries. These are the geometries of differentiable (connected) homogeneous spaces under specified symmetry groups.

Definition 3.3.1 (Klein Geometries).

A (smooth) homogeneous space is a smooth manifold equipped with a transitive (smooth Lie) group action.

A (smooth) Klein space is a (smooth) homogeneous space which is connected.

Let (M_1, G_1) and (M_2, G_2) be two Klein spaces (the second component represents the group action). A (geometric) isomorphism between (M_1, G_1) and (M_2, G_2) is a diffeomorphism $M_1 \xrightarrow{\phi} M_2$ with a group isomorphism $G_1 \xrightarrow{\sim} G_2$ such that ϕ is equivariant in the sense that the following diagram commutes [Sha97]:

$$\begin{array}{ccc} G_1 & \xrightarrow{\sim} & G_2 \\ \downarrow & & \downarrow \\ \text{Diff}(M_1) & \xrightarrow{\text{Ad}_\phi} & \text{Diff}(M_2) \end{array}$$

A Klein geometry is defined as an isomorphism class of Klein spaces.

Let G be a Lie group and H a closed subgroup such that the coset space G/H is connected. The Klein geometry associated to $H \subset G$ is the space G/H , provided with the symmetry group $G \subset \text{Aut}(G/H)$.

An isomorphism of Klein geometries specified by two pairs $(G_1/H_1, G_1)$ and $(G_2/H_2, G_2)$ is given by a group isomorphism $G_1 \xrightarrow{\phi} G_2$ such that $\phi(H_1) = H_2$:

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi} & H_2 \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow{\phi} & G_2 \end{array}$$

In particular, adjoint actions of elements of G on H do not change the Klein geometry of G/H .

Conversely, all Klein geometries can be presented by such a coset space: choosing a basepoint x_0 in (M, G) , the orbital map establishes an isomorphism $G/G_{x_0} \xrightarrow{\sim} M$, with G_{x_0} the isotropy (or

stabiliser) subgroup. The isotropy group is defined up to (at least) conjugacy, which corresponds to changing the base point. The Klein geometry associated to $(G/H, G)$ is often written as $(H \curvearrowright G)$ or (G, H) , we will use the latter notation (not to be confused with the notation (M, G) which we will not use anymore).

In this way, we see an equivalence between coset spaces and pointed Klein geometries.

Here are a few examples of Klein geometries – the first one will be of particular interest to us.

Example 3.3.2.

- The Euclidean n -plane \mathbb{R}^n with symmetry group $SO_n \times \mathbb{R}^n$ and isotropy groups SO_n .
- The oriented n -sphere S^n with symmetry group SO_{n+1} and isotropy groups SO_n .
- The unoriented n -sphere S^n with symmetry group O_{n+1} and isotropy groups O_n .
- The (orthogonal) Grassmannian of k planes of \mathbb{R}^n with symmetry group O_n and isotropy groups $O_k \times O_{n-k}$.
- The space of inner products on \mathbb{R}^n with symmetry group GL_n and isotropy groups O_n .
- The Riemann sphere CP^1 with symmetry group $PSL_2(\mathbb{C})$ and isotropy groups $Aff_{\mathbb{C}}(\mathbb{C})$.
- The conformal sphere which is the reunion of the isotropic lines in $\mathbb{P}(\mathbb{R}^{n+1,1})$ with symmetry group $PO_{n+1,1}$ and isotropy groups conjugated to the so-called *Poincaré conformal group* [CS09].

We bring to attention the fact that the tangent spaces to a Klein geometry (G, H) are naturally modelled by $\mathfrak{g}/\mathfrak{h}$, as a quotient of $T_e G$. The coset space G/H being homogeneous all tangent spaces *can* be identified with $\mathfrak{g}/\mathfrak{h}$. There is an H worth of ambiguity however, which we come back to in Section 3.3.4 and Example 3.3.10. All linear structures on $\mathfrak{g}/\mathfrak{h}$ (e.g. an inner product) which are invariant under H can be extended to a tangent structure on G/H . This motivates the following definition for “infinitesimal Klein spaces”:

Definition 3.3.3 (Klein pair). A *Klein pair* is a Lie algebra \mathfrak{g} provided with a given Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ [Sha97].

Not all Klein pairs can be integrated into Klein spaces because \mathfrak{h} can integrate into a nonclosed subgroup (they are called “virtual Lie subgroups” [CS09]).

Klein geometries serve as the reference models for Cartan geometries, introduced in the next section.

3.3.2 Cartan geometries

In Cartan geometry one wants to “tie together” parts of identical Klein geometries, in a similar fashion to how affine spaces are glued into differentiable manifolds. One main difference as we will see is that the identification with the reference geometry only occurs *infinitesimally* at each point, in many cases at first order, which allows Cartan geometries to have an intrinsic curvature.

Let M be a Hausdorff paracompact smooth manifold and (G, H) a reference Klein geometry.

Definition 3.3.4. A (G, H) -Cartan geometry on M is given by the data of:

- A (right) H -principal bundle P on M , which we will call the *Cartan principal bundle*,

- A non-degenerate \mathfrak{g} -valued 1-form ϖ on P which is normalized on the vertical action of \mathfrak{h} : for $\bar{\xi}$ representing $\xi \in \mathfrak{h}$ on P ,

$$\varpi(\bar{\xi}) = \xi$$

and equivariant under H , which we will call the *Cartan connection form*.

Note that the definition only depends on H and its action on \mathfrak{g} but not directly on the choice of the group G . This leads to the concept of *model mutation* which we will touch upon in Section 3.3.6.

Example 3.3.5 (Klein geometries as Cartan geometries). Let G be a Lie group and \mathfrak{g} its Lie algebra, identified with the tangent space at identity $T_e G$.

Write L_g for the left translation by any element g of G . The following map defines a parallelism:

$$\varpi_G : \begin{cases} TG \rightarrow T_e G \xrightarrow{\sim} \mathfrak{g} \\ (g, X) \mapsto (g, (dL_g)|_e^{-1} X) \end{cases}$$

The map can be seen as a \mathfrak{g} -valued 1-form on G : $\varpi_G \in \Omega^1(G, \mathfrak{g})$. It is called the *Maurer-Cartan form* of G and satisfies the following *Maurer-Cartan equation*:

$$d\varpi_G + \frac{1}{2} [\varpi_G \wedge \varpi_G] = 0 \quad (3.8)$$

Vector fields with a constant image by ϖ_G are the *left-invariant vector fields* on G . They are the vector fields by which \mathfrak{g} acts on G *on the right*.

The fibration $G \rightarrow G/H$ can then be viewed as an H -principal bundle provided with a Cartan 1-form ϖ_G .³

Correspondingly, it is possible to define a “left” Maurer-Cartan form ϖ_G^L on G using the right action of G (the previous one can be called right Maurer-Cartan form and we will use the notation ϖ_G^R to avoid confusion when needed). It satisfies a different Maurer-Cartan equation⁴:

$$d\varpi_G^L - \frac{1}{2} [\varpi_G^L \wedge \varpi_G^L] = 0$$

The two Maurer-Cartan forms can be related in two ways. First, let $X \in T_g \mathfrak{g}$. Then

$$\varpi_G^L(X) = DR_g^{-1}(X) = D(\text{Aut}_g \circ L_g^{-1})(X) = \text{Ad}_g \varpi_G^R(X) \quad (3.9)$$

Next, let us write $i : G \rightarrow G$ for the group inversion. Then $\forall \xi \in \mathfrak{g}, \forall g \in G$,

$$Di(g \cdot \xi) = -\xi \cdot g^{-1}$$

Therefore

$$i^* \varpi_G^L = -\varpi_G^R \quad (3.10)$$

³Note that the construction of the H -principal bundle over the homogeneous space is dependent of the choice of an origin point of the homogeneous space: the elements of G above a given point m are the elements sending the origin point to m . The group acting on the bundle’s fibres is the isotropy group of the chosen origin point.

⁴It can be seen as the usual Maurer-Cartan equation for the opposite Lie bracket.

3.3.3 Curvature

On a principal bundle, equivariant horizontal forms are naturally identified with differential forms on the base manifold:

$$\Omega_{\text{hor}}^{\bullet}(P)^H \simeq \Omega^{\bullet}(M)$$

More generally, for any representation E of H , equivariant horizontal E -valued forms are identified with differential forms on the basis manifold with value in the associated vector bundle:

$$\Omega_{\text{hor}}^{\bullet}(P; E)^H \simeq \Omega^{\bullet}(M; P[E])$$

Definition 3.3.6. Let M be a manifold provided with a (G, H) -Cartan geometry (P, ϖ) .

Its (total, or Cartan) *curvature* 2-form is the equivariant horizontal \mathfrak{g} -valued 2-form on P defined by

$$\Omega = d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \quad (3.11)$$

The projection of Ω to $\mathfrak{g}/\mathfrak{h}$ is called the *torsion* of the Cartan geometry (or of the Cartan connection). The Cartan geometry is said to be (locally) *flat* when the total curvature identically vanishes [CS09].

Example 3.3.7. Riemannian geometry with affine metric connections is the Cartan geometry associated to the standard action of the Euclidean group $O_n \ltimes \mathbb{R}^n$ on \mathbb{R}^n ($G = O_n \ltimes \mathbb{R}^n$, $H = O_n$). The reduced curvature and torsion constructions match with the usual curvature and torsion associated to the connection. In particular the *Riemannian curvature* corresponds to the reduced curvature of a torsionless Cartan connection.

Example 3.3.8. The Maurer-Cartan equation (3.8) implies that the canonical Cartan geometries on the (connected) coset spaces are flat. Beware that this is usually a different notion of flatness than the commonly used one in (pseudo-)Riemannian geometry.

Conversely, when a (G, H) -Cartan geometry is flat, the Cartan 1-form satisfies the Maurer-Cartan equation (in the Lie algebra \mathfrak{g}). We will prove in Section 7.2.1 that this defines on P a structure of a free transitive \mathfrak{g} -manifold which is compatible over \mathfrak{h} with the existing action of H .

Example 3.3.9 (Affine manifolds). An affine structure on a n -manifold is equivalently a *flat* $(GL_n \ltimes \mathbb{R}^n, GL_n)$ -Cartan geometry structure on the manifold.

Next section introduces the notion of *mutation* which is helpful to relate Cartan geometry modelled on different but close Klein geometries.

3.3.4 $(H, \mathfrak{g}/\mathfrak{h})$ -structure and reductive Cartan geometries

To a (G, H) -Cartan geometry on a manifold it is associated an $(H, \mathfrak{g}/\mathfrak{h})$ -structure, in a way we describe now.

On the total space P of a fibre bundle $P \rightarrow M$, we call *horizontal forms* (also called semi-basic forms) the differential forms that have a vanishing contraction with any vertical vector fields. Recall the notation for the algebra of horizontal differential forms:

$$\Omega_{\text{hor}}^{\bullet}(P) := \{\alpha \in \Omega^{\bullet}(P) \mid \forall X \in \ker(dp : TP \rightarrow TM), X \lrcorner \alpha = 0\}$$

A Cartan principal bundle has an H -principal fibration $P \xrightarrow{P} M$ with an equivariant 1-form ϖ which is non-degenerate with value in \mathfrak{g} . Furthermore, ϖ is normalised so that it provides an

identification between the vertical tangent bundle and \mathfrak{h} . Its projection α to $\mathfrak{g}/\mathfrak{h}$ hence provides an equivariant identification

$$p^*TM \simeq TP/VP \xrightarrow{\alpha := \varpi^{\mathfrak{g}/\mathfrak{h}}} \mathfrak{g}/\mathfrak{h} \times P$$

The 1-form α can as well be interpreted as an H -equivariant mapping to the bundle of $\mathfrak{g}/\mathfrak{h}$ -indexed (co-)frames of TM :

$$P \rightarrow \text{Iso}(TM, \mathfrak{g}/\mathfrak{h}) \simeq \text{Iso}(\mathfrak{g}/\mathfrak{h}, TM)$$

hence P is identified with an $(H, \mathfrak{g}/\mathfrak{h})$ -structure on M . The frames can be identified with standard \mathbb{R}^n -frames by choosing a reference frame in $\mathfrak{g}/\mathfrak{h}$. The archetypical example is the following:

Example 3.3.10 (Canonical $(H, \mathfrak{g}/\mathfrak{h})$ -structure on G/H). As mentioned the Lie group G forms an H -principal fibre bundle $G \xrightarrow{p} G/H$. The Maurer-Cartan 1-form induces an associated $(H, \mathfrak{g}/\mathfrak{h})$ -structure, which we now describe by the bundle mapping from G to the bundle of $\mathfrak{g}/\mathfrak{h}$ -indexed frames $\text{Iso}(\mathfrak{g}/\mathfrak{h}, T(G/H))$.

Write by L the left action of G on itself:

$$L_g : x \in G \mapsto g \cdot x$$

The projection p is invariant under right action by H and equivariant under left action by G . Its kernel can be written as $G \cdot T_e H$ by which we mean the image of the following map:

$$\begin{aligned} G \times T_e H &\rightarrow TG \\ (g, \xi) &\mapsto dL_g(\xi) \end{aligned}$$

or in other words the left-invariant distribution generated by $T_e H \subset T_e G$. It is then possible to use an element g of G to map $\mathfrak{g}/\mathfrak{h}$ to $T_{[g]}(G/H)$: the map

$$\begin{aligned} G \times T_e G &\rightarrow T(G/H) \\ (g, u) &\mapsto dp \circ dL_g(u) \end{aligned}$$

vanishes on $G \times T_e H$ hence factorises to $G \times (T_e G/T_e H) \simeq G \times \mathfrak{g}/\mathfrak{h}$. The partial applications (with fixed $g \in G$) on $\mathfrak{g}/\mathfrak{h}$ are linear isomorphisms $\mathfrak{g}/\mathfrak{h} \xrightarrow{\sim} T_{[g]}(G/H)$. It thus gives a bundle map $G \rightarrow \text{Iso}(\mathfrak{g}/\mathfrak{h}, T(G/H))$.

The bundle map commutes with the action of H according to the following computation:

$$dp(dL_{gh}(u)) = dp(dR_h \circ dL_g(u)) = dp(DL_g(\text{Ad}_h(u)))$$

With extra structure on the Klein geometry it is possible to produce from ϖ an H -principal connection on the $(H, \mathfrak{g}/\mathfrak{h})$ -structure:

Definition 3.3.11. A Klein geometry (G, H) is *reductive* when \mathfrak{h} has a supplementary (adjoint) subrepresentation of H in \mathfrak{g} :

$$\mathfrak{g} \simeq_H \mathfrak{h} \oplus \mathfrak{m}$$

We will consider this supplementary subrepresentation, isomorphic to $\mathfrak{g}/\mathfrak{h}$, fixed as *part of the reductive Klein geometry structure*.

A Klein pair $(\mathfrak{g}, \mathfrak{h})$ is *reductive* when \mathfrak{h} has a supplementary subrepresentation of \mathfrak{h} in \mathfrak{g} :

$$\mathfrak{g} \simeq_{\mathfrak{h}} \mathfrak{h} \oplus \mathfrak{m}$$

A reductive structure on a Klein geometry (G, H) can equivalently be represented as an H -equivariant projection $\mathfrak{g} \rightarrow_H \mathfrak{h}$: the supplementary subspace \mathfrak{m} corresponds to the kernel.

Example 3.3.12 (Reductive structure on affine Klein geometries). Let V be a finite dimensional real vector space and H a Lie group acting on V . The associated affine Klein geometry on V can be presented as $(H \ltimes V, V)$. It has the natural following reductive structure:

$$\mathfrak{h} \ltimes V \simeq_H \mathfrak{h} \oplus V$$

In particular, this applies to Euclidean Geometry.

As a principal H -bundle above V , $H \ltimes V$ is trivialisable but one has to be careful. The projection to V identified as a the coset space $H \ltimes V/H$ differs from the manifold-level factor projection on V :

$$\begin{aligned} H \ltimes V &\rightarrow (H \ltimes V)/H \simeq \{0\} \times V \simeq V \\ (h, v) &\mapsto h \cdot v \end{aligned}$$

so that the corresponding trivialisaton of the principal bundle $H \ltimes V$ is

$$\begin{aligned} H \ltimes V &\rightarrow H \times V \\ (h, v) &\rightarrow (h, h \cdot v) \end{aligned}$$

which intertwines the right action of H on $H \ltimes V$ with the right action of H on the H factor of $H \times V$.

When the reference geometry is reductive, the Cartan curvature decomposes into a “reduced” curvature $\Omega_{\mathfrak{h}}$ and a torsion Θ components:

$$\Omega = \underbrace{\Omega_{\mathfrak{h}}}_{\mathfrak{h}} \oplus \underbrace{\Theta}_{\mathfrak{g}/\mathfrak{h}} \quad (3.12)$$

Theorem 3.3.13 ([CS09]). *Let (G, H) be a Klein geometry. Then there is a bijection between left G -equivariant H -principal connections on*

$$G \rightarrow G/H$$

and reductive structures

$$\mathfrak{g} \simeq_H \mathfrak{h} \oplus \mathfrak{m}$$

on (G, H) . This bijection is affine when the reductive structure is represented by the projection

$$\mathfrak{g} \xrightarrow{P_{\mathfrak{h}}} \mathfrak{h}$$

Furthermore, the curvature 2-form of the corresponding connection is the left-invariant tensor on G corresponding to the following map at identity:

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{P_{\mathfrak{m}} \otimes P_{\mathfrak{m}}} \mathfrak{m} \otimes \mathfrak{m} \xrightarrow{-[\cdot, \cdot]} \mathfrak{g} \xrightarrow{P_{\mathfrak{h}}} \mathfrak{h}$$

In particular, the connection is flat if and only \mathfrak{m} is an ideal of \mathfrak{g} .

If (G, H) is a reductive Klein geometry, all (G, H) -Cartan geometries inherit a $(H \xrightarrow{\text{Ad}} \text{End}(\mathfrak{g}/\mathfrak{h}))$ -structure *with principal connection*: the Cartan one-form ϖ on P decomposes $(H$ -

equivariantly) into

$$\varpi = \underbrace{\omega}_{\mathfrak{h}} \oplus \underbrace{\alpha_{\mathfrak{m}}}_{\mathfrak{m}} \simeq \underbrace{\omega}_{\mathfrak{h}} \oplus \underbrace{\alpha}_{\mathfrak{g}/\mathfrak{h}}$$

with ω defining an H -principal connection 1-form on P and α a non-degenerate equivariant horizontal form with value in $\mathfrak{g}/\mathfrak{h}$. The 1-form α can be interpreted as a soldering form: it factors on M to an isomorphism $TM \xrightarrow{\sim} P \times_H (\mathfrak{g}/\mathfrak{h})$.

One geometrically simple case is when the adjoint action of H on $\mathfrak{g}/\mathfrak{h}$ is faithful:

Definition 3.3.14. A Klein geometry (G, H) is said to be of *first order* if the adjoint action of H on $\mathfrak{g}/\mathfrak{h}$ is faithful. It is of *higher order* otherwise [Sha97].

When a Klein geometry (G, H) is of first order and effective (that is to say the action of G on G/H is effective), $(H, \mathfrak{g}/\mathfrak{h})$ -structures are given by subbundles of the frame bundles. If it is of higher order, $(H, \mathfrak{g}/\mathfrak{h})$ -structures are given in a similar way by subbundles of higher frame bundles [Slo96].

3.3.5 Associated bundles and tractor connections

Associated G -principal bundle

Let M be a smooth manifold equipped with a (G, H) -Cartan geometry defined by a principal bundle $P \rightarrow M$ and a connection form ϖ .

Considering the left action of H on G , it is possible to construct the associated G -principal bundle

$$Q := P[G] = P \times_H G$$

called *extended principal bundle* [CS09].

As explained earlier, since the left action of H commutes with the right action on G , we obtain a right G -space which forms indeed a G -principal bundle over M . Furthermore, there is a natural G -principal connection on $P[G]$ which we now construct [CS09]. Define the following 1-form on $P \times G$:

$$\text{Ad}_g^{-1} \varpi + \varpi_G \in \Omega^1(P \times G, \mathfrak{g}) \quad (3.13)$$

with ϖ_G the right Maurer-Cartan form on G . In order for it to factor into a G -principal connection 1-form on $P \times_H G$, it needs to satisfy the following properties:

1. It is of maximal rank.
2. It vanishes on vectors $(p \cdot \xi, -\xi \cdot g) \in T(P \times G)$ for $\xi \in \mathfrak{h}$.
3. It is *invariant* under the “diagonal” right action of H given as follows : $\Delta_h(p, g) = (p \cdot h, h^{-1} \cdot g)$.
4. It is equivariant under the right action of G on the factor G .

The first condition is naturally satisfied since the 1-form coincides with ϖ_G on $P \times TG$ which is already of maximal rank. For the second condition, we will need the following equation relating the left and right Maurer-Cartan forms on G (Equation (3.9)):

$$\varpi_G^L|_g = \text{Ad}_g \circ \varpi_G|_g$$

The condition then follows from the following computation:

$$\text{Ad}_g^{-1} \varphi + \varpi_G(p \cdot \bar{\xi}, -\xi \cdot g) = \text{Ad}_g^{-1}(\xi) + \text{Ad}_g^{-1}(\varpi_G^L(-\xi \cdot g)) = \text{Ad}_g^{-1}(\xi - \xi) = 0$$

The third condition results from the following:

$$\begin{aligned}\Delta_h^*(\text{Ad}_g^{-1}\varpi + \varpi_G) &= \text{Ad}_{h^{-1}g}^{-1}\Delta_h^*\varpi + \Delta_h^*\varpi_G \\ &= \text{Ad}_g^{-1}\text{Ad}_h\text{Ad}_{h^{-1}}\varpi + \varpi_G \\ &= \text{Ad}_g^{-1}\varpi + \varpi_G\end{aligned}$$

As for the fourth condition, it is straightforward from Expression (3.13).

Note furthermore that, differently from the situation of principal connections, the G -principal connection 1-form on Q restricts to a *non-degenerate* 1-form on P , or equivalently $HQ|_P \oplus TP = TG|_P$. We can thus formulate the following theorem:

Theorem 3.3.15. *Cartan geometries on M modelled over (G, H) can be equivalently represented as G -principal bundles $Q \rightarrow M$ equipped with a G -principal connection and a reduction of structure group to H such that the horizontal distribution is transverse to the H -principal subbundle (this is called the Cartan condition).*

In particular, this implies that the curvature of the Cartan connection 1-form satisfies a Bianchi identity over Q , therefore over P as well:

$$d\Omega + \frac{1}{3}[\varpi \wedge \Omega] = 0 \in \Omega_{\text{hor}}^3(Q, \mathfrak{g})^G \simeq \Omega^3(P, \mathfrak{g})^H$$

For the same reason, there is a Ricci identity similar to Equation 3.4.

Jet space interpretation

In this section we assume for simplicity that (G, H) models a Klein geometry of order 1. In particular the natural bundle $G \rightarrow G/H$ is isomorphic to a frame bundle. In this section, it will be convenient to use left coset spaces as well – everything done with homogeneous spaces and principal bundles is true as well for left group actions. Consequently, we will consider $G \rightarrow H \backslash G$, equipped with the opposite left Maurer-Cartan form $-\varpi_G^L$, image of the right Maurer-Cartan form under inversion (Equation 3.10).⁵

Let M be a smooth manifold equipped with a (G, H) -Cartan geometry $(P \rightarrow M, \varpi)$. As mentioned in Section 3.3.4, the Cartan geometry defines an $(H, \mathfrak{g}/\mathfrak{h})$ -structure on M : a class of identifications of $\mathfrak{g}/\mathfrak{h}$ with TM at each point, or in other words a class of invertible 1-jets of maps $\mathfrak{g}/\mathfrak{h} \rightarrow M$ from the origin of $\mathfrak{g}/\mathfrak{h}$ to each point of M . Naturally, it is the same for $G \rightarrow H \backslash G$. Therefore, the extended principal bundle

$$Q = P \times_H G$$

can be understood as follows: it is the space of invertible 1-jets of maps from M to $H \backslash G$ which preserve the $(H, \mathfrak{g}/\mathfrak{h})$ -structure, namely which send $\mathfrak{g}/\mathfrak{h}$ -valued frames belonging to P to $\mathfrak{g}/\mathfrak{h}$ -valued frames belonging to G (as a $(H, \mathfrak{g}/\mathfrak{h})$ -structure over $H \backslash G$). Indeed this gives us an embedding

$$P \times_H G \hookrightarrow \mathcal{J}_{\text{diff}}^1(M, G)$$

⁵It is possible to avoid resorting to left coset spaces by rewriting the construction of $P \times_H G$ accordingly, but we think the construction is clearer this way, with a different presentation of the Klein geometry.

and $P \times_H G$ naturally projects to both M and $H \backslash G$:

$$\begin{array}{ccc} & P \times_H G & \\ \swarrow \pi_M & & \searrow \pi_{H \backslash G} \\ M & & H \backslash G \end{array}$$

with a fibre above $M \times G$ which is *diffeomorphic* to H .

From this perspective, the reduction of structure group to H corresponds to restricting to the fibre above $\{e\} \in H \backslash G$.

The Cartan connection is given by an Ehresmann connection over $Q \rightarrow M$ and is required to satisfy both the Cartan condition and equivariance. Equivariance states that the horizontal distribution is equivariant over $H \backslash G$: it respects completely the symmetry of $H \backslash G$. The Cartan connection means that the horizontal distribution is transverse to the fibre above $\{e\} \in H \backslash G$, and under equivariance, this is equivalent to the horizontal distribution being transverse to all fibres above $H \backslash G$. Furthermore the horizontal distribution has the same dimension as M and $H \backslash G$. As a consequence: the tangent projection $TQ \xrightarrow{\pi_{H \backslash G}^*} T(H \backslash G)$ establishes linear isomorphisms

$$HQ \xrightarrow{\sim} \pi_{H \backslash G}^* T(H \backslash G)$$

Of course, since it is an Ehresmann connection over M , the same goes for the projection to M :

$$HQ \xrightarrow{\sim} \pi_M^* TM$$

In particular, this gives an isomorphism

$$\begin{array}{ccc} & HQ & \\ \swarrow \sim & & \searrow \sim \\ \pi_M^* TM & \xrightarrow{\sim} & \pi_{H \backslash G}^* T(H \backslash G) \end{array}$$

which is constructed *from the Cartan connection*. The natural question is how this relates to the embedding $P \times_H G \hookrightarrow \mathcal{J}_{\text{diff}}^1(M, G)$ which is *not* dependent on the Cartan connection.

To approach the question, we need to look at the contact forms. Let $(p, g) \in P \times G$. As stated earlier, p defines a $\mathfrak{g}/\mathfrak{h}$ -valued coframe

$$T_{\pi_M(p)} M \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h}$$

and g as well:

$$T_{[g]} H \backslash G \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h}$$

In fact, we know explicit expressions: the $\mathfrak{g}/\mathfrak{h}$ -valued coframes correspond exactly to the $\mathfrak{g}/\mathfrak{h}$ -valued soldering forms:

$$\begin{aligned} \pi_M^* TM &\simeq TP/VP \xrightarrow{\alpha = \varpi \bmod \mathfrak{h}} \mathfrak{g}/\mathfrak{h} \\ \pi_{H \backslash G}^* T(H \backslash G) &\simeq TG/VG \xrightarrow{-\omega_{\mathfrak{g}/\mathfrak{h}}^L = \varpi_G^L \bmod \mathfrak{h}} \mathfrak{g}/\mathfrak{h} \end{aligned}$$

Therefore at the point $(p, g) \in P \times G$, using an arbitrary coframe e^a at $T_{[g]}(H \backslash G)$, the pullback

to $P \times G$ of the contact forms on $P \times_H G$ take the form

$$e^a - e^a \circ (-\omega_{\mathfrak{g}/\mathfrak{h}}^L)^{-1} \circ \alpha : T_{(p,g)}P \times G \rightarrow \mathbb{R}^n$$

or using $\omega_{\mathfrak{g}/\mathfrak{h}}^L$ as a more natural $\mathfrak{g}/\mathfrak{h}$ -valued coframe on $T_{[g]}(H \backslash G)$ we obtain the $\mathfrak{g}/\mathfrak{h}$ -valued contact form

$$\omega_{\mathfrak{g}/\mathfrak{h}}^L + \alpha : T_{(p,g)}P \times G \rightarrow \mathfrak{g}/\mathfrak{h}$$

This is, up to a Aut_g automorphism of $\mathfrak{g}/\mathfrak{h}$, the $\mathfrak{g}/\mathfrak{h}$ component of the pullback to $P \times G$ of the principal connection 1-form on Q (3.13). Therefore, the contact 1-forms on Q are generated by the $\mathfrak{g}/\mathfrak{h}$ component of the principal connection 1-form. In particular, the horizontal distribution is included in the Cartan distribution (the distribution of vectors annihilated by all contact 1-forms).

To summarise, we showed that the isomorphism

$$\begin{array}{ccc} & HQ & \\ \swarrow \sim & & \searrow \sim \\ \pi_M^* TM & \xrightarrow{\sim} & \pi_{H \backslash G}^* T(H \backslash G) \end{array}$$

matches with the one defined by the embedding into the jet space $P \times_H G \hookrightarrow \mathcal{J}_{\text{diff}}^1(M, G)$, independently of the Cartan connection.

Since the curvature of the Cartan connection is identified with the curvature of the horizontal distribution on Q , the Cartan connection is flat if and only if the horizontal distribution is locally integrable. In this case, since a leaf of HQ locally projects diffeomorphically both on M and on $H \backslash G$, it defines a local diffeomorphism between M and $H \backslash G$ which preserves the $\mathfrak{g}/\mathfrak{h}$ -structure. Furthermore, since the contact 1-forms vanish on a leaf, the whole contact ideal necessarily vanishes as well. This implies that the leaf is the 1st order prolongation of a n -dimensional submanifold of $M \times H \backslash G$ which is locally the graph of a diffeomorphism preserving the $\mathfrak{g}/\mathfrak{h}$ -structure.

In this way, integral manifolds of the horizontal distribution give (possibly partial) maps between M and $H \backslash G$ which preserve the $\mathfrak{g}/\mathfrak{h}$ -structure. Another common example is that of distinguished curves in M [CS09]. They are a specific class of curves defined on M in correspondence with a specific class of curves on the Klein model $H \backslash G$. A simple example for a reductive Klein geometry $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{m}$ is that of \mathfrak{m} -canonical curves: these are curves on $H \backslash G$ which are the projections of integral curves on G of left-invariant vectors $\bar{\xi}^R$ for $\xi \in \mathfrak{m}$. The corresponding notion of canonical curves on M is that of curves which correspond to \mathfrak{m} -canonical curves under the following procedure: lift the curve to a horizontal curve in Q and project it to $H \backslash G$.

From this perspective, M and $H \backslash G$ essentially have a symmetrical role, however (G, H) being the model geometry we see this construction as a construction over M . Nonetheless, this point of view allows studying the problem of equivalence of Cartan geometry in generality and not only the equivalence with the “flat” geometry that the Klein model is. Of course, if $H \backslash G$ is replaced with a less symmetrical space, the construction draws away from the convenient framework of principal bundles.

Tractor connections

As mentioned earlier, the G -principal bundle Q has a natural G -principal connection. This implies that for any G -manifold X , there is an associated connection on the associated fibre bundle $Q[X]$. When X is a linear representation of G , these associated bundles are called *tractor bundles* and the associated linear connections are called *tractor connections*. Since the tractor connections are

associated to a principal connection on Q , they satisfy the Ricci identity

Notice that $Q[X]$ can be constructed from P as follows:

$$Q[X] = Q \times_G X \simeq P \times_H G \times_G X \simeq P \times_H X$$

In particular, even if for a generic H -manifold Y the associated bundle $P[Y]$ is not naturally equipped with a connection, when Y is the restriction to H of a G -manifold then $P[Y]$ has a natural connection. However, when (G, H) has a reductive structure, P obtains an H -principal connection, hence every associated bundle $P[Y]$ has a natural connection. In this situation, the bundle associated to a G -manifold X can be equipped with a connection as a bundle associated to P or as a bundle associated to Q . These connections are different in general.

Example 3.3.16 (Vector bundle associated to an affine Cartan geometry [Bry]). Let M be an n -manifold equipped with a $(\mathrm{GL}_n \times \mathbb{R}^n, \mathrm{GL}_n)$ -Cartan geometry, which is exactly the structure of a principal connection on the bundle of linear frames. The group $\mathrm{GL}_n \times \mathbb{R}^n$ is isomorphic to the affine group of \mathbb{R}^n . It has a linear representation of dimension \mathbb{R}^{n+1} as follows:

$$\begin{aligned} \mathrm{GL}_n \times \mathbb{R}^n &\rightarrow \mathrm{GL}_{n+1} \\ (g, v) &\mapsto \left[\begin{array}{c|c} 1 & 0 \\ \hline g \cdot v & g \end{array} \right] \end{aligned}$$

There is a corresponding tractor bundle $Q[\mathbb{R}^{n+1}]$. Since under GL_n , \mathbb{R}^{n+1} decomposes as $\underbrace{\mathbb{R}}_{\mathrm{triv.}} \oplus_{\mathrm{GL}_n} \mathbb{R}^n$, there is an isomorphism of vector bundle

$$Q[\mathbb{R}^{n+1}] \simeq P[\mathbb{R} \oplus \mathbb{R}^n] \simeq \mathbb{R} \times TM$$

The tractor bundle has a natural connection induced from the connection on $\mathrm{GL}_n \times \mathbb{R}^n$. The action of the curvature on $Q[\mathbb{R}^{n+1}]$ can be expressed in terms of the “linear” curvature endomorphism $R \in \Omega^2(M, \mathrm{End}(TM))$ and the torsion $T \in \Omega^2(M, TM)$ as follows:

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline T & R \end{array} \right] \in \Omega^2(M, \mathrm{End}(\mathbb{R} \times TM))$$

Example 3.3.17 (Tractor bundles for affine Cartan geometries). Let $(H \ltimes V, V)$ be an affine Klein geometry with V a linear representation of H . Then there is a retract of $H \ltimes V$ to the subgroup H :

$$\begin{array}{c} \xrightarrow{\mathrm{id}} \\ H \longleftarrow H \ltimes V \longrightarrow H \simeq (H \ltimes V)/V \end{array}$$

As a consequence, every representation of H can be seen as a representation of $H \ltimes V$ with a trivial action of V . In particular, for a representation W of H , there are both a tractor bundle and a bundle associated to the H -principal bundle. It turns out that in this case, both connections match. In fact, every affine Klein geometry is equipped with a reductive structure:

$$\mathfrak{h} \ltimes V \simeq_{\mathfrak{h}} \mathfrak{h} \oplus V$$

and the H -principal bundle P_H can be constructed from the $(H \ltimes V)$ -principal bundle Q as follows:

$$P_H \simeq Q[H \ltimes V/V] \simeq Q/V$$

There is one specific associated bundle which is useful in the study of Cartan geometries: the so-called *Cartan space* \mathcal{S} [CS09]:

$$\mathcal{S} = Q[G/H]$$

Its fibres can be identified with the homogeneous space G/H up to the left action of an element of H . The reduction of structure group $P \hookrightarrow Q$ defines a canonical global section $e \in \Gamma(M, \mathcal{S})$.

Let $c : I \rightarrow M$ be a smooth curve starting from a point m . We will construct its *development*, which is a Cartan-geometrical “horizontal” equivalent to holonomy and parallel transport. Since Cartan connections, in contrary to principal connections, do take into account horizontal directions it is possible to obtain transport which is sensitive to horizontal directions in the principal bundle.

There is a time-parametrised parallel transport associated to c :

$$t \in I \mapsto \text{PT}_{c|_{[0,t]}} \in \text{Hom}_G(Q_m, Q_{c(t)})$$

It acts on \mathcal{S} to give parallel transports

$$t \in I \mapsto \text{PT}_{c|_{[0,t]}}^{\mathcal{S}} \in \text{Hom}_G(\mathcal{S}_m, \mathcal{S}_{c(t)})$$

The *development* of c is defined as the following curve in \mathcal{S}_m ⁶:

$$\begin{cases} I & \rightarrow \mathcal{S}_m \\ t & \mapsto \left(\text{PT}_{c|_{[0,t]}}^{\mathcal{S}} \right)^{-1} (e(c(t))) \end{cases} \quad (3.14)$$

In other words, it is the “trace” of the section of \mathcal{S} in \mathcal{S}_m under parallel transport along c . Since \mathcal{S}_m is identified with G/H up to an element of H , the development of c can be identified with a curve in G/H which starts from the origin, up to the global left action of a (constant) element of H .

Example 3.3.18 (Development in affine Cartan geometries). Let $(H \ltimes V, H)$ be an affine Klein geometry (V is a finite dimensional real vector space on which H acts linearly) and M a differentiable manifold with an $(H \ltimes V, H)$ -Cartan geometry. The Klein model $H \ltimes V/H$ is an affine space isomorphic to V . In particular, the Cartan space is naturally isomorphic to the tangent space of M :

$$\mathcal{S} = Q[V] \simeq P[V] \simeq TM$$

and the canonical section is identified with the zero section.

As a consequence, developments of curves on M take values in TM . They can be represented in \mathbb{R}^n with ambiguity up to the linear action of an element of H .

The adjoint tractor bundle

One very useful natural tractor bundle is the so-called *adjoint tractor bundle*. On P , the Cartan connection 1-form ϖ defines an H -equivariant \mathfrak{g} -valued coframe. Namely, it defines an H -equivariant isomorphism

$$\varpi : TP \xrightarrow{\sim} {}_H P \times \mathfrak{g}$$

(recall that \rightarrow_H denotes an H -equivariant map). In particular, H -equivariant vector fields on P correspond to H -equivariant \mathfrak{g} -valued fields.

⁶[Sha97] defines a development with value in G/H but the treatment is inexact due to an ambiguity over the starting point of the developments.

The *adjoint tractor bundle* is the tractor bundle associated to the adjoint action $\text{Ad}_G : G \rightarrow \text{GL}(\mathfrak{g})$:

$$P[\mathfrak{g}] = P \times_H \mathfrak{g} \simeq Q \times_G \mathfrak{g}$$

It is a bundle of Lie algebras over M .

We just explained that sections of $P[\mathfrak{g}]$ correspond to H -equivariant vector fields on P . This can be equivalently formulated as the following isomorphism:

$$P[\mathfrak{g}] = P \times_H \mathfrak{g} \xrightarrow[\varpi^{-1}]{\simeq} TP/H$$

Similarly, ϖ induces an equivariant isomorphism between *vertical* vector fields on P and \mathfrak{h} -valued fields:

$$\varpi|_{VP} : VP \xrightarrow[\simeq]{\simeq} P \times \mathfrak{h}$$

Accordingly, the associated bundle to $\text{Ad}_H : H \rightarrow \text{GL}(\mathfrak{h})$ admits a representation as a quotient bundle:

$$P[\mathfrak{h}] = P \times_H \mathfrak{h} \xrightarrow[\varpi^{-1}]{\simeq} VP/H$$

Like the adjoint tractor bundle, $P[\mathfrak{h}]$ is a bundle of Lie algebras.

Since there is a natural short sequence of representations of H :

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{h}$$

there is a corresponding short sequence of associated bundles:

$$\begin{array}{ccccc} P[\mathfrak{h}] & \hookrightarrow & P[\text{Ad}_G] & \twoheadrightarrow & P[\mathfrak{g}/\mathfrak{h}] \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ VP/H & \hookrightarrow & TP/H & \twoheadrightarrow & TM \end{array}$$

Let V be a vector space with an action of G . There is in particular a Lie algebra action of $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$ which defines a G -invariant element of $\text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{gl}(V))$. Accordingly, there is a section of $\text{Hom}_{\text{Lie}}(P[\mathfrak{g}], P[\mathfrak{gl}(V)]) \simeq \text{Hom}_{\text{Lie}}(P[\mathfrak{g}], \text{End}(P[V]))$. In other words, the adjoint tractor bundle is a bundle of Lie algebras which has a natural action on every tractor bundle. This construction actually still holds when V is replaced by a non-linear representation of G (a G -manifold).

The curvature 2-form Ω of the Cartan connection can be constructed as an element of $\Omega^2(P, \mathfrak{g})^H$. Therefore, it can be naturally interpreted as a $P[\mathfrak{g}]$ -valued 2-form. In particular, this implies that it has an action on all tractor bundles.

The fundamental derivative

Let M be a smooth manifold equipped with a (G, H) -Cartan geometry $(P \rightarrow M, \varpi)$. Let V be a representation of H . In general the Cartan connection does not induce a covariant derivative on $P[V]$: a vector on M does not have enough data to act on a section in a uniquely defined manner. Instead, there is what is called the *fundamental derivative* which uses vectors in the adjoint tractor bundle [CS09].

Let $S \in \Gamma(P, V)^H$ correspond to a section $\sigma \in \Gamma(M, P[V])$. Then dS is an H -equivariant V -valued 1-form on P . We define *the fundamental derivative of S*

$$dS \in \Omega^1(P, V)^H \xrightarrow[\varpi^{-1}]{\simeq} \Gamma(P, \text{Hom}_{\mathbb{R}}(\mathfrak{g}, V))^H$$

It corresponds to an associated section

$$D\sigma \in \Gamma((TP/H)^* \otimes P[V]) \simeq \Gamma(\text{Hom}(P[\mathfrak{g}], P[V]))$$

This definition can be directly generalised to sections of $\bigotimes^k TP/H^* \otimes P[V]$ by replacing V with $\bigotimes^k \mathfrak{g}^* \otimes V$

Let us write $D^2 : \Gamma(P[V]) \rightarrow \Gamma(\Lambda^2 P[\mathfrak{g}]^* \otimes V)$ the square of the fundamental derivative, which is reminiscent of the covariant second derivative from Definition 4.2.12. The fundamental derivative satisfies Ricci and Bianchi identities as follows: let A_1, A_2, A_3 be sections of the adjoint tractor bundle which respectively project to vector fields X_1, X_2, X_3 and σ a section of an tractor bundle $P[V]$.

$$\begin{aligned} D_{A_1, A_2}^2 \sigma - D_{A_2, A_1}^2 \sigma &= D_{\Omega(X_1, X_2)} \sigma + D_{[A_1, A_2]^{P[\mathfrak{g}]}} \sigma \\ &+ [A_1, \Omega(X_2, X_3)] - \Omega([X_1, X_2], A_3) + \Omega(d\pi(\Omega(X_1, X_2)), X_3) + D_{A_1} \Omega(X_2, X_3) \\ &+ [A_2, \Omega(X_3, X_1)] - \Omega([X_2, X_3], A_1) + \Omega(d\pi(\Omega(X_2, X_3)), X_1) + D_{A_2} \Omega(X_3, X_1) \\ &+ [A_3, \Omega(X_1, X_2)] - \Omega([X_3, X_1], A_2) + \Omega(d\pi(\Omega(X_3, X_1)), X_2) + D_{A_3} \Omega(X_1, X_2) \end{aligned}$$

For tractor bundles $P[V] \simeq Q[V]$, the fundamental derivative can be related to the tractor connection and the action of the adjoint tractor bundle on vector adjoint bundles [CS09].

3.3.6 Relating different Cartan geometries

There is an obvious notion of morphism between Cartan geometries of the same type: a morphism of principal bundles which preserves the connection 1-form. The underlying map between basis manifolds is necessarily a local diffeomorphism.

There are however more ways to related Cartan geometries. In this section we describe two ways one can create new Cartan geometries from an existing one. General references are [CS09; Žád19].

Model mutations

The definition of a $(H \hookrightarrow G)$ -Cartan geometry on a manifold M does not actually involve the group structure of G but only the Lie group structure of H and the structure of \mathfrak{g} as a representation of H . It is thus possible to establish a correspondence between Cartan geometries modelled over (G_1, H) and (G_2, H) if $\mathfrak{g}_1 \simeq_H \mathfrak{g}_2$ as representations of H [Sha97]. Formally, given an $(H \hookrightarrow G_1)$ -Cartan geometry (P, ϖ) on a manifold M and an H -equivariant vector space isomorphism $\mu : \mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{g}_2$, the \mathfrak{g}_2 -valued form $\mu_* \varpi$ on P defines an $(H \hookrightarrow G_2)$ -Cartan connection on P hence an $(H \hookrightarrow G_2)$ -Cartan geometry on M .

Definition 3.3.19 (Model mutations). Let (G_1, H_1) and (G_2, H_2) be two Klein geometries. A *mutation* from (G_1, H_1) to (G_2, H_2) is the data of a Lie group isomorphism

$$H_1 \xrightarrow[f]{} H_2$$

and an f -equivariant linear isomorphism $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ extending the induced isomorphism between the subalgebras $\mathfrak{h}_1 \xrightarrow[df]{} \mathfrak{h}_2$:

$$\begin{array}{ccc} \mathfrak{h}_1 & \xrightarrow{\sim_{df}} & \mathfrak{h}_1 \\ \downarrow & & \downarrow \\ \mathfrak{g}_1 & \xrightarrow{\sim_{\mu}} & \mathfrak{g}_2 \end{array}$$

such that

$$\forall (u, v) \in \mathfrak{g}_1, \mu([u, v]) = [\mu(u), \mu(v)] \pmod{\mathfrak{h}_2} \quad (3.15)$$

Mutations are invertible and two Klein geometries which are related by a mutation are called *mutants* of each other.

Example 3.3.20 (Geometric isomorphisms). The geometric isomorphisms (Definition 3.3.1) form a class of auto-mutations which are “trivial” in the sense that the Lie algebra structure of \mathfrak{g} , and even the Lie group structure of G does not change. This confirms that the notion of model mutation indeed depends on the underlying Klein geometry and not on the specific representative coset space chosen.

If Cartan geometries modelled on mutant Klein geometries are defined by equivalent geometrical data, namely a principal bundle with a Cartan connection, the choice of model Klein geometry matters in the *geometric interpretation* of the data. In particular, the Cartan curvature does actually depend on the Lie algebra structure of \mathfrak{g} :

$$\Omega = d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \in \Omega^2(P, \mathfrak{g})$$

so that flatness depends on the chosen Klein model. Condition 3.15 however implies that the torsion component does not change under mutation.

Example 3.3.21 (Mutant models of Riemannian geometry, [Sha97]). The three following Riemannian Klein geometries of dimension n are mutant of each other:

- Euclidean geometry: $(O_n \ltimes \mathbb{R}^n, O_n)$
- Spherical geometry: (O_{n+1}, O_n)
- Hyperbolic geometry: $(SO_{n,1}, O_n)$

According to Example 3.3.8, the sphere and the hyperbolic space have *flat* canonical Cartan geometries, although they are obviously curved from the point of view of Riemannian geometry. Mutation allows us to compare curvature between mutant models: although the sphere has a flat *spherical* geometry, mutation to a Euclidean geometry gives the sphere a *curved* Euclidean (Cartan) geometry.

Homotheties (dilations of the \mathbb{R}^n component by a fixed factor in \mathbb{R}^{n*}) form automutations of Euclidean geometry, hence correspond to automutations of all these three geometries. In the case of Euclidean geometry they can be integrated into Klein space automorphisms. In dimension $n \geq 4$ these are the only automutations.

Some reductive structures on Klein geometries can be understood under the perspective of model mutations. A reductive structure on a Klein geometry (G, H) can be identified with a section

$$\mathfrak{g} \xrightarrow{\sim_H} \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$$

of the projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$. The supplementary \mathfrak{h} -subrepresentation \mathfrak{m} to \mathfrak{h} is the inverse image of $\mathfrak{g}/\mathfrak{h}$. They form mutations when Condition 3.15 is satisfied. Note that auto-mutations of $\mathfrak{h} \ltimes \mathfrak{g}/\mathfrak{h}$

which preserve the reductive decomposition are necessarily Lie algebra automorphisms [Sha97] (Lemma 5.6.4).

We explained in Section 3.3.4 how an $(H \hookrightarrow G)$ -Cartan geometry on a manifold M determines an $(H, \mathfrak{g}/\mathfrak{h})$ -structure. When the Klein geometry has a reductive structure, the $(H, \mathfrak{g}/\mathfrak{h})$ -structure inherits a principal connection. Conversely, given a group H and a representation V , an (H, V) -structure on the manifold equipped with a principal connection defines an $(H, H \ltimes V)$ -Cartan geometry. On the H -principal bundle $P \rightarrow M$ associated to the (H, V) -structure, the solder form α defines an equivariant horizontal V -valued 1-form while the connection 1-form defines an \mathfrak{h} -normalized equivariant \mathfrak{h} -valued 1 form, so that they combine into an $\mathfrak{h} \ltimes V$ -valued 1-form:

$$\varpi = \omega \oplus \alpha \in \Omega^1(P, \mathfrak{h} \ltimes V)^H$$

which is non-degenerate and defines a Cartan connection on P .

Note that the Cartan geometries thus obtained are necessarily modelled over “affine” reductive Klein geometries of the type $(H \ltimes V, H)$ with V a representation of H . In other words, V has to be a torsionless abelian normal subgroup of $H \ltimes V$. Starting from a reductive Klein geometry: $\mathfrak{g} =_H \mathfrak{h} \oplus \mathfrak{m}$ a Cartan geometry induces an $(H, \mathfrak{g}/\mathfrak{h})$ -structure with connection, from which one can construct back an $(H \ltimes \mathfrak{g}/\mathfrak{h}, H)$ -Cartan geometry in which $\mathfrak{g}/\mathfrak{h}$ is to be understood *as a vector space representation of \mathfrak{h}* . One can say that the $(H, \mathfrak{g}/\mathfrak{h})$ -structure loses the data of the component $\mathfrak{m} \times \mathfrak{m} \xrightarrow{[\cdot, \cdot]} \mathfrak{g}$ of the bracket. In this way, the (H, V) -structures with connection on M are identified with $(H \ltimes V, H)$ -Cartan geometries on M .

This is summed up in the following diagram for a given reference Klein geometry (G, H) :

$$\begin{array}{ccc} \{(G, H)\text{-Cartan geometries}\} & \xrightarrow[\omega = \varpi^{\mathfrak{h}}]{\alpha = \varpi^{\mathfrak{g}/\mathfrak{h}}} & \{(H, \mathfrak{g}/\mathfrak{h})\text{-structures with connection}\} \\ \text{maps} \uparrow & & \swarrow \\ \mathfrak{g} \xrightarrow{\sim} \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} & & \\ \downarrow & & \\ \{(H \ltimes \mathfrak{g}/\mathfrak{h}, H)\text{-Cartan geometries}\} & & \end{array}$$

in which the morphisms commute when the same reductive structure is chosen for the map $\mathfrak{g} \xrightarrow{\sim} \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ and to define the component $\varpi^{\mathfrak{h}}$.

When (G, H) admits a reductive structure $\mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, the Klein geometry is said to be *of symmetric type* [Lot04]. In this case such a reductive structure seen as a morphism

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$$

actually forms a mutation. Gathering all Cartan geometries in a given mutation class $H \ltimes V$ on a manifold M with the same dimension as V , we obtain the following diagram

$$\begin{array}{ccc} \{(G, H)\text{-Cartan geometries on } M\} & \xrightarrow{\mathfrak{g}/\mathfrak{h} \xrightarrow{\sim} \mathfrak{h} \oplus V} & \{(H, V)\text{-structures on } M \text{ with connection}\} \\ \text{mutations} \downarrow & & \swarrow \\ \mathfrak{g} \xrightarrow{\sim} \mathfrak{h} \oplus V & & \\ \uparrow & & \\ \{(H \ltimes V, H)\text{-Cartan geometries on } M\} & & \end{array}$$

in which horizontal and vertical arrows are not unique but labelled by the reductive structures on the Klein geometries (G, H) such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Model mutations fit into the more general framework of *Cartan extension* in which one considers certain maps between infinitesimal Klein geometries.

Correspondence spaces

Let M be a differentiable manifold with $P \rightarrow M$ a H -principal bundle equipped with an (G, H) -Cartan connection ϖ . On P the form ϖ defines a \mathfrak{g} -valued coframe. The data of H is involved in two related geometric properties:

- H acts freely on P and $P/H \simeq M$ is a differentiable manifold.
- ϖ is equivariant with respect to the action of H .

In particular these properties hold for every (closed) Lie subgroup K of H as well.

Definition 3.3.22 (Correspondence space). Let (G, H) be a Klein geometry. Let M be a differentiable manifold equipped with a (G, H) -Cartan geometry $(P \rightarrow M, \varpi)$. Let K be a closed Lie subgroup of H .

Then the *correspondence space* associated to (G, K) is the differentiable manifold P/K equipped with the (G, K) -Cartan geometry specified by $(P \rightarrow P/K, \varpi)$.

The structure of correspondence spaces can be represented in the following commutative diagram:

$$\begin{array}{ccc}
 K & \longleftrightarrow & H \\
 \swarrow & & \downarrow \\
 & & P \\
 \swarrow & & \searrow \\
 & & P/K \\
 \downarrow & & \swarrow \\
 M \simeq P/H & &
 \end{array}$$

$\begin{array}{l} \text{K-orbits} \\ \text{H-orbits} \\ \text{K-principal} \\ \text{H-principal} \end{array}$

Since the Cartan connection 1-forms over M and P/K are the same, the curvature 2-forms are the same as well.

A section of $P/K \rightarrow M$ is equivalent to a reduction of structure group of P to K : it is equivalent to specifying above each point of M a K -orbit of P :

$$\begin{array}{ccccc}
 K\text{-orbits} & \longleftrightarrow & P & \longrightarrow & P/K \\
 \swarrow & & \downarrow & & \swarrow \\
 & & M \simeq P/H & &
 \end{array}$$

$\begin{array}{l} \text{K-principal} \\ \text{H-principal} \end{array}$

Let us call P_K such a K -principal bundle over M which is a reduction of P . The question of whether it defines a Cartan geometry over M is hard to answer without further structure. There is however a case in which the question becomes simple: the case of affine Klein geometry, or H -structures.

Theorem 3.3.23 ([RS17]). Let $(\mathfrak{g} \simeq \mathfrak{h} \ltimes V, H)$ be an affine Klein geometry with V a finite dimensional real representation of H . Let M be a differentiable manifold with a (G, H) -Cartan geometry (P, ϖ) . Let K be a closed subgroup of H and $P_K \xrightarrow{i} P$ a reduction of structure group of G to K .

Assume that $\varpi(TP_K) = (\mathfrak{k} \ltimes V) \times P_K$. Then the pullback $i^*\varpi$ of ϖ to P_K defines a K -equivariant \mathfrak{g} -valued 1-form which splits as follows, according to

$$\mathfrak{g} \simeq_K \mathfrak{k} \ltimes V \oplus_K \mathfrak{h}/\mathfrak{k}$$

1. A $(K \ltimes V, K)$ -Cartan connection ϖ_K on P_K
2. A $\mathfrak{h}/\mathfrak{k}$ -valued basic 1-form on P_K .

Example 3.3.24. Let M be a differentiable n -manifold with its canonical GL_n -structure, endowed with linear connection which has an associated affine connection. Assume M is furthermore equipped with a positive metric g , which defines a reduction of structure to O_n of the frame bundle.

The $(\mathrm{GL}_n \ltimes \mathbb{R}^n, \mathrm{GL}_n)$ -Cartan connection is compatible with the reduction to $\mathrm{O}_n \ltimes \mathbb{R}^n$ if and only if the linear connection is compatible with the reduction to O_n , which can be proved [RS17] to be equivalent to the metric g being parallel. Therefore, there is an induced $(\mathrm{O}_n \ltimes \mathbb{R}^n, \mathrm{O}_n)$ -Cartan connection if and only if the connection is metric, as can be expected.

Another property of interest regarding reduction of structure groups is *variation of the reduction of the structure group*. Let us assume the Klein pair $(\mathfrak{g}, \mathfrak{k})$ is equipped with a reductive structure. Then $P \rightarrow P/\mathfrak{k}$ has an induced K -principal connection. Let $\sigma : [0, 1] \times M \rightarrow P/K$ be a smoothly parametrised section of $P/K \rightarrow M$. Consider the following pullback K -principal bundle:

$$\begin{array}{ccc} \sigma^*P & \hookrightarrow & P \\ \downarrow & & \downarrow \\ [0, 1] \times M & \xrightarrow{\sigma} & P/K \end{array}$$

The pullback bundle σ^*P can be equipped with the pullback K -principal connection. Then the vector field $\frac{\partial}{\partial t}$ lifts to an horizontal vector field on σ^*P . As a consequence we obtain identifications between $\sigma_{t_1}^*P$ and $\sigma_{t_2}^*P$ for any two instants of $[0, 1]$.

Chapter 4

Spinors and Clifford algebras

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In this Section we will present the main features of spinors. Spinors are defined as objects on which a Clifford algebra acts, which allows defining a Dirac operator which is a first-order differential operator which squares to the D'Alembertian (or a Laplacian of any signature), according to the Weitzenböck-Lichnerowicz-Schrödinger formula.

We first define the Clifford algebras and introduce their main structures. All the algebras we consider are unital but are not supposed to be commutative. General references for this chapter are [Mei13; Var04; LS19].

4.1 Algebra

4.1.1 Clifford algebras

In this section, \mathbb{K} is either the field of real numbers or the field of complex numbers, and V is a \mathbb{K} -vector space of finite dimension n equipped with a non-degenerate symmetric bilinear form $\eta = \langle \cdot | \cdot \rangle$ and an associated quadratic form q .

Definition 4.1.1 (Clifford algebra). The Clifford algebra of (V, q) is the unital \mathbb{K} -algebra spanned by elements v of V subject to the anti-commutation relation

$$uv + vu = 2 \langle u | v \rangle 1$$

We write it $\text{Cl}(V, q)$, sometimes omitting the mention of q .

It has dimension 2^n and can be constructed as the quotient of the tensor algebra of V by the ideal generated by the following elements: for $(v_1, v_2) \in V^2$

$$v_1 \otimes v_2 + v_2 \otimes v_1 - 2 \langle u|v \rangle 1$$

As such it satisfies the following universal property: for any linear map from V to a \mathbb{K} -algebra A such that

$$\forall (u, v) \in V \times V, f(u)f(v) + f(v)f(u) = 2 \langle u|v \rangle 1$$

there exists a unique algebra morphism $\text{Cl}(V, q) \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} V & & A \\ \downarrow f & \searrow & \uparrow \\ \text{Cl}(V, q) & \dashrightarrow & A \end{array}$$

Remark. Some authors use the following anti-commutation relation instead:

$$uv + vu = -2 \langle u|v \rangle 1$$

We discuss these considerations of convention in Section 4.3.

Example 4.1.2 (Clifford algebras of one-dimensional spaces). The Clifford algebra of \mathbb{R} with a negative inner product is isomorphic to \mathbb{C} , since a normed vector i of \mathbb{R} satisfies

$$i^2 = -1$$

in $\text{Cl}(\mathbb{R})$.

The Clifford algebra of \mathbb{R} with a positive inner product has an element e such that

$$e^2 = 1$$

As a consequence, there is a decomposition of the identity into *central* idempotents:

$$1 = \frac{1+e}{2} + \frac{1-e}{2}$$

and

$$\text{Cl}(\mathbb{R}, (-)) \simeq \frac{1+e}{2} \text{Cl}(\mathbb{R}, (-)) \oplus \frac{1-e}{2} \text{Cl}(\mathbb{R}, (-)) \simeq \mathbb{R} \oplus \mathbb{R}$$

Since the Clifford algebra of V is generated by elements of V , it has further structure:

Theorem 4.1.3. *The canonical filtration of the tensor algebra of V*

$$\mathbb{K} \subset \mathbb{K} \oplus V \subset \mathbb{K} \oplus V \oplus V \otimes V \subset \bigoplus_{0 \leq k \leq 3} T^k V \subset \dots$$

induces an \mathbb{N} -filtration on the Clifford algebra.

The \mathbb{N} -grading of the tensor algebra of V induces a \mathbb{Z}_2 -grading on the Clifford algebra, giving it the structure of a superalgebra.

Alternatively, the grading can be defined by requiring that the generators $v \in V \rightarrow \text{Cl}(V)$ are of odd degree and the filtration by requiring that the subspace of filtration degree 1 is $\mathbb{R} \oplus V$.

We will be using the notations $\text{Cl}^+(V)$ and $\text{Cl}^-(V)$ respectively for the even part and the odd part of the Clifford algebra.

Theorem 4.1.4. *Let us write $\text{Cl}^{(k)}(V)$ for the subspace of elements of filtration degree k or less. The $(\mathbb{Z}-)$ graded algebra associated to the filtration of $\text{Cl}(V)$ is defined as follows. Construct the following graded vector space:*

$$\text{gr}(\text{Cl}(V)) = \bigoplus_{k \in \mathbb{N}} \text{Cl}^{(k)}(V) / \text{Cl}^{(k-1)}(V)$$

Then the product of $\text{Cl}(V)$ induces a graded product on $\text{gr}(\text{Cl}(V))$.

There is an isomorphism of graded algebra from the associated graded algebra to the exterior algebra of V :

$$\text{gr}(\text{Cl}(V)) \xrightarrow{\sim} \Lambda^\bullet V$$

In particular, given any basis (e_i) of V , the family of elements

$$\left\{ \prod_{\substack{i \in I \\ \text{increasing order}}} e_i \mid I \subset \llbracket 1, n \rrbracket \right\}$$

defines a vector basis of $\text{Cl}(V)$.

In particular, V embeds linearly into $\text{Cl}(V)$ and can be identified with $\text{Cl}^{(1)}(V) \cap \text{Cl}^-(V)$.

We define the following notations: write $\mathbb{R}^{p,q}$ for \mathbb{R}^{p+q} equipped with the inner product for which the canonical basis is orthogonal, the first p vectors have squared norm 1 and the last q vectors have squared norm -1 . Their Clifford algebras are written as follows:

$$\text{Cl}_{p,q} := \text{Cl}(\mathbb{R}^{p,q})$$

Every real Clifford algebra is isomorphic to such one, according to the following theorem:

Theorem 4.1.5 (Functoriality). *Let W be a finite dimensional \mathbb{K} -vector space equipped with a nondegenerate symmetric bilinear form.*

Let $f : V \rightarrow W$ be a (possibly non-surjective) isometry. Then there exists a unique (graded and filtered) algebra morphism $\text{Cl}(V) \rightarrow \text{Cl}(W)$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow & & \downarrow \\ \text{Cl}(V) & \xrightarrow{\text{Cl}(f)} & \text{Cl}(W) \end{array}$$

In particular, the sign change automorphism $v \in V \mapsto -v$ induces an algebra automorphism of $\text{Cl}(V)$, which acts as -1 on $\text{Cl}^-(V)$ and as identity on $\text{Cl}^+(V)$. We will call it the *grading isomorphism* and write it α :

$$\alpha : \begin{cases} \text{Cl}(V) \rightarrow \text{Cl}(V) \\ \text{Cl}^+(V) \xrightarrow{\text{id}} \text{Cl}^+(V) \\ \text{Cl}^-(V) \xrightarrow{-\text{id}} \text{Cl}^-(V) \end{cases}$$

On \mathbb{C}^n , any two nondegenerate symmetric bilinear form are isomorphic, since multiplying vectors by i multiply their squared norm by -1 . As a consequence, Clifford algebras of complex spaces of the same dimension are isomorphic. We write $\mathbb{C}l_n$ for the Clifford algebra of \mathbb{C}^n equipped with the (complexified) canonical real inner product.

Theorem 4.1.6 (Monoidal structure). *Let W be a finite dimensional \mathbb{K} -vector space equipped with a nondegenerate symmetric bilinear form.*

There is a unique graded and filtered algebra isomorphism between $\text{Cl}(V \oplus W)$ with the sum bilinear form, and the graded tensor product algebra¹ $\text{Cl}(V) \hat{\otimes} \text{Cl}(W)$ such that the following diagram commutes:

$$\begin{array}{ccc} & V \oplus W & \\ \swarrow & & \searrow \\ \text{Cl}(V \oplus W) & \xrightarrow{\sim} & \text{Cl}(V) \hat{\otimes} \text{Cl}(W) \end{array} \quad \begin{array}{l} \\ \\ (v,w) \mapsto v \otimes 1 + 1 \otimes w \end{array}$$

Example 4.1.7 (Clifford algebras of real 2-dimensional spaces).

- $\text{Cl}_{2,0}$

Let (e_1, e_2) be an orthonormal basis of $(\mathbb{R}^2, ++)$. We are going to build a more convenient albeit non-homogeneous presentation of the Clifford algebra. Define

$$\begin{aligned} E_+ &= \frac{1 + e_1}{2} \\ E_- &= \frac{1 - e_1}{2} \end{aligned}$$

We know that they satisfy the following relations:

$$\begin{aligned} E_+ + E_- &= 1 \\ E_+^2 &= E_+ \\ E_-^2 &= E_- \\ E_+ E_- &= E_- E_+ = 0 \end{aligned}$$

Since

$$e_2 e_1 = -e_1 e_2$$

the products of E_+ and E_- with e_2 behave as follows:

$$\begin{aligned} e_2 E_+ &= E_- e_2 \\ e_2 E_- &= E_+ e_2 \end{aligned}$$

As a consequence, e_2 decomposes as follows:

$$e_2 = (E_+ + E_-)e_2(E_+ + E_-) = E_+ e_2 E_- + E_- e_2 E_+ = e_2 E_- + e_2 E_+$$

with

$$(e_2 E_-)^2 = (e_2 E_+)^2 = 0$$

¹We write $\hat{\otimes}$ for the tensor product of graded algebras, while we use \otimes for the ungraded tensor products.

We have obtained a vector basis $(E_+, E_-, e_2 E_-, e_2 E_+)$ of $\text{Cl}_{2,0}$ which exactly satisfies the relations of $\text{M}_2(\mathbb{R})$ according to the linear morphism defined as follows:

$$\begin{aligned} E_+ &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ E_- &\mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ E_+ e_2 &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ E_- e_2 &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

- $\text{Cl}_{0,2}$

It is generated by i and j such that

$$\begin{aligned} i^2 &= j^2 = -1 \\ ij &= -ij \end{aligned}$$

Furthermore, defining $k = ij$, we have

$$k^2 = ijij = -i^2 j^2 = -1$$

and k anticommutes with both i and j . One recognises the quaternion algebra \mathbb{H} .

- $\text{Cl}_{1,1}$

It is generated by i and c such that

$$\begin{aligned} i^2 &= -1 \\ c^2 &= 1 \\ ci &= -ic \end{aligned}$$

Left multiplication by i defines a structure of complex vector space on $\text{Cl}_{1,1}$, for which c defines a real (conjugation) structure. As a consequence, we obtain an injective morphism $\text{Cl}_{1,1} \rightarrow \text{End}_{\mathbb{R}}(\mathbb{C}) \simeq \text{M}_2(\mathbb{R})$ which is necessarily an algebra isomorphism for dimension reasons.

Theorem 4.1.8 (Complexification). *Assume that V is a real vector space. Then there is a natural graded and filtered algebra isomorphism between the complexification of the Clifford algebra of V and the Clifford algebra of the complexified vector space associated with V , equipped with the bilinear form extended by \mathbb{C} -bilinearity:*

$$\mathbb{C} \otimes_{\mathbb{R}} \text{Cl}(V) \xrightarrow{\sim} \text{Cl}(\mathbb{C} \otimes_{\mathbb{R}} V)$$

Principal antiautomorphism and opposite algebras

Definition 4.1.9 (Principal antiautomorphism). There is a uniquely defined (graded, filtered) algebra antiautomorphism $x \mapsto T x$ defined on $\text{Cl}(V)$ such that for every family of vectors (v_1, \dots, v_k) the following holds in $\text{Cl}(V)$:

$$T(v_1 \cdots v_k) = v_k \cdots v_1$$

It is called the *principal antiautomorphism*, or *canonical antiautomorphism*, or *transposition* (which we already use with a different meaning).

The *ungraded opposite* algebra of $\text{Cl}(V)$ which we write $\text{Cl}(V)^{\text{opp}}$ is defined as the same vector space with the following bilinear product:

$$a \odot b = b \cdot a$$

The principal antiautomorphism of $\text{Cl}(V)$: $a \mapsto {}^T a$ defines a graded and filtered isomorphism

$$\text{Cl}^{\text{opp}}(V) \xrightarrow{\sim} \text{Cl}(V)$$

Since it is a graded algebra anti-automorphism, the principal antiautomorphism commutes with the gradation automorphism α and with inversion on invertible elements.

Using the graded structure it is possible to relate $\text{Cl}(V)$ and $\text{Cl}(-V)$:

Theorem 4.1.10 (Opposite Clifford superalgebra, [Del99]). *The opposite superalgebra to $\text{Cl}(V)$ is defined as the same vector space with a bilinear product which behaves as follows between homogeneous elements a and b :*

$$a \hat{\odot} b = (-1)^{|b||a|} b \cdot a$$

The identity of V induces a graded and filtered algebra isomorphism between $\text{Cl}(-V)$ and the opposite superalgebra to $\text{Cl}(V)$

$$\text{Cl}(-V) \simeq \text{Cl}(V)^{\text{sopp}}$$

In particular, since $\text{Cl}^+(V)$ is an even subalgebra of $\text{Cl}(V)$, there is the following chain of isomorphisms:

$$\text{Cl}^+(-V) \simeq \text{Cl}^+(V)^{\text{sopp}} \simeq \text{Cl}^+(V)^{\text{opp}} \simeq \text{Cl}^+(V)$$

If (e_i) is a pseudo-orthonormal basis of V , this isomorphism takes the following form [Ber+01]:

$$\begin{aligned} \text{Cl}^+(-V) &\xrightarrow{\sim} \text{Cl}^+(V) \\ e_{i_1} \dots e_{i_{2k}} &\mapsto (-1)^{\frac{2k(2k-1)}{2}} e_{i_{2k}} \dots e_{i_1} = (-1)^k e_{i_{2k}} \dots e_{i_1} \end{aligned}$$

Clifford algebra of the dual vector space

The inner product on V defines a linear isomorphism

$$v \in V \xrightarrow[\eta]{\sim} {}^t v \in V^*$$

The inner product itself is transported to V^* to the *inverse inner product*

$$\eta^{-1} \in \text{Hom}_{\mathbb{K}}(V^*, V) \simeq V \otimes V$$

as described in Section 1.3. Therefore the inner product defines an isometry

$$(V, \eta) \xrightarrow{\sim} (V^*, \eta^{-1})$$

which induces an isomorphism of Clifford algebras

$$\text{Cl}(V, \eta) \xrightarrow{\sim} \text{Cl}(V^*, \eta^{-1})$$

Let (e_i) be a pseudo-orthonormal basis of V and (e^{*i}) the dual basis. We have the relation

$$e^{*i} = \eta^{ij} ({}^t e_j)$$

Therefore the isomorphism between the two Clifford algebras takes the following form:

$$\begin{array}{ccc} \text{Cl}(V) & : & e_j \quad , \quad e^i = \eta^{ij} e_j \\ \text{Cl}(\eta) \updownarrow \text{Cl}(\eta^{-1}) & & \downarrow \uparrow \quad \downarrow \uparrow \\ \text{Cl}(V^*) & : & {}^t e_j \quad , \quad e^{*i} = \eta^{ij} ({}^t e_j) \end{array} \quad (4.1)$$

The notation (e^i) for elements of V and not of the dual V^* breaks our convention but we shall understand them as representations in $\text{Cl}(V, \eta)$ of covectors of the dual basis. In particular, we have the following relation:

$$e^i e_j + e_j e^i = 2\delta_j^i$$

which, assuming (e_i) is pseudo-orthonormal, relates the dual basis with the inverses of the basis vectors:

$$e^i = \|e_i\| (e_i)^{-1}$$

There is no summation implied since this formula breaks the covariance of indices.

There is a natural framework for working with both vectors and covectors in a common Clifford algebra: one can consider the Clifford algebra of $V \oplus V^*$ equipped with the natural split product [Mei13]

$$\langle (x_1, \alpha_1) | (x_2, \alpha_2) \rangle = \frac{1}{2} (\alpha_1(x_2) + \alpha_2(x_1))$$

Then the relation

$$e^i e_j + e_j e^i = \delta_j^i$$

is naturally satisfied – we now expound on the differing factor of 2. In order to account for the product η however, one wants to identify vectors x with the associated linear form ${}^t x$. The most natural way to do this is to use *isometric embeddings* of V and V^* into $V \oplus V^*$:

$$\begin{aligned} V &\rightarrow V \oplus V^* \\ v &\mapsto (v, {}^t v) \end{aligned}$$

$$\begin{aligned} V^* &\rightarrow V \oplus V^* \\ \alpha &\mapsto (\eta^{-1}(\alpha), \alpha) \end{aligned}$$

Then the images in $\text{Cl}(V \oplus V^*)$ of $e_j \in V$ and $e^i \in V^*$ satisfy the expected anti-commutation relations with the correct factor.

4.1.2 Classification of the Clifford algebras

Clifford algebras as filtered superalgebras are readily classified thanks to Theorem 4.1.6 which allows decomposing any Clifford algebra into a graded tensor product of copies of $\text{Cl}_{1,0}$ and $\text{Cl}_{0,1}$. It is however of interest to identify their structure as ungraded algebras. Theorem 4.1.6 does not allow concluding for the ungraded structure.

For dealing with the ungraded structure, the so-called “chirality element” will be very convenient.

Definition 4.1.11 (Chirality element). Equip V with an orientation. Along with the inner product, the orientation specifies a nonzero volume element $\text{vol} \in \Lambda^n V$ such that for every *direct* (pseudo-)orthonormal basis (e_i) ,

$$e_1 \wedge \cdots \wedge e_n = \text{vol}$$

Then there is a unique element $\omega \in \text{Cl}(V)$ such that in $\text{Cl}(V)$ for every *direct* (pseudo-)orthonormal basis (e_i) ,

$$e_1 \cdots e_n = \omega$$

It is called the *chirality element* of $\text{Cl}(V)$ and satisfies, with (p, q) the signature of V ,

$$\omega^2 = (-1)^{\frac{n(n-1)}{2}} (-1)^q \in \mathbb{R} \subset \text{Cl}(V)$$

Furthermore

- If V is even-dimensional, ω belongs to the supercenter of $\text{Cl}(V)$.
- If V is odd-dimensional, ω belongs to the ungraded center of $\text{Cl}(V)$.

Remark. Depending on conventions, signature and choice of field of scalars, the chirality element is sometimes multiplied by i in order to obtain an element which squares to 1. This amounts to using a non-real volume element in the complex(ified) vector space. We choose not to do this in order to have a treatment which does not rely on the complex structure and stays valid in all signatures and dimensions. This furthermore allows representing the Hodge operator on $\Lambda^\bullet V$ by the action of the chirality operator (via the isomorphism $\text{gr}(\text{Cl}(V)) \simeq \Lambda^\bullet V$).

In dimension 4 the chirality element is often written γ_5 and since it anti-commutes with all vectors it can be used to define a morphism from the Clifford algebra of a 5-dimensional space (see Corollary 4.1.19).

Corollary 4.1.12. Assume V is odd-dimensional: ω belongs to the ungraded center of $\text{Cl}(V)$.

1. If V is a complex vector space, there is an integer s such that $i^s \omega$ squares to 1. As a consequence $\text{Cl}(V)$ decomposes as a direct sum of algebras

$$\text{Cl}(V) \simeq \frac{1 + i^s \omega}{2} \text{Cl}(V) \oplus \frac{1 - i^s \omega}{2} \text{Cl}(V)$$

2. If $\frac{n-1}{2} + q$ is even, ω squares to 1 and $\text{Cl}(V)$ decomposes as a direct sum of algebras

$$\text{Cl}(V) \simeq \frac{1 + \omega}{2} \text{Cl}(V) \oplus \frac{1 - \omega}{2} \text{Cl}(V)$$

3. If $\frac{n-1}{2} + q$ is odd, ω squares to -1 and defines a complex structure on $\text{Cl}(V)$ (which commutes with the existing scalar action of \mathbb{K}).

This chirality element is very helpful to build morphisms with ungraded tensor products of Clifford algebras. It allows expressing Clifford algebras of direct sums as an ungraded tensor product, at the cost of omitting part of the grading or filtration structure. Reducing down to low dimensional Clifford algebras will give a classification, but having a direct expression for the isomorphism can prove to be useful in itself. It is also the underlying principle behind most isomorphism theorems between Clifford algebras of close signatures.

Theorem 4.1.13. *Let V_1, V_2 be finite dimensional vector spaces of respective dimensions n_1 and n_2 . Assume they are respectively equipped with a non-degenerate inner product of respective signatures (p_1, q_1) and (p_2, q_2) . Assume furthermore that V_1 is oriented and thus equipped with a positive normed volume element. Write ω_1 for the corresponding chirality element in $\text{Cl}(V)$.*

- *If V_1 is of even dimension, there is a graded algebra isomorphism*

$$\begin{aligned} \text{Cl}(V_1 \oplus \omega_1^2 V_2) &\xrightarrow{\sim} \text{Cl}(V_1) \otimes \text{Cl}(V_2) \\ a \in V_1 &\mapsto a \otimes 1 \\ b \in V_2 &\mapsto \omega_1 \otimes b \end{aligned}$$

with $\omega_1^2 V_2$ being V_2 equipped with the inner product multiplied by the scalar ω_1^2 . The signature can be changed on the right-side of the isomorphism:

$$\text{Cl}(V_1 \oplus V_2) \xrightarrow{\sim} \text{Cl}(V_1) \otimes \text{Cl}(\omega_1^2 V_2)$$

The morphism preserves the grading but not the natural filtration of Clifford algebras.

- *If V_1 is of odd dimension, there is an algebra isomorphism*

$$\begin{aligned} \text{Cl}(V_1 \oplus -\omega_1^2 V_2) &\xrightarrow{\sim} \text{Cl}(V_1) \hat{\otimes} \text{Cl}(V_2) \\ a \in V_1 &\mapsto a \otimes 1 \\ b \in V_2 &\mapsto \omega_1 \otimes b \end{aligned}$$

Here as well the signature can be changed on the right-side of the isomorphism:

$$\text{Cl}(V_1 \oplus V_2) \xrightarrow{\sim} \text{Cl}(V_1) \hat{\otimes} \text{Cl}(-\omega_1^2 V_2)$$

The morphism does not preserve the Clifford algebra grading but intertwines the grading of $\text{Cl}(V_1 \oplus -\omega_1^2 V_2)$ with that induced by the $\text{Cl}(V_1)$ factor:

$$\begin{aligned} \text{Cl}^+(V_1 \oplus -\omega_1^2 V_2) &\longrightarrow \text{Cl}^+(V_1) \hat{\otimes} \text{Cl}(V_2) \\ \oplus & \qquad \qquad \qquad \oplus \\ \text{Cl}^-(V_1 \oplus -\omega_1^2 V_2) &\longrightarrow \text{Cl}^-(V_1) \hat{\otimes} \text{Cl}(V_2) \end{aligned}$$

Proof. All the identities are proved in a similar way so we only prove the first one. Assume V_1 is of even dimension n_1 . Consider the application

$$\begin{aligned} V_1 \oplus V_2 &\rightarrow \text{Cl}(V_1) \otimes \text{Cl}(V_2) \\ a \in V_1 &\mapsto a \otimes 1 \\ b \in V_2 &\mapsto \omega_1 \otimes b \end{aligned}$$

The images of elements of V_1 and V_2 satisfy the following anticommutation relations ($\{ \cdot, \cdot \}$ is the ungraded anticommutator):

$$\begin{aligned} \{a_1 \otimes 1, a_2 \otimes 1\} &= \{a_1, a_2\} 1 \otimes 1 = 2 \langle a_1 | a_2 \rangle 1 \otimes 1 \\ \{\omega_1 \otimes b_1, \omega_1 \otimes b_2\} &= \omega_1^2 \otimes \{b_1, b_2\} = \omega_1^2 2 \langle b_1 | b_2 \rangle 1 \otimes 1 \\ \{a \otimes 1, \omega_1 \otimes b\} &= \{a, \omega_1\} \otimes b = 0 \end{aligned}$$

Therefore, according to the universal property of the Clifford algebras, the linear maps extends to an algebra homomorphism

$$\text{Cl}(V_1 \oplus \omega_1^2 V_2) \rightarrow \text{Cl}(V_1) \otimes \text{Cl}(V_2)$$

Since the image of $\omega_1 b$ is $\omega_1^2 b \in \text{Cl}(V_2)$, the image contains both $\text{Cl}(V_1)$ and $\text{Cl}(V_2)$ so the algebra morphism is surjective. Since both algebras have dimension $2^{\dim V_1 + \dim V_2}$, the homomorphism is an algebra isomorphism. Furthermore, the image of $V_1 \oplus V_2$ in $\text{Cl}(V_1) \oplus \text{Cl}(V_2)$ is odd and since $V_1 \oplus V_2$ generates $\text{Cl}(V_1 \oplus \omega_1^2 V_2)$, the morphism preserves the grading. \square

As a consequence, we obtain the usual factorisation properties of Clifford algebras:

Corollary 4.1.14. Let (p, q) be a signature. On $\mathbb{R}^{2,0}$, $\omega^2 = -1$ thus there is an ungraded algebra isomorphism

$$\text{Cl}_{p+2,q} \simeq \text{Cl}_{2,0} \otimes \text{Cl}_{q,p} \simeq \text{M}_2(\text{Cl}_{q,p})$$

On $\mathbb{R}^{1,1}$, $\omega^2 = 1$ thus there is an ungraded algebra isomorphism

$$\text{Cl}_{p+1,q+1} \simeq \text{Cl}_{1,1} \otimes \text{Cl}_{p,q} \simeq \text{M}_2(\text{Cl}_{p,q})$$

On $\mathbb{R}^{0,2}$, $\omega^2 = -1$ thus there is an ungraded algebra isomorphism

$$\text{Cl}_{p,q+2} \simeq \text{Cl}_{0,2} \otimes \text{Cl}_{p,q} \simeq \mathbb{H} \otimes \text{Cl}_{p,q}$$

These rules allow identifying the ungraded (and unfiltered) algebra isomorphism classes of all real Clifford algebras.

Another corollary concerns the even part of the Clifford algebra of an odd-dimensional vector space: it is a subalgebra and is isomorphic to a Clifford algebra.

Corollary 4.1.15 (Even parts of odd Clifford algebras are isomorphic to Clifford algebras). Let V be a vector space of odd dimension equipped with a non-degenerate inner product. Assume V has an element e such that $\langle e|e \rangle = 1$. Then

$$V \simeq \mathbb{K}e \oplus e^\perp$$

and there is an ungraded algebra isomorphism

$$\text{Cl}^+(V) \simeq \text{Cl}^+(\mathbb{K}e) \hat{\otimes} \text{Cl}(e^\perp) \simeq \text{Cl}(e^\perp, -\langle \cdot | \cdot \rangle)$$

Assume now that V instead has an element f such that $\langle f|f \rangle = -1$. Then

$$V \simeq \mathbb{K}f \oplus f^\perp$$

and there is an ungraded algebra isomorphism

$$\text{Cl}^+(V) \simeq \text{Cl}^+(\mathbb{K}f) \hat{\otimes} \text{Cl}(f^\perp) \simeq \text{Cl}(f^\perp)$$

When ω is an (ungraded-)central idempotent (of odd degree) and decomposes $\text{Cl}(V)$ into a product of two subalgebras, each of these subalgebras can be identified with $\text{Cl}^+(V)$. Indeed, the ungraded morphisms

$$\begin{aligned} \text{Cl}^+(V) &\rightarrow \frac{1 \pm \omega}{2} \text{Cl}(V) \\ x &\mapsto \frac{1 \pm \omega}{2} x \end{aligned}$$

are injective, since their respective kernel ideals are $\text{Cl}^+(V) \cap \frac{1 \pm \omega}{2} \text{Cl}(V)$ which both are the zero ideal. This allows us to deduce the following corollary:

Corollary 4.1.16. Assume that V is odd-dimensional and that its chirality element ω squares to 1.

Then there is an ungraded algebra isomorphism:

$$\begin{aligned} \text{Cl}^+(V) \oplus \text{Cl}^-(V) &\xrightarrow{\sim} \frac{1+\omega}{2} \text{Cl}(V) \oplus \frac{1-\omega}{2} \text{Cl}(V) \xrightarrow{\sim} \text{Cl}(V) \\ (x_1, x_2) &\longmapsto \frac{1+\omega}{2} x_1 \oplus \frac{1-\omega}{2} x_2 \longmapsto \frac{1+\omega}{2} x_1 + \frac{1-\omega}{2} x_2 \end{aligned}$$

We explained that acting with the chirality element on the Clifford algebra represents Hodge duality on the Clifford algebra. However in even dimension there is another way to act with the chirality element on vectors so as to define an algebra isomorphism:

Theorem 4.1.17 ([LS19], Prop 1.6). Assume that V is of even dimension and oriented so that $\text{Cl}(V)$ has a super-central chirality element ω . Consider the following map:

$$f_\omega : v \in V \mapsto \omega v \in \text{Cl}(V)$$

- If $\omega^2 = -1$ then f_ω extends to a graded unfiltered algebra automorphism

$$\begin{aligned} \beta : \text{Cl}(V) &\xrightarrow{\sim} \text{Cl}(V) \\ a^+ \in \text{Cl}^+(V) &\mapsto a^+ \in \text{Cl}^+(V) \\ a^- \in \text{Cl}^-(V) &\mapsto \omega a^- \in \text{Cl}^-(V) \end{aligned}$$

The automorphism β squares to the grading automorphism $\alpha = \text{id}_{\text{Cl}^+(V)} - \text{id}_{\text{Cl}^-(V)}$.

- If $\omega^2 = 1$ then f_ω extends to a graded unfiltered algebra isomorphism

$$\beta : \text{Cl}(-V) \xrightarrow{\sim} \text{Cl}(V)$$

with $-V$ representing V equipped with the opposite bilinear product.

The automorphism β squares to the identity.

Example 4.1.18. In Lorentzian signature $(1, 3)$, $\omega^2 = -1$ and β defines a graded automorphism of the Clifford algebra.

These automorphisms allow us to relate several isomorphisms we presented so far:

Corollary 4.1.19. Let V be an oriented even-dimensional vector space and $\omega \in \text{Cl}(V)$ the corresponding chirality element. We define $\epsilon = \omega^2 \in \{\pm 1\}$. Consider the vector space

$$V \oplus_{\perp} \epsilon \mathbb{R}$$

with $\epsilon \mathbb{R}$ denoting \mathbb{R} equipped with an inner product of signature (ϵ) . Let $e \in \epsilon \mathbb{R}$ be a pseudonormed vector. Then ωe is an (ungraded)-central idempotent of $\text{Cl}(V \oplus_{\perp} \epsilon \mathbb{R})$ and

$$\text{Cl}(V \oplus \epsilon \mathbb{R}) / (e - \pm \omega) = \text{Cl}(V \oplus \epsilon \mathbb{R}) / (\omega e - \pm \epsilon) \simeq \frac{\omega e + \pm \epsilon}{2} \text{Cl}(V \oplus \epsilon \mathbb{R})$$

so that there is a commutative square of ungraded algebra isomorphisms:

$$\begin{array}{ccc} \mathrm{Cl}^+(V \oplus \epsilon\mathbb{R}) & \xrightarrow{\sim} & \mathrm{Cl}(-\epsilon V) \\ \downarrow \wr & & \downarrow \beta \\ \mathrm{Cl}(V \oplus \epsilon\mathbb{R}) / (e - \pm\omega) & \xrightarrow[\substack{v \rightarrow v \\ e \rightarrow \pm\omega}]{\sim} & \mathrm{Cl}(V) \end{array}$$

with the top morphism being induced from the ungraded isomorphism

$$\mathrm{Cl}(V \oplus \epsilon\mathbb{R}) \simeq \mathrm{Cl}(-\epsilon V) \hat{\otimes} \mathrm{Cl}(\epsilon\mathbb{R})$$

Remark. The morphism $\mathrm{Cl}(V \oplus \epsilon\mathbb{R}) \rightarrow \mathrm{Cl}(V \oplus \epsilon\mathbb{R}) / (e - \pm\omega) \rightarrow \mathrm{Cl}(V)$ justifies the notation γ_5 often used in physics literature for the chirality element of the Clifford algebra in Lorentzian signature. Indeed if the elements corresponding to a basis are named e_1, e_2, e_3, e_4 then the chirality element can be used as the image of a fifth vector from a pseudo orthonormal basis of $\mathrm{Cl}_{1,4}$.

There is another isomorphism theorem which is important in many applications of Clifford algebras:

Theorem 4.1.20 (Bott periodicity of real Clifford algebras). *The following two real Clifford algebras have a super-central chirality element which squares to 1:*

$$\begin{aligned} \mathrm{Cl}_{1,1} &\simeq \mathrm{M}_2(\mathbb{R}) \\ \mathrm{Cl}_8 &\simeq \mathrm{M}_{16}(\mathbb{R}) \end{aligned}$$

As a consequence, for integers $p \geq q$, there are ungraded isomorphisms

$$\begin{aligned} \mathrm{Cl}_{p,q} &\simeq \mathrm{M}_{2^q}(\mathbb{R}) \otimes \mathrm{Cl}_{p-q} \\ \mathrm{Cl}_{q,p} &\simeq \mathrm{M}_{2^q}(\mathbb{R}) \otimes \mathrm{Cl}_{0,p-q} \end{aligned}$$

and for integers p, q, r, s there is an ungraded isomorphism

$$\mathrm{Cl}_{p+8r, q+8s} \simeq \mathrm{M}_{16^{r+s}}(\mathbb{R}) \otimes \mathrm{Cl}_{p,q}$$

A similar albeit simpler theorem holds for complex Clifford algebras:

Theorem 4.1.21 (Bott periodicity of complex Clifford algebras). *The Clifford algebra Cl_2 is isomorphic to $\mathrm{M}_2(\mathbb{C})$ and has a super-central chirality element which after multiplication by i squares to 1. As a consequence, for integers p, s , there is an ungraded isomorphism:*

$$\mathrm{Cl}_{p+2s} \simeq \mathrm{M}_{2^s}(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Cl}_p$$

In particular, for $n = 2k$ the (ungraded) complex Clifford algebra of \mathbb{C}^{2k} takes the form

$$\mathrm{Cl}_{2k} \simeq \mathrm{M}_{2^k}(\mathbb{C})$$

and for $n = 2k + 1$ the (ungraded) complex Clifford algebra of \mathbb{C}^{2k+1} takes the form

$$\mathrm{Cl}_{2k+1} \simeq \mathrm{M}_{2^k}(\mathbb{C}) \oplus \mathrm{M}_{2^k}(\mathbb{C})$$

We close the section by presenting a variation of Theorem 4.1.13 with two chiral elements:

Theorem 4.1.22. *Let V_1, V_2 be finite dimensional spaces of respective dimensions n_1 and n_2 . Assume they are respectively equipped with a non-degenerate inner product of respective signatures (p_1, q_1) and (p_2, q_2) . Assume furthermore that they are oriented and thus equipped with positive normed volume elements $\omega_i \in \text{Cl}(V_i)$.*

If $V_1 \oplus V_2$ is of odd dimension, there is an algebra isomorphism

$$\begin{aligned} \text{Cl}(\omega_2^2 V_1 \oplus \omega_1^2 V_2) &\xrightarrow{\sim} \text{Cl}(V_1) \otimes \text{Cl}(V_2) \\ a \in V_1 &\mapsto a \otimes \omega_2 \\ b \in V_2 &\mapsto \omega_1 \otimes b \end{aligned}$$

with $\omega_1^2 V_2$ being V_2 equipped with the inner product multiplied by the scalar ω_1^2 . It is possible to change the signature on the right-side of the isomorphism:

$$\text{Cl}(V_1 \oplus V_2) \xrightarrow{\sim} \text{Cl}(\omega^2 V_1) \otimes \text{Cl}(\omega_1^2 V_2)$$

The morphism does not preserve the Clifford algebra grading but it intertwines the grading of $\text{Cl}(\omega_2^2 V_1 \oplus \omega_1^2 V_2)$ with the grading induced on the tensor product by the Clifford algebra of the odd-dimensional vector space.

If V_1 and V_2 are both of even dimension, there is a graded algebra isomorphism

$$\begin{aligned} \text{Cl}(-\omega_2^2 V_1 \oplus -\omega_1^2 V_2) &\xrightarrow{\sim} \text{Cl}(V_1) \hat{\otimes} \text{Cl}(V_2) \\ a \in V_1 &\mapsto a \otimes \omega_2 \\ b \in V_2 &\mapsto \omega_1 \otimes b \end{aligned}$$

Here as well the signature can be changed on the right-side of the isomorphism:

$$\text{Cl}(V_1 \oplus V_2) \xrightarrow{\sim} \text{Cl}(-\omega_2^2 V_1) \hat{\otimes} \text{Cl}(-\omega_1^2 V_2)$$

The morphism preserves the grading but does not preserve the natural filtration.

4.1.3 Spin groups

Specific subgroups of the Clifford algebra will be of particular relevance for working with spinors. Let V be a \mathbb{K} -vector space of finite dimension n equipped with a nondegenerate symmetric bilinear form.

