HAL
open science

Regularity and stability in shape optimization under geometric constraint

Raphaël Prunier

To cite this version:

Raphaël Prunier. Regularity and stability in shape optimization under geometric constraint. Optimization and Control [math.OC]. Sorbonne Université, 2023. English. NNT : 2023SORUS138 . tel-04155961

HAL Id: tel-04155961

https://theses.hal.science/tel-04155961v1

Submitted on 7 Jul 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Raphaël Prunier

Régularité et stabilité en optimisation de forme sous contrainte géométrique

---

dirigée par Jimmy Lamboley et Dorin Bucur

Soutenue le 22 juin 2023 devant le jury composé de :

M. Dorin Bucur	Université de Savoie	co-directeur
M. Guillaume Carlier	Université Paris Dauphine	président du jury
M. Nicola Fusco	Université de Naples	rapporteur
M. Michaël Goldman	Ecole Polytechnique	rapporteur
Mme Flaviana Iurlano	Sorbonne Université	examinatrice
M. Jimmy Lamboley	Sorbonne Université	directeur
M. Vincent Millot	Université Paris-Est Créteil Val de Marne	examinateur

Institut de mathématiques de Jussieu-Paris Rive gauche. UMR 7586.
Boîte courrier 247
4 place Jussieu
75 252 Paris Cedex 05

Sorbonne Université.
École doctorale de sciences mathématiques de Paris centre.
Boîte courrier 290
4 place Jussieu
75 252 Paris Cedex 05

Tout ce que nous devons décider c'est que faire
du temps qui nous est imparti.

Gandalf.

À Delphine et Marie.

Remerciements

J'adresse mes plus sincères remerciements à Jimmy, pour son encadrement au cours de ces trois années. Nous nous sommes rencontrés lorsqu'il enseignait un cours d'analyse en Master de préparation à l'agrégation, et croisés de nouveau à l'occasion d'un cours d'optimisation en Master de recherche l'année suivante. Lorsque je lui ai demandé de faire un stage sous sa direction, j'étais donc convaincu que le stage (et sa suite éventuelle) se passerait très bien, ce qui a été déterminant dans mon choix de poursuivre avec lui en thèse. J'ai énormément appris à ses côtés au cours de ces trois années. Merci Jimmy pour ta rigueur scientifique et ton honnêteté, qui m'ont toujours tiré vers le haut. Merci pour ton investissement dans mon travail, nos longues heures de rendez-vous à Jussieu, le temps infini que tu as passé à relire consciencieusement mes maths. Merci pour ton optimisme, qui a permis d'apporter des idées clées sans lesquelles les maths n'auraient pas abouti. Merci également pour avoir été très présent (à distance) lors des multiples confinements. Merci pour ton organisation et ton anticipation dans tout, qui ont largement contribué à diminuer mon stress à tout un tas de moments de la thèse. J'admire la clarté avec laquelle tu parles de maths, merci de faire qu'elles sont simples. Je suis fier d'avoir été sous ton encadrement pendant ces trois années. Pour terminer sur une note non mathématique, merci pour tes conseils en matière d'escalade et ta suggestion d'asso dans laquelle l'ambiance est si sympa.

Merci infiniment Dorin pour ton encadrement au cours de ces trois années. Merci pour ton accueil à Chambéry à de nombreuses reprises, et pour les journées intensives de maths que nous y avons passées. J'ai beaucoup appris au cours de ces semaines savoyardes. Merci pour avoir partagé toutes tes intuitions et ton expérience sur des problèmes que je ne connaissais pas. Je suis très heureux d'avoir travaillé avec toi pendant ces trois années.

I am deeply grateful to Nicola and Michaël G. for having reported my thesis. Thanks to both of you for your very careful readings of the manuscript and your helpful remarks and comments on it. I am truely honoured that you have accepted to read my work.

Merci beaucoup à Flaviana, Guillaume et Vincent d'avoir accepté de faire partie de mon jury. Je suis très honoré que vous assistiez à ma soutenance.

Un grand merci à Mickaël N.. Merci pour ton accueil à Leipzig et à Chambéry lorsque j'y suis passé. Merci pour la collaboration que nous avons faite ensemble. Merci pour ta patience à répondre à toutes mes questions de régularité, parfois même quand elles étaient redondantes... Encore toutes mes félicitations pour ton poste.

Je souhaite remercier toutes les personnes avec qui j'ai eu des échanges mathématiques au cours de ma thèse. Merci à Berardo, Matteo et Michaël G. pour les discussions que nous avons eues sur les calculs "à la" Fuglede. Merci pour vos intuitions. Je remercie égalemement Idriss pour son enthousiasme mathématique et les discussions que nous avons eues à Paris ou en conférence. Merci à Ilias pour nos discussions d'optimisation sous contrainte de convexité, et ses conseils sur la thèse ; merci aussi pour ton sourire permanent. Merci Romain pour nos échanges sur l'optimisation de forme, c'était super de partager avec toi là-dessus. Thanks also to David J. for the nice email exchange we had regarding one of his works. Merci aux nombreuses personnes qui ont contribué à l'organisation de conférences auxquelles j'ai eu la

chance de participer. Merci pour les super conditions mises en place pour faire des maths. Merci donc à Andrea, Antoine H., Antoine L., Bohzidar, Benoît M., Berardo, Charles, Ilaria L., Jimmy, Jean-François B., Jules, Marc, Matteo, Michaël G., and thanks to Gloria, Gianpaolo, Simone. Merci à ceux que j'aurais oublié de citer. Merci également à Elise, Maxime et Paul pour l'organisation du GT Calva ; merci pour leur effort pour réunir la communauté parisienne de calcul des variations.

Merci à Antoine Henrot et Michel Pierre pour leur textbook dans lequel j'ai tout appris sur les bases de l'optimisation de forme. Merci aussi à Guillaume pour nous avoir fait parvenir son travail préliminaire sur le calcul de variations sous contrainte de convexité, sans lequel nous n'aurions rien pu faire. Merci également pour les échanges que nous avons eus à ce sujet.

Je remercie Marco M. et Arnau pour avoir fait parti de mon comité de suivi. Merci également à Marco M. pour avoir été mon tuteur.

Je remercie tous les membres de l'équipe administrative de l'IMJ pour leur aide tout au long de la thèse. Merci pour tous les efforts que vous faites pour que nos conditions de travail soient les meilleures possibles. Je remercie particulièrement l'équipe informatique, Cyril, Evariste, Sakina, les gestionnaires avec qui j'ai été en contact, Gaëlle, Patrick, Sitti et les administratrices Ayano et Emilie. Un merci tout particulier à Sitti pour sa disponibilité et sa gentillesse.

Merci à Corentin pour sa disponibilité lors du rendu de thèse et la préparation de la soutenance.

Je n'oublie pas Eric, que je remercie pour toutes les discussions matinales qui ont ponctué mes journées de travail au labo. Merci pour ta gentillesse.

Je remercie toutes les personnes avec qui nous avons organisé le séminaire des doctorants de l'IMJ. Merci tout d'abord aux permanents, Bram, Elba, Mathieu F., Sébastien pour leur conseils avisés lors des répétitions de séminaire, les coups sympas qu'on a bus ensemble après, et bien sûr leur effort constant pour obtenir la carte de l'IMJ. Merci ensuite à mes collègues doctorants co-organisateurs, Antoine S., Christophe, Francesca, Thiago, avec qui j'ai pris un réel plaisir à co-organiser le séminaire. Merci Antoine pour ton assiduité au séminaire et aux pots qui ont suivi, ton investissement dans les exposés, nos discussions escalade et politique. Merci Christophe pour ta fidélité implacable aux répétitions et aux séminaires. Merci pour les fous rires qu'on a eus, merci pour ta générosité et ton écoute attentive. Merci Thiago pour ta détermination à simplifier les exposés malgré le fait que tu connaissais déjà les maths.

Un immense merci à tous les doctorant.es (et affilié.es ;)) qui sont passés par le 5ème étage du couloir 15-16 de Jussieu. Je ne peux pas imaginer de meilleur environnement pour faire une thèse que celui-là. Parmi tant d'autres choses, merci pour nos sorties en gang au RA, nos descentes café au 4ème, nos longues pauses de midi (et de 16h, 17h, 18h...), nos débats, merci pour tous les coups qu'on a bus ensemble, les repas qu'on a partagés, et bien évidemment les soirées karaoké. Merci pour tout le fun qu'on a eu. Votre présence dans mon quotidien va me manquer.

Je commence la tournée de ces remerciements par tous mes co-bureaux et passés et présents, et avant tout bien sûr par mon acolyte Jacques. Merci pour ta présence à mes côtés tout au long de cette thèse, elle n'aurait pas eu la même saveur sans toi. Merci pour ta fidélité sans faille, tes encouragements constants, ta générosité. Merci pour nos sorties cinés, restos, raclette... Je n'oublie bien sûr pas les crêpes (avec beaucoup de) nutella de Pinocchio et notre amour commun pour les pizzas. Merci pour tous les fous rires qu'on a eus. Merci enfin bien sûr pour ta rigueur scientifique à tous les niveaux et ton inspirant enthousiasme pour les maths. Merci maintenant à Vadim. Merci de nous avoir accueillis Jacques et moi lorsque nous étions exilés du 519. Merci pour nos sorties running, tous les rires qu'on a eus, ta sensibilité, merci pour ta passion contagieuse des jeux de société. Merci à Nelson pour avoir fait que notre bureau était tellement mieux organisé (nos noms sur les portes ! :O). Merci pour ta bonne humeur et ton humour. Merci Odylo pour ta bienveillance. Thanks Haohao for your kindness and your modesty. Merci enfin (ici aussi) Thiago pour ta sympathie et toutes les histoires incroyables qui te sont arrivé. Merci à tous ceux

qui sont passés par le 509. Merci bien sûr à Carla pour les manifs (on lâche rien) ; merci pour ta gentillesse et ta curiosité, merci pour la joie que tu as apportée au bureau. Thanks also to Carol, Lu and Nastaran.

Merci Xénia pour ton affection et ta gentillesse. Merci pour toutes les super soirées qu'on a passés ensembles, toutes les sorties parisiennes, notre passion commune de la nourriture. Merci pour les vacances qu'on a partagées, et tous les moments de rire qu'on a eus. Merci Arnaud M. pour tes invitations raclette, tes jeux de mots sans fin, ton humour, ta connaissance imbattable des films et de la chanson français.es. Merci Anna pour ta délicatesse et ton écoute, pour toutes nos discussions. Merci Camille d'être un soleil pour le couloir, pour ta générosité, pour ton amour de la danse ; merci aussi de comprendre autant mon rapport à la thèse :p Merci Germain pour ta passion des polytopes et du RER C, ton Jacob von Uexküll qui prend la poussière sur mon étagère (promis je le lis pour ta soutenance !). Merci Christina pour les moments de rire qu'on a partagés et notre inquiétude commune pour l'organisation de la soutenance. Merci Perla de nous régaler avec tes gâteaux, merci de croire que le BDD peut changer le monde. Merci Eva pour ton engagement et nos discussions écologie ou rando. Merci Joaco pour ton chill à toute épreuve. Merci João pour ta curiosité des autres et nos discussions escalade. Merci Pietro pour ta générosité et ton effort constant d'inclure tout le monde. Merci Haowen pour m'avoir fait découvrir la cuisine sichuanaise et nos discussions en tous genre. Merci Felipe pour notre complicité musicale et ta passion du karaoké (même si les gens savent plus utiliser le mot "passion"...). Merci Tristan pour ton amour des grands cardinaux et notre duo bancal sur "Sous le vent". Merci Lucas pour m'avoir fait comprendre la signification de "IMJ-PRG", tes histoires à dormir debout de tango argentin en haut de la tour Zamansky ou de soi-disant "frère jumeau" ; merci bien sûr pour ton émerveillement pour les maths. Merci Chiara pour ta bonne humeur et ta gentillesse. Merci Mahya pour ton rire légendaire. Merci Mathieu D. pour toutes nos discussions sport, escalade et desserts, et ta connaissance exhaustive des meilleures pâtisseries parisiennes. Merci Adrien R. pour ta finesse et pour ton engagement pour l'enseignement. Merci Adrien C. pour ton honnêteté dans les débats et nos discussions Philharmonie. Merci Mattias pour ta folie. Merci Etienne pour ton ouverture d'esprit, merci pour ton partage sur la culture bretonne. Merci Antoine T. pour tes explications sur les échecs et nos discussions éducation nationale. Merci Thomas L. F. pour l'ambiance au labo. Merci Maxime pour nos discussions ciné. Merci Virgile pour ton assiduité irréductible au SdT. Merci Philomène pour ta bienveillance et ton humilité. Merci Coline pour ta gentillesse. Merci Thomas F. d'être si sympa. Merci Jean-Michel pour tes convictions. Merci Thomas G. pour tes propositions de grimpe. Merci Chenyu pour ton smile permanent. Merci Francesco pour ton engagement. Merci Elvaz pour ta fidélité à la salle info. Merci aussi à Arnaud E., Benoît, Fabrizio, Gabriel, Grace, Pierre, Thibaut, Victor. Pour terminer je remercie le pack brésilien dans son ensemble (ceux déjà cités et ceux qui ne l'auraient pas été), pour l'ambiance et la bonne humeur qu'ils ont installées au labo depuis deux ans.

Merci également aux doctorants de Chambéry que j'ai eu la chance de rencontrer à l'occasion de mes passages à Chambéry, Peio, Tristan, Yoann.

Je remercie les potes matheux que j'ai rencontrés en conférence ou en séminaire. Merci donc à Elise (merci d'avoir pensé à moi pour parler au GT !), Giulio, Jules, Roméo, Rodrigo, Thomas.

Merci à mes copains et copine de calcul des variations et optim de forme Gabriele, Kathie, Marco, Romain, que j'ai beaucoup de plaisir à retrouver sur Paris. Merci pour tous les coups (au Papagallo bien évidemment !). Merci à Luca pour tous les super moments de fun et de complicité qu'on a passés à Chambéry, à Roscoff, à Naples. Merci d'être si cool et si humble.

Pour les enseignants de maths qui m'ont marqué, merci à Ayman Moussa pour son cours d'approximation de fonctions absolument génial. Merci à Anne-Laure Dalibard pour son cours d'équations d'évolution qui a été sans aucun doute l'un des plus beaux cours de ma vie.

Merci à mon professeur de spécialité de Terminale M. Baudet pour m'avoir montré que les maths c'était fun. Merci plus encore à ma grand-mère pour son esprit tellement logique ; merci à elle et à mère pour m'avoir amené vers les maths.

J'en viens maintenant à remercier mes ami.es. non liés à la thèse. Merci aux potes de l'escalade à

l'ASSU. Merci pour ce super camp de Pâques dans des conditions exceptionnelles. Merci donc à Elliot, Elsa, Grégoire, Jérôme, Jimmy, Keren, Marie, Pierre, Salomé, Sophie, Vincent. Merci aussi à Francesco.

Merci à mes copains d'agreg (encore des matheux), Benji, Elie, (Jacky ici aussi), Oscar, Quentin, que j'ai grand bonheur à retrouver à chaque fois. Merci pour nos discussions maths. Merci Benjamin et Elie pour votre débat mémorable Beyoncé VS Mozart, et notre bataille interminable au billard. Merci Quentin d'être là malgré la distance et pour tous tes passages en métropole.

Merci maintenant à mes ami.es du lycée, mes copains de toujours, Alexia, Alexis, Arthur, qui comptent plus que tout pour moi. Merci d'être tellement présent.es dans tous les moments de ma vie et pour toute la joie que vous y apportez. Merci aussi à Flo et Max que je vois trop peu et qui font bien plaiz ; merci à Flo pour sa gentillesse, merci à Max pour son humour. Merci enfin à Naïs que j'ai beaucoup de plaisir à retrouver après plusieurs années.

Merci à mes ami.es de la Double Licence. Merci à mes compères Alice, Charlie, Minh Thi que j'aime très fort. Merci pour tous les délires qu'on partage ensemble et toute notre complicité. Merci à Juju et Kenza que j'ai toujours beaucoup de plaisir à retrouver. Merci à Victoria pour sa fidélité et sa délicatesse, merci pour toute son affection. Merci aussi à Cèd pour sa folie. Merci à Marie et son soleil à chaque fois que je la vois. Merci enfin à Léna, qui garde toujours une place particulière malgré le fait que nos chemins se soient séparés.

Et *last but not least* : un merci infini à ma famille. Merci pour votre amour et votre soutien sans faille.

Merci d'abord à ma famille Appas et Vicq ; merci à mes cousines Cléo, Eléonore, Julia, Sybil et à mon cousin Félicien, que je connais par coeur et que je prends tellement de plaisir à voir. Merci pour tout ce qu'on a partagé et tout ce qu'on partage encore. Merci à ma tantine Anne qui nous régale à chaque fois et à mon oncle Alex qui nous fait beaucoup rire. Merci à ma chère Friquie de toujours penser à nous. Merci aussi à François, Quiterie (et Marguerite !) et Sylvie.

Merci ensuite à ma famille Prunier, mes cher.es Fabien et Céline, et à ma cousine matheuse Bérénice, à qui je pense beaucoup même s'ils sont loins. Merci pour toute votre bienveillance. Merci bien sûr aussi à l'ami Bruno pour toute son affection, merci à Matts et à Adrien pour tous les moments de fun qu'on a en vacances.

Merci à Titou pour sa gentillesse et sa générosité. Et je termine par mes personnes préférées : merci à ma mère et à ma soeur, Delphine et Marie, à qui je dois tout. Merci pour votre amour inconditionnel, merci d'être mon noyau et ma stabilité. Merci à ma soeur que j'admire tant et qui m'inspire chaque jour. Merci à ma mère qui m'a tout donné. Ce manuscrit vous est dédié.

Résumé

Dans cette thèse on étudie la stabilité de la boule pour certains problèmes d'optimisation de forme. On se concentre en particulier sur des problèmes isopérimétriques et de type Faber-Krahn, et l'objectif est donc de montrer que la boule est toujours un minimum d'une certaine perturbation du problème de minimisation du périmètre d'une part, et de la 1ère valeur propre du Laplacien Dirichlet d'autre part, à volume fixé. Pour démontrer la stabilité de la boule, on met en place la procédure classique suivante : on montre d'abord la stabilité de la boule pour des perturbations régulières pour montrer ensuite la stabilité en général. Dans cette perspective, on est amené à s'intéresser en particulier aux questions de (i) régularité des formes optimales et (ii) minimisation sous contrainte de convexité.

Le Chapitre 1 est consacré à une presentation des problèmes spécifiques de la thèse. Nous détaillons les résultats obtenus et les méthodes utilisées.

Le Chapitre 2 constitue nos résultats de régularité sous contrainte de convexité. Après avoir défini une notion appropriée de quasi-minimiseur du périmètre sous contrainte de convexité, nous montrons qu'un tel quasi-minimiseur a son bord qui est $C^{1,1}$. Nous prouvons également qu'une large classe de problèmes de minimisation de la somme du périmètre et d'un terme perturbatif sous contraintes de convexité et de volume rentrent dans ce cadre. Enfin, nous montrons que de nombreux exemples pour lesquels le terme perturbatif est issu de la théorie des EDP ont leurs minimiseurs qui sont des quasi-minimiseurs en le sens introduit, obtenant ainsi la régularité $C^{1,1}$ dans ces cas.

Le Chapitre 3 est dédié à nos résultats de stabilité pour des problèmes isopérimétriques. Nous étudions la minimisation de fonctionnelles de la forme $P(\Omega) + \varepsilon R(\Omega)$ pour un $\varepsilon > 0$ fixé, où $\Omega \mapsto P(\Omega)$ est le périmètre et $\Omega \mapsto R(\Omega)$ est un terme perturbatif, dans l'objectif de montrer que la boule est un minimiseur (éventuellement local) parmi les ensembles convexes et sous contrainte de volume, lorsque ε est assez petit. On s'intéresse plus particulièrement à deux problèmes indépendants où R est une énergie de type EDP, le cas de la capacité et de la 1ère valeur propre Dirichlet, en prouvant une forme faible ou forte de stabilité dans chacun des cas.

Le Chapitre 4 contient les résultats que nous avons obtenus sur la stabilité du spectre de Dirichlet de la boule. Nous étudions la possibilité de contrôler le spectre de Dirichlet d'un ouvert Ω par la distance de sa fréquence fondamentale $\lambda_1(\Omega)$ à celle d'une boule de volume correspondant $\lambda_1(B)$. On prouve des inégalités quantitatives de la forme $|\lambda_k(\Omega) - \lambda_k(B)| \leq (\lambda_1(\Omega) - \lambda_1(B))^\alpha$ avec un exposant α différent selon que $\lambda_k(B)$ est simple ou non, et montrons qu'il est optimal dans les deux cas.

Mots clés : optimisation de forme, problème isopérimétrique, optimisation spectrale, problème à frontière libre, convexité, dérivée de forme, régularité des formes optimales, stabilité de la boule.

Abstract

This thesis studies the stability of the ball for certain shape optimization problems. We focus in particular on isoperimetric and Faber-Krahn type problems, and thus aim at showing that the ball remains a minimizer of a given perturbation of the minimization of the perimeter on the one hand, and of the 1st eigenvalue of the Dirichlet Laplacian on the other hand, at fixed volume. In order to show the stability of the ball we use the following classical strategy: we prove first the stability of the ball for smooth perturbations, and we then build on this first result to prove stability for all shapes. In this perspective, we are led to study in particular the questions of (i) regularity of optimal shapes and (ii) minimization under a convexity constraint.

Chapter 1 is dedicated to a general introduction of the specific problems we tackle in this thesis. We also detail extensively the results we obtained and the methods we used.

Chapter 2 is constitued by our regularity results under convexity constraint. After defining a suitable notion of quasi-minimizer of the perimeter under convexity constraint, we show that any such quasi-minimizer is $C^{1,1}$. We also show that an important class of minimization problems of the sum of the perimeter and a perturbative term under convexity and volume constraints enters in this framework. We finally prove that numerous examples where the perturbative term stems from PDE-theory fall into this category, thus getting $C^{1,1}$-regularity in these cases.

Chapter 3 consists of our stability results for isoperimetric problems. We thus study minimization problems of the form $P(\Omega) + \varepsilon R(\Omega)$ for some $\varepsilon > 0$ where $\Omega \mapsto P(\Omega)$ is the perimeter and $\Omega \mapsto R(\Omega)$ is a perturbative term, and aim at showing that the ball is a (local or global) minimizer among convex sets of fixed volume, for small enough ε. We investigate independently two cases where R is an energy of PDE-type, namely the capacity and the 1st Dirichlet eigenvalue, and prove either a weak or a strong form of stability in both cases.

Chapter 4 contains the results we obtained about the stability of the Dirichlet spectrum of the ball. We study the possibility of controlling the whole Dirichlet spectrum of an open set Ω by the distance of its fundamental frequency $\lambda_1(\Omega)$ to the one of a ball of corresponding volume $\lambda_1(B)$. We thus prove quantitative inequalities of the form $|\lambda_k(\Omega) - \lambda_k(B)| \leq (\lambda_1(\Omega) - \lambda_1(B))^\alpha$ with a different exponent α depending on whether $\lambda_k(B)$ is simple, and show that it is sharp in both cases.

Keywords: shape optimization, isoperimetric problem, spectral optimization, free boundary problem, convexity, shape derivative, regularity of optimal shapes, stability of the ball.

Structure of the thesis

This thesis relies on the three following works, which all deal with the issues of regularity and stability in shape optimization. They correspond to three distinct papers.

Published paper

1. *Regularity in shape optimization under convexity constraint*, written in collaboration with J. Lamboley, published in "Calc. Var. Partial Differential Equations", 62(3):Paper No. 101, 44, 2023.

Submitted papers

2. *Fuglede-type arguments for isoperimetric problems and applications to stability among convex shapes*, written as a single author.

3. *Sharp quantitative stability of the Dirichlet spectrum near the ball*, written in collaboration with D. Bucur, J. Lamboley, and M. Nahon.

The manuscript is organized around these three papers in the following way:

- Chapter 1 is dedicated to a general introduction. After having set the context, we present the specific problems of the thesis and detail our main contributions. We finally give a possible extension of some of our results in the form of an open problem.

- Chapter 2 constitutes our regularity results under convexity contraint. It corresponds to the paper *Regularity in shape optimization under convexity constraint*.

- Chapter 3 consists of our stability results for isoperimetric problems. It corresponds to the paper *Fuglede-type arguments for isoperimetric problems and applications to stability among convex shapes*.

- Chapter 4 contains the results we obtained concerning the stability of the Dirichlet spectrum of the ball. It corresponds to the paper *Sharp quantitative stability of the Dirichlet spectrum near the ball*.

Let us give a few comments about the relations between the different chapters. The four chapters are independent from one another. In particular the Introduction from Chapter 1 can be read independently of the other chapters, and we have tried to write it with a sufficient amount of details so that it provides some insights of the questions, motivations, results and methods from the three other chapters. On the other hand, as Chapters 2, 3 and 4 correspond to separate papers, they are self-contained. This also means that they carry their own internal notations, which can sometimes be different from the ones adopted in the Introduction. The bibliography is however common for the four chapters.

Contents

1 Introduction **14**
1.1 Context and preliminaries . . . 14
1.1.1 General context of shape optimization. Isoperimetric problem. . . . 14
1.1.2 Spectrum of an open set. Faber-Krahn and Saint-Venant problems. . . . 15
1.2 Problems of the thesis . . . 16
1.2.1 Main goal of the thesis: stability of the ball in certain classes of shapes . . . 17
1.2.2 Regularity in shape optimization . . . 20
1.2.3 The convexity constraint . . . 23
1.3 Main contributions of the thesis . . . 26
1.3.1 Contributions of Chapter 2: regularity under convexity constraint . . . 26
1.3.2 Contributions of Chapter 3: Fuglede-type arguments for isoperimetric problems and applications to stability among convex shapes . . . 30
1.3.3 Contributions of Chapter 4: sharp quantitative stability for the Dirichlet spectrum near the ball . . . 37
1.4 Open problem: strong stability of the ball in more general classes . . . 43

2 Regularity in shape optimization under convexity constraint **45**
2.1 Introduction . . . 45
2.2 Regularity in shape optimization . . . 49
2.2.1 Regularity for quasi-minimizers of the perimeter under convexity constraint . . . 50
2.2.2 Regularity with volume constraint . . . 60
2.3 Examples and applications . . . 63
2.3.1 First examples . . . 63
2.3.2 PDE and Spectral examples . . . 65
2.3.3 Optimality of the $C^{1,1}$ regularity . . . 79
2.4 Appendix . . . 80

3 Fuglede-type arguments for isoperimetric problems and applications to stability among convex shapes **83**
3.1 Introduction . . . 83
3.1.1 Stability in shape optimization . . . 83
3.1.2 Stability of the ball under convexity constraint . . . 84
3.1.3 Strategy of proof . . . 87
3.1.4 Plan of the paper . . . 88
3.2 Stability of the ball for an isoperimetric problem with convexity constraint involving capacity. Proof of Theorem 3.1.1 . . . 89
3.2.1 Weak (IT) property . . . 90
3.2.2 Proof of Theorems 3.2.1 and 3.1.1 . . . 93
3.2.3 Further stability results . . . 94
3.3 Minimality of the ball in a $C^{1,\alpha}$ neighborhood . . . 95

3.3.1 Continuity in the domain Ω. 97
3.3.2 Second derivative of λ_1. 104
3.3.3 Estimate of $\lambda_1''(t)$ and proof of the main results. 106
3.4 *Selection principle*: minimality of the ball among convex sets. Proof of Theorem 3.1.2. . . . 109
3.4.1 Regularity theory for the quasi-minimizer of the perimeter under convexity constraint 110
3.4.2 Proof of Theorem 3.1.2 114
3.5 Appendix 117
3.5.1 Quantified Inverse Function Theorem 117
3.5.2 Fractional Sobolev spaces 118
3.5.3 Compactness in classes of convex sets 119

4 Sharp quantitative stability of the Dirichlet spectrum near the ball 121
4.1 Introduction 121
4.1.1 Presentation of the problem 121
4.1.2 Main results 125
4.1.3 Outline of the paper 127
4.2 Some preliminary estimates 127
4.3 Proof of Theorem 4.1.1: the square root bound 129
4.4 Proof of Theorem 4.1.2: the linear bound 131
4.4.1 Existence of a minimizer 132
4.4.2 Blow-ups and viscosity solutions 138
4.4.3 Minimizers are nearly spherical 144
4.4.4 Minimality of the ball among nearly spherical sets. 148
4.4.5 Conclusion. 153
4.5 Proof of Theorem 4.1.3: linear bound on clusters of multiple eigenvalues 154
4.5.1 Harnack inequality 156
4.5.2 Flatness improvement 162
4.5.3 Minimality of the ball among nearly spherical sets. 169
4.5.4 Conclusion 171
4.6 Discussion and consequences 171
4.6.1 About the sharpness of the results 171
4.6.2 Proof of the reverse Kohler-Jobin inequality: Corollary 4.1.5 172
4.6.3 Proof of Theorem 4.1.4 173

Chapter 1

Introduction

1.1 Context and preliminaries

In this first short section we give a presentation of the general context of shape optimization, describing the usual questions associated and introducing the model problems to have in mind (such as the *isoperimetric*, *Faber-Krahn*, *Saint-Venant* problems).

1.1.1 General context of shape optimization. Isoperimetric problem.

This thesis takes place in the general framework of shape optimization, which consists in the study of minimization problems of the form

$$\inf \{J(\Omega),\ \Omega \in \mathcal{S}_{\text{ad}}\}$$

where $\mathcal{S}_{\text{ad}}$ is a given class of measurable subsets of $\mathbb{R}^N$ (for some integer $N \geq 2$) and $J : \mathcal{S}_{\text{ad}} \to \mathbb{R} \cup \{+\infty\}$ is a functional. The class $\mathcal{S}_{\text{ad}}$ can consist of all measurable subsets of $\mathbb{R}^N$, but in many relevant shape optimization problems, it can also be defined through a volume constraint, or some more specific geometric constraint such as convexity (as will be most important in this thesis), simple connectedness and so on. The functional J often involves geometric quantities (volume, perimeter, diameter...) or PDE-type energies (see below Section 1.1.2). The questions which are related to this minimization problem are the usual questions in calculus of variations, among which the most important ones are:

- Existence of an optimal shape,
- Uniqueness of the optimal shape,
- Properties of an optimal shape: symmetries, regularity of its boundary, explicit identification...

For classical introductions to shape optimization we refer the reader to [HP18], [BB05] and the references therein. The most famous example of shape optimization problem is probably the *isoperimetric problem*, which consists in the minimization of the perimeter among measurable sets of fixed volume, that is

$$\inf \left\{P(\Omega),\ \Omega \subset \mathbb{R}^N \text{ measurable},\ |\Omega| = v_0\right\} \tag{1.1}$$

where $0 < v_0 < \infty$, $|\Omega| = \int_{\mathbb{R}^N} \mathbf{1}_\Omega$ is the volume of Ω and

$$P(\Omega) := \sup \left\{ \int_{\mathbb{R}^N} \operatorname{div}(\theta),\ \theta \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N),\ \|\theta\|_{L^\infty(\mathbb{R}^N)} \leq 1 \right\} \tag{1.2}$$

is the classical De Giorgi perimeter. In this model shape optimization problem one can identify the class of minimizers: Euclidean balls of volume v_0 are the only minimizers (up to modification on a zero-measure

set). The homogeneity of the volume and of the perimeter leads to the scale-invariant *isoperimetric inequality*:

$$\frac{P(\Omega)}{|\Omega|^{\frac{N-1}{N}}} \geq \frac{P(B)}{|B|^{\frac{N-1}{N}}} \tag{1.3}$$

for any measurable $\Omega \subset \mathbb{R}^N$ with $0 < |\Omega| < \infty$ and where $B \subset \mathbb{R}^N$ is any ball, with equality occuring only for sets $\Omega \overset{\text{a.e.}}{=} B$ up to a translation. For references around the *isoperimetric inequality* we refer for instance to the works quoted in [HP18], among which we can cite the survey [Oss78].

1.1.2 Spectrum of an open set. Faber-Krahn and Saint-Venant problems.

It happens very frequently in shape optimization that one has to deal with functionals involving the spectrum of a shape Ω. One classically defines the spectrum of the Dirichlet Laplacian of an open set $\Omega \subset \mathbb{R}^N$ with finite measure as follows. We let $R_\Omega : H^{-1}(\Omega) \to H^1_0(\Omega)$ be the resolvent operator over Ω, which associates to any $f \in H^{-1}(\Omega)$ the unique distributional solution $u \in H^1_0(\Omega)$ to the Laplace equation $-\Delta u = f$ in Ω, by which we mean that

$$\forall \phi \in \mathcal{C}^\infty_c(\Omega),\ \int_\Omega \nabla u \cdot \nabla \phi = \langle f, \phi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

We refer for instance to [HP18, Proposition 4.5.1] for existence and uniqueness of such a solution for any $f \in H^{-1}(\Omega)$, but let us mention that this classically relies on a Poincaré-type inequality over Ω: there exists a dimensional constant $C_N > 0$ such that

$$\forall u \in H^1_0(\Omega),\ \|u\|_{L^2(\Omega)} \leq C_N |\Omega|^{2/N} \|\nabla u\|_{L^2(\Omega)}$$

(see [HP18, Lemma 4.5.3]), ensuring that the semi-norm $u \in H^1_0(\Omega) \mapsto \|\nabla u\|_{L^2(\Omega)}$ is in fact a norm. In the general case where Ω is unbounded with finite measure, this inequality is itself usually derived from rearrangement arguments relying on the *Polya-Szegö* inequality (see [HP18, Theorem 6.1.4]).

Since $|\Omega| < \infty$, we further have that R_Ω is compact when seen as an operator $L^2(\Omega) \to L^2(\Omega)$. As a consequence, it is a classical fact stemming from the theory of positive self-adjoint compact operators that the spectrum of the Dirichlet Laplacian over Ω is discrete and only accummulates at $+\infty$, so that one can sort in nondecreasing order its eigenvalues to get

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \cdots \nearrow +\infty$$

Of particular importance is the first eigenvalue $\lambda_1(\Omega)$, which is simple, associated to an eigenfunction $u > 0$ in Ω verifying the variational principle

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \inf\left\{ \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2},\ v \in H^1_0(\Omega) \setminus \{0\} \right\}. \tag{1.4}$$

Higher eigenvalues also enjoy a variational characterization: in general, for $k \in \mathbb{N}^*$ one has the min-max principle or *Courant-Fischer* formulae (see for instance [Hen06])

$$\lambda_k(\Omega) = \min_{\substack{V_k \subset H^1_0(\Omega) \\ V_k \text{ has dimension } k}} \max\left\{ \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2},\ v \in V_k \setminus \{0\} \right\}.$$

Instead of being open, there is a milder standard condition under which one can define the spectrum of the Laplacian over a subset $\Omega \subset \mathbb{R}^N$ of finite measure, namely the hypothesis that Ω is quasi-open, which means that $\Omega = \{\tilde{u} > 0\}$ for some $u \in H^1(\mathbb{R}^N)$, where $\tilde{u}$ is the quasi-continuous representative of u defined by

$$\tilde{u}(x) = \lim_{r \to 0} \fint_{B_r(x)} u,\ \text{a.e.}$$

The same arguments allow to define the spectrum in this more general setting (see again [HP18, Section 4.5]). The class of quasi-open sets is more suited for obtaining general existence theorems related to shape optimization problems involving the eigenvalues λ_k, among which we can quote the result by Buttazzo and Dal Maso from [BDM93a] regarding functionals nonincreasing for the inclusion.

One of the most famous inequalities in spectral geometry is the so-called *Faber-Krahn* inequality, which asserts that the only minimizers (up to modification on sets of zero capacity) of λ_1 in the class of open (or quasi-open) sets with finite fixed measure are balls. In its scale-invariant formulation it states that

$$\lambda_1(\Omega)|\Omega|^{\frac{2}{N}} \geq \lambda_1(B)|B|^{\frac{2}{N}} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is an open set with $0 < |\Omega| < \infty$ and $B \subset \mathbb{R}^N$ is any ball, with equality occuring only if (up to translation) $\Omega = B$ up to a set of zero capacity. It can be seen as an immediate consequence of the *Polya-Szegö* inequality. There is a second related inequality which is of central importance in shape optimization for spectral functionals: the so-called *Saint-Venant inequality*. It concerns the *Dirichlet energy* $E(\Omega)$ with right-hand side $f = 1$, which is defined for open (or quasi-open) sets $\Omega \subset \mathbb{R}^N$ of finite measure using the same tools as for the spectrum. It is defined through the variational principle

$$E(\Omega) := \inf \left\{ \int_\Omega \frac{|\nabla u|^2}{2} - \int_\Omega u, \ u \in H_0^1(\Omega) \right\}. \tag{1.6}$$

It can therefore be written $E(\Omega) = -\frac{1}{2}\int_\Omega |\nabla w_\Omega|^2 = -\frac{1}{2}\int_\Omega w_\Omega$ where $w_\Omega \in H_0^1(\Omega)$ is the torsion function associated to Ω, meaning that it is the unique solution to

$$\begin{cases} -\Delta w_\Omega = 1 \text{ in } \Omega \\ w_\Omega \in H_0^1(\Omega) \end{cases}$$

in the distributional sense. The *Saint-Venant inequality* states that the ball is the unique minimizer (up to translation and modification on zero-capacity sets) of the *Dirichlet energy* above among open sets Ω of fixed positive measure, *i.e.*

$$E(\Omega)|\Omega|^{-\frac{N+2}{N}} \geq E(B)|B|^{-\frac{N+2}{N}}$$

in its scale-invariant formulation. An equivalent point of view on this inequality comes from considering instead the *torsion energy* defined by

$$T(\Omega) := -2E(\Omega) = \int_\Omega |\nabla w_\Omega|^2 = \int_\Omega w_\Omega.$$

Hence, the *Saint-Venant inequality* ensures that the ball is the unique maximizer (up to translation and modification on zero-capacity sets) of the *torsion energy* among open sets Ω of fixed positive measure:

$$T(\Omega)|\Omega|^{-\frac{N+2}{N}} \leq T(B)|B|^{-\frac{N+2}{N}}.$$

For references on the *Faber-Krahn* and *Saint-Venant* inequalities and more general shape optimization problems related to the spectrum of sets we refer to [Hen06] and the references therein.

1.2 Problems of the thesis

This section is dedicated to a presentation of the problems concerned by this thesis. It is split in three parts. We present the main goals of this thesis and give a presentation of the associated problems in Section 1.2.1. We then give an overview of the issues of regularity among shapes and the convexity constraint (in respectively Sections 1.2.2 and 1.2.3), which are the two main tools of this thesis.

1.2.1 Main goal of the thesis: stability of the ball in certain classes of shapes

Main goals of the thesis. One of the main goals of this thesis is to identify relevant classes $\mathcal{S}_{\text{ad}} \subset \mathcal{P}(\mathbb{R}^N)$ of subsets of $\mathbb{R}^N$ such that the usual first and second optimality conditions at the ball for a functional $F : \mathcal{S}_{\text{ad}} \to \mathbb{R}$ ensure local minimality of the ball in $\mathcal{S}_{\text{ad}}$. Roughly speaking, we want to study whether assuming that

$$\text{for all } \xi : \mathbb{R}^N \to \mathbb{R}^N \text{ sufficiently smooth, } \left.\frac{d}{dt}\right|_{t=0} F((\mathrm{Id}+t\xi)(B)) = 0, \ \left.\frac{d^2}{dt^2}\right|_{t=0} F((\mathrm{Id}+t\xi)(B)) > 0 \text{ if } \xi \neq 0 \tag{1.7}$$

ensures that

$$\forall \Omega \in \mathcal{S}_{\text{ad}}, \ |\Omega \Delta B| \ll 1, \ F(\Omega) \geq F(B). \tag{1.8}$$

A closely related question is whether minimality for smooth perturbations $(\mathrm{Id}+\xi)(B)$ of the ball is enough for obtaining local (in a L^1 sense) minimality of F inside $\mathcal{S}_{\text{ad}}$. It was observed in [DL19, Proposition 6.1] that this is not true in general if $\mathcal{S}_{\text{ad}}$ consists of all shapes, so that the question of finding reasonable classes $\mathcal{S}_{\text{ad}}$ and relevant examples of F for which the implication

$$\textit{local minimality of the ball for smooth shapes} \Longrightarrow \textit{local minimality of the ball in } \mathcal{S}_{\text{ad}} \tag{1.9}$$

holds, is interesting. Apart from the choice of class $\mathcal{S}_{\text{ad}}$, the degree of "smoothness" of the shapes for which one is able to prove minimality of the ball in the first place is of major importance in this matter, as will become clearer further in this section. As a consequence, we can summarize the main questions and goals of this thesis as follows:

1. *Deriving local minimality of the ball for a functional F among classes of "smooth" perturbations $\mathcal{S}_{\text{smooth}} \subset \mathcal{S}_{\text{ad}}$.*
2. *Does (1.7) imply (1.8) for a functional F and a class $\mathcal{S}_{ad}$?*
3. *Does the implication (1.9) hold for a functional F and a class $\mathcal{S}_{ad}$?*

They are closely related to the question of stability of the ball, which we introduce below.

Let us sketch the plan of Section 1.2.1. We start by defining and discussing a general notion of stability and its link with quantitative inequalities. We then describe in details a central strategy for proving stability inequalities, which relies on the principle (1.9). We finally motivate the issues of regularity and convexity, which can be considered as the two mains tools of this thesis for proving stability among certain classes of shapes.

Stability in shape optimization. Given a class $\mathcal{S}_{\text{ad}}$ of measurable subsets of $\mathbb{R}^N$ of volume $|A| = 1$, and functionals $J, R : \mathcal{S}_{\text{ad}} \to \mathbb{R}$, we consider the minimization problem

$$\inf \{J(\Omega) + \varepsilon R(\Omega), \ \Omega \in \mathcal{S}_{\text{ad}}\}$$

where $\varepsilon > 0$ is a small parameter. Let us further assume that J and R are translation invariant (but the following discussion easily adapts without this assumption). We make the hypothesis that balls of volume 1 are the unique minimizers of J in $\mathcal{S}_{\text{ad}}$, meaning that we have a rigid inequality $J(\Omega) \geq J(B)$ for any $\Omega \in \mathcal{S}_{\text{ad}}$ where B is any ball of volume 1 (which is verified for instance for $J = P$ or $J = \lambda_1$). We say that the ball is (globally or locally) *weakly stable* for the problem $J + R$ provided for sufficiently small ε

$$B \textit{ is a (global or local) minimizer of } J + \varepsilon R \textit{ in } \mathcal{S}_{\text{ad}}. \tag{1.10}$$

By locality we mean here that minimality holds for sets Ω which are close to B in a rough sense. This is often given by the fact that the L^1 distance with the ball $|\Omega \Delta B|$ is small, which is always what we will

mean in the sequel. Let us refer to (1.16) below where we introduce a different (and somehow stronger) notion of stability of the ball. In the sense (1.10), stability of the ball means that the ball is still (locally or globally) minimal for perturbations of the minimization of J by R. There is an equivalent point of view on stability which is also interesting. It consists in writing

$$\forall \Omega \in \mathcal{S}_{\mathrm{ad}},\ J(\Omega) - J(B) \geq \varepsilon(R(B) - R(\Omega)) \tag{1.11}$$

(holding locally if stability is asummed to be local). When the ball is maximal for R inside $\mathcal{S}_{\mathrm{ad}}$ (which is often verified for relevant examples) the right-hand-side is non-negative, thus implying that the deficit of J quantifies the deficit of R. This is the point of view of *quantitative inequalities*. There is an extensive literature on such inequalities, which concerns in particular functionals $R(\Omega) = \mathcal{A}(\Omega)$ where $\mathcal{A}(\Omega)$ is some asymmetry translating the fact that Ω is more or less far from being a ball. A famous example goes back to Bonnesen in [Bon24] which proves that if $\Omega \subset \mathbb{R}^2$ is convex and such that $B_{r_1} \subset \Omega \subset B_{r_2}$ where B_{r_i} are concentric and have radius r_i

$$\frac{P(\Omega) - P(B)}{P(B)} \geq (r_2 - r_1)^2 \tag{1.12}$$

(see [Fus15] for more details on this inequality). Note that the asymmetry $r_2 - r_1$ gives information about the Hausdorff distance between Ω and B, and is made possible by the convexity assumption. In order to obtain *quantitative inequalities* for all sets, one has to weaken the asymmetry. The most famous example in this direction is given by the *sharp quantitative isoperimetric inequality*, which was first proven in [FMP08]. Setting

$$\mathcal{A}_F(\Omega) := \inf\left\{|\Omega \Delta (B + x)|,\ x \in \mathbb{R}^N\right\} \tag{1.13}$$

the Fraenkel asymmetry of Ω, it states that the ball is globally stable for $P - \mathcal{A}_F^2$, or in other words that for all $\Omega \subset \mathbb{R}^N$ with $|\Omega| = 1$ it holds

$$P(\Omega) - P(B) \geq c_N \mathcal{A}_F(\Omega)^2 \tag{1.14}$$

for some dimensional $c_N > 0$. Sharpness refers to the exponent 2 on the asymmetry, as it can be shown that both sides of the inequality are of the same order for particular sequences of ellipsoids converging to the ball. Prior to this sharp inequality, non-sharp results were available (see for instance to [Hal92] in which the author proves an inequality with exponent 4). Inequality (1.14) was obtained in [FMP08] by symmetry arguments, and it was later applied by the same authors in [FMP09] to obtain quantitative versions of other shape inequalities, such as the Faber-Krahn inequality (1.5) and the isocapacitary inequality (see below (1.41)). The idea behind their approach was to follow the classical proofs of the same inequalities but to use the quantitative inequality (1.14) in place of (1.3), so that by refining the analysis one could obtain quantitative versions of the inequalities. However their method was expected not to be optimal, as it provided in both cases the exponent 4 (the expected sharp exponent being again 2).

In [CL12] the authors provided a new proof of (1.14) by introducing a general method - the so-called *selection principle* - which turns out to be sufficiently flexible to be applied to other functionals. In fact, it has been successfully applied since for proving local or global stability (1.10) for many other shape inequalities, among which we can quote [AFM13], [FFM+15] for $J = P$, [BDPV15], [AKN21, AKN22] for $J = \lambda_1$, [DPMM21] for $J = \mathrm{Cap}$ (see (1.39) below for a definition). Let us highlight in particular the *sharp quantitative Faber-Krahn inequality*, which is the analogue of (1.14) with λ_1 in place of P, *i.e.*

$$\forall \Omega \subset \mathbb{R}^N \text{ open with } |\Omega| = 1,\ \lambda_1(\Omega) - \lambda_1(B) \geq c_N \mathcal{A}_F(\Omega)^2. \tag{1.15}$$

Stability for smooth shapes $\Longrightarrow$ stability for all shapes. The *selection principle*. Relying on the idea expressed by (1.9), there are numerous examples of works where the authors prove stability in two steps: (i) prove first local minimality of the ball among smooth perturbations and (ii) then prove stability for general shapes $\Omega \in \mathcal{S}_{\mathrm{ad}}$. Before going into the details of this method, let us just say that step (i) relies

on differential calculus on the functionals involved, while step (ii) uses a regularity theory associated to the functional.

In order to prove minimality of the ball for $J_\varepsilon := J + \varepsilon R$ one proves the following two independent steps.

- *(i) Stability for smooth shapes:* proving minimality of the ball for J_ε inside a subclass $\mathcal{S}_X \subset \mathcal{S}_{\rm ad}$ of smooth perturbations of the ball, by which we mean that any $\Omega \in \mathcal{S}_X$ can be written $\Omega = B_\xi := ({\rm Id} + \xi)(B)$ where $\xi : \mathbb{R}^N \to \mathbb{R}^N$ is a vector field lying in a space of smoothness X, with in addition $\|\xi\|_X \ll 1$. This is achieved by performing differential calculus over $J_\varepsilon(B_\xi)$, meaning that by differentiating $j_\varepsilon(\xi) := J_\varepsilon(B_\xi)$ in $\xi \in X$ at $\xi = 0$ one proves that $j'_\varepsilon(0) = 0$ and $j''_\varepsilon(0)$ is coercive (in some sense taking into account the volume constraint and invariance by translation). In the literature this is refered to as *Fuglede-type computations*, in reference to the seminal work [Fug89] where the author made such computations for the perimeter. This can be achieved for geometric functionals F where $F(B_\xi)$ can be explicitly written in terms of ξ (such as $F = P$ or $F = |\cdot|$) as well as for less explicit energies involving PDE theory (see for instance [Dam02], [AFM13], [BDPV15], [DL19]). Note that this first step of the strategy is not so specific to the functional under study, and it is by now very well understood. We refer to [DL19] for a detailed study; in particular the authors prove a general stability result inside $\mathcal{S}_X$ under first and second order optimality conditions.

- *(ii) Stability for all shapes:* recovering minimality of the ball for general perturbations $\Omega \in \mathcal{S}_{\rm ad}$ by relying on the first step and on a regularity theory for the functional under study. In the classical setting where $\mathcal{S}_{\rm ad}$ consists of all shapes of given volume, this step relies on fine arguments related to regularity theories associated to the functionals. In the case $J = P$, it is the theory of quasi-minimizers of the perimeter (see (1.23)), while for $J = \lambda_1$ or $J = {\rm Cap}$ it is the free boundary regularity theory (see (1.27)). The idea is then to prove that minimizers $\Omega_\varepsilon \in \mathcal{S}_{\rm ad}$ of $J + \varepsilon R$ in $\mathcal{S}_{\rm ad}$ are smooth in the sense of X, meaning that they can be written $\Omega_\varepsilon = B_{\xi_\varepsilon}$ for some $\xi_\varepsilon \in X$ with $\|\xi_\varepsilon\|_X \ll 1$. Note that stability is then obtained by letting $\varepsilon \to 0$, as $\varepsilon \to 0$ usually enforces that minimizers $\Omega_\varepsilon \to B$ in the L^1 sense, so that regularity of Ω_ε then enables to write $\Omega_\varepsilon = B_{\xi_\varepsilon}$ with a strengthening of convergence $\|\xi_\varepsilon\|_X \to 0$, thus allowing to apply the first step of the method to get stability in $\mathcal{S}_{\rm ad}$. This second step of the strategy highlights the importance of the space X found in step (i), and the aim of proving stability for the "lowest degree of smoothness possible".

Let us mention several works relying on this regularity strategy: [KM13, KM14] for $J = P$ and $R = V_\alpha$ the Riesz energy, [GNR18] for $J = P$ and $R = \mathcal{I}$ the logarithmic capacity, and [MR21] for $J = \lambda_1$ and $R = V_\alpha$. The authors prove that minimizers of the associated problems have in fact the appropriate degree of smoothness required in order to apply step (i). However, as in [BDPV15], there are cases where minimizers are not expected to have too much regularity, so that one needs to regularize them in some sense in order to improve regularity and apply step (i). This is the *selection principle* procedure introduced in [CL12], which is a tool for proving step (ii). Although it can be employed to prove *weak stability* (1.10), it also adapts well to a different notion of stability which we introduce now, closer to the question (1.7) $\Longrightarrow$ (1.8). We say that the ball is *strongly stable* for the problem $J + R$ if there exists $c^* > 0$ such that:

$$\begin{aligned} &B \textit{ is a local minimizer of } J + cR \textit{ in } \mathcal{S}_{\rm ad} \textit{ for } c \in (0, c^*) \textit{ while} \\ &B \textit{ is not a local minimizer of } J + cR \textit{ in } \mathcal{S}_{\rm ad} \textit{ for } c > c^* \end{aligned} \tag{1.16}$$

where locality is again meant in a L^1 sense. The value c^* is an optimal threshold fully describing the situation of local minimality of the ball. For this strong notion of stability we always have in mind that c^* is given by (1.7), meaning that $F = J + cR$ ceases to verify the second-order optimality condition from (1.7) exactly at c^*. In this *strong stability* setup the *selection principle* goes as follows: proceeding by contradiction for some $0 < c < c^*$, one assumes the existence of a sequence Ω_i such that

$$\begin{cases} J_c(\Omega_i) < J_c(B), \ \forall i \in \mathbb{N} \\ a_i := |\Omega_i \Delta B| \to 0 \end{cases} \tag{1.17}$$

One then introduces for each $i \in \mathbb{N}$ a minimizer $\widetilde{\Omega}_i$ of

$$\inf \left\{ J_c(\Omega) + C \left| |\Omega \Delta B| - a_i \right| ,\ \Omega \in \mathcal{S}_{\text{ad}} \right\} \tag{1.18}$$

for some penalizing constant $C > 0$. If $\widetilde{\Omega}_i$ exists it satisfies for free $J_c(\widetilde{\Omega}_i) < J_c(B)$, while the condition $|\widetilde{\Omega}_i \Delta B| \to 0$ is expected to hold thanks to the penalization by a_i, so that overall the sequence $\widetilde{\Omega}_i$ still verifies the contradiction hypothesis (1.17). In order to obtain a contradiction the goal is then to show that $\widetilde{\Omega}_i = B_{\xi_i}$ for a sequence $\xi_i \in X$ with $\|\xi_i\|_X \to 0$. The idea is to interpret minimizers of (1.18) as minimizers of a perturbation (by a volume term) of the problem of minimizing J_c, which enters in the frameworks we describe in details in Section 1.2.2 when either $J = P$ or $J = \lambda_1$ (see the first two paragraphs in Section 1.2.2).

Regularity and convexity. We have understood in the two steps strategy described above that regularity of minimizers for the problem

$$\inf\{J(\Omega) + cR(\Omega),\ \Omega \in \mathcal{S}_{\text{ad}}\}$$

can be of main importance for proving (local or global) minimality of the ball. Note that an underlying regularity theory is a regularity theory inside $\mathcal{S}_{\text{ad}}$, which is therefore very specific to the chosen set of admissible shapes.

As mentionned above, the study of classes $\mathcal{S}_{\text{ad}}$ where the principle (*first and second-order optimality conditions* $\Longrightarrow$ L^1 *local minimality*) (*i.e.* (1.7) $\Longrightarrow$ (1.8)) can hold is of interest, and we thus study in this thesis the particular case of convex shapes:

$$\mathcal{S}_{\text{ad}} = \mathcal{S}_{\text{conv}} := \{\Omega \subset \mathbb{R}^N \text{ convex}\}$$

As a consequence, we see convex shapes as an interesting class to examine in itself, but also as a useful tool for illustrating the robustness of the strategy. We are thus led to investigate the possibility of regularity theories inside $\mathcal{S}_{\text{conv}}$.

This motivates the studies of (i) regularity and (ii) the convexity contraint, in shape optimization. The remaining of Section 1.2 is thus organized as follows: we introduce and discuss in details regularity among all shapes in Section 1.2.2, and we decribe the convexity constraint in Section 1.2.3.

1.2.2 Regularity in shape optimization

In this section we give a general presentation of the problem of proving regularity for solutions of shape optimization problems. By regularity of an open set $\Omega \subset \mathbb{R}^N$ we will always mean that it can be locally parametrized as the hypergraph of a function bearing some regularity. Precisely, we say that Ω is $\mathcal{C}^{k,\alpha}$ for some $k \in \mathbb{N}^*$ and $\alpha \in (0,1]$ if for any $x_0 \in \partial\Omega$ there exists a $(N-1)$-dimensional open ball B_{x_0}, $a > 0$ and a function $g := g_{x_0} \in \mathcal{C}^{k,\alpha}(\overline{B_{x_0}})$ such that up to a rotation

$$\begin{aligned} \Omega \cap (B_{x_0} \times (-a,a)) &= \left\{ x = (x', x_N) \in \mathbb{R}^N,\ x_N > g(x') \right\}, \\ \partial\Omega \cap (B_{x_0} \times (-a,a)) &= \left\{ x = (x', x_N) \in \mathbb{R}^N,\ x_N = g(x') \right\}. \end{aligned} \tag{1.19}$$

We say that Ω is Lipschitz when the functions g are Lipschitz continuous.

Regularity for perturbed isoperimetric problems. Quasi-minimizers of the perimeter. In reference to the standard *isoperimetric problem* (1.1), we call perturbed isoperimetric problem a shape optimization problem of the form

$$\inf \{P(\Omega) + R(\Omega),\ \Omega \in \mathcal{S}_{\text{ad}}\} \tag{1.20}$$

where $\mathcal{S}_{\text{ad}}$ is a given class of measurable subsets of $\mathbb{R}^N$, P is the De Giorgi perimeter (as defined in (1.2)) and $R : \mathcal{S}_{\text{ad}} \to \mathbb{R}$ is a perturbative functional. We see this problem as a perturbation of the *isoperimetric*

problem when the functional R is of lower order than the perimeter, in the sense that it does not change its regularizing effect. Before giving a more precise idea of what we mean by *perturbed*, let us give some informal considerations about the fact that the perimeter tends to regularize optimal shapes.

There are several ways of giving sense to the differentation of P and R at an optimal shape $\Omega^* \in \mathcal{S}_{\rm ad}$ for (1.20). Provided Ω^* has some regularity (say it is at least Lipschitz) a simple idea is to perturb locally Ω^* around a point $x_0 \in \partial\Omega^*$, meaning that we differentiate in t the quantity $(P+R)(\Omega^*_t)$ where $\partial\Omega^*_t$ is given around x_0 by the graph of some function $g + t\phi$ where g parametrizes $\partial\Omega^*$ as in (1.19) and $\phi \in \mathcal{C}^\infty_c(B_{x_0})$. Since the portion of $P(\Omega^*_t)$ changing around x_0 is given by the $(N-1)$-dimensional area of the Lipschitz graph of $g + t\phi$ (see for instance [AFP00, Remark 2.72]), one classically obtains

$$\frac{d}{dt}\bigg|_{t=0} P(\Omega^*_t) = -\int_{B_{x_0}} \operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right)\phi \tag{1.21}$$

where the divergence is taken in the sense of distributions. As a consequence, provided one can differentiate $R(\Omega^*_t)$ and if (say) the class $\mathcal{S}_{\rm ad}$ consists of open sets with fixed volume, one obtains an optimality condition which is a (non-linear) elliptic equation in g. If the derivative of R in this sense has some low regularity, classical theory of elliptic PDE's allows to improve the regularity of g. For a rigorous presentation of this type of argument for the isoperimetric problem we refer to [BB05, Section 1.2].

A typical example of R which we think of as being of lower order than the perimeter is given by $R(\Omega) = \int_\Omega f$ where $f \in L^\infty_{\rm loc}(\mathbb{R}^N)$ (the resulting problem is the so-called *prescribed mean curvature* problem, see for instance [FM11]). More generally we have in mind any R which is of volume type, meaning that R verifies a (local) Lipschitz-type property with respect to the volume distance: for any bounded $D \subset \mathbb{R}^N$ there exists $C_D > 0$ such that

$$\forall \Omega, \tilde\Omega \subset D,\ |R(\Omega) - R(\tilde\Omega)| \le C_D |\Omega \Delta \tilde\Omega|. \tag{1.22}$$

These examples fall into the general category of *quasi-minimizers* of the perimeter, which is the rigorous notion formalizing the fact that minimizers of perturbed isoperimetric problems (1.20) share some regularity. Given a bounded open set $D \subset \mathbb{R}^N$ and $\Lambda > 0$, $r_0 > 0$, a measurable set with finite perimeter $\Omega \subset \mathbb{R}^N$ is called a (Λ, r_0)-minimizer of the perimeter in D if for any $0 < r < r_0$, $x \in \mathbb{R}^N$ and $\tilde\Omega \Delta \Omega \Subset B_r(x) \cap D$ it holds

$$P(\Omega, B_r(x)) \le P(\tilde\Omega, B_r(x)) + \Lambda r^N \tag{1.23}$$

where $P(E, B_r(x))$ is the perimeter of E inside $B_r(x)$, that is

$$P(E, B_r(x)) := \sup\left\{\int_E \operatorname{div}(\theta),\ \theta \in \mathcal{C}^\infty_c(B_r(x), \mathbb{R}^N),\ \|\theta\|_{L^\infty(B_r(x))} \le 1\right\}.$$

Note that Λr^N is to be thought of as an error term (which in this case is a volume term), since $|\Omega\Delta\tilde\Omega|$ is smaller (up to a constant) than r^N in the minimality property (1.23). A deep result of geometric measure theory states that quasi-minimizers in D are $\mathcal{C}^{1,\frac12}$ regular inside D up to some residual set of dimension less than $N-8$ (see [Mag12, Theorem 21.8] or [Tam84, Theorem 1.9]). There are numerous references studying this type of perturbed isoperimetric problem by relying on this quasi-minimality property to infer regularity, and we can quote among many others [AFM13], [KM13, KM14], [CL12] or [FFM+15] for some non-local perimeter.

Regularity for perturbed Faber-Krahn problems. Free boundary problems. In reference to the standard *Faber-Krahn inequality* (1.5) we call perturbed Faber-Krahn problem a shape optimization problem of the form

$$\inf\{\lambda_1(\Omega) + R(\Omega),\ \Omega \in \mathcal{S}_{\rm ad}\} \tag{1.24}$$

where $\mathcal{S}_{\rm ad} \subset \mathcal{P}(\mathbb{R}^N)$ is a given class of open subsets of $\mathbb{R}^n$ and $R : \mathcal{S}_{\rm ad} \to \mathbb{R}$ is a perturbative term. Analogously to the case of the perimeter in the previous paragraph, before getting more precise about

the functionals R and the associated problems we have in mind, we start by examining the first order optimality condition associated to the usual *Faber-Krahn* problem.

If Ω^* is a bounded open minimizer of λ_1 among all open sets of unit volume, then for any $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$, setting $\Omega_t^* := (\mathrm{Id} + t\xi)(\Omega^*)$ one has the optimality condition

$$\frac{d}{dt}\bigg|_{t=0} (\lambda_1 + \Lambda|\cdot|)(\Omega_t^*) = 0$$

for some Lagrange multiplier $\Lambda \in \mathbb{R}_+^*$ associated to the volume constraint. Assuming enough regularity on Ω^* ($\mathcal{C}^2$ is enough, see for instance [HP18]) this reads

$$\partial_{\nu_{\Omega^*}} u = \sqrt{\Lambda} \text{ over } \partial\Omega^* \tag{1.25}$$

where $\nu_{\Omega^*} : \partial\Omega^* \to \mathbb{R}^N$ is the outward unit normal of Ω^* and u is the first eigenfunction of Ω^* (chosen positive and normalized as $\|u\|_{L^2(\Omega)} = 1$). Together with the equations verified by u as being an eigenfunction with constant sign we obtain a typical example of an *overdetermined problem*:

$$\begin{cases} -\Delta u = \lambda_1(\Omega^*)u \text{ in } \Omega^*, \\ u > 0 \text{ in } \Omega^*, \\ u = 0 \text{ over } \partial\Omega^*, \\ \partial_{\nu_{\Omega^*}} u = \sqrt{\Lambda} \text{ over } \partial\Omega^*. \end{cases} \tag{1.26}$$

Serrin studied this type of problems in his seminal work [Ser71] and proved, by relying on the so-called moving plane method, that under smoothness assumptions over Ω^* the system (1.26) implies that Ω^* is a ball. This enters the class of *free boundaries* problems, in reference to the condition (1.25) which holds on the unknown boundary $\partial\Omega^*$. For references on *overdetermined problem* we refer to [FG08] and the references therein.

Going back to (1.24), an important class of functionals R which can be considered as perturbations of λ_1 are functionals of volume-type, meaning that the variation $|R(\Omega) - R(\widetilde{\Omega})|$ is comparable to the volume difference $|\Omega\Delta\widetilde{\Omega}|$ (as in (1.23)). In this case, there is an extensive literature stemming from the pioneering work [AC81] which ensures that one might expect regularity. In [AC81] the authors study the *free boundary* problem

$$\inf\left\{\int_D |\nabla u|^2 + \Lambda|\{u > 0\} \cap D|,\ u \in H^1(D),\ u = g \text{ on } \partial D\right\} \tag{1.27}$$

where D is a Lipschitz bounded open set, and $g \in H^1(D)$ is nonnegative. They proved $\mathcal{C}^{1,\alpha}$ regularity of minimizers $u \geq 0$ around flat points of the free boundary $\partial\{u > 0\} \cap D$. More precisely, setting $\Omega := \{u > 0\}$, they decompose $\partial\Omega$ into a relatively open $\mathcal{C}^{1,\alpha}$ part (characterized by some flatness property of the associated points $x \in \partial\Omega$), and a singular part having less than $(N-1)$ Hausdorff dimension (the existence of a critical exponent $N^* \in \mathbb{N}$ with a classification of the possible behaviours of the singular set depending on whether $N \geq N^*$ was proven by [Wei99], and it was later obtained by several authors that $N^* \in [5, 7]$, see [Vel19] for references). Relying on hodograph transform techniques from [KN77], they proved higher regularity of the regular part of the free boundary. For an introduction to the problem (1.27) and to the so-called *one-phase Bernoulli problem* we refer to the very nice lecture notes [Vel19].