Definition 4.1.23 (Norm of an element). The *norm* of an element x of $\text{Cl}(V)$ is defined as follows:

$$N(x) = {}^T x x$$

In particular, for every family of vectors (v_1, \dots, v_k) it takes the following real value:

$$N(v_1 \cdots v_k) = \langle v_1 | v_1 \rangle \langle v_2 | v_2 \rangle \cdots \langle v_k | v_k \rangle$$

Definition 4.1.24 (Twisted adjoint action of invertible elements). Let a be a homogeneous invertible element of $\text{Cl}(V)$. Its *twisted adjoint action* on an element x of $\text{Cl}(V)$ is defined as follows:

$$\widetilde{\text{Ad}}_a(x) = \alpha(a) x a^{-1}$$

It defines a group action of the homogeneous invertible elements of $\text{Cl}(V)$ on $\text{Cl}(V)$.

Remark. Contrary to the standard conjugation action, twisted conjugation action does not always act by algebra automorphisms:

$$\alpha(a)xya^{-1} = \alpha(a)x\alpha(a^{-1})\alpha(a)ya^{-1} = \alpha(a\alpha(x)a^{-1})\widetilde{\text{Ad}}_a(y)$$

Non-isotropic vectors of V act by reflection along their orthogonal hyperplane.

Definition 4.1.25 (Clifford group). The Clifford group of a Clifford algebra is the group of *homogeneous* elements such that the twisted adjoint action preserves the subspace V :

$$\Gamma(V) = \{a \in \text{Cl}^\times(V) \cap (\text{Cl}^+(V) \cup \text{Cl}^-(V)) \mid \widetilde{\text{Ad}}_a(V) \subseteq V\}$$

Its even subgroup will be denoted $\Gamma^+(V)$.

Given an element a of $\Gamma(V)$ and a vector v of V , the following relation holds

$$\alpha(a)va^{-1} = {}^T(\alpha(a)va^{-1}) = {}^T(a^{-1})v{}^T(\alpha(a)) = \alpha({}^T\alpha(a)^{-1})v({}^T\alpha(a)^{-1})^{-1} = \widetilde{\text{Ad}}_{\alpha(a)^{-1}}(v)$$

therefore $\widetilde{\text{Ad}}_{\alpha(a)^{-1}} = \widetilde{\text{Ad}}_a$. Furthermore, $\Gamma(V)$ is stable under α , since all its elements are homogeneous. As a consequence, $\Gamma(V)$ is stable under transposition and

$$\forall a \in \Gamma(V), \widetilde{\text{Ad}}_{\alpha(a)} = \widetilde{\text{Ad}}_{\alpha(a)^{-1}}$$

Theorem 4.1.26. *The twisted adjoint action of the Clifford group on V defines a group morphism:*

$$\Gamma(V) \rightarrow \text{O}(V)$$

Its kernel is the group of invertible scalars $\mathbb{K}^\times \subset \text{Cl}(V)$.

Remark. When V has even dimension and has a chirality element ω which squares to -1 , the twisted adjoint action can be related to the standard adjoint action using the automorphism β defined in Theorem 4.1.17. Indeed, for $a \in \text{Cl}^+(V)$ and invertible, $\beta(a) = a$ and $\widetilde{\text{Ad}}_{\beta(a)} = \widetilde{\text{Ad}}_a = \text{Ad}_a$. For $a \in \text{Cl}^-(V)$ invertible and $v \in V$, the following holds:

$$\omega av \alpha(\omega a)^{-1} = \omega av \alpha(a)^{-1} \alpha(\omega)^{-1} = \omega av \alpha(a)^{-1} (-\omega) = \omega(-1)^2 \omega av \alpha(a)^{-1} = -av \alpha(a)^{-1}$$

This proves that for a in $\Gamma(V)$, $\omega a \in \Gamma(V)$ and that $\widetilde{\text{Ad}}_{\beta(a)}(v) = \text{Ad}_a(v)$. Namely, the following diagram commutes:

$$\begin{array}{ccc} \Gamma(V) & & \text{O}(V) \\ \uparrow & \searrow \widetilde{\text{Ad}} & \\ \Gamma(V) & \xrightarrow{\text{Ad}} & \end{array}$$

The degree of freedom \mathbb{K}^\times is often unneeded and one wishes to work with smaller groups.

Definition 4.1.27 (Pin and Spin groups). The Pin *group* associated with V is the subgroup of $\Gamma(V)$ on which N is worth ± 1 :

$$\text{Pin}(V) = N^{-1}\{\pm 1\} \cap \Gamma(V)$$

The induced representation

$$\text{Pin}(V) \rightarrow \text{O}(V)$$

has $\{\pm 1\}$ as kernel.

Similarly, we define the *spin group* associated with V [Var04] as

$$\text{Spin}(V) = N^{-1}\{\pm 1\} \cap \Gamma^+(V)$$

It is the inverse image of $\text{SO}(V)$ under the representation of $\Gamma(V)$ on V

Last, we define the *orthochronous spin group* associated with V as

$$\text{Spin}^+(V) = \text{Ad}^{-1}(\text{SO}^+(V)) \cap \text{Pin}(V)$$

It differs from the spin group in the case V has mixed signature (neither positive nor negative).

Remark. Some authors call $\text{Spin}^+(V)$ the *spin group*. It is the simply-connected Lie group integrating the Lie algebra $\mathfrak{spin}(V)$ and as such there is a correspondence (Cartan-Lie theorem) between $\text{Spin}^+(V)$ and $\mathfrak{spin}(V)$ (as well as between their representations).

Remark. Elements of $\text{Spin}(V)$ are *even* so that their twisted adjoint action coincide with their standard adjoint action.

Theorem 4.1.28.

- Every element of $\Gamma(V)$ can be written as the product in $\text{Cl}(V)$ of non-isotropic vectors of V .
- Every element of $\text{Pin}(V)$ can be written as the product in $\text{Cl}(V)$ of vectors of V of norm ± 1 .
- Every element of $\text{Spin}(V)$ can be written as the product in $\text{Cl}(V)$ of an even number of vectors of V of norm ± 1 .
- Every element of $\text{Spin}^+(V)$ can be written as the product in $\text{Cl}(V)$ of an even number of vectors of V of norm 1 and an even number of vectors of V of norm -1 .

Theorem 4.1.29. *The groups $\text{Pin}(V)$, $\text{Spin}(V)$ and $\text{Spin}^+(V)$ are natural twofold coverings of subgroups of $\text{O}(V)$:*

$$\begin{aligned} \mathbb{Z}_2 &\hookrightarrow \text{Pin}(V) \twoheadrightarrow \text{O}(V) \\ \mathbb{Z}_2 &\hookrightarrow \text{Spin}(V) \twoheadrightarrow \text{SO}(V) \\ \mathbb{Z}_2 &\hookrightarrow \text{Spin}^+(V) \twoheadrightarrow \text{SO}^+(V) \end{aligned}$$

with $\text{SO}^+(V)$ standing for the proper orthochronous orthogonal group of V .

We will use the following notations:

$$\begin{aligned} \text{Spin}_{p,q}^+ &:= \text{Spin}^+(\mathbb{R}^{p,q}) \\ \text{Spin}_n &:= \text{Spin}_{n,0} \end{aligned}$$

and similarly for the Pin and $\text{Spin}_{p,q}$ groups.

Theorem 4.1.30. *The fundamental group of $\text{Spin}_{p,q}^+ = \text{Spin}^+(\mathbb{R}^{p,q})$ can be computed as follows:*

$$\pi_1(\text{Spin}_{p,q}^+) \simeq \pi_1(\text{Spin}_p \times \text{Spin}_q) \simeq \pi_1(\text{Spin}_p) \times \pi_1(\text{Spin}_q)$$

with

$$\begin{aligned}\pi_1(\mathrm{Spin}_1) &\simeq 1 \\ \pi_1(\mathrm{Spin}_2) &\simeq \mathbb{Z} \\ \pi_1(\mathrm{Spin}_n) &\simeq 1 \quad \text{for } n \geq 3\end{aligned}$$

Spin Lie algebra and sigma matrices

The Spin group defines a Lie subgroup of the group of invertible elements of $\mathrm{Cl}(V)$. Therefore its Lie algebra, constructed as the tangent space at 0, can be represented inside $\mathrm{Cl}(V)$, which is naturally isomorphic to its own tangent space at 0. Since the composition law of $\mathrm{Spin}(V)$ comes from the product of $\mathrm{Cl}(V)$, the Lie algebra bracket will be represented as the commutator of the associative algebra, similarly to the case of matrix groups.

First, the Lie algebra is constructed from $\mathrm{Spin}^+(V)$ thus we can restrict our consideration to the even Clifford algebra $\mathrm{Cl}^+(V)$.

Theorem 4.1.31 ([Var04]). *Assume $\dim(V) \geq 3$. Define the following subspace of $\mathrm{Cl}^+(V)$:*

$$\mathrm{Cl}^2(V) = \mathrm{span}(\{ab - ba\}_{a,b \in V})$$

It is a Lie subalgebra of $\mathrm{Cl}^+(V)$ for the ungraded commutator

$$[a, b] = ab - ba$$

Furthermore the Lie group (non-exact) sequence

$$\mathrm{Spin}^+(V) \hookrightarrow \Gamma^+(V) \twoheadrightarrow \mathrm{SO}(V)$$

which differentiates to

$$\mathfrak{spin}(V) \hookrightarrow \mathrm{Lie}(\Gamma^+(V)) \twoheadrightarrow \mathfrak{so}(V)$$

restricts to isomorphisms:

$$\mathfrak{spin}(V) \xrightarrow{\sim} \mathrm{Cl}^2(V) \xrightarrow{\sim} \mathfrak{so}(V)$$

In particular, the morphism $\mathrm{Cl}^2(V) \rightarrow \mathfrak{so}(V)$ can be constructed by restriction to $V \subset \mathrm{Cl}(V)$ of the infinitesimal adjoint representation

$$\mathrm{ad}|_{\mathrm{Cl}^2(V)} : \mathrm{Cl}^2(V) \rightarrow \mathrm{End}_{\mathbb{K}}(\mathrm{Cl}(V))$$

Let e_1, e_2 be two vectors of norm ± 1 . We are looking for the image in $\mathfrak{so}(V)$ of $[e_1, e_2]$. First,

$$[e_1, e_2] = 2e_1e_2$$

so we only need to compute $[e_1e_2, \cdot]$. Next, for $v^\perp \in \mathrm{Vect}(e_1, e_2)^\perp$, we compute

$$e_1e_2v^\perp = v^\perp e_1e_2$$

thus

$$[e_1e_2, v^\perp] = 0$$

hence $[e_1 e_2, \cdot]$ vanishes on $\text{Vect}(e_1, e_2)^\perp$. We only need to compute

$$\begin{aligned} [e_1 e_2, e_1] &= e_1 e_2 e_1 - e_1 e_1 e_2 = -2N(e_1)e_2 \\ [e_1 e_2, e_2] &= e_1 e_2 e_2 - e_2 e_1 e_2 = 2N(e_2)e_1 \end{aligned}$$

Notice the factor 2 which is related to the \mathbb{Z}_2 -extension $\text{Spin}^+(V) \rightarrow \text{SO}^+(V)$.

Therefore we conclude that the correspondence is

$$\begin{aligned} \text{Cl}^2(V) &\leftrightarrow \mathfrak{so}(V) \\ \frac{1}{4}[e_1, e_2] &\leftrightarrow e_1 \otimes {}^t e_2 - e_2 \otimes {}^t e_1 \end{aligned} \tag{4.2}$$

with ${}^t e_i : v \in V \mapsto \langle e_i | v \rangle$. Using the $\mathfrak{so}(V)$ -module isomorphism $\mathfrak{so}(V) \simeq \Lambda^2 V$ given by the inner product, we can also write

$$\begin{aligned} \text{Cl}^2(V) &\leftrightarrow \Lambda^2 V \\ \frac{1}{4}[e_1, e_2] &\leftrightarrow e_1 \wedge e_2 = e_1 \otimes e_2 - e_2 \otimes e_1 \end{aligned}$$

Accordingly, when using indices i for $\mathfrak{so}(V)$ and a, b for V , we will be using both notations $1/2\sigma_i$ and $1/2\sigma_{ab}$ for a basis of $\mathfrak{spin}(V)$ embedded in $\text{Cl}(V)$. The notation is meant to be reminiscent of the Pauli matrices (which represent a 3-dimensional Clifford algebra), using the $1/2$ factor but not the i factor common in the physics literature (used to turn the sigma elements into hermitian operators). Explicitly, writing in components ρ_i^{ab} the morphism $\mathfrak{so}(V) \xrightarrow{\sim} \Lambda^2 V$ the two notations are related by

$$\frac{1}{2}\rho_i^{ab}\sigma_{ab} = \sigma_i \tag{4.3}$$

4.1.4 Clifford modules

Spinors are elements of a Clifford module, namely a representation of a Clifford algebra. In order to work with them, we need to understand their structure.

Theorem 4.1.32. *Let V be an n -dimensional vector space equipped with a nondegenerate symmetric bilinear form.*

If n is even, then the Clifford algebra $\text{Cl}(V)$ is a simple algebra.

If n is odd, then

- *If there exists a chirality element (ungraded central, since n is odd) which squares to 1 (namely, when $q - p = 3 \pmod{4}$), $\text{Cl}(V)$ is the direct sum of two isomorphic simple algebras.*
- *Otherwise $\text{Cl}(V)$ is a simple algebra.*

Remark. The *Wedderburn-Artin Theorem* ([Pro07] Section 6.1.9) states in our particular case that finite-dimensional simple real algebras are matrix rings over a real division algebra.

We have already identified the ungraded structure of the complex Clifford algebras. There is a finer structure in the real case:

Theorem 4.1.33 ([Del99]). *Let (p, q) be a signature. Depending on $p - q \pmod{8}$, the real Clifford algebra $\text{Cl}_{p,q}$ is a matrix algebra or the sum of two matrix algebras over the following real division algebras:*

$q - p \pmod{8}$	0, 6	1, 5	2, 4	3	7
Division algebra	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{H}	\mathbb{R}

Notice the symmetry: couples of signatures $q_1 - p_1 \pmod 8$ and $q_2 - p_2 \pmod 8$ which sum to $6 = -2 \pmod 8$ have the same division algebra since

$$\mathrm{Cl}_{p+2,q} \simeq \mathrm{M}_2(\mathbb{R}) \otimes \mathrm{Cl}_{q,p}$$

Definition 4.1.34. Let $\mathrm{Cl}(V)$ be a Clifford algebra over the field \mathbb{K} .

A *Clifford module* over $\mathrm{Cl}(V)$ is a (finite-dimensional) \mathbb{K} -linear representation of $\mathrm{Cl}(V)$.

A *spinorial module* over $\mathrm{Cl}(V)$ is a (finite-dimensional) \mathbb{K} -linear representation of $\mathrm{Cl}^+(V)$. Equivalently, it is a (finite-dimensional) \mathbb{K} -linear representation of $\mathrm{Spin}^+(V)$ such that $-1 \in \mathrm{Spin}^+(V)$ acts as the scalar $-1 \in \mathbb{K}$ ([LM89], Section I.5).

Remark. This definition differs from [Var04] which defines Clifford modules as $\mathrm{Cl}^+(V)$ -modules.

Remark. Since $\Gamma(V)$, $\mathrm{Pin}(V)$, $\mathrm{Spin}(V)$ and $\mathrm{Spin}^+(V)$ are groups, their representations have natural tensor products. However algebras such as $\mathrm{Cl}(V)$ and $\mathrm{Cl}^+(V)$ need a *coproduct* $\mathrm{Cl}(V) \rightarrow \mathrm{Cl}(V) \otimes \mathrm{Cl}(V)$ in order to define a general tensor product between modules. It is reasonable to furthermore require that the two tensor products coincide. However this is impossible since the linearity of a coproduct

$$-1 \in \Gamma(V) \subset \mathrm{Cl}^+(V) \mapsto -1 \in \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V)$$

is incompatible with the diagonal embedding

$$-1 \in \Gamma(V) \rightarrow (-1, -1) \in \Gamma(V) \times \Gamma(V)$$

used to construct tensor products of group representations.

Example 4.1.35. Consider the underlying vector space to the even Clifford algebra $\mathrm{Cl}^+(V)$.

It has a structure of spinorial module, corresponding to the left action of $\mathrm{Spin}^+(V) \subset \mathrm{Cl}^+(V)$.

On another hand, there is a natural action of $\mathrm{O}(V)$ on $\mathrm{Cl}^+(V)$ (according to Theorem 4.1.5). It induces an action of $\mathrm{Spin}^+(V)$ on $\mathrm{Cl}^+(V)$ but this action does not give a structure of spinorial module, for example $-1 \in \ker(\mathrm{Spin}^+(V) \rightarrow \mathrm{O}(V))$ acts trivially.

We now have all that is needed to talk about the structure of spin modules.

Theorem 4.1.36 ([Var04]). *Equip V with a volume element and call the corresponding chirality element $\omega \in \mathrm{Cl}(V)$. We call (p, q) the signature of V . The algebras $\mathrm{Cl}(V)$ and $\mathrm{Cl}^+(V)$ have one or two classes of irreducible modules depending on the residue of $q - p$ modulo 4. We call them respectively Σ_V or Σ_V^\pm and S_V or S_V^\pm :*

$q - p \pmod 4$	0	1	2	3
$\mathrm{Cl}(V)$ -modules	Σ_V	Σ_V	Σ_V	Σ_V^+, Σ_V^-
$\mathrm{Cl}^+(V)$ -modules	S_V^+, S_V^-	S_V	S_V	S_V

In the complex case, the structure is more straightforward:

Theorem 4.1.37 ([Del99]). *For any integer n there is an ungraded isomorphism*

$$\mathrm{Cl}_{n+1}^+ \simeq \mathrm{Cl}_n$$

For even n , Cl_n is a simple algebra and has a unique irreducible module $\Sigma_n^{\mathbb{C}}$. It decomposes under Cl_n^+ as the sum of two irreducible “semi-spin” modules:

$$\Sigma_n^{\mathbb{C}} \simeq_{\mathrm{Cl}_n^+ \text{-mod}} S_n^{\mathbb{C}+} \oplus S_n^{\mathbb{C}-}$$

with $S_n^{\mathbb{C}\pm}$ being eigenspaces of any idempotent chirality element. Defining a \mathbb{Z}_2 -grading on $\Sigma_n^{\mathbb{C}}$ such that the homogeneous components are $S_n^{\mathbb{C}+}$ and $S_n^{\mathbb{C}-}$ (there are two possibilities), $\Sigma_n^{\mathbb{C}}$ forms a graded module over $\mathbb{C}l_n$.

We will need *hermitian structures* on the complex Clifford modules, which for us will be of arbitrary signature (not always definite positive). We can obtain them using the following theorem:

Theorem 4.1.38 ([Rob88]). *Assume V is real of even dimension $n = 2k$ and signature (p, q) . Consider the complex Clifford algebra $\mathbb{C}l(V) := \mathbb{C}l(\mathbb{C} \otimes V)$: it is a complex matrix algebra isomorphic to $\text{End}_{\mathbb{C}}(\Sigma_V)$.*

Every hermitian form (of arbitrary signature) on Σ_V defines an adjoint operation on $\mathbb{C}l(V)$ which is an involutive anti-linear anti-automorphism. Conversely, every such involutive anti-linear anti-automorphism of algebra of $\mathbb{C}l(V)$ is the adjoint operation associated to an hermitian form on Σ_V defined up to a (nonzero) real factor.

Theorem 4.1.39. *Assume V is real. Let S be a $\mathbb{C}l^+(V)$ -module equipped with a hermitian form h . The form h is invariant under the action of $\text{Spin}(V)$ if and only if the adjoint of the endomorphism representing $a \in \mathbb{C}l^+(V)$ is the endomorphism representing ${}^T a$.*

Theorem 4.1.40 ([Rob88]). *Assume V is real, of even dimension and signature (p, q) . Consider the complex Clifford algebra $\mathbb{C}l(V) := \mathbb{C}l(\mathbb{C} \otimes_{\mathbb{R}} V)$. Consider the antilinear anti-automorphism defined on vectors as follows:*

$$v \in \mathbb{C} \otimes V \mapsto -\bar{v}$$

It takes on a generic element a on $\mathbb{C}l(V)$ the form $a \mapsto {}^t \overline{\alpha(a)}$.

The irreducible complex Clifford module $\Sigma_V^{\mathbb{C}}$ decomposes under $\mathbb{C}l^+(V)$ as a sum of two “semi-spin” modules:

$$\Sigma_V^{\mathbb{C}} \simeq_{\mathbb{C}l^+} S_V^{\mathbb{C}+} \oplus S_V^{\mathbb{C}-}$$

Any hermitian form h on $\Sigma_V^{\mathbb{C}}$ associated to the antiautomorphism defined above is invariant under $\text{Spin}(V)$ and behaves as follows:

- *If p and q are odd then $S_V^{\mathbb{C}+}$ and $S_V^{\mathbb{C}-}$ are both isotropic for h and are as consequence in (hermitian) duality.*
- *If p and q are even then $S_V^{\mathbb{C}+}$ and $S_V^{\mathbb{C}-}$ are orthogonal for h .*
 - *If $\langle \cdot | \cdot \rangle$ is positive definite then h is definite on $S_V^{\mathbb{C}+}$ and on $S_V^{\mathbb{C}-}$ with opposite signatures.*
 - *If $\langle \cdot | \cdot \rangle$ is negative definite then h is definite on $\Sigma_V^{\mathbb{C}}$.*
 - *If $\langle \cdot | \cdot \rangle$ is indefinite then the restrictions of h on both $S_V^{\mathbb{C}+}$ and $S_V^{\mathbb{C}-}$ have split signature.*

Definition 4.1.41 (Spinorial product). *Let V be a real vector space. Let Σ be a Clifford module over $\mathbb{C}l(V)$. We will call a *spinorial product* on Σ a hermitian product such that vectors of V act on Σ by anti-selfadjoint endomorphisms:*

$$\forall (s_1, s_2) \in \Sigma \times \Sigma, \forall v \in V, \langle s_1 | v \cdot s_2 \rangle = -\langle v \cdot s_1 | s_2 \rangle$$

Remark. Our *spinorial product* are not to be confused with what is called “spinorial metric” in [Bou+15], which is equivalent to the data of a metric over the manifold M along with a so-called Spin structure, which we introduce in the next section.

The compatibility requirement of the spinorial product with the action of the Clifford algebra can be reformulated as follows: the spinorial product, interpreted as a \mathbb{C} -linear morphism

$$\bar{\Sigma} \rightarrow \Sigma^*$$

is required to be intertwining with the previously defined antilinear anti-automorphism of the complex Clifford algebra: the following diagram commutes:

$$\begin{array}{ccc} \text{Cl}(V) \otimes_{\mathbb{C}} \bar{\Sigma} & \xrightarrow{(a,s) \mapsto (\overline{T\alpha(a)}, \langle s |)} & \text{Cl}(V) \otimes_{\mathbb{C}} \Sigma^* \\ \downarrow & & \downarrow \\ \bar{\Sigma} & \xrightarrow{\langle \cdot |} & \Sigma^* \end{array}$$

Alternatively, the spinorial product is also naturally interpreted as a linear map

$$\bar{\Sigma} \otimes_{\mathbb{C}} \Sigma^* \xrightarrow{\langle \cdot | \cdot \rangle} \mathbb{C}$$

As stated earlier, there is no natural tensor product action of the Clifford algebra but there is an action of the Clifford group. The compatibility condition can then be written as another intertwining requirement, namely the following diagram commutes:

$$\begin{array}{ccc} \Gamma(\mathbb{C} \otimes_{\mathbb{R}} V) \times \bar{\Sigma} \otimes_{\mathbb{C}} \Sigma^* & \xrightarrow{N \times \langle \cdot | \cdot \rangle} & \mathbb{C}^* \times \mathbb{C} \\ (g, \bar{s}_1 \otimes s_2) \mapsto (g \cdot \bar{s}_1 \otimes g \cdot s_2) \downarrow & & \downarrow (\lambda, z) \mapsto \lambda z \\ \bar{\Sigma} \otimes_{\mathbb{C}} \Sigma^* & \xrightarrow{\langle \cdot | \cdot \rangle} & \mathbb{C} \end{array}$$

4.2 Spinors on manifolds

4.2.1 Spin structures

In general relativity spacetime does not have a vector space structure, but it is a differentiable 4-manifold equipped with a metric of signature $(1, 3)$ (see Section 4.3 for a discussion about the signature). The considered vector space will be replaced by the family of tangent spaces at every point.

Definition 4.2.1. Let M be a differentiable manifold. A *Lorentzian structure* on M is an $\text{SO}_{1,3}^+$ -structure on M . Namely, it is the data of a metric g of signature $(1, 3)$ and a compatible space and time-orientation.

Definition 4.2.2 (Clifford algebra bundle). The real Clifford algebras of all tangents spaces of M equipped with the metric gather into a locally trivial bundle in \mathbb{Z}_2 -graded filtered algebras which we call $\text{Cl}(M)$. There is a linear embedding

$$TM \hookrightarrow \text{Cl}(M)$$

We will write $\text{Cl}(M)$ for the bundle of complexified Clifford algebras.

We want to construct a vector bundle over M which has an action of the Clifford algebra bundle which is on each fibre a Clifford module. This will require extra structure on M called a *Spin structure*.

Definition 4.2.3 (Spin structure). Let M be an oriented n -manifold and let $\mathrm{GL}^+(M)$ be its GL_n^+ -principal bundle of positive linear tangent frames. Let $\widetilde{\mathrm{GL}}_n^+ \rightarrow \mathrm{GL}_n^+$ be the non-trivial twofold covering of GL_n^+ .

A Spin-structure on M is a lift of structure group of $\mathrm{GL}^+(M)$ through $\widetilde{\mathrm{GL}}_n^+ \rightarrow \mathrm{GL}_n^+$ (Section 3.2.2). Namely, it is a $\widetilde{\mathrm{GL}}_n^+$ -principal bundle $P \rightarrow M$ equipped with one of the two following equivalent structures:

- A smooth bundle map $P \rightarrow \mathrm{GL}^+(M)$ which is $\widetilde{\mathrm{GL}}_n^+ \rightarrow \mathrm{GL}_n^+$ -intertwining in the sense that the following action diagram commutes:

$$\begin{array}{ccc} P \times \widetilde{\mathrm{GL}}_n^+ & \longrightarrow & P \\ \downarrow & & \downarrow \\ \mathrm{GL}^+(M) \times \mathrm{GL}_n^+ & \longrightarrow & \mathrm{GL}^+(M) \end{array}$$

- A solder form: an \mathbb{R}^n -valued 1-form α on P which is
 1. of maximal rank and vanishes on vertical vectors (vectors which project to 0 in TP)
 2. equivariant for the action $\widetilde{\mathrm{GL}}_n^+ \rightarrow \mathrm{GL}_n^+ \hookrightarrow \mathrm{End}(\mathbb{R}^n)$.

We will call P the “spinor frame bundle”. A (4-)manifold equipped with a spin structure is called a *spin (4-)manifold*.

Theorem 4.2.4. *Let M be a spin 4-manifold. Given any metric g of signature (p, q) and compatible space and time-orientation of M , we denote $\mathrm{SO}^+(g)$ the corresponding bundle of space and time-oriented orthonormal frames of M . Then the following pullback defines a $\mathrm{Spin}_{p,q}^+$ -principal bundle P on M equipped with a solder form for the natural action of $\mathrm{Spin}_{p,q}^+$ on $\mathbb{R}^{p,q}$:*

$$\begin{array}{ccc} P & \longleftarrow & \widetilde{\mathrm{GL}}^+(M) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{SO}^+(g) & \longleftarrow & \mathrm{GL}^+(M) \end{array} \tag{4.4}$$

Conversely, equip M with a metric g of signature (p, q) and a compatible space and time orientation. Assume furthermore that there exists a $\mathrm{Spin}_{p,q}^+$ -principal bundle P on M equipped with a $\mathrm{Spin}_{p,q}^+$ -equivariant solder form and a $\mathrm{Spin}_{p,q}^+ \rightarrow \mathrm{SO}_{p,q}^+$ -equivariant bundle map

$$P \rightarrow \mathrm{SO}^+(g)$$

Then one can construct a $\widetilde{\mathrm{GL}}_4^+$ -principal bundle as the following associated bundle:

$$\widetilde{\mathrm{GL}}^+(M) := P \times_{\mathrm{Spin}_{1,3}^+} \widetilde{\mathrm{GL}}_4^+$$

which can be equipped with a $\widetilde{\mathrm{GL}}_4^+$ -equivariant solder form such that P is identified with the pullback described in Diagram (4.4).

Remark. We use the most common name in the literature, but calling such a structure Spin-structure would be more consistent if we called $\mathrm{Spin}(V)$ the connected groups we call $\mathrm{Spin}^+(V)$.

We will call a manifold equipped with both a Lorentzian structure and a Spin structure a *Lorentzian spin manifold*.

Spin structures allow defining the spinor bundles in which spinor fields take value.

Definition 4.2.5. Let M be a Lorentzian spin 4-manifold with metric g . Let Σ be a real or complex module over $\text{Cl}_{1,3}$.

The *associated spinor bundle*

$$P[\Sigma] := P \times_{\text{Spin}_{1,3}^+} \Sigma$$

is a vector bundle over M with fibre Σ and a natural action of the Clifford algebra bundle $\text{Cl}(M)$.

If Σ is equipped with a bilinear or sesquilinear product then so does $P[\Sigma]$. If $\mathbb{R}^{1,3}$ acts by selfadjoint operators (resp. antiselfadjoint operators) on Σ then so does $TM \subset \text{Cl}(M)$ on $P[\Sigma]$.

Definition 4.2.6 (Universal spinor bundle, [GMS09]). Let M be a spin 4-manifold and Σ a $\text{Cl}_{1,3}$ -module. Define the *universal Σ -bundle* constructed as follows:

$$\widetilde{\text{GL}}^+(M) \times_{\text{Spin}_{1,3}^+} \Sigma$$

Then the Σ -bundle associated to any Lorentzian structure can be constructed as the following pullback:

$$\begin{array}{ccc} P[\Sigma] & \longleftarrow & \widetilde{\text{GL}}^+(M) \times_{\text{Spin}_{1,3}^+} \Sigma \\ \downarrow & \lrcorner & \downarrow \\ \text{SO}^+(g) & \longleftarrow & \text{GL}^+(M) \end{array}$$

Theorem 4.2.7 (Universal Clifford module bundle). *Let M be a Lorentzian spin 4-manifold equipped with metric g . Define the universal Clifford module bundle constructed as follows [LM89]:*

$$P[\text{Cl}_{1,3}] = P \times_{\text{Spin}_{1,3}^+} \text{Cl}_{1,3}$$

It is the associated bundle to P for the left action of $\text{Spin}_{1,3}^+$ on $\text{Cl}_{1,3}$. It is not an algebra bundle but it is a \mathbb{Z}_2 -graded vector bundle with a right action of $\text{Cl}_{1,3}$ and a left action of $\text{Cl}(M)$ which are both graded, faithful and transitive on each fibre.

Given a $\text{Cl}_{1,3}$ -module Σ , the associated Σ -bundle can be constructed as follows:

$$P[\Sigma] \simeq P[\text{Cl}_{1,3}] \otimes_{\text{Cl}_{1,3}} \Sigma$$

Remark.

- One should not confuse the universal Clifford module bundle with the associated bundle for the *adjoint action* of $\text{Spin}_{1,3}^+$ on $\text{Cl}_{1,3}$:

$$P \times_{\text{Ad}_{\text{Spin}_{1,3}^+}} \text{Cl}_{1,3} \simeq \text{SO}(g) \times_{\text{SO}_{1,3}^+} \text{Cl}_{1,3} \simeq \text{Cl}(M)$$

In what follows $P[\text{Cl}_{1,3}]$ shall always denote the universal Clifford module bundle.

- The Clifford algebra bundle does not have the structure of an associated bundle $P[\text{Cl}_{1,3}] \otimes_{\text{Cl}_{1,3}} \Sigma$ but has nonetheless the structure of a left Clifford module bundle. This is due to the fact that although it can be constructed from P as an associated bundle of fibre $\text{Cl}_{1,3}$, the action of $\text{Spin}_{1,3}^+$ differs from the restriction to $\text{Spin}_{1,3}^+$ of the left action of the Clifford algebra. In

other terms, the considered action of $\text{Spin}_{1,3}^+$ is extra structure on top of the $\text{Cl}_{1,3}$ -module structure of the fibre.

- The two “universal” bundles are universal in different senses: the universal Σ -bundle is universal with respect to all Lorentzian structures, while the universal Clifford module bundle is universal with respect to all $\text{Cl}_{1,3}$ -modules.

The second part of the theorem comes from the following associativity-like chain of isomorphisms:

$$P[\text{Cl}_{1,3}] \otimes_{\text{Cl}_{1,3}} \Sigma \simeq P \times_{\text{Spin}_{1,3}^+} \text{Cl}_{1,3} \otimes_{\text{Cl}_{1,3}} \Sigma \simeq P \times_{\text{Spin}_{1,3}^+} \Sigma = P[\Sigma]$$

Definition 4.2.6 and Theorem 4.2.7 can be used in general dimension and signature.

Theorem 4.2.8 (Topological obstruction and classification, [LM89; Bou+15; Ger68]). *Let M be a 4-manifold. Assume it is orientable, namely its first Stiefel-Whitney class vanishes:*

$$w_1(TM) = 0$$

Then M admits spin structures if and only if its second Stiefel-Whitney class vanishes as well:

$$w_2(TM) = 0$$

If M is noncompact, this is equivalent to M being parallelisable.

In this case, spin structures up to isomorphism have a free and transitive action of each of the following equivalent structures, classified up to isomorphism by $H(M, \mathbb{Z}_2)$;

1. principal \mathbb{Z}_2 -bundles over M
2. 2-sheeted (surjective) covering spaces over M
3. real line bundles over M .

The action of a \mathbb{Z}_2 -bundle $Q \rightarrow M$ on $\widetilde{\text{GL}}^+(M)$ is as follows:

$$Q \otimes \widetilde{\text{GL}}^+(M) := Q \times_{\mathbb{Z}_2} \widetilde{\text{GL}}^+(M)$$

with the subgroup $\mathbb{Z}_2 = \ker(\widetilde{\text{GL}}_4^+ \rightarrow \text{GL}_4^P) \subset \widetilde{\text{GL}}_4^+$.

4.2.2 The Dirac operator

One major feature of spinors on a Lorentzian spin manifold is the *Dirac operator*, which we now introduce. It is in particular involved in the *Dirac equation* governing the relativistic spinor fields.

Theorem 4.2.9 (Spin connection). *Let M be a spin manifold equipped with a metric g of signature (p, q) . Assume furthermore that M is equipped with a metric connection ∇ .*

Then there is a unique principal connection on the spinor frame bundle $P \rightarrow M$ such that the associated connection on

$$TM \simeq P[\mathbb{R}^{p,q}]$$

coincides with ∇ . It is called the spin connection (sometimes spinor connection) associated with ∇ .

For any module Σ over $\text{Cl}_{p,q}$, there is an associated connection on the associated spinor bundle $P[\Sigma]$. It can be related to the associated connection on the universal Clifford module bundle as

follows: let a be a local section of $P[\text{Cl}_{p,q}]$ and ψ a local map from M to Σ . Then the associated connection on $P[\Sigma] \simeq P[\text{Cl}_{p,q}] \otimes_{\text{Cl}_{p,q}} \Sigma$ satisfies the following Leibniz rule:

$$\nabla^{P[\Sigma]}(a \otimes \psi) = \nabla^{P[\text{Cl}_{p,q}]}a \otimes \psi + a \otimes d\psi$$

with ds the trivial flat derivative on the trivial vector bundle $\Sigma \times M \rightarrow M$.

Proof. We prove the Leibniz rule. Let $A \in \Gamma(P, \text{Cl}_{1,3})^{\text{Spin}_{1,3}^+}$ and $\Psi \in \Gamma(P, \Sigma)^{\text{Spin}_{1,3}^+}$ -invariant be $\text{Spin}_{1,3}^+$ -equivariant (resp. invariant) fields on P , corresponding respectively to the associated section a and the Σ -valued field ψ on M . Let ω be the connection 1-form on P .

The essential point is that the action of $\text{Spin}_{1,3}^+$ on $\text{Cl}_{1,3} \otimes_{\text{Cl}_{1,3}} \Sigma$ corresponding to the isomorphism $\text{Cl}_{1,3} \otimes_{\text{Cl}_{1,3}} \Sigma \simeq \Sigma$ is simply the left action on the factor $\text{Cl}_{1,3}$.

The covariant derivative of $a \otimes \psi$ is associated to

$$d^\omega(A \otimes \Psi) = d(1 \otimes \Psi) + \omega \cdot (A \otimes \Psi) = (dA) \otimes \Psi + A \otimes d\Psi + (\omega \cdot A) \otimes \Psi = (d^\omega A) \otimes \Psi + A \otimes d\Psi$$

The former term corresponds on M to $(\nabla a) \otimes \psi$ and the latter to $a \otimes d\psi$, which proves the Leibniz rule. \square

When there is no mention of ∇ , we will implicitly be working with the Levi-Civita connection associated with g .

Definition 4.2.10 (Dirac operator). Let M be a manifold equipped with a metric g of signature (p, q) and a compatible connection ∇ . Let $E \rightarrow M$ be a vector bundle which is a module over $\text{Cl}(M)$ equipped with a connection.

The *Dirac operator* \not{D} is a first-order differential operator acting on sections ψ of E as follows:

$$\begin{array}{c} \Gamma(E) \\ \downarrow \nabla \\ \Gamma(T^*M \otimes E) \\ \downarrow g^{-1} \times \text{id} \\ \Gamma(TM \otimes E) \\ \downarrow \\ \Gamma(\text{Cl}(M) \otimes E) \\ \downarrow \\ \Gamma(E) \end{array} \quad \not{D}$$

If we use the so-called covariant “gamma matrices” associated with E :

$$\gamma : T^*M \xrightarrow{g^{-1}} TM \rightarrow \text{End}(E)$$

the Dirac operator acting on a section ψ takes the following form:

$$\not{D}\psi = \gamma^i \nabla_i \psi$$

Remark. The Dirac operator is sometimes restricted to *complex* Clifford modules and has an extra i factor. Such considerations of conventions are discussed in Section 4.3.

Theorem 4.2.11 ([LM89], II.5.3). *Let M be an oriented spin manifold equipped with a metric g of signature (p, q) and a compatible connection ∇ . Let vol be the positive (pseudo-)Riemannian volume. Let $E \rightarrow M$ be a vector bundle which is a module over $\text{Cl}(M)$ and which is equipped with a hermitian inner product for which TM acts by anti-selfadjoint operators.*

Then the Dirac operator on $P[\Sigma]$ is formally self-adjoint, namely

$$\forall \psi_1, \psi_2 \in \Gamma_0(P[\Sigma]), \int_M (\langle \not{\nabla} \psi_1 | \psi_2 \rangle + \langle \psi_1 | \not{\nabla} \psi_2 \rangle) \text{vol} = 0$$

with Γ_0 standing for the space of compactly supported sections.

Remark. The Dirac operator uses multiplication by odd elements and therefore requires the structure of a Clifford bundle. However the equation

$$\not{\nabla} \psi = 0$$

can be written in an equivalent way using any non vanishing and nowhere isotropic vector field u , for example a global nonvanishing timelike vector field. It gives an *invertible* section of $\text{Cl}(M)$ and

$$u \not{\nabla} \psi = u e^i \nabla_i \psi$$

with $(u e^i) \in \text{Cl}^+(V)$. As such, the equation can be written on spinor bundles, which only have an action of $\text{Cl}^+(V)$.

However the Dirac equation used in physics usually has a “mass parameter”, namely one considers the eigenvalue equation

$$(\not{\nabla} - m)\psi = 0$$

which *does* change when $\not{\nabla}$ is replaced by $u \not{\nabla}$. Only the massless case, called the *Weyl equation* can be formulated in spinor bundles². Sections of (usually complex) spinor bundles are sometimes accordingly called “Weyl spinors”, while sections of a (complex) Clifford bundle are called “Dirac spinors”.

The Bochner formula

The crucial property of the Dirac operator is that it is a first-order differential operator whose square has the symbol of a Laplacian. We will establish the formulas in presence of torsion. We first define the connection Laplacian.

Definition 4.2.12 (Covariant second derivative, [LM89], II.8). Let M be a manifold with a tangent connection ∇ . Let $E \rightarrow M$ be a vector bundle equipped with a linear connection ∇^E .

The *covariant second derivative* on E associated to ∇^E is the following second-order differential operator:

$$\begin{array}{c} \xrightarrow{\hspace{10em}} \\ \nabla^2 : \Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M \otimes E) \xrightarrow{\nabla^{T^*M \otimes E}} \Gamma(T^*M \otimes T^*M \otimes E) \end{array}$$

On a section A of E , ∇^2 acts as follows:

$$\nabla_{X,Y}^2 A = \nabla_X \nabla_Y A - \nabla_{\nabla_X Y} A$$

²There exists however a variant for real spinors in which the mass is not a scalar [Del+00].

In particular, the curvature of ∇ on E can be expressed using the following formula:

$$\nabla_{X,Y}^2 - \nabla_{Y,X}^2 = \mathcal{R}(X,Y) - \nabla_{T(X,Y)}$$

with T the torsion of ∇ .

Definition 4.2.13 (Connection Laplacian, [LM89] Section II.8). Let M be a manifold with a metric g and a connection ∇ . Let $E \rightarrow M$ be a vector bundle equipped with a linear connection ∇^E .

The connection Laplacian on E is a differential operator on E defined as follows:

$$\nabla^* \nabla = - \langle g^{-1} | \nabla^2 \rangle$$

with g^{-1} the inverse metric.

On a section A of E , using a local frame (e_i) the connection Laplacian takes the following form:

$$\nabla^* \nabla A = -g^{ij} (\nabla_{e_i} \nabla_{e_j} A - \nabla_{\nabla_{e_i} e_j} A)$$

Remark. As the notation suggests, ∇ has a formal adjoint ∇^* on the space of compact supported sections $\Gamma_0(E)$ such that the connection Laplacian indeed takes the form $\nabla^* \nabla$.

We can now state the so-called *Bochner formula* relating the squared Dirac operator to the connection Laplacian.

Theorem 4.2.14 (Bochner formula, [LM89], II.8.2). *Let M be a manifold with a metric g of signature (p, q) and a compatible connection ∇ .*

Let $E \rightarrow M$ be a vector bundle with an action of $\text{Cl}(M)$ and a connection ∇^E which is compatible with the action of $\text{Cl}(M)$ in the following sense:

$$\forall \psi \in \Gamma(E), \forall X \in \Gamma(TM), \nabla^E(X \cdot \psi) = (\nabla X) \cdot \psi + X \cdot \nabla^E \psi$$

Then the square of the Dirac operator on E differs from the connection Laplacian as follows, with (e_i) a local orthonormal frame:

$$\nabla^2 = -\nabla^* \nabla + \tilde{\mathcal{R}} - \frac{1}{2} e_i e_j \cdot \nabla_{T(e_i, e_j)}$$

with $\tilde{\mathcal{R}} \in \text{Cl}(M)$ taking the following form:

$$\tilde{\mathcal{R}}^E = \frac{1}{2} e_i e_j \cdot \mathcal{R}^E(e_i, e_j)$$

Proof. Let (e_i) be a local orthonormal frame.

$$\nabla^2 = e_i \nabla_{e_i} e_j \nabla_{e_j}$$

Let us write for a local section ψ of $P[\Sigma]$:

$$c(\cdot) \nabla_{(\cdot)} \psi : (u, v) \in TM \otimes TM \mapsto u \cdot \nabla_v \psi$$

Then

$$e_j \nabla_{e_j} \psi = \langle g^{-1} | c(\cdot) \nabla_{(\cdot)} \psi \rangle$$

and

$$\begin{aligned}
\nabla_{e_i} (e_j \nabla_{e_j} \psi) &= \nabla_{e_i} \langle g^{-1} | c(\cdot) \nabla(\cdot) \psi \rangle \\
&= \langle g^{-1} | \nabla_{e_i} c(\cdot) \nabla(\cdot) \psi \rangle \\
&= \langle g^{-1} | c(\cdot) \nabla_{e_i} \nabla(\cdot) \psi \rangle \\
&= e_j \nabla_{e_i, (\cdot)}^2 \psi
\end{aligned}$$

using the compatibility of the spin connection with the action of the Clifford module to obtain the third line.

As a consequence

$$\begin{aligned}
e_i \nabla_{e_i} e_j \nabla_{e_j} &= e_i e_j \nabla_{e_i, e_j}^2 \\
&= \frac{1}{2} (\{e_i, e_j\} + e_i e_j - e_j e_i) \nabla_{e_i, e_j}^2 \\
&= g_{ij} \nabla_{e_i, e_j}^2 + \frac{1}{2} e_i e_j (\nabla_{e_i, e_j}^2 - \nabla_{e_j, e_i}^2) \\
&= -\nabla^* \nabla + \frac{1}{2} e_i e_j (\mathcal{R}_{ij}^E - \nabla_{T_{ij}}) \\
&= -\nabla^* \nabla + \tilde{\mathcal{R}} - \frac{1}{2} e_i e_j \nabla_{T_{ij}}
\end{aligned}$$

□

Example 4.2.15 (Dirac operator over the Minkowski space). Consider the Minkowski space with its flat (affine) structure, its constant metric η and the trivial spin structure. Using the gamma matrices introduced earlier, the Dirac operator takes the following form:

$$\not{D} = \gamma^i \partial_i$$

and squares to

$$\not{D}^2 = \frac{1}{2} \{\gamma^i, \gamma^j\} \partial_i \partial_j = \eta^{ij} \partial_i \partial_j$$

Therefore the Dirac operator gives a first order differential operator square root of the D'Alembertian, although not acting on scalar-valued fields but spinor fields.

The operator $\tilde{\mathcal{R}}$ has a very simple expression when working with spinorial bundles. In order to establish such a formula, we need a generic formula for specific contractions of the curvature happening when working with spinor bundles.

Lemma 4.2.16 ([Bou+15]). Let M be a manifold with a metric g of signature (p, q) and a compatible connection ∇ . Let us write \mathcal{R} the curvature endomorphism of ∇ and use it both as an $\text{End}(TM)$ -valued 2-form and a 4-tensor:

$$\mathcal{R}(X, Y, Z, T) = \langle \mathcal{R}(X, Y) Z | T \rangle$$

Let us define the *spinorial curvature* \mathcal{R} associated with ∇ as follows: for (e_i) a local pseudo-orthonormal frame of M , then

$$\forall X, Y \in \Gamma(TM), \quad \mathcal{R}(X, Y) := \frac{1}{4} \eta^{ij} (\mathcal{R}(X, Y) e_i) \cdot e_j = -\frac{1}{4} \mathcal{R}(X, Y, e_i, e_j) e^i e^j$$

For any spinor bundle $P[\Sigma]$ equipped with the spin connection, the tensor $\mathcal{R}^{P[\Sigma]}$ matches with the action of \mathcal{R} on $P[\Sigma]$.

The following identity holds:

$$\sum_{i,j} e_i e_j \mathcal{R}(e_i, e_j) = -\frac{1}{2} \text{Scal} + \frac{1}{2} \text{Ric}_{mn} (e^m e^n - e^n e^m) - \frac{1}{4} \mathcal{R}_{[ijkl]} e^i e^j e^k e^l$$

with Scal the scalar curvature of ∇ , Ric its Ricci curvature and $\mathcal{R}_{[ijkl]}$ the purely antisymmetric part of its total curvature. When ∇ is the Levi-Civita connection, the following identity holds for any vector field X :

$$\sum_j e_j \cdot \mathcal{R}(X, e_j) = -\frac{1}{2} g^{-1}(\text{Ric}(X)) \in \Gamma(\text{Cl}(M))$$

Remark. The covariant vectors (e^i) suggest an interpretation using the (isomorphic) Clifford algebra of the cotangent bundle.

Proof. Let Σ be a Clifford module and let $\Omega \in \Omega_{\text{hor}}^2(P, \mathfrak{spin}_{p,q})^{\text{Spin}_{p,q}^+}$ be the curvature of the spin connection. Since Σ is a Clifford module, $\text{Spin}_{p,q}^+$ acts through its embedding in $\text{Cl}_{p,q}$, therefore we are interested in the image in $\Omega_{\text{hor}}^2(P, \text{Cl}_{p,q})^{\text{Spin}_{p,q}^+}$ of Ω . Let us write

$$(\rho_i)^a_b : \mathfrak{spin}_{p,q} \simeq \mathfrak{so}_{p,q} \hookrightarrow \mathfrak{gl}(\mathbb{R}^{p,q})$$

the Lie algebra action of $\mathfrak{spin}_{p,q}$ on $\mathbb{R}^{p,q}$. Then the curvature endomorphism $\mathcal{R} \in \Omega^2(M, \mathfrak{so}(TM))$ is associated to the equivariant form $(\rho_i)^a_b \Omega^i \in \Omega_{\text{hor}}^2(M, \mathfrak{so}_{p,q})^{\text{Spin}_{p,q}^+}$. Recall that the Lie algebra isomorphism $\mathfrak{spin}_{p,q} \simeq \mathfrak{so}_{p,q}$ sends the product $\frac{1}{4} e_1 e_2$ with e_1, e_2 orthogonal pseudo-normed vectors to the following element of $\mathfrak{so}_{p,q} \hookrightarrow \mathfrak{gl}(\mathbb{R}^{p,q})$:

$$\frac{1}{2} e_1 e_2 \mapsto e_1 \otimes {}^t e_2 - e_2 \otimes {}^t e_1$$

Let (e_a) be a pseudo-orthonormal basis of $\mathbb{R}^{p,q}$. Then

$$\rho_i \Omega^i = \frac{1}{2} \rho_i^a_b \Omega^i \eta^{bb'} (e_a \otimes {}^t e_{b'} - e_{b'} \otimes {}^t e_a)$$

Therefore the corresponding element in $\Omega_{\text{hor}}^2(P, \text{Cl}_{p,q})^{\text{Spin}_{p,q}^+}$ is

$$\frac{1}{4} \rho_i^a_b \Omega^i \eta^{bb'} e_a e_{b'} = \frac{1}{4} (\Omega^i \cdot e_b) \eta^{bb'} e_{b'} \in \Omega_{\text{hor}}^2(P, \text{Cl}_{p,q})^{\text{Spin}_{p,q}^+}$$

The associated form in $\Omega^2(M, \text{Cl}(M))$ is exactly \mathcal{R} . The second expression derives from the identity

$$e_l = \langle e_l | e_l \rangle e_l^{-1}$$

and the fact that e_k and e_l anticommute for $k \neq l$.

Let X be a vector field on M and (e_i) a local orthonormal frame. The crucial arguments will be the Bianchi identity 3.7:

$$d^\nabla T = \mathcal{R} \wedge \text{Id}$$

namely in components

$$d^\nabla T(e_i, e_j, e_k) = 2(\mathcal{R}(e_i, e_j) e_k + \mathcal{R}(e_j, e_k) e_i + \mathcal{R}(e_k, e_i) e_j)$$

and the following identity for 3-vectors in the Clifford algebra:

$$3e^i e^j e^k = e^i e^j e^k + e^j e^k e^i + e^k e^i e^j + 2(g^{ij}e^k - 2g^{ik}e^j + g^{jk}e^i)$$

as well as the following identity [For12]:

$$\begin{aligned} \frac{1}{4}(e^i e^j - e^j e^i)(e^k e^l - e^l e^k) &= \epsilon^{ijkl}\omega + \frac{1}{2}\epsilon^{ij}{}^m{}_n \epsilon^{kl}{}_{no}(\gamma^m \gamma^n - \gamma^n \gamma^m) + g^{jk}g^{il} - g^{ik}g^{jl} \\ &= \epsilon^{ijkl}\omega + \frac{1}{2}\delta_{[k'l'n]}^{[ijm]}g^{k'k}g^{l'l}(\gamma_m \gamma^n - \gamma^n \gamma_m) + g^{jk}g^{il} - g^{ik}g^{jl} \end{aligned}$$

with $\delta_{[k'l'n]}^{[ijm]}$ the normalised totally antisymmetric Kronecker symbol.

We can now compute

$$\begin{aligned} e^i e^j \mathcal{R}(e^i, e^j) &= -\frac{1}{4}\mathcal{R}_{ijkl}e^i e^j e^k e^l \\ &= -\frac{1}{4}\mathcal{R}_{ijkl}\frac{1}{4}(e^i e^j - e^j e^i)(e^k e^l - e^l e^k) \\ &= -\frac{1}{4}\mathcal{R}_{ijkl}\left(\epsilon^{ijkl}\omega + \frac{1}{2}\delta_{[k'l'n]}^{[ijm]}g^{k'k}g^{l'l}(e_m e^n - e^n e_m) + g^{jk}g^{il} - g^{ik}g^{jl}\right) \\ &= -\frac{1}{4}\mathcal{R}_{[ijkl]}e^i e^j e^k e^l \\ &\quad - \frac{1}{8}\mathcal{R}_{ij}{}^{k'l'}\left(-\delta_k^i \delta_n^j \delta_{l'}^m + \delta_{l'}^i \delta_n^j \delta_k^m - \delta_n^i \delta_k^j \delta_{l'}^m + \delta_n^i \delta_{l'}^j \delta_k^m\right)(e_m e^n - e^n e_m) - \frac{1}{2}\text{Scal} \\ &= -\frac{1}{4}\mathcal{R}_{[ijkl]}e^i e^j e^k e^l + \frac{1}{2}\text{Ric}_{mn}(e^m e^n - e^n e^m) - \frac{1}{2}\text{Scal} \end{aligned}$$

Now assume that torsion vanishes. The Bianchi identity then implies the following equations:

$$\begin{aligned} \mathcal{R}_{ijkl} &= \mathcal{R}_{klij} \\ \mathcal{R}_{ijlk} &= -\mathcal{R}_{klji} \\ \text{Ric}_{ji} &= -\text{Ric}_{ij} \end{aligned}$$

Using these, we can perform the following computation:

$$\begin{aligned} e^j \mathcal{R}(X, e^j) &= -\frac{1}{4}\mathcal{R}(X, e^j, e^k, e^l)e^j e^k e^l \\ &= \frac{1}{4}\mathcal{R}(e^k, e^l, e^j, X)\frac{1}{3}(e^j e^k e^l + e^k e^l e^j + e^l e^j e^k + 2(g^{jk}e^l - 2g^{jl}e^k + g^{kl}e^j)) \\ &= \frac{1}{12}\left(\mathcal{R}(e^k, e^l, e^j, X) + \mathcal{R}(e^j, e^k, e^l, X) + \mathcal{R}(e^l, e^j, e^k, X)\right)e^j e^k e^l \\ &\quad + 2(-\text{Ric}(e^l, X)e^l - 2\text{Ric}(e^k, X)e^k + 0) \\ &= -\frac{1}{2}\text{Ric}(X, e^i)e^i = -\frac{1}{2}g^{-1}\text{Ric}(X) \end{aligned}$$

□

Remark. The various antisymmetrical parts of \mathcal{R} and Ric can be expressed in term of the covariant exterior derivative of the torsion.

Theorem 4.2.17 (Bochner-Weitzenböck-Lichnerowicz-Schrödinger formula). *Let M be a spin*

manifold with a metric g of signature (p, q) and a compatible connection ∇ . Then for any spinor bundle $P[\Sigma]$ equipped with the associated spin connection, the Bochner formula takes the following form in a local orthonormal frame (e_i) :

$$\nabla^2 = -\nabla^* \nabla - \frac{1}{4} \text{Scal} + \frac{1}{4} e^i e^j (\text{Ric}_{ij} - \text{Ric}_{ji} - 2\nabla_{T_{ij}}) - \frac{1}{8} e^i e^j e^k e^l \mathcal{R}_{[ijkl]}$$

In particular, when the connection is torsionless, only the scalar curvature term is left.

Considering the Clifford module structure of $\text{Cl}(M)$ with the Levi-Civita connection, the Dirac operator satisfies on $TM \subset \text{Cl}(M)$ the following Bochner formula:

$$\forall X \in \Gamma(TM), \nabla^2 X = -\nabla^* \nabla X - g^{-1}(\text{Ric}(X))$$

Furthermore, when M is oriented, $\text{Cl}(M)$ equipped with the Levi-Civita Dirac operator is isomorphic to $\wedge^* T^* M$ equipped with the following operator $d + d^*$.

Proof. The first assertion is a straightforward conclusion from Theorem 4.2.14 and 4.2.16.

Let us now assume ∇ is torsionless. We do not show the correspondence with $\wedge^* T^* M$ with the operator $d + d^*$. We need to compute the action of $e_i e_j \mathcal{R}_{ij}$ on $TM \subset \text{Cl}(M)$. We will use the following property: for all sections A of $\text{Cl}(M)$,

$$\mathcal{R}(e_i, e_j) \cdot A = \left[-\frac{1}{4} \mathcal{R}(e_i, e_j, e_k, e_l) e_k e_l, A \right]$$

This allows expressing the action of $e_i e_j \mathcal{R}_{ij}$ using \mathcal{R} . Let X be a vector field seen as a section of $\text{Cl}(M)$.

$$\begin{aligned} e_i e_j \mathcal{R}_{ij} X &= e_i e_j [\mathcal{R}_{ij}, X] \\ &= [e_i e_j \mathcal{R}_{ij}, X] - [e_i e_j, X] \mathcal{R}_{ij} \\ &= \left[-\frac{1}{2} \text{Scal}, X \right] - (e_i \{e_j, X\} - \{e_i, X\} e_j) \mathcal{R}_{ij} \\ &= 0 - 2(\langle e_j | X \rangle e_i - \langle e_i | X \rangle e_j) \mathcal{R}_{ij} \\ &= -2(e_i \mathcal{R}_{e_i, X} - e_j \mathcal{R}_{X, e_j}) \\ &= -2g^{-1} \text{Ric}(X) \end{aligned}$$

according to Lemma 4.2.16. Thus we obtain the formula

$$\nabla^2 X = -\nabla^* \nabla X - g^{-1} \text{Ric}(X)$$

□

4.2.3 Lie Derivative of Spinors

Background

In this section we illustrate the idea of correspondence spaces (Section 3.3.6) by giving a geometric description of the so-called Lie derivative of spinors [Kos72; GM05]. Let us first present the physical motivation. It comes from the variational formulation of field theories. In General Relativity, spacetime is modelled by a Lorentzian spin 4-manifold M with a $\text{Spin}_{1,3}^+$ -structure $P \rightarrow M$. Let Σ be a $\text{Cl}_{1,3}$ -module. A spinor field is a section on M of the associated spinor bundle $P[\Sigma]$.

The common formulation of field theories use a *variational principle*, namely the field equations amount to the field being critical points of a functional called action. In particular, they are derived and usually studied using *independent* variations of every dynamical field of the theory. This brings us to the following problem: in General Relativity the metric g is a dynamical field. However when it is varied, the bundle of orthonormal frame necessarily changes. As a consequence, the spin frame bundle needs to change as well. This means that the very bundle in which the spinor fields live cannot stay unvaried, so that we cannot consider “variations of the metric with fixed matter fields”.

This technicality is made more manifest when considering diffeomorphisms of spacetime. Invariance under diffeomorphism is an essential principle of General Relativity, and diffeomorphisms generally act non-trivially on the metric. In particular, identifying the so-called “stress-energy tensor” requires defining an action of the infinitesimal diffeomorphisms (which may not preserve the Lorentzian structure) on matter fields such as spinor fields.

The usual solution [Heh+76; Tra06; Pop20] is as follows. One fixes an auxiliary vector bundle $V \rightarrow M$ which is isomorphic to TM – but one does not fix such an isomorphism. It is equipped with a Lorentzian metric η , a space and time-orientation and its orthonormal frame bundle is equipped with a lift of the structure group from $SO_{1,3}^+$ to $Spin_{1,3}^+$. Spinor fields on spacetime are then sections of the associated spinor bundle. The metric is replaced by a “tetrad” e , which is a linear bundle isomorphism $TM \xrightarrow{\sim} V$, inducing the metric $g = e^*\eta$. In this way, spinor fields live in a bundles independent of g , and the tetrad relates the spinor bundles to the geometry of spacetime. The idea is that a tetrad keep track of more data than the metric: they have an extra $SO_{1,3}^+$ degree of freedom, which makes it easier to define a transformation law for spinors. We will expound on this approach in Section 11.1, within the formalism of Sciama-Kibble theory.

However effective this approach may be, it is arguably roundabout. In particular, the matter fields living in an arbitrary bundle adds a layer of complexity to the geometrical framework.

An alternative approach to the issue is to define some kind of “connection” which allows the spinor fields to vary following a variation in the Lorentzian structure of spacetime. It is put to use in the so-called “spinor Lie derivative”, pioneered by Kosmann [Kos72]. The idea comes from the observation that Killing vector fields, namely those who preserve the metric, act naturally on the spinor fields. It is possible to define an action of vector fields on spinors by isolating a “Killing part” in every vector field. The underlying geometry is beautifully exposed in [BG92]: it is related to a “transport” of orthonormal bases between different inner products which already exists at the linear level.

We now present the construction using the structure of correspondence spaces.

The covariant approach to spinors on manifolds

Let M be a spin 4-manifold with a \widetilde{GL}_4^+ -structure $\widetilde{GL}^+(M) \rightarrow GL(M)$. A Lorentzian structure, namely a Lorentzian metric with a space and time-orientation, is exactly a section of $GL(M)/SO_{1,3}^+$, in other words a reduction of the structure group to $SO_{1,3}^+$. If we require it to be compatible with the orientation of M , it has to be a section of the subbundle $GL^+(M)/SO_{1,3}^+$.

In order to define spinor fields, we need spin structures. They are defined consistently for all Lorentzian structures according to the following bundle isomorphism

$$GL^+(M)/SO_{1,3}^+ \xrightarrow{\sim} \widetilde{GL}^+(M)/Spin_{1,3}^+$$

the sections of which are exactly Lorentzian structures.

Any (space and time-oriented) Lorentzian structure $\sigma : M \rightarrow \widetilde{GL}^+(M)/Spin_{1,3}^+$ thus defines a $Spin_{1,3}^+$ -structure through pullback. Similarly, spinor bundles associated to a spinor module Σ

can be constructed as pullbacks of the universal spinor bundle (Definition 4.2.6).

Let

$$\sigma : [0, 1] \times M \rightarrow \widetilde{\mathrm{GL}}^+(M) / \mathrm{Spin}_{1,3}^+$$

be a parametrised Lorentzian structure. For every value of $t \in [0, 1]$ it defines a Lorentzian structure on M . We now explain how it can also define an *identification* between each of them, namely a principal bundle isomorphism between Lorentzian structures corresponding to different parameters. Before this, let us explain how it addresses the problem of identifying spinor bundles along σ . This is due to the following general fact: an isomorphism between G -principal bundles induces isomorphisms between associated bundles for any G -space. In our case, an (equivariant) identification between the Lorentzian structures $\sigma_{t_1}^* \widetilde{\mathrm{GL}}^+(M)$ and $\sigma_{t_2}^* \widetilde{\mathrm{GL}}^+(M)$ for two parameters $t_1, t_2 \in [0, 1]$ induces an identification between associated bundles

$$\sigma_{t_1}^* \widetilde{\mathrm{GL}}^+(M) \times_{\mathrm{Spin}_{1,3}^+} \Sigma \xrightarrow{\sim} \sigma_{t_2}^* \widetilde{\mathrm{GL}}^+(M) \times_{\mathrm{Spin}_{1,3}^+} \Sigma$$

This allows transporting spinor fields above σ_{t_1} to spinor fields above σ_{t_2} in a smooth and transitive fashion.

As explained in Section 3.3.6, a reductive structure on the Klein pair $(\mathfrak{gl}_4, \mathfrak{spin}_{1,3})$ would define a $\mathrm{Spin}_{1,3}^+$ -principal connection on

$$\widetilde{\mathrm{GL}}^+(M) \rightarrow \widetilde{\mathrm{GL}}^+(M) / \mathrm{Spin}_{1,3}^+$$

which would allow identifying the different Lorentzian structures along σ . It is essentially what is presented in [BG92]. The reductive structure is the following:

$$\mathfrak{gl}_4 \simeq_{\mathfrak{spin}_{1,3}} \mathfrak{spin}_{1,3} \oplus \mathbb{S}_2$$

with \mathbb{S}_2 standing for the self-adjoint elements of \mathfrak{gl}_4 for the Lorentzian inner product.

This construction brings the following question: do we obtain a consistent identification between all pullback spin structures? Unfortunately this is not the case, as is explained in [BG92]. The reason is that the $\mathrm{Spin}_{1,3}^+$ -principal connection on $\widetilde{\mathrm{GL}}_4^+ \rightarrow \widetilde{\mathrm{GL}}_4^+ / \mathrm{Spin}_{1,3}^+$ associated to the reductive structure is not flat. The identification between two sets of direct orthonormal frames is therefore generally dependent on the path used.

Is it possible to find alternative reductive structures with different geometrical properties? Reductive structures are identified with $\mathfrak{spin}_{1,3}$ -equivariant sections of $\mathfrak{gl}_4 \rightarrow \mathfrak{gl}_4 / \mathfrak{spin}_{1,3}$. We know that \mathfrak{gl}_4 decomposes into irreducible $\mathfrak{spin}_{1,3}$ -modules as follows:

$$\mathfrak{gl}_4 \xrightarrow{\sim}_{\mathfrak{spin}_{1,3}} \mathfrak{spin}_{1,3} \oplus \mathbb{R} \oplus \mathbb{S}_2 / \mathbb{R}$$

We are looking for sections

$$\mathbb{R} \oplus \mathbb{S}_2 / \mathbb{R} \rightarrow_{\mathfrak{spin}_{1,3}} \mathfrak{gl}_4$$

They form an affine space over

$$\mathrm{Hom}_{\mathfrak{spin}_{1,3}}(\mathbb{R} \oplus \mathbb{S}_2 / \mathbb{R}, \mathfrak{spin}_{1,3})$$

However there is no such $\mathfrak{spin}_{1,3}$ -equivariant map since the irreducible representation $\mathfrak{spin}_{1,3}$ is not present in $\mathbb{R} \oplus \mathbb{S}_2 / \mathbb{R}$. As a consequence, the reductive structure is unique.

Having defined a way to transport spinor fields along a path of Lorentzian structures, we can now construct the Lie derivative of spinor fields. Let X be a complete (for simplicity) vector field

on M . Its flow gives a 1-parameter group of diffeomorphisms of M which can be used to pull back a fixed Lorentzian structure: $\sigma_t = (\phi_t)^* \sigma_0$. Let

$$\psi \in \Gamma \left(M, \sigma_0^* \widetilde{\text{GL}}^+(M) \times_{\text{Spin}_{1,3}^+} \Sigma \right)$$

be a spinor field associated to the Lorentzian structure σ_0 . Then $\psi_t = \phi_t^* \psi$ is a spinor field associated to the Lorentzian structure σ_t . Namely at every point $m \in M$, $t \mapsto \sigma_t(m)$ traces a smooth path in $\widetilde{\text{GL}}^+(M)/\text{Spin}_{1,3}^+$ and $\psi_t(m)$ is a lift of $\sigma_t(m)$ to $\widetilde{\text{GL}}^+(M) \times_{\text{Spin}_{1,3}^+} \Sigma$. The $\text{Spin}_{1,3}^+$ -principal connection on $\widetilde{\text{GL}}^+(M) \rightarrow \widetilde{\text{GL}}^+(M)/\text{Spin}_{1,3}^+$ allows defining a covariant derivative of the field $\psi_t(m)$ in the direction $\partial_t \sigma_t(m)$ at $t = 0$. The covariant derivative is an element of $\widetilde{\text{GL}}^+(M) \times_{\text{Spin}_{1,3}^+} \Sigma|_{\sigma(m)}$, hence an element of the spinor bundle associated to σ at the point m .

Constructed over the whole manifold M , these sections form the Lie derivative of ψ along X .

4.2.4 Pin structures in Lorentzian signature

When working with field theory, it is often required to have an action of spatial parity and time reversal. In particular, this implies working with the full bundle of orthonormal frames which are not necessarily compatible with a time or a space orientation. Regarding spinor fields, this requires working with twofold covers not of the proper orthochronous orthogonal group $\text{SO}_{p,q}^+$, but twofold covers of the whole orthogonal group $\text{O}_{p,q}$ (or possibly of an intermediate group, depending on the considered orientation structure).

In order to simplify the discussion, we will only discuss the case of Lorentzian signature $(1, 3)$ in this section and will refer to $\mathbb{R}^{1,3}$ as “spacetime”. General references for this Section are [Tra05; Jan20].

The full orthogonal group

Let us first discuss about the whole orthogonal group $\text{O}_{1,3}$. It is the total isometry group of an inner product of signature $(1, 3)$ on \mathbb{R}^4 . Since all (nonzero) proportional metrics have the same isometry group, it is also the isometry group of an inner product of signature $(3, 1)$:

$$\text{O}_{1,3} \xrightarrow{\sim} \text{O}_{3,1}$$

The isometry group has four connected components. Its neutral component is $\text{SO}_{1,3}^+$. In particular, this gives a group isomorphism (compatible with the natural action on \mathbb{R}^4):

$$\text{SO}_{1,3}^+ \xrightarrow{\sim} \text{SO}_{3,1}^+$$

The isometry group fits into the following short exact sequence [Ser13; MT86]:

$$\text{SO}_{1,3}^+ \hookrightarrow \text{O}_{1,3} \twoheadrightarrow \pi_0(\text{O}_{1,3}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

For any decomposition of $\mathbb{R}^{1,3}$ into timelike line \mathbb{R} and spacelike 3-plane \mathbb{R}^3 , elements of $\text{O}_{1,3}$ sent in $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the sign (\pm) of the determinants of their diagonal blocks $\text{End}(\mathbb{R}^1)$ and $\text{End}(\mathbb{R}^3)$ (this is independent of the chosen splitting of $\mathbb{R}^{1,3}$) [MT86].

Furthermore, composing with the product $\{\pm 1\} \times \{\pm 1\} \rightarrow \{\pm 1\}$ gives the spacetime determinant of elements of $\text{O}_{1,3}$.

In addition, the short exact sequence splits as follows, with T the time reversal and P the

space parity operator associated to the $3 + 1$ decomposition $\mathbb{R}^{1,3} = \mathbb{R}^1 \oplus_{\perp} \mathbb{R}^3$:

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_2 &\rightarrow O_{1,3} \\ (a \bmod 2, b \bmod 2) &\mapsto T^a P^b \end{aligned}$$

As a consequence, $O_{1,3}$ has the structure of a semi-direct product

$$O_{1,3} \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes SO_{1,3}^+$$

Orientation structures

The set of space and time-orientations on $\mathbb{R}^{1,3}$ can be identified as the following homogeneous space:

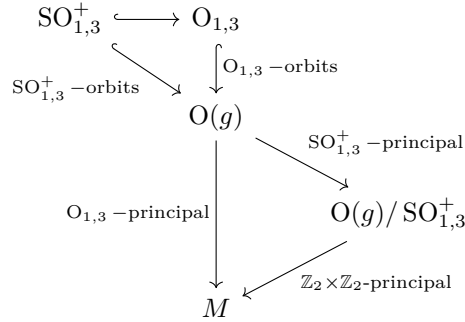
$$O_{1,3} / SO_{1,3}^+$$

on which the action of $O_{1,3}$ factors through the quotient $O_{1,3} / SO_{1,3}^+ \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and gives the set of orientations a structure of free homogeneous $\mathbb{Z}_2 \times \mathbb{Z}_2$ -space.

Let M be a 4-manifold equipped with a metric g of signature $(1, 3)$. It defines a frame bundle $O(g)$ of orthonormal frames, which is a principal $O_{1,3}$ -bundle. The bundle of space and time-orientations on M can be constructed as the quotient bundle

$$O(g) / SO_{1,3}^+$$

It is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -principal bundle over M . In particular, it is a fourfold covering of M , and fits into the following diagram (with a similar structure to that of correspondence spaces, see Section 3.3.6):



A space and time-orientation of M is equivalent to a global section of $O(g) / SO_{1,3}^+ \rightarrow M$ (which is subject to topological obstruction). Alternatively, it is possible to work directly on the fourfold cover $O(g) / SO_{1,3}^+$ on which $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts: time reversal and space parity then take a more geometrical character. Constructions on M become $\mathbb{Z}_2 \times \mathbb{Z}_2$ -invariant (or equivariant) constructions but working on $O(g) / SO_{1,3}^+$ allows for non-invariant constructions as well.

Pin coverings

The group $\text{Spin}_{1,3}^+$ was constructed from $\text{Cl}_{1,3}$. It is a specific connected twofold cover of $\text{SO}_{1,3}^+$. In particular, the isomorphism $\text{SO}_{1,3}^+ \simeq \text{SO}_{3,1}^+$ induces an isomorphism

$$\text{Spin}_{1,3}^+ \simeq \text{Spin}_{3,1}^+$$

The group $\text{Pin}_{1,3}$ was constructed as well from $\text{Cl}_{1,3}$. It is a \mathbb{Z}_2 -extension of $\text{O}_{1,3}$. In fact, there are two *different* natural covering maps to $\text{O}_{1,3}$: the twisted adjoint action map and the adjoint action map. However, according to the remark following Theorem 4.1.26, the two covering maps are isomorphic in the following sense: there exists a group isomorphism $\beta : \text{Pin}_{1,3} \rightarrow \text{Pin}_{1,3}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & \text{Pin}_{1,3} & \\
 \nearrow & \uparrow & \searrow \widetilde{\text{Ad}} \\
 \mathbb{Z}_2 & & \text{O}_{1,3} \\
 \searrow & \downarrow \beta & \nearrow \text{Ad} \\
 & \text{Pin}_{1,3} &
 \end{array}$$

As a consequence, it is equivalent to consider any of the two covering maps $\text{Pin}_{1,3} \rightarrow \text{O}_{1,3}$.

The subtlety about Pin groups is the following fact: $\text{Pin}_{1,3}$ is *not isomorphic* to $\text{Pin}_{3,1}$. This means that $\text{Pin}_{1,3}$ and $\text{Pin}_{3,1}$ are two different \mathbb{Z}_2 -extensions of $\text{O}_{1,3}$ and there is room for the choice of \mathbb{Z}_2 -extension. This raises the question of possible \mathbb{Z}_2 -extensions of $\text{O}_{1,3}$.

There are 16 central \mathbb{Z}_2 -extensions of $\text{O}_{1,3}$ [Tra05], half of which restrict to the connected covering

$$\text{Spin}_{1,3}^+ \rightarrow \text{SO}_{1,3}^+$$

They can be classified as follows: according to the decomposition $\mathbb{R}^{1,3} = \mathbb{R}^1 \oplus_{\perp} \mathbb{R}^3$ there is a time reversal T and a space parity P in $\text{O}_{1,3}$ such that $\{1, P, T, PT\}$ each belongs to a different connected component of $\text{O}_{1,3}$. Let \tilde{P} and \tilde{T} be arbitrary lifts in the considered central \mathbb{Z}_2 -extension of $\text{O}_{1,3}$. Then the \mathbb{Z}_2 -extension is classified by the following three signs:

$$\tilde{P}\tilde{T} = \underbrace{\pm 1}_{\lambda} \tilde{T}\tilde{P} \quad \tilde{P}^2 = \underbrace{\pm 1}_{\mu}, \quad \tilde{T}^2 = \underbrace{\pm 1}_{\nu}$$

The elements $\{\pm 1, \pm \tilde{P}, \pm \tilde{T}, \pm \tilde{P}\tilde{T}\}$ form a subgroup $H_{\lambda\mu\nu}$ such that the \mathbb{Z}_2 -extension can be constructed as

$$(H \times \text{Spin}_{1,3}^+) / \{(1, 1), (-1, -1)\}$$

The extension is generally called $\text{Pin}_{\lambda\mu\nu}$ and fits into the following diagram of group extensions:

$$\begin{array}{ccccc}
 \text{Spin}_{1,3}^+ & \hookrightarrow & \text{Pin}_{\lambda\mu\nu} & \longrightarrow & H_{\lambda\mu\nu} \\
 \downarrow \text{Ad} & & \downarrow & & \downarrow \det_{\mathbb{R}^1} \times \det_{\mathbb{R}^3} \\
 \text{SO}_{1,3}^+ & \hookrightarrow & \text{O}_{p,q} & \longrightarrow & \mathbb{Z}_2 \times \mathbb{Z}_2
 \end{array}$$

The Pin groups constructed from the Clifford algebras, namely $\text{Pin}_{1,3}$ and $\text{Pin}_{3,1}$ are identified as follows:

$$\begin{aligned}
 \text{Pin}_{1,3} &\simeq \text{Pin}_{-++} \\
 \text{Pin}_{3,1} &\simeq \text{Pin}_{---}
 \end{aligned}$$

We now discuss how this ties in with the Spin structures introduced in Section 4.2.1.

Spin structures

We recall the definition of a Spin structure on an 4-manifold: it is the data of a $\widetilde{\text{GL}}^+$ -principal bundle $\text{GL}^+(M)$ with a $\widetilde{\text{GL}}_4^+ \rightarrow \text{GL}_4$ intertwining bundle map to the bundle of linear frames

$$\text{GL}^+(M) \rightarrow \text{GL}(M)$$

We explained how it induces for each choice of Lorentzian structure a $\text{Spin}_{1,3}^+$ -structure. The point of this notion of spin structure is that it is independent of the Lorentzian structure and allows the induced $\text{Spin}_{1,3}^+$ -structure to follow the variations of the Lorentzian structure.

Let us mention first that given an actual $\text{Spin}_{1,3}^+$ -structure, therefore a metric and a space and time-orientation beside the Spin structure, then for any $\text{Pin}_{\lambda\mu\nu}$ group it is possible to build a corresponding Pin-structure. Namely, if $P \rightarrow M$ is the $\text{Spin}_{1,3}^+$ -principal spinor frame bundle, one can consider

$$P \times_{\text{Spin}_{1,3}^+} \text{Pin}_{\lambda\mu\nu}$$

which defines a $\text{Pin}_{\lambda\mu\nu}$ -structure over M . In other words, when the manifold has enough orientation structure, the used Pin group can be freely chosen.

We wish to work without orientation structure on M . Therefore we need to work with a \mathbb{Z}_2 -central extension of the whole linear group GL_4 . However it is not unique: there are exactly two such central extensions which coincide with $\widetilde{\text{GL}}_4^+$ over GL_4^+ . They can be classified using the value of the square of any inverse image of an involution of negative determinant, namely the square of an inverse image in $\widetilde{\text{GL}}_4^+$ of a section of the following short exact sequence:

$$\text{GL}_4^+ \hookrightarrow \text{GL}_4 \xrightarrow{\det} \mathbb{Z}_2$$

Let us call \widetilde{T} an element of the central extension mapping to $T \in \text{O}_{1,3} \subset \text{GL}_4$ and -1 the nontrivial element mapped to Id_4 in GL_4 . Then

- If $\widetilde{T}^2 = 1$ then the central extension is isomorphic to

$$\mathbb{Z}_2 \times_{\text{Ad}(\widetilde{T})} \widetilde{\text{GL}}_4^+$$

The central extension is called Gin_4^+ .

- If $\widetilde{T}^2 = -1$ then the central extension is isomorphic to

$$\left(\mathbb{Z}/4\mathbb{Z} \times_{\text{Ad}(\widetilde{T})} \widetilde{\text{GL}}_4^+ \right) / \{(0 \bmod 4, 1), (2 \bmod 4, -1)\}$$

The central extension is called Gin_4^- .

Assume M is equipped with a Gin^+ or a Gin^- structure, namely a Gin_4^+ -principal (resp. Gin_4^- -principal) bundle $\text{Gin}^\pm(M)$ with a $\text{Gin}_4^\pm \rightarrow \text{GL}_4$ -intertwining bundle map to the linear frame bundle:

$$\text{Gin}^\pm(M) \rightarrow \text{GL}(M)$$

Applying the same construction as with spin structures, one defines a putative Pin-structure

as the following pullback:

$$\begin{array}{ccc} \text{Pin}(g) & \hookrightarrow & \text{Gin}^\pm(M) \\ \downarrow & & \downarrow \\ \text{O}(g) & \hookrightarrow & \text{GL}(M) \end{array}$$

They form principal bundles over M for the group Pin^\pm defined as the similar group-level pullback:

$$\begin{array}{ccc} \text{Pin}^\pm & \hookrightarrow & \text{Gin}_4^\pm \\ \downarrow & & \downarrow \\ \text{O}_{1,3} & \hookrightarrow & \text{GL}_4 \end{array}$$

The remaining question is to identify the groups Pin^\pm among the central \mathbb{Z}_2 -extensions of $\text{O}_{1,3}$. We know that in Pin^\pm , $\tilde{T}^2 = \pm 1$. The idea in order to compute the other signs is to use a “Wick rotation” thanks to the embedding

$$\{\pm 1, \pm PT\} \subset \text{O}(\mathbb{R}^1) \times \text{O}(\mathbb{R}^3) \cap \text{SL}_4 \subset \text{SO}(\mathbb{R}^4)$$

because we *know* that the \mathbb{Z}_2 -extensions restricted to SO_4 form a connected covering which has to be Spin_4 . In particular in Spin_4 we now that PT can be represented as a chirality element ω which squares to 1:

$$\left(\tilde{P}\tilde{T}\right)^2 = 1$$

Once we know the sign λ , this equation will allow us to obtain the last sign μ . Before this, notice that in both $\text{Cl}_{1,3}$ and $\text{Cl}_{3,1}$, the chirality element squares to -1 :

$$\left(\tilde{P}\tilde{T}\right)^2 = -1$$

which means that the Pin-structures induced by Gin^\pm structure do not correspond to $\text{Pin}_{1,3}$ or $\text{Pin}_{3,1}$ -structures. This justifies enlarging the spectrum of considered “Pin” structures beside the $\text{Pin}_{1,3}$ and the $\text{Pin}_{3,1}$ groups.

We now compute λ . It is computed using the following formula:

$$\tilde{T}\tilde{P} = \text{Ad}_{\tilde{T}} \tilde{P}\tilde{T}$$

Since $\tilde{P}\tilde{T}$ belongs to Spin_4 and $\text{Ad}_{\tilde{T}}|_{\text{Spin}_4}$ is the lift of time reversal $\text{Ad}_T : \text{SO}_4 \rightarrow \text{SO}_4$ which preserves the identity element of Spin_4 , it necessarily corresponds to the restriction to Spin_4 of the action of T on Cl_4 . As such it multiplies chirality elements by -1 hence

$$\tilde{T}\tilde{P} = -\tilde{P}\tilde{T}$$

It follows that in Gin^\pm

$$P^2 = \mp 1$$

Let us summarise the results:

Theorem 4.2.18. *The Pin groups constructed from the Clifford algebras $\text{Cl}_{1,3}$ and $\text{Cl}_{3,1}$ and the Pin^\pm groups constructed from the Gin_4^\pm groups belong to four different classes of isomorphism.*

They fit in the $\text{Pin}_{\lambda\mu\nu}$ classification as follows:

$$\begin{aligned}\text{Pin}_{1,3} &\simeq \text{Pin}_{++-} \\ \text{Pin}_{3,1} &\simeq \text{Pin}_{---} \\ \text{Pin}^+ &\simeq \text{Pin}_{+--+} \\ \text{Pin}^- &\simeq \text{Pin}_{-++}\end{aligned}$$

They are characterised among the $\text{Pin}_{\lambda\mu\nu}$ by the property that

$$(\tilde{P}\tilde{T})^2 = -1$$

In particular, the choice of a Pin group affects the nature of the spinor bundles.

Theorem 4.2.19 ([Tra05]). *The four identified Pin groups each admit an irreducible faithful real representation such that:*

- for $\text{Pin}_{1,3}$, the representation admits a quaternionic structure of dimension 2 commuting with $\text{Pin}_{1,3}$.
- for $\text{Pin}_{3,1}$, the representation is of real dimension 4.
- for Pin^\pm , the representation admits a complex structure of dimension 4 commuting with Pin^\pm .

Spin groups

In fact we shall prove that the disconnected groups $\text{Spin}_{1,3}$ and $\text{Spin}_{3,1}$ are isomorphic [Ber+01]: recall the isomorphism

$$\begin{aligned}\text{Cl}_{3,1}^+ &\xrightarrow{\sim} \text{Cl}_{1,3}^+ \\ e_{i_1} \dots e_{i_{2k}} &\mapsto (-1)^{\frac{2k(2k-1)}{2}} e_{i_{2k}} \dots e_{i_1} = (-1)^k e_{i_{2k}} \dots e_{i_1}\end{aligned}$$

with (e_i) a pseudo-orthonormal basis of $\mathbb{R}^{1,3}$, coming from $\text{Cl}_{1,3}^{\text{sopp}} \simeq \text{Cl}_{3,1}$. The (graded-anti-)isomorphism between the Clifford algebras preserves the subspace $\mathbb{R}^{1,3} \simeq \mathbb{R}^{3,1}$ as well as sends Ad_a to $\text{Ad}_{\pm\tau a} = \text{Ad}_{\alpha(a)^{-1}} = \text{Ad}_a^{-1}$. Therefore the isomorphism also preserves the Clifford group and its intersection with the even part of the algebra: we obtain a group isomorphism [Ber+01]

$$\text{Spin}_{3,1} \xrightarrow{\sim} \text{Spin}_{1,3}$$

The group $\text{Spin}_{1,3}$ fits into the following group extension

$$\text{Spin}_{1,3}^+ \hookrightarrow \text{Spin}_{1,3} \rightarrow \mathbb{Z}_2$$

Chirality elements ω belong to $\text{Spin}_{1,3} \setminus \text{Spin}_{1,3}^+$, square to -1 and are super-central, hence in the centre of $\text{Spin}_{1,3}$. Therefore $\text{Spin}_{1,3}$ can be constructed from $\text{Spin}_{1,3}^+$ and the subgroup $\mathbb{Z}/4\mathbb{Z}$ generated by a chirality element as follows:

$$\text{Spin}_{1,3} \simeq (\mathbb{Z}/4\mathbb{Z} \times \text{Spin}_{1,3}^+) / \{(0 \bmod 4, 1), (2 \bmod 4, -1)\}$$

Since $\text{Ad}_\omega = PT$, we proved the following: the pullback group

$$\begin{array}{ccc} \text{Spin}_{\lambda\mu\nu} & \hookrightarrow & \text{Pin}_{\lambda\mu\nu} \\ \downarrow & & \downarrow \\ \text{SO}_{1,3} & \hookrightarrow & \text{O}_{1,3} \end{array}$$

is isomorphic to $\text{Spin}_{1,3}$ if and only if $(\widetilde{PT})^2 = -1$. This is the case for the two groups $\text{Pin}_{1,3}$, $\text{Pin}_{3,1}$ but not for the groups Pin^+ and Pin^- .

Furthermore, the groups $\text{Pin}_{1,3}$ and $\text{Pin}_{3,1}$ are also easily decomposed as semi-direct products using $\text{Spin}_{1,3}$ [Ber+01]:

$$\begin{aligned} \text{Pin}_{1,3} &\simeq \mathbb{Z}_2 \rtimes_T \text{Spin}_{1,3} \\ \text{Pin}_{3,1} &\simeq \mathbb{Z}_2 \rtimes_{\text{Ad}(e_1)} \text{Spin}_{1,3} \end{aligned}$$

with e_1 any vector in $\mathbb{R}^{3,1}$ which has norm 1.

In conclusion, in Lorentzian signature, there are several possibilities for “pinors”, that is fields taking value in vector bundles associated to Pin-structures. The covariant approach to Spin-structures is only compatible with two of them which are notably not the Pin groups constructed from the Clifford algebras.

4.3 Sign conventions

In this section, we discuss briefly the different coexisting sign conventions for Clifford algebra and spinors in Lorentzian signature. There are mostly related and incoherent changes in the conventions can lead to a change in algebraic properties.

Clifford algebra conventions

We consider the vector space V equipped with the nondegenerate quadratic form q and the associated bilinear product $\langle \cdot | \cdot \rangle$.

There are two conventions for the Clifford algebra of (V, q) : imposing either the equation

$$uv + vu = 2 \langle u|v \rangle 1$$

like we did or the opposite convention

$$uv + vu = -2 \langle u|v \rangle 1$$

which is used for example in [LM89; Bou+15]. Obviously, changing the convention is equivalent to replacing q by $-q$. In particular according to Theorem 4.1.10 there is an isomorphism $\text{Cl}(-V) \simeq \text{Cl}(V)^{\text{opp}}$ which can be used to relate the two conventions; especially, the *even* subalgebras are isomorphic as filtered algebra.

Changing conventions implies changing signs in many formulas. Recall that for two orthogonal pseudo-normed vectors (e_1, e_2) in V , the infinitesimal adjoint action of $\frac{1}{2}e_1e_2$

$$\text{ad} \left(\frac{1}{2}e_1e_2 \right) : a \in \text{Cl}(V) \mapsto \frac{e_1e_2a - ae_1e_2}{2}$$

corresponds to the endomorphism

$$e_1 \otimes {}^t e_2 - e_2 \otimes {}^t e_1 \in \mathfrak{so}(V)$$

As a consequence changing the metric sign changes ${}^t e_i$ by a sign and changes the image in $\mathfrak{so}(V)$ of $\frac{1}{2}e_1 e_2$ by a sign as well. Therefore the identification $\mathrm{Cl}(V)^{\mathrm{soPP}} \simeq \mathrm{Cl}(-V)$ restricted to the spin Lie algebras fits into the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{spin}_{1,3} & \longrightarrow & \mathfrak{spin}_{3,1} \\ \downarrow \mathrm{ad} & & \downarrow \mathrm{ad} \\ \mathfrak{so}_{1,3} & \xrightarrow{A \mapsto -A} & \mathfrak{so}_{3,1} \end{array}$$

with the bottom morphism reversing the Lie algebra bracket.

In fact the graded antiautomorphism between the Clifford algebras, restricted to the spin groups, fits into the following commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}_{1,3} & \longrightarrow & \mathrm{Spin}_{3,1} \\ \downarrow \mathrm{Ad} & & \downarrow \mathrm{Ad} \\ \mathrm{SO}_{1,3} & \xrightarrow{g \mapsto g^{-1}} & \mathrm{SO}_{3,1} \end{array}$$

with the bottom morphism corresponding to the inversion of elements in $\mathrm{GL}(\mathbb{R}^4)$.

When working over \mathbb{C} , a global multiplication by i (akin to a Wick rotation) gives an isometry between q and $-q = i^2 q$ so that it defines a Clifford algebra isomorphism $\mathrm{Cl}(V) \xrightarrow{\sim} \mathrm{Cl}(-V)$.

Lorentzian signature

The phrase ‘‘Lorentzian signature’’ is often used to refer to two different signatures: $(+---)$ (the ‘‘mostly minus’’ convention) [Del+00; IZ80] and $(-+++)$ (the ‘‘mostly plus’’ convention) [Car19b; MTW73; Wei95]. The former tends to be found in the literature of high-energy physics and the latter in the literature on general relativity.

Changing the signature amounts to changing the sign of the considered metric and is therefore subject to the same considerations. It may change the structure of the real Clifford algebra but does not change the structure of complex Clifford algebras.

Notice that in the two cases the even parts of the Clifford algebra are isomorphic:

$$\begin{aligned} \mathrm{Cl}_{1,3}^+ &\simeq \mathrm{Cl}_{3,0} \\ \mathrm{Cl}_{3,1}^+ &\simeq \mathrm{Cl}_{3,0} \end{aligned}$$

with the isomorphism depending on the choice of a chirality element. In particular, there are the ‘‘same’’ *spinorial modules* in the two different signatures.

On the other hand, $\mathrm{Pin}_{p,q}$ groups do depend on the used signature. However as discussed earlier the choice of a Pin group is a consideration which has to be made independently of signature choice.

Dirac operator

There are two (formally) different differential operators which are called Dirac operator. Using a local orthonormal frame (e_i) , the first acts on sections σ of real Clifford modules:

$$\not{\nabla} = e_j \nabla_{e_j} \sigma$$

The second acts on sections σ of complex Clifford modules (more generally of Spin^c modules):

$$i e_j \nabla_{e_j} \sigma$$

with i the distinguished square root of -1 .

The real convention allows working with real Clifford modules, in which the so-called ‘‘Majorana spinors’’ take value. When working with complex Clifford modules, adding an i factor ∇ is equivalent to multiplying the real vector space by i :

$$iV^{\mathbb{R}} \subset V^{\mathbb{C}} \simeq \mathbb{C} \otimes_{\mathbb{R}} V^{\mathbb{R}}$$

In other words, it can be interpreted as performing a Wick rotation $\text{Cl}(V^{\mathbb{R}}) \simeq \text{Cl}(-V^{\mathbb{R}})$ before constructing the Dirac operator.

The factor of i changes the sign of the square of the Dirac operator:

$$(i\not{\nabla})^2 = -\not{\nabla}^2$$

It also affects the formal selfadjointness of the Dirac operator, which is required in order to obtain the masses as real eigenvalues of the Dirac operator. Since given a vector X the operator ∇_X is formally anti-selfadjoint, the metric on the spinor bundle $P[\Sigma]$ has to make the action on $P[\Sigma]$ of X anti-selfadjoint in order for $\not{\nabla}$ to be formally selfadjoint.

On the other hand, if vectors act on $P[\Sigma]$ by selfadjoint operators, $\not{\nabla}$ is formally anti-selfadjoint and $i\not{\nabla}$ becomes formally selfadjoint. Both possibilities are consistent with $\mathfrak{so}(TM)$ acting with infinitesimal isometries of the metric. Here again the two possibilities can be related by the global Wick rotation: it changes the action of real vectors from selfadjoint to anti-selfadjoint and conversely. It is also possible to work with a non-(formally) self-adjoint Dirac operator, in which case one will often consider the symmetric square

$$\not{\nabla}^* \not{\nabla}$$

instead of the usual square $\not{\nabla}^2$.

Finally, when working in Riemannian signature it is often convenient to have $\not{\nabla}^2$ being a positive operator. In this case, signature is usually taken as $(n, 0)$ and with the convention $uv + vu = 2\langle u|v\rangle$ the suitable Dirac operator has no i factor while it does with the opposite convention for the Clifford algebra.

Chapter 5

A few formulas in Riemann-Cartan Geometry

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Riemann-Cartan geometry is the geometry of manifolds equipped with a metric and a metric connection. While (pseudo-)Riemannian geometry deals with manifold equipped with only a metric, which specifies uniquely a torsionless connection called the Levi-Civita connection, Riemann-Cartan geometry allows for the extra degree of freedom that is torsion. As a consequence, many equations of Riemannian geometry acquire a torsion contribution. In this section, we present a few of these formulas which we will need in later parts.

Since standard differential geometry cursus often involves Riemannian geometry but rarely get students used to deal with connections presenting torsion, one has to be very careful when using differential geometry formulas and make sure they do not rely on a torsionlessness assumption. These formulas include

$$\text{Ric}_{\mu\nu} = \text{Ric}_{\nu\mu}$$

and

$$d(i_X \text{vol}) = \text{tr}(\nabla X) \text{vol}$$

for X a vector field and vol the riemannian volume. The latter reveals an ambiguity¹ in the definition of covariant divergence in presence of torsion and is often involved in integrations by parts.

Throughout this section we will consider an oriented n -manifold M equipped with a metric g of signature (p, q) and a metric connection ∇ . Orientation defines a Riemannian volume vol and g defines a bundle of proper (pseudo-)orthonormal frames $\text{SO}^+(M) \rightarrow M$, which is equipped with a soldering form $\alpha \in \Omega^1(\text{SO}^+(M), \mathbb{R}^{p,q})^{\text{SO}^+_{p,q}}$ and connection 1-form $\omega \in \Omega^1(\text{SO}^+(M), \mathfrak{so}_{p,q})$. Calculations in this section will not use the Cartan geometry point of view therefore “curvature” will mean curvature of the principal connection 1-form ω and torsion will be considered as a distinct object.

¹Arguably the more geometric definition is the one relying on the volume form and the exterior differential.

The Ricci curvature

The Ricci curvature is a component of the curvature of the connection which can be defined for any linear tangent connection – it is easy in our definition below to replace $\mathrm{SO}_{p,q}^+$ by another Lie group acting on \mathbb{R}^n . On the frame bundle, it is represented by the Ricci curvature form which is a tensorial covector-valued 1-form:

$$\mathrm{Ric} \in \Omega^1(\mathrm{SO}^+(M), \mathbb{R}^{p,q*})$$

It is obtained from Ω by representing $\mathfrak{so}_{p,q}$ in $\mathrm{End}(\mathbb{R}^{p,q})$ then taking the trace *with respect to the first 2-form index*, using the connection to identify $\mathbb{R}^{p,q}$ with the horizontal space.

Let us write

$$\rho : \mathfrak{so}_{p,q} \rightarrow \mathrm{End}(\mathbb{R}^{p,q})$$

the natural representation of $\mathfrak{so}_{p,q}$ and

$$u : \mathbb{R}^{p,q} \rightarrow \Gamma(T\mathrm{SO}^+(M))$$

the horizontal vector fields such that at each point of $\mathrm{SO}^+(M)$ (frames of $T_x M$ for $x \in M$), u_a is the horizontal lift to $T\mathrm{SO}^+(M)$ of the vector in $T_x M$ corresponding to $e_a \in \mathbb{R}^{p,q}$. In other words,

$$\langle \alpha | u \rangle : \mathbb{R}^{p,q} \rightarrow \Gamma(\mathrm{SO}^+(M), \mathbb{R}^{p,q})$$

is the natural embedding into constant sections.

Then the Ricci curvature form can be expressed by the following formula, using I, J for coordinates of $\mathrm{SO}^+(M)$, i for indices in $\mathfrak{so}_{p,q}$ and a, b for indices in $\mathbb{R}^{p,q}$:

$$\mathrm{Ric}_{J,b} = \Omega_{IJ}^i \rho_{i,b}^a u_a^I \quad (5.1)$$

As a tensorial $\mathbb{R}^{p,q*}$ -valued 1-form on $\mathrm{SO}^+(M)$, it is associated to an (a priori non-symmetric) bilinear form on M .

5.1 The Bianchi identity

The curvature obeys the so-called (*algebraic, or first*) *Bianchi identity*:

$$d\Theta + \omega \cdot \Theta = \Omega \cdot \alpha \quad (5.2)$$

We are interested in its consequences for the Ricci curvature, so we compute the contraction

$$\begin{aligned} (\Omega \cdot \alpha)_{IJK}^a u_a^I &= (\Omega^i \rho_{ib}^a \wedge \alpha^b)_{IJK} u_a^I \\ &= u_a^I (\Omega_{IJ}^i \rho_{ib}^a \wedge \alpha_K^b + \Omega_{JK}^i \rho_{ib}^a \wedge \alpha_I^b + \Omega_{KI}^i \rho_{ib}^a \wedge \alpha_J^b) \\ &= \mathrm{Ric}_{J,b} \wedge \alpha_K^b + \Omega_{JK}^i \rho_{ia}^a - \mathrm{Ric}_{K,b} \wedge \alpha_J^b \\ (\Omega \cdot \alpha)_{IJK}^a u_a^I &= \mathrm{Ric}_{J,b} \wedge \alpha_K^b - \mathrm{Ric}_{K,b} \wedge \alpha_J^b \end{aligned} \quad (5.3)$$

in which $\Omega^i \rho_{ia}^a$ vanished because $\mathfrak{so}_{p,q}$ acts by traceless endomorphisms; this term corresponds to the action of Ω on the determinant line bundle of M . Applying the same contraction to the left side of (5.2) we obtain

$$(d\Theta + \omega \cdot \Theta)_{IJK}^a u_a^I = \mathrm{Ric}_{J,b} \wedge \alpha_K^b - \mathrm{Ric}_{K,b} \wedge \alpha_J^b \quad (5.4)$$

which can be read as: the antisymmetric part of the Ricci curvature is equal to the covariant exterior divergence of the torsion.

We want to express the left hand term with a covariant divergence, working in the vector bundle trivializing *vector-valued 2-forms*. Let us state the final formula beforehand. We write it over the base manifold, using $\text{tr}(T)_\mu = T^\nu{}_{\nu\mu}$ and $\text{div}^\nabla T_{\mu\nu} = \nabla_\rho T^\rho{}_{\mu\nu}$:

$$\text{Ric}_{\mu\nu} - \text{Ric}_{\nu\mu} = \text{div}^\nabla(T)_{\mu\nu} - d \text{tr}(T) \quad (5.5)$$

Preliminary definitions

We need to introduce a somewhat cumbersome notation in order to differentiate between *horizontal forms* and $\Lambda^\bullet \mathbb{R}^{p,q^*}$ -valued forms on the frame bundle. This is because although they are identified through the solder form, the covariant exterior differential acts differently on each of them. This distinction is harder to keep track of when working on the base manifold, but working on the frame bundle makes it clearer.

We now define the morphism relating forms of both kind. Let E be an arbitrary (finite-dimensional) representation of $\mathfrak{so}_{p,q}$. Let σ be an equivariant horizontal k -form with values in E . We define $\hat{\sigma}$ the equivariant section of $\Lambda^k \mathbb{R}^{p,q^*} \otimes E$ defined by the isomorphism that the connection establishes between the horizontal distribution and $\mathbb{R}^{p,q}$:

$$\hat{\sigma}_{ab\dots k}^A := \sigma_{I_1 I_2 \dots I_k}^A u_a^{I_1} u_b^{I_2} \dots u_k^{I_k} \quad (5.6)$$

We introduce two avatars of the covariant differential: the first one is the standard covariant exterior differential

$$d^\omega A = dA + \omega \cdot A$$

for A an equivariant horizontal differential form with values in a representation E of $\mathfrak{so}_{p,q}$. For example the definition of the torsion can be written as:

$$\Theta = d^\omega \alpha \quad (5.7)$$

The second one is an antisymmetrised covariant derivative. We write it $d^\omega \wedge$ and *only applies to equivariant sections of representations* $\Lambda^k \mathbb{R}^{p,q^*} \otimes E$ of $\mathfrak{so}_{p,q}$ (0-forms) with E a representation of $\mathfrak{so}_{p,q}$. Let $\hat{\sigma}$ be an equivariant section of $\Lambda^k \mathbb{R}^{p,q^*} \otimes E$ (corresponding to an equivariant section $\sigma \in \Omega^k(\text{SO}^+(M), E)$). We consider

$$d^\omega \hat{\sigma} \in \Omega^1(\text{SO}^+(M), \Lambda^k \mathbb{R}^{p,q^*} \otimes E) \xrightarrow[\rightarrow]{\sim} \Gamma(\text{SO}^+(M), \mathbb{R}^{p,q^*} \otimes \Lambda^k \mathbb{R}^{p,q^*} \otimes E)$$

and compose with the antisymmetrisation $\mathbb{R}^{p,q^*} \otimes \Lambda^k \mathbb{R}^{p,q^*} \rightarrow \Lambda^{k+1} \mathbb{R}^{p,q^*}$ to obtain

$$d^\omega \wedge : \Gamma(\text{SO}^+(M), \Lambda^k \mathbb{R}^{p,q^*} \otimes E) \rightarrow \Gamma(\text{SO}^+(M), \Lambda^{k+1} \mathbb{R}^{p,q^*} \otimes E) \quad (5.8)$$

We denote $\hat{\wedge}$ for the product in the exterior algebra $\Lambda^* \mathbb{R}^{p,q^*}$, to distinguish it from the wedge product \wedge of $\Omega^*(\text{SO}^+(M))$. We also introduce the following *trace* of vector-valued horizontal k -forms:

$$\text{tr}(\sigma)_{JK\dots} := u_a^I \sigma_{IJK\dots}^a \in \Omega^{k-1}(\text{SO}^+(M)) \quad (5.9)$$

and will also use the natural trace on $\Lambda^k \mathbb{R}^{p,q^*} \otimes \mathbb{R}^{p,q}$. The contracted Bianchi identity (5.4) takes the form

$$\text{tr}(\widehat{d^\omega \Theta})_{ab} = \widehat{\text{Ric}}_{ab} - \widehat{\text{Ric}}_{ba} \quad (5.10)$$

The last definition we need is that of the contraction of an E -valued k -form with the $\mathbb{R}^{p,q}$ -valued

2-form Θ . Let $A \in \Omega^k(\mathrm{SO}^+(M), E)$. We define the following equivariant E -valued $(k+1)$ -form

$$(\Theta \lrcorner A)_{I_0 \dots I_k} := \sum_{0 \leq i < j \leq k} (-1)^{i+j} \Theta_{I_i, I_j}^b u_b^J A_{J, I_0, \dots, I_i, \dots, I_j, \dots, I_k}$$

Alternatively, the contraction can be defined using the trace:

$$\widehat{\Theta} \lrcorner \widehat{A} = \mathrm{tr} \left(\widehat{\Theta} \widehat{\wedge} \widehat{A} \right) - \mathrm{tr}(\widehat{\Theta}) \widehat{\wedge} \widehat{A}$$

The contracted Bianchi identity

We want to re-express $\mathrm{tr} \circ d^\omega$ in

$$\mathrm{tr} (d^\omega \Theta)_{JK} = u_a^I (d\Theta_{IJK} + \omega \cdot \Theta) \quad (5.11)$$

We will need the following formula

Lemma 5.1.1. Let A be a E -valued equivariant horizontal k -form. Then

$$\widehat{d^\omega A} = d^\omega \wedge \widehat{A} + \widehat{\Theta} \lrcorner \widehat{A}$$

The proof simply uses the fact that $u_a \lrcorner \omega = 0$ and one computes the components of the exterior differential:

$$\begin{aligned} d^\omega A(u_{a_0}, u_{a_1} \dots u_{a_k}) &= dA(u_{a_0}, u_{a_1} \dots u_{a_k}) \\ &= \sum_i (-1)^i u_{a_i} (A(u_{a_0} \dots \widehat{u}_{a_i} \dots u_{a_k})) \\ &\quad + \sum_{i < j} (-1)^{i+j} A([u_{a_i}, u_{a_j}], u_{a_0} \dots \widehat{u}_{a_i} \dots \widehat{u}_{a_j} \dots u_{a_k}) \\ &= \sum_i (-1)^i u_{a_i} (\widehat{A}_{a_0 \dots \widehat{a_i} \dots a_k}) \\ &\quad + \sum_{i < j} (-1)^{i+j} A(u_b \Theta^b(u_{a_i}, u_{a_j}), u_{a_0} \dots \widehat{u}_{a_i} \dots \widehat{u}_{a_j} \dots u_{a_k}) \\ &= \sum_i (-1)^i d^\omega \widehat{A}_{a_0 \dots \widehat{a_i} \dots a_k}(u_{a_i}) + \sum_{i < j} (-1)^{i+j} \Theta^b(u_{a_i}, u_{a_j}) \widehat{A}_{b, a_0 \dots \widehat{a_i} \dots \widehat{a_j} \dots a_k} \end{aligned}$$

which we write as

$$\widehat{d^\omega A} = d^\omega \wedge \widehat{A} + \widehat{\Theta} \lrcorner \widehat{A} \quad (5.12)$$

We can now rewrite (5.10):

$$\mathrm{tr}(d^\omega \wedge \widehat{\Theta} + \widehat{\Theta} \lrcorner \widehat{\Theta})_{ab} = \widehat{\mathrm{Ric}}_{ab} - \widehat{\mathrm{Ric}}_{ba} \quad (5.13)$$

We extract a divergence term with the help of the following lemma:

Lemma 5.1.2. Let \widehat{A} be an equivariant section of $\mathbb{R}^{p,q} \otimes \wedge^k \mathbb{R}^{p,q*}$. Then $\mathrm{tr} (d^\omega \wedge \widehat{A})$ decomposes as follows

$$\mathrm{tr} (d^\omega \wedge \widehat{A}) = \mathrm{tr} d^\omega \widehat{A} - d^\omega \wedge \mathrm{tr} \widehat{A}$$

It follows from decomposing the indices over which the trace is taken:

$$\begin{aligned}\mathrm{tr}\left(d^\omega \wedge \hat{A}\right)_{a_1 \dots a_k} &= \delta_b^{a_0} \left(\sum_i (-1)^i d^\omega \hat{A}_{a_0 \dots \hat{a}_i \dots a_k}^b(u_{a_i}) \right) \\ \mathrm{tr}\left(d^\omega \wedge \hat{A}\right)_{a_1 \dots a_k} &= d^\omega \hat{A}_{a_1 \dots \hat{a}_i a_k}^b(u_b) - \sum_i (-1)^{i-1} d^\omega \hat{A}_{ba_1 \dots \hat{a}_i \dots a_k}^b(u_{a_i})\end{aligned}\tag{5.14}$$

Using Lemma 5.1.2 then once again Lemma 5.1.1, we perform the following computation:

$$\begin{aligned}\mathrm{tr}(d^\omega \wedge \hat{\Theta}) &= \mathrm{tr} d^\omega \hat{\Theta} - d^\omega \wedge \mathrm{tr} \hat{\Theta} \\ &= \mathrm{tr} d^\omega \hat{\Theta} - \left(\widehat{d^\omega \mathrm{tr} \Theta} - \hat{\Theta} \lrcorner \mathrm{tr} \Theta \right)\end{aligned}$$

but $\mathrm{tr} \Theta$ is simply an (equivariant) 1-form so that

$$d^\omega \mathrm{tr} \Theta = d \mathrm{tr} \Theta$$

and Equation (5.10) now takes the form:

$$\left(\mathrm{tr}(d^\omega \hat{\Theta}) - \widehat{d \mathrm{tr} \Theta} + \mathrm{tr}(\hat{\Theta} \lrcorner \Theta) + \hat{\Theta} \lrcorner \mathrm{tr} \Theta \right)_{ab} = \widehat{\mathrm{Ric}}_{ab} - \widehat{\mathrm{Ric}}_{ba}\tag{5.15}$$

with $\mathrm{tr} d^\omega \hat{\Theta}$ corresponding to the usual covariant divergence². Finally we prove that the quadratic term in Θ vanishes:

Lemma 5.1.3.

$$\mathrm{tr}(\hat{\Theta} \lrcorner \hat{\Theta}) + \hat{\Theta} \lrcorner \mathrm{tr} \Theta = 0$$

The computation is straightforward:

$$\begin{aligned}\mathrm{tr}(\hat{\Theta} \lrcorner \hat{\Theta})_{ab} &= \delta_e^c \left(\hat{\Theta}_{dc}^e \hat{\Theta}_{ab}^d + \hat{\Theta}_{da}^e \hat{\Theta}_{bc}^d + \hat{\Theta}_{db}^e \hat{\Theta}_{ca}^d \right) \\ &= \hat{\Theta}_{dc}^c \hat{\Theta}_{ab}^d + \hat{\Theta}_{da}^c \hat{\Theta}_{bc}^d + \hat{\Theta}_{db}^c \hat{\Theta}_{ca}^d \\ &= -\mathrm{tr}(\hat{\Theta})_d \hat{\Theta}_{ab}^d + \hat{\Theta}_{da}^c \hat{\Theta}_{bc}^d - \hat{\Theta}_{ca}^d \hat{\Theta}_{bd}^c \\ &= -\widehat{\Theta} \lrcorner \mathrm{tr} \hat{\Theta} + 0\end{aligned}$$

Our final rewriting of (5.10) is

$$\boxed{\mathrm{tr}(d^\omega \hat{\Theta})_{ab} - d(\mathrm{tr} \hat{\Theta})_{ab} = \widehat{\mathrm{Ric}}_{ab} - \widehat{\mathrm{Ric}}_{ba}}\tag{5.16}$$

which corresponds on M to Equation (5.5).

5.2 Variation of the Ricci curvature

We want to compare the Ricci curvature of two connections. Let ω and $\omega + \tau$ be two connection 1-forms: $\tau \in \Omega_{\mathrm{hor}}^1(\mathrm{SO}^+(M), \mathfrak{so}_{p,q})^{\mathrm{SO}^+_{p,q}}$. We denote their respective curvature 2-forms Ω and Ω^τ .

²It corresponds to the quantity $\nabla_\mu T^\mu{}_{\nu\rho}$ on M but it can be argued that covariant divergence should have an additional term when ∇ has torsion.

They are related by the following equation

$$\Omega^\tau = d\omega + d\tau + \frac{1}{2}[\omega \wedge \omega] + [\omega \wedge \tau] + \frac{1}{2}[\tau \wedge \tau] = \Omega + d^\omega \tau + \frac{1}{2}[\tau \wedge \tau] \quad (5.17)$$

Using the representation embedding: $\mathfrak{so}_{p,q} \hookrightarrow \mathbb{R}^{p,q} \otimes \mathbb{R}^{p,q*}$, we can take the trace and we obtain

$$\text{Ric}(\omega + \tau)_{J,b} = \rho_{i,b}^a u_a^I \left(\Omega + d^\omega \tau + \frac{1}{2}[\tau \wedge \tau] \right)_{I,J}^i = \text{Ric}(\omega)_{J,b} + \text{tr} \left(d^\omega \tau + \frac{1}{2}[\tau \wedge \tau] \right)_{J,b}$$

or in terms of $\mathbb{R}^{p,q*} \otimes \mathbb{R}^{p,q}$ -valued field on the frame bundle

$$\widehat{\text{Ric}}(\omega + \tau) = \widehat{\text{Ric}}(\omega) + \text{tr} \left(\widehat{d^\omega \tau} + \frac{1}{2}[\widehat{\tau \wedge \tau}] \right) \quad (5.18)$$

Using the lemmas, it is also possible to reformulate it in a similar way to Equation (5.16):

$$\widehat{\text{Ric}}(\omega + \tau) = \widehat{\text{Ric}}(\omega) + \text{tr} d^\omega \widehat{\tau} - d^\omega \widehat{\text{tr} \tau} + \text{tr} \left(\widehat{\Theta^\omega \lrcorner \tau} + \frac{1}{2}[\widehat{\tau \wedge \tau}] \right) \quad (5.19)$$

5.3 The torsion contribution to the Dirac operator

We want to compare the Dirac operator of a connection with the Dirac operator associated to the Levi-Civita connection. For this reason, we have to change the setting from a SO^+ -principal frame bundle to a Spin^+ -principal frame bundle.

The Dirac operator on the spin frame bundle

In this section, the spin frame bundle is equipped with a principal connection ω . We consider a Clifford module Σ and $\psi : \text{Spin}^+(M) \rightarrow \Sigma$ an equivariant map. We write

$$\gamma : \mathbb{R}^{p,q*} \rightarrow \text{End}(\Sigma)$$

the “gamma matrices”. The Dirac operator applied to ψ can be identified with

$$d^\omega \psi := \gamma^a u_a \lrcorner d^\omega \psi = \gamma^a \mathcal{L}_{u_a} \psi$$

The torsion contribution

Let us call ω_{LC} the connection 1-form of the Levi-Civita connection. There is an equivariant horizontal $\mathfrak{spin}_{p,q}$ -valued 1-form κ such that

$$\omega_{LC} = \omega + \kappa$$

The form κ is called *contorsion*. It will be useful to decompose it as a horizontal 1-form with values in $\mathbb{R}^{p,q} \otimes \mathbb{R}^{p,q*}$:

$$\kappa^a_b = \kappa^a_{cb} \alpha^c$$

Since the Levi-Civita connection is by definition torsion-free, the torsion of ω is readily computed as

$$\begin{aligned}\Theta^a &= -[\kappa \wedge \alpha]^a \\ &= -\kappa^a_b \wedge \alpha^b \\ &= -\kappa^a_{cb} \alpha^c \wedge \alpha^b \\ &= \frac{1}{2} (\kappa^a_{bc} - \kappa^a_{cb}) \alpha^b \wedge \alpha^c\end{aligned}$$

Thus

$$\Theta^a_{bc} = \kappa^a_{bc} - \kappa^a_{cb}$$

This relation is readily reversed to obtain, using the metric η^{ab} to freely lower indices [Sha02; Ham02]:

$$\kappa_{abc} = \frac{1}{2} (\Theta_{abc} - \Theta_{cab} + \Theta_{bca})$$

Note that the vectors u_a associated to ω are no longer horizontal for the Levi-Civita connection but they satisfy

$$\alpha(u_a) = e_a$$

which is independent of the connection 1-form. Recall the embedding from Section 4.1.3:

$$(x^a_b) \in \mathfrak{so}_{p,q} \mapsto \frac{1}{4} x^a_b [\gamma_a, \gamma^b] = \frac{1}{2} x^a_b \gamma_a \gamma^b \in \mathfrak{spin}_{p,q} \subset \text{Cl}_{p,q}$$

We can thus compute the Dirac operator applied to ψ as follows:

$$\begin{aligned}\mathfrak{d}^\omega \psi &= \gamma^a u_a \lrcorner \mathfrak{d}^\omega \psi \\ &= \gamma^a u_a \lrcorner (\mathfrak{d}_{LC}^\omega - \kappa \cdot) \psi \\ &= \mathfrak{d}_{LC}^\omega \psi - \gamma^a \frac{1}{2} \kappa^b_{ac} \frac{1}{2} \gamma_b \gamma^c \psi\end{aligned}$$

We are interested in computing the extra term

$$-\gamma^a \frac{1}{2} \kappa^b_{ac} \frac{1}{2} \gamma_b \gamma^c = -\frac{1}{4} \kappa_{bac} \gamma^a \gamma^b \gamma^c$$

We proceed to the following decomposition of the components

$$\begin{aligned}2\kappa_{bac} &= \Theta_{bac} - \Theta_{cba} + \Theta_{acb} \\ &= \frac{1}{3} (\Theta_{bac} + \Theta_{cba} + \Theta_{acb}) + \frac{4}{3} (\Theta_{bac} + \Theta_{cab}) - \frac{2}{3} (\Theta_{bac} + \Theta_{abc}) \\ &= \frac{1}{6} (\Theta_{bac} + \Theta_{cba} + \Theta_{acb} - \Theta_{bca} - \Theta_{cab} - \Theta_{abc}) + \frac{4}{3} (\Theta_{bac} + \Theta_{cab}) - \frac{2}{3} (\Theta_{bac} + \Theta_{abc}) \\ &= \Theta_{[bac]} - \frac{8}{3} \Theta_{(bc)a} - \frac{4}{3} \Theta_{(ab)c}\end{aligned}$$

where the brackets $[\cdot]$ stand for the normalised antisymmetrised tensor and the parentheses (\cdot)

for the normalised symmetrised tensor. The contraction with the gamma matrices gives

$$\begin{aligned}
2\kappa_{bac}\gamma^a\gamma^b\gamma^c &= \left(\Theta_{[bac]} - \frac{8}{3}\Theta_{(bc)a} - \frac{4}{3}\Theta_{(ab)c} \right) \gamma^a\gamma^b\gamma^c \\
&= \Theta_{[abc]}\gamma^a\gamma^b\gamma^c - \frac{8}{3}\Theta_{bca}\gamma^a\gamma^{(b}\gamma^c) - \frac{4}{3}\Theta_{abc}\gamma^{(a}\gamma^b)\gamma^c \\
&= \Theta_{[abc]}\gamma^a\gamma^b\gamma^c - \frac{8}{3}\Theta_{bca}\gamma^a\eta^{bc} - \frac{4}{3}\Theta_{abc}\eta^{ab}\gamma^c \\
&= \Theta_{[abc]}\gamma^a\gamma^b\gamma^c - 4\Theta_{abc}\eta^{ab}\gamma^c
\end{aligned}$$

and

$$-\frac{1}{4}\kappa_{bac}\gamma^a\gamma^b\gamma^c = -\frac{1}{8}\Theta_{[abc]}\gamma^a\gamma^b\gamma^c + \frac{1}{2}\Theta_{abc}\eta^{ab}\gamma^c \quad (5.20)$$

is the contribution of the torsion to the Dirac operator:

$$d^\mu\psi = d^\mu_{LC}\psi + \left(\frac{1}{2}\Theta_{abc}\eta^{ab}\gamma^c - \frac{1}{8}\Theta_{[abc]}\gamma^a\gamma^b\gamma^c \right) \psi \quad (5.21)$$

Chapter 6

Generalities on foliations, Lie groupoids and orbifolds

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In this section, we define Lie groupoids and foliations and present essential results and examples.

6.1 Lie Groupoids

Lie groupoids are sometimes presented as structures encompassing the geometry of differentiable manifolds and the structure of Lie groups. We will present a somewhat different perspective. General references for this section are [Mac05; MM03].

Let us first discuss groupoids more generally. They can be thought of as a generalisation of groups. Most groups arise as groups of transformations: a set of invertible transformations of some space. Elements of the group correspond to “global”, or “rigid”, transformations of the space: this rigidity is vividly illustrated by the example of the group of rotations of a three-dimensional solid. One could say that this picture relies on the finite dimensionality of the group of rotations. We

will see below how some infinite-dimensional groups can be naturally constructed as a byproduct of a Lie groupoid structure.

Seen along similar lines, a groupoid contains both the data of the transformations and of the “space” being acted on. An element, or a *morphism*, of a groupoid consists in the data of a starting point and an ending point, both belonging to the space, and “morphism data” describing in a way how the points are related. One can say that while a group acts globally on *one single object*, a groupoid acts *between points*. From this perspective, a group G acting on a set X can be readily represented as a groupoid: rather than interpreting an element g of G as a global transformations of X , one would consider a copy of g for each point of x , as a “morphism” connecting x to $g \cdot x$. We will describe in more detail this construction called *action groupoid* below.

More abstractly, a group is the abstract algebraic structure of a group of transformation of some object. As such, it can be made to “act” on various spaces. Comparatively, a groupoid is the abstract algebraic structure of transformations and isomorphisms of a *family of objects*. This idea will be formalised with the notion of action of a Lie groupoid defined in Section 6.1.5. This should give a more precise picture of what is meant by “the geometry of differentiable manifolds and the structure of Lie groups”: in a groupoid, morphisms have a “spatial”, sometimes called “horizontal”, extension, which is supported by an underlying space (a manifold, in the case of Lie groupoids).

Definition 6.1.1 (Groupoids). A groupoid Γ is defined by:

- A set Γ_0 with elements called *objects*, or *points*
- A set Γ_1 with elements called *morphisms*.
- A source map s and a target map $t: \Gamma_1 \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} \Gamma_0$. For $(a, b) \in (\Gamma_0)^2$ we define

$$\Gamma(a, b) = \{\gamma \in \Gamma_1 \mid (s(\gamma), t(\gamma)) = (a, b)\}$$

- An identity section:

$$e: \begin{cases} \Gamma_0 & \rightarrow \Gamma_1 \\ a & \mapsto e_a \end{cases}$$

which is a section of both s and t :

$$s \circ e = t \circ e = \text{id}_{\Gamma_0}$$

- Composition maps for all $(a, b, c) \in (\Gamma_0)^3$ which are maps

$$\circ: \Gamma(b, c) \times \Gamma(a, b) \rightarrow \Gamma(a, c)$$

such that the following associativity diagram commutes for all points a, b, c, d :

$$\begin{array}{ccccc}
 & & \Gamma(c, d) \times \Gamma(b, c) \times \Gamma(a, b) & & \\
 & \swarrow & & \searrow & \\
 & & \circ \times \text{id} & & \text{id} \times \circ \\
 \Gamma(b, d) \times \Gamma(a, b) & & & & \Gamma(c, d) \times \Gamma(a, c) \\
 & \searrow & & \swarrow & \\
 & & \circ & & \circ \\
 & & \Gamma(a, d) & &
 \end{array}$$

and such that the identity section satisfies the following:

$$\begin{aligned}(\cdot) \circ e(a) &= \text{id}_{\Gamma(a,b)} \\ e(a) \circ (\cdot) &= \text{id}_{\Gamma(c,a)}\end{aligned}$$

- Inversion maps for all $(a, b) \in (\Gamma_0)^2$:

$$i : \begin{cases} \Gamma(a, b) & \xrightarrow{\sim} \Gamma(b, a) \\ \gamma & \mapsto \gamma^{-1} \end{cases}$$

such that for all $\gamma \in \Gamma(a, b)$,

$$\begin{aligned}\gamma \circ \gamma^{-1} &= e_b \\ \gamma^{-1} \circ \gamma &= e_a\end{aligned}$$

This abstract definition is very general: for example there is a groupoid formed by all the (small enough sets) and bijections between them. Indeed a groupoid is exactly a category whose arrows are all invertible and therefore given any (small enough) category there is a groupoid composed of all invertible arrows.

The groupoids that will be of interest to us will be “concrete” in the following sense: they represent morphisms between objects living in a common “geometrical context”. In particular, the groupoid we will consider will have a *manifold* of objects, and the morphisms between objects will also have a structure of manifold. A convenient framework which will be suitable for our purposes is that of *Lie groupoids*.

Definition 6.1.2 (Lie groupoids). A *Lie groupoid* Γ is a groupoid $(\Gamma^0, \Gamma^1, s, t, i, \circ)$ such that:

1. Γ^0 has a structure of smooth manifold
2. Γ^1 has a structure of possibly non-Hausdorff, possibly non-paracompact smooth manifold.
3. s, t, i, \circ are all smooth maps.
4. s and t are (surjective) submersions.

We may allow Γ^0 to be non-Hausdorff, in which case we will talk of a *non-Hausdorff Lie groupoid*.

A Lie groupoid *over a manifold* M is a Lie groupoid which has M as object manifold.

In particular, each pair of points of the groupoid has a *manifold* of morphisms between them. For the sake of convenience, we will sometimes talk indistinctly of Lie groupoids and of their morphism manifolds when there is no risk of confusion about the structure maps.

We now introduce some general terminology useful for working with groupoids:

Definition 6.1.3. Let Γ be a Lie groupoid.

1. An *automorphism* is a morphism from a point to itself. The automorphisms at a point $a \in \Gamma_0$ form the *automorphism group of a* , or *isotropy group of a* , written $\text{Aut}_\Gamma(a)$ or $\text{Aut}(a)$. They have natural structures of Lie groups.
2. The *orbit* of a point $a \in \Gamma_0$ is the set of points $b \in \Gamma_0$ such that there exists a morphism between a and b . They form a partition of Γ_0 into (immersed) submanifolds and the corresponding quotient space is called the *orbit space of Γ* . A groupoid with only one orbit is called *transitive*.

3. Given a point a , the set of morphisms between points of the orbit of a is a submanifold of Γ_1 and defines a transitive Lie groupoid over the orbit of a . It is called the *transitive component of a* . A groupoid is partitioned into its transitive components.

There is a natural notion of morphism between Lie groupoids which is very strict but still very useful:

Definition 6.1.4 (Strict morphisms of Lie groupoids). Let Γ and Γ' be two Lie groupoids. A (strict)¹ morphism is the data of two smooth maps $\phi_0 : \Gamma_0 \rightarrow \Gamma'_0$ and $\phi_1 : \Gamma_1 \rightarrow \Gamma'_1$ which preserve the groupoid structure in the sense that the following diagrams commute:

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\phi_1} & \Gamma'_1 \\ \downarrow s & & \downarrow s' \\ \Gamma_0 & \xrightarrow{\phi_0} & \Gamma'_0 \end{array}$$

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\phi_1} & \Gamma'_1 \\ \downarrow t & & \downarrow t' \\ \Gamma_0 & \xrightarrow{\phi_0} & \Gamma'_0 \end{array}$$

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\phi_1} & \Gamma'_1 \\ e \uparrow & & e' \uparrow \\ \Gamma_0 & \xrightarrow{\phi_0} & \Gamma'_0 \end{array}$$

$$\begin{array}{ccc} \Gamma_1 \times_{(s,t)} \Gamma_1 & \xrightarrow{\phi_1 \times_{(s,t)} \phi_1} & \Gamma'_1 \times_{(s',t')} \Gamma'_1 \\ \downarrow \circ & & \downarrow \circ \\ \Gamma_1 & \xrightarrow{\phi_1} & \Gamma'_1 \end{array}$$

Morphisms of Lie groupoids can be naturally composed in an associative manner.

We now present a few examples.

6.1.1 Examples

Pair groupoid and discrete groupoid

Let M be a smooth manifold. There are two natural ways in which it can be understood as a Lie groupoid.

First, the *discrete groupoid* over M is the Lie groupoid with M as manifold of points, no morphism between two different points and only the identity morphism from each point to itself. The manifold M corresponds to the space of orbits of the groupoid; no morphism of the groupoid relate points of the manifold in a non-trivial way.

On the other hand, the *pair groupoid* over M is the Lie groupoid with

$$M \times M \begin{array}{c} \xrightarrow{\pi_1} \\ \rightrightarrows \\ \xleftarrow{\pi_2} \end{array} M$$

as source and target maps. Since there is exactly one morphism between two points, identity, composition and inversion can be defined in only one way. Here, the manifold is seen as one cohesive, symmetric object: there is exactly one orbit. For any point $m \in M$, the manifold of morphisms starting from M is diffeomorphic to M . In other words, the degrees of freedom of M appear in the “direction” of morphisms and not in the orbit space. There is however no isotropy: all isotropy groups are trivial.

¹In opposition to the homotopical notions of generalised morphism or of anafunctor.

Lie groupoids over M all come with a Lie groupoid morphism to the pair groupoid, given on the morphism space by (s, t) . In fact, a Lie groupoid can be “decomposed” into its subgroupoid of automorphisms (the “internal part”) and its image in the pair groupoid:

$$\text{Aut}(\Gamma) \hookrightarrow \Gamma \rightarrow \text{Pair}(M)$$

Jet groupoids

Let M be a smooth manifold and k an integer. The pair groupoid can be straightforwardly generalised to include “ k -contact data” on how the points of M are related. Indeed the space of k -jets of diffeomorphisms of M , which we wrote $\mathcal{J}_{\text{diff}}^k(M)$, has naturally a structure of Lie groupoid above M : we explained in Section 2.5 and the previous sections how they can be composed and inverted.

Disjoint union of Lie groups

Let $(G_i)_{i \in I}$ a finite set of Lie groups. Within the group framework, there are two standard ways to “combine” them:

- The *direct product* in which elements from groups with different indices *commute*.
- The *free product* in which the only relation between elements of groups with different indices is that all neutral elements are made equal.

However, in the framework of Lie groupoids, there is another alternative, which preserves the individuality of each group. This is made possible because groupoids allow for “spatial” extension. The Lie groups can be organised into a Lie groupoid $\coprod_{i \in I} G_i \rightrightarrows I$ with no morphism between distinct elements of I and a group G_i of automorphisms above each point $i \in I$.

The transitive components of the disjoint union are exactly the groups G_i .

Action groupoids

Let G be a Lie group and M a G -manifold. The action of G on M can be described with a Lie groupoid, which is one of our model cases for Lie groupoids. The idea is that an element g in G can be interpreted as morphisms starting from each point m and ending on the point $g \cdot m$. In other words, the action map $G \times M \rightarrow M$ can be interpreted as the target map of a Lie groupoid:

Example 6.1.5 (Action groupoids). Let M be a smooth manifold with an action of a Lie group G . There is an associated groupoid with $\Gamma_1 = G \times M$ and $\Gamma_0 = M$ and as structure maps:

$$\begin{aligned} s : (g, m) \in G \times M &\mapsto m & e : m \in M &\mapsto (e, m) \\ t : (g, m) \in G \times M &\mapsto g \cdot m & i : (g, m) \in G \times M &\mapsto (g^{-1}, m) \\ & & \circ : [(g_2, g_1 \cdot m), (g_1, m)] &\mapsto (g_2 \cdot g_1, m) \end{aligned}$$

It is called the *action groupoid* and is sometimes written $G \ltimes M$.

From the groupoid perspective on the action groupoid, morphisms starting from different points are unrelated but “happen” to have a component living in G which allows for comparison between morphisms with unrelated endpoints. Indeed, the Lie groupoid structure of the action groupoid in itself does not have enough internal structure to identify it as an action groupoid. One need to work with extra structure, the analysis of which will be carried in Section 8.1.1.

In particular, G as a group acting on M can be interpreted as a group of “global sections” of the groupoid, namely sections of $s : G \times M \rightarrow M$. It is then natural to consider more general sections of the form $m \mapsto g(m)$ and therefore actions of the form

$$m \in M \mapsto g(m) \cdot m$$

possibly only locally defined. This is very reminiscent of actions of gauge groups and we will see below how it is related.

A simple generalisation of action groupoids is groupoids associated with a “local group action”, in that only “small” elements of a group are allowed to act on the points of a manifold. A common example is the local flow of a vector field on a manifold, understood as a local action of the group \mathbb{R} . In particular, Lie groupoids allow for the group elements acting to depend on the point of the manifold.

From the perspective of action groupoids, one could say that rather than a generalisation of Lie groups, Lie groupoids are a generalisation of G -manifolds, with “the acting group allows to vary depending on the initial point”. They can include “non-effective” data in the sense that there may exist different morphisms between the same pair of points of M .

Let us also record the action groupoid for a *right* group action, since it will be useful for our considerations:

Example 6.1.6 (Action groupoids). Let P a smooth manifold with a right action of a Lie group H . There is an associated groupoid with $\Gamma_1 = P \times H$ and $\Gamma_0 = P$ and as structure maps:

$$\begin{aligned} s : (p, h) \in P \times H &\mapsto p & e : p \in P &\mapsto (p, e) \\ t : (p, h) \in P \times H &\mapsto p \cdot h & i : (p, h) \in P \times H &\mapsto (p, h^{-1}) \\ & & \circ : [(p \cdot h_1, h_2), (p, h_1)] &\mapsto (p, h_1 \cdot h_2) \end{aligned}$$

It is called the *action groupoid* and is sometimes written $P \rtimes H$.

Atiyah-Lie groupoids

Let M be a smooth manifold and $P \rightarrow M$ a G -principal bundle with G a Lie group. It can be interpreted as a “manifold of G -torsors”. The *Atiyah-Lie groupoid* associated with P , written $\text{At}(P)$, is a groupoid over M defined as follows: the morphisms from a point a to a point b are the G -equivariant diffeomorphisms $P_a \xrightarrow{\sim} P_b$. There is a construction which gives it a manifest structure of smooth manifold:

$$\text{At}(P) \simeq (P \times P) / \{(p_1g, p_2g) \sim (p_1, p_2)\}_{g \in G}$$

with a Cartesian product (and not a fibred product over M). Identities, compositions and inversions are their natural realisations for diffeomorphisms.

The Atiyah-Lie groupoid is an example of an “internal” groupoid over M in the following sense: there is no restriction on which points are related by a morphism, but given two points there is a remaining degree of freedom on the morphisms between the points. The extra degree of freedom does not manifest on M itself but can be seen on (associated) *bundles over M* .

Example 6.1.7 (Atiyah-Lie groupoid of the frame bundle). Assume M is an n -dimensional manifold. Consider its frame bundle $\text{GL}(M)$.

The morphisms of the associated Atiyah-Lie groupoid consists in GL_n -equivariant diffeomorphisms between fibres. But a GL_n -equivariant diffeomorphism between the linear frames of $T_{m_1}M$ and the linear frames of $T_{m_2}M$ is exactly the same as a linear isomorphism $T_{m_1}M \xrightarrow{\sim} T_{m_2}M$.

Furthermore, such a linear isomorphism is exactly the same as an invertible 1st order contact element from m_1 to m_2 .

From this, we can conclude (we omit checking the correspondence of the structure maps) that there is a Lie groupoid diffeomorphism

$$\text{At}(\text{GL}_n(M)) \simeq \mathcal{J}_{\text{diff}}^1(M)$$

We define another groupoid on M which is closer to the idea of “points of the space with a data of how they are related”. It will however not be a Lie groupoid. The morphisms from a point a to b are the classes of smooth paths from a to b with *sitting instants* up to *thin homotopy*. A path is said to have *sitting instants* if it is locally constant in the neighbourhood of each end. A *thin homotopy* is a smooth homotopy such that the two-parameter map has rank at most one at every point; they are furthermore required to have sitting instants at every time. Composition is given by the concatenation, which is seamlessly possible since the path have sitting instants. Identities are given by constant paths and inversion is given by reversing the parametrisation.

We call it the (*smooth*) *path groupoid* of M^2 and write it $\mathcal{P}(M)$.

The path groupoid gives a convenient framework to work with parallel transport: given a G -principal connection on $P \rightarrow M$, the corresponding parallel transport along paths defines a groupoid morphism

$$\mathcal{P}(M) \xrightarrow{\text{PT}} \text{At}(P)$$

In fact, any such groupoid morphism has to be the parallel transport associated with a smooth G -principal connection [Dum09; SW07]. Therefore the groupoid language allow handling the whole parallel transport as an algebraic structure, instead of restricting to holonomy groups, for example: the holonomy groups are the isotropy groups of the image groupoid in $\text{At}(P)$.

6.1.2 Étale groupoids

Before defining étale groupoids, let us introduce a familiar example.

Let M be a smooth manifold and \widetilde{M} a covering space. Given a point m in M and two points x, y in the fibre \widetilde{M}_m , there is a smooth map defined in a neighbourhood of x which sends x to y and preserves the fibration over M . According to covering space theory this map extends to a global covering automorphism iff $\pi_1(\widetilde{M}, x)$ and $\pi_1(\widetilde{M}, y)$ have the same image in $\pi_1(M, m)$.

This can be encoded in a groupoid over \widetilde{M} : $\widetilde{M} \times_M \widetilde{M} \rightrightarrows \widetilde{M}$. The property described above means that to each element of the groupoid of source x and target y is associated a germ of diffeomorphisms which sends x to y .

Note that this germ of diffeomorphisms can be *directly obtained from the groupoid itself*. Indeed both source and target maps

$$\widetilde{M} \times_M \widetilde{M} \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} \widetilde{M}$$

are themselves covering maps. As such they may locally be inverted. Given $(x, y) \in \widetilde{M} \times_M \widetilde{M}$, one can choose a neighbourhood on which both s and t are diffeomorphisms. Considering the map $t \circ s^{-1}$ between the images give a representative of the germ of covering automorphisms.

Note that the action groupoid of $\text{Aut}_M(\widetilde{M})$ has a natural embedding into $\widetilde{M} \times_M \widetilde{M}$ but this is an isomorphism only when the covering is *Galois*: the groupoid $\widetilde{M} \times_M \widetilde{M}$ has more morphisms in the general case. Here again, groupoids appear has a generalisation of group actions, with the

²This is a different definition than in [SW07].

acting manifolds over each point which are not always related to one absolute global manifold (an acting Lie group).

This is an example of the so-called *étale Lie groupoids*:

Definition 6.1.8 (Étale Lie groupoids). A Lie groupoid is *étale* when its source map, or equivalently its target map, is a local diffeomorphism.

Note that étale Lie groupoids have a morphism manifold and a base manifold of same dimension.

The mechanism described for covering manifolds works more generally for étale groupoids, which usually also have morphisms between different points. The construction is totally similar: let Γ be an étale Lie groupoid over a manifold M and γ a morphism from m_1 to m_2 . There is a neighbourhood \mathcal{U} of γ on which the restrictions of s and t are diffeomorphisms. Therefore one can construct a local diffeomorphism from $s(\mathcal{U})$ to $t(\mathcal{U})$:

$$\begin{array}{ccc}
 & \mathcal{U} & \\
 s|_{\mathcal{U}} \swarrow \cong & & \searrow \cong t|_{\mathcal{U}} \\
 s(\mathcal{U}) & \xrightarrow{\sim} & t(\mathcal{U}) \\
 m_1 & \xrightarrow{\quad} & m_2
 \end{array}$$

Two different choices of \mathcal{U} will give local diffeomorphism which coincide on a smaller neighbourhood so that they have the same *germ* at m_1 . This construction is naturally compatible with composition and inversion, and on identity morphisms the germ can be represented by identities.

Although it is not of finite dimension, there is a tautological example of étale groupoid over a manifold M : the *groupoid of differentiable germs*, or *Haefliger groupoid* [Car12], of M : $\text{Diff}_{\text{germs}}(M)$. Its space of morphisms between m_1 and m_2 is the space of germs of locally defined diffeomorphisms sending m_1 to m_2 .

The mechanism we described above implies that all étale groupoids above M come with a natural map to $\text{Diff}(M)$. They also come with natural maps to the Lie groupoids of jets of diffeomorphisms $\mathcal{J}_{\text{diff}}^k(M)$.

6.1.3 Proper Lie groupoids

Properness is a regularity property of groupoids, akin to properness of group actions. It will be recurrent throughout the text.

We first recall the definition of a proper map between topological spaces.

Definition 6.1.9 (Proper map). Let $f : X \rightarrow Y$ be a continuous map between topological spaces.

The map f is said to be *proper* if for all compact subsets K of Y the inverse image $f^{-1}(K)$ is compact.

A proper map is a map that “tends to infinity” when its argument “goes to infinity”. This can be formalised for example using one point compactifications.

The following properties will be useful:

Theorem 6.1.10 ([Lee03], Appendix A).

- A continuous closed map with compact fibres is proper.
- A continuous proper map to a locally compact Hausdorff space is closed (and has compact fibres).

Informally speaking, a proper Lie groupoid is a Lie groupoid such that as a morphism goes to infinity, either its source or its target as to go to infinity as well. Here is a proper definition:

Definition 6.1.11 (Proper Lie groupoid, [MM03; Mes16]). A Lie groupoid $\Gamma \overset{s}{\rightrightarrows} \underset{t}{M}$ is called *proper* if ³ the map

$$\Gamma \xrightarrow{(s,t)} M \times M$$

is a proper map.

If M is Hausdorff (which we usually assume), this is equivalent to requiring that (s, t) be closed and have compact fibres.

Since the composition with any given morphism from y to x gives a diffeomorphism $\Gamma(x, y) \xrightarrow{\sim} \Gamma(x, x)$, for (s, t) to have compact fibres is equivalent to require that the isotropy group of each point is compact.

Example 6.1.12 (Proper group action). Let G be a Lie group acting continuously on a smooth manifold X . Then the action is proper if and only if the action groupoid $G \ltimes X$ is proper.

6.1.4 Sections and bisections of a groupoid

If a Lie groupoid is a space of morphisms between points, in order to consider morphisms defined over the whole base manifold, one needs to look at *sections* of the Lie groupoid:

Definition 6.1.13 (Global section of a Lie groupoid). A global section of a Lie groupoid is a global section of its source map. We write $\text{Sect}(\Gamma)$ the space of (smooth) sections of the Lie groupoid Γ .

In other words, a section attaches to each point a morphism starting from that point. Let us assume Γ is a groupoid over a manifold M . In particular, to a section $\sigma \in \text{Sect}(\Gamma)$ is associated a map

$$t \circ \sigma : M \rightarrow M$$

Furthermore, sections can be associatively composed as follows:

$$\sigma_2 \circ \sigma_1 : m \mapsto \sigma_2 [t(\sigma_1(m))] \circ \sigma_1(m)$$

The identity section defines a neutral element for the composition so that $\text{Sect}(\Gamma)$ forms a *monoid*, or *semi-group* with identity.

The monoid of sections fits into a short exact sequence:

Theorem 6.1.14. *Let Γ be a Lie groupoid over a manifold M . Then the monoid of sections of Γ fits into the following short exact sequence of monoids:*

$$\text{Sect}(\text{Aut}(\Gamma)) \hookrightarrow \text{Sect}(\Gamma) \rightarrow \mathcal{C}^\infty(M, M)$$

Notice that sections belonging to $\text{Sect}(\text{Aut}(\Gamma))$ are invertible. In fact, a global section σ is invertible as soon as $t \circ \sigma$ is. Such invertible sections are called (global) bisections:

Theorem 6.1.15. *Let Γ be a Lie groupoid over a manifold M . Invertible sections form a group $\text{Bisect}(\Gamma)$, called the group of global bisections of Γ . It fits into the following short exact sequence of groups:*

$$\text{Sect}(\text{Aut}(\Gamma)) \hookrightarrow \text{Bisect}(\Gamma) \rightarrow \text{Diff}(M)$$

³Some authors require furthermore that the manifold of morphisms be Hausdorff.

We now illustrate this construction with a few examples of Lie groupoids previously introduced:

Example 6.1.16. Let M be a smooth manifold.

- Global sections of the pair groupoid $\text{Pair}(M)$ are exactly smooth maps $M \rightarrow M$. Global bisections are diffeomorphisms of M . The image of a section is the graph of the map in $M \times M$: the pair groupoid gives algebraic structure to this space. In particular, this space is the natural support for working with kernels and their convolution is naturally associated with the composition law of the pair groupoid.
- Let G be a Lie group acting on M . Global sections of the action groupoid $G \ltimes M$ are identified with smooth maps $M \rightarrow G$, but the composition depends on the action of G . There is an embedding $G \hookrightarrow \text{Bisect}(G \ltimes M)$. In the Lie groupoid framework, it is “natural” to consider the action of an element $g(m)$ which is dependent on points of the base manifold.
- Let P be a G -principal bundle over M and consider the associated Atiyah-Lie groupoid $\text{At}(P)$. Global bisections can be identified with diffeomorphisms of M equipped with a G -equivariant lift to P . The group of global bisections is often called the *gauge group of P* . Bisections can be shown to act on every associated bundle to P .

In particular, notice that the group of bisections is in general of infinite dimension. Lie groupoids provide a finite-dimensional support to these infinite-dimensional groups which allows working with these infinite-dimensional groups with finite-dimensional methods.

6.1.5 Action of a Lie Groupoid

An action of a group G is given by the data of an object X and a family of transformations of X which satisfy the same composition relations as the elements of G . This generalises to a groupoid as follows: an action of a groupoid Γ is given by a *family of objects* $(X_a)_{a \in \Gamma_0}$ and families of transformations between the objects which satisfy the same composition relations as the morphisms of Γ . In a way, it may be more appropriate here to think of elements of Γ_0 as *objects*, possibly gathered in a geometrical context, than “atomic” points.

In the case of Lie groupoids, we want the objects of the Lie groupoid to be represented by manifolds, gathered in a “manifold of manifolds”, namely a (non-necessarily locally trivial) bundled manifold. The morphisms are represented by diffeomorphisms between the fibre manifolds indexed by points of the groupoid:

Definition 6.1.17 (Action of a Lie groupoid [MM03]). Let $\Gamma : \Gamma_1 \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} \Gamma_0$ be a Lie groupoid. A (left) action of Γ on a manifold M along a map $\phi : M \rightarrow \Gamma_0$ is given by a map

$$\begin{aligned} \mu : \Gamma_1 \times_{(s,\phi)} M &\rightarrow M \\ (\gamma, m) &\mapsto \gamma \cdot m \end{aligned}$$

such that

1. $\forall (\gamma, m) \in \Gamma_1 \times_{(s,\phi)} M, \phi(\gamma \cdot m) = t(\gamma)$
2. $\forall m \in M, e_{\phi(m)} \cdot m = m$
3. $\forall (\gamma', \gamma, m) \in \Gamma_1 \times_{(s,t)} \Gamma_1 \times_{(s,\phi)} M, (\gamma' \circ \gamma) \cdot m = \gamma' \cdot (\gamma \cdot m)$

In other words, μ defines a Lie groupoid structure on

$$\Gamma_1 \times_{(s,\phi)} M \begin{array}{c} \xrightarrow{\pi_M} \\ \xrightarrow{\mu} \end{array} M$$

with the composition and neutral elements given by the corresponding structure maps in Γ .

This definition may seem very abstract, but we have already introduced a few examples:

Example 6.1.18. • Let M be a G -manifold and $X \rightarrow M$ a fibre bundle. Then an action of $G \times M$ on $X \rightarrow M$ is equivalent to an action of G on X which gives it a structure of G -equivariant bundle.

- Let $P \rightarrow M$ be a G -principal bundle. The Atiyah-Lie groupoid $\text{At}(P)$ naturally acts on P along the fibration map $P \rightarrow M$. Furthermore this action commutes with the action of G : the map μ is G -equivariant. In fact, this allows constructing an action of $\text{At}(P)$ on *all the associated bundles*.
- If P is equipped with a principal connection, its parallel transport induces an action of the path groupoid of M on P . This action is associated with the Lie groupoid morphism from the path groupoid to $\text{At}(P)$ defined by the parallel transport. Recall however that the path groupoid is not a Lie groupoid.
- Let M be a foliated manifold (all terms will be introduced in Section 6.2). Linear holonomy defines an action of the holonomy groupoid on the normal bundle to the foliation.

6.1.6 Weak equivalence of Lie groupoids

Lie groupoids have a notion of strict morphism, and correspondingly of strict isomorphisms.

There are more general notions of equivalences between Lie groupoids, which are essential in the homotopical theory of Lie groupoids. Here we will simply present the notion of weak equivalence, which will rely on two notions we define now.

Definition 6.1.19 (Full and faithful Lie groupoid morphism). Let Γ and Γ' be two Lie groupoids and $\Phi : \Gamma \rightarrow \Gamma'$ be a strict Lie groupoid morphism. Then there is a commutative diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\Phi_1} & \Gamma'_1 \\ \downarrow (s,t) & & \downarrow (s',t') \\ \Gamma_0 \times \Gamma_0 & \xrightarrow{\Phi_0 \times \Phi_0} & \Gamma'_0 \times \Gamma'_0 \end{array}$$

Then Φ is *full and faithful* if this diagram defines a pullback:

$$\Gamma_1 \xrightarrow[\Phi_1]{\sim} (\Gamma_0 \times \Gamma_0) \times_{\Gamma'_0 \times \Gamma'_0} \Gamma'_1$$

In particular this implies that the fibres of (s, t) are sent by diffeomorphisms to fibres of (s', t') .

Definition 6.1.20 (Essentially surjective Lie groupoid morphism). Let Γ and Γ' be two Lie groupoids and $\Phi : \Gamma \rightarrow \Gamma'$ be a strict Lie groupoid morphism.

It is possible to pullback through Φ_0 outgoing morphisms in Γ' as follows:

$$\begin{array}{ccc} \Gamma_0 \times_{(\Phi_0, t)} \Gamma'_1 & \longrightarrow & \Gamma'_1 \\ \downarrow & & \downarrow s \\ \Gamma_0 & \xrightarrow{\Phi_0} & \Gamma'_0 \end{array}$$

Elements of the space $\Gamma_0 \times_{(\Phi_0, t)} \Gamma'_1$ are points a of Γ_0 along with a morphism in Γ'_1 starting from $\Phi_1(a)$.

Then Φ is *essentially surjective* if the target map on $\Gamma_0 \times_{(\Phi_0, t)} \Gamma'_1$:

$$\Gamma_0 \times_{(\Phi_0, t)} \Gamma'_1 \rightarrow \Gamma'_1 \xrightarrow{t} \Gamma'_0$$

is a surjective submersion. Surjectivity means that $\Phi_0(\Gamma_0)$ crosses all orbits of Γ' .

Definition 6.1.21 (Weak equivalence of Lie groupoids). A weak equivalence of Lie groupoids is a strict Lie groupoid morphism which is both *full and faithful* and *essentially surjective*.

Example 6.1.22. Let G be a Lie group and H a closed subgroup. Then G acts transitively on the coset space G/H . In the corresponding action groupoid $G \ltimes G/H$, the isotropy group of $[e] \in G/H$ is identified to H .

Then the inclusion

$$\left(\text{Aut}([e]) \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} [e] \right) \simeq \left(H \times [e] \rightrightarrows [e] \right) \hookrightarrow G \ltimes G/H$$

is a weak equivalence of Lie groupoids. Indeed, fullness and faithfulness are straightforward since the diagram

$$\begin{array}{ccc} H & \xleftarrow{\Phi_1} & G \ltimes G/H \\ \downarrow & & \downarrow \\ [e] \times [e] & \hookrightarrow & G/H \times G/H \end{array}$$

is nothing more than the inclusion of the fibre above $[e] \times [e]$, which is a pullback. Essential surjectivity is a consequence of the transitivity of the action of G on G/H : the map

$$\begin{array}{l} [e] \times G \rightarrow G/H \\ ([e], g) \mapsto g \cdot [e] \end{array}$$

is manifestly a submersion.

Theorem 6.1.23 (Morita equivalence of Lie groupoids). *The equivalence relation on Lie groupoids generated by the relation “There exists a weak equivalence from Γ to Γ' ” is called Morita equivalence. A Morita equivalence from a Lie groupoid Γ to a Lie groupoid Γ' is the data of a Lie groupoid Γ'' with two weak equivalences:*

$$\begin{array}{ccc} & \Gamma'' & \\ \swarrow \simeq & & \searrow \simeq \\ \Gamma & & \Gamma' \end{array}$$

Two Lie groupoids are Morita equivalent if and only if there exists a Morita equivalence between them.

Morita equivalence is the suitable notion of equivalence when working with Lie groupoids as presentations of their “orbit spaces”, effectively understood as differentiable stacks [Mes16].

6.2 Foliated manifolds

In this section we present fundamental notions and constructions related to foliated manifolds. We will only consider regular foliations. In this whole section, M will be a differentiable manifold of dimension n .

The notion of immersed submanifold will be relevant:

Definition 6.2.1 (Embedded and immersed submanifolds [Mic08; RS13]). An *embedded submanifold* of M is the data of a subset N of M equipped with a structure of manifold such that the inclusion is a smooth topological embedding.

An *immersed submanifold* of M is the data of a differentiable manifold N and of an injective immersion into M :

$$i_N : N \rightarrow M$$

Two immersed submanifolds are identified if they have the same image and the corresponding bijection is a diffeomorphism.

The notion of *embedded submanifolds* is stronger in that the image of an immersed submanifold can have different immersed submanifold structures. If an embedded submanifold admits an equations locally with respect to the topology of the ambient manifold, immersed submanifolds $i(N)$ may only admit local equations that identify open subsets of the immersed submanifolds *with respect to the finer topology of N* .

Example 6.2.2. Consider an irrational action of \mathbb{R} on the 2-torus \mathbb{T}^2 . Then the orbits are immersed submanifolds diffeomorphic to \mathbb{R} , but they are not locally closed and therefore cannot be embedded submanifolds.

6.2.1 Regular foliations

We first define regular foliations of codimension k on M :

Definition 6.2.3 (Foliations [Mic08]). A *regular foliation* on M of codimension $k \in \mathbb{N}$ is a partition of M into *connected immersed submanifolds* $(L_i)_{i \in I}$ of codimension k , called the *leaves* of the foliation, which satisfy one of the two following equivalent properties:

- M admits a local basis of open subsets (\mathcal{U}_j) on which there exists submersions $\varphi_j : \mathcal{U}_j \rightarrow \mathbb{R}^k$ such that fibres of φ_j are exactly *connected components of the intersections of leaves with \mathcal{U}_j* : connected components of $L_i \cap \mathcal{U}_j$ for any $i \in I$.
- M admits a local basis of open subsets (\mathcal{U}_j) on which there exists diffeomorphisms $\phi_j : \mathcal{U}_j \xrightarrow{\sim} \mathbb{R}^{n-k} \times \mathbb{R}^k$ such that inverse images of submanifolds $\mathbb{R}^{n-k} \times \{x\}$ are exactly connected components of $L_i \cap \mathcal{U}_j$ for any $i \in I$. Such a chart is called a *foliation chart*.

In both cases, given such an open subset \mathcal{U}_j , a connected component of the intersection of a leaf with \mathcal{U}_j is called a *plaque*.

A manifold equipped with a foliation is called a *foliated manifold*. Since we will only work with regular foliations, we will most often omit the qualifier *regular*.

Remark. Since each leaf is an immersed submanifold, a foliation of codimension k defines an immersed submanifold of codimension k

$$\coprod_{i \in I} L_i \rightarrow M$$

such that the immersion is a bijection [Mic08]. From this perspective, a k -codimensional foliated n -manifold can be understood as a $(n - k)$ -manifold (which is paracompact although it usually has an uncountable number of connected components) with additional *transversal* structure relating different connected components. This manifold will be written $M_{\mathcal{F}}$.

Up to now, the definition could have been done without the requiring that the leaves are connected. However, Frobenius's theorem gives an equivalence between such foliations and local structure on M . Before stating the theorem, we need a couple of definitions:

Definition 6.2.4 (Involutive distributions).

- A (smooth) distribution of codimension k on M is a vector subbundle of TM of codimension k .
- A (smooth) distribution is called *involutive* if the Lie bracket of two sections is another section of the distribution.

Theorem 6.2.5 (Frobenius Theorem [Mic08]). *There is a bijection between foliations of codimension k on M and (smooth) involutive distributions of codimension k on M which can be constructed as follows:*

$$\{\text{Foliation of codimension } k\} \leftrightarrow \{\text{Involutive distribution of codimension } k\}$$

$$\left(\coprod_{i \in I} L_i \right) \mapsto \left(\coprod_{i \in I} TL_i \hookrightarrow TM \right)$$

The simplest example of foliations, which also somewhat plays the role of standard model, are the so-called *simple foliations*.

Definition 6.2.6 (Simple foliations). Let $f : M \rightarrow X$ be a submersion. Then the *connected components of the fibres* of f form a foliation on M , which has for codimension $\dim X$.

When the fibres of f are all connected, X gives a very convenient model for the “space of leaves” of the foliation. A central question in foliation theory is to characterise the structure of the space of leaves of given foliations. In the following section, we describe its naive topological structure.

6.2.2 Topological leaf space of a foliation

The leaf space is defined as the space of leaves of the foliation on M . As such, it is a set of equivalence classes for the equivalence relation of belonging to a common leaf. Thus it can be constructed as a quotient of M by an equivalence relation, and can be equipped with the quotient topology. We will write \mathcal{F} for the foliation structure and $T\mathcal{F}$ for the tangent involutive distribution.

Let us write M/\mathcal{F} for the space of leaves. It has a quotient map $M \xrightarrow{\pi} M/\mathcal{F}$. The quotient topology is defined so that a map $M/\mathcal{F} \rightarrow X$ to any topological space is continuous if and only if

its pullback to M :

$$\begin{array}{ccc}
 M & \overset{\text{-----}}{\longrightarrow} & X \\
 & \searrow & \nearrow \\
 & M/\mathcal{F} &
 \end{array}$$

is continuous.

In fact, the same approach allows defining a *quotient smooth structure* (Section 1.1) on M/\mathcal{F} :

Definition 6.2.7 (Quotient smooth structure). Let P be a topological space with a smooth structure (e.g. a smooth manifold) and $\pi : P \rightarrow Q$ a quotient map (a continuous surjection which induces the topology of Q as a final topology).

Then the quotient smooth structure on Q is defined as follows: let \mathcal{U} be an open subset of Q . Then a map defined on \mathcal{U} is smooth if and only if its pullback under π to $\pi^{-1}(\mathcal{U})$ is smooth.

It satisfies the following universal property: for every smooth space X , any smooth map $P \rightarrow X$ which is constant on equivalence classes factors uniquely to a map $Q \rightarrow X$ which is smooth. In particular a map $f : Q \rightarrow X$ is smooth *if and only if* its pullback $f \circ \pi : P \rightarrow X$ is a smooth map.

This smooth structure is not necessarily locally Euclidean: the quotient space may not be a manifold, even locally. Open subsets of M/\mathcal{F} are in correspondence with open subsets of M which are “saturated” in the following sense: they contain the leaf of any element they contain

$$\begin{aligned}
 \{\text{Open subsets of } M/\mathcal{F}\} &\rightarrow \{\text{Saturated open subsets of } M\} \\
 \mathcal{U} &\mapsto \pi^{-1}\{\mathcal{U}\}
 \end{aligned}$$

Smooth functions on \mathcal{U} are exactly smooth functions on $\pi^{-1}(\mathcal{U})$ which are constant on leaves.

This raises the question of what does the set of these functions look like. Such questions can be approached using the notion of *transversal*:

Definition 6.2.8. Let m be a point of M and k the codimension of the foliation. A transversal to the foliation at x is any submanifold S_x of dimension k which contains x which is *transversal* to the foliation in the following sense:

$$TS_x \oplus T|_{S_x} \mathcal{F} = T|_{S_x} M$$

In other words, S is transversal to every leaf of the foliation it intersects.

Note that with this definition a transversal at x contains x but needs to be transversal to the foliation at every point. The idea is that when a transversal crosses at most once each leaf, it can be used as a model for part of the leaf space. In order to justify this, we will need the following result:

Theorem 6.2.9. *Let S be a submanifold of M which is at every point transversal to \mathcal{F} . Then projection map to the leaf space*

$$\pi : S \rightarrow M \rightarrow M/\mathcal{F}$$

is an open map.

The proofs makes use of holonomy, which will be introduced in Section 6.2.3.

Proof. We need to prove the following: for all open subsets \mathcal{U} of S , the subset $\bar{\mathcal{U}} = \pi^{-1}(\pi(\mathcal{U}))$ is an open subset. By definition, $\bar{\mathcal{U}}$ is the following subset:

$$\{m \in M \mid \exists s \in S, s \text{ and } m \text{ belong to the same leaf.}\}$$

Namely, if we write L_x for the leaf x belongs to,

$$\bar{U} = \bigcup_{s \in \mathcal{U}} L_s$$

We will prove the following: if T is a transversal to \mathcal{F} at a point x , then x admits a neighbourhood basis of open subsets (U_x) such that all points of U_x belong to leaves of elements of $U_x \cap T$. Let us first justify why it implies the theorem.

Let x be a point in \bar{U} . There exists a point s in S belong to the same leaf. As a consequence, there exists a subtransversal S' at s and a holonomy from (s, S') to a transversal Q at x . In particular, $Q \subset \bar{U}$. Now applying the assumed result we know that x has a neighbourhood V_x such that all elements belong to leaves of elements of Q . Therefore they belong to leaves of elements of \bar{U} hence they belong to \bar{U} . We have shown that x has a neighbourhood included in \bar{U} , and since this holds for every x , \bar{U} is an open subset of X .

We now prove the claimed result. Let T be a transversal to \mathcal{F} at a point x . Let $\varphi : U_x \subset X \rightarrow \mathbb{R}^k$ be a submersion on a neighbourhood of x such that fibres are the plaques of U_x (k is the codimension of \mathcal{F}).

The restricted map

$$T \cap U_x \rightarrow \mathbb{R}^k$$

is by hypothesis a local diffeomorphism at x therefore it is open. Let \mathcal{W} be a neighbourhood of $\varphi(x)$ included in $\varphi(T \cap U_x)$. Then $\varphi^{-1}(\mathcal{W})$ is an open neighbourhood of x such that all elements belong to leaves of elements of T . Since the supports of foliated charts form a basis of open subsets of X , we obtain the claimed result. \square

We can now assert the following:

Theorem 6.2.10. *Let S be a submanifold of M which is transversal to \mathcal{F} at every point. Then the restriction to S of the quotient map*

$$S \rightarrow M/\mathcal{F}$$

is open and continuous and defines an embedding of S/\mathcal{F} in M/\mathcal{F} such that the smooth structure of S/\mathcal{F} is the restriction of that of M/\mathcal{F} . In particular, if S crosses at most once each leaf, then we obtain a smooth embedding

$$S \hookrightarrow M/\mathcal{F}$$

In this way, transversals are candidates to build charts on the leaf space. This brings us to the following criterion on the simplicity of foliations:

Theorem 6.2.11 (Simple foliations, [Fer21]). *The quotient smooth space structure on the leaf space is locally Euclidean if and only if the foliation is proper in the following sense: around every point m , M admits a foliated chart such that plaques are the intersections of the support with the leaves. In other words the intersections of leaves with the support of the chart are connected.*

Such foliated charts give transversals which define charts on the leaf space, since the transversals are openly embedded.

This theorem is strongly tied to Godement's criterion for quotient manifolds, which we record here:

Theorem 6.2.12 (Godement's criterion, [Fer21], 9.3, [Bou07], 5.9.5, p.51). *Let M be a manifold (non necessarily Hausdorff or paracompact). Let $R \subset M \times M$ be the graph of an equivalence relation. There exists at most one manifold structure on M/R such that the quotient map*

$M \rightarrow M/R$ is a submersion. The quotient manifold structure exists if and only if the following two conditions are satisfied:

1. R is an (embedded) submanifold of $M \times M$
2. The projection on the first factor $R \rightarrow M$ is a submersion

In this case, the quotient manifold is Hausdorff if and only if R is closed in $M \times M$.

The proof is very close to the construction for proper foliations, but equivalence relations allow for non-connected equivalence classes. In particular, the quotient manifold structure matches with the quotient smooth structure as a smooth space.

However, many naturally occurring foliations do not satisfy the properness property, as the following example shows:

Example 6.2.13 (A singular leaf space). Consider a constant vector field $(1, \alpha)$ on the 2-torus \mathbb{T}^2 with α irrational. The orbits form a foliation of \mathbb{T}^2 since they are integral manifold of the vector field generating the action. This foliation is far from being proper: in fact each leaf is dense in \mathbb{T}^2 .

As a consequence, any smooth map on \mathbb{T}^2 which is constant on leaves is globally constant. In particular, smooth maps on the leaf space are constant.

This example suggests that the structure of smooth space is not fine enough to capture the structure of the leaf space in some cases. Note however that “structure of the leaf space” is very vague. A more precise wording is as follows: the quotient smooth structure of the leaf space may fail to distinguish the orbit spaces of transitive actions and non-transitive actions.

Finer structure of the leaf space is captured by the *holonomy groupoid* which we introduce in the next section. Although we will use the holonomy groupoid for different purposes, it gives a Lie groupoid model for the leaf space which, along with the suitable notion of weak equivalences (and suitable morphisms), allows dealing with leaf spaces in a more practical manner.

6.2.3 Holonomy of a regular foliation

Holonomy of a foliated path

In this section, M is equipped with a regular foliation \mathcal{F} of codimension k . Holonomy is a structure of transport along leaves of foliation. It requires foliated paths in order to be defined:

Definition 6.2.14 (Foliated paths). A *foliated path* in M is a continuous path which is included inside a leaf of \mathcal{F} . When the path is smooth, it is equivalently a path which is tangent to $T\mathcal{F}$ at all times. It is equivalently a continuous path for the topology of $M_{\mathcal{F}}$.

When not specified, the paths we use shall be parametrised by $[0, 1]$.

In order to work with concatenation of smooth paths and homotopy, we will be using the following notion:

Definition 6.2.15 (Homotopy with sitting instants). Let X and Y be topological spaces and $h : X \times [0, 1] \rightarrow Y$ a continuous map. The map h is said to *have sitting instants* if there exists $\epsilon \in]0, 1]$ such that $h|_{X \times [0, \epsilon]}$ and $h|_{X \times [1-\epsilon, 1]}$ are constant maps with respect to the second argument.

The point of this notion is that the concatenation of two smooth homotopies is smooth (and has sitting instants) as soon as they both have sitting instants. The following simple result makes it easy to convert homotopies into homotopies with sitting instants:

Lemma 6.2.16. Any smooth homotopy is (smoothly) homotopic, with fixed end maps, to a smooth homotopy with sitting instants.

In other words, homotopy classes of homotopies are all representable by smooth homotopies with sitting instants.

We will be using the following notation for germs of maps (Section 2.6): the germ of a smooth map from a neighbourhood of a point x in X to Y which sends x to y will be written as follows:

$$(x, X) \rightarrow (y, Y)$$

and $(x, X) \xrightarrow{\sim} (y, Y)$ in case of a germ of local diffeomorphism.

We now describe the fundamental local principle behind holonomy. Let $\phi : \mathcal{U} \xrightarrow{\sim} L \times \mathbb{R}^k$ be a foliation chart with \mathbb{R}^k -component φ . Let m_1 and m_2 be any two points belonging to the same plaque, namely the same fibre above \mathbb{R}^k . Let S_1 and S_2 be any two transversals to \mathcal{F} respectively at m_1 and m_2 . Since S_1 and S_2 are transversal to the fibres of φ , φ restricts on each to a local diffeomorphism, which we call respectively φ_1 and φ_2 .

In particular there exist a neighbourhood \mathcal{V}_1 of m_1 in S_1 and a neighbourhood \mathcal{V}_2 of m_2 in S_2 such that φ_i is a diffeomorphism on \mathcal{V}_i and $\varphi_1(\mathcal{V}_1) = \varphi_2(\mathcal{V}_2)$. This implies that no two elements of \mathcal{V}_1 (resp. \mathcal{V}_2) belong to the same plaque. This allows constructing a diffeomorphism between \mathcal{V}_1 and \mathcal{V}_2 such that corresponding elements in \mathcal{V}_1 and \mathcal{V}_2 are in the same plaque:

$$\begin{array}{ccc}
 & \mathcal{V} = \varphi_i(\mathcal{V}_i) & \\
 \varphi_1|_{\mathcal{V}_1} \nearrow & & \nwarrow \varphi_2|_{\mathcal{V}_2} \\
 \mathcal{V}_1 & \xrightarrow{\sim} & \mathcal{V}_2 \\
 m_1 \longleftarrow & & \longrightarrow m_2
 \end{array}$$

Since the diffeomorphism between \mathcal{V}_1 and \mathcal{V}_2 matches elements of the same plaques, it is uniquely defined as soon as \mathcal{V}_1 and \mathcal{V}_2 are small enough: it does not depend on the choice of φ . In particular, the *germ* (Section 2.6) of the diffeomorphism between m_1 and m_2 is uniquely defined. Furthermore, if one considers a third point m_3 , equipped with a transversal S_3 , since the germ of diffeomorphism between two transversals is uniquely defined, the composition of germs of diffeomorphisms from S_1 to S_2 to S_3 coincides with the germ of diffeomorphism from S_1 to S_3 :

$$(m_1, S_1) \xrightarrow{\sim} (m_2, S_2) \xrightarrow{\sim} (m_3, S_3) = (m_1, S_1) \xrightarrow{\sim} (m_3, S_3)$$

One consequence is that changing the transversal at m_1 results in a uniquely defined change in the germ of diffeomorphism: if S'_1 is another transversal at m_1 , there is an identification of germs

$$(m_1, S'_1) \xrightarrow{\sim} (m_2, S_2) = (m_1, S'_1) \xrightarrow{\sim} (m_1, S_1) \xrightarrow{\sim} (m_3, S_3)$$

Furthermore for any two points m_1 and m_2 of \mathcal{U} , even belonging to different leaves, it is possible to consider the transversals $S_i = \phi^{-1}(\phi_L(m_i) \times \mathbb{R}^k)$. They satisfy the following property: they both are projected diffeomorphically to S under φ , which induces a diffeomorphism

$$\begin{array}{ccc}
 & \mathbb{R}^k & \\
 \phi_1|_{S_1} \nearrow & & \nwarrow \phi_2|_{S_2} \\
 S_1 & \xrightarrow{\sim} & S_2 \\
 m_1 \longleftarrow & & \longrightarrow m_2
 \end{array}$$

The conclusion is that even for two points in \mathcal{U} belonging to different leaves, it is possible to find transversals at those points which are in global diffeomorphism under a map which sends a point to a point of the same plaque.

Let c be a possibly non-smooth continuous foliated path starting from a point m_1 and ending on a point m_2 . Let S_1 and S_2 be respectively transversals to the foliation at m_1 and m_2 . We will construct a uniquely defined germ of diffeomorphism from (m_1, S_1) to (m_2, S_2) which maps points of (an open subset of) S_1 to elements of S_2 belonging to the same leaf. It will be called *the holonomy of c between S_1 and S_2* .

Since the image of c is compact it can be covered by a finite number of foliated charts

$$\phi_j : (\mathcal{U}_j) \rightarrow L \times \mathbb{R}^k$$

(and $\varphi_j = (\phi_j)_{\mathbb{R}^k}$) for $1 \leq j \leq N$. The role of c essentially stops there: the remaining of the construction purely depend on the string of foliated charts. It is possible to decompose the segment I over which c is defined in subsegments $I_j = [t_j, t_{j+1}]$ such that $c(I_j) \subset \mathcal{U}_j$. Let us choose a transversal S^j at each point $c(t_j)$, with $S^0 = S_1$ and $S^{N+1} = S_2$. The construction previously described gives germs of diffeomorphisms:

$$(c(t_j), S^j) \xrightarrow{\sim} (c(t_{j+1}), S^{j+1})$$

which can be composed together to construct a germ of diffeomorphism:

$$(m_1 = c(t_0), S^0) \xrightarrow{\sim} (m_2 = c(t_{N+1}), S^{N+1})$$

This diffeomorphism can be represented by a local diffeomorphism which *matches elements of the same leaves*.

Let $\mathcal{V}^j \subset S^j$ be open subsets such that the germs of diffeomorphisms can be represented by diffeomorphisms $\mathcal{V}^j \xrightarrow{\sim} \mathcal{V}^{j+1}$. Since the foliated charts project \mathcal{V}^j diffeomorphically in \mathbb{R}^k , it is possible to restrict the open subsets \mathcal{U}^j to

$$\mathcal{U}'^j = \phi_j^{-1}(\varphi_j(\mathcal{V}^j))$$

on which the foliated charts take the form

$$\phi'_j : \mathcal{U}'^j \xrightarrow{\sim} L \times \mathcal{V}^j$$

The point is that this allows to extend $c|_{I_j}$ into a *foliated homotopy* starting from \mathcal{V}^j and ending on \mathcal{V}^{j+1} :

$$h_j : \begin{cases} \mathcal{V}^j \times I_j & \rightarrow \mathcal{U}'^j \\ (s, t) & \mapsto (\phi'_j)^{-1}((\phi'_j)_L(c(t)), s) \end{cases}$$

It satisfies the following properties:

- For every $s \in \mathcal{V}^j$, $h_j(s, \cdot)$ defines a foliated path in \mathcal{U}'^j .
- $h_j(c(t_j), \cdot) = c|_{I_j}$
- $h_j(\cdot, t_{j+1})$ is the diffeomorphism $\mathcal{V}^j \xrightarrow{\sim} \mathcal{V}^{j+1}$ matching together elements in the same plaque of \mathcal{U}'^j .

Concatenating these homotopies, we obtain a foliated homotopy $h : S^0 \times I \rightarrow M$ which satisfies the following properties:

- For every $s \in \mathcal{V}^0$, $h(s, \cdot)$ defines a foliated path.
- $h(c(m_1), \cdot) = c$
- $h(\cdot, 1)$ is the previously constructed diffeomorphism $\mathcal{V}^0 \xrightarrow{\sim} \mathcal{V}^{N+1}$.

Now, the germ of diffeomorphism $(m_1, S_1) \xrightarrow{\sim} (m_2, S_2)$ is independent of the chosen transversals at each $c(t_j)$. If one were to change the transversal at $c(t_j)$ (neither the first of the last), then due to the following commutative diagram of germs of diffeomorphisms, the germ of diffeomorphism $(m_1, S_1) \xrightarrow{\sim} (m_2, S_2)$ would not change:

$$\begin{array}{ccccc}
 & & (c(t_j), S^j) & & \\
 & \nearrow \sim & & \searrow \sim & \\
 \cdots & \xrightarrow{\sim} & (c(t_{j-1}), S^{j-1}) & & (c(t_{j+1}), S^{j+1}) \xrightarrow{\sim} \cdots \\
 & \searrow \sim & & \nearrow \sim & \\
 & & (c(t_j), S^j) & &
 \end{array}$$

Let us prove this construction is independent from the chosen charts. If there are two systems of charts covering the image of c , there exists another system of charts refining both which still covers c . It is therefore enough to prove that the construction is invariant under refinement of the system of charts. Let \mathcal{U} be the support of a foliated chart which is covered by more foliated charts (\mathcal{U}_i) and assume c is contained inside \mathcal{U} . Then the germ of diffeomorphism

$$(m_1, S_1) \xrightarrow{\sim} (m_2, S_2)$$

can be represented by a local diffeomorphism which matches elements of the *leaves for the foliation of \mathcal{U}* , namely elements of the same plaque in \mathcal{U} . This diffeomorphism thus necessarily has the same germ that the one directly constructed from the chart supported by \mathcal{U} .

This proves that the obtained germ of diffeomorphism is invariant under refined of the covering of c and thus only depends on c : it is called *the holonomy of c* between S_1 and S_2 and will be written

$$\text{Hol}_{S_1}^{S_2}(c)$$

Remark. As asserted earlier, the holonomy of c can be defined as a pure function of the covering string of foliated charts. But it is actually independent of the choice of string of foliated charts covering c . This justifies talking of the holonomy of the path c .

Unicity of holonomy proves the following: given an open subset $\mathcal{U} \subset S_1$, if there exists a foliated homotopy $h : S_1 \times I \rightarrow M$ such that

- For every $s \in S_1$, $h(s, \cdot)$ defines a foliated path.
- $h(c(m_1), \cdot) = c$
- $h(\cdot, 1)$ defines a smooth injection of S_1 into S_2 .

then $\text{Hol}_{S_1}^{S_2}(c)$ can be represented by $h(\cdot, 1) : S_1 \rightarrow h(S_1, 1) \subset S_2$.

We now show a stability property of holonomy. Let c be a foliated path parametrised by I and let us use the \mathcal{U}^j constructed in the construction of h : there are sub-segments $I_j = [t_j, t_{j+1}]$ of I such that $c(I_j) \subset \mathcal{U}^j$ and there are transversals \mathcal{V}^j to \mathcal{F} at $c(t_j)$ such that the holonomy of

$c|_{I_j}$ is represented by a diffeomorphism $V^j \xrightarrow{\sim} V^{j+1}$. Let d be another foliated path parametrised by I such that $d(I_j) \subset \mathcal{U}^{j+1}$. We define at $d(t_j)$ the following transversal:

$$T^j = \phi_j'^{-1} ((\phi_j')_L(d(t_j)) \times \mathbb{R}^k)$$

Then inside \mathcal{U}^{j+1} matching elements of the same plaques gives the following commutative diagram of holonomies:

$$\begin{array}{ccc} T^j & \xrightarrow{\sim} & T^{j+1} \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{V}^j & \xrightarrow{\sim} & \mathcal{V}^{j+1} \end{array}$$

By concatenation, we obtain a commutative diagram with the holonomies of d and c :

$$\begin{array}{ccc} T^0 & \xrightarrow[\sim]{\text{Hol}_{T^0}^{T^{N+1}}(d)} & T^{N+1} \\ \downarrow \wr & & \downarrow \wr \\ S_1 & \xrightarrow[\text{Hol}_{S_1}^{S_2}(c)]{\sim} & S_2 \end{array} \quad (6.1)$$

with holonomies in \mathcal{U}^0 and in \mathcal{U}^N as vertical legs.

In particular, we obtain the following theorem:

Theorem 6.2.17. *Let c be a foliated path from m_1 to m_2 with S_1 and S_2 respective transversals at m_1 and m_2 such that the holonomy of c can be represented as a diffeomorphism $S_1 \xrightarrow{\sim} S_2$. Then there exists a neighbourhood of c in the compact open topology on $\mathcal{C}(I, M)$ in which any foliated path starting from S_1 and ending on S_2 has an holonomy which can be represented by*

$$\text{Hol}_{S_1}^{S_2}(c) : S_1 \rightarrow S_2$$

Since homotopies of paths are parametrised by a connected interval, we obtain the particularly important corollary:

Corollary 6.2.18. The holonomy of a foliated path is invariant under homotopy of foliated paths (with fixed ends).

In particular, this implies that

- Holonomy is invariant under oriented reparametrisation of the path.
- Holonomy of the reversed path of c ($t \in [a, b] \mapsto c(b + a - t)$) is the inverse of the holonomy of c between the same transversals.

Furthermore, we defined holonomy of *continuous* foliated paths, but every continuous path is *homotopic* to a smooth path [Hir12] (Chapter 5, Lemma 1.5). As a consequence, all holonomies can be constructed from smooth paths. Further homotopies can be used to ensure they have sitting instants, so that they have smooth concatenations.

From the construction of holonomy, it is manifest that it behaves well with respect to concatenation, which we write with the symbol \odot :

Theorem 6.2.19. *Let m_1, m_2 and m_3 be three points of the same leaf of \mathcal{F} and S_1, S_2, S_3 respectively transversals at m_1, m_2 and m_3 . Let c_1 be a foliated path from m_1 to m_2 and c_2 a foliated path from m_2 to m_3 . Then the following relation holds:*

$$\text{Hol}_{S_1}^{S_3}(c_2 \odot c_1) = \text{Hol}_{S_2}^{S_3}(c_2) \circ \text{Hol}_{S_1}^{S_2}(c_1)$$

Let us now deal with the question of the dependency on the transversals. If S and S' are two transversals at m , then the holonomy of a sitting path at m , which we write $[m]$, defines a germ of diffeomorphism $(m, S) \xrightarrow{\sim} (m, S')$. Therefore, if one changes the transversal at the end point of the foliated path, the germ of diffeomorphism changes according to the following relation:

$$\text{Hol}_{S_1}^{S'_2}(c) = \text{Hol}_{S_2}^{S'_2}([m_2]) \circ \text{Hol}_{S_1}^{S_2}(c)$$

Naturally, if S'_2 matches with S_2 in a neighbourhood of m_2 then the holonomy of the sitting path can be represented by the identity on a (transversal) neighbourhood of m_2 . There is a similar law for changing the transversal at the starting point. In particular, the question of whether two foliated paths have the same holonomy is independent of the choice of transversals: it is possible to talk of the *holonomy class of a foliated path*.

Linear holonomy and Bott connection

Let m_1 and m_2 be two points belonging to the same leaf with a transversal S_1 (resp. S_2) at m_1 (resp. S_2). Let c be a foliated path from m_1 to m_2 . Its holonomy defines the germ of a diffeomorphism

$$(m_1, S_1) \xrightarrow{\sim} (m_2, S_2)$$

As such it can be differentiated to obtain the *linear holonomy* of c [MM03]:

$$d\text{Hol}_{S_1}^{S_2}(c)|_{m_1} : T_{m_1}S_1 \xrightarrow{\sim} T_{m_2}S_2$$

Definition 6.2.20 (Normal bundle). The *normal bundle of the foliation* is the following quotient vector bundle:

$$\mathcal{N} = TM/T\mathcal{F}$$

Its vectors are “transverse” vectors to the foliation, in the sense that they only have a component transverse to the foliation. The normal bundle is very convenient since it gives a unique model for the tangent space to the transversals: if S is a transversal at a point m , then the projection $TM \rightarrow \mathcal{N}$ restricts to an isomorphism on T_mS :

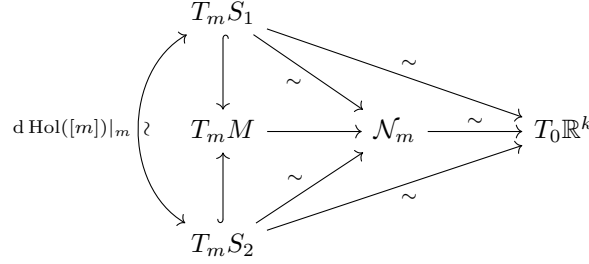
$$\begin{array}{ccc} T_mS & & \\ \downarrow & \searrow \sim & \\ T_mM & \longrightarrow & \mathcal{N}_m \end{array}$$

These isomorphisms are all consistent with the holonomy of sitting paths as we now prove. Let $\phi : (m, \mathcal{U}) \rightarrow (0, \mathbb{R}^k)$ be a submersion on a foliated neighbourhood of m such that its fibres are the plaques. Its differential vanishes on $T\mathcal{F}$ and therefore factors through \mathcal{N} . Restricted to T_mS_i , the differential gives isomorphisms:

$$T_mS_i \xrightarrow{\sim} T_0\mathbb{R}^k$$

But these are the differential at m of the maps to \mathbb{R}^k used to defined the holonomy from S_1 to

S_2 . This is summed up in the following commutative diagram:



As a consequence, the linear holonomy of c can be uniquely represented on the normal bundle as a map

$$\mathcal{N}_{m_1} \xrightarrow{\sim} \mathcal{N}_{m_2}$$

This defines a *parallel transport* on the normal bundle $\mathcal{N}_m \rightarrow M_{\mathcal{F}}$. It is associated with a covariant derivative, called *the Bott connection*, which is simply constructed as follows. Let X be a vector field on $M_{\mathcal{F}}$ and $Y_{\mathcal{N}}$ a section of $\Gamma(M_{\mathcal{F}}, \mathcal{N})$. It can be represented by an element $Y \in \Gamma(M_{\mathcal{F}}, TM)$. The covariant derivative of $Y_{\mathcal{N}}$ can be constructed as follows:

$$\nabla_X^{\text{Bott}} Y = \mathcal{L}_X Y \quad \text{mod } T\mathcal{F}$$

The parallel transport along a path can be constructed using linear holonomy; the parallel transport along flow lines of a vector field on $M_{\mathcal{F}}$ can be constructed using the differential of the flow on a representative of $X_{\mathcal{N}}$ in TM [Mor76]. The property that linear holonomy is invariant under homotopy means that the Bott connection is flat.

When \mathcal{F} is a simple foliation with a submersion $\pi : M \rightarrow M/\mathcal{F}$, there is a vector bundle isomorphism

$$\mathcal{N} \xrightarrow{\text{d}\pi} \pi^* T(M/\mathcal{F})$$

It turns out that this defines a bijection between vectors on M/\mathcal{F} and Bott-parallel sections of \mathcal{N} over leaves of \mathcal{F} [Vit18; BW97]. In terms of fields, the following holds:

$$\Gamma(M/\mathcal{F}, T(M/\mathcal{F})) \simeq \Gamma(M, \mathcal{N})^{\text{Bott-parallel}}$$

This suggests using Bott-parallel normal vector fields to model vector fields on the leaf space even when it does not have a quotient manifold structure.

Definition 6.2.21 (Basic vector fields). A *basic vector field* on M is a vector field X such that

$$\mathcal{L}_X T\mathcal{F} \subset T\mathcal{F}$$

This is equivalent to its normal component being Bott-parallel.

A *basic normal field* on M is a Bott-parallel section of \mathcal{N} .

The Bott parallelism can be used similarly to define basic (normal) differential forms on M , which are pullbacks of differential forms on M/\mathcal{F} when it is a quotient smooth manifold.

The holonomy groupoid

Holonomy of foliated paths is very suggestive of a groupoid formalism: it is compatible with concatenation and reversion of paths, and invariant under reparametrisation which allows for

associate concatenation of paths. Indeed it can be organised into the so-called *holonomy groupoid of \mathcal{F}* :

Definition 6.2.22 (Holonomy groupoid of a foliation). Let M be a smooth manifold equipped with a foliation \mathcal{F} . The *holonomy groupoid of \mathcal{F}* , written $\text{Hol}_{\mathcal{F}}$, is defined as follows:

- $(\text{Hol}_{\mathcal{F}})_0 = M$.
- $(\text{Hol}_{\mathcal{F}})_1$ is the set of holonomy classes⁴ of (smooth) paths contained in any leaf, with as source map the initial point and as target map the final point of the path.
- The identity section is given by the identity map on the transversals, which is the holonomy of sitting paths.
- Composition is given by the composition of germs of diffeomorphisms which corresponds to the holonomy class of the concatenation of paths.
- Inversion is given by the inversion of diffeomorphism germs, which are the holonomy of reversed paths.