Relying on the definition of λ_1 as a minimum (see (1.4)), one can intepret the minimization of $\lambda_1(\Omega) + \Lambda|\Omega|$ as a problem of type (1.27). This observation enabled many authors to apply the techniques from [AC81] in order to prove $\mathcal{C}^{1,\alpha}$ (or even higher) regularity of minimizers for problems of type (1.24) where R is of volume type. In [BDPV15], regularity was successfully obtained as a crucial step for proving the *sharp quantitative Faber-Krahn inequality* (see (1.15)) for minimizers of $E + R$, with E the Dirichlet energy and $R = \Lambda|\Omega| + \alpha_F$ some regularized Fraenkel asymmetry α_F (see further (1.13) for the definition of the

usual Fraenkel asymmetry $\mathcal{A}_F$). In this work the *Kohler-Jobin inequality* (1.52) enabled to reduce the study of minimizers of $\lambda_1 + R$ to the study of minimizers of $E + R$, thus bringing a simpler free boundary problem. Along the same lines, we can quote [MR21] where the authors followed the same strategy for the functional $\lambda_1 + \Lambda|\cdot| + V_\alpha$ with $V_\alpha(\Omega) := \iint_\Omega |x-y|^{\alpha-N} dxdy$ the Riesz energy, and [DPMM21] for $\mathrm{Cap} + \Lambda|\cdot| + \alpha_F$. We also refer to [Buc12] where non-degeneracy properties of some one-sided minimizers of $E + \Lambda|\cdot|$ were used as a means to prove existence of minimizers of λ_k among quasi-open sets of fixed measure.

Let us mention a second approach for proving regularity of the free boundary around flat points for problem (1.27), which was introduced in [DS11]. The author studies continuous nonnegative solutions of

$$\begin{cases} \Delta u = f \text{ in } \Omega, \\ |\nabla u| = g \text{ over } \partial\Omega, \end{cases}$$

where the equations are verified in the viscosity sense (roughly meaning that one can write conditions like $\Delta\phi(x_0) \leq f$ and $|\nabla\phi(x_0)| \leq g$ for smooth ϕ, provided $\phi \leq u$ around $x_0 \in \Omega$ and $\phi(x_0) = u(x_0)$, see for instance [Vel19, Definition 7.6] for a rigorous definition). It was proven in [DS11] that if

$$\|f\|_{L^\infty(B_{x_0})},\ \|g\|_{\mathcal{C}^{0,\beta}(B_{x_0})} \ll 1$$

for a fixed ball centered at $x_0 \in \partial\{u > 0\}$, then provided $\{u > 0\} \cap B_{x_0}$ is close enough to a hyperplane near x_0 it is in fact $\mathcal{C}^{1,\alpha}$ in a smaller ball centered at x_0. The idea is based on a partial Harnack inequality, used to prove an improvement of flatness for flat solutions u of the above problem. It turned out to be quite flexible, especially regarding adaptations to multi-phase free boundary problems (see for instance [MTV21], [KL18]).

In this thesis (see Chapter 4) we have specifically considered problems (1.24) where the functional R is of same order as λ_1, by which we mean that it is also given by some energy term of the same type as λ_1. In this direction we can mention the works [KL18, KL19], [MTV17] and [CSY18], which prove regularity results for functionals of the form $F(\lambda_1(\Omega), \dots, \lambda_n(\Omega)) + \Lambda|\Omega|$, where F is nondegenerate in some sense, by which we mean that it leads to a vectorial free boundary problem associated to the boundary condition

$$a_1(\partial_{\nu_\Omega} u_1)^2 + \cdots + a_n(\partial_{\nu_\Omega} u_n)^2 = \text{cst over } \partial\Omega^*$$

for some $a_i \geq 0$, for a minimizer Ω^*. In a different direction, in [OK13] and [OK14] the authors study numerically and theoretically convex combinations of respectively two and three successive Dirichlet eigenvalues. Their results apply in particular to the minimizations $\lambda_1 + t(\lambda_2 - \lambda_1)$ for $t \in [0,1]$ and $\lambda_1 + t_1(\lambda_2 - \lambda_1) + t_2(\lambda_3 - \lambda_1)$ for $t_1, t_2 \geq 0$ with $t_1 + t_2 \leq 1$.

1.2.3 The convexity constraint

Interest of the convexity constraint. We speak of shape optimization problems (or more generally calculus of variations problems) under convexity constraint when the admissible sets (or functions) are assumed to be convex. One of the motivations for studying such problems is that it can be physically relevant in some situations, the most famous example being perhaps Newton's problem of the body of minimal resistance, which consists in the minimization

$$\inf\left\{\int_D \frac{1}{1+|\nabla u|^2},\ u : D \to [-M, 0],\ u \text{ is convex}\right\} \tag{1.28}$$

for $D \subset \mathbb{R}^2$ a smooth open convex set and $M > 0$. The energy one wants to minimize represents the resistance of a convex body (given by the graph of u, imposed to have M as a maximal height) as it moves in a fluid with constant velocity. We refer for instance to [BFK95], [LRP01b], [BB05] for an introduction to this problem and details about the physical relevance of its modelling.

On the other hand, the set of convex shapes (or of convex functions) can also appear as a natural class to investigate when (i) either there is no existence for the considered problem without constraint, or (ii) when one knows that the optima are not convex (thus giving rise to interesting new problems). Concerning the first issue, on the contrary enforcing a convexity constraint is expected to give good compactness properties so as to ensure existence in many situations where existence fails in general (see Chapter 2 in [BB05]). Apart from Newton's problem (1.28) above for which this is the case (see [BB05]), let us quote the famous Gamow's liquid drop model, which consists in the minimization

$$\inf\left\{P(E)+\iint_{E\times E}\frac{dxdy}{|x-y|},\ E\subset\mathbb{R}^3,\ |E|=v_0\right\} \tag{1.29}$$

While it is known that existence holds for small volumes $v_0>0$ (and in fact that balls are the unique minimizers, see [KM14]), there is no existence for large volumes and it is expected that the behaviour of minimizing sequences is to spread mass into small pieces (see [LO14]). On the other hand the convexity constrained version of (1.29) has a solution for all volumes (see further Proposition 2.3.1), thus motivating the convexity constraint.

As an example of the second issue (ii), let us note that there is numerical evidence that for $k=2,5,6,7$ the minimization of $P+c\lambda_k$ (for $c>0$) brings non-convex shapes as optima in three dimensions (see [BO16, Figure 2]), so that the study of the minimization of this functional inside the class of convex shapes is meaningful to examine.

Regularity under convexity constraint. Let us now describe in more details the methods and specific issues related to proving regularity for shape optimization problems with a convexity constraint. The situation is very different from the situation without convexity, as any convex optimizer Ω^* has its boundary which can be locally parametrized by convex (hence Lipschitz) functions. Therefore minimizers have a bit of regularity, and the hard part is to move from this first regularity property to higher regularity, as we shall explain now. When the shape functional $J(\Omega)$ can be (locally) represented by the integral over a convex set $D\subset\mathbb{R}^{N-1}$ of some Lagrangian $L(x,u(x),\nabla u(x))$ (which often happens for geometric functionals, such as P or $|\cdot|$), we are naturally led to study the minimization problem

$$\inf\left\{\int_D L(x,u(x),\nabla u(x))dx,\ u:D\to\mathbb{R}\text{ is convex}, u=u_0\text{ over }\partial D\right\} \tag{1.30}$$

for some fixed $u_0:D\to\mathbb{R}$ enforcing boundary conditions on u. Such problems have been studied by several authors for Lagrangians $L(x,u,p)$ verifying some strict convexity assumptions in p (see [Car02] and [CLR01] for $\mathcal{C}^1$ regularity results, and [CCL13] for $\mathcal{C}^{1,\alpha}$ regularity results). There are at least two specific difficulties linked to the convexity constraint for proving regularity of optimal shapes.

- The first difficulty is shared with problems in calculus of variations such as (1.30), and concerns the derivation of optimality conditions. In fact, although the initial Lipschitz regularity of convex minimizers is often enough to give meaning to differentiations of the type (1.21), due to the convexity constraint one cannot write that expressions like (1.21) vanish for all ϕ. It is still possible to write optimality conditions (see [Car02] for first order conditions in any dimensions, and [LN10] for first and second order conditions in two dimensions) but it is in general very difficult to exploit, especially for $N=3$ or higher dimensions. Let us mention however the works [Car02] and [LNP12] where the authors respectively obtain $\mathcal{C}^1$ and $\mathcal{C}^{1,1}$ regularity for $N=2$ by relying on such optimality conditions, the first for problems of type (1.30), the second for more general functionals which can also include terms $R(\Omega)$ not (locally) written as integrals of Lagrangians (including for instance *perturbed isoperimetric problems* of type (1.20)).

- The second difficulty is specific to convexity constrained shape optimization problems, as it concerns the possibility of interpreting such problems inside the calculus of variations framework. In fact,

the convexity constraint being fairly rigid and somehow global, this prevents to localize so easily the arguments by perturbing the boundary of an open convex set Ω near each $x_0 \in \partial\Omega$. One of the main contributions of this thesis consists in proposing a setup enabling to do so (see further the methods employed for proving Theorem 1.3.3).

These two difficulties are closely linked to the fact that we do not know whether minimizers are strictly convex, and in fact in general there is no reason why this should happen. Newton's problem of body of minimal resistance (1.28) is a striking example in this direction, as it is proven in [LRP01a] that there does not exist any open set over which a minimizer is strictly convex. This is due to the intrinsic non-convex nature of the problem, which forces the minimizers to saturate the convexity constraint in some sense. Let us also refer to [LNP12] and [LNP22] where the authors prove respectively that minimizers of functionals of the type $R(\Omega) - P(\Omega)$ and $R(\Omega) - \lambda_1(\Omega)$ in the class of convex sets are polygons. There are less "extreme" examples of minimization among convex shapes for which one can expect minimizers to have flat parts in their boundary, even for functionals including some strictly convex energy such as the perimeter. It is proven for instance in [AFN11] (see also [BCH17, Theorem 1.2]) that the best constant among convex shapes in the *quantitative isoperimetric inequality* (1.14) is achieved by a stadium shape (the convex hull of two disjoint disks of same radius). Also, it is proven in [HO03] that a minimizer of λ_2 with volume and convexity constraint must have at least two segments in its boundary.

In view of these two difficulties, a general strategy for proving regularity of minimal convex shapes is the so-called *cutting procedure*. Given a minimizer Ω, it consists in building competitors in the form $\Omega \cap H$ where H is some well-chosen hyperplane (which are thus straightforwardly convex). More specifically, if $x_0 \in \partial\Omega$, one builds a family of appropriate hyperplanes $(H_r)_{r>0}$ such that $\Omega \cap H_r$ is obtained by cutting a piece of Ω around x_0 (which may not be localized around x_0) with furthermore $|\Omega \cap H_r| \to |\Omega|$. The goal is then to translate conditions $J(\Omega) \leq J(\Omega \cap H_r)$ into regularity of Ω around x_0. For instance, due to convexity, proving that Ω is $\mathcal{C}^1$ amounts to proving that it has no corner. Let us quote two works relying on this cutting procedure to obtain regularity for minimization among convex shapes of fixed volume: in [Buc03] the author proves $\mathcal{C}^1$ regularity of minimizers of λ_k, and in [GNR18] the authors prove $\mathcal{C}^{1,1}$ regularity of minimizers of $P(\Omega) + \mathcal{I}(\Omega)$ in two dimensions with $\mathcal{I}$ the logarithmic capacity.

Stability under convexity constraint. One of the main goals of this thesis was to apply the two steps strategy relying on the idea that stability for smooth shapes implies stability in the class of convex shapes (see the corresponding paragraph in Section 1.2.1), and in particular to perform the *selection principle* in a convexity constrained situation. It turns out that this regularization procedure adapts well to convexity constrained problems, and we refer further to Section 1.3.1 for the regularity results and to Section 1.3.2 for stability results. Let us give hereafter a couple of more specific comments about stability under convexity constraint.

- By enforcing a strong geometric constraint such as convexity of the admissible sets, a lot more is possible concerning quantitative inequalities. In fact, while the exponents in general quantitative inequalities like (1.14) are still sharp (since they are usually achieved by sequences of ellipsoids), there is room for proving inequalities with stronger notions of asymmetry. In the spirit of the Bonnesen inequality in two dimensions (1.12), in [Fug89] the author proved quantitative isoperimetric inequalities (with sharp exponents) with a Hausdorff-type asymmetry $\mathcal{A}_H(\Omega) := \inf\{d_H(\Omega, (B + x)),\ x \in \mathbb{R}^N\}$ (where d_H is the Hausdorff distance).

- Since a convex set already has some regularity (in particular it is Lipschitz), it can also happen that one can prove stability directly without having to prove higher regularity (or increase it through the *selection principle*). For instance, as Lipschitz continuity is enough to perform the Fuglede-type computations over $P(B_\xi)$ from [Fug89], the quantitative isoperimetric inequality (1.14) restricted to the class of convex sets can be obtained in this way (see [Fug89]). We have used the same idea for proving a stability result for the isoperimetric problem with capacity (see further Theorem 1.3.6).

1.3 Main contributions of the thesis

This section is dedicated to introducing the main contributions and results of the thesis, while describing the originality of the methods we used. It is split into three parts: we first present the regularity results under convexity constraint (corresponding to Chapter 2), then we present the stability results for isoperimetric problems (corresponding to Chapter 3), and we finally present the stability results for the Dirichlet spectrum (corresponding to Chapter 4).

1.3.1 Contributions of Chapter 2: regularity under convexity constraint

Main questions and main results. The regularity results we obtained in Chapter 2 deal with *perturbed isoperimetric problems* set with a convexity constraint, that is problems of the form (1.20) where $\mathcal{S}_{\text{ad}}$ only contains convex shapes and R is seen as a perturbative term. More precisely, if $D \subset \mathbb{R}^N$ is measurable (not necessarily having finite measure) and $\mathcal{K}^N$ denotes the class of convex bodies of $\mathbb{R}^N$ (compact convex sets with non-empty interior), we are interested in the problem

$$\inf\left\{P(K) + R(K),\ K \in \mathcal{K}^N, K \subset D\right\} \tag{1.31}$$

and its volume-constrained version

$$\inf\left\{P(K) + R(K),\ K \in \mathcal{K}^N, K \subset D,\ |K| = v_0\right\} \tag{1.32}$$

for some $0 < v_0 < |D|$. Here the perimeter of a convex body $P(K)$ is still the classical De Giorgi perimeter from (1.2), or equivalently $P(K) = \mathcal{H}^{N-1}(\partial K)$ (since K is Lipschitz). As in the case of unconstrained perturbations of the isoperimetric problem (see the corresponding paragraph in Section 1.2.2), we have in mind functionals R satisfying a Lipschitz-type property for the volume distance (as in (1.22)). Put in a slightly different way, we want to define an appropriate notion of quasi-minimizer (analogous to the unconstrained notion (1.23)) ensuring regularity. The first two questions in this direction which we addressed can thus be formulated as follows.

(i) Can one identify reasonable hypotheses on R ensuring regularity of problems (1.31) and (1.32) ?

(ii) Equivalently, can one define a suitable notion of quasi-minimizer of the perimeter under convexity constraint ensuring regularity of quasi-minimizers ?

One of the main contributions of Chapter 2 consists in bringing answers to these questions. As described in Section 1.2.1, an important motivation for both questions comes from applying the (*local minimality for smooth perturbations $\Longrightarrow$ local minimality for all shapes*) strategy from (1.9), and in particular the *selection principle* procedure (see Theorem 1.3.9 below, corresponding to Theorem 3.1.2 from Chapter 3). Let us then define the sufficient hypotheses we have found. For $D' \subset D \in \mathcal{K}^N$ we let

$$\mathcal{K}^N_{D',D} := \{K \in \mathcal{K}^N,\ D' \subset K \subset D\}.$$

Definition 1.3.1 (Perturbed isoperimetric problem)**.** *Let $R : \mathcal{K}^N \to \mathbb{R}$. We define the two following notions of Lipschitz continuity for R with respect to the volume distance: a 'weak" notion:*

$$\forall K \in \mathcal{K}^N, \exists (C_K, \varepsilon_K) \in (0,\infty)^2,\ \forall \widetilde{K} \in \mathcal{K}^N \quad s.t.\ \widetilde{K} \subset K \ \ and\ \ |K \backslash \widetilde{K}| \le \varepsilon_K,\ \ R(\widetilde{K}) - R(K) \le C_K |K \backslash \widetilde{K}| \tag{1.33}$$

and a "strong" notion:

$$\forall D' \subset D \in \mathcal{K}^N,\ \exists C_{D',D} > 0,\ \forall\big(K_1, K_2 \in \mathcal{K}^N_{D',D},\ K_1 \subset K_2\big),\ \ |R(K_2) - R(K_1)| \le C_{D',D} |K_2 \backslash K_1|. \tag{1.34}$$

A short argument about distances for convex sets (see further Proposition 2.2.9) enables to see that the strong Lipschitz property implies in fact the weak one. We now define quasi-minimizers of the perimeter under convexity constraint.

Definition 1.3.2 ((Λ,ε)-quasi-minimizer)**.** *Let $\Lambda,\varepsilon>0$. We say that $K\in\mathcal{K}^N$ is a (Λ,ε)-quasi-minimizer of the perimeter under convexity constraint (q.m.p.c.c. for short) if*

$$\forall \widetilde{K}\in\mathcal{K}^N \text{ such that } \widetilde{K}\subset K \text{ and } |K\setminus\widetilde{K}|\le\varepsilon,\quad P(K)\le P(\widetilde{K})+\Lambda|K\setminus\widetilde{K}|. \tag{1.35}$$

Let us comment on this notion of quasi-minimizer in comparison with the usual notion of quasi-minimizers (1.23).

- First, the local minimality of a q.m.p.c.c. is meant in the sense that the volume difference $|K\Delta\widetilde{K}|$ is small, which is milder than requiring that $K\Delta\widetilde{K}$ is located in a ball of small radius $B_r(x)$. This is due to the fact that this latter condition is much too restrictive for convex sets, as one sees by noticing that if K is a square there are no convex sets $\widetilde{K}$ other than K which perturb K around a point of its boundary located inside a segment. The possibility of perturbating a convex set around a point $x_0\in\partial K$ is in fact directly related to some kind of strict convexity of K around x_0.
- Here we merely require the minimality condition for perturbations $\widetilde{K}$ from the inside, as the cutting procedure we will use only relies on competitors $\widetilde{K}\subset K$. Note that enforcing minimality also for outward perturbations would not increase in general the regularity of K, as shows the counterexample in Proposition 2.3.18.

The first main regularity result is then the following (it corresponds to Theorem 2.2.3 and Corollary 2.2.4 from Chapter 2).

Theorem 1.3.3. *Let $N\ge 2$. Any q.m.p.c.c. K is $\mathcal{C}^{1,1}$. As a consequence, if $R:\mathcal{K}^N\to\mathbb{R}$ satisfies (1.33), any minimizer of (1.31) is $\mathcal{C}^{1,1}$.*

In this general form problem (1.31) does not always have minimizers (taking for instance $R=0$), and we refer further to Theorem 2.3.4 for an existence result under various hypotheses over D and R. We also prove (in Proposition 2.3.18) that Theorem 1.3.3 is optimal in general. Precisely, we show that we can find a q.m.p.c.c. in two dimensions which is $\mathcal{C}^{1,1}$ and not $\mathcal{C}^2$.

Let us also mention a similar regularity result obtained in two dimensions in [LNP12] under different conditions upon R. If K^* is a given minimizer of (1.31), the authors rather assume that R admits a shape derivative at K^* which is in $L^\infty(\partial K^*)$ (see Remark 2.2.6). They obtain $\mathcal{C}^{1,1}$ regularity of K^* in this case. Their proof is based on a radically different approach from the one we use (which we detail further in the methods), as it is based on the analysis of an Euler-Lagrange equation under convexity constraint associated to the minimization. Our result is more general and more flexible than this result (compare to Theorem 2.1.1), as it provides regularity in any dimension $N\ge 2$ and moreover does not assume the existence of a shape derivative of R at K^* (which is not verified in general for Dirichlet and Neumann eigenvalues for instance).

The second main result of Chapter 2 (corresponding to Theorem 2.2.10) deals with the second family of problems, that is volume-constrained problems (1.32).

Theorem 1.3.4. *Let $N\ge 2$. Let $D\in\mathcal{K}^N$, $0<v_0<|D|$ and suppose R satisfies (1.34). Then any solution K^* to (1.32) is a q.m.p.c.c. and is therefore $\mathcal{C}^{1,1}$.*

These two general regularity theorems at hand, an important question is to find explicit and interesting examples which enter in this framework. This is the third question we addressed.

(iii) Finding relevant examples of convexity constrained perturbed isoperimetric problems, i.e. proving (1.33) and (1.34) for standard shape functionals R.

In the third result of Chapter 2 (Theorem 2.3.2) we prove that Dirichlet and Neumann eigenvalues (rigorously defined for Lipschitz sets further in Section 2.3.2) as well as the Dirichlet energy verify these two hypotheses.

Theorem 1.3.5. *Let $k \in \mathbb{N}^*$, $N \geq 2$. Then any $R \in \{\lambda_k, \mu_k, E\}$ satisfies (1.33) and (1.34).*

Note that this strongly uses the fact that we are working with convex sets, and it no longer holds among general shapes as one sees by making a crack in a shape, which does not change the volume while it increases λ_k and E. This third result enables to build many examples of functionals R verifying (1.33) and (1.34). As a Corollary we thus obtain regularity for minimizers of a large class of perturbed isoperimetric problems (see further Theorem 2.1.1). Let us pick up on some specific examples.

First, we obtain $\mathcal{C}^{1,1}$ regularity for minimizers of the problems

$$\inf \left\{ P(K) + c\lambda_k(K),\ K \in \mathcal{K}^N \right\}$$

for some $c > 0$, $k \in \mathbb{N}^*$ (see further Proposition 2.3.5). This is the convexity-constrained version of a problem for which one expects non-convex minimizers (for $k = 2, 5, 6, 7$ and $N = 3$, see [BO16, Figure 2] for numerical evidence). We also obtain $\mathcal{C}^{1,1}$ regularity of minimizers of the problems

$$\inf \left\{ P(K) \pm \mu_k(K),\ K \in \mathcal{K}^N,\ |K| = v_0 \right\}$$

$$\inf \left\{ P(K) + \lambda_k(K),\ K \in \mathcal{K}^N,\ |K| = v_0 \right\}$$

for any $k \in \mathbb{N}^*$ (see Proposition 2.3.6).

Methods. We now decribe separately the methods we employed for proving the results.

Methods for Theorem 1.3.3. This result is inspired from regularity results of [CCL13], where the authors prove $\mathcal{C}^{1,1-\frac{N}{p}}$ regularity for solutions of the convex minimization

$$\inf \left\{ \int_D \frac{|\nabla u|^2}{2} + fu,\ u \in H^1(D), u \text{ is convex} \right\} \tag{1.36}$$

where $D \subset \mathbb{R}^N$ is a bounded convex open set and $f_+ \in L^p(D)$ for some $p > N$. Our idea is to see locally ∂K as the graph of a function g which can be interpreted as the solution of a calculus of variations problem of the type (1.36). However, due to difficulties linked to the localization of the convexity constraint, there are additional issues arising in the shape optimization context. Let us be a bit more specific: writing for any convex $g : \overline{B_{x_0}} \to \mathbb{R}$ with $g \geq g_0$ and $(g - g_0)_{|\partial B_{x_0}} = 0$ (where g_0 parametrizes K near x_0, see (1.19))

$$K_g = \left(K \cap (\overline{B_{x_0}}^c \times \mathbb{R}) \right) \cup \left(K \cap \mathrm{Epi}(g) \right)$$

where $\mathrm{Epi}(g) := \{(x,t) \in \overline{B_{x_0}} \times \mathbb{R},\ t \geq g(x)\}$ stands for the epigraph of g, it is not difficult to see that any such K_g is convex with $K_g \subset K$. On the other hand, there are not necessarily many such K_g if one is not careful enough regarding the choice of the neighborhood B_{x_0}, and one can easily construct convex sets K such that $g = g_0$ and $K_g = K$ for any such g, so that one cannot build any competitors to minimality in this way. This happens for instance in three dimensions for a prism with triangular base, by looking at a point on any edge linking the two triangular faces. This issue concerns "flat" parts of ∂K, by which we mean points around which K is not strictly convex. In order to illustrate this latter fact, see that if the function g_0 parametrizing K near x_0 is strongly convex then for any $\phi \in \mathcal{C}_c^\infty(B_{x_0})$, provided $t \in \mathbb{R}$ is taken sufficiently small the function $g + t\phi$ is still convex; $K_{g+t\phi}$ is therefore also convex in this case, thus providing many relevant competitors.

We therefore have to build an appropriate framework in order to be able to use ideas similar to [CCL13]. Letting K be a q.m.p.c.c., the method we employ is a cutting-type procedure, which consists in building competitors in the form $K_r := K \cap H_r = K \cap \mathrm{Epi}(\sigma_r)$ (for a parameter $r > 0$) where σ_r is an affine function defined on some hyperplane. Apart from showing how to adapt the regularity results obtained by [CCL13] for a general Lagrangian $L(x, u(x), \nabla u(x))$ strongly convex in ∇u (in our case, the underlying

Lagrangian is $L(x,u,p)=\sqrt{1+|p|^2}$), we believe that our method enables to get a clearer perspective on how one might expect to insert shape optimization problems under convexity constraint in a calculus of variations framework. More precisely, we have understood that localizing the arguments by parametrizing ∂K around some $x_0\in\partial K$ by $\{(x,g(x)),\ x\in D\}$ (for some $(N-1)$-dimensional open set D) is only possible when taking D maximal in some sense, and that this forces to perform global estimates on the functionals which are difficult in general. In our case we do it for the perimeter functional, and it is one of the most involved step in the proof of Theorem 1.3.3. Precisely, if K is a q.m.p.c.c. an important part of the proof consists in estimating from below

$$P(K)-P(K_r)\geq\int_{D_r}\left(\sqrt{1+|\nabla g|^2}-\sqrt{1+|\nabla\sigma_r|^2}\right)d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure and $D_r\subset D$ is an open set depending on $r>0$ (see (2.29)). While the right-hand-side is a difference of convex energies closer to calculus of variations, we have to deal with the additional difficulty that the domain of integration D_r depends on r, which is tightly linked to the fact that we have taken D maximal in some sense. Apart from this estimate from below, we estimate from above $|K\setminus K_r|$ (in (2.20) and (2.43)) and have to handle the fact that the condition $g\leq\sigma_r$ does not exactly encode $K\setminus K_r$, meaning that in general the inclusion $K\setminus K_r\subset\{(x,t)\in D\times\mathbb{R},\ g(x)\leq t\leq\sigma_r(x)\}$ is strict.

Methods for Theorem 1.3.4. The strategy consists in applying Theorem 1.3.3 by showing that any minimizer of (1.32) is a q.m.p.c.c. The main idea is to penalize the volume constraint (this is done in Lemma 2.2.11), meaning that we show that if K^* minimizes (1.32) then there exists $\varepsilon=\varepsilon_{K^*}>0$ and $\Lambda=\Lambda_{K^*}>0$ such that it also minimizes

$$\inf\left\{P(K)+R(K)+\Lambda||K|-v_0|,K\in\mathcal{K}^N,\ K\subset K^*,\ |K^*\setminus K|\leq\varepsilon\right\}.$$

The penalization idea is classical (see for instance [DPLPV18, Lemma 4.5]) and consists in building, for each $K\subset K^*$ with $|K^*\setminus K|\ll 1$, some $K\subset\widetilde{K}\subset D$ satisfying the volume constraint $|\widetilde{K}|=v_0$ and such that the increase in energy is comparable to the increase in volume, *i.e.* $G(\widetilde{K})\leq G(K)+\Lambda||K|-v_0|$ where $G:=P+R$. The novelty of our method lies in the fact that $\widetilde{K}$ has to be convex, which prevents from localizing the argument by building $\widetilde{K}$ as a perturbation of K around a point (as is done in [DPLPV18, Lemma 4.5]). Instead we construct $\widetilde{K}$ as a Minkowski sum $(1-t)K+tD$ between K and D for some apropriate $t\in[0,1]$, and use then Minkowski's theory of mixed-volumes to estimate the perimeter and volume of $\widetilde{K}$.

Methods for Theorem 1.3.5. The result is inspired from a strategy laid out in [BL07, BL08]. The idea is to rely on the variational characterizations of λ_k,μ_k,E in order to estimate the variation of energy between two nested convex sets $K_1\subset K_2$. In the case of the Dirichlet energy E (in 2.3.8), letting w_{K_2} be the torsion function of K_2 we write

$$0\leq E(K_1)-E(K_2)\leq\frac{1}{2}\int_{K_2}\left(|\nabla(w_{K_2}\circ\phi)|^2-|\nabla w_{K_2}|^2\right)-\int_{K_2}(w_{K_2}\circ\phi-w_{K_2})$$

where $\phi:=\phi_{K_1,K_2}:K_1\to K_2$ is a Lipschitz change of variables which is built in order to be the identity on a "large" part of K_1, meaning that it is the identity on some $K_3\subset K_1$ where $|K_1\setminus K_3|\leq C|K_2\setminus K_1|$ for some constant $C>0$ independent of $K_1,K_2\in\mathcal{K}^N_{D',D}$. The result then follows from a uniform control of the Lipschitz norm of ϕ_{K_1,K_2} for $K_1,K_2\in\mathcal{K}^N_{D',D}$ (in Lemma 2.3.9), and a Lipschitz estimate of w_{K_2} uniform in $\mathcal{K}^N_{D',D}$ (in Lemma 2.3.10). Our main input in this matter concerns a natural and simple idea for building the change of variables ϕ_{K_1,K_2}, based on the representation of the boundary of a convex set K as a graph over the sphere of an interior ball.

In the case of the Dirichlet eigenvalues λ_k we rely on an idea from [Buc03, Theorem 3.4] which enables to compare the variation in λ_k of two nested bounded open sets $\Omega' \subset \Omega$ with the variation in Dirichlet energy:

$$\left|\lambda_k(\Omega) - \lambda_k(\Omega')\right| \leq C_k \lambda_k(\Omega)^{N/2+1} \lambda_k(\Omega') |E(\Omega) - E(\Omega')|$$

for some contant $C_k > 0$, thus enabling to apply the Dirichlet energy case.

For Neumann eigenvalues (in Proposition 2.3.13) we again use the variational characterization of μ_k, this time building rather an extension operator $\Pi_K : W^{1,\infty}(K) \to W^{1,\infty}(\mathbb{R}^N)$ with uniformly controlled norm and proving uniform Lipschitz estimates for Neumann eigenfunctions for $K \in \mathcal{K}^N_{D',D}$.

1.3.2 Contributions of Chapter 3: Fuglede-type arguments for isoperimetric problems and applications to stability among convex shapes

Main questions. In Chapter 3 we are interested in stability issues for perturbed isoperimetric problems. We study and prove stability for different classes of sets written as "smooth" perturbations of the ball, which constitutes the so-called *Fuglede-type computations*, and obtain as applications of these results stability inside the class of convex sets. Let us then set the problems in their convexity constrained form, *i.e.* problems (1.31) and (1.32):

$$\inf \left\{ P(K) + R(K),\ K \in \mathcal{K}^N, K \subset D \right\},$$

$$\inf \left\{ P(K) + R(K),\ K \in \mathcal{K}^N, K \subset D,\ |K| = v_0 \right\}.$$

We aim at showing stability in the weak form (1.10) as well as in the strong form (1.16) for problems where R is an energy associated to a PDE linked to K, meaning a PDE set either on K or on $\mathbb{R}^N \setminus K$. We are mostly concerned with problems for which stability fails among all shapes, so that convex sets becomes a relevant class to consider. In the remaining of this paragraph and before going into more details, we very briefly introduce the two independent problems for which we obtain stability results, in order to highlight the common features of the methods we employ in both cases.

We are interested in the minimizations

$$\inf \left\{ P(K) + \varepsilon \mathrm{Cap}(K)^{-1},\ K \in \mathcal{K}^N,\ |K| = 1 \right\} \tag{1.37}$$

for $N \geq 3$ and a parameter $\varepsilon > 0$ ($\mathrm{Cap}(K)$ is defined below (1.39)), and

$$\inf \left\{ P(K) - c\lambda_1(K),\ K \in \mathcal{K}^N,\ |K| = 1 \right\} \tag{1.38}$$

for $N \geq 2$ and a parameter $c > 0$.

The results of Chapter 3 about convex shapes are the following: global minimality of the ball for $\varepsilon \ll 1$ for (1.37) (*i.e. weak stability* (1.10), see Theorem 1.3.6) and local minimality of the ball for an optimal range of c for (1.38) (*i.e. strong stability* (1.16), see Theorem 1.3.9). Although the two results are independent, the strategy we employ for proving them constitutes a general scheme going beyond convexity. We rely on the following steps

1. **(IT)** property for the functional (Lemma 1.3.8 and Theorem 1.3.10).
2. Minimality of the ball for the functional among "smooth" perturbations of the ball (Theorem 1.3.7 and Theorem 1.3.11).
3. Minimality of the ball for convex sets (Theorems 1.3.6 and 1.3.9).

The smoothness for which we prove local minimality in Step 2 is in direct link with the **(IT)** property of Step 1. Roughly speaking, what we call **(IT)** property denotes a Taylor expansion at a certain order of the functional, so that it allows to prove coercivity of the second derivative of the functional for perturbations

of the ball with enough regularity, and hence local minimality for such perturbations in Step 2. On the other hand, Steps 1 and 3 for problems (1.37) and (1.38) are handled very differently.

Let us also emphasize the fact that only Step 3 uses convexity of the considered sets, so that the results from both Step 1 and 2 for (1.37) and (1.38) are relevant for the minimization of the corresponding functionals among general shapes. In fact, they constitute *Fuglede-type computations* for the corresponding isoperimetric functionals, and could thus be used as a first step for proving stability in different or more general classes than the class of convex sets. On the other hand, apart from bearing interest in its own, we believe that obtaining a full stability result for the class of convex sets illustrates the robustness of the strategy.

We now describe separately and in details the motivations for the two problems, as well as the results we have obtained and the methods we have used.

Weak stability for $\inf\left\{P(K)+\varepsilon\mathbf{Cap}(K)^{-1},\ K\in\mathcal{K}^N,\ |K|=1\right\}$**.**

Motivations and results. For $N\geq 3$ we define the capacity functional $\mathrm{Cap}:\mathcal{K}^N\to\mathbb{R}$ as the usual *electrostatic capacity*:

$$\mathrm{Cap}(K):=\inf\left\{\int_{\mathbb{R}^N}|\nabla u|^2,\ u\in\mathcal{C}_c^\infty(\mathbb{R}^N),\ u\geq 1 \text{ over } K\right\}. \tag{1.39}$$

Provided K is sufficiently smooth (see [CS03, Theorem 2 and 3]) this minimization is uniquely solved by the so-called *capacitary function* u_K which verifies the following PDE set over $\mathbb{R}^N\setminus K$:

$$\begin{cases}-\Delta u_K=0 \text{ in } \mathbb{R}^N\setminus K\\ u_K=1 \text{ over } K,\ u_K\in\mathcal{C}^0(\mathbb{R}^N)\\ u_K(x)\to 0 \text{ as } |x|\to+\infty\end{cases} \tag{1.40}$$

The *isocapacitary inequality* (see for instance [DPMM21]) which states that

$$\forall\Omega\subset\mathbb{R}^N \text{ open},\ |\Omega|=1,\ \mathrm{Cap}(\Omega)\geq\mathrm{Cap}(B) \tag{1.41}$$

together with the *isoperimetric inequality* are in competition in the minimization (1.37), thus making the problem non trivial. It is shown in [GNR15, Theorem 3.2 and Theorem 6.2] that existence among all shapes of volume 1 fails for all $\varepsilon>0$ and in all dimensions $N\geq 2$, while (1.37) always admits minimizers (see [GNR18, Theorem 1.1]), thus motivating the convexity constraint. Note that (1.37) with the definition of capacity (1.39) is ill-posed for $N=2$, since (1.39) is always equal to 0, as one sees by looking at the energy of the (truncated) fundamental solution of the Laplacian. Therefore one has to proceed differently to define capacity in two dimensions, and we mention that stability for convex sets for the equivalent version of (1.37) for $N=2$ was obtained in [GNR18].

The first question of Chapter 3 is the following.

(i) Proving stability for $P+Cap^{-1}$ for convex shapes, i.e. proving that balls are the only minimizers of (1.37) for small $\varepsilon>0$.

The following result addresses this first question (it corresponds to Theorem 3.1.1 from Chapter 3).

Theorem 1.3.6 (Weak stability of the ball for the capacity)**.** *Let $N\geq 3$. There exists $\varepsilon_0=\varepsilon_0(N)>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$ balls of unit volume are the unique minimizers of (1.37). Equivalently, if B is a ball of unit volume*

$$\forall K\in\mathcal{K}^N,\ P(K)-P(B)\geq\frac{\varepsilon_0}{\mathrm{Cap}(B)}\left(\frac{\mathrm{Cap}(K)-\mathrm{Cap}(B)}{\mathrm{Cap}(K)}\right)$$

This Theorem is thus the $N \geq 3$ version of the two dimensional result obtained in [GNR18]. It is called *weak* as the value ε_0 is non-optimal (it is not even explicit), meaning that contrarily to Theorem 1.3.9 below our method does not enable to show that the ball is no longer a minimizer for $\varepsilon > \varepsilon_0$ (even in a local volume sense, *i.e.* for $K \in \mathcal{K}^N$ such that $|K \Delta B| \ll 1$).

As mentionned above and will be described in details further in the methods, the proof of Theorem 1.3.6 relies on a Fuglede-type argument which seems relevant independently of convexity issues. The result is as follows (see Theorem 3.2.1 in Chapter 3).

If $h \in W^{1,\infty}(\partial B)$ we will denote B_h the Lipschitz perturbation of the ball by h given by

$$B_h := \{tx(1 + h(x)),\ t \in [0, 1),\ x \in \partial B\}. \tag{1.42}$$

Theorem 1.3.7 (Fuglede-type computations for $P + \varepsilon \mathrm{Cap}^{-1}$: minimality for Lipschitz perturbations)**.** *Let $N \geq 3$. There exists $\eta > 0$ and $\varepsilon_0 > 0$ such that for all $h \in W^{1,\infty}(\partial B)$ verifying $\|h\|_{W^{1,\infty}(\partial B)} \leq \eta$ with $|B_h| = |B|$ and such that B_h has barycenter at the origin, and for all $\varepsilon \in (0, \varepsilon_0)$, then*

$$P(B_h) + \varepsilon \mathrm{Cap}(B_h)^{-1} \geq P(B) + \varepsilon \mathrm{Cap}(B)^{-1}$$

with equality only if $B_h = B$.

This stability result for Lipschitz perturbations is an improvement of previous results from [GNR15, Corollary 5.6], where the authors show the same result inside a class of $\mathcal{C}^{1,1}$ sets with curvature uniformly bounded from above.

Methods.

1. **Methods for Theorem 1.3.7.** The idea is to prove some weak $\mathbf{(IT)}_{H^1,W^{1,\infty}}$ property for the capacity, which consists in a second-order estimate of the increase in energy $\mathrm{Cap}(B_h) - \mathrm{Cap}(B)$ for some $h \in W^{1,\infty}(\partial B)$ with $|B_h| = |B|$. The main step for proving Theorem 1.3.7 thus consists in the following Lemma.

   Lemma 1.3.8 (Weak $\mathbf{(IT)}_{H^1,W^{1,\infty}}$ for the capacity)**.** *Let $N \geq 3$. There exists $C_N > 0$ such that if $h \in W^{1,\infty}(\partial B)$ with B_h of volume $|B_h| = |B|$ and $\|h\|_{L^\infty(\partial B)} \leq 1/2$ then*

   $$\mathrm{Cap}(B_h) - \mathrm{Cap}(B) \leq C_N \|h\|^2_{H^1(\partial B)}.$$

   This second-order estimate is a *weak* form (adaptated to the capacity) of the **(IT)** property from Theorem 1.3.10. In fact, while Theorem 1.3.10 consists in estimating the third-order derivative, in this Lemma we rather control from above the second derivative. Moreover it is non-optimal in terms of the norm employed (the H^1 norm), since as in the case of λ_1 the second derivative of the capacity is continuous for the $H^{1/2}$ norm (for this fact see the proof of Theorem 2.8 in [DPMM21]). However it allows every Lipschitz perturbation, which is a much bigger class than in other similar results, and will drastically simplify the other steps of the proof of Theorem 1.3.7 (compare to [GNR18] where the authors have to prove $C^{1,1}$ regularity of optimal shapes). This H^1 norm is indeed sufficient to conclude in our case thanks to the presence of the perimeter functional in the minimization. We believe that this result is quite interesting as it holds for low regularity assumptions on the perturbation h.

   Let us quickly describe how this lemma is obtained. Relying on the formulation of the capacity (1.39) as an infimum, we build a good competitor in the form $u_B \circ \phi_h$ where u_B is the solution of (1.40) for the ball and $\phi_h(x) = \frac{x}{1+h(x/|x|)}$ is a natural Lipschitz change of variables sending B_h onto B. Getting

   $$\mathrm{Cap}(B_h) \leq \int_{\mathbb{R}^N} |\nabla(u_B \circ \phi_h)|^2$$

we then carefully estimate the energy of $u_B \circ \phi_h$ to control it from above by $\mathrm{Cap}(B) + C_N \|h\|^2_{H^1(\partial B)}$, where we importantly use the fact that u_B is radially symmetric while ϕ_h has the special structure $f(x)x$ with $f : \mathbb{R}^N \to \mathbb{R}^N$ being tangential. The construction of this competitor is a rather simple idea, and we believe that it can be of use in various contexts for estimating energies defined as infima, thus getting such **(IT)** properties for low regularity of the perturbation h. As a matter of fact, it turns out that one can use the very same competitor in order to obtain a similar weak $\mathbf{(IT)}_{H^1,W^{1,\infty}}$ property for λ_1 (see Proposition 3.2.5 (i)).

Theorem 1.3.7 then follows from combining the lemma with Fuglede's computations in [Fug89], where he proved that if B_h has unit volume and barycenter at the origin, and if $\|h\|_{W^{1,\infty}(\partial B)} \ll 1$, then

$$P(B_h) - P(B) \geq c_N \|h\|^2_{H^1(\partial B)} \tag{1.43}$$

for some explicit constant $c_N > 0$.

2. **Methods for Theorem 1.3.6.** Theorem 1.3.6 is directly obtained from Theorem 1.3.7 by relying on regularity properties of convex shapes, as the condition $\|h\|_{W^{1,\infty}(\partial B)} \ll 1$ for a convex minimizer K^* of (1.37) is obtained by letting $\varepsilon \to 0$. In fact, if K_ε minimizes (1.37), the condition $\varepsilon \to 0$ enforces $P(K_\varepsilon) - P(B) \to 0$. This classically ensures that the convex sets K_ε (which are already Lipschitz) converge to the ball in a Lipschitz sense, meaning that $K_\varepsilon = (\mathrm{Id} + h_\varepsilon \nu_B)(B)$ with $\|h_\varepsilon\|_{W^{1,\infty}(\partial B)} \to 0$. Contrarily to the method we employ for proving Theorem 1.3.9, as convex shapes are already smooth enough we do not need here to apply a regularization procedure in order to use the corresponding *Fuglede-type computations*. This idea is similar to some results from [Fug89] (see also the associated paragraph in Section 1.2.3).

Strong stability for $\inf\big\{P(K) - c\lambda_1(K),\ K \in \mathcal{K}^N,\ |K| = 1\big\}$**.**

Motivations. Due to the *isoperimetric inequality* (1.3) and the *Faber-Krahn inequality* (1.5), there is a competition in (1.38). As such the minimization of $P - c\lambda_1$ at fixed volume is ill-posed, even when considered in a convexity constrained form as in (1.38) (the infimum is $-\infty$, as one sees by looking at a sequence of long thin rectangles of unit volume). On the other hand, even in a volume local sense, a ball of unit volume is never a minimizer among all shapes, since one asymptotically decreases the energy by perforating the ball by a small hole (see for instance [DL19, Proposition 6.1]). As a consequence, one has to restrict to smaller classes in order to study stability issues related to $P - \lambda_1$. This has been investigated by many authors in various contexts, also as a means to prove quantitative inequalities in the form (1.11) (which is equivalent to stability). Let us quote the main works in this direction.

- As a consequence of the Payne-Weinberger inequality proven in [PW61], one has *weak stability* (1.10) of the ball for simply connected sets with bounded perimeter:

$$\forall \Omega \subset \mathbb{R}^2 \text{ open and simply connected with } |\Omega| = 1,\ P(\Omega) - P(B) \geq \varepsilon\,(\lambda_1(\Omega) - \lambda_1(B)) \tag{1.44}$$

  provided $P(\Omega) \leq C$, for some $\varepsilon = \varepsilon(C) > 0$.

- In dimensions $N \geq 3$ an analogous form of *weak stability* has been proven in [BNT10] using the Brunn-Minkowski theory. Indeed, the result [BNT10, Theorem 1.1] implies the quantitative inequality

$$\forall \Omega \subset \mathbb{R}^N \text{ open and convex with } |\Omega| = 1,\ P(\Omega) - P(B) \geq \varepsilon\,(\lambda_1(\Omega) - \lambda_1(B))$$

  provided $\lambda_1(\Omega), P(\Omega) \leq C$, for some $\varepsilon = \varepsilon(C) > 0$.

- In [Nit14, Theorem 1.2] (see also [DL19, Proposition 5.5 (ii)]) the author performed *Fuglede-type computations* and provided the optimal value c^* such that stability holds for smooth perturbations

of the ball, *i.e.* for the sets $B_\xi := (\mathrm{Id}+\xi)(B)$ with $\xi \in \mathcal{C}^\infty_c(\mathbb{R}^N, \mathbb{R}^N)$ and $\|\xi\|_{\mathcal{C}^\infty} \ll 1$. The underlying idea is to compute the interval of values c such that the second derivative of the functional $P - c\lambda_1$ at the ball is coercive (we refer to the *selection principle* paragraph in Section 1.2.1 for more details regarding the general idea).

Main results. As a consequence, a natural problem arising is then

(ii) Proving (local) minimality of the ball for $P - c\lambda_1$ for $c \in (0, c^*)$ for a suitable class of sets.

One of the main results of Chapter 3 is *strong stability* (1.16) of the ball for $P - \lambda_1$ in the class of convex sets (corresponding to Theorem 3.1.2).

Theorem 1.3.9 (Strong stability of the ball for λ_1)**.** *Let $N \geq 2$. Let ω_N be the volume of a ball of radius 1, and $p_N := N\omega_N$, $l_N := j^2_{N/2-1}$ be respectively the perimeter and first eigenvalue of a ball of radius 1 ($j_{N/2-1}$ is the first zero of the Bessel function of the first kind of order $N/2-1$). Set*

$$c^* := \frac{N(N+1)p_N}{4l_N(l_N - N)\omega_N^{\frac{N+1}{N}}}. \tag{1.45}$$

Let B be a ball of unit volume.

- *Let $0 < c < c^*$. Then there exists $\delta_c > 0$ such that*

$$\forall K \in \mathcal{K}^N, |K| = 1 \text{ with } |K\Delta B| \leq \delta_c,\ (P - c\lambda_1)(K) \geq (P - c\lambda_1)(B).$$

- *Let $c > c^*$. There exists a sequence of smooth convex bodies $(K_{j,c})_{j\in\mathbb{N}}$ of unit volume for which $|K_{j,c}\Delta B| \to 0$ and*

$$(P - c\lambda_1)(K_{j,c}) < (P - c\lambda_1)(B) \text{ for each } j \in \mathbb{N}.$$

This Theorem is sharp in the sense that it provides the optimal range of values of c for which B is a L^1-local minimizer of $P - c\lambda_1$ among convex shapes of fixed volume. As will be detailed further in the methods (see also the *selection principle* paragraph above in Section 1.2.1), we prove this Theorem using the *selection principle* technique. As a consequence we prove as a first step *Fuglede-type computations*, which we think are of independent interest. More precisely, we start by proving a second-order expansion of λ_1 for which we estimate the third-order term as the product of the $H^{1/2}$ norm squared of the perturbation by a modulus of continuity of the $\mathcal{C}^{1,\alpha}$ norm (in Theorem 1.3.10 below, corresponding to Theorem 3.3.5 in Chapter 3).

For any $h \in W^{1,\infty}(\partial B)$ and $t \in [0,1]$, we set $\lambda_1(t) := \lambda_1(B_{th})$ (B_{th} is defined by (1.42)).

Theorem 1.3.10 (**(IT)**$_{H^{1/2},\mathcal{C}^{1,\alpha}}$ for λ_1)**.** *Let $N \geq 2$. There exists $\alpha \in (0,1)$ such that the functional λ_1 satisfies an* **(IT)**$_{H^{1/2},\mathcal{C}^{1,\alpha}}$ *condition at the ball B, i.e. there exists $\delta > 0$ and a modulus of continuity ω_{λ_1} such that for any $h \in \mathcal{C}^{1,\alpha}(\partial B)$ with $\|h\|_{\mathcal{C}^{1,\alpha}(\partial B)} \leq \delta$ we have*

$$\forall t \in [0,1],\ \lambda_1(B_{th}) = \lambda_1(B) + \lambda_1'(0)t + \frac{\lambda_1''(0)}{2}t^2 + \omega_{\lambda_1}(\|h\|_{\mathcal{C}^{1,\alpha}(\partial B)})\|h\|^2_{H^{1/2}(\partial B)}$$

Note that this result does not assume convexity of the set B_h, so that it holds for general perturbations $h \in \mathcal{C}^{1,\alpha}(\partial B)$. We will explain its meaning in more details further in the methods, but let us just highlight the important features: $\lambda_1''(0)$ (which depends on h) is a quadratic form in h which is naturally continuous and coercive in $H^{1/2}$ norm. As a consequence, this type of result gives an account of the smoothness of h needed to control the remainder of this Taylor expansion by the second derivative (here $\mathcal{C}^{1,\alpha}$). In comparison with [Nit14, Theorem 1.2], and in general with results of third-order expansions for similar

energies, our result is original in the sense that it goes below second derivatives spaces as $\mathcal{C}^{2,\alpha}$ or $W^{2,p}$ spaces, which are the usual spaces for which this kind of property is obtained (see for instance [Dam02] or [AFM13]). Concerning specifically λ_1, this result is an improvement of [DL19, Theorem 1.4], which was itself an improvement of [Dam02] and [DP00], in which the authors prove a similar expansion with respectively $\|h\|_{W^{2,p}(\partial B)}$ (for any $p > N$) and $\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)}$ instead. This refinement is motivated by the fact that our regularity theory from Chapter 2 does not provide enough convergence for applying the latter $W^{2,p}$ result (see instead our convergence result, Theorem 1.3.12 below).

Relying on Theorem 1.3.10 we are able to reach local minimality in a $\mathcal{C}^{1,\alpha}$ neighborhood of the ball, thus completing the first step of the *selection principle* strategy (see the corresponding paragraph in Section 1.2.1). The result is as follows (see Theorem 3.3.1).

Theorem 1.3.11 (Fuglede-type computations for $P - c\lambda_1$: minimality in a $\mathcal{C}^{1,\alpha}$ neighborhood)**.** *Let $N \geq 2$. For $c > 0$ set $\mathcal{J}_c := P - c\lambda_1$ and let c^* be given by (1.45). There exists $\alpha \in (0,1)$ such that for any $0 < c < c^*$ there exists $\delta_c > 0$ such that for each $h \in \mathcal{C}^{1,\alpha}(\partial B)$ with $|B_h| = |B|$ then*

$$\mathcal{J}_c(B_h) \geq \mathcal{J}_c(B)$$

with equality only if (up to translating) $B_h = B$.

Note again that this result holds for general $\mathcal{C}^{1,\alpha}$ perturbations h with no convexity assumption, so that it is of interest independently of being an important step for proving Theorem 1.3.9.

With $\mathcal{C}^{1,\alpha}$ local minimality at hand we are now led to perform the second step of the *selection principle*, which is the contradiction argument in itself. As described in details in Section 1.2.1, this is done by relying on a regularity theory, which in our case is the regularity theory under convexity constraint for perturbed isoperimetric problem laid out in Chapter 2. The following result thus importantly relies on Theorem 1.3.4 (see Theorem 3.4.2 and Corollary 3.4.3 for the corresponding results in Chapter 3).

Theorem 1.3.12 (Uniform regularity and convergence of q.m.p.c.c)**.** *Let $N \geq 2$ and $\Lambda > 0$, $\varepsilon > 0$. Set $\nu_B(x) := x$ for $x \in \mathbb{R}^N$.*

(i) (Regularity) Let $0 < m < M$. If K is a (Λ, ε)-q.m.p.c.c. verifying $B_m(z) \subset K \subset B_M(z)$ for some $z \in \mathbb{R}^N$, then there exists $h \in \mathcal{C}^{1,1}(\partial B)$ such that (up to translation) K can be written

$$K = (Id + h\nu_B)(B), \text{ with } \|h\|_{\mathcal{C}^{1,1}(\partial B)} \leq C$$

with $C = C(N, \Lambda, \varepsilon, m, M) > 0$ only depending on the indicated parameters.

(ii) (Convergence) If (K_j) is a sequence of (Λ, ε)-q.m.p.c.c. such that $|K_j \Delta B| \to 0$, then there exists a sequence $h_j \in \mathcal{C}^{1,1}(\partial B)$ such that

$$\forall j \in \mathbb{N}, \ K_j = (Id + h_j \nu_B)(B),$$

and for all $\alpha \in (0,1)$ it holds (up to subsequence) $h_j \to 0$ in $\mathcal{C}^{1,\alpha}$.

The second item is a useful convergence version of the regularity result. It is an analogue of a similar result for classical quasi-minimizers (1.23) in the usual setting: if a sequence (E_j) of (Λ, r_0)-quasi-minimizers converges to the ball in a L^1 sense, then E_j is $\mathcal{C}^{1,1/2}$ for large j and it converges (up to subsequence) to the ball in $\mathcal{C}^{1,\alpha}$ for each $\alpha \in (0, 1/2)$ (see for instance [AFM13, Theorem 4.2] for a rigorous statement).

Methods

1. **Methods for Theorems 1.3.10 and 1.3.11.** These two results constitute the first step of the *selection principle* strategy described in details in Section 1.2.1. In the notations and vocabulary of Section 1.2.1, Theorem 1.3.11 precisely corresponds to the *Fuglede-type computations* for the

functional $\mathcal{J}_c$ (for the appropriate values of c) with the space $X = \mathcal{C}^{1,\alpha}$, while Theorem 1.3.10 is the main ingredient we use for achieving the computations. In fact, once Theorem 1.3.10 is proven, we apply a general stability result from [DL19, Theorem 1.3] ensuring that local minimality is deduced from the optimality conditions

$$\forall h \in \mathcal{C}^{1,\alpha}(\partial B),\ \left.\frac{d}{dt}\right|_{t=0} \left(\mathcal{J}_c - \mu|\cdot|\right)(B_{th}) = 0,$$

$$\forall h \in \mathcal{C}^{1,\alpha}(\partial B) \text{ with } \int_{\partial B} h = 0 \text{ and } \int_{\partial B} h\vec{x} = \vec{0},\ \left.\frac{d^2}{dt^2}\right|_{t=0} \left(\mathcal{J}_c - \mu|\cdot|\right)(B_{th}) \geq c_N \|h\|^2_{H^{1/2}(\partial B)} \quad (1.46)$$

where μ is some appropriate Lagrange multiplier associated to the volume constraint, $c_N > 0$ is dimensional, and the conditions

$$\int_{\partial B} h = 0,\ \int_{\partial B} h\vec{x} = \vec{0}$$

take into account the volume constraint of the problem and the invariance by translation of the functional $\mathcal{J}_c$. The second-order optimality condition (1.46) comes from the computations of [Nit14] (the more explicit expression (1.45) of the threshold c^* was obtained in [DL19]).

To prove Theorem 1.3.10 we rely on an idea from [DL19]: we prove the stronger $\textbf{(IC)}_{H^{1/2},\mathcal{C}^{1,\alpha}}$ property

$$\forall t \in [0,1],\ |\lambda_1''(t) - \lambda_1''(0)| \leq \omega_{\lambda_1}(\|h\|_{\mathcal{C}^{1,\alpha}(\partial B)})\|h\|^2_{H^{1/2}(\partial B)} \quad (1.47)$$

for some modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}$ independent of $h \in \mathcal{C}^{1,\alpha}(\partial B)$, from which Theorem 1.3.10 follows by usual Taylor expansion with integral remainder. The authors [DL19] prove this property with a $W^{2,p}$ norm instead (see [DL19, Theorem 1.4]). We improve their result by refining the analysis of certain estimates (see Lemma 3.3.7) and introducing new methods for others (see Lemma 3.3.6).

2. **Methods for Theorem 1.3.12.** To prove the regularity statement (i) of Theorem 1.3.12, we rely on the regularity results from Chapter 2. More specifically, we use the ideas from Theorem 1.3.4 where $\mathcal{C}^{1,1}$ regularity of (Λ, ε)-q.m.p.c.c. is obtained, but have to follow the constants in the proof in order to show that this regularity only depends on Λ, ε and relevant geometric constants such as the inradius and outradius of the convex set. Precisely, this step consists in estimating the $\mathcal{C}^{1,1}$ norm of some $g : B_{x_0} \to \mathbb{R}$ parametrizing a (Λ, ε)-q.m.p.c.c. K near a point $x_0 \in \partial K$ (see (1.19)), and to show

$$\|g\|_{\mathcal{C}^{1,1}(B_{x_0})} \leq C(N, \Lambda, \varepsilon, m, M).$$

From this quantified local cartesian $\mathcal{C}^{1,1}$ regularity we obtain the announced global spherical estimate by standard arguments. Although this passage often comes as classical in the literature, it does not seem to be so well referenced and we believe that a careful examination of all the arguments can be of use. The convergence assertion (ii) in any $\mathcal{C}^{1,\alpha}$ norm then comes from a standard application of the Arzela-Ascoli Theorem.

3. **Methods for Theorem 1.3.9.** Theorem 1.3.9 comes as a combination of Theorems 1.3.11 and 1.3.12 together with the *selection principle* technique from [CL12] (see Section 1.2.1). Supposing that Theorem 1.3.9 does not hold first provides a sequence (K_j) such that

$$\begin{cases} \forall j \geq 0,\ \mathcal{J}_c(K_j) < \mathcal{J}_c(B), \\ |K_j \Delta B| \to 0. \end{cases} \quad (1.48)$$

We find a convex body $D \in \mathcal{K}^N$ such that $K_j \subset D$, and then regularize each K_j into $\widetilde{K}_j$ defined as a minimizer of

$$\inf\left\{\mathcal{J}_c(K) + \mu\left||K\Delta B| - |K_j\Delta B|\right|,\ |K| = |B|,\ K \subset D,\ K \in \mathcal{K}^N\right\}$$

for some $\mu \gg 1$. We show that $\widetilde{K}_j \to B$ in the Hausdorff sense and still verifies the contradiction hypothesis (1.48), while by relying on a refined version of Theorem 1.3.5 we prove that $\widetilde{K}_j$ are (Λ, ε)-quasi-minimizers under convexity constraint for some uniform Λ, ε. We therefore apply Theorem 1.3.12 to deduce that $\widetilde{K}_j$ converges in any $\mathcal{C}^{1,\alpha}$ sense to the ball. Theorem 1.3.11 finally provides the desired contradiction.

1.3.3 Contributions of Chapter 4: sharp quantitative stability for the Dirichlet spectrum near the ball

Main questions. In Chapter 4 we study stability of the ball for the Dirichlet spectrum. If $\Omega \subset \mathbb{R}^N$ is an open set of finite measure, B is a ball of same measure and $k \in \mathbb{N}^*$, the general question is the following

If $\lambda_1(\Omega)$ is close to $\lambda_1(B)$, does it imply that $\lambda_k(\Omega)$ is close to $\lambda_k(B)$?

More precisely, the issue is to quantify the distance of the whole spectrum of a shape Ω to the spectrum of the ball by the closeness of the fundamental frequency to the one of the ball. This raises the questions of quantitative inequalities of the type

$$\forall \Omega \subset \mathbb{R}^N \text{ open with } |\Omega| = \omega_N,\ |\lambda_k(\Omega) - \lambda_k(B)| \leq C_{N,k} \left(\lambda_1(\Omega) - \lambda_1(B)\right)^{\gamma} \tag{1.49}$$

for some exponent $\gamma > 0$ and constant $C_{N,k} > 0$, for open sets Ω with $\lambda_1(\Omega) \leq M$ for a fixed constant $M > 0$, where $\omega_N > 0$ is the measure of a ball of unit radius. Quantitative inequalities of this form have been studied by several authors: it was proven respectively in [BC06] and [MP19] that for all open $\Omega \subset \mathbb{R}^N$ with $|\Omega| = \omega_N$

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{N,k} \left(\lambda_1(\Omega) - \lambda_1(B)\right)^{1/80N}, \text{ if } \lambda_1(\Omega) \text{ is close to } \lambda_1(B)$$
$$-C_{N,k} \left(\lambda_1(\Omega) - \lambda_1(B)\right)^{\frac{1}{6}-\kappa} \leq \lambda_k(\Omega) - \lambda_k(B) \leq C_{N,k} \left(\lambda_1(\Omega) - \lambda_1(B)\right)^{\frac{1}{12}-\kappa}, \text{ if } \lambda_1(\Omega) \text{ is bounded}$$

for each $N \geq 2$, for some constant $C_{N,k} > 0$, and each $\kappa > 0$ (for the second series of inequalities, the authors have better but still non sharp exponents in the case $N = 2$). Our initial goal was to find the best possible exponent $\gamma > 0$ in (1.49).

Examining the first order optimality condition, two distinct behaviours occur depending on the multiplicity of $\lambda_k(B)$: either (i) $\lambda_k(B)$ is simple, so that λ_k is differentiable in the sense of shape derivatives at B (meaning that $t \mapsto \lambda_k\big((\mathrm{Id} + t\xi)(B)\big)$ is differentiable for smooth vector fields ξ), the associated eigenfunction is radial and the ball B is a critical point (under volume constraint) of λ_k, or (ii) $\lambda_k(B)$ is multiple and this is no longer the case. Since on the other hand B is a critical point for λ_1 and the second derivative of λ_1 is coercive at the ball, one can expect that the optimal exponents in (1.49) are $1/2$ for multiple eigenvalues and 1 for simple eigenvalues. Refining this analysis, when looking at a whole cluster of eigenvalues $(\lambda_k(B), \dots, \lambda_l(B))$ (*i.e.* such that $\lambda_{k-1}(B) < \lambda_k(B) = \cdots = \lambda_l(B) < \lambda_{l+1}(B)$) the ball is again a critical point of the sum $\sum_{i=k}^{l} \lambda_i$ (see for instance [LLdC06, Proposition 2.30]), and one can therefore make the hypothesis that we recover $\gamma = 1$ as optimal exponent for quantitative inequalities (1.49) with $\sum_{i=k}^{l} \lambda_i$ in place of λ_k. This latter general case would also account for the simple eigenvalue case. Said differently, we expect that the ball is a minimizer of

$$\{\Omega \subset \mathbb{R}^N \text{ open}, |\Omega| = \omega_N\} \mapsto \lambda_1(\Omega) + \delta \sum_{i=k}^{l} \lambda_i(\Omega) \tag{1.50}$$

provided $\delta \in \mathbb{R}$ is sufficiently close to 0, both for $\delta > 0$ and $\delta < 0$.

Partly due to technical reasons explained further in the methods (but this will also imply new interesting inequalities, see Corollary 1.3.16) we replace λ_1 by T^{-1} in this minimization, where $T(\Omega) = -2E(\Omega)$ is

the *torsional energy* given by $T(\Omega) = \int_\Omega w_\Omega$ with $w_\Omega \in H^1_0(\Omega)$ the torsion function verifying $-\Delta w_\Omega = 1$ in Ω. We thus rather study minimality of the ball for

$$\{\Omega \subset \mathbb{R}^N \text{ open}, |\Omega| = \omega_N\} \mapsto T(\Omega)^{-1} + \delta \sum_{i=k}^{l} \lambda_i(\Omega). \tag{1.51}$$

Minimality of the ball for this functional will imply minimality in (1.50), as the classical *Kohler-Jobin inequality* states minimality of the ball for $T(\Omega)\lambda_1(\Omega)^{\frac{N+2}{2}}$ among all open sets of measure ω_N, thus yielding for some $C_N > 0$

$$T(\Omega)^{-1} - T(B)^{-1} \leq C_N \lambda_1(\Omega)^{\frac{N}{2}} (\lambda_1(\Omega) - \lambda_1(B)) \tag{1.52}$$

for the same Ω.