It is given a smooth structure generated by the *smoothly parametrised* smooth foliated path. Namely, for every smoothly parametrised smooth foliated path $c : N \times [0, 1] \mapsto M$ (parametrisation is smooth with respect to the n -dimensional structure of M), its parametrised holonomy is a smooth map $N \rightarrow \text{Hol}_{\mathcal{F}}$ and a map on $(\text{Hol}_{\mathcal{F}})_1$ is smooth if and only if its composition with all these smoothly parametrised holonomies is smooth.

We now prove that this defines a Lie groupoid structure with $\dim(\text{Hol}_{\mathcal{F}})_1 = \dim M + \dim \mathcal{F}$. First, the source and target maps are manifestly smooth, since smoothly parametrised paths have smoothly parametrised endpoints.

Charts can be constructed as follows: let c be a smooth foliated path from m_1 to m_2 . Consider two foliated charts around m_1 and m_2 . They can be reduced so that

1. Each of them has a transverse S_i which crosses each plaque exactly once.
2. The supports each have a smooth *foliated* retract by deformation to the transverse:

$$r_i : \mathcal{U}_i \times [0, 1] \rightarrow S_i$$

It may furthermore be assumed to have *sitting instants*, i.e. it is stationary at small enough and high enough times.

3. The holonomy of c can be represented as a diffeomorphism between the transverses $S_1 \xrightarrow{\sim} S_2$ with a smooth foliated homotopy $h : [0, 1] \times S_1 \rightarrow M$ such that
 - $h(\cdot, 0) = \text{id}_{S_1}$.
 - $h(s_1, \cdot)$ are smooth foliated paths.
 - $h(\cdot, 1) = \text{Hol}_{S_1}^{S_2}(c)$.

It may furthermore be assumed to have sitting instants.

⁴Namely we omit the details of the path and only keep the holonomy data.

The foliated charts take the form

$$m_i \in \mathcal{U}_i \xrightarrow[\phi_i]{\sim} (s_0, 0) \in L \times S \subset L \times \mathbb{R}^k$$

with $\phi_i^{-1}(L \times \{0\})$ the transversals respectively at m_i and $S \xrightarrow{\sim} S_1 \xrightarrow{\sim} S_2$ parametrising the plaques of \mathcal{U}_1 and \mathcal{U}_2 . Given a point $s \in S$, the plaques $\phi_i^{-1}(L \times \{s\})$ are matching under $\text{Hol}_{S_1}^{S_2}(c)$. Consider the subset

$$R = \{(x_1, x_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \mid x_1 \text{ and } x_2 \text{ belong to plaques matching under } \text{Hol}_{S_1}^{S_2}(c)\}$$

It is the image of the following smooth embedding:

$$\begin{aligned} L \times L \times S &\rightarrow \mathcal{U}_1 \times \mathcal{U}_1 \\ (l_1, l_2, s) &\mapsto (\phi_1^{-1}(l_1, s), \phi_2^{-1}(l_2, s)) \end{aligned}$$

and as such, it is a submanifold of $\mathcal{U}_1 \times \mathcal{U}_2$.

This allows us to define a smoothly parametrised foliated path for each $l_1, l_2 \in L$ and $S \in L$, from $\varphi_1^{-1}(l_1, s)$ to $\varphi_2^{-1}(l_2, s)$, using \check{r} for the reverse homotopy:

$$\check{r}(\varphi_2^{-1}(l_2, s), \cdot) \odot h(s, \cdot) \odot r(\varphi_1^{-1}(l_1, s), \cdot)$$

Their holonomies give a smooth injection

$$R \simeq L \times L \times S \xrightarrow{i} (\text{Hol}_{\mathcal{F}})_1$$

Furthermore, this injection is a section over R of (s, t) , which is smooth:

$$\begin{array}{ccc} i(R) & \hookrightarrow & (\text{Hol}_{\mathcal{F}})_1 \\ \begin{array}{c} \uparrow \\ i \\ \downarrow \end{array} \begin{array}{c} (s,t) \\ \downarrow \\ (s,t) \end{array} & & \downarrow (s,t) \\ R & \hookrightarrow & M \times M \end{array}$$

There is thus a diffeomorphism $i : R \xrightarrow{\sim} i(R)$.

However, we still need to prove that the image in $(\text{Hol}_{\mathcal{F}})_1$ is open, in order to use i as a chart. Namely, we need to prove that for any smoothly parametrised smooth foliated path $H : N \times I \rightarrow M$, the subset $\text{Hol}(H)^{-1}(i(R)) \subset N$ is open. Let there be such an H ; we can assume without loss of generality that $H(n_0, \cdot) = c$ and we want to show that, on a neighbourhood of n_0 , $\text{Hol}(H)$ takes value in R .

According to the stability result 6.2.17 (more precisely Diagram 6.1), there exists in N a neighbourhood \mathcal{U}_N of n_0 such that there exist transversals T_0^n and T_1^n respectively at $H(n, 0)$ and at $H(n, 1)$ for all $n \in N$ and there are commutative diagrams

$$\begin{array}{ccc} T_0^n & \xrightarrow[\sim]{\text{Hol}(H(n, \cdot))} & T_1^n \\ \downarrow \wr & & \downarrow \wr \\ S_0 & \xrightarrow[\sim]{\text{Hol}(H(n_0, \cdot))} & S_1 \end{array}$$

with the holonomies inside \mathcal{U}_1 and \mathcal{U}_2 as vertical legs.

In particular for $n \in \mathcal{U}_N$, $H(n, 0)$ and $H(n, 1)$ belong to matching plaques under $\text{Hol}(H(n_0, \cdot)) =$

$\text{Hol}(c)$. In other words

$$\forall n \in \mathcal{U}_N, (H(n, 0), H(n, 1)) \in R$$

But there is a similar commutative diagram for images of i so that for every $n \in \mathcal{U}_N$ we have the following commutative diagram:

$$\begin{array}{ccc}
 T_0^n & \xrightarrow[\sim]{\text{Hol}(H(n, \cdot))} & T_1^n \\
 \wr \downarrow & & \downarrow \wr \\
 S_0 & \xrightarrow[\text{Hol}(c)]{\sim} & S_1 \\
 \wr \uparrow & & \uparrow \wr \\
 T_0^n & \xrightarrow[\sim]{i(H(n, 0), H(n, 1))} & T_1^n
 \end{array}
 \begin{array}{l}
 \text{id} \curvearrowright \\
 \text{id} \curvearrowleft
 \end{array}$$

We thus proved that for all n in a neighbourhood \mathcal{U}_N of n_0 , $\text{Hol}(H(n, \cdot))$ takes value in $i(R)$, which allows concluding that $i(R)$ is open.

Under the product map

$$(\text{Hol}_{\mathcal{F}})_1 \xrightarrow{(s,t)} M \times M$$

the chart described above is sent to

$$R \hookrightarrow \mathcal{U}_1 \times \mathcal{U}_2$$

so that (s, t) is an immersion. In fact, the smooth structure could be defined from this immersion, once the topology of $(\text{Hol}_{\mathcal{F}})_1$ is well established: smooth functions are continuous functions which restrict to smooth functions on the charts for the smooth structure defined by the embedding in $M \times M$.

Functoriality of the Holonomy groupoid

A natural question about the holonomy groupoid construction is that of its functoriality: does a map between foliated manifolds induces a Lie groupoid morphism between the holonomy groupoids? This raises the prior question: what is a suitable notion of map between foliated manifolds. One answer is given by the following functoriality theorem:

Theorem 6.2.23. *Let $f : M \rightarrow N$ be a smooth map between foliated manifolds which is a submersion. Assume that it is “foliated” in the following sense: the image of the leaves of M are included in leaves of N . Equivalently, the image tangent distribution to the leaves of M is included in that of N .*

*Then for any foliated path c in M from x to y the holonomy of f_*c only depends on $\text{Hol}(c)$. Furthermore for small enough transversals at x (resp. y) S_x (resp. S_y) it is possible to find submanifolds $S'_x \subset S_x, S'_y \subset S_y$ on which f is an embedding and such that the images $f(S'_x)$ and*

$f(S'_y)$ respectively form transversals at $f(x)$ and $f(y)$ and the following diagram commutes:

$$\begin{array}{ccc}
 S_x & \xrightarrow{\text{Hol}(c)} & S_y \\
 \uparrow & & \uparrow \\
 S'_x & \xrightarrow{\text{Hol}(c)|_{S'_x}} & S'_y \\
 \downarrow f|_{S'_x} \wr & & \downarrow f|_{S'_y} \wr \\
 f(S_x) & \xrightarrow{\text{Hol}(f_*c)} & f(S_y)
 \end{array}$$

As a consequence, f lifts from $f : M \rightarrow N$ to a Lie groupoid morphism between the holonomy groupoids $\text{Hol}(M) \rightarrow \text{Hol}(N)$.

Proof. We will call \mathcal{F}_M (resp. \mathcal{F}_N) the foliation of the manifold \mathcal{F}_M (resp. \mathcal{F}_N) and $T\mathcal{F}_M$ (resp. $T\mathcal{F}_N$) its tangent distribution. Let c be a foliated path in M from a point x to a point y . It has an associated holonomy in M from x to y , and its image f_*c under f is a foliated path and has an associated holonomy from $f(x)$ to $f(y)$.

Let us first show the existence of transversals such as asserted. Let S_x be a transversal to \mathcal{F}_M at x such that the holonomy of c can be represented as a foliated holonomy starting from S_x . Namely, we require the existence of a smooth map

$$\phi_0 : S_x \times [0, 1] \rightarrow M$$

such that

- $\phi_0(x, \cdot) = c$.
- $\forall a \in S_x, \phi_0(a, 0) = a$.
- $\phi_0(\cdot, 1)$ is a diffeomorphism of S_x onto a transverse to \mathcal{F}_M at x .
- For all $a \in S_x$, the path

$$t \in [0, 1] \mapsto \phi_0(a, t)$$

is a foliated path.

Since $Df(T_x M) = T_{f(x)} N$ and $Df(T_x \mathcal{F}_M) \subset T_{f(x)} \mathcal{F}_N$, we conclude that $Df(T_x S)$ has to contain a supplementary subspace to $T_{f(x)} \mathcal{F}_N$. As a consequence, there exists a submanifold $S'_x \subset S_x$ which contains x , on which f is injective and which is embedded through f into a transversal to \mathcal{F}_N at $f(x)$.

Since it may be the case that f is not an embedding on $\phi(S', 1)$, we need to construct similarly a transversal S_y to \mathcal{F}_M at y with a submanifold S'_y which is embedded under f into a transversal to \mathcal{F}_N at $f(y)$. Reducing S_x and S_y (as well as S'_x and S'_y) if need be, the holonomy of c can be represented as a smooth map

$$\phi_1 : S_x \times [0, 1] \rightarrow M$$

such that

- $\phi_1(x, \cdot) = c$.
- $\forall a \in S_x, \phi_1(a, 0) = a$.

- $\phi_1(\cdot, 1)$ is a diffeomorphism of S_x onto S_y .

- $\phi_1(S'_x, 1) = S'_y$.

- For all $a \in S_x$, the path

$$t \in [0, 1] \mapsto \phi_1(a, t)$$

is a foliated path.

In order to relate with the holonomy in N , consider the map

$$\phi_N : \begin{cases} S'_x \times [0, 1] \rightarrow N \\ (a, t) \mapsto \phi_1(a, t) \end{cases}$$

By assumption, $\phi_N(\cdot, 0)$ (resp. $\phi_N(\cdot, 1)$) coincides with the isomorphism $S'_x \xrightarrow{\sim} f(S'_x)$ with the transversal to \mathcal{F}_N at x (resp. with the isomorphism $S'_y \xrightarrow{\sim} f(S'_y)$ with the transversal at y). Furthermore for all $a \in S'_x$ the path

$$t \in [0, 1] \mapsto \phi_N(a, t)$$

is a foliated path and $\phi_N(\cdot, f(x)) = f_*c$. As a consequence, the holonomy of f_*c can be represented as in the following commutative diagram:

$$\begin{array}{ccc} S'_x & \xrightarrow{\text{Hol}_{\mathcal{F}_M}(c)|_{S'_x}} & S'_y \\ \wr \downarrow f|_{S'_x} & & \wr \downarrow f|_{S'_y} \\ f(S'_x) & \xrightarrow[\text{Hol}_{\mathcal{F}_N}(f_*c)]{\sim} & f(S'_y) \end{array}$$

Let us now show that the holonomy class of f_*c only depends on the holonomy class of c . It is equivalent to show the following: if c is a loop and has trivial holonomy, then f_*c has trivial holonomy. From now, assume that c is a loop based at x with trivial holonomy.

Let S be a transversal to \mathcal{F}_M at x such that the trivial holonomy of c can be represented on S . Namely, we require the existence of a smooth map

$$\phi : S \times [0, 1] \rightarrow M$$

such that

- $\phi(x, \cdot) = c$.

- $\forall a \in S, f(a, 0) = f(a, 1) = a$.

- For all $a \in S$, the path

$$t \in [0, 1] \mapsto \phi(a, t)$$

is a foliated path.

Similarly to previously, we consider a submanifold $S' \subset S$ on which f is an embedding onto a

transversal to \mathcal{F}_N at $f(x)$. Then the commutative diagram takes the following form:

$$\begin{array}{ccc}
 & S' & \\
 f|_{\nu} \swarrow \sim & & \searrow \sim f|_{\nu} \\
 f(S') & \xrightarrow{\sim} & f(S') \\
 & \text{Hol}_{\mathcal{F}_N}(f_*c) &
 \end{array}$$

Therefore, the holonomy of f_*c is trivial and $c \mapsto f_*c$ factors to a map between the holonomy groupoids.

Smoothness of the morphism is a direct consequence of the following observation: the image under f of a smoothly parametrised smooth foliated path on M is a smoothly parametrised smooth foliated path on N . \square

Remark. One may hope that the image under a foliated submersion of a transversal be a transversal, or that the inverse image is a transversal. This is however not the case without stronger assumptions. Indeed according to the definition we gave, nothing prevents a foliated map from sending transversal directions to directions tangent to the foliation. Similarly, the submersion may have invariant directions (directions in which the map is constant) along the fibration, which implies that inverse images will contain these directions which are tangent to the foliation.

When the map is a local diffeomorphism, it induces bijections between directions tangent to the foliations and bijections between directions non-tangent to the foliation, so that it sends transversals to transversals.

6.3 Orbifolds

Let M be a smooth manifold and G a Lie group acting smoothly on M . A recurring question is studying the *equivariant geometry* of M , that is to say G -equivariant geometrical properties and constructions on M . As is very common in geometry, it is very convenient to have a “geometrical space” such that its geometry correspond in a sense to the equivariant geometry of M . This is the idea behind *orbit spaces*. The following theorem gives a reference case in which the orbit space has a natural smooth manifold structure:

Theorem. *If the action of G is free and proper, then the quotient topological space M/G has a smooth manifold structure such that the quotient map*

$$M \rightarrow M/G$$

is a G -principal bundle fibration map.

Note in particular that the orbit space comes with the geometric structure of a G -principal bundle. The hypotheses are however restrictive, in particular freeness prevents the orbits from having different types.

(Smooth) orbifolds are a generalisation of smooth manifolds which is meant to encompass “nice” orbit spaces. In particular they are allowed “singularities” corresponding to quotients under finite isotropy groups. We first give a definition which involves atlases and is very similar to that of smooth manifolds. We next present an alternative approach which uses Lie groupoids. General references are [Car19a; ALR07; Moe02].

6.3.1 Classical orbifolds and atlases

Smooth manifolds are spaces provided with a smooth structure which are locally isomorphic to open subsets of the Euclidean spaces \mathbb{R}^n . Orbifolds are meant to extend the notion of manifold to spaces which are locally isomorphic to *quotients of open subsets of \mathbb{R}^n by finite groups*.

Definition 6.3.1 ([Mes16]). Let X be a topological space equipped with a smooth structure. It is called an n -dimensional (classical, or effective) *orbifold* if it satisfies the following local property:

- It is locally diffeomorphic to quotients of open subset \mathbb{R}^n by actions of finite groups.

as well as the following two global topological requirements:

- It is Hausdorff.
- It is paracompact.

A smooth map $\phi : \mathcal{U} \rightarrow X$ from an open subset of \mathbb{R}^n with a finite group $G_{\mathcal{U}}$ acting effectively on \mathcal{U} which is G -invariant and induces a diffeomorphism from $\mathcal{U}/G_{\mathcal{U}} \xrightarrow{\sim} \phi(\mathcal{U})$ is called an (*orbifold*) *chart*. A family of orbifold charts which cover X is called a (classical) *orbifold atlas*.

If connected smooth manifolds are homogeneous under their automorphism group, it is not the case for orbifolds. Similarly to manifolds with boundaries or corners or disconnected manifolds, orbifolds have a natural “decomposition” which is inherent to their orbifold structure hence is necessarily preserved by automorphisms.

Lemma 6.3.2 ([ALR07]). Let X be an orbifold and $x \in X$. Let (ϕ_1, G_1) and (ϕ_2, G_2) be two orbifold charts around x . Then for $x_1 \in \mathcal{U}_1$ and $x_2 \in \mathcal{U}_2$ such that $\phi_1(x_1) = \phi_2(x_2) = x$, the isotropy groups of x_1 and x_2 respectively under G_1 and G_2 are isomorphic.

Definition 6.3.3 (Local group at a point). Let X be an orbifold and $x \in X$. Then the isomorphism class of isotropy groups of inverse images of x in orbifold charts is called the *local group at x* . It is usually written Γ_x .

Theorem 6.3.4 (Canonical Stratification of an orbifold, [Car19a; Mes16]). *Let X be an orbifold. It is partitioned into*

$$\bigcup_G \Sigma_G \quad \text{with} \quad \Sigma_G = \{x \in X, \mid \Gamma_x \simeq G\}$$

The connected components of the Σ_G are manifolds called strata. The decomposition of X into strata is called the canonical stratification of X . The closure of each stratum is a reunion of strata of lower dimensions. Σ_e is a connected and dense open submanifold called regular stratum. Its complementary subset is called the singular set or singular locus of X .

Example 6.3.5 (Conical singularity of degree two). Consider a linear plane $M = \mathbb{R}^2$ with the action of parity

$$P : m \in M \mapsto -m$$

Then the quotient M/P has a smooth structure of orbifold. Globally, it is Hausdorff and second-countable. Furthermore, the quotient map $\mathbb{R}^2 \rightarrow M/P$ defines a global orbifold chart. Smooth functions on M/P are exactly *even smooth functions* on M . In particular, they necessarily have vanishing derivatives of odd order at the origin.

6.3.2 Orbifold groupoids

Another approach to orbifolds is possible using groupoids. The local groups are replaced by a cleaner transitive Lie groupoid and this allows for a geometrical notion of morphism, which we will however not touch upon. A more important feature is that the groupoid data allows for *non-effective local groups*.

The suitable class of groupoids is the so-called orbifold groupoids:

Definition 6.3.6. An *orbifold groupoid* is a Lie groupoid which is both proper and étale ⁵.

An orbifold groupoid has finite isotropy groups, since they have to be both compact and discrete. The term is justified by the following result:

Theorem 6.3.7 ([Moe02]). *The orbit space of an orbifold groupoid is a Hausdorff space and has a natural structure of orbifold, namely it admits an atlas identifying open subsets with quotients of Euclidean spaces by the isotropy groups of the orbifold.*

The restriction to an orbit of the orbifold groupoid is a transitive groupoid. As such all isotropy groups are isomorphic. However using groupoids allows using the isotropy groups as local groups. Let $\Gamma \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} M$ be an orbifold groupoid. The action of the isotropy groups on charts is essentially constructed from the groupoid morphism $\Gamma \rightarrow \text{Diff}_{\text{germs}}(M)$. In particular, it need not be effective.

Furthermore, the notion of weak equivalence of orbifold groupoids is adapted to the underlying orbifold geometry. We say that an orbifold groupoid is *effective* if at every point $m \in M$ the morphism $\Gamma(m, m) \rightarrow \text{Diff}_{\text{germs}}(M)(m, m)$ is injective. In this case the associated orbifold structure on the orbit space is effective: the local groups defined using charts are isomorphic to the isotropy groups of the orbit in the groupoid.

Theorem 6.3.8 ([ALR07]). *Let $\Phi : \Gamma \rightarrow \Gamma'$ be a weak equivalence of effective orbifold groupoids. Then the induced map between the orbit spaces is a isomorphism of effective orbifolds.*

Conversely, if the orbit spaces of effective orbifold groupoids are isomorphic then there exists a Morita equivalence between the orbifold groupoids.

Accordingly, this suggests the following definition of orbifolds:

Theorem 6.3.9 (Orbifold structure, [ALR07; Mes16]). *Let X be a paracompact Hausdorff topological space. An orbifold structure, or orbifold atlas on X is an orbifold groupoid Γ with a homeomorphism between its orbit space and X .*

⁵This definition is a bit more restrictive than the one in [Moe02].

Chapter 7

Generalised Cartan geometries

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7.1 A toy model

This chapter deals with the structure of “generalised Cartan bundle” which is a generalisation of principal bundles equipped with a Cartan connection. We depict in this brief section a simple and amenable case of this kind of structure constructed by performing a “twist” on a frame bundle.

The orthonormal direct frame bundle of \mathbb{R}^2

We will consider the plane as a smooth manifold $M = \mathbb{R}^2$ which we will represent as an open disk. It is equipped with its standard oriented Euclidean structure. This allows us to define its (orthonormal direct) frame bundle P , which is the space of direct orthonormal bases of the tangent space at each point. It forms a smooth SO_2 -principal bundle over M :

$$\text{SO}_2 \hookrightarrow P \xrightarrow{\pi} M$$

This bundle turns out to be trivialisable in many ways (for example because M is contractible) and can thus be represented as an open solid torus $M \times \text{SO}_2$, generated by the revolution of M . Since P is a space of frames, each point p defines a frame of $T_{\pi(p)}M$. Different frames above $\pi(p)$ are related by the action of SO_2 . This is represented as a cylinder in Figure 7.1: the top and bottom face should be identified. In particular, the trivialisised frame bundle comes with a

parallelisation, namely smooth vector fields which constitute at each point a linear frame of P . At each point, its first two vectors are the two frame vectors along the section (in red and green in Figure 7.1), we call them e_1 and e_2 . The third vector is the normalised vector transverse to the section, generator of the revolution action of SO_2 (in purple on Figure 7.1). We will call it ξ .

The structure of the frame bundle imposes that the frame (e_1, e_2) is *equivariant* under rotation around the revolution axis. If we call $R(\theta)$ the revolution of angle $\theta \in \mathbb{R}$ this means:

$$\begin{cases} R(\theta)^* e_1 = \cos(\theta)e_1 + \sin(\theta)e_2 & (7.1a) \\ R(\theta)^* e_2 = -\sin(\theta)e_1 + \cos(\theta)e_2 & (7.1b) \end{cases}$$

Effectively, the frame of M is rotating when progressing along the revolution. Since ξ generates the action of SO_2 , there is an infinitesimal version of the equivariance equations:

$$\begin{cases} [\xi, e_1] = e_2 & (7.2a) \\ [\xi, e_2] = -e_1 & (7.2b) \end{cases}$$

The two sets of equations are equivalent since $R(\theta) = \exp(\theta\xi)$. Now, we want to present a different representation of this situation. The frame rotation can be “untwisted” by applying a diffeomorphism of the solid torus (with nontrivial mapping class). Using the standard coordinates (x, y) on the section M and a cyclic coordinate z on SO_2 , the diffeomorphism takes the following form:

$$\begin{cases} M \times \text{SO}_2 \rightarrow M \times \text{SO}_2 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos(z)x + \sin(z)y \\ -\sin(z)x + \cos(z)y \\ z \end{pmatrix} \end{cases}$$

In this representation the fibre above a given point $m \in M$ is no longer a straight circle but twists around the torus. This is depicted in Figure 7.2.

The equivariance equations still hold, but in this representation they do not express the rotation of the frame vector but rather the “twist” of the action of SO_2 . What we are interested in are spaces which *locally* present the same structure as a frame bundle, thus we will focus on Equations (7.2). We ask the following question: if we have a solid torus P with a frame field (e_1, e_2, ξ) satisfying Equations (7.2), can it always be identified with the frame bundle of a smooth manifold?

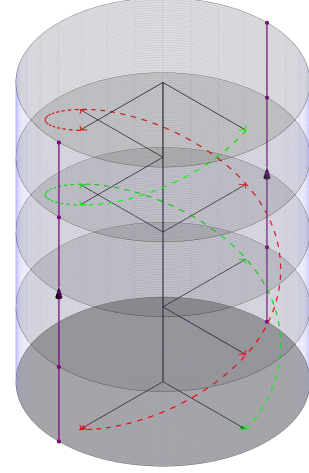


Figure 7.1: The frame bundle of \mathbb{R}^2 as an (unfolded) bulk torus

A twisted frame bundle

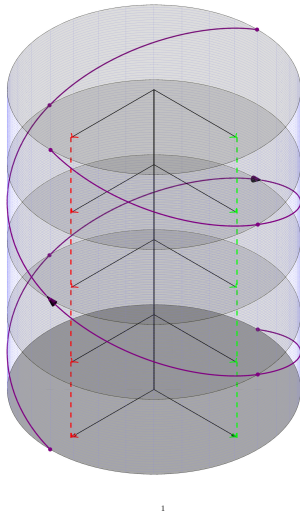


Figure 7.2: The frame bundle of \mathbb{R}^2 as a twisted torus

Counterexamples are readily found: it is possible that the action of ξ does not integrate to an action of SO_2 which is free (all isotropy groups are trivial). More precisely it is possible for the orbits under the action of ξ to have different lengths (finite or not). A simple example is depicted in Figure 7.3 – the top and the bottom face should be identified here as well. It can be seen as a $(2 + 1)$ -dimensional version of the Möbius strip. Here the infinitesimal equivariance holds with a factor $\frac{1}{2}$ for the generator ξ ¹:

$$\begin{cases} [\xi, e_1] = \frac{1}{2}e_2 \\ [\xi, e_2] = -\frac{1}{2}e_1 \end{cases}$$

The orbit of the center of M closes over one revolution, but the other orbits (in purple on Figure 7.3) require two revolutions to close. Although the action of ξ can be integrated into a group action of \mathbb{R} on $M \times \text{SO}_2$, since points have different isotropy groups it is not possible to factor the action to a quotient SO_2 of \mathbb{R} such that all isotropy groups are trivial.

In this example if P were to be interpreted as a frame bundle, it would be over its orbit space, which can be identified as $\mathbb{R}^2/(x \sim -x)$. This is a singular space, with a conic singularity at origin (which is a manifestation of the varying size of the orbits). Therefore in this example P equipped with the transverse vectors (e_1, e_2) and the generator ξ , although it cannot be identified with the frame bundle of a smooth manifold, may still be interpreted as a frame bundle over a

¹It may seem as though any real factor could be used but $\frac{1}{2}$ is of specific relevance as it can be generalised to the *projective* quotients of the higher dimension orthogonal groups. This is discussed in Section 7.3

singular space.

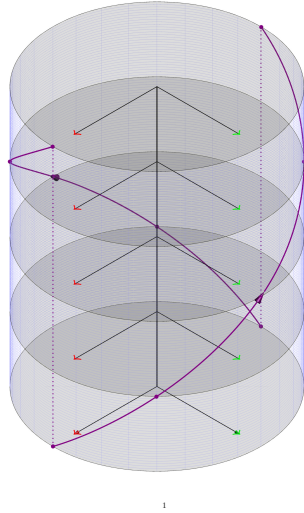


Figure 7.3: The torus as a “twisted” frame bundle

This example is “proper” in a technical sense, which ensures the singularities are tame. One could study cases in which the “twist” of the orbits under ξ is irrational so that the orbits do not close (except the central orbit) but are dense on the surface of smaller tori. These cases have harsher singularities.

In the upcoming sections we want to investigate such “generalised frame bundles” which have the local structure of frame bundles but are not a priori actual frame bundles over smooth manifolds. Our study will generalise the case of the solid torus to a more general Lie group with a linear representation of any dimension. In particular we will have to deal with the problems of *properness* of the action of a Lie group, of *completeness* of the action of a Lie algebra as well as a property pertaining to the integration of an infinitesimal action of a Lie group which is called *univalence*.

The structure of a connection

The astute reader will have noticed that the trivialisation of the orthonormal frame bundle is not uniquely defined by the frame bundle structure. It is equivalent to the additional structure of a (metric and torsionless) *flat* connection. The Euclidean plane inherits one induced by its affine space structure. In particular, the frame field with which P is equipped depends on this choice of a connection. The structure we want to study on P is more properly called “generalised frame bundle with connection” but we will use “generalised frame bundle” as a shorthand.

Indeed a more convenient way to encode the parallelisation (e_1, e_2, ξ) of P is by using the dual coframe, which is a family of three 1-forms $(\varpi^1, \varpi^2, \varpi^\xi)$. If P has the structure of an actual frame bundle, the forms ϖ^1 and ϖ^2 correspond to the so-called “solder form” of the frame bundle and the form ϖ^ξ is a connection 1-form. They are gathered in what is called a *Cartan connection (1-)form*. Using this 3-components 1-form ϖ is very convenient to formulate the infinitesimal equivariance equations (7.2) (this is detailed in Section 7.2.1).

Because the frame bundle is the space supporting tangent connections on the base space, this implies that covariant physical field theories can be formulated on the frame bundle. By extension, it is possible to formulate them on generalised frame bundles: they have all the required geometrical structure. This is of particular interest as a mean to formulate field theories on singular spaces. For example in higher dimension, given the Cartan connection form ϖ it is very easy to construct the associated Riemann curvature tensor or the Einstein tensor, so that one can study Einstein’s field equations (see our example Section 7.3.3).

7.2 Generalised Cartan geometries

7.2.1 Definition and basic constructions

Definition

Let $P \rightarrow M$ be an H -principal bundle equipped with a solder form α and a connection form ω . Gathering the two forms in a $\mathfrak{h} \oplus \mathbb{R}^n$ -valued 1-form gives a *coframe* on P , which we write $\varpi := \omega \oplus \alpha$. The coframe ϖ is H -equivariant.

As a coframe, ϖ can be used to find the vector fields representing the action of \mathfrak{h} : if $\xi \in \mathfrak{h}$ is represented by $\bar{\xi}$, the following holds:

$$\bar{\xi} = \varpi^{-1}(\xi, 0)$$

The idea can be straightforwardly generalised to a (G, H) -Cartan geometry with Cartan connection 1-form ϖ .

From there, the idea of *generalised Cartan geometry* is to start from the object ϖ and use it to define the action of \mathfrak{h} and the vertical and horizontal tangent spaces. The base manifold M is to be reconstructed as an orbit space. Of course, ϖ cannot be any arbitrary coframe: it has to be H -equivariant, and the vector fields $\bar{\xi}$ have to form a representation of \mathfrak{h} . As we will see, equivariance is already a stronger requirement than compatibility with the bracket. The coframe ϖ only defines an action of the Lie algebra \mathfrak{h} so we will be looking at equivariance under the Lie algebra action.

Let us now derive an intrinsic characterisation of the \mathfrak{h} -equivariance of ϖ .

Cartan 1-forms

First, given $\xi \in \mathfrak{h} \subset \mathfrak{g}$ we define the following vector field

$$\bar{\xi} = \varpi^{-1}(\xi) \in \Gamma(TP)$$

For a general $\zeta \in \mathfrak{g}$ we will use the same notation:

$$\bar{\zeta} = \varpi^{-1}(\zeta) \in \Gamma(TP)$$

Recall the solder form α associated with ϖ :

$$\alpha : TP \xrightarrow{\varpi} \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$$

We will use indices $i, j \dots$ for vectors of \mathfrak{h} , $a, b \dots$ for vectors of $\mathfrak{g}/\mathfrak{h}$ and $A, B \dots$ for vectors of \mathfrak{g} .

Equivariance under \mathfrak{h} can be stated as:

$$\mathcal{L}_{\bar{\xi}}\varpi + \text{ad}_{\xi}\varpi = 0 \tag{7.4}$$

The left hand term can be reformulated as follows:

$$\mathcal{L}_{\bar{\xi}}\varpi + \text{ad}_{\xi}\varpi = (i_{\bar{\xi}}d + di_{\bar{\xi}})\varpi + [\xi, \varpi] = i_{\bar{\xi}}d\varpi + d\xi + [\varpi(\bar{\xi}), \varpi] = i_{\bar{\xi}}\left(d\varpi + \frac{1}{2}[\varpi \wedge \varpi]\right)$$

Thus Equation (7.4) can be reformulated as

$$\forall \xi \in \mathfrak{h}, \quad i_{\bar{\xi}}\left(d\varpi + \frac{1}{2}[\varpi \wedge \varpi]\right) = 0 \tag{7.5}$$

Since the vectors $(\bar{\xi})_{\xi \in \mathfrak{h}}$ span the vertical directions and the forms α^a form a basis of the horizontal 1-forms, this is also equivalent to the existence of variable coefficients Ω_{bc}^A such that

$$d\varpi^A + \frac{1}{2} [\varpi \wedge \varpi]^A = \frac{1}{2} \Omega_{bc}^A \alpha^b \wedge \alpha^c \quad (7.6)$$

Another reformulation of (7.5) is obtained by contraction with $\bar{\zeta}$ for $\zeta \in \mathfrak{g}$:

$$\forall \xi \in \mathfrak{h}, \forall \zeta \in \mathfrak{g}, \quad d\varpi(\bar{\xi}, \bar{\zeta}) + [\varpi(\bar{\xi}), \varpi(\bar{\zeta})] = 0$$

As $\varpi(\bar{\zeta}) = \zeta$ is a constant \mathfrak{g} -valued field, the equation can be rephrased as

$$[\bar{\xi}, \bar{\zeta}] = \overline{[\xi, \zeta]} \quad (7.7)$$

Thus Equation (7.6) equivalent to a bracket compatibility condition on the vector fields $\bar{\zeta}$ which in particular implies that the vector fields $\bar{\xi}$ form a free representation of \mathfrak{h} on P .

This motivates the following definition:

Definition 7.2.1 (Cartan 1-forms). Let P be a manifold and (G, H) a Klein geometry with G of the same dimension as P .

A $(\mathfrak{g}, \mathfrak{h})$ -Cartan 1-form (or Cartan form) on P is a \mathfrak{g} -valued coframe ϖ^A such that there exists (variable) coefficients Ω_{bc}^A such that:

$$d\varpi^A + \frac{1}{2} [\varpi \wedge \varpi]^A = \frac{1}{2} \Omega_{bc}^A \alpha^b \wedge \alpha^c \quad (7.8)$$

with α^a the projection of ω^A to $\mathfrak{g}/\mathfrak{h}$.

We will say that a $(\mathfrak{g}, \mathfrak{h})$ -Cartan form on P defines a *generalised $(\mathfrak{g}, \mathfrak{h})$ -Cartan geometry* on P and a manifold equipped with a generalised $(\mathfrak{g}, \mathfrak{h})$ -Cartan geometry will be called a *generalised Cartan bundle*.

The notion appeared in [AM95] as ‘‘Cartan connection’’; they propose a notion of ‘‘generalized Cartan connection’’ which requires a pre-existing action of the Lie algebra but allows for a degenerate 1-form ϖ . Indeed a good part of our formal manipulation will not require vector fields and will apply to degenerate forms as well but in this case ϖ is not sufficient to define the action of \mathfrak{h} , which we want to be able to do. Cartan 1-forms are also a case of what would be called \mathfrak{h} -flatness of ϖ in [GW13] (with $K = 1$).

As justified above, a Cartan 1-form defines on the manifold an action of \mathfrak{h} for which it is equivariant. A familiar example of the phenomenon is the case of *Maurer-Cartan forms*, introduced in Example 3.3.5. It generalises directly to manifolds:

Example 7.2.2 (Maurer-Cartan forms on manifolds). Let P be an m -dimensional manifold and \mathfrak{g} an m -dimensional Lie algebra. A \mathfrak{g} -valued Maurer-Cartan form on P is a \mathfrak{g} -valued coframe ϖ satisfying the following Maurer-Cartan equation:

$$d\varpi + \frac{1}{2} [\varpi \wedge \varpi] = 0 \quad (7.9)$$

It can be interpreted as a $(\mathfrak{g}, \mathfrak{g})$ -Cartan 1-form on P .

Such a coframe defines a *transitive* action of \mathfrak{h} on P , and conversely transitive Lie algebra actions are associated to such coframes.

Remark (Curvature form and symmetry breaking). As is suggested by the Maurer-Cartan forms,

the curvature form $d\varpi + \frac{1}{2}[\varpi \wedge \varpi]$ quantifies how far the vector fields $\bar{\zeta}$ are from forming a representation of the Lie algebra \mathfrak{g} .

The structure discussed here can be approached starting from Ω rather than the fixed Klein geometry (G, H) : we have an abstract Lie algebra \mathfrak{g} and we are looking for a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that Ω is horizontal in the sense of having only components along the $\mathfrak{g}/\mathfrak{h}$ directions. From this perspective, the corresponding *Cartan geometry* can be understood as geometry with symmetry broken down from \mathfrak{g} to \mathfrak{h} [Wis12; Wis10; GW13].

We will call a manifold equipped with a $(\mathfrak{g}, \mathfrak{h})$ -Cartan 1-form a *generalised Cartan bundle* modelled over (G, H) . In the case of an affine Klein geometry $(H \ltimes \mathbb{R}^n, H)$, we talk of a *generalised frame bundle with connection*, often shortened to *generalised frame bundle*. One benefit of using the structure of Cartan bundle is that it is a purely local structure. This fact is used in a recent paper by the author [Pie22] to obtain this structure from solutions to differential equations coming from a variational principle. We come back to this in Section 7.3.3. We have justified the following result:

Theorem 7.2.3. *Let P be a generalised Cartan bundle with a $(\mathfrak{g}, \mathfrak{h})$ -Cartan 1-form ϖ . Then the map*

$$\xi \in \mathfrak{h} \mapsto \varpi^{-1}(\xi, 0) \in \Gamma(TP)$$

defines a free action of the Lie algebra \mathfrak{h} on P for which ϖ is an equivariant \mathfrak{g} -valued 1-form.

We now extend the constructions used in the previous sections to the case of generalised frame bundles.

Basic fields on Generalised Cartan Bundles

Let P be a generalised Cartan bundle with Cartan 1-form ϖ . The coframe defines an \mathfrak{h} -equivariant identification

$$TP \simeq P \times \mathfrak{g}$$

We define the horizontal and vertical distributions:

$$VP := \ker \alpha = \varpi^{-1}(\mathfrak{h}) \tag{7.10}$$

Vectors will be called vertical if they belong to VP . Conversely, differential forms will be called *horizontal* if they have vanishing contraction with vectors of VP . The space of horizontal differential forms on P will be written $\Omega_{\text{hor}}^\bullet$.

The structure equation

$$d\alpha^a + [\varpi \wedge \alpha]^a = \frac{1}{2} \Omega_{bc}^a \alpha^b \wedge \alpha^c$$

implies that the ideal spanned by the (α^a) is a differential ideal and as a consequence the vertical distribution VP is involutive: it integrates to a foliation on P .

The Lie algebra \mathfrak{h} acts on P and the action naturally lifts to natural vector bundles such as TP or $\Lambda^\bullet TP$. The action on T^*P preserves the horizontal forms:

$$\mathcal{L}_\xi \alpha = -\text{ad}_\xi \alpha$$

is a horizontal form for all $\xi \in \mathfrak{h}$. Indeed this is equivalent to the action of \mathfrak{h} preserving VP , which is a consequence of the involutivity of VP since the fundamental vector fields of \mathfrak{h} span VP .

Definition 7.2.4 (Basic fields). Corresponding to the vertical foliation, there is a notion of *basic tensors* and *basic differential forms* (Section 6.2.3).

We define (local) *basic vector fields* on P as local sections of TP/VP which are \mathfrak{h} -invariant. Note that a horizontal vector field is identified through ϖ to a $\mathfrak{g}/\mathfrak{h}$ -valued field. Since ϖ is equivariant, it is equivalent to ask for the horizontal vector field to be invariant or the $\mathfrak{g}/\mathfrak{h}$ -valued field to be equivariant.

Similarly we define *basic tensor fields*, respectively *basic differential forms*, as fields with value in a tensor product of $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}^*$, resp. *horizontal differential forms*, which are \mathfrak{h} -equivariant. Similarly to the situation of standard frame bundles, a horizontal differential form can be identified with a $\Lambda^\bullet \mathfrak{g}/\mathfrak{h}^*$ -valued field.

Finally, given a representation V of \mathfrak{g} , a *basic section* of V is an \mathfrak{h} -equivariant V -valued field. They generalise the sections of tractor bundles. We denote as follows the space of basic V -valued differential forms:

$$\Omega_{bas}^\bullet(P, V) := \Omega_{hor}^\bullet(P, V)^{\mathfrak{h}} \quad (7.11)$$

Covariant derivation

One important remark is that Lemma 3.1.15 generalises to this generalised Cartan geometry framework:

Theorem 7.2.5. *Let V be a representation of \mathfrak{g} . A V -valued horizontal k -form ψ is basic if and only if the following form*

$$d^\varpi \psi = d\psi + \varpi \cdot \psi$$

is a horizontal $(k+1)$ -form, with $\varpi \cdot \psi$ denoting an action of the \mathfrak{g} component as well as a wedge product.

The proof is identical:

$$\begin{aligned} \mathcal{L}_{\bar{\xi}} \psi + \xi \cdot \psi &= (i_{\bar{\xi}} d + d i_{\bar{\xi}}) \psi + \varpi(\bar{\xi}) \cdot \psi \\ &= i_{\bar{\xi}} d\psi + 0 + i_{\bar{\xi}}(\varpi \cdot \psi) \\ &= i_{\bar{\xi}}(d\psi + \varpi \cdot \psi) \end{aligned}$$

Definition 7.2.6 (Covariant derivation). Let V be a representation of \mathfrak{g} and ψ a basic section of V . Its *covariant derivative*, or *tractor derivative* is defined as

$$d^\varpi \psi = d\psi + \varpi \cdot \psi \in \Omega_{bas}^1(P, V) \quad (7.12)$$

which is a basic V -valued 1-form. More generally for a basic V -valued k -form ψ , its covariant exterior differential, or tractor exterior differential is defined as

$$d^\varpi \psi = d\psi + \omega \cdot \psi \in \Omega_{bas}^{k+1}(P, V) \quad (7.13)$$

which is a basic V -valued $(k+1)$ -form.

It is important to note that although horizontal differential forms are identified with $\Lambda^\bullet \mathfrak{g}/\mathfrak{h}^*$ -valued maps, horizontal differential forms and $\Lambda^\bullet \mathfrak{g}/\mathfrak{h}^*$ have different exterior covariant differentials.

Reductive geometry

When $(\mathfrak{g}, \mathfrak{h})$ has a reductive structure, namely an \mathfrak{h} -equivariant splitting

$$\mathfrak{g} \simeq_{\mathfrak{h}} \mathfrak{h} \oplus \mathfrak{m}$$

it is possible to split the coframe ϖ into a $\mathfrak{g}/\mathfrak{h}$ part α and an \mathfrak{h} part which we write ω . In this situation, the tangent bundle is split into a horizontal part and a vertical part:

$$TP = HP \oplus VP$$

with

$$HP := \ker \omega = \varpi^{-1}(\mathfrak{m}) \quad (7.14)$$

A vector belonging to HP will be called *horizontal* and a differential form will be called *vertical* if they have vanishing contraction with all vectors of HP .

By hypothesis, the action of \mathfrak{h} preserves the decomposition $\mathfrak{h} \oplus \mathfrak{m}$, therefore the action of \mathfrak{h} on T^*P preserves vertical forms:

$$\mathcal{L}_{\xi}\omega = -\text{ad}_{\xi}\omega$$

is a vertical form. As a consequence, \mathfrak{h} also preserves the horizontality of vectors.

The horizontal distribution defines an \mathfrak{h} -equivariant section $TP/VP \xrightarrow{\sim} HP \hookrightarrow TP$ and as such, it allows a simpler definition of basic tensors fields: a basic tensor field is an \mathfrak{h} -invariant section of some tensor product of HP and HP^* .

Curvature and torsion forms

The *curvature 2-form* of the Cartan 1-form is defined similarly to the frame bundle case:

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] \in \Omega_{bas}^2(P, \mathfrak{g}) \quad (7.15)$$

The *torsion* is defined in a similar fashion:

$$\Theta = d\alpha + [\omega \wedge \alpha] \in \Omega_{bas}^2(P, \mathfrak{g}/\mathfrak{h}) \quad (7.16)$$

Let ψ be a basic V -valued k -form. The following Ricci-type identity holds:

$$d^{\varpi}d^{\varpi}\psi = \Omega \cdot \psi \quad (7.17)$$

The curvature satisfies a Bianchi identity:

$$d^{\varpi}\Omega = 0 \quad (7.18)$$

7.2.2 Principal bundles

Let P be a generalised Cartan bundle with a Cartan form ϖ . We want to generalise the structure of principal bundles with a structure Lie group K to the framework of generalised Cartan bundles. There are two approaches.

The first is to generalise H -equivariant K -principal bundles on P . Of course, P does not have a group action but a Lie algebra action of \mathfrak{h} so that we can only consider \mathfrak{h} -equivariant bundles.

Definition 7.2.7 (\mathfrak{h} -equivariant K -principal bundles). An \mathfrak{h} -equivariant K -principal bundle on P is a fibre bundle $E \xrightarrow{\phi} P$ with a lift of the action of \mathfrak{h} to K -invariant vector fields on E .

Let E be such a bundle. We shall assume it is provided with an \mathfrak{h} -invariant K -principal connection 1-form which we call A (which is always possible for a pullback principal bundle as described in Section 3.2.3). The connection form A defines on E horizontal and vertical distributions; A defines a \mathfrak{k} -valued coframe on the vertical distribution, while the Cartan form ϖ

pulls back to a coframe $\phi^*\varpi$ of the horizontal distribution of E . They can be gathered into a $\mathfrak{k} \oplus \mathfrak{g}$ -valued coframe on:

$$A \oplus \phi^*\varpi \in \Omega^1(E, \mathfrak{k} \oplus \mathfrak{g})$$

which is both K -equivariant and \mathfrak{h} -equivariant.

For $\xi \in \mathfrak{h}$ we define

$$\hat{\xi} := (A \oplus \phi^*\varpi)^{-1}(0 \oplus \xi)$$

which represents ξ on E . The invariance of A under the fields $\hat{\xi}$ for $\xi \in \mathfrak{h}$ is formulated as

$$0 = \mathcal{L}_{\hat{\xi}}A = \left(di_{\hat{\xi}} + i_{\hat{\xi}}d \right) A = d0 + i_{\hat{\xi}}dA$$

Since $A(\hat{\xi}) = 0$, this is equivalent to

$$i_{\hat{\xi}} \left(dA + \frac{1}{2} [A \wedge A] \right) = 0$$

We also define for $u \in \mathfrak{k}$ the vector fields \hat{u} by which \mathfrak{k} acts on E . Similarly to Equation (7.5), equivariance of A is equivalent to

$$\forall u \in \mathfrak{k}, i_{\hat{u}} \left(dA + \frac{1}{2} [A \wedge A] \right) = 0$$

The conclusion is that the equivariance of A under $\mathfrak{k} \oplus \mathfrak{h}$ is equivalent to the existence of coefficients F_{ab}^I , with I superscripts associated to the space \mathfrak{k} , such that

$$dA^I + \frac{1}{2} [A \wedge A]^I = \frac{1}{2} F_{bc}^I \alpha^b \wedge \alpha^c \quad (7.19)$$

The \mathfrak{h} -equivariant K -principal connection thus satisfies an equation similar to (7.8), and the $\mathfrak{k} \oplus \mathfrak{g}$ -valued coframe is a Cartan 1 form: using a superscript B for $\mathfrak{k} \oplus \mathfrak{g}$, we have

$$d(A \oplus \varpi)^B + \frac{1}{2} [A \oplus \varpi \wedge A \oplus \varpi]^B = \frac{1}{2} \Omega_{bc}^B \alpha^b \wedge \alpha^c \quad (7.20)$$

To sum up, the \mathfrak{h} -equivariant K -principal bundle E has the structure of a generalised Cartan bundle modelled on $(K \times G, K \times H)$.

This brings us to the second approach, which is to consider the K -principal fibration as directions in a generalised bundle instead of an actual K -principal bundle. This requires the generalised bundle to have the data of a K -principal connection which will be integrated into a coframe used to define the generalised Cartan bundle structure. A generalised Cartan bundle modelled on $\mathfrak{k} \oplus \mathfrak{g}$ is equipped with commuting actions of \mathfrak{k} and \mathfrak{h} . If \mathfrak{k} only acts trivially on $\mathfrak{g}/\mathfrak{h}$, it can be used to define basic sections of vector bundle associated to representations of \mathfrak{k} . In this sense, such a generalised Cartan bundle can support “internal degrees of freedom” with infinitesimal variations corresponding to \mathfrak{k} .

From this perspective, principal bundles are readily integrated in the formalism of generalised frame bundles: they correspond to a Lie-subalgebra direct factor $\mathfrak{k} \subset \mathfrak{k} \oplus \mathfrak{g}$ which acts trivially on $\mathfrak{g}/\mathfrak{h}$.

7.2.3 Isotropy groups

Contrary to standard Cartan bundles, generalised Cartan bundles do not a priori have a group action. Indeed, in our definition the group H is not even specified, there is only a Lie algebra \mathfrak{h} . In this section we give a first approach to the construction of isotropy groups.

Let P be a (right) \mathfrak{h} -manifold. All paths we will consider are smooth, parametrised by $[0, 1]$ with *sitting instants*, that is to say they are locally constant around their ends. This will allow for seamless concatenation of smooth paths.

Let H be a connected Lie group integrating \mathfrak{h} ; we write ω_H its Maurer-Cartan form (Example 3.3.5). We want to act with an element $h \in H$ on a point $p \in P$. Let γ be a path on H from e to h . A natural construction is to try to integrate the following differential equation:

$$c'(t) = c(t) \cdot \omega_H(\gamma'(t)) \quad (7.21)$$

$$c(0) = p \quad (7.22)$$

The equation does not always have a complete solution but when it does the final end of c provides a candidate for $x \cdot h$. We will write such a complete solution c_p^γ . Of course, for this to hold as a definition we have to make sure that the final end is independent of the choice of γ . We will set up the general geometric approach to this question in Section 8.1.1 while the constructions in the present section are more hands-on. Let us define the following concatenation between paths in H starting from e (notice the unusual order!):

$$\gamma_0 \star \gamma_1 : \begin{cases} t \in [0, 1/2] & \mapsto \gamma_0(2t) \\ t \in [1/2, 1] & \mapsto \gamma_0(1) \cdot \gamma_1(2t - 1) \end{cases}$$

The path $\gamma_0 \star \gamma_1$ is a path from e to $\gamma_0(1) \cdot \gamma_1(1)$. Concatenation is only associative up to homotopy but it will not matter for our purposes. Then $c_p^{\gamma_0 \star \gamma_1}$ exists if and only if $c_p^{\gamma_0}$ exists and $c_{[c_p^{\gamma_0}(1)]}^{\gamma_1}$ exists, in which case

$$c_p^{\gamma_0 \star \gamma_1} = c_{[c_p^{\gamma_0}(1)]}^{\gamma_1} \odot c_p^{\gamma_0}$$

with \odot denoting the standard concatenation of smooth paths at a coinciding end. Notice the opposite orders of the two concatenations.

We also define the following path reversion:

$$\check{\gamma} : t \in [0, 1] \rightarrow \gamma(1)^{-1} \cdot \gamma(1 - t)$$

which is a path from e to $\gamma(1)^{-1}$. It is defined such that $c_p^{\check{\gamma}}$ exists if and only if there exists p_0 and $c_{p_0}^\gamma$ with $c_{p_0}^\gamma(1) = p$. In this case

$$c_p^{\check{\gamma}}(t) = c_{p_0}^\gamma(1 - t)$$

Even if H does not have a defined action on P , it is possible to have an idea of the type of the hypothetical orbit of p by considering its H -isotropy group, which we now define:

$$\Gamma^H(p) := \left\{ h \in H \mid \exists \gamma \in \mathcal{C}^\infty([0, 1], H), \begin{cases} \gamma(0) = e \\ \gamma(1) = h \\ c_p^\gamma \text{ exists and } c_p^\gamma(1) = p \end{cases} \right\} \quad (7.23)$$

It is a subgroup of H : the constant path on e shows that $e \in \Gamma^H(p)$, concatenation of paths with final ends h_1 and h_2 gives a path with final end $h_1 \cdot h_2$, and the reverse of a path with final end h

gives a path with final end h^{-1} .

When the action of \mathfrak{h} integrates to a Lie group action of H then $\Gamma^H(p)$ is exactly the isotropy group of p under the action of H . In this case all paths γ have associated solutions c_p^γ starting from every point p .

Let \tilde{H} be the simply-connected integration of \mathfrak{h} : there is a unique covering map $\tilde{H} \xrightarrow{f} H$, which is associated with the identity on \mathfrak{h} . It can be used to lift and project paths starting from the neutral elements, establishing a bijection

$$\{\text{Paths in } H \text{ starting from } e_H\} \xrightleftharpoons[f_*]{f^*} \{\text{Paths in } \tilde{H} \text{ starting from } e_{\tilde{H}}\}$$

In particular, given $\gamma \in \mathcal{C}^\infty([0, 1], \tilde{H})$ with a tangent vector field γ' ,

$$\forall t \in [0, 1], \quad \omega_H(f_*\gamma'(t)) = \omega_{\tilde{H}}(\gamma'(t))$$

so that given $p \in P$ a solution c_p^γ is equivalently a solution $c_{f_*\gamma}^p$. One concludes that h_0 is in $\Gamma^H(p)$ if and only if $f(h_0)$ is in $\Gamma^{\tilde{H}}(p)$. Thus all isotropy groups can be determined from the \tilde{H} -isotropy groups, as their projections under the natural covering map. Conversely the \tilde{H} -isotropy group is the inverse image under the covering map of the H -isotropy group for any Lie group integration H . We have thus proved the following:

Theorem 7.2.8. *Let P be an \mathfrak{h} -manifold and \tilde{H} the simply-connected Lie group integration of \mathfrak{h} . Then for any connected Lie group integration H of \mathfrak{h} with a covering map and $\tilde{H} \xrightarrow{f} H$ and any point p in P , the following holds:*

$$f^{-1}(\Gamma^H(p)) = \Gamma^{\tilde{H}}(p)$$

As a direct consequence

$$\Gamma^H(p) = f(\Gamma^{\tilde{H}}(p))$$

The H -isotropy groups share with the usual isotropy groups the following property: they are all conjugated along orbits. Let $p_0, p_1 \in P$ connected by a $c_{p_0}^{\gamma_0}$ for γ_0 a smooth path in H from e to g_0 with sitting instants. Let $c_{p_1}^\gamma$ which is a loop based at p_1 . There is a conjugated loop based at p_0 constructed by concatenation $(c_{p_0}^{\gamma_0})^{-1} \odot c_{p_1}^\gamma \odot c_{p_0}^{\gamma_0}$. It is associated with a path $\gamma_0 \star \gamma \star \gamma_0^{-1}$ from e to $\gamma_0(1) \cdot \gamma(1) \cdot \gamma_0(1)^{-1}$. Thus we have the following inclusion

$$\text{Ad}_{\gamma_0(1)} \Gamma^H(p_1) \subset \Gamma^H(p_0)$$

Finally, as γ_0 can be reversed the opposite inclusion holds and we have the following theorem:

Theorem 7.2.9. *Let P be an \mathfrak{h} -manifold and H a Lie group integration of \mathfrak{h} . Let p_0, p_1 be two points of P with γ a smooth path in H starting from e such that $c_{p_0}^\gamma(1) = p_1$. Then²*

$$\Gamma^H(p_1) = \text{Ad}_{\gamma(1)^{-1}} \Gamma^H(p_0) \tag{7.24}$$

Another property is that the H -isotropy groups are preserved by equivariant maps between \mathfrak{h} -manifold:

²The unusual conjugation is due to the Lie algebra acting *on the right*.

Theorem 7.2.10. *Let P and Q be two \mathfrak{h} -manifolds and H a Lie group integration of \mathfrak{h} . Let $\phi : P \rightarrow Q$ be a \mathfrak{h} -equivariant smooth map.*

Then for all p in P ,

$$\Gamma^H(p) \subset \Gamma^H(\phi(p))$$

The proof is straightforward: if c_p^γ is a complete solution in P then $\phi(c_p^\gamma)$ is a complete solution for the same path γ starting from $\phi(p)$:

$$\phi(c_p^\gamma) = c_{\phi(p)}^\gamma$$

In particular for a vector space V with a representation of H , the following corollary holds:

Corollary 7.2.11. *Let P be an \mathfrak{h} -manifold, H a Lie group integration of \mathfrak{h} and V a representation of H .*

Let $\phi : P \rightarrow V$ be an \mathfrak{h} -equivariant field. Then at each point p of P , $\phi(p)$ is invariant under $\Gamma^H(p)$.

Example 7.2.12. Consider a special orthochronous orthogonal group $\mathrm{SO}_{p,q}^+$. Its 2-covering by the corresponding spin group induces a Lie algebra isomorphism:

$$\mathfrak{spin}_{p,q} \xrightarrow{\sim} \mathfrak{so}_{p,q}$$

The $\mathrm{Spin}_{p,q}$ -isotropy groups are

$$\Gamma^{\mathrm{Spin}_{p,q}^+}(x) = \ker(\mathrm{Spin}_{p,q}^+ \rightarrow \mathrm{SO}_{p,q}^+) \simeq \mathbb{Z}/2\mathbb{Z}$$

Let Σ be an irreducible spinor representation of $\mathrm{Spin}_{p,q}^+$. Let $\psi : \mathrm{SO}_{p,q}^+ \rightarrow \Sigma$ be a field which is equivariant under $\mathfrak{spin}_{p,q} \simeq \mathfrak{so}_{p,q}$. Then the values of ψ have to be invariant under $\mathbb{Z}/2\mathbb{Z}$ which acts as -1 on spinor representations: ψ is necessarily identically zero.

This fact will be used in the examples presented in Section 7.3.

Recurrence orbits

Let γ be a path in H from e to e and $p \in P$ such that c_p^γ exists. The path c_p^γ may not be a loop. When this property holds for all paths γ and starting points p , the \mathfrak{h} -manifold is called H -univalent (more in Section 8.1.1). A simple example is a nontrivial connected covering $H_1 \rightarrow H$: there are loops based at e in H which lift to non-loops.

The points of P which can be connected to p by paths integrating the action of loops of H based at e form the *recurrence orbit*³ of p . More formally, it is the set defined as follows:

$$p \cdot I^H := \left\{ p' \in P \mid \exists \gamma \in \mathcal{C}^\infty([0, 1], H), \left\{ \begin{array}{l} \gamma(0) = \gamma(1) = e \\ c_p^\gamma \text{ exists and } c_p^\gamma(1) = p' \end{array} \right\} \right\} \quad (7.25)$$

We call these sets recurrence *orbits* because they define an equivalence relation on P : for all p, p' and p'' in P ,

- $p \in p \cdot I^H$
- $p' \in p \cdot I^H \Leftrightarrow p \in p' \cdot I^H$

³It is adapted from *recurrence set* suggested in [MK22].

$$\bullet \begin{cases} p' \in p \cdot I^H \\ p'' \in p' \cdot I^H \end{cases} \implies p'' \in p \cdot I^H$$

as can be proved using a sitting path at e , concatenation and inversion of paths. In a way, the recurrence orbits can be understood as the orbits under the neutral element of H .

The recurrence orbits $p \cdot I^H$ are dependent on the choice of the Lie group integration H : for a covering $H_1 \rightarrow H$, the orbit $p \cdot I^H$ can be identified as the points connected to p by paths c_p^γ integrating paths γ going from $e \in H_1$ to any element of the discrete subgroup $\ker(H_1 \rightarrow H)$.

The recurrence orbits are all trivial when the action of \mathfrak{h} integrates to an action of the Lie group H . We will see in Section 8.1.2 to what extent the converse holds.

Here again there is a functoriality property:

Theorem 7.2.13. *Let P and Q be two \mathfrak{h} -manifolds and H a Lie group integration of \mathfrak{h} . Let $\phi : P \rightarrow Q$ be an \mathfrak{h} -equivariant smooth map.*

Then for all p in P ,

$$\phi(p \cdot I^H) \subset \phi(p) \cdot I^H$$

In particular, the following corollary holds:

Corollary 7.2.14. *Let P be an \mathfrak{h} -manifold, H a Lie group integration of \mathfrak{h} and V a representation of H .*

Let $\phi : P \rightarrow V$ be a \mathfrak{h} -equivariant field defined on P . Then at each point p of P ,

$$\forall p' \in p \cdot I^H, \quad \phi(p') = \phi(p)$$

7.3 Examples of Generalised Frame Bundles

In this section we list a few examples in which the structure of generalised frame bundle with connection becomes relevant. A first class of examples actually have a group action but with varying orbit types.

7.3.1 Conical singularity with restricted chirality

Consider the group Spin_4 . It can be decomposed as a direct product of groups:

$$\text{Spin}_4 \simeq \text{Spin}_3 \times \text{Spin}_3 \simeq \text{Sp}_1 \times \text{Sp}_1$$

with Sp_1 the group of unitary quaternions. The action of Spin_4 on \mathbb{R}^4 by projection to SO_4 can be represented using the quaternionic structure: $\text{Sp}_1 \times \text{Sp}_1$ acts on the quaternion space \mathbb{H} by:

$$\begin{aligned} \text{Spin}_4 \times \mathbb{H} &\simeq \text{Sp}_1 \times \text{Sp}_1 \times \mathbb{H} \rightarrow \mathbb{H} \\ (g, z) &\simeq (p, q, z) \quad \mapsto \bar{p}zq \end{aligned}$$

Indeed the Clifford algebra Cl_4 is isomorphic to $M_2(\mathbb{H})$ and the subspace of vectors \mathbb{R}^4 can be identified with a quaternionic line within Cl_4 on which Spin_4 has the described action.

Consider now the semi-direct product $\text{Spin}_4 \ltimes \mathbb{R}^4$: it is the product manifold equipped with the following product structure:

$$(g_1, x_1) \cdot (g_2, x_2) = (g_1 g_2, g_2^{-1} \cdot x_1 + x_2)$$

It can be interpreted as the space of “spinorial frames” above the affine (coset) space

$$\begin{aligned} (\mathrm{Spin}_4 \ltimes \mathbb{R}^4) / \mathrm{Spin}_4 &\simeq \mathbb{R}^4 \\ [g, x] &\mapsto g \cdot x \end{aligned}$$

Let γ^5 be a fixed chirality element of Spin_4 : it is the (ordered) product in the Clifford algebra Cl_4 of vectors of an orthonormal direct basis (its sign depends on the chosen orientation). It squares to identity and takes the form

$$\begin{aligned} \mathrm{Spin}_4 &\simeq \mathrm{Sp}_1 \times \mathrm{Sp}_1 \\ \gamma^5 &\leftrightarrow (1, -1) \end{aligned}$$

The chirality element generates a central \mathbb{Z}_2 subgroup of Spin_4 :

$$\{(1, 1), (1, -1)\} \subset \mathrm{Sp}_1 \times \mathrm{Sp}_1 \simeq \mathrm{Spin}_4$$

Let us consider the quotient of $\mathrm{Spin}_4 \ltimes \mathbb{R}^4$ under the corresponding left action of \mathbb{Z}_2 :

$$P := (\mathbb{Z}_2) \backslash (\mathrm{Spin}_4 \ltimes \mathbb{R}^4)$$

with the generator of \mathbb{Z}_2 acting as

$$\gamma^5 \cdot (g, x) = (\gamma^5 g, x)$$

The quotient map

$$\mathrm{Spin}_4 \ltimes \mathbb{R}^4 \rightarrow (\mathbb{Z}_2) \backslash (\mathrm{Spin}_4 \ltimes \mathbb{R}^4)$$

is a 2-fold covering map.

As a subgroup of $\mathrm{Spin}_4 \ltimes \mathbb{R}^4$, it is no longer central and not even a normal subgroup so that P is not a quotient group but it still has a left action of Spin_4 .

The Maurer-Cartan form ϖ on $\mathrm{Spin}_4 \ltimes \mathbb{R}^4$ is invariant under left action by $\mathrm{Spin}_4 \ltimes \mathbb{R}^4$ hence factors to P . The manifold P is naturally fibred above $(\mathbb{Z}/2\mathbb{Z}) \backslash \mathbb{R}^4$ which has a conical singularity at the origin.

The perspective of generalised frame bundles suggests defining spinor fields on $(\mathbb{Z}/2\mathbb{Z}) \backslash \mathbb{R}^4$ as basic spinor fields on $(\mathbb{Z}_2) \backslash (\mathrm{Spin}_4 \ltimes \mathbb{R}^4)$.

Computing the isotropy groups, one finds that for x in $\mathbb{R}^4 \setminus \{0\}$, $[e, x] \in P$ has orbits of type Spin_4 but $(e, 0)$ has an orbit of type $\mathrm{Spin}_3 \times \mathrm{SO}_3$. That is, the Spin_4 -isotropy groups of the orbit of $[e, 0]$ correspond to the subgroup $1 \times \mathbb{Z}_2 \subset \mathrm{Spin}_3 \times \mathrm{Spin}_3$. The decomposition $\mathrm{Spin}_4 \simeq \mathrm{Spin}_3 \times \mathrm{Spin}_3$ corresponds to the decomposition of Dirac spinors under respectively left-handed and right-handed spinors. For a spinor, being invariant under $1 \times \mathbb{Z}_2$ is equivalent to having a vanishing right-handed part. As a consequence, basic spinor fields on $(\mathbb{Z}_2) \backslash (\mathrm{Spin}_4 \ltimes \mathbb{R}^4)$ necessarily have a vanishing right-handed component above the origin of $(\mathbb{Z}_2) \backslash \mathbb{R}^4$.

In other words, the geometry of the generalised frame bundle P requires the spinor fields on $(\mathbb{Z}_2) \backslash \mathbb{R}^4$ to have a right-handed part which vanishes at the origin.

7.3.2 Locally Klein geometries

This example generalises the previous one and gives a large family of *flat* frame bundles above singular spaces.

Let H be a Lie group acting on a vector space V . Consider the Lie group $H \ltimes V$: it has a (right) Maurer-Cartan form ϖ (Example 3.3.5). The action of the group H on the left on $H \ltimes V$

preserves ϖ while it is equivariant under the right action of H .

The group $H \times V$ can be seen as a frame bundle as follows:

$$\begin{array}{ccc} H \times V & \longrightarrow & (H \times V) / H \xrightarrow{\sim} V \\ (h, v) & \longmapsto & [h, v] \longmapsto h \cdot v \end{array}$$

The *diffeomorphism* $(H \times V) / H \simeq V$ is a consequence of the following section:

$$v \in V \mapsto (e, v) \in H \times V$$

with $\{e\} \times V$ crossing exactly once each (right) orbit under H . Since H is not a normal subgroup, the isomorphism $(H \times V) / H \simeq V$ is not a group homomorphism. Furthermore the splitting gives a section of the frame bundle over V so that the frame bundle is trivialisable:

$$\begin{array}{ccc} H \times V & \xrightarrow{(h,v) \mapsto (h,h \cdot v)} & H \times V \\ & \searrow & \swarrow \\ & V & \\ & \swarrow & \searrow \\ & & V \end{array}$$

$(h,v) \mapsto h \cdot v$ $(h,v) \mapsto v$

with the trivial right action of $h_1 \in H$ on $H \times V$:

$$(h, v) \cdot h_1 = (hh_1, v)$$

On the trivialisised bundle, the Maurer-Cartan form takes the form

$$\omega_H \oplus h^{-1} \cdot \omega_V$$

Now let K be a discrete subgroup of H . It defines a left coset manifold

$$P := K \backslash (H \times V)$$

of which $H \times V$ is a covering. Furthermore the left action of K commutes with the right action of H so that H has an induced right action on P .

Since ϖ is invariant under the left action of H , it factors to P to a 1-form ϖ_P which still satisfies the Maurer-Cartan equation:

$$d\varpi_P + \frac{1}{2} [\varpi_P \wedge \varpi_P] = 0$$

To understand the underlying manifold, let us construct the orbit space of P under the right action of P : it is the double coset space

$$K \backslash (H \times V) / H$$

Since we know $(H \times V) / H$ is isomorphic to V as a left H -space, we can conclude the following:

$$K \backslash (H \times V) / H \simeq K \backslash V$$

As a conclusion, $K \backslash (H \times V)$ models a H -structure above the quotient space $K \backslash V$. These examples are generalisations of what is called a *locally Klein geometry*, which are (connected, regular) quotients of homogeneous spaces by discrete groups [Sha97].

The example presented in the introductory section 7.1 can be interpreted as a case of this construction. The torus is a representation of the frame bundle $\mathrm{SO}_2 \ltimes \mathbb{R}^2$. Dually to the $\mathfrak{so}_2 \ltimes \mathbb{R}^2$ -valued coframe there is a frame (Z, e_1, e_2) . The construction of the twisted frame bundle is akin to considering the quotient under the left action of $\{\pm 1\} \subset \mathrm{SO}_2$ and renormalising Z as $\frac{1}{2}Z$. The underlying manifold can be constructed as

$$\begin{aligned} \{\pm 1\} \backslash (\mathrm{SO}_2 \ltimes \mathbb{R}^2) &\rightarrow \{\pm 1\} \backslash \mathbb{R}^2 \\ [h, v] &\mapsto [h \cdot v] \end{aligned}$$

There is an exceptional PSO_2 fibre above the origin of $\{\pm 1\} \backslash \mathbb{R}^2$ but the other fibres are of type SO_2 .

7.3.3 Dynamical generalised frame bundle structures

Generalised frame bundles with connections are spaces with the same local structure as frame bundles with connection. Therefore they serve as a useful generalisation when one expects to produce a frame bundle structure or more generally a Cartan geometry structure from local equations. In General Relativity the causal and gravitational structure of spacetime are encoded in a Lorentzian structure on a four-dimensional spacetime. Depending on the variant, the metric may be supplanted by a coframe field (a so-called *tetrad*) and the metric connection may have supplementary degrees of freedom. The point is that this geometry corresponds to a G -structure with connection over the spacetime. Generalised frame bundles with connection provide a new frame for dynamically defined G -structures, more precisely a generalisation thereof. In particular, the solder form is the frame bundle equivalent of the tetrad field.

Generalised frame bundles have sufficient structure to define curvature, torsion and matter fields which are all basic fields. This was already put to use in [NR78a] for the specific case of “group manifolds”. For example, the Einstein tensor can be defined as follows. We write η_{ab} for a \mathfrak{g} -invariant metric on \mathbb{R}^4 and ρ_{ia}^b for the components of the representation (the i index corresponds to \mathfrak{so}_4)

$$\rho : \mathfrak{g} \rightarrow \mathfrak{so}(\mathbb{R}^4, \eta)$$

Recall the curvature 2-form:

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] \in \Omega_{bas}^2(P, \mathfrak{g})$$

The curvature tensor is constructed as

$$\mathcal{R} = \Omega^i \rho_i \in \Omega_{bas}^2(P, \mathfrak{so}(\mathbb{R}^4, \eta))$$

The associated Ricci tensor is a basic \mathbb{R}^{4*} -valued 1-form:

$$\mathrm{Ric}_d = \Omega_{bc}^i \rho_{id}^b \alpha^c \in \Omega_{bas}^1(P, \mathbb{R}^{4*})$$

and the scalar curvature is a basic scalar:

$$\mathrm{Scal} = \mathrm{Ric}_a(\zeta^a) \in \Omega_{bas}^0(P)$$

The (tetradic) Einstein tensor is defined as

$$\mathrm{Ein}_a = \mathrm{Ric}_a - \frac{1}{2} \mathrm{Scal} \eta_{ab} \alpha^b \in \Omega_{bas}^1(P, \mathbb{R}^{4*})$$

and with a matter field ψ on which a stress-energy tensor $T \in \Omega_{bas}^1(P, \mathbb{R}^{4*})$ depends, Einstein's field equations can be formulated as

$$\text{Ein} = T \in \Omega_{bas}^1(P, \mathbb{R}^{4*}) \tag{7.26}$$

Following an idea from [Tol78] and revisited in [HV16] along lines described in Section 10.3, we construct in Chapter 11 a Lagrangian field theory on a 10-dimensional manifold with an $\mathfrak{so}_4 \ltimes \mathbb{R}^4$ -valued coframe ϖ and a field ψ with value in a spinor representation of Spin_4 as dynamical fields. The theory uses a kind of generalised Lagrange multipliers, and the equations of motion constrain ϖ to be a Cartan 1-form and impose equations of Einstein-Cartan type and Dirac type on ϖ and ψ , which also involve the Lagrange multipliers. We proceed to show that in the case the generalised \mathfrak{so}_4 -structure is a standard Spin_4 -structure the field equations can be decoupled from the Lagrange multipliers and the usual Einstein-Cartan-Dirac field equations are recovered on the underlying spacetime.

7.4 Universal Formulas for \mathfrak{g} -valued 1-forms

In this section we establish general formulas which hold for spaces equipped with a Lie algebra-valued 1-form. Let \mathcal{P} be a differentiable manifold and \mathfrak{g} a unimodular Lie algebra (i.e. the adjoint action is traceless), for example a semisimple Lie algebra. We consider the space

$$Q := T^*\mathcal{P} \otimes \mathfrak{g}$$

of \mathfrak{g} -valued covectors on \mathcal{P} . A \mathfrak{g} -valued 1-form is exactly a section of $Q \xrightarrow{\pi} \mathcal{P}$, hence equations holding on Q will be true when pulled back to \mathcal{P} by any \mathfrak{g} -valued 1-form.

The standard situation is the case in which \mathcal{P} is an H -principal bundle over some base manifold and it is equipped with a \mathfrak{g} -valued coframe which is a Cartan 1-form for some Klein model (G, H) . A direct generalisation is the situation of generalised Cartan geometries. However the computations we present here require no assumption on the \mathfrak{g} -valued 1-form.

We require that \mathfrak{g} is equipped with a volume form $\text{vol}_{\mathfrak{g}}$, which then induces linear isomorphisms

$$\begin{aligned} \Lambda^k \mathfrak{g} &\xrightarrow{\sim} \Lambda^{n-k} \mathfrak{g}^* \\ \tau &\mapsto i_{\tau} \text{vol}_{\mathfrak{g}} \end{aligned}$$

with $n = \dim \mathfrak{g}$, as detailed in Section 1.4. In particular we will be using superscripts $A, B, C \dots$ for vectors of \mathfrak{g} and the aforementioned morphisms can be written

$$(\text{vol}^{(n-k)})_{AB\dots K} : \Lambda^k \mathfrak{g} \rightarrow \Lambda^{n-k} \mathfrak{g}^*$$

The space Q has a canonical \mathfrak{g} -valued 1-form which we call λ . It can be constructed as follows:

$$\lambda : T(T^*\mathcal{P} \otimes \mathfrak{g}) \xrightarrow{d\pi} \pi^*T\mathcal{P} \simeq T\mathcal{P} \times_{\mathcal{P}} T^*\mathcal{P} \otimes \mathfrak{g} \xrightarrow{(u, \phi) \mapsto \phi(u)} \mathfrak{g}$$

As an element of $\Omega^1(Q, \mathfrak{g})$, λ induces a graded algebra morphism

$$\Lambda^{\bullet} \mathfrak{g}^* \rightarrow \Omega^{\bullet}(Q)$$

and in particular this defines $(n - k)$ -forms which are parametrised by $\Lambda^k \mathfrak{g}^*$:

$$\lambda_{AB\dots K}^{(n-k)}$$

We define the following quantity

$$\Lambda := d\lambda + \frac{1}{2} [\lambda \wedge \lambda]$$

which represents a “universal curvature 2-form”: when \mathcal{P} is a principal bundle with a Cartan connection 1-form, the pullback to \mathcal{P} of Λ is the curvature 2-form.

Our formulas will also involve maps with value in a representation of \mathfrak{g} : let Σ be a \mathfrak{g} -module and Σ^* the dual module. We will consider the trivial bundle

$$\Sigma \times Q \rightarrow Q$$

as well as its dual trivial bundle, and define the fibre coordinates

$$s : \Sigma \times Q \rightarrow \Sigma$$

and

$$\bar{s} : \Sigma^* \times Q \rightarrow \Sigma^*$$

We define the corresponding universal covariant differential

$$d^\lambda s = ds + \lambda \cdot s$$

which are simply the second factor projections. Let ψ be a Σ -valued k -form on Q . We define the following exterior covariant differential:

$$d^\lambda \psi = d\psi + \lambda \cdot \psi$$

with an implicit wedge product in the notation $\lambda \cdot \psi$.

We will show that in this “universal” framework, the following formulas hold:

$$d\lambda^{(10)} = \Lambda^A \wedge \lambda_A^{(9)} \tag{7.27a}$$

$$d\lambda_A^{(9)} = \Lambda^C \wedge d\lambda_{AC}^{(8)} = d^\lambda \lambda_A^{(9)} \tag{7.27b}$$

$$d\lambda_{AB}^{(8)} = \Lambda^C \wedge d\lambda_{ABC}^{(7)} - c_{AB}^C \lambda_C^{(9)} \tag{7.27c}$$

$$d(\bar{s}_\alpha s^\alpha) = (d^\lambda \bar{s})_\alpha s^\alpha + \bar{s}_\alpha d^\lambda s^\alpha \tag{7.27d}$$

$$d^\lambda(\psi^\alpha \wedge \mu) = (d^\lambda \psi^\alpha) \wedge \mu + (-1)^k \psi^\alpha \wedge d\mu \tag{7.27e}$$

$$d^\lambda \Lambda = 0 \tag{7.27f}$$

$$d^\lambda(d^\lambda s)^\alpha = \Lambda \cdot s^\alpha \tag{7.27g}$$

with μ an arbitrary differential form on Q .

The first computation goes as follows:

$$\begin{aligned} d\lambda^{(10)} &= (d\lambda^A) \wedge \lambda_A^{(9)} \\ &= \left(\Lambda - \frac{1}{2} [\lambda \wedge \lambda] \right)^A \wedge \lambda_A^{(9)} \\ &= \Lambda^A \wedge \lambda_A^{(9)} \end{aligned}$$

In a similar fashion

$$\begin{aligned} d\lambda_A^{(9)} &= (d\lambda^B) \wedge \lambda_{AB}^{(8)} \\ &= \left(\Lambda - \frac{1}{2} [\lambda \wedge \lambda] \right)^B \wedge \lambda_{AB}^{(8)} \\ &= \Lambda^B \wedge \lambda_{AB}^{(8)} - \frac{1}{2} c_{CD}^B \lambda^C \wedge \lambda^D \wedge \lambda_{AB}^{(8)} \\ &= \Lambda^B \wedge \lambda_{AB}^{(8)} - c_{AB}^B \lambda^{(10)} \\ &= \Lambda^B \wedge \lambda_{AB}^{(8)} \end{aligned}$$

with $c_{AB}^B = 0$ by *unimodularity* of the Lie algebra. To relate $d\lambda_A^{(9)}$ to $d^\lambda \lambda_A^{(9)}$ one just has to notice that

$$\lambda \cdot \lambda_A^{(9)} = -c_{BA}^C \lambda^B \wedge \lambda_C^{(9)} = -c_{BA}^B \lambda^{(10)} = 0$$

according to the same argument, so that

$$d^\lambda \lambda_A^{(9)} = d\lambda_A^{(9)}$$

The next computation is similar:

$$\begin{aligned} d\lambda_{AB}^{(8)} &= (d\lambda^C) \wedge \lambda_{ABC}^{(7)} \\ &= \left(\Lambda - \frac{1}{2} [\lambda \wedge \lambda] \right)^C \wedge \lambda_{ABC}^{(7)} \\ &= \Lambda^C \wedge \lambda_{ABC}^{(7)} - \frac{1}{2} c_{DE}^C \lambda^D \wedge \lambda^E \wedge \lambda_{ABC}^{(7)} \\ &= \Lambda^C \wedge \lambda_{ABC}^{(7)} - \left(c_{BC}^C \lambda_A^{(9)} - c_{AC}^C \lambda_B^{(9)} + c_{AB}^C \lambda_C^{(9)} \right) \\ &= \Lambda^C \wedge \lambda_{ABC}^{(7)} - c_{AB}^C \lambda_C^{(9)} \end{aligned}$$

Next we check the compatibility with equivariant contractions:

$$\begin{aligned} d(\bar{s}_\alpha s^\alpha) &= (d\bar{s}_\alpha) s^\alpha + \bar{s}_\alpha ds^\alpha \\ &= (d^\lambda \bar{s} - \lambda \cdot \bar{s})_\alpha s^\alpha + \bar{s}_\alpha (d^\lambda s - \lambda \cdot s)^\alpha \\ &= (d^\lambda \bar{s}_\alpha) s^\alpha + \bar{s}_\alpha (d^\lambda s)^\alpha - (\lambda \cdot \bar{s})_\alpha s^\alpha + \bar{s}_\alpha (d^\lambda s - \lambda \cdot s)^\alpha \\ &= (d^\lambda \bar{s}_\alpha) s^\alpha + \bar{s}_\alpha (d^\lambda s)^\alpha \end{aligned}$$

In particular, for ψ a Σ -valued k -form over Q and $\bar{\psi}$ a Σ^* -valued l -form, the following holds

$$d(\bar{\psi}_\alpha \wedge \psi^\alpha) = (d^\lambda \bar{\psi}_\alpha) \wedge \psi^\alpha + (-1)^l \bar{\psi}_\alpha \wedge (d^\lambda \psi)^\alpha \quad (7.28)$$

For μ a differential form, we obtain the following

$$d^\lambda(\psi^\alpha \wedge \mu) = d(\psi^\alpha \wedge \mu) + \lambda \cdot \psi^\alpha \wedge \mu = (d^\lambda \psi^\alpha) \wedge \mu + (-1)^k \psi^\alpha \wedge d\mu$$

Next is the *Bianchi identity*, which will make use of the *Jacobi identity* $[[\lambda \wedge \lambda] \wedge \lambda] = 0$:

$$\begin{aligned} d^\lambda \Lambda^A &= d \left(d\lambda + \frac{1}{2} [\lambda \wedge \lambda] \right)^A + \left[\lambda \wedge \left(d\lambda + \frac{1}{2} [\lambda \wedge \lambda] \right) \right]^A \\ &= \frac{1}{2} ([d\lambda \wedge \lambda] - [\lambda \wedge d\lambda])^A + [\lambda \wedge d\lambda]^A \\ &= 0 \end{aligned}$$

And finally,

$$\begin{aligned} d^\lambda(d^\lambda s)^\alpha &= (d + \lambda \cdot)(ds + \lambda \cdot s)^\alpha \\ &= (d(\lambda \cdot s) + \lambda \cdot ds + \lambda \wedge \lambda \cdot s)^\alpha \\ &= \left((d\lambda) \cdot s + \frac{1}{2} [\lambda \wedge \lambda] \cdot s \right)^\alpha \\ &= \Lambda \cdot s^\alpha \end{aligned}$$

Note that the two last equations can as well be written using d :

$$\begin{aligned} d\Lambda &= [\Lambda \wedge \lambda] \\ d(d^\lambda s)^\alpha &= (\Lambda \cdot s - \lambda \cdot d^\lambda s)^\alpha \end{aligned}$$

Chapter 8

Integration of Generalised Cartan geometries

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8.1 Integration of a Lie algebra action

Generalised Cartan bundles have the action of a Lie algebra, but no group action. We want to study how a generalised Cartan bundle can be related to a standard principal bundle with a Cartan connection, and get a more precise idea of the possibilities offered by this generalisation.

In this section we deal with the question of integrating the action of a finite dimensional (real) Lie algebra \mathfrak{h} on a manifold P into a group action. The general theory (on non-Hausdorff nor paracompact manifolds)¹ is presented in detail in [Pal57], see also [KM04] for a more concise presentation. We introduce a Lie groupoid perspective on the question and show its relevance for dealing with Lie algebra actions, bringing the study in a different direction as was done in [Bla14]. The question we study can be phrased as followed, in terms of Lie groupoid theory: we want to integrate the *action Lie algebroid* (or *transformation Lie algebroid*) [FC06; MM00] associated to the action of a Lie algebra, and study the resulting structure. In particular, we prove a result

¹What they call *differentiable manifolds* are in our language smooth spaces which are locally isomorphic to a manifold.

of weak equivalence between the Lie groupoid approach and a different completion approach (Theorem 8.1.34).

8.1.1 Integration of a Lie algebra action into a local \mathfrak{h} group action

The action foliation on $P \times H$

Let P be a manifold on which a Lie algebra \mathfrak{h} of dimension r acts *on the right*, i.e. a right \mathfrak{h} -manifold [KM04]. The vector fields representing the Lie algebra are commonly called *fundamental vector fields* (also: *Killing vector fields*). We want to integrate the action into a Lie group action. In other words given a *connected* Lie group H integrating \mathfrak{h} , we want to define an action of H such that the associated infinitesimal action corresponds with the existing \mathfrak{h} action. Lie's third theorem ensures us we can always find a connected Lie group integrating \mathfrak{h} :

Theorem 8.1.1 (Third theorem of Lie, [Ser92]). *Any finite-dimensional real Lie algebra is isomorphic to the Lie algebra of a real simply-connected Lie group.*

A first obstruction is that the vector fields may not be complete. An immediate example is H minus any point. In this case the problem will be to *globalise* the Lie algebra action, by which we mean *embedding* P into a larger manifold on which the group H acts, such that the Lie algebra actions correspond.

The property identified as necessary and sufficient for a globalisation to exist, in a sufficiently general setting, is called *univalence* of the infinitesimal action [Pal57]. It formalises the idea that the action of an element h of H on a point of P can be constructed by integrating the infinitesimal action along a smooth path in H from e to h and is independent from the chosen path. We now define it in more detail.

Consider the product $P \times H$, we will write $p_H : P \times H \rightarrow H$ the factor projection which is a fibration over H . The Lie algebra \mathfrak{h} acts *on the right* on both P and H thus on the product manifold $P \times H$. It acts by a Lie subalgebra $\mathfrak{h} \hookrightarrow \Gamma(T(P \times H))$ of constant rank r :

$$(x, h) \cdot \xi = (x \cdot \xi, h \cdot \xi)$$

Indeed the TH component is already of rank r . Furthermore, as a Lie subalgebra, it is closed under vector bracket, so that it defines an involutive distribution, which we call D . Invoking the Frobenius theorem, D integrates into a foliation, which we call the *action foliation* and write F (or F_D to avoid ambiguities). Its leaves project to H under p_H by local diffeomorphisms: naming a leaf L the differential of the projection is the parallelism

$$\begin{array}{ccccc} & & \text{id}_{\mathfrak{h}} \times p_H & & \\ & \nearrow & & \searrow & \\ \mathfrak{h} \times L \simeq TL & \hookrightarrow & T(P \times H) & \xrightarrow{dp_H} & TH \xrightarrow{\omega_H} \mathfrak{h} \times H \end{array}$$

However the projections of the leaves on H do not have to be onto or one-to-one. The following property will be central in our considerations.

Definition 8.1.2 (Univalence). The action of \mathfrak{h} on P is called *univalent* (for H) if on each leaf L of D the restriction of projection $p_H: L \hookrightarrow P \times H \rightarrow H$ is *injective*.

Note that the projection

$$D \begin{array}{c} \xrightarrow{\quad} \\ \longleftarrow \quad \longrightarrow \\ \xrightarrow{\quad} \end{array} T(P \times H) \longrightarrow TH$$

being by linear isomorphisms, the distribution D can also be interpreted as a connection on the product fibre bundle $P \times H \rightarrow H$. Integrability of D is then equivalent to the flatness of the connection and univalence is equivalent to having trivial holonomy. Given a path γ in H , the procedure of solving the differential equation

$$c'(t) = c(t) \cdot \omega_H(\gamma'(t))$$

amounts to building a parallel lift of the path γ in $P \times H$.

Furthermore, since the distribution D is invariant under the *left* action of H , the same goes for the action foliation. We now turn to building a “local group action” for a given connected Lie group H integrating \mathfrak{h} .

Local group actions

In order to build a group action from a Lie algebra action, we want to build a structure which can encode the action of certain elements of the group, typically built by integrating along a smooth path in the group starting from e .

Definition 8.1.3. Let H be a Lie group. A (right) *local group action* of H on a manifold P is the data of an open subset \mathcal{U} of $P \times H$ provided with a map

$$\begin{cases} P \times H \rightarrow P \\ (p, h) \mapsto p \cdot h \end{cases}$$

such that

- $P \times \{e\} \subset \mathcal{U}$ and $\forall p \in P, p \cdot e = p$
- $\forall (p, h_1, h_2) \in P \times H \times H,$

$$\begin{cases} (p, h_1) \in \mathcal{U} \\ (p \cdot h_1, h_2) \in \mathcal{U} \end{cases} \implies (p, h_1 h_2) \in \mathcal{U}$$

and

$$(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$$

Remark. There are two perspectives on how a local group action is “local”. First, only a neighbourhood of the identity element of the group may act on a specified point of P . Second, given an element of the group, it may only act on part of the manifold. In this sense a local group action is local with respects to both the base manifold and the Lie group.

Remark. This notion is related but different from the notion of “local group of transformations” which can be found in [Pal57; Olv86], also called “local action of H ” in [Bla14]. In fact it is very close to the notion of “maximum local group of transformations”². The essential difference is

²An additional hypothesis on local groups of transformations is that the subset of group elements acting on each point of P has to be connected.

that in local action groups as we defined, composition of morphisms is always defined. A local group of transformations may only capture the action of “small” elements of the group, while larger elements may be build as composition of smaller elements *possibly in multiple ways*.

Definition 8.1.3 can be rephrased concisely using the notion of Lie groupoids (Section 6.1): a local group action of H on P is exactly the data of a Lie groupoid on an open subset $\Gamma_1 \subset P \times H$ and $\Gamma_0 = P$ with

- As source map $(p, h) \mapsto p$.
- As target map $(p, h) \mapsto p \cdot h$.
- As identity section $p \mapsto (p, e)$.
- As composition $[(p', h_2), (p, h_1)] \mapsto (p, h_1 h_2)$.

These definitions rely on the fact that the space of morphisms is a subset of $P \times H$. It is more convenient to develop a more intrinsic characterisation, which we do in the subsequent section.

Action groupoids and local pseudo-actions

We start by recalling the definition of action groupoids:

Definition 8.1.4. Let P be a smooth manifold with an action of a Lie group H . The corresponding action hroupoid has a manifold of points $\Gamma_0 = P$, a manifold of morphisms $\Gamma_1 = P \times H$ and as structure maps:

$$\begin{aligned} s : (p, h) \in P \times H &\mapsto p & e : p \in P &\mapsto (p, e) \\ t : (p, h) \in P \times H &\mapsto p \cdot h & i : (p, h) \in P \times H &\mapsto (p, h^{-1}) \\ & & \circ : [(p \cdot h_1, h_2), (p, h_1)] &\mapsto (p, h_1 \cdot h_2) \end{aligned}$$

It is called the *action groupoid* and is sometimes written $P \rtimes H$.

This construction gives an embedding of the notion of group action into the theory of groupoids: a map $t : P \times H \rightarrow P$ such that $P \times H$ equipped with the supplementary structure maps (s, e, i, \circ) defined as in the definition forms a groupoid is exactly a (right) action of H on P .

We want to characterise the structure of action groupoid as extra structure on a Lie groupoid. The action groupoid of an H -manifold comes with a (projection) map $\Gamma_1 \rightarrow H$ which is compatible with identity, inversion and composition and which is a diffeomorphism on each fibre of s (or t):

$$\{s = p\} \simeq \{p\} \times H \xrightarrow{\sim} H$$

Such a map gives a diffeomorphism $\Gamma_1 \xrightarrow{\sim} P \times H$ which is compatible with the fibration over P by s and the factor projection $P \times H \rightarrow P$. Namely, the following diagram commutes

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\sim} & P \times H \\ & \searrow s & \swarrow p_1 \\ & & P \end{array}$$

The hypotheses ensure that if we transport the Lie groupoid structure from Γ to $P \times H \rightarrow P$ its structure maps (s, e, i, \circ) are of the form described in Example 6.1.6, so that the target map defines an action of H on P and $P \times H \rightrightarrows P$ is an action groupoid which is isomorphic to Γ .

Seeing the Lie group H as a Lie groupoid with a point $1 = \{*\}$ as base manifold and H as manifold of morphisms, the map from Γ_1 to H defines a morphism of Lie groupoids and we can formulate the following theorem, essentially inspired from [Arm07]:

Theorem 8.1.5. *Let $\Gamma = \Gamma_1 \begin{smallmatrix} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{smallmatrix} P$ be a Lie groupoid over a manifold P and H a Lie group. There is an equivalence between the two following structures:*

1. *An isomorphism of Lie groupoids between Γ_1 and the action groupoid associated to an action of H on P .*
2. *A Lie groupoid morphism $\Gamma \rightarrow \left(H \begin{smallmatrix} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{smallmatrix} 1 \right)$ such that for all $x \in P$ the partial maps*

$$\{s = x\} \mapsto H$$

are diffeomorphisms.

The following example shows that not only group actions are entirely captured by their action groupoids, but the morphisms between them are as well:

Example 8.1.6. Let P and Q be two manifold with a smooth action of a Lie group H .