Let us conclude this paragraph by giving a slightly different perspective upon quantitative inequalities (1.49). In the equivalent form (1.51), this enters the general framework of *perturbed Faber-Krahn problems* of the form

$$\inf \left\{ \lambda_1(\Omega) + \varepsilon F\big(\lambda_2(\Omega), \dots, \lambda_k(\Omega)\big),\ \Omega \subset \mathbb{R}^N \text{ open }, \ |\Omega| = \omega_N \right\} \tag{1.53}$$

where $k \in \mathbb{N}$ with $k \geq 2$, for some $F : \mathbb{R}^{N-1} \to \mathbb{R}$ and $\varepsilon > 0$, and the question is whether the ball B is a minimizer for $\varepsilon \ll 1$ (*i.e.* whether it is *weakly stable*, see the corresponding paragraph in Section 1.2.1). Note that this problem consists in minimizing $\lambda_1 + R$ for some energy R of same order than λ_1 (on the contrary to situations where R is a volume term).

Main results. We now present the main results of Chapter 4, together providing a complete picture of sharp quantitative inequalities (1.49) (see Proposition 4.6.1 for sharpness of the exponents). The first result holds for all eigenvalues λ_k, independently of $\lambda_k(B)$ being simple or multiple. It is sharp in general when $\lambda_k(B)$ is multiple. From now on we consider an integer $N \geq 2$ and set

$$\mathcal{A} := \left\{ \Omega \subset \mathbb{R}^N \text{ open}, \ |\Omega| = \omega_N \right\}.$$

The result is as follows (see Theorem 4.1.1).

Theorem 1.3.13. *For every $k \in \mathbb{N}^*$ there exists $C_{N,k} > 0$ such that for $\Omega \in \mathcal{A}$,*

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{N,k} \lambda_1(\Omega)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \left(T(\Omega)^{-1} - T(B)^{-1}\right)^{\frac{1}{2}}.$$

The inequality is written in a scaling-free form, thus holding for all open sets Ω with $B = B_\Omega$ any ball of same measure as Ω. The second main result is as follows (it corresponds to Theorem 4.1.2). It applies to simple eigenvalues of the ball, the first occurences in two dimensions being given by $k = 1, 6, 15, 30$.

Theorem 1.3.14. *For every $k \in \mathbb{N}^*$ with $\lambda_k(B)$ simple there exists $C_{N,k} > 0$ such that for any open set $\Omega \in \mathcal{A}$,*

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{N,k} |\Omega| \left(T(\Omega)^{-1} - T(B)^{-1}\right)$$

The last result treats the general case of the cluster of eigenvalues and generalizes Theorem 1.3.14 (see Theorem 4.1.3).

Theorem 1.3.15. *For every $k, l \in \mathbb{N}^*$ with $k \leq l$ satisfying*

$$\lambda_{k-1}(B) < \lambda_k(B) = \dots = \lambda_l(B) < \lambda_{l+1}(B),$$

there exists $C_{N,k} > 0$ such that for any open set $\Omega \in \mathcal{A}$,

$$\left| \sum_{i=k}^{l} \Big[\lambda_i(\Omega) - \lambda_i(B) \Big] \right| \leq C_{N,k} |\Omega| \left(T(\Omega)^{-1} - T(B)^{-1}\right)$$

As a consequence of Theorem 1.3.14, we are able to answer a conjecture raised in [vdBBP21]. Indeed, taking $k = 1$, our result specializes into a reverse form of the *Kohler-Jobin* inequality (1.52), giving

$$T(\Omega)^{-1} - T(B)^{-1} \geq c_N(\lambda_1(\Omega) - \lambda_1(B))$$

for all $\Omega \in \mathcal{A}$ and some $c_N > 0$. It gives as a corollary a full *reverse Kohler-Jobin* inequality, stated in next result (see Corollary 4.1.5 in Chapter 4).

Corollary 1.3.16. *There exists $p_N > 1$ such that for any $p > p_N$, $\Omega \in \mathcal{A} \mapsto T(\Omega)\lambda_1(\Omega)^{\frac{1}{p}}$ is maximal on the ball.*

This result was obtained for $\mathcal{C}^{2,\alpha}$ perturbations of the ball by a Fuglede-type argument in [BBGLB22, Theorem 6.2], and we prove it here in the class $\mathcal{A}$.

Finally, we obtain as an application of Theorem 1.3.15 a result about general functionals of eigenvalues $F(\lambda_1, \dots, \lambda_k)$ where F is symmetric in its arguments when $\lambda_i(B) = \lambda_j(B)$. The precise result is as follows (see Theorem 4.1.4).

Theorem 1.3.17. *Let $k \in \mathbb{N}^*$ such that $\lambda_k(B) < \lambda_{k+1}(B)$. Let $F \in \mathcal{C}^2((\mathbb{R}_+^*)^k, \mathbb{R})$ be verifying*

$$|F(\lambda)| \leq C(1 + |\lambda|) \text{ for some } C > 0$$
$$\forall i, j \in \{1, \dots, k\} \text{ with } \lambda_i(B) = \lambda_j(B), \ \frac{\partial F}{\partial \lambda_i} = \frac{\partial F}{\partial \lambda_j} \text{ at } (\lambda_1(B), \dots, \lambda_k(B))$$

Then there exists $\delta_F > 0$ such that the functional

$$\Omega \in \mathcal{A} \mapsto T(\Omega)^{-1} + \delta F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$$

is minimal on the ball for any $\delta \in \mathbb{R}$ such that $|\delta| < \delta_F$.

Methods. Let us now decribe separately the methods we employed to prove the results.

Methods for Theorem 1.3.13. To prove Theorem 1.3.13 we rely on several distinct ideas. First, using a result of [Buc03, Theorem 3.4] one can always control the increase in eigenvalues of two nested open sets $\omega \subset \Omega$ by the increase in torsional energy, thus giving for some $C_{N,k} > 0$

$$\frac{1}{\lambda_k(\Omega)} - \frac{1}{\lambda_k(\Omega \cap B)} \leq C_{N,k}\lambda_1(\Omega)^{\frac{N}{2}} \left[T(\Omega) - T(\Omega \cap B)\right].$$

By using $w_\Omega \wedge w_B$ as a test function for the energy $T(\Omega \cap B)$, we then prove the estimate (which we think is of interest in itself)

$$T(\Omega) - T(\Omega \cap B) \leq C_N|\Omega \setminus B|$$

for some other constant $C_N > 0$, valid for any $\Omega \in \mathcal{A}$ (in Lemma 4.3.2). Combining these two estimates with the *sharp quantitative Saint-Venant inequality* we arrive at the inequality from Theorem 1.3.13.

Methods for Theorem 1.3.14. Let us remind that showing the quantitative inequalities from Theorem 1.3.14 amounts to proving that B minimizes

$$\Omega \in \mathcal{A} \mapsto T^{-1}(\Omega) + \delta\lambda_k(\Omega) \tag{1.54}$$

for sufficiently small $\delta \in \mathbb{R}$. The proof of this result is the first involved and technical part of Chapter 4, as it goes through regularity theory for vectorial-type free boundary problems. This comes from the fact that the expected optimality condition for this minimization is

$$(\partial_\nu w)^2 + T(\Omega^*)^2\delta(\partial_\nu u_k)^2 = Q \text{ over } \partial\Omega^* \tag{1.55}$$

for some $Q = Q(\delta) > 0$ (with $Q(\delta) \to \frac{1}{n^2}$ as $\delta \to 0$), where w and u_k are respectively the torsion function and the kth eigenfunction of a minimizer Ω^*. This is seen as a vectorial version of the free boundary condition $(\partial_\nu u)^2 = 1$ associated to the usual *one-phase Bernoulli problem* pioneered in [AC81]. When $\delta > 0$, this type of free boundary condition was studied by [KL18], [MTV17] and [CSY18] in the form

$$\sum_{i=1}^{n} a_i (\partial_\nu u_i)^2 = 1 \text{ over } \partial\Omega^*$$

for some $a_i \geq 0$. On the other hand, for $\delta < 0$ the condition (1.55) does not seem to have been studied in the literature. Instead, we interpret (1.57) in the framework of [MTV21], where the authors investigate the boundary condition

$$(\partial_\nu v_1)(\partial_\nu v_2) = Q \text{ over } \partial\Omega^* \tag{1.56}$$

for positive functions v_1, v_2. Setting the phases $v_1 := w - T(\Omega)|\delta|^{1/2} u_k$ and $v_2 := w + T(\Omega)|\delta|^{1/2} u_k$ in the regime $|\delta| \to 0$, we expect the condition $(\partial_\nu v_1)(\partial_\nu v_2) = Q$ on the free boundary $\partial\Omega^*$.

Let us give a final general comment before going into the steps of the proof: we take advantage of the fact that the torsion T^{-1} dominates λ_k in the minimization (1.54) in the regime $\delta \to 0$, relying on the fact that the functions w and u_k verify that u_k/w is uniformly controlled from above (see Lemma 4.2.2). This technical aspect explains the replacing of λ_1 by T^{-1}, thus passing from the minimization of $\lambda_1 + \delta\lambda_k$ to the minimization of $T^{-1} + \delta\lambda_k$.

The general strategy goes in the four following successive steps, which we first state as such before describing them in details below. We proceed as follows.

1. We start by proving existence of a minimizer Ω of (1.54) for $|\delta| \ll 1$ inside the class of quasi-open sets of measure ω_N (in Proposition 4.4.5).

2. We then prove first regularity properties of minimizers: any minimizer Ω of (1.54) is open, bounded with controlled diameter, the associated torsion function w_Ω and k^{th} eigenfunction $u_{k,\Omega}$ are Lipschitz continuous with uniform Lipschitz norm. Moreover w_Ω is non-degenerate, meaning that is has at least linear growth near the free boundary $\partial\Omega = \partial\{w_\Omega > 0\}$ (see Lemma 4.4.6 for these properties). Furthermore minimizers Ω verify (up to translation) that $\Omega \to B$ in a volume sense when $|\delta| \to 0$ (in Lemma 4.4.3).

3. We first infer uniform $\mathcal{C}^{1,\alpha}$ regularity of minimizers (in Lemma 4.4.14), and uniform $\mathcal{C}^{2,\alpha}$ regularity (in Lemma 4.4.15).

4. We perform a Fuglede-type argument, by proving that the ball is a local minimum of (1.54) among $\mathcal{C}^{2,\alpha}$ perturbations (in Proposition 4.4.17).

The conclusion of Theorem 1.3.14 then comes as a combination of Steps 3 and 4.

Let us detail how each step is obtained and highlight the difficulties associated. Once existence for (1.54) is ensured by Step 1, we will call Ω a minimizer.

1. The difficulties for deriving existence are mostly linked to the case $\delta < 0$, since the case $\delta > 0$ enjoys the useful structural condition that the studied functional (1.54) is nonincreasing for inclusion. In this latter case, a combination of the classical existence result from [BDM93b] and concentration-compactness arguments from [Buc00] would enable to proceed more directly. In the case $\delta < 0$ on the other hand, we also rely on concentration-compactness methods from [Buc00], but have to exclude some homogenization behaviour of the minimizing sequences (prevented for $|\delta| \ll 1$) by using capacitary measures.

2. Lipschitz regularity is proven for w_Ω, from which Lipschitz regularity of the kth eigenfunction $u_{k,\Omega}$ directly follows from the fact that w_Ω dominates $u_{k,\Omega}$. This is obtained by techniques derived from [AC81] (see [BHP05]): one proves that Ω is a minimizer of T^{-1} up to a volume term for outward perturbation $\tilde{\Omega} \supset \Omega$ (with $|\tilde{\Omega} \setminus \Omega| \leq 2$)

$$T(\Omega)^{-1} \leq T(\tilde{\Omega})^{-1} + \Lambda_N |\tilde{\Omega} \setminus \Omega|.$$

For non-degeneracy, we employ again methods introduced in [AC81]: in the fashion of [Buc12, Lemma 1], one proves that Ω is a minimizer of T^{-1} up to a volume term for inward perturbation $\tilde{\Omega} \subset \Omega$ (with $|\tilde{\Omega}| \geq \frac{1}{2}|\Omega|$):

$$T(\Omega)^{-1} + \Lambda_N |\Omega \setminus \tilde{\Omega}| \leq T(\tilde{\Omega})^{-1}.$$

3. This is the most technical part of the proof of Theorem 1.3.14. The general idea is to prove that $(w, u_k) := (w_\Omega, u_{k,\Omega})$ is a viscosity solution of

$$\begin{cases} -\Delta w = 1, \ -\Delta u_k = \lambda_k u_k \text{ in } \Omega \\ w = u_k = 0 \text{ over } \partial\Omega \\ (\partial_\nu w)^2 + T(\Omega)^2 \delta (\partial_\nu u_k)^2 = Q \end{cases} \tag{1.57}$$

for some $Q > 0$, where the free boundary condition translates the fact that Ω minimizes (1.54). Suppose for a moment that this is achieved. We then obtain $\mathcal{C}^{1,\alpha}$ regularity around flat points of the free boundary $\partial\Omega = \partial\{w > 0\}$ using [KL18] in the case $\delta > 0$, and [MTV21] in the case $\delta < 0$, where in the latter case we interpret the free boundary condition from (1.57) as (1.56).

The proof of (1.57) is rather technical. The boundary condition in the viscosity sense is obtained by using a blow-up argument which we can describe as follows: one proves the existence of blow-ups $\alpha(x \cdot \nu)_+$ and $\beta(x \cdot \nu)_+$ ($\alpha > 0$ and $\beta \in \mathbb{R}$) of respectively w and u_k at any contact point of the free boundary $z \in \partial\Omega$, by which we mean that there exists an interior ball $B_{z+R\nu,R} \subset \Omega$ or an exterior ball $B_{z-R\nu,R} \subset \mathbb{R}^N \setminus \Omega$ with

$$\left(\frac{w(z + rx)}{r}, \frac{u_k(z + rx)}{r} \right)$$

converging locally uniformly (up to subsequence) towards $\alpha(x \cdot \nu)_+$ and $\beta(x \cdot \nu)_+$ as $r \to 0$. Furthermore, the gradients (α, β) verify the desired boundary condition, *i.e.* $\alpha^2 + T(\Omega)^2 \delta \beta^2 = Q$. The existence of such blow-ups is obtained following techniques from [CS05], by relying on harmonic functions arguments and Harnack inequality. In the case $\delta < 0$ (and only in this case) we use the optimality of Ω in order to exclude the possibility of a blow-up $\overline{w} = \alpha^+(x \cdot \nu)_+$ on $\{x \cdot \nu > 0\}$ and $\overline{w} = \alpha^-(x \cdot \nu)_-$ on $\{x \cdot \nu < 0\}$, for some $\alpha^\pm > 0$. The optimality condition over (α, β) is then obtained by performing a blow-up argument inside the shape derivative of $T^{-1} + \delta\lambda_k$ at Ω, for which we rely on low regularity expressions of shape derivatives of T and λ_k.

The passage from $\mathcal{C}^{1,\alpha}$ to $\mathcal{C}^{2,\alpha}$ of Ω is obtained by partial hodograph transform techniques, first introduced in [KN77].

4. The last step relies on the following coercivity property for the torsion: for each $\Omega = B_h$ where

$$B_h := \{tx(1 + h(x)), \ t \in [0, 1), \ x \in \partial B\} \tag{1.58}$$

for some $h \in \mathcal{C}^{2,\alpha}(\partial B)$ with $\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)} \ll 1$, $|B_h| = |B|$ and B_h having barycenter at the origin, it holds

$$T(B) - T(B_h) \geq c_N \|h\|^2_{H^{1/2}(\partial B)}$$

(proven in [BDPV15], see also [Dam02]). The main step for proving minimality of B for (1.54) among $\mathcal{C}^{2,\alpha}$ perturbations thus consists in estimating

$$|\lambda_k(B_h) - \lambda_k(B)| \leq C_{N,k} \|h\|^2_{H^{1/2}(\partial B)}$$

for some $C_{N,k} > 0$ whenever $h \in \mathcal{C}^{2,\alpha}(\partial B)$ with $\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)} \ll 1$ and $|B_h| = |B|$ (in Proposition 4.4.19). This can be seen as some kind of weak $\mathbf{(IT)}_{H^{1/2},\mathcal{C}^{2,\alpha}}$ property (compare with the weak **(IT)** property from Lemma 1.3.8), consisting in a second-order estimate of $\lambda_k(B_h) - \lambda_k(B)$. The idea is to bound from above the second derivative of λ_k at a set B_{th} ($t \in [0,1]$) by the $H^{1/2}$ norm of the perturbation h.

Methods for Theorem 1.3.15. We follow the same four steps strategy as for proving Theorem 1.3.14, the underlying idea being again that the ball is a critical point of $\sum_{i=k}^{l} \lambda_i$. We only detail steps 3 and 4 as the two first are completely analogous. This time we call Ω a minimizer of (1.51).

3. Going from $\mathcal{C}^{1,\alpha}$ to higher regularity is obtained likewise. As for obtaining $\mathcal{C}^{1,\alpha}$ estimates, instead of (1.55) this time we are rather led to study the free boundary condition

$$(\partial_\nu w)^2 + T(\Omega)^2 \delta \sum_{i=k}^{l} (\partial_\nu u_i)^2 = Q \text{ over } \partial\Omega^*$$

for $\delta \in \mathbb{R}$, $|\delta| \ll 1$, which w and the successive eigenfunctions $u_k, \dots, u_l$ are expected to verify. While in the case $\delta > 0$ we can still apply the results from [KL18] about viscosity solutions $(u_1, \dots, u_m)$ to

$$\begin{cases} -\Delta u_i = f_i, \text{ in } \Omega, & \text{for } i \in \{1, \dots, m\}, \\ u_i = 0 \text{ over } \partial\Omega, & \text{for } i \in \{1, \dots, m\}, \\ \sum_{i=1}^{m} a_i (\partial_\nu u_i)^2 = Q \end{cases} \tag{1.59}$$

for positive numbers a_i, on the other hand, in the case $\delta < 0$ we do not have at hand a result of the type [MTV21] for nonnegative flat solution $(\underline{u_1}, \overline{u_1}, \dots, \underline{u_m}, \overline{u_m})$ of

$$\begin{cases} -\Delta \underline{u_i} = \underline{f_i}, \ -\Delta \overline{u_i} = \overline{f_i} \text{ in } \Omega, & \text{for } i \in \{1, \dots, m\}, \\ \underline{u_i} = \overline{u_i} = 0 \text{ over } \partial\Omega, & \text{for } i \in \{1, \dots, m\}, \\ \sum_{i=1}^{m} (\partial_\nu \underline{u_i})(\partial_\nu \overline{u_i}) = Q \end{cases}$$

for $\Omega = \{\underline{u_i} > 0\} = \{\overline{u_i} > 0\}$ when $m \geq 2$. We thus build a framework containing both the cases $\delta > 0$ and $\delta < 0$: we introduce a space linked to the viscosity solutions $(v, \underline{v}_1, \overline{v}_1, \dots, \underline{v}_m, \overline{v}_m)$ of

$$\begin{cases} -\Delta v = f, \ -\Delta \underline{v_i} = \underline{f_i}, \ -\Delta \overline{v_i} = \overline{f_i} \text{ in } \Omega, & \text{for } i \in \{1, \dots, m\}, \\ \underline{v_i} = \overline{v_i} = 0 \text{ over } \partial\Omega, & \text{for } i \in \{1, \dots, m\}, \\ \alpha^2 + \sum_{i=1}^{m} \frac{\underline{\alpha_i}^2 + \overline{\alpha_i}^2}{2} = 1 \text{ if } \delta > 0, \ \alpha^2 + \sum_{i=1}^{m} \underline{\alpha_i}\overline{\alpha_i} = 1 \text{ if } \delta < 0, \end{cases} \tag{1.60}$$

where the free boundary condition is again supposed to hold at any contact point $z \in \partial\Omega$, for some blow-ups $(\alpha(x \cdot \nu)_+, \underline{\alpha_1}(x \cdot \nu)_+, \overline{\alpha_1}(x \cdot \nu)_+, \dots, \underline{\alpha_m}(x \cdot \nu)_+, \overline{\alpha_m}(x \cdot \nu)_+)$ (in Definition 4.5.7). We then prove in the same spirit as in Theorem 1.3.14 that some phases $(w, \underline{v}_1, \overline{v}_1, \dots, \underline{v}_m, \overline{v}_m)$ with the $\underline{v}_i$ involving a combination of w and the u_i verify (1.60) in all cases $\delta > 0$ and $\delta < 0$.

We therefore work on viscosity solution to (1.60), and aim at proving $\mathcal{C}^{1,\alpha}$ regularity of Ω for flat solutions. We follow the general improvement of flatness argument strategy introduced in [DS11]. We thus prove first a partial Harnack inequality at fixed scale for flat solutions $(v, \underline{v}_1, \overline{v}_1, \dots, \underline{v}_m, \overline{v}_m)$ of (1.60) (in Proposition 4.5.12). Let us give a rough idea of the argument in the case where there is only one phase u: one proves that if u is ε-flat in the direction $e \in \mathbb{S}^{N-1}$ around a point $z \in \partial\{u > 0\}$ (set to be $z = 0$ here), meaning that there exists $|a|, |b| \leq \varepsilon$ with

$$\begin{aligned} 0 \leq & b - a \leq \varepsilon \\ (x \cdot e + a)_+ \leq & u(x) \leq (x \cdot e + b)_+ && \text{in } B_1, \\ & |\Delta u| < \varepsilon^2 && \text{in } B_1 \cap \{v > 0\} \\ & (\partial_\nu u)^2 = 1 && \text{in } \partial\{u > 0\} \end{aligned}$$

then u is $\kappa\varepsilon$-flat (for some universal $\kappa \in (0,1)$) in the same direction e, in a smaller ball B_{r_N} with universal radius $r_N > 0$. This partial Harnack inequality is obtained by harmonic functions arguments, where in our case an additional difficulty stems from the fact that the problem is multiphase. For $\delta > 0$ we derive it by following the arguments of [KL18] while paying attention to the constants, and for $\delta < 0$ we use as in [MTV21] the observation that $(v, \sqrt{\underline{v_1}\overline{v_1}}, \dots, \sqrt{\underline{v_m}\overline{v_m}})$ is a supersolution of a problem of the type (1.59).

A second step consists in obtaining the so-called improvement of flatness for sufficiently flat solutions (in Proposition 4.5.15): there exists universal $\tau \in (0,1), \overline{\varepsilon} > 0$ such that if u is ε-flat in the direction $e \in \mathbb{S}^{N-1}$ around $z := 0 \in \partial\{u > 0\}$ with $\varepsilon < \overline{\varepsilon}$, then $u_\tau := u(\tau x)/\tau$ is $\varepsilon/2$-flat in a direction $e' \in \mathbb{S}^{N-1}$ with $|e' - e| \leq C_N\varepsilon$. This is obtained by a contradiction argument, thus letting $\varepsilon \to 0$ and examining the equations verified at the limit by the functions $u_\varepsilon := \frac{u - x\cdot e}{\varepsilon}$. Relying on the partial Harnack inequality, one proves convergence (up to subsequence) of u_ε in a Hölder sense towards a function U, first proven to be harmonic in the upper ball $B_1 \cap \{x \cdot e > 0\}$ and then extended to a harmonic function inside the whole ball B_1 thanks to a boundary condition $\partial_e U = 0$ on $B_1 \cap \{x \cdot e = 0\}$. Standard growth properties of harmonic functions then provide a contradiction. Unlike in [DS11] and as in [MTV21] and [KL18], we face here the additional difficulty of having to handle several phases. In comparison with these three works, one of the novelties of our improvement of flatness result concerns uniform estimates in terms of the multiplicity m. In fact, we also examine the case $m \to \infty$ in the contradiction argument, and treat it by finding a way of reordering the functions $(\underline{u_i})_\varepsilon := \frac{\underline{u_i} - x\cdot e}{\varepsilon}, (\overline{u_i})_\varepsilon := \frac{\overline{u_i} - x\cdot e}{\varepsilon}$ in order to ensure uniform convergence of the graphs $\{(x, (\underline{u_i})_\varepsilon(x))\}$.

4. The strategy is similar to the simple eigenvalue case. Here instead we prove the second-order estimate (in Proposition 4.5.20)

$$\left| \sum_{i=k}^{l} (\lambda_i(B_h) - \lambda_i(B)) \right| \leq C_{N,k} \|h\|^2_{H^{1/2}(\partial B)}$$

for some $C_{N,k} > 0$, whenever $h \in \mathcal{C}^{2,\alpha}(\partial B)$ with $\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)} \ll 1$ and $|B_h| = |B|$. The sum $\sum_{i=k}^{l} \lambda_i$ being smooth as a symmetric function of the eigenvalues (see [LLdC06, Theorem 2.6]), this is obtained again by bounding from above the second derivative of $\sum_{i=k}^{l} \lambda_i$ at a set B_{th} ($t \in [0,1]$). However, we face here a difficulty specific to the fact that the eigenvalues are mutliple, which prevents from differentiating them individually. This is dealt with by making use of the idea that eigenvalues can be followed analytically in one variable, meaning that one can reorder $(\lambda_k(B_t), \dots, \lambda_l(B_t))$ around $t = 0$ (with the notation $B_t := B_{th}$ for $t \in [0,1]$) so that each reordered eigenvalue $\lambda(B_t)$ is analytic in t (see Lemma 4.4.18). We thus differentiate individually the eigenvalues $\lambda(B_t)$ twice and estimate the second-order derivative of the sum (which is left unchanged by the reordering). Here again we cannot estimate individually the $\lambda''(B_t)$, due to the fact that some of the terms appearing in $\lambda''(B_t)$ are implicitly defined because of the reordering (this is the case for instance for the derivatives of the eigenfunctions). We prove on the contrary that these implicit terms vanish in the sum.

1.4 Open problem: strong stability of the ball in more general classes

In Theorem 1.3.9 we prove strong stability of the ball for $P - c\lambda_1$ among convex shapes in any dimension $N \geq 2$, that is we find a threshold $c^* > 0$ such that (i) the ball is a L^1-local minimum among convex sets for $c \in (0, c^*)$ and (ii) the ball is not a L^1-local minimum among convex sets for $c \in (c^*, \infty)$. One can wonder about the optimal (or at least more general) classes for which this strong stability result could hold. This question is supported by the following weak stability results: for $N = 2$, the ball is L^1-locally minimal among simply connected sets for $P - \varepsilon\lambda_1$ and $\varepsilon \ll 1$ (thanks to the Payne-Weinberger inequality, see (1.44)); for $N \geq 3$, the ball is L^1-locally minimal among Lipschitz perturbations of the ball $P - \varepsilon\lambda_1$ and $\varepsilon \ll 1$ (see further Proposition 3.2.4). On the other hand the ball is never locally minimal for simply

connected sets in dimensions $N \geq 3$ (see [DL19, Proposition 6.1]). Hence, to sum up the available results concerning the local minimality of the ball for $P - c\lambda_1$ are the following

1. For $N = 2$: weak stability among simply connected sets, strong stability among convex sets.
2. For $N \geq 3$: weak stability among Lipschitz sets, strong stability among convex sets.

This raises the two following interesting questions:

1. *For $N = 2$: is the ball a L^1-local minimizer among simply connected sets of $P - c\lambda_1$ for $c \in (0, c^*)$?*
2. *For $N \geq 3$: is the ball a L^1-local minimizer among Lipschitz sets of $P - c\lambda_1$ for $c \in (0, c^*)$?*

A possible strategy to tackle these two problems would be to try to apply the same principle as the one we followed for proving strong stability among convex sets, that is by relying on (i) minimality of the ball for $P - c\lambda_1$ among sufficiently smooth perturbations and (ii) a regularity theory for quasi-minimizers enabling to prove sufficient regularity of minimizers of $P - c\lambda_1$. Let us mention that we prove step (i) for $\mathcal{C}^{1,\alpha}$ perturbations of the ball in Theorem 1.3.11. On the other hand, proving step (ii) would require substantial input, as it would demand to build a regularity theory of quasi-minimizers of the perimeter among simply connected sets for $N = 2$, and among Lipschitz sets for $N \geq 3$. The possiblity of such regularity theories seems to us to have interest in itself.

Chapter 2

Regularity in shape optimization under convexity constraint

This Chapter is a reprint of the paper *Regularity in shape optimization under convexity constraint* [LP23], written by the author of this thesis in collaboration with J. Lamboley, published in "Calc. Var. Partial Differential Equations".

Abstract

This paper is concerned with the regularity of shape optimizers of a class of isoperimetric problems under convexity constraint. We prove that minimizers of the sum of the perimeter and a perturbative term, among convex shapes, are $C^{1,1}$-regular. To that end, we define a notion of quasi-minimizer fitted to the convexity context and show that any such quasi-minimizer is $C^{1,1}$-regular. The proof relies on a cutting procedure which was introduced to prove similar regularity results in the calculus of variations context. Using a penalization method we are able to treat a volume constraint, showing the same regularity in this case. We go through some examples taken from PDE theory, that is when the perturbative term is of PDE type, and prove that a large class of such examples fit into our $C^{1,1}$-regularity result. Finally we provide a counter-example showing that we cannot expect higher regularity in general.

2.1 Introduction

In this paper, we study the regularity properties of minimizers in shape optimization under convexity constraint for a large class of problems of isoperimetric type.

In the classical framework of shape optimization (classical in the sense that there is no convexity constraint), the question of regularity has a long-standing history, with strong interactions with the fields of geometric measure theory and free boundary problems. More specifically, the study of various problems involving the classical De-Giorgi perimeter P leads to the notion of quasi-minimizer of the perimeter (see (2.9)) which is very useful to prove regularity for many problems of the form

$$\min\Big\{P(\Omega)+R(\Omega),\ \ \Omega\in\mathcal{A}\Big\} \tag{2.1}$$

Here R is considered to be a perturbative term, and $\mathcal{A}$ is a given class of measurable sets, for example the class of sets of given volume V_0, or the class of sets included in a fixed box D, or a mix of both:

$$\mathcal{A}_1=\{\Omega\subset\mathbb{R}^N,\ \ |\Omega|=V_0\},\quad \mathcal{A}_2=\{\Omega\subset\mathbb{R}^N,\Omega\subset D\},\quad \mathcal{A}_3=\{\Omega\subset\mathbb{R}^N,\ \Omega\subset D,\ |\Omega|=V_0\}$$

though many other examples are possible (note that $|\Omega|$ denotes the volume of Ω). It would be impossible to refer to every work in this direction, but we refer to [Mag12] for a nice introduction to the concept of

quasi-minimizer of the perimeter and for several examples, and to [SZ97, ACKS01, Lan07, KM14, DPV14, DPLPV18, Peg21] for a short sample of applications.

In this paper, we are interested in a similar class of problems, where we add a convexity assumption to the admissible shapes. More precisely, we replace (2.1) by

$$\min\Big\{P(K)+R(K),\ \ K\in\mathcal{A}\cap\mathcal{K}^N\Big\} \tag{2.2}$$

where $\mathcal{K}^N$ denotes the class of convex bodies of $\mathbb{R}^N$ (convex compact sets with nonempty interior). As before, $P(K)$ denotes the perimeter of K, and as K is a convex body (and therefore is a Lipschitz set) we have $P(K)=\mathcal{H}^{N-1}(\partial K)$ where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. Again R is a shape functional that is considered as a perturbative term, and we will make assumptions on R so that the term driving the regularity of optimal shapes is the perimeter term.

Before going into more details about our motivations and our strategy, let us start by giving a consequence of the three main results of the paper that are Theorems 2.2.3, 2.2.10 and 2.3.2:

Theorem 2.1.1. *Let $n\in\mathbb{N}^*$, $N\geq 2$, $F:(0,+\infty)\times(0,+\infty)\times(0,+\infty)^n\times\mathbb{R}^n_+\to\mathbb{R}$ be locally Lipschitz. Let $R:\mathcal{K}^N\to\mathbb{R}$ be defined by the formula*

$$R(K):=F\Big(|K|,\tau(K),\lambda_1(K),\dots,\lambda_n(K),\mu_1(K),\dots,\mu_n(K)\Big) \tag{2.3}$$

where $\tau(K)$ is the torsional rigidity of K, $\lambda_1(K),\dots,\lambda_n(K)$ are the n first Dirichlet eigenvalues of K, and $\mu_1(K),\dots,\mu_n(K)$ the n first Neumann eigenvalues of K (see Section 2.3 for precise definitions).

(i) Let $D\subset\mathbb{R}^N$ be measurable (non-necessarily with $|D|<\infty$). Then any solution to the minimization

$$\inf\big\{P(K)+R(K),\ K\in\mathcal{K}^N,\ K\subset D\big\}$$

is of class $C^{1,1}$.

(ii) Let $D\in\mathcal{K}^N$ and $0<V_0<|D|$. Then any solution to the minimization

$$\inf\big\{P(K)+R(K),\ K\in\mathcal{K}^N,\ K\subset D,\ |K|=V_0\big\}$$

is of class $C^{1,1}$.

In other words, we have identified a large class of functions R so that solutions to (2.2) are smooth up to the $C^{1,1}$-regularity. We refer to Section 2.3.3 for an example of a shape optimization problem of the kind (2.2) leading to an optimal shape that is a stadium, and is therefore not C^2, thus showing that our result is sharp in general. Note that existence is not always ensured for the two problems above (taking for instance $R=0$), and we thus prove in Theorem 2.3.4 the existence of solutions under various general hypotheses on R and D.

Motivations: Let us give a few motivations for such a result: shape optimization under convexity constraint goes back to the study of Newton's problem of the body of minimal resistance, that can be formulated as

$$\min\left\{J(u)=\int_D\frac{dx}{1+|\nabla u|^2},\ \ u:D\to[-M,0]\ \text{ convex}\right\} \tag{2.4}$$

where D is a smooth convex set in $\mathbb{R}^2$, and $M>0$ is given. In this formulation, the graph of u represents the form of a 3-dimensional convex body, and the energy J models the resistance experienced by the body as it moves through a homogeneous fluid with constant velocity in the direction orthogonal to D (in the negative direction in this formulation). The constant M gives a maximal height for the body under study. We refer to [BB05, LRP01a] for more details about this problem, but it is worth noticing that while one

can prove that this problem admits a solution (despite the energy not having a convexity property in $|\nabla u|$), even when D is a disk the solutions are not explicitly known (see [LWZ22]) and even their regularity is not known. It is nevertheless understood that the problem contains a non-convexity structure and that solutions cannot be locally smooth, see for example [LRP01a]. Other models with various backgrounds may also involve a convexity constraint, see for example [RC98] with a model in economics.

In the framework of shape optimization, it is interesting to notice that (2.1) might have optimizers that are convex (for example the euclidean ball), and in this case the study of (2.2) is not relevant. Nevertheless, in many situations, (2.1) may lead to non-convex solutions or even absence of an optimal shape. Let us give two examples of these situations:

- in [DPV14] (see also [DPLPV18]), the authors study (as in Theorem 2.1.1, λ_k denotes the kth-Dirichlet eigenvalue of the Laplace operator in Ω):

$$\min\left\{P(\Omega)+c\lambda_k(\Omega),\ \ \Omega\subset\mathbb{R}^N\right\}$$

  for $k\in\mathbb{N}^*$ and $c>0$, and show that optimal shapes are smooth up to a residual set of co-dimension less than 8 (see [DPV14, Remark 3.6] where it is shown that this problem is equivalent to a constrained formulation). When $N=2$ we have $(P+c\lambda_k)(\mathrm{Conv}(\Omega))\leq(P+c\lambda_k)(\Omega)$ so optimal shapes are necessarily convex, but when $N\geq 3$ this argument is not valid anymore. In [BO16, Figure 2] some numerical computations of optimal shapes are done when $N=3$, and one can observe that for some values of k the optimal shapes are not convex, so that the same problem with a convexity constraint is of interest (see Section 2.3.2 for more details about this problem).

- the famous Gamow's liquid drop model leads to the shape optimization problem

$$\inf\left\{P(\Omega)+\int_\Omega\int_\Omega\frac{dxdy}{|x-y|},\ \ \Omega\subset\mathbb{R}^3,\ |\Omega|=V_0\right\} \tag{2.5}$$

  where $V_0\in(0,+\infty)$. It is conjectured that there is a threshold $V^*>0$ such that the ball is a solution if $V_0<V^*$, and that there is no solution if $V_0>V^*$, see [KM14, Jul14, LO14] for partial results in this direction. Let us note that the non-existence phenomenon (which is proven for V_0 large enough, see [LO14]) is expected to be due to the splitting of the mass into pieces, and the convexity constraint is thus violated for such minimizing sequences. It is therefore interesting to wonder about a version of (2.5) within the class of convex bodies. In general, as it is easier to get existence within the class of convex bodies (see for example Theorem 2.3.4) hence many problems of the type (2.2) will be of interest if (2.1) has no solution.

Let us also quote two areas of applications to motivate our regularity results:

- in the study of Blaschke-Santaló diagrams for $(P,\lambda_1,|\cdot|)$ in the class of convex planar sets (see [FL21]), that is to say describing the set

$$\left\{(x,y)\in\mathbb{R}^2,\ \exists K\in\mathcal{K}^2,\ \ P(K)=x,\ \lambda_1(K)=y,\ |K|=1\right\}$$

  The authors of [FL21] use some regularity theory in shape optimization under convexity constraint in the proof of their main result [FL21, Theorem 1.2]. Similar results in higher dimension (replacing $\mathcal{K}^2$ with $\mathcal{K}^N$ for $N\geq 3$) are still open problems, and we believe that the tools we develop in this paper can be of help for further investigation in this direction.

- since the work of Cicalese and Leonardi [CL12], it is known that regularity theory can help to prove a quantitative version of classical isoperimetric inequalities, see also [BDPV15] where this strategy is the only one (we know of) giving the optimal exponent. It will be worth investigating if one could get new quantitative isoperimetric inequalities in the class of convex sets thanks to our regularity results.

State of the art about regularity theory with convexity constraint: In the framework of Calculus of variations, one can wonder about the regularity properties of solutions to the following generalization of (2.4):

$$\min\left\{\int_\Omega L(x,u(x),\nabla u(x))dx,\ \ u\in X,\ \ u \text{ convex}\right\} \tag{2.6}$$

where Ω is a convex set in $\mathbb{R}^N$, $L:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Lagrangian and X is a suitable functional space, possibly including boundary constraints. In [Car02] the author obtains in particular a C^1-regularity result when L is locally uniformly convex in the third variable, and when $N=1$. In [CLR01], the authors study the same case (L locally uniformly convex), but this time when $N\geq 1$, and X includes a Dirichlet boundary condition: they identify conditions on Ω, L and X so that solutions are C^1.

These results were not sharp in general, therefore Caffarelli, Carlier and Lions studied in [CCL13][1] the model case

$$\min\left\{J(u):=\frac{1}{2}\int_\Omega|\nabla u|^2dx+\int_\Omega fudx,\ \ u\in H^1(\Omega),\ \ u \text{ convex}\right\}, \tag{2.7}$$

and proved that a minimizer u^* is locally $C^{1,1-N/p}$ in Ω if $f_+\in L^p(\Omega)$ with $p>N$, and that this regularity is optimal.

In the framework of shape optimization, D. Bucur proved in [Buc03] a C^1-regularity result for shape optimization problems with convexity constraint, for functionals involving λ_k and the volume; this result can be applied for example to

$$\min\Big\{\lambda_k(K),\ \ |K|=V_0,\ K\in\mathcal{K}^N\Big\}.$$

A sharp regularity result for this problem is still an open problem, though it is expected that optimal shapes are $C^{1,1/2}$ (and that this result is sharp when $k=2$), see [Lam11].

For problems of the kind (2.2), [LNP12] shows that under some assumption on R (see Remark 2.2.6) and assuming $N=2$, solutions must be $C^{1,1}$. Comparing it to Theorem 2.1.1 above, this result applies to $R(K)=F(|K|,\lambda_1(K),\tau(K))$. Therefore, the results we show in the current paper are a generalization of [LNP12, Theorem 1] to the higher-dimensional case, and to a wider class of functional as well.

Finally, in [GNR18] we can find another $C^{1,1}$-regularity result for solutions to the following 2-dimensional version of a model for charged liquid drop at an equilibrium state:

$$\min\Big\{P(K)+\mathcal{I}(K),\ K\in\mathcal{K}^2,\ |K|=V_0\Big\}$$
$$\text{where}\quad \mathcal{I}(K)=\inf\left\{\int_{K\times K}\log\left(\frac{1}{|x-y|}\right)d\mu(x)d\mu(y),\mu\in\mathcal{P}(K)\right\} \tag{2.8}$$

where $\mathcal{P}(K)$ denotes the set of probabily measures supported on K. Here $\mathcal{I}(K)$ can be seen as a capacity term, and it is not so far from the functionals involved in (2.3), though it is related to a PDE in the exterior of K. Our result does not apply directly to (2.8) or to its higher-dimensional generalizations, but it will be the subject of future work to adapt our tools to this context.

Strategy of proof and plan of the paper: When dealing with regularity theory for (2.2) or (2.6), we already have a mild regularity property, namely that solutions are necessarily locally Lipschitz. This is a big difference with (2.1) where the most difficult part is to prove that solutions are a bit regular, further regularity being obtained usually through an Euler-Lagrange optimality condition.

In [Car02] as well as in [LNP12], the proofs of the regularity results also rely on the writing and the use of an Euler-Lagrange equation, taking into account the convexity constraint, which involves a Lagrange

[1] At the time we are writing this paper, the work [CCL13] is not published. Let us say here that we will use several ideas from this paper, though we will reproduce them for the convenience of the reader. We try to make it as clear as possible when these ideas are used in our proofs. We warmly thank G. Carlier for providing us a version of [CCL13].

multiplier (infinitely dimensional). It does not seem easy to adapt this method to higher dimensional cases.

In [Buc03, CLR01, CCL13, GNR18], the method is rather different, and consists in building test functions or shapes using a cutting procedure.

In this paper, we obtain three main results, which together lead to the proof of Theorem 2.1.1:

1. in the spirit of what is done without convexity constraint, we introduce a new notion of quasi-minimizer of the perimeter under convexity constraint (see Definition 2.2.1). We show in Theorem 2.2.3 that these sets are $C^{1,1}$ adapting the ideas of [CCL13]. A first important observation is that when writing the perimeter term as a function on the graph, we obtain a Lagrangian of the form $\int_D L(\nabla u)dx$ with a uniform convexity property, which explains that the ideas for (2.7) can be adapted to this case. However, the main difficulty here is to be careful on how a convex body can be seen as the graph of a convex function: it is not possible to have a local point of view, because this would lead to constraints that are too restrictive (see Remark 2.2.2). As an application, we show in Corollary 2.2.4 that if R satisfies a suitable Lipschitz property with respect to the volume metric (see (2.13)), then solutions to (2.2) (a priori with no other constraints) are quasi-minimizer of the perimeter under convexity constraint, and are therefore $C^{1,1}$. These results are described in Section 2.2.1.

2. then in Section 2.2.2, and in the same spirit with what is done in the *classical* case (without convexity constraint, see [Tam88] and [DPLPV18] for example), we show how one can handle volume constraint (see Theorem 2.2.10). To that end we show that the volume constraint can be penalized using Minkowski sums (see Lemma 2.2.11).

3. finally in Section 2.3 we focus on examples, and in particular we show that the functional (2.3) satisfies a Lipschitz property with respect to the volume metric, so that Theorem 2.2.3 can be applied (see Theorem 2.3.2).

2.2 Regularity in shape optimization

In the classical context of sets minimizing perimeter (without convexity constraint), the concept of quasi-minimizer of the perimeter has proved to be very convenient: denoting P the classical De-Giorgi perimeter, we say that $\Omega \subset \mathbb{R}^N$ is a local quasi-minimizer of the perimeter if there exists $\alpha \in (0,1]$, $C > 0$ and $r_0 > 0$ such that for every $r \in (0, r_0)$ and $x \in \mathbb{R}^N$ we have:

$$P(\Omega) \leq P(\widetilde{\Omega}) + Cr^{N-1+\alpha}, \quad \text{for every measurable } \widetilde{\Omega} \subset \mathbb{R}^N \text{ such that } \Omega \Delta \widetilde{\Omega} \Subset B_r(x). \tag{2.9}$$

The regularity theory then shows that quasi-minimizers of the perimeter are $C^{1,\alpha/2}$, up to a possibly singular set of dimension less than $N-8$ (see for instance [Tam88]). This regularity can even be strengthened to $C^{1,\alpha}$ for every $\alpha \in (0,1)$ if there exists $\Lambda > 0$ such that

$$P(\Omega) \leq P(\widetilde{\Omega}) + \Lambda |\Omega \Delta \widetilde{\Omega}|, \quad \text{for every measurable } \widetilde{\Omega} \subset \mathbb{R}^N \tag{2.10}$$

(see [Amb97, Theorem 4.7.4]). In order to take advantage of these results, when studying a shape optimization problem involving the perimeter in the energy functional, one tries to show that a minimizer of our problem must be a quasi-minimizer of the perimeter. To that end, one needs to handle the different terms in the energy, as well as the various constraints.

In this section we therefore introduce a new notion of quasi-minimizer of the perimeter, under a convexity constraint. We study the regularity property it leads to, and then show how one can deal with various constraints and energy terms.

Throughout this section we denote by $\mathcal{K}^N$ the class of convex bodies of $\mathbb{R}^N$ (convex compact sets with nonempty interior). Note that (as convex bodies are Lipschitz set), we have $P(K) = \mathcal{H}^{N-1}(\partial K)$ for any $K \in \mathcal{K}^N$, *i.e.* the perimeter of a convex body is the $N-1$-dimensional Hausdorff measure of its boundary.

2.2.1 Regularity for quasi-minimizers of the perimeter under convexity constraint

Definition 2.2.1. *We say that $K \in \mathcal{K}^N$ is a quasi-minimizer of the perimeter under convexity constraint if there exist $\varepsilon_K > 0, \Lambda_K \geq 0$ such that*

$$\forall \widetilde{K} \in \mathcal{K}^N \text{ such that } \widetilde{K} \subset K \text{ and } |K \setminus \widetilde{K}| \leq \varepsilon_K, \quad P(K) \leq P(\widetilde{K}) + \Lambda_K |K \setminus \widetilde{K}| \tag{2.11}$$

Remark 2.2.2. This notion of quasi-minimizer is not the mere restriction to convex perturbations of the standard notion of quasi-minimizer recalled in (2.9) and (2.10):

- first, here we ask that K be minimal in a volume-neighborhood instead of asking it only for sets $\widetilde{K}$ verifying $\widetilde{K} \Delta K \subset B_r(x)$ for some $x \in \mathbb{R}^N$ and for small enough $r > 0$. This is due to the fact that the latter condition is much too restrictive for convex sets, as it is not always possible to perturbate a convex set K into $\widetilde{K} \in \mathcal{K}^N$ only over some ball $B_r(x)$. For instance, if $K \in \mathcal{K}^2$ is a square with $x \in \partial K$ located inside a segment of ∂K, then we can see that if r is small enough any $\widetilde{K} \in \mathcal{K}^2$ such that $\widetilde{K} \Delta K \subset B_r(x)$ must be K itself. In fact, the possibility of perturbating a convex set K locally around $x \in \partial K$ is somehow directly connected to some kind of strict convexity of K around x. As a consequence, the error term is replaced by the volume of $K \Delta \widetilde{K}$, similarly to what is done in [Mag12].
- on the other hand we merely require optimality for the sets $\widetilde{K}$ which perturbate K from the inside, as this will be sufficient to obtain regularity properties (see the proof of Theorem 2.2.3, where the competitors $K_r \subset K$ are obtained by cutting K by a well-chosen hyperplane). Note that improving the quasi-minimality property by allowing also outward perturbations of K does not lead to better regularity properties in general (as shows the counter-example in Section 2.3.3).

The regularity result for quasi-minimizers proved in this section is the following.

Theorem 2.2.3. *Let K be a quasi-minimizer of the perimeter under convexity constraint. Then K is $C^{1,1}$.*

As mentioned in the introduction, this leads to a regularity result for minimizer of certain energy having a perimeter term: letting $D \subset \mathbb{R}^N$, we consider the following shape optimization problem

$$\inf \left\{ P(K) + R(K), \ K \in \mathcal{K}^N, \ K \subset D \right\} \tag{2.12}$$

where R is a shape functional satisfying

$$\forall K \in \mathcal{K}^N, \exists C_K > 0, \ \exists \varepsilon_K > 0, \ \forall \widetilde{K} \in \mathcal{K}^N \quad \text{s.t. } \widetilde{K} \subset K \ \text{ and } \ |K \setminus \widetilde{K}| \leq \varepsilon_K, \quad R(\widetilde{K}) - R(K) \leq C_K |K \setminus \widetilde{K}| \tag{2.13}$$

Then we have the following easy consequence of Theorem 2.2.3:

Corollary 2.2.4. *Assume that $R : \mathcal{K}^N \to \mathbb{R}$ satisfies (2.13). Then any solution K^* of (2.12) is $C^{1,1}$.*

Remark 2.2.5. Note that it may happen that (2.12) has no solution even if D is bounded: it is the case for example if $R \equiv 0$. See Theorem 2.3.4 (i) for an existence result when D is a convex body and there is an additional volume constraint on K.

Remark 2.2.6. In [LNP12] is proved a result similar to Corollary 2.2.4 in the case $N = 2$: more precisely, it is proved (see [LNP12, Corollary 1]) that if K^* is a solution of (2.12) and if R admits a shape derivative at K^* (see [HP18, Section 5.9.1]) which can be represented in $L^p(\partial K^*)$ with $p \in [1, \infty]$, which means that for every $\xi \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$,

$$R'(K^*)(\xi) = \lim_{t \to 0} \frac{\big(R((\mathrm{Id} + t\xi)(K^*)) - R(K^*)\big)}{t} = \int_{\partial K^*} g \, \xi \cdot \nu_{\partial K^*} d\sigma \tag{2.14}$$

for some function $g \in L^p(\partial K^*)$, then $\partial K^* \cap D$ is $W^{2,p}$. In particular when $p = \infty$, this leads to the $C^{1,1}$-regularity as in Corollary 2.2.4.

The authors also prove (see [LNP12, Section 3.2]) that the function $R : K \mapsto F(|K|, \lambda_1(K), \tau(K))$ (where $F : \mathbb{R}^3 \to \mathbb{R}$ is smooth, and the PDE functionals λ_1 and τ are defined at the beginning of Section 2.3.2) satisfies (2.14) for some $g \in L^\infty(\partial K^*)$, leading to a $C^{1,1}$-optimal shape in that case. Let us conclude with two comments:

1. The proof of [LNP12, Theorem 1,Corollary 1] is completly different from the one of Theorem 2.2.3 and Corollary 2.2.4, as it relies on an Euler-Lagrange equation for (2.12) (see [LNP12, Proposition 1]), and we believe that these ideas are restricted to the 2-dimensional case.

2. As we will see in Section 2.3, assumption (2.13) is much more flexible than (2.14) and applies to much more examples, in particular it does not require the existence of a shape derivative.

Ideas of the proof of Theorem 2.2.3: As mentioned in the Introduction, the proof consists in building a framework enabling to use the ideas of [CCL13], where the authors prove $C^{1,1}$-regularity of the minimizers to the calculus of variation problem (2.7). For any $K \in \mathcal{K}^N$ we locally write ∂K near some point $\widehat{x_0} \in \partial K$ as the graph of a convex function $u : \Omega \to \mathbb{R}$, so that the perimeter of K "near this point" is seen as a Lagrangian $\int_\Omega L(\nabla u)$ where L is locally strongly convex, meaning that $L : \mathbb{R}^N \to \mathbb{R}$ is smooth and verifies

$$\forall M > 0,\ \exists \alpha > 0,\ \forall \left(|p| \leq M, |p'| \leq M\right),\ L(p') - L(p) \geq \langle \nabla L(p), p' - p\rangle + \frac{\alpha}{2}|p' - p|^2$$

This gives hope that we can use the procedure from [CCL13], as it is natural to expect that such an energy behaves like (2.7). A main difficulty however is to show that the geometrical context actually allows to build competitors in a similar fashion to [CCL13]. If K is a quasi-minimizer in the sense of Definition 2.2.1, such competitors will be obtained by setting

$$K_v := K \cap \mathrm{Epi}(v)$$

for well-chosen convex functions $v : \Omega \to \mathbb{R}$ with $v \geq u$, using the notation $\mathrm{Epi}(v)$ for the epigraph of v. It is important to notice that it is not possible to work locally (*i.e.* picking Ω as a small neighborhood) but we rather have to choose Ω maximal, and this leads to new difficulties in comparison with [CCL13] (mostly linked to the case $N \geq 3$, see also Remark 2.2.7). An important part of Step **(ii)** of the proof is concerned with addressing this issue.

Proof of Theorem 2.2.3: Let K be a quasi-minimizer of the perimeter under convexity constraint in the sense of Definition 2.2.1.

Representation of K as a graph: Let $\widehat{x_0} \in \partial K$; applying Proposition 2.4.3, we get that there exists a hyperplane $H \subset \mathbb{R}^N$ and a unit vector $\xi \in \mathbb{R}^N$ normal to H such that, denoting by (x, t) a point in $H \times \mathbb{R}\xi$ coordinates (and hence denoting $\widehat{x_0} := (x_0, 0)$):

- The set $\Omega := \{x \in H,\ (x + \mathbb{R}\xi) \cap \mathrm{Int}(K) \neq \emptyset\}$ is open, bounded and convex, and the function

$$\begin{aligned} u : \Omega &\to \mathbb{R} \\ x &\mapsto \min\{t \in \mathbb{R},\ (x, t) \in K\} \end{aligned}$$

  is well-defined and convex.

- It holds

$$\begin{aligned} &\{(x, u(x)),\ x \in \Omega\} \subset \partial K \\ &\quad K \cap (\Omega \times \mathbb{R}\xi) \subset \{(x, t) \in \Omega \times \mathbb{R}\xi,\ u(x) \leq t\} \end{aligned}$$

- There exists a $\beta > 0$ and $c := c(\beta) > 0$ such that $B_\beta := B_\beta(x_0) \Subset \Omega$ verifies

$$\{(x,t) \in B_\beta \times \mathbb{R}\xi,\ u(x) \le t \le u(x) + c\} \subset K. \tag{2.15}$$

Throughout the proof the coordinates (x,t) are thought in the orthogonal decomposition $H \times \mathbb{R}\xi$. Moreover the notation $B_r(x)$ for some $x \in H$ and $r > 0$ will denote a ball lying in H.

Since $B_\beta \Subset \Omega$ we have that u is globally Lipschitz in B_β. Setting some $y \in B_{\beta/2}(x_0)$ and $p \in \partial u(y)$ (where $\partial u(y)$ denotes the subdifferential of the convex function u at y) we let $l(x) := u(y) + \langle p, x - y\rangle$ for $x \in H$ and $M_r := \sup_{B_r(y)}(u - l)$. We aim to prove that there exists $C > 0$ and $r_0 > 0$ (both independent on y) such that $M_r \le Cr^2$ for any $y \in B_{\beta/2}(x_0)$ and $0 < r < r_0$. This classically ensures that u is $C^{1,1}$ over $B_{\beta/2}(x_0)$ (see for instance Lemma 3.2 in [DPF15]). As the case $M_r = 0$ is trivial, from now on we fix $y \in B_{\beta/2}(x_0)$, $0 < r < \beta/2$ and assume $M_r > 0$. Note that p, l, M_r (and other objects we will introduce along the proof) depend on y, although for simplicity it does not appear in the notations. We will also set $y = 0$ for simplicity, while paying attention to the fact that the estimates we make in the proof do not depend on y.

Construction of a competitor: Let q_r be some unit vector such that $M_r = (u - l)(rq_r)$. We set

$$\forall x \in H,\ \sigma_r(x) := l(x) + \frac{M_r}{2r}(\langle q_r, x\rangle + r),\ \widehat{\sigma}_r(x) := (x, \sigma_r(x)),$$

$$H_r := \widehat{\sigma}_r(H),\ H_r^+ := \{(x,t) \in \mathbb{R}^{N-1} \times \mathbb{R},\ t \ge \sigma_r(x)\},$$

and finally we define:

$$K_r := K \cap H_r^+$$

Notice that $K_r \subset K$ is convex and compact. As we will show in section **(i)** of the proof, the construction of K_r ensures that $\mathrm{Int}(K_r) \neq \emptyset$ and $|K \setminus K_r| \le \varepsilon_K$ for $r \le r(\mathrm{diam}(\Omega), \varepsilon_K, \|\nabla u\|_{L^\infty(B_\beta)})$ (see (2.20) and the end of section **(i)**). Therefore from (2.11) we will get

$$P(K) - P(K_r) \le \Lambda_K |K \setminus K_r| \tag{2.16}$$

for such r. With (2.16) we are left to estimate **(i)** the volume variation from above and **(ii)** the perimeter variation from below. We first provide some central estimates on the size of the set $\{u \le \sigma_r\}$ which were proven in [CCL13]. Note that these estimates only use convexity of u and do not depend on the particular kind of energy introduced in the problem (2.7). We reproduce the proof of [CCL13] below for the convenience of the reader.

Estimate of $\{u \le \sigma_r\}$: Let us prove

$$B^+_{r/2}(0) \subset \{u \le \sigma_r\} \subset \{|\langle q_r, \cdot\rangle| \le r\} \tag{2.17}$$

where we set $B^+_{r/2}(0) = B_{r/2}(0) \cap \{\langle q_r, \cdot\rangle \ge 0\}$.

- $\{u \le \sigma_r\} \subset \{\langle q_r, \cdot\rangle \ge -r\}$: if $\langle q_r, x\rangle < -r$, then

$$u(x) \ge l(x) > l(x) + \frac{M_r}{2r}(\langle q_r, x\rangle + r) = \sigma_r(x)$$

  whence we deduce $x \notin \{u \le \sigma_r\}$.

- $\{u \le \sigma_r\} \subset \{\langle q_r, \cdot\rangle \le r\}$: over the interval $I := \Omega\ \cap \{tq_r, t > r\}$ we know that $u > l + M_r$ thanks to the convexity of u. Therefore one can separate the convex sets I and $\{u \le l + M_r\}$ by some hyperplane Π. Since by definition $B_r(0) \subset \{u \le l + M_r\}$, Π must also seperate $B_r(0)$ and I, implying that $\Pi = \{x \in \mathbb{R}^N,\ \langle q_r, x\rangle = r\}$. This yields in particular $(u - l) \ge M_r$ over $\Pi \cap \Omega$. Given now $x \in \Omega$ such that $\langle q_r, x\rangle > r$ set $z \in \Pi \cap [0, x]$: from the two informations $(u - \sigma_r)(0) = -M_r/2$ and $(u - \sigma_r)(z) = (u - l)(z) - M_r \ge 0$ we deduce $(u - \sigma_r)(x) > 0$ using convexity of $u - \sigma_r$, so that $x \notin \{u \le \sigma_r\}$.

- $B^+_{r/2}(0) \subset \{u \le \sigma_r\}$: given $x \in B_{r/2}(0)$, we have $2x \in B_r(0)$ hence

$$(u-l)(x) \le \frac{1}{2}(u-l)(2x) + \frac{1}{2}\underbrace{(u-l)(0)}_{=0} \le \frac{M_r}{2}$$

using convexity. If in addition $\langle q_r, x\rangle \ge 0$ then

$$(u-\sigma_r)(x) = (u-l)(x) - \frac{M_r}{2r}(\langle q_r, x\rangle + r) \le \frac{M_r}{2} - \frac{M_r}{2} \le 0$$

so that $x \in \{u \le \sigma_r\}$.

(i) Estimate from above

We have

$$\begin{aligned} |K \setminus K_r| &= |K \cap (\mathbb{R}^N \setminus H_r^+)| \\ &= |\{(x,t) \in K,\ t < \sigma_r(x)\}| \\ &\le |\{(x,t) \in \Omega \times \mathbb{R}^+,\ u(x) \le t \le \sigma_r(x)\}| \\ &= \int_{\{u \le \sigma_r\}} (\sigma_r - u) d\mathcal{H}^{N-1} \end{aligned} \tag{2.18}$$

using Fubini's Theorem.

If $x \in \{u \le \sigma_r\}$ we have thanks to the right-hand-side inclusion of (2.17)

$$0 \le (\sigma_r - u)(x) = \underbrace{(l-u)(x)}_{\le 0} + \frac{M_r}{2r}\left(\langle q_r, x\rangle + r\right) \le M_r$$

i.e.

$$0 \le \sigma_r - u \le M_r \text{ over } \{u \le \sigma_r\} \tag{2.19}$$

Injecting (2.19) into (2.18) yields

$$|K \setminus K_r| \le M_r \mathcal{H}^{N-1}(\{u \le \sigma_r\}) \tag{2.20}$$

We will refine further (2.20) (into (2.43)), but this estimate is sufficient for now. It gives in particular that there exists $r_0(\mathrm{diam}(\Omega), \varepsilon_K, \|\nabla u\|_{L^\infty(B_\beta)}) > 0$ such that $|K \setminus K_r| \le \varepsilon_K$ for any $r < r_0(\mathrm{diam}(\Omega), \varepsilon_K, \|\nabla u\|_{L^\infty(B_\beta)})$ hence also that $\mathrm{Int}(K_r) \ne \emptyset$ for such r. Indeed, as $|p| \le \|\nabla u\|_{L^\infty(B_\beta)}$ it holds for $0 < r < \beta/2$:

$$M_r \le \sup_{x \in B_r(0)} |u(x) - u(0)| + \sup_{x \in B_r(0)} |\langle p, x\rangle| \le 2r\|\nabla u\|_{L^\infty(B_\beta)} \tag{2.21}$$

so that from (2.20) we get

$$|K \setminus K_r| \le 2r\|\nabla u\|_{L^\infty(B_\beta)} \mathcal{H}^{N-1}(\{u \le \sigma_r\}) \le 2r\|\nabla u\|_{L^\infty(B_\beta)} \mathcal{H}^{N-1}(\Omega)$$

which yields $|K \setminus K_r| \le \varepsilon_K$ for $r < \varepsilon_K / (2\mathcal{H}^{N-1}(\Omega)\|\nabla u\|_{L^\infty(B_\beta)})$.

(ii) Estimate from below

We now deal with estimating from below the perimeter variation. In the view of (2.20), provided that $r \leq r_0(\mathrm{diam}(\Omega), \varepsilon_K, \|\nabla u\|_{L^\infty(B_\beta)})$ then K_r has non-empty interior, which we will suppose from now on. Let us start by showing that

$$P(K) - P(K_r) \geq \mathcal{H}^{N-1}(\widehat{u}(\Omega)) - \mathcal{H}^{N-1}(\widehat{v_r}(\Omega) \cap \partial K_r) \tag{2.22}$$

where we set

$$\forall x \in \Omega,\ \widehat{u}(x) := (x, u(x)),\ v_r(x) = \max\{u, \sigma_r\}(x),\ \widehat{v_r}(x) := (x, v_r(x)).$$

We have

$$\begin{aligned} P(K) - P(K_r) &= \mathcal{H}^{N-1}(\partial K) - \mathcal{H}^{N-1}(\partial K_r) \\ &= \mathcal{H}^{N-1}(\partial K \cap \widehat{u}(\Omega)) + \mathcal{H}^{N-1}(\partial K \cap \widehat{u}(\Omega)^c) - \mathcal{H}^{N-1}(\partial K_r \cap \widehat{v_r}(\Omega)) \\ &\quad -\mathcal{H}^{N-1}(\partial K_r \cap \widehat{v_r}(\Omega)^c) \\ &= \mathcal{H}^{N-1}(\widehat{u}(\Omega)) - \mathcal{H}^{N-1}(\partial K_r \cap \widehat{v_r}(\Omega)) + \mathcal{H}^{N-1}(\partial K \cap \widehat{u}(\Omega)^c) \\ &\quad -\mathcal{H}^{N-1}(\partial K_r \cap \widehat{v_r}(\Omega)^c) \end{aligned}$$

using in the third line that $\widehat{u}(\Omega) \subset \partial K$. If we show

$$\partial K_r \cap \widehat{v_r}(\Omega)^c \subset \partial K \cap \widehat{u}(\Omega)^c \tag{2.23}$$

then we obtain (2.22). Therefore, let $\widehat{x} \in \partial K_r \cap \widehat{v_r}(\Omega)^c$. As $\partial K_r \subset K$, we first want to show that $\widehat{x} \notin \mathrm{Int}(K)$. Assume by contradiction that $\widehat{x} \in \mathrm{Int}(K)$, then as

$$\partial K_r = \partial(K \cap H_r^+) = (K \cap \partial H_r^+) \cup (\partial K \cap H_r^+) = (K \cap H_r) \cup \big(\partial K \cap H_r^+ \setminus H_r\big) \tag{2.24}$$

(the second equality comes from the fact that K_r and H_r^+ are closed) we must have $\widehat{x} \in H_r$. But then $\widehat{x} \in H_r \cap \mathrm{Int}(K)$ and we deduce that there exists $x \in \Omega$ such that $\widehat{x} = (x, \sigma_r(x))$ with $\sigma_r(x) > u(x)$, thus getting $v_r(x) = \sigma_r(x)$ and $\widehat{x} = \widehat{v_r}(x)$, which is a contradiction. Now, as $\widehat{x} \in \partial K_r \subset H_r^+$, assuming by contradiction that there exists $x \in \Omega$ such that $\widehat{x} = (x, u(x))$ leads to $u(x) \geq \sigma_r(x)$, implying again the contradiction $\widehat{x} = \widehat{v_r}(x)$. This concludes the proof of (2.23) and (2.22).

Let us rewrite

$$\widehat{v_r}(\Omega) \cap \partial K_r = \widehat{v_r}(\Omega) \cap K = \widehat{v_r}(\Omega_r) \tag{2.25}$$

by setting $\Omega_r := \widehat{v_r}^{-1}(K) \subset \Omega$ (see Figure 2.1). The first equality of (2.25) is justified in the following way: first, as $\partial K_r \subset K$, then $\widehat{v_r}(\Omega) \cap \partial K_r \subset \widehat{v_r}(\Omega) \cap K$. Second, if $\widehat{x} \in \widehat{v_r}(\Omega) \cap K$, let us write $\widehat{x} = (x, v_r(x))$ for some $x \in \Omega$. Then either $\sigma_r(x) \geq u(x)$, giving $\widehat{x} = \widehat{v_r}(x) = \widehat{\sigma_r}(x) \in H_r$; as $\widehat{x} \in K$, we get that $\widehat{x} \in K \cap H_r \subset \partial K_r$ thanks to (2.24). Else, $u(x) > \sigma_r(x)$ so that $\widehat{x} = \widehat{u}(x) \in \partial K \cap H_r^+ \setminus H \subset \partial K_r$ using again (2.24).
We then get from (2.22)

$$P(K) - P(K_r) \geq \mathcal{H}^{N-1}(\widehat{u}(\Omega)) - \mathcal{H}^{N-1}(\widehat{v_r}(\Omega_r)) \geq \mathcal{H}^{N-1}(\widehat{u}(\Omega_r)) - \mathcal{H}^{N-1}(\widehat{v_r}(\Omega_r)) \tag{2.26}$$

We will rewrite the right hand side of (2.26) using the classical formula for the perimeter of a Lipschitz graph, but we start by showing two importants features of Ω_r. Let us note here that the introduction of Ω_r is not necessary if $N = 2$, while it is meaningful for $N \geq 3$ (see Remark 2.2.7).
Ω_r **is convex:** Let us show that $\Omega_r = \pi_H(K_r)$, where $\pi_H(K_r)$ is the orthogonal projection over H of the convex set K_r, which will give right away that Ω_r is convex. First, if $x \in \Omega_r$ then $x \in \Omega$ with $(x, v_r(x)) \in K$, and $t = v_r(x) \geq \sigma_r(x)$ satisfies that $(x, t) \in K$ with $t \geq \sigma_r(x)$, providing $(x, t) \in K_r$ hence $x \in \pi_H(K_r)$. Conversely, let $x \in H$ be such that there exists $t \in \mathbb{R}$ with $(x, t) \in K_r$, implying that $(x, t) \in K$ with $t \geq \sigma_r(x)$. Note that $x \in \Omega$ by definition of Ω and using that $K_r \subset K$. If $u(x) \geq \sigma_r(x)$ then $\widehat{v_r}(x) = \widehat{u}(x) \in K$ which means that $x \in \Omega_r$. Else, t is such that $t \geq \sigma_r(x) \geq u(x)$, which gives that

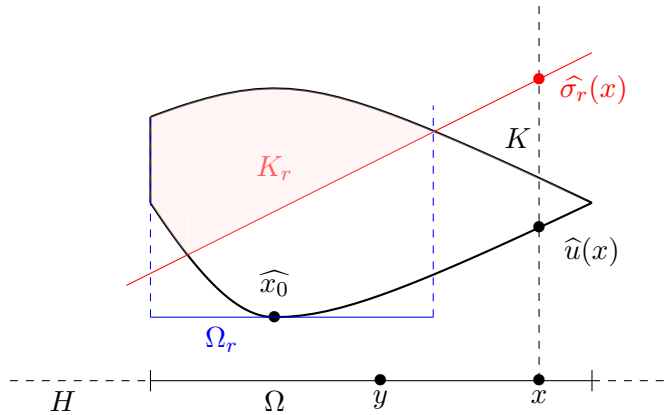


Figure 2.1: Cutting procedure

$\widehat{v_r}(x) = \widehat{\sigma_r}(x) \in K$ by convexity of K, since $(x,t) \in K$ and $(x, u(x)) \in K$. Thus $\Omega_r = \pi_H(K_r)$ and Ω_r is convex.

Ω_r has non-empty interior: We now prove that Ω_r has non-empty interior with a ball which has size uniform in y (which we set to be $y = 0$), *i.e.* that there exists $\widetilde{\beta} \in (0, \frac{\beta}{2})$ such that

$$\forall r \in (0, \widetilde{\beta}),\ B_{\widetilde{\beta}}(0) \subset \Omega_r \tag{2.27}$$

Given $\widetilde{\beta} \in (0, \beta/2)$ that will be chosen later, using (2.21) and $|p| \leq \|\nabla u\|_{L^\infty(B_\beta)}$ we get for any $x \in B_{\widetilde{\beta}}(0)$ and $r \in (0, \widetilde{\beta})$,

$$\begin{aligned}\sigma_r(x) &= u(0) + \langle p, x\rangle + \frac{M_r}{2r}(\langle q_r, x\rangle + r)\\ &\leq u(0) + \|\nabla u\|_{L^\infty(B_\beta)}\left(2\widetilde{\beta} + r\right) \leq u(x) + 4\widetilde{\beta}\|\nabla u\|_{L^\infty(B_\beta)}\end{aligned}$$

If we choose $\widetilde{\beta}$ such that $4\widetilde{\beta}\|\nabla u\|_{L^\infty(B_\beta)} \leq c(\beta)$ where $c(\beta)$ satifies (2.15), we deduce

$$\forall r \in (0, \widetilde{\beta}),\ \sigma_r \leq u + c \text{ in } B_{\widetilde{\beta}}(0) \tag{2.28}$$

We are now in a position to prove (2.27). Let $x \in B_{\widetilde{\beta}}(0)$: if $u(x) \geq \sigma_r(x)$, then $\widehat{v_r}(x) = \widehat{u}(x) \in K$ hence $x \in \Omega_r$. Otherwise $u(x) \leq \sigma_r(x)$, and then $u(x) \leq \sigma_r(x) \leq u(x) + c$ thanks to (2.28); using (2.15) we get in fact $\widehat{v_r}(x) = (x, \sigma_r(x)) \in K$, meaning that $x \in \Omega_r$.

Rewriting (2.26) with Lipschitz graphs: We claim now that (2.26) rewrites

$$\begin{aligned}P(K) - P(K_r) &\geq \int_{\Omega_r}\left[\sqrt{1+|\nabla u|^2} - \sqrt{1+|\nabla v_r|^2}\right] d\mathcal{H}^{N-1}\\ &= \int_{\mathrm{Int}(\Omega_r)}\left[\sqrt{1+|\nabla u|^2} - \sqrt{1+|\nabla v_r|^2}\right] d\mathcal{H}^{N-1}\end{aligned} \tag{2.29}$$

If (2.29) holds true, the second line comes from the fact that $|\Omega_r| = |\mathrm{Int}(\Omega_r)|$, since Ω_r is convex. Now, from [AFP00, Remark 2.72] one has

$$\mathcal{H}^{N-1}(\widehat{u}(\omega)) = \int_\omega \sqrt{1+|\nabla u|^2} d\mathcal{H}^{N-1}$$

if $\omega \subset \Omega_r$ with $u_{|\omega}$ Lipschitz. As u is not necessarily Lipschitz over the whole of Ω_r, let us take an increasing sequence (Ω_n) of open subsets of Ω with $\Omega_n \Subset \Omega$ for each n and $\cup_n \Omega_n = \Omega$. Then setting

$\Omega_r^n := \Omega_n \cap \Omega_r$, the sequence (Ω_r^n) is still increasing with $\cup_n \Omega_r^n = \Omega_r$. As $u_{|\Omega_r^n}$ is now Lipschitz we can write

$$\mathcal{H}^{N-1}(\widehat{u}(\Omega_r^n)) = \int_{\Omega_r^n} \sqrt{1+|\nabla u|^2} d\mathcal{H}^{N-1}$$

The monotonous convergence theorem applies on each side of the equation, yielding at the limit

$$\mathcal{H}^{N-1}(\widehat{u}(\Omega_r)) = \int_{\Omega_r} \sqrt{1+|\nabla u|^2} d\mathcal{H}^{N-1} \tag{2.30}$$

The same goes for $\mathcal{H}^{N-1}(\widehat{v_r}(\Omega_r))$, thus getting (2.29).

Estimate from below of (2.29): Let

$$\omega_r := \{x \in \mathrm{Int}(\Omega_r),\ u(x) \le \sigma_r(x)\} \subset \{u \le \sigma_r\}$$

and $\widetilde{\omega_r}$ be the projection of ω_r onto $\Gamma_r := \{\langle q_r, \cdot\rangle = 0\}$.

Thanks to (2.17) we have

$$\omega_r \subset \{|\langle q_r, \cdot\rangle| \le r\} \tag{2.31}$$

On the other hand, if $0 < r < \widetilde{\beta}$ we have $B_r(0) \subset \Omega_r$ using (2.27), so that (2.17) again gives

$$B^+_{r/2}(0) \subset \omega_r \tag{2.32}$$

for such r.

The local strong convexity of the function $\xi \in \mathbb{R}^N \mapsto \sqrt{1+\xi^2}$ combined with the fact that for any $r \in (0, \beta/2)$

$$|\nabla \sigma_r| = \left| p + \frac{M_r}{2r} q_r \right| \le 2\|\nabla u\|_{L^\infty(B_\beta)}$$

(where we used (2.21)) enable to find $\alpha = \alpha(\|\nabla u\|_{L^\infty(B_\beta)}) > 0$ such that for any $r \in (0, \beta/2)$,

$$\sqrt{1+|\nabla u|^2} - \sqrt{1+|\nabla v_r|^2} \ge \frac{\nabla v_r \cdot (\nabla u - \nabla v_r)}{\sqrt{1+|\nabla v_r|^2}} + \frac{\alpha}{2}|\nabla u - \nabla v_r|^2 \text{ over } B_\beta$$

Note also the weaker (but global) estimate

$$\sqrt{1+|\nabla u|^2} - \sqrt{1+|\nabla v_r|^2} \ge \frac{\nabla v_r \cdot (\nabla u - \nabla v_r)}{\sqrt{1+|\nabla v_r|^2}} \quad \text{over } \Omega_r$$

Since $\mathcal{H}^{N-1}(\widehat{u}(\Omega_r)) \le \mathcal{H}^{N-1}(\partial K) < +\infty$, (2.30) implies in particular that $\sqrt{1+|\nabla u|^2} \in L^1(\Omega_r)$, hence also that $|\nabla u| \in L^1(\Omega_r)$. This ensures $u \in W^{1,1}(\Omega_r)$ as we also have $u \in L^\infty(\Omega_r)$. Let us integrate the two previous estimates and use (2.29) to get

$$P(K) - P(K_r) \ge \int_{\omega_r} \frac{\nabla \sigma_r \cdot (\nabla u - \nabla \sigma_r)}{\sqrt{1+|\nabla \sigma_r|^2}} d\mathcal{H}^{N-1} + \frac{\alpha}{2} \int_{\omega_r \cap B_\beta} |\nabla u - \nabla \sigma_r|^2 d\mathcal{H}^{N-1} \tag{2.33}$$

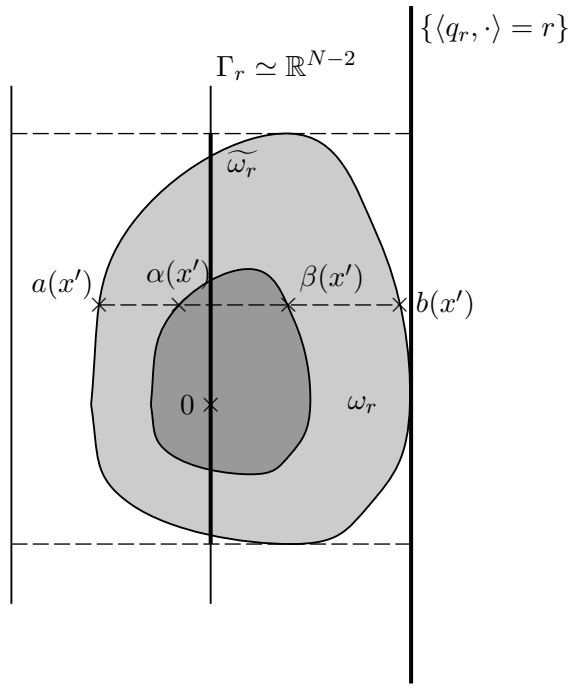
Recalling (2.19) we have,

$$0 \le \sigma_r - u \le M_r \text{ over } \omega_r. \tag{2.34}$$

As $\nabla \sigma_r$ is a fixed vector we get by integrating by parts

$$-\int_{\omega_r} \frac{\nabla \sigma_r \cdot (\nabla u - \nabla \sigma_r)}{\sqrt{1+|\nabla \sigma_r|^2}} d\mathcal{H}^{N-1} = -\int_{\partial\omega_r} \frac{\nabla \sigma_r \cdot n}{\sqrt{1+|\nabla \sigma_r|^2}} (u - \sigma_r) d\mathcal{H}^{N-2} \le \int_{\partial\omega_r} (\sigma_r - u) d\mathcal{H}^{N-2}$$

Figure 2.2: Localization of ω_r



where n denotes the outer unit normal to $\partial\omega_r$. Observing that (2.34) must also hold on $\partial\omega_r$, and since $u = \sigma_r$ on $\partial\omega_r \cap \mathrm{Int}(\Omega_r)$ we deduce

$$-\int_{\omega_r} \frac{\nabla\sigma_r \cdot (\nabla u - \nabla\sigma_r)}{\sqrt{1+|\nabla\sigma_r|^2}} d\mathcal{H}^{N-1} \leq M_r \mathcal{H}^{N-2}\big[\partial\omega_r \cap \partial\Omega_r\big]$$

Moreover, let us show that,

$$\mathcal{H}^{N-2}\big[\partial\omega_r \cap \partial\Omega_r\big] \leq \frac{(N-1)\mathcal{H}^{N-1}(\omega_r)}{d(y, \partial\Omega_r)} \tag{2.35}$$

so that for $r \in (0, \widetilde{\beta})$ we get

$$-\int_{\omega_r} \frac{\nabla\sigma_r \cdot (\nabla u - \nabla\sigma_r)}{\sqrt{1+|\nabla\sigma_r|^2}} d\mathcal{H}^{N-1} \leq M_r \frac{(N-1)\mathcal{H}^{N-1}(\omega_r)}{d(y, \partial\Omega_r)} \leq M_r \frac{(N-1)}{\widetilde{\beta}} \mathcal{H}^{N-1}(\omega_r) \leq C M_r r \mathcal{H}^{N-2}(\widetilde{\omega_r}) \tag{2.36}$$

with $C = \frac{2(N-1)}{\widetilde{\beta}} > 0$, where we used (2.27) and (2.31).

Estimate (2.35) was proven in [CCL13]; we reproduce the argument for the convenience of the reader: from Stokes formula,

$$\begin{aligned}\mathcal{H}^{N-1}(\omega_r) = \frac{1}{N-1}\int_{\omega_r} \mathrm{div}(x) d\mathcal{H}^{N-1}(x) &= \frac{1}{N-1}\int_{\partial\omega_r} \underbrace{\langle x, n(x)\rangle}_{\geq 0} d\mathcal{H}^{N-2}(x) \\ &\geq \frac{1}{N-1}\int_{\partial\omega_r \cap \partial\Omega_r} \langle x, n(x)\rangle d\mathcal{H}^{N-2}(x)\end{aligned}$$

and notice that for any $x \in \partial\omega_r \cap \partial\Omega_r$ such that $n(x)$ is well-defined we have $\langle x, n(x)\rangle = d(0, H_x)$ where H_x is the tangent hyperplane at x to $\partial\omega_r$, which is also the tangent hyperplane at x to $\partial\Omega_r$. This gives $\langle x, n(x)\rangle \geq d(0, \partial\Omega_r)$ using the convexity of Ω_r, implying (2.35).

L^2 Estimate from below of $\nabla(u - \sigma_r)$: We now estimate from below the term

$$\int_{\omega_r \cap B_\beta} |\nabla u - \nabla\sigma_r|^2 d\mathcal{H}^{N-1} \geq \int_{\omega_r \cap B_{\widetilde{\beta}}(0)} |\nabla u - \nabla\sigma_r|^2 d\mathcal{H}^{N-1}$$

We shall prove

$$\forall r \in (0, \widetilde{\beta}/2), \ \int_{\omega_r \cap B_{\widetilde{\beta}}(0)} |\nabla u - \nabla \sigma_r|^2 \geq \delta \frac{M_r^2}{r} \mathcal{H}^{N-2}(\widetilde{\omega_r}) \tag{2.37}$$

for some $\delta = \delta(\widetilde{\beta}, \mathrm{diam}(\Omega))$. This estimate was proved in [CCL13] but we reproduce the proof for the convenience of the reader.

Let $\gamma = \frac{\widetilde{\beta}}{2\mathrm{diam}(\Omega)} > 0$ be such that if $r \in (0, \widetilde{\beta}/2)$ we have thanks to (2.27)

$$[-r, r]q_r \times \gamma\widetilde{\omega_r} \Subset B_{\widetilde{\beta}}(0) \subset \mathrm{Int}(\Omega_r). \tag{2.38}$$

where $[-r, r]q_r \times \gamma\widetilde{\omega_r} := \{tq_r + \gamma x', \ t \in [-r, r], \ x' \in \widetilde{\omega_r}\}$. Setting

$$A_{\gamma/2} := \frac{\gamma}{2}\widetilde{\omega_r},$$

then for any $x' \in A_{\gamma/2}$ we can write

$$\begin{cases} (x' + \mathbb{R}q_r) \cap \omega_r = [(a(x'), x'), (b(x'), x')] \\ (x' + \mathbb{R}q_r) \cap \frac{1}{2}\omega_r = [(\alpha(x'), x'), (\beta(x'), x')] \end{cases}$$

for some functions a, b and α, β defined over $A_{\gamma/2}$, with $(a(x'), x')$ and $(b(x'), x') \in \partial\omega_r \cap B_{\widetilde{\beta}}(0)$ thanks to (2.38) and the right inclusion of (2.17) (see Figure 2.2). In particular it holds $(u - \sigma_r)(a(\cdot), \cdot) = (u - \sigma_r)(b(\cdot), \cdot) = 0$.

Since $u \leq \sigma_r$ over ω_r and since $u - \sigma_r$ is convex we get for every $h \in \omega_r$

$$(u - \sigma_r)(\frac{h}{2}) \leq \frac{1}{2}(u - \sigma_r)(0) + \frac{1}{2}(u - \sigma_r)(h) \leq \frac{1}{2}(u - \sigma_r)(0) = -\frac{M_r}{4},$$

that is

$$u - \sigma_r \leq -M_r/4 \text{ over } \frac{1}{2}\omega_r \tag{2.39}$$

Set $0 < r < \widetilde{\beta}/2$ and $x' \in A_{\gamma/2}$. We apply the inequality

$$\text{For } t_0 \leq t_1 \text{ and } f \in H^1([t_0, t_1]), \ \int_{t_0}^{t_1} f'^2(t)dt \geq \frac{(f(t_1) - f(t_0))^2}{t_1 - t_0},$$

to $(u - \sigma_r)(\cdot, x')$ over the segments $[a(x'), \alpha(x')]$ and $[\beta(x'), b(x')]$, each of which has length smaller than $2r$ (see (2.17)), obtaining

$$\begin{aligned} \int_{a(x')}^{b(x')} (\partial_{q_r}(u - \sigma_r))^2 dt &\geq \int_{a(x')}^{\alpha(x')} (\partial_{q_r}(u - \sigma_r))^2 dt + \int_{\beta(x')}^{b(x')} (\partial_{q_r}(u - \sigma_r))^2 dt \\ &\geq \frac{[u(\alpha(x'), x') - \sigma_r(\alpha(x'), x')]^2}{\alpha(x') - a(x')} + \frac{[u(\beta(x'), x') - \sigma_r(\beta(x'), x')]^2}{b(x') - \beta(x')} \geq \frac{M_r^2}{16r} \end{aligned}$$

since $(u - \sigma_r)(a(x'), x') = (u - \sigma_r)(b(x'), x') = 0$, and where we also used (2.39) in the last inequality.

We write $|\nabla(u - \sigma_r)| \geq |\partial_{q_r}(u - \sigma_r)|$ and integrate the above estimate over $x' \in A_{\gamma/2}$; this yields

$$\begin{aligned} \int_{\omega_r \cap B_{\widetilde{\beta}}(0)} |\nabla(u - \sigma_r)|^2 d\mathcal{H}^{N-1} &\geq \int_{A_{\gamma/2}} \left(\int_{a(x')}^{b(x')} (\partial_{q_r}(u - \sigma_r))^2 dt \right) d\mathcal{H}^{N-2}(x') \\ &\geq \frac{M_r^2 \mathcal{H}^{N-2}(A_{\gamma/2})}{16r} \\ &\geq \delta \frac{M_r^2 \mathcal{H}^{N-2}(\widetilde{\omega_r})}{r} \end{aligned}$$

for some constant $\delta = \delta(\widetilde{\beta}, \mathrm{diam}(\Omega)) > 0$, getting (2.37).

Conclusion of the estimate from below: Plugging (2.37) and (2.36) into (2.33) gives

$$\forall r \in (0, \widetilde{\beta}/2),\ \frac{M_r^2}{r}\mathcal{H}^{N-2}(\widetilde{\omega_r}) \leq C\big(P(K) - P(K_r) + M_r r \mathcal{H}^{N-2}(\widetilde{\omega_r})\big) \tag{2.40}$$

where $C = C(N, d(x_0, \partial\Omega), c(\beta), \|\nabla u\|_{L^\infty(B_\beta)}, \mathrm{diam}(\Omega))$, which completes the proof of the estimate from below.

Conclusion

We claim that for all $r \in (0, \widetilde{\beta})$ it holds

$$\begin{aligned}\mathcal{H}^{N-1}(\{u \leq \sigma_r\}) &= \mathcal{H}^{N-1}(\omega_r) + \mathcal{H}^{N-1}(\{u \leq \sigma_r\} \setminus \omega_r) \\ &\leq (1 + C)\mathcal{H}^{N-1}(\omega_r)\end{aligned} \tag{2.41}$$

with $C = \gamma^{1-N}$, where $\gamma = \frac{\widetilde{\beta}}{2\mathrm{diam}(\Omega)}$ was introduced to obtain (2.38). Indeed, denoting by $\kappa_r := \{u \leq \sigma_r\} \setminus \omega_r$, it suffices to show

$$\forall r \in (0, \widetilde{\beta}),\ \gamma\kappa_r \subset \omega_r \tag{2.42}$$

Let $x \in \kappa_r$; we have $(u - \sigma_r)(\gamma x) \leq 0$ using the convexity of $u - \sigma_r$, and furthermore $\gamma x \in B_{\widetilde{\beta}}(0) \subset \mathrm{Int}(\Omega_r)$ (see (2.27)). This provides $\gamma x \in \omega_r$, allowing to conclude that (2.42) holds.

Injecting (2.41) into (2.20) and using (2.31) we get that there is a constant $C = C(\widetilde{\beta}, \mathrm{diam}(\Omega))$ such that

$$\forall r \in (0, \widetilde{\beta}),\ |K \setminus K_r| \leq C M_r r \mathcal{H}^{N-2}(\widetilde{\omega_r}) \tag{2.43}$$

Gathering (2.40) and (2.43) and recalling (2.16), this finally provides the existence of some positive $r_0 = r_0(\widetilde{\beta}, \mathrm{diam}(\Omega), \varepsilon_K, \|\nabla u\|_{L^\infty(B_\beta)})$ such that

$$\forall r \in (0, r_0),\ \frac{M_r^2}{r}\mathcal{H}^{N-2}(\widetilde{\omega_r}) \leq C M_r r \mathcal{H}^{N-2}(\widetilde{\omega_r}) \tag{2.44}$$

where $C = C(N, \Lambda_K, d(x_0, \partial\Omega), c(\beta), \|\nabla u\|_{L^\infty(B_\beta)}, \mathrm{diam}(\Omega))$. Thanks to (2.32) we can simplify by $\mathcal{H}^{N-2}(\widetilde{\omega_r})$ in (2.44), to get that $M_r \leq Cr^2$. This completes the proof. □

Remark 2.2.7.

- If $N = 2$, the proof can be simplified, as one can show that there exists some r_0 such that $\Omega_r = \Omega$ for $r < r_0$. Indeed, since H is one-dimensional, the right inclusion of (2.17) reads $\{x \in \Omega, u(x) \leq \sigma_r(x)\} \subset [y - r, y + r]$, which shows that $\Omega_r = \Omega$ for small r using (2.15).

  On the other hand, for $N \geq 3$ it is not hard to find convex bodies K such that $\Omega_r \subsetneq \Omega$ for each (small) r. For instance, let us consider $K := C \cap B$ the intersection of the unit ball $B \subset \mathbb{R}^N$ with the cylinder $C := [-1, 1] \times B_{\mathbb{R}^{N-1}}((0, \ldots, 0, 1), 1)$. Although it is possible to compute explicitly Ω let us just notice that $\Omega \supset (-1, 1) \times \{0\}^{N-2} =: S$. We consider the situation where $x_0 = y = 0$ with $p = 0 \in \partial u(x_0)$, so that $l \equiv 0$ in this case. Then we see that $q_r = (0, 1, 0, \ldots, 0)$ satisfies $M_r = \sup_{B_r(0)} u = u(rq_r)$. Note that $u \equiv 0$ along S, while on the other hand $\sigma_r = M_r/2 > 0$ over S. As a consequence, since $K \cap P = B \cap P$ where P denotes the (x_1, x_N) plan, we find $x = (x_1, 0, \ldots, 0) \in S$ close enough from $(-1, 0, \ldots, 0)$ such that $\widehat{v_r}(x) = \widehat{\sigma_r}(x) \notin K$, thus getting that $\Omega_r \subsetneq \Omega$ in this case.

- Using the same ideas as in [CCL13] where they study the regularity of u solution of (2.7) when $f \in L^p(\Omega)$ (for some $p > N$), one can prove the $C^{1,\alpha}$ regularity of a convex body K satisfying

$$\forall \widetilde{K} \in \mathcal{K}^N \text{ such that } \widetilde{K} \subset K \text{ and } |K \setminus \widetilde{K}| \leq \varepsilon,\quad P(K) \leq P(\widetilde{K}) + \Lambda|K \setminus \widetilde{K}|^\gamma \tag{2.45}$$

for some $\varepsilon > 0, \Lambda \geq 0$ and $\gamma \in (1 - 1/N, 1]$. In this case, instead of (2.44), we derive with the same arguments

$$\frac{M_r^2}{r}\mathcal{H}^{N-2}(\widetilde{\omega_r}) \leq C\big(|K\Delta K_r|^\gamma + M_r r\mathcal{H}^{N-2}(\widetilde{\omega_r})\big) \leq CM_r^\gamma r^\gamma \mathcal{H}^{N-2}(\widetilde{\omega_r})^\gamma$$

Using (2.32) we get

$$M_r^{2-\gamma} \leq Cr^{(N-2)(\gamma-1)+\gamma+1} \tag{2.46}$$

Direct computation gives that $(N-2)(\gamma-1)+\gamma+1 > 2-\gamma$ whenever $\gamma \in (1-1/N, 1]$, so that the classical result [DPF15, Lemma 3.1] gives that K is $C^{1,\alpha}$ with $\alpha = (N(\gamma-1)+1)/(2-\gamma)$.

2.2.2 Regularity with volume constraint

We now focus on problems having a volume constraint, as they often appear in applications. We thus consider the problem

$$\inf\big\{P(K) + R(K),\ K \in \mathcal{K}^N, K \subset D, |K| = V_0\big\} \tag{2.47}$$

for some convex body $D \in \mathcal{K}^N$ and $0 < V_0 < |D|$. Existence for this problem can be shown under the assumption (2.51) below made upon R (see Theorem 2.3.4 (i)). In this section we prove that under suitable assumptions on R, minimizers of this problem are $C^{1,1}$. We use a penalization method to prove that these solutions are quasi-minimizer of the perimeter under convexity constraint.

Preliminaries

Before introducing the hypothesis which we will make upon R, let us recall the notion of Hausdorff distance between sets. If A and B are non-empty compact subsets of $\mathbb{R}^N$, the Hausdorff distance $d_H(A, B)$ between A and B is defined as the quantity

$$d_H(A, B) := \max\left\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(A, x)\right\}$$

where $d(\cdot,\cdot)$ denotes the euclidean distance. The Hausdorff distance d_H is a distance over the class of non-empty compact sets of $\mathbb{R}^N$.

Let us recall two classical facts about d_H, whose proof is given in the Appendix:

Proposition 2.2.8. *Let $D \in \mathcal{K}^N$, (K_n) be a sequence of convex bodies verifying $K_n \subset D$ for any $n \geq 0$, and let $K \subset D$ be a non-empty compact convex set. Then*

1. *We have the equivalence:*
$$d_H(K_n, K) \to 0 \Longleftrightarrow |K_n \Delta K| \to 0 \tag{2.48}$$

2. *If $d_H(K_n, K) \to 0$, and $C \in \mathcal{K}^N$ is such that $C \subset Int(K)$ then*
$$C \subset K_n \quad \textit{for large } n. \tag{2.49}$$

We now introduce a new assumption on R which is slightly stronger than (2.13), as seen in Proposition 2.2.9 below: for any $D' \in \mathcal{K}^N$ with $D' \subset D$ we set

$$\mathcal{K}^N_{D',D} := \{K \in \mathcal{K}^N,\ D' \subset K \subset D\} \tag{2.50}$$

and we assume

$$\forall D' \subset D \in \mathcal{K}^N,\ \exists C_{D',D} > 0,\ \forall\big(K_1, K_2 \in \mathcal{K}^N_{D',D},\ K_1 \subset K_2\big),\ \ |R(K_2) - R(K_1)| \leq C_{D',D}|K_2 \setminus K_1| \tag{2.51}$$

Proposition 2.2.9. *If R satifies (2.51) then R satisfies (2.13).*

Proof. Letting $K \in \mathcal{K}^N$ be fixed, there exists $D' \in \mathcal{K}^N$ such that $D' \subset \mathrm{Int}(K)$. Thanks to (2.49) we know that there exists $\delta > 0$ such that if $\widetilde{K} \in \mathcal{K}^N$ verifies $d_H(K, \widetilde{K}) \leq \delta$ then $\widetilde{K} \supset D'$. Thanks to (2.48) we can find $\varepsilon_K > 0$ such that if $|K \setminus \widetilde{K}| \leq \varepsilon_K$, $d_H(K, \widetilde{K}) \leq \delta$. Putting these together and applying hypothesis (2.51) with the class $\mathcal{K}^N_{D',K}$ gives that R satisfies (2.13). □

Note that on the other hand condition (2.51) is genuinely stronger than (2.13). In fact, (2.51) is double-sided while it is not the case for (2.13), but there is a deeper difference which boils down to the fact that in (2.13) the constants (ε_K, C_K) depend on K, while in (2.51) the constant $C_{D',D}$ is locally uniform. In this sense, (2.13) somehow says that R is "differentiable" everywhere while (2.51) means that R is locally Lipschitz; one can build an example of R verifying a double-sided (2.13) and not (2.51) by setting $R(K) := f(|K|)$ for some $f : \mathbb{R}^+ \to \mathbb{R}$ differentiable everywhere but not locally Lipschitz.

Main result

The main result of this section is the following.

Theorem 2.2.10. *Let K^* be a solution of problem (2.47), with R satisfying (2.51) and $0 < V_0 < |D|$. Then K^* is a quasi-minimizer of the perimeter in the sense of Definition 2.2.1, and is therefore $C^{1,1}$.*

The proof of Theorem 2.2.10 relies on the following important lemma, which allows the use of the results of section 2.2.1 over an auxiliary problem for which K^* is still optimal. For any $K \in \mathcal{K}^N$ and $\varepsilon > 0$ we set the class $\mathcal{O}_\varepsilon(K)$ of convex bodies which are $\varepsilon-$close perturbations of K from the inside:

$$\mathcal{O}_\varepsilon(K) := \{\widetilde{K} \in \mathcal{K}^N,\ \widetilde{K} \subset K,\ |K \setminus \widetilde{K}| \leq \varepsilon\}.$$

Lemma 2.2.11. *Let K^* be a solution of problem (2.47), with R satisfying (2.51) and $0 < V_0 < |D|$. Then there exists $\Lambda > 0$ and $\varepsilon > 0$ such that K^* is a solution of*

$$\min\left\{P(K) + R(K) + \Lambda\big||K| - V_0\big|,\ K \in \mathcal{O}_\varepsilon(K^*)\right\} \tag{2.52}$$

We will use in the proof of this lemma the following classical result concerning Minkowski sums and mixed volume (see for instance [Sch14, Theorem 5.1.7]).

Theorem 2.2.12 (Mixed volume)**.** *For any $m \in \mathbb{N}^*$ and $K_1, ..., K_m \in \mathcal{K}^N$, the map $(t_1, ..., t_m) \in (\mathbb{R}^+)^m \mapsto |t_1K_1 + ... + t_mK_m|$ is a homogeneous polynomial of degree N, i.e. there exists a symmetric function $V : (\mathcal{K}^N)^N \to \mathbb{R}$ (called mixed volume) such that for any $t_1, t_2, ..., t_m \geq 0$*

$$|t_1K_1 + ... + t_mK_m| = \sum_{i_1,...,i_N=1}^{m} t_{i_1}...t_{i_N} V(K_{i_1}, ...K_{i_N})$$

Furthermore V is nondecreasing in each coordinate for the inclusion of sets, and continuous for the Hausdorff distance.

Proof of Lemma 2.2.11: Set $G(K) := P(K) + R(K)$ for any $K \in \mathcal{K}^N$. We use a classical strategy (see for example [DPLPV18, Lemma 4.5], though our construction will be adapted to the convexity constraint): for any $K \subset K^*$ with $|K^* \setminus K| \leq \varepsilon$ (for sufficiently small ε) we build a convex body $\widetilde{K} \subset D$ such that $|\widetilde{K}| = V_0$ and $G(\widetilde{K}) \leq G(K) + \Lambda\big||K| - V_0\big|$ (for sufficiently large Λ). Writing then

$$G(K^*) \leq G(\widetilde{K}) \leq G(K) + \Lambda\big||K| - V_0\big|$$

yields the conclusion.

For $K \subset D$ a convex body and $t \in [0,1]$ we set the Minkowski sum

$$K_t := (1-t)K + tD$$

and note that K_t is a convex body and $K_t \subset D$. We first claim that there exist $\varepsilon_0 > 0, c > 0, t_0 > 0$ such that

$$\forall K \in \mathcal{O}_{\varepsilon_0}(K^*),\ \forall t \in [0,t_0],\ |K_t| - |K| \geq ct \tag{2.53}$$

Let $f_K(t) := |K_t|$. By Theorem 2.2.12, f_K is polynomial with degree N and more precisely

$$f_K(t) = \sum_{k=0}^{N} \binom{N}{k} (1-t)^k t^{N-k} V(K[k], D[N-k])$$

where $K[p]$ stands for $(K, K, \cdots, K)$ with p repetitions.

Now, as the class $\{L \subset \mathbb{R}^N \text{ compact convex},\ L \subset D\}$ is compact for d_H and since V is continuous for d_H, we deduce that the coefficients of f_K are uniformly bounded for $K \in \mathcal{K}^N, K \subset D$. To conclude that the claim holds it therefore suffices to show that $f'_K(0)$ is bounded from below by a positive constant uniform in $K \in \mathcal{O}_\varepsilon(K^*)$ for some small ε.

One has

$$\begin{aligned} f'_K(0) &= N\big(V(K[N-1], D) - V(K[N])\big) \\ &= N\big(V(K, ..., K, D) - |K|\big) \end{aligned}$$

which is nonnegative by monotonicity of the mixed volume. Moreover, as soon as we have $K \subsetneq D$, we can apply [Sch14, Theorem 7.6.17] to get $V(K[N-1], D) - |K| > 0$; equality would in fact imply that D is a 0-tangential body of K, hence that $K = D$. This gives in particular $f'_{K^*}(0) > 0$ (since $V_0 < |D|$). Since $K \mapsto f'_K(0)$ is continuous for d_H, we therefore have $f'_K(0) \geq f'_{K^*}(0)/2$ for any convex body $K \subset D$ with $d_H(K, K^*)$ small enough. Thanks to (2.48), we deduce the existence of $\varepsilon_0 > 0$ such that $f'_K(0) \geq f'_{K^*}(0)/2 > 0$ for any $K \in \mathcal{O}_{\varepsilon_0}(K^*)$. This yields (2.53) for $c := f'_{K^*}(0)/4$ and for some small $t_0 = t_0(D, K^*)$.

We now show that a reverse inequality holds for the perimeter: there exists $C = C(D) > 0$ such that

$$\forall t \in [0,1],\ \forall K \in \mathcal{K}^N \text{ such that } K \subset D,\ P(K_t) - P(K) \leq Ct \tag{2.54}$$

Since for any $L \in \mathcal{K}^N$,

$$P(L) = NV(L[N-1], B) \tag{2.55}$$

where B is the ball of unit radius (see for instance [Sch14, p.294, (5.43) to (5.45)]) the mapping $t \mapsto P(K_t) = NV(K_t[N-1], B)$ is a polynomial function whose coefficients are continuous quantities of $(V(K[i], D[N-1-i], B))_{0 \leq i \leq N-1}$. Hence the continuity of V for d_H and the compactness of the class $\{L \subset \mathbb{R}^N \text{ compact convex},\ L \subset D\}$ for d_H give (2.54).

Putting together (2.54) with (2.53) and setting $C' := C/c$ provides

$$\forall K \in \mathcal{O}_{\varepsilon_0}(K^*),\ \forall t \in [0,t_0],\ P(K_t) - P(K) \leq C'\big||K_t| - |K|\big| \tag{2.56}$$

On the other hand, there exists $D' \in \mathcal{K}^N$ such that $D' \subset \mathrm{Int}(K)$. Then arguing as in the proof of Proposition 2.2.9 ensures that for ε_1 small enough, any $K \in \mathcal{O}_{\varepsilon_1}(K^*)$ verifies $K \supset D'$. Therefore by (2.51) there exists $C_{D',D} > 0$ such that

$$\forall t \in [0,1],\ \forall K \in \mathcal{O}_{\varepsilon_1}(K^*),\ R(K_t) - R(K) \leq C_{D',D}|K_t \setminus K| = C_{D',D}\big||K_t| - |K|\big|. \tag{2.57}$$

Let $\varepsilon = \min\{\varepsilon_0, \varepsilon_1, ct_0\}$ and $K \in \mathcal{O}_\varepsilon(K^*)$.

We deduce from (2.53) that

$$|K_{t_0}| - |K^*| = |K_{t_0}| - |K| + |K| - |K^*| \geq ct_0 - \varepsilon \geq 0$$

By continuity of $t \mapsto |K_t|$, this ensures that

there exists $t \in [0, t_0]$ such that $|K_t| = |K^*|$. With (2.56) and (2.57) we get that the set $\widetilde{K} := K_t$ satisfies all the requirements laid out at the beginning of the proof with $\Lambda := C' + C_{D',D}$. □

Proof of Theorem 2.2.10: Thanks to Lemma 2.2.11, an optimal shape K^* for (2.47) is solution of (2.52) for some $\varepsilon > 0$ and $\Lambda > 0$. Therefore we have

$$\forall K \in \mathcal{O}_\varepsilon(K^*), \quad P(K^*) \leq P(K) + R(K) - R(K^*) + \Lambda |K^* \setminus K|,$$

By Proposition 2.2.9, R verifies hypothesis (2.13), and as a consequence there exists $C_{K^*} > 0$ such that

$$\forall K \in \mathcal{O}_\varepsilon(K^*), \quad P(K^*) \leq P(K) + (C_{K^*} + \Lambda)|K^* \setminus K|$$

Hence K^* is a quasi-minimizer of the perimeter under convexity constraint in the sense of Definition 2.2.1. We can therefore apply Theorem 2.2.3 to get that K^* is $C^{1,1}$. □

Remark 2.2.13. As in Remark 2.2.7, there is an analogous result to Theorem 2.2.10 if R is merely γ-Hölder for some $\gamma \in (1 - 1/N, 1]$, meaning that (2.51) is replaced with

$$\forall D' \subset D \in \mathcal{K}^N, \ \exists C_{D',D} > 0, \ \forall \big(K_1, K_2 \in \mathcal{K}^N_{D',D}, \ K_1 \subset K_2\big), \ \ |R(K_2) - R(K_1)| \leq C_{D',D} |K_2 \setminus K_1|^\gamma \quad (2.58)$$

In this case, keeping the same notations as in the proof of Lemma 2.2.11, the same arguments show

$$G(\widetilde{K}) \leq G(K) + \Lambda \big||K| - V_0\big|^\gamma,$$

and with the additionnal remark that

$$P(K_t) - P(K) \leq \Lambda \big||K_t| - |K|\big| \leq \Lambda' \big||K_t| - |K|\big|^\gamma$$

with $\Lambda' := \Lambda \times |D|^{1-\gamma}$, we conclude in this case that the optimal shape is $C^{1,\alpha}$ for the same α as in Remark 2.2.7.

2.3 Examples and applications

This section is dedicated to applications of the results of Section 2.2.2. We therefore provide examples of functionals R satisfying hypothesis (2.51) (and therefore (2.13) as well), Theorem 2.2.10 then implying that the minimizers of the corresponding problem are $C^{1,1}$-regular.

2.3.1 First examples

Let us start by giving two examples taken from the literature of minimization of $P + R$ for which proving that the functional R satisfies hypotheses (2.13) and (2.51) is quite easy.

A first example of relevant R is given through the following model of a liquid drop subject to the action of a potential: it consists in minimizing the energy

$$P(E) + \int_E g$$

among bounded subsets E of $\mathbb{R}^N$ with given volume, where $g : \mathbb{R}^N \to \mathbb{R}$ is a fixed function in $L^1_{\text{loc}}(\mathbb{R}^N)$, see for example [FM11]. An optimal shape may not be convex, and in this case it is interesting to study the counterpart of this problem with an additional convex constraint: under reasonable hypotheses on g

we can prove existence and regularity for a minimizer of this problem in the class of convex shapes (see Proposition 2.3.1 below).

We can also consider the following generalization of the Gamow model (2.5), which consists in the minimization

$$\inf\left\{P(E)+\int_E\int_E\frac{dxdy}{|x-y|^{N-\alpha}},\ E\subset\mathbb{R}^N,\ |E|=V_0\right\}$$

for a given mass $V_0\in(0,+\infty)$ and parameter $\alpha\in(0,N)$. The interaction term $V_\alpha(E):=\int_E\int_E|x-y|^{\alpha-N}dxdy$, called *Riesz potential*, is maximized by any ball of volume m by virtue of the Riesz inequality, so that there is a competition in the above minimization. As in the case of (2.5) it is known that for small masses m the ball of corresponding volume is the unique (up to translation) solution to the minimization problem (see the Introduction of [FFM+15] for a review of these results), while it is proven in [KM13, Theorem 2.5] and [KM14, Theorem 3.3] for $\alpha\in(N-2,N)$ that beyond a certain threshold of mass m there is no existence for this problem (and for $\alpha=N-2$ in [Fra21, Theorem 3]). On the contrary, the convexity constraint will enforce existence for all masses ; furthermore, applying Theorem 2.2.10 we are able to show regularity for the problem when considered under a convexity constraint, see Proposition 2.3.1 below.

We thus have the following proposition.

Proposition 2.3.1. *Let $N\geq 2$ and $V_0\in(0,+\infty)$.*
Let $g\in L^\infty_{loc}(\mathbb{R}^N)$ be coercive, that is to say $\lim_{|x|\to\infty}g(x)=+\infty$. Then there exists a solution to the problem

$$\inf\left\{P(K)+\int_K g,\ K\in\mathcal{K}^N,\ |K|=V_0\right\}$$

and any such solution $K^*$is $C^{1,1}$.
Let $\alpha\in(0,N)$. Then there exists a solution to the problem

$$\inf\left\{P(K)+V_\alpha(K),\ K\in\mathcal{K}^N,\ |K|=V_0\right\}$$

and any such solution K^* is $C^{1,1}$.

Proof. **Existence:**

1. Let (K_n) be a minimizing sequence for the first problem. As g is coercive, there exists a bounded set A such that $g\geq 0$ outside A. Therefore, as there exists $C>0$ such that $P(K_n)+\int_{K_n}g\leq C$ by definition of (K_n), we can write

$$P(K_n)\leq C-\int_{K_n}g\leq C-\int_{K_n\cap A}g\leq C+\|g\|_{L^1(A)}$$

so the perimeters $P(K_n)$ are uniformly bounded. We thus use the inequality

$$\mathrm{diam}(K)\leq C(N)\frac{P(K)^{N-1}}{|K|^{N-2}} \tag{2.59}$$

valid for any convex body K (see [EFT05, Lemma 4.1]) to get that the $\mathrm{diam}(K_n)$ are also uniformly bounded, recalling also that $|K_n|=V_0$. As a consequence, using the coercivity of g we now show that there is no loss of generality in assuming that the K_n are uniformly bounded: let $r>0$ be such that B_r the ball centered at 0 of radius r verifies $\mathrm{diam}(K_n)\leq\mathrm{diam}(B_r)$ for every n, and set $m:=\operatorname{ess\,sup}_{B_r}g$. Thanks to the fact that g is coercive we can find $r'>r$ such that $g\geq m$ outside $B_{r'}$. For any fixed $n\in\mathbb{N}$, either $K_n\subset B_{2r'}$ and we set $\widetilde{K_n}:=K_n$, or else there exists $x\in K_n\cap(B_{2r'})^c$ so that $K_n\subset(B_{r'})^c$ thanks to the bound on the diameters. In this latter case, we thus have $g\geq m$ over K_n while $K_n-x\subset B_{r'}$ gives that $g\leq m$ over K_n-x , so that

$$P(K_n-x)+\int_{K_n-x}g\leq P(K_n)+m|K_n|\leq P(K_n)+\int_{K_n}g$$

using also the translation invariance of the perimeter. We then rather set $\widetilde{K_n} := K_n - x$. This argument ensures that the sequence $(\widetilde{K_n})$ is still minimizing with the additionnal property that $\widetilde{K_n} \subset B_{2r'}$ for each n. We will keep denoting it K_n.

Now, $K \mapsto \int_K g$ satisfies (2.51): if $D \in \mathcal{K}^N$ and $K' \subset K \subset D$ are convex bodies, then

$$\left| \int_K g - \int_{K'} g \right| \leq \|g\|_{L^\infty(D)} |K \setminus K'|. \tag{2.60}$$

Using the Blaschke selection theorem and (2.48) we can extract a subsequence (still denoted (K_n)) and a compact convex K^* such that $K_n \to K^*$ for the Hausdorff distance and in volume. In particular $|K^*| = V_0$. Thanks to (2.60) we have that $\int_{K_n} g \to \int_{K^*} g$ and $P(K_n) \to P(K^*)$ by continuity of the perimeter for convex domains (see for instance [BB05, Proposition 2.4.3, (ii)]). We thus get existence.

2. Let (K_n) be a minimizing sequence for the second problem. Since V_α is nonnegative we immediately get that $(P(K_n))$ is bounded, getting thus from (2.59) that the sequence $\mathrm{diam}(K_n)$ is bounded as well. By translation invariance of the perimeter and of V_α there is not loss of generality in assuming that there exists a compact set D such that $K_n \subset D$ for each n.

   Let us now show that V_α verifies (2.51). This was done in [KM14, Equation (2.11)], but we reproduce hereafter the short argument for the convenience of the reader. Let $D \in \mathcal{K}^N$ and $K' \subset K \subset D$ be convex bodies. We set $v_E(x) := \int_E |x-y|^{\alpha-N} dy$ for any compact set E and write $f(x,y) := |x-y|^{\alpha-N}$. We have

$$\begin{aligned} 0 \leq V_\alpha(K) - V_\alpha(K') &= \int_K \int_K f - \int_K \int_{K'} f + \int_{K \setminus K'} \int_{K'} f \\ &= \int_{K \setminus K'} v_K + v_{K'} \\ &\leq 2|K \setminus K'| \left(\int_{B_D} \frac{dy}{|y|^{N-\alpha}} \right) \end{aligned} \tag{2.61}$$

   with B_D a ball of volume $|D|$, where we used that

$$v_{K'}(x) \leq v_K(x) = \int_K \frac{dy}{|x-y|^{N-\alpha}} = \int_{x-K} \frac{dy}{|y|^{N-\alpha}} \leq \int_{B_D} \frac{dy}{|y|^{N-\alpha}}$$

   thanks to the Hardy-Littlewood inequality and since $|K| \leq |D|$.

   Since V_α verifies (2.51) and $K_n \subset D$ for all $n \in \mathbb{N}$ we conclude to existence as before using the Blaschke selection theorem.

Regularity: We proved respectively in (2.60) and (2.61) that $K \mapsto \int_K g$ and V_α satisfy (2.51). We can therefore apply Theorem 2.2.10 to get that any minimizer is $C^{1,1}$. □

2.3.2 PDE and Spectral examples

We now focus on more difficult examples, which will lead to the proof of Theorem 2.1.1 given in the introduction. Let us first set some notations and definitions.

If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz open set we denote respectively by

$$\begin{aligned} 0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_n(\Omega) \leq \cdots \nearrow +\infty \\ 0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \leq \mu_n(\Omega) \leq \cdots \nearrow +\infty \end{aligned}$$

the nondecreasing sequence of the Dirichlet and Neumann Laplacian eigenvalues associated to Ω (see for example [Hen20] for more details). We also define $\tau(\Omega)$ the torsional rigidity of Ω as

$$\tau(\Omega) = \int_\Omega u_\Omega dx = -2\min\left\{\int_\Omega \frac{|\nabla u|^2}{2} - \int_\Omega fu,\ u \in H_0^1(\Omega)\right\}$$

where u_Ω is the unique solution of

$$\begin{cases} -\Delta u = 1 \text{ in } \Omega \\ u \in H_0^1(\Omega) \end{cases} \tag{2.62}$$

For any convex body $K \in \mathcal{K}^N$ we will frequently use the notation $\Omega_K := \mathrm{Int}(K)$, and then define $\lambda_n(K) := \lambda_n(\Omega_K)$, $\mu_n(K) := \mu_n(\Omega_K)$ for $n \in \mathbb{N}^*$, and $\tau(K) := \tau(\Omega_K)$.

We are now ready to state the main result of this section, which will be proved later on in Sections 2.3.2 and 2.3.2.

Theorem 2.3.2. *Let $n \in \mathbb{N}^*$, $N \geq 2$. Then any $R \in \{\lambda_n, \mu_n, \tau\}$ satisfy (2.13) and (2.51), namely for every $D' \subset D \subset \mathbb{R}^N$ convex bodies there exists $C = C(D', D, R)$ such that for any $K' \subset K$ lying in $\mathcal{K}^N_{D',D}$ (defined in (2.50))*

$$\big|R(K) - R(K')\big| \leq C|K \setminus K'|.$$

Remark 2.3.3. The fact that $K' \subset K$ is not essential to ensure that Lipschitz estimates hold. In fact, one has that

$$\exists C > 0, \forall (K, K') \in \mathcal{K}^N_{D',D},\ \ |R(K) - R(K')| \leq C|K\Delta K'|$$

by applying Theorem 2.3.2 with K and $K \cup K'$ on the one hand, K' and $K \cup K'$ on the other hand.

As a consequence, combined with Corollary 2.2.4 and Theorem 2.2.10, we are able to prove Theorem 2.1.1.

Proof of Theorem 2.1.1: Recall that $R(K) := F(|K|, \tau(K), \lambda_1(K), \dots, \lambda_n(K), \mu_1(K), \dots, \mu_n(K))$ for some $F : (0, +\infty) \times (0, +\infty) \times (0, +\infty)^n \times \mathbb{R}^n_+ \to \mathbb{R}$ locally Lipschitz. Let us show that R satifies (2.51), so that it also satisifes (2.13) (thanks to Proposition 2.2.9) and Corollary 2.2.4 and Theorem 2.2.10 give the results.

Let $D_1 \subset D_2 \in \mathbb{R}^N$ be convex bodies and let $K, K' \in \mathcal{K}^N_{D_1,D_2}$ with $K' \subset K$. Set $L = K$ or $L = K'$. Then from monotonicity of Dirichlet eigenvalues and torsion, for any $k \in \mathbb{N}^*$ it holds

$$\lambda_k(D_2) \leq \lambda_k(L) \leq \lambda_k(D_1)$$
$$\tau(D_1) \leq \tau(L) \leq \tau(D_2)$$

Moreover, since $\mu_k(L) \leq \lambda_k(L) \leq \lambda_k(D_1)$ we have for any $k \in \mathbb{N}^*$

$$\mu_k(L) \leq \lambda_k(D_1)$$

Also,

$$|D_1| \leq |L| \leq |D_2|$$

Putting these four estimates together and using that F is locally Lipschitz we find $C(F, D_1, D_2)$ such that

$$\begin{aligned} |R(K) - R(K')| \ \leq\ & C(F, D', D)\big(\sum_{k=1}^n |\lambda_k(K) - \lambda_k(K')| \\ & + \sum_{k=1}^n |\mu_k(K) - \mu_k(K')| + |\tau(K) - \tau(K')| + \big||K| - |K'|\big|\big) \end{aligned}$$

Applying Theorem 2.3.2 for λ_k, μ_k and τ and noticing that $||K| - |K'|| = |K \setminus K'|$ ensures that R satisfies (2.51). The result follows. □

A general existence result

In this short section we show a general existence result for the minimization among convex sets of a functional of the type $P + R$, where R is mostly thought of as a PDE-type functional. Using mild continuity of R, we show existence of a minimizer under additional box and volume constraints, and we also show existence in the unconstrained case with coercivity assumptions of R. The statement is as follows.

Theorem 2.3.4. *(i) Let $R : \mathcal{K}^N \to \mathbb{R}$ be lower-semi-continuous for the Hausdorff convergence of convex bodies. Let $D \in \mathcal{K}^N$ and $0 < V_0 < |D|$. Then there exists a minimizer to the problem*

$$\inf \left\{ P(K) + R(K),\ K \in \mathcal{K}^N,\ K \subset D,\ |K| = V_0 \right\}$$

(ii) Let $V_0 > 0$. Let $n \in \mathbb{N}^*$ and $F : (\mathbb{R}^+)^{2n+2} \to \mathbb{R}$ be coercive (meaning $\lim_{|x|\to\infty} F(x) = +\infty$) and lower-semi-continuous, and set

$$R(K) := F(|K|, \tau(K), \lambda_1(K), \dots, \lambda_n(K), \mu_1(K), \dots, \mu_n(K))$$

Then there exists minimizers to the problems

$$\inf \left\{ P(K) + R(K),\ K \in \mathcal{K}^N \right\}$$

$$\inf \left\{ P(K) + R(K),\ K \in \mathcal{K}^N,\ |K| = V_0 \right\}$$

Proof. In both cases existence is proved using the direct method: let (K_i) be a minimizing sequence.

(i) Since $K_i \subset D$ for each n, then thanks to the Blaschke selection theorem and (2.48) we can extract a subsequence (still denoted (K_i)) and a compact convex K^* such that $K_i \to K^*$ for the Hausdorff distance and in volume. We can pass to the limit in $|K_i| = V_0$ to get $|K^*| = V_0 > 0$, so that K^* has non-empty interior. We deduce that $\underline{\lim}\, R(K_i) \geq R(K^*)$ thanks to the hypothesis made on R and that $P(K_i) \to P(K^*)$ by continuity of the perimeter for convex domains (see for instance [BB05, Proposition 2.4.3, (ii)]), thus getting existence.

(ii) We start with existence for the first of the two problems. Thanks to John's ellipsoid Lemma, there exists $c_i \in \mathbb{R}^N$ and ellipsoids E_i such that

$$E_i \subset K_i \subset c_i + N(E_i - c_i)$$

We have by monotonicity of the perimeter for convex bodies $P(K_i) \geq P(E_i)$, while we also have $\mathrm{diam}(K_i) \leq N\mathrm{diam}(E_i)$. As a consequence, if we assume by contradiction that (up to subsequence) $\mathrm{diam}(K_i) \to +\infty$, then we first deduce $\mathrm{diam}(E_i) \to +\infty$ so that $P(E_i) \to +\infty$, whence $P(K_i) \to +\infty$. The function F being coercice and lower-semi-continuous it is therefore bounded from below, and we get the contradiction $P(K_i) + R(K_i) \to +\infty$. Therefore $\mathrm{diam}(K_i)$ is bounded and we can assume by translation invariance of P and R that there exists a compact set $D \subset \mathbb{R}^N$ such that $K_i \subset D$ for each i. Thanks to the Blaschke selection theorem and (2.48) we can extract a subsequence (still denoted (K_i)) and a compact convex K^* such that $K_i \to K^*$ for the Hausdorff distance and in volume. The case $|K^*| = 0$ is excluded, since it would lead to $|E_i| \to 0$ and then $+\infty \leftarrow \lambda_1(E_i) \leq N^2\lambda_1(K_i)$ by monotonicity of λ_1, which yields $R(K_i) \to +\infty$ by coercivity of F hence the contradiction $P(K_i) + R(K_i) \to +\infty$. As a consequence $|K^*| > 0$, which means that K^* has non-empty interior, and in particular there exists $D' \in \mathcal{K}^N$ such that $D' \subset \mathrm{Int}(K^*)$. Since $d_H(K_i, K^*) \to 0$ we know thanks to (2.49) that $K_i \in \mathcal{K}^N_{D',D}$ for large enough i. As a consequence, the K_i are uniformly Lipschitz in the sense that they verify the ε-cone condition for some ε independent of i (see Definition 2.4.1 and Remark 2.4.2). We thus have continuity $\lambda_k(K_i) \to \lambda_k(K^*)$ and $\tau(K_i) \to \tau(K^*)$ (see [Hen06, Theorem 2.3.18]) and $\mu_k(K_i) \to \mu_k(K^*)$ (see [Hen06, Theorem 2.3.25]). Recalling the lower-semi-continuity of F we deduce $\underline{\lim}\, R(K_i) \geq R(K^*)$, and by continuity of the perimeter for convex domains we also have $P(K_i) \to P(K^*)$. This finishes the proof of existence for the first problem.

Existence for the second problem is shown with the same argument, by noticing that the volume constraint passes to the limit.

□

Selected examples

Before moving on to the proof of Theorem 2.3.2 (which is the object of sections 2.3.2 and 2.3.2), we discuss here some specific examples where R involves spectral functionals and for which we can prove existence without a box constraint (in the spirit of Theorem 2.3.4 (ii)). We will make use of Theorem 2.3.2 in this section.

We start by considering spectral problems with a perimeter constraint, which have been studied in the literature without the additionnal convexity constraint (see for instance [BBH09], [DPV14], [Bog19], [BO16]). Namely, given $p_0 > 0$, we are interested in the minimization problems

$$\inf\{\lambda_n(K),\ K \in \mathcal{K}^N,\ P(K) = p_0\} \tag{2.63}$$

In [BBH09], the authors use convexity for proving existence as well as C^∞ regularity and some qualitative properties of minimizers of λ_2 under perimeter constraint in two dimensions. In fact, although their problem is set without a convexity constraint, they are able to show that solutions are in fact convex, thus yielding a bit of regularity to start with. On the other hand, in dimension $N = 3$ there are eigenvalues for which the expected solutions are not convex (see [BO16, Figure 2]), so that the convexity constraint would thus be meaningful in the minimization.

In our case we can prove existence together with $C^{1,1}$ regularity of minimizers. This is the object of next result.

Proposition 2.3.5. *Let $n \in \mathbb{N}^*$, $N \geq 2$ and $p_0 > 0$. Then there exists a solution to problem (2.63) and any such solution is $C^{1,1}$.*

Proof. The proof is divided into proof of existence and proof of regularity.
Existence: We use the direct method. Let (K_i) be a minimizing sequence for (2.63). By definition there exists $C > 0$ such that $\lambda_n(K_i) \leq C$ for each i. Since $\lambda_1(K_i) \leq \lambda_n(K_i) \leq C$, we deduce using Faber-Krahn inequality

$$C|K_i|^{2/N} \geq \lambda_1(K_i)|K_i|^{2/N} \geq \lambda_1(B)|B|^{2/N} =: C_N$$

with B the unit ball, giving

$$|K_i| \geq \left(\frac{C_N}{C}\right)^{N/2} \tag{2.64}$$

On the other hand the perimeters $P(K_i)$ are bounded from above (in fact $P(K_i) = p_0$), yielding that the K_i are uniformly bounded (up to translation), using (2.59). We therefore get existence by proceeding as in the proof of Theorem 2.3.4 (i): thanks to the Blaschke selection theorem and (2.48) we thus find a subsequence (still denoted (K_i)) converging to some compact convex set K^* in the Hausdorff sense and in volume. The lower bound on volumes (2.64) thus ensures that $|K^*| > 0$, so that the convex K^* has nonempty interior. Hence there exists $D' \in \mathcal{K}^N$ such that $D' \subset \mathrm{Int}(K^*)$. Since $d_H(K_i, K^*) \to 0$ we know thanks to (2.49) that $K_i \in \mathcal{K}^N_{D',D}$ for large enough i. We deduce that $\lambda_n(K_i) \to \lambda_n(K^*)$ using that λ_n satisfies (2.51) thanks to Theorem 2.3.2, and that $P(K_i) \to P(K^*)$ by continuity of the perimeter for convex domains. This finishes the proof of the existence part.
Regularity: Let K^* be any minimizer for (2.63). Following [DPV14, Remark 3.6] we can show that there exists $\mu > 0$ such that K^* minimizes

$$\inf\{\lambda_n(K) + \mu P(K),\ K \in \mathcal{K}^N\} \tag{2.65}$$

As a consequence we can apply Corollary 2.2.4 to get that K^* is $C^{1,1}$. □

We now move on to problems of the kind (2.47) with a volume constraint and with R of spectral type. These type of problems are related to the study of Blaschke-Santalo diagrams, see [FL21] and the numerical results in [Fto21]. Again, we can drop the box constraint and still get existence:

Proposition 2.3.6. *Let $N \geq 2$, $V_0 > 0$ and $n \in \mathbb{N}^*$. There exist minimizers to the problems*

$$\inf\{P(K) + \lambda_n(K),\ K \in \mathcal{K}^N, |K| = V_0\} \tag{2.66}$$

$$\inf\{P(K) \pm \mu_n(K),\ K \in \mathcal{K}^N,\ |K| = V_0\} \tag{2.67}$$

and any minimizer is $C^{1,1}$.

Proof. The regularity assertion is a consequence of Theorem 2.1.1. Let us prove existence of a solution for the two family of problems:

1. Existence is obtained by applying Theorem 2.3.4 (ii).

2. For the minimization of $P + \mu_n$ we can directly apply Theorem 2.3.4 (ii) to get existence. For the minimization of $P - \mu_n$, first note the inequalities

$$\operatorname{diam}(K) \leq C(N)\frac{P(K)^{N-1}}{|K|^{N-2}} \qquad \mu_n(K) \leq \frac{C_n(N)}{\operatorname{diam}(K)^2} \tag{2.68}$$

for any convex body K, for some constants $C(N)$ and $C_n(N)$ only depending on the indicated parameters (for the first, recall (2.59) and see for instance [Ros04, Proposition 2.1 (b)] for the second). Let (K_i) be some minimizing sequence for problem (2.67). The sequence $(P(K_i) - \mu_n(K_i))$ being bounded from above by definition, we find $C > 0$ such that

$$P(K_i) \leq C + \mu_n(K_i) \leq C'(N)(1 + \operatorname{diam}(K_i)^{-2})$$

for some dimensional constant $C'(N)$, using the second inequality of (2.68). Now, for fixed i we either have $\operatorname{diam}(K_i) \geq 1$, in which case we deduce $P(K_i) \leq 2C'(N)$, or $\operatorname{diam}(K_i) \leq 1$. Thanks to the first inequality of (2.68) this yields

$$\operatorname{diam}(K_i) \leq \max\left\{1, \frac{C(N)(2C'(N))^{N-1}}{V_0^{N-2}}\right\}$$

Therefore, using the translation invariance of P and μ_n we can find a compact set D such that $K_i \subset D$ for each i. Recalling that $R(K) := \mu_n(K)$ verifies (2.51) thanks to Theorem 2.3.2, the rest of the proof of existence is as in Theorem 2.3.4 (i).