Then a Lie groupoid morphism between the corresponding action groupoids *which preserves the H component of the morphisms* is necessarily of the form

$$\begin{array}{ccc} P \times H & \xrightarrow{\phi \times \text{id}_H} & Q \times H \\ \downarrow \pi_P & & \downarrow \pi_Q \\ P & \xrightarrow{\phi} & Q \end{array}$$

with ϕ an H -equivariant map from P to Q .

We reformulate the notion a local group action from this perspective:

Theorem 8.1.7. *Let P be a smooth manifold and H a Lie group acting (on the right) on P . A local group action of H on P is equivalent to the data of a Lie groupoid with $\Gamma_0 = P$ and a Lie groupoid morphism*

$$\Gamma \rightarrow \left(H \begin{smallmatrix} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{smallmatrix} 1 \right)$$

such that it defines an open embedding on each fibre for p in P :

$$\{s = p\} \hookrightarrow \{p\} \times H \xrightarrow{\sim} H$$

This perspective suggests a generalisation which will be useful in the following sections:

Definition 8.1.8. A (right) *local group pseudo-action* of H on P is the data of a Lie groupoid over P : $\Gamma \begin{smallmatrix} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{smallmatrix} P$ equipped with a morphism τ to the Lie groupoid $H \begin{smallmatrix} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{smallmatrix} 1$ such that the map

$$\Gamma \xrightarrow{(s,\tau)} P \times H$$

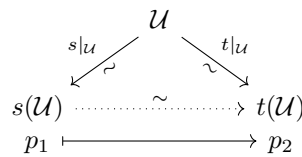
is *locally* a diffeomorphism. Equivalently, this is a morphism τ such that the restricted maps

$$\{s = p\} \rightarrow H$$

are local diffeomorphisms for every p in P .

In this generalised structure, different morphisms starting from a given point p in P can be mapped to the same element h in H . In other words, a given element h of H may “act” in multiple ways on the same point p in P . Given a specific element γ of Γ from $p_1 = s(\gamma)$ to $p_2 = t(\gamma)$ with $\tau(\gamma) = h$, it is possible to construct the “local action” of γ on a neighbourhood of p_1 as we now describe.

First, notice that $\Gamma \xrightarrow{(t,\tau)} P \times H$ is a local diffeomorphism as well, since inversion is a diffeomorphism of Γ . As a consequence, when restricted to $\{\tau = h\} \subset \Gamma$, both maps $\{\tau = h\} \xrightarrow[s]{t} P$ are local diffeomorphisms. Thus given an open neighbourhood \mathcal{U} of γ in $\{\tau = h\}$ on which s and t are injective, we obtain diffeomorphisms with respective neighbourhoods of p_1 and p_2 . Therefore one can construct a local diffeomorphism from $s(\mathcal{U})$ to $t(\mathcal{U})$:



In this way a local group pseudo-action allows to define the action of an element h of H on the neighbourhood of a point p of P by selecting a morphism in $\{\tau = h\}$ starting from p . As stated earlier there may be different germs of diffeomorphisms associated to the same h and the same p .

Remark. This name comes from the notion of *pseudoaction* of a groupoid defined in [Bla14] as a multiplicative foliation on the manifold of morphisms such that the leaves are projected by local diffeomorphisms under the source and target maps. In other words, there exists locally a notion of “constant” element of the groupoid acting on a neighbourhood of any given starting point, which is furthermore compatible with the groupoid composition. It allows associating locally defined diffeomorphisms to elements of the groupoid, similarly to the construction we just presented.

Remark. Let us emphasise here the difference with the approach from [Pal57], which will illustrate the purpose of the structure we just defined. Their approach is to try to *complete* the manifold in order to obtain a well defined group action. They relate the existence of such a completion with the existence of a “maximum local group of transformations” which is very close to what we call local group actions.

Our approach in this section looks to directly build the local group action, without considering for now the problem of completion (which will be discussed in Section 8.1.2). A similar idea is presented in [Bla14] in which they start from (or build from the action of the Lie algebra) the action of “small” group elements and use chains of such small elements. In our approach we build the action of paths in the group H (starting from e), so that composition is an internal composition law between paths. In order to obtain a local group action there will be a consistency condition which boils down to ensuring that the action of a path is uniquely determined by its end.

Put in another way, “local” means that only specific elements of the group may act on a given point of the space (the maps $\{s = p\} \rightarrow H$ fail to be surjective) while “pseudo” means that an element of the group may act on a given point in several distinct ways (the maps fail to be injective).

With these tools in hand, we will look to build a *Lie groupoid* integration of a Lie algebra action. We present in the next section the construction, which will rely on the holonomy groupoid introduced in Section 6.2.3.

Integration of the Lie algebra action into a local group pseudo-action

In this section we construct a groupoid integrating the action of a Lie algebra \mathfrak{h} on a manifold P . We present a construction different from that which can be found in [Pal57; Bla14]. The former is interested in “maximal local group of transformations” (see the remark following Definition 8.1.3) and the latter uses chains of “small” group elements. In contrast, we use holonomy which exactly embodies the data we are interested in. Our approach is also very specific to the structure at hand compared to the general construction detailed in [FC06]. The construction was tersely mentioned in [Daz97], without proof. We give an detailed analysis of the structure we obtain.

Let H be a Lie group integration of \mathfrak{h} . There is a holonomy groupoid (Section 6.2.3) associated to the action foliation F on $P \times H$, we call it Hol_F . The subspaces $P \times \{h\}$ give local transverses to the foliation, so that holonomy can be represented as a germ of diffeomorphism between open subsets of P . The *left* action of H preserves the action foliation so that by functoriality of the holonomy groupoid construction (Theorem 6.2.23) H has a left action on Hol_F by Lie groupoid automorphisms.

Let p_1 and p_2 be two points of P and (c^γ, γ) a foliated path in $P \times H$ from (p_1, h_1) to (p_2, h_2) . Let \mathcal{U}_1 and \mathcal{U}_2 be two respective neighbourhoods of p_1 and p_2 in P such that $\text{Hol}(c^\gamma, \gamma)$ can be represented as a diffeomorphism from $\mathcal{U}_1 \times \{h_1\}$ to $\mathcal{U}_2 \times \{h_2\}$. Then invariance under H implies that for all h in H the holonomy $\text{Hol}(c^{h \cdot \gamma}, h \cdot \gamma)$ is defined from $\mathcal{U}_1 \times \{h \cdot h_1\}$ to $\mathcal{U}_2 \times \{h \cdot h_2\}$ and the following diagram of diffeomorphisms is commutative:

$$\begin{array}{ccc} \mathcal{U}_1 \times \{h_1\} & \xrightarrow[\sim]{\text{Hol}(c^\gamma, \gamma)} & \mathcal{U}_2 \times \{h_2\} \\ \wr \downarrow h \cdot & & \wr \downarrow h \cdot \\ \mathcal{U}_1 \times \{h \cdot h_1\} & \xrightarrow[\sim]{\text{Hol}(c^{h \cdot \gamma}, h \cdot \gamma)} & \mathcal{U}_2 \times \{h \cdot h_2\} \end{array}$$

As a consequence the diffeomorphism $\mathcal{U}_1 \xrightarrow{\sim} \mathcal{U}_2$ remains unchanged when γ is replaced with $h \cdot \gamma$.

The construction described in 7.2.3 using the paths c_p^γ can be conveniently described using the holonomy groupoid. A candidate for the action of an element $h \in H$ on a point $p_1 \in P$ was constructed using a path γ in H from e to h and trying to integrate the following differential equation

$$c'(t) = c(t) \cdot \omega_H(\gamma'(t))$$

with initial condition $c(0) = p_1$. Building such a path c amounts to lifting the path γ under the projection

$$P \times H \rightarrow H$$

into a path (c, γ) tangent to the action foliation F starting from (p_1, e) . To such a path (c, γ) it is associated an element of Hol_F with source (p_1, e) and target (p_2, h) . The component p_2 of the target is the candidate for $p_1 \cdot h$. The holonomy between open subsets of P is the candidate for the corresponding action of h on a neighbourhood of p_1 .

The holonomy groupoid contains the holonomy of paths starting from every (p, h) in $P \times H$ but we only need paths starting from points (p, e) in order to build an action on P . Indeed given any path γ in H which has a parallel lift (c^γ, γ) in $P \times H$ starting from a point $(p, \gamma(0))$, then for all h in H , the path $(c^\gamma, h \cdot \gamma)$ is a parallel lift of γ starting from $(p, h \cdot \gamma(0))$. An alternative perspective is the observation that the differential equation

$$c'(t) = c(t) \cdot \omega_H(\gamma'(t))$$

purely depends on $\gamma'(t)$. The conclusion is that the groupoid Hol_F contains a redundancy in describing the construction of paths on P . Let us define the objects (paths) we will be interested

in, before organising them into a Lie groupoid.

Definition 8.1.9. Let H be a connected Lie group with Lie algebra \mathfrak{h} and P an \mathfrak{h} -manifold.

- An H -path on P is the data of a smooth path c on P and a smooth path γ on H starting from e such that the differential equation

$$c'(t) = c(t) \cdot \omega_H(\gamma'(t))$$

is satisfied at all times. The γ component will sometimes be omitted.

- An H -loop on P is an H -path (c, γ) on P such that c is a loop (γ need not be a loop).
- When the differential equation has a complete solution c starting at a point $p_1 \in P$ and ending at a point p_2 , there are complete solutions for each initial condition in a neighbourhood of p_1 . The germ of the corresponding flow between neighbourhoods of p_1 and p_2 will be called the *local action* of the H -path c .

Remark. • The data of an H -path can be defined using only the structure of \mathfrak{h} and the derivative path γ' with value in \mathfrak{h} . Specifying the Lie group H suggests keeping in mind the endpoint of γ which does live in H .

- As explained above, the local action of an H -path represents the holonomy of the corresponding foliated path in $P \times H$.

The local actions of H -paths are structured into a Lie groupoid *above* P , which can be constructed from the holonomy groupoid of the action foliation F :

Definition 8.1.10 (Local pseudo-action groupoid associated to a Lie algebra action). Let P be a (right) \mathfrak{h} -manifold and H a Lie group integration of \mathfrak{h} . The *local pseudo-action groupoid* of H on P will be written $P \rtimes H_{loc}$ and is defined as follows:

- The base manifold is P .
- The morphism manifold is the submanifold of $(\text{Hol}_F)_1$ of holonomies starting from a point of $P \times \{e\}$:

$$s^{-1}(P \times \{e\}) \subset (\text{Hol}_F)_1$$

It has dimension $\dim P + \dim H$.

- The source and target maps are respectively the P components of the starting and ending point of the holonomy.
- Composition of the holonomy of two paths (c_1, γ_1) and (c_2, γ_2) such that $c_1(1) = c_2(0)$ is defined as the holonomy of the following foliated path

$$(c_2 \odot c_1, [\gamma_1(1) \cdot \gamma_2] \odot \gamma_1)$$

- The identity section is the class of germs of the identity diffeomorphisms.
- The inversion map maps the holonomy of a path (c, γ) to that of the reversed path $(\check{c}, \gamma(1)^{-1} \cdot \check{\gamma})$.

A thorough definition would also require checking that composition and inversion are well defined on holonomy classes. Fixing the H component of the starting point of the holonomies allows eliminating the redundancy due to considering arbitrary paths in H . Another way to eliminate the redundancy which is equivalent is to construct a quotient of the holonomy groupoid which is tailor-made to get rid of the superfluous degrees of freedom. The previous construction can then be understood as a *section* from the quotient to the holonomy groupoid, effectively a kind of gauge fixing.

Definition 8.1.11 (Local pseudo-action groupoid associated to a Lie algebra action, quotient construction). Let P be a (right) \mathfrak{h} -manifold and H a Lie group integration of \mathfrak{h} . The *local pseudo-action groupoid* of H on P can be constructed as follows:

- The base manifold is P .
- The morphism manifold is the free quotient $H \backslash \text{Hol}_F$.
- The source and target maps are the factorisations to $H \backslash \text{Hol}_F$ of the projection to P of the source and target maps of Hol_F :

$$\begin{array}{ccc} (\text{Hol}_F)_1 & \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} & P \times H \\ \downarrow & & \downarrow \\ H \backslash (\text{Hol}_F)_1 & \begin{array}{c} \xrightarrow{\text{---}s\text{---}} \\ \xrightarrow{\text{---}t\text{---}} \end{array} & P \end{array}$$

- Composition is defined as the class in $H \backslash (\text{Hol}_F)_1$ of the composition in Hol_F of representatives. Since the composition of Hol_F is H -equivariant it factors to $H \backslash (\text{Hol}_F)_1$.
- The identity section is the class of germs of the identity diffeomorphisms.
- The inversion map is the factorisation to $H \backslash (\text{Hol}_F)_1$ of the inversion map of Hol_F , which is H -equivariant.

Since the projection $P \times H \rightarrow P$ can be identified with the quotient map $P \times H \rightarrow H \backslash (P \times H)$, the local pseudo-action groupoid can be understood as the quotient of Hol_F under the left action of H .

Theorem 8.1.12. *The two constructions of the local pseudo-action groupoid associated to a Lie algebra action yield isomorphic Lie groupoids: the isomorphism is given by the following maps*

$$\begin{array}{ccc} s^{-1}(P \times \{e\}) & \begin{array}{c} \xrightarrow{[c,\gamma] \mapsto [c,\gamma]} \\ \xrightarrow{\text{mod } H} \end{array} & H \backslash (\text{Hol}_F)_1 \\ \begin{array}{c} \searrow t \\ \swarrow s \end{array} & & \begin{array}{c} \swarrow t \\ \searrow s \end{array} \\ & & P \end{array}$$

The proof is a matter of checking that $s^{-1}(P \times \{e\})$ indeed crosses each coset exactly once so that there is a diffeomorphism

$$(\text{Hol}_F)_1 \xrightarrow{\sim} s^{-1}(P \times \{e\}) \times H$$

which intertwines the left action of H on $(\text{Hol}_F)_1$ with the left action on the factor H and that the map is compatible with the groupoid structure maps.

Let us present an example which illustrates the purpose of the construction:

Example 8.1.13. Let H be a connected Lie group with Lie algebra \mathfrak{h} and P a right H -manifold. Then the action of H on P differentiates to an action of \mathfrak{h} .

Let γ be a smooth path in H starting from e and ending on an element h . Let p be any point of P . Then the differential equation

$$\begin{cases} c'(t) &= c(t) \cdot \omega_H(\gamma'(t)) \\ c(0) &= p \end{cases}$$

is solved by the path

$$c(t) = p \cdot \gamma(t)$$

ending on $p \cdot h$. Thus it corresponds in the local pseudo-action groupoid to a morphism from p to $p \cdot h$.

Since what precedes holds for any point of P , we have foliated lifts of γ starting from any point of P . Consider the product space $P \times H$. We can choose $P \times \{e\}$ as a transversal to F at the point (p, e) and $P \times \{h\}$ as a transversal at the point (p, h) . Then the holonomy of (c, γ) is given by

$$P \times \{e\} \xrightarrow{\text{id} \times \cdot h} P \times \{h\}$$

In particular, given a starting point p there is exactly one morphism for each element h in H , which ends on $p \cdot h$. This can be used to construct a Lie groupoid isomorphism with the action Lie groupoid $P \rtimes H$ for the action of H on P .

We now describe a system of charts for the smooth structure of $(P \rtimes H_{loc})_1$. Let (c, γ) be an H -path with sitting instants in P parametrised from 0 to 1 and $[c, \gamma]$ the associated element of $(P \rtimes H_{loc})_1$. Define $p_0 = c(0)$ and $h_0 = \gamma(0)^{-1}\gamma(1)$.

Consider the following parametrised path in H for $\xi \in \mathfrak{h}$:

$$t \in [0, 1] \mapsto \exp(t\xi)$$

We define e^ξ as this path *reparametrised in order to have sitting instants*. We define γ_h the concatenation $(\gamma(1) \cdot e^\xi) \odot \gamma$, reparametrised in order to have a parameter varying from 0 to 1. Then the differential equation

$$\begin{cases} c'(t) &= c(t) \cdot \omega_H(\gamma'_\xi(t)) \\ c(0) &= p \end{cases}$$

admits a global solution (on $[0, 1]$) for $\xi = 0$ and $p = p_0$. As a consequence, there are open neighbourhoods $\mathcal{U} \subset P$ of p_0 and $\mathcal{V} \subset \mathfrak{h}$ of 0 such that for all p in \mathcal{U} and ξ in \mathcal{V} the equations admit a complete solution ([Mic08] Section I.3.7). We write c_p^ξ these solutions. We can reduce \mathcal{V} to a subneighbourhood of 0 if needed in order to ensure that the exponential map is injective over \mathcal{V} .

The associated chart around $[c, \gamma]$ is the map

$$\begin{aligned} \mathcal{U} \times \mathcal{V} \subset P \times \mathfrak{h} &\rightarrow (P \rtimes H_{loc})_1 \\ (p, \xi) &\mapsto [c_p^\xi, \gamma_\xi] \end{aligned} \tag{8.1}$$

We will prove that this map is

1. smooth for the submanifold structure
2. open

3. an embedding

which will prove that it is a chart for the smooth structure.

First, it is naturally smooth as a map into Hol_F since it is the holonomy of a smoothly parametrised foliated path. Furthermore, we know that the map

$$\begin{aligned} \mathcal{U} \times \mathcal{V} \times H &\rightarrow s(\text{Hol}_F)_1 \\ (p, \xi, h) &\mapsto [c_p^\xi, h \cdot \gamma_\xi] \end{aligned} \tag{8.2}$$

defines a chart on $(\text{Hol}_F)_1$ (Section 6.2.3). Considering $(P \rtimes H_{loc})_1$ as an embedded submanifold, the image of $\mathcal{U} \times \mathcal{V} \times \{e\}$ is exactly the intersection of $(P \rtimes H_{loc})_1$ with the image of the chart of $(\text{Hol}_F)_1$. As a consequence the map $(p, \xi) \mapsto [c_p^\xi, \gamma_\xi]$ is an open embedding in $(P \rtimes H_{loc})_1$. Thus we proved that we have charts for the smooth structure of $(P \rtimes H_{loc})_1$.

The local pseudo-action groupoid of H comes with a natural Lie groupoid morphism to $H \begin{smallmatrix} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{smallmatrix} 1$: it is constructed by the following factorisation

$$\begin{array}{ccc} \text{Hol}_F & & \\ \downarrow & \searrow^{[\gamma, c^\gamma] \mapsto \gamma(0)^{-1}\gamma(1)} & \\ H_{loc} \times P & \xrightarrow{\tau} & H \end{array}$$

and associates to a holonomy from a pair (p_0, h_0) to a pair (p_1, h_1) the element $h_0^{-1}h_1$ such that the right translation in H relates h_0 to h_1 .

With this map to H , the local pseudo-action groupoid has the structure of a local group pseudo-action of H :

Theorem 8.1.14. *Let H be a connected Lie group with Lie algebra \mathfrak{h} and P an \mathfrak{h} -manifold.*

Then the local pseudo-action groupoid $P \rtimes H_{loc}$ equipped with the above defined map τ to H defines a local group pseudo-action of H on P .

This implies in particular that τ is a submersion.

Proof. Consider the charts as defined in Expression (8.1), associated to an H -path (c, γ) . The map $(P \rtimes H_{loc})_1 \xrightarrow{(s, \tau)} P \times H$ takes the following form in the local coordinates (p, ξ) :

$$(p, \xi) \in \mathcal{U} \times \mathcal{V} \mapsto (p, \gamma(1) \exp(\xi)) \in P \times H$$

It is manifestly a local diffeomorphism. □

Remark. A careful examination of the construction of the charts shows that the smooth structure of $(P \rtimes H_{loc})_1$ is essentially defined by requiring that the map

$$(P \rtimes H_{loc})_1 \xrightarrow{(s, \tau)} P \times H$$

is a local diffeomorphism. In this way all the charts can be gathered into one geometrical map (which however does not encode the topology of $(P \rtimes H_{loc})_1$).

Recall that a neighbourhood \mathcal{U} of a point p in P defines transversals $\mathcal{U} \times \{h\}$ to F at the points (p, h) . The holonomy between such transversals corresponds to the local action of elements of $P \rtimes H_{loc}$ in the following sense:

Theorem 8.1.15. Let $[c, \gamma]$ be an element of $P \rtimes H_{loc}$ going from a point p_1 in P to a point p_2 . Let $h = \tau([c, \gamma])$.

Let \mathcal{U}_1 be a neighbourhood of p_1 in P and \mathcal{U}_2 a neighbourhood of p_2 such that the holonomy of (c, γ) can be realised as a diffeomorphism

$$\mathcal{U}_1 \times \{e\} \xrightarrow[\sim]{\text{Hol}(c, \gamma)} \mathcal{U}_2 \times \{h\}$$

In other words, assume there is a smooth map

$$\phi : \mathcal{U}_1 \times \{e\} \times [0, 1] \rightarrow P \times H$$

such that

- $\phi(\cdot, 0) = \text{id}_{\mathcal{U}_1 \times \{e\}}$
- $\phi(\cdot, 1)$ is the holonomy of (c, γ) from $\mathcal{U}_1 \times \{e\}$ to $\mathcal{U}_2 \times \{h\}$
- For all x in \mathcal{U}_1 , the path

$$t \in [0, 1] \mapsto \phi(x, t)$$

is a foliated path.

Then there exists an open neighbourhood \mathcal{V} of $[c, \gamma]$ in the submanifold $\{\tau = h\}$ such that the following diagram of local diffeomorphisms commutes:

$$\begin{array}{ccc} & \mathcal{V} & \\ s|_{\mathcal{V}} \swarrow & & \searrow t|_{\mathcal{V}} \\ \mathcal{U}_1 & \xrightarrow[\sim]{\text{Hol}(c, \gamma)} & \mathcal{U}_2 \end{array}$$

with \mathcal{U}_1 and \mathcal{U}_2 respectively identified with their embeddings $\mathcal{U}_1 \times \{e\}$ and $\mathcal{U}_2 \times \{h\}$.

Proof. Let there be such a map ϕ . Consider the following map:

$$\sigma : x \in \mathcal{U}_1 \mapsto [\phi(x, \cdot)] \in P \rtimes H_{loc}$$

It satisfies the following properties:

- $s(\sigma(x)) = x$
- $t(\sigma(x)) = \phi(x, 1) = \text{Hol}(c, \gamma)(x)$
- $\tau(\sigma(x)) = h$

Furthermore, it is straightforward to prove that it is continuous using an atlas associated with the transversal $\mathcal{U}_1 \times \{e\}$.

Since $(s, \tau) \circ \sigma$ is smooth, σ is a smooth section of s . As a consequence its image is a submanifold of $(P \rtimes H_{loc})_1$ of dimension n , which is furthermore contained inside the n -submanifold $\{\tau = h\}$. It is thus an open neighbourhood of $[c, \gamma]$ in $\{\tau = h\}$ satisfying the properties asserted in the theorem. \square

The recurrence groupoid

Univalence for H can be formulated as follows: given any starting point $p \in P$, the restriction

$$\tau|_{\{s=p\}} : \{s = p\} \rightarrow H$$

has to be *injective*. A formulation closer to the idea that loops in H should have trivial action on P is the triviality of the kernel subgroupoid $\ker(\tau) = \{\tau = e\}$:

$$\ker(\tau) = \text{id}_{P \times H_{loc}} \begin{matrix} \xrightarrow{s} \\ \xrightarrow{t} \end{matrix} P$$

This should make it clear why we started with the data of holonomy: the kernel subgroupoid $\ker(\tau)$ contains the *holonomy* of foliated lifts to $P \times H$ of loops in H based at e . They correspond to local actions of the loops of H on P . For such an H -path starting from a point p to have trivial local action, it is not sufficient to be a loop: all H -paths with the same H component (the γ path) starting in a neighbourhood of p need to be loops. The subgroupoid $\ker(\tau)$ will be called the *recurrence groupoid*. Since τ is a submersion, it is a Lie subgroupoid of the local action groupoid.

Now recall that the morphisms of the local action groupoid are holonomy classes of paths. Requiring that every loop in H has foliated lifts to $P \times H$ which only have trivial holonomy, a condition on the whole holonomy germs, is equivalent to requiring that all the foliated lifts are loops, a 0-order condition on the lifts. Therefore univalence can be formulated as follows: the recurrence groupoid $\ker(\tau)$ has no morphism between distinct points. This 0-order condition is indeed very close to the original Definition 8.1.2 cited from [Pal57].

Theorem 8.1.16 (Characterisation of univalence). *Let H be a connected Lie group with Lie algebra \mathfrak{h} and P an \mathfrak{h} -manifold. Then the action of \mathfrak{h} on P is univalent for H if and only if one of the following equivalent conditions is satisfied:*

- The recurrence groupoid $\ker(\tau)$ is trivial, namely it only contains identity morphisms.
- The recurrence groupoid $\ker(\tau)$ only contains automorphisms.
- The map τ is injective on all submanifolds of morphisms with fixed source.

The obstruction to univalence is readily quantified in the recurrence orbits introduced in Section 7.2.3. They are naturally defined using the local pseudo-action groupoid:

Definition 8.1.17. Let $p \in P$. We introduce the following notation for the recurrence groupoid:

$$P \times I^H := \left(I^H \begin{matrix} \xrightarrow{s} \\ \xrightarrow{t} \end{matrix} P \right) = \ker(\tau) \subset P \times H_{loc}$$

It can be realised as the following subgroupoid of Hol_{F_D} :

$$I^H = (s, t)^{-1}(P \times \{e\} \times P \times \{e\}) \subset (\text{Hol}_{F_D})_1$$

Its orbits are called *recurrence orbits* (under H) and we write them $p \cdot I^H$ for points p in P .

The previous theorem can be reformulated as follows: the action of \mathfrak{h} is univalent for H if and only if all the recurrence orbits are trivial.

We can formulate the following theorem, which is a consequence of the definition and the remarks in Section 8.1.1:

Theorem 8.1.18. *Let H be a connected Lie group with Lie algebra \mathfrak{h} and P an \mathfrak{h} -manifold. Then the local pseudo-action groupoid $P \rtimes H_{loc}$ defines a local group action of H if and only if the action of \mathfrak{h} is univalent for H .*

The theorem is very close to part of Theorem 3.X in [Pal57], although in a different language.

Since $(s, \tau) : (P \rtimes H_{loc})_1 \rightarrow P \times H$ is a local diffeomorphism and $P \rtimes I^H$ is the inverse image of e under τ , we conclude that the source map of the recurrence groupoid defines a (surjective) local diffeomorphism onto P . The same goes for the target map, as can be proved by precomposing the source map with the inversion diffeomorphism of $(P \rtimes H_{loc})_1$. Using the notion of étale groupoid defined in Section 6.1, we can state the following:

Theorem 8.1.19. *The recurrence groupoid is an étale Lie groupoid.*

In particular, this implies that to each morphism in $P \rtimes I^H$ it is associated a germ of diffeomorphism of P . We reproduce here the argument. Given any morphism $[c, \gamma]$ in $P \rtimes I^H$ from a point p_1 to a point p_2 , there is a neighbourhood $\mathcal{U} \subset I^H$ on which the restrictions of the source and the target maps are diffeomorphisms. Therefore one can construct a local diffeomorphism from $s(\mathcal{U})$ to $t(\mathcal{U})$:

$$\begin{array}{ccc}
 & \mathcal{U} & \\
 s|_{\mathcal{U}} \swarrow & & \searrow t|_{\mathcal{U}} \\
 s(\mathcal{U}) & \overset{\sim}{\dashrightarrow} & t(\mathcal{U}) \\
 p_1 \longleftarrow & & \longrightarrow p_2
 \end{array}$$

Since

$$P \rtimes I^H = \{\tau = e\} \subset (P \rtimes H_{loc})_1$$

this indeed corresponds to the local action of elements of $\{\tau = e\}$. As a consequence, the local diffeomorphism obtained is a representative of the holonomy of $[c, \gamma]$ between transversals of F in $P \times H$ corresponding to open subsets of P .

In a similar way, the H -isotropy groups are naturally defined using the local pseudo-action groupoid:

Definition 8.1.20. Let $p \in P$. Its H -isotropy group is defined as

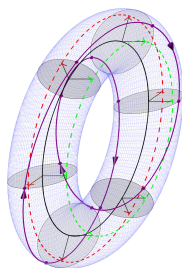
$$\Gamma^H(p) = \tau(\text{Aut}_{P \times_{\mathfrak{h}} H}(p))$$

Remark. More generally, given $k \in \mathbb{N}$, is it possible to define an H -isotropy group of order k constructed as the image under τ of the subgroup of automorphisms of p with holonomy trivial to the order k .

Non-functoriality of the local action groupoid construction

One may expect the local action groupoid construction to behave functorially with respect to \mathfrak{h} -equivariant maps between \mathfrak{h} -manifolds. Indeed Theorem 6.2.23 is a positive result in this direction and functorially holds under similar conditions. However we present here a simple example showing that an arbitrary equivariant map does not always induce a map between holonomy classes.

Consider the twisted torus $T \simeq \mathbb{R}^2 \times \mathbb{S}^1$ introduced in Section 7.1 (one could consider instead a lower dimensional Moebius strip).



It is equipped with a vector field Z which acts by torus translations “twisted” by a rotation (in purple on the figure). Using (e_1, e_2) as basis of the \mathbb{R}^2 slices transversal to Z we had:

$$\begin{cases} [Z, e_1] = \frac{1}{2}e_2 \\ [Z, e_2] = -\frac{1}{2}e_1 \end{cases}$$

Figure 8.1: The torus as a “twisted” frame bundle

The vector field Z defines an action of the Lie algebra \mathfrak{so}_2 on T . As mentioned in Section 8.1.1, the holonomy classes of foliated paths in $T \times \text{SO}_2$ correspond to germs of diffeomorphisms on T . Consider now the 1-dimensional subspace $Q = \{0\} \times \mathbb{S}^1$ represented as a black circle on the figure. It is stable under the action of Z thus is a

\mathfrak{so}_2 -submanifold. We shall exhibit a SO_2 -loop in Q with trivial local action in Q but non-trivial local action in T .

Now, as a \mathfrak{so}_2 -manifold, Q is isomorphic to SO_2 therefore the local action of SO_2 -paths is very rigid. The local action of loops of SO_2 is always trivial.

Going back to T , consider a SO_2 -loop in Q of winding number 1, for example

$$\theta \in [0, 2\pi] \mapsto \exp(\theta Z)(0, 0)$$

By the bracket relation with e_1 and e_2 , we know that

$$\begin{cases} \exp(2\pi Z)^* e_1 = -e_1 \\ \exp(2\pi Z)^* e_2 = -e_2 \\ \exp(2\pi Z)^* Z = Z \end{cases}$$

thus the local action of the loop corresponds to the following map:

$$(x, z) \in \mathbb{R}^2 \times \mathbb{S}^1 \mapsto (-x, z)$$

which is not trivial.

There exists another Lie groupoid very similar to the holonomy groupoid which is called the *monodromy groupoid*. Instead of holonomy classes of paths it contains the additional data of the *homotopy* classes of foliated paths. Homotopy classes behave more conveniently with respect to smooth maps thus the construction has better functoriality properties. However, although the holonomy class of a path only depends on its foliated holonomy class, several homotopy classes may have the same holonomy class (even when H is simply connected). Since we want to identify the groupoid elements with the corresponding element in the Lie group H and their holonomy, elements we want to identify may not be identical in the monodromy groupoid (even in the case H is simply connected).

When both \mathfrak{h} -manifolds are univalent for H , the local action of an H -path is uniquely specified by the corresponding element of H so that an \mathfrak{h} -equivariant map induces a map between H -paths which indeed factors to the holonomy groupoid and to the local action groupoid. More precisely, the following theorem holds:

Theorem 8.1.21. *Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P and Q be two \mathfrak{h} -manifolds and $f : P \rightarrow Q$ an \mathfrak{h} -equivariant map. Assume that Q is univalent for H .*

Then f induces a Lie groupoid morphism

$$\begin{array}{ccc} P \rtimes H_{loc} & \dashrightarrow & Q \rtimes H_{loc} \\ \Downarrow & & \Downarrow \\ P & \xrightarrow{f} & Q \end{array}$$

which is compatible with the τ maps.

Proof. First, to prove that f induces a morphism $\text{Hol}_{F(P)} \rightarrow \text{Hol}_{F(Q)}$ it is sufficient to show that any foliated loop in $P \times H$ with trivial holonomy is sent under $f \times \text{id}_H$ to a foliated loop in $Q \times H$ with trivial holonomy. Since f is equivariant, the image is a foliated loop. But any foliated loop of $Q \times H$ has holonomy in the recurrence subgroupoid which is trivial as a consequence of univalence (Theorem 8.1.16). Thus f induces a morphism between the holonomy groupoids. It is readily proved to be equivariant under the left action of H so that it factors to a morphism between the local action groupoids.

Since $f \times \text{id}_H$ preserves the H component, the induced map preserves the image under τ of the morphisms. \square

As mentioned, when the equivariant map is a local diffeomorphism there is an induced map between the local action groupoids, along the lines of Theorem 6.2.23:

Theorem 8.1.22. *Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P and Q be two \mathfrak{h} -manifolds and $f : P \rightarrow Q$ an \mathfrak{h} -equivariant map. Assume that f is a local diffeomorphism.*

Then f induces a Lie groupoid morphism

$$\begin{array}{ccc} P \rtimes H_{loc} & \dashrightarrow & Q \rtimes H_{loc} \\ \Downarrow & & \Downarrow \\ P & \xrightarrow{f} & Q \end{array}$$

such that the map between the manifolds of morphisms is a local diffeomorphism which is compatible with the τ maps.

Proof. The map $f \times \text{id}_H : P \times H \rightarrow Q \times H$ is a local diffeomorphism which preserves the action foliation and which is left H -equivariant. Therefore according to Theorem 6.2.23 there is an induced map between the holonomy groupoids of $P \times H$ and $Q \times H$:

$$\text{Hol}(f) : \text{Hol}_{F(P \times H)} \rightarrow \text{Hol}_{F(Q \times H)}$$

This map is equivariant under the left action of H hence factors to a map between the local action groupoids.

The obtained map is compatible with τ since $f \times \text{id}_H$ preserves the H component. The obtained map fits into the following commutative diagram

$$\begin{array}{ccc} P \rtimes H_{loc} & \dashrightarrow & Q \rtimes H_{loc} \\ \downarrow (s_P, \tau) & & \downarrow (s_Q, \tau) \\ P \times H & \xrightarrow{f \times \text{id}_H} & Q \times H \end{array}$$

in which the three morphisms written with solid arrows are local diffeomorphism. Therefore so does the remaining morphism. \square

We now study the question of completing the manifold P into a manifold with a global group action of H . We first deal with the simpler case of a complete \mathfrak{h} -manifold.

8.1.2 Completion of an \mathfrak{h} -manifold

Univalence in the complete case

If the fundamental vector fields on P are *complete* then so do the fundamental vector fields on $P \times H$. In this case, starting from each point $(p, h) \in P \times H$ belonging to a leaf L , the flows of the fundamental fields of $P \times H$ define a mapping $\mathfrak{h} \rightarrow L \subset P \times H$. Its image projects under p_H to the neighbourhood $h \exp(\mathfrak{h})$ of h . Hence the image of the projection of L to H is invariant under (right) multiplication by $\exp \mathfrak{h}$. But the subset $\exp \mathfrak{h} \subset H$ generates the whole (connected) group so that any non-empty subset which is invariant under multiplication by $\exp \mathfrak{h}$ has to be H . Thus the projection is onto. It is actually a covering map, and there is an equivalence:

Lemma 8.1.23 ([Pal57], Lemma IV.2.d, Theorem III.6.XVII). Let H be a connected Lie group and \mathfrak{h} its Lie algebra. Let P be a Hausdorff \mathfrak{h} -manifold. Let D be the integrable distribution on $P \times H$ spanned by the vectors

$$(p, h) \cdot \xi = (p \cdot \xi, h \cdot \xi)$$

Then the action of \mathfrak{h} is complete if and only if for each leaf L of D the projection $L \subset P \times H \rightarrow H$ is a smooth covering map.

In [Pal57] only the following implication is stated: if the action is complete then it is “uniform” which is proved to be equivalent to the smooth covering condition. The converse is readily proved since H is a complete \mathfrak{h} -manifold and the flows of fundamental vector fields lift under equivariant covering maps.

Recall that the distribution D can be interpreted as a connection on the bundle fibration $P \times H \rightarrow H$. Completeness of the action of \mathfrak{h} means that the connection is complete: parallel transport exists above any path starting from any point in the fibre. From the point of view of the local pseudo-action groupoid, it means that the map $P \times H_{loc} \rightarrow H$ is surjective on each subspace of morphisms with fixed source $\{s = p\}$. If the action is furthermore univalent, these maps are diffeomorphisms

$$\{s = p\} \xrightarrow{\sim} H$$

which are bundled together in a mapping

$$(P \times H_{loc})_1 \xrightarrow{\sim} P \times H$$

The target map $P \times H_{loc} \xrightarrow{t} P$ thus defines a map $P \times H \rightarrow P$ which gives to $P \times H$ a structure of Lie groupoid with $(p, h) \mapsto p$ as source map and $(p_2, h_2) \cdot (p_1, h_1) \mapsto (p, h_1 h_2)$ (restricted to composable pairs) as composition map. This is exactly the data of a group action of H on P , as explained in Theorem 8.1.5.

Theorem 8.1.24. *Let H be a connected Lie group with Lie algebra \mathfrak{h} and P an \mathfrak{h} -manifold.*

The action of \mathfrak{h} on P integrates to a group action of H on P if and only if it is complete and univalent for H .

This theorem is essentially contained in [Pal57], although not stated as such (equivalence between “properness” and “uniformity and univalence”).

If we choose H to be the simply-connected Lie group integration of \mathfrak{h} then each leaf of F has to be diffeomorphic to H under the projection $P \times H \rightarrow H$. The Lie algebra action is then necessarily H -univalent and it is readily globalisable on P to a group action of H .

For an arbitrary (connected) group integration H of \mathfrak{h} , univalence does not necessarily hold: the projection of each leaf to H is only a covering map. The non-univalent case will be more generally discussed in the next section.

Example 8.1.25 (Complete but non-univalent action). Let P be a smooth manifold with a complete vector field X . X can be interpreted as an action of the Lie algebra \mathfrak{u}_1 for example for the infinitesimal generator i of U_1 with period 2π . But the action of X is univalent for \mathfrak{u}_1 if and only if 2π is a period for the flow of X .

For example, the vector field ∂_x on \mathbb{R} defines a non-univalent action of \mathfrak{u}_1 on \mathbb{R} .

An example with a compact Lie group and a compact manifold can be constructed using the Lie group U_1 itself as manifold. Consider the generator $i \in \mathfrak{u}_1$ with $s \in \mathbb{R} \mapsto \exp(si) \in U_1$ of period 2π . There is a linear map:

$$i \in \mathfrak{u}_1 \rightarrow \frac{1}{2}\partial_\theta \in \Gamma(TU_1)$$

It defines an action of \mathfrak{u}_1 on U_1 which has period 4π and is therefore non-univalent for the integration U_1 of \mathfrak{u}_1 with a generator of period 2π (or any period which is not a multiple of 4π).

Writing \tilde{H} for the simply-connected integration group of \mathfrak{h} , a general (connected) group integration of \mathfrak{h} is a quotient \tilde{H}/K by a discrete normal subgroup:

$$K \hookrightarrow \tilde{H} \twoheadrightarrow H$$

It is possible to relate the groupoid $P \times H_{loc}$ to the groupoid $P \times \tilde{H}_{loc}$ as we now explain. Let us consider the respective action foliations F on $P \times H$ and \tilde{F} on $P \times \tilde{H}$. The projection $P \times \tilde{H} \rightarrow P \times H$ is \mathfrak{h} -equivariant, namely the following diagram commutes:

$$\begin{array}{ccc} P \times \tilde{H} \times \mathfrak{h} & \longrightarrow & P \times H \times \mathfrak{h} \\ \downarrow (p, \tilde{h}, \xi) \mapsto (p, \xi, \tilde{h} \cdot \xi) & & \downarrow (p, h, \xi) \mapsto (p, \xi, h \cdot \xi) \\ T(P \times \tilde{H}) & \longrightarrow & T(P \times H) \end{array}$$

As a consequence, it preserves the foliation. Since it is a local diffeomorphism, it induces a map between the holonomy groupoids (Theorem 6.2.23). It fits in the following commutative diagram:

$$\begin{array}{ccc} \text{Hol}_{\tilde{F}} & \xrightarrow{\tilde{\tau}: [c, \gamma] \mapsto \gamma(0)^{-1} \gamma(1)} & \tilde{H} \\ \downarrow & & \downarrow \\ \text{Hol}_F & \xrightarrow{\tau: [c, \gamma] \mapsto \gamma(0)^{-1} \gamma(1)} & H \end{array}$$

The morphism $\text{Hol}_{\tilde{F}} \rightarrow \text{Hol}_F$ is a surjection, since any H -path in P can be lifted to an \tilde{H} -path by lifting the γ component from H to \tilde{H} (starting from the identity element of \tilde{H}).

The morphism $\text{Hol}_{\tilde{F}} \rightarrow \text{Hol}_F$ is $(\tilde{H} \rightarrow H)$ -intertwining for the action on the left, so that it factors to the local action groupoids:

$$\begin{array}{ccccc} \text{Hol}_{\tilde{F}} & \longrightarrow & P \times \tilde{H}_{loc} & \xrightarrow{\tilde{\tau}} & \tilde{H} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hol}_F & \longrightarrow & P \times H_{loc} & \xrightarrow{\tau} & H \end{array}$$

The condition to obtain an action of H is H -univalence: the kernel of τ should consist only of identity morphisms. But the kernel of τ is exactly the image in $P \times H_{loc}$ of $\tilde{\tau}^{-1}(K)$. Therefore we can assert the following:

Theorem 8.1.26. *Let \tilde{H} be a simply connected Lie group and \mathfrak{h} its Lie algebra. Let P be a complete \mathfrak{h} -manifold.*

Let K be a discrete normal subgroup of \tilde{H} . Then P is \tilde{H}/K -univalent if and only if $\tilde{\tau}^{-1}(K)$ is reduced to the identity subgroupoid. Namely, it is the case if and only if for any path γ in \tilde{H} from e to an element k of K and any point $p \in P$, the solution of the time-dependent differential equation

$$c' = c \cdot \omega_{\tilde{H}}^R(\gamma')$$

starting from p is a loop: $c(1) = c(0) = p$.

Combining with Theorem 8.1.24 we obtain the following result:

Theorem 8.1.27. *Let \tilde{H} be a simply connected Lie group and \mathfrak{h} its Lie algebra. Let P be a complete \mathfrak{h} -manifold.*

For any discrete normal subgroup K of \tilde{H} , the action of \mathfrak{h} can be integrated to a smooth group action of \tilde{H}/K if and only if for each point $p \in P$ and any smooth path γ in \tilde{H} from e to an element k of K , the solution of the time dependent differential equation

$$c' = \omega_{\tilde{H}}^L(\gamma') \cdot c$$

starting from p is a loop: $c(1) = c(0) = p$.

In particular, the Lie algebra action always integrates to a group action of \tilde{H} .

When the action of \mathfrak{h} is not free, there can be yet smaller quotients of \tilde{H} to which the group action would factorise.

When the Lie algebra action integrates to a group action, Theorem 8.1.21 takes the following form:

Theorem 8.1.28. *Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P and Q be two complete \mathfrak{h} -manifolds univalent for H .*

Then every map $f : P \rightarrow Q$ which is \mathfrak{h} -equivariant is H -equivariant as well.

The induced morphism between local action groupoids preserves the map τ to H . It is thus equivalent to a morphism between the corresponding action groupoids which preserves the H component and defines an H -equivariant map (Example 8.1.6).

As a particular case, consider a representation V of the connected group H . If an \mathfrak{h} -manifold P is equipped with a V -valued field ϕ equivariant under the Lie algebra action, and if the action of \mathfrak{h} on P integrates to an action of H , ϕ is naturally equivariant under the action of H .

Completion in the general case

Now when the fundamental vector fields are not complete, it is to be expected that the \mathfrak{h} -manifold P needs to be extended in order to have a well defined action of each element of the group H . We thus want to build such an extension in a similar way to associated (right) G -spaces $X \times_H G$ or induced (right) representations $M \otimes_{k[H]} k[G]$. We will construct a quotient of $P \times H$ along the direction given by \mathfrak{h} : this direction is given by an integrable distribution and the space we are looking for is the leaf space.

Consider the product manifold $P \times H$. There is a right action of \mathfrak{h} defined as follows:

$$(p, h) \cdot \xi = (p \cdot \xi, -\xi \cdot h), \quad \xi \in \mathfrak{h} \tag{8.5}$$

Of course, it is isomorphic to the product right action used in Section 8.1.1: the intertwining isomorphism is $(p, h) \mapsto (p, h^{-1})$. However it will be useful to consider them as distinct foliations. Geometrically, this expression of the action of \mathfrak{h} makes it more explicit that the corresponding vector fields on $P \times H$ are directions in which $p \cdot h$ ought to be constant. Furthermore note that this action commutes with the natural *right* action of H on the factor H .

We call Δ the distribution spanned by the fundamental vector fields defined in (8.5). It is involutive therefore integrable and invariant under the right action of H . The corresponding foliation is called the *graph foliation* ([KM04]) and is denoted F_Δ . Recall the *left* Maurer-Cartan form on H written ω_H^L (Example 3.3.5). Two points (p_1, h_1) and (p_2, h_2) belong to the same leaf of F_Δ when there is a smooth path γ in H from h_1 to h_2 and a path c in P from p_1 to p_2 such that

$$\forall t, \quad c'(t) = c(t) \cdot (-\omega_H^L(\gamma'(t)))$$

or in terms of the opposite path γ^{-1} from h_1^{-1} to h_2^{-2} :

$$\forall t, \quad c'(t) = c(t) \cdot \omega_H^R(\gamma^{-1}'(t))$$

As was the case for the action foliation, the distribution Δ defines an H -equivariant connection on the product bundle $P \times H \rightarrow H$. Let us write $P \times_{\mathfrak{h}} H$ for the leaf space. It is provided with a quotient smooth structure (Definition 6.2.7). In the case at hand, a locally defined function on $P \times_{\mathfrak{h}} H$ is smooth if and only if its natural pullback to $P \times H$ is. This smooth structure may fail to correspond to a smooth manifold structure.

The leaf space is furthermore naturally equipped with an action of H since the graph foliation is stable under the right action of H . Since the action of H on $P \times H$ is smooth, the induced action on $P \times_{\mathfrak{h}} H$ can be proved to be smooth as follows. The following diagram is commutative:

$$\begin{array}{ccc} P \times H \times H & \xrightarrow{\text{id}_P \times m_H} & P \times H \\ (P \times H \rightarrow P \times_{\mathfrak{h}} H) \times \text{id}_H \downarrow & & \downarrow \\ P \times_{\mathfrak{h}} H \times H & \longrightarrow & P \times_{\mathfrak{h}} H \end{array}$$

with $m_H : H \times H \rightarrow H$ the product law of H . Since the diagonal morphism is smooth, according to the universal property of quotient smooth structures, the lower horizontal map is smooth.

The manifold P can be embedded in $P \times H$ as the fibre above $e \in H$:

$$p \in P \mapsto (p, e) \in P \times H$$

and can be further projected to $P \times_{\mathfrak{h}} H$:

$$P \rightarrow P \times H \rightarrow P \times_{\mathfrak{h}} H$$

Since $P \times \{e\}$ generates the whole of $P \times H$ under the right action of H , the same goes for its image in $P \times_{\mathfrak{h}} H$.

Univalence means that for every h in H , $P \times \{h\}$ intersects each leaf of F_Δ at most once (this is a direct translation of Definition 8.1.2). Since F_Δ is invariant under the right action of H , the property for $h = e$ is equivalent to the property for every h in H . As a consequence the action of \mathfrak{h} on P is H -univalent if and only if the composition

$$P \rightarrow P \times H \rightarrow P \times_{\mathfrak{h}} H$$

is injective.

The map $P \rightarrow P \times_{\mathfrak{h}} H$ can be proved to be open (Theorem 6.2.9) so that when univalence holds P embeds into $P \times_{\mathfrak{h}} H$. Since $P \times_{\mathfrak{h}} H$ can be covered with such embeddings of $P \times \{h\}$ and P is a manifold, we obtain the following theorem:

Theorem 8.1.29. *Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P be a (Hausdorff) \mathfrak{h} -manifold which is univalent for H .*

Then the smooth structure of the H -completion $P \times_{\mathfrak{h}} H$ of P is locally isomorphic to that of a manifold. It may however not be Hausdorff nor paracompact as a topological space.

When P fails to be univalent, the open maps $P \xrightarrow{p \mapsto [p,h]} P \times_{\mathfrak{h}} H$ are not injective, so that we obtain “charts” which identify open subsets of $P \times_{\mathfrak{h}} H$ with a quotient of P , as explained in [KM04]. In fact the points of $P \times \{h\}$ with the same image in $P \times_{\mathfrak{h}} H$ are those such that there exists a foliated path (for the graph foliation) connecting them. Recall the aforementioned intertwining isomorphism between the action foliation and the graph foliation:

$$(p, h) \in P \times H \mapsto (p, h^{-1}) \in P \times H$$

Then there exists a foliated path between two points (p_1, h) and (p_2, h) if and only if there is a morphism from p_1 to p_2 in the recurrence groupoid. Therefore the suitable quotient of P with which the open subsets of $P \times_{\mathfrak{h}} H$ are identified is the orbit space of the recurrence groupoid. For example when it is a groupoid orbifold (Section 6.3), $P \times_{\mathfrak{h}} H$ locally has the smooth structure of an orbifold, a possibility already envisioned in [KM04] (see also Section 8.1.2 below).

The H -completion satisfies the following universal property:

Theorem 8.1.30 ([Pal57], Theorem III.4.X and [KM04], Theorem 5.3). *Let H be a Lie group and \mathfrak{h} its Lie algebra. Let P be a (Hausdorff) \mathfrak{h} -manifold. Assume that the action of \mathfrak{h} is univalent for H .*

Then there exists a smooth H -space P_H , such that P embeds \mathfrak{h} -equivariantly as a smooth open subspace of P_H :

$$P \hookrightarrow_{\mathfrak{h}} P_H$$

and which satisfies the following universal property:

for any (non necessarily Hausdorff) H -manifold X and \mathfrak{h} -equivariant mapping $P \rightarrow X$ there is a unique smooth H -equivariant extension to P_H :

$$\begin{array}{ccc} P & \xrightarrow{\forall} & X \\ \downarrow & \nearrow \exists! & \\ P_H & & \end{array}$$

The space P_H is locally isomorphic to a smooth manifold. It may not be Hausdorff nor paracompact, but it always has closed points (it is a T_1 space). We will call the embedding $P \hookrightarrow_{\mathfrak{h}} P_H$ the (H -)completion map.

Remark. As stated, the H -completion of a univalent \mathfrak{h} -manifold is not always Hausdorff. As a quotient of a topological space under an equivalence relation, there is a simple criterion for its Hausdorffness: $P \times_{\mathfrak{h}} H$ is Hausdorff if and only if the graph of the equivalence relation, as a subset of $P \times_{\mathfrak{h}} H \times P \times_{\mathfrak{h}} H$, is closed (see Theorem 6.2.12 stated later). In our case, it boils down to the following: two pairs (p_1, h_1) and (p_2, h_2) which do not belong to the same leaf need to have neighbourhoods such that there is no leaf intersecting both neighbourhoods.

Without loss of generality, it is sufficient to consider the case with $h_1 = e$. Then given a small enough neighbourhood \mathcal{V} of p_1 in P , it is possible to construct a neighbourhood $\mathcal{U}_P \times \mathcal{U}_H$ of (p_1, e) such that any leaf intersecting $\mathcal{U}_P \times \mathcal{U}_H$ has to intersect $\mathcal{V} \times \{e\}$. In other words, Hausdorffness of the H -completion is equivalent to the following: for all p_1, p_2 in P and h in H such that there exists no smooth path γ in H from e to h such that there exists a complete solution $c_{p_2}^\gamma$ to

$$\begin{cases} c'(t) = c(t) \cdot \omega_H^L(-\gamma'(t)) \\ c(0) = p_2 \end{cases}$$

ending on p_1 , there exists a neighbourhood \mathcal{V}_1 of p_1 and \mathcal{V}_2 of p_2 such that there exists no smooth path γ in H from e to h such that there exists a complete solution starting from a point in \mathcal{V}_2 and ending in \mathcal{V}_1 .

We now present a simple example in which the H -completion is not Hausdorff.

Example 8.1.31 (Non-Hausdorff \mathbb{R} -completion). Consider the space $P = \mathbb{R}^2 \setminus \{0\}$ with an action of the Lie algebra $\mathfrak{h} = \mathbb{R}$ given on the generator by the vector field $X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The action is incomplete. Let $H = \mathbb{R}$.

The diagonal action of \mathfrak{h} which generates the graph foliation is as follows:

$$(p, t) \cdot u = (uX|_p, -u|_t) \in P \times H$$

As a consequence the leaves of the graph foliation are as follows:

1. $\{(p - tX, tX)\}_{t \in \mathbb{R}}$ for $p \in \mathbb{R}^* \times \mathbb{R}$
2. $\{(p - tX, tX)\}_{-\infty \leq t < y}$ for $p = (0, y)$ with $y > 0$
3. $\{(p - tX, tX)\}_{-y < t \leq \infty}$ for $p = (0, -y)$ with $y > 0$.

In particular, the action is univalent for H .

Now consider the two smooth curves in $P \times H$:

$$\begin{aligned} a : t \in \mathbb{R} &\mapsto \left(\begin{pmatrix} -t \\ 1 \end{pmatrix}, 0 \right) \\ b : t \in \mathbb{R} &\mapsto \left(\begin{pmatrix} -t \\ -1 \end{pmatrix}, 2 \right) \end{aligned}$$

At any time $t \neq 0$ they belong to the same leaf, but at $t = 0$ they belong to different leaves. As a consequence, the graph of equivalence relation associated to the graph foliation is not closed inside $(P \times H) \times (P \times H)$ and the leaf space is not Hausdorff.

Remark. This example suggest that it may be relevant to consider the quotient of $P \times H$ under the closure of the leaves. This is equivalent to consider the Hausdorff quotient of $P \times_{\mathfrak{h}} H$. In the case of Example 8.1.31 one would obtain \mathbb{R}^2 with \mathbb{R} acting by upwards vertical translations. This alternative completion has already been proposed in [KM04], but more work is required to assess in which cases it is relevant.

In [Pal57] (Section III.8) it is presented a similar univalent counterexample on $\mathbb{R}^2 \setminus \{0\}$, with the difference that the vector field X has fixed points and does not admit a smooth (or continuous) extension to \mathbb{R}^2 .

These examples illustrate well one of the obstructions to the Hausdorffness of the H -completion: the completion of incomplete orbits may be ill-separated when they are next to orbits which are “less incomplete”.

The second global property usually required from manifolds is *paracompactness*. It can sometimes be inherited from $P \times H$ to $P \times_{\mathfrak{h}} H$, according to the following lemma:

Lemma 8.1.32 ([Bou98] Theorem 9.10.5). Let X be a locally compact (Hausdorff) topological space. Then X is paracompact if and only if X is the disjoint union of locally compact σ -compact topological spaces.

The point is that the continuous image of a σ -compact space is σ -compact. If P is σ -compact (for example paracompact and connected) then $P \times_{\mathfrak{h}} H$ is σ -compact (possibly non-Hausdorff). If it is furthermore Hausdorff and P is H -univalent, so that $P \times_{\mathfrak{h}} H$ is locally Euclidean, $P \times_{\mathfrak{h}} H$ is in particular locally compact and it is paracompact.

Let \tilde{H} be the simply-connected integration group of \mathfrak{h} . A general (connected) group integration of \mathfrak{h} is a quotient $H = \tilde{H}/K$ by a discrete normal subgroup. There is a smooth covering map $P \times \tilde{H} \rightarrow P \times H$. It is equivariant with respect to the diagonal action of \mathfrak{h} hence it sends the graph foliation of $P \times \tilde{H}$ to that of $P \times H$. Therefore it induces a smooth map between the leaf spaces:

$$\begin{array}{ccc} P \times \tilde{H} & \longrightarrow & P \times H \\ \downarrow & & \downarrow \\ P \times_{\mathfrak{h}} \tilde{H} & \longrightarrow & P \times_{\mathfrak{h}} H \end{array}$$

If an \mathfrak{h} -manifold P is not \tilde{H} -univalent it cannot be \tilde{H}/K -univalent. Indeed in the following commutative diagram

$$\begin{array}{ccc} & P & \\ p \mapsto [p,e] \swarrow & & \searrow p \mapsto [p,e] \\ P \times_{\mathfrak{h}} \tilde{H} & \longrightarrow & P \times_{\mathfrak{h}} H \end{array}$$

if the left morphism is not injective, the right one cannot be injective. As stated earlier, injectivity of the completion map is equivalent to univalence.

When the action of \mathfrak{h} is complete, any path in H can be lifted to $P \times H$ along F_{Δ} . As a consequence, the projection $L \hookrightarrow P \times H \rightarrow H$ of any leaf L is surjective, and every leaf has a point of the form (p, e) . Thus when the action of \mathfrak{h} is complete and H -univalent, the map $P \hookrightarrow P \times_{\mathfrak{h}} H$ is a diffeomorphism.

Assume now that P is an \tilde{H} -univalent \mathfrak{h} -manifold. Translating the criterion given in Theorem 8.1.26 in terms of the graph foliation using the inversion, we obtain the following: P is H -univalent if and only if for all (p_1, p_2, k) in $P \times P \times K$,

$$(p_1, e) \in P \times \tilde{H} \underset{F_{\Delta}}{\rightsquigarrow} (p_2, k) \implies p_1 = p_2$$

When P is furthermore complete, so that the leaves of the graph foliation of $P \times \tilde{H}$ project by diffeomorphisms to P , this is equivalent to requiring that

$$\forall (p, h) \in P \times \tilde{H}, \forall k \in K, \quad (p, h) \underset{F_{\Delta}}{\rightsquigarrow} (p, hk)$$

Namely, P is \tilde{H}/K -univalent if and only if the right action of K preserves the leaves of the action foliation. In this case, the induced action of $K \subset \tilde{H}$ on $P \times_{\mathfrak{h}} \tilde{H}$ is trivial and the action of \tilde{H} factors to H . This is to be expected, since for a complete H -univalent \mathfrak{h} -manifold, the H -completion of P is identified with P itself.

Notice how the approach described here is different from the one in Section 8.1.2 in which we

constructed the action of H using paths in H : here we construct from P a new space which has a natural action of H but which is not always identified with P .

Relation between the local action and the completion constructions

As can be expected, the two constructions can be related. Since the groupoid structure on $P \rtimes H_{loc}$ describes a local action, the right object to compare it to is the action groupoid $(P \times_{\mathfrak{h}} H) \rtimes H$. When univalence holds, $P \times_{\mathfrak{h}} H$ is a possibly non-Hausdorff manifold, so that the action groupoid is a possibly non-Hausdorff Lie groupoid. We shall assume univalence in this section.

Recall the natural completion map

$$P \xrightarrow{p \mapsto [p, e]} P \times_{\mathfrak{h}} H$$

The map can be extended as a morphism between Lie groupoids as according to the following diagram, with (c, γ) a path from (p_0, h_0) to (p_1, h_1) foliated for the action foliation F_D :

$$\begin{array}{ccc} \text{Hol}_{F_D} & \xrightarrow{[c, \gamma] \mapsto (\tau([c, \gamma]), [p_0, e])} & H \rtimes (P \times_{\mathfrak{h}} H) \\ \downarrow & \dashrightarrow & \\ P \rtimes H_{loc} & & \end{array}$$

More precisely, the map

$$\begin{aligned} (\text{Hol}_{F_D})_1 &\rightarrow (P \times_{\mathfrak{h}} H) \rtimes H \\ [c, \gamma] &\mapsto ((p_0, e), \tau([c, \gamma])) \end{aligned}$$

is smooth and manifestly invariant under the left action of H , therefore it factors to $(P \rtimes H_{loc})_1$. Consider the completion map $P \hookrightarrow P \times_{\mathfrak{h}} H$. The following diagram with source maps is manifestly commutative:

$$\begin{array}{ccc} P \rtimes H_{loc} & \xrightarrow{[c, \gamma] \mapsto ((p_0, e), \tau([c, \gamma]))} & (P \times_{\mathfrak{h}} H) \rtimes H \\ \downarrow s & & \downarrow s \\ P & \hookrightarrow & P \times_{\mathfrak{h}} H \end{array}$$

Let us prove it is compatible with the target maps as well. We can actually prove more:

Lemma 8.1.33. The points (p_1, h) and (p_2, e) of $P \times H$ belong to the same leaf of the graph foliation F_{Δ} if and only if there exists in $P \times H$ a path from (p_1, e) to (p_2, h) which is foliated for the action foliation F_D .

Proof. Indeed the aforementioned foliated isomorphism:

$$\begin{aligned} P \times H &\rightarrow P \times H \\ (p, h) &\mapsto (p, h^{-1}) \\ F_{\Delta} &\rightarrow F_D \end{aligned}$$

sends (p_1, e) to (p_1, e) and (p_2, h) to (p_2, h^{-1}) . Since F_D is stable under the left action of H , the following chain of equivalences holds, with $x \overset{F_{\Delta}}{\rightsquigarrow} y$ (resp. $x \overset{F_D}{\rightsquigarrow} y$) standing for the existence of an

F_Δ -foliated (resp. F_D -foliated) path from x to y :

$$(p_1, e) \underset{F_\Delta}{\rightsquigarrow} (p_2, h) \overset{(p, h) \mapsto (p, h^{-1})}{\overset{\cong}{\rightleftarrows}} (p_1, e) \underset{F_D}{\rightsquigarrow} (p_2, h^{-1}) \overset{h}{\overset{\cong}{\rightleftarrows}} (p_1, h) \underset{F_D}{\rightsquigarrow} (p_2, e)$$

□

As a consequence, given an F_D -foliated path (c, γ) from (p_1, e) to (p_2, h) , the points (p_1, h) and (p_2, e) belong to the same leaf of F_Δ thus are projected to the same point in $P \times_{\mathfrak{h}} H$. Therefore, the following diagram with the target maps commutes:

$$\begin{array}{ccc} P \times H_{loc} & \xrightarrow{[c, \gamma] \mapsto ((p_1, e), h = \tau([c, \gamma]))} & (P \times_{\mathfrak{h}} H) \times H \\ t : [c, \gamma] \mapsto p_2 \downarrow & & \downarrow t : ([p, h_1], h_2) \mapsto [p, h_1 h_2] \\ P & \xrightarrow{p \mapsto [p, e]} & P \times_{\mathfrak{h}} H \end{array}$$

The compatibility with the identity and the composition maps is straightforward, using the morphism property of τ . Thus we constructed a Lie groupoid morphism:

$$P \times H_{loc} \rightarrow (P \times_{\mathfrak{h}} H) \times H$$

Lemma 8.1.33 has one further implication: let p_1 and p_2 be two points in P and $[p_1, e]$ and $[p_2, e]$ their respective images in $P \times_{\mathfrak{h}} H$. Assume they are related by the action of an element h of H :

$$[p_2] = [p_1] \cdot h$$

This means that $(p_1, h) \in P \times H$ belongs to the same leaf as (p_2, e) . The lemma then implies the existence of a morphism in $P \times H_{loc}$ from p_1 to p_2 with image h under τ . In other words: given p_1 and p_2 in P , the groupoid morphism $P \times H_{loc} \rightarrow H \times (P \times_{\mathfrak{h}} H)$ gives a surjection from the morphisms between p_1 and p_2 to the morphisms between $[p_1, e]$ and $[p_2, e]$.

Injectivity of the map from the space of morphisms between p_0 and p_1 to the space of morphisms between $[p_1, e]$ and $[p_0, e]$ is equivalent to the injectivity of τ on the submanifolds of morphisms in $P \times H_{loc}$ with fixed source and target. This is equivalent to the injectivity of τ of submanifolds of morphisms from each point to itself. According to Theorem 8.1.16, this is equivalent to the univalence of the action of \mathfrak{h} on P for the Lie group H .

To sum up, the restricted maps between the submanifolds of morphisms with fixed source and target

$$P \times H_{loc}(p_1, p_2) \rightarrow (P \times_{\mathfrak{h}} H) \times H([p_1, e], [p_2, e])$$

are smooth bijections. It turns out that this defines a pullback diagram of smooth manifolds:

$$\begin{array}{ccc} (P \times H_{loc})_1 & \longrightarrow & (P \times_{\mathfrak{h}} H) \times H \\ \downarrow (s, t) & & \downarrow (s, t) \\ P \times P & \longrightarrow & P \times_{\mathfrak{h}} H \times P \times_{\mathfrak{h}} H \end{array}$$

This is a consequence of the fact that the product of the completion maps $P \times P \hookrightarrow P \times_{\mathfrak{h}} H \times P \times_{\mathfrak{h}} H$ is a submersion [Mic08]. Such a Lie groupoid morphism is said to be *full and faithful* 6.1.6.

The image of P in $P \times_{\mathfrak{h}} H$ meets each orbit under H (every point is of the form $[p, h] = [p, e] \cdot h$). The charts

$$P \xrightarrow{\sim} P \times \{h\} \mapsto P \times_{\mathfrak{h}} H$$

are diffeomorphisms onto their images thus are submersions. As a consequence, the map

$$(P \rtimes H_{loc})_0 \times H \simeq P \times H \rightarrow P \times_{\mathfrak{h}} H$$

is a surjective submersion. This fact is formulated in the language of Lie groupoids by saying that the morphism $P \rtimes H_{loc} \rightarrow (P \times_{\mathfrak{h}} H) \rtimes H$ is “essentially surjective” 6.1.6.

A morphism of Lie groupoids which is essentially surjective, full and faithful is called a *weak equivalence of Lie groupoids* 6.1.6. We obtain a new consequence of univalence:

Theorem 8.1.34. *Let P be an \mathfrak{h} -manifold and H a Lie group integration of \mathfrak{h} . If the action of \mathfrak{h} is univalent for H then the Lie groupoid morphism $P \rtimes H_{loc} \rightarrow (P \times_{\mathfrak{h}} H) \rtimes H$ is a weak equivalence of Lie groupoids.*

A careful analysis of the argument shows that univalence is only involved at two points of the discussion. First, it was needed in order for $P \rtimes H_{loc}$ to be a (possibly non-Hausdorff manifold), so that we could talk about Lie groupoids. Second, it was essentially equivalent to the property that the map between morphism manifolds be injective, at fixed source and target. In a framework with a more general notion of Lie groupoids, one should be able to state an equivalence between H -univalence and the property that the morphism $P \rtimes H_{loc} \rightarrow (P \times_{\mathfrak{h}} H) \rtimes H$ is a weak equivalence.

Non-univalent actions

Even in the case of a non-univalent action, the leaf space construction produces an H -space, whether the \mathfrak{h} -manifold P is complete or not. However, when forming the leaf space, the manifold P will be reduced to a suitable quotient such that the action becomes univalent (effectively identifying points that are connected by a loop based at identity in H). This construction can be applied with *any* Lie group integration of \mathfrak{h} – in particular *non-simply connected* ones – but the larger the fundamental group is, the smaller (and possibly more singular) the quotient of P will be. The leaf space of $P \times H$ is called the H -completion of P and satisfies a universal property:

Theorem 8.1.35 ([KM04], Theorem 5.3). *Let H be a Lie group and \mathfrak{h} its Lie algebra. Let P be a Hausdorff \mathfrak{h} -manifold.*

There exists an H -space P_H provided with a smooth structure but which is not necessarily a smooth manifold nor Hausdorff, with a smooth \mathfrak{h} -equivariant map from P :

$$P \rightarrow_{\mathfrak{h}} P_H$$

and which has the following universal property:

For any (possibly non-Hausdorff) H -manifold X and \mathfrak{h} -equivariant mapping $P \rightarrow X$ there is a unique smooth H -equivariant “extension” to P_H :

$$\begin{array}{ccc} P & \xrightarrow{\forall} & X \\ \downarrow & \nearrow \exists! & \\ P_H & & \end{array}$$

Even when H is simply connected, it is possible for H -univalence to fail:

Example 8.1.36 (Non-univalence with simply-connected groups, [Pal57] Section III.9). Let H be a simply-connected (non-necessary hypothesis) Lie group of dimension $n \geq 2$. Consider a neighbourhood of e diffeomorphic to \mathbb{R}^n , and extract an open sub-neighbourhood \mathcal{U} of e

diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{n-1}$. \mathcal{U} has an infinitesimal action of \mathfrak{h} , so does its universal covering $\bar{\mathcal{U}}$. But a non-contractile smooth loop of \mathcal{U} based at e lifts to $\bar{\mathcal{U}}$ to a non-closed path so that the action of \mathfrak{h} on $\bar{\mathcal{U}}$ is not univalent.

For H simply-connected non-univalence cannot be solved by considering a further covering of H , so in order to integrate the action of \mathfrak{h} on $\bar{\mathcal{U}}$ into a group action it is unavoidable to take a quotient of $\bar{\mathcal{U}}$ (which will be \mathcal{U} if H is chosen as the Lie group integration).

The role of simply-connectedness of H can be understood as follows: two H -paths starting from the same point $p \in P$ with an identical image h under τ can be proven to end on the same point of P if they are homotopical as H -paths. Simply-connectedness of H implies that there is no homotopical obstruction to this argument on H , but there may still be obstruction due to the incompleteness of P as a \mathfrak{h} -manifold.

The univalent quotient

The considered quotient of P can also be understood from the point of view of the local action groupoid. Recall that the groupoid $P \times H_{loc}$ comes with a map $\tau : P \times H_{loc} \rightarrow H$. The action of \mathfrak{h} is univalent when the kernel of this map, the recurrence groupoid, is a discrete groupoid (namely it has no nontrivial morphisms). When it is nontrivial, one wants to use the quotient of P by the recurrence groupoid. Since it is in a sense a normal subgroupoid, it is possible to build a quotient groupoid which will have univalence for H . However the quotient groupoid may be singular.

Recall that the maps $I^H \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} P$ are local diffeomorphisms. As a consequence, taken together they define an immersion

$$I^H \xrightarrow{(s,t)} P \times P$$

The image C is the graph of the equivalence relation on P under which we wish to take the quotient. Using Godement’s criterion (Theorem 6.2.12) for quotient manifolds, we have to check the following:

1. C is an (embedded) submanifold of $P \times P$
2. The projection on the first factor $C \rightarrow P$ is a submersion

Remark. In the case P/C is a quotient manifold, the quotient map $\pi : P \rightarrow P/C$ is a submersion and the submanifold C can be constructed as the following pullback

$$\begin{array}{ccc} C & \longrightarrow & P/C \\ \downarrow & & \downarrow x \mapsto (x,x) \\ P \times P & \xrightarrow{\pi \times \pi} & P/C \times P/C \end{array}$$

Assuming that P is connected, P/C is as well and has a global dimension. Since the map $\pi \times \pi$ is a surjective submersion, C has the same codimension in $P \times P$ as the diagonal $\Delta_{P/C}$ does in $P/C \times P/C$, namely the dimension of P/C . Thus we derived the following identity:

$$\dim(P/C) = \text{codim}_P(C) \tag{8.6}$$

In our case, C is the image of I^H in $P \times P$. The composition

$$I^H \rightarrow P \times P \xrightarrow{p_1} P$$

is a submersion (this is one of the Lie groupoid axioms). The restrictive condition is the first one. Suppose the map $I^H \rightarrow P \times P$ is injective, namely that there is at most one morphism between two points, or equivalently no non-trivial morphism from a point to itself. Then the embedded (and closed) submanifold requirement holds in particular when the map $I^H \rightarrow P \times P$ is proper ([Lee03] Theorem 4.22). In this case the recurrence orbit is then said to be *proper*, see Section 6.1.3. One simple case is when I^H is compact.

When the map $I^H \rightarrow P \times P$ is not injective but is nonetheless a proper immersion, the recurrence groupoid has the structure of an *orbifold groupoid* (presented in Section 6.3). In this case, the quotient space P/I^H has the smooth structure of an *orbifold*: it is locally isomorphic to a quotient of a Euclidean space by the action of a finite group.

The remark following Lemma 8.1.32 regarding paracompactness holds here as well: if P is σ -compact and P/I^H is Hausdorff and locally compact, P/I^H is paracompact.

Example 8.1.37. Recall Example 8.1.36 of non-univalent \mathfrak{h} -manifolds for a simply-connected Lie group H . In these examples, P is a nontrivial connected covering of an open subset \mathcal{U} of H . The action of \mathfrak{h} on P is transitive and the recurrence orbits are the fibres above \mathcal{U} . As a consequence, the quotient map of $P \rightarrow P/I^H$ is identified with the covering map to \mathcal{U} .

In [KM04] (Section 6) it is constructed an example of action of the Lie algebra \mathbb{R}^2 on $(\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}$ which is non-univalent and presents a different type of pathology from the one illustrated in Example 8.1.31. Orbits in $(\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}$ accumulate around the (punctured) plane $z = 0$ and the corresponding leaves in $((\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}) \times \mathbb{R}^2$ accumulate around specific leaves contained in the plane $z = 0$. As a consequence, the leaf space is not Hausdorff.

In fact in this example the recurrence orbits are not closed. As explained in Section 8.1.2, the leaf space is covered by open subsets smoothly identified with the orbit space $((\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}) / I^{\mathbb{R}^2}$. Theorem 6.2.12 shows that if the recurrence orbits are not closed, the orbit space $((\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}) / I^{\mathbb{R}^2}$ is not closed so the quotient topology of $((\mathbb{R}^2 \setminus \{(0, 0)\}) \times \mathbb{R}) \times_{\text{Lie}(\mathbb{R}^2)} \mathbb{R}^2$ is not Hausdorff.

We call the quotient space P/I^H the *H-univalent quotient* of P . When it is a Hausdorff differentiable manifold, it is an *H-univalent \mathfrak{h} -manifold*, as can be expected:

Theorem 8.1.38. *Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P be an \mathfrak{h} -manifold such that its H -univalent quotient is a Hausdorff³ manifold.*

Then the quotient map

$$P \rightarrow P/I^H$$

is a local diffeomorphism and the H -univalent quotient of P has a unique structure of \mathfrak{h} -manifold such that the quotient map is \mathfrak{h} -equivariant. This action of \mathfrak{h} on P/I^H is furthermore H -univalent.

Proof. We first prove that the quotient map is a local diffeomorphism. We know that $s : I^H \rightarrow P$ is a local diffeomorphism so that I^H has the same dimension as P . The map $I^H \xrightarrow{(s,t)} P \times P$ is an immersion. By assumption it can be expressed as a surjective immersion to its image R . This implies that (s, t) is necessarily a submersion ([Lee03], Theorem 4.14). As a consequence, R is of the same dimension as P . Using Equation (8.6) we conclude that

$$\dim(P/I^H) = \dim P$$

Therefore the quotient map $P \rightarrow P/I^H$ which is a submersion between manifolds of the same dimension, is a local diffeomorphism.

We now construct the action of the Lie algebra \mathfrak{h} on P/I^H . Since we wish to avoid describing explicit charts, we will construct the vectors fields on P/I^H through their action on the smooth

³We do not wish to tackle the question of foliations and differential equations on non-Hausdorff manifolds.

functions. Let f be a smooth function on P/I^H . It corresponds to a smooth function $\bar{f} = \pi^* f$ on P which is constant on recurrence orbits. Let $\xi \in \mathfrak{h}$ and $\bar{\xi}$ the corresponding fundamental vector field on P .

Let us prove that $\mathcal{L}_{\bar{\xi}}\bar{f}$ is constant on the recurrence orbits. Since $\mathcal{L}_{\bar{\xi}}$ clearly satisfies the Leibniz relation with respect to functions constant on recurrence orbits, this will define a vector field on P/I^H which is related to ξ on P through $\pi : P \rightarrow P/I^H$. Furthermore since the vectors fields $\bar{\xi}$ satisfy the commutation relation of the Lie algebra \mathfrak{h} , the same will hold for the vector fields on P/I^H .

Let p_1 and p_2 be two point in P belonging to the same recurrence orbit. Then for small enough times the flow of $\bar{\xi}$ is simultaneously defined starting from p_1 and p_2 . The derivative of \bar{f} along $\bar{\xi}$ at p_1 (resp. p_2) is then defined as

$$\mathcal{L}_{\bar{\xi}}\bar{f}(p_1) = \partial_t|_{t=0}\bar{f}(\exp(t\bar{\xi})(p_1))$$

respectively

$$\mathcal{L}_{\bar{\xi}}\bar{f}(p_2) = \partial_t|_{t=0}\bar{f}(\exp(t\bar{\xi})(p_2))$$

But since p_1 and p_2 belong to the same recurrence orbit, when defined $\exp(t\bar{\xi})(p_1)$ and $\exp(t\bar{\xi})(p_2)$ belong to the same recurrence orbit, so that at small enough times

$$\bar{f}(\exp(t\bar{\xi})(p_1)) = \bar{f}(\exp(t\bar{\xi})(p_2))$$

As a consequence

$$\mathcal{L}_{\bar{\xi}}\bar{f}(p_1) = \mathcal{L}_{\bar{\xi}}\bar{f}(p_2)$$

which proves that $\mathcal{L}_{\bar{\xi}}\bar{f}$ is constant on recurrence orbits.

Uniqueness of the action of \mathfrak{h} on P/I^H is a direct consequence of the quotient map $P \rightarrow P/I^H$ being a surjective submersion with the requirement that it is \mathfrak{h} -equivariant. \square

As a consequence the local action groupoid of H on P/I^H defines a local group action of H on P/I^H .

The H -univalent quotient satisfies the following universal property:

Theorem 8.1.39. *Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P be an \mathfrak{h} -manifold and Q an H -univalent \mathfrak{h} -manifold. Then any smooth \mathfrak{h} -equivariant map $P \rightarrow Q$ factors through P/I^H .*

Indeed any \mathfrak{h} -equivariant map preserves the recurrence orbits as asserted in Theorem 7.2.13, thus factors as a set-theoretical map from P/I^H :

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{h}\text{-equivariant}} & Q \\ & \searrow & \nearrow \\ & P/I^H & \end{array}$$

Since the pullback of the map to P is smooth by hypothesis, the factored map $P/I^H \rightarrow Q$ is smooth by definition of the quotient smooth structure.

Remark. The construction of the univalent quotient can be proved to be functorial with a proof along the same lines.

A direct consequence of Theorem 8.1.39 is that any \mathfrak{h} -equivariant smooth map to an H -manifold, for example a linear representation of V , factors through the H -univalent quotient.

As stated earlier, it is this univalent quotient which is identified with patches of the H -completion $P \times_{\mathfrak{h}} H$. Since we have an \mathfrak{h} -equivariant map from P to P/I^H which is a local diffeomorphism, Theorem 8.1.22 ensures that it induces a morphism between the local action groupoids with the map between morphism manifolds a local diffeomorphism:

Theorem 8.1.40. *Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P be an \mathfrak{h} -manifold such that its H -univalent quotient is a Hausdorff manifold.*

Then the quotient map to the H -univalent quotient lifts to a morphism of local action groupoids:

$$\begin{array}{ccc} (P \times H_{loc})_1 & \longrightarrow & ((P/I^H) \times H_{loc})_1 \\ \begin{array}{c} \downarrow t \\ \downarrow s \end{array} & & \begin{array}{c} \downarrow t \\ \downarrow s \end{array} \\ P & \longrightarrow & P/I^H \end{array}$$

such that both maps between base manifolds and morphism manifolds are local diffeomorphisms.

It is the quotient of $P \times H_{loc}$ by the subgroupoid I^H in the following sense: given any Lie groupoid $\Gamma : \Gamma_1 \xrightarrow[t]{s} M$ with a morphism ϕ from $P \times H_{loc}$ such that all morphisms of I^H are sent to identities, there exists a unique Lie groupoid morphism from $(P \times H_{loc})_1$ to Γ such that the following diagram commutes:

$$\begin{array}{ccc} P \times H_{loc} & \xrightarrow{\phi} & \Gamma \\ \downarrow & \nearrow \text{dashed} & \\ (P/I^H) \times H_{loc} & & \end{array}$$

The proof of the universal property will rely on the following lemma:

Lemma 8.1.41. Let be $[c, \gamma]$ a morphism in $(P/I^H) \times H_{loc}$. Then it is the image of a morphism from $P \times H_{loc}$. Furthermore given two morphisms $[c_1, \gamma]$ and $[c_2, \gamma]$ in the preimage of $[c, \gamma]$ in $P \times H_{loc}$, there exist morphisms $[\eta_1, \ell_1]$ and $[\eta_2, \ell_2]$ in $P \times I^H$ such that the following equation is well defined and holds:

$$[\eta_1, \ell_1] \circ [c_1, \gamma] = [c_2, \gamma] \circ [\eta_2, \ell_2]$$

Proof. We name π the quotient map $P \xrightarrow{\pi} P/I^H$.

Let (c, γ) be an H -path in P/I^H , parametrised on $[0, 1]$. Since π is an \mathfrak{h} -equivariant local diffeomorphism, at every point t of $[0, 1]$ there exists a connected open neighbourhood \mathcal{U}_t on which (c, γ) can be lifted to an H -path in P . By compactness of the segment we can cover $[0, 1]$ by a finite family of such open subsets $(\mathcal{U}_{t_i})_{1 \leq i \leq n}$. We furthermore assume without loss of generality that there is no inclusion of one of the open subsets into another one. We order them by increasing order of lower bound. Since the higher bounds are also in increasing order, we conclude that for each i in $\llbracket 1, n-1 \rrbracket$ there is an $s_i \in \mathcal{U}_{t_i} \cap \mathcal{U}_{t_{i+1}}$. We define $s_0 = 0$ and $s_n = 1$.

By hypothesis for all i in $\llbracket 1, n \rrbracket$ there exists a path in P which lifts $c|_{\mathcal{U}_{t_i}}$. It does so in an \mathfrak{h} -equivariant manner so that it is an H -path associated with $\gamma|_{\mathcal{U}_{t_i}}$. We call c_i the restriction of this lift to $[s_{i-1}, s_i]$. Then for all i in $\llbracket 1, n-1 \rrbracket$, we know that

$$\pi(c_i(s_i)) = \pi(c_{i+1}(s_i))$$

so that $c_i(s_i)$ and $c_{i+1}(s_i)$ belong to the same recurrence orbit. As a consequence there exist H -paths (d_i, ℓ_i) in P from the points $c_i(s_i)$ to $c_{i+1}(s_i)$ which define morphisms of the recurrence groupoid. Since they belong to the recurrence groupoid, the morphisms in $(P/I^H) \times H_{loc}$

associated with $(\pi_*\ell_i)$ are the identities of the $c(s_i)$. We furthermore require that the paths (d_i, ℓ_i) have sitting instants.

We replace the paths $(c_i, \gamma|_{[s_{i-1}, s_i]})$ with reparametrised paths which have sitting instants. Thus in $(P/I^H) \rtimes H_{loc}$ the following identities hold:

$$\begin{aligned} & [\pi_*(c_1 \circ d_1 \circ c_2 \circ \cdots \circ d_{n-1} \circ c_n), \gamma|_{[0, s_1]} \circ \ell_1 \circ \gamma|_{[s_1, s_2]} \circ \cdots \circ \ell_{n-1} \circ \gamma|_{[s_{n-1}, 1]}] \\ &= [\pi_*c_1, \gamma|_{[0, s_1]}] \circ [\pi_*d_1, \ell_1] \circ [\pi_*c_2, \gamma|_{[s_1, s_2]}] \circ \cdots \circ [\pi_*d_{n-1}, \ell_{n-1}] \circ [\pi_*c_n, \gamma|_{[s_{n-1}, 1]}] \\ &= [c|_{[0, s_1]}, \gamma|_{[0, s_1]}] \circ \text{id}_c(s_1) \circ [c|_{[s_1, s_2]}, \gamma|_{[s_1, s_2]}] \circ \cdots \circ \text{id}_c(s_{n-1}) \circ [c|_{[s_{n-1}, 1]}, \gamma|_{[s_{n-1}, 1]}] \\ &= [c, \gamma] \end{aligned}$$

We now prove the second part of the lemma. Let $[c_1, \gamma]$ and $[c_2, \gamma]$ be two such lifts. Then $\pi(c_1(0)) = \pi(c_2(0))$ so there exists a morphism $[\eta_1, \ell_1]$ in the recurrence groupoid from $c_2(0)$ to $c_1(0)$. Now the morphism

$$[c_1, \gamma] \circ [\eta_1, \ell_1] \circ [c_2, \gamma]^{-1} = [c_1 \circ \eta_1 \circ \check{c}_2, \gamma \circ \ell_1 \circ \check{\gamma}]$$

belongs to the recurrence groupoid since $\gamma \circ \ell_1 \circ \check{\gamma}$ is a loop in H . \square

We now prove the theorem.

Proof. Let ϕ be a Lie groupoid morphism from $P \rtimes H_{loc}$ to a Lie groupoid $\Gamma = \Gamma_1 \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} \Gamma_0$ which sends morphisms of I^H to identities. In particular ϕ_0 sends recurrence orbits to points of Γ_0 and factors through P/I^H by a map we call f_0 .

We now show that the image of a morphism of $(P/I^H) \rtimes H_{loc}$ in Γ can be uniquely defined using a lift in P . Let $[c, \gamma]$ be a morphism in $(P/I^H) \rtimes H_{loc}$ and $[c_1, \gamma]$ and $[c_2, \gamma]$ the classes of two lifts in P . Then there exist morphisms $[\eta_1, \ell_1]$ and $[\eta_2, \ell_2]$ in $P \rtimes I^H$ such that

$$[\eta_1, \ell_1] \circ [c_1, \gamma] = [c_2, \gamma] \circ [\eta_2, \ell_2]$$

As a consequence, $[c_1, \gamma]$ and $[c_2, \gamma]$ have the same image under ϕ_1 . Therefore all the morphisms in the preimage of $[c, \gamma]$ have the same image under ϕ_1 : we define $f_1([c, \gamma])$ as this image. This is the only choice possible to factor ϕ_1 through $((P/I^H) \rtimes H_{loc})_1$ which justifies uniqueness.

Since f_1 fits into the following commutative diagram

$$\begin{array}{ccc} (P \rtimes H_{loc})_1 & \xrightarrow{\phi_1} & \Gamma_1 \\ \downarrow & \nearrow f_1 & \\ (P/I^H) \rtimes H_{loc} & & \end{array}$$

with the vertical map being a local diffeomorphism and the horizontal map being smooth, f_1 is necessarily smooth. \square

As a consequence, under the hypotheses of the theorem, there is a factorisation of the natural morphism

$$P \rtimes H_{loc} \rightarrow (P \times_{\mathfrak{h}} H) \rtimes H$$

through $(P/I^H) \rtimes H_{loc}$:

$$P \rtimes H_{loc} \longrightarrow (P/I^H) \rtimes H_{loc} \dashrightarrow (P \times_{\mathfrak{h}} H) \rtimes H$$

To summarize, when the action is not univalent, the \mathfrak{h} -manifold needs to be *quotiented*, and when the Lie algebra action is not complete, the \mathfrak{h} -manifold needs to be *extended* in order to obtain an H -space. In any case, the resulting H -space is constructed as a leaf space and may be singular, in particular it may not be a manifold or may not be Hausdorff, but it does satisfy a universal property (it is however always a differentiable stack).

The local action groupoid construction gives a unified approach to \mathfrak{h} -manifolds and carries more information than the H -completion: this information is captured by the recurrence subgroupoid which describes the possible actions of the identity element of H . When univalence holds, there is a natural weak equivalence map between the two constructions. When univalence does not hold, the univalent quotient embodies the data of P which is left in the H -completion.

Isotropy subgroups

In order to understand the structure of the H -completion of an \mathfrak{h} -manifold P , it is useful to look at its isotropy groups, which we will write $H_{[p,h]}$.

Let H be a connected Lie group with Lie algebra \mathfrak{h} . Let P be an \mathfrak{h} -manifold, and $P \times_{\mathfrak{h}} H$ its H -completion. First, recall that P spans the whole of $P \times_{\mathfrak{h}} H$ under the (right) action of H . Since along a group orbit the isotropy groups are conjugated, the question of the isotropy groups of points of $P \times_{\mathfrak{h}} H$ is reduced to that of the points of the image of P . More precisely, the following holds:

$$H_{[p,h]} = H_{[p,e] \cdot h} = hH_{[p,e]}h^{-1}$$

Let us adopt the notation $L_{p,h}$ for the leaf of the graph foliation F_D containing a point (p, h) : it corresponds to a point $[p, h]$ in the H -completion.

Let p be a point in P . Its image in $P \times_{\mathfrak{h}} H$ corresponds the leaf $L_{p,e}$, so its isotropy group in $P \times_{\mathfrak{h}} H$ is simply

$$\{h \in H \mid L_{p,e} \cdot h = L_{p,e}\}$$

Since the action of H preserves the graph foliation, we know that

$$\forall p \in P, \forall h \in H, L_{p,e} \cdot h = L_{p,h}$$

Therefore $H_{[p,e]}$ is the set of elements h of H such that

$$L_{p,h} = L_{p,e}$$

Namely, with ω_H^L the left Maurer-Cartan form of H (Example 3.3.5), there exists in H a smooth path γ from e to h such that the time dependent differential equation

$$c' = \omega_H^L(\gamma') \cdot c$$

has a global solution starting from p which is a loop: $c(1) = c(0) = p$.

Recall the morphism $P \times H_{loc} \rightarrow (P \times_{\mathfrak{h}} H) \times H$ which gives the element of H corresponding to each morphism in $P \times H_{loc}$. It naturally induces a morphism from the isotropy group of a point p in P to the isotropy group of its image $[p, e]$ in $P \times_{\mathfrak{h}} H$:

$$(P \times H_{loc})(p, p) \rightarrow H_{[p,e]}$$

Moreover, we proved the fullness of the groupoid morphism: this map is a surjection. The image is, by definition, the groups $\Gamma^H(p)$ defined in Section 7.2.3. We thus identified the isotropy groups of the H -completion, which are the image under τ of the isotropy groups of $P \times H_{loc}$. When

univalence holds for H , the groupoid morphism is faithful as well: the isotropy groups of the local action groupoid are in bijection with those of the H -completion.

Let V be a representation of H . As justified in Corollary 7.2.11, in the presence of a V -valued field ϕ which is equivariant under the Lie algebra action, there is the obvious condition on the isotropy groups of the points of P that, at each point, ϕ has to be invariant under the isotropy group:

$$\forall p \in P, \forall h \in H_{[p,e]}, \quad \phi(p) \cdot h = \phi(p) \quad (8.7)$$

Non-connected Lie groups

The construction discussed up to now only gives an action of a connected Lie group integration H^0 of the Lie algebra \mathfrak{h} . A general Lie group is an extension of its group of connected components by its neutral component subgroup:

$$H^0 \hookrightarrow H \twoheadrightarrow H/H^0 \simeq \pi_0(H)$$

with H/H^0 having the topology of a *discrete* group. The Lie algebra of H is naturally identified with that of H^0 :

$$\mathfrak{h}_e \xrightarrow{\sim} \mathfrak{h}$$

Hence in general a Lie group with \mathfrak{h} as a Lie algebra will be an extension of a discrete group by a connected Lie group integrating \mathfrak{h} .

Let P be an H -space. In order to construct the action of a non-connected Lie group H one has to take a representative of each class of H/H^0 , then find for each a diffeomorphism of P such that the composition between them and with the action of elements of H^0 respects the composition in H .

More concretely, let us consider a semi-direct product, which amounts to choosing representatives which form a splitting of the exact sequence⁴:

$$H^0 \hookrightarrow H^0 \rtimes C \twoheadrightarrow C$$

The adjoint action of C on H^0 is written

$$\beta : c \in C \mapsto \beta_c \in \text{Aut}(H^0)$$

To extend the action of H^0 to $H^0 \rtimes C$ one needs to find an action of the discrete group C on P such that

$$\forall (h, c) \in H^0 \times C, \forall p \in P, \quad p \cdot c \cdot h = p \cdot \beta_c(h) \cdot c$$

Example 8.1.42 (Extension from SO_{2n+1} to O_{2n+1}). Let P be an SO_{2n+1} -manifold. The element $-\text{Id}$ in O_{2n+1} generates a *central* subgroup of order 2 which defines a splitting of the exact sequence

$$\text{SO}_{2n+1} \hookrightarrow \text{O}_{2n+1} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$$

Thus extending the action of SO_{2n+1} to an action of O_{2n+1} is equivalent to finding an involutive diffeomorphism of P which commutes to the action of SO_{2n+1} .

In even dimension there is no central splitting so that one has to take into account the semi-direct product structure.

⁴Not every (second-countable) Lie group is a semi-direct product of its identity component by its group of connected components.

If there is a field ϕ on P with value in a representation V of H^0 which is equivariant under H^0 , one may wish for equivariance under the non-connected group H . In this case one has to make sure the action of H^0 can be extended to an action of H on V . Then H -equivariance of ϕ amounts to equivariance under the action of elements of each connected component.