□

Remark 2.3.7.

- One can also wonder about the minimization

$$\inf\{P(K) - \lambda_n(K),\ K \in \mathcal{K}^N,\ |K| = V_0\}$$

In this case the problem is ill-posed, as the box constraint is needed to ensure existence. In fact, one can see that the infimum is $-\infty$, choosing the sequence of long thin rectangle $R_\varepsilon := [0, V_0\varepsilon^{-1}] \times [0, \varepsilon] \times [0, 1]^{N-2}$ for which $P(R_\varepsilon) \leq C_N\varepsilon^{-1}$ for some dimensional constant $C_N > 0$ while $\lambda_n(R_\varepsilon) \sim V_0^{-2}\pi^2\varepsilon^{-2}$.

- Thanks to the isoperimetric inequality and the Faber-Krahn inequality (respectively the Szego-Weinberger inequality), it is known that the unique solution up to translation to the minimization of $P + \lambda_1$ (respectively of $P - \mu_2$) is any ball B of volume V_0. On the other hand, if $n \geq 2$ (respectively $n \geq 3$) the problem (2.66) (respectively (2.67)) has $C^{1,1}$ solutions which are not analytically known.

- Inspired by [FL21], one could wonder about the regularity properties of solutions to

$$\min\{P(K),\ K \in \mathcal{K}^N,\ |K| = V_0, \lambda_n(K) = \ell_0\}, \quad \max\{\lambda_n(K),\ K \in \mathcal{K}^N,\ |K| = V_0,\ P(K) = p_0\},$$

where $p_0 > 0, \ell_0 > 0$. In [FL21, Corollary 3.13] it is proven when $N = 2$ and $n = 1$ that these problems are equivalent (for suitable choices of p_0 and ℓ_0) and that solutions are $C^{1,1}$. Nevertheless, we were not able to apply our regularity result to these cases, so the regularity of solutions of these problems remains open in other cases ($N \geq 3$ or $n \geq 2$), up to our knowledge.

Torsional rigidity and Dirichlet eigenvalues

If $D' \subset D \subset \mathbb{R}^N$ are convex bodies, we still denote by $\mathcal{K}^N_{D',D}$ the set

$$\mathcal{K}^N_{D',D} := \{K \in \mathcal{K}^N,\ D' \subset K \subset D\}$$

Let us state now the main result of this section, which basically restates Theorem 2.3.2 for τ. Indeed as we will see below, the proof of Theorem 2.3.2 for λ_n will be a consequence of the same result for τ.

Proposition 2.3.8. *Let $N \geq 2$ and $D' \subset D \subset \mathbb{R}^N$ be convex bodies. Then there exists $C = C(D', D)$ such that for any $K' \subset K$ lying in $\mathcal{K}^N_{D',D}$*

$$0 \leq \tau(K) - \tau(K') \leq C|K \setminus K'| \tag{2.69}$$

The proof of Proposition 2.3.8 is based on two preliminary lemmas: for convex bodies $K' \subset K$,

1. we construct a "change of variable" operator $T_{K,K'} : W^{1,\infty}(\Omega_K) \to W^{1,\infty}(\Omega_{K'})$ whose norm is uniformly bounded in $\mathcal{K}^N_{D',D}$, and which is the identity on a large part of K', see Lemma 2.3.9 (recall that Ω_K denotes the interior of K).

2. we show uniform $W^{1,\infty}$-estimates of the torsion function of Ω_K, see Lemma 2.3.10.

Lemma 2.3.9 (Change of variables)**.** *There exists $C = C(D', D) > 0$ such that for any $K, K' \in \mathcal{K}^N_{D',D}$ with $K' \subset K$, there exists a bi-Lipschitz homeomorphism $\phi := \phi_{K',K} : \mathbb{R}^N \to \mathbb{R}^N$ such that the operator $T := T_{K,K'}$ defined by*

$$\begin{aligned} T_{K,K'} : L^1(\Omega_K) &\to L^1(\Omega_{K'}) \\ f &\mapsto f \circ \phi \end{aligned}$$

satisfies the requirements:

- *There exists $K'' \subset K'$ such that $|K' \setminus K''| \leq C|K \setminus K'|$ and*

$$Tf(x) = f(x) \text{ a.e. in } K'', \text{ for any } f \in L^1(\Omega_K)$$

- *For all f_1 and f_2 respectively in $H^1_0(\Omega_K)$ and $W^{1,\infty}(\Omega_K)$, Tf_1 and Tf_2 belongs to $H^1_0(\Omega_{K'})$ and $W^{1,\infty}(\Omega_{K'})$ respectively, with furthermore*

$$\begin{aligned} \|Tf_1\|_{H^1_0(\Omega_{K'})} &\leq C\|f_1\|_{H^1_0(\Omega_K)} \\ \|Tf_2\|_{W^{1,\infty}(\Omega_{K'})} &\leq C\|f_2\|_{W^{1,\infty}(\Omega_K)} \end{aligned}$$

Note that this result is similar to [BL08, Theorem 4.23] but for a different class of sets, namely $\mathcal{K}^N_{D',D}$: it is unclear whether [BL08, Theorem 4.23] implies Lemma 2.3.9, so we prefered to make our own proof of this result.

Let us recall that any $K \in \mathcal{K}^N$ has its boundary ∂K naturally parametrized as a graph over the sphere. More precisely, we can assume up to translating that 0 is contained in Ω_K, and then set $\rho(x) := \sup\{\lambda \geq 0, \lambda x \in K\}$ for any $x \in \partial B$, called the radial function of K. Then the set K is globally parametrized by ρ:

$$K = \{\lambda x \rho(x), x \in \partial B,\ \lambda \in [0,1]\} \tag{2.70}$$

It is classical that $\rho \in W^{1,\infty}(\partial B)$ and moreover one can estimate $\nabla_\tau \rho$ in terms of ρ

$$\|\nabla_\tau \rho\|_{L^\infty(\partial B)} \leq \frac{(\sup \rho)^2}{\inf \rho} \tag{2.71}$$

(see for instance the computations leading to (3.13) in [Fus15]).

Proof of Lemma 2.3.9: We will assume up to translating that $0 \in \mathrm{Int}(D)$. Let $K, K' \in \mathcal{K}^N_{D',D}$ with $K' \subset K$. The proof consists in building a bi-Lipschitz change of variables $\phi : K' \to K$ which is the identity on a large part of K', and such that

$$\|\phi\|_{W^{1,\infty}(\Omega_{K'})}, \|\phi^{-1}\|_{W^{1,\infty}(\Omega_K)} \leq C \tag{2.72}$$

for some constant $C = C(D', D) > 0$ independent of K and K'.

Let ρ and ρ' denote respectively the radial functions of K and K'. Let α be defined over ∂B by the relation

$$\rho' - \alpha = c(\rho - \rho') \tag{2.73}$$

for some $c > 0$ that will be chosen later. Then $\alpha \leq \rho'$ and we get the estimate

$$\alpha = ((c+1)\rho' - c\rho) \geq (c+1)\mathrm{inr}(D') - c\,\mathrm{diam}(D)$$

where $\mathrm{inr}(D')$ is the inradius of D'. For $c = c(D', D) = \frac{\mathrm{inr}(D')}{2[\mathrm{diam}(D) - \mathrm{inr}(D')]}$ we get

$$\alpha \geq \frac{\mathrm{inr}(D')}{2} \tag{2.74}$$

which is a lower bound independent of $K, K' \in \mathcal{K}^N_{D',D}$.

If $u \in \mathbb{R}^N \setminus \{0\}$ we denote by $x_u = u/|u|$. Let ϕ be defined over $\mathbb{R}^N$ by the formulae

$$\forall u \in \mathbb{R}^N \setminus \{0\},\ \phi(u) := \begin{cases} \phi_1(u) := x_u \left(\frac{c+1}{c}|u| - \frac{\alpha(x_u)}{c}\right), \text{ if } |u| \geq \alpha(x_u) \\ \phi_2(u) := u, \text{ if } |u| < \alpha(x_u) \end{cases}$$

and $\phi(0) := 0$. Observe that the function ϕ is continuous and increasing along any normal direction $x \in \partial B$, and it verifies by construction that

$$\phi(0) = 0,\ \phi(u) = \rho(x_u)x_u \text{ if } |u| = \rho'(x_u)$$

This ensures that ϕ is a bijection from K' to K.

Define K'' as the (non necessarily convex) set on which ϕ is the identity, *i.e.*

$$K'' := \{\lambda x \alpha(x),\ x \in \partial B,\ 0 \leq \lambda \leq 1\}$$

The mapping $u \in \mathbb{R}^N \setminus K'' \mapsto x_u$ having Lipschitz constant $2(\min_{\mathbb{R}^N \setminus K''} |u|)^{-1} = 2(\min \alpha)^{-1}$, then

$$u \in \mathbb{R}^N \setminus K'' \mapsto x_u \alpha(x_u)$$

has Lipschitz constant only depending on $\min \alpha$ and $\|\nabla_\tau \alpha\|_{L^\infty(\partial B)}$. From the definition (2.73) of α and recalling (2.71), we deduce that ϕ_1 has Lipschitz constant L only depending on c, $\mathrm{inr}(D')$ and $\mathrm{diam}(D)$, hence only on $\mathrm{inr}(D')$ and $\mathrm{diam}(D)$. Now, ϕ_2 is Lipschitz over $\mathbb{R}^N$ (with Lipschitz constant 1), so that we deduce that ϕ is globally Lipschitz over $\mathbb{R}^N$. Indeed: let $u_0 \in K''$, $u_1 \in \mathbb{R}^N \setminus K''$ and pick $t \in [0,1]$ such that the point $u_t := (1-t)u_0 + tu_1$ verifies $u_t \in \partial K''$ with $[u_t, u_1] \subset \mathbb{R}^N \setminus K''$. Since $\phi(u_t) = \phi_1(u_t) = \phi_2(u_t)$ we can write

$$\begin{aligned} |\phi(u_0) - \phi(u_1)| &= |\phi_2(u_0) - \phi_2(u_t) + \phi_1(u_t) - \phi_1(u_1)| \\ &\leq 1 \times |u_0 - u_t| + L|u_t - u_1| \\ &\leq \max(1, L)|u_0 - u_1| \end{aligned}$$

Hence ϕ is globally Lipschitz and its Lipschitz constant only depends on D' and D. The same arguments can be applied to ϕ^{-1}, thus getting (2.72). Let us now show that the operator defined by

$$\forall f \in L^1_{\text{loc}}(\Omega_K),\ Tf := f \circ \phi \tag{2.75}$$

satisfies the expected requirements. For $p \in \{2, \infty\}$, any $f \in W^{1,p}(\Omega_K)$ satifies $f \circ \phi \in W^{1,p}(\Omega_{K'})$ and the weak derivatives of $f \circ \phi$ can be expressed with the classical formula for the derivative of a composition (see for instance [Maz85, Theorem 1.1.7]); furthermore, if $f \in H^1_0(\Omega_K)$ then $f \circ \phi$ verifies $f \circ \phi = 0$ a.e. outside $\Omega_{K'}$, giving that $Tf \in H^1_0(\Omega_{K'})$ since $\Omega_{K'}$ is Lipschitz (see [HP18, 3.2.16]). Together with (2.72) we deduce that T satisfies the second requirement.

By construction $\phi(u) = u$ if $u \in K''$, so that it only remains to show

$$|K' \setminus K''| \leq C|K \setminus K'| \tag{2.76}$$

with C uniform in the class $\mathcal{K}^N_{D',D}$.

It is classical that $|K| = \frac{1}{N}\int_{\partial B} \rho^N d\mathcal{H}^{N-1}$ (see for instance [Sch14, (1.53)]) and similarly for K' and K'' with ρ' and α in place of ρ, respectively. Therefore

$$|K' \setminus K''| = \frac{1}{N}\int_{\partial B} \left(\rho'^N - \alpha^N\right) d\mathcal{H}^{N-1},\ |K \setminus K'| = \frac{1}{N}\int_{\partial B} \left(\rho^N - \rho'^N\right) d\mathcal{H}^{N-1} \tag{2.77}$$

Using the identity

$$x^N - y^N = (x-y)\sum_{k=0}^{N-1} x^k y^{N-1-k}$$

we obtain:

$$|K' \setminus K''| \leq N(\text{diam}(D))^{N-1}\int_{\partial B} \left(\rho' - \alpha\right) d\mathcal{H}^{N-1} = cN(\text{diam}(D))^{N-1}\int_{\partial B} \left(\rho - \rho'\right) d\mathcal{H}^{N-1}$$

recalling (2.73). Likewise we get

$$|K \setminus K'| \geq \left(N\frac{\text{inr}(D')^{N-1}}{2^{N-1}}\right)\int_{\partial B} \left(\rho - \rho'\right) d\mathcal{H}^{N-1}$$

recalling (2.74). This proves (2.76) for some $C = C(N, D, D')$ and completes the proof. □

Next lemma is a control of the torsion function $\|u_{\Omega_K}\|_{W^{1,\infty}(\Omega_K)}$ uniformly in the class $\mathcal{K}^N_{D',D}$:

Lemma 2.3.10. *Let $D \in \mathcal{K}^N$. There exists $C = C(D) > 0$ such that for all $K \in \mathcal{K}^N$ with $K \subset D$, then*

$$\|u_K\|_{W^{1,\infty}(\Omega_K)} \leq C$$

where $u_K = u_{\Omega_K}$ is the torsion function defined in (2.62).

Proof. • **L^∞ estimate of u_K:** We apply a standard maximum principle argument. We note first that $u_K \in C^0(\overline{\Omega})$ (see [GT01, Theorem 6.13]). We choose $x_0 \in \Omega_K$ and let

$$w(x) := \frac{1}{2N}\left(\text{diam}(\Omega_K)^2 - |x - x_0|^2\right)$$

The construction of w ensures

$$\begin{cases} -\Delta w &= -\Delta u_K & \text{in} & \Omega_K \\ w &\geq u_K & \text{over} & \partial\Omega_K \end{cases}$$

The maximum principle then writes

$$0 \leq u_K \leq w \leq \frac{\text{diam}(\Omega_K)^2}{2N} \text{ in } \Omega$$

so that

$$\|u_K\|_{L^\infty(\Omega_K)} \leq \frac{\text{diam}(\Omega_K)^2}{2N} \tag{2.78}$$

- **L^∞ estimate of ∇u_K:** This is obtained in [CF10, Lemma 1] ; we reproduce the proof for sake of completeness.

  Since Ω_K is convex the corresponding torsion function u_K is 1/2-concave (see [Ken85, Theorem 4.1]), yielding also that the level sets $\{u_K > c\}$ are convex for any $c \in \mathbb{R}$. Let $x_0 \in \Omega_K$ and take a supporting hyperplane H to the convex set $A := \{u_K > u_K(x_0)\}$, which we can assume to be $H = \{x_N = 0\}$ without loss of generality. The convexity of A ensures that A is located on one side of H, say that $A \subset \{x_N \geq 0\}$. As $x_0 \in H$ we must also have $A \subset \{x_N \leq d\}$ where $d := \mathrm{diam}(\Omega_K)$. Hence

  $$A \subset \{0 \leq x_N \leq d\}. \tag{2.79}$$

  This construction provides a natural barrier to u_K at x_0. In fact, denoting by $F := \{0 < x_N < d\}$, we let $w : F \to \mathbb{R}$ be defined for $x \in F$ by,

  $$w(x) = \frac{1}{2} x_N (d - x_N) + u_K(x_0). \tag{2.80}$$

  Then see that w verifies

  $$\begin{cases} -\Delta w = 1 & \text{in } F, \\ \quad w = u_K(x_0) & \text{over } \partial F. \end{cases}$$

  Furthermore it holds

  $$\forall x \in F,\ |\nabla w(x)| = |\partial_N w(x)| \leq \frac{d}{2}. \tag{2.81}$$

  We can now estimate the gradient of u_K at x_0. Noting that $u_K \in C^0(\overline{\Omega})$ (see [GT01, Theorem 6.13]), we have thanks to (2.79) and (2.80) that $w \geq c \geq u_K$ over $\partial A \subset \{u_K \geq u_K(x_0)\}$. We can therefore apply the maximum principle in the open set A to get

  $$\begin{aligned} \sup_{x \in \Omega} \frac{u_K(x) - u_K(x_0)}{|x - x_0|} &\leq \sup_{x \in A} \frac{u_K(x) - u_K(x_0)}{|x - x_0|} \\ &\leq \sup_{x \in A} \frac{w(x) - w(x_0)}{|x - x_0|} \\ &\leq \sup_{x \in F} \frac{w(x) - w(x_0)}{|x - x_0|}. \end{aligned}$$

  Using (2.81) we finally obtain

  $$\sup_{x \in \Omega} \frac{u_K(x) - u_K(x_0)}{|x - x_0|} \leq \frac{d}{2},$$

  that is

  $$\|\nabla u_K\|_{L^\infty(\Omega)} = \sup_{x, y \in \Omega} \frac{|u_K(x) - u_K(y)|}{|x - y|} \leq \frac{1}{2} \mathrm{diam}(\Omega).$$

□

Remark 2.3.11. We did not use the convexity of Ω_K for estimating $\|u_K\|_{L^\infty(\Omega_K)}$ in terms of $\mathrm{diam}(\Omega_K)$, so that estimate (2.78) holds for any bounded open set Ω. Moreover, one can also obtain a finer estimate relying on a symmetrization argument due to G. Talenti: if v is the solution to

$$\begin{cases} -\Delta v = 1 \text{ in } \Omega_K^\sharp \\ v \in H_0^1(\Omega_K^\sharp) \end{cases}$$

where $\Omega_K^\sharp$ is the ball centered at the origin having the same volume than Ω_K, then [Tal76, Theorem 1 (iv)] implies

$$u_K^\sharp \leq v \text{ in } \Omega_K^\sharp$$

with $u_K^\sharp$ denoting the symmetric decreasing rearrangement of u_K. This provides a Volume-type control of $\|u_K\|_{L^\infty(\Omega_K)}$:

$$\|u_K\|_{L^\infty(\Omega_K)} = \|u_K^\sharp\|_{L^\infty(\Omega_K^\sharp)} \leq \|v\|_{L^\infty(\Omega_K^\sharp)} \leq C(N)|\Omega_K|^{2/N}$$

with $C(N)$ a dimensional constant[2]. This estimate implies (2.78) up to a dimensional multiplicative constant, as Ω_K can always be included in some cube of size $\mathrm{diam}(\Omega_K)$. It can reveal to be very convenient for controlling $\|u_K\|_{L^\infty(\Omega_K)}$ when we can bound $|\Omega_K|$ while having $\mathrm{diam}(\Omega_K) \longrightarrow +\infty$.

We are now in a position to prove Proposition 2.3.8:

Proof of Proposition 2.3.8: Let $K' \subset K$ be convex bodies. Recall that Ω_K and $\Omega_{K'}$ denote the interiors of K and K' respectively. Let $T : \Omega_K \to \Omega_{K'}$ the change of variables given by Lemma 2.3.9. Let $u_K := u_{\Omega_K}$ the torsion function, solution of (2.62). Using $Tu_K \in H_0^1(\Omega_{K'})$ as a test function in the variational formulation of $\tau(K')$ we can write

$$0 \leq \tau(K) - \tau(K') \leq \int_K \Big(|\nabla(Tu_K)|^2 - |\nabla u_K|^2\Big) - 2\int_K (Tu_K - u_K)$$

Using the properties of T given by Lemma 2.3.9 we get

$$\begin{aligned} 0 \leq \tau(K) - \tau(K') &\leq \int_{K\setminus K''} (|\nabla(Tu_K)|^2 - |\nabla u_K|^2) - 2\int_{K\setminus K''} (Tu_K - u_K) \\ &\leq |K \setminus K''|\Big(\|\nabla(Tu_K)\|^2_{L^\infty(\Omega_{K'})} + 2\|Tu_K\|_{L^\infty(\Omega_{K'})} + 2\|u_K\|_{L^\infty(\Omega_K)}\Big) \\ &\leq C \max\Big(\|u_K\|_{L^\infty(\Omega_K)}, \|\nabla u_K\|^2_{L^\infty(\Omega_K)}\Big)\, |K \setminus K'| \end{aligned}$$

for some $C = C(D', D)$. Lemma 2.3.10 then yields the result. □

We will now show Theorem 2.3.2 for λ_n. To that end we will use the following result that was proved in [Buc03, Theorem 3.4]:

Theorem 2.3.12. *Let $\Omega' \subset \Omega$ be bounded open sets. For any $n \in \mathbb{N}^*$ it holds*

$$\big|\lambda_n(\Omega) - \lambda_n(\Omega')\big| \leq C(n)\lambda_n(\Omega)^{N/2+1}\lambda_n(\Omega')|\tau(\Omega) - \tau(\Omega')|$$

where $C(n) = 2n^2 e^{1/4\pi}$.

Proof of Theorem 2.3.2 for λ_n: Theorem 2.3.12 gives that for any $K \in \mathcal{K}^N_{D',D}$

$$\big|\lambda_n(K) - \lambda_n(K')\big| \leq C(n)\lambda_n(K)^{N/2+1}\lambda_n(K')|\tau(K) - \tau(K')|$$

From monotonicity of the Dirichlet eigenvalues we have $\lambda_n(K)^{N/2+1}\lambda_n(K') \leq \lambda_n(D')^{2+N/2}$, and therefore the estimate from Proposition 2.3.8 gives the result. □

Neumann eigenvalues

The purpose of this section is to prove Theorem 2.3.2 in the case of Neumann eigenvalues. We actually get a slightly better result (with no assumption of inclusion between K and K', see also Remark 2.3.3), more precisely:

Proposition 2.3.13. *Let $N \geq 2$. Let $D' \subset D \subset \mathbb{R}^N$ be convex bodies. For any $n \in \mathbb{N}^*$ there exists $C_n = C_n(D', D) > 0$ such that for each $K, K' \in \mathcal{K}^N_{D',D}$*

$$|\mu_n(K) - \mu_n(K')| \leq C_n|K\Delta K'| \tag{2.82}$$

[2]This was pointed out to us by D. Bucur.

Remark 2.3.14. This result is stronger than [Ros04, Theorem 4.2], proven in the same class but for a different distance between sets (namely, the L^∞ distance between the radial functions). More precisely, it is proven in [Ros04] that there exists $C_n = C_n(D', D)$ such that for all $K', K \in \mathcal{K}^N_{D',D}$ with respective radial functions ρ_K and $\rho_{K'}$

$$|\mu_n(K) - \mu_n(K')| \leq C_n \|\rho_K - \rho_{K'}\|_\infty.$$

This latter result is not enough to provide (2.82), as one sees for example by taking $K = [0,1]^3$ and $K_i = K \cap \{x \in \mathbb{R}^3, x_3 > 1/i - x_1\}$ (K_i is built by cutting the neighborhood of an edge). It is not hard to see that $|K \setminus K_i| \leq C\|\rho_K - \rho_{K_i}\|^2_\infty$ (we fixed an origin inside K, for example $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$), thus contradicting the possibility of controlling $\|\rho_K - \rho_{K'}\|_\infty$ by $|K \Delta K'|$. On the other hand this result is implied by (2.82), recalling the expression of volume in terms of radial functions (see for instance [Sch14, (1.53)])

$$|K \Delta K'| = \frac{1}{N} \int_{\partial B} |\rho_K^N - \rho_{K'}^N| \leq C\|\rho_K - \rho_{K'}\|_\infty$$

with C a constant only depending on D' and D.

As in the Dirichlet case we follow the general strategy of [BL07, BL08], as we were not able to apply their result, namely [BL07, Theorem 6.11]. The proof of Proposition 2.3.13 relies on the two independent steps:

1. we construct an extension operator $\Pi_K : W^{1,\infty}(\Omega_K) \to W^{1,\infty}(\mathbb{R}^N)$ whose norm uniformly bounded in $\mathcal{K}^N_{D',D}$.
2. we provide $W^{1,\infty}$-estimates of Neumann eigenfunctions.

Note that unlike in the Dirichlet case, we cannot rely on a statement like Theorem 2.3.12 and we have to directly work with the variational formulation of eigenvalues (we could actually apply the same strategy for proving Theorem 2.3.2 for λ_n, though we thought it was more elegant to use Theorem 2.3.12).

The following lemma deals with the first item of this strategy:

Lemma 2.3.15 (Extension operator)**.** *Let $N \geq 2$ and $D' \subset D \subset \mathbb{R}^N$ be convex bodies. There exists $C = C(D', D) > 0$ such that for any $K \in \mathcal{K}^N_{D',D}$ there exists a bounded operator*

$$\Pi_K : L^1(\Omega_K) \to L^1(\mathbb{R}^N)$$

satisfying the requirements:

- *for any $f \in L^1(\Omega_K)$, $\Pi_K f(x) = f(x)$ for a.e. x in Ω_K,*
- *if $f \in W^{1,\infty}(\Omega_K)$ then $\Pi_K f \in W^{1,\infty}(\mathbb{R}^N)$ with*

$$\|\Pi_K f\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C\|f\|_{W^{1,\infty}(\Omega_K)}$$

Remark 2.3.16. This result could be seen as a consequence of [Che77, Theorem II.1] which asserts the same result in the wider class of sets satisfying the $\varepsilon-$cone condition (see Definition 2.4.1). Nevertheless, using that the domains we consider are convex, we are able to give a shorter proof of this result.

Proof. The ideas are similar from the ones in the proof of Lemma 2.3.9. We again assume up to translating that $0 \in \mathrm{Int}(D')$, and let ρ be the radial function associated to K.

If $u \in \mathbb{R}^N \setminus \{0\}$ we set $x_u := u/|u|$. We let

$$\widetilde{K} := \{\lambda x(\rho(x) + 1),\ x \in B,\ \lambda \in [0,1]\}$$

$$\forall u \in \mathbb{R}^N,\ \phi_1(u) := \begin{cases} u & \text{if } u \in K \\ \rho(x_u)x_u & \text{if } u \notin K \end{cases}$$

and

$$\forall u \in \mathbb{R}^N,\ \phi_2(u) := \begin{cases} 1 & \text{if } u \in K \\ \rho(x_u) - |u| + 1 & \text{if } |u| \in [\rho(x_u), \rho(x_u)+1] \\ 0 & \text{if } |u| > \rho(x_u) + 1 \end{cases}$$

Functions ϕ_1 and ϕ_2 are built by gluing continuously Lipschitz functions; as in the proof of Lemma 2.3.9 we thus deduce that ϕ_1 and ϕ_2 are Lipschitz with Lipschitz constant only depending on $\|\nabla_\tau \rho\|_{L^\infty(B)}$ and $\min \rho$, hence only on D' and D. Therefore there exists $C(D', D)$ such that

$$\begin{cases} \|\phi_2\|_{L^\infty(\mathbb{R}^N)} \leq 1,\ \|\nabla \phi_2\|_{L^\infty(\mathbb{R}^N)} \leq C(D', D) \\ \|D\phi_1\|_{L^\infty(\mathbb{R}^N)} \leq C(D', D) \end{cases} \tag{2.83}$$

We finally let, for any $f \in L^1(\Omega_K)$ and $u \in \mathbb{R}^N$

$$\Pi_K f(u) := \begin{cases} f(\phi_1(u))\phi_2(u) & \text{if } u \in \widetilde{K} \\ 0 & \text{if } u \notin \widetilde{K} \end{cases}$$

By construction $\Pi_K f(u) = f(u)$ for $u \in \Omega_K$, and $\|\Pi_K f\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^\infty(\Omega_K)}$. Since ϕ_1 and ϕ_2 are Lipschitz we have that $\Pi_K f \in W^{1,\infty}(\mathbb{R}^N)$ if $f \in W^{1,\infty}(\Omega_K)$ and further $\nabla \Pi_K f = (f \circ \phi_1)\nabla\phi_2 + \nabla(f \circ \phi_1)\phi_2$ a.e.. Using (2.83) we deduce

$$\begin{aligned} \|\nabla \Pi_K f\|_{L^\infty(\mathbb{R}^N)} = \|\nabla \Pi_K f\|_{L^\infty(\widetilde{K})} &\leq \|\phi_2 \nabla(f \circ \phi_1)\|_{L^\infty(\widetilde{K})} + \|(f \circ \phi_1)\nabla\phi_2\|_{L^\infty(\widetilde{K})} \\ &\leq \|\phi_2\|_{W^{1,\infty}(\mathbb{R}^N)} \big(\|\nabla(f \circ \phi_1)\|_{L^\infty(\widetilde{K})} + \|f \circ \phi_1\|_{L^\infty(\widetilde{K})}\big) \\ &\leq (1 + C(D', D))^2 \|f\|_{W^{1,\infty}(\Omega_K)} \end{aligned}$$

This completes the proof of the lemma. □

We now state a $W^{1,\infty}$-estimate for Neumann eigenfunctions:

Lemma 2.3.17. *Let $N \geq 2$, $D' \subset D \subset \mathbb{R}^N$ be convex bodies, and $n \in \mathbb{N}^*$. There exists $C_n = C_n(N, D', D) > 0$ such that for all $K \in \mathcal{K}^N_{D',D}$,*

$$\|v_{K,n}\|_{W^{1,\infty}(\Omega_K)} \leq C_n$$

where $v_{K,n}$ is any Neumann eigenfunction associated to $\mu_n(K)$ and such that $\|v_{K,n}\|_{L^2(\Omega_K)} = 1$.

Proof. We fix $n \in \mathbb{N}$ and denote more simply $v_n := v_{K,n}$.

- **L^∞ estimate of v_n:** By [Ros04, Proposition 3.1] (and the remark following), it holds

  $$\|v_n\|_{L^\infty(\Omega_K)} \leq C_1\big((1 + \sqrt{\mu_k(K)})C_2\big)^r$$

  where

  $$\begin{cases} C_1 = C_1(N) \\ C_2 = C_2(D', D) \\ r = r(N) \end{cases}$$

  Since $\mu_n(K) \leq \lambda_n(K) \leq \lambda_n(D')$, we get the estimate

  $$\|v_n\|_{L^\infty(\Omega_K)} \leq C_n(N, D', D) \tag{2.84}$$

- **L^∞ estimate of ∇v_n:** It is proved in [Maz09] in any dimension $N \geq 2$ that

$$\|\nabla v_n\|_{L^\infty(\Omega_K)} \leq C(N)|K|^{1/N}\mu_n(K)C_{\Omega_K}^{-1}\|v_n\|_{L^\infty(\Omega_K)}$$

where C_{Ω_K} is the isoperimetric constant relative to Ω_K, *i.e.* if $\Omega \subset \mathbb{R}^N$ is a bounded open set

$$C_\Omega := \inf_{\substack{E\subset\Omega \\ 0<|E|\leq|\Omega|/2}} \frac{P(E,\Omega)}{|E|^{1-1/N}} = \inf_{\substack{E\subset\Omega \\ 0<|E|<|\Omega|}} \frac{P(E,\Omega)}{\min\{|E|,|\Omega\setminus E|\}^{\frac{N-1}{N}}}$$

with $P(\cdot,\Omega)$ denoting the relative perimeter in Ω. Together with (2.84) this provides

$$\|\nabla v_n\|_{L^\infty(\Omega_K)} \leq C_n(N,D',D)C_{\Omega_K}^{-1} \tag{2.85}$$

Now it is shown in [Tho15, Corollary 2] that $C_\Omega \geq \delta(\varepsilon,b,N) > 0$ for any Lipschitz domain Ω with diameter $\leq b$ and satisfying the ε-cone condition (2.94). As there exists $\varepsilon = \varepsilon(D',D)$ such that (2.94) is satisfied for any $K \in \mathcal{K}^N_{D',D}$ (see Remark 2.4.2) we can conclude from (2.85) that

$$\|\nabla v_n\|_{L^\infty(\Omega_K)} \leq C_n(N,D',D)$$

□

We are now in a position to prove Proposition 2.3.13. The following is a combination of the proofs of Theorems 3.2 and 4.20 of [BL07] adapted to the particular case of the Neumann Laplace operator, which we reproduce for the convenience of the reader.

Proof of Proposition 2.3.13: We denote by $v_1,...,v_n$ n first eigenfunctions of the Neumann Laplace operator on Ω_K normalized for the L^2 norm (*i.e.* $\|v_k\|_{L^2(\Omega_K)} = 1$). Recall that functions v_k are orthogonal for the L^2 scalar product. Let $f := \sum_{k=1}^n \alpha_k v_k$ with $\|f\|_{L^2(\Omega_K)} = 1$, which means $\sum_{k=1}^n \alpha_k^2 = 1$. Set

$$T := R_{K'} \circ \Pi_K : L^1(\Omega_K) \to L^1(\Omega_{K'})$$

where Π_K is the extension operator given by Lemma 2.3.15 and $R_{K'}$ is the restriction onto $\Omega_{K'}$. Note first that it suffices to prove estimate (2.82) under the additionnal condition $|K\Delta K'| \leq \varepsilon_n$ for some small $\varepsilon_n(D',D) > 0$ depending on n, D', D; indeed, using $\mu_n(K) \leq \lambda_n(K) \leq \lambda_n(D')$ for any $K \in \mathcal{K}^N_{D',D}$, we have

$$|\mu_n(K) - \mu_n(K')| \leq 2\lambda_n(D') \leq \frac{2\lambda_n(D')}{\varepsilon_n}|K\Delta K'|$$

if $|K\Delta K'| \geq \varepsilon_n$.

Estimate from below of $\|Tf\|_{L^2(\Omega_{K'})}$: We will first prove

$$\|Tf\|^2_{L^2(\Omega_{K'})} \geq 1 - C_n|\Omega_K \setminus \Omega_{K'}| \tag{2.86}$$

for some $C_n = C_n(N,D',D) > 0$, which immediately provides

$$\|Tf\|^{-2}_{L^2(\Omega_{K'})} \leq 1 + 2C_n|\Omega_K \setminus \Omega_{K'}| \tag{2.87}$$

whenever $|\Omega_K \setminus \Omega_{K'}| \leq 1/2C_n$, using the inequality $(1-x)^{-1} \leq 1 + 2x$ if $0 \leq x \leq 1/2$.

We have, as $R_{K'} \circ \Pi_K(f) = f$ on $\Omega_K \cap \Omega_{K'}$,

$$\|Tf\|^2_{L^2(\Omega_{K'})} \geq \|Tf\|^2_{L^2(\Omega_K\cap\Omega_{K'})} = \|f\|^2_{L^2(\Omega_K\cap\Omega_{K'})} = \|f\|^2_{L^2(\Omega_K)} - \|f\|^2_{L^2(\Omega_K\setminus\Omega_{K'})} \tag{2.88}$$

But thanks to Lemma 2.3.17 and recalling that $\sum_{k=1}^n \alpha_k^2 = 1$

$$\|f\|^2_{L^2(\Omega_K\setminus\Omega_{K'})} \le \Big(\sum_{k=1}^n \alpha_k \|v_k\|_{L^2(\Omega_K\setminus\Omega_{K'})}\Big)^2 \le \sum_{k=1}^n \|v_k\|^2_{L^2(\Omega_K\setminus\Omega_{K'})}$$
$$\le \Big(\sum_{k=1}^n \|v_k\|^2_{L^\infty(\Omega_K)}\Big)|\Omega_K\setminus\Omega_{K'}| \le C_n(N,D',D)|\Omega_K\setminus\Omega_{K'}|$$

Pluging this into (2.88) yields (2.86).

Estimate from above of $\|\nabla Tf\|_{L^2(\Omega_{K'})}$: We now prove

$$\|\nabla Tf\|^2_{L^2(\Omega_{K'})} \le \mu_n(K) + C_n|\Omega_{K'}\setminus\Omega_K| \tag{2.89}$$

for some $C_n = C_n(N,D',D)$.

Note that

$$\begin{aligned}\|\nabla Tf\|^2_{L^2(\Omega_{K'})} &= \|\nabla Tf\|^2_{L^2(\Omega_{K'}\cap\Omega_K)} + \|\nabla Tf\|^2_{L^2(\Omega_{K'}\setminus\Omega_K)}\\ &= \|\nabla f\|^2_{L^2(\Omega_{K'}\cap\Omega_K)} + \|\nabla Tf\|^2_{L^2(\Omega_{K'}\setminus\Omega_K)}\\ &\le \|\nabla f\|^2_{L^2(\Omega_K)} + \|\nabla Tf\|^2_{L^2(\Omega_{K'}\setminus\Omega_K)}\end{aligned} \tag{2.90}$$

The Neumann eigenfunctions being orthogonal for the L^2 scalar product, we also have $\int_{\Omega_K}\nabla v_k\cdot\nabla v_{k'} = \mu_k(K)\int_{\Omega_K} v_k v_{k'} = 0$ for $k\neq k'$. Furthemore $\|\nabla v_k\|^2_{L^2(\Omega_K)} = \mu_k(K) \le \mu_n(K)$ and we thus get

$$\|\nabla f\|^2_{L^2(\Omega_K)} = \sum_{k=1}^n \alpha_k^2\|\nabla v_k\|^2_{L^2(\Omega_K)} \le \mu_n(K) \tag{2.91}$$

On the other hand, denoting by $C_1 = C_1(D',D)$ and $C_2 = C_2(N,D',D)$ the constants respectively given by Lemmas 2.3.15 and 2.3.17, we obtain

$$\begin{aligned}\|\nabla Tf\|^2_{L^2(\Omega_{K'}\setminus\Omega_K)} &\le \|\nabla Tf\|^2_{L^\infty(\Omega_{K'})}|\Omega_{K'}\setminus\Omega_K|\\ &\le C_1\|f\|^2_{W^{1,\infty}(\Omega_K)}|\Omega_{K'}\setminus\Omega_K|\\ &\le C_1\Big(\sum_{k=1}^n \|v_k\|^2_{W^{1,\infty}(\Omega_K)}\Big)|\Omega_{K'}\setminus\Omega_K|\\ &\le C_1 n C_2^2|\Omega_{K'}\setminus\Omega_K|\end{aligned} \tag{2.92}$$

With estimates (2.91) and (2.92), (2.90) gives (2.89) for $C_n(N,D',D) = C_1 n C_2^2$.

Min-max principle and conclusion: Let us remind the following min-max principle:

$$\mu_n(K') = \min_{\dim(V)=n}\ \max_{\substack{g\in V\\ g\neq 0}} \frac{\|\nabla g\|^2_{L^2(\Omega_{K'})}}{\|g\|^2_{L^2(\Omega_{K'})}} \tag{2.93}$$

where the minimum is taken over all n-dimensional subspaces $V\subset H^1(\Omega_{K'})$.

In the view of (2.86), if $|\Omega_K\setminus\Omega_{K'}|\le\varepsilon_n$ for some $\varepsilon_n = \varepsilon_n(N,D',D)$, we deduce $\|Tf\|_{L^2(\Omega_{K'})} > 0$ for any $f = \sum_{k=1}^n \alpha_k v_k$ with $\|f\|_{L^2(\Omega_K)} = 1$, thus getting that $Tv_1,\dots,Tv_n$ are linearly independent as $(v_1,\dots,v_n)$ also are. Setting $L_n := \mathrm{Vect}(v_1,...,v_n)$, formula (2.93) therefore implies

$$\mu_n(K') \le \max_{\substack{g\in TL_n\\ g\neq 0}} \frac{\|\nabla g\|^2_{L^2(\Omega_{K'})}}{\|g\|^2_{L^2(\Omega_{K'})}} = \max_{\substack{f\in L_n\\ \|f\|_{L^2(\Omega_K)}=1}} \frac{\|\nabla Tf\|^2_{L^2(\Omega_{K'})}}{\|Tf\|^2_{L^2(\Omega_{K'})}}$$

Putting estimates (2.87) and (2.89) provides

$$\begin{aligned}\mu_n(K') &\leq \big(\mu_n(K) + C_n|\Omega_{K'} \setminus \Omega_K|\big)\big(1 + 2C_n|\Omega_K \setminus \Omega_{K'}|\big)\\ &\leq \mu_n(K) + \widetilde{C}_n|\Omega_K \Delta \Omega_{K'}|\end{aligned}$$

for some constant $\widetilde{C}_n(N, D', D)$, if $|\Omega_K \setminus \Omega_{K'}| \leq \varepsilon_n$, where we used $\mu_n(K) \leq \lambda_n(K) \leq \lambda_n(D')$. Switching the roles played by K and K' we get (2.82). □

2.3.3 Optimality of the $C^{1,1}$ regularity

We show in this short section that the Hölder regularity obtained in Theorem 2.2.3 is optimal. Precisely we prove:

Proposition 2.3.18. *In $\mathbb{R}^2$, there exists a quasi-minimizer of the perimeter under convexity constraint which is $C^{1,1}$ but not C^2.*

Let us first introduce some notations. If Ω is a measurable set we define its (scaling invariant) Fraenkel asymmetry $\alpha(\Omega)$, which is scaling invariant:

$$\alpha(\Omega) := \inf\left\{\frac{|\Omega \Delta B|}{|\Omega|},\ B \subset \mathbb{R}^2 \text{ a ball }, \ |B| = |\Omega|\right\}.$$

We also denote $D(\Omega) := (P(\Omega) - P(B))/P(B)$ the normalized isoperimetric deficit, where B is any ball with same volume than Ω. We call *stadium* a set $\Omega \subset \mathbb{R}^2$ which is obtained as the convex envelope of two disjoint disks of same radius.

The following result is proved in [AFN11] (see also [BCH17, Theorem 1.2], and the introduction of [CL13]).

Theorem 2.3.19. *It holds*

$$\inf_{K \in \mathcal{K}^2} \frac{D(K)}{\alpha(K)^2} \simeq 0.405585 > 0$$

and equality is achieved at a particular stadium K^*.

Proof of Proposition 2.3.18: Let us note that any stadium is $C^{1,1}$ but not C^2, as the curvature jumps from value 0 on a flat part to a positive value on a semi-circle. Call c^* the value of the infimum above, and set $V_0 := |K^*|$ and B_{V_0} a ball of volume V_0. Then Theorem 2.3.19 implies

$$P(K) - P(B_{V_0}) - c^* P(B_{V_0})\alpha(K)^2 \geq 0$$

for any planar convex body K of volume V_0, with equality at K^*. In other words, K^* minimizes the functional $P - P(B_{V_0}) - c\alpha^2$ among planar convex sets of volume V_0, with $c := P(B)c^*$. If one proves that α^2 satisfies hypothesis (2.51), then we deduce that K^* is a quasi-minimizer of the perimeter under convexity constraint, which concludes the proof. Let $(K, K') \in \mathcal{K}^2$, and with no loss of generality (as α is invariant with scaling) let us assume that $|K| = |K'| = 1$. We denote B an optimal ball in the definition of $\alpha(K)$: we have

$$\alpha^2(K') - \alpha^2(K) \leq 2(\alpha(K') - \alpha(K)) \leq 2\left(|K'\Delta B| - |K\Delta B|\right) \leq 2|K\Delta K'|$$

where the last inequality is obtained by easily checking that $K'\Delta B \subset (K\Delta K') \cup (K\Delta B)$. Inverting the roles played by K and K' we deduce that α verifies (2.51), thus completing the proof. □

2.4 Appendix

Parametrization of convex bodies in cartesian graphs Let us start by covering a few preliminaries about Lipschitz sets. The following definition, first given by D. Chenais in [Che75], is a very convenient way of considering "uniformly" Lipschitz sets.

Definition 2.4.1. *Let $\varepsilon > 0$. We say that an open set $\Omega \subset \mathbb{R}^N$ satisfies the ε-cone condition if for any $x \in \partial\Omega$ there exists a unit vector ξ_x such that*

$$\forall y \in B_\varepsilon(x) \cap \overline{\Omega},\ C(y, \xi_x, \varepsilon) \subset \Omega \tag{2.94}$$

where we set

$$C(y, \xi_x, \varepsilon) := \{z \in \mathbb{R}^N, \langle z - y, \xi_x \rangle > \cos(\varepsilon)|z - y|, 0 < |z - y| < \varepsilon\}.$$

Remark 2.4.2. For any fixed $M \geq m > 0$, it can be shown that any open convex set Ω such that there exists $x \in \mathbb{R}^N$ with

$$B_m(x) \subset \Omega \subset B_M(x)$$

verifies the ε-cone condition for some $\varepsilon = \varepsilon(m, M)$ (see for instance [HP18, Proposition 2.4.4]).

The following proposition shows that one can see a convex set as the graph of a Lipschitz function with specific additional properties that will be used in the proof of Theorem 2.2.3 (see Figure 2.3 for an illustration).

Proposition 2.4.3. *Let $K \in \mathcal{K}^N$. For any $\widehat{x_0} \in \partial K$, there exists*

- *A hyperplane $H \subset \mathbb{R}^N$ containing $\widehat{x_0}$,*
- *A unit vector $\xi \in \mathbb{R}^N$ normal to H,*

such that, denoting by (x, t) a point in $H \times \mathbb{R}\xi$ coordinates (and hence denoting $\widehat{x_0} := (x_0, 0)$), it holds

1. *The set $\Omega := \{x \in H,\ (x + \mathbb{R}\xi) \cap Int(K) \neq \emptyset\}$ is open, bounded and convex, and the function*

$$\begin{aligned} u : \Omega &\to \mathbb{R} \\ x &\mapsto \min\{t \in \mathbb{R},\ (x, t) \in K\} \end{aligned}$$

   is well-defined and convex. Furthermore, if ε is such that $Int(K)$ satifies the 2ε-cone condition (see Definition 2.4.1), then $B^\varepsilon := B_{\varepsilon \tan(\varepsilon)}(x_0) \subset \Omega$ and $u_{|B^\varepsilon}$ is $\tan(\varepsilon)^{-1}$-Lipschitz.

2. *It holds*

$$\begin{aligned} \{(x, u(x)),\ x \in \Omega\} &\subset \partial K \\ K \cap (\Omega \times \mathbb{R}\xi) &\subset \{(x, t) \in \Omega \times \mathbb{R}\xi,\ u(x) \leq t\} \end{aligned}$$

3. *For any open set $\omega \Subset B^\varepsilon$, there exists $c > 0$ such that*

$$\{(x, t) \in \omega \times \mathbb{R}\xi,\ u(x) \leq t \leq u(x) + c\} \subset K$$

   Furthermore we can choose c only depending on $d(\omega, \partial B^\varepsilon)$ and ε.

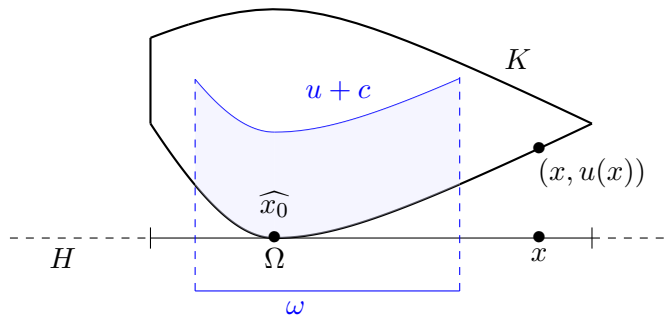


Figure 2.3: Convex body in cartesian graph

Proof. The proof is inspired from [HP18, Theorem 2.4.7], with a few adaptations due to convexity. Since $K \in \mathcal{K}^N$, then $\mathrm{Int}(K)$ satisfies the cone condition (see Definition 2.4.1 and Remark 2.4.2). We can assume without loss of generality that it satisfies the 2ε-cone condition for some $\varepsilon > 0$ with $\tan(\varepsilon) \leq 1$. Let then $\xi := \xi_{\widehat{x_0}}$ be a unit vector associated to $\widehat{x_0}$ and the 2ε-cone condition, that is

$$\forall \widehat{x} \in K \cap B_{2\varepsilon}(\widehat{x_0}),\ C(\widehat{x}, \xi, 2\varepsilon) \subset \mathrm{Int}(K)$$

We set $H := \{\widehat{x} \in \mathbb{R}^N, \langle \widehat{x} - \widehat{x_0}, \xi \rangle = 0\}$.

The function u is well-defined by construction of Ω. The convexity of K gives immediately that u is convex: if $\lambda \in [0,1]$ and $x, y \in \Omega$, the point $(1-\lambda)(x, u(x)) + \lambda(y, u(y)) \in K$ since K is convex, giving that $u((1-\lambda)x + \lambda y) \leq (1-\lambda)u(x) + \lambda u(y)$ by definition of u.

For the rest of the proof we will write (y, y_ξ) for the $H \times \mathbb{R}\xi$ coordinates of a point $\widehat{y} \in \mathbb{R}^N$. Any cone $C(\widehat{x}, \xi, 2\varepsilon)$ can be written

$$C(\widehat{x}, \xi, 2\varepsilon) = \left\{ \widehat{y} \in B_{2\varepsilon}(\widehat{x}),\ y_\xi - x_\xi > \frac{1}{\tan(\varepsilon)} |y - x| \right\} \tag{2.95}$$

Indeed, if $\widehat{y} \in \mathbb{R}^N$ and $\widehat{x} \in \mathbb{R}^N$ are such that $y_\xi - x_\xi \geq 0$, then

$$\begin{aligned} y_\xi - x_\xi > \cos(\varepsilon)|\widehat{y} - \widehat{x}| &\iff (y_\xi - x_\xi)^2 > \cos^2(\varepsilon)\big(|y-x|^2 + (y_\xi - x_\xi)^2\big) \\ &\iff y_\xi - x_\xi > \frac{1}{\tan(\varepsilon)}|y - x| \end{aligned}$$

Recalling that $B^\varepsilon = \{x \in H,\ |x - x_0| < \varepsilon \tan(\varepsilon)\}$ we claim that

$$\forall x \in B^\varepsilon,\ \begin{cases} (x, \varepsilon) \in \mathrm{Int}(K) \\ (x, -\varepsilon) \notin \mathrm{Int}(K) \end{cases} \tag{2.96}$$

Indeed, let $x \in B^\varepsilon$. Let us first show that $(x, \varepsilon) \in \mathrm{Int}(K)$. Since $C(\widehat{x_0}, \xi, 2\varepsilon) \subset \mathrm{Int}(K)$ by the 2ε-cone condition, then it suffices to prove that $(x, \varepsilon) \in C(\widehat{x_0}, \xi, 2\varepsilon)$. But as $|x - x_0| < \varepsilon$ it holds $(x, \varepsilon) \in B_{2\varepsilon}(\widehat{x_0})$, and furthermore $\tan(\varepsilon)^{-1}|x - x_0| < \varepsilon$, so that we deduce $(x, \varepsilon) \in C(\widehat{x_0}, \xi, 2\varepsilon)$ thanks to (2.95) and hence $(x, \varepsilon) \in \mathrm{Int}(K)$. For the second assertion it is sufficient to prove that $C(\widehat{x_0}, -\xi, 2\varepsilon) \subset \mathbb{R}^N \setminus \mathrm{Int}(K)$, since in any case $(x, -\varepsilon) \in C(\widehat{x_0}, -\xi, 2\varepsilon)$ using again (2.95), as $(x, -\varepsilon) \in B_{2\varepsilon}(\widehat{x_0})$ with $\tan(\varepsilon)^{-1}|x - x_0| < \varepsilon = (-\varepsilon) \times (-1)$. Suppose then by contradiction that there exists $\widehat{x} \in C(\widehat{x_0}, -\xi, 2\varepsilon) \cap \mathrm{Int}(K)$; since $\widehat{x} \in B_{2\varepsilon}(\widehat{x_0})$, it holds that $C(\widehat{x}, \xi, 2\varepsilon) \subset \mathrm{Int}(K)$ by the 2ε-cone property. But then $\widehat{x_0} \in C(\widehat{x}, \xi, 2\varepsilon)$, yielding $\widehat{x_0} \in \mathrm{Int}(K)$, which is a contradiction. This finishes the proof of (2.96).

Thanks to (2.96), it holds $B^\varepsilon \Subset \Omega$. Let us show that $u_{|B^\varepsilon}$ is Lipschitz continuous with Lipschitz constant $\tan(\varepsilon)^{-1}$. If $x \in B^\varepsilon$ then $(x, \varepsilon) \in \mathrm{Int}(K)$ thanks to (2.96) and $(x, u(x)) \in \partial K$ so that $[(x, u(x)), (x, \varepsilon)] \subset K$. This ensures $-\varepsilon < u(x) < \varepsilon$ using again (2.96), so that in particular $(x, u(x)) \in B_{2\varepsilon}(\widehat{x_0})$. Let now $x, y \in B^\varepsilon$.

As $(x, u(x)) \in B_{2\varepsilon}(\widehat{x_0})$ we must have $C((x, u(x)), \xi, 2\varepsilon) \subset \mathrm{Int}(K)$ by the 2ε-cone property; but then, as $(y, u(y)) \in \partial K$ we get $(y, u(y)) \notin C((x, u(x)), \xi, 2\varepsilon)$, giving

$$u(y) - u(x) \leq \frac{1}{\tan(\varepsilon)}|y - x|$$

Reversing the roles played by x and y, we deduce in fact that $u_{|B^\varepsilon}$ is $\tan(\varepsilon)^{-1}$-Lipschitz. This finishes the proof of the first requirement.

The construction of u ensures that the second requirement is verified. As for the third let δ be such that $d(\omega, \partial B^\varepsilon) \geq \delta > 0$. Set $c := \delta \tan(\varepsilon)^{-1}$ and let $x \in B^\varepsilon$ and $u(x) \leq t \leq u(x) + c$. As $x \in B^\varepsilon$, we have $(x, \varepsilon) \in \mathrm{Int}(K)$ and $(x, u(x)) \in \partial K$ so that $[(x, u(x)), (x, \varepsilon)] \subset K$, hence it suffices to show that $t < \varepsilon$ to get the claim. As it holds that $|x - x_0| < \varepsilon \tan(\varepsilon) - \delta$, then recalling that $u_{|B^\varepsilon}$ is $\tan(\varepsilon)^{-1}$-Lipschitz we get

$$t \leq u(x) + c < \tan(\varepsilon)^{-1} \times (\varepsilon \tan(\varepsilon) - \delta) + c = \varepsilon$$

This finishes the proof of the third point and hence the proof of the Proposition. □

Proof of Proposition 2.2.8

Proof. Let (K_n) be a sequence of convex bodies verifying $K_n \subset D$ where $D \in \mathcal{K}^N$.

- Let us first focus on

$$d_H(K_n, K) \to 0 \Longleftrightarrow |K_n \Delta K| \to 0$$

  The direct sense is proved in [BB05, Proposition 2.4.3, (ii)]. As for the converse, suppose $|K_n \Delta K| \to 0$, and assume by contradiction that (K_n) does not converge to K for d_H. Up to extracting we can therefore suppose that there exists $\varepsilon > 0$ such that

$$\forall n \in \mathbb{N},\ d_H(K_n, K) \geq \varepsilon \tag{2.97}$$

  Thanks to the Blaschke selection theorem which states that $\{L \text{ compact convex of } \mathbb{R}^N,\ L \subset D\}$ is compact for the Hausdorff distance, up to further extraction there exists K_∞ compact convex such that $K_n \to K_\infty$ for d_H. Using again [BB05, Proposition 2.4.3, (ii)] and since $|K_n \Delta K| \to 0$ we must have $K_\infty = K$, contradicting (2.97).

- We now assume that $d_H(K_n, K) \to 0$, and let $C \in \mathcal{K}^N$ be such that $C \subset \mathrm{Int}(K)$. We want to prove

$$C \subset K_n \quad \text{for large } n.$$

  There exists $\varepsilon > 0$ such that $d(C, \partial K) \geq \varepsilon$. Since $K_n \to K$ for d_H, we also have $\partial K_n \to \partial K$ for d_H (see [Sch14, Lemma 1.8.1]). This gives $d(C, \partial K_n) \geq \varepsilon/2$ for large n. Let us assume to get a contradiction that we do not have $C \subset K_n$ for n large enough. Up to extraction we can therefore suppose that $C \cap (\mathbb{R}^N \setminus K_n) \neq \emptyset$ for each n. But then $C \subset \mathbb{R}^N \setminus K_n$ thanks to the convexity of C, since otherwise there would exist $x \in C \cap \partial K_n$ which is in contradiction with $d(C, \partial K_n) \geq \varepsilon/2$. This rewrites $K_n \subset \mathbb{R}^N \setminus C$ for each n, yielding $K \subset \overline{(\mathbb{R}^N \setminus C)}$ at the limit. This contradicts the hypothesis $\mathrm{Int}(C) \subset K$.

□

Acknowledgements: The authors would like to thank Dorin Bucur and Guillaume Carlier for very helpful discussions about this work. The authors also thank Guillaume Carlier for providing a version of [CCL13]. This work was partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR).

Chapter 3

Fuglede-type arguments for isoperimetric problems and applications to stability among convex shapes

This Chapter is a reprint of the submitted paper *Fuglede-type arguments for isoperimetric problems and applications to stability among convex shapes* written by the author of this thesis.

Abstract

This paper is concerned with stability of the ball for a class of isoperimetric problems under convexity constraint. Considering the problem of minimizing $P + \varepsilon R$ among convex subsets of $\mathbb{R}^N$ of fixed volume, where P is the perimeter functional, R is a perturbative term and $\varepsilon > 0$ is a small parameter, stability of the ball for this perturbed isoperimetric problem means that the ball is the unique (local, up to translation) minimizer for any ε sufficiently small. We investigate independently two specific cases where $\Omega \mapsto R(\Omega)$ is an energy arising from PDE theory, namely the capacity and the first Dirichlet eigenvalue of a domain $\Omega \subset \mathbb{R}^N$. While in both cases stability fails among all shapes, in the first case we prove (non-sharp) stability of the ball among convex shapes, by building an appropriate competitor for the capacity of a perturbation of the ball. In the second case we prove sharp stability of the ball among convex shapes by providing the optimal range of ε such that stability holds, relying on the *selection principle* technique and a regularity theory under convexity constraint.

3.1 Introduction

3.1.1 Stability in shape optimization

In this article we are interested in the question of stability of the ball for isoperimetric-type problems under a convexity constraint. It takes place in the framework of shape optimization problems involving the perimeter functional P, which consists in the minimization problems

$$\inf \{P(A) + R(A),\ A \in \mathcal{A}\}$$

where $\mathcal{A}$ is a class of measurable subsets $A \subset \mathbb{R}^N$ of volume $|A| = 1$, $P(A)$ is the perimeter of A in the usual De Giorgi sense, and $R : \mathcal{A} \to \mathbb{R}$ is a functional thought of as a perturbative term. Due to the well-known *isoperimetric inequality*, any ball $B \subset \mathbb{R}^N$ of unit volume is minimal for the minimization of $P+R$ among all sets in $\mathcal{A}$ when $R = 0$: for all measurable sets $A \subset \mathbb{R}^N$ with volume $|A| = 1$, $P(A) \geq P(B)$ with equality if and only if A is a ball of unit volume. By stability of the ball for the problem $P + R$ we

mean that, considering $R_\varepsilon := \varepsilon R$ for a small parameter $\varepsilon > 0$, then provided ε is close enough to 0:

The ball B is a local minimizer of $P + \varepsilon R$ in $\mathcal{A}$

where by locality we mean that the L^1 distance of $\Omega \in \mathcal{A}$ with B is small (*i.e.* $|\Omega \Delta B| \ll 1$). In other words, this notion of stability states that the ball is still a minimizer of the perimeter functional when it is perturbed by another functional R. Another way of putting it comes from rewriting the minimality of B as

$$P(A) - P(B) \geq \varepsilon \left(R(B) - R(A)\right)$$

so that, assuming moreover that $R(B) \geq R(A)$ for each $A \in \mathcal{A}$ (which is always verified for the cases we have in mind), then the deficit of perimeter quantifies the deficit of the functional R. Let us mention that this point of view on stability encompasses what is usually refered to in the literature as quantitative inequalities, the most famous one being the *sharp quantitative isoperimetric inequality*, proven in [FMP08]. It claims that by setting

$$\delta_F(A) := \inf\{|A\Delta(B + x)|,\ x \in \mathbb{R}^N\}$$

the Fraenkel asymmetry of a set $A \subset \mathbb{R}^N$ of unit volume, then there exists $c_N > 0$ such that

$$P(A) - P(B) \geq c_N \delta_F(A)^2$$

In our stability setting this can be rephrased into stability of $P - \delta_F^2$ among all sets of unit volume. The literature on quantitative inequalities in shape optimization is very prolific, and we refer for instance among many others to [FMP08], [BDPV15], [FFM+15], [AFM13], [CL12], [BNT10], [FMP10] and to [Fus15] for a nice review of stability results linked to the isoperimetric inequality.

3.1.2 Stability of the ball under convexity constraint

We are more specifically interested in shape optimization problems where $\mathcal{A}$ only contains convex shapes, that is

$$\inf\left\{P(K) + \varepsilon R(K),\ K \in \mathcal{K}^N,\ |K| = 1\right\}$$

where $\mathcal{K}^N$ denotes the class of convex bodies of $\mathbb{R}^N$ (that is, compact convex sets with non-empty interior). The addition of the convexity constraint is interesting since stability among all shapes fails for the functionals we will consider. This happens for some problems where R is of PDE-type, by which we mean that $R(K)$ is an energy associated to a PDE which is set on K or $\mathbb{R}^N \setminus K$. In this paper we investigate independently two specific problems falling into this category. Let us now introduce them and state the stability results associated.

Weak stability for $P + \mathbf{Cap}^{-1}$

We are interested in a first problem which involves a PDE set on the exterior of the domain. For $N \geq 3$ we introduce the capacity functional $\mathrm{Cap} : \mathcal{K}^N \to \mathbb{R}$ which we define as the usual *electrostatic capacity*:

$$\mathrm{Cap}(K) := \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2,\ u \in C_c^\infty(\mathbb{R}^N),\ u \geq 1 \text{ over } K\right\} \tag{3.1}$$

When $N = 2$, one can see by looking at the energy of the fundamental solution of the Laplacian that the infimum in (3.1) is always 0. Therefore one must proceed differently to define the capacity for $N = 2$ (see Section 3.2).

We now set the problem: letting $\varepsilon > 0$ be a small parameter, we are interested in the minimization

$$\inf\left\{P(K) + \varepsilon \mathrm{Cap}(K)^{-1},\ K \in \mathcal{K}^N,\ |K| = 1\right\} \tag{3.2}$$

for $N \geq 3$. Before giving motivations and context for this problem let us state the stability result which we obtained.

Theorem 3.1.1 (Weak stability of the ball for the capacity)**.** *Let $N \geq 3$. There exists $\varepsilon_0 = \varepsilon_0(N) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the centered ball of unit volume B is the unique (up to translation) minimizer of* (3.2)*.*

We call this result *weak* stability in the sense that the ε_0 found is not optimal (contrarily to Theorem 3.1.2 below), and is in fact not even explicit. Note that it is the $N \geq 3$ version of the two dimensional result [GNR18, Corollary 1.3], where the authors prove weak stability of the ball with the logarithmic capacity instead. Our approach is however very different, see Section 3.1.3.

Due to the *isocapacitary inequality* (see for instance [DPMM21]) which states that

$$\forall \Omega \subset \mathbb{R}^N \text{ open}, \ |\Omega| = 1, \ \mathrm{Cap}(\Omega) \geq \mathrm{Cap}(B)$$

and the *isoperimetric inequality*, there is a competition in the minimization (3.2) which makes the problem non trivial. The introduction of the convexity constraint in the problem comes from the fact that existence does not hold without any additional geometric assumption (for non-existence for any $\varepsilon > 0$ and in all dimensions $N \geq 2$ see [GNR15, Theorem 3.2 and Theorem 6.2]). On the other hand (3.2) admits minimizers for any $\varepsilon > 0$ (see [GNR18, Theorem 1.1]).

Strong stability for $P - \lambda_1$

If $\Omega \subset \mathbb{R}^N$ is an open set with finite volume we let $\lambda_1(\Omega)$ be its first Dirichlet eigenvalue, defined as the smallest number $\lambda \in \mathbb{R}$ such that there exists a non trivial function u verifying

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u \in H^1_0(\Omega) \end{cases}$$

where the first equation holds in the distributional sense in Ω. It has a variational characterization:

$$\lambda_1(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}, \ u \in H^1_0(\Omega) \right\}.$$

For $K \in \mathcal{K}^N$ we set $\lambda_1(K) := \lambda_1(\mathrm{Int}(K))$. Consider then the minimization problems

$$\inf \left\{ P(\Omega) - c\lambda_1(\Omega), \ \Omega \subset \mathbb{R}^N \text{ open}, |\Omega| = 1 \right\}, \quad \inf \left\{ P(K) - c\lambda_1(K), \ K \in \mathcal{K}^N, \ |K| = 1 \right\}$$

for any fixed parameter $c > 0$. There is a competition between the perimeter and λ_1, as it is known from the *isoperimetric* and *Faber-Krahn inequalities* that a ball B of volume 1 minimizes them both among shapes of unit volume. Intuitively, we expect that the perimeter is the dominant term for small values of c while we expect that this is no longer the case in the regime $c \to +\infty$, so that B might be a local minimizer for small values of c and not for large values of c. As such there is no global minimizer to any of the two problems, taking for instance a sequence of long thin rectangles of unit volume. However, even in a loose local sense there is no stability of the ball for the first problem, meaning that for any $c > 0$ there exists a sequence $(\Omega_{j,c})_{j \in \mathbb{N}}$ of open sets with

$$|\Omega_{j,c}| = 1, \ |\Omega_{j,c} \Delta B| \to 0 \text{ and } (P - c\lambda_1)(\Omega_{j,c}) < (P - c\lambda_1)(B) \text{ for each } j \in \mathbb{N}$$

as one sees by comparing the energy of the ball to the energy of the ball perforated by a small hole at its center (see for instance [DL19, Proposition 6.1]). A strong geometric constraint such as convexity of the admissible sets forbids this kind of behaviour, so that one might expect stability in this case. This is the object of the second main result of this article, which can be seen as sharp stability of the ball under convexity constraint for the functional $P - \lambda_1$. The result is as follows.

Theorem 3.1.2 (Sharp stability of the ball for λ_1)**.** *Let $N \geq 2$. Let ω_N be the volume of a ball of radius 1, and $p_N := N\omega_N$, $l_N := j_{N/2-1}^2$ be respectively the perimeter and first eigenvalue of a ball of radius 1 ($j_{N/2-1}$ is the first zero of the Bessel function of the first kind of order $N/2-1$). Set*

$$c^* := \frac{N(N+1)p_N}{4l_N(l_N - N)\omega_N^{\frac{N+1}{N}}} \tag{3.3}$$

Let B be a ball of unit volume.

- *Let $0 < c < c^*$. Then there exists $\delta_c > 0$ such that*

$$\forall K \in \mathcal{K}^N, |K| = 1 \text{ with } |K\Delta B| \leq \delta_c,\ (P - c\lambda_1)(K) \geq (P - c\lambda_1)(B). \tag{3.4}$$

- *Let $c > c^*$. There exists a sequence of smooth convex bodies $(K_{j,c})_{j\in\mathbb{N}}$ of unit volume for which $|K_{j,c}\Delta B| \to 0$ and*

$$(P - c\lambda_1)(K_{j,c}) < (P - c\lambda_1)(B) \text{ for each } j \in \mathbb{N}. \tag{3.5}$$

Note that the novelty of this result comes from the first item (inequality (3.4)), as (3.5) was already obtained by [Nit14] (see the second point below). We thus give an answer to the question of local minimality of the ball for the problem $P - c\lambda_1$ under convexity constraint for any value $c > 0$ (except $c = c^*$). Let us place it among existing results in the literature.

- First, in a weak form the stability of the ball for $P - \lambda_1$ (in the sense of (3.4)) was already known. It was first obtained in two dimensions by Payne and Weinberger in [PW61] for the larger class of simply connected domains. More precisely, the Payne-Weinberger inequality states that for any $\Omega \subset \mathbb{R}^2$ open, simply connected with unit volume it holds

$$\lambda_1(\Omega) - \lambda_1(B) \leq \lambda_1(B)\left(J_1(j_{01})^{-2} - 1\right)\left(\frac{P(\Omega)^2}{4\pi|\Omega|} - 1\right) \tag{3.6}$$

  where J_1 is the Bessel function of the first kind of order one, and j_{01} is the first zero of the Bessel function of the first kind and of order zero. While this inequality is much more general since it gives a control of the Faber-Krahn deficit by the isoperimetric deficit for any simple connected set, it implies in particular stability of the ball among simply connected sets Ω for which $P(\Omega)$ is bounded from above. One can in fact derive the inequality

$$\forall \Omega \subset \mathbb{R}^2 \text{ open and simply connected with } |\Omega| = 1,\ (P(\Omega) - P(B)) \geq \frac{\varepsilon_2}{P(B) + C}\left(\lambda_1(\Omega) - \lambda_1(B)\right)$$

  where $\varepsilon_2 := 4\pi\lambda_1(B)^{-1}(J_1(j_{01})^{-2} - 1)^{-1}$, provided $P(\Omega) \leq C$. Note that the constant $\varepsilon_2/(P(B) + C)$ becomes $\varepsilon_2(2P(B))^{-1} = \left(\sqrt{\pi}j_{01}^2(J_1(j_{01})^{-2} - 1)\right)^{-1} \approx 0.036$ as $C \to P(B)$, while the optimal constant given by (3.3) equals $c^* = \frac{3}{\sqrt{\pi}j_{01}^2(j_{01}^2-2)} \approx 0.077$.

  On the other hand, a Payne-Weinberger type inequality for convex sets was proven in any dimensions $N \geq 2$ by Brandolini, Nitsch and Trombetti [BNT10, Theorem 1.1] using the Brunn-Minkowski theory. They prove that for any open convex set of unit volume $\Omega \subset \mathbb{R}^N$ it holds

$$\frac{\lambda_1(\Omega) - \lambda_1(B^*)}{\lambda_1(\Omega)} \leq C_N\left(\frac{P(\Omega)^{\frac{N}{N-1}} - P(B)^{\frac{N}{N-1}}}{P(\Omega)^{\frac{N}{N-1}}}\right)$$

  for some explicit constant $C_N > 0$, where B^* is a ball with same perimeter as Ω. This again implies a non-optimal local stability of the ball for convex sets in the sense given by (3.4).

- Second, in [Nit14, Theorem 1.2] (see also [DL19, Proposition 5.5 (ii)]) Nitsch conjectured the optimal value $c^* > 0$ by proving that (i) if $c < c^*$, the ball B is a minimizer of $P - c\lambda_1$ among unit volume perturbations of the ball by a smooth vector field ξ *i.e.* for $B_\xi := (\mathrm{Id} + \xi)(B)$ with $\|\xi\|_{C^\infty} \ll 1$ and (ii) if $c > c^*$, there exists $\|\xi_j\|_{C^\infty} \to 0$ with $(P - c\lambda_1)(B_{\xi_j}) < (P - c\lambda_1)(B)$.

- Finally, one can interpret this result in the context of Blaschke-Santalo diagrams. In fact, it provides the exact value of the tangent at $(x_0, y_0) := (P(B), \lambda_1(B))$ of the upper boundary of the diagram for $(P, \lambda_1, |\cdot|)$ in the class of planar convex sets, that is of the set

$$\mathcal{D} := \left\{(x, y) \in \mathbb{R}^2,\ \exists K \in \mathcal{K}^2,\ P(K) = x,\ \lambda_1(K) = y,\ |K| = 1\right\} \tag{3.7}$$

  It was proven in [FL21] that this diagram lies between two continuous increasing functions, meaning that

$$\mathcal{D} = \left\{(x, y) \in [x_0, +\infty[\times\mathbb{R}^+,\ f(x) \le y \le g(x)\right\}$$

  for some continuous increasing $f, g : [x_0, +\infty) \to \mathbb{R}^+$. Relying on non minimality of the ball for $c > c^*$, the authors proved that $\limsup_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \ge \frac{1}{c^*}$ (see [FL21, Corollary 3.17]). On the other hand, minimality for any $c < c^*$ from Theorem 3.1.2 ensures that the reverse inequality holds, so that the function g admits a tangent at (x_0, y_0) with coefficient $(c^*)^{-1}$. The precise result is thus the following.

  Corollary 3.1.3. *Let $B \subset \mathbb{R}^2$ be a ball of unit volume and set $(x_0, y_0) := (P(B), \lambda_1(B))$. Let c^* be given by (3.3). Then the upper boundary of the diagram (3.7) admits a tangent at (x_0, y_0) with coefficient $(c^*)^{-1}$, i.e. $g'(x_0) = (c^*)^{-1}$.*

Although the convexity constraint is a natural class for proving this strong form of stability of the ball, our result opens up the question as to finding the more general class for which this could hold. Since a weak form of stability holds in two dimensions for perturbations of the ball which are simply connected (by the Payne-Weinberger inequality), we believe that it would be interesting to investigate whether the sharp stability we obtained could be extended to this class. As shown in [DL19, Remark 6.2], one cannot hope for such a general class when $N \ge 3$, but one can however make the same conjecture for the class of Lipschitz perturbations of the ball (see also further (ii) from Proposition 3.2.5).

3.1.3 Strategy of proof

Although the two results are independent, the strategy we employ for proving them follows a general scheme, which is recurrent in the literature and not specific to convexity (see [DL19] for a detailed description). We rely on the following steps:

1. *Fuglede-type computations:* minimality of the ball for the functional among "smooth" perturbations of the ball (Theorems 3.2.1 and 3.3.1).

2. Local minimality of the ball for convex sets (Theorems 3.1.1 and 3.1.2).

The first step of this strategy refers to the seminal work [Fug89] of B. Fuglede, where the author obtained it for the perimeter functional. By "smooth" perturbations of the ball B in the first step we mean that minimality holds for domains $\Omega = (\mathrm{Id} + \xi)(B)$ with ξ lying in some normed space of smoothness X and $\|\xi\|_X$ is small enough. Since they are independent of convexity, the respective results Theorems 3.2.1 and 3.3.1 constituting the first step bear interest in themselves. On the other hand, in the second step one studies the regularity of minimizers of the associated problem, aiming to prove that each minimizer Ω can be written $\Omega = (\mathrm{Id} + \xi)(B)$ with $\xi \in X$ and X is the space obtained in the first step. The two parts of the strategy are thus closely linked to each other through the choice of the space X.

Let us now explain separately how we proceed for proving Theorems 3.1.2 and 3.1.1.

Weak stability with Lipschitz regularity

For proving Theorem 3.1.1 we first perform *Fuglede-type computations* for Lipschitz perturbations of the ball. This is done in Theorem 3.2.1. It is an improvement of previous results from [GNR15, Corollary 5.6], where the authors perform *Fuglede-type computations* for a class of $C^{1,1}$ sets with curvature uniformly bounded from above. The proof of Theorem 3.2.1 relies on a second-order estimate of the variation of the capacity for Lipschitz perturbations of the ball, shown in Lemma 3.2.2. To obtain this latter bound we take advantage of the fact that the capacity is defined as a minimum, thus enabling us to estimate it from above by providing a natural competitor, for which only low regularity is needed.

Theorem 3.1.1 is then obtained from Theorem 3.2.1 by using Lipschitz regularity of convex sets as in [Fug89]. In reference to the two steps strategy described above, note that in this case the passage from the *Fuglede-type computations* to local minimality of the ball in the class of convex sets is quite direct, due to the fact that we are able to perform these computations for a space X with low regularity.

Strong stability with a $C^{1,\alpha}$ regularity theory

Since it was proven in [Nit14, Theorem 1.2] that minimality in (3.4) holds among smooth perturbations of the ball for any $c \in (0, c^*)$, the idea of Theorem 3.1.2 is to pass from smooth to non-smooth convex perturbations of the ball in the minimality. The strategy we will use is the so-called *selection principle*, which was first introduced by Cicalese and Leonardi in [CL12] as a means to give a new proof to the sharp quantitative isoperimetric inequality. The robustness of their method allowed it to be employed in many other contexts for proving various inequalities for shapes, among which we can quote the *sharp quantitative Faber-Krahn inequality* proven in [BDPV15]. The strategy consists in a refinement of the two steps method described above. Roughly speaking, if one wants to prove local minimality of the ball of unit volume B for a functional $\mathcal{J}$ among a class $\mathcal{A}$ of shapes, the idea is to reduce the proof of the inequality in $\mathcal{A}$ to the inequality in a class of smooth shapes through a regularizing procedure. This is usually based on a regularity theory related to the functional $\mathcal{J}$ under study.

In order to apply this *selection principle* method we first need to prove *Fuglede-type computations* for the functional $P - c\lambda_1$ for $C^{1,\alpha}$ perturbations of the ball (in Theorem 3.3.1), which are finer than all the available results in the literature (in [Nit14, DL19, Dam02, DP00]; see Section 3.3 for a justification of these refinements). On the other hand, to perform the second step of the strategy we prove a convergence result for quasi-minimizers of the perimeter under convexity constraint (Corollary 3.4.3), which uses the regularity theory from [LP23].

Note that the proof of Theorem 3.1.2 is much more involved than the proof of Theorem 3.1.1. This is related to the fact that it relies on a regularity theory among convex shapes, but is also because in order to prove the *Fuglede-type computations* we are led to perform very technical computations (see the proof of Theorem 3.3.5).

3.1.4 Plan of the paper

Section 3.2 is dedicated to the proof of Theorem 3.1.1. Sections 3.3 and 3.4 are independent of this first section, and deal with proving Theorem 3.1.2: in Section 3.3 we show the first step of the *selection principle* method by proving minimality of the ball for $P - c\lambda_1$ in a $C^{1,\alpha}$ neighborhood; then, in Section 3.4 we perform the regularizing procedure in itself. We provide a small appendix in Section 3.5.

Acknowledgements: The author is deeply grateful to J. Lamboley for very helpful discussions and careful readings of previous versions of this document. The author also thanks M. Goldman, M. Novaga and B. Ruffini for valuable discussions about this work. The author warmly thanks R. Petit for interesting discussions about convergence of smooth sets. This work was partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR).

3.2 Stability of the ball for an isoperimetric problem with convexity constraint involving capacity. Proof of Theorem 3.1.1

In this section we prove Theorem 3.1.1, which is the stability result associated to (3.2). We defined in (3.1) the capacity functional in dimensions $N \geq 3$. One also has

$$\mathrm{Cap}(K) = \min\left\{\int_{\mathbb{R}^N} |\nabla u|^2,\ u \in \overline{C_c^\infty(\mathbb{R}^N)}^{|\cdot|_{H^1}},\ u \geq 1 \text{ over } K\right\} \tag{3.8}$$

where $\overline{C_c^\infty(\mathbb{R}^N))}^{|\cdot|_{H^1}}$ denotes the closure of $C_c^\infty(\mathbb{R}^N)$ inside $H^1(\mathbb{R}^N)$ for the H^1 semi-norm $|u|_{H^1} := \|\nabla u\|^2_{L^2(\mathbb{R}^N)}$. It is proven in [CS03, Theorem 2 and 3] that provided K is sufficiently smooth, this minimization is uniquely solved by the so-called *capacitary function* u_K which verifies

$$\begin{cases} -\Delta u_K = 0 \text{ in } \mathbb{R}^N \setminus K \\ u_K = 1 \text{ over } K,\ u_K \in C^0(\mathbb{R}^N) \\ u_K(x) \to 0 \text{ as } |x| \to +\infty \end{cases} \tag{3.9}$$

The number $\mathrm{Cap}(K)$ also appears in the asymptotic expansion of u_K:

$$u_K(x) \sim |x|^{2-N}\left(\mathrm{Cap}(K)\sigma_N(N-2)^{-(N-1)/(N-2)}\right) \quad \text{as } |x| \to +\infty$$

with σ_N denoting the $(N-1)$-dimensional measure of the unit sphere in $\mathbb{R}^N$. Finally, let us also note that $\mathrm{Cap}(K) = (I_{N-2}(K))^{-1}$ where I_{N-2} is the Riesz potential energy which is given by

$$I_{N-2}(K) := \inf\left\{\iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^{2-N} d\mu(x)d\mu(y),\ \mu \in \mathcal{P}(K)\right\}$$

with $\mathcal{P}(K)$ denoting the set of probabilities supported on K (see [GNR15, Remark 2.5]).

When $N = 2$, one can see by looking at the energy of the fundamental solution of the Laplacian that the infimum in (3.1) is always 0. Therefore one must proceed differently to define the capacity for $N = 2$. We first set the Robin constant V_K of some $K \in \mathcal{K}^2$, which can be defined in three equivalent ways (see [Bag67, Theorem 4] and section 3 in [GNR18]): first, letting

$$\mathrm{Cap}_R(K) := \inf\left\{\int_{\mathbb{R}^2} |\nabla u|^2,\ u \in H_0^1(B_R(0)),\ u \geq 1 \text{ over } K\right\} \tag{3.10}$$

then we set $V_K := \lim\left(2\pi\mathrm{Cap}_R(K)^{-1} - \log(R)\right)$ as $R \to \infty$. Equivalently if $K \in \mathcal{K}^2$ there is a unique function u_K such that

$$\begin{cases} -\Delta u_K = 0 \text{ in } \mathbb{R}^2 \setminus K \\ u_K = 0 \text{ over } \partial K \\ z \mapsto (u_K(z) - \log|z|) \text{ admits a finite limit } \alpha \text{ as } |z| \to +\infty \end{cases}$$

and $\alpha = V_K$. Finally, one also has

$$V_K := \inf\left\{-\iint_{K\times K} \log(|x-y|) d\mu(x)d\mu(y),\ \mu \in \mathcal{P}(K)\right\}.$$

Then the *logarithmic capacity* is given by

$$\mathrm{Cap}(K) := e^{-V_K}$$

In two dimensions, instead of (3.2) we set the minimization problem

$$\inf\left\{P(K)-\varepsilon\log\left(\mathrm{Cap}(K)\right),\ K\in\mathcal{K}^N,\ |K|=1\right\} \tag{3.11}$$

where $\varepsilon>0$ is a small parameter. Stability of the ball among convex sets in two dimensions was obtained in [GNR18, Corollary 1.3] using the two steps strategy we described in the Introduction, by proving (i) *Fugldede-type computations* in a certain class of "smooth" perturbations and (ii) regularity of minimizers of (3.11). Theorem 3.1.1 is the $N\geq 3$ version of this result. Instead, here we prove (i) for Lipschitz perturbations (in Theorem 3.2.1 below), which will be enough in order to obtain local minimality of the ball for convex sets without having to prove regularity of the minimizers. The result is as follows.

Theorem 3.2.1 (Fuglede-type computations for $P+\varepsilon\mathrm{Cap}^{-1}$: minimality for Lipschitz perturbations)**.** *Let $N\geq 3$, and let B denote the centered unit ball. For $h\in W^{1,\infty}(\partial B)$ we denote $B_h:=\{tx(1+h(x)),\ t\in[0,1),\ x\in\partial B\}$. There exists $\eta>0$ and $\varepsilon_0>0$ such that for all $h\in W^{1,\infty}(\partial B)$ verifying $\|h\|_{W^{1,\infty}(\partial B)}\leq\eta$ with $|B_h|=|B|$ and such that B_h has barycenter at the origin, and for all $\varepsilon\in(0,\varepsilon_0)$, then*

$$P(B_h)+\varepsilon\mathrm{Cap}(B_h)^{-1}\geq P(B)+\varepsilon\mathrm{Cap}(B)^{-1}$$

with equality only if $B_h=B$.

The proof of Theorem 3.2.1 importantly relies on Lemma 3.2.2 below, which consists in a weak **(IT)** property (see the statement of Theorem 3.3.4 for a strong **(IT)** property).

3.2.1 Weak (IT) property

For $h\in W^{1,\infty}(\partial B)$ with $\|h\|_{L^\infty(\partial B)}\leq 1/2$ the notation B_h denotes the Lipschitz open set

$$B_h:=\{tx(1+h(x)),\ t\in[0,1),\ x\in\partial B\}$$

In the following Lemma we estimate from above the variation of Cap for a Lipschitz perturbation B_h of B in terms of the H^1 norm of h. Since the Lemma does not use the convexity of the sets B_h it is stated for general $h\in W^{1,\infty}(\partial B)$.

Lemma 3.2.2 (Weak **(IT)**$_{H^1,W^{1,\infty}}$)**.** *Let $N\geq 3$. There exists $C_N>0$ such that if $h\in W^{1,\infty}(\partial B)$ with B_h of volume $|B_h|=|B|$ and $\|h\|_{L^\infty(\partial B)}\leq 1/2$ then*

$$\mathrm{Cap}(B_h)-\mathrm{Cap}(B)\leq C_N\|h\|^2_{H^1(\partial B)}$$

Proof of Lemma 3.2.2. Fix $h\in W^{1,\infty}(\partial B)$ with $\|h\|_{L^\infty(\partial B)}\leq 1/2$ and $|B_h|=|B|$. We extend h over $\mathbb{R}^N\setminus\{0\}$ by setting $h(x):=h(x/|x|)$, thus getting a 0-homogenous function $h:\mathbb{R}^N\to\mathbb{R}^N$. Let then ϕ_h be the Lipschitz homeomorphism

$$\begin{aligned}\phi_h:\quad \mathbb{R}^N &\longrightarrow \mathbb{R}^N\\ x &\longmapsto \frac{x}{(1+h(x))}\end{aligned} \tag{3.12}$$

Let us denote more simply $\widetilde{h}:=(1+h)^{-1}$, so that $\phi_h(x)=\widetilde{h}(x)x$. We have $D\phi_h(x)=\widetilde{h}(x)\mathrm{Id}+x\otimes\nabla\widetilde{h}(x)$, so that using the formula $\det(\mathrm{Id}+a\otimes b)=1+a\cdot b$ for vectors a,b and since $\nabla\widetilde{h}(x)\cdot x=0$ it holds $\det(D\phi_h)=\widetilde{h}^N$. As a consequence, thanks to the change of variable $y=\phi_h(x)$ and using polar coordinates, the hypothesis $|B_h|=|B|$ reads

$$\int_{\partial B}(1+h)^N=\int_{\partial B}1 \tag{3.13}$$

Expanding $(1+h)^N - 1 = Nh + \sum_{i=2}^N \binom{N}{i} h^i$ we thus get that there exists $C_N > 0$ such that

$$\text{if } \|h\|_{L^\infty(\partial B)} \le 1/2, \ \int_{\partial B} h \le C_N \|h\|^2_{L^2(\partial B)} \tag{3.14}$$

We set $u_B(x) := \min\{1, |x|^{2-N}\}$ the capacitary function for B, which is the unique solution to (3.9) for $K = B$.

Step 1[1]: $\mathrm{Cap}(B_h) \le \int_{\mathbb{R}^N} |\nabla(u_B \circ \phi_h)|^2$. For $R \ge 1$ let $\theta_R : \mathbb{R}^N \to \mathbb{R}$ be some cut-off function such that

$$\begin{cases} \theta_R \equiv 1 \text{ on } B_R(0), \ \theta_R \equiv 0 \text{ on } \mathbb{R}^N \setminus B_{2R}(0) \\ \|\theta_R\|_{L^\infty(\mathbb{R}^N)} \le 1, \ \|\nabla \theta_R\|_{L^\infty(\mathbb{R}^N)} \le \frac{1}{R} \end{cases}$$

Then $v_R(x) := \theta_R(x)(u_B \circ \phi_h)(x) \in W^{1,\infty}(\mathbb{R}^N)$ and has compact support, so that by standard mollification we have $v_R \in \overline{C_c^\infty(\mathbb{R}^N)}^{|\cdot|_{H^1}}$. As a consequence by (3.8) we get

$$\mathrm{Cap}(B_h) \le \int_{\mathbb{R}^N} |\nabla v_R|^2 \tag{3.15}$$

We now verify that $\nabla v_R \to \nabla(u_B \circ \phi_h)$ in $L^2(\mathbb{R}^N)$ as $R \to +\infty$.

We have

$$\begin{aligned} \|\nabla v_R - \nabla(u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N)} &= \|\nabla v_R - \nabla(u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N \setminus B_R)} \\ &\le \|\nabla(u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N \setminus B_R)} + \|\theta_R \cdot \nabla(u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N \setminus B_R)} \\ &\quad + \|\nabla \theta_R \cdot (u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N \setminus B_R)} \qquad (3.16) \\ &\le 2\|\nabla(u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N \setminus B_R)} + \|\nabla \theta_R \cdot (u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N \setminus B_R)} \qquad (3.17) \end{aligned}$$

Since $\nabla(u_B \circ \phi_h) = (D\phi_h)^T \nabla u_B \circ \phi_h$ with $\phi_h \in W^{1,\infty}(\mathbb{R}^N)$ and $\nabla u_B \in L^2(\mathbb{R}^N)$, it holds

$$\int_{\mathbb{R}^N} |\nabla(u_B \circ \phi_h)|^2 \le \|D\phi_h\|^2_{L^\infty(\mathbb{R}^N)} \|\det(D\phi_h)\|^{-1}_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla u_B|^2 < \infty$$

Hence $\|\nabla(u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N \setminus B_R)} \to 0$. On the other hand, as $\|1+h\|_{L^\infty(\mathbb{R}^N)} \le 3/2$ we have for $|x| \ge 3/2$,

$$|u_B \circ \phi_h(x)| = |\phi_h(x)| \le \left(\frac{3}{2}\right)^{N-2} |x|^{2-N}$$

Since $|\nabla \theta_R| \le R^{-1}$, using polar coordinates this gives for $R \ge 3/2$

$$\begin{aligned} \int_{B_{2R} \setminus B_R} |\nabla \theta_R \cdot (u_B \circ \phi_h)|^2 &\le P(B) \left(\frac{3}{2}\right)^{N-2} R^{-2} \int_R^{2R} r^{2(2-N)} r^{N-1} dr \\ &= P(B) \left(\frac{3}{2}\right)^{N-2} \begin{cases} (4-N)^{-1} R^{-2}((2R)^{4-N} - R^{4-N}) \text{ if } N \ne 4 \\ R^{-2} \log(2) \text{ if } N = 4 \end{cases} \end{aligned}$$

In any case we thus get

$$\int_{B_{2R} \setminus B_R} |\nabla \theta_R \cdot (u_B \circ \phi_h)|^2 \to 0 \text{ as } R \to +\infty.$$

Pluging this into (3.17) we deduce in fact

$$\|\nabla v_R - \nabla(u_B \circ \phi_h)\|_{L^2(\mathbb{R}^N)} \to 0, \text{ as } R \to +\infty$$

[1]This argument for admissibility of $u_B \circ \phi_h$ was suggested to us by M. Goldman, M. Novaga and B. Ruffini.

so that

$$\mathrm{Cap}(B_h) \leq \int_{\mathbb{R}^N} |\nabla(u_B \circ \phi_h)|^2$$

thanks to (3.15).

Step 2: Estimate of the energy of $u_B \circ \phi_h$. Recall that

$$\forall x \in \mathbb{R}^N,\ D\phi_h(x) = \widetilde{h}(x)\mathrm{Id} + x \otimes \nabla\widetilde{h}(x) \tag{3.18}$$

so that

$$\begin{aligned}|\nabla(u_B \circ \phi_h)|^2 &= \left|{}^t(D\phi_h) \cdot \nabla u_B \circ \phi_h\right|^2 \\ &= \left|\widetilde{h} \cdot (\nabla u_B \circ \phi_h) + \nabla\widetilde{h}\left((\nabla u_B \circ \phi_h) \cdot x\right)\right|^2\end{aligned}$$

Since u_B is radial, the vector $\nabla u_B(\phi_h(x))$ is thus proportionnal to $\phi_h(x)$ and hence to x, which yields $\nabla u_B(\phi_h(x)) \cdot \nabla\widetilde{h}(x) = 0$ (because $\widetilde{h}$ is constant in the direction x). We deduce

$$|\nabla(u_B \circ \phi_h)|^2 = \widetilde{h}^2|\nabla u_B \circ \phi_h|^2 + ((\nabla u_B \circ \phi_h) \cdot x)^2|\nabla\widetilde{h}|^2$$

We can therefore write the energy of $u_B \circ \phi_h$ as follows

$$\int_{\mathbb{R}^N} |\nabla(u_B \circ \phi_h)|^2 = \int_{\mathbb{R}^N} \widetilde{h}^2|\nabla u_B \circ \phi_h|^2 + \int_{\mathbb{R}^N} ((\nabla u_B \circ \phi_h) \cdot x)^2|\nabla\widetilde{h}|^2 \tag{3.19}$$

Let us first deal with the term $\int_{\mathbb{R}^N} \widetilde{h}^2|\nabla u_B(\phi_h)|^2$. Recalling that $\det(D\phi_h) = \widetilde{h}^N$ we get by the change of variable $y = \phi_h(x)$

$$\begin{aligned}\int_{\mathbb{R}^N} \widetilde{h}^2|\nabla u_B \circ \phi_h|^2 &= \int_{\mathbb{R}^N} (1+h)^{-2}|\nabla u_B \circ \phi_h|^2 \\ &= \int_{\mathbb{R}^N} (1+h)^{N-2}|\nabla u_B|^2 \\ &= \int_{\mathbb{R}^N} |\nabla u_B|^2 + \sum_{i=1}^{N-2} \binom{N-2}{i} \int_{\mathbb{R}^N} h^i|\nabla u_B|^2 \\ &= \mathrm{Cap}(B) + \sum_{i=1}^{N-2} \binom{N-2}{i} \int_{\mathbb{R}^N} h^i|\nabla u_B|^2\end{aligned}$$

Now using polar coordinates we have for $i \geq 1$

$$\begin{aligned}\int_{\mathbb{R}^N} h^i|\nabla u_B|^2 &= \int_0^\infty dr \int_{|\theta|=r} h^i(\theta)v'(r)^2 d\theta \\ &= \left(\int_0^\infty v'(r)^2 r^{N-1} dr\right)\left(\int_{\partial B} h^i(\theta)d\theta\right) \\ &= a_N \int_{\partial B} h^i\end{aligned}$$

where we set $a_N := \int_0^\infty v'(r)^2 r^{N-1} dr = P(B)^{-1}\int_{\mathbb{R}^N} |\nabla u_B|^2$. For $i \geq 2$, since $\|h\|_{L^\infty(\partial B)} \leq 1/2$ we thus get

$$\int_{\mathbb{R}^N} h^i|\nabla u_B|^2 \leq C_N\|h\|^2_{L^2(\partial B)},$$

while if $i = 1$ thanks to (3.14) we have $\int_{\partial B} h \leq C_N\|h\|^2_{L^2(\partial B)}$ so that

$$\int_{\mathbb{R}^N} h^i|\nabla u_B|^2 \leq C'_N\|h\|^2_{L^2(\partial B)}.$$

These two give

$$\int_{\mathbb{R}^N} \widetilde{h}^2 |\nabla u_B(\phi_h)|^2 \leq \mathrm{Cap}(B) + C_N \|h\|^2_{L^2(\partial B)} \tag{3.20}$$

for some dimensional constant $C_N > 0$.

We now turn to the estimate of $\int_{\mathbb{R}^N} ((\nabla u_B \circ \phi_h) \cdot x)^2 |\nabla \widetilde{h}|^2$. Denoting by $a(x) := x/|x|$, then for $x \neq 0$

$$\nabla \widetilde{h}(x) = -\widetilde{h}^2 Da(x)^T \nabla_\tau h(a(x))$$

where ∇_τ is the tangential gradient. As each coefficient of $Da(x)$ is controlled by $2|x|^{-1}$, since $\|\widetilde{h}\|_{L^\infty(\partial B)} \leq 2$ this yields

$$((\nabla u_B \circ \phi_h) \cdot x)^2 |\nabla \widetilde{h}(x)|^2 \leq 8 |\nabla u_B \circ \phi_h|^2 |\nabla_\tau h(a(x))|^2$$

Changing variables and using polar coordinates in the same fashion as before, this ensures that

$$\int_{\mathbb{R}^N} ((\nabla u_B \circ \phi_h) \cdot x)^2 |\nabla \widetilde{h}|^2 \leq C_N \|\nabla_\tau h\|^2_{L^2(\partial B)} \tag{3.21}$$

for some $C_N > 0$. Injecting this estimate together with (3.20) into (3.19) finally yields

$$\int_{\mathbb{R}^N} |\nabla (u_B \circ \phi_h)|^2 \leq \mathrm{Cap}(B) + C_N \|h\|^2_{H^1(\partial B)}$$

for some $C_N > 0$.

Conclusion: Thanks to Step 1 it holds

$$\mathrm{Cap}(B_h) \leq \int_{\mathbb{R}^N} |\nabla (u_B \circ \phi_h)|^2$$

from which we get using Step 2

$$\mathrm{Cap}(B_h) \leq \mathrm{Cap}(B) + C_N \|h\|^2_{H^1(\partial B)}$$

This finishes the proof of the Lemma. □

Remark 3.2.3. We tried to apply the same strategy in the case $N = 2$ in order to retrieve the result from [GNR18]. However, due to the specificity of the definition of capacity in the two dimensional case, this brings additional difficulties and we do not know whether the argument could work.

3.2.2 Proof of Theorems 3.2.1 and 3.1.1

Relying on Lemma 3.2.2, we first prove the *Fuglede-type computations* for Lipschitz perturbations from Theorem 3.2.1.

Proof of Theorem 3.2.1. Let $h \in W^{1,\infty}(\partial B)$ with $\|h\|_{L^\infty(\partial B)} \leq 1/2$, such that $|B_h| = |B|$ and B_h has barycenter at the origin. It is proven in [Fus15, Theorem 3.1] that there exists $\eta > 0$ such that

$$\text{if } \|h\|_{W^{1,\infty}(\partial B)} \leq \eta, \ P(B_h) - P(B) \geq \frac{1}{4} \|\nabla_\tau h\|^2_{L^2(\partial B)}$$

Thanks to (3.13) one has the Poincaré type inequality (see also [Fus15, Proof of Theorem 3.1])

$$\exists \delta > 0, \text{ if } \|h\|_{L^\infty(\partial B)} \leq \delta, \ \|h\|^2_{L^2(\partial B)} \leq 2 \|\nabla_\tau h\|^2_{L^2(\partial B)}$$

Setting $\widetilde{\eta} := \min\{\eta, \delta\}$, we thus deduce thanks to Lemma 3.2.2 that if $\|h\|_{W^{1,\infty}(\partial B)} \leq \widetilde{\eta}$ then

$$\begin{aligned}
\mathrm{Cap}(B)^{-1} - \mathrm{Cap}(B_h)^{-1} = \frac{\mathrm{Cap}(B_h) - \mathrm{Cap}(B)}{\mathrm{Cap}(B)\mathrm{Cap}(B_h)} &\leq \frac{C_N}{\mathrm{Cap}(B)^2}\|h\|^2_{H^1(\partial B)} \\
&\leq \frac{3C_N}{\mathrm{Cap}(B)^2}\|\nabla_\tau h\|^2_{L^2(\partial B)} \\
&\leq \frac{12C_N}{\mathrm{Cap}(B)^2}\left(P(B_h) - P(B)\right)
\end{aligned}$$

where we also used the isocapacitary inequality in the first line. Taking $\varepsilon_0 := \frac{\mathrm{Cap}(B)^2}{12C_N}$ we get the conclusion of the Theorem (note also that by following the chain of inequalities one must have a strict inequality whenever $h \neq 0$). □

We are now ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let us first note that (3.2) admits a solution for any $\varepsilon > 0$ (it was proven in [GNR18, Theorem 1.1]).

Due to convexity, any $K \in \mathcal{K}^N$ with barycenter at the origin can always be written

$$K = \{tx(1 + h_K(x)),\ t \in [0,1],\ x \in \partial B\}$$

where $h_K : \partial B \to \mathbb{R}$ is the distance function to the origin.

Let $K \in \mathcal{K}^N$ be minimizing (3.2), and suppose that K has barycenter at the origin. Let $\eta > 0$ and $\varepsilon_0 > 0$ be given by Theorem 3.2.1. Now, by minimality

$$\begin{aligned}
P(K) - P(B) &\leq \varepsilon\frac{\mathrm{Cap}(K) - \mathrm{Cap}(B)}{\mathrm{Cap}(K)\mathrm{Cap}(B)} \\
&\leq \frac{\varepsilon}{\mathrm{Cap}(B)}
\end{aligned}$$

Thanks to [Fus15, Lemma 3.3] there exists $\delta_\eta > 0$ such that provided $P(K) - P(B) \leq \delta_\eta$ then $\|h\|_{W^{1,\infty}(\partial B)} \leq \eta$. We therefore deduce that if $\varepsilon \in (0, \mathrm{Cap}(B)\delta_\eta)$, we have in fact $\|h\|_{W^{1,\infty}(\partial B)} \leq \eta$, so that taking $\varepsilon \in (0, \min\{\varepsilon_0, \delta_\eta \mathrm{Cap}(B)\})$ we can apply Theorem 3.2.1 to get that K is a ball. This finishes the proof. □

3.2.3 Further stability results

The strategy we employed for proving Theorem 3.1.1 can be adapted to the case of λ_1. In fact, one can proceed likewise to get a result analogous to the Lemma 3.2.2 below in the case of λ_1 (see (i) in Proposition 3.2.5), leading to a result of the same type as Theorem 3.1.1. Let us state the stability result for the sake of clarity.

Proposition 3.2.4 (Weak stability of the ball for λ_1)**.** *Let $N \geq 2$. There exists $\varepsilon_0 > 0$ and $\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\forall K \in \mathcal{K}^N, |K| = 1 \text{ with } |K \Delta B| \leq \delta,\ (P - \varepsilon\lambda_1)(K) \geq (P - \varepsilon\lambda_1)(B).$$

Although this gives a simple proof of the stability of the ball in the case of λ_1, this Proposition is strictly weaker than the stronger result we prove in Theorem 3.1.2, since the range of $\varepsilon > 0$ for which the ball is locally minimal is not optimal (and was already known, see Section 3.1.2). On the other hand, since the analogues of Lemma 3.2.2 and Theorem 3.2.1 in the case of λ_1 do not explicitly appear in the literature (up to our knowledge), we think it might be of interest to state them rigorously. This is the object of the next result.

Proposition 3.2.5. *Let $N \geq 2$.*

(i) (Weak (IT)$_{H^1,W^{1,\infty}}$). There exists $C_N > 0$ such that if $h \in W^{1,\infty}(\partial B)$ with B_h of volume $|B_h| = |B|$ and $\|h\|_{L^\infty(\partial B)} \leq 1/2$ then

$$\lambda_1(B_h) - \lambda_1(B) \leq C_N \|h\|^2_{H^1(\partial B)}.$$

(ii) (Stability of the ball for Lipschitz perturbations). There exists $\eta > 0$ and $\varepsilon_0 > 0$ such that for all $h \in W^{1,\infty}(\partial B)$ verifying $\|h\|_{W^{1,\infty}(\partial B)} \leq \eta$ with $|B_h| = |B|$ and such that B_h has barycenter at the origin, and for all $\varepsilon \in (0, \varepsilon_0)$, then

$$P(B_h) - \varepsilon\lambda_1(B_h) \geq P(B) - \varepsilon\lambda_1(B)$$

with equality only if $B_h = B$.

Let us comment on the second item of this Proposition. While this non-optimal stability of the ball for Lipschitz perturbations is implied by the Payne-Weinberger inequality (3.6) in two dimensions, it does not seem to be known in the case $N \geq 3$. It opens up the question regarding a natural class of sets for which the optimal stability from Theorem 3.1.2 might hold: can one prove optimal stability of the ball for Lipschitz perturbations?

3.3 Minimality of the ball in a $C^{1,\alpha}$ neighborhood

The goal of this section consists in proving the first step of the *selection principle* strategy as we described it in the Introduction, namely that the ball is a strict (up to translation) minimum in a $C^{1,\alpha}$ neighborhood of the functional $P - c\lambda_1$ for $c \in (0, c^*)$. This is stated in next result.

Let us set a few preliminary notations. The notation B refers to the open ball of unit volume centered at 0. In this section, any function $h : \partial B \to \mathbb{R}$ defined on the sphere is extended to the whole of $\mathbb{R}^N$ by setting $h(x) := \theta(x)h(\frac{x}{|x|})$ (for some smooth θ with $\theta \equiv 1$ near ∂B) into a compactly supported function as smooth as h and which is constant near ∂B along directions normal to ∂B. Note that this extension is different from the one we make in Section 3.2 (in Lemma 3.2.2). For any $\alpha \in (0, 1]$ and $h \in C^{1,\alpha}(\partial B)$ we denote $\xi_h(x) := h(x)x$, so that $\|\xi_h\|_{C^{1,\alpha}(\mathbb{R}^N)^N} \leq C\|h\|_{C^{1,\alpha}(\partial B)}$, for some $C > 0$. We set

$$B_h = \{tx(1 + h(x)),\ t \in [0, 1),\ x \in \partial B\} = (\mathrm{Id} + \xi_h)(B),$$

which is a bounded Lipschitz open set provided $\|\xi_h\|_{W^{1,\infty}(\mathbb{R}^N)^N} < 1$, with boundary $\partial B_h = \{x(1 + h(x)),\ x \in \partial B\}$.

Theorem 3.3.1 (Fuglede-type computations for $P - c\lambda_1$: minimality for $C^{1,\alpha}$ perturbations)**.** *Let $N \geq 2$. For $c > 0$ set $\mathcal{J}_c := P - c\lambda_1$ and let c^* be given by (3.3). Then there exists $\alpha \in (0, 1)$ such that for any $0 < c < c^*$ there exists $\delta_c > 0$ such that for each $h \in C^{1,\alpha}(\partial B)$ with $|B_h| = |B|$ and $\|h\|_{C^{1,\alpha}(\partial B)} \leq \delta_c$ then*

$$\mathcal{J}_c(B_h) \geq \mathcal{J}_c(B)$$

with equality only if (up to translating) $B_h = B$.

Note that this result is of interest in itself, in particular no convexity constraint of the sets B_h is assumed. Let us emphasize on the importance of the space $C^{1,\alpha}$ in which we obtain minimality of the ball, regarding the general goal of proving minimality of $\mathcal{J}_c$ for all convex shapes (see Theorem 3.1.2). In fact, Theorem 3.3.1 is an improvement of [DL19, Proposition 5.5], where the authors get minimality of the ball for the same interval of c's but in a $W^{2,p}$ neighborhood (for any $p > N$), which was itself an improvement of previous works for $C^{2,\alpha}$ perturbations from [DP00, Dam02]. These latter results are not enough to apply the *selection principle* strategy, as this procedure does not give more than convergence in any $C^{1,\alpha}$ sense of quasi-minimizers (see Corollary 3.4.3 and Remark 3.4.4).

One of the main ingredients of the proof of Theorem 3.3.1 consists in proving a so-called **(IT)** property for the functional

$$\mathcal{J}_c := P - c\lambda_1.$$

This is achieved in Theorem 3.3.4. This property was introduced in [DL19, p. 3012], and describes a suitable second-order Taylor expansion at the ball B for the functional $\mathcal{J}_c$, where one identifies the remainder as the product of some "weak" Sobolev norm of the perturbation by something which goes to 0 as the perturbation goes to 0 in a much stronger sense (see Theorem 3.3.4 for a precise statement).

Let us first define the notion of shape differentiability for a shape functional. If $\Omega \subset \mathbb{R}^N$ is a bounded open set and $\xi \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ we denote by $\Omega_\xi := (\mathrm{Id} + \xi)(\Omega)$ the open Lipschitz deformation of Ω by ξ.

Definition 3.3.2. *Let $N \geq 2$. Let $\mathcal{J} : \{\Omega \subset \mathbb{R}^N,\ \Omega$ open bounded$\} \to \mathbb{R}$ be a functional. Let $\Omega \subset \mathbb{R}^N$ be open bounded and let $X \subset W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ be a normed space. For $k \in \{1, 2\}$ we say that $\mathcal{J}$ is k-times shape differentiable around Ω (for the space X) in the direction $\xi \in X$ if the function*

$$\mathcal{J}_\Omega : \xi \in X \mapsto \mathcal{J}(\Omega_\xi)$$

is k-times differentiable in a neighborhood of 0. We denote by $\mathcal{J}'_\Omega(\xi) \in \mathcal{L}^1(X, \mathbb{R})$ (respectively $\mathcal{J}''_\Omega(\xi) \in \mathcal{L}^2(X \times X, \mathbb{R})$) the first (respectively second) derivative at $\xi \in X$.

Remark 3.3.3. Note that although $\mathcal{J}'_\Omega(\xi)$ and $\mathcal{J}''_\Omega(\xi)$ are *a priori* linear and bilinear continuous forms over X, provided the set Ω enjoys some regularity properties it happens very often that they can be naturally extended to spaces of much lower regularity; for instance, the perimeter functional P has its first derivative continuous for the L^2 norm, while its second derivative can be continuously extended in H^1. In the case of λ_1 it is respectively the L^2 and $H^{1/2}$ spaces over which the first and second derivatives can be defined (see for instance [DL19, Lemma 2.8]).

We can now state the second main result of this section.

Theorem 3.3.4 (**(IT)** property for $\mathcal{J}_c$)**.** *Let $N \geq 2$. For $c > 0$ set $\mathcal{J}_c := P - c\lambda_1$. There exists $\alpha \in (0, 1)$ such that for any $c > 0$ there exists $\delta_c > 0$ and a modulus of continuity ω_c such that for all $h \in C^{1,\alpha}(\partial B)$ with $\|h\|_{C^{1,\alpha}(\partial B)} \leq \delta_c$ it holds*

$$\mathcal{J}_c(B_h) = \mathcal{J}_c(B) + (\mathcal{J}_c)'_B(0) \cdot (\xi_h) + \frac{1}{2}(\mathcal{J}_c)''_B(0) \cdot (\xi_h, \xi_h) + \omega_c(\|h\|_{C^{1,\alpha}(\partial B)})\|h\|^2_{H^1(\partial B)}$$

This condition was defined in [DL19] as the $\mathbf{(IT)}_{H^1,C^{1,\alpha}}$ condition, meaning that the functional $\mathcal{J}_c$ verifies a second-order Taylor expansion with the remainder term behaving as the product of $\|h\|^2_{H^1}$ with a modulus of continuity of $\|h\|_{C^{1,\alpha}}$. It was shown by Fuglede in [Fug89] that the perimeter satisfies a stronger $\mathbf{(IT)}_{H^1,W^{1,\infty}}$ condition (see for instance [DL19, Proposition 4.5]: there exists ω_P a modulus of continuity and $\delta_P > 0$ such that for all $h \in W^{1,\infty}(\partial B)$ with $\|h\|_{W^{1,\infty}(\partial B)} \leq \delta_P$ it holds

$$P(B_h) = P(B) + P'_B(0) \cdot (\xi_h) + \frac{1}{2}P''_B(0) \cdot (\xi_h, \xi_h) + \omega_P\left(\|h\|_{W^{1,\infty}(\partial B)}\right)\|h\|^2_{H^1} \tag{3.22}$$

As a consequence, proving Theorem 3.3.4 reduces to proving an $\mathbf{(IT)}_{H^1,C^{1,\alpha}}$ condition result for λ_1 (see the proof of Theorem 3.3.4 for this reduction), and in fact we will prove a stronger $\mathbf{(IT)}_{H^{1/2},C^{1,\alpha}}$ condition for λ_1. In order to do so we follow the strategy laid out by [DL19]: it will be convenient to show that λ_1 verifies the so-called condition $\mathbf{(IC)}_{H^{1/2},C^{1,\alpha}}$, as stated in next result, which constitutes the core result of this section.

Theorem 3.3.5 (**(IC)** property for λ_1)**.** *Let $N \geq 2$. For any $t \in [0, 1]$ and $h \in C^{1,\alpha}(\partial B)$ let $\lambda_1(t) := \lambda_1(B_{th})$. There exists $\alpha \in (0, 1)$ such that the functional λ_1 satisfies an $\mathbf{(IC)}_{H^{1/2},C^{1,\alpha}}$ condition at the ball B, i.e. there exists $\delta > 0$ and a modulus of continuity ω_{λ_1} such that for any $h \in C^{1,\alpha}(\partial B)$ with $\|h\|_{C^{1,\alpha}(\partial B)} \leq \delta$ we have*

$$\forall t \in [0, 1],\ |\lambda''_1(t) - \lambda''_1(0)| \leq \omega_{\lambda_1}(\|h\|_{C^{1,\alpha}(\partial B)})\|h\|^2_{H^{1/2}(\partial B)}. \tag{3.23}$$

The proof of this result is inspired by the strategy of [DL19, Theorem 1.4] for proving that λ_1 satisfies an $\mathbf{(IC)}_{H^{1/2},W^{2,p}}$ condition. Nevertheless, as the space $C^{1,\alpha}$ is strictly weaker than the space $W^{2,p}$, some estimates require a refined analysis (see Lemma 3.3.7) and new methods (see Lemma 3.3.6). We believe that this result is of independent interest, since it goes strictly below spaces with second derivatives as $C^{2,\alpha}$ or $W^{2,p}$ spaces, which are the usual spaces for which this kind of property is obtained (see for instance [Dam02] or [AFM13]).

Let us mention that in order to prove Theorem 3.3.5 we will first prove it for functions $h \in C^{1,1}(\partial B)$ instead of $h \in C^{1,\alpha}(\partial B)$, as it will allow us to consider the second-order geometric quantities of B_h (such as the mean curvature and second fundamental form) in the classical sense as functions of $L^\infty(\partial B_h)$, thus easing the computations (in particular in Lemmas 3.3.7, 3.3.9 and 3.3.10). We then remove this additional regularity assumption by a density argument.

The expression of $\lambda_1''(t)$ (see Lemma 3.3.10) involves both PDE-type terms and geometric terms. We start this section by proving three preparatory Lemmas in Section 3.3.1, which provide continuity-type estimates in the domain Ω of the quantities involved in the expression of $\lambda_1''(t)$.

3.3.1 Continuity in the domain Ω.

In this section we prove three preparatory Lemmas. Lemmas 3.3.6 and 3.3.8 will be useful for us to estimate the PDE terms in the variation $|\lambda_1''(t) - \lambda_1''(0)|$, while Lemma 3.3.7 will enable us to estimate the geometric terms. Note that we will also use them as a means to justify the expression of $\lambda_1''(t)$ from Lemma 3.3.10. Let us set some notations for this section.

Geometric notation. Let Ω be a $C^{1,1}$ bounded open set. For any $\xi \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ we set Ω_ξ the $C^{1,1}$ open set $\Omega_\xi := (\mathrm{Id} + \xi)(\Omega)$. The operator ∇_{τ_ξ} denotes the tangential gradient over $\partial\Omega_\xi$, div_{τ_ξ} and D_{τ_ξ} respectively the tangential divergence and jacobian. Setting $n_\xi \in C^{0,1}(\partial\Omega_\xi)$ the outer unit normal of Ω_ξ (in particular n_0 denotes the outer unit normal of Ω), we set $H_\xi := \mathrm{div}_{\tau_\xi}(n_\xi) \in L^\infty(\partial\Omega_\xi)$ (respectively $b_\xi := D_{\tau_\xi} n_\xi \in L^\infty(\partial\Omega_\xi \times \partial\Omega_\xi)$) the mean curvature (respectively, second fundamental form) on $\partial\Omega_\xi$.

Letting ϕ_ξ be the Lipschitz homeomorphism $\mathrm{Id} + \xi$, when a function f_ξ is defined on Ω_ξ (respectively $\partial\Omega_\xi$) we denote $\widehat{f_\xi}$ the function $f_\xi \circ \phi_\xi$ defined over Ω (repectively $\partial\Omega$). We also introduce $\tilde{J}_\xi$ the surface Jacobian from $\partial\Omega$ to $\partial\Omega_\xi$ given by the expression $\tilde{J}_\xi = \det(D\phi_\xi)|D\phi_\xi^{-T} n_0|$, meaning that for a $C^1(\partial\Omega_\xi)$ function f_ξ we have

$$\int_{\partial\Omega_\xi} f_\xi = \int_{\partial\Omega} \tilde{J}_\xi \widehat{f_\xi} \tag{3.24}$$

PDE notation. For $\xi \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$, we denote by v_ξ the first L^2 normalized and nonnegative Dirichlet eigenfunction over Ω_ξ, and set $\lambda_\xi := \lambda_1(\Omega_\xi)$.

We start by proving a continuity-type estimate of v_ξ and λ_ξ in the spirit of [DL19, Lemma 4.8].

Lemma 3.3.6. *Let Ω be a $C^{1,1}$ bounded open set. Let $0 < \alpha' < \alpha < 1$. There exists a modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ only depending on α, α' and Ω such that for all $\xi \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ it holds*

$$\|\widehat{v_\xi} - v_0\|_{C^{1,\alpha'}(\overline{\Omega})} \leq \omega(\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)}). \tag{3.25}$$

Moreover, it holds

$$\lambda_\xi \to \lambda_0 \text{ as } \left(\|\xi\|_{W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)} \text{ is bounded and } \|\xi\|_{L^\infty(\mathbb{R}^N,\mathbb{R}^N)} \to 0\right). \tag{3.26}$$

To prove (3.25) we adapt the method used by [DP00, Proposition 4.1], which is based on a compactness argument itself relying on a bound for an appropriate norm of $\widehat{v_\xi}$.

Proof. **Proof of (3.26).** The condition on ξ ensures that the Ω_ξ are uniformly Lipschitz, and the result therefore follows for instance from [Hen06, Theorem 2.3.18].

Proof of estimate (3.25). The proof is divided in two steps.

Step 1: $C^{1,\alpha}$ bound of $\widehat{v_\xi}$. Let us first prove that provided $\xi \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ verifies $\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)} \leq C_1$ for some $C_1 > 0$ then it holds

$$\|\widehat{v_\xi}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_2 \tag{3.27}$$

for some constant C_2 independent of ξ.

This bound relies on standard elliptic estimates. In fact, the equation verified by v_ξ

$$\begin{cases} -\Delta v_\xi = \lambda_\xi v_\xi \quad \text{in } \Omega_\xi, \\ v_\xi \in H^1_0(\Omega_\xi). \end{cases} \tag{3.28}$$

translates into the following elliptic equation for $\widehat{v_\xi}$

$$\begin{cases} -\mathrm{div}(A_\xi \nabla \widehat{v_\xi}) = \lambda_\xi J_\xi \widehat{v_\xi} \quad \text{over } \Omega \\ \widehat{v_\xi} \in H^1_0(\Omega) \end{cases} \quad \text{where} \quad \begin{cases} J_\xi := \det(\mathrm{Id} + D\xi) \\ A_\xi := J_\xi (\mathrm{Id} + D\xi)^{-1} \left((\mathrm{Id} + D\xi)^{-1}\right)^T \end{cases} \tag{3.29}$$

We now apply first order Schauder estimates (see [Sch12, Theorems 3.3.11, 3.3.12]) to get

$$\|\widehat{v_\xi}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \left(\|\widehat{v_\xi}\|_{L^\infty(\Omega)} + \|\lambda_\xi J_\xi \widehat{v_\xi}\|_{C^{0,\alpha}(\overline{\Omega})} \right) \tag{3.30}$$

where $C = C_N(\gamma_\xi, \|A_\xi\|_{C^{0,\alpha}(\overline{\Omega})})$ with γ_ξ the ellipticity constant of A_ξ. Now, there exists $C > 0$ such that for all ξ it holds

$$|J_\xi|, \|A_\xi\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \left(1 + \|\xi\|_{C^{1,\alpha}(\overline{\Omega})}\right) \tag{3.31}$$

Note that Ω satisfies a uniform exterior sphere condition since it is $C^{1,1}$. Equation (3.28) ensures that $v_\xi \in C^\infty(\Omega_\xi) \cap C^0(\overline{\Omega_\xi})$ (since Ω_ξ is Lipschitz). Suppose for a moment that $\xi \in C^2(\mathbb{R}^N, \mathbb{R}^N)$. Then $\widehat{v_\xi} = v_\xi \circ (\mathrm{Id} + \xi) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and we can therefore apply [GT01, Theorem 15.9] to deduce that

$$\|\nabla \widehat{v_\xi}\|_{L^\infty(\Omega)} \leq C \tag{3.32}$$

for some $C = C(N, \gamma_\xi, \|A_\xi\|_{L^\infty(\Omega)}, \|J_\xi\|_{L^\infty(\Omega)}, |\lambda_\xi|, \|\widehat{v_\xi}\|_{L^\infty(\Omega)})$. By assuming that $\|\xi\|_{W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)} \leq \delta$ for some δ small enough we can suppose that $\gamma_\xi \geq 1/2$, and also that Ω_ξ contains a fixed ball for any ξ, thus ensuring that λ_ξ is bounded thanks to the monotonicity of λ_1. Moreover, we have the L^∞ bound (see [Dav89, Example 2.1.8])

$$\|\widehat{v_\xi}\|_{L^\infty(\Omega)} = \|v_\xi\|_{L^\infty(\Omega_\xi)} \leq e^{1/8\pi} \lambda_\xi^{N/4} \tag{3.33}$$

Inserting (3.31), (3.32), and (3.33) into (3.30) provides the desired estimate (3.27). This concludes the proof of the Lemma in the case $\xi \in C^2(\mathbb{R}^N, \mathbb{R}^N)$. For a general $\xi \in C^{1,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ we can argue by density: thanks to a standard mollification we find $\xi_j \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ a sequence converging to ξ in $C^1(\overline{\Omega})$ norm with $\|\xi_j\|_{C^{1,\alpha}(\mathbb{R}^N)^N} \leq \|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$. Then as $\|\widehat{v_{\xi_j}}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_2$ thanks to estimate (3.27) proved in the C^2 case, we deduce thanks to the Arzela-Ascoli theorem that up to subsequence $\widehat{v_{\xi_j}} \to v$ in $C^1(\overline{\Omega})$ for some $v \in C^{1,\alpha}(\overline{\Omega})$. On the other hand, J_{ξ_j} and A_{ξ_j} go respectively to J_ξ and A_ξ in $C^0(\overline{\Omega})$, and furthermore $\lambda_{\xi_j} \to \lambda_\xi$ (thanks to (3.26)). We can therefore pass to the limit in the sense of distribution in (3.29) to get

$$\begin{cases} -\mathrm{div}(A_\xi \nabla v) = \lambda_\xi J_\xi v, \quad \text{in } \Omega \\ v \in H^1_0(\Omega) \end{cases}$$

Now, since we also have $v \geq 0$ and $\|v J_\xi^{1/2}\|_{L^2(\Omega)} = 1$ we deduce that $v = \widehat{v_\xi}$. Thanks to the C^1 convergence $\widehat{v_{\xi_j}} \to \widehat{v_\xi}$ we finally have

$$\|\widehat{v_\xi}\|_{C^{1,\alpha}(\overline{\Omega})} \leq \underline{\lim} \|\widehat{v_{\xi_j}}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_2$$

thus concluding the proof of (3.27) in the general case.

Step 2. We proceed by contradiction, therefore assuming that there exists $\varepsilon_0 > 0$ and a sequence $\|\xi_j\|_{C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)} \to 0$ such that

$$\forall j \geq 0,\ \|\widehat{v_{\xi_j}} - v_0\|_{C^{1,\alpha'}(\overline{\Omega})} \geq \varepsilon_0 \tag{3.34}$$

Thanks to the bound (3.27) we can use the Arzela-Ascoli theorem to infer the existence of $v \in C^{1,\alpha}(\overline{\Omega})$ such that up to subsequence

$$\widehat{v_{\xi_j}} \to v \text{ in } C^{1,\alpha'}(\overline{\Omega})$$

Now, since $\|\xi_j\|_{C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)} \to 0$ we have that A_{ξ_j} and J_{ξ_j} go respectively to Id and 1 in $C^{0,\alpha}(\overline{\Omega})$. Proceeding as in the previous step we identify v with v_0, which enters in contradiction with (3.34). This concludes the proof of estimate (3.25) and hence the proof of the Lemma.

□

In the following Lemma we prove continuity type-estimates of several geometric quantities associated to a set Ω. We denote $\alpha_\xi := n_\xi \cdot n_0$ and $\beta_\xi := \alpha_\xi n_\xi - n_0$.

Lemma 3.3.7. *Let Ω be a $C^{1,1}$ bounded open set. For any $\alpha \in (0,1)$ there exists $C = C(\alpha) > 0$ and $\delta = \delta(\alpha) > 0$ independent of $\xi \in C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)$ such that if $\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)} \leq \delta$ then*

- $\|\widetilde{J}_\xi - 1\|_{C^{0,\alpha}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$.
- $\|\widehat{n_\xi} - n_0\|_{C^{0,\alpha}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$, $\|\widehat{\alpha_\xi} - 1\|_{C^{0,\alpha}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$, $\|\widehat{\beta}_\xi\|_{C^{0,\alpha}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$.

Let $s \in (0,1)$ and $p \in (1,\infty)$ and denote by p' its conjugate exponent. For each $\alpha \in (\max\{s, 1-s\}, 1)$ there exists $\delta > 0$ such that if $\xi \in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$ with $\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N} \leq \delta$, we have the following expansions

$$\widehat{H_\xi} - H_0 = \omega_{s,p}(\xi), \quad \widehat{b_\xi} - b_0 = \omega_{s,p}(\xi),$$

$$\widehat{\nabla_{\tau_\xi}\alpha_\xi} = \omega_{s,p}(\xi)$$

where the notation $\omega_{s,p}(\xi)$ means that there exists $a_{1,\xi}$, $a_{2,\xi}$, $b_{1,\xi}$, $b_{2,\xi}$ (independent of s and p) such that $\omega_{s,p}(\xi) = a_{1,\xi}b_{1,\xi} + a_{2,\xi}b_{2,\xi}$ with

$$\|a_{1,\xi}\|_{W^{-s,p'}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}, \quad \|b_{1,\xi}\|_{W^{s,p}(\partial\Omega)} \leq C,$$

$$\|a_{2,\xi}\|_{W^{-s,p'}(\partial\Omega)} \leq C, \qquad \|b_{2,\xi}\|_{W^{s,p}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}.$$

Proof. All of these estimates rely on an appropriate expression for n_ξ. Following [DL19, Lemma 4.3] we write $\Omega = \{w < 0\}$ for some $w \in C^{1,1}(\mathbb{R}^N)$ with ∇w not vanishing in a neighborhood of $\partial\Omega$, so that $\Omega_\xi = \{w \circ \phi_\xi^{-1} < 0\}$ and

$$n_\xi = \frac{\nabla(w \circ \phi_\xi^{-1})}{|\nabla(w \circ \phi_\xi^{-1})|} = \frac{D\phi_\xi^{-T}\nabla w(\phi_\xi^{-1})}{|D\phi_\xi^{-T}\nabla w(\phi_\xi^{-1})|}.$$

Notice that n_ξ only involves $\nabla w \in C^{0,1}(\partial\Omega)$ and $D\xi \in C^{0,\alpha}(\mathbb{R}^N,\mathbb{R}^N)$. As a consequence, expanding the maps $A \mapsto (A^{-1})^T$ and $A \mapsto \det(A)$ around Id and $y \mapsto |y|$ around n_0 we get in fact $\|\widetilde{J}_\xi - 1\|_{C^{0,\alpha}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$. As for $\widehat{n_\xi}$, $\widehat{\alpha_\xi}$ and $\widehat{\beta}_\xi$, we expand $x \mapsto \frac{x}{|x|}$ around $\frac{\nabla w}{|\nabla w|}$ and get likewise the announced estimates.

The case of $\widehat{H_\xi}$, $\widehat{b_\xi}$ and $\widehat{\nabla_{\tau_\xi}\alpha_\xi}$ is more involved as the second derivatives of ξ and w come into play. As the argument is analogous in the three cases we only prove the estimate for $\widehat{H_\xi}$. Write $a(x) := \frac{x}{|x|}$ and $\psi_\xi := D\phi_\xi^{-T}\nabla w(\phi_\xi^{-1})$. Then

$$\widehat{H_\xi} = \operatorname{div}(a \circ \psi_\xi) \circ \phi_\xi = Da(\psi_\xi \circ \phi_\xi) : D\psi_\xi^T(\phi_\xi)$$

where : is the matrix dot product. In particular, one has

$$H_0 = \operatorname{div}(a \circ \nabla w) = Da(\nabla w) : D^2 w.$$

Writing $z_\xi = (D\phi_\xi^{-T}(\phi_\xi) - \mathrm{Id})\nabla w$, we let $c_1 := D\psi_\xi^T(\phi_\xi) - D^2 w$ and $c_2 := Da(\nabla w + z_\xi) - Da(\nabla w)$. We therefore rewrite $\widehat{H_\xi} = (Da(\nabla w) + c_2) : (D^2 w + c_1)$ and we thus want to estimate

$$\begin{aligned} \widehat{H_\xi} - H_0 &= (Da(\nabla w) + c_2) : (D^2 w + c_1) - Da(\nabla w) : D^2 w \\ &= Da(\nabla w) : c_1 + c_2 : D^2 w + c_2 : c_1. \end{aligned} \tag{3.35}$$

By expanding c_1 at $\xi = 0$, we see that it is a sum of terms of the form $d_{ijk}\partial_{ij}\xi_k$ where d_{ijk} only involves first derivatives of w and ξ, and of terms of the form $d'_{ij}\partial_{ij}w$ where d'_{ij} only involves first derivatives of w and ξ with $\|d'_{ij}\|_{C^{0,\alpha}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$. Using (iii) and the embedding $C^{0,\alpha}(\partial\Omega) \subset W^{1-s,p'}(\partial\Omega)$ from (ii) of Proposition 3.5.2, for each $\alpha \in (1-s, 1)$ there exists $C > 0$ such that

$$\begin{aligned} \|\partial_{ij}\xi_k\|_{W^{-s,p'}(\partial\Omega)} &\leq C\|\nabla\xi_k\|_{W^{1-s,p'}(\partial\Omega)} \\ &\leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}. \end{aligned}$$

Using again the embedding $C^{0,\alpha}(\partial\Omega) \subset W^{s,p}(\partial\Omega)$ provides $\|d_{ijk}\|_{W^{s,p}(\partial\Omega)} \leq C\|d_{ijk}\|_{C^{0,\alpha}(\partial\Omega)} \leq C$ for each $\alpha \in (s, 1)$. Proceeding likewise for the terms $d'_{ij}\partial_{ij}w$ we deduce that c_1 has the form $c_1 = \omega_{s,p}(\xi)$.