Extending equivariant fields to the completion

When the manifold P is equipped with an equivariant field under the Lie algebra action, for example a k -form ϕ with value in a representation V of H , we want to “extend” it to the completion of P (recall that the completion map may not be injective). We are interested in objects on $P \times H$ which correspond to differential forms on $P \times_{\mathfrak{h}} H$ when the leaf space has a quotient manifold structure. We will use a general construction, which allows talking about vector fields and differential forms on the leaf space even when it does not have a natural manifold structure.

Forms on the leaf space correspond to forms of $P \times H$ which are *normal* to the leaves of F_D and parallel under the *Bott connection* (see Section 6.2.3). Furthermore, we are looking for forms which are equivariant under the right action of H on $P \times_{\mathfrak{h}} H$ so we will be considering forms on $P \times H$ which are equivariant under the right action of H on $P \times H$.

The Bott connection is a natural connection on the normal bundle to the *leaves* of the integrable distribution D :

$$\mathcal{N}D := T(P \times H)/D$$

The associated parallel transport along a vector field tangent to a leaf can be computed using the differentiable flow of the vector field on any representative vector in $T(H \times P)$, so that the covariant derivative is given by the Lie derivative:

$$\forall X \in \Gamma(D), Y \in \Gamma(TP), \quad \nabla_X^{Bott}[Y] = [\mathcal{L}_X(Y)] \quad \text{mod } D$$

In our case, there is a preferred generating family of vector fields tangent to the leaves given by the following diagonal fundamental vector fields on P :

$$X_{\xi}|_{(p,h)} = (p, h) \cdot \xi = (p \cdot \xi, -\xi \cdot h)$$

for $\xi \in \mathfrak{h}$. We want to extend ϕ to $P \times H$ into a k -form which is normal to D and invariant under the diagonal action of \mathfrak{h} .

Write

$$\pi_P : P \times H \rightarrow P$$

for the second factor projection. The pullback tangent bundle π_P^*TP is naturally identified to a subbundle of $T(P \times H)$:

$$\pi_P^*TP \simeq TP \times H \subset T(P \times H)$$

We will make implicit use of this identification. We want to relate $\mathcal{N}D$ to π_P^*TP .

The field ϕ is defined over P which is embedded in $P \times H$ as $P \times \{e\}$. Now π_P^*TP is a supplementary distribution to D so that there is a natural vector bundle isomorphism

$$\pi_P^*TP \oplus D \simeq T(P \times H)$$

The first factor projection $\pi_H : P \times H \rightarrow H$ intertwines the diagonal action of \mathfrak{h} on $P \times H$ with the inverse action on the left on H : namely

$$\xi_L|_h := -\xi \cdot h$$

for $\xi \in \mathfrak{h}$. Since the subbundle π_P^*TP is the kernel of the differential $d\pi_H$, one concludes that π_P^*TP is equivariant with respect to the diagonal action of \mathfrak{h} . This also holds for the involutive distribution D thus the decomposition $\pi_P^*TP \oplus D$ is equivariant with respect to the diagonal action of \mathfrak{h} and we obtain a H -equivariant bundle isomorphism:

$$\pi_P^*TP \simeq_{\mathfrak{h}} \mathcal{N}D$$

We want to use π_P^*TP as representing $\mathcal{N}D$ and then extend ϕ by defining its value on $\Lambda^k \pi_P^*TP$. In order to make sure it corresponds to a Bott-parallel form we need to identify the connection on π_P^*TP corresponding to the Bott connection on $\mathcal{N}D$. Recall that the Bott parallel transport on $\mathcal{N}D$ along the fundamental vector fields X_ξ is simply the action of their flows on representative vectors. Since the subbundle $\pi_P^*TP \subset T(P \times H)$ is preserved under the action of \mathfrak{h} , the parallel transport along fundamental vector fields in π_P^*TP associated to the Bott connection can thus be identified to the natural flow action of \mathfrak{h} on π_P^*TP . This can be summed up in the following diagram, with $\exp(X_\xi)$ the locally defined flow of a fundamental vector field and $e^\xi : t \mapsto \exp(tX_\xi)$ the associated flow lines:

$$\begin{array}{ccccc} & & \sim & & \\ & \swarrow & & \searrow & \\ \pi_P^*TP & \longrightarrow & T(P \times H) & \xrightarrow{\text{mod } D} & \mathcal{N}D \\ & \downarrow D \exp(X_\xi) & \downarrow D \exp(X_\xi) & & \downarrow \text{PT}_{e^\xi}^{\text{Bott}} \\ \pi_P^*TP & \longrightarrow & T(P \times H) & \xrightarrow{\text{mod } D} & \mathcal{N}D \\ & \swarrow & & \searrow & \\ & & \sim & & \end{array}$$

For $\xi \in \mathfrak{h}$ we write $\bar{\xi} \in \Gamma(TP)$ for the corresponding fundamental vector field on P . We write

$$n_D : \begin{cases} T(P \times H) \rightarrow \pi_P^*TP \\ (u, \xi_L)|_{(p,h)} \mapsto (u - \bar{\xi}|_p, 0) \end{cases}$$

for the linear projection to π_P^*TP along D . Then n_D is invariant under the diagonal action of \mathfrak{h} . The adjoint operator

$$n_D^* : \pi_P^*T^*P \rightarrow T^*(P \times H)$$

extends to graded algebra morphisms between the exterior algebras:

$$n_D^* : \Lambda^\bullet \pi_P^*T^*P \rightarrow \Lambda^\bullet T^*(P \times H)$$

We want to extend ϕ to $P \times H$ into a V -valued k -form which is at the same time normal to D , Bott-parallel and *right* H -equivariant. Let us extend ϕ from P to $P \times H$ as follows:

$$\phi_H : \begin{cases} P \times H \rightarrow \Lambda^k T_p^*P \otimes V \subset \Lambda^k T_{(p,h)}^*(P \times H) \otimes V \\ (p, h) \mapsto h^{-1} \cdot (n_D^* \phi)|_p \end{cases}$$

The extended ϕ_H has vanishing contraction with vectors of D :

$$\forall X \in D, i_X \phi_H = 0$$

spanned by the diagonal action of \mathfrak{h} by definition of n_D and is clearly equivariant under the right

action of H . Bott-parallelism is equivalent to invariance under the Lie derivative of the diagonal fundamental vector fields (ξ, ξ_L) . The following calculation shows why it is satisfied:

$$\begin{aligned} \mathcal{L}_{(\xi, \xi_L)}(\phi_H)(p, h) &= h^{-1} \cdot \xi \cdot (n_D^* \phi)|_{(p, h)} + h^{-1} \cdot \mathcal{L}_{(\xi, -\xi_L)}(n_D)^* \phi|_{(p, h)} + h^{-1} \cdot (n_D^* \mathcal{L}_{\xi} \phi)|_{(p, h)} \\ &= h^{-1} \cdot n_D^* (\xi \cdot \phi + 0 + \mathcal{L}_{\xi} \phi)|_{(p, h)} \\ &= 0 \end{aligned}$$

under the assumption of equivariance of ϕ under the right action of \mathfrak{h} , with the middle term vanishing because n_D is invariant under the diagonal action of \mathfrak{h} .

In this way equivariant k -forms on P can be extended to the completion:

Theorem 8.1.43. *Let P be an \mathfrak{h} -manifold and H a Lie group integration of \mathfrak{h} . Let V be a representation of H and $\phi \in \Omega^k(P, V)$ equivariant under \mathfrak{h} . Assume that the H -completion P_H of P has the quotient structure of a smooth manifold.*

Then there exists on P_H an H -equivariant V -valued k -form ϕ_H which pulls back to ϕ under the completion map $P \rightarrow P_H$:

$$\begin{aligned} P &\rightarrow P_H \\ \Omega^\bullet(P, V) &\leftarrow \Omega^\bullet(P_H, V) \\ \phi &\leftarrow \phi_H \end{aligned}$$

We can then extend our previous remark on the action of isotropy group on equivariant fields. In the presence of an equivariant k -form ϕ with value in a representation V of H , there is the obvious condition on the isotropy groups of the points of P that at each point ϕ has to be invariant under the isotropy group as an element of $\Lambda^k T^*P \otimes V$:

$$\forall p \in P, \forall h \in H_{[p, e]}, \quad \phi(p) \cdot h = \phi(p) \quad (8.8)$$

This extension procedure preserves H -invariant equations. In particular, the following result will be of interest to us:

Theorem 8.1.44. *Let P be a generalised Cartan bundle equipped with a $(\mathfrak{g}, \mathfrak{h})$ -valued Cartan 1-form ϖ . P has an associated Lie algebra action of \mathfrak{h} . Let H be a Lie group integration of \mathfrak{h} such that the H -completion of P is a smooth manifold.*

Then the H -equivariant extension of ϖ to $P \times_{\mathfrak{h}} H$ is a (G, H) -Cartan 1-form.

Proof. Notice that Theorem 8.1.43 implies that ϕ_H pulls back to $h \cdot \phi$ under the map

$$p \in P \mapsto [p, h] \in P \times_{\mathfrak{h}} H$$

Since these maps are open, ϕ_H is a Cartan 1-form on the image of each of these maps. But for h going through H , these maps cover $P \times_{\mathfrak{h}} H$. Therefore ϕ_H is a Cartan 1-form on the whole of $P \times_{\mathfrak{h}} H$. \square

A corollary of Theorem 8.1.43 is that equivariant vector-valued differential forms factor to the recurrence quotient:

Corollary 8.1.45. *Let P be an \mathfrak{h} -manifold and H a Lie group integration of \mathfrak{h} . Let V be a representation of H and $\phi \in \Omega^k(P, V)$ equivariant under \mathfrak{h} . Assume that the H -univalent quotient P/I^H of P has the quotient structure of a smooth manifold.*

Then there exists on P/I^H an \mathfrak{h} -equivariant V -valued k -form ϕ_{rec} which pulls back to ϕ under the quotient map $P \rightarrow P/I^H$:

$$\begin{aligned} P &\rightarrow P/I^H \\ \Omega^\bullet(P, V) &\leftarrow \Omega^\bullet(P/I^H, V) \\ \phi &\leftarrow \phi_{rec} \end{aligned}$$

The corollary is immediate since P/I^H embeds as an open submanifold of the H -completion.

8.2 The structure of H -spaces

8.2.1 Slices and Cartan property

In this section, we consider a *Hausdorff* manifold P on which a Lie group H acts smoothly *on the right*. We discuss sufficient properties for the action to have *slices* (defined below). The main references will be [Pal61] which discuss the existence of slices for action of non compact groups, and [Bre72] which discuss the structure of *compact* group actions but mainly uses the existence of slices for many results. We state some of the results in a more general setting, mainly the case of (Hausdorff) completely regular spaces.

We will be interested in the existence of a special type of charts on H -manifolds called *tubes*. If charts on a smooth manifold are local identifications with Euclidean spaces, tubes are local identifications with G -manifolds of a specific type:

Definition 8.2.1 (Tubes). A *tube* about an orbit O of type $K \backslash H$ is an H -equivariant *embedding*

$$A \times_K H \rightarrow \mathcal{U} \supset O$$

onto an *open neighbourhood* of O with A some K -space.

If A is a Euclidean space with isometric action, the tube is said to be *linear*. An action (or an H -space) which has linear tubes around each orbit is called *locally smooth* [Bre72] (note that it needs not be differentiable).

Definition 8.2.2 (Tubular Neighbourhoods). Assume the action is smooth and let O be an orbit of type $K \backslash H$. An *open invariant tubular neighbourhood* about O is the data of a smooth H -equivariant vector bundle E_O over the *coset manifold* $K \backslash H$ with an equivariant diffeomorphism χ from an open H -invariant neighbourhood \mathcal{U}_E of the zero section of E_O to an open H -invariant neighbourhood of O [Bre72; RS13]:

$$\begin{array}{ccc} K \backslash H & \xrightarrow{h \mapsto h \cdot p} & O \\ \downarrow & & \downarrow \\ \mathcal{U}_E & \xrightarrow[\chi]{\sim} & \chi(\mathcal{U}_E) \\ \downarrow & & \downarrow \\ E_O & & P \end{array}$$

A simple image to have in mind is the case $H = \mathbb{R}$ and $K = \mathbb{Z} \subset \mathbb{R}$ acting on some space A by some homeomorphism f . The tube can be represented by a cylinder $A \times [0, 1]$ to which the two ends are glued according to f . For example with f the parity on $A = \mathbb{R}$, the tube is a (non-compact) Möbius band.

We now define the strongly related notion of slice:

Definition 8.2.3 (Slices). A *slice* for H at a point p of P is an H_p -stable subset S of P containing p such that $S \cdot H$ is an open subset and has an equivariant mapping $f : S \cdot H \rightarrow H_p \backslash H$ such that $f^{-1}([e]) = S$. Such an f is necessarily unique [Pal61].

When S is an H_p -space isomorphic to a Euclidean space with isometric action, the slice is said to be *linear* [Bre72].

The relation between tubes and slices is given by the following lemma:

Lemma 8.2.4 ([Bre72], Theorem 2.4.2). Let $p \in S \subset P$ with S stable under H_p , then S is a slice at p if and only if the map

$$\begin{aligned} S \times_{H_p} H &\rightarrow P \\ [h, s] &\mapsto s \cdot h \end{aligned}$$

is a tube about $p \cdot H$.

In a way, a slice contains the data of a tube augmented with a specific kind of section, fixing some “gauge ambiguity” in the identification of the tube with a space of the form $A \times_K H$.

For *smooth* Lie group actions, smooth linear tubes provide open invariant tubular neighbourhoods, since

$$A \times_K H \rightarrow H$$

is naturally a smooth vector bundle above $K \backslash H$. The converse holds with A the intersection of the neighbourhood of the zero section with any given fibre of the vector bundle.

Definition 8.2.5 (Cartan action). The action of H on P has the *Cartan property*, or is a *Cartan action* if every point admits a neighbourhood \mathcal{U} such that

$$\{h \in H \mid \mathcal{U} \cdot h \cap \mathcal{U} \neq \emptyset\}$$

has compact closure in H . Such a subset \mathcal{U} is called *thin* [Pal61].

The Cartan property prevents orbits “accumulation” so that their subset topology matches that of the suitable coset spaces:

Theorem 8.2.6 ([Pal61], Proposition 1.1.5). *Let P be a completely regular Cartan H -space and $p \in P$. Then the orbital map $h \in H_p \backslash H \rightarrow h \cdot p$ is a homeomorphism onto the orbit $p \cdot H$ of p (in particular the induced topology on the quotient matches the coset topology).*

When P is furthermore a differentiable manifold and H is a Lie group acting smoothly, the orbital map gives the orbit the structure of an embedded submanifold.

The Cartan property is strongly related to the existence of slices as the following characterisation shows:

Theorem 8.2.7 (Characterisation of Cartan actions, [Pal61] Theorem 2.3.3). *The action of a Lie group H on a completely regular space P has the Cartan property if and only if it admits a slice at each point and all isotropy groups are compact.*

Theorem 8.2.8 ([Pal61], Proposition 2.2.2). *A differentiable Cartan action on a differentiable manifold admits open invariant tubular neighbourhoods about each orbit.*

We give a rough sketch of the proof adapted from [Pal61], Sections 2.2, 2.3:

Proof. Let p be a point in P and call O its orbit. Its isotropy group H_p is compact according to Theorem 8.2.7. Therefore one can build a Riemannian metric on P which is invariant under H_p . Let us call⁵ TO the distribution

$$T_pO := TO|_p = p \cdot \mathfrak{h}$$

and NO its normal bundle, which can be embedded in $TP|_O$ as TO^\perp which is a supplementary subbundle to TO .

The point is to build a transversal S to the orbit O through p in the following way: the geodesic exponential map on T_pP restricted to a small enough neighbourhood of the origin is a diffeomorphism. It is H_p -equivariant because the metric is H_p -invariant. When further restricted to N_pO its image is a submanifold transversal to T_pO , which we call S_0 .

It can then be proved that there exists a local section $\chi : \mathcal{U} \subset H_p \backslash H \rightarrow H$ on an open neighbourhood of $[e]$ in $H_p \backslash H$ and an open submanifold S_1 of S_0 such that

$$\begin{aligned} S_1 \times \mathcal{U} &\rightarrow P \\ (s, u) &\mapsto s \cdot \chi(u) \end{aligned}$$

is a *local* diffeomorphism onto an open neighbourhood of p in P . Such an S_1 is called a *near-slice*, and it is proved in [Pal61] (Proposition 2.1.7) that for a Cartan action any near-slice at p contains a slice S_2 as a sub-transversal to T_pO . The inverse image $\mathcal{U}_2 := \exp_p^{-1}(S_2)$ of S_2 by the geodesic exponential in $N_p \subset T_pP$ is an open neighbourhood of the origin which is stable under the compact group H_p . Hence S_2 , or at least a sub-slice of it, is a linear slice.

A tubular neighbourhood is built by replacing the metric on the tube by an H -equivariant extension of the metric on S_2 and using the geodesic exponential map on $\mathcal{U}_2 \cdot H \subset NO \subset T|_OP$. \square

Definition 8.2.9 (Proper actions in the sense of Palais, [Pal61]). The action of H on P is *proper* if each point p of P admits a neighbourhood \mathcal{U} such that every point y of P admits a neighbourhood \mathcal{U}_y such that

$$\{h \in H \mid \mathcal{U}_y \cdot h \cap \mathcal{U} \neq \emptyset\}$$

has compact closure in H .

As long as P is locally compact (and Hausdorff) this is equivalent to the usual definition (Example 6.1.12), namely the following map is proper ([Kar16], Remark 5.2.4):

$$\begin{aligned} P \times H &\rightarrow P \times P \\ (p, h) &\mapsto (p \cdot h, p) \end{aligned}$$

In particular, in this case, proper group actions on P have closed orbits since they are the image of the closed subsets $\{p\} \times H$ (this can be proved under weaker hypotheses, see [Pal61]).

We gather in the following theorem results relating properness and the Cartan property.

Theorem 8.2.10 (Properness and Cartan property, [Pal61] Theorems 1.2.3, 1.2.9, [Kar16] Remark 5.2.4).

- All actions of compact groups on completely regular spaces are proper.
- A proper action has the Cartan property.
- If P is a locally compact and completely regular topological space, the action of H is proper if and only if it has the Cartan property and the quotient topology on P/H is Hausdorff.

⁵We do not know yet whether O is a submanifold of P .

- An H -space P is Cartan if and only if each point admits a neighbourhood \mathcal{U} such that $G \cdot \mathcal{U}$ is a proper H -space.

8.2.2 Principal orbits

Let P be an Hausdorff differentiable manifold on which a Lie group H acts smoothly. In this section we assume that P is *connected* and *has linear slices at each point*.

Note that when an action does not have the Cartan property, the induced topology (or smooth structure) on an orbit does not necessarily match with the quotient structure on the coset space associated to the isotropy groups. A standard example is the orbits of an action of \mathbb{R} of irrational slope on the 2-torus, which are not locally connected.

Orbit types form a pre-ordered set: we define for two orbits O_1 and O_2 the following pre-order⁶ relation:

$$[O_1] \leq [O_2]$$

if and only if for any $p_1 \in O_1, p_2 \in O_2$, H_{p_2} is conjugated to a subgroup of H_{p_1} . Roughly speaking, an orbit is larger when its isotropy group is smaller.

This relation may or may not be antisymmetric. The following lemma justifies it is under some assumptions we state below.

Lemma 8.2.11 ([Mes16], Lemma 3.15). Let H be a Lie group and K a closed subgroup which has a finite number of connected components (e.g. compact).

For any h in H , $hKh^{-1} \subset K$ implies that $hKh^{-1} = K$.

Proof. Let h be an element of H such that $\text{Ad}_h K \subset K$.

At the level of the Lie algebra, since Ad_h is a linear automorphism, dimension considerations imply that $\text{Ad}_h \mathfrak{k} \subset \mathfrak{k} \implies \text{Ad}_h \mathfrak{k} = \mathfrak{k}$. The subgroups $\text{Ad}_h K$ and K thus have the same neutral component which we call K_0 and which is preserved by Ad_h . The action of h thus factors to the quotient group K/K_0 , which is finite by assumption.

As an injection of the finite set K/K_0 into the subset

$$\text{Ad}_h(K/K_0) = (\text{Ad}_h K)/K_0 \subset K/K_0$$

the map Ad_h necessarily defines a bijection of K/K_0 . One concludes that $\text{Ad}_h K$ has a point in each connected component of K , thus is equal to K . □

A consequence of this lemma is that when H is compact, so that all its closed subgroups are compact, the pre-order relation on orbit types is antisymmetric hence an order relation. More generally, when the action of H on P is proper, so that all isotropy groups are compact, the pre-order relation *on the orbit types of P* is antisymmetric.

Borrowing the terminology from representation theory, we make the following definition:

Definition 8.2.12 (Isotypic components). An H -space which has a unique orbit type is called *isotypic*.

Let $[O]$ be an orbit type under H and P an H -space. The reunion of the orbits in P of type $[O]$ is called the (*isotypic*) *part* of type $[O]$ of P , we write it $P_{[O]}$. We call *isotypic components* their connected components.

When an action is locally smooth, isotypic parts are submanifolds:

⁶A transitive and reflexive relation.

Theorem 8.2.13 ([Bre72], Theorem IV.3.3). *Let P be a locally smooth H -manifold. Let $[O]$ be an orbit type present in P . Then the part of P of type $[O]$, $P_{[O]}$, is an embedded submanifold of P .*

The partition of a proper H -space P into the isotypic components is called *the stratification by orbit types*. It satisfies the following property, which already appeared in the section on orbifolds (Section 6.3):

Theorem 8.2.14 (Frontier condition, [Mes16]). *The closure of an isotypic component X is the reunion of X with isotypic components of strictly lower dimension.*

A dense part of the manifold belongs to one isotypic part, which is called the *principal part* of the manifold:

Theorem 8.2.15 (Principal orbit type, [Bre72] Theorem 3.1). *If P has linear slices at each point then it has a maximal orbit type for the partial order relation. The isotypic part of maximal type is a dense open subset of P and its image in the orbit space is connected. The maximal orbit type is called the principal orbit type, the orbits are called principal orbits and the isotropy groups principal isotropy groups.*

Principal orbits are easily characterized by the trivial transversal action of their isotropy groups:

Theorem 8.2.16 (Characterisation of the principal orbits, [Bre72] Theorem 3.2.iii). *Let p be a point of P with a linear slice S and compact isotropy group. The orbit through p is a principal orbit if and only if H_p acts trivially on S . In particular, the linear tube generated by S is naturally isomorphic to $H_p \backslash H \times S$.*

In particular at any point of a principal orbit, the infinitesimal action of the isotropy group on the normal space to the orbit is trivial.

An orbit is principal when its isotropy type is maximal, but at every point of a slice at p the isotropy groups are subgroups of H_p . Lemma 8.2.11 ensures they are all equal to H_p .

8.2.3 Principal H -spaces

In this section, we state necessary conditions for a free H -space to have an H -principal bundle structure over its orbit space.

We are interested by how P fibrates over its orbit space P/H – in particular, when is it a principal bundle fibration? A first condition is to have a unique orbit type. It is not enough, as shown for example by irrational quotients of the 2-torus.

When the action is locally smooth, a manifold with unique orbit type has a structure of principal bundle above its orbit space:

Theorem 8.2.17 ([Bre72] Theorems IV.3.3, VI.2.5). *Let H be a Lie group and P a connected Cartan H -manifold which is isotypic with isotropy groups conjugated to $K \subset H$. Then the orbit space has a (non-necessarily Hausdorff) quotient manifold structure, and the quotient map $P \rightarrow P/H$ is a fibre bundle with fibre $k \backslash H$ and structure group $\text{Aut}_H(K \backslash H) \simeq \text{Norm}(K)/K$.*

We reproduce here the proof from [Bre72] which is stated for H compact but directly applies to a Cartan action.

Proof. Let us write

$$P \xrightarrow{\pi} P/H$$

the projection to the orbit space.

We first endow P/H with the following smooth structure (as defined in Section 1.1): a function $\mathcal{U} \subset P/H \rightarrow \mathbb{R}$ is smooth if and only if its pullback to $\pi^{-1}(\mathcal{U})$ is a smooth function. We want to prove that it is locally isomorphic to that of a (finite dimensional) vector space (using a global structure saves us from having to check compatibility conditions).

Consider an open linear tube $S \times_K H \subset P$. Isotropy groups are all compact (Theorem 8.2.7) so applying Theorem 8.2.16 isotropy groups of elements of S are all identical to K . Hence the tube is diffeomorphic to $H/K \times S$.

Composing the tube embedding

$$S \times_K H \simeq_H S \times K \backslash H \rightarrow P$$

with the orbit space projection $P \rightarrow P/H$ we obtain a continuous bijection from S and the subset $\mathcal{U} = \pi(S \times_K H) = \pi(S) \subset P/H$ it projects onto. Because $\pi^{-1}(\mathcal{U}) = S \times_K H$ is open in P , \mathcal{U} is open in P/H for the quotient topology. Since the smooth (local) functions on P/H are the local functions which pullback to P to smooth functions, smooth functions on \mathcal{U} are exactly smooth functions on $S \times_K H$ which are H -invariant, which are identified with smooth functions on S . Since S is a local slice the smooth structure on P/H is locally isomorphic to that of a Euclidian vector space i.e. P/H is a quotient smooth (possibly non-Hausdorff) manifold.⁷

Such open subsets $\pi(S) \subset P/H$ cover the orbit space (and form a topology basis as S can be arbitrarily reduced). The inverse image of $\pi(S)$ is by construction diffeomorphic to $S \times K \backslash H$ as an H -space, hence $P \rightarrow P/H$ forms a principal bundle with fibre $K \backslash H$ with the action of H . The structure group of the principal bundle is the group of diffeomorphisms of $K \backslash H$ commuting with the action of H , $\text{Aut}_H(K \backslash H)$, which is readily shown to be identified with the quotient of the normaliser of K by K itself: $\text{Norm}(K)/K$. \square

The structure we are looking for is that of a principal fibre bundle. In this case the Cartan property characterises exactly principal bundle actions:

Theorem 8.2.18 (Locally trivial H -spaces, [Pal61] Section 4.1). *Let H be a Lie group and P a completely regular connected free H -space. Then P is a locally trivial principal H -bundle over its orbit space if and only if the action of H has the Cartan property.*

In particular, for a subspace $Q \subset P$ that is a reunion of orbits of the same type, Q is a Cartan H -space and fibrates locally trivially above its orbit space. In particular, for P_{prin} the principal isotypic component we obtain the H -principal fibration of a dense subset of P over its orbit space.

Part of the theorem can be slightly generalised in the following manner:

Theorem 8.2.19. *Let H be a Lie group and P a completely regular connected H -space. If the action admits linear slices and is isotypic with normal isotropy groups, then P is a locally trivial H/K -principal bundle over its orbit space, for K the isotropy group of the action.*

8.2.4 Manifold structure on the orbit space

When there are several orbit types, one cannot hope to have a principal bundle fibration over the whole orbit space. Here we are interested in the local structure of the orbit space. According

⁷Alternatively it is possible to use the transition functions between two open subsets of P/H which are the image of two tubes in P to build an atlas on P/H .

to Ehresmann's fibration theorem [Dun18], when H is compact if P/H is a smooth quotient manifold then all orbits are necessarily H -equivariantly diffeomorphic, hence have the same type.

First, the orbit space of a proper H -manifold is stratified in a similar way as the H -manifold itself is:

Theorem 8.2.20 (Stratification of the orbit space by orbit types, [Mes16; Mei03]). *Let H be a Lie group and P a proper H -manifold.*

For a compact subgroup $K \subset H$, define the subset

$$(P/H)_{[K]} : P_{[H/K]}/H$$

the orbit space of orbits of type H/K .

Then the connected components of the subsets $(P/H)_K$ satisfy the frontier condition 8.2.14.

Furthermore the quotient smooth structures on the connected components of $(P/H)_{[K]}$ give them the structure of smooth quotient manifolds.

Thus even when the orbit space is singular, it decomposes as a reunion of smooth manifolds.

If we allow for manifold with boundaries, a large part of the orbit space has a quotient manifold structure.

Theorem 8.2.21 ([Bre72] Theorems IV.4.3, 4.6, [Mei03]). *Let H be a Lie group and P a connected proper H -manifold. Then there exists a closed subset $C \subset P/H$ of empty interior such that $(P/H) \setminus C$ is a manifold (of the same dimension as the linear slices) with boundary.*

In general C can be chosen with (covering) codimension at least 3. If the orbits are connected and $H_1(P, \mathbb{Z}_2) = 0$ then C can be chosen with (covering) codimension at least 4.

Globally, the structure of the orbit space of a proper H -manifold is captured by the notion of *orbispace* [Mes16]. Let us simply describe what is the considered local structure on P/H . Tubes and slices on P are directly related with charts on P/H . Let $A \times_K H \hookrightarrow P$ be a linear tube in P . The projection to the orbit space identify its image with

$$(A \times_K H) / H \simeq A/K$$

Since the projection to the orbit space is an open map ([Bre72] Section I.3), the image is an open subset of P/H identified with A/K . Since A is a linear slice, K a compact group and P can be covered with linear tubes, the smooth structure of P/H is locally identified with that of a quotient of a Euclidean space by a compact group (with specific properties). In the case the action is infinitesimal and the isotropy groups are discrete thus finite, P/H has a structure of *orbifold*.

We close this section with a remark about compactness. Lemma 8.1.32 asserts that a locally compact Hausdorff σ -compact space is paracompact. As soon as P is σ -compact, P/H is as well. When P is proper so that it admits linear tubes around every orbit, P/H is covered by quotients of Euclidean spaces by compact groups. Let A be a finite dimensional real vector space with the action of a compact group K . Since the quotient map $A \rightarrow A/K$ is open and the Euclidean space A is locally compact, A/K is locally compact as well. This proves that P/H is locally compact and as a consequence paracompact. This gives a simple sufficient condition for P/H to be paracompact.

8.3 Integration of a Cartan 1-form

In this section we finally deal with the question of building a principal bundle structure starting from a manifold P with a Cartan 1-form ϖ with value in a Lie algebra \mathfrak{g} . Recall the following

Maurer-Cartan-like equation the Cartan form ϖ is required to satisfy:

$$d\varpi + \frac{1}{2}[\varpi \wedge \varpi] = \frac{1}{2}\Omega_{bc}\alpha^a \wedge \alpha^b \quad (8.9)$$

with $\alpha = \varpi_{\mathfrak{g}/\mathfrak{h}}$ the $\mathfrak{g}/\mathfrak{h}$ component of ϖ and Ω_{bc} unconstrained coefficients.

8.3.1 Integration of a Cartan 1-form

Cartan 1-forms

We showed in Section 7.2.1 that P is equipped with an action of the Lie algebra \mathfrak{h} and that ϖ is equivariant under this action.

If the vector fields are complete the Lie algebra action is readily integrated into a Lie group action of the simply connected Lie group integration \tilde{H} of \mathfrak{h} . If not, one needs to *complete* the manifold P as described in Section 8.1.2. According to Theorem 8.1.44, the Cartan form extends to a Cartan form on the completion. As the isotropy algebras are trivial, the isotropy groups are necessarily discrete subgroups.

Let us introduce the following intermediary notion between Cartan 1-form and actual Cartan connection 1-forms on principal bundles:

Definition 8.3.1 (Generalised Cartan connections). Let $(\mathfrak{g}, \mathfrak{h})$ be a Klein geometry. Let P be an \mathfrak{h} -manifold. A *generalised $(\mathfrak{g}, \mathfrak{h})$ -Cartan connection* on P is a nondegenerate \mathfrak{g} -valued 1-form ϖ which is normalised for the action of \mathfrak{h} and satisfies the equivariance equation:

$$\forall \xi \in \mathfrak{h}, \quad \begin{cases} \varpi(\bar{\xi}) = \xi & (8.10) \\ \mathcal{L}_{\bar{\xi}}\varpi + \text{ad}_{\xi}\varpi = 0 & (8.11) \end{cases}$$

Remark. Note that our definition differs from the one in [AM95], as theirs does not impose that the form is nondegenerate, but is close to their definition of *principal Cartan connection* (which requires a structure of principal bundle).

Our construction is summed up in the following theorem:

Theorem 8.3.2. *Let P be a Hausdorff smooth manifold equipped with a \mathfrak{g} -valued Cartan 1-form ϖ . The fundamental vector fields defined by*

$$\xi \in \mathfrak{h} \mapsto \bar{\xi} := \varpi^{-1}(\xi, 0)$$

form a free action of the Lie algebra \mathfrak{h} . Let H be a (connected) Lie group integration of \mathfrak{h} such that the H -completion P_H of P is a smooth manifold.

Then there exists a generalised Cartan connection ϖ_H on P_H which pulls back to ϖ on P . Furthermore, P_H has discrete isotropy groups.

Principal orbits and bundle fibration

In this section, P is a connected Hausdorff manifold with a Cartan form ϖ , which defines a (free) infinitesimal action of \mathfrak{h} . We now assume that the infinitesimal action integrates to a *group* action of H . Namely we require univalence and completeness, according to Theorem 8.1.24. We want to construct from P a principal bundle with a Cartan connection. For this we need the action to have suitable isotropy groups, as described in Theorem 8.2.19

We further require the action to be *Cartan*, so that it has linear slices. Note that when \mathfrak{g} is provided with an Ad_H -invariant inner product, the non-degenerate coframe ϖ which is equivariant

allows us to pull it back to TP into a global H -invariant metric. Since the action is Cartan the isotropy groups are compact, and the action of \mathfrak{h} being free they are discrete, hence finite:

Lemma 8.3.3. Let H be a Lie group and P a Cartan H -manifold. If the infinitesimal action of \mathfrak{h} is free then the isotropy groups of P are finite.

We restrict our attention to the principal isotypic component of P , P_{prin} , which is a dense open subspace (see Theorem 8.2.13).

We want an effective group action on P_{prin} in order to construct a principal fibre bundle. For this, the necessary condition is to have all isotropy groups identical, so that there is one uniquely defined quotient group of H acting on P_{prin} . In particular, the isotropy groups have to be normal subgroups of H , although this is no sufficient condition.

Theorem 8.3.4. Let P be a connected manifold equipped with a \mathfrak{g} -valued generalised Cartan connection to which the associated action of \mathfrak{h} integrates to an action of a Lie group H . Assume that the action of H on P is isotypic and that the action is Cartan. The isotropy groups are then finite.

Assume furthermore that the isotropy groups are normal subgroups of H . They are then all identical. Let us call K the isotropy group. The manifold P has a free and effective action of $K \backslash H$ and forms a principal $K \backslash H$ -bundle over the quotient $P/(K \backslash H)$ which has a quotient differentiable manifold structure (Hausdorff when the action of H is proper). The principal bundle is equipped with a solder form and a $(K \backslash H)$ -principal connection.

Proof. The action naturally factors to a free action of $K \backslash H$. Theorem 8.2.17 applies and the orbit space $P/(K \backslash H) \simeq P/H$ is a manifold over which P forms a principal $K \backslash H$ -bundle. Since K is discrete, the respective Lie algebras of $K \backslash G$ and $K \backslash H$ are identified with \mathfrak{g} (resp. \mathfrak{h}). The vertical distribution integrates to the fibres of the principal bundle fibration. The $\mathfrak{g}/\mathfrak{h}$ component of the generalised Cartan connection on P defines a solder form and the \mathfrak{h} component a $K \backslash H$ -principal connection on P . \square

The Cartan connection imposes constraints on the isotropy subgroups. Because we restrict to the principal isotypic component, isotropy groups act trivially on the normal tangent bundle to the orbits (Theorem 8.2.16). The tangent spaces to the orbits constitute the (integrable) distribution ($\ker \alpha$), so α^a forms a coframe of the normal bundle to the orbits. Furthermore α is by hypothesis equivariant under H so that it is equivariant, hence invariant, under the isotropy groups of the principal orbits (Equation (8.7)). One concludes that the principal isotropy groups act trivially on $\mathfrak{g}/\mathfrak{h}$:

Lemma 8.3.5. Let P be a Cartan H -manifold with a generalised $(\mathfrak{g}, \mathfrak{h})$ -Cartan connection ϖ . The principal isotropy groups of P are subgroups of the kernel of the action

$$H \xrightarrow{\text{Ad}} \text{End}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{h})$$

In particular, when the action of \mathfrak{h} on $\mathfrak{g}/\mathfrak{h}$ is faithful, the principal isotropy groups have to be trivial.

Example 8.3.6. Let P be a differentiable manifold.

Assume P has a smooth action of SO_n with a generalised $\mathfrak{so}_n \ltimes \mathbb{R}^n$ -valued Cartan connection ϖ . Then the action is both proper and free over the principal isotypic component P_{prin} (due to Lemma 8.3.5). It thus defines a principal bundle $P_{prin} \rightarrow P_{prin}/\text{SO}_n$ over the orbit space which is a smooth Hausdorff manifold, and ϖ defines an SO_n -principal connection.

The examples of locally Klein geometries introduced in Section 7.3 are examples of this construction. In particular, Example 7.3.1 as an action of Spin_4 which is free except on one single exceptional orbit which induces a localised singularity on the orbit space.

8.3.2 $(\mathfrak{g}, \mathfrak{h})$ -Cartan 1-form on a compact manifold with compact H

Under compactness assumptions, many hypotheses of the construction become automatically satisfied. Let P be a compact (Hausdorff) manifold. Let (G, H) be a reductive Klein geometry with H compact and simply-connected. Assume P is equipped with a $(\mathfrak{g}, \mathfrak{h})$ -valued Cartan 1-form ϖ .

The Cartan 1-form defines a free action of \mathfrak{h} on P according to Theorem 7.2.3. Since the manifold P is compact, the action of \mathfrak{h} is complete. Hence the infinitesimal action integrates to a Lie group action of the simply-connected integration \tilde{H} of \mathfrak{h} , as stated in Section 8.1.2.

In the case \tilde{H} is compact, the action is then necessarily *proper*, in particular it has the Cartan property (see Section 8.2.1 for more detail). Hence it has linear slices and a principal orbit type. If we consider the principal isotypic component of P , which is a dense open subset (Theorem 8.2.15), it is a fibre bundle above its orbit space, which is a smooth manifold according to Theorem 8.2.17. All the isotropy groups are necessarily finite (Lemma 8.3.3).

Lemma 8.3.7. Let H be a simply connected compact Lie group with Lie algebra \mathfrak{h} and P a compact \mathfrak{h} -manifold. Then the action integrates into a group action of H and there is a finite subgroup $K \subset H$ and a dense open subset $\mathcal{U} \subset P$ stable under H such that

$$\mathcal{U} \rightarrow \mathcal{U}/H$$

is a fibre bundle with typical fibre $K \backslash H$ which is an homogeneous H -space covered by H .

When the action of \mathfrak{h} on $\mathfrak{g}/\mathfrak{h}$ is faithful, Lemma 8.3.5 implies that the principal isotropy groups are necessarily trivial.

Theorem 8.3.8. Let P be a compact (Hausdorff) manifold. Let H be a simply connected compact Lie group with a Lie algebra \mathfrak{h} and an action on $\mathfrak{g}/\mathfrak{h}$. Let $\varpi \in \Omega^1(P, \mathfrak{g})$ be a Cartan 1-form on P and $\alpha \in \Omega^1(P, \mathfrak{g}/\mathfrak{h})$ its projection to $\mathfrak{g}/\mathfrak{h}$.

Then there exists a dense open subset $\mathcal{U} \subset P$ stable under H such that $\mathcal{U} \rightarrow \mathcal{U}/H$ is a smooth manifold,

$$\mathcal{U} \rightarrow \mathcal{U}/H$$

is an H -principal bundle and ϖ defines a (G, H) -Cartan connection on the fibre bundle.

Example 8.3.9. Let P be a differentiable manifold.

Assume P is compact and has a smooth action of \mathfrak{spin}_n with a $\mathfrak{spin}_n \times \Sigma^n$ -valued generalised Cartan connection ϖ , with Σ^n a faithful spinorial representation of \mathfrak{spin}_n . The action of \mathfrak{spin}_n is complete and integrates to a group action of Spin_n , under which ϖ is equivariant. The group action is proper and since Σ^n is a faithful representation of Spin_n the principal isotypic component P_{prin} is free under the group action.

One concludes that P_{prin} defines a principal bundle $P_{prin} \rightarrow P_{prin}/\text{Spin}_n$ over its orbit space, which is a smooth Hausdorff manifold, and ϖ defines a solder form and a Spin_4 -principal connection.

Now, in the general case, the total orbit space is an orbifold (Section 8.2.4). Even in this case, the structure we obtain can be understood as a principal connection on the frame bundle. The theory of frame bundles and connections on orbifolds is exposed in [Alf21].

Part II

Field Theory

Chapter 9

Covariant formalism for first order field theory

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We discuss the general geometrical framework in which classical field theories are formulated. The different formulations are centered around different geometrical structures. We explain in particular how the pre-multisymplectic geometry naturally emerges in the study of variational theories. General references are [GIM97; RW19; H el11; Gie21].

9.1 Variational field theories

Let us first define what we mean by field theories. We want a source space \mathcal{E} and a target bundle $Q \rightarrow \mathcal{E}$. Throughout the whole section, \mathcal{E} will be a smooth manifold of dimension n . Fields will be sections of Q , usually subject to “boundary conditions” defining a subset $\Gamma_{\text{b.c.}}(Q) \subset \Gamma(Q)$. When \mathcal{E} is one-dimensional and represents a time coordinate, fields are trajectories and the field theory is usually referred to as “mechanics”. The field theory will then specify a subspace of the space of fields $\Gamma_{\text{b.c.}}(Q)$ which will be the space of solutions of a system of equations, or the space of classical trajectories. Realistic field theories have much more structure, for example they have a specified symmetry group, or should satisfy physical principles such as locality or determinism.

A *variational field theory* is defined by a source space \mathcal{E} , a target bundle Q , boundary conditions

defining a subset of sections $\Gamma_{\text{b.c.}}$ and an action

$$S : \Gamma_{\text{b.c.}} \rightarrow \mathbb{R}$$

More generally, a *local variational field theory* is defined by

1. A source space \mathcal{E}
2. A target bundle $Q \xrightarrow{\pi} \mathcal{E}$
3. A subsheaf of the local sections of \mathcal{E} : that is above any open subset $\mathcal{U} \subset \mathcal{E}$ a space of local sections satisfying the boundary conditions which we write

$$\Gamma_{\text{b.c.}}(\mathcal{U}, Q)$$

such that the question of whether $\sigma \in \Gamma(\mathcal{U}, Q)$ belongs to $\Gamma_{\text{b.c.}}$ can be answered locally.

4. A local action S well defined on compactly supported fields:

$$S : \Gamma_c(\mathcal{E}, Q) \rightarrow \mathbb{R}$$

which is differentiable in some sense.

The solution fields are the ones at which the differential of S vanishes:

$$\delta S|_{\sigma} = 0$$

This is called the *equation of motion*.

9.2 Lagrangian field theories

A *local Lagrangian* is a map from the stalks of sections of Q , gathered in the bundle $\acute{\text{E}}t(Q) \rightarrow \mathcal{E}$, to the bundle of densities over \mathcal{E} :

$$\mathcal{L} : \acute{\text{E}}t(Q) \rightarrow \text{Dens}(\mathcal{E})$$

To a local section $\sigma \in \Gamma(\mathcal{U}, Q)$ it is associated the local section $\bar{\sigma}$ of $\acute{\text{E}}t(Q)$ defined by

$$\bar{\sigma}_x = \text{germ}_x(\sigma)$$

The local Lagrangian defines a local action functional

$$S_{\mathcal{U}}[\sigma] = \int_{\mathcal{U}} \mathcal{L} \circ \bar{\sigma}$$

under general integrability conditions. A section is said to *satisfy the equations of motion* when it is locally an extremum of the local action functional among sections of \mathcal{E} . A local variational theory is called *Lagrangian* when it can be expressed with a Lagrangian.

In the physically relevant cases, the Lagrangian will be of *finite order*, namely it depends on the k -order jet of the sections:

$$\mathcal{L} : \mathcal{J}^k(Q) \rightarrow \text{Dens}(\mathcal{E})$$

for some integer k and does so in a smooth manner. In this case, given a (local) section σ , one can find around each point on which it is defined a neighbourhood \mathcal{U} such that the Lagrangian density

is integrable on \mathcal{U} for sections in a neighbourhood of $\sigma_{\mathcal{U}}$. The Lagrangian being assumed smooth, one can then differentiate $S_{\mathcal{U}}$ at $\sigma_{\mathcal{U}}$. In this framework, the variational principle is modified in the following way: a section s defined on an open subset \mathcal{W} is said to satisfy the equations of motion if it is locally a *critical point* of the action functional among sections of \mathcal{E} satisfying the boundary conditions:

$$\forall x \in \mathcal{W}, \exists \mathcal{U} \in \mathcal{V}(x), \delta S_{\mathcal{U}}[s_{\mathcal{U}}] = 0 \quad (9.1)$$

with δ denoting the “functional” derivative of the local action $S_{\mathcal{U}}$ and $\mathcal{V}(x)$ the set of open neighbourhoods of x (included in \mathcal{W}). From now on we will assume all Lagrangians to be of finite order.

If one were interested in finding critical points among *all sections of the jet space*, the corresponding *equations of motion* would simply be the vanishing of the vertical differential of \mathcal{L} , but the so-called *holonomic* sections, namely the ones that are associated to a section of Q , are only a subspace of these sections. The different components of their jets are related in a local way so that the equations of motion are weaker than the vanishing of the whole vertical differential of \mathcal{L} .

The local Lagrangian densities we will consider will be n -forms on a n -manifold, either initially oriented or provided with an orientation by the fields.

Since the central question of classical field theories is to study the solutions and the space they form, we can consider different field theories to be equivalent: two variational field theories with the same space of fields and boundary conditions will be called *equivalent* when the difference of the local actions is independent of the chosen local field. Two Lagrangian field theories with the same space of fields and boundary conditions will be called *equivalent* when over any local field, the Lagrangian densities differ by an exact term. In particular equivalent Lagrangian field theories define equivalent variational field theories.

9.3 Variational principle

The variational principle associated to a local action is the equation of motion stating that the field is a critical point for the local action. The *boundary conditions* usually imposed on the accepted field configurations consequently restrict the allowed variations. The local actions are differentiated with respect to variations with *compact support*, which ensures the integrability of the local action and its differentiability on the variation support as well as preserve all kinds of boundary conditions. From now on we will restrict our attention to Lagrangian theories of first order. In particular, we will use the notation

$$\mathcal{J}(Q) \xrightarrow{\pi^1} Q$$

for the bundle $\mathcal{J}^1(Q) \rightarrow Q$, with the jet order left implicit. Let us recall the notation for local coordinates we will use: x^i are local coordinates on Q , y^A are fibre coordinates on \mathcal{P} and v_i^A are the associated 1st order coordinates on $\mathcal{J}(Q)$ (fibre coordinates with respect to $\mathcal{J}(Q) \rightarrow Q$).

Let \mathcal{U} be the interior of a codimension 0 submanifold with boundary – there exists a topology basis of such open subsets. Let $\sigma \in \Gamma(\mathcal{U}, Q)$ be a local section and $X \in \Gamma(\mathcal{U}, \sigma^*VQ)$ a compactly supported (vertical) variation. X is extended arbitrarily to a neighbourhood of $\sigma(\mathcal{U})$ in $Q|_{\mathcal{U}}$ and the corresponding flow lifted to $\mathcal{J}(Q|_{\mathcal{U}})$ to a flow generated by vector we write jX , it is the so-called *prolongation* of X (introduced in Section 2). Using the flow of the extended X we define a one-parameter family of sections (σ_t) for t in a neighbourhood of 0 in \mathbb{R} coinciding with σ in 0.

The variational principle on \mathcal{U} takes the form:

$$\partial_t \left(\int_{x \in \mathcal{U}} \mathcal{L}(x, j\sigma_t(x)) \right) \Big|_{t=0} = 0 \quad (9.2)$$

which can be expressed with a vertical derivative which we write ∂_V :

$$\int_{x \in \mathcal{U}} (\partial_V \mathcal{L})(x, j\sigma(x))(X) dx$$

We will use a formulation with the Lie derivative. Post-composing the Lagrangian

$$\mathcal{J}(Q) \xrightarrow{\mathcal{L}} \pi^* \Lambda^n T^* \mathcal{E}$$

with the pullback map

$$\pi^* \Lambda^n T^* \mathcal{E} \xrightarrow{\pi^*} \Lambda^n T^* Q$$

the Lagrangian can be represented as a $\pi^* \Lambda^n T^* \mathcal{E} \subset \Lambda^n T^* Q$ -valued map and can as such be pulled back along $j\sigma$. In this way, the application $x \mapsto \mathcal{L}(x, j\sigma(x))$ is identified with $j\sigma^*(\mathcal{L})$ and the variational principal can be written

$$\partial_t \Big|_{t=0} \left(\int_{\mathcal{U}} j\sigma_t^* \mathcal{L} \right) = 0$$

which is expressed with the Lie derivative as:

$$\int_{\mathcal{U}} j\sigma^*(\mathcal{L}_{jX} \mathcal{L}) = 0 \quad (9.3)$$

Furthermore the local flow of jX on $\mathcal{J}(Q)$ projects to the flow of X on Q hence to 0 on \mathcal{E} , which means that the Lie derivative can be simplified to

$$\mathcal{L}_{jX} \mathcal{L} = (di_{jX} + i_{jX} d) \mathcal{L} = d0 + i_{jX} d \mathcal{L} = i_{jX} d \mathcal{L}$$

Thus the equation of motion is

$$\int_{\mathcal{U}} j\sigma^*(i_{jX} d \mathcal{L}) = 0 \quad (9.4)$$

for all variations X of s . But $jX|_{j\sigma}$ does not take all (vertical) values for varying X , moreover it is not a $\mathcal{C}^\infty(\mathcal{E})$ -linear function of X so we cannot conclude that the integrand $j\sigma^*(i_{jX} d \mathcal{L})$ has to vanish for all X . The prolongation of X can be expressed in local coordinates [Olv86]:

$$j\sigma^*(jX) = \sigma^* X^A \partial_{y^A} + \partial_{x_i} (\sigma^* X)^A \partial_{v_i^A} \quad (9.5)$$

which shows that $j\sigma^*(jX)$ is independent of the extension outside of the image of σ . Notice that the non-linearity with respect to $\mathcal{C}^\infty(\mathcal{E})$ is due to the term $\partial_{x_i} (\sigma^* X)^A \partial_{v_i^A}$.

Since the considered field variations are compactly supported, any exact term in the Lagrangian has no influence on the variational equations (although it may affect the global action value depending on the boundary conditions). It is then seamless to perform an integration by part to obtain an integrand $\mathcal{C}^\infty(\mathcal{E})$ -linear in $\sigma^* X$. The geometric realisation of this integration by part leads to the *Poincaré-Cartan form*.

9.4 The Poincaré-Cartan form

We considered a Lagrangian that is purely horizontal, namely a section of $\pi^*(\Lambda^n T^* \mathcal{E}) \rightarrow \mathcal{J}(Q)$. Now the local action does not change if we add any contact term to the Lagrangian form. The idea is then to add a suitable contact term $\theta^B \wedge p_B$ such that

$$\partial_{v_i^A} \lrcorner d(\mathcal{L} + \theta^B \wedge p_B) \in \mathcal{I}$$

Using the following two properties of the local contact 1-form

$$\begin{aligned} \partial_{v_i^A} \lrcorner \theta^B &= \partial_{v_i^A} \lrcorner (dy^B - v_i^B dx^i) = 0 \\ \partial_{v_i^A} \lrcorner d\theta^B &= \partial_{v_i^A} \lrcorner (-dv_i^B \wedge dx^i) = -\delta_A^B dx^i \end{aligned}$$

the computation goes

$$\begin{aligned} \partial_{v_i^A} \lrcorner d(\mathcal{L} + \theta^B p_B) &= \partial_{v_i^A} \lrcorner d\mathcal{L} + (\partial_{v_i^A} \lrcorner d\theta)^B \wedge p_B - (\partial_{v_i^A} \lrcorner \theta^B) \wedge dp_B \\ &\quad + d\theta^B \wedge \partial_{v_i^A} \lrcorner p_B + \theta^B \wedge \partial_{v_i^A} \lrcorner dp_B \\ &= \mathcal{L}_{\partial_{v_i^A}} \mathcal{L} - dx^i \wedge p_A + \left(d\theta^B \wedge \partial_{v_i^A} \lrcorner p_B - \theta^B \wedge \partial_{v_i^A} \lrcorner dp_B \right) \end{aligned} \quad (9.6)$$

(we used $\partial_{v_i^A} \lrcorner d\mathcal{L} = \mathcal{L}_{\partial_{v_i^A}}$). The local coordinates x^i define a local volume form $dx^{(n)} = dx^1 \wedge \cdots \wedge dx^n$ and the Lagrangian can be decomposed as $\mathcal{L} = Ldx^{(n)}$. Using the dual form convention (Section 1.4)

$$dx^j \wedge dx_i^{(n-1)} = \delta_i^j dx^{(n)}$$

we decompose

$$p_A = p_A^i dx_i^{(n-1)} + dy^B \wedge p_{B,A} + dv_j^B \wedge p_{B,A}^j$$

The term (9.6) is a contact term when

$$\partial_{v_i^A} Ldx^{(n)} \equiv p_A^i dx^{(n)} + dx^i \wedge \left(dy^B \wedge p_{B,A} + dv_j^B \wedge p_{B,A}^j \right) \pmod{\mathcal{I}}$$

which is solved by

$$p_A \equiv \partial_{v_i^A} Ldx_i^{(n-1)} \pmod{\mathcal{I}} \quad (9.7)$$

There is only one solution if we require p_A to be horizontal forms:

$$p_A = \partial_{v_i^A} Ldx_i^{(n-1)} \quad (9.8)$$

Contact terms in p have no influence on the variational equation (9.4) since adding a term from \mathcal{I}^2 to \mathcal{L} yields only an extra contact term in $i_{jX} d\mathcal{L}$ ¹.

The *Poincaré-Cartan form* associated to \mathcal{L} is locally defined as

$$\Theta_{PC} = \mathcal{L} + \theta^A \wedge \partial_{v_i^A} Ldx_i^{(n-1)} \in \Lambda^n \pi^{1*} T^* Q \quad (9.9)$$

This definition is shown to be coordinate independent, since the second term in the right hand side can be constructed from the coordinate independent tensor $\theta^A \otimes \partial_{x_i} \otimes \partial_{v_i^A}$ and \mathcal{L} , with θ^A composed with the exterior product, ∂_{x_i} with the interior product, and $\partial_{v_i^A}$ with the vertical

¹However they may affect the symplectic form on the solution space, which is constructed by the integration on a Cauchy surface of the contraction of $d\Theta_{PC}$ with *two* variation fields.

derivative. We already stated that θ is a geometrically defined 1-form with value in $(\pi^1)^*V(Q/\mathcal{E})$. Vectors ∂_{x_i} are elements of $(\pi \circ \pi^1)^*T\mathcal{E}$ and $\partial_{v_i^A}$ correspond to the (pulled back) affine action of $\pi^*T^*\mathcal{E} \otimes VQ$ on $\mathcal{J}(Q) \rightarrow Q$.

The tensor product $\theta^A \otimes \partial_{x_i} \otimes \partial_{v_i^A}$ can be interpreted as the linear mapping

$$\begin{array}{c}
T\mathcal{J}(Q) \\
\downarrow \theta \\
\pi^*(V(Q/\mathcal{E})) \\
\downarrow \cdot \otimes \text{id}_{T\mathcal{E}} \\
\underbrace{(\pi^1)^*(V(Q/\mathcal{E})) \otimes (\pi \circ \pi^1)^*T^*\mathcal{E} \otimes (\pi \circ \pi^1)^*T\mathcal{E}}_{(\pi^1)^*(\pi^*T^*\mathcal{E} \otimes VQ)} \\
\downarrow \wr \\
V(\mathcal{J}(Q)/Q) \otimes (\pi \circ \pi^1)^*T\mathcal{E}
\end{array}$$

Hence the element is geometrically defined and the term $\theta^A \wedge \partial_{v_i^A} L dx_i^{(n-1)}$ is as well.

Remark. The uniqueness of the Poincaré-Cartan is guaranteed for first order theories but it is not the case for higher order theories ([Vit10]).

The variational equation becomes

$$\int_{\mathcal{U}} j\sigma^*(\mathcal{L}_{jX}\Theta_{PC}) = 0 \quad (9.10)$$

The Lie derivative is decomposed using the homotopy formula:

$$\mathcal{L}_{jX}\Theta_{PC} = \text{di}_{jX}\Theta_{PC} + i_{jX}d\Theta_{PC}$$

The integral of the first term vanishes as X and jX extend smoothly by 0 on $\partial\mathcal{U}$:

$$\int_{\mathcal{U}} j\sigma^*(\text{di}_{jX}\Theta_{PC}) = \int_{\mathcal{U}} dj\sigma^*(i_{jX}\Theta_{PC}) = \int_{\partial\mathcal{U}} j\sigma^*(i_{jX}\Theta_{PC}) = 0$$

The second term $i_{jX}d\Theta_{PC}$ is $C^\infty(\mathcal{E})$ -linear in X by construction since in the contraction the component $\partial_{x_i}(\sigma^*X)^A \partial_{v_i^A}$ vanishes (see (9.5)). We conclude that the variational equation is equivalent to

$$j\sigma^*(i_{jX}d\Theta_{PC}) = 0 \quad (9.11)$$

Given the Poincaré-Cartan form, \mathcal{L} can be obtained back by identifying the purely horizontal part of Θ_{PC} modulo contact terms. In this sense, the Lagrangian form and the Poincaré-Cartan form can be thought of as two realisations of the same object: the Lagrangian form is the purely horizontal component (with respect to $\mathcal{J}(Q) \rightarrow \mathcal{E}$) and the Poincaré-Cartan form is a realisation convenient for expressing the variational equations.

In a local trivialisation the differential of the Poincaré-Cartan form is

$$d\Theta_{PC} = d\mathcal{L} + d\theta^A \wedge \partial_{v_i^A} L dx_i^{(n-1)} - \theta^A \wedge d\left(\partial_{v_i^A} L dx_i^{(n-1)}\right) \quad (9.12)$$

so the interior product with $jX = jX^A \partial_{y^A} + jX_i^A \partial_{v_i^A}$ gives

$$\begin{aligned}
i_{jX} d\Theta_{PC} &= i_{jX} d\mathcal{L} + i_{jX} d\theta^A \wedge \partial_{v_i^A} L dx_i^{(n-1)} + d\theta^A \wedge i_{jX} \partial_{v_i^A} L dx_i^{(n-1)} \\
&\quad - i_{jX} \theta^A d \left(\partial_{v_i^A} L dx_i^{(n-1)} \right) + \theta^A \wedge i_{jX} d \left(\partial_{v_i^A} L dx_i^{(n-1)} \right) \\
&\equiv i_{jX} d\mathcal{L} + i_{jX} d\theta^A \wedge \partial_{v_i^A} L dx_i^{(n-1)} - i_{jX} \theta^A d \left(\partial_{v_i^A} L dx_i^{(n-1)} \right) \quad \text{mod } \mathcal{I} \\
&\equiv jX^A \partial_{y^A} \mathcal{L} + jX_j^A \partial_{v_j^A} \mathcal{L} - jX_j^A dx^j \wedge \partial_{v_i^A} L dx_i^{(n-1)} - jX^A d \left(\partial_{v_i^A} L dx_i^{(n-1)} \right) \quad \text{mod } \mathcal{I} \\
&\equiv jX^A \left(\partial_{y^A} \mathcal{L} - d \left(\partial_{v_i^A} L dx_i^{(n-1)} \right) \right) \quad \text{mod } \mathcal{I}
\end{aligned} \tag{9.13}$$

with $jX^A = X^A$ taking all possible values.

The variational equation takes the form of the *Euler-Lagrange equation*:

$$\sigma^* (\partial_{y^A} L) - \partial_i \sigma^* \left(\partial_{v_i^A} L \right) = 0 \tag{9.14}$$

9.5 The Noether theorems

The Noether theorems provide explicit structure to Lagrangian field theories with infinitesimal symmetries. Let there be a Lagrangian field theory with a target bundle $Q \xrightarrow{\pi} \mathcal{E}$ and a first-order Lagrangian \mathcal{L} . Let X be a vector field on $\mathcal{J}(Q)$ and J a differential form on $\mathcal{J}(Q)$ such that

$$\mathcal{L}_X \Theta_{PC} = dJ \quad \text{mod } \mathcal{I}$$

Then the homotopy formula implies that

$$i_X d\Theta_{PC} = d(J - i_X \Theta_{PC}) \quad \text{mod } \mathcal{I}$$

therefore for any solution σ of the Euler-Lagrange equation,

$$d(j\sigma)^* (J - i_X \Theta_{PC}) = 0$$

Hence to a *symmetry* of the Poincaré-Cartan form is associated a *conserved current* $J - i_X \Theta_{PC}$. This is the principle behind *Noether's first theorem*.

In certain cases, X is part of a *gauge symmetry*. This implies that there is a family of symmetries X which can be constructed from sections of a vector bundle over \mathcal{E} using a differential operator of finite order. In other words, the symmetries can be (locally) parametrised by a finite family of real functions over \mathcal{E} . In this case, the *second Noether theorem* asserts that the conserved currents associated to the symmetries are already *exact* on solutions of the Euler-Lagrange equations. Furthermore, the existence of symmetries which admit continuous parameters (with compact support) shows that the Euler-Lagrange equations are underdetermined, and the Noether theorem gives a differential relation between the Euler-Lagrange equations [Gie21].

9.6 The covariant pre-multisymplectic formalism

Non-holonomic sections

The contact term omitted in the computation (9.13) becomes involved if we try to apply the variational principle to *non-holonomic* (rigorously: *non-necessarily holonomic*) sections of $\mathcal{J}(Q) \rightarrow$

\mathcal{E} . Naturally the corresponding variations would not need to preserve the contact ideal. Let us compute exactly the term $i_{jX}d\Theta_{PC}$ for a generic vertical jet variation $jX = jX^A\partial_{y^A} + jX^A_i\partial_{v^A_i}$. We keep the notation p_A for the purpose of conciseness – recalling it is a purely horizontal form:

$$\begin{aligned} i_{jX}d\Theta_{PC} &= jX^A(\partial_{y^A}\mathcal{L} - dp_A) + d\theta^A \wedge i_{jX}p_A + \theta^A \wedge i_{jX}dp_A \\ &= jX^A(\partial_{y^A}\mathcal{L} - dp_A) + \theta^A \wedge i_{jX}dp_A \\ &= jX^A(\partial_{y^A}\mathcal{L} - dp_A) + \theta^A \wedge (jX^B\partial_{y^B}p_A + jX^B_j\partial_{v^B_j}p_A) \\ &= jX^A(\partial_{y^A}\mathcal{L} - dp_A + \theta^B \wedge \partial_{Y^A}p_B) + jX^A_j\theta^B \wedge \partial_{v^A_j}p_B \end{aligned} \quad (9.15)$$

As stated earlier, the vector fields $\partial_{v^A_j}$ express the affine action of $VQ \otimes T^*\mathcal{E}$. The form $\partial_{v^A_j}p_B$ can be interpreted as an element of $\underbrace{(VQ \otimes T^*\mathcal{E})^*}_{(A,j)} \otimes \underbrace{VQ}_B \otimes \Lambda^{(n-1)}T^*\mathcal{E}$. The Lagrangian \mathcal{L} is said *regular* or *non-degenerate* if at every point of $\mathcal{J}(Q)$ it induces a bijective mapping $VQ \otimes T^*\mathcal{E} \rightarrow VQ \otimes \Lambda^{(n-1)}T^*\mathcal{E}$. In this case, the variational principle applied to a non-holonomic section $\sigma : \mathcal{E} \rightarrow \mathcal{J}(Q)$ gives the following set of equations:

$$\begin{cases} \sigma^*\theta^B = 0 \\ \sigma^*(\partial_{y^A}\mathcal{L} - dp_A + \theta^B \wedge \partial_{Y^A}p_B) \equiv \phi^*(\partial_{y^A}\mathcal{L} - dp_A) = 0 \end{cases} \quad (9.16)$$

$$\quad (9.17)$$

But since the vanishing of the pullback of the contact forms is equivalent to the jet section being holonomic, (9.16) is equivalent to ϕ being the 1st jet of a section of $Q \rightarrow \mathcal{E}$ satisfying the Euler-Lagrange equations (9.11).

To summarize: if \mathcal{L} is a non-degenerate Lagrangian, the associated Poincaré-Cartan form defines a variational principle over *all sections* of $\mathcal{J}(Q) \rightarrow \mathcal{E}$ the solutions to which are exactly jets of sections of $Q \rightarrow \mathcal{E}$ which are critical points of the local actions.

As a consequence, one can omit the contact structure of the jet space and simply consider $\mathcal{J}(Q)$ as the target bundle of the field theory, with variational equations defined by Θ_{PC} . One advantage of this point of view is that less structure on the target bundle allows for more transformations hence larger symmetry groups and isomorphisms between more geometries – we will briefly say more about it later in 9.10.

In particular, the Poincaré-Cartan form itself defines a bundle mapping $\mathcal{J}(Q) \xrightarrow{\Theta_{PC}} \Lambda^n T^*Q$. The forms are of a specific form: recall the local definition

$$\Theta_{PC} = \mathcal{L} + (dy^A - v^A_j dx^j) \wedge \partial_{v^A_i} L dx_i^{(n-1)} \quad (9.9)$$

We define for a vector space E with a “vertical” subspace $V \subset E$ the subspace of n -forms that are “at most k times vertical”:

$$\Lambda_k^n E^* := \{\omega \in \Lambda^n E^* / (\Lambda^{k+1}V) \lrcorner \omega = 0\} \quad (9.18)$$

Then Θ_{PC} takes value in $\Lambda_1^n T^*Q$ (with the vertical direction given by the vertical tangent subspace $VQ \subset TQ$). We wish to formulate the field theory with $\Lambda_1^n T^*Q$ as target bundle. A local basis for $(\pi^1)^*(\Lambda_1^n T^*Q)$ is given by $(dx^{(10)}, \theta^A)$: we see that for a non-degenerate Lagrangian the mapping $\mathcal{J}(Q) \xrightarrow{\Theta_{PC}} \Lambda_1^n T^*Q$ is of maximal rank but its (local) image has codimension one since $\Lambda_1^n T^*Q$ is a dimension $n + 1$ bundle.

The Legendre transformation

The forms p_A are called *momenta*. For a non-degenerate Lagrangian they locally define (by definition) a complete system of fibre coordinates of $\mathcal{J}(Q) \rightarrow Q$. Therefore we obtain a local diffeomorphism $\mathcal{J}(Q) \rightarrow \Lambda_1^n T^*Q / \Lambda_0^n T^*Q$:

$$\begin{array}{ccc} \mathcal{J}(Q) & \xrightarrow{\quad} & \Lambda_1^n T^*Q / \Lambda_0^n T^*Q \\ & \searrow \Theta_{PC} & \nearrow \\ & \Lambda_1^n T^*Q & \end{array}$$

Let us use finite dimensional vector spaces in order to set definitions: $V \subset E$ with a quotient we write B . V is the vertical vector space while B is the base vector space, of dimension n . The space of linear sections $B \rightarrow E$ which we write $\Gamma(B, E)$ is an affine space with underlying vector space $\text{Hom}(E, V)$. We are interested in maps from the space of sections to the vector line of volume elements $\Lambda^n B^*$. We write $\text{Aff}(\Gamma(B, E), \Lambda^n E^*)$ the linear space of affine mapping $\Gamma(B, E) \rightarrow \Lambda^n B^*$. The *twisted affine dual* of $\Gamma(B, E)$ is the quotient of $\text{Aff}(\Gamma(B, E), \Lambda^n E^*)$ by the subspace of constant maps: we write $\Gamma(B, E)^*$ (the construction can be done with any vector line, which is implicit in this notation).

These spaces are related to the $\Lambda_k^n E^*$ spaces: to begin with, $\Lambda_0^n E^* \simeq \Lambda^n B^*$. Then one can prove that the pullback operation

$$\begin{aligned} \Gamma(B, E) \otimes \Lambda^n E^* &\rightarrow \Lambda^n B^* \\ (\sigma, \Theta) &\rightarrow \sigma^* \Theta \end{aligned} \tag{9.19}$$

gives an isomorphism $\Lambda_1^n E^* \simeq \text{Aff}(\Gamma(B, E), \Lambda^n E^*)$, so that $\Gamma(B, E)^* \simeq \Lambda_1^n E^* / \Lambda_0^n E^*$ [RW19; Vit13].

In the case at hand we consider $VQ \hookrightarrow TQ \twoheadrightarrow T\mathcal{E}$. The vector line bundle is $\Lambda^n T^* \mathcal{E}$, the fibre of $\mathcal{J}(Q)$ above $y \in Q$ can be identified with the space of sections $T_{\pi(y)} \mathcal{E} \rightarrow T_y Q$ and $\Lambda_1^n T^* Q$ can be interpreted as the bundle of affine maps from $\mathcal{J}(Q)$ to $\pi^* \Lambda^n T \mathcal{E}$. The quotient $\Lambda_1^n T^* \mathcal{E} / \Lambda_0^n \pi^* T^* \mathcal{E}$ is the (twisted) *affine dual jet bundle*. From what precedes the Poincaré-Cartan form defines a mapping from the jet space to the space of horizontal n -forms depending affinely on the 1-jet. Taking the quotient by the bundle $\Lambda^n \pi^* T^* \mathcal{E}$ of forms constant with respect to the 1-jet, we obtain a *local diffeomorphism* to the affine dual jet bundle, which we write $\mathcal{J}^*(Q) \xrightarrow{\pi^1} Q$. These maps are respectively called (*extended*) *Legendre transformation* and *restricted Legendre transformation*.

$$\begin{array}{ccc} \mathcal{J}(Q) & \xrightarrow{\widetilde{\mathbb{F}\mathcal{L}}} & \Lambda_1^n T^*Q \\ & \searrow \mathbb{F}\mathcal{L} & \downarrow \\ & & \mathcal{J}^*(Q) \end{array}$$

In a way dependent on the used fibre coordinates, the Poincaré-Cartan form, hence the Legendre transformation, can be written as follows:

$$\Theta_{PC} = \left(L - v_j^A \partial_{v_j^A} L \right) dx^{(n)} + \partial_{v_j^A} L \wedge dy^A \wedge dx_j^{(n-1)}$$

One identifies the canonical *Hamiltonian* $-(L - v_j^A \partial_{v_j^A} L)$ here defined over the first order jet space (and dependent on the choice of coordinates y^A).

The space $\Lambda_1^n T^*Q$ has a structure of affine line bundle above $\mathcal{J}^*(Q)$. A section h such that

the composed map

$$\mathcal{J}(Q) \rightarrow \mathcal{J}^*(Q) \xrightarrow{h} \Lambda_1^n T^*Q$$

is equal to the extended Legendre transformation is called a *Hamiltonian section*. In local coordinates, if we write the tautological form of $\Lambda_1^n T^*Q$ as follows:

$$\Theta = -edx^{(n)} + p_A^i dy^A \wedge dx_i^{(n-1)}$$

then a Hamiltonian section takes the form

$$-h(p, y, x)dx^{(n)} + p_A^i dy^A \wedge dx_i^{(n-1)}$$

9.7 Hamilton-de Donder-Weyl equations

We are close to a formulation of the variational equations on $\mathcal{J}^*(Q)$. The Legendre transformation pulls back by definition the tautological form of $\Lambda_1^n T^*Q$ to Θ_{PC} hence the variational equation (9.11) can be transported to $\Lambda_1^n T^*Q$ however one has to require that the vector fields involved are tangent to the image of the Legendre transformation.

Assume given a Hamiltonian section h . Using it to pull back the tautological form of $\Lambda_1^n T^*Q$ defines an n -form (called “presymplectic potential current” in [Sch]) which we denote Θ^h . It is pulled back under the Legendre transformation to Θ_{PC} . Given a section ψ of $\mathcal{J}(Q) \xrightarrow{\pi \circ \pi^1} \mathcal{E}$ there is a corresponding section $\varphi = \mathbb{F}\mathcal{L} \circ \psi$ of $\Lambda_1^n T^*Q$ and for each variation jX of ψ (prolongation of a vector field X on Q) there is a corresponding variation $\mathbb{F}\mathcal{L}_* jX$ of $\mathbb{F}\mathcal{L} \circ \psi$. Then ψ satisfies the variational equation

$$\forall jX, \quad \psi^*(i_{jX} d\Theta_{PC}) = 0$$

if and only if $\mathbb{F}\mathcal{L} \circ \psi$ satisfies the corresponding equation for Θ^h . The Hamiltonian section defines a premultisymplectic form ²:

$$\Omega = d\Theta^h \tag{9.20}$$

The equation satisfied by the Legendre transform φ of a section satisfying the variational equation is

$$\forall K \in \varphi^*V(\mathcal{J}(Q)^*/\mathcal{E}), \quad \varphi^*(i_K \Omega) = 0$$

which can also be written

$$(\varphi_* \Lambda^n T\mathcal{E}) \lrcorner \Omega \subset (\pi \circ \bar{\pi}^1)^* T^* \mathcal{E} \tag{9.21}$$

This is the *Hamilton-de Donder-Weyl equation*. Using local coordinates (x^i, y^A, v_i^A) , it takes the following form:

$$\begin{aligned} \partial_{x^i} y^A &= -\partial_{p_A^i} h \\ \partial_{x^i} v_i^A &= \partial_{y^A} h \end{aligned}$$

Furthermore, given a vector K in $\varphi_* T\mathcal{E}$, the equation

$$\varphi^*(i_K \Omega) = i_{\varphi_* K} \varphi^* \Omega = 0$$

holds since \mathcal{E} is of dimension n and $\varphi^* \Omega$ is an $(n+1)$ -form. Since $\varphi_* T\mathcal{E}$ is a supplementary tangent subspace to $V(\mathcal{J}^*(Q)/\mathcal{E})$, the Hamilton-de Donder-Weyl equation is equivalent to the

²There are varying sign conventions, see [Wen15] for a discussion in the case of the canonical symplectic form on the cotangent bundle.

following equation:

$$\forall K \in T\mathcal{J}^*(Q), \quad \varphi^*(i_K\Omega) = 0$$

As long as the Lagrangian is non-degenerate the restricted Legendre transformation is a local diffeomorphism, and the variational equations are local. This means that even when the Legendre transformation is not a global diffeomorphism, sections of $\mathcal{J}^*(Q) \rightarrow \mathcal{E}$ which satisfy the variational equation have as local inverse images in $\mathcal{J}(Q)$ local sections which satisfy the (non-holonomic) variational equation. As such these local inverse images are necessarily holonomic, hence the 1-jet of the projection of the section to Q . In conclusion, for a section φ of $\mathcal{J}^*(Q)$, φ satisfies the Hamilton-de Donder-Weyl equation if and only if $\bar{\pi}^1 \circ \varphi \in \Gamma(\mathcal{E}, Q)$ satisfies the Euler-Lagrange equation.

9.8 Covariant Hamiltonian field theories

A Hamiltonian field theory is defined by a target bundle $Q \xrightarrow{\pi} \mathcal{E}$ and a further fibre bundle

$$E \xrightarrow{\bar{\pi}^1} Q$$

which is equipped with a closed $(n + 1)$ -form Ω , called the pre-multisymplectic form.

The space of fields is a subspace of $\Gamma(\mathcal{E}, E)$. The associated Hamilton-de Donder-Weyl equation on a field φ is

$$(\varphi_* \wedge^n T\mathcal{E}) \lrcorner \Omega \subset (\pi \circ \bar{\pi}^1)^* T^* \mathcal{E}$$

An equivalence between two Hamiltonian field theories with the same space of fields is a bundle isomorphism which preserves the pre-multisymplectic form.

Hamiltonian field theories have a Noether theory which is structured by a homotopy Poisson bracket which we introduce now [Rog11; Mit21].

Definition 9.8.1 (Observables). Let E be a smooth manifold equipped with a closed $(n + 1)$ -form Ω . The set of *observables* is defined as follows:

$$\mathfrak{Ham}^0(\Omega) = \{(X, F) \in \Gamma(TE) \times \Omega^{n-1}(E) \mid -\Omega(X, \cdot) = dF\}$$

An element of $\mathfrak{Ham}^0(\Omega)$ corresponds to an observable in the following sense: given an (oriented) hypersurface Σ of \mathcal{E} , an observable (X, F) can be evaluated on a section $\varphi : \mathcal{E} \rightarrow E$ as follows (under suitable integrability properties):

$$\int_{\Sigma} \varphi^* F$$

Assume now that φ is a solution to the Hamilton-de Donder-Weyl equation. Let Σ_+ and Σ_- be two hypersurfaces which form (with suitable orientations) the boundary of a codimension 0

submanifold K . Then the observable can be indifferently evaluated over Σ_+ or Σ_- :

$$\begin{aligned} \int_{\Sigma_+} \phi^* F - \int_{\Sigma_-} \phi^* F &= \int_K \phi^* dF \\ &= \int_K -\phi^* (i_X \Omega) \\ &= \int_K 0 \\ &= 0 \end{aligned}$$

In particular when Σ is a Cauchy surface, the (admissible) initial conditions at Σ can be taken as a representation of the solution space, hence to every observable it is associated a function on the solution space. In this sense, observables are “geometrically” defined quantities over the space of solutions to the field equations.³

Observables are structured in a homotopy Lie algebra \mathfrak{Ham}^i as follows:

$$\begin{aligned} \mathfrak{Ham}^0(\Omega) \\ \mathfrak{Ham}^i(\Omega) &= \Omega^{n-1+i}(E) \quad \text{for } -(n-1) \leq i \leq -1 \\ \mathfrak{Ham}^{i \leq -n}(\Omega) &= 0 \end{aligned}$$

equipped with the following differential:

$$d : \begin{cases} \mathfrak{Ham}^0 \rightarrow 0 \\ \mathfrak{Ham}^{i \leq -1} \xrightarrow{d_{dR}} \mathfrak{Ham}^{i+1} \end{cases}$$

and the following brackets for each integer k between 2 and $n+1$:

$$\begin{aligned} [\dots]_2 &: \begin{cases} (\mathfrak{Ham}^0)^{\otimes 2} \rightarrow \mathfrak{Ham}^0 \\ (X_1, F_1) \otimes (X_2, F_2) \mapsto ([X_1, X_2], -\Omega(X_1, X_2, \cdot)) \end{cases} \\ \text{for } k \geq 3, [\dots]_k &: \begin{cases} (\mathfrak{Ham}^0)^{\otimes k} \rightarrow \mathfrak{Ham}^{-(k-2)} \\ (X_1, F_1) \otimes (X_2, F_2) \otimes \dots \otimes (X_k, F_k) \mapsto (-1)^{k-1} \Omega(X_1, X_2, \dots, X_k, \cdot) \end{cases} \end{aligned}$$

They satisfy a homotopical version of the Jacobi identity.

9.9 Lagrange multipliers

In Lagrangian field theory, Lagrange multipliers are a practical way to impose constraints on a system. Let $Q \rightarrow \mathcal{E}$ be a fibre bundle with a 1st order Lagrangian \mathcal{L}^0 . We write Θ_{PC}^0 the associated Poincaré-Cartan form. Assume for convenience that \mathcal{E} is equipped with a closed volume form vol (this is always locally possible).

Holonomic Lagrange multipliers

Let f be a real function on Q such that 0 is not a singular value. We want to restrict the Lagrangian field theory associated to \mathcal{L}^0 to the submanifold $\{f = 0\}$. Of course it is possible

³There are approaches allowing for a dependence in higher coordinates, see e.g. [Vit09].

to arbitrarily restrict allowed fields to this submanifold. However Lagrange multipliers allow to formulate a restricted theory as an unrestricted theory with fields taking value in a larger target bundle, with an additional field.

Let us replace the target bundle Q by $Q \times \underbrace{\mathbb{R}}_{\tau}$. We define the following Lagrangian on $\mathcal{J}(Q \times \mathbb{R})$:

$$\mathcal{L} = \mathcal{L}^0 - \tau f \text{vol}$$

with vol an arbitrary volume form on \mathcal{E} . Note that the dependency with respect to τ is only at first order, so that \mathcal{L} can be formulated on the bundle $\mathcal{J}(Q) \times \mathbb{R}$. Notably, the term $\tau f \text{vol}$ does not depend on 1st order coordinates and since the Legendre transform is linear, the associated Poincaré-Cartan form is

$$\Theta_{PC}^0 - \tau f \text{vol}$$

Therefore the Euler-Lagrange terms of this new Lagrangian are

$$i_{\partial_{Y^A}} d(\Theta_{PC}^0 - \tau f \text{vol}) = i_{\partial_{Y^A}} d\Theta_{PC}^0 - \tau \partial_{Y^A} f \text{vol}$$

and

$$i_{\partial_{\tau}} d(\Theta_{PC}^0 - \tau f \text{vol}) = f \text{vol}$$

Namely, the Euler-Lagrange equations on a section $\phi = (\sigma, T) \in \Gamma(\mathcal{E}, Q \times \mathbb{R})$ take the following form:

$$\begin{cases} j\sigma^* i_{\partial_{Y^A}} d\Theta_{PC}^0 = T\sigma^* \partial_{Y^A} f \text{vol} \\ \sigma^* f \text{vol} = 0 \end{cases}$$

In other words, solutions of the Euler-Lagrange equations are required to take value in $\{f = 0\}$ and the Euler-Lagrange equations associated with Θ_{PC}^0 are relaxed. For variations not respecting the equation $f = 0$, the Euler-Lagrange term associated with Θ_{PC}^0 need not be 0, it can be proportional to $\partial_{Y^A} f$ with a single factor common to all directions given by T . For variations preserving the equation $f = 0$, the Euler-Lagrange equation is left unchanged.

If we want to impose more than one constraint, we need one multiplier τ^a for each constrained degree of freedom. Namely, we add a term

$$\tau_a f^a \text{vol}$$

to the Lagrangian. It can be geometrically modelled by a dynamical section τ of a vector bundle Q_{cons} (for example $\mathbb{R}^m \times \mathcal{E} \rightarrow \mathcal{E}$) and a fixed constraint $f^a \text{vol}$ which is a differential form-valued section of the dual vector bundle, namely an element of $\Gamma(Q, Q_{cons}^* \otimes \pi^* \Lambda^n T^* \mathcal{E})$.

When a Lagrangian theory has Lagrange multipliers, the Legendre transform is degenerate: the momenta dual to the multipliers τ_a vanish since they are only involved in the Lagrangian at order 0. Indeed we know that the Poincaré-Cartan form takes the form

$$\Theta_{PC}^0 - \tau_a f^a \text{vol}$$

Therefore the Legendre transform takes value in $\mathcal{J}^*(Q) \times \mathbb{R}^m$:

$$\begin{array}{ccc} \mathcal{J}(Q \times \mathbb{R}^m) & \dashrightarrow & \mathcal{J}^*(Q) \times \mathbb{R}^m \\ \downarrow & \searrow \widetilde{\mathbb{F}\mathcal{L}} & \downarrow \\ \mathcal{J}(Q) \times \mathbb{R}^m & \dashrightarrow & \mathcal{J}^*(Q \times \mathbb{R}^m) \end{array}$$

The canonical Hamiltonian receives a term $\tau_a f^a$:

$$H(x^i, y^A, v_i^A, \tau_a) = H^0(x^i, y^A, v_i^A) + \tau_a f^a$$

while the momenta are not affected by the Lagrange multipliers. The corresponding Hamiltonian field theory is thus naturally formulated on $\mathcal{J}(Q) \times \mathbb{R}^m$. In particular, if the original Lagrangian theory is non-degenerate, the Euler-Lagrange equations are equivalent to the Hamilton-de Donder-Weyl equations on $\mathcal{J}(Q) \times \mathbb{R}^m$ with two subtleties:

- Since the momenta dual to τ_a are restricted to zero, there is no $d\tau_a \wedge \text{vol}_i$ term in the premultisymplectic form hence no Hamilton-de Donder-Weyl equation on the variations of τ .
- The constraints equations $\{f^a = 0\}$ correspond to the components τ_a of the Hamilton-de Donder-Weyl equations, which would involve the momenta dual to τ_a if they were free. Alternatively, these equations define the subspace of $\mathcal{J}(Q \times \mathbb{R}^m)$ which is stable under evolution according to the Hamilton-de Donder-Weyl equations: such equations are called “secondary constraints” in opposition to the *primary constraint* of restricting of $\mathcal{J}^*(Q \times \mathbb{R}^m)$ to the subspace $\mathcal{J}^*(Q) \times \mathbb{R}^m$ with vanishing momenta dual to τ_a [HT92].

It is also possible to use Lagrange multipliers directly starting from a Hamiltonian field theory. As observed earlier, the multiplier term $-\tau_a f^a \text{vol}$ is added as such to the Poincaré-Cartan form. For a Hamiltonian theory constructed from a Hamiltonian section h , it can be naturally shifted to a Hamiltonian section

$$h(x, y, p) \text{vol} \mapsto (h(x, y, p) - \tau_a f^a) \text{vol}$$

(beware that this writing depends on the choice of coordinates y^A). More generally, adding multipliers amounts to adding an \mathbb{R}^m degree of freedom to the fields and a $d(\tau f^a) \wedge \text{vol}$ term to the premultisymplectic form.

Non-holonomic Lagrange multipliers

The same computations are still valid when $f^a \text{vol}$ are replaced by more general n -forms (F^a) – we will consider elements of $\Omega^n(Q)$, which will be sufficient for our purposes. One can consider the Euler-Lagrange term associated to

$$\mathcal{L}^{\text{cons}} = T_a F^a(z, y)$$

with $(T_a)_{1 \leq a \leq m}$ being additional free scalar-valued fields (or one field with value in a suitable vector bundle). We write τ_a for the corresponding added coordinate in the configuration space: $Q \times \mathbb{R}^m$. Note that $\mathcal{L}^{\text{cons}}$ may be a “Lagrangian” of a more general type than defined in Section 9.2 since it may not be purely horizontal. A more precise definition would be to define $\mathcal{L}^{\text{cons}}$ as the purely horizontal part of $T_a F^a$ modulo contact forms. In any case we will use the

corresponding Poincaré-Cartan form:

$$\Theta^{cons} = \tau_a F^a \tag{9.22}$$

which satisfies the equation

$$\partial_{v_j^A} \lrcorner d\Theta^{cons} = 0$$

If F^a does not belong to $\Lambda_1^n T^*Q$ then $d\Theta^{cons}$ may fall outside of $\Lambda_1^n(\pi^1)^*T^*Q$ but since the contacts terms of higher degree have no influence on the variational equations, this poses no problem. In fact only the class of F^a modulo \mathcal{I}^2 matters, any \mathcal{I}^2 part may be changed or cut out without consequence on the Euler-Lagrange equation (on *holonomic sections*).

Remark. This is the very reason for which we chose F^a in $\Omega^n(Q)$: by not allowing for further dependency on the first order coordinate, the Legendre transformation is a trivial operation. However as mentioned, up to contact terms, F^a can be represented in $\Gamma(\mathcal{J}(Q), \pi^{1*} \Lambda^n T^* \mathcal{E})$ with a possibly non-trivial albeit very specific (antisymmetric) dependency in the coordinates v_i^A . Using such horizontal forms instead of forms in $\Omega^n(Q)$ would give similar results after the Legendre transformation.

The corresponding premultisymplectic form is

$$d\Theta^{cons} = d\tau_a \wedge F^a + \tau_a dF^a \tag{9.23}$$

The contribution to the Euler-Lagrange forms is

$$EL^{cons,a} = \partial_{\tau_a} \lrcorner d\Theta^{cons} = F^a \tag{9.24}$$

$$EL_A^{cons} = \partial_{y^A} \lrcorner d\Theta^{cons} = -d\tau_a \wedge i_{\partial_A} F^a + \tau_a i_{\partial_A} dF^a \tag{9.25}$$

hence for a field $\phi = (\sigma, T) : \mathcal{E} \rightarrow Q \times \mathbb{R}^m$ the Euler-Lagrange equations associated to the whole Lagrangian $\mathcal{L}_0 - \mathcal{L}_{cons}$ are

$$\begin{cases} \sigma^* F^a = 0 & (9.26) \\ \sigma^* EL_A^0 = \phi^* (\tau_a i_{\partial_A} dF^a - d\tau_a \wedge i_{\partial_A} F^a) & (9.27) \end{cases}$$

with EL^0 the Euler-Lagrange form associated to \mathcal{L}^0 .

The first equation gives the constraints one intends to impose on the fields. In the second equation τ_a can compensate for some nonzero components of EL_A^0 . Indeed in the case $F^a = f^a \text{vol}$, the right-hand term is $T_a \sigma^* (\partial_{y^A} f^a \text{vol})$. In the case the f^a have independent differentials (in the vertical y^A directions), T_a parametrise non-trivial components of $\sigma^* EL_A^0$ along the $d_y f^a$ that are allowed by the variational principle *restricted to sections satisfying the constraint condition* $\sigma^* f^a = 0$.

We dealt with n -forms so far, we now explain how to use Lagrange multipliers with lower degree forms. For any form $f \in \Omega^k(Q)$ with $k \leq n$, assuming we are given a local frame of Q , we can consider the forms

$$F_I = f \wedge \text{vol}_I^{(k)} \tag{9.28}$$

for I a multi-index of size k parametrising the basis of $(n - k)$ -forms $\text{vol}_I^{(k)}$ (as described in Section 1.4). We consider free multiplier fields T^I which we gather in a term

$$T^I F_I = f \wedge T^I \text{vol}_I^{(k)} = f \wedge T$$

with $T \in \Omega^{n-k}(Q)$. The term encodes the constraints

$$(\sigma^* f) \wedge \text{vol}_I^{(k)} = 0$$

for all multi-indices I hence

$$\sigma^* f = 0$$

which is very similar to the mechanism we will use in our application with the term (10.10). The difference is that we only required *specific components* of $d\lambda + \frac{1}{2}[\lambda \wedge \lambda]$ to vanish, by selecting specific values for the multi-index I .

Such constraints can be of non-holonomic nature: consider $F_i = df \wedge \text{vol}_i$ for $f \in \Omega^0(Q)$. Coupling them to Lagrange multipliers τ^i we obtain a term

$$\tau^i df \wedge \text{vol}_i = df \wedge \tau$$

with $\tau \in \Lambda^{n-1}(T^*Q)$. Considering the Lagrangian $\mathcal{L}_0 - df \wedge \tau$ we obtain the following Euler-Lagrange equations on a field (σ, τ) with vertical variations $X \in j\sigma^*V(\mathcal{J}^1(Q)/\mathcal{E})$

$$\begin{cases} \sigma^* df = 0 & (9.29) \\ j\sigma^*(\text{EL}_X^0) + \sigma^*(\mathcal{L}_X(f)) \wedge dT = 0 & (9.30) \end{cases}$$

which is to compare with the equations associated with a single holonomic Lagrange multiplier $\tau f(z, y)\text{vol}$:

$$\begin{cases} \sigma^* f = 0 & (9.31) \\ j\sigma^*(\text{EL}_X^0) - T\sigma^*(\mathcal{L}_X(f))\text{vol} = 0 & (9.32) \end{cases}$$

The constraint enforced by the term $df \wedge \tau$ is

$$d\sigma^* f = 0 \quad (9.33)$$

or in another words $\sigma^* f$ has to be (locally) constant. The difference with the holonomic constraint is that *we do not specify the constant*. Indeed we only require σ to be tangent to the leaves of the foliation defined by df , namely the level hypersurfaces of f . Hence the leaf is allowed to change between the fields, however this is a *non-local* variation.

Here again, the term $f \wedge T$ is invariant under Legendre transformation, although the momentum component of the Legendre transform can be affected by the term $f \wedge T$. Therefore it is possible to perform a similar Legendre transformation on a Hamiltonian field theory. One needs to extend the field configuration space from $E \rightarrow Q \rightarrow \mathcal{E}$ to

$$E \times_Q \prod_{a=1}^m \underbrace{\Lambda^{n-k_a} \pi^* T^* \mathcal{E}}_{\tau_a = \tau_a^I \text{vol}_I} \rightarrow Q \rightarrow \mathcal{E}$$

and add the term

$$-d(F^a \wedge \tau_a)$$

to the pre-multisymplectic form. Doing so, the obtain pre-multisymplectic field theory will have

$$\phi^*(F^a) = 0$$

as components of the Hamilton-de Donder-Weyl equations which are dual to τ_a .

Exact Euler-Lagrange terms

We now explain how, using specific non-local variations of the dynamic fields, it is possible to gather Lagrange multiplier contributions in an exact term. By performing an integration it is then possible to obtain a non-local Euler-Lagrange equation which has no multiplier contribution. Let us start with a simple example.

We consider a real line bundle $Q = \mathbb{R} \times Q \rightarrow Q$ with a fibre coordinate y and provided with a Lagrangian \mathcal{L}^0 . We enforce the constraint $dy = 0$ by adding a term $-dy \wedge \tau$ to the Lagrangian with τ a free variable in $\Lambda^{n-1}T^*Q$:

$$\mathcal{L} = \mathcal{L}^0 - dy \wedge \tau$$

Equation (9.27) on a field $\phi = (\sigma, T) : Q \rightarrow \Lambda^{n-1}T^*Q \times \mathbb{R}$ associated to the variation field ∂_y becomes

$$\sigma^* \text{EL}_y^0 = dT \tag{9.34}$$

asserting that $\phi^* \text{EL}_y^0$ is an exact form of primitive T . In a similar fashion enforcing a constraint $dy \wedge \text{vol}_i = 0$ with a term $\tau dy \wedge \text{vol}_i$, the Euler-Lagrange term would be “exact in the i direction”, in the sense that $\phi^* \text{EL}^0 = \partial_i T \text{vol}$.