We now expand c_2 at $\xi = 0$: one has

$$c_2 = Da(\nabla w + z_\xi) - Da(\nabla w) = \int_0^1 D^2 a(\nabla w + t z_\xi) \cdot z_\xi dt$$

As $\|z_\xi\|_{C^{0,\alpha}(\mathbb{R}^N)^N} \leq C|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$, using the same ideas we get

$$\begin{aligned} \left\| \int_0^1 D^2 a(\nabla w + t z_\xi) \cdot z_\xi dt \right\|_{W^{s,p}(\partial\Omega)} &\leq C \left\| \int_0^1 D^2 a(\nabla w + t z_\xi) dt \right\|_{C^{0,\alpha}(\partial\Omega)} \|z_\xi\|_{C^{0,\alpha}(\mathbb{R}^N)^N} \\ &\leq C \left(\|w\|_{C^{1,\alpha}(\partial\Omega)} + \|z_\xi\|_{C^{0,\alpha}(\mathbb{R}^N)^N} \right) \|z_\xi\|_{C^{0,\alpha}(\mathbb{R}^N)^N} \\ &\leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N} \end{aligned}$$

for some $C > 0$, for each $\alpha \in (s, 1)$. As a consequence $\|c_2\|_{W^{s,p}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$.

Since $c_1 = \omega_{s,p}(\xi)$ and $\|c_2\|_{W^{s,p}(\partial\Omega)} \leq C\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$, we deduce from (3.35) the announced expansion for $\widehat{H_\xi}$, thus finishing the proof of the Lemma. □

We now prove a final preparatory Lemma, which consists in proving a continuity estimate in terms of ξ and θ for the H^1 norm of the derivative of $t \mapsto v_{(\mathrm{Id}+t\theta)(\Omega_\xi)}$

Lemma 3.3.8. *Let Ω be a $C^{1,1}$ bounded open set. For any $\xi \in C^{1,\alpha}\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and $\theta \in C^{1,\alpha}\left(\mathbb{R}^N, \mathbb{R}^N\right)$ we let $v'_{\xi,\theta}$ be the derivative at 0 of $t \mapsto v_{(Id+t\theta)(\Omega_\xi)}$. Then there exists $\alpha \in (0, 1)$ and ω a modulus of continuity such that for all ξ and θ with $\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$ and $\|\theta\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$ sufficiently small it holds*

$$\|v'_{0,\theta}\|_{H^1(\Omega)} \leq C\|\theta\|_{H^{1/2}(\partial\Omega)^N} \tag{3.36}$$

and

$$\|\widehat{v'_{\xi,\theta}} - v'_{0,\theta}\|_{H^1(\Omega)} = \omega_{C^{1,\alpha},H^{1/2}}(\xi, \theta) \tag{3.37}$$

where

$$\omega_{C^{1,\alpha},H^{1/2}}(\xi, \theta) := \omega(\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N})\|\theta\|_{H^{1/2}(\partial\Omega)^N} + \omega(\|\theta\|_{C^{1,\alpha}(\mathbb{R}^N)^N})\|\xi\|_{H^{1/2}(\partial\Omega)^N}.$$

Proof. We denote $\lambda'_{\xi,\theta} := (\lambda_1)'_{\Omega_\xi}(0)\cdot(\theta)$. It is classical (see for instance [HP18, Theorem 5.3.1]) that the functions $v'_{\xi,\theta}$ satisfies the following equations

$$\begin{cases} -\Delta v'_{\xi,\theta} = \lambda_\xi v'_{\xi,\theta} + \lambda'_{\xi,\theta} v_\xi, \text{ in } \Omega_\xi \\ v'_{\xi,\theta} = -(\partial_{n_\xi} v_\xi)\theta\cdot n_\xi, \quad \text{over } \partial\Omega_\xi \\ \int_{\Omega_\xi} v'_{\xi,\theta} v_\xi = 0 \end{cases}.$$

Let $\mathbf{H}_{\xi,\theta}$ be the harmonic extension on Ω_ξ of $(\partial_{n_\xi} v_\xi)\theta\cdot n_\xi$. Then recalling the expression of $\lambda'_{\xi,\theta}$ (see for instance [HP18, Section 5.9.3]) we can write

$$\lambda'_{\xi,\theta} = -\int_{\partial\Omega_\xi} (\partial_{n_\xi} v_\xi)^2 \theta\cdot n_\xi = -\int_{\partial\Omega_\xi} (\partial_{n_\xi} v_\xi)\mathbf{H}_{\xi,\theta} = \lambda_\xi \int_{\Omega_\xi} v_\xi \mathbf{H}_{\xi,\theta}$$

where we used Green's formula and $\int_{\Omega_\xi} \nabla v_\xi \nabla \mathbf{H}_{\xi,\theta} = 0$. We decompose $v'_{\xi,\theta} = -\pi_\xi \mathbf{H}_{\xi,\theta} + w_{\xi,\theta}$ where π_ξ is the orthogonal projection onto $\{v_\xi\}^\perp$ for the L^2 scalar product on $L^2(\Omega_\xi)$. Thanks to the above expression for $\lambda'_{\xi,\theta}$ we know that $w_{\xi,\theta}$ solves

$$\begin{cases} (-\Delta - \lambda_\xi) w_{\xi,\theta} = -\lambda_\xi \pi_\xi \mathbf{H}_{\xi,\theta}, \text{ in } \Omega_\xi \\ w_{\xi,\theta} = 0, \text{ over } \partial\Omega_\xi \\ \int_{\Omega_\xi} v_\xi w_{\xi,\theta} = 0. \end{cases}, \tag{3.38}$$

We estimate separately $\mathbf{H}_{\xi,\theta}$ and $w_{\xi,\theta}$.

- **Estimate of $\mathbf{H}_{\xi,\theta}$.** Since $\mathbf{H}_{\xi,\theta}$ is harmonic inside Ω_ξ, $\widehat{\mathbf{H}_{\xi,\theta}}$ satisfies $-\mathrm{div}(A_\xi \nabla \widehat{\mathbf{H}_{\xi,\theta}}) = 0$ in Ω, where A_ξ was defined in (3.29). As a consequence, we have that

$$\Delta(\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}) = \Delta\widehat{\mathbf{H}_{\xi,\theta}} = -\mathrm{div}((A_\xi - \mathrm{Id})\nabla\widehat{\mathbf{H}_{\xi,\theta}})$$

so that by using standard elliptic estimates (see for instance [GT01]) we get

$$\|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^1(\Omega)} \le C_N \left(\|(A_\xi - \mathrm{Id})\nabla\widehat{\mathbf{H}_{\xi,\theta}}\|_{L^2(\Omega)} + \|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^{1/2}(\partial\Omega)} \right) \tag{3.39}$$

for some $C_N > 0$. There exists $C > 0$ such that

$$\begin{aligned} \|(A_\xi - \mathrm{Id})\nabla\widehat{\mathbf{H}_{\xi,\theta}}\|_{L^2(\Omega)} &\le \|A_\xi - \mathrm{Id}\|_{L^\infty(\Omega)} \left(\|\nabla\widehat{\mathbf{H}_{\xi,\theta}} - \nabla\mathbf{H}_{0,\theta}\|_{L^2(\Omega)} + \|\nabla\mathbf{H}_{0,\theta}\|_{L^2(\Omega)} \right) \\ &\le C\|\xi\|_{W^{1,\infty}(\mathbb{R}^N)^N} \left(\|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^1(\Omega)} + \|\theta\cdot n_0\|_{H^{1/2}(\partial\Omega)} \right) \end{aligned}$$

where we used that

$$\|\nabla\mathbf{H}_{0,\theta}\|_{L^2(\Omega)} \le \|\mathbf{H}_{0,\theta}\|_{H^{1/2}(\partial\Omega)} = \|\partial_{n_0} v_0(\theta\cdot n_0)\|_{H^{1/2}(\partial\Omega)} \le C\|\theta\cdot n_0\|_{H^{1/2}(\partial\Omega)}$$

using $v_0 \in C^{1,\alpha}(\overline{\Omega})$ for each $\alpha \in (0,1)$ and the product law $C^{0,\alpha}(\partial\Omega)\cdot H^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ for some $\alpha \in (0,1)$ close to 1 (see (iv) of Proposition 3.5.2). Assuming that $\|\xi\|_{W^{1,\infty}(\mathbb{R}^N)^N} \le \frac{1}{2C_N C}$ we thus get from (3.39) that there exists $C > 0$ such that

$$\|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^1(\Omega)} \le C \left(\|\xi\|_{W^{1,\infty}(\mathbb{R}^N)^N} \|\theta\|_{H^{1/2}(\partial\Omega)^N} + \|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^{1/2}(\partial\Omega)} \right). \tag{3.40}$$

Now, to estimate $\|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^{1/2}(\partial\Omega)} = \|\widehat{\partial_{n_\xi} v_\xi}(\widehat{\theta\cdot n_\xi}) - \partial_{n_0} v_0(\theta\cdot n_0)\|_{H^{1/2}(\partial\Omega)}$, we write

$$\widehat{\partial_{n_\xi} v_\xi}(\widehat{\theta\cdot n_\xi}) - \partial_{n_0} v_0(\theta\cdot n_0) = \widehat{\partial_{n_\xi} v_\xi}\left((\widehat{\theta\cdot n_\xi}) - (\theta\cdot n_0) \right) + (\theta\cdot n_0)\left(\widehat{\partial_{n_\xi} v_\xi} - \partial_{n_0} v_0 \right) \tag{3.41}$$

Let us decompose

$$\begin{aligned}\widehat{\partial_{n_\xi} v_\xi} - \partial_{n_0} v_0 &= (\widehat{\nabla v_\xi} - \nabla v_0)\cdot n_0 + \widehat{\nabla v_\xi}\cdot(\widehat{n_\xi} - n_0)\\ &= \left(D\phi_\xi^{-T}\nabla\widehat{v_\xi} - \nabla v_0\right)\cdot n_0 + D\phi_\xi^{-T}\nabla\widehat{v_\xi}\cdot(\widehat{n_\xi} - n_0)\end{aligned}$$

Using Lemmas 3.3.6 and 3.3.7 we thus get $\|\widehat{\partial_{n_\xi} v_\xi} - \partial_{n_0} v_0\|_{C^{0,\alpha'}(\partial\Omega)} \leq \omega\left(\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}\right)$ for any $\alpha' < \alpha$. Using again the product law $C^{0,\alpha'}(\partial\Omega)\cdot H^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ for $\alpha' < \alpha$ close enough to 1 we deduce

$$\begin{aligned}\|\left(\widehat{\partial_{n_\xi} v_\xi} - \partial_{n_0} v_0\right)(\theta\cdot n_0)\|_{H^{1/2}(\partial\Omega)} &\leq C\|\widehat{\partial_{n_\xi} v_\xi} - \partial_{n_0} v_0\|_{C^{0,\alpha'}(\partial\Omega)}\|\theta\cdot n_0\|_{H^{1/2}(\partial\Omega)}\\ &\leq \omega(\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N})\|\theta\|_{H^{1/2}(\partial\Omega)^N}\end{aligned} \tag{3.42}$$

for any α sufficiently close to 1. On the other hand, we have

$$\widehat{\theta\cdot n_\xi} - \theta\cdot n_0 = (\widehat{\theta} - \theta)\cdot\widehat{n_\xi} + \theta\cdot(\widehat{n_\xi} - n_0)$$

With the same tools as before we get $\|\theta\cdot(\widehat{n_\xi} - n_0)\|_{H^{1/2}(\partial\Omega)} \leq C\|\theta\|_{H^{1/2}(\partial\Omega)^N}\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$ for the same $\alpha \in (0,1)$. For the other term we write

$$\widehat{\theta}(x) - \theta(x) = \int_0^1 \nabla\theta(x + t\xi(x))\cdot\xi(x)dt$$

so that the product law $C^{0,\alpha}(\partial\Omega)\cdot H^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ again gives

$$\begin{aligned}\|\widehat{\theta} - \theta\|_{H^{1/2}(\partial\Omega)} &\leq C\left\|\int_0^1 \nabla\theta(x + t\xi(x))\right\|_{C^{0,\alpha}(\mathbb{R}^N)^N}\|\xi\|_{H^{1/2}(\partial\Omega)^N}\\ &\leq C\|\theta\|_{C^{1,\alpha}(\mathbb{R}^N)^N}\left(1 + \|\xi\|_{C^{0,\alpha}(\mathbb{R}^N)^N}\right)\|\xi\|_{H^{1/2}(\partial\Omega)^N}\\ &\leq C\|\theta\|_{C^{1,\alpha}(\mathbb{R}^N)^N}\|\xi\|_{H^{1/2}(\partial\Omega)^N}\end{aligned}$$

yielding $\|(\widehat{\theta} - \theta)\cdot\widehat{n_\xi}\|_{H^{1/2}(\partial\Omega)} \leq C\|\xi\|_{H^{1/2}(\partial\Omega)^N}\|\theta\|_{C^{1,\alpha}(\mathbb{R}^N)^N}$. Combining the two estimates we thus get

$$\|\widehat{\theta\cdot n_\xi} - \theta\cdot n_0\|_{H^{1/2}(\partial\Omega)} \leq \omega_{C^{1,\alpha},H^{1/2}}(\xi,\theta)$$

This estimate together with (3.42) enable to estimate $\|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^{1/2}(\partial\Omega)}$ thanks to the decomposition (3.41), so that (3.40) becomes

$$\|\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}\|_{H^1(\Omega)} \leq \omega_{C^{1,\alpha},H^{1/2}}(\xi,\theta)\,. \tag{3.43}$$

This finishes the proof of the estimate of $\mathbf{H}_{\xi,\theta}$.

- **Estimate of $w_{\xi,\theta}$.** To estimate $\|\widehat{w_{\xi,\theta}} - w_{0,\theta}\|_{H^1(\Omega)}$ let us write the equation verified by $\widehat{w_{\xi,\theta}} - w_{0,\theta}$. If one writes $\mathcal{L}_\xi := \mathrm{div}(A_\xi\nabla\cdot)$, then for a function $f_\xi : \Omega_\xi \to \mathbb{R}$ one has $\widehat{\Delta f_\xi} = \mathcal{L}_\xi\widehat{f_\xi}$. Recalling (3.38) we therefore have

$$\begin{cases}(-\Delta - \lambda_0)(\widehat{w_{\xi,\theta}} - w_{0,\theta}) = \left[(-\Delta - \lambda_0) - (-\mathcal{L}_\xi - \lambda_\xi)\right]\widehat{w_{\xi,\theta}} - \lambda_\xi\widehat{\pi_\xi\mathbf{H}_{\xi,\theta}} + \lambda_0\pi_0\mathbf{H}_{0,\theta} \text{ in } \Omega\\ \widehat{w_{\xi,\theta}} - w_{0,\theta} = 0 \text{ over } \partial\Omega\end{cases}$$

Using that $(-\Delta - \lambda_0)^{-1} : H^{-1}(\Omega) \to \{v_0\}^\perp \cap H^1(\Omega)$ is an isomorphism, there exists $C_N > 0$ such that

$$\begin{aligned}\|\widehat{w_{\xi,\theta}} - w_{0,\theta} - \gamma_{\xi,\theta}v_0\|_{H^1(\Omega)} \leq C_N\big(&\|(A_\xi - \mathrm{Id})\nabla\widehat{w_{\xi,\theta}}\|_{L^2(\Omega)} + |\lambda_\xi - \lambda_0|\|\widehat{w_{\xi,\theta}}\|_{L^2(\Omega)}\\ &+\|\lambda_\xi\widehat{\pi_\xi\mathbf{H}_{\xi,\theta}} - \lambda_0\pi_0\mathbf{H}_{0,\theta}\|_{L^2(\Omega)}\big)\end{aligned}$$

where $\gamma_{\xi,\theta} \in \mathbb{R}$ is chosen so that $\widehat{w_{\xi,\theta}} - w_{0,\theta} - \gamma_{\xi,\theta} v_0 \in \{v_0\}^\perp$. Now, since $\|A_\xi - \mathrm{Id}\|_{L^\infty(\Omega)} \leq C\|\xi\|_{W^{1,\infty}(\mathbb{R}^N)^N}$ and $|\lambda_\xi - \lambda_0| \leq \omega(\|\xi\|_{L^\infty(\mathbb{R}^N)^N})$ (thanks to Lemma 3.3.6) we get

$$\|\widehat{w_{\xi,\theta}} - w_{0,\theta} - \gamma_{\xi,\theta} v_0\|_{H^1(\Omega)} \leq C\|\xi\|_{W^{1,\infty}(\mathbb{R}^N)^N} \left(\|\widehat{w_{\xi,\theta}}\|_{H^1(\Omega)} + \|\lambda_\xi \widehat{\pi_\xi \mathbf{H}_{\xi,\theta}} - \lambda_0 \pi_0 \mathbf{H}_{0,\theta}\|_{L^2(\Omega)} \right) \quad (3.44)$$

If we denote $\langle \cdot, \cdot \rangle_\xi$ the scalar product in $L^2(\Omega_\xi)$, we write

$$\widehat{\pi_\xi \mathbf{H}_{\xi,\theta}} - \pi_0 \mathbf{H}_{0,\theta} = (\widehat{\mathbf{H}_{\xi,\theta}} - \mathbf{H}_{0,\theta}) - (\langle \mathbf{H}_{\xi,\theta}, v_\xi \rangle_\xi v_\xi - \langle \mathbf{H}_{0,\theta}, v_0 \rangle_0 v_0)$$

Using estimate (3.25) from Lemma 3.3.6 and the Harmonic estimate (3.43) we have

$$|\langle \mathbf{H}_{\xi,\theta}, v_\xi \rangle_\xi - \langle \mathbf{H}_{0,\theta}, v_0 \rangle_0| = \left| \int_\Omega \widehat{\mathbf{H}_{\xi,\theta}} \widehat{v_\xi} J_\xi - \int_\Omega \mathbf{H}_{0,\theta} v_0 \right| \leq \omega_{C^{1,\alpha}, H^{1/2}}(\xi, \theta)$$

(J_ξ was defined in (3.29)) so that using again (3.25), (3.43) and (3.26) we deduce

$$\|\lambda_\xi \widehat{\pi_\xi \mathbf{H}_{\xi,\theta}} - \lambda_0 \pi_0 \mathbf{H}_{0,\theta}\|_{L^2(\Omega)} \leq \omega_{C^{1,\alpha}, H^{1/2}}(\xi, \theta) \quad (3.45)$$

Estimate (3.44) thus becomes

$$\|\widehat{w_{\xi,\theta}} - w_{0,\theta} - \gamma_{\xi,\theta} v_0\|_{H^1(\Omega)} \leq C \left(\|\xi\|_{W^{1,\infty}(\mathbb{R}^N)^N} \|\widehat{w_{\xi,\theta}}\|_{H^1(\Omega)} + \omega_{C^{1,\alpha}, H^{1/2}}(\xi, \theta) \right) \quad (3.46)$$

Now, since $\int_{\Omega_\xi} w_{\xi,\theta} v_\xi = \int_\Omega w_{0,\theta} v_0 = 0$ we have

$$|\gamma_{\xi,\theta}| = \left| \int_\Omega (\widehat{w_{\xi,\theta}} - w_{0,\theta}) v_0 \right| = \left| \int_\Omega \widehat{w_{\xi,\theta}} (\widehat{v_\xi} J_\xi - v_0) \right| \leq \omega(\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}) \|\widehat{w_{\xi,\theta}}\|_{L^2(\Omega)}$$

using again Lemma 3.3.6, which gives

$$\begin{aligned} \|\widehat{w_{\xi,\theta}} - w_{0,\theta}\|_{H^1(\Omega)} &\leq \omega(\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}) \|\widehat{w_{\xi,\theta}}\|_{H^1(\Omega)} + \omega_{C^{1,\alpha}, H^{1/2}}(\xi, \theta) \\ &\leq \omega(\|\xi\|_{C^{1,\alpha}(\mathbb{R}^N)^N}) \left(\|\widehat{w_{\xi,\theta}} - w_{0,\theta}\|_{H^1(\Omega)} + \|w_{0,\theta}\|_{H^1(\Omega)} \right) + \omega_{C^{1,\alpha}, H^{1/2}}(\xi, \theta) \quad (3.47) \end{aligned}$$

Using $\|w_{0,\theta}\|_{H^1(\Omega)} \leq C\|\mathbf{H}_{0,\theta}\|_{L^2(\Omega)} \leq C\|\theta\|_{H^{1/2}(\partial\Omega)^N}$ based on (3.38), and taking $\omega_{C^{1,\alpha}, H^{1/2}}(\xi, \theta) \leq \frac{1}{2}$ finally yields

$$\|\widehat{w_{\xi,\theta}} - w_{0,\theta}\|_{H^1(\Omega)} = \omega_{C^{1,\alpha}, H^{1/2}}(\xi, \theta). \quad (3.48)$$

Putting together (3.43) and (3.48) provides the desired estimate (3.37). As for (3.36), we have seen that $\|w_{0,\theta}\|_{H^1(\Omega)} \leq C\|\theta\|_{H^{1/2}(\partial\Omega)^N}$ based on (3.38). On the other hand, $\|\pi_0 \mathbf{H}_{0,\theta}\|_{L^2(\Omega)} \leq \|\mathbf{H}_{0,\theta}\|_{L^2(\Omega)}$ by definition and

$$\|\nabla \pi_0 \mathbf{H}_{0,\theta}\|^2_{L^2(\Omega)} = \|\nabla \mathbf{H}_{0,\theta}\|^2_{L^2(\Omega)} - \langle \mathbf{H}_{0,\theta}, v_0 \rangle^2$$

since $\int_\Omega \nabla \mathbf{H}_{0,\theta} \nabla v_0 = 0$, thus yielding also

$$\|\nabla \pi_0 \mathbf{H}_{0,\theta}\|_{L^2(\Omega)} \leq \|\mathbf{H}_{0,\theta}\|_{H^1(\Omega)} \leq C\|\mathbf{H}_{0,\theta}\|_{H^{1/2}(\partial\Omega)} \leq C\|\theta\|_{H^{1/2}(\partial\Omega)^N}$$

Hence $\|\pi_0 \mathbf{H}_{0,\theta}\|_{H^1(\Omega)} \leq C\|\theta\|_{H^{1/2}(\partial\Omega)^N}$, so that we finally have $\|v'_{0,\theta}\|_{H^1(\Omega)} \leq \|\pi_0 \mathbf{H}_{0,\theta}\|_{H^1(\Omega)} + \|w_{0,\theta}\|_{H^1(\Omega)} \leq C\|\theta\|_{H^{1/2}(\partial\Omega)^N}$, thus giving (3.36). □

3.3.2 Second derivative of λ_1.

A final preparatory step to estimate $\lambda_1''(t)-\lambda_1''(0)$ is to justify that the expression (3.49) below of the second derivative $(\lambda_1)''_\Omega(0)$ is valid when $\Omega = (\mathrm{Id}+\xi)(B)$ for some vector field ξ which is only $C^{1,1}$. Formula (3.49) is indeed well-known for C^3 domains (see for instance [HP18, Theorem 5.9.2 and Section 5.9.6]), but for $C^{1,1}$ domains it does not seem to have been justified in the literature. As a matter of fact, the expression (3.49) has been implicitly used in [DL19] without further justification for domains $(\mathrm{Id}+\xi)(B)$ with $\xi \in W^{2,p}(\mathbb{R}^N,\mathbb{R}^N)$. From (3.49) we will immediately deduce a corresponding expression for $\lambda_1''(t)$, see Lemma 3.3.10.

In this paragraph, if ξ is a Lipschitz vector field and f_ξ is defined on Ω_ξ or $\partial\Omega_\xi$ we still write $\widehat{f_\xi}$ the function $f_\xi \circ \phi_\xi$ defined on Ω or $\partial\Omega$.

Lemma 3.3.9. *[Second derivative of λ_1] Let $\Omega \subset \mathbb{R}^N$ be a bounded open set given by $\Omega = (Id+\zeta)(B)$ for some $\zeta \in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$. Let n, H and b denote respectively its outer unit normal, curvature and second fundamental form. We set $\lambda_\Omega := \lambda_1(\Omega)$. Then there exists $\alpha \in (0,1)$ such that for $\theta \in C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)$ it holds*

$$\lambda_\Omega''(0)\cdot(\theta,\theta) = 2\left(\int_\Omega |\nabla v'|^2 - \lambda_\Omega \int_\Omega |v'|^2\right) + \int_{\partial\Omega} (\partial_n v)^2 \left[H(\theta\cdot n)^2 - b(\theta_\tau,\theta_\tau) + 2\nabla_\tau(\theta\cdot n)\cdot\theta_\tau\right] \tag{3.49}$$

where θ_τ denotes the tangential component of θ, v is the first L^2 normalized and nonnegative eigenfunction of Ω and v' is uniquely determined by the equations

$$\begin{cases} -\Delta v' = \lambda_\Omega v' + (\lambda_\Omega'(0)\cdot\theta)v, & \text{in } \Omega \\ v' = -\partial_n v(\theta\cdot n), & \text{over } \partial\Omega \\ \int_\Omega v'v = 0. \end{cases}$$

Note that when Ω is C^3 the term $\int_\Omega |v'|^2 - \lambda_\Omega \int_\Omega |v'|^2$ is more commonly written as the boundary term $\int_{\partial\Omega} v'\partial_n v'$ (which we cannot justify if Ω is merely $C^{1,1}$).

In our case one does not have enough regularity over Ω to perform the classical integration by parts leading to expression (3.49). As a consequence, in order to prove (3.49) in the $C^{1,1}$ case we rely on (3.49) in the smooth case combined with a low-regularity formula of $(\lambda_1)''_\Omega(0)\cdot(\theta,\theta)$ which holds true for bounded Lipschitz open sets, proven in [BB22, Theorem 2.1] (see also [Lau20] for an expression for the Dirichlet energy in the same spirit).

Proof of Lemma 3.3.9. One can apply the second derivative formula from [BB22, Theorem 2.1] which holds true for any Lipschitz domain and $\theta \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$:

$$\begin{aligned}\lambda_\Omega''(0)\cdot(\theta,\theta) = \int_\Omega \left(-2|\nabla\dot v_\theta|^2 + 2\lambda_\Omega|\dot v_\theta|^2 + 2(\mathbf{S}_\Omega : D\theta)\mathrm{div}(\theta) + (\lambda_\Omega|v|^2 - |\nabla v|^2)(\mathrm{div}(\theta)^2 + D\theta^T : D\theta)\right) \\ + \int_\Omega \left(2(2D\theta^2 + D\theta D\theta^T)|\nabla v|^2 - 2\lambda_\Omega'(0)\cdot(\theta)\mathrm{div}(\theta)|v|^2\right)\end{aligned} \tag{3.50}$$

where : is the matrix dot product, $\mathbf{S}_\Omega := (|\nabla v|^2 - \lambda_\Omega|v|^2)\mathrm{Id} - 2\nabla v\otimes\nabla v$, and $\dot v_\theta$ denotes the material derivative of v in the direction θ, meaning the derivative at 0 in the direction θ of the function $\theta \in W^{1,\infty} \mapsto v_\theta \circ (\mathrm{Id}+\theta) \in H_0^1(\Omega)$ where v_θ is the first L^2 normalized eigenfunction of Ω_θ. Note that $\dot v_\theta$ verifies the elliptic equation

$$\begin{cases} -\Delta\dot v_\theta - \lambda_\Omega\dot v_\theta = \lambda_\Omega v\mathrm{div}(\theta) + (\lambda_\Omega'(0)\cdot(\theta))\,v + \mathrm{div}(A_\theta'\nabla v), \text{ in } \Omega \\ \dot v_\theta = 0 \text{ over } \partial\Omega \\ \int_\Omega \dot v_\theta v = -\frac{1}{2}\int_\Omega |v|^2\mathrm{div}(\theta) \end{cases}$$

with $A'_\theta := \mathrm{div}(\theta)\mathrm{Id} - D\theta - D\theta^T$.

Recall that $\Omega = B_\zeta = (\mathrm{Id}+\zeta)(B)$. Since $\zeta \in C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)$, there exists $\zeta_j \in C^\infty(\mathbb{R}^N,\mathbb{R}^N)$ converging locally to ζ in $C^{1,\beta}$ for each $\beta \in (0,1)$. Letting $\Omega_j := B_{\zeta_j} = (\mathrm{Id}+\zeta_j)(B)$, we have $\Omega_j = (\mathrm{Id}+\zeta_j)\circ(\mathrm{Id}+\zeta)^{-1}(B_\zeta)$ so that $\Omega_j = (\mathrm{Id}+\xi_j)(\Omega)$ with $\xi_j := (\mathrm{Id}+\zeta_j)\circ(\mathrm{Id}+\zeta)^{-1} - \mathrm{Id}$ converging locally to 0 in $C^{1,\beta}$ for each $\beta \in (0,1)$. As Ω_j is smooth, $\lambda''_{\Omega_j}(0)\cdot(\theta,\theta)$ both equals (3.49) (see [HP18, Theorem 5.9.2 and Section 5.9.6]) and (3.50) and we will pass to the limit in both expressions.

Through the proof we give an additionnal index j to the notations linked to Ω_j: v_j is the first eigenfunction of Ω_j, $\lambda'_j := \lambda'_{\Omega_j}(0)\cdot(\theta)$, $\dot v_j$ is the material derivative of v_j in direction θ and $v'_j := v'_{\xi_j,\theta}$(the notation was introduced in Lemma 3.3.8). As for the geometric quantites related to Ω_j, n_j denotes the outer unit normal to Ω_j, and so on. The proof of the Lemma is divided into two steps.

Step 1: continuity of $\lambda''_\Omega(0)\cdot(\theta,\theta)$ in Ω. In this step one can assume that $\theta \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N)$. Let us prove that $\lambda''_{\Omega_j}(0)\cdot(\theta,\theta) \to \lambda''_\Omega(0)\cdot(\theta,\theta)$.

Since in particular $\Omega_j \to \Omega$ in the Hausdorff sense, there exists $D \subset \mathbb{R}^N$ open bounded such that $\Omega_j, \Omega \subset D$ for every j. Setting $\gamma_j := \int_D \dot v_j v_j$ we have

$$\|\dot v_j - \gamma_j v_j\|_{H^1_0(D)} \le \|\lambda_j v_j \mathrm{div}(\theta)\|_{H^{-1}(\Omega_j)} + \|\lambda'_j v_j\|_{H^{-1}(\Omega_j)} + \|\mathrm{div}(A'_\theta \nabla v_j)\|_{H^{-1}(\Omega_j)}$$

Now, since $\lambda'_j = \int_{\Omega_j} \mathbf{S}_j : D\theta$ thanks to [BB22, Theorem 2.1], and using that λ_j is bounded and v_j is bounded in H^1 (thanks to Lemma 3.3.6) we deduce that $\dot v_j - \gamma_j v_j$ is bounded in $H^1_0(D)$, yielding in turn that $\dot v_j$ is bounded in $H^1_0(D)$ as $\gamma_j = -\frac{1}{2}\int_{\Omega_j}|v_j|^2\mathrm{div}(\theta)$. As a consequence, $\dot v_j$ converges (up to subsequence) towards some $\tilde v \in H^1_0(D)$ weakly in H^1 and strongly in L^2. Using again Lemma 3.3.6 we have that $\lambda_j \to \lambda_\Omega$, $\lambda'_j \to \lambda'_\Omega(0)\cdot(\theta)$ and $v_j \to v$ in H^1, so that relying also on the Hausdorff convergence $\Omega_j \to \Omega$ one can pass to the limit in the sense of distributions in

$$\begin{cases} -\Delta \dot v_j - \lambda_j \dot v_j = \lambda_j v_j \mathrm{div}(\theta) + \lambda'_j v_j + \mathrm{div}(A'_\theta \nabla v_j), \text{ in } \Omega_j \\ \dot v_j = 0 \text{ over } \partial\Omega_j \\ \int_{\Omega_j} \dot v_j v_j = -\frac{1}{2}\int_{\Omega_j}|v_j|^2\mathrm{div}(\theta) \end{cases}$$

to deduce that $\tilde v$ verifies

$$\begin{cases} -\Delta \tilde v - \lambda_\Omega \tilde v = \lambda_\Omega v \mathrm{div}(\xi) + (\lambda'_\Omega(0)\cdot(\xi))\, v + \mathrm{div}(A'_\theta \nabla v), \text{ in } \Omega \\ \int_\Omega \tilde v v = -\frac{1}{2}\int_\Omega |v|^2 \mathrm{div}(\theta) \end{cases}$$

Since Ω is Lipschitz it suffices to prove that $\tilde v = 0$ a.e. outside Ω to deduce that $\tilde v \in H^1_0(\Omega)$ (see for instance [HP18, Proposition 3.2.16]) and therefore that $\tilde v = \dot v$. But this is seen directly by passing to the limit a.e. in the identity $\mathbf{1}_{D\setminus\Omega_j}\dot v_j = 0$. Now, the convergence of $\dot v_j$ towards $\dot v$ is strong in H^1, since by multiplying the equation by $\dot v_j$ and integrating by parts we get $\|\nabla \dot v_j\|_{L^2(D)} \to \|\nabla \dot v\|_{L^2(D)}$. We can therefore pass to the limit as $j \to +\infty$ in (3.50) to deduce that we have in fact $\lambda''_{\Omega_j}(0)\cdot(\theta,\theta) \to \lambda''_\Omega(0)\cdot(\theta,\theta)$.

Step 2: continuity of (3.49) in Ω. In this step we rather assume $\theta \in C^{1,\alpha}(\mathbb{R}^N,\mathbb{R}^N)$ for some α given by Lemma 3.3.8. We want to pass to the limit in the expression

$$2\left(\int_{\Omega_j}|\nabla v'_j|^2 - \lambda_j \int_{\Omega_j}|v'_j|^2\right) + \int_{\partial\Omega_j}(\partial_{n_j} v_j)^2\left[H_j(\theta\cdot n_j)^2 - b_j(\theta_{\tau_j},\theta_{\tau_j}) + 2\nabla_{\tau_j}(\theta\cdot n_j)\cdot\theta_{\tau_j}\right] \tag{3.51}$$

Thanks to a change of variable, the integral on Ω_j can be written

$$2\int_\Omega \left(|\widehat{\nabla v'_j}|^2 - \lambda_j|\widehat{v'_j}|^2\right) J_j = 2\int_\Omega \left(\langle A_j \nabla\widehat{v'_j}, \nabla\widehat{v'_j}\rangle - \lambda_j J_j |\widehat{v'_j}|^2\right)$$

where $J_j := J_{\xi_j}$ and $A_j := A_{\xi_j}$ are defined in (3.29). We have that A_j and J_j go respectively to Id and 1 in $L^\infty\Omega)$ as $j \to +\infty$, and also $\lambda_j \to \lambda_\Omega$, while on the other hand $\widehat{v'_j}$ converges to $v'_{0,\theta}$ in $H^1(\Omega)$ thanks to Lemma 3.3.8. As a consequence the integral on Ω_j converges to $2\left(\int_\Omega |\nabla v'|^2 - \lambda_\Omega \int_\Omega |v'|^2\right)$.

For the convergence of the integral on $\partial\Omega_j$, we show that $\int_{\partial\Omega_j}(\partial_{n_j}v_j)^2H_j(\theta\cdot n_j)^2 \to \int_{\partial\Omega}(\partial_n v)^2H(\theta\cdot n)^2$ and the other terms can be treated similarly. We can write

$$\begin{aligned}\int_{\partial\Omega_j}(\partial_{n_j}v_j)^2H_j(\theta\cdot n_j)^2 &= \int_{\partial\Omega}\tilde{J}_j(\widehat{\partial_{n_j}v_j})^2\widehat{H_j}(\widehat{\theta\cdot n_j})^2\\ &= \int_{\partial\Omega_j}\widehat{H_j}\widehat{g_j}\end{aligned}$$

for $\widehat{g_j} := \tilde{J}_j(\widehat{\partial_{n_j}v_j})^2(\widehat{\theta\cdot n_j})^2$, where $\tilde{J}_j := \tilde{J}_{\xi_j}$ is the surface jacobian defined in (3.24). We claim that $\widehat{g_j} \to (\partial_n v)^2(\theta\cdot n)^2 =: g$ in $C^{0,\alpha}$. This comes from Lemmas 3.3.6 and 3.3.7, and moreover one estimates $\left((\widehat{\theta\cdot n_j}) - \theta\cdot n\right)$ as follows

$$\begin{aligned}\|(\widehat{\theta\cdot n_j}) - \theta\cdot n\|_{C^{0,\alpha}(\partial\Omega)} &\le \|\theta\cdot(\widehat{n_j} - n)\|_{C^{0,\alpha}(\partial\Omega)} + \|(\widehat{\theta}-\theta)\widehat{n_j}\|_{C^{0,\alpha}(\partial\Omega)}\\ &\le \|\theta\|_{C^{0,\alpha}(\partial\Omega)}\|\xi_j\|_{C^{1,\alpha}(\partial\Omega)} + C\|\theta\|_{C^{1,\alpha}(\partial\Omega)}\|\xi_j\|_{C^{0,\alpha}(\partial\Omega)}\end{aligned}$$

where the estimate of $\widehat{\theta}-\theta$ comes from writting $\widehat{\theta}-\theta = \int_0^1 \nabla\theta((1-t)\cdot + t\phi_{\xi_j})\cdot\xi_j dt$. As a consequence $\widehat{g_j} \to g$ in any $W^{s,p}$ for $s\in[0,\alpha)$ and $p\in[1,\infty)$, thanks to the embedding $C^{0,\alpha}(\partial\Omega)\subset W^{s,p}(\partial\Omega)$ (see (ii) from Proposition 3.5.2). On the other hand $\widehat{H_j} \to H$ in $W^{-s,p}$ for any $s\in(0,1)$ and $p\in(1,\infty)$ thanks to Lemma 3.3.7. We therefore use a $W^{-s,p'}\cdot W^{s,p}$ duality estimate (see (i) from Proposition 3.5.2) to deduce that

$$\int_{\partial\Omega}\widehat{H_j}\widehat{g_j} \to \int_{\partial\Omega} Hg$$

This proves the convergence of (3.51) towards the corresponding expression for Ω, and thus concludes Step 2.

Conclusion. Since each Ω_j is smooth we have that $\lambda''_{\Omega_j}(0)\cdot(\theta,\theta)$ both equals (3.51) and (3.50) for each j. The two previous steps ensure that we can pass to the limit on both sides to deduce that (3.49) holds for Ω. □

3.3.3 Estimate of $\lambda_1''(t)$ and proof of the main results.

Relying on the expression of the second derivative given by Lemma 3.3.9 we can now tackle estimating the variation of the second derivative of $\lambda_1(t) = \lambda_1((\mathrm{Id}+t\xi_h)(B))$. We first obtain an expression for $\lambda_1''(t)$, which we state in the next Lemma.

Let us first recall and set some notations for the remainder of this section. We use the notations from Theorems 3.3.1, 3.3.4 and 3.3.5: any $h:\partial B\to\mathbb{R}$ is extended to the whole of $\mathbb{R}^N$ into some h locally constant in normal directions around ∂B, and we then set $\xi_h(x) := h(x)x$ so that $\|\xi_h\|_{C^{1,\alpha}(\mathbb{R}^N)^N} \le C\|h\|_{C^{1,\alpha}(\partial B)}$ for any $\alpha\in(0,1]$ and $h\in C^{1,\alpha}(\partial B)$. We let $B_h := (\mathrm{Id}+\xi_h)(B)$ and $B_t := B_{th}$ for $t\in[0,1]$. The notations n_t, H_t and b_t refer respectively to the outer unit normal, curvature and second fundamental form of B_t.

Lemma 3.3.10. *Let $h\in C^{1,1}(\partial B)$. For all $t\in[0,1]$ it holds*

$$\lambda_1''(t) = 2\left(\int_{B_t}|\nabla v_t'|^2 - \lambda_1(t)\int_{B_t}|v_t'|^2\right) + \int_{\partial B_t}(\partial_{n_t}v_t)^2\left[H_t(\xi_h\cdot n_t)^2 - b_t((\xi_h)_{\tau_t},(\xi_h)_{\tau_t}) + 2\nabla_{\tau_t}(\xi_h\cdot n_t)\cdot(\xi_h)_{\tau_t}\right] \tag{3.52}$$

where $(\xi_h)_{\tau_t}$ is the tangential (over ∂B_t) component of $\nabla\xi_h$, v_t is the first L^2 normalized eigenfunction of B_t and v_t' is determined by the equations

$$\begin{cases}-\Delta v_t' = \lambda_1(t)v_t' + \lambda_1'(t)v_t, & \text{in } B_t\\ v_t' = -\partial_{n_t}v_t(\xi_h\cdot n_t), & \text{over } \partial B_t\\ \int_{B_t}v_t'v_t = 0.\end{cases}$$

Proof of Lemma 3.3.10. By Definition 3.3.2 it holds

$$\lambda_1''(t) = \lambda_B''(t\xi_h) \cdot (\xi_h, \xi_h)$$

but since $B_{s+t} = (\mathrm{Id} + (s+t)\xi_h)(B) = (\mathrm{Id} + s\xi_h)(B_t)$ when $\|h\|_{L^\infty(\partial B)}$ is small we immediately get

$$\lambda_B''(t\xi_h) \cdot (\xi_h, \xi_h) = \lambda_{B_t}''(0) \cdot (\xi_h, \xi_h)$$

The result then follows from applying Lemma 3.3.9 with $\Omega = B_t$. □

Relying on the expression of $\lambda_1''(t)$ given by this Lemma we are now ready to prove Theorem 3.3.5.

Proof of Theorem 3.3.5. Let us first suppose that $h \in C^{1,1}(\partial B)$. We can therefore use the expression of $\lambda_1''(t)$ from Lemma 3.3.10 which we rewrite in the following way

$$\begin{aligned} \lambda_1''(t) &= 2\int_{B_t} \left(|\nabla v_t'|^2 - \lambda_1(t) \int_{B_t} |v_t'|^2 \right) + \int_{\partial B_t} (\partial_{n_t} v_t)^2 \left[H_t \alpha_t^2 - b_t(\beta_t, \beta_t) - 2\nabla_{\tau_t} \alpha_t \cdot \beta_t \right] h^2 \\ &\quad -2\int_{\partial B_t} (\partial_{n_t} v_t)^2 \alpha_t (\beta_t \cdot \nabla_{\tau_t} h) h \qquad (3.53) \\ &:= \mathcal{T}_1(t) + \mathcal{T}_2(t) + \mathcal{T}_3(t) \end{aligned}$$

where we put $\alpha_t := n_t \cdot n$ and $\beta_t := \alpha_t n_t - n$. We thus prove for each $1 \le i \le 3$

$$\forall t \in [0,1],\ |\mathcal{T}_i(t) - \mathcal{T}_i(0)| \le \omega(\|h\|_{C^{1,\alpha}(\partial B)}) \|h\|^2_{H^{1/2}(\partial B)}. \qquad (3.54)$$

Estimate of $\mathcal{T}_1(t)$. See that $v_t' = v'_{t\xi_h, \xi_h}$ in the notations of Lemma 3.3.8. Writing $\int_{B_t} |\nabla v_t'|^2 = \int_B \langle A_t \nabla \widehat{v_t'}, \nabla \widehat{v_t'} \rangle$ where $A_t := A_{t\xi_h}$ is defined in (3.29), we get

$$\begin{aligned} \left| \int_{B_t} |\nabla v_t'|^2 - \int_B |\nabla v_0'|^2 \right| &= \left| \int_B \langle (A_t - \mathrm{Id}) \nabla \widehat{v_t'}, \nabla \widehat{v_t'} \rangle + (\nabla \widehat{v_t'} - \nabla v_0') \cdot (\nabla \widehat{v_t'} + \nabla v_0') \right| \\ &\le \omega(\|h\|_{C^{1,\alpha}(\partial B)}) \|h\|^2_{H^{1/2}(\partial B)} \end{aligned}$$

thanks to Lemma 3.3.8. On the other hand,

$$\begin{aligned} \left| \lambda_1(t) \int_{B_t} |v_t'|^2 - \lambda_1(0) \int_B |v_0'|^2 \right| &= \left| (\lambda_1(t) - \lambda_1(0)) \int_{B_t} |v_t'|^2 + \lambda_1(0) \int_B (J_t - 1) |\widehat{v_t'}|^2 + (\widehat{v_t'} - v_0)(\widehat{v_t} + v_0) \right| \\ &\le \omega(\|h\|_{C^{1,\alpha}(\partial B)}) \|h\|^2_{H^{1/2}(\partial B)} \end{aligned}$$

using Lemmas 3.3.8 and 3.3.6. Putting these two together finally yields

$$|\mathcal{T}_1(t) - \mathcal{T}_1(0)| \le \omega(\|h\|_{C^{1,\alpha}(\partial B)}) \|h\|^2_{H^{1/2}(\partial B)}$$

thus finishing the proof of the estimate of $\mathcal{T}_1(t)$.

Estimate of $\mathcal{T}_2(t)$. Thanks to a surface change of variables we have $\mathcal{T}_2(t) = \int_{\partial B} \widehat{\sigma_t} h$ where

$$\widehat{\sigma_t} := (\widehat{\partial_{n_t} v_t})^2 \left[\widehat{H_t} \widehat{\alpha_t}^2 - \widehat{b_t}(\widehat{\beta_t}, \widehat{\beta_t}) - 2\widehat{\nabla_{\tau_t} \alpha_t} \cdot \widehat{\beta_t} \right] \tilde{J}_t$$

with $\tilde{J}_t$ the surface Jacobian. We have

$$|\mathcal{T}_2(t) - \mathcal{T}_2(0)| = \left| \int_{\partial B} (\widehat{\sigma_t} - \sigma_0) h^2 \right| \qquad (3.55)$$

Notice that $\widehat{\sigma}_t$ is a sum of terms of the form $\widehat{y}_t \times \widehat{z}_t$ for some $\widehat{y}_t \in \{\widehat{H_t}, (\widehat{b}_t)_{ij}, (\widehat{\nabla_{\tau_t}\alpha_t})_i\}$ with $\widehat{z}_t$ which is a product of terms in $\{\widehat{\alpha}_t, \widehat{\partial_{n_t} v_t}, \widehat{\beta}_t, \tilde{J}_t\}$. By Lemma 3.3.7 we have that $\widehat{y}_t - y_0 = \omega_{s,p}(\xi_h)$ (uniformly in t), which we denote more simply $\omega_{s,p}(h)$. On the other hand, thanks to Lemmas 3.3.6 and 3.3.7, for all $s \in (0,1)$, $p \in (1,\infty)$ and $\alpha \in (s,1)$ there exists ω such that $\|\widehat{z}_t - z_0\|_{W^{s,p}(\partial B)} \leq C\|\widehat{z}_t - z_0\|_{C^{0,\alpha}(\partial B)} \leq \omega(\|h\|_{C^{1,\alpha}(\partial B)})$ (using also the embedding $C^{0,\alpha}(\partial\Omega) \subset W^{s,p}(\partial\Omega)$, see (ii) from Proposition 3.5.2). As a consequence we also have that $\widehat{\sigma}_t - \sigma_0 = \omega_{s,p}(h)$, which we write $\widehat{\sigma}_t - \sigma_0 = a_{1,h}b_{1,h} + a_{2,h}b_{2,h}$ in the notations of Lemma 3.3.7.

Let now $s \in (0,1)$ and $p \in (1,\infty)$ be such that the couple (s,p) satisfies (v) from Proposition 3.5.2. This allows the product law $H^{1/2}(\partial B) \cdot H^{1/2}(\partial B) \subset W^{s,p}(\partial B)$, and we thus have

$$\|b_{i,h}h^2\|_{W^{s,p}(\partial B)} \leq C\|b_{i,h}h\|_{H^{1/2}(\partial B)}\|h\|_{H^{1/2}(\partial B)}.$$

Using the product law $C^{0,\alpha}(\partial\Omega) \cdot H^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ from (iv) of Proposition 3.5.2 we get

$$\|b_{1,h}h\|_{H^{1/2}(\partial B)} \leq C\|h\|_{H^{1/2}(\partial B)},$$

$$\|b_{2,h}h\|_{H^{1/2}(\partial B)} = \omega\left(\|h\|_{C^{1,\alpha}(\partial B)}\right)\|h\|_{H^{1/2}(\partial B)}.$$

Using a duality estimate $W^{-s,p'} \cdot W^{s,p}$ (see (i) from Proposition 3.5.2) we can therefore estimate (3.55)

$$\begin{aligned}|\mathcal{T}_2(t) - \mathcal{T}_2(0)| &\leq \|a_{1,h}\|_{W^{-s,p'}(\partial B)}\|b_{1,h}h^2\|_{W^{s,p}(\partial B)} + \|a_{2,h}\|_{W^{-s,p'}(\partial B)}\|b_{2,h}h^2\|_{W^{s,p}(\partial B)} \\ &= \omega(\|h\|_{C^{1,\alpha}(\partial B)})\|h\|^2_{H^{1/2}(\partial B)}.\end{aligned}$$

Estimate of $\mathcal{T}_3(t)$. Using a surface change of variables we have $\mathcal{T}_3(t) = \int_{\partial B} h(\widehat{\rho}_t \cdot \nabla_{\widehat{\tau}_t} h)$ where

$$\widehat{\rho}_t := (\widehat{\partial_{n_t} v_t})^2 \widehat{\alpha}_t \widehat{\beta}_t \tilde{J}_t, \ \nabla_{\widehat{\tau}_t} h := \nabla h - (\nabla h \cdot \widehat{n}_t)\widehat{n}_t.$$

We write

$$|\mathcal{T}_3(t) - \mathcal{T}_3(0)| = \left|\int_{\partial B} h\widehat{\rho}_t \cdot (\nabla_\tau h - \nabla_{\widehat{\tau}_t} h) + \int_{\partial B} h(\widehat{\rho}_t - \rho_0) \cdot \nabla_\tau h\right|.$$

Now, $\nabla_\tau h - \nabla_{\widehat{\tau}_t} h = (\nabla h \cdot \widehat{n}_t)\widehat{n}_t = \nabla_\tau h \cdot (\widehat{n}_t - n)\widehat{n}_t$ since $\nabla h \cdot n = 0$. On the other hand $\|\widehat{\rho}_t - \rho_0\|_{C^{0,\alpha'}(\partial B)} \leq \omega(\|h\|_{C^{1,\alpha}(\partial B)})$ for any $0 < \alpha' < \alpha < 1$ thanks to Lemmas 3.3.6 and 3.3.7. Using a duality estimate $H^{-1/2} \cdot H^{1/2}$, a product law $C^{0,\alpha'} \cdot H^{1/2} \subset H^{1/2}$ and the fact that $\|\nabla_\tau h\|_{H^{-1/2}(\partial B)} \leq C\|h\|_{H^{1/2}(\partial B)}$ (see (i), (iii) and (iv) from Proposition 3.5.2) we thus get

$$\begin{aligned}|\mathcal{T}_3(t) - \mathcal{T}_3(0)| &\leq \|h\widehat{\rho}_t(\widehat{n}_t - n)\widehat{n}_t\|_{H^{1/2}(\partial B)}\|\nabla_\tau h\|_{H^{-1/2}(\partial B)} + \|h(\widehat{\rho}_t - \rho_0)\|_{H^{1/2}(\partial B)}\|\nabla_\tau h\|_{H^{-1/2}(\partial B)} \\ &\leq \omega(\|h\|_{C^{1,\alpha}(\partial B)})\|h\|^2_{H^{1/2}(\partial B)}\end{aligned}$$

using Lemma 3.3.7. This finishes the proof of the $\mathcal{T}_3(t)$ estimate.

We have thus proved (3.54) and hence Theorem 3.3.5 in the case where $h \in C^{1,1}(\partial B)$. We then reduce the regularity hypothesis made over h to $h \in C^{1,\alpha}(\partial B)$ with a density argument. Recall that $\lambda_1''(t) = \lambda''_{B_{th}}(0) \cdot (\xi_h, \xi_h)$ since $B_{s+t} = (\mathrm{Id} + (s+t)\xi_h)(B) = (\mathrm{Id} + s\xi_h)(B_t)$ when $\|h\|_{L^\infty(\partial B)}$ is small. Let then h_j be smooth and converging to h in $C^{1,\beta}(\partial B)$ for each $0 < \beta < \alpha$ with $\|h_j\|_{C^{1,\alpha}(\partial B)} \leq \|h\|_{C^{1,\alpha}(\partial B)}$. With an argument similar to **Step 1** of the proof of Lemma 3.3.9 we can pass to the limit in the expression $\lambda''_{B_{th_j}}(0) \cdot (\xi_{h_j}, \xi_{h_j})$ to get that $\lambda''_{B_{th_j}}(0) \cdot (\xi_{h_j}, \xi_{h_j}) \to \lambda_1''(t)$. We can thus let $j \to +\infty$ in

$$\left|\lambda''_{B_{th_j}}(0) \cdot (\xi_{h_j}, \xi_{h_j}) - \lambda''_B(0) \cdot (\xi_{h_j}, \xi_{h_j})\right| \leq \omega(\|h_j\|_{C^{1,\alpha}(\partial B)})\|h_j\|^2_{H^{1/2}(\partial B)}$$

and get the desired estimate. This finishes the proof in the general case.

□

Theorem 3.3.4 is now a consequence of Theorem 3.3.5. The way to pass from an **(IC)** to an **(IT)** condition was shown in [DL19] (see [DL19, p.3014]), but we reproduce the short proof for the convenience of the reader.

Proof of Theorem 3.3.4 . Fix $c > 0$. Thanks to Theorem 3.3.5 we find $\delta > 0$, $\alpha \in (0,1)$ and a modulus of continuity ω_{λ_1} such that for all $h \in C^{1,\alpha}(\partial B)$ with $\|h\|_{C^{1,\alpha}(\partial B)} \leq \delta$ it holds

$$\forall t \in [0,1],\ |\lambda_1''(t) - \lambda_1''(0)| \leq \omega_{\lambda_1}(\|h\|_{C^{1,\alpha}(\partial B)})\|h\|^2_{H^{1/2}(\partial B)} \tag{3.56}$$

We can write

$$\lambda_1(B_h) = \lambda_1(B) + (\lambda_1)'_B(0)\cdot(\xi_h) + \frac{1}{2}(\lambda_1)''_B(0)\cdot(\xi_h,\xi_h) + \int_0^1 (\lambda_1''(t) - \lambda_1''(0))(1-t)dt$$

using a second-order Taylor expansion with integral remainder. Using (3.56) we thus get

$$\lambda_1(B_h) = \lambda_1(B) + (\lambda_1)'_B(0)\cdot(\xi_h) + \frac{1}{2}(\lambda_1)''_B(0)\cdot(\xi_h,\xi_h) + \omega_{\lambda_1}(\|h\|_{C^{1,\alpha}(\partial B)})\|h\|^2_{H^{1/2}(\partial B)}$$

hence that λ_1 satisfies an $\mathbf{(IT)}_{H^{1/2},C^{1,\alpha}}$ condition. Combining this together with the expansion for P (see (3.22)) we get the expansion for $\mathcal{J}_c = P - c\lambda_1$:

$$\mathcal{J}_c(B_h) = \mathcal{J}_c(B) + (\mathcal{J}_c)'_B(0)\cdot(\xi_h) + \frac{1}{2}(\mathcal{J}_c)''_B(0)\cdot(\xi_h,\xi_h) + \omega_c(\|h\|_{C^{1,\alpha}(\partial B)})\|h\|^2_{H^1(\partial B)}$$

with $\omega_c := \omega_P - c\omega_{\lambda_1}$. This concludes the proof. □

We are now able to prove Theorem 3.3.1, relying on the stability results proved in [DL19].

Proof of Theorem 3.3.1. We proved in Theorem 3.3.4 that the functional $\mathcal{J}_c$ satisfies an $\mathbf{(IT)}_{H^1,C^{1,\alpha}}$ condition in the sense of [DL19, Theorem 1.3]. It also satisfies a $(\mathbf{C}_{H^1})$ hypothesis (see [DL19, Lemma 2.8]). On the other hand it was proven in [Nit14, Theorem 1.2] (see also [DL19, Proposition 5.5 (ii)]) that $\mathcal{J}_c$ is a critical and strictly stable shape under volume constraint and up to translations whenever $c \in (0, c^*)$. For any such c we therefore apply [DL19, Theorem 1.3] and get that there exists $\delta_c > 0$ such that for any $h \in C^{1,\alpha}(\partial B)$ with $\|h\|_{C^{1,\alpha}(\partial B)} \leq \delta_c$ and $|B_h| = |B|$ it holds

$$\mathcal{J}_c(B_h) \geq \mathcal{J}_c(B)$$

with equality only if B_h is a ball. This gives strict minimality (up to translations) of B in a $C^{1,\alpha}$ neighborhood, thus concluding the proof of the Theorem. □

3.4 *Selection principle*: minimality of the ball among convex sets. Proof of Theorem 3.1.2.

This section is dedicated to the second part of the *selection principle* strategy which we described in the Introduction, namely the regularizing procedure which enables to reduce the proof of the inequality from Theorem 3.1.2

$$\forall K \in \mathcal{K}^N \text{ with } |K\Delta B| \leq \delta_c,\ (P - c\lambda_1)(K) \geq (P - c\lambda_1)(B)$$

for general convex perturbations K of B to the same inequality for $C^{1,\alpha}$ perturbations of B. As is usual in this procedure (as was originally done by [CL12], see also among many others [AFM13], [BDPV15], [AKN21]) the argument goes by contradiction: we assume that (3.4) does not hold, meaning that there exists a sequence (K_j) converging to the ball in the L^1 sense but for which $\mathcal{J}_c(K_j) < \mathcal{J}_c(B)$. The strategy is then to replace the sequence K_j by a sequence $\widetilde{K_j}$ also converging to B, for which (3.4) is still not verified

and in which each $\widetilde{K_j}$ is meant to be much smoother than K_j. The sets $\widetilde{K_j}$ are built as minimizers of an auxiliary minimization problem which is constructed in a way that any minimizer satisfies the first two properties. The regularity of $\widetilde{K_j}$ then comes from the fact that it is a minimizer of an isoperimetric problem under convexity constraint: it was shown in [LP23] that such minimizers are $C^{1,1}$, and we will provide a uniform version of this result (see Theorem 3.4.2). This will enable us to apply the result of minimality in a smooth neighborhood proven in Section 3.3 (see Theorem 3.3.1) to finally get a contradiction.

In this section $B \subset \mathbb{R}^N$ still denotes the open centered unit ball.

3.4.1 Regularity theory for the quasi-minimizer of the perimeter under convexity constraint

This subsection is dedicated to the regularity theorem which is central to the *selection principle* we perform in Section 3.4. When working in the framework of quasi-minimizer of the perimeter without convexity constraint, a very useful type of results concerns the strengthening of convergence for a sequence of quasi-minimizers converging to the ball: if a sequence (E_j) of (uniform) quasi-minimizers converges to the ball in a L^1 sense, then E_j is $C^{1,1/2}$ for large j and it converges (up to subsequence) to the ball in $C^{1,\alpha}$ for each $\alpha \in (0, 1/2)$ (see for instance [AFM13, Theorem 4.2] for a rigorous statement). This is a compactness-type result, which is a direct consequence of the $C^{1,1/2}$ regularity of quasi-minimizers and an estimate of their norm. We want here to prove an analogous result in our convexity constrained case. The regularity result we state below (Theorem 3.4.2) importantly relies on the $C^{1,1}$ regularity results from [LP23] (see [LP23, Theorem 2.3]). Nevertheless, in comparison with [LP23, Theorem 2.3] we have to follow the constants in the proof in order to show that a quasi-minimizer is locally parametrized in cartesian graphs by $C^{1,1}$ functions with norm only depending on the relevant constants. We then pass from this quantified local cartesian $C^{1,1}$ regularity to a global spherical estimate. Although this passage often comes as classical in the literature, it does not seem to be so well referenced and we believe that a careful examination of all the arguments can be of use (see also [Pet22, Appendix B] for similar arguments).

Let us first define the notion of quasi-minimizer of the perimeter under convexity constraint which was introduced in [LP23, Definition 2.1].

Definition 3.4.1 ((Λ, ε)-q.m.p.c.c.)**.** *Let $N \geq 2$. Let $\Lambda > 0$, $\varepsilon > 0$. We say that $K \in \mathcal{K}^N$ is a (Λ, ε)-quasi-minimizer of the perimeter under convexity constraint (or (Λ, ε)-q.m.p.c.c. for short) if*

$$\forall \widetilde{K} \in \mathcal{K}^N \text{ such that } \widetilde{K} \subset K \text{ and } |K \setminus \widetilde{K}| \leq \varepsilon, \quad P(K) \leq P(\widetilde{K}) + \Lambda |K \setminus \widetilde{K}|. \tag{3.57}$$

For $x \in \mathbb{R}^N$, let $\nu_B(x) := x$. Any function $h : \partial B \to \mathbb{R}$ is extended to $\mathbb{R}^N$ as in Section 3.3 (see the beginning of Section 3.3). For $r > 0$ and $z \in \mathbb{R}^N$, the notation $B_r(z)$ denotes the ball of radius r centered at z. The q.m.p.c.c. regularity result is the following.

Theorem 3.4.2 (Regularity of q.m.p.c.c)**.** *Let $N \geq 2$, $\Lambda > 0$, $\varepsilon > 0$ and $0 < m < M$. Let K be a (Λ, ε)-q.m.p.c.c. verifying $B_m(z) \subset K \subset B_M(z)$ for some $z \in \mathbb{R}^N$. Then there exists $h \in C^{1,1}(\partial B)$ such that (up to translation) K can be written*

$$K = (Id + h\nu_B)(B) = \{tx(1 + h(x)),\ t \in [0, 1],\ x \in \partial B\}, \text{ with } \|h\|_{C^{1,1}(\partial B)} \leq C$$

with $C = C(N, \Lambda, \varepsilon, m, M) > 0$ only depending on the indicated parameters.

Let us postpone the proof of this Theorem and show first how we deduce from this result a convergence type result of quasi-minimizers (as in the classical setting).

Corollary 3.4.3 (Convergence of q.m.p.c.c)**.** *Let $N \geq 2$, $\Lambda > 0$, $\varepsilon > 0$. If (K_j) is a sequence of (Λ, ε)-q.m.p.c.c. such that $|K_j \Delta B| \to 0$, then there exists a sequence $h_j \in C^{1,1}(\partial B)$ such that*

$$\forall j \in \mathbb{N},\ K_j = (Id + h_j \nu_B)(B),$$

and for all $\alpha \in (0, 1)$ it holds $h_j \to 0$ in $C^{1,\alpha}$.

Remark 3.4.4. Let us note here that this $C^{1,\alpha}$ convergence for all $\alpha \in (0,1)$ is optimal in the sense that one cannot hope for more than $C^{1,1}$ regularity for a q.m.p.c.c. (see [LP23, Proposition 3.18] for a counter-example to higher Hölder regularity in two dimensions).

Proof of Corollary 3.4.3. Since $|K_j \Delta B| \to 0$, one also has that $K_j \to B$ in the Hausdorff sense thanks to Proposition 3.5.4. As a consequence, there exists $z \in \mathbb{R}^N$ and $0 < m < M$ such that

$$\forall j \in \mathbb{N},\ B_m(z) \subset K_j \subset B_M(z).$$

The existence of the upper ball follows directly from the definition of the Hausdorff convergence; we refer for instance to [LP23, Proposition 2.8, 2.] for the existence of a lower ball. Therefore, thanks to Theorem 3.4.2 we deduce that there exists $h_j : \partial B \to \mathbb{R}$ such that

$$\forall j \in \mathbb{N},\ K_j = (\mathrm{Id} + h_j \nu_B)(B) \text{ and } \|h_j\|_{C^{1,1}(\partial B)} \leq C$$

for some $C > 0$ independent of j. From this bound on the $C^{1,1}$ norms we deduce for each $\alpha \in (0,1)$ the convergence (up to subsequence) of h_j in $C^{1,\alpha}$ norm to some $h \in C^{1,1}(\partial B)$, using the Arzela-Ascoli Theorem. Since $K_j \to B$ in the Hausdorff sense we must have $h = 0$, which ensures also that the whole sequence (h_j) converges to 0. This finishes the proof of Corollary 3.4.3. □

We can now pass to the proof of Theorem 3.4.2.