In the case Q is a compact manifold, the variation of the action $\int_Q \phi^* \mathcal{L}$ under a global translation of ϕ by a *constant* value $\delta\phi$ is

$$\left(\int_Q \phi^* \text{EL}_y^0 \right) \delta\phi$$

so that asserting that $\phi^* \text{EL}_y^0$ is exact is equivalent to asserting that the action has a trivial variation under this non-local field variation.

In order to formalise this observation let (F^a) be a family of homogeneous forms on Q of respective degrees k_a : $F^a \in \Omega^{k_a}(Q)$ and consider Lagrange multipliers $\tau_a \in \Lambda^{n-k_a}T^*Q$. We therefore have a Lagrangian

$$\mathcal{L} = \mathcal{L}_0 - F^a \wedge \tau_a$$

Now let X be a vector field on Q (hence which does not act on the multipliers) such that \mathcal{L}_X preserves the ideal generated by the F^a . Beware that this ideal of constraints is not necessarily generated in degree one, hence is not necessarily generated by the annihilator of a plane distribution. Consider the Euler-Lagrange term corresponding to X :

$$\begin{aligned} \text{EL}_X &= i_{jX} d(\Theta^0 - F^a \wedge \tau_a) = \text{EL}_X^0 - d(i_X F^a \wedge \tau_a) + \mathcal{L}_X(F^a \wedge \tau_a) \\ &= \text{EL}_X^0 - d(F^a \wedge \tau_a) + \mathcal{L}_X(F^a) \wedge \tau_a \end{aligned} \tag{9.35}$$

so that, under the constraint Euler-Lagrange equations (9.24)

$$\phi^* \mathcal{L}_X F^a \equiv 0$$

and the Euler-Lagrange equation corresponding to X is equivalent to

$$\sigma^* \text{EL}_X^0 \equiv d(\sigma^* i_X F^a \wedge T_a) \pmod{(F^a)} \tag{9.36}$$

in a process very reminiscent of Noether’s theorem on conserved currents. The degrees of freedom τ_a then take the role of coefficients parametrising a differential primitive of EL_X^0 . Suppose that it is possible to find a set of such vector *fields* (X^I) preserving the constraints (which is a local

property of the first order prolongations of X^I) which spans the vertical directions of Q . Then we can express the whole Euler-Lagrange equation system as

$$\begin{cases} \phi^* F^a = 0 & (9.37) \\ \phi^* \text{EL}_{X^I}^0 \equiv d(F_I^a \wedge T_a) & (9.38) \end{cases}$$

in which we wrote $F_I^a := i_{X^I} F^a$. Note a possible gauge freedom since $F_I^a \wedge T_a$ is only involved through its exterior differential, so that variations of T_a inducing closed variations of all the $F_I^a \wedge T_a$ are symmetries of the equations.

Now if the source space Q is compact, it is possible to integrate over Q to obtain *non-local equations*

$$\int_Q \sigma^* \text{EL}_{X^I}^0 = 0 \quad (9.39)$$

When Q is not compact, one may try to find lower dimension compact submanifolds and to build from the Euler-Lagrange lower degree forms which are exact. This will be our approach in Section 11.4. We will integrate along 6-dimensional orbits, and for that purpose we will need to factor out a *closed* 4-form from both $\sigma^* \text{EL}_X^0$ and the exact multiplier term.

9.10 More geometrical perspectives on Lagrangian field theory

Variational problem and exterior differential systems

An approach to Euler-Lagrange equations, developed by Bryant and Griffiths [BGG02], does away with the fibration over the base space and focuses on the internal structure of the jet space itself. The data is then the contact ideal \mathcal{I} and the Lagrangian as well as a second ideal \mathcal{J} spanned by the horizontal forms $V(\mathcal{J}(Q)/Q)^\perp$ which contains the information corresponding to the fibration over Q . A Lagrangian is a section of $\Lambda^n V(\mathcal{J}(Q)/Q)^\perp$. It generalises the definition from Section 9.2 by allowing for Q/\mathcal{E} -vertical factors – indeed this framework is based on the total space of the Grassmannian $\text{Gr}_n(TQ)$ which extends the jet space by forgetting the section (transversality) condition on the n -dimensional planes.

The Poincaré-Cartan form (their Poincaré-Cartan form is the exterior differential of ours, which they call *Betounes form*) is defined by requiring that

$$d(\mathcal{L} - \theta^A \wedge \beta^A) \in \mathcal{I} \quad (9.40)$$

with $\theta^A \wedge \beta^A$ the local decomposition of a term belonging to the *algebraic ideal* generated by the contact 1-forms. Eliminating the ambiguity in the definition takes some work but the result is that using $\Theta_{PC} = \mathcal{L} - \theta^A \wedge \beta^A$ the differential can be written

$$d\Theta_{PC} = \theta^A \wedge \Psi_A$$

so that the forms Ψ_A can be identified with our $\partial_{y^A} \lrcorner d\Theta_{PC} \pmod{\mathcal{I}}$ and the Euler-Lagrange equations are gathered in a differential system generated by

$$\mathcal{E}_{\mathcal{L}} := (\theta^A, d\theta^A, \Psi_A, d\Psi_A)$$

which turns out to be independent of the choice of β^A (as long as (9.40) is satisfied).

This approach uses a stronger geometrical structure than the pre-multisymplectic approaches however omitting the fibration over the source space \mathcal{E} allows for more equivalences than keeping the whole jet bundle fibrations. As a consequence there can be more symmetries as well as equivalences between relatively different geometrical situations.

Local Lagrangian field theory as a higher abelian gauge theory

In Schreiber’s *prequantum field theory* [Sch], the action functional is back at the center of the stage. It is the geometrically meaningful quantity, and the (local) Lagrangians are only means to define it. The action functional is geometrically understood as the parallel transport of a *higher line bundle with connection* on the configuration space. The variation of the action is related to the curvature of the connection by a higher equivalent of the Ambrose-Singer theorem so that the (higher) curvature defines the premultisymplectic form.

With an eye toward quantization, one can consider an action which has U_1 values instead of real values. The higher line bundle then has to be complex and can be topologically nontrivial, which enables one to use a global action in some situations in which the Lagrangians can only be defined locally. Since the Lagrangians are connection coefficients (components of a de Rham-Čech cocycle) they are only defined up to gauge transformation, which embodies the fact that to a given variational principle can correspond different local Lagrangians. In this sense, the “homotopy” equivalence of Lagrangian systems acquire a geometrical meaning in this framework. In particular the homotopy Lie algebra of observables is naturally associated with the homotopical structure of the higher line bundle with connection.

The case of mechanics is particularly simple to present ⁴ since the action takes value in a real or complex line bundle (more accurately: in the associated Atiyah-Lie groupoid) – this perspective can already be found in [Del+00]. A (local) trivialisation of the line bundle gives a representation of the connection as a 1-form: this is a Lagrangian representation of the action so that

$$S_{t_0}^{t_1}[\sigma] = \int_{[t_0, t_1]} \sigma^* \mathcal{L}$$

Adding an extra term to \mathcal{L} is akin to changing the selected gauge. Given a vector field X preserving the curvature ω on E , a lift to an infinitesimal automorphism of the line bundle with connection is exactly the data of a real-valued function f such that

$$i_X \omega + df = 0$$

and the standard Poisson bracket corresponds to the Lie bracket of infinitesimal automorphisms. When computing the finite flow of such an infinitesimal automorphism in a trivialisation, the Legendre transform of f appears in a path integral over the line flows of X [Bla92].

In this approach, the natural symplectic manifold (for mechanics and field theory) is the space of solutions of the field equations, the so-called covariant phase space [CW87; Zuc86; Kha14]. The structure of this covariant phase space has also been formalised using more algebraic methods [And92; Vit09]

⁴We proceed on a heuristic level: a careful exposition would need to discuss contact terms, or be formulated on the space of solutions.

Chapter 10

Hélein-Vey's generalisation of the Einstein-Cartan gravitation

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10.1 Cartan geometry and field theories

Gauge theories are geometrically modelled by principal bundles, principal connections and associated bundles. For example, in Maxwell's theory of electromagnetism, the electromagnetic field can be understood as the curvature of a U_1 -principal connection. When working in a quantum mechanical framework, this interpretation becomes inevitable as Dirac's quantization condition requires the electromagnetic field to be an "integral 2-form" (with respect to a dimensioned unity charge). More practically, the standard model is a gauge theory with a gauge group isomorphic to $U_1 \times SU_2 \times SU_3$.

On another hand, covariant formulations of field theories, incentivised by the relativistic perspective, are sometimes considered as "gauge theories with respect to groups of diffeomorphisms". A much celebrated example is the covariant modelling of spacetime by a smooth manifold, and of the gravitational field by the Levi-Civita connection associated to a metric. The connection is associated with a covariant derivative which is essential in order to manipulate differential operations in a coordinate-independent fashion. Actually in a covariant framework, gauge field theories as well use covariant derivatives in order to build gauge-independent and coordinate independent differential expressions.

Systems of coordinates define a lot of structure on manifolds so that coordinate independence can be realised in different frameworks according to the type of dependence on the coordinates.

In particular, constructions depending on a (co)frame data can generally be realised “covariantly” on frame bundles as is illustrated in the example of Section 7.3.3. The takeaway is that gauge fields as well as matter field with gauge degrees of freedom can be modelled on principal bundles and/or frame bundles. As explained in 3.2.3, an H -principal bundle can be combined with a G -structure into an $H \times G$ -structure. The same mechanism was described in Section 7.2.2 for Cartan geometries: a K -principal bundle can be combined with the H -principal bundle of a (G, H) -Cartan geometry into a $(K \times G, K \times H)$ -Cartan geometry. In this sense, Cartan geometry can support both spacetime geometry and the geometry of internal gauge symmetries.

Gauge field theories lifted to the principal bundle

Let us illustrate concretely what we mean by formulating a field theory in terms of the Cartan geometry. We first freeze the gauge fields, namely the connections, in order to consider them as part of the fixed Cartan geometry.

Let M be an n -manifold with a *static* (G, H) -Cartan geometry $P_H \xrightarrow{\pi} M$ with Cartan 1-form $\varpi \in \Omega^1(P_H, \mathfrak{g})$ and a K -principal bundle $K \rightarrow M$ with a static connection 1-form $A \in \Omega^1(P, \mathfrak{k})$. Assume given a field theory specified by:

1. A linear representation V_K of K and a field $\psi_1 \in \Gamma(M, P_K[V_K])$.
2. A linear representation V_G of G and a field $\psi_2 \in \Gamma(M, P_H[V_G])$.
3. A first order Lagrangian density $\mathcal{L}_{A, \varpi} \in \Gamma(\Lambda_1^n(\mathcal{J}^1(P_K[V_K] \times_M P_H[V_G])))$ with an associated Poincaré-Cartan form $\Theta_{A, \varpi}$.

Using ∂_{Y^A} as basis of variations of ψ_1 and ∂_{Z^B} as basis of variations of ψ_2 , the Euler-Lagrange equations are:

$$\begin{cases} j^1(\psi_1, \psi_2)^*(i_{\partial_{Y^A}} \Theta_{A, \varpi}) = 0 \\ j^1(\psi_1, \psi_2)^*(i_{\partial_{Z^B}} \Theta_{A, \varpi}) = 0 \end{cases}$$

Gathering both principal bundles into a product principal bundle, we obtain a $K \times H$ -principal bundle

$$P_{K \times H} := P_K \times_M P_H$$

which is naturally equipped with a $(K \times G, K \times H)$ -Cartan connection 1-form $\varpi_{K \times G}$; it is the sum of the pullbacks of A and ϖ . The vector bundle $P_K[V_K] \times P_H[V_G]$ can then be constructed using the action of $K \times G$ on $V_K \times V_G$ with G acting trivially on V_K and conversely:

$$P_K[V_K] \times P_H[V_G] \simeq P_{K \times H}[V_K \times V_G]$$

Therefore, there is no loss of generality in considering uniquely the bundle P_H with its (G, H) -Cartan geometry and a field taking value in the associated vector bundle with fibre V_G . Thus we hereon assume our field theory only depends on the Cartan geometry of M and there is only a field taking value in $P_H[V_G]$.

We want to lift the field theory to P_H . It is straightforward to lift ψ : it is a section naturally associated with an H -equivariant V_G -valued function Ψ . Similarly, $\mathcal{L}_{A, \varpi}[\psi]$ can be lifted into a $\pi^* \Lambda^n T^* M$ -valued function depending on the 1-jet of Ψ : we call it $\mathcal{L}_{A, \varpi}^{P_H}$. In order to formulate the field theory on P_H , we need to turn it into a top form on P_H . To this aim, we equip \mathfrak{h} with a volume form. It can be pulled back to a k -form $\varpi^{(k)}$ with k the dimension of H so that

$$\mathcal{L}_{A, \varpi}^{P_H} \wedge \varpi^{(k)}$$

is a top form on P_H . Since $\mathcal{L}_{A,\varpi}^{P_H}$ is a horizontal form of maximum degree and ϖ is vertical, the mapping $\mathcal{L}_{A,\varpi}^{P_H} \mapsto \mathcal{L}_{A,\varpi}^{P_H} \wedge \varpi^{(k)}$ is injective.

It is therefore possible to formulate a Lagrangian field theory on P_H with an H -equivariant V_G -valued field Ψ and the Lagrangian form $\mathcal{L}_{A,\varpi}^{P_H} \wedge \varpi^{(k)}$. The natural question is whether the theory is still equivalent to the initial theory. It brings the following difficulty: variations of Ψ which preserve H -equivariance are *not compactly supported* when G is not compact. Even when G is compact, they are not *local*, which invalidates the standard derivation of the Euler-Lagrange equations.

One solution is not to turn the pulled back Lagrangian into a $(k+n)$ -degree form but to keep it as an n -form, and to integrate over a *variable* section of $P_H \rightarrow M$, which is a method used in the group manifold approach to supergravity [Cas18; DAu20]. The solution we will pursue is using *Lagrange multipliers*, like described in Section 9.9, as we now explain.

Let us assume the group H is connected, so that H -equivariance is equivalent to \mathfrak{h} -equivariance. Let us furthermore equip (G, H) with a reductive structure: there is an H -equivariant splitting

$$\mathfrak{g} \simeq_H \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$$

Let us correspondingly split ϖ into an \mathfrak{h} -valued part ω^i and a $\mathfrak{g}/\mathfrak{h}$ -valued part α^a . Then according to Theorem 7.2.5, equivariance of Ψ is equivalent to the existence of coefficients D_a with value in $V_G \otimes (\mathfrak{g}/\mathfrak{h})^*$ such that

$$d\Psi + \varpi \cdot \Psi = D_a \alpha^a \in \Omega^1(P_H, V_G)$$

Since these are local equations, we can use Lagrange multipliers in order to impose them. We add a V_G^* -valued $(n+k-1)$ -form P to the field. Using indices β for V_G , we can construct a term

$$P_\beta \wedge (d\Psi + \varpi \cdot \Psi)^\beta$$

to add to the Lagrangian $\mathcal{L}_{A,\varpi}^{P_H}$. It is not per se an element of $\Lambda^{n+k} T^* P_H$ parametrised by the jet of Ψ and P but it is the pullback under (P, Ψ) of an element of the following bundle:

$$\underbrace{\Lambda^{n+k-1} (T^* P_H \otimes V_G^*)}_P \times \underbrace{V_G}_\Psi \times_{P_H} \Lambda^{n+k} T^* P_H$$

which can be naturally added to the Poincaré-Cartan form corresponding to the initial Lagrangian. If we call

$$v : \Lambda^{n+k-1} T^* P_H \otimes V_G^* \times V_G \rightarrow V_G$$

the projection on the V_G component and

$$p : \Lambda^{n+k-1} T^* P_H \otimes V_G^* \times V_G \rightarrow \Lambda^{n+k-1} T^* P_H \otimes V_G^*$$

the tautological $(n+k-1)$ -form, we can add the following term to the Poincaré-Cartan form:

$$p_\beta \wedge (dv + \varpi \cdot v)^\beta$$

It is then straightforward to check that for the Euler-Lagrange equations corresponding to variations δp_β of p_β are

$$\delta p_\beta \wedge \Psi^* (dv + \varpi \cdot v)^\beta = 0$$

which should hold for all δp_β , whence

$$\Psi^* (dv + \varpi \cdot v) = 0$$

Now, the Euler-Lagrange equations corresponding to variations of Ψ gain a contribution of p_β , dealing with which will take more work. However, we want to bring attention to another point: the field theory obtained on P_H only uses the Cartan 1-form and not the global action of H . This means that it can be straightforwardly generalised to generalised Cartan geometries! The formulas can be used without any change.

The last trick, which is already at work in [HV16], is that these generalised Cartan geometries can be obtained as solutions of Euler-Lagrange equations. The mechanism is very similar to what we did for Ψ : a (G, H) -Cartan 1-form is nothing more than a \mathfrak{g} -valued coframe ϖ which satisfies the following property: there exist coefficients Ω_{ab}^A in $\mathfrak{g} \otimes \Lambda^2 \mathfrak{g}/\mathfrak{h}^*$ such that

$$d\varpi^A + \frac{1}{2} [\varpi \wedge \varpi]^A = \frac{1}{2} \Omega_{ab}^A \alpha^a \wedge \alpha^b$$

These can be imposed using Lagrange multipliers too. Another point which will be relevant when studying the application to gravitation, is that this quantity is exactly the total curvature tensor of the Cartan connection. This means that on one hand we can impose the vanishing of certain components with Lagrange multipliers, and on the other hand include a scalar curvature term in the action with the same kind of term.

The mechanism will be explained in detail in Section 10.3.

10.2 The Einstein-Cartan theory of gravitation

We start with a very succinct presentation of the Einstein-Cartan theory of gravitation. In Einstein's original theory of gravitation, gravitation is modelled by *the curvature* of a Lorentzian metric on the differentiable manifold modelling the spacetime. More accurately, the Newtonian gravitational field is replaced by the torsion-free Levi-Civita connection associated to the metric, which defines the geodesic equation governing inertial trajectories. Newton's formula for the interaction between massive bodies and Poisson's equation for the gravitational potential are replaced by Einstein's field equation, which is a second order differential equation, and the geodesic equation. It was then realised that a first order formulation is possible if one was to consider as possible field a couple gathering the metric itself and an a priori independent affine connection. This is called the *Palatini formalism* and exist in several variations, requiring the connection to be metric or not (see [HK78; HLS81; DP12]). In the case the connection is neither assumed to be metric nor torsion-free an extra gauge freedom appears, the so-called projective symmetry [HK78; DP12].

Within the Palatini formalism, it is possible to make explicit a Lorentzian gauge symmetry: this is the so-called *tetradic Palatini formalism*, in which the metric field is replaced by a field of linear frames, or equivalently (in our case) *coframes*. The linear frame is called a *tetrad*, or *vierbein* (*vielbein* in general dimension). The *Einstein-Cartan theory* (more precisely Einstein-Cartan-Sciama-Kibble) is the case in which the connection is assumed to be metric but not torsion-free.

The tetrad will be particularly useful when dealing with spinors on dynamical spacetimes in Section 11. This will be discussed in Section 11.1. We will also discuss in Section 11.4 how torsion is involved in the dynamics of spinors in a curved spacetime.

We start with a differentiable 4-manifold \mathcal{E} that is meant to embody spacetime. The fields will be :

- A *generalized tetrad* $e : T\mathcal{E} \rightarrow V$ which identifies the tangent bundle with a reference space and time-oriented lorentzian vector bundle (V, η) . We do not assume V to be a trivialisable vector bundle as may implicitly be the case in standard tetrad formalism.

- A dynamic *metric connection* ∇ on V , or equivalently the tangent connection $\tilde{\nabla} = e^*\nabla$ on $T\mathcal{E}$, which is compatible with the inverse pullback metric $e^*\eta$.

The tetrad is a (nondegenerate) V -valued 1-form written locally as $(e^a)_{0 \leq a \leq 3}$ in which the a superscript corresponds to any reference basis of V . With (x^μ) a local coordinate system of \mathcal{E} the vielbein decomposes with components $e^a = e_\mu^a dx^\mu$.

The (adimensional)¹ *Einstein-Cartan action* (hereafter EC action) with vanishing cosmological constant is then given by [HV16; Yoo18] :

$$S_{EC}[e, \nabla] = \int_{\mathcal{E}} \mathcal{L}_{EC}[e, \nabla] = \int_{\mathcal{E}} \mathcal{R}_c^a \eta^{cb} \wedge e_{ab}^{(2)} \tag{10.1}$$

with \mathcal{R} the endomorphism curvature 2-form of ∇ , η^{ac} representing the *inverse* metric and $e_{ab}^{(2)}$ the dual exterior 2-forms as described in Section 1.4. In the case \mathcal{E} is noncompact, so that the integral can be ill-defined, the action has to be understood as a motivation for the corresponding *variational* (Euler-Lagrange) equation, which is a local equation on the fields: even when the action is not globally defined it is locally, as well as its first variation.

10.3 Hélein-Vey's model

We now introduce the model due to Hélein and Vey in [HV16]. It is a model defined on a 10-dimensional space, which is meant to acquire the structure of the Lorentzian frame bundle under the field equation. Fields then define a principal connection which has to satisfy Einstein-Cartan's field equation in vacuum. In order to explain the construction of the Lagrangian, we lift the usual Einstein-Cartan Lagrangian over the frame bundle of spacetime. In a second step, we explain how it is possible to then omit the frame bundle structure and generalise the field theory thus obtained to all 10-dimensional manifolds.

Although the theory we study originates from [HV16], the language and the approach we present is original (and was presented in [Pie22]). The point is to clarify the structure of the theory in order to better understand the extent to which it is a generalisation of Einstein-Cartan's theory as well as to make it clear how to adapt the theory in order to allow for matter fields. Indeed in Chapter 11 we will explain how to handle spinor fields within the same formalism. In this section we only derive the Euler-Lagrange terms but we postpone the study of the solutions to Section 11.4 in which we will deal at once with gravitational terms and spinor terms.

Conventions and notations

We will write η_{ab} for the Minkowski metric on the Minkowski space \mathfrak{m} , used at the same time as a pseudo-Euclidean affine space, as an abelian Lie group and as an abelian Lie algebra. Our convention for the Lorentzian signature is $(+ - - -)$ and for the Clifford algebras $u \cdot v + v \cdot u = -2 \langle u|v \rangle$ (more detail in Section 4.3).

We will be working with the connected *proper orthochronous Lorentz group* $\mathfrak{L} = \text{SO}_{1,3}^+$ which we will just call *Lorentz group*. Its Lie algebra is $\mathfrak{l} = \mathfrak{so}_{1,3}$. The *Poincaré group* is isomorphic to the semi-direct product of \mathfrak{m} with \mathfrak{L} :

$$\mathfrak{P} \simeq \mathfrak{L} \ltimes \mathfrak{m}$$

and there is an isomorphism between the associated Lie algebras:

$$\mathfrak{p} \simeq \mathfrak{l} \ltimes \mathfrak{m}$$

¹The properly dimensioned action has a factor $\frac{1}{2\kappa}$ with $\kappa = 8\pi G$.

A *Lorentzian structure* on a 4-manifold is an SO^+ -structure, namely the data of:

1. A metric g of Lorentzian signature.
2. A space and time-orientation compatible with g .

10.3.1 Formulating gravitation on the frame bundle

We wish to formulate the Einstein-Cartan theory on the *frame bundle of V* , or to be precise on the *proper orthochronous orthonormal frame bundle of V* , which we write $\text{SO}^+(V)$ (hereafter referred to as the frame bundle). The tetrad field on spacetime is represented by a *solder form* on $\text{SO}^+(V)$ (defined below). In this way the tetrad data is integrated in the geometrical setting and we consider all coframes in an equivariant manner. Anticipating on Section 11.2, working on a frame bundle will allow us to consider spinor fields with value in a trivialised bundle, in the same way the tetrad does. For this purpose, we will lift the Lagrangian form defining the action (10.1) to the frame bundle, according to the procedure described in 10.1.

The space of fields

The frame bundle of V depends on the metric and space and time-orientation structures on V ; the connection is then an extra structure on the frame bundle. Write

$$\pi : \text{SO}^+(V) \rightarrow \mathcal{E}$$

the principal fibration. Vectors in the kernel of $d\pi$ are called *vertical* and differential forms which have a trivial contraction with all vertical vectors are called *horizontal*.

The frame bundle has the structure of an \mathfrak{L} -principal bundle on \mathcal{E} . The tetrad field is replaced by a *solder form* α : a nondegenerate \mathfrak{L} -equivariant horizontal 1-form with values in \mathfrak{m} (more detail is to be found in Section 3.2). The solder form establishes an isomorphism between V which is the associated vector bundle of fibre \mathfrak{m} and $T\mathcal{E}$, or a injection of $\text{SO}^+(V)$ into the frame bundles of $T\mathcal{E}$. Hence choosing an isomorphism class for V amounts to selecting a frame bundle and it is in this sense that α plays the role of the tetrad e . The Lorentz gauge symmetry on e is geometrically realised as the action on α of the (equivariant) automorphisms of the frame bundle.

The (right) action of elements of \mathfrak{L} on $\text{SO}^+(V)$ induces an action of the Lie algebra \mathfrak{l} as follows: to an element ξ in \mathfrak{l} is associated a vector field

$$\bar{\xi} = \partial_t (R_{\exp(t\xi)})|_{t=0} \in \Gamma(T\text{SO}^+(V))$$

A metric connection on V corresponds to a nondegenerate \mathfrak{L} -equivariant \mathfrak{l} -valued 1-form ω on $\text{SO}^+(V)$ which is normalized with respect to the action of the Lorentz algebra \mathfrak{l} . Normalisation means that the following holds:

$$\forall \xi \in \mathfrak{l}, \quad \omega(\bar{\xi}) = \xi$$

Its kernel is a horizontal distribution which is the corresponding *Ehresmann connection*. The data of a metric connection together with the solder form gives rise to a *Cartan connection* $\omega \oplus \alpha \in \Omega^1(\text{SO}^+(V), \mathfrak{l} \ltimes \mathfrak{m})^{\mathfrak{L}}$ (the superscript \mathfrak{L} denotes the subset of the \mathfrak{L} -equivariant forms). The space of fields introduced in Section 10.2 can then be described as the set of $(\mathfrak{L} \ltimes \mathfrak{m}, \mathfrak{L})$ -Cartan connection forms on the principal bundle $\text{SO}^+(V)$.

Lifting the Lagrangian

We want to express the action (10.1) lifted to $\mathrm{SO}^+(V)$ as a function of the Cartan connection. Since the signature is non-Riemannian, the structure group $\mathfrak{L} = \mathrm{SO}_{1,3}^+$ is noncompact hence so is the bundle space. We shall consider the action as a formal integral and a motivation to derive variational equations over local variations, and forget about any domain of integration. As a remark note that a structure group reduction to maximal compact subgroups SO_3 always *exists* but would involve extra physical data (although topologically trivial): a nowhere vanishing timelike vector field (it is called a *field of observers* in [GW13]).

The curvature 2-form associated with the connection is expressed as

$$\Omega = d\omega + \frac{1}{2} [\omega \wedge \omega]$$

It takes value in the Lie algebra \mathfrak{l} . Since \mathfrak{l} acts on \mathfrak{m} , we will allow ourself to consider the associated $\mathrm{End}(\mathfrak{m})$ -valued 2-form without changing the notation. We write π^* for the pullback to $\mathrm{SO}^+(V)$ of any tensorial-valued differential form, which is then identified with a (horizontal) differential form of the same degree with values in a *trivial* bundle and which is equivariant.

In order to relate the curvature 2-form to the curvature tensor, for a basis ξ_i of \mathfrak{l} we introduce the components $\rho_{i,d}^b$ of the action $\mathfrak{l} \rightarrow \mathrm{End}(\mathfrak{m})$ so that

$$(\xi_i \cdot x)^b = \rho_{i,d}^b x^d$$

for $x \in \mathfrak{m}$. They satisfy the antisymmetry relation

$$\rho_{i,d}^b \eta^{dc} + \rho_{i,d}^c \eta^{db} = 0$$

Then the curvature tensor \mathcal{R} is defined on \mathcal{E} so that $e^* \mathcal{R} = \tilde{\mathcal{R}}$ and

$$(\pi^* \mathcal{R})_c^a = \Omega^i \rho_{i,c}^a$$

as explained in Section 3.2, using implicitly the identification

$$\pi^* \mathrm{End}(TM) \simeq \mathbb{R}^{p,q} \otimes \mathbb{R}^{p,q*} \times \mathrm{SO}^+(V)$$

We equip the Minkowski vector space \mathfrak{m} with a space and time orientation in addition to its metric structure, so that a duality operator $\Lambda^k \mathfrak{m} \simeq \Lambda^{4-k} \mathfrak{m}^*$ is defined (as in Section 1.4). We can then lift the EC Lagrangian form (10.1):

$$\pi^* \left(\mathcal{R}_c^a \eta^{cb} \wedge e_{ab}^{(2)} \right) = \Omega^i \rho_{i,c}^a \eta^{cb} \wedge \alpha_{ab}^{(2)} \quad (10.2)$$

which gives a horizontal (scalar-valued) 4-form on the frame bundle. It is not a top form on the bundle space yet. In order to turn it into an \mathfrak{L} -invariant top form it has to be wedge-multiplied with a (right-)invariant volume form on \mathfrak{L} . Such a volume form is not unique, but we can specify it in a consistent way for all fibres using the Lie algebra action, as we explain now.

As an \mathfrak{l} -valued 1-form, ω establishes a map $\mathfrak{l}^* \rightarrow \Omega_{vert}^1(\mathrm{SO}^+(V))^{\mathfrak{L}}$ which extends to a graded algebra morphism

$$\omega^* : \Lambda^\bullet \mathfrak{l}^* \rightarrow \Omega_{vert}^\bullet(\mathrm{SO}^+(V))^{\mathfrak{L}} \quad (10.3)$$

so that specifying a volume element in $\Lambda^6 \mathfrak{l}^*$ gives a vertical 6-form on $\mathrm{SO}^+(V)$, which we will write $\omega^{(6)}$. Although $\omega^{(6)}$ effectively depends on the connection, detailed computations show that

$h \wedge \omega^{(6)}$ does not for h any specified *horizontal* 4-form. We thus define the following Lagrangian 10-form on $\text{SO}^+(V)$:

$$\mathcal{L} = \Omega^i \rho_{i,c}^a \eta^{cb} \wedge \alpha_{ab}^{(2)} \wedge \omega^{(6)} \quad (10.4)$$

Even if we do not explicitly specify the volume element of \mathfrak{l}^* , it is still possible to discuss about coupling constants and relative signs when considering a Lagrangian with matter components as long as the same volume element is used for all terms. These considerations remain however out of the scope of this paper.

Using the whole Cartan connection form we can in a similar way consider the morphism

$$(\omega \oplus \alpha)^* : \Lambda^\bullet(\mathfrak{l} \oplus \mathfrak{m})^* \rightarrow \Omega^\bullet(\text{SO}^+(V))^{\mathfrak{L}} \quad (10.5)$$

and providing \mathfrak{l} with an arbitrary volume element we use the vector-forms duality (as described in Section 1.4) on $\mathfrak{l} \times \mathfrak{m}$ so that we can write

$$\mathcal{L} = \Omega^i \rho_{i,c}^a \eta^{cb} \wedge (\omega \oplus \alpha)_{ab}^{(8)} \quad (10.6)$$

We can now formulate the variational problem on $\text{SO}^+(V)$: the field is a Cartan connection 1-form $\omega \oplus \alpha$, in other words a 1-form with value in $\mathfrak{l} \oplus \mathfrak{m}$, more precisely in the Lie algebra $\mathfrak{l} \times \mathfrak{m}$, which is a nondegenerate 1-form, (i.e. of constant rank 10), equivariant and normalised for the principal action of \mathfrak{l} in the following sense:

$$\forall \xi \in \mathfrak{l}, (\omega \oplus \alpha)(\bar{\xi}) = \xi \oplus 0$$

It has to be an extremal point of the locally defined action

$$S_{EC}[\alpha, \omega] = \int \Omega^i \rho_{i,c}^a \eta^{cb} \wedge (\omega \oplus \alpha)_{ab}^{(8)} \quad (10.7)$$

for compactly-supported variations. The integral is taken with respect to the orientation given by $\alpha^{(4)} \wedge \omega^{(6)}$. There is a problem however: the fibre of $\text{SO}^+(V)$ are noncompact, and ω is required to be equivariant. A compactly supported variation which preserves equivariance is necessarily trivial. This calls for a different treatment of the constraints on the field ω . The constraint on $\omega \oplus \alpha$ of being a Cartan connection 1-form can be written in the following way, writing $\bar{\xi}$ for the (left-invariant) vector field representing $\xi \in \mathfrak{l}$ and R_g for the action of $g \in \mathfrak{L}$:

$$\begin{cases} \alpha^{(4)} \wedge \omega^{(6)} \text{ is nowhere vanishing} \\ i_{\bar{\xi}}(\omega \oplus \alpha) = \xi \oplus 0 \\ R_g^*(\omega \oplus \alpha) + g \cdot (\omega \oplus \alpha) = 0 \end{cases}$$

Now since the group \mathfrak{L} is connected, \mathfrak{l} -equivariance is equivalent to \mathfrak{L} -equivariance so that the constraint can be written as local equations as follows:

$$\begin{cases} \alpha^{(4)} \wedge \omega^{(6)} \text{ is nowhere vanishing} & (10.8a) \\ i_{\bar{\xi}}(\omega \oplus \alpha) = \xi \oplus 0 & (10.8b) \\ \mathcal{L}_{\bar{\xi}}(\omega \oplus \alpha) + \xi \cdot (\omega \oplus \alpha) = 0 & (10.8c) \end{cases}$$

We want to derive the variational Euler-Lagrange equation for (10.7) under the constraints (10.8a-10.8c) but because the equivariance constraint (10.8c) is non-holonomic, and in a sense non-local,

the usual derivation does not directly apply. Moreover, the action (10.7) cannot be used as such since the domain is non compact (and requiring equivariance along the noncompact fibres definitely prevents any nontrivial field variation from having a compact support, or even from decaying at infinity). The central question of the present Section is the derivation and the treatment of the variational equations under such constraints. This will involve translating the constraints into Lagrange multipliers terms (as presented in Section 9.9) in the Lagrangian as is explained in the following discussion.

10.3.2 Generalised frame bundle structure with connection as a dynamical field

In Section 10.3.1 we described a formulation of Einstein-Cartan gravitation with one field ϖ which is defined over the frame bundle of spacetime. The idea of the model proposed by Hélein and Vey in [HV16] is to forget any a priori structure of frame bundle and simply study the field ϖ defined over a structure-less 10-dimensional manifold \mathcal{P} . Indeed, turning around the constraints (10.8b, 10.8c), they can be seen as defining a “generalised frame bundle structure” from ϖ , as presented in Section 7.2. A similar mechanism assuming only part of the fibration structure is studied in [GW13].

As a consequence, the structure of generalised frame bundle, namely the generalised Cartan connection ϖ is what we wish to obtain from the Euler-Lagrange equations.

Lagrange multiplier terms

The previous discussion motivated dropping the constraint (10.8b) so as to take it as a definition of the fields $\bar{\xi}$ instead. Constraint (10.8a) is an open (and algebraic) condition so we will simply keep it as restraining the configuration space to an open subset.

However Equation (10.8c) is different. It is both a differential constraint and a (topologically) closed constraint. We want to incorporate it into the Lagrangian by means of Lagrange multipliers. For this, the convenient formulation of (10.8c) is

$$d\varpi^A + \frac{1}{2} [\varpi \wedge \varpi]^A = \frac{1}{2} \Omega_{bc}^A \alpha^b \wedge \alpha^c$$

with Ω_{bc}^A arbitrary (non-constant) coefficients which are antisymmetric in b, c (the derivation is given in Section 7.2.1). As the coefficients Ω_{bc}^A are arbitrary, the equation only means that the components along $\omega \wedge \omega$ and along $\alpha \wedge \omega$ vanish. According to Section 1.4 if we use again the notation $\varpi_{BC}^{(8)}$ for the 8-form which is dual to $\varpi^B \wedge \varpi^C$, we can rewrite the equation as

$$\left(d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \right) \wedge \varpi_{jk}^{(8)} = 0 \tag{10.9a}$$

$$\left(d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \right) \wedge \varpi_{bk}^{(8)} = 0 \tag{10.9b}$$

with the index b corresponding to a basis of \mathfrak{m} and indices j, k corresponding to a basis of \mathfrak{l} .

Under this form, it is straightforward to impose the conditions (10.9) using Lagrange multipliers: we add to the theory free fields P_A^{jk} and P_A^{bk} and consider the following term to add to the Lagrangian:

$$\left(d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \right)^A \wedge \frac{1}{2} P_A^{jk} \varpi_{jk}^{(8)} + \left(d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \right)^A \wedge P_A^{bk} \varpi_{bk}^{(8)} \tag{10.10}$$

which would impose the constraint (10.9) through the equations of motion corresponding to variations of P_A^{jk} and P_A^{bk} . Note the fundamental difference with *holonomic* Lagrange multipliers which would be coupled to a term such as $f(\varpi, v)\varpi^{(10)}$. The Lagrange multipliers we use serve to impose an (exterior) differential constraint. Such Lagrange multipliers are presented in more detail in Section 9.9, which explains how to use the variational terms derived from them.

The $\varpi_{BC}^{(8)}$ forms being antisymmetric in BC , only the antisymmetric part of P_A^{BC} is involved in the term (10.10). We thus constrain the multipliers P_A^{BC} (with $BC = bk$ or $BC = jk$) to be antisymmetric in BC , effectively using the 8-forms

$$\frac{1}{2}P_A^{BC} \left(d\varpi + \frac{1}{2}[\varpi \wedge \varpi] \right)^A \wedge \varpi_{BC}^{(8)}$$

as Lagrange multiplier fields. If one wanted to impose a torsion-freeness constraint on the connection, one could in a similar fashion use a free p_a^{bc} term, as is for example done in three dimensions in [DOS10].

The Lagrangian

The Lagrangian (10.6) is

$$\mathcal{L}[\varpi] = \Omega^i \rho_{i,d}^b \eta^{dc} \wedge (\omega \oplus \alpha)_{bc}^{(8)} = \Omega^i \rho_{i,d}^b \eta^{dc} \wedge \varpi_{bc}^{(8)}$$

and can be written under the following form:

$$\mathcal{L}[\varpi] = \left(d\varpi + \frac{1}{2}[\varpi \wedge \varpi] \right)^i \rho_{i,d}^b \eta^{dc} \wedge \varpi_{bc}^{(8)}$$

Note how as a linear function of the curvature 2-form it is very similar to the terms (10.10).

Let \mathcal{P} a 10-manifold and $\mathfrak{p} \simeq \mathfrak{l} \times \mathfrak{m}$ the Poincaré Lie algebra. Let us denote $\text{Iso}(T\mathcal{P}, \mathfrak{p}) \subset T^*\mathcal{P} \otimes \mathfrak{p}$ the subbundle of \mathfrak{p} -valued coframes.² Note that it is an \mathfrak{L} -principal bundle above \mathcal{P} , although this structure will not be relevant for our considerations. We consider the following configuration bundle over \mathcal{P} :

$$Q = \underbrace{\text{Iso}(T\mathcal{P}, \mathfrak{p})}_{\varpi^A} \times \underbrace{(\mathfrak{m} \wedge \mathfrak{l} \oplus \Lambda^2 \mathfrak{l})}_{P_A^{bk}, P_A^{jk}} \otimes \mathfrak{p}^*$$

with $\mathfrak{m} \wedge \mathfrak{l} \subset \Lambda^2 \mathfrak{p}$ the image of $\mathfrak{m} \otimes \mathfrak{l} \subset \mathfrak{p}^{\otimes 2}$.

On $T^*\mathcal{P} \otimes \mathfrak{p}$ there is a tautological section: writing $f : T^*\mathcal{P} \otimes \mathfrak{p} \rightarrow \mathcal{P}$ the fibration map, it takes the form of a diagonal section

$$T^*\mathcal{P} \otimes \mathfrak{p} \rightarrow T^*\mathcal{P} \otimes \mathfrak{p} \times_{\mathcal{P}} T^*\mathcal{P} \otimes \mathfrak{p} \simeq f^*(T^*\mathcal{P} \otimes \mathfrak{p})$$

Dual to the tangent projection $T(T^*\mathcal{P} \otimes \mathfrak{p}) \xrightarrow{Tf} T\mathcal{P}$ there is a pullback map $T^*\mathcal{P} \rightarrow T^*(T^*\mathcal{P} \otimes \mathfrak{p})$ so that the tautological section can be pulled back to a \mathfrak{p} -valued 1-form on $T^*\mathcal{P} \otimes \mathfrak{p}$:

$$T^*\mathcal{P} \otimes \mathfrak{p} \rightarrow f^*(T^*\mathcal{P} \otimes \mathfrak{p}) \simeq (f^*T^*\mathcal{P}) \otimes \mathfrak{p} \xrightarrow{f^*} T^*(T^*\mathcal{P} \otimes \mathfrak{p}) \otimes \mathfrak{p}$$

We call it *the canonical \mathfrak{p} -valued 1-form* on $T^*\mathcal{P} \otimes \mathfrak{p}$ and write it λ . On $\text{Iso}(T\mathcal{P}, \mathfrak{p})$ it can be

²Our manipulations will actually be meaningful on all of $T^*\mathcal{P} \otimes \mathfrak{p}$ including degenerated points, with due adjustments since $T\mathcal{P}$ and \mathfrak{p} are no longer identified.

identified with a solder form (defined in Section 3.2). We will use the notation $\lambda_{A_1 \dots A_k}^{(10-k)}$ for the dual $(10-k)$ -forms defined according to Section 1.4. We also use the notations p_A^{bk} and p_A^{jk} for the (trivial) fibre coordinates of the component in $(\mathfrak{m} \wedge \mathfrak{l} \oplus \Lambda^2 \mathfrak{l}) \otimes \mathfrak{p}^*$, in order to establish a clear distinction with the corresponding components P_A^{bk} and P_A^{jk} of sections on Q :

$$\begin{array}{ccc} Q & \xrightarrow{p_A^{bk}, p_A^{jk}} & (\mathfrak{m} \wedge \mathfrak{l} \oplus \Lambda^2 \mathfrak{l}) \otimes \mathfrak{p}^* \\ \updownarrow \varphi & \nearrow & \\ \mathcal{P} & & P_A^{bk}, P_A^{jk} \end{array}$$

The explicit expression is $P_A^{bk} = \varphi^* p_A^{bk}$, $P_A^{jk} = \varphi^* p_A^{jk}$.

The Lagrangian form gathering both the lifted Einstein-Cartan Lagrangian and the Lagrange multiplier fields is

$$\begin{aligned} \mathcal{L}[\varpi, P] &= \rho_{i,d}^b \eta^{dc} \left(d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \right)^i \wedge \varpi_{bc}^{(8)} \\ &\quad + \frac{1}{2} P_A^{jk} \left(d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \right)^A \wedge \varpi_{jk}^{(8)} + P_A^{bk} \left(d\varpi + \frac{1}{2} [\varpi \wedge \varpi] \right)^A \wedge \varpi_{bk}^{(8)} \end{aligned} \quad (10.11)$$

To obtain a local expression in terms of coordinates, let (z^I) be a local system of coordinates on \mathcal{P} . It induces local coordinates (z^I, λ_I^A) on $T^*\mathcal{P} \otimes \mathfrak{p}$ and fibre coordinates on the 1-jet bundle $\mathcal{J}^1(T^*\mathcal{P} \otimes \mathfrak{p})$: we write them $v_{I,J}^A$, defined so that

$$v_{I,J}^A(\varpi) = \partial_J (\lambda_I^A(\varpi))$$

Then the Lagrangian (10.11) can be expressed as a 10-form on $\mathcal{J}^1(Q)$ in terms of the local coordinates (there is no 1st order contribution from P):

$$\begin{aligned} \mathcal{L} &= \rho_{i,d}^b \eta^{dc} \left(v_{I,J} dz^J \wedge dz^I + \frac{1}{2} [\lambda \wedge \lambda] \right)^i \wedge \lambda_{bc}^{(8)} \\ &\quad + \frac{1}{2} p_A^{jk} \left(v_{I,J} dz^J \wedge dz^I + \frac{1}{2} [\lambda \wedge \lambda] \right)^A \wedge \lambda_{jk}^{(8)} + p_A^{bk} \left(v_{I,J} dz^J \wedge dz^I + \frac{1}{2} [\lambda \wedge \lambda] \right)^A \wedge \lambda_{bk}^{(8)} \end{aligned} \quad (10.12)$$

such that $\mathcal{L}[\varpi, P] = (\varpi, P)^* \mathcal{L}$. In this form, there is no constraint extraneous to the Lagrangian other than the open constraint of nondegeneracy of ϖ . Hence we can use the usual Legendre transform formula (9.8) (described in Section 9.4) to compute the Poincaré-Cartan form. If we define

$$p_A^{bc} = 2\delta_A^i \rho_{i,d}^b \eta^{dc} \quad (10.13)$$

then \mathcal{L} takes the concise form

$$\mathcal{L} = \frac{1}{2} p_A^{BC} \left(v_{I,J} dz^J \wedge dz^I + \frac{1}{2} [\lambda \wedge \lambda] \right)^A \wedge \lambda_{BC}^{(8)} \quad (10.14)$$

We can thus consider that we have a field $p_A = p_A^{BC} \lambda_{BC}^{(8)}$ which is subject to the holonomic constraint (10.13).

The Poincaré-Cartan form

Introduce the following notation: for a p -form u on $T^*\mathcal{P} \otimes \mathfrak{p}$ with values in a \mathfrak{p} -module we will write

$$d^\lambda u := du + \lambda \wedge u \quad (10.15)$$

with $\lambda \wedge u$ including the action of \mathfrak{p} (in our discussion it will mainly be about products of adjoint and coadjoint representations of $\mathfrak{l} \simeq \mathfrak{p}/\mathfrak{m}$). Define also

$$\Lambda := d\lambda + \frac{1}{2} [\lambda \wedge \lambda] = d^\lambda \lambda - \frac{1}{2} [\lambda \wedge \lambda] \quad (10.16)$$

The operator d^λ is meant to model a covariant differential while Λ models a universal curvature 2-form. They satisfy the expected equations (proved in Section 7.4)

$$d^\lambda d^\lambda u = \Lambda \wedge u \quad (10.17)$$

$$d^\lambda \Lambda = 0 \quad (10.18)$$

$$d^\lambda(u^A \wedge v_A) = (d^\lambda u^A) \wedge v_A + (-1)^{|u|} u^A \wedge d^\lambda v_A \quad (10.19)$$

for u^A and v_A homogeneous differential forms with values in dual \mathfrak{p} -modules.

To compute the Poincaré-Cartan form we will use the following formula from Section 9.4:

$$\partial_{v_{I,J}^A} \lrcorner d(\mathcal{L} + \pi_B^J \wedge \chi_J^B) = 0 \quad \text{mod } [\theta_J^A]$$

with π_B^J 9-form fields on \mathcal{P} and $\chi_J^A = d\lambda_J^A - v_{I,J}^A dz^I$. Using this formula rather than Formula (9.8) will save us some back and forth between the coframes dz^I and λ^A . We determine the value of π_B^J as a function of p_A^{BC} :

$$\begin{aligned} \partial_{v_{I,J}^A} \lrcorner d(\mathcal{L} + \pi_D^J \wedge \chi_J^D) &= \partial_{v_{I,J}^A} \lrcorner d\left(\frac{1}{2} p_D^{BC} \left(v_{K,L} dz^L \wedge dz^K + \frac{1}{2} [\lambda \wedge \lambda]\right)^D \wedge \lambda_{BC}^{(8)} + \pi_D^K \wedge \chi_K^D\right) \\ &= \frac{1}{2} p_A^{BC} (dz^J \wedge dz^I - dz^I \wedge dz^J) \wedge \lambda_{BC}^{(8)} \\ &\quad + \left(\partial_{v_{I,J}^A} \lrcorner d\pi_D^J\right) \wedge \chi_J^D - \pi_D^K \wedge \partial_{v_{I,J}^A} \lrcorner (-dv_{K,L}^D \wedge dz^L) \\ &= p_A^{BC} dz^J \wedge dz^I \wedge \lambda_{BC}^{(8)} + \pi_A^I \wedge dz^J + \left(\partial_{v_{I,J}^A} \lrcorner d\pi_D^J\right) \wedge \chi_J^D \\ &= dz^J \wedge (-p_A^{BC} dz^I \wedge \lambda_{BC}^{(8)} + \pi_A^I) + \left(\partial_{v_{I,J}^A} \lrcorner d\pi_D^J\right) \wedge \chi_J^D \end{aligned}$$

Since the term $\left(\partial_{v_{I,J}^A} \lrcorner d\pi_D^J\right) \wedge \chi_J^D$ is a contact term the momentum forms π_A^I are defined by

$$dz^J \wedge (-p_A^{BC} dz^I \wedge \lambda_{BC}^{(8)} + \pi_A^I) = 0$$

Since dz^J form a basis of 1-forms on \mathcal{P} , we conclude that the momenta forms are directly parameterised by the Lagrange multipliers p_A^{BC}

$$\pi_A^I = p_A^{BC} dz^I \wedge \lambda_{BC}^{(8)}$$

and the corresponding contact term in the Poincaré-Cartan form is

$$p_A^{BC} dz^I \wedge \lambda_{BC}^{(8)} \wedge (d\lambda_I^A - v_{J,I}^A dz^J) = p_A^{BC} \wedge \lambda_{BC}^{(8)} \wedge (d\lambda^A - v_{J,I}^A dz^I \wedge dz^J) \quad (10.20)$$

The Poincaré-Cartan form is

$$\begin{aligned} \mathcal{L} + p_A^{BC} \wedge \lambda_{BC}^{(8)} \wedge (d\lambda^A - v_{J,I}^A dz^I \wedge dz^J) &= \frac{1}{2} p_A^{BC} \left(v_{I,J} dz^J \wedge dz^I + \frac{1}{2} [\lambda \wedge \lambda] \right)^A \wedge \lambda_{BC}^{(8)} \\ &\quad + p_A^{BC} \wedge \lambda_{BC}^{(8)} \wedge (d\lambda^A - v_{J,I}^A dz^I \wedge dz^J) \\ &= \frac{1}{2} p_A^{BC} \left(d\lambda + \frac{1}{2} [\lambda \wedge \lambda] \right)^A \wedge \lambda_{BC}^{(8)} \end{aligned}$$

We write the Poincaré-Cartan form as follows:

$$\overline{\Theta}_{EC} = \frac{1}{2} p_A^{BC} \Lambda^A \wedge \lambda_{BC}^{(8)} \quad (10.21)$$

It goes with the holonomic constraint (10.13) on the p_A^{bc} . We identify two components:

$$\Theta_{EC} = \frac{1}{2} 2\rho_{i,d}^b \eta^{dc} \Lambda^i \wedge \lambda_{bc}^{(8)} \quad (10.22)$$

$$\Theta_{EC}^{cons} = \frac{1}{2} p_A^{jk} \Lambda^A \wedge \lambda_{jk}^{(8)} + p_A^{bk} \Lambda^A \wedge \lambda_{bk}^{(8)} \quad (10.23)$$

Note that since

$$\frac{1}{2} [\lambda \wedge \lambda]^i \wedge \lambda_{ab}^{(8)} = \frac{1}{2} c_{DE}^i \lambda^D \wedge \lambda^E \lambda_{ab}^{(8)} = c_{ab}^i \lambda^{(10)} = 0$$

the form Θ_{EC} can also be expressed with $d\lambda$ replacing Λ . However the expression Λ has an interpretation as the curvature, furthermore if we generalise from $\mathfrak{p} \simeq \mathfrak{l} \ltimes \mathfrak{m}$ to other Lie algebras then $\frac{1}{2} [\lambda \wedge \lambda]^i \wedge \lambda_{ab}^{(8)}$ may not vanish.

We introduce the notation \widehat{BC} for pairs of indices of \mathfrak{p} *except pairs which correspond to $\mathfrak{m} \otimes \mathfrak{m}$* (i.e. only pairs which correspond to $(\mathfrak{l} \otimes \mathfrak{l}) \oplus (\mathfrak{m} \otimes \mathfrak{l}) \oplus (\mathfrak{l} \otimes \mathfrak{m})$). The Poincaré-Cartan form can then be expressed as

$$\overline{\Theta}_{EC} = \underbrace{\delta_A^i \rho_{i,c}^a \eta^{bc} \Lambda^A \wedge \lambda_{ab}^{(8)}}_{\Theta_{EC}} + \frac{1}{2} \underbrace{p_A^{\widehat{BC}} \Lambda^A \wedge \lambda_{\widehat{BC}}^{(8)}}_{\Theta_{EC}^{cons}} \quad (10.24)$$

The Poincaré-Cartan form takes value in the *affine dual* of $\mathcal{J}^1(Q)$ (described in Section 9.6). It is the fibre bundle of 10-forms on Q that have a vanishing contraction with all 2-vectors of Q that are purely vertical with respect to the fibration above the source space \mathcal{P} . This affine dual is usually written:

$$\Lambda_1^{10} T^* Q$$

Note that the coefficients of the Poincaré-Cartan form have no dependency on the first order component of the 1-jet, so that the only dependency comes from the factor

$$\Lambda^A = d\lambda^A + \frac{1}{2} [\lambda \wedge \lambda]^A$$

We can therefore restrict the momentum space from the whole affine dual to Q itself, on which $\overline{\Theta}_{EC}$ can still be expressed as a 10-form, along with the supplementary constraint (10.13) (for

details on the equivalence between the Lagrangian and the Hamiltonian formulations see the last section of [De +05] on affine Lagrangians). One can also say that $\frac{1}{2}p_A^{\widehat{BC}}\lambda_{BC}^{(8)}$ correspond to specific elements of the linear dual space of $\mathcal{J}^1(Q)$ of the form $\Lambda^A \wedge \frac{1}{2}p_A^{\widehat{BC}}\lambda_{BC}^{(8)}$, according to (10.10). In fact, the Lagrange multiplier terms are derived in [HV16] as free momenta obtained by a Legendre transformation under constraint.

In fact it could be argued that (10.21) is the natural formulation on $\Lambda^8 T^* \mathcal{P} \otimes \mathfrak{p}^* \times_{\mathcal{P}} Q$ of the Lagrangian given in (10.11), rather than (10.12). We chose to go for the “naive” formulation of the Lagrangian on the 1-jet bundle to illustrate the systematic approach to the Legendre transformation.

We derive the variational equations in the next section.

10.3.3 Variational equations for gravitation

In this section we compute the variational equations corresponding to the Poincaré-Cartan form (10.21). The *Euler-Lagrange form* is defined for vertical vector fields X on Q as

$$\text{EL}_X := i_X d\overline{\Theta}_{EC} \quad (10.25)$$

Since the Poincaré-Cartan form is defined on Q there is no need to use the 1st prolongation of X as described in Section 9.4.

The Euler-Lagrange equations consist in the vanishing of all the components of the Euler-Lagrange form with respect to X under the pullback by a field $(\varpi, P) : \mathcal{P} \rightarrow \text{Iso}(T\mathcal{P}, \mathfrak{p}) \times (\mathfrak{m} \wedge \mathfrak{l} \oplus \mathfrak{l} \wedge \mathfrak{l}) \otimes \mathfrak{p}^*$. Namely,

$$\forall X, \quad (\varpi, P)^* \text{EL}_X = 0$$

A local basis of the vertical vector fields is given by the fields $\partial_{\lambda_I^A}$ and $\partial_{p_A^{\widehat{BC}}}$ dual to fibre coordinates. For convenience, we define $\partial_{\lambda_B^A} = \lambda_I^B \partial_{\lambda_I^A}$ so that

$$\begin{aligned} \partial_{\lambda_B^A} \lrcorner \Lambda^C &= \partial_{\lambda_B^A} \lrcorner d\lambda^C + \partial_{\lambda_B^A} \lrcorner \frac{1}{2} [\lambda \wedge \lambda]^C \\ &= (\lambda_I^B \partial_{\lambda_I^A}) \lrcorner d\lambda^C + \frac{1}{2} c_{DE}^C \partial_{\lambda_B^A} \lrcorner \lambda^D \wedge \lambda^E = \delta_A^C \lambda_I^B dz^I + 0 \\ &= \delta_A^C \lambda^B \end{aligned} \quad (10.26)$$

We impose on the field space that (λ^A) form a nondegenerate family (coframe) at each point of \mathcal{P} , consequently we can substitute $\partial_{\lambda_B^A}$ for $\partial_{\lambda_I^A}$ in the basis of vertical vector fields. One can interpret $\partial_{\lambda_B^A}$ as representing the infinitesimal action of $\text{End}(\mathfrak{p})$ on $\text{Iso}(T\mathcal{P}, \mathfrak{p})$.

We will make use of the following identities, proved in Section 7.4 :

$$\begin{aligned} d\lambda^{(10)} &= \Lambda^D \wedge \lambda_D^{(9)} \\ d\lambda_A^{(9)} &= \Lambda^D \wedge \lambda_{AD}^{(8)} \\ d\lambda_{AB}^{(8)} &= \Lambda^D \wedge d\lambda_{ABD}^{(7)} - c_{AB}^D \lambda_D^{(9)} \\ d^\lambda \Lambda^A &= 0 \end{aligned}$$

10.3.4 Variation of the multiplier $P_A^{\widehat{BC}}$

We check that the variation of the Lagrange multipliers yields the expected constraint equations. Recall that we do not allow BC to take the form bc in $\partial_{p_A^{BC}}$. We use the notation $(\text{EL}_{EC})_{BC}^A$ for $(\text{EL}_{EC})_{\partial_A^{BC}}$:

$$(\text{EL}_{EC})_{BC}^A = \partial_{p_A^{BC}} \lrcorner d \left(\frac{1}{2} p_D^{EF} \Lambda^D \wedge \lambda_{EF}^{(8)} \right) = \frac{1}{2} \left(\partial_{p_A^{BC}} \lrcorner dp_D^{EF} \right) \Lambda^D \wedge \lambda_{EF}^{(8)} = \frac{1}{2} \Lambda^A \wedge \lambda_{BC}^{(8)} \quad (10.27)$$

We write $\Omega := \varpi^* \Lambda \in \Omega^2(\mathcal{P}, \mathfrak{p})$ which has as components

$$\Omega^A = \frac{1}{2} \Omega_{BC}^A \varpi^B \wedge \varpi^C$$

A critical field (P, ϖ) satisfies

$$\Omega_{BC}^A = 0 \quad (10.28)$$

which is equivalent to Equations (10.9). Thus a solution ϖ of the Euler-Lagrange equations defines a Cartan 1-form therefore a generalised frame bundle structure on \mathcal{P} .

10.3.5 Variation of the coframe ϖ

Instead of using $\partial_{\lambda_B^A}$, it will be convenient to use a vertical vector field X on $\text{Iso}(T\mathcal{P}, \mathfrak{p})$ which has variable coefficients ϵ_B^A as follows:

$$X = \epsilon_B^A \partial_{\lambda_B^A}$$

These coefficients are gathered into a \mathfrak{p} -valued 1-form:

$$\epsilon^A = \epsilon_B^A \lambda^B$$

so that

$$X \lrcorner d\lambda^A = \mathcal{L}_X \lambda^A = \epsilon^A \quad (10.29)$$

The correspondence $X \leftrightarrow \epsilon$ corresponds to the usual identification between the vertical tangent bundle to a vector bundle and the Whitney sum of the vector bundle with itself.

We mention one more identity from Section 7.4:

$$d(\bar{s}_{[A]} s^{[A]}) = (d\bar{s})_\alpha \wedge s^{[A]} + \bar{s}_\alpha d\lambda^{[A]}$$

for $[A]$ any index, or index list, in a \mathfrak{l} -module. We now compute EL_X :

$$\begin{aligned} X \lrcorner d \left(\frac{1}{2} p_A^{BC} \Lambda^A \wedge \lambda_{BC}^{(8)} \right) &= X \lrcorner \left(d\lambda \Lambda^D \wedge \frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} + \Lambda^A \wedge d\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) \right) \\ &= X \lrcorner \left(0 + \Lambda^A \wedge d\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) \right) \\ &= (X \lrcorner \Lambda^A) \wedge d\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) + \Lambda^A \wedge X \lrcorner d\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) \end{aligned}$$

On one hand

$$X \lrcorner \Lambda = X \lrcorner \left(d\lambda + \frac{1}{2} [\lambda \wedge \lambda] \right) = \epsilon + 0$$

and on the other hand

$$\begin{aligned} X \lrcorner \left(d^\lambda \left(\frac{1}{2} p_D^{BC} \lambda_{BC}^{(8)} \right) \right) &= X \lrcorner \left(\left(d^\lambda \frac{1}{2} p_D^{BC} \right) \wedge \lambda_{BC}^{(8)} \right) + X \lrcorner \frac{1}{2} p_D^{BC} \left(d^\lambda \lambda_{BC}^{(8)} \right) \\ &= 0 + X \lrcorner \frac{1}{2} p_D^{BC} \left(\Lambda^A \wedge \lambda_{BCA}^{(7)} \right) \\ &= \epsilon^A \wedge \frac{1}{2} p_D^{BC} \lambda_{BCA}^{(7)} \end{aligned}$$

Gathering the two terms, we obtain

$$\boxed{\text{EL}_X = \epsilon^A \wedge \left(\Lambda^D \wedge \frac{1}{2} p_D^{BC} \lambda_{BCA}^{(7)} + d^\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) \right)} \quad (10.30)$$

The corresponding Euler-Lagrange equations on a field $(\varpi, P) \in \Gamma(\mathcal{P}, Q)$ are then:

$$\forall \epsilon \in \Omega^1(\mathcal{P}, \mathfrak{p}), \quad \epsilon^A \wedge \left(\Omega^D \wedge \frac{1}{2} P_D^{BC} \varpi_{ABC}^{(7)} + d^\varpi \left(\frac{1}{2} P_A^{BC} \varpi_{BC}^{(8)} \right) \right) = 0$$

where ϵ_B^A are now defined over \mathcal{P} since the X we consider here are variations of the field ϖ . The Euler-Lagrange equation corresponding to variations of ϖ are equivalent to the following equation:

$$\Omega^D \wedge \frac{1}{2} P_D^{BC} \varpi_{ABC}^{(7)} + d^\varpi \left(\frac{1}{2} P_A^{BC} \varpi_{BC}^{(8)} \right) = 0 \quad (10.31)$$

The Einstein term

We now explain how the usual Einstein tensor can be identified in (10.31). First, we assume that Equation (10.28) is satisfied, so that ϖ defines a generalised frame bundle structure. We will identify tensors built out of ϖ which correspond to the various curvature and the torsion tensors in the standard frame bundle case. For more detail on curvature on the frame bundle, see Section 3.2.

Let us isolate the part depending on the fixed momenta $p_D^{bc} = 2\delta_D^l \rho_{l,e}^b \eta^{ec}$: we obtain

$$\begin{aligned} \Omega^l \wedge \frac{1}{2} p_l^{bc} \varpi_{Abc}^{(7)} + \delta_A^i d^\varpi \left(\frac{1}{2} p_i^{bc} \varpi_{bc}^{(8)} \right) &= \Omega^l \wedge \frac{1}{2} p_l^{bc} \varpi_{Abc}^{(7)} + \delta_A^i \frac{1}{2} p_i^{bc} d^\varpi \left(\varpi_{bc}^{(8)} \right) \\ &= \Omega^i \wedge \frac{1}{2} p_i^{bc} \varpi_{Abc}^{(7)} + \delta_A^i \frac{1}{2} p_i^{bc} \Omega^D \wedge \varpi_{bcD}^{(7)} \\ &= (\delta_A^D \Omega^i + \delta_A^i \Omega^D) \wedge \frac{1}{2} p_i^{bc} \varpi_{bcD}^{(7)} \end{aligned}$$

Now assuming that Equation (10.28) is satisfied so that $\Omega^A = \frac{1}{2} \Omega_{bc}^A \alpha^a \wedge \alpha^b$, we compute the

wedge product:

$$\begin{aligned}
& (\delta_A^D \Omega^i + \delta_A^i \Omega^D) \wedge \frac{1}{2} p_i^{bc} \varpi_{bcD}^{(7)} \\
&= \frac{1}{2} p_i^{bc} \left(\delta_A^D \left(\Omega_{cD}^i \varpi_b^{(9)} - \Omega_{bD}^i \varpi_c^{(9)} + \Omega_{bc}^i \varpi_D^{(9)} \right) + \delta_A^i \left(\Omega_{cD}^D \varpi_b^{(9)} - \Omega_{bD}^D \varpi_c^{(9)} + \Omega_{bc}^D \varpi_D^{(9)} \right) \right) \\
&= \frac{1}{2} p_i^{bc} \left(2\Omega_{Ab}^i \varpi_c^{(9)} + \Omega_{bc}^i \varpi_A^{(9)} + \delta_A^i \left(2\Omega_{db}^d \varpi_c^{(9)} + \Omega_{bc}^D \varpi_D^{(9)} \right) \right)
\end{aligned}$$

We want to separate the terms according to their dependency on A (of type a or i) and $\varpi^{(9)}$ (with a subscript c or j):

$$\begin{aligned}
(\delta_A^D \Omega^i + \delta_A^i \Omega^D) \wedge \frac{1}{2} p_i^{bc} \varpi_{bcD}^{(7)} &= \left(\delta_A^a p_i^{bc} \Omega_{ab}^i + \delta_A^c \frac{1}{2} p_i^{de} \Omega_{de}^i \right) \varpi_c^{(9)} \\
&\quad + \delta_A^i \left(\left(p_i^{bc} \Omega_{db}^d + \frac{1}{2} p_i^{de} \Omega_{de}^c \right) \varpi_c^{(9)} + \left(\frac{1}{2} p_i^{de} \Omega_{de}^j + \frac{1}{2} p_k^{bc} \Omega_{bc}^k \delta_i^j \right) \varpi_j^{(9)} \right)
\end{aligned}$$

Now we just have to rewrite the factors in front of $\varpi_c^{(9)}$ so as to get rid of p . Recall the definition

$$p_i^{bc} = 2\rho_{i,d}^b \eta^{dc}$$

The first term is δ_A^a contracted with the following expression:

$$p_i^{bc} \Omega_{ab}^i + \delta_a^c \frac{1}{2} p_i^{de} \Omega_{de}^i = 2\rho_{i,f}^b \eta^{fc} \Omega_{ab}^i + \delta_a^c \rho_{i,f}^d \eta^{fe} \Omega_{de}^i$$

We recognize the contractions of Ω^i : these are the components of the tensor

$$-2 \operatorname{Ric}_{a,f} \eta^{fc} + \delta_a^c \operatorname{Scal} \tag{10.32}$$

which is (minus twice) the Einstein tensor.

For the second term, we will use the following property: for any tensor field A and any list of indices $[D]$, we have

$$p_i^{bc} A_{bcD} = 0 \Leftrightarrow A_{bc[D]} - A_{cb[D]} = 0$$

This is a consequence of the definition $p_i^{bc} = 2\rho_{i,e}^b \eta^{ec}$ interpreted as an isomorphism $\mathfrak{t} \xrightarrow{\sim} \Lambda^2 \mathfrak{m}$. We have

$$p_i^{bc} \Omega_{db}^d + \frac{1}{2} p_i^{de} \Omega_{de}^c = p_i^{de} \left(\delta_e^c \Omega_{fd}^f + \frac{1}{2} \Omega_{de}^c \right) = \frac{1}{2} p_i^{de} \left(\Omega_{de}^c + \delta_e^c \Omega_{fd}^f - \delta_d^c \Omega_{fe}^f \right)$$

in which the antisymmetric term

$$\Omega_{de}^c + \delta_e^c \Omega_{fd}^f - \delta_d^c \Omega_{fe}^f$$

corresponds to the components of the tensor field

$$T + \operatorname{tr}(T) \wedge \operatorname{Id} \tag{10.33}$$

This is a contraction of the torsion which is quite similar to the Ricci curvature. Its divergence is non zero and actually equates (twice) the antisymmetric part of the Ricci tensor.

These tensors are to be equated in (10.31) with quantities dependent on the multipliers $P_A^{\widehat{BC}}$.

We will see in Section 11.4 how to get rid of the multipliers in order to extract meaningful field equations building on Section 9.9.

Comparison with gravitation on “soft Poincaré manifolds”

In [NR78b; Cas18] it is presented a very similar term for the so-called gravitation on “soft Poincaré manifolds”. In our language, they take the following form:

$$\frac{1}{2}p_i^{ab} \left(\delta_A^c \Omega^i \wedge \alpha_{cab}^{(1)} + \delta_A^i \Omega^c \wedge \alpha_{cab}^{(1)} \right)$$

and its vanishing is equivalent to

$$\begin{cases} \frac{1}{2}p_i^{ab} \Omega^i \wedge \alpha_{cab}^{(1)} = 0 \\ \frac{1}{2}p_i^{ab} \Omega^c \wedge \alpha_{cab}^{(1)} = 0 \end{cases}$$

These equations imply the Einstein field equation, the vanishing of the term (10.33) *as well as Equation* (10.28). However, as our theory is formulated with 10-forms, there are in our equations factors of ω^i due to the term $\lambda_{bcD}^{(7)}$ in (10.31). They weaken the constraint imposed by the equation on the non-horizontal components $\Omega_{bk}^A, \Omega_{jk}^A$. For this reason, we needed to add the Lagrange multipliers $p_A^{\widehat{BC}}$ in Section 10.3.2 in order to enforce Equation (10.28).

Chapter 11

Coupling Dirac Spinors with Hélein-Vey's model

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In this chapter, we present a model similar to the one from Section 10.3 which includes a coupling with Dirac spinors.

11.1 The Dirac Lagrangian

Let us first present the standard Dirac Lagrangian as well as the tetrad approach to spinor fields on a dynamical spacetime.

The Dirac Lagrangian

We start with a discussion on the standard framework for Dirac spinors on a curved spacetime. Let M be a 4-manifold equipped with a Lorentzian structure, in particular a metric g . In order to work with spinors, we equip it with a spin structure as defined in Section 4.2.1. Therefore M has a $\text{Spin}_{1,3}^+$ -structure representable by a principal bundle $P \rightarrow \text{GL}(M)$.

Assume furthermore that M is equipped with a spinorial connection, which can be represented by a $\text{Spin}_{1,3}^+$ -principal connection over P . It may be induced from the Levi-Civita connection or may present torsion. We also choose an irreducible $\mathbb{Cl}_{1,3}$ -module Σ (which is unique up to isomorphism), which will be the model for our spinor bundle. To keep in line with the traditional treatment of spinors, we will make use of the “holomorphic” and “anti-holomorphic” directions in Σ . We equip Σ with a spinorial metric (Definition 4.1.41) which maps Σ to the dual space Σ^* in an anti-linear fashion and we will be writing the spinor contraction implicitly:

$$\bar{\psi}_1 \psi_2 := \bar{\psi}_{1\alpha} \psi_2^\alpha$$

We will be using freely the gamma matrices, in their covariant

$$\gamma : T^*M \xrightarrow{g^{-1}} TM \rightarrow \text{End}(\Sigma)$$

and their contravariant

$$\gamma : TM \rightarrow \text{End}(\Sigma)$$

incarnations. We will use a right (transposed) action of gamma matrices on Σ^* as follows:

$$\bar{\psi} \gamma := -\gamma \cdot \bar{\psi}$$

such that

$$\bar{\psi}_1 (\gamma \psi_2) = (\bar{\psi}_1 \gamma) \psi_2$$

A *spinor field* ψ is a section of the associated bundle $P[\Sigma]$. There is a Dirac operator (more in Section 4.2.2) $\overleftrightarrow{\nabla}$ which acts on sections of $P[\Sigma]$ in a formally self-adjoint way. Given a real scalar $m \in \mathbb{R}$, called the *mass*, the *Dirac Equation* is the following eigenvalue equation:

$$\overleftrightarrow{\nabla} \psi - m\psi = 0$$

It can be derived as the Euler-Lagrange equation associated to a Lagrangian on M , called the *Dirac Lagrangian*. Its real version uses the symmetrized Dirac operator $\overleftrightarrow{\nabla}$, defined as follows:

$$\bar{\psi}_1 \overleftrightarrow{\nabla} \psi_2 = \bar{\psi}_1 \gamma^\mu \nabla_\mu \psi_2 - (\nabla_\mu \bar{\psi}_1) \gamma^\mu \psi_2 = 2\Re(\bar{\psi}_1 \overleftrightarrow{\nabla} \psi_2)$$

The (adimensional)¹ Dirac Lagrangian [Wei95; FR03] is

$$L[\psi] = \frac{1}{2} \bar{\psi} \overleftrightarrow{\nabla} \psi - m \bar{\psi} \psi \tag{11.1}$$

It corresponds to the Lagrangian form

$$\mathcal{L}[\psi] = \left(\frac{1}{2} \bar{\psi} \overleftrightarrow{\nabla} \psi - m \bar{\psi} \psi \right) \text{vol} \tag{11.2}$$

with vol the positive Lorentzian volume form.

The absence of an i factor is due to our conventions for Clifford algebra, Lorentzian signature and spinorial metric (see Section 4.3 for a discussion).

¹The dimensioned Lagrangian has a factor $\hbar c$ in front of the kinetic term and a factor c^2 in front of the mass term.

Tetrads and Sciama-Kibble Theory

We have explained the Lagrangian formulation of Dirac spinors in a nondynamical spacetime. When the Lorentzian structure and the connection are dynamical fields however there are complications: the spinor bundle $P[\Sigma]$ varies accordingly to the Lorentzian structure! If the Lorentzian structure, the spin connection and the spinor field are all gathered in a single field, this does not pose any real problem: variational equations still correspond to criticality of the action. It is however a problem when one wants to consider variations of the Lorentzian structure *distinct from the variations of the spinor field*.

One solution is given by using a principal connection over the correspondence space $\widetilde{\text{GL}}^+(M) \rightarrow \widetilde{\text{GL}}^+(M)/\text{Spin}_{1,3}^+$ (Section 4.2.3). It associates to a change in the metric a morphism between the corresponding spin structures, which induces a morphism between the spinor bundles. The identification between the spin structures is however dependent on a *path used to join the metrics*. In particular, computing the transformed spinor field under a change of metric requires solving a differential equation.

The solution used by the Sciama-Kibble theory is the following: Lorentzian structures are replaced by (co)frame fields, called *vierbein* or *tetrad*. We already introduced them in Section 10.2 but here we expound more on their role in handling spinors. There are two immediate consequences to using a tetrad field:

- There is an extra $\text{SO}_{1,3}^+$ gauge degree of freedom (when the action only depends on the induced Lorentzian structure).
- Two frames at one point are related by a unique linear transformation, which acts on the frames as a GL_4 -equivariant automorphism.

This solves part of the problem: two tetrads are related by a unique GL_4 -equivariant automorphism of $\text{GL}(M)$. However, the spin structures corresponding to the tetrads only get identified with an ambiguity of \mathbb{Z}_2 , which is the kernel of $\text{Spin}_{1,3}^+ \rightarrow \text{SO}_{1,3}^+$.

A standard global tetrad field can be understood as a global parallelism of M

$$\text{GL}(M) \xrightarrow{\sim} M \times \text{GL}(\mathbb{R}^n)$$

This suggests two solutions:

- It is possible to omit the prior spin structure on M and transport through the tetrad a trivial spin structure

$$M \times \text{Spin}_{1,3}^+ \rightarrow M \times \text{GL}(\mathbb{R}^n) \simeq \text{GL}(M)$$

- The tetrad may be supplanted by a “spinorial tetrad” which is not a trivialisation of $\text{GL}(M)$ but a trivialisation of $\widetilde{\text{GL}}^+(M)$ thus adding the needed bit of information to identify the corresponding structures.

It is the first option which is used in the Sciama-Kibble approach.

The set of global tetrad fields is non-empty if and only if the spacetime is parallelisable, which is a stronger condition than admitting spin structures. There is however a straightforward geometrical generalisation: one can consider an auxiliary oriented Lorentzian vector bundle $V \rightarrow M$ equipped with a lift of structure group $\text{Spin}^+(V) \rightarrow \text{SO}^+(V)$. A V -valued tetrad is then an isomorphism $e : TM \xrightarrow{\sim} V$. Each such tetrad induces a Lorentzian spin structure on M

and between two such tetrads there is a well defined morphism between the spin structures as according to the following commutative diagram:

$$\begin{array}{ccccc}
 & & \overset{\sim}{\curvearrowright} & & \\
 e^* \text{Spin}^+(V) & \xrightarrow{\sim} & \text{Spin}^+(V) & \xleftarrow{\sim} & (e')^* \text{Spin}^+(V) \\
 \downarrow & & \downarrow & & \downarrow \\
 e^* \text{SO}^+(V) & \xrightarrow{\sim} & \text{SO}^+(V) & \xleftarrow{\sim} & (e')^* \text{SO}^+(V) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{GL}(M) & \xrightarrow[\sim]{e} & \text{GL}(V) & \xleftarrow[\sim]{e} & \text{GL}(M)
 \end{array}$$

As a consequence, a section of $\text{Spin}^+(V) \times_{\text{Spin}_{1,3}^+} \Sigma$ can be naturally interpreted as a spinor field over M for both Spin structures $e^* \text{Spin}^+(V)$ and $(e')^* \text{Spin}^+(V)$.

Recall that not only the tetrad, but also the metric connection is to be a dynamical field. They can be gathered into an $(\text{SO}_{1,3}^+ \times \mathbb{R}^{1,3}, \text{SO}_{1,3}^+)$ -Cartan connection over $\text{SO}^+(V)$, or equivalently a $(\text{Spin}_{1,3}^+ \times \mathbb{R}^{1,3}, \text{Spin}_{1,3}^+)$ -Cartan connection over $\text{Spin}^+(V)$. This remark will play an important role later.

Using an auxiliary vector bundle V requires adapting the expressions which rely on the soldering of the tangent bundle. There are two approaches: either transporting everything on M using the tetrad field or expressing as much as possible over V and associated bundles to $\text{Spin}^+(V)$, using the tetrad field only when unavoidable. The latter approach, which we followed in Section 10.2, makes clearer the respective dependencies with respect to the tetrad and the other fields. For example, using indices μ, ν for TM and a, b for V , given a tetrad (e_a^μ) , a section ψ of $\text{Spin}^+(V)[\Sigma]$ and a linear connection ∇ on V , the Dirac operator acting on ψ takes the following form:

$$\not{D}\psi = \gamma^a e_a^\mu \nabla_\mu \psi$$

with $\gamma^a : V^* \rightarrow \text{End}(\text{Spin}^+(V)[\Sigma])$. If we want to express it over $e^* \text{Spin}^+(V) \rightarrow M$ then we need to use the image spinor field $\tilde{\psi} = e^* \psi$ and the image connection $\tilde{\nabla} = e^* \nabla$:

$$\tilde{\not{D}}\tilde{\psi} = \gamma^\mu \tilde{\nabla}_\mu \tilde{\psi}$$

with $\gamma^\mu : T^*M \rightarrow \text{End}(e^* \text{Spin}^+(V)[\Sigma])$. We will opt for formulations similar to the first type.

Using the same language, it is possible to reformulate the Dirac Lagrangian form. We use the conventions from Section 1.4. The volume form takes the form $e^{(4)}$ and the Lagrangian can be expressed as follows:

$$\mathcal{L}[\psi, e, \nabla] = \frac{1}{2} (\bar{\psi} \gamma^a \nabla \psi - \nabla \bar{\psi} \gamma^a \psi) \wedge e_a^{(3)} - m \bar{\psi} \psi e^{(4)} \quad (11.3)$$

In Sciama-Kibble theory, this is added to the Einstein-Cartan Lagrangian formulated in terms of the tetrad, namely the Lagrangian from Expression 10.1, in order to have a theory with dynamical spacetime along with the dynamical spinor fields.

The role of torsion The Sciama-Kibble theory fits into the framework of Einstein-Cartan gravitation. It departs from Einstein's General Relativity by allowing for torsion in the connection over spacetime. This has many geometrical implications some of which are presented in Section 5.

From the physical perspective, the two main implications are:

- The Ricci tensor may have an antisymmetrical part, which is coupled to antisymmetrical terms in the energy-momentum tensor.
- There is a variational equation corresponding to variations of the torsion, which couples the torsion to a *spin density tensor* [Tra06].

Replacing the Levi-Civita connection with a metric connection presenting torsion implies a “geometrical” coupling between the torsion and the fields on which the covariant derivative is made to act.

The impact of torsion on the field equations for our model for Dirac spinors in a dynamical spacetime is briefly analysed in Section 11.4.3.

11.2 The Dirac Lagrangian in the Hélein-Vey formalism

11.2.1 Building the spinor Lagrangian

Lift to the spin frame bundle

On the spin frame bundle, spinors are represented by $\text{Spin}_{1,3}^+$ -equivariant Σ -valued fields so that the configuration space of the spinor field is $\Sigma \times \text{Spin}^+(\mathcal{E})$. We will use $s = (s^\alpha)$ as coordinates on the factor Σ and (\bar{s}_α) for dual coordinates; the latter can be read as the spinor metric $\bar{s} : \Sigma \rightarrow \Sigma^*$.

We write $\gamma_a \in \text{End}(\Sigma)$ for the action of vectors $e_a \in \mathbb{R}^{1,3}$, use $\sigma_i \in \text{End}(\Sigma)$ for the action of $\xi_i \in \mathfrak{l} \simeq \mathfrak{spin}_{1,3}$ (resp. $\bar{\sigma}_i \in \text{End}(\Sigma^*)$) and $\psi : \text{Spin}^+(\mathcal{E}) \rightarrow \Sigma$. More detail on the notation and the action of the Spin group on Clifford modules will be found in Section 4.1. Given a system (z^J) of local coordinates on $\text{Spin}^+(\mathcal{E})$, we call ζ_I^α the associated coordinates on the 1-jet bundle $\mathcal{J}^1(\text{Spin}^+(\mathcal{E}), \Sigma)$:

$$\zeta_I^\alpha \circ \psi = \partial_I \psi^\alpha$$

The covariant derivative of the spinor field formulated on the frame bundle takes the form

$$d^\omega \psi = d\psi + \omega^i \sigma_i \cdot \psi$$

Therefore the Dirac Lagrangian pulled back to $\mathcal{J}^1(\text{Spin}^+(\mathcal{E}, \Sigma))$ can be expressed as:

$$\frac{1}{2} (\bar{\psi} \gamma^a d^\omega \psi - (d^\omega \bar{\psi}) \gamma^a \psi) \wedge \lambda_a^{\mathfrak{m}(3)} - m \bar{\psi} \psi \lambda^{\mathfrak{m}}$$

where $\lambda^{\mathfrak{m}}$ (resp. $\lambda^{\mathfrak{l}}$) denotes the pullbacks of volume elements of \mathfrak{m} (resp. \mathfrak{l}) by the canonical form λ and $\lambda_a^{\mathfrak{m}(3)}$ is the 3-form dual to λ^a in \mathfrak{m} .

Adding a factor $\lambda^{\mathfrak{l}}$ in order to obtain a volume form, the pulled back Dirac Lagrangian corresponds to $j\phi^* \mathcal{L}_{Dirac}$ with:

$$\mathcal{L}_{Dirac} = \left(\frac{1}{2} (\bar{s} \gamma^a (\zeta_J dz^J + \lambda^i \sigma_i s) - (\bar{\zeta}_J dz^J + \lambda^i \bar{\sigma}_i \bar{s}) \gamma^a s) \wedge \lambda_a^{\mathfrak{m}(3)} - m \bar{s} s \lambda^{\mathfrak{m}} \right) \wedge \lambda^{\mathfrak{l}} \quad (11.4)$$

Note that in this expression the contribution of the term $\lambda^i \sigma_i$ (resp. $\lambda^i \bar{\sigma}_i$) actually vanishes due to the $\lambda_a^{\mathfrak{m}(3)} \wedge \lambda^{\mathfrak{l}} = \lambda_a^{(9)}$ factor which already selects the horizontal directions in $\zeta_{\alpha, J} dz^J$ (resp. $\bar{\zeta}_J^\alpha dz^J$).

Omitting the frame bundle structure and Lagrange multipliers

As in the previous section, we consider as source space for the fields a differentiable 10-manifold \mathcal{P} . We will consider a total Lagrangian composed of \mathcal{L}_{Dirac} , \mathcal{L}_{EC} and Legendre multipliers terms in order to make the $\text{Spin}_{1,3}^+$ structure emerge dynamically on the space \mathcal{P} . As mentioned earlier, the structure obtained from Equations (10.9) on ϖ induces an action of the Lie algebra $\mathfrak{so}_{1,3}$, which is naturally isomorphic to $\mathfrak{spin}_{1,3}$.

In particular, it is enough to define the equivariance of the spinor fields under $\mathfrak{spin}_{1,3}$. Indeed, the difference between the usual linear frame bundle and spinor frame bundles only appears at the “global” level, when there are complete orbits under the group action. Thus we do not have to adapt the notion of generalised frame bundle in order to accommodate for spin structures.

A spinor field $\psi : \mathcal{P} \rightarrow \Sigma$ will have to satisfy the equivariance condition

$$\forall \xi \in \mathfrak{l}, \mathcal{L}_{\xi}\psi + \xi \cdot \psi = 0 \quad (11.5)$$

with \mathfrak{l} acting via $\sigma : \mathfrak{l} \rightarrow \text{End}(\Sigma)$, we write $\xi_i \cdot \psi = \sigma_i \psi$. We will formulate the equivariance in a similar way to previously: recall the notation

$$d^\lambda s := ds + \lambda^i \sigma_i s \quad (11.6a)$$

$$d^\lambda \bar{s} := d\bar{s} + \lambda^i \bar{\sigma}_i \bar{s} \quad (11.6b)$$

with the operators σ_i being anti-selfadjoint. This notation allows us to write the condition (11.5) (writing separately \mathbb{C} -linear and \mathbb{C} -antilinear directions although they correspond to the same degree of freedom)

$$\begin{cases} \psi^* \left(d^\lambda s \wedge \lambda_i^{(9)} \right) = 0 \\ \psi^* \left(d^\lambda \bar{s} \wedge \lambda_i^{(9)} \right) = 0 \end{cases} \quad (11.7a)$$

$$\quad (11.7b)$$

We consider the following Lagrange multiplier term (using a similar notation $d^\lambda \bar{s}$)

$$\frac{i}{2} \left(\bar{\kappa}_\alpha^i d^\lambda s^\alpha - \kappa^{\alpha i} d^\lambda \bar{s}_\alpha \right) \wedge \lambda_i^{(9)}$$

with $\bar{\kappa}^i$ conjugate to κ^i (so that the constraint term is real), which added to \mathcal{L}_{Dirac} makes up the following Lagrangian

$$\begin{aligned} \overline{\mathcal{L}_{Dirac}} = & \left(\frac{1}{2} (\bar{s} \gamma^a (\zeta_J dz^J + \lambda^i \sigma_i s) - (\bar{\zeta}_J dz^J + \lambda^i \bar{\sigma}_i \bar{s}) \gamma^a s) \wedge \lambda_a^{m(3)} - m \bar{s} s \lambda^m \right) \wedge \lambda^l \\ & + \frac{i}{2} (\bar{\kappa}^i (\zeta_J dz^J + \lambda^j \sigma_j s) - \kappa^i (\zeta_J dz^J + \lambda^j \bar{\sigma}_j \bar{s})) \wedge \lambda_i^{(9)} \quad (11.8) \end{aligned}$$

defined over

$$\left(\underbrace{\Sigma}_{s^\alpha} \oplus \underbrace{\Sigma \otimes \mathfrak{l}}_{\kappa^{\alpha i}} \right) \times \underbrace{\text{Iso}(T\mathcal{P}, \mathfrak{p})}_{\lambda_I^A}$$

11.2.2 The Poincaré-Cartan form

We now compute the Poincaré-Cartan form. The Lagrangian (11.8) being affine in the 1st-order jets, the Legendre transformation is straightforward. The image of the Legendre transform,

namely the *momentum space*, is a subspace of

$$\begin{aligned} \Lambda_1^{10}T^* [(\Sigma \oplus \Sigma \otimes \mathfrak{l}) \times \text{Iso}(T\mathcal{P}, \mathfrak{p})] \\ \simeq (\Sigma \oplus \Sigma \otimes \mathfrak{l}) \times [(\Sigma^* \oplus \Sigma^* \otimes \mathfrak{l}^*) \otimes \Lambda^9 T^* \mathcal{P} \oplus_{\mathcal{P}} \Lambda^{10} T^* \mathcal{P}] \times_{\mathcal{P}} \Lambda_1^{10} T^* \text{Iso}(T\mathcal{P}, \mathfrak{p}) \end{aligned}$$

We use the Legendre transformation formula (9.8) (see Section 9.4):

$$\begin{aligned} \overline{\Theta}_{Dirac} &= \overline{\mathcal{L}_{Dirac}} + \frac{\partial \overline{\mathcal{L}_{Dirac}}}{\partial v_{I,J}^A} (d\lambda_I^A - v_{I,L}^A dz^L) \wedge dz_J^{(9)} \\ &\quad + \frac{\partial \overline{\mathcal{L}_{Dirac}}}{\partial \zeta_J^\alpha} (ds^\alpha - \zeta_L^\alpha dz^L) \wedge dz_J^{(9)} + \frac{\partial \overline{\mathcal{L}_{Dirac}}}{\partial \zeta_{\alpha,J}} (d\bar{s}_\alpha - \zeta_{\alpha,L} dz^L) \wedge dz_J^{(9)} \\ &= \frac{1}{2} \left(\bar{s} \gamma^a (d^\lambda s) - (d^\lambda \bar{s}) \gamma^a s \right) \wedge \lambda_a^{(9)} + i \frac{1}{2} \left(\bar{\kappa}^i (d^\lambda s) - (d^\lambda \bar{s}) \kappa^i \right) \wedge \lambda_i^{(9)} - m \bar{s} s \lambda^{(10)} \quad (11.9) \end{aligned}$$

The momentum dual to κ vanishes as κ only appears in the Lagrangian at order 0 so that the momenta have trivial component in $(\mathfrak{l}^* \otimes \Sigma) \otimes \Lambda^9 T^* \mathcal{P}$.

We are interested in a model coupling the Dirac spinor with the Einstein-Cartan gravitational fields. We thus consider a Lagrangian which is the sum of the two Lagrangians and is defined over

$$Q = \text{Iso}(T\mathcal{P}, \mathfrak{p}) \times \Sigma \times [(\mathfrak{m} \wedge \mathfrak{l} \oplus \mathfrak{l} \wedge \mathfrak{l}) \otimes \mathfrak{p}^* \oplus (\Sigma \otimes \mathfrak{l})]$$

the whole Poincaré-Cartan form decomposes as follows

$$\bar{\Theta} = \Theta_{EC} + \Theta_{Dirac} + \Theta_{EC}^{cons} + \Theta_{Dirac}^{cons} = \overline{\Theta}_{EC} + \overline{\Theta}_{Dirac} \quad (11.10)$$

with

$$\overline{\Theta}_{EC} = \frac{1}{2} p_A^{BC} \Lambda^A \lambda_{BC}^{(8)} \quad (11.11)$$

$$\overline{\Theta}_{Dirac} = \frac{1}{2} \left(\bar{s} \gamma^a (d^\lambda s) - (d^\lambda \bar{s}) \gamma^a s \right) \lambda_a^{(9)} + i \frac{1}{2} \left(\bar{\kappa}^i (d^\lambda s) - (d^\lambda \bar{s}) \kappa^i \right) \lambda_i^{(9)} - m \bar{s} s \lambda^{(10)} \quad (11.12)$$

with the line over $\bar{\Theta}$ denoting the inclusion of the Lagrange multiplier terms.

Here as well, the Poincaré-Cartan form is defined on the configuration space to which we added the Lagrange multipliers (the pullback by a section only depends on the 0-order jet). In accordance with the results from Section 10.3.3 the momenta dual to λ_I^A are restricted to the subspace $\Lambda^8 T^* P \otimes \mathfrak{p}^* \times_{\mathcal{P}} \text{Iso}(T\mathcal{P}, \mathfrak{p})$ of $\Lambda_1^{10} T^* \mathcal{P} \otimes \mathfrak{p}$. For convenience, we will work with complex spinor indices and we therefore enlarge the component $\Sigma \otimes \Lambda^9 T^* \mathcal{P}$ to a factor $(\Sigma \oplus \Sigma^*) \otimes \Lambda^9 T^* \mathcal{P}$. We define fibre coordinates on the momentum space using the components of the canonical 10-form:

$$j_A^{BC} \Lambda^A \wedge \lambda_{BC}^{(8)} + \bar{\Phi}_\alpha^A d^\lambda s^\alpha \wedge \lambda_A^{(9)} + \Phi^{\alpha A} d^\lambda \bar{s}_\alpha \wedge \lambda_A^{(9)} - h \lambda^{(10)}$$

Each coordinate corresponds to a factor of the momentum space, as follows:

$$\underbrace{(\text{Iso}(T\mathcal{P}, \mathfrak{p}))}_{\lambda_I^A} \times \underbrace{\Sigma}_{s^\alpha} \oplus \underbrace{\Sigma \otimes \mathfrak{l}}_{\kappa^{\alpha j}} \times_{\mathcal{P}} \underbrace{\Lambda^8 T^* P \otimes \mathfrak{p}^*}_{j_A^{BC}} \times_{\mathcal{P}} \underbrace{[(\Sigma \oplus \Sigma^*) \otimes \Lambda^9 T^* \mathcal{P}]}_{\Phi^{\alpha A}, \bar{\Phi}_\alpha^A} \oplus_{\mathcal{P}} \underbrace{\Lambda^{10} T^* \mathcal{P}}_h$$

The image of the Legendre transform is a subspace defined by (holonomic) constraints, which take the following form

$j_a^{bc} = 0$ (11.13a)	$j_i^{bc} = 2\rho_{id}^b \eta^{dc}$ (11.13b)
$j_A^{bj} = p_A^{bj}$ (11.13c)	$j_A^{ij} = p_A^{ij}$ (11.13d)
$\Phi^{\alpha a} = -\frac{1}{2}(\gamma^a s)^\alpha$ (11.13e)	$\Phi^{\alpha j} = -\frac{i}{2}\kappa^{\alpha j}$ (11.13f)
$\bar{\Phi}_\alpha^a = -\frac{1}{2}(\bar{\gamma}^a \bar{s})_\alpha$ (11.13g)	$\bar{\Phi}_\alpha^j = \frac{i}{2}\bar{\kappa}_\alpha^j$ (11.13h)
$h = m\bar{s}_\alpha s^\alpha$ (11.13i)	

The momentum space can hence be identified as

$$\left[\underbrace{\text{Iso}(T\mathcal{P}, \mathfrak{p})}_{\lambda_i^A} \times \underbrace{\Sigma}_{s^\alpha} \right] \times \left[\underbrace{(\mathfrak{m} \wedge \mathfrak{l} \oplus \mathfrak{l} \wedge \mathfrak{l}) \otimes \mathfrak{p}^*}_{p_A^{BC}} \oplus \underbrace{(\Sigma \otimes \mathfrak{l})}_{\kappa^{\alpha i}} \right] = Q \quad (11.14)$$

We compute the variational equations in the next section.

11.3 Variational equations for a spinor on a generalised frame bundle

We already have the Euler-Lagrange forms corresponding to $\overline{\Theta_{EC}}$, we have to compute the term corresponding to $\overline{\Theta_{Dirac}}$. Recall its expression

$$\overline{\Theta_{Dirac}} = \frac{1}{2} \left(\bar{s} \gamma^a (d^\lambda s) - (d^\lambda \bar{s}) \gamma^a s \right) \wedge \lambda_a^{(9)} + \frac{i}{2} \left(\bar{\kappa}^i (d^\lambda s) - (d^\lambda \bar{s}) \kappa^i \right) \wedge \lambda_i^{(9)} - m \bar{s} s \lambda^{(10)} \quad (11.15)$$

We will make use of the following identities, proven in Section 7.4:

$$\begin{aligned} d^\lambda (d^\lambda s)^\alpha &= (\Lambda \cdot s)^\alpha \\ d^\lambda (d^\lambda \bar{s})_\alpha &= (\Lambda \cdot \bar{s})_\alpha \end{aligned} \quad (10.17)$$

Let ϕ be a section of the phase space. We denote its different components as follows:

$$\left[\underbrace{T^* \mathcal{P} \otimes \mathfrak{p}}_{\varpi_i^A} \times \underbrace{\Sigma}_{\psi^\alpha} \right] \times_{\mathcal{P}} \left[\underbrace{(\Lambda^8 T^* \mathcal{P} \otimes \mathfrak{l})}_{P_A^{BC}} \oplus \underbrace{(\Lambda^9 T^* \mathcal{P} \otimes \Sigma \otimes \mathfrak{l})}_{K^{\alpha i}} \right] \quad (11.16)$$

11.3.1 Variation of the multipliers $K^{\alpha i}$

We check here that the variation of the Lagrange multipliers K_α^i yields the expected constraint equations. There are independent variations in holomorphic directions corresponding to the index α , and in the anti-holomorphic direction corresponding to $\bar{K}^{\alpha i}$.

$$\begin{aligned}
& (\text{EL}_{Dirac})_{\alpha i} \\
&= \partial_{\kappa^{\alpha i}} \lrcorner d \left(\frac{1}{2} \left(\bar{s}_\beta \gamma^a (d^\lambda s)^\beta - (d^\lambda \bar{s})_\beta \gamma^a s^\beta \right) \wedge \lambda_a^{(9)} \right. \\
&\quad \left. + \frac{i}{2} \left(\bar{\kappa}_\beta^j (d^\lambda s)^\beta - (d^\lambda \bar{s})_\beta \kappa^{\beta j} \right) \wedge \lambda_j^{(9)} - m \bar{s} s \lambda^{(10)} \right) \\
&= -\frac{i}{2} d^\lambda \bar{s}_\alpha \wedge \lambda_i^{(9)}
\end{aligned} \tag{11.17}$$

Since $\overline{\Theta_{Dirac}}$ is real the two Euler-Lagrange terms are conjugate under the antilinear correspondence $\Sigma \rightarrow \Sigma^*$:

$$(\text{EL}_{Dirac})_i^\alpha = \frac{i}{2} d^\lambda s^\alpha \wedge \lambda_i^{(9)} \tag{11.18}$$

From now on we will only use the Euler-Lagrange terms coupled to anti-holomorphic $\partial_{\kappa^{\alpha i}}$ variations. The corresponding Euler-Lagrange equations on a field ϕ are thus:

$$\phi^* \left(d^\lambda s^\alpha \wedge \lambda_i^{(9)} \right) = d^\omega \psi^\alpha \wedge \varpi_i^{(9)} = 0 \tag{11.19}$$

Therefore, the vanishing of the pullback of the Euler-Lagrange form $\phi^* \text{EL}_i^\alpha$ is equivalent to requiring the equivariance of the corresponding section ψ of Σ with respect to the action of \mathbb{I} defined by ϖ . Namely, in the case there is a spin frame bundle structure derived from the variational equations derived in Section 10.3.3, it means that ψ is associated to a section of the associated spinor bundle.

11.3.2 Variation of the spinor field and of the coframe

We start with the variational equations with respect to the variation on the spinor fields. We have to compute $d\overline{\Theta_{Dirac}}$. For the sake of clarity we will compute

$$d \left(\bar{s} \gamma^a (d^\lambda s) \wedge \lambda_a^{(9)} + i \bar{\kappa}^i (d^\lambda s) \wedge \lambda_i^{(9)} - m \bar{s} s \lambda^{(10)} \right)$$

and obtain $d\overline{\Theta_{Dirac}}$ as the real part. Let us start with the term dependent of κ^i :

$$\begin{aligned}
d \left(\bar{\kappa}^i (d^\lambda s) \wedge \lambda_i^{(9)} \right) &= \left(d^\lambda \bar{\kappa}^i \wedge (d^\lambda s) + \bar{\kappa}^i d^\lambda (d^\lambda s) \right) \wedge \lambda_i^{(9)} - \bar{\kappa}^i (d^\lambda s) \wedge d^\lambda \lambda_i^{(9)} \\
&= \left(d^\lambda \bar{\kappa}^i \wedge (d^\lambda s) + \bar{\kappa}^i \Lambda \cdot s \right) \wedge \lambda_i^{(9)} - \bar{\kappa}^i (d^\lambda s) \wedge \Lambda^B \wedge \lambda_{iB}^{(8)} \\
&= d^\lambda \bar{\kappa}^i \wedge d^\lambda s \wedge \lambda_i^{(9)} + \bar{\kappa}^i (\sigma_j s) \wedge \Lambda^j \wedge \lambda_i^{(9)} - \bar{\kappa}^i (d^\lambda s) \wedge \Lambda^B \wedge \lambda_{iB}^{(8)}
\end{aligned}$$

An identical calculation replacing $\kappa^{\alpha i} \lambda_i^{(9)}$ with $\gamma^a s^\alpha \lambda_a^{(9)}$ gives

$$d \left(\bar{s} \gamma^a (d^\lambda s) \wedge \lambda_a^{(9)} \right) = \left(d^\lambda (\bar{s} \gamma^a) \wedge d^\lambda s + \bar{s} \gamma^a (\sigma_j s) \Lambda^j \right) \wedge \lambda_a^{(9)} - \bar{s} \gamma^a (d^\lambda s) \wedge \Lambda^B \wedge \lambda_{aB}^{(8)}$$

Recall that γ^a is parallel in the following sense:

$$(d^\lambda \gamma^a) \wedge \lambda_b^{(9)} = (d\gamma^a + \lambda^j [\sigma_j, \gamma^a]) \wedge \lambda_b^{(9)} = 0 + 0$$

We can then compute

$$\begin{aligned} d\left(\bar{s}\gamma^a(d^\lambda s) \wedge \lambda_a^{(9)}\right) &= \left((d^\lambda \bar{s})\gamma^a + \bar{s}d^\lambda \gamma^a\right) \wedge d^\lambda s \wedge \lambda_a^{(9)} \\ &\quad + \bar{s}\gamma^a(\sigma_j s)\Lambda^j \wedge \lambda_a^{(9)} - \bar{s}\gamma^a(d^\lambda s) \wedge \Lambda^B \wedge \lambda_{aB}^{(8)} \\ &= \left(d^\lambda \bar{s}\gamma^a \wedge d^\lambda s + \bar{s}\gamma^a \sigma_j s \Lambda^j\right) \wedge \lambda_a^{(9)} - \bar{s}\gamma^a(d^\lambda s) \wedge \Lambda^B \wedge \lambda_{aB}^{(8)} \end{aligned}$$

Last, corresponding to the mass term

$$\begin{aligned} d\left(m\bar{s}_\alpha s^\alpha \lambda^{(10)}\right) &= md(\bar{s}_\alpha s^\alpha) \wedge \lambda^{(10)} + m\bar{s}sd\lambda^{(10)} \\ &= m\left((d^\lambda \bar{s})_\alpha s^\alpha + \bar{s}_\alpha d^\lambda s^\alpha\right) \wedge \lambda^{(10)} + m\bar{s}s\Lambda^B \wedge \lambda_B^{(9)} \end{aligned} \quad (11.20)$$

Recall that both γ^a and σ_i are anti-selfadjoint:

$$\begin{aligned} \overline{\gamma^a s s} &= -\bar{s}\gamma^a s \\ \overline{\bar{s}\sigma_i s} &= -\bar{s}\sigma_i s \end{aligned}$$

We obtain the total exterior differential by taking the real part of the sum of the three terms. We use curly braces $\{\cdot, \cdot\}$ for anticommutators :

$$\begin{aligned} \overline{d\Theta_{Dirac}} &= \frac{1}{2} \left(\left(2d^\lambda \bar{s} \wedge \gamma^a d^\lambda s + \bar{s}\{\sigma_j, \gamma^a\} s \Lambda^j \right) \wedge \lambda_a^{(9)} - \left(\bar{s}\gamma^a(d^\lambda s) - (d^\lambda \bar{s})\gamma^a s \right) \wedge \Lambda^B \wedge \lambda_{aB}^{(8)} \right) \\ &\quad + \frac{i}{2} \left[\left(d^\lambda \bar{\kappa}^i \wedge d^\lambda s + (d^\lambda \bar{s}) \wedge d^\lambda \kappa^i \right) \wedge \lambda_i^{(9)} + (\bar{\kappa}^i(\sigma_j s) - (\bar{s}\sigma_j)\kappa^i) \Lambda^j \wedge \lambda_i^{(9)} \right. \\ &\quad \left. - \left(\bar{\kappa}^i(d^\lambda s) - (d^\lambda \bar{s})\kappa^i \right) \wedge \Lambda^B \wedge \lambda_{iB}^{(8)} \right] - m \left((d^\lambda \bar{s})s + \bar{s}d^\lambda s \right) \wedge \lambda^{(10)} - m\bar{s}s\Lambda^B \wedge \lambda_B^{(9)} \end{aligned} \quad (11.21)$$

Here too we adopt the notation

$$EL^\alpha := EL_{\partial_{\bar{s}_\alpha}} \quad (11.22)$$

Start with the Euler-Lagrange terms corresponding to variations of the spinor field:

$$\begin{aligned} EL_{Dirac}^\alpha &= \partial_{\bar{s}_\alpha} \lrcorner \overline{d\Theta_{Dirac}} \\ &= \frac{1}{2} \left(2(\gamma^a(d^\lambda s))^\alpha \wedge \lambda_a^{(9)} + (\gamma^a s)^\alpha \Lambda^B \wedge \lambda_{aB}^{(8)} \right) \\ &\quad + \frac{i}{2} \left(d^\lambda \kappa^{\alpha i} \wedge \lambda_i^{(9)} + \kappa^{\alpha i} \Lambda^B \wedge \lambda_{iB}^{(8)} \right) - ms^\alpha \lambda^{(10)} \end{aligned}$$

We introduce the notation \equiv for equality which holds up to “constraint” terms (10.27, 11.17). This is justified by the fact that the analysis in Section 11.4 will proceed by first assuming these constraint equations satisfied.

$$EL_{Dirac}^\alpha \equiv \frac{1}{2} \left(2(\gamma^a d^\lambda s)^\alpha \wedge \lambda_a^{(9)} + (\gamma^a s)^\alpha \Lambda^c \wedge \lambda_{ac}^{(8)} + id^\lambda \left(\kappa^i \lambda_i^{(9)} \right)^\alpha \right) - ms^\alpha \lambda^{(10)} \quad (11.23)$$

We now compute the Euler-Lagrange terms corresponding to variations of the coframe ϖ , which will govern the interaction of the spinors with the spacetime geometry. Recall the notation

$$X = \epsilon_B^A \partial_{\lambda_B^A}$$

and

$$\epsilon^A = \epsilon_B^A \lambda^B$$

We obtain

$$\begin{aligned} (\text{EL}_{Dirac})_X &= X \lrcorner \overline{d\Theta_{Dirac}} \\ &= \frac{1}{2} \left(\bar{s} \{ \sigma_j, \gamma^a \} s \epsilon^j \wedge \lambda_a^{(9)} + \left(\bar{s} \gamma^a (d^\lambda s) - (d^\lambda \bar{s}) \gamma^a s \right) \wedge \epsilon^B \wedge \lambda_{aB}^{(8)} \right) \\ &\quad + \frac{i}{2} \left((\bar{\kappa}^i (\sigma_j s) - (\bar{s} \bar{\sigma}_j) \kappa^i) \epsilon^j \wedge \lambda_i^{(9)} + \left(\bar{\kappa}^i (d^\lambda s) - (d^\lambda \bar{s}) \kappa^i \right) \wedge \epsilon^B \wedge \lambda_{iB}^{(8)} \right) - m \bar{s} s \epsilon^B \wedge \lambda_B^{(9)} \\ &= -\epsilon^B \wedge \left(\frac{1}{2} \left(\bar{s} \gamma^a (d^\lambda s) - (d^\lambda \bar{s}) \gamma^a s \right) \wedge \lambda_{aB}^{(8)} + \frac{i}{2} \left(\bar{\kappa}^i (d^\lambda s) - (d^\lambda \bar{s}) \kappa^i \right) \wedge \lambda_{iB}^{(8)} + m \bar{s} s \lambda_B^{(9)} \right) \\ &\quad + \epsilon^j \wedge \frac{1}{2} \left(\bar{s} \{ \sigma_j, \gamma^a \} s \lambda_a^{(9)} + i (\bar{\kappa}^i (\sigma_j s) - (\bar{s} \bar{\sigma}_j) \kappa^i) \lambda_i^{(9)} \right) \end{aligned} \quad (11.24)$$

11.3.3 The total Euler-Lagrange terms

Gathering the expressions (10.27,11.17,10.30,11.24), the total Euler-Lagrange terms corresponding to the Poincaré-Cartan form (11.10) are then, using again a vertical vector field $X = \epsilon_B^A \partial_{\lambda_B^A}$,

$$(\text{EL})_{\widehat{BC}}^A = \frac{1}{2} \Lambda^A \wedge \lambda_{\widehat{BC}}^{(8)} \quad (11.25a)$$

$$(\text{EL})_i^\alpha = \frac{i}{2} d^\lambda s^\alpha \wedge \lambda_i^{(9)} \quad (11.25b)$$

$$\begin{aligned} \text{EL}_X &= \epsilon^A \wedge \left[\Lambda^D \wedge \frac{1}{2} p_D^{BC} \lambda_{BCA}^{(7)} + d^\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\bar{s} \gamma^b (d^\lambda s) - (d^\lambda \bar{s}) \gamma^b s \right) \wedge \lambda_{bA}^{(8)} - \frac{i}{2} \left(\bar{\kappa}^j (d^\lambda s) - (d^\lambda \bar{s}) \kappa^j \right) \wedge \lambda_{jA}^{(8)} - m \bar{s} s \lambda_A^{(9)} \right] \\ &\quad + \epsilon^j \wedge \frac{1}{2} \left(\bar{s} \{ \sigma_j, \gamma^a \} s \lambda_a^{(9)} + i (\bar{\kappa}^i (\sigma_j s) - (\bar{s} \bar{\sigma}_j) \kappa^i) \lambda_i^{(9)} \right) \end{aligned} \quad (11.25c)$$

$$\text{EL}^\alpha = (\gamma^a d^\lambda s)^\alpha \wedge \lambda_a^{(9)} + \frac{1}{2} (\gamma^a s)^\alpha \Lambda^c \wedge \lambda_{ac}^{(8)} - m s^\alpha \lambda^{(10)} + \frac{i}{2} d^\lambda (\kappa^i \lambda_i^{(9)})^\alpha \quad (11.25d)$$

The term EL_X can be decomposed according to the different components of ϵ : ϵ_b^a , ϵ_b^i , ϵ_j^a , ϵ_j^i . Each one corresponds to a variation of a different part of the structure of generalised frame bundle:

- ϵ_b^a corresponds to variations of the tetrad
- ϵ_b^i corresponds to variations of the connection
- ϵ_j^a corresponds to variations of the orbits
- ϵ_j^i corresponds to variations of the action of \mathfrak{l} .

Unfortunately, the very presence of the Lagrange multipliers $p_A^{\widehat{BC}}$ and $(\kappa^{\alpha i}, \bar{\kappa}_\alpha^i)$ makes the corresponding differential equations $\phi^* \text{EL} = 0$ hard to study beyond the geometric structure of a Cartan 1-form and a section of the associated spinor bundle. Since the Lagrange multipliers

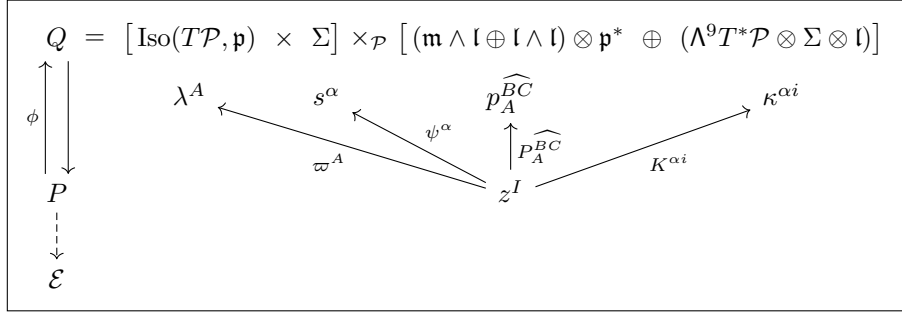
need not be equivariant, they have to be studied on the total bundle space. Note however the dependency on the multipliers: they appear in d^λ -exact terms

$$d^\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) \\ d^\lambda \left(\kappa^i \lambda_i^{(9)} \right)^\alpha$$

and in the following three terms:

$$\epsilon^A \wedge \Lambda^D \wedge \frac{1}{2} p_D^{\widehat{BC}} \lambda_{\widehat{BC}A}^{(7)} \\ \epsilon^A \wedge \left(\bar{\kappa}^j (d^\lambda s) - (d^\lambda \bar{s}) \kappa^j \right) \wedge \lambda_{jA}^{(8)} \\ \epsilon^j \wedge \left(\bar{\kappa}^i (\sigma_j s) - (\bar{s} \sigma_j) \kappa^i \right) \lambda_i^{(9)}$$

If we assume that (11.25a,11.25b) vanish, these three terms are shown to be dependent only on the vertical component $\epsilon_k^A \lambda^k$. This will be useful in the treatment of the Euler-Lagrange equations in Section 11.4. We close this section with a diagram describing the different coordinates introduced and the corresponding fields, with the hypothetical underlying spacetime \mathcal{E} .



11.4 Derivation of the Einstein-Cartan-Dirac equations on spacetime in Riemannian signature

So far we have derived the following Euler-Lagrange terms:

$$(\text{EL})_{\widehat{BC}}^A = \frac{1}{2} \Lambda^A \wedge \lambda_{\widehat{BC}}^{(8)} \quad (11.25a)$$

$$(\text{EL})_i^\alpha = \frac{i}{2} d^\lambda s^\alpha \wedge \lambda_i^{(9)} \quad (11.25b)$$

which constrain the coframe ϖ to define a generalised frame bundle structure and the field ψ to define a basic spinor field, as well as the following Euler-Lagrange terms:

$$\begin{aligned} \text{EL}_X = \epsilon^A \wedge \left[\Lambda^D \wedge \frac{1}{2} p_D^{BC} \lambda_{BCA}^{(7)} + d^\lambda \left(\frac{1}{2} p_A^{BC} \lambda_{BC}^{(8)} \right) \right. \\ \left. - \frac{1}{2} \left(\bar{s} \gamma^b (d^\lambda s) - (d^\lambda \bar{s}) \gamma^b s \right) \wedge \lambda_{bA}^{(8)} - \frac{i}{2} \left(\bar{\kappa}^j (d^\lambda s) - (d^\lambda \bar{s}) \kappa^j \right) \wedge \lambda_{jA}^{(8)} - m \bar{s} s \lambda_A^{(9)} \right] \\ + \epsilon^j \wedge \frac{1}{2} \left(\bar{s} \{ \sigma_j, \gamma^a \} s \lambda_a^{(9)} + i \left(\bar{\kappa}^i (\sigma_j s) - (\bar{s} \sigma_j) \kappa^i \right) \lambda_i^{(9)} \right) \end{aligned} \quad (11.25c)$$

$$\text{EL}^\alpha = (\gamma^a d^\lambda s)^\alpha \wedge \lambda_a^{(9)} + \frac{1}{2} (\gamma^a s)^\alpha \Lambda^c \wedge \lambda_{ac}^{(8)} - m s^\alpha \lambda^{(10)} + \frac{i}{2} d^\lambda \left(\kappa^i \lambda_i^{(9)} \right)^\alpha \quad (11.25d)$$

relating geometric quantities depending on ϖ , matter quantities depending on ψ and the Lagrange multipliers p_i^{bc} and $\kappa^{\alpha i}$.

We assume in this section that the generalised frame bundle structure is an actual frame bundle structure, that is

1. The Lie algebra action on \mathcal{P} integrates into a Lie group action
2. The orbit space \mathcal{E} has a Hausdorff quotient manifold structure
3. The fibration $\mathcal{P} \rightarrow \mathcal{E}$ forms a principal bundle over the orthogonal group or over the spin group.

We will further need to modify our problem by replacing the Lorentzian signature with a *Riemannian signature*. In this case, we show that the Euler-Lagrange equations on \mathcal{P} imply field equations on \mathcal{E} which match with the usual Einstein-Cartan-Dirac field equations.

The generalised frame bundle structure cannot always be identified with an actual frame bundle. In order to make it possible, two global properties are required, according to the results of Chapter 8. Let us briefly recall them.

First, the Lie algebra action has to integrate to a group action. If the Lie algebra acts by non-complete vector fields, the manifold needs to be completed. The problem cannot always be circumvented by restricting to “maximal solutions”. The necessary condition is *univalence* which means that the action of an element of the group does not depend on the path leading from identity to the element used to construct it (so that orbital mappings $x \mapsto g \cdot x$ can be unambiguously defined, although partially). The Lie algebra action is then “globalisable”, which means the manifold can be embedded into a larger (possibly non-Hausdorff) manifold on which the Lie group acts, with an embedding equivariant under the infinitesimal action. On a compact manifold the vector fields are complete and the action readily integrates into a group action of the simply-connected Lie group integration of the Lie algebra.

The second property is *properness* of the group action which ensures sufficient separation of the orbit space. In particular, it holds for all compact Lie group actions.

When these two properties are satisfied, there is a dense open subset of \mathcal{P} which can be identified with an orthonormal frame bundle, or a spin frame bundle, over its orbit space. In particular, all orbits on this open subset are either isomorphic to the Spin group or to the SO^+ group.

The derivation of the spacetime equations will proceed as follows. First we assume that the “constraint” Equations (11.25a,11.25b) are satisfied and that they define a frame bundle structure with a spinor field on the underlying spacetime. Next, we identify variations of the field which preserve the constraints, so that by the mechanism introduced in Section 9.9 we obtain Euler-Lagrange equations with all Lagrange multiplier contributions gathered in an exact

term. This is to be compared with the approach in [HV16] in which they perform an explicit change of coordinates depending on the choice of a local section on the frame bundle and after some algebraic manipulations manage to identify exact divergence terms in the Euler-Lagrange equations. In particular, the use of explicit coordinate systems make the computations quite heavy.

We want to get rid of the exact term by integrating over the orbits under the Spin group, which are compact *in Riemannian signature*. For this purpose we need an extra step in order to obtain equations on 6-forms along the orbits such that all the Lagrange multipliers are gathered in an exact 6-form. Only then we can proceed to integration. Identifying an integrand which is invariant under the action of Spin_4 and has a vanishing integral, we conclude that it must identically vanish. We thus obtain an equivariant equation on \mathcal{P} which we interpret as an equation on the underlying spacetime \mathcal{E} , equation from which the Lagrange multipliers are absent. We conclude the section with a brief analysis of the field equations, which can already be found in the literature.

As we replace the Lorentzian signature with the Riemannian signature, we need to replace the Poincaré algebra with the Lie algebra of the Euclidean group $\mathfrak{iso}_4 = \mathfrak{so}_4 \ltimes \mathbb{R}^4$ as well as the metric η by the Euclidean metric on \mathbb{R}^4 . The corresponding spin group is $\text{Spin}_{4,0}$, usually denoted Spin_4 . The algebraic manipulations involving the spinors we made in Section 11.3 still hold, see Section 4.1 for more detail. For the sake of brevity we will assume we have a spin frame bundle and will write about the action of Spin_4 but when the spinor field ψ is identically zero the principal bundle can be a mere metric SO_4 -structure.

We keep the indices conventions corresponding to the decomposition $\mathfrak{iso}_4 = \mathfrak{so}_4 \ltimes \mathbb{R}^4$. We will make implicit use of the isomorphism $\mathfrak{so}_4 \simeq \mathfrak{spin}_4$.

11.4.1 Exact terms in the Euler-Lagrange equations

Given a Lagrangian involving general Lagrange multipliers, we explained in Section 9.9 how it is possible, using suitable vector fields, to obtain Euler-Lagrange equations which have all the dependency in Lagrange multipliers gathered in an exact term. In our case, we have Lagrange multipliers $p_A^{\widehat{BC}}$ involved in a Lagrangian term

$$p_A^{\widehat{BC}} \Lambda^A \wedge \lambda_{\widehat{BC}}^{(8)}$$

as well as Lagrange multipliers $\kappa^{\alpha i}$ involved in a term

$$\frac{i}{2} \left(\bar{\kappa}^i d\lambda_s - d\lambda_s \bar{\kappa}^i \right) \wedge \lambda_i^{(9)}$$

We want to study solutions to Equations (11.25). For this we want to get rid of the non-physical Lagrange multipliers fields. The solution is to gather them in an exact term and to make it vanish by integration. The integration cannot be made on the total space \mathcal{P} which is not assumed to be compact. We need to find suitable compact submanifolds on which the Euler-Lagrange Equations (11.25c,11.25d) can be “restricted” in a meaningful way while preserving the exactness of the Lagrange multipliers term. First, let us present the general idea we use in this section with a focus on only the Euler-Lagrange equation corresponding to the Dirac equation.

The Dirac term

As an illustration of the general phenomenon with a simple case, let us focus first on the Euler-Lagrange equation coming from the variation of the spinor fields. Recall the corresponding

Euler-Lagrange form:

$$\text{EL}^\beta = (\gamma^a d^\lambda s)^\beta \wedge \lambda_a^{(9)} + \frac{1}{2}(\gamma^a s)^\beta \Lambda^c \wedge \lambda_{ac}^{(8)} - m s^\beta \lambda^{(10)} + \frac{i}{2} d^\lambda \left(\kappa^i \lambda_i^{(9)} \right)^\beta \quad (11.25d)$$

We assumed that \mathcal{P} has a structure of Spin_4 -principal bundle over a base manifold on which ϖ defines a Cartan connection 1-form.

We focus on one orbit which we will call O . It is by hypothesis isomorphic to Spin_4 (as a Spin_4 -space). We want to study the consequences of the equation $\phi^*(\text{EL}^\beta) = 0$ on O . First, we need to obtain an equation on a vertical 6-form. To this aim, we want to factor out a 4-form $\alpha^{(4)}$. We handle each term separately:

$$\begin{aligned} \phi^* \left((\gamma^a d^\lambda s)^\beta \wedge \lambda_a^{(9)} \right) &= (\gamma^a \partial_a \psi)^\beta \omega^{(6)} \wedge \alpha^{(4)} \\ \phi^* \left(\frac{1}{2} (\gamma^a s)^\beta \Lambda^c \wedge \lambda_{ac}^{(8)} \right) &= \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c \omega^{(6)} \wedge \alpha^{(4)} \\ \phi^* \left(m s^\beta \lambda^{(10)} \right) &= m \psi^\beta \omega^{(6)} \wedge \alpha^{(4)} \\ \phi^* \left(\frac{i}{2} d^\lambda \left(\kappa^i \lambda_i^{(9)} \right)^\beta \right) &= \frac{i}{2} d^\omega \left(K^i \varpi_i^{(9)} \right)^\beta \\ &= \frac{i}{2} d^\omega \left(K^i \omega_i^{(5)} \right)^\beta \wedge \alpha^{(4)} - \frac{i}{2} K^{\alpha i} \omega_i^{(5)} \wedge d\alpha^{(4)} \\ &= \frac{i}{2} d^\omega \left(K^i \omega_i^{(5)} \right)^\beta \wedge \alpha^{(4)} \end{aligned}$$

with ∂_a standing for the derivation on \mathcal{P} in the direction dual to α^a . In the last computation we used Formula (7.27e)

$$d^\omega(u^\beta \wedge \mu) = d^\omega u^\beta \wedge \mu + (-1)^{|u|} u^\beta \wedge d\mu$$

for μ a real valued differential form. All together, we obtain

$$\phi^* \text{EL}^\beta = \left((\gamma^a \partial_a \psi)^\beta \omega^{(6)} + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c \omega^{(6)} + m \psi^\beta \omega^{(6)} + \frac{i}{2} d^\omega \left(K^i \omega_i^{(5)} \right)^\beta \right) \wedge \alpha^{(4)}$$

Thus the equation

$$\phi^* \text{EL}^\beta = 0$$

is equivalent to

$$(\gamma^a \partial_a \psi)^\beta \omega^{(6)} + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c \omega^{(6)} + m \psi^\beta \omega^{(6)} + \frac{i}{2} d^\omega \left(K^i \omega_i^{(5)} \right)^\beta = 0 \quad \text{mod } [\alpha^a]$$

Now, the structure of a solder form on a frame bundle ensures that the orbits are exactly the leaves of the vertical distribution, which is the subset of tangent vectors annihilated by all α^a . Therefore the previous equation implies that

$$(\gamma^a \partial_a \psi)^\beta \omega^{(6)} + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c \omega^{(6)} + m \psi^\beta \omega^{(6)} + \frac{i}{2} d^\omega \left(K^i \omega_i^{(5)} \right)^\beta \quad (11.26)$$

has a vanishing restriction to any orbit. We now explain how this equation can be decoupled in two separate equations: one on ψ , the other one on K .

Since the orbit O is by assumption isomorphic to Spin_4 , there exists Spin_4 -equivariant Σ^* -

valued fields on O . In fact they are uniquely specified by their value at any given point of the orbit. There is no further constraint on the value at the chosen point. Let $\bar{\chi}$ be such a field: $\bar{\chi} \in \Omega^0(O, \Sigma^*)$. Since the α^a vanish on O , equivariance of $\bar{\chi}$ implies that

$$d^\omega \bar{\chi}_\beta = 0$$

on O , according to Theorem 7.2.5. We can thus contract the expression (11.26) with $\bar{\chi}$ to obtain

$$\begin{aligned} & \bar{\chi} \left((\gamma^a \partial_a \psi) \omega^{(6)} + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c \omega^{(6)} + m \psi^\beta \omega^{(6)} \right) + \frac{i}{2} \bar{\chi} d^\omega \left(K^i \omega_i^{(5)} \right)^\beta \\ &= \bar{\chi} \left((\gamma^a \partial_a \psi) + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c + m \psi^\beta \right) \omega^{(6)} + \frac{i}{2} d \left(\bar{\chi} K^i \omega_i^{(5)} \right) - \frac{i}{2} d^\omega \bar{\chi} d^\omega \wedge K^i \omega_i^{(5)} \\ &= \bar{\chi} \left((\gamma^a \partial_a \psi) + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c + m \psi^\beta \right) \omega^{(6)} + \frac{i}{2} d \left(\bar{\chi} K^i \omega_i^{(5)} \right) - 0 \end{aligned}$$

The Euler-Lagrange equation $\bar{\chi}_\beta \phi^* \text{EL}^\beta = 0$ now takes the form

$$\bar{\chi} \left((\gamma^a \partial_a \psi) + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c + m \psi^\beta \right) \omega^{(6)} = -\frac{i}{2} d \left(\bar{\chi} K^i \omega_i^{(5)} \right)$$

Notice that the left hand term is manifestly *invariant* under the action of Spin_4 , as a (real-valued) contraction of equivariant forms. On the other hand, the right hand term is manifestly exact, so that integrating over the (compact) orbit O , we obtain

$$\int_O \bar{\chi} \left((\gamma^a \partial_a \psi) + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c + m \psi^\beta \right) \omega^{(6)} = 0$$

Since the integrand is an invariant 6-form, it is in particular a volume form on O , so that a vanishing integral implies

$$\bar{\chi} \left((\gamma^a \partial_a \psi) + (\gamma^a \psi)^\beta \Omega_{ac}^c + m \psi^\beta \right) \omega^{(6)} = 0$$

Now, such an equation has to hold at each point of O , and this for each choice of equivariant $\bar{\chi}$. Since for each given point of O the values of $\bar{\chi}$ at this point are unrestricted, we obtain the following field equation:

$$(\gamma^a \partial_a \psi) + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c + m \psi^\beta = 0 \quad (11.27)$$

Finally, since the directions ∂_a are the horizontal vectors such that $\alpha^b(\partial_a) = \delta_a^b$, the term

$$\gamma^a \partial_a \psi$$

corresponds exactly to the Dirac operator applied to ψ . Since Ω_{ab}^c represents the torsion tensor, Ω_{ac}^c is a trace of the torsion tensor. Above all, all the terms in this equation are equivariant, which means that this equation can be interpreted on the underlying spacetime \mathcal{E} . Writing T for the torsion tensor and keeping the notation ψ for the spinor field on \mathcal{E} , the corresponding field equation is the Cartan-Dirac equation:

$$\not{D}\psi - \frac{1}{2} T_{\mu\nu}^\mu \gamma^\nu \psi + m \psi = 0 \quad (11.28)$$

Recall the Euler-Lagrange term

$$(\gamma^a \partial_a \psi)^\beta \omega^{(6)} + \frac{1}{2} (\gamma^a \psi)^\beta \Omega_{ac}^c \omega^{(6)} + m \psi^\beta \omega^{(6)} + \frac{i}{2} d^\omega \left(K^i \omega_i^{(5)} \right)^\beta$$

We now know that the sum of the first three terms has to vanish. As a consequence the same goes for the last term:

$$d^\omega \left(K^i \omega_i^{(5)} \right) = 0$$

which is an equation on K decoupled from the field ψ .

Hopefully this section illustrated the mechanism under which the equations on ψ decouple from the Lagrange multipliers K and induce equivariant field equations which can then be interpreted on the underlying spacetime manifold.

One key step in the calculation was to consider the Euler-Lagrange equation $\bar{\chi}_\beta \phi^* \text{EL}^\beta = 0$. This term can be interpreted as the Euler-Lagrange equation associated with a variation of ϕ directed by the vector field $\bar{\chi}$. In other words, it was crucial to find appropriate variations of the fields. In order to proceed similarly with the other Euler-Lagrange term, let us find those suitable vector fields. We want vertical variations of the field which preserve the constraints (11.25a,11.25b) on-shell.

Infinitesimal variations preserving the constraints

Once a frame bundle structure is given by a field $\phi = (\varpi, P, \psi, K) : \mathcal{P} \rightarrow Q$ satisfying the constraint equation (11.25a), it is natural to consider variations of the structure corresponding to variations of the tetrad or of the connection in the usual spacetime formalism. These are given by *equivariant* variations of $\omega^i = \phi^* \lambda^i$ and $\alpha^a = \phi^* \lambda^a$ as we will show. This is similar to the equivariant variations of the spinor field $\psi^\alpha = \phi^* s^\alpha$ discussed in the previous section. Note however the difference with the situation described in Section 9.9: equivariance is formulated using the principal bundle structure hence only makes sense on \mathcal{P} and is a notion dependent on the field ϕ . Instead of assuming our variations equivariant from the start we will derive this condition.

Variations of the coframe

Let us start with variations of the coframe ϖ^A . The field ϕ provides us with a nondegenerate equivariant 1-form: the coframe itself $\varpi^A = \phi^* \lambda^A$. The variation of ϖ will be given by a vertical vector field X on the image of ϖ :

$$X \in \Gamma(\mathcal{P}, \varpi^* [V(\text{Iso}(T\mathcal{P}, \mathfrak{p}))])$$

Using here again the isomorphism

$$V(T^*\mathcal{P} \otimes \mathfrak{iso}_4) \simeq T^*\mathcal{P} \otimes \mathfrak{iso}_4 \times_{\mathcal{P}} T^*\mathcal{P} \otimes \mathfrak{iso}_4$$

a vertical variation of ϖ^A is equivalent to a 1-form $\epsilon^A \in \Omega^1(\mathcal{P}, \mathfrak{iso}_4)$. We will use for convenience Lie derivatives \mathcal{L}_X but they will not depend on the chosen 1st order extension of X since we will work on the image of ϕ . The main property we will use is

$$\phi^* \mathcal{L}_X \lambda^A = \epsilon^A \tag{11.29}$$

Let us decompose ϵ^A into a \mathfrak{so}_4 -valued component $\tau^i \in \Omega(\mathcal{P}, \mathfrak{so}_4)$ and a \mathbb{R}^4 -valued component β^a :

$$\epsilon = \tau \oplus \beta$$

To study the action of X on the constraints, we will have it act on $(d\lambda + \frac{1}{2}[\lambda \wedge \lambda])\lambda_{BC}^{(8)}$. We can then compute

$$\begin{aligned}\phi^* \mathcal{L}_X \left(d\lambda + \frac{1}{2}[\lambda \wedge \lambda] \right)^i &= d\phi^* \mathcal{L}_X \lambda^i + 2\phi^* \frac{1}{2} [\mathcal{L}_X \lambda \wedge \lambda]^i = d\tau^i + [\omega \wedge \tau]^i \\ \phi^* \mathcal{L}_X \left(d\lambda + \frac{1}{2}[\lambda \wedge \lambda] \right)^a &= d\phi^* \mathcal{L}_X \lambda^a + 2\phi^* \frac{1}{2} [\mathcal{L}_X \lambda \wedge \lambda]^a = d\beta^a + [\omega \wedge \beta]^a + [\tau \wedge \alpha]^a\end{aligned}$$

and

$$\phi^* \mathcal{L}_X \lambda_{BC}^{(8)} = \phi^* \left((\mathcal{L}_X \lambda^D) \wedge \lambda_{BCD}^{(7)} \right) = \epsilon^D \wedge \varpi_{BCD}^{(7)} \quad (11.30)$$

which gather to give

$$\phi^* \mathcal{L}_X \left(\left(d\lambda + \frac{1}{2}[\lambda \wedge \lambda] \right)^i \wedge \lambda_{BC}^{(8)} \right) = \left(d\tau^i + [\omega \wedge \tau]^i \right) \wedge \lambda_{BC}^{(8)} + \Omega^i \wedge \epsilon^D \wedge \varpi_{BCD}^{(7)} \quad (11.31a)$$

$$\phi^* \mathcal{L}_X \left(\left(d\lambda + \frac{1}{2}[\lambda \wedge \lambda] \right)^a \wedge \lambda_{BC}^{(8)} \right) = \left(d\beta^a + [\omega \wedge \beta]^a + [\tau \wedge \alpha]^a \right) \wedge \lambda_{BC}^{(8)} + \Omega^a \wedge \epsilon^D \wedge \varpi_{BCD}^{(7)} \quad (11.31b)$$

In order for these terms to vanish under the constraint equation (11.25a), we will require the following three conditions on ϵ :

1. ϵ to be purely horizontal: $\epsilon^A = \epsilon_b^A \alpha^b$. This will prevent $\epsilon^D \wedge \varpi_{BCD}^{(7)}$ from having purely horizontal components (of type $\lambda_{bc}^{(8)}$) so that the term

$$\Omega^A \wedge \epsilon^D \wedge \varpi_{BCD}^{(7)}$$

necessarily vanishes.

2. There exist coefficients r_{bc}^i such that

$$d\tau^i + [\omega \wedge \tau]^i = \frac{1}{2} r_{bc}^i \alpha^b \wedge \alpha^c \quad (11.32a)$$

Namely, the term $(d\tau^i + [\omega \wedge \tau]^i) \wedge \lambda_{BC}^{(8)}$ vanishes. But now that we assumed that τ is horizontal, this equation exactly means that τ is *equivariant*

3. There exist coefficients t_{bc}^a such that

$$d\beta^a + [\omega \wedge \beta]^a = \frac{1}{2} t_{bc}^a \alpha^b \wedge \alpha^c \quad (11.32b)$$

Since $[\tau \wedge \alpha]$ is now assumed to be purely horizontal, the term $(d\beta^a + [\omega \wedge \beta]^a + [\tau \wedge \alpha]^a) \wedge \lambda_{BC}^{(8)}$ vanishes. But this is exactly requiring that β^a is *equivariant*.

Under these three conditions, each term of the right hand sides of (11.31) vanish.

We also need to check whether such variations preserve the constraint (11.25b):

$$\begin{aligned}\mathcal{L}_X \left((ds + \omega^i \sigma_i s) \wedge \lambda_i^{(9)} \right) &= (ds + \omega^i \sigma_i s) \wedge \mathcal{L}_X \lambda^B \wedge \lambda_{iB}^{(8)} = (ds + \omega^i \sigma_i s) \epsilon_c^B \lambda^c \wedge \lambda_{iB}^{(8)} \\ &= (ds + \omega^i \sigma_i s) \epsilon_c^c \lambda_i^{(9)}\end{aligned}\quad (11.33)$$

Therefore the constraint is preserved without condition.

As a conclusion, any equivariant horizontal 1-form ϵ represents a variation preserving the constraints (11.25a,11.25b). They can be identified with families of equivariant coefficients ϵ_b^A on \mathcal{P} with value in $\mathbb{R}^{4*} \otimes \mathfrak{iso}_4$.

Unfortunately, we cannot find coefficients ϵ_i^A such that the right hand sides of (11.31) vanish in a similar manner for two reasons. First, to a nonzero ϵ^i it will correspond a term

$$\Omega^A \wedge \epsilon_i^D \varpi^i \wedge \varpi_{BCD}^{(7)} = \Omega^A \wedge \left(\epsilon_i^i \varpi_{BC}^{(8)} - \delta_C^i \epsilon_i^D \varpi_{BD}^{(8)} + \delta_B^i \epsilon_i^D \varpi_{CD}^{(8)} \right)$$

which can contain nonzero components of Ω^A when there are non-vanishing components ϵ_i^d .

Second, for a non-horizontal 1-form ϵ , equivariance is no longer equivalent to

$$d\epsilon^A + [\omega \wedge \epsilon]^A = \frac{1}{2} E_{bc}^A \alpha^b \wedge \alpha^c$$

because the following equation holds, with $\epsilon_i = \epsilon(\xi_i) \in \mathfrak{p}$:

$$(\mathcal{L}_{\bar{\xi}_i} + \xi_i \cdot) \epsilon = i_{\xi_i} (d\epsilon + [\omega \wedge \epsilon]) + d\epsilon_i + [\omega \wedge \epsilon_i]$$

Thus arbitrary equivariant 1-forms are not necessarily solutions of $i_{\xi_i} (d\epsilon + [\omega \wedge \epsilon]) = 0$. Indeed, ω itself is equivariant but satisfies a different equation:

$$d\omega^i + [\omega \wedge \omega]^i = \Omega^i + \frac{1}{2} [\omega \wedge \omega]^i = \frac{1}{2} \Omega_{bc}^i \alpha^a \wedge \alpha^b + \frac{1}{2} c_{jk}^i \omega^j \wedge \omega^k$$

We therefore proceed only with the horizontal variations of ϖ and general variations of ψ . Insofar as the fibration above the spacetime does not vary, these variations can be viewed as moving the direct orthonormal frame bundle inside the general linear frame bundle of the spacetime. Note however that beside metric and connection variations, there is an extra gauge freedom due to the action of the spin group Spin_4 on $T^*\mathcal{P} \otimes \mathfrak{iso}_4$ by bundle automorphisms [Hél].

Variations of the spinor field

If the structure group is Spin_4 , the same computation can be done with variations X of ψ , identified with Σ -valued fields χ over \mathcal{P} (while keeping the generalised frame structure unvaried). In this case, the compatibility with the constraint (11.25b) requires

$$\mathcal{L}_X (ds + \omega^i \sigma_i \cdot s) = d\chi + \omega^i \sigma_i \chi = \varsigma_a \alpha^a \quad (11.34)$$

which here again expresses the local equivariance of χ over \mathcal{P} .

It is obvious that variations of ψ preserve the constraint term $(d\lambda + \frac{1}{2} [\lambda \wedge \lambda]) \lambda_{BC}^{(8)}$.

A basis for the constraint-preserving variations

We want to construct a local basis of equivariant \mathfrak{so}_4 -valued horizontal forms, as well as equivariant Σ -valued fields. Here we will make use of the structure of principal bundle of \mathcal{P} , or more exactly

the existence of slices. A slice S is the image of a local section $\mathcal{P}/\text{Spin}_4 \rightarrow \mathcal{P}$. Thus equivariant fields on $\text{Spin}_4 \cdot S$ are uniquely identified by their value on S .

We write $\beta^{\mathbf{a},\mu}$ (resp. $\tau^{\mathbf{i},\mu}$) for a basis of horizontal \mathbb{R}^4 -valued forms (resp. horizontal \mathfrak{so}_4 -valued forms) on S : the superscript μ corresponds to a basis of horizontal scalar 1-forms on S , while \mathbf{a} and \mathbf{i} correspond to a basis of \mathfrak{so}_4 (resp. \mathbb{R}^4)-valued maps on S (basis under $\mathcal{C}^\infty(S)$). Similarly we index equivariant spinor variations by their value on S and write $\bar{\chi}_\alpha$ for a basis of such vectors (variations of the adjoint spinor giving the spinor field equation), which can be identified with Σ^* -valued maps over S .

Euler-Lagrange terms corresponding to the variations

Recall the decomposition of the Poincaré-Cartan form $\bar{\Theta} = \Theta + \Theta^{\text{cons}}$:

$$\Theta = \frac{1}{2} 2\rho_{i,c}^a \eta^{cb} \Lambda^i \wedge \lambda_{ab}^{(8)} + \frac{1}{2} \left(\bar{s} \gamma^a (d\lambda_s) - (d\lambda_{\bar{s}}) \gamma^a s \right) \wedge \lambda_a^{(9)} - m \bar{s} s \lambda^{(10)} \quad (11.35)$$

$$\Theta^{\text{cons}} = \frac{1}{2} p_D^{\widehat{BC}} \Lambda^D \wedge \lambda_{\widehat{BC}}^{(8)} + \frac{i}{2} \left(\bar{\kappa}^i (d\lambda_s) - (d\lambda_{\bar{s}}) \kappa^i \right) \wedge \lambda_i^{(9)} \quad (11.36)$$

Let X be a vertical variation of ϕ which consists in an equivariant horizontal variation ϵ of ϖ and an equivariant variation χ of ψ . According to the principle presented in Section 9.9, the Euler-Lagrange term corresponding to a variation X and Poincaré-Cartan form Θ^{cons} is exact up to a term which vanish under the constraint:

$$i_X d\Theta^{\text{cons}} = \mathcal{L}_X(\Theta^{\text{cons}}) - di_X \Theta^{\text{cons}} \equiv -di_X \Theta^{\text{cons}} \equiv 0 \quad (11.37)$$

We define the unconstrained Euler-Lagrange forms:

$$(\text{EL}^0)_X = i_X d\Theta \quad (11.38)$$

The form EL^0 has no dependency in $P_A^{\widehat{BC}}$ nor in $K^{\alpha i}$.

A field ϕ satisfying the Euler-Lagrange equations (11.25a,11.25b) satisfies the following:

$$\phi^*(\text{EL}^0)_X = -\phi^* di_X \Theta^{\text{cons}} \quad (11.39)$$

The dependency in $P_A^{\widehat{BC}}$ and in $K^{\alpha i}$ is gathered in the exact term. In order to make use of the exactness, we want to perform an integral.

11.4.2 Integration into variational equations on the spacetime

Unconstrained Euler-Lagrange terms

The unconstrained Euler-Lagrange form corresponds to the terms in (11.25a-11.25d) not involving the Lagrange multipliers $P_A^{\widehat{BC}}$, $K^{\alpha i}$:

$$\begin{aligned} \phi^* \text{EL}_X^0 = & \epsilon^A \wedge \phi^* \left[\Lambda^i \wedge \frac{1}{2} p_i^{bc} \lambda_{bcA}^{(7)} - \frac{1}{2} \left(\bar{s} \gamma^b (d\lambda_s) - (d\lambda_{\bar{s}}) \gamma^b s \right) \wedge \lambda_{bA}^{(8)} - m \bar{s} s \lambda_A^{(9)} \right] \\ & + \epsilon^j \wedge \phi^* \frac{1}{2} \left(p_j^{bc} \Lambda^D \wedge \lambda_{bcD}^{(7)} + \bar{s} \{ \sigma_j, \gamma^a \} s \lambda_a^{(9)} \right) \\ & + \bar{\chi}_\alpha \phi^* \left(\frac{1}{2} \left(2(\gamma^a d\lambda_s)^\alpha \wedge \lambda_a^{(9)} + (\gamma^a s)^\alpha \Lambda^c \wedge \lambda_{ac}^{(8)} \right) - m s^\alpha \lambda^{(10)} \right) \end{aligned} \quad (11.40)$$

The idea is to perform a “partial integration”, or fibre integration, of these Euler-Lagrange equations over the orbits under Spin_4 . Since they are compact, the exact terms will vanish and the resulting equations will only involve EL^0 . Furthermore, Spin_4 -equivariance of the variation X implies Spin_4 -invariance of the form $\phi^* \text{EL}_X^0$, so that a vanishing integral over orbits implies that $\phi^* \text{EL}_X^0$ vanishes at each point. To this aim, we need to transform (11.40) into an equation on 6-forms so that it can be integrated along the 6-dimensional orbits of Spin_4 .

To this end we want to “factor out” a factor $\alpha^{(4)}$ while keeping the exactness of the right-hand term in (11.39). The computation will be easier if we use explicit 10-forms on \mathcal{P} . Doing so we will not need to keep track of contact terms and constraint terms.

Let us thus reexpress the different terms in (11.40):

$$\phi^* \left(\Lambda^i \wedge \epsilon^A \wedge \lambda_{bcA}^{(7)} \right) = \Omega^i \wedge \epsilon^A \wedge \varpi_{bcA}^{(7)} = \frac{1}{2} \Omega_{de}^i \alpha^d \wedge \alpha^e \wedge \epsilon_f^A \alpha^f \wedge \varpi_{bcA}^{(7)} = \frac{1}{2} \Omega_{de}^i \epsilon_f^a \delta_{bca}^{[def]} \varpi^{(10)} \quad (11.41a)$$

$$\begin{aligned} \phi^* \left(\epsilon^A \wedge \left(\bar{s} \gamma^b (d^\lambda s) - (d^\lambda \bar{s}) \gamma^b s \right) \wedge \lambda_{bA}^{(8)} \right) &= - (\bar{\psi} \gamma^b d^\omega \psi - d^\omega \bar{\psi} \gamma^b \psi) \epsilon^A \wedge \varpi_{bA}^{(8)} \\ &= - (\bar{\psi} \gamma^b d^\omega \psi - d^\omega \bar{\psi} \gamma^b \psi) \epsilon_c^A \wedge \left(\delta_A^c \varpi_b^{(9)} - \delta_b^c \varpi_A^{(9)} \right) \\ &= - \epsilon_c^a (\bar{\psi} \gamma^b \partial_a \psi - \partial_a \bar{\psi} \gamma^b \psi) (\delta_a^c \delta_b^d - \delta_b^c \delta_a^d) \varpi^{(10)} \end{aligned} \quad (11.41b)$$

$$\phi^* \left(\epsilon^A \wedge \bar{s} s \lambda_A^{(9)} \right) = \epsilon_A^A \bar{\psi} \psi \varpi^{(10)} = \epsilon_a^a \bar{\psi} \psi \varpi^{(10)} \quad (11.41c)$$

$$\phi^* \left(\epsilon^j \wedge \Lambda^D \wedge \lambda_{bcD}^{(7)} \right) = \frac{1}{2} \epsilon_e^j \Omega_{fg}^d \delta_{bcd}^{[efg]} \varpi^{(10)} \quad (11.41d)$$

$$\phi^* \left(\epsilon^j \wedge \bar{s} \{ \sigma_j, \gamma^a \} s \lambda_a^{(9)} \right) = \epsilon_a^j \bar{\psi} \{ \sigma_j, \gamma^a \} \psi \varpi^{(10)} \quad (11.41e)$$

$$\bar{\chi}_\alpha \phi^* \left((\gamma^a d^\lambda s)^\alpha \wedge \lambda_a^{(9)} + \frac{1}{2} (\gamma^a s)^\alpha \Lambda^c \wedge \lambda_{ac}^{(8)} \right) = \bar{\chi}_\alpha \left(\gamma^a \partial_a \psi^\alpha + \frac{1}{2} \gamma^a \psi^\alpha \Omega_{ac}^c \right) \varpi^{(10)} \quad (11.41f)$$

The non-normalised *antisymmetric Kronecker symbol* $\delta_{bcd}^{[efg]}$ is defined as follows:

$$\delta_{bcd}^{[efg]} = \delta_b^e \left(\delta_c^f \delta_d^g - \delta_d^f \delta_c^g \right) + \delta_c^e \left(\delta_d^f \delta_b^g - \delta_b^f \delta_d^g \right) + \delta_d^e \left(\delta_b^f \delta_c^g - \delta_c^f \delta_b^g \right)$$

From these it is straightforward to factor out $\alpha^{(4)}$ in each term. We will write

$$\phi^* \text{EL}_X^0 = \phi^* E_X \wedge \alpha^{(4)}$$

with E_X defined as a purely horizontal 6-form on the image of ϕ (but it can also be expressed as a form on the jet bundle: $E_X \in \Gamma(\mathcal{J}^1(Q), \pi^* \Lambda^6 T^* \mathcal{P})$).

Factorisation and exactness of the constraint terms

We need to factorise the exact term $\phi^*(di_X \Theta^{cons})$. Extracting the relevant terms from Equations (11.25) we have

$$\begin{aligned} i_X \Theta^{cons} &= i_X \left(\frac{1}{2} p_D^{BC} \Lambda^D \wedge \lambda_{BC}^{(8)} + \frac{i}{2} \left(\bar{\kappa}^i (d\lambda_s) - (d\lambda_{\bar{s}}) \kappa^i \right) \wedge \lambda_i^{(9)} \right) \\ &= \frac{1}{2} p_D^{BC} \epsilon^D \wedge \lambda_{BC}^{(8)} - \frac{i}{2} \bar{\chi}_\alpha \kappa^{\alpha i} \wedge \lambda_i^{(9)} \\ &= p_D^{jc} \epsilon_c^D \lambda_j^{(9)} - \frac{i}{2} \bar{\chi}_\alpha \kappa^{\alpha i} \wedge \lambda_i^{(9)} \\ &= \left(p_D^{jc} \epsilon_c^D \lambda_j^{so_4(5)} - \frac{i}{2} \bar{\chi}_\alpha \kappa^{\alpha i} \lambda_i^{so_4(5)} \right) \wedge \lambda^{m(4)} \end{aligned}$$

Note how essential it is here again that ϵ is a purely horizontal form. This is what allows us to factor $\epsilon^D \wedge \lambda_{BC}^{(8)}$ by $\lambda^{m(4)}$.

Considering the exterior differential of the pullback, we obtain

$$\begin{aligned} \phi^* di_X \Theta^{cons} &= d\phi^* i_X \Theta^{cons} \\ &= d \left(\left(p_D^{jc} \epsilon_c^D \omega_j^{(5)} - \frac{i}{2} \bar{\chi}_\alpha K^{\alpha i} \omega_i^{(5)} \right) \wedge \alpha^{(4)} \right) \\ &= d \left(p_D^{jc} \epsilon_c^D \omega_j^{(5)} - \frac{i}{2} \bar{\chi}_\alpha K^{\alpha i} \omega_i^{(5)} \right) \wedge \alpha^{(4)} + \left(p_D^{jc} \epsilon_c^D \omega_j^{(5)} - \frac{i}{2} \bar{\chi}_\alpha K^{\alpha i} \omega_i^{(5)} \right) \wedge d\alpha^{(4)} \\ &= d \left(p_D^{jc} \epsilon_c^D \omega_j^{(5)} - \frac{i}{2} \bar{\chi}_\alpha K^{\alpha i} \omega_i^{(5)} \right) \wedge \alpha^{(4)} + 0 \end{aligned}$$

We used $d\alpha^{(4)} = 0$ which is a consequence of Equation (11.25a):

$$d\alpha^a + [\omega \wedge \alpha]^a = \frac{1}{2} \Omega_{bc}^a \alpha^b \wedge \alpha^c$$

We call E_X^{cons} the term which is differentiated:

$$\phi^* E_X^{cons} := p_D^{jc} \epsilon_c^D \omega_j^{(5)} - \frac{i}{2} \bar{\chi}_\alpha K^{\alpha i} \omega_i^{(5)} \quad (11.42)$$

We finally obtain the following equation, which holds for each X vertical variation of ϕ which is equivariant:

$$\boxed{\phi^* EL_X^0 \wedge \alpha^{(4)} = d\phi^* E_X^{cons} \wedge \alpha^{(4)}} \quad (11.43)$$

Integration along the orbits

The tangent space to the orbits under Spin_4 is exactly the kernel of $\alpha^{(4)}$. This means that on any orbit $\text{Spin}_4 \cdot x \subset \text{Spin}_4 \cdot S$, Equation (11.43) implies

$$\phi^* E_X^0|_{\text{Spin}_4 \cdot x} = d\phi^* E_X^{cons}|_{\text{Spin}_4 \cdot x} \quad (11.44)$$

The orbit being compact, we can integrate along the orbit:

$$\int_{\text{Spin}_4 \cdot x} \phi^* E_X^0|_{\text{Spin}_4 \cdot x} = \int_{\text{Spin}_4 \cdot x} d\phi^* E_X^{\text{cons}}|_{\text{Spin}_4 \cdot x} = 0 \quad (11.45)$$

by virtue of the Stokes theorem.

We want to conclude that $\phi^* E_X|_{\text{Spin}_4 \cdot x} = 0$. It will follow if we can show that $\phi^* E_X$ is Spin_4 -invariant on \mathcal{P} . Indeed since Spin_4 preserves the orientation of the orbit and acts transitively, any invariant form with vanishing integral is necessarily identically zero.

But Equations (11.41) express $\phi^* E_X$ as the product of $\varpi^{(10)}$, which is Spin_4 -invariant by construction, and a factor which is Spin_4 -invariant as a complete contraction of equivariant quantities (including ϵ and $\bar{\chi}$). Thus factoring out an equivariant $\alpha^{(4)}$ leaves us with an equivariant $\phi^* E_X^0$.

Thus we proved that

$$\boxed{\phi^* E_X^0 = 0} \quad (11.46)$$

for X which are variations of ψ which are equivariant and variations of ϖ which are *horizontal and* equivariant, over $\text{Spin}_4 \cdot x$ for any orbit. Therefore it holds at each point.

Equation (11.46) a priori only holds for 1-forms ϵ which are *constant* linear combinations of the fields $\beta^{a,\mu}$, $\tau^{i,\mu}$ and $\bar{\chi}_\alpha$. However it is a tensorial equation which holds at each point. Thus it still holds if X was multiplied by any real function on $\text{Spin}_4 \cdot S$. One concludes that Equation (11.46) holds for X which is *any variation of ψ and any horizontal variation of ϖ* . Let us recall the coefficient of $\omega^{(6)}$ in $\phi^* E_X^0$, as expressed in (11.41):

$$\begin{aligned} & \epsilon_b^a \left(\frac{1}{2} p_i^{cd} \frac{1}{2} \Omega_{ef}^i \delta_{cda}^{[bef]} + \frac{1}{2} (\delta_a^b \delta_c^d - \delta_c^b \delta_a^d) (\bar{\psi} \gamma^c \partial_a \psi - \partial_a \bar{\psi} \gamma^c \psi) - m \delta_a^b \bar{\psi} \psi \right) \\ & + \epsilon_b^i \left(\frac{1}{2} p_i^{ef} \frac{1}{2} \Omega_{cd}^g \delta_{efg}^{[bcd]} + \bar{\psi} \{ \sigma_i, \gamma^b \} \psi \right) + \bar{\chi}_\alpha \left(\gamma^a \partial_a \psi^\alpha + \frac{1}{2} \gamma^a \psi^\alpha \Omega_{ac}^c - m \psi^\alpha \right) \end{aligned} \quad (11.47)$$

Since it has to vanish for *any* ϵ_b^A and $\bar{\chi}_\alpha$, it means that each coefficient vanishes:

$$\frac{1}{4} p_i^{cd} \Omega_{ef}^i \delta_{cda}^{[bef]} + \frac{1}{2} (\delta_a^b \delta_c^d - \delta_c^b \delta_a^d) (\bar{\psi} \gamma^c \partial_a \psi - \partial_a \bar{\psi} \gamma^c \psi) - m \delta_a^b \bar{\psi} \psi = 0 \quad (11.48a)$$

$$\frac{1}{4} p_i^{ef} \Omega_{cd}^g \delta_{efg}^{[bcd]} + \bar{\psi} \{ \sigma_i, \gamma^b \} \psi = 0 \quad (11.48b)$$

$$\gamma^a \partial_a \psi^\alpha + \frac{1}{2} \gamma^a \psi^\alpha \Omega_{ac}^c - m \psi^\alpha = 0 \quad (11.48c)$$

We obtained *tensorial equations* on \mathcal{P} . By assumption, \mathcal{P} is a (spin) frame bundle above a spacetime \mathcal{E} . We now express these tensorial equations on \mathcal{E} .

Expression in spacetime coordinates

Since all fields involved in Equations (11.48) are equivariant, they are all associated to sections of the associated principal bundles on spacetime. As a consequence, the equations can be formulated on spacetime. For this purpose, we need a local system of coordinates, with a local trivialisation of the spinor frame bundle. We use greek indices $\mu, \nu \dots$ for coordinates. We keep the notation ψ for the spinor field as well as the superscript α for spinor fields. On \mathcal{E} there is a metric g corresponding to $\eta_{ab} \alpha^a \otimes \alpha^b$; it is compatible with the Spin_4 -structure defined by $\mathcal{P} \rightarrow \mathcal{E}$.

Dealing with the factors p_i^{ab} requires care. Let us recall their definition:

$$p_i^{bc} = 2\rho_{i,d}^b \eta^{dc}$$

We also stated that they correspond to an isomorphism

$$\mathfrak{iso}_4 \xrightarrow{\sim} \Lambda^2 \mathbb{R}^{4*}$$

First,

$$\frac{1}{2} p_i^{de} \Omega_{ab}^i \text{ corresponds to } \mathcal{R}^o{}_{\mu\nu\pi} g^{\pi\chi}$$

(with indices corresponding in alphabetic order) as explained in Chapter 5.

The Ω term in Equation (11.48a) becomes:

$$\frac{1}{2} \mathcal{R}^\chi{}_{\pi\rho\nu} g^{\nu\sigma} \delta_{\chi\sigma\mu}^{[\nu\sigma\tau]} = \mathcal{R}^\nu{}_{\sigma\mu\nu} g^{u\sigma} - \mathcal{R}^\chi{}_{\chi\mu\nu} g^{u\nu} + \mathcal{R}^\chi{}_{\chi\sigma\nu} g^{u\sigma} \delta_\mu^\nu = \delta_\mu^\nu \text{Scal} - 2g^{\nu\pi} \text{Ric}_{\mu\pi}$$

For Equation (11.48b) we recall as well that σ_i can be written as a function of γ^a in the following way:

$$\sigma_i = \frac{1}{4} p_i^{ab} \gamma_a \gamma_b = \frac{1}{8} p_i^{ab} [\gamma_a, \gamma_b]$$

as explained in Section 4.1.3 (p is twice the map ρ described there). Equation (11.48b) is thus equivalent to

$$0 = \frac{1}{2} \Omega_{cd}^g \delta_{efg}^{[bcd]} + \frac{1}{4} \bar{\psi} \{[\gamma_e, \gamma_f], \gamma^b\} \psi = \Omega_{fg}^g \delta_e^b + \Omega_{ef}^g \delta_g^b - \Omega_{eg}^g \delta_f^b + \frac{1}{4} \bar{\psi} \{[\gamma_e, \gamma_f], \gamma^b\} \psi$$

The spacetime tensor corresponding to Ω_{bc}^a is the torsion tensor $T_{\nu\chi}^\mu$. We define its trace

$$\text{tr}(T)_\chi = T_{\sigma\chi}^\sigma$$

The equation on spacetime corresponding to (11.48b) is:

$$- \text{tr}(T)_\chi \delta_\nu^\mu + \text{tr}(T)_\nu \delta_\chi^\mu + T_{\nu\chi}^\mu + \frac{1}{4} \bar{\psi} \{[\gamma_\nu, \gamma_\chi], \gamma^\mu\} \psi = 0 \quad (11.49)$$

Last, the horizontal derivatives ∂_a turn to covariant derivatives on spacetime, as they correspond to deriving the fields on the frame bundle along the horizontal directions defined by the connection 1-form. In particular, $\gamma^a \partial_a$ corresponds to the (covariant) Dirac operator. We summarize the correspondence in a table:

Frame bundle	Spacetime
∂_a	∇_a
$\gamma^a \partial_a$	$\not{\nabla}$
η_{ab}	$g_{\mu\nu}$
$p_i^{cd} \Omega_{ef}^i$	$\mathcal{R}^\chi{}_{\mu\nu}{}^o$
Ω_{bc}^a	$T_{\nu\chi}^\mu$
ψ^α	ψ^α

For convenience, we convert (11.48a) to a totally covariant equation. Separating geometry

terms and matter terms in Equations (11.48), we obtain:

$$\left\{ \begin{array}{l} 2 \operatorname{Ric}_{\mu\nu} - g_{\mu\nu} \operatorname{Scal} = g_{\mu\nu} \left(\frac{1}{2} \bar{\psi} \overleftrightarrow{\nabla} \psi - m \bar{\psi} \psi \right) - \frac{1}{2} \bar{\psi} \gamma_{\mu} \overleftrightarrow{\nabla}_{\nu} \psi \quad (11.50a) \\ T_{\nu\chi}^{\mu} - \operatorname{tr}(T)_{\chi} \delta_{\nu}^{\mu} + \operatorname{tr}(T)_{\nu} \delta_{\chi}^{\mu} = -\frac{1}{4} \bar{\psi} \{[\gamma_{\nu}, \gamma_{\chi}], \gamma^{\mu}\} \psi \quad (11.50b) \\ \overleftrightarrow{\nabla} \psi - \frac{1}{2} \operatorname{tr}(T)_{\mu} \gamma^{\mu} \psi - m \psi = 0 \quad (11.50c) \end{array} \right.$$

These are the Einstein-Cartan-Dirac field equations (Equation (11.50a) is usually presented with an extra factor 1/2). Equation (11.50b) defines algebraically the tensor $T - \operatorname{Id} \wedge \operatorname{tr} T$ as a function of the spinor field, hence the torsion as well (because $\operatorname{tr}(T - \operatorname{Id} \wedge \operatorname{tr} T) = (1 - 4 + 1) \operatorname{tr}(T)$).

Untreated variational equations

We obtained equations which correspond to the Euler-Lagrange equations for equivariant variations of ψ and horizontal equivariant variations of ϖ . We proved that in this case the term $\phi^* \operatorname{EL}_X^0$ has to vanish. This is only a consequence of the Euler-Lagrange equations and is by no means equivalent to those. There are two parts of the Euler-Lagrange which we do not use.

First, for *non-equivariant variations* of ψ and *non-equivariant horizontal variations* of ϖ , we proved that $\phi^* E_X = 0$. But the Euler-Lagrange equation (11.39) is

$$\phi^*(\operatorname{EL}_X^0 + i_X d\Theta^{\operatorname{cons}}) = 0$$

Thus the Euler-Lagrange equation is equivalent to

$$\phi^*(i_X d\Theta^{\operatorname{cons}}) = 0 \quad (11.51)$$

which is an equation involving the Lagrange multipliers p_i^{ab} and $\kappa^{\alpha i}$ as well as the fields ϖ^A and ψ^{α} .

Second, we did not consider at all *vertical* variations of ϖ . This is because we could not find such variations which preserve the constraints. Furthermore, vertical components would also prevent factoring an $\alpha^{(4)}$ factor out of the exact term in (11.43). Vertical variations of α^a correspond to variations of the vertical distribution which integrates into the orbits. Vertical variations of ω^i correspond to variations of the vectors fields representing the Lie algebra \mathfrak{spin}_4 .

We only note that these equations are subject to some degeneracy, according to Noether's second theorem. Indeed the Poincaré-Cartan form (11.35,11.36) is invariant by diffeomorphisms of \mathcal{P} [Hé].

Since the method that allowed us to obtain variational equations on spacetime does not apply to these equation, their study falls outside of the scope of our work.

11.4.3 Decomposing the field equations

For completion, we present a brief mathematical analysis of the structure of the Einstein-Cartan-Dirac equations. It is classical and can be found in the literature, as well as its physical implications [HD71; Tra06; Sha02].

We first present a couple formulas we will need to carry to analysis. We will use the parenthesis $(\mu\nu)$ notation for symmetrisation and bracket notation $[\mu\nu]$ for antisymmetrisation (normalized by a 1/2 factor, or 1/n! if n indices are involved).

Chiral current

Seen as a morphism $\mathbb{R}^{p,q} \rightarrow \text{End}(\Sigma_{p,q})$, γ is $\text{Spin}_{p,q}$ -equivariant ($\text{Spin}_{p,q}$ being represented by $\text{SO}_{p,q}$). The hermitian metric allows us to define a “dual” $\Gamma : \overline{\Sigma_{p,q}} \otimes \Sigma_{p,q} \rightarrow \mathbb{R}^{p,q*}$ which is equivariant as well:

$$(a, s_1, s_2) \in \mathbb{R}^{p,q} \otimes \overline{\Sigma_{p,q}} \otimes \Sigma_{p,q} \mapsto \langle s_1 | \gamma(a) \cdot s_2 \rangle \quad (11.52)$$

Notice how the action of the $\text{Pin}_{p,q}$ group is *twisted* under the morphism, as γ takes antihermitian values. In a similar way, one can define tensor-valued hermitian forms by using products of γ .

We are interested in the element $\{\sigma_{\mu\nu}, \gamma_\tau\}$ as it appears in the Equation (11.72). By definition $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. Since the commutator bracket is a Poisson bracket, one has a Jacobi-like (derivation) identity with the anti-commutator, so that

$$\{[\gamma_\mu, \gamma_\nu], \gamma_\tau\} = [\gamma_\mu, \{\gamma_\nu, \gamma_\tau\}] - \{\gamma_\nu, [\gamma_\mu, \gamma_\tau]\} = \{\gamma_\nu, [\gamma_\tau, \gamma_\mu]\} \quad (11.53)$$

hence the $\text{End}(\Sigma_{p,q})$ -valued 3-form $\{\sigma_{\mu\nu}, \gamma_\tau\}$ is antisymmetric in two pairs of indices, therefore totally antisymmetric because two transpositions span the whole symmetric group. As a consequence it is equal to its antisymmetric part:

$$\{[\gamma_\mu, \gamma_\nu], \gamma_\tau\} = \{[\gamma_{[\mu}, \gamma_{\nu]}], \gamma_\tau\} = 4\gamma_{[\mu}\gamma_{\nu]}\gamma_\tau$$

where the bracket between indices can be expressed using the chirality element and the Levi-Civita symbol (see Section 1.4), using the method described in [Pal07] (with a chirality element different by an i factor):

$$\frac{1}{2}\{\sigma_{\mu\nu}, \gamma_\tau\} = \gamma_{[\mu}\gamma_{\nu]}\gamma_\tau = \varepsilon_{\nu\mu\tau} \gamma^\nu \gamma^5 \quad (11.73)$$

Since γ_μ have antihermitian values, $\frac{1}{2}\{\sigma_{\mu\nu}, \gamma_\tau\}$ takes value in *hermitian* operators.

Let us also record the following formula, also proved (using the Lorentzian signature) in [Pal07]:

$$\frac{1}{2}\epsilon_{\chi\nu\tau\chi}\epsilon^{v\tau\chi\mu} = (-1)^q (\delta_\chi^v \delta_\nu^\mu - \delta_\chi^\mu \delta_\nu^v) \quad (11.74)$$

with $(-1)^q$ corresponding to the norm of the positive volume element vol .

In order to carry out the analysis, we want to decompose the tensor equations into components with different index symmetries (sub-representations under SO_4).

Pure axiality of the torsion

Starting with torsion, it is convenient to set all indices to have the same type (covariant): we write $T_{\tau\mu\nu} = g_{\tau\pi} T^\pi_{\mu\nu}$. Let us define the trace of the torsion as follows:

$$\text{tr}(T)_\mu = T^\nu_{\nu\mu} = g^{\tau\mu} T_{\tau\mu\nu}$$

The torsion can be decomposed as [Sha02; AP12; Ham02]

$$T_{\tau\mu\nu} = \frac{1}{3} (\text{tr}(T)_\nu g_{\tau\mu} - \text{tr}(T)_\mu g_{\tau\nu}) + \mathcal{A}_{\tau\mu\nu} + \mathcal{T}_{\tau\mu\nu} \quad (11.54)$$

with $\text{tr}(\mathcal{T}) = 0$, the component \mathcal{A} purely antisymmetric and $\mathcal{T}_{\tau\mu\nu} + \mathcal{T}_{\mu\nu\tau} + \mathcal{T}_{\nu\tau\mu} = 0$ (\mathcal{T} is called the *pure torsion* part). We then express the torsion term of (11.50b) in terms of these components:

$$T_{\tau\mu\nu} - (g_{\tau\mu} \text{tr}(T)_\nu - g_{\tau\nu} \text{tr}(T)_\mu) = -\frac{2}{3} (\text{tr}(T)_\nu g_{\tau\mu} - \text{tr}(T)_\mu g_{\tau\nu}) + \mathcal{A}_{\tau\mu\nu} + \mathcal{T}_{\tau\mu\nu} \quad (11.55)$$

Now the matter term $-\frac{1}{2}\bar{\psi}\{\sigma_{\mu\nu},\gamma_\tau\}\psi$ is totally antisymmetric (see Section 11.4.3). Equation (11.50b) hence decomposes into the following three equations

$$\text{tr}(T) = 0 \quad (11.56a)$$

$$\mathcal{A}_{\tau\mu\nu} = -\frac{1}{4}\bar{\psi}\{\sigma_{\mu\nu},\gamma_\tau\}\psi \quad (11.56b)$$

$$\mathcal{T} = 0 \quad (11.56c)$$

The equations require the torsion to be reduced to its so-called *axial* part. Notably, the trace term appearing in (11.50c) has to vanish [Ham02] so that

$$T^\mu{}_{\nu\chi} = -\frac{1}{4}\bar{\psi}\{[\gamma_\nu,\gamma_\chi],\gamma^\mu\}\psi \quad (11.57)$$

The 3-form \mathcal{A} which is part of the spacetime geometry is (algebraically) coupled to the spinor field – one can say that this degree of freedom is what separates Einstein-Cartan theory from Einstein's theory of General Relativity. Equation (11.50b) corresponds to variations of the connection and as such the matter term (the ψ part) is identified with the angular momentum current [FR03]. Torsion hence couples the angular momentum current with the various fields of the theory (ψ in the present case).

As a purely antisymmetric 3-form, we can express \mathcal{A} by its dual (pseudo-)vector A , also called *axial vector*:

$$\mathcal{A}_{\tau\mu\nu} = (A \lrcorner \text{vol})_{\tau\mu\nu} = A^X \epsilon_{X\tau\mu\nu}$$

with $\epsilon_{X\tau\mu\nu}$ the Levi-Civita symbol, which is the components of a volume form in a basis of determinant 1. Equation (11.56b) can be reexpressed using the chirality operator

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

defined in Section 11.4.3:

$$A^X \epsilon_{X\tau\mu\nu} = -\frac{1}{2}\epsilon_{X\mu\nu\tau}\bar{\psi}\gamma^X\gamma_5\psi$$

from which we obtain

$$A^X = -\frac{1}{2}\bar{\psi}\gamma^X\gamma_5\psi \quad (11.58)$$

The equation of motion of the spinor (11.50c) and its conjugate then take the form

$$(\not{\nabla} - m)\psi = 0 \quad (11.59a)$$

$$\left(\overleftarrow{\nabla} - m\right)\bar{\psi} = 0 \quad (11.59b)$$

and one concludes that the term $-g_{\mu\nu}\bar{\psi}\left(\frac{1}{2}\overleftrightarrow{\nabla} - m\right)\psi$ in (11.50a) has to vanish. This is a general property of homogeneous Lagrangians (as is the Dirac Lagrangian): they take the value zero on-shell.

Symmetric and antisymmetric parts of the curvature-energy relation

The first equation (11.50a) corresponds to Equation (11.48a) hence to the coefficient ϵ_b^a in variations, that is to say it is the Euler-Lagrange term corresponding to horizontal variations of

α . It can be decomposed into a symmetric and an antisymmetric parts:

$$2 \operatorname{Ric}_{(\mu\nu)} - g_{\mu\nu} \operatorname{Scal} = g_{\mu\nu} \left(\frac{1}{2} \bar{\psi} \overleftrightarrow{\nabla} \psi - m \bar{\psi} \psi \right) - \frac{1}{2} \bar{\psi} \gamma_{(\mu} \overleftrightarrow{\nabla}_{\nu)} \psi \quad (11.60a)$$

$$2 \operatorname{Ric}_{[\mu\nu]} = -\frac{1}{2} \bar{\psi} \gamma_{[\mu} \overleftrightarrow{\nabla}_{\nu]} \psi \quad (11.60b)$$

We first look at Equation (11.60b). It is the variational term corresponding to “antisymmetric” variations in the solder form, that is to say variations which preserve the metric. In other words, they correspond to infinitesimal automorphisms of the frame bundle.

The Bianchi identity on the Ricci curvature (5.5) (in Section 5.1) relates the antisymmetric part of the Ricci curvature to the exterior divergence of the torsion

$$2 \operatorname{Ric}_{[\mu\nu]} = \nabla_{\pi} T_{\mu\nu}^{\pi} - \operatorname{d} \operatorname{tr}(T)_{\mu\nu} = \nabla_{\pi} T_{\mu\nu}^{\pi} \quad (11.61)$$

This allows to rewrite (11.60b) as an equation on torsion:

$$\nabla_{\pi} T_{\mu\nu}^{\pi} = -\frac{1}{2} \bar{\psi} \gamma_{[\mu} \overleftrightarrow{\nabla}_{\nu]} \psi \quad (11.62)$$

The right-hand term, which corresponds to (twice) the antisymmetric part of the “canonical energy-momentum tensor” (the terms in ψ in (11.50a)), is the so-called *Belinfante (improvement) tensor* [Wei95; HLS81; Pop20].

The Cartan geometry, through the Bianchi identity, then imposes the following equation

$$\boxed{\frac{1}{2} \bar{\psi} \gamma_{[\mu} \overleftrightarrow{\nabla}_{\nu]} \psi = \frac{1}{4} \nabla_{\pi} (\bar{\psi} \{ \sigma_{\mu\nu}, \gamma^{\pi} \} \psi)} \quad (11.63)$$

which relates the Belinfante tensor to the *covariant* divergence of the angular momentum current (also called *spin density* [Tra06] or *spin current* [HK78]). In theories without torsion, the connection has to follow the solder form variations, so that Equation (11.63) would directly take the place of (11.60b) as in [Wei95] (as variational equation for variations of the solder form which preserve the metric).

The symmetric component (11.60a) corresponds to complementary variations of the solder form hence to variations of the metric (symmetric variations of a frame have been considered as the natural complement to isometric variations [BG92]). The corresponding matter term is then identified with the symmetric energy-momentum tensor [FR03; HK78]. It is Einstein’s field equation binding spacetime’s (Ricci symmetric) curvature to the distribution of energy-momentum.

Taking into account the fact that the term corresponding to the Dirac Lagrangian vanishes due to (11.59), Equations (11.60a) simplifies to

$$\boxed{2 \operatorname{Ric}_{(\mu\nu)} - g_{\mu\nu} \operatorname{Scal} = \frac{1}{2} \bar{\psi} \gamma_{(\mu} \overleftrightarrow{\nabla}_{\nu)} \psi} \quad (11.64)$$

11.4.4 Expression in terms of the Levi-Civita connection

To compare the Einstein-Cartan theory with Einstein’s General Relativity, we relate the connection to the Levi-Civita connection by means of its *contorsion*, which is defined as its difference with the Levi-Civita connection:

$$\nabla = \nabla^{LC} + K \quad (11.65)$$

with K a 1-form on \mathcal{P} with values in $\mathfrak{so}(T\mathcal{P})$. According to the previous section, the field equations (11.50b) require the torsion to be purely axial. Since contorsion is uniquely defined by the torsion (assuming metricity) (see Section 5.3 or [AP12; Sha02]), in our case K has to be $\frac{1}{2}T$:

$$K^\pi{}_{\mu\nu} = \frac{1}{2}T^\pi{}_{\mu\nu}$$

Ricci and scalar curvatures

According to Equation (5.18) from Section 5.2, the Ricci curvature of the connection can be related to the Ricci curvature of the Levi-Civita connection by the following equation:

$$\begin{aligned} \text{Ric}_{\mu\nu} &= \text{Ric}_{\mu\nu}^{LC} + \delta_\pi^\tau (\nabla_\tau^{LC} K_{\mu\nu}^\pi - \nabla_\mu^{LC} K_{\tau\nu}^\pi + K_{\tau\kappa}^\pi K_{\mu\nu}^\kappa - K_{\mu\kappa}^\pi K_{\tau\nu}^\kappa) \\ &= \text{Ric}_{\mu\nu}^{LC} + \frac{1}{2} (\text{div}^{LC} T)_{\mu\nu} - 0 + 0 - \frac{1}{4} T_{\mu\kappa}^\pi T_{\pi\nu}^\kappa \end{aligned} \quad (11.66)$$

We see that the difference between the Ricci curvatures difference has an antisymmetric term $(\text{div}^{LC} T)_{\mu\nu}$ and a symmetric term $-\frac{1}{4}T_{\mu\kappa}^\pi T_{\pi\nu}^\kappa$, because the torsion is purely axial. Since the Ricci curvature of the Levi-Civita connection is symmetric (due to the Bianchi identity (11.61) for a torsion-free connection), the antisymmetric difference between the Ricci curvatures is exactly the antisymmetric part of the Ricci curvature of the connection, as expressed in the Bianchi identity.

We express the Ricci curvature difference in term of the axial vector. Start with the quadratic term:

$$\begin{aligned} T_{\mu\kappa}^\pi T_{\pi\nu}^\kappa &= g^{\pi\tau} g^{\kappa\rho} T_{\tau\mu\kappa} T_{\rho\pi\nu} = g^{\pi\tau} g^{\kappa\rho} A^\chi A^\nu \varepsilon_{\chi\tau\mu\kappa} \varepsilon_{\nu\rho\pi\nu} \\ &= A^\chi A^\nu (g_{\chi\nu} g_{\mu\nu} - g_{\chi\nu} g_{\mu\nu}) \\ &= A^\chi A_\chi g_{\mu\nu} - A_\nu A_\mu \end{aligned}$$

The change in scalar curvature is directly derived:

$$g^{\mu\nu} \left(-\frac{1}{4} T_{\mu\kappa}^\pi T_{\pi\nu}^\kappa \right) = -\frac{3}{4} A^\chi A_\chi \quad (11.67)$$

Identifying the totally antisymmetric torsion with a 3-form, its divergence for the torsion-free Levi-Civita connection corresponds to (minus) the codifferential for the Hodge duality structure [Pet06]. It can hence be identified with the Hodge dual of the exterior differential of the axial 1-form $A_\pi = g_{\pi\chi} A^\chi$:

$$(\text{div}^{LC} T)_{\mu\nu} = -(\star dA)_{\mu\nu} \quad (11.68)$$

The left-hand terms of (11.60a-11.60b) are expressed as

$$\begin{aligned} 2 \text{Ric}_{(\mu\nu)} - g_{\mu\nu} \text{Scal} &= 2 \text{Ric}_{\mu\nu}^{LC} - g_{\mu\nu} \text{Scal}^{LC} - \frac{2}{4} (A^\chi A_\chi g_{\mu\nu} - A_\mu A_\nu) + g_{\mu\nu} \frac{3}{4} A^\chi A_\chi \\ &= 2 \text{Ric}_{\mu\nu}^{LC} - g_{\mu\nu} \text{Scal}^{LC} + \frac{1}{4} (A^\chi A_\chi g_{\mu\nu} + 2A_\mu A_\nu) \end{aligned} \quad (11.69)$$

and

$$2 \text{Ric}_{[\mu\nu]} = -(\star dA)_{\mu\nu} \quad (11.70)$$

The spin connection

We also need to express the covariant derivative of spinor fields in terms of K . It is straightforward from (11.65):

$$\nabla_\mu \psi = \nabla_\mu^{LC} \psi + \frac{1}{2} K_{\mu\nu}^\pi \frac{1}{2} \sigma_\pi{}^\nu \psi = \nabla_\mu^{LC} \psi + \frac{1}{8} T_{\tau\mu\nu} \sigma^{\tau\nu} \psi \quad (11.71)$$

We can re-express the kinetic term $\bar{\psi} \gamma_\mu \overleftrightarrow{\nabla}_\nu \psi$:

$$\begin{aligned} \bar{\psi} \gamma_\mu \overleftrightarrow{\nabla}_\nu \psi &= \bar{\psi} \gamma_\mu \left(\nabla_\nu^{LC} + \frac{1}{8} T_{\nu\pi\tau} \sigma^{\tau\pi} \right) \psi - \left(\nabla_\nu^{LC} + \frac{1}{8} T_{\nu\pi\tau} \bar{\sigma}^{\tau\pi} \right) \bar{\psi} \gamma_\mu \psi \\ &= \bar{\psi} \left(\gamma_\mu \overleftrightarrow{\nabla}_\nu^{LC} + \frac{1}{8} T_{\nu\pi\tau} \{ \sigma^{\pi\tau}, \gamma_\mu \} \right) \psi \end{aligned} \quad (11.72)$$

and in terms of the axial (pseudo)-vector, using the formulae (11.73-11.74) from Section 11.4.3:

$$\frac{1}{2} \{ \sigma_{\pi\tau}, \gamma_\mu \} = \varepsilon_{\nu\pi\tau\mu} \gamma^\nu \gamma^5 \quad (11.73)$$

$$\frac{1}{2} \varepsilon^{\chi\nu\pi\tau} \varepsilon_{\nu\pi\tau\mu} = (\delta_\mu^\chi \delta_\nu^\nu - \delta_\mu^\nu \delta_\nu^\chi) \quad (11.74)$$

to obtain:

$$\begin{aligned} \bar{\psi} \gamma_\mu \overleftrightarrow{\nabla}_\nu \psi &= \bar{\psi} \left(\gamma_\mu \overleftrightarrow{\nabla}_\nu^{LC} + \frac{1}{4} T_{\nu\tau\kappa} \varepsilon^{\nu\tau\kappa}{}_\mu \gamma_\nu \gamma_5 \right) \psi \\ &= \bar{\psi} \left(\gamma_\mu \overleftrightarrow{\nabla}_\nu^{LC} + \frac{1}{4} A^\chi \varepsilon_{\chi\nu\tau\kappa} \varepsilon^{\nu\tau\kappa}{}_\mu \gamma_\nu \gamma_5 \right) \psi \\ &= \bar{\psi} \left(\gamma_\mu \overleftrightarrow{\nabla}_\nu^{LC} + \frac{1}{2} (A^\chi \gamma_\chi g_{\mu\nu} - A_\mu \gamma_\nu) \gamma_5 \right) \psi \end{aligned} \quad (11.75)$$

The Dirac operator is readily rewritten as well, using (11.73) and the total antisymmetry mentioned right above. We directly quote Section 5.3:

$$\begin{aligned} \not{D} &= \not{D}^{LC} + \frac{1}{4} \gamma^\mu K_{\pi\mu\nu} \sigma^{\pi\nu} \\ &= \not{D}^{LC} + \frac{1}{2} \text{tr}(T)_\mu \gamma^\mu - \frac{1}{8} \mathcal{A}_{\tau\mu\nu} \gamma^\tau \gamma^\mu \gamma^\nu \\ &= \not{D}^{LC} - \frac{1}{8} A^\chi \varepsilon_{\chi\tau\mu\nu} \gamma^\tau \gamma^\mu \gamma^\nu \\ &= \not{D}^{LC} - \frac{1}{8} A^\chi 3! \gamma_\chi \gamma^5 \\ &= \not{D}^{LC} - \frac{3}{4} A^\chi \gamma_\chi \gamma_5 \end{aligned} \quad (11.76)$$

As a side note, in the general case the (vanishing here) trace term exactly compensates the trace term of the Cartan-Dirac Equation (11.50c) so that the difference between the two Dirac equations is always a contribution of the purely axial part of the torsion [Sha02].

The field equations (11.50a-11.50c) can then be reformulated as a Levi-Civita connection +

axial (pseudo)-vector theory:

$$\left\{ \begin{array}{l} 2 \operatorname{Ric}_{\mu\nu}^{LC} - g_{\mu\nu} \operatorname{Scal}^{LC} + \frac{1}{4} (A^\chi A_\chi g_{\mu\nu} + 2A_\mu A_\nu) \\ \qquad \qquad \qquad = \frac{1}{2} \bar{\psi} \left(\gamma_{(\mu} \overleftrightarrow{\nabla}_{\nu)}^{LC} + \frac{1}{2} (A^\chi \gamma_\chi g_{\mu\nu} - A_{(\mu} \gamma_{\nu)}) \gamma_5 \right) \psi \qquad (11.77a) \\ (\star dA)_{\mu\nu} = \frac{1}{2} \bar{\psi} \left(\gamma_{[\mu} \overleftrightarrow{\nabla}_{\nu]}^{LC} - A_{[\mu} \gamma_{\nu]} \gamma_5 \right) \psi \qquad (11.77b) \\ \qquad \qquad \qquad A^\chi = -\frac{1}{2} \bar{\psi} \gamma^\chi \gamma_5 \psi \qquad (11.77c) \\ \overleftrightarrow{\nabla}^{LC} \psi - \frac{3}{4} A^\chi \gamma_\chi \gamma_5 \psi - m\psi = 0 \qquad (11.77d) \end{array} \right.$$

The axial pseudo-vector is algebraically defined by (11.77c) according to which it corresponds the chiral currents. It can be integrated away from the field equations, giving for example in (11.77d) a cubic interaction term

$$+\frac{3}{8} (\bar{\psi} \gamma^\chi \gamma_5 \psi) \gamma_\chi \gamma_5 \psi$$

This being said, bringing back the dimensional constants in the equations, one notice that the right-hand side of (11.77a) has the gravitational constant as a factor [Heh14]. Hence the cubic term tends to be very weak in standard matter. In (11.77a), the (axial) torsion has a contribution to both the Einstein curvature term and the (matter) energy-momentum term.

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