Proof of Theorem 3.4.2. **Step 1: cartesian estimates of K.** Let $\widehat{x_0} \in \partial K$ be fixed. We claim that there exists

- a hyperplane $H \subset \mathbb{R}^N$ containing $\widehat{x_0}$ and a unit vector $\xi \in \mathbb{R}^N$ normal to H,
- a $(N-1)$ dimensional ball B_β centered at $\widehat{x_0}$ and of radius $\beta = \beta(m, M) > 0$ with $B_\beta \subset H$

such that, denoting by (x,t) a point in $H \times \mathbb{R}\xi$ coordinates and defining $u : B_\beta \to \mathbb{R}$ by the formula $u(x) := \min\{t \in \mathbb{R}, (x,t) \in K\}$ we have

$$\{(x, u(x)),\ x \in B_\beta\} \subset \partial K \tag{3.58}$$

$$K \cap (B_\beta \times \mathbb{R}\xi) \subset \{(x,t) \in B_\beta \times \mathbb{R}\xi,\ u(x) \leq t\} \tag{3.59}$$

and $u \in C^{1,1}(\overline{B_\beta})$ with

$$\|u\|_{C^{1,1}(\overline{B_\beta})} \leq C, \text{ where } C = C(N, \Lambda, \varepsilon, m, M) \tag{3.60}$$

In this Step, for any $z \in H$ and $r > 0$, we denote by $B_r(z) \subset H$ the $(N-1)$-dimensional ball of radius r centered at z.

The existence of H, ξ, β', u such that

$$\beta' = \beta'(m, M)$$

$$u \in C^{0,1}(B_{\beta'}(\widehat{x_0})), \text{ with } \|\nabla u\|_{L^\infty(B_{\beta'}(\widehat{x_0}))} \leq C(m, M)$$

and such that (3.58) and (3.59) are satisfied for $B_{\beta'}(\widehat{x_0})$ comes from the convexity of K (it is proven for instance in [LP23, Proposition 4.3]). We now prove (3.60) for $\beta := \beta'/2$. Let $y \in B_\beta(\widehat{x_0})$, $p \in \partial u(y)$ and set for each $r \in (0, \beta)$

$$M_r(y) := \sup_{B_r(y)} (u - (u(y) + \langle p, \cdot - y\rangle)$$

Using quasi-minimality of K and the estimates of β' and $\|\nabla u\|_{L^\infty(B_{\beta'}(\widehat{x_0}))}$ above, we can use [LP23, Theorem 2.3] (see last equation of the proof of Theorem 2.3) to deduce that there exists

$$C = C(N, \Lambda, \varepsilon, m, M)$$

$$r_0 = r_0(\varepsilon, m, M)$$

such that

$$\forall y \in B_\beta(\widehat{x_0}),\ \forall r \in (0, r_0),\ M_r(y) \leq Cr^2$$

We now apply Lemma 3.2 in [DPF15] which ensures that $u \in C^{1,1}(\overline{B_\beta(\widehat{x_0})})$. More precisely, it is proven in [DPF15, Lemma 3.2] that there exists $\rho_0 > 0$ and $\eta > 0$ only depending on r_0 and the Lipschitz character of $B_\beta(\widehat{x_0})$ (hence only on β) such that

$$\forall x \in B_\beta(x_0),\ \forall y \in B_\beta(\widehat{x_0}) \cap B_{\rho_0}(x),\ |\nabla u(x) - \nabla u(y)| \leq C'|x - y|$$

where $C' = 6\eta^{-1}$. As we also have

$$\forall x \in B_\beta(\widehat{x_0}),\ \forall y \in B_\beta(\widehat{x_0}) \text{ with } |y - x| \geq \rho_0,$$
$$|\nabla u(x) - \nabla u(y)| \leq 2\|\nabla u\|_{L^\infty(B_\beta(x_0))}\rho_0^{-1}|x - y|$$

then gathering the two we get that

$$\forall x, y \in B_\beta(\widehat{x_0}), |\nabla u(x) - \nabla u(y)| \leq \widetilde{C}|x - y|$$

by setting $\widetilde{C} := \max\left\{C', 2\|\nabla u\|_{L^\infty(B_\beta(x_0))}\rho_0^{-1}\right\}$. This together with the bound on $\|\nabla u\|_{L^\infty(B_{\beta'}(\widehat{x_0}))}$ above, and

$$\|u\|_{L^\infty(B_\beta(\widehat{x_0}))} \leq \operatorname{diam}(K)$$

yield the desired estimate on $\|u\|_{C^{1,1}(\overline{B_\beta})}$.

Step 2: local spherical estimates of ∂K. This step and the next one are similar to [Pet22, Appendix B]. Fix $\widehat{x_0} \in \partial K$. We apply Step 1 at $\widehat{x_0}$, and up to translating and rotating we assume without loss of generality that $z = 0$ (so that $B_m(0) \subset K \subset B_M(0)$) and $\xi = e_N$ is the Nth canonical direction. In this step we consider z as the origin, so that the coordinates $(x, t) \in H \times \mathbb{R}$ will now take this into account. As a consequence, $\widehat{x_0}$ is now written $\widehat{x_0} = (0, t_0)$ for some $t_0 < 0$, and if $\Omega := B_\beta$ denotes the $(N-1)$-dimensional ball found in Step 1, we have

$$\forall x \in \Omega,\ \widehat{u}(x) := (x, u(x) + t_0) \in \partial K$$

Since H is orthogonal to $\xi = e_N$ and contains $\widehat{x_0}$ we have

$$H = \widehat{x_0} + \{x_N = 0\}$$

We write more simply $B_m := B_m(0)$. Let θ be the map

$$\theta : \Omega \to \partial B_m$$
$$x \mapsto m\frac{\widehat{u}(x)}{|\widehat{u}(x)|}$$

which associates to $x \in \Omega$ the spherical coordinates corresponding to $\widehat{u}(x)$. Let now $\rho_K : \partial B_m \to (0, \infty)$ be the distance function of the convex set K, meaning that for any $\phi \in \partial B_m$, $\rho_K(\phi)$ is the unique $t > 0$ such that $t\phi \in \partial K$. Then for each $x \in \Omega$, it holds $\theta(x)\rho_K(\theta(x)) = \widehat{u}(x)$, so that

$$\rho_K(\theta(x)) = \frac{|\widehat{u}(x)|}{m} \tag{3.61}$$

As a consequence, $\rho_K \circ \theta \in C^{1,1}(\Omega)$ with norm only depending on the $C^{1,1}$ norm of u. The rest of Step 2 consists in showing that ρ_K itself is $C^{1,1}$ next to $\theta(x_0)$ and to estimate its norm, by using a suitable version of the inverse function Theorem.

We let θ' be the map

$$\begin{aligned}\theta' : \Omega \times \mathbb{R} &\to \mathbb{R}^N \\ (x,t) &\mapsto \theta(x)(1+t)\end{aligned}$$

Note that $\theta' \in C^{1,1}(\Omega \times \mathbb{R})$ and $\theta'_{|\Omega\times\{0\}} = \theta$. Then it holds

$$D\theta'(\widehat{x_0}) = m \left(\begin{array}{c|c} |t_0|^{-1} I_{N-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots\cdots\cdots 0 & 1 \end{array}\right) \tag{3.62}$$

Using a quantitative version of the inverse function Theorem (see Theorem 3.5.1) we deduce the existence of a radius $r_0 = r_0\left(\|\theta'\|_{C^{1,1}(V_0)}, |D\theta'(x_0)^{-1}|, \beta\right)$, V_0 and W_0 respectively open neighborhoods of $\widehat{x_0}$ and $\theta'(\widehat{x_0})$ such that θ' is a $C^{1,1}$-diffeomorphism from V_0 onto W_0, with

$$B_{r_0}(\widehat{x_0}) \subset V_0, B_{r_0}(\theta'(\widehat{x_0})) \subset W_0 \tag{3.63}$$

and

$$\|(\theta')^{-1}\|_{C^{1,1}(W_0)} \leq C\left(\|\theta'\|_{C^{1,1}(V_0)}, |D\theta'(x_0)^{-1}|\right) \tag{3.64}$$

Now, by definition of θ' and θ, and since $|\widehat{u}(x)| \geq m$, it holds $\|\theta'\|_{C^{1,1}(V_0)} \leq C\left(\|u\|_{C^{1,1}(\Omega)}, m^{-1}\right)$. On the other hand, by (3.62) we have $|(D\theta')(x_0)^{-1}| \leq C(|t_0|, m^{-1})$. As $K \subset B_M(0)$ we have $|t_0| \leq M$ so that (3.63) and (3.64) become respectively

$$B_{r_0}(\widehat{x_0}) \subset V_0, B_{r_0}(\theta'(\widehat{x_0})) \subset W_0, \text{ with } r_0 = r_0\left(\beta, m, M, \|u\|_{C^{1,1}(\Omega)}\right) \tag{3.65}$$

and

$$\|(\theta)'^{-1}\|_{C^{1,1}(W_0)} \leq C\left(m, M, \|u\|_{C^{1,1}(\Omega)}\right)$$

Recalling that $\theta'_{|\Omega\times\{0\}} = \theta$ there exists a constant $C > 0$ such that $\|\theta^{-1}\|_{C^{1,1}(W_0\cap\partial B_m)} \leq \|(\theta')^{-1}\|_{C^{1,1}(W_0)}$. Hence, by (3.61) and the latter estimate of θ' we get

$$\begin{aligned}\|\rho_{K_j}\|_{C^{1,1}(W_0\cap\partial B_m)} &\leq \|\rho_{K_j}\circ\theta\|_{C^{1,1}(V_0\cap H)}\|\theta^{-1}\|_{C^{1,1}(W_0\cap\partial B_m)} \\ &\leq C, \text{ with } C = C\left(m, M, \|u\|_{C^{1,1}(\Omega)}\right)\end{aligned} \tag{3.66}$$

Using the estimates of β and $\|u\|_{C^{1,1}(\Omega)}$ found in Step 1 we finally get that the radius r_0 and constant C respectively from (3.65) and (3.66) only depend on $N, \Lambda, \varepsilon, m, M$.

Step 3: global estimate of ρ_K: relying on the local estimate (3.66) of ρ_K proven in Step 2 we now estimate $\|\rho_K\|_{C^{1,1}(\partial B_m)}$.

According to Step 2, for any $\phi \in \partial B_m$ there exists $W_\phi \subset \mathbb{R}^N$ an open neighborhood of ϕ such that

$$B_{r_0}(\phi) \subset W_\phi$$

$$\|\rho_K\|_{C^{1,1}(W_\phi\cap\partial B_m)} \leq C$$

where $r_0 = r_0(N, \Lambda, \varepsilon, m, M)$ and $C = C(N, \Lambda, \varepsilon, m, M)$. Using a standard compactness argument over ∂B_m we find $\eta > 0$ only depending on r_0 such that for any $\phi, \psi \in \partial B_m$ with $|\phi - \psi| \leq \eta$

$$\frac{|\rho_K(\phi) - \rho_K(\psi)|}{|\phi - \psi|} \leq C, \ \frac{|\nabla\rho_K(\phi) - \nabla\rho_K(\psi)|}{|\phi - \psi|} \leq C$$

Combining these with a global bound $\|\rho_K\|_{W^{1,\infty}(\partial B_m)} \leq C$ we deduce that the same estimates hold for $|\phi - \psi| \geq \eta$ so that we finally have

$$\|\rho_K\|_{C^{1,1}(\partial B_m)} \leq C \tag{3.67}$$

for some $C = C(N, \Lambda, \varepsilon, m, M)$.

Conclusion. Let $h(\phi) := \rho_K(m\phi) - 1$ for each $\phi \in \partial B$. Then we have that $K = (\mathrm{Id} + hn_B)(B)$ and (3.67) ensures that

$$||h||_{C^{1,1}(\partial B)} \leq C$$

for some $C = C(N, \Lambda, \varepsilon, m, M) > 0$. This concludes the proof.

□

3.4.2 Proof of Theorem 3.1.2

In this subsection we perform the *selection principle*, relying both on the convergence result for quasi-minimizers (Corollary 3.4.3) and the strict minimality of the ball in a $C^{1,\alpha}$ neighborhood shown in Theorem 3.3.1.

We will use the fact that λ_1 satisfies some kind of Lipschitz hypothesis for the Volume distance. This is stated in next Proposition.

Proposition 3.4.5. *Let $N \geq 2$. Let $D \in \mathcal{K}^N$ and $0 < V_0 < |D|$. There exists $C = C(V_0, D)$ such that for all convex bodies $K_1, K_2 \subset D$ with $|K_1|, |K_2| \geq V_0$ it holds*

$$|\lambda_1(K_1) - \lambda_1(K_2)| \leq C|K_1 \Delta K_2|. \tag{3.68}$$

Note that this Lipschitz type property is an improvement of the result obtained for λ_1 in [LP23, Theorem 3.2 and Remark 3.3].

Proof of Proposition 3.4.5. It was proven in [LP23, Theorem 3.2 and Remark 3.3] that for any $D' \subset D \in \mathcal{K}^N$ there exists $C = C(D', D)$ such that for all $D' \subset K_1 \subset D$, $D' \subset K_2 \subset D$ it holds

$$|\lambda_1(K_1) - \lambda_1(K_2)| \leq C|K_1 \Delta K_2| \tag{3.69}$$

Let then $K_1, K_2 \subset D$ with $|K_1|, |K_2| \geq V_0$. Assume first that $|K_1 \Delta K_2| \geq V_0/2$. Thanks to (ii) in Proposition 3.5.3 we can find $\varepsilon = \varepsilon(V_0, D) > 0$ independent of K_1, K_2 such that the inradii satisfy $r_{K_1}, r_{K_2} \geq \varepsilon$. As a consequence, for $i = 1, 2$ there exists $x_i \in \mathbb{R}^N$ such that $B_\varepsilon(x_i) \subset K_i$. By monotonicity of λ_1 we deduce

$$|\lambda_1(K_1) - \lambda_1(K_2)| \leq 2\lambda_1(B_\varepsilon(0)) \leq 4V_0^{-1}\lambda_1(B_\varepsilon(0))|K_1 \Delta K_2| \tag{3.70}$$

Assume otherwise that $|K_1 \Delta K_2| \leq V_0/2$. Then we have

$$|K_1 \cap K_2| = |K_1| - |K_1 \setminus K_2| \geq V_0/2$$

Using again (ii) from Proposition 3.5.3 we can therefore find $\varepsilon' = \varepsilon'(V_0, D) > 0$ such that the inradius of the convex body $K_1 \cap K_2 \subset D$ satisfies $r_{K_1 \cap K_2} \geq \varepsilon'$. Hence, there exists $x \in D$ such that $B_{\varepsilon'}(x) \subset K_1 \cap K_2 \subset K_i$, $i = 1, 2$. Letting $R > 0$ be such that $B_R(x) \supset D$, we have $B_{\varepsilon'}(0) \subset K_i - x \subset B_R(0)$ for $i = 1, 2$ and we therefore use property (3.69) to deduce

$$\begin{aligned} |\lambda_1(K_1) - \lambda_1(K_2)| &= |\lambda_1(K_1 - x) - \lambda_1(K_2 - x)| \\ &\leq C|(K_1 - x)\Delta(K_2 - x)| \\ &= C|K_1 \Delta K_2| \end{aligned}$$

This estimate together with (3.70) gives the conclusion. □

We can now pass to the proof of Theorem 3.1.2.

For convex bodies K_1 and K_2 the notation $d_H(K_1, K_2)$ refers to the usual Hausdorff distance between K_1 and K_2 (see Section 3.5.3 in the Appendix for some facts about the Hausdorff distance for convex sets).

Proof of Theorem 3.1.2. **Step 1: penalization.** As a preparation of the selection procedure from Step 2 below, we prove in this step that if we let $D \in \mathcal{K}^N$, $0 < V_0 < |D|$, $a \in \mathbb{R}, \mu \geq 0$ and set $R = -\lambda_1 + \mu||K\Delta B| - a|$, then a minimizer K^* of

$$\inf\left\{P(K) + R(K), K \in \mathcal{K}^N,\ K \subset D,\ |K| = V_0\right\} \tag{3.71}$$

is a (Λ, ε)-q.m.p.c.c. (see Definition 3.4.1) for some $\Lambda = \Lambda\left(V_0, d(K^*, \partial D), \mu, D\right)$ and $\varepsilon = \varepsilon\left(V_0, d(K^*, \partial D), \mu, D\right)$. This result is an adaptation of [LP23, Lemma 2.11].

Let $0 < v_0 < V_0$ and $\delta > 0$. We introduce the class

$$\mathcal{A}_{v_0,\delta} := \{K \in \mathcal{K}^N,\ K \subset D,\ |K| \geq v_0,\ d(K, \partial D) \geq \delta\}$$

which is compact for d_H, by Proposition 3.5.3 and continuity of the volume for the Hausdorff distance. Note that the set K^* belongs to the class for $v_0 = V_0$. Set

$$\forall \varepsilon > 0,\ \mathcal{O}_\varepsilon(K^*) := \{K \in \mathcal{K}^N,\ K \subset K^*, |K^* \setminus K| \leq \varepsilon\}$$

and let for any convex body $K \subset D$ and $t \in [0, 1]$

$$K_t := (1 - t)K + tD$$

We first claim that there exists constants $(\varepsilon_0, c, t_0) \in (0, \infty)^3$ depending only on V_0, δ and D (hence independent of the minimizer K^*) such that

$$\forall K \in \mathcal{O}_{\varepsilon_0}(K^*),\ \forall t \in [0, t_0],\ |K_t| - |K| \geq ct \tag{3.72}$$

Setting $f_K(t) := |K_t|$, it is shown in [LP23, Lemma 2.11] that f_K is a polynomial in t with coefficients continuous in K for d_H, and that $f'_K(0)$ is positive whenever $K \subsetneq D$. Set $\varepsilon_0 := V_0/2$. By compactness of $\mathcal{A}_{\varepsilon_0,\delta}$ inside $\{K \in \mathcal{K}^N,\ K \subsetneq D\}$ and continuity of $K \mapsto f'_K(0)$ for d_H, then one can find $c = c(V_0, \delta, D) > 0$ such that for any $K \in \mathcal{A}_{\varepsilon_0,\delta}$ it holds $f'_K(0) \geq c$. Any $K \in \mathcal{O}_{\varepsilon_0}(K^*)$ verifies $|K| \geq V_0/2$, so that we deduce $f'_K(0) \geq c$ for such K. Since the coefficients of the polynomial $f_K(t)$ are continuous in K for d_H, they are uniformly bounded for $K \in \mathcal{A}_{\varepsilon_0,\delta}$. This together with the lower bound on $f'_K(0)$ yields the above estimate.

The existence of $C = C(D)$ such that

$$\forall t \in [0, 1], \forall K \in \mathcal{K}^N \text{ with } K \subset D,\ P(K_t) - P(K) \leq Ct$$

is proven in [LP23, equation (54)]. As a consequence, this together with (3.72) gives

$$\forall K \in \mathcal{O}_{\varepsilon_0}(K^*),\ \forall t \in [0, t_0],\quad P(K_t) - P(K) \leq C'||K_t| - |K|] \tag{3.73}$$

with $C' := C/c$.

Since any $K \in \mathcal{O}_{\varepsilon_0}(K^*)$ verifies $|K| \geq V_0/2$ we can apply Proposition 3.4.5 to get the existence of $C = C(V_0, D)$ such that for all $t \in [0, 1]$ and $K \in \mathcal{O}_{\varepsilon_0}(K^*)$

$$\begin{aligned} R(K_t) - R(K) &= \lambda_1(K) - \lambda_1(K_t) + \mu\left(||K_t\Delta B| - a| - ||K\Delta B| - a|\right) \\ &\leq C|K_t \setminus K| + \mu||K_t\Delta B| - |K\Delta B|| \\ &\leq (C + \mu)|K_t \setminus K| \end{aligned} \tag{3.74}$$

Let us now show that for $\varepsilon := \min\{\varepsilon_0, ct_0\}$ there exists $\Lambda = \Lambda(V_0, \delta, D, \mu)$ such that a minimizer K^* of (3.71) is a minimizer of

$$\inf\{P + R + \Lambda||K| - V_0|,\ K \in \mathcal{O}_\varepsilon(K^*)\} \tag{3.75}$$

Since $|K_{t_0}| - |K^*| = |K_{t_0}| - |K| + |K| - |K^*| \geq ct_0 - \varepsilon \geq 0$ then by continuity of $t \mapsto |K_t|$ there exists $t \in [0, t_0]$ such that $|K_t| = |K^*| = V_0$. Hence by minimality and using (3.73) and (3.74) we get

$$P(K^*) + R(K^*) \leq P(K_t) + R(K_t) \leq P(K) + R(K) + \Lambda||K| - V_0||$$

for some $\Lambda = \Lambda(V_0, \delta, D, \mu)$, which ensures the minimality of K^* for (3.75).

Therefore, if $K \in \mathcal{K}^N$ with $K \subset K^*$ and $|K^* \setminus K| \leq \varepsilon$ then the computation leading to (3.74) with (K^*, K) in place of (K_t, K) gives

$$P(K^*) - P(K) \leq (C + \mu + \Lambda)\, |K^* \setminus K|$$

so that K^* is a (Λ', ε)-q.m.p.c.c where $\Lambda' := \Lambda + C + \mu$ and ε only depend on V_0, δ, μ and D. This finishes the proof of the first step.

Step 2: selection procedure. Although the *selection principle* was first introduced in [CL12], the way we display the argument in this step is more inspired of [AFM13]. Let $0 < c < c^*$. Recall the notation $\mathcal{J}_c := P - c\lambda_1$. Let us assume in order to obtain a contradiction that the conclusion of Theorem 3.1.2 is false. Then there exists a sequence of convex bodies $(K_j)_{j\in\mathbb{N}}$ such that

$$\begin{cases} \forall j \in \mathbb{N},\ \mathcal{J}_c(K_j) < \mathcal{J}_c(B) \\ |K_j \Delta B| \to 0 \end{cases}$$

Thanks to Proposition 3.5.4, $K_j \to B$ in the Hausdorff sense, so that there exists $D \in \mathcal{K}^N$ such that $K_j \subset D$ for every j. We can assume without loss of generality that $B \Subset \mathrm{Int}(D)$. Thanks to Proposition 3.4.5 the functional λ_1 is lower-semi-continuous for the volume distance, and we can apply the existence result [LP23, Theorem 3.4 (i)] to get that for any fixed $\mu > 0$ there exists for each j a solution to the problem

$$\inf \left\{ \mathcal{J}_c(K) + \mu \left| |K\Delta B| - |K_j \Delta B| \right|,\ |K| = |B|,\ K \subset D,\ K \in \mathcal{K}^N \right\} \tag{3.76}$$

Assuming that the value of μ has been fixed (we choose μ later on), we let $\widetilde{K_j}$ be a solution.

Thanks to (i) from Proposition 3.5.3 there exists a convex body $\widetilde{K} \subset D$ with $|\widetilde{K}| = |B|$ such that (up to subsequence) $\widetilde{K_j} \to \widetilde{K}$ in the Hausdorff sense and in measure. We have $\lambda_1(\widetilde{K_j}) \to \lambda_1(\widetilde{K})$ using Proposition 3.4.5, and $P(\widetilde{K_j}) \to P(\widetilde{K})$ by [BB05, Proposition 2.4.3 (ii)]. Now, from optimality and recalling $\mathcal{J}_c(K_j) < \mathcal{J}_c(B)$ we can write

$$\mathcal{J}_c(\widetilde{K_j}) + \mu \left| |\widetilde{K_j}\Delta B| - |K_j \Delta B| \right| \leq \mathcal{J}_c(K_j) < \mathcal{J}_c(B) \tag{3.77}$$

so that we get at the limit

$$\mathcal{J}_c(\widetilde{K}) + \mu |\widetilde{K}\Delta B| \leq \mathcal{J}_c(B) \tag{3.78}$$

Thanks to Proposition 3.4.5 and using the isoperimetric inequality we get

$$\begin{aligned} \mathcal{J}_c(\widetilde{K}) - \mathcal{J}_c(B) &= \Big(P(\widetilde{K}) - P(B)\Big) + c\Big(\lambda_1(B) - \lambda_1(\widetilde{K})\Big) \\ &\geq c(\lambda_1(B) - \lambda_1(\widetilde{K})) \\ &\geq -C|\widetilde{K}\Delta B| \end{aligned}$$

where the constant C only depends on c and D. Injecting this into (3.78) provides

$$-C|\widetilde{K}\Delta B| + \mu|\widetilde{K}\Delta B| \leq 0$$

so that we get $\widetilde{K} = B$ if μ is chosen bigger than C in (3.76).

Therefore $\widetilde{K_j} \to B$ in measure and Hausdorff distance. Since we have chosen $B \Subset \mathrm{Int}(D)$ we find a convex body $\widetilde{D} \subset D$ with $d(\widetilde{D}, \partial D) > 0$ and such that for j sufficiently large

$$D' \subset \widetilde{K_j} \subset \widetilde{D}$$

By construction of the $\widetilde{K_j}$, we deduce from Step 1 that each $\widetilde{K_j}$ is a (Λ, ε)-q.m.p.c.c. with parameters independent of j. We can therefore apply Corollary 3.4.3 to get the existence of $h_j \in C^{1,1}(\partial B)$ such that up to subsequence

$$\widetilde{K_j} = (\mathrm{Id} + h_j \nu_B)(B) \text{ and } \|h_j\|_{C^{1,\alpha}(\partial B)} \to 0$$

for α chosen to satisfy Theorem 3.3.1. We can therefore apply Theorem 3.3.1 to deduce that for sufficiently large j,

$$\mathcal{J}_c(\widetilde{K_j}) \geq \mathcal{J}_c(B)$$

But this enters in contradiction with (3.77), thus concluding the proof of the Theorem. □

3.5 Appendix

3.5.1 Quantified Inverse Function Theorem

Theorem 3.5.1 (Quantified IFT)**.** *Let $N \in \mathbb{N}^*$. Let $V := B_r(\widehat{x_0}) \subset \mathbb{R}^N$ for some $\widehat{x_0} \in V$ and $r > 0$. Let $f \in C^{1,1}(V, \mathbb{R}^N)$ with $Df(\widehat{x_0})$ invertible. Then there exists $V_0 \subset V, W_0 \subset \mathbb{R}^N$ and $\rho = \rho\left(\|f\|_{C^{1,1}}, |Df(x_0)^{-1}|, r\right)$ only depending on the indicated parameters such that f is a $C^{1,1}$ diffeomorphism from V_0 onto W_0 and*

$$B_\rho(\widehat{x_0}) \subset V_0,\ B_\rho(f(\widehat{x_0})) \subset W_0$$

$$\|f^{-1}\|_{C^{1,1}(W_0,V_0)} \leq C\left(\|f\|_{C^{1,1}(V,\mathbb{R}^N)}, |Df(x_0)^{-1}|, r\right)$$

Proof. **Step 1:** By following the usual proof of the Inverse function Theorem we first show that f is a C^1 diffeomorphism from $V_0' \subset V$ to $W_0' := B_\delta(f(\widehat{x_0}))$ with

$$\delta = \delta\left(\|Df\|_{C^{0,1}(V)}, |Df(\widehat{x_0})^{-1}|\right).$$

Since the proof is classical we only emphasize on the details needed to quantify the size of the neighborhood W_0'.

We will keep the notation V for the set $V := B_r(\widehat{x_0})$. Let for any $y \in \mathbb{R}^N$ the function $\phi_y : V \to \mathbb{R}^N$ be defined by

$$\forall x \in V,\ \phi_y(x) := x - (Df(\widehat{x_0}))^{-1}(f(x) - y).$$

Then ϕ_y is $C^{1,1}$, for $x \in V$ its differential $D\phi_y(x) = \mathrm{Id} - (Df(\widehat{x_0}))^{-1} Df(x)$ is independent of y, and we have for $x \in V$,

$$\begin{aligned}|D\phi_y(x)| &\leq |Df(\widehat{x_0})^{-1}||Df(x) - Df(\widehat{x_0})| \\ &\leq |Df(\widehat{x_0})^{-1}|\|Df\|_{C^{0,1}(V)}|x - \widehat{x_0}|\end{aligned}$$

As a consequence there exists

$$r' = r'\left(|Df(\widehat{x_0})^{-1}|, \|Df\|_{C^{0,1}(V)}, r\right) > 0$$

such that for all $x \in B_{r'}(\widehat{x_0})$ it holds $|D\phi_y(x)| \leq 1/2$. We thus get that for all $y \in \mathbb{R}^N$, ϕ_y is 1/2-Lipschitz over $B_{r'}(\widehat{x_0})$. Now, we have

$$|Df(\widehat{x_0})^{-1}(f(\widehat{x_0}) - y)| \leq |Df(\widehat{x_0})^{-1}||f(\widehat{x_0}) - y| \leq r'/2$$

when $y \in B_\delta(f(\widehat{x_0}))$ with $\delta := \frac{r'}{2}|Df(\widehat{x_0})^{-1}|$. Hence,

$$\begin{aligned}\forall y \in B_\delta(f(\widehat{x_0})),\ \forall x \in \overline{B_{r'}}(\widehat{x_0}),\ |\phi_y(x) - \widehat{x_0}| &\leq |\phi_y(x) - \phi_y(\widehat{x_0})| + |\phi_y(\widehat{x_0}) - \widehat{x_0}| \\ &\leq |x - \widehat{x_0}|/2 + r'/2 \\ &\leq r'\end{aligned}$$

As a consequence, for all $y \in B_\delta(f(\widehat{x_0}))$, ϕ_y sends $\overline{B_{r'}}(\widehat{x_0})$ into itself and is 1/2-Lipschitz. Therefore, for such y the mapping ϕ_y has a unique fixed point in $\overline{B_{r'}}(\widehat{x_0})$, meaning that $f(x) = y$ for a unique $x \in \overline{B_{r'}}(\widehat{x_0})$.

This gives the existence of $f^{-1}: W_0' \to V_0'$ with $W_0' := B_\delta(f(\widehat{x_0}))$ and $V_0' := f^{-1}(W_0')$, and one classically shows that f is C^1 diffeomorphism from V_0' to W_0' with furthermore $Df^{-1} = (Df(f^{-1}))^{-1}$.

Step 2: Using an explicit expansion of the inverse mapping about $Df(\widehat{x_0})$ we have that

$$|Df(x)^{-1}| \leq 2|Df(\widehat{x_0})^{-1}| \tag{3.79}$$

whenever $|Df(x) - Df(\widehat{x_0})| \leq 1/(2|Df(\widehat{x_0})^{-1}|)$. But this latter condition is fulfilled if $x \in V_0 := B_{\widetilde{r}}(\widehat{x_0})$ for some $\widetilde{r}$ depending on r', $|Df(\widehat{x_0})^{-1}|$ and $\|f\|_{C^{1,1}(V)}$ only. Moreover, f is a C^1 diffeomorphism from V_0 to $W_0 := f(V_0)$, and one can find $\widetilde{\delta} = \widetilde{\delta}\left(\widetilde{r}, \|f^{-1}\|_{C^{0,1}(W_0)}\right)$ such that $W_0 \supset B_{\widetilde{\delta}}(f(\widehat{x_0}))$. Now, thanks to (3.79) and $Df^{-1} = (Df(f^{-1}))^{-1}$ we have

$$\|f^{-1}\|_{L^\infty(W_0)} \leq r, \ \|Df^{-1}\|_{L^\infty(W_0)} \leq 2|Df(\widehat{x_0})^{-1}| \tag{3.80}$$

Letting $y, y' \in W_0$, since $f^{-1}(y), f^{-1}(y') \in V_0$ we can use (3.79) and (3.80) to get

$$\begin{aligned} |Df^{-1}(y) - Df^{-1}(y')| &\leq \|(Df)^{-1}\|^2_{L^\infty(V_0)} |Df(f^{-1}(y)) - Df(f^{-1}(y'))| \\ &\leq C|y - y'| \end{aligned}$$

for some $C = C\left(|Df(\widehat{x_0})^{-1}|, \|f\|_{C^{1,1}(V)}, r'\right)$. Combined with (3.80), and recalling the estimate of r' from Step 1 we thus get

$$\|f^{-1}\|_{C^{1,1}(W_0)} \leq C\left(|Df(\widehat{x_0})^{-1}|, \|f\|_{C^{1,1}(V)}, r\right)$$

Setting $\rho := \min\{\widetilde{\delta}, \widetilde{r}\}$ we have proved the Theorem. □

3.5.2 Fractional Sobolev spaces

In this paragraph we state some standard facts about Sobolev spaces on the boundary of a $C^{1,1}$ open set $\Omega \subset \mathbb{R}^N$.

Proposition 3.5.2. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a $C^{1,1}$ bounded open set.*

(i) Let $s \in (0,1)$ and $p \in (1,\infty)$. There exists $C > 0$ such that for any $a, b \in C^{1,1}(\partial\Omega)$

$$\left|\int_{\partial\Omega} ab\right| \leq C\|a\|_{W^{-s,p'}(\partial\Omega)}\|b\|_{W^{s,p}(\partial\Omega)}.$$

(ii) Let $s \in (0,1)$. Then for all $\alpha \in (s,1)$ and $p \in [1,\infty)$ it holds

$$C^{0,\alpha}(\partial\Omega) \subset W^{s,p}(\partial\Omega), \ C^{1,\alpha}(\partial\Omega) \subset W^{1+s,p}(\partial\Omega),$$

with continuous injections.

(iii) Let $s \in (0,1)$ and $p \in (1,\infty)$. Then there exists $C > 0$ such that for any $a \in C^{1,1}(\partial\Omega)$ it holds

$$\|\nabla_\tau a\|_{W^{-s,p}(\partial\Omega)} \leq C\|a\|_{W^{1-s,p}(\partial\Omega)}.$$

(iv) There exists $\alpha \in (0,1)$ such that the product law $C^{0,\alpha}(\partial\Omega) \cdot H^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ holds, meaning that for any $a, b \in C^{1,1}(\partial\Omega)$

$$\|ab\|_{H^{1/2}(\partial\Omega)} \leq C\|a\|_{C^{0,\alpha}(\partial\Omega)}\|b\|_{H^{1/2}(\partial\Omega)},$$

for some constant $C > 0$ independent of a and b.

(v) *There exists* $s \in (0,1)$ *and* $p \in (1,\infty)$ *for which the product law* $H^{1/2}(\partial\Omega) \cdot H^{1/2}(\partial\Omega) \subset W^{s,p}(\partial\Omega)$ *holds, meaning that for any* $a, b \in C^{1,1}(\partial\Omega)$

$$\|ab\|_{W^{s,p}(\partial\Omega)} \leq C\|a\|_{H^{1/2}(\partial\Omega)}\|b\|_{H^{1/2}(\partial\Omega)},$$

for some constant $C > 0$ *independent of* a *and* b.

Proof. All of the statements are deduced from the same statements over $W^{s,p}(\mathbb{R}^{N-1})$ spaces by working in local charts. We give hereafter brief indications or references for the $\mathbb{R}^{N-1}$ case.

(i) For the same statement over $\mathbb{R}^{N-1}$ see for instance [RS11, Proposition p.20].

(ii) If $f \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^{N-1})$, then if $A \subset \mathbb{R}^{N-1}$ is bounded

$$\forall x, y \in A,\ \frac{|f(x) - f(y)|^p}{|x-y|^{N-1+sp}} \leq |f|^p_{C^{0,\alpha}(A)}|x-y|^{-(N-1-p(\alpha-s))}$$

where $|\cdot|_{C^{0,\alpha}(A)}$ is the $C^{0,\alpha}$ semi-norm on A. The local integrability of

$$(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} \mapsto |x-y|^{-(N-1-p(\alpha-s))}$$

then ensures that $C^{0,\alpha}_{\text{loc}}(\mathbb{R}^{N-1}) \subset W^{s,p}_{\text{loc}}(\mathbb{R}^{N-1})$.

(iii) For the same fact over $\mathbb{R}^{N-1}$, see for instance [Bha12, Remark 8.10.14].

(iv) For $(N-1)/p < s < 1$ one has the product law $W^{s,p}(\mathbb{R}^{N-1}) \cdot H^{1/2}(\mathbb{R}^{N-1}) \subset H^{1/2}(\mathbb{R}^{N-1})$ (see [BH21, Lemma 7.2]). Using (ii) we obtain the announced product law.

(v) Applying [RS11, Theorem 1 p.176] we first have the product law $H^{1/2}(\mathbb{R}^{N-1}) \cdot H^{1/2}(\mathbb{R}^{N-1}) \subset F^{1/2}_{\tilde{p},2}(\mathbb{R}^{N-1})$ for any $\tilde{p} > 1$ verifying $\frac{1}{\tilde{p}} > 1 - \frac{1}{2N}$, where $F^{1/2}_{\tilde{p},2}(\mathbb{R}^{N-1})$ denotes the usual Triebel-Lizorkin space of corresponding indices. Applying then [RS11, Theorem p. 31] we have $F^{1/2}_{\tilde{p},2}(\mathbb{R}^{N-1}) \subset W^{s,p}(\mathbb{R}^{N-1})$ where $p > \tilde{p}$ is close to $\tilde{p}$ and $s = 1/2 + N(1/p - 1/\tilde{p}) \in (0, 1/2)$, so that we obtain in fact $H^{1/2}(\mathbb{R}^{N-1}) \cdot H^{1/2}(\mathbb{R}^{N-1}) \subset W^{s,p}(\mathbb{R}^{N-1})$.

□

3.5.3 Compactness in classes of convex sets

In this paragraph we gather some classical facts about Hausdorff distance and compactness in some classes of convex bodies.

If C_1 and C_2 are non-empty compact subsets of $\mathbb{R}^N$, the Hausdorff distance $d_H(C_1, C_2)$ between C_1 and C_2 is defined as the quantity

$$d_H(C_1, C_2) := \max\left\{\sup_{x \in C_1} d(x, C_2), \sup_{x \in C_2} d(x, C_1)\right\}$$

where $d(\cdot,\cdot)$ denotes the euclidean distance. The Hausdorff distance d_H is a distance over the class of non-empty compact sets of $\mathbb{R}^N$. We say that a sequence $A_j \subset \mathbb{R}^N$ of non-empty compact sets converges in the Hausdorff sense when it converges for d_H.

Proposition 3.5.3. *Let* $D \in \mathcal{K}^N$ *and* $0 < V_0 < |D|$ *and set*

$$\mathcal{C}_1 := \left\{K \text{ compact convex of } \mathbb{R}^N, K \subset D\right\}$$

$$\mathcal{C}_2 := \left\{K \in \mathcal{K}^N,\ |K| = V_0,\ K \subset D\right\}$$

(i) The classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are compact for the Hausdorff distance.

(ii) There exists $\varepsilon = \varepsilon(V_0, D) > 0$ such that for each $K \in \mathcal{C}_2$ the inradius r_K of K satisfies $r_K \geq \varepsilon$.

Proof. (i) The Blaschke selection Theorem states that the class $\mathcal{C}_1$ is compact for the Hausdorff distance (see for instance [Sch14, Theorem 1.8.7]). Compactness of $\mathcal{C}_2$ then follows from the fact that $\mathcal{C}_2$ is a closed subset of $\mathcal{C}_1$, thanks to the continuity of the volume for d_H.

(ii) Let us show that the inradius mapping $K \in \mathcal{K}^N \mapsto r_K$ is l.s.c. for d_H. Let $K \in \mathcal{K}^N$ and $K_j \in \mathcal{K}^N$ with $K_j \to K$ in Haudorff distance, and let $r > 0$ and $x \in \mathbb{R}^N$ be such that $B_r(x) \Subset \mathrm{Int}(K)$. Thanks to [LP23, Proposition 2.8, 2.], we have that $K_j \supset B_r(x)$ for large enough j, so that $\liminf r_{K_j} \geq r$. This is valid for any $r < r_K$, so that $\liminf r_{K_j} \geq r_K$, thus showing that $K \in \mathcal{K}^N \mapsto r_K$ is l.s.c. Since furthermore $\mathcal{C}_2$ is compact for d_H, we deduce that $K \in \mathcal{C} \mapsto r_K$ has a minimum $\varepsilon > 0$, thus finishing the proof.

□

Proposition 3.5.4. *Let $B \subset \mathbb{R}^N$ be the centered unit ball. Let $K_j \in \mathcal{K}^N$ be a sequence of convex bodies such that $|K_j \Delta B| \to 0$. Then $K_j \to B$ in the Hausdorff sense.*

Proof. It suffices to show that there exists a bounded set $D \subset \mathbb{R}^N$ such that $K_j \subset D$ for each j, since the Blaschke selection Theorem then applies to provide compactness of the sequence K_j for d_H and therefore the convergence of the whole sequence K_j to B. Fix $j \in \mathbb{N}$; since $|K_j \Delta B| \to 0$ we can suppose j large enough so that $K_j \cap B \neq \emptyset$.

Suppose now that $K_j \not\subset B_2(0)$. Since $K_j \cap B \neq \emptyset$, by convexity of K_j we can find $x^j \in K_j$ with $|x^j| = 2$, which we will suppose (up to changing coordinates) to be written $x^j := x = (0, \dots, 0, 2)$. Let $x_0 := (-1, 0 \dots, 0)$ and set $x_1 := (1, 0 \dots, 0)$, $x_2 = (0, 1, 0 \dots, 0)$ until $x_{N-1} := (0, \dots, 0, 1, 0)$. Let finally $C := \mathrm{conv}\{x, x_0, x_1 \dots, x_{N-1}\}$ and some ball $B' \Subset \mathrm{Int}(C \setminus B)$.

Let $f : \mathbb{R}^{N+1} \to \mathcal{K}^N$ be defined by

$$f(y_1, \dots, y_{N+1}) = \mathrm{conv}\{y_1, \dots, y_{N+1}\} = \left\{ \sum_{i=1}^{N+1} \lambda_i y_i, \ \sum_{i=1}^{N+1} \lambda_i = 1, \ \lambda_i \geq 0 \right\}$$

Then f is continuous for the Hausdorff distance, so that there exists $\varepsilon > 0$ such that if for all $0 \leq i \leq N-1$, $|z_i - x_i| \leq \varepsilon$ then $\mathrm{conv}\{x, z_1, \dots, z_N\} \supset B'$ (see for instance [LP23, Proposition 2.8, 2.]). Now, let $\delta := \min_i\{|B \cap B_\varepsilon(x_i)|\} > 0$. Taking j sufficiently large so that $|K_j \Delta B| < \delta$, this implies that for such j there exists for each $i = 0, \dots, N-1$ some $y_i^j \in K_j \cap B_\varepsilon(x_i)$. By convexity, $K_j \supset \mathrm{conv}\{x, y_0^j \dots, y_{N-1}^j\}$, which itself contains B', thus giving $|K_j \setminus B| \geq |B'|$. This does not happen for sufficiently large j, and as a consequence there exists $j_0 \geq 0$ such that $K_j \subset B_2(0)$ for $j \geq j_0$. This proves the claim. □

Chapter 4

Sharp quantitative stability of the Dirichlet spectrum near the ball

This Chapter is a reprint of the submitted paper *Sharp quantitative stability of the Dirichlet spectrum near the ball*, written by the author of this thesis in collaboration with D. Bucur, J. Lamboley, and M. Nahon.

Abstract

Let $\Omega \subset \mathbb{R}^n$ be an open set with same volume as the unit ball B and let $\lambda_k(\Omega)$ be the k-th eigenvalue of the Laplace operator of Ω with Dirichlet boundary conditions on $\partial\Omega$. In this paper, we answer the following question:

If $\lambda_1(\Omega) - \lambda_1(B)$ is small, how large can $|\lambda_k(\Omega) - \lambda_k(B)|$ be ?

We establish quantitative bounds of the form $|\lambda_k(\Omega) - \lambda_k(B)| \leq C(\lambda_1(\Omega) - \lambda_1(B))^\alpha$ with sharp exponents α depending on the multiplicity of $\lambda_k(B)$. We first show that such an inequality is valid with $\alpha = 1/2$ for any k, improving previous known results; through the study of a vectorial free boundary problem, we then show that one can achieve the better exponent $\alpha = 1$ if $\lambda_k(B)$ is simple. We prove sharpness of the exponents in both cases. As a consequence of these results, we also obtain the persistence of the ball as a minimizer for a large class of spectral functionals which are small perturbations of the fundamental frequency on the one hand, and a full reverse Kohler-Jobin inequality on the other hand, solving an open problem formulated by M. Van Den Berg, G. Buttazzo and A. Pratelli.

4.1 Introduction

4.1.1 Presentation of the problem

Let $\mathbb{R}^n$ be the Euclidean space for some $n \geq 2$, and let ω_n denote the measure of the unit ball in $\mathbb{R}^n$. We set

$$\mathcal{A} = \{\Omega \subset \mathbb{R}^n \text{ open set of measure } \omega_n\},$$

and $B(= B_1)$ the unit ball of $\mathbb{R}^n$ centered at the origin. For Ω an open set of finite volume, we write

$$\lambda_k(\Omega) := \inf\left\{\sup_{v\in V} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2},\ V \subset H_0^1(\Omega) \text{ of dimension } k\right\}$$

the k-th eigenvalue of the Laplacian on Ω with Dirichlet boundary conditions on $\partial\Omega$ (counting multiplicities). The associated eigenfunctions, normalized in $L^2(\Omega)$, are denoted $(u_k)_{k\geq 1}$ and verify for each $k \in \mathbb{N}^*$

$$u_k \in H_0^1(\Omega),\ -\Delta u_k = \lambda_k(\Omega) u_k \text{ in } \Omega.$$

For every $\Omega \in \mathcal{A}$, the Faber-Krahn inequality states that $\lambda_1(\Omega) \geq \lambda_1(B)$, with equality if and only if Ω coincides with a ball (up to a set of zero capacity). Several recent works point out that Ω must be close to B in some sense when $\lambda_1(\Omega)$ is close to $\lambda_1(B)$. We refer to [BDPV15, AKN21] and the references therein for the most recent results and a history of the quantitative Faber-Krahn inequality. Roughly speaking, the variation of the first eigenvalue $\lambda_1(\Omega) - \lambda_1(B)$ controls (the square of) the Fraenkel asymmetry of Ω and of the L^2 norm of the variation of the eigenfunction.

The main purpose of this paper is to get a sharp control of the variation of the whole spectrum in terms of $\lambda_1(\Omega) - \lambda_1(B)$. Precisely, we seek inequalities of the form[1]

$$\left|\sum_{i=k}^{l} \big(\lambda_i(\Omega) - \lambda_i(B)\big)\right| \leq C_{n,k}\lambda_1(\Omega)^{1-\alpha}(\lambda_1(\Omega) - \lambda_1(B))^{\alpha}, \tag{4.1}$$

for which the power $\alpha > 0$ is sharp.

Heuristics about sharp power α. In some particular cases, inequality (4.1) has already been studied in the literature. In a first paper [BC06], Bertrand and Colbois established the inequality

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{n,k}\,(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{80n}}\,,$$

for sets Ω with $\lambda_1(\Omega)$ bounded from above. Later, relying on the quantitative Faber-Krahn inequality from [BDPV15], Mazzoleni and Pratelli improved the exponents into (see [MP19])

$$-c_{n,k}\,(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{6}-\varepsilon} \leq \lambda_k(\Omega) - \lambda_k(B) \leq C_{n,k}\,(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{12}-\varepsilon}\,, \tag{4.2}$$

for any $\varepsilon > 0$ (with better exponents in dimension $n = 2$) when $\lambda_1(\Omega)$ is bounded from above, but the authors naturally expected these exponents not to be optimal.

Indeed, the following observation is in order. When looking at domains which are volume-preserving smooth perturbations of the ball, one may see B as a non-degenerate stable critical point of λ_1 under volume constraint. On the other hand, for $k \geq 2$ the condition for B to be a critical point of λ_k is that the associated eigenfunction u_k has constant gradient on the boundary. This is the case for eigenvalues associated to radial eigenfunctions, which precisely correspond to the simple eigenvalues. In conclusion, when $\lambda_k(B)$ is simple one may expect a *sharp* bound of the type

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{n,k}(\lambda_1(\Omega) - \lambda_1(B)). \tag{4.3}$$

On the contrary, when $\lambda_k(B)$ is degenerate (multiple), then λ_k has only directional derivatives at B which, in general, are non-zero. Consequently, we cannot expect a better bound than

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{n,k}(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{2}}. \tag{4.4}$$

Nevertheless, as observed in [MP19], although $\lambda_2(B)$ is multiple one still has

$$\lambda_2(\Omega) - \lambda_2(B) \leq C(\lambda_1(\Omega) - \lambda_1(B)) \tag{4.5}$$

as a consequence of Ashbaugh-Benguria's inequality which asserts that the ball maximizes the ratio λ_2/λ_1. More generally, for a whole cluster associated to a multiple eigenvalue

$$\lambda_{k-1}(B) < \lambda_k(B) = \cdots = \lambda_l(B) < \lambda_{l+1}(B),$$

[1] Note that this formulation with $\lambda_1(\Omega)^{1-\alpha}$ is just a way to avoid assuming that $\lambda_1(\Omega)$ is bounded from above as in the previous references.

while each individual λ_i is not differentiable at B (for $k \le i \le l$), any smooth symmetric function of $(\lambda_k, \dots, \lambda_l)$ is differentiable and has a critical point at B. Therefore, one can still hope for a result better than (4.4), namely a linear bound on the sum

$$\left| \sum_{i=k}^{l} [\lambda_i(\Omega) - \lambda_i(B)] \right| \le C_{n,k}(\lambda_1(\Omega) - \lambda_1(B)) \tag{4.6}$$

generalizing the estimate for simple eigenvalues.

The goal of this paper is to show that (4.3), (4.4) and (4.6) indeed hold, and that the above observations turn out to provide the sharp exponents in (4.1).

Strategy. As a first result (see Theorem 4.1.1 below), we will show that one can obtain (4.1) for $\alpha = \frac{1}{2}$ (valid for simple and for multiple eigenvalues), by making use of suitable test functions and of the quantitative Faber-Krahn inequality. This improves the previous results from [BC06, MP19].

To obtain a sharper result with exponent $\alpha = 1$ (the case of a simple eigenvalue or of a whole cluster of multiple eigenvalues) our proof appeals to a deep analysis of a new type of vectorial free boundary problem which falls out from the situations already studied in the literature.

Indeed, let us consider $\lambda_{k-1}(B) < \lambda_k(B) = \cdots = \lambda_l(B) < \lambda_{l+1}(B)$: inequality (4.1) with $\alpha = 1$ becomes

$$\left| \sum_{i=k}^{l} [\lambda_i(\Omega) - \lambda_i(B)] \right| \le C_{n,k}(\lambda_1(\Omega) - \lambda_1(B)).$$

Its proof is equivalent to the fact that, for some $\varepsilon > 0$ small enough, the ball is the unique solution of both shape optimization problems below (i.e. for both signs $+$ and $-$)

$$\min \left\{ \lambda_1(\Omega) - \varepsilon \sum_{i=k}^{l} \lambda_i(\Omega) : \Omega \in \mathcal{A} \right\}, \qquad \min \left\{ \lambda_1(\Omega) + \varepsilon \sum_{i=k}^{l} \lambda_i(\Omega) : \Omega \in \mathcal{A} \right\}. \tag{4.7}$$

The strategy to prove this assertion is based on regularity theory, and proceeds by a series of steps, some of them being rather technical. We prove first the existence of an optimal domain and the Lipschitz regularity of the associated eigenfunctions. In a second step, we prove the regularity of the boundary and that, in some strong $\mathcal{C}^{2,\gamma}$ sense, the optimal domain is close to the ball. Finally, we use a second order shape derivative argument to conclude that the optimal domain is the ball, provided ε is small enough. These steps have been followed for example in [KM13, KM14] for the study of Gamow's model, which can be seen as a perturbation of the classical isoperimetric problem. Similar ideas can be found in [CL12], where the authors prove the quantitative isoperimetric inequality. On the other hand, we are dealing here with a perturbation of λ_1 instead of the perimeter functional, so that as in [BL09] and in the proof of the quantitative Faber-Krahn inequality from [BDPV15] (see also [MR21]) one needs to use the regularity tools developed in the field of free boundary problems.

Although the strategy to solve the shape optimization problems (4.7) follows the same main lines as [BDPV15], the nature of our problem raises a series of new technical difficulties, mostly in the case of the negative sign. First, in this case the shape functional is not decreasing for inclusion, so that the existence of a solution is not guaranteed by the general result of Buttazzo-Dal Maso [BDM93b]. Second, the optimality condition formally reads

$$\left(\frac{\partial u_1}{\partial \nu_\Omega} \right)^2 - \varepsilon \sum_{i=k}^{l} \left(\frac{\partial u_i}{\partial \nu_\Omega} \right)^2 = \text{constant on } \partial\Omega,$$

where ν_Ω is the outward normal vector at the boundary $\partial\Omega$. The presence of the negative sign falls out from all the situations studied in the literature [KL18, MTV17, CSY18, MTV17], including the degenerate case from [KL19]. The regularity analysis of this situation requires most of the technicalities. We will

use some key ideas from [MTV21] for the analysis of our problem: more precisely, when $k = l$ (*i.e.* for simple eigenvalues), we will be able to apply the results from [MTV21] (see Section 4.4.3), but when $k < l$ (case of multiple eigenvalues), we will have to show the same type of results in more general situations (see Sections 4.5.1 and 4.5.2).

As a global picture, our analysis will require to study a generalization of the vectorial Alt-Caffarelli problems in the wider setting

$$\begin{cases} -\Delta v_i = f_i & \text{in } \Omega,\ \forall i = 1, \dots, m, \\ v_i = 0 & \text{in } \partial\Omega,\ \forall i = 1, \dots, m, \\ q\left(\frac{\partial v_1}{\partial \nu_\Omega}, \dots, \frac{\partial v_m}{\partial \nu_\Omega}\right) = 1 & \text{in } \partial\Omega, \end{cases}$$

where Ω is the common domain of $(v_i)_{i=1,\dots,m}$, $f_i \in L^\infty_{\text{loc}}(\Omega)$ and q is a quadratic form on $\mathbb{R}^m$. The outward normal derivatives $\frac{\partial v_i}{\partial \nu_\Omega}$ at the boundary are understood in some weak sense - variational or viscosity (this will become clear later) - and the states (v_i) are assumed to be "flat" (in a sense precised in Sections 4.4 and 4.5 and Corollary 4.5.17):

- The case $m = 1$, $q(x) = x^2$ corresponds to the classical Alt-Caffarelli problem of [AC81].
- The case $m \geq 2$, $q(x_1, \dots, x_m) = \sum_{i=1}^m c_i x_i^2$ is the one treated in [KL18, KL19] with uniform estimates in (c_i) as long as $c_i \geq 0$, $\sum_{i=1}^m c_i = 1$. Similar results (obtained through different methods) may also be found in [CSY18, MTV17] in the case $c_i = 1$.
- The case $m = 2$, $q(x_1, x_2) = x_1 x_2$ under the additional hypothesis that u_1, u_2 are positive is treated in [MTV21].

Our problem may be seen as

- $m \geq 2$ with $q(x_1, \dots, x_m) = x_1^2 + b(x_2, \dots, x_m)$,

where b is a quadratic form on $\mathbb{R}^{m-1}$ with no positivity assumption. However we shall make use of the hypothesis that the function v_1 "dominates" all the others, meaning that $\left|\frac{v_i}{v_1}\right|$ is not too large for every $i \geq 2$ (for precise statements we refer to Definition 4.5.7). This hypothesis holds for free in some situations, for instance when v_1 is the torsion function and v_i is a small multiple of the eigenfunction u_i of Ω, where we recall that the torsion function w_Ω is the unique solution to

$$\begin{cases} \Delta w_\Omega = -1 & \text{in } \Omega, \\ w_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Applications in spectral geometry and a reverse Kohler-Jobin inequality. It has been observed numerically in [OK13, Fig 5.4] that the set minimizing $\lambda_k(\Omega)$ in $\mathcal{A}$ is also minimizing $\lambda_k(\Omega) + \varepsilon\lambda_{k-1}(\Omega)$, provided $\varepsilon > 0$ is small (the computations were performed for $3 \leq k \leq 6$).

This phenomenon of persistence of minimizers for perturbed functionals has also been conjectured in [vdBBP21] for a functional involving the first Dirichlet eigenvalue and the torsional rigidity which are interacting in a competing way. Recall that the torsional ridigidy is defined by

$$T(\Omega) := \int_\Omega w_\Omega = \int_\Omega 2w_\Omega - |\nabla w_\Omega|^2 = \max\left\{\int_\Omega 2v - |\nabla v|^2, v \in H^1_0(\Omega)\right\},$$

where w_Ω is the torsion function. While the Saint-Venant inequality states that the set with *maximal* torsional rigidity in $\mathcal{A}$ is the ball, the conjecture from [vdBBP21] reads

$$\exists p_n > 0, \forall \Omega \in \mathcal{A}, \qquad T(\Omega)\lambda_1(\Omega)^{\frac{1}{p_n}} \leq T(B)\lambda_1(B)^{\frac{1}{p_n}}. \tag{4.8}$$

As this inequality becomes Saint-Venant inequality when $p_n \to \infty$, the challenge is to prove that the ball B remains a maximizer of $T(\Omega)\lambda_1(\Omega)^{1/p}$ for finite values of p.

If $p_n = \frac{2}{n+2}$, the inequality above occurs in the opposite sense and is due to Kohler-Jobin [KJ78]. This is why, for p_n large, inequality (4.8) can be seen as a reverse Kohler-Jobin inequality. In [BBGLB22] it has been proved to hold locally for some p_n large, in the class of $\mathcal{C}^{2,\gamma}$ nearly spherical domains.

The main consequence of our analysis is the occurence of the persistance phenomenon of the ball as minimizer for spectral functionals which are either small perturbations of the first Dirichlet eigenvalue (for instance as in (4.7)) or of the (reciprocal of the) torsional rigidity. In particular we will prove the validity of the full reverse Kohler-Jobin inequality (4.8), see Corollary 4.1.5.

4.1.2 Main results

Inequality (4.1) for sharp exponents α will actually be proved in a stronger version, with the torsional deviation $T(\Omega)^{-1} - T(B)^{-1}$ on the right-hand side in place of $\lambda_1(\Omega) - \lambda_1(B)$.

Indeed, as noted just above, Kohler-Jobin's inequality from [KJ78] states that $\mathcal{A} \ni \Omega \mapsto T(\Omega)\lambda_1(\Omega)^{\frac{n+2}{2}}$ is minimal on the ball, which directly implies the bound

$$\begin{aligned} T(\Omega)^{-1} - T(B)^{-1} &\leq C_n \left(\lambda_1(\Omega)^{\frac{n+2}{2}} - \lambda_1(B)^{\frac{n+2}{2}}\right) \\ &\leq \frac{n+2}{2} C_n \lambda_1(\Omega)^{\frac{n}{2}} \left(\lambda_1(\Omega) - \lambda_1(B)\right) \end{aligned} \tag{4.9}$$

for any $\Omega \in \mathcal{A}$, with $C_n := \left(T(B)\lambda_1(B)^{\frac{n+2}{2}}\right)^{-1}$. Relying on this inequality and growth estimates of the type $\lambda_k(\Omega) \leq C_{n,k}\lambda_1(\Omega)$ (see below Proposition 4.2.1), in order to obtain inequalities of the type (4.1) it is enough to prove them for the torsional deviation in the right hand side instead.

One of the reasons why we replace the first eigenvalue with the torsion energy is of technical nature. In our problem, which involves simultaneously several eigenfunctions, we have a clear advantage to do this, since some uniform regularity estimates on these eigenfunctions may be directly deduced from the same estimates on the torsion function (see for instance Lemma 4.2.2). On the other hand, a second advantage is that inequality (4.1) gets a nontrivial meaning even for $k = 1$.

For the sake of clarity, we split the proof of inequality (4.1) with sharp exponents α in the three results. The first one applies to every eigenvalue, and is sharp when $\lambda_k(B)$ is degenerate.

Theorem 4.1.1. *There exists $C_n > 0$ such that for any $\Omega \in \mathcal{A}$,*

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_n k^{2+\frac{4}{n}} \lambda_1(\Omega)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \left(T(\Omega)^{-1} - T(B)^{-1}\right)^{\frac{1}{2}}.$$

Thanks to (4.9), this result improves the inequalities (4.2) from [MP19]. Note that we wrote this inequality in a scale invariant version, so that it holds for any open set Ω with a corresponding ball of same measure.

Note also that in this inequality we estimate the dependence in k of the constants. In order to do the same for the next results, we introduce for any $k \geq 1$ the spectral gap

$$\begin{aligned} g_n(k) = \min\Big\{1, \lambda_k(B) - \sup\{\lambda_i(B),\ i : \lambda_i(B) < \lambda_k(B)\}, \\ \inf\{\lambda_i(B),\ i : \lambda_i(B) > \lambda_k(B)\} - \lambda_k(B)\Big\}. \end{aligned} \tag{4.10}$$

It is a positive bounded function of k. We state first the case of a simple eigenvalue of the ball which gives a sharper estimate than the one from Theorem 4.1.1.

Theorem 4.1.2. *There exists $C_n > 0$ such that for every $k \in \mathbb{N}^*$ with $\lambda_k(B)$ simple and for any open set $\Omega \in \mathcal{A}$ it holds*

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_n \frac{k^{4+\frac{8}{n}}}{g_n(k)} |\Omega| \left(T(\Omega)^{-1} - T(B)^{-1}\right).$$

The constant C_n is not known explicitly since there are two implicit arguments in the proof (the flatness improvement of proposition 4.5.15 which is obtained by contradiction, and the application of the quantitative Faber-Krahn inequality from [BDPV15]).

In dimension 2 the valid choices of k are

$$k = 1, 6, 15, 30, 51, 74, 105, 140, 175, 222, 269, 326, 383, 446, 517, 588, ...$$

Let us mention again here that the crucial argument making the previous result works is that the ball is a critical point of λ_k when $\lambda_k(B)$ is simple.

Consider now $k \leq l$ such that

$$\lambda_{k-1}(B) < \lambda_k(B) = \cdots = \lambda_l(B) < \lambda_{l+1}(B).$$

The function $\Omega \mapsto \sum_{i=k}^{l} \lambda_i(\Omega)$ has a critical point at the ball (see for instance [LLdC06, Proposition 2.30]) and a result analogous to Theorem 4.1.2 holds.

Theorem 4.1.3. *There exists $C_n > 0$ such that for every $k, l \in \mathbb{N}^*$ with $k \leq l$ satisfying*

$$\lambda_{k-1}(B) < \lambda_k(B) = \cdots = \lambda_l(B) < \lambda_{l+1}(B),$$

and for any open set $\Omega \in \mathcal{A}$,

$$\left| \sum_{i=k}^{l} \left[\lambda_i(\Omega) - \lambda_i(B) \right] \right| \leq C_n \frac{k^{6+\frac{10}{n}}}{g_n(k)} |\Omega| \left(T(\Omega)^{-1} - T(B)^{-1}\right)$$

As a consequence of Theorems 4.1.1 and 4.1.3, we also obtain a general result on the stability of the Saint-Venant (and Faber-Krahn) inequality under perturbation by a spectral functional having enough symmetries.

Theorem 4.1.4. *Let $k \in \mathbb{N}^*$ be such that $\lambda_k(B) < \lambda_{k+1}(B)$. Let $F \in \mathcal{C}^2((\mathbb{R}_+^*)^k, \mathbb{R})$ be verifying*

$$|F(\lambda)| \leq C(1 + |\lambda|) \text{ for some } C > 0,$$
$$\forall i, j \in \{1, \ldots, k\} \text{ with } \lambda_i(B) = \lambda_j(B), \ \frac{\partial F}{\partial \lambda_i} = \frac{\partial F}{\partial \lambda_j} \text{ at } (\lambda_1(B), \ldots, \lambda_k(B))$$

Then there exists $\delta_F > 0$ such that for any $\delta \in \mathbb{R}$ with $|\delta| < \delta_F$, the functional

$$\Omega \in \mathcal{A} \mapsto T(\Omega)^{-1} + \delta F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) \tag{4.11}$$

is minimal only on balls.

In particular, the full reverse Kohler-Jobin inequality holds.

Corollary 4.1.5. *There exists $p_n > 1$ such that $\mathcal{A} \ni \Omega \mapsto T(\Omega)\lambda_1(\Omega)^{\frac{1}{p_n}}$ is maximal on the ball.*

4.1.3 Outline of the paper

In Section 4.2, we give some classical estimates of Dirichlet eigenvalues and eigenfunctions. In Section 4.3, we prove Theorem 4.1.1 as well as several useful lemmas on eigenfunctions and the torsion function. This is established by combining estimates from [Buc03] on eigenvalues of nested domains, some estimates with explicit test functions and the quantitative Faber-Krahn inequality of [BDPV15].

The next two sections are devoted to the proof of Theorems 4.1.2 and 4.1.3; while the second result is strictly stronger than the first, for expository reasons we shall first give a full proof of Theorem 4.1.2 in Section 4.4 and then adapt this proof to the vectorial case in Section 4.5, while pointing out the differences.

Precisely, in Section 4.4, we start by restating Theorem 4.1.2 as a shape optimization problem in the spirit of (4.7). In a first step, we prove the existence of a relaxed minimizer among capacitary measures and, in a second step that this measure corresponds to an open set which is a smooth perturbation of the ball in an increasingly stronger sense. The key passage from an open set to a $\mathcal{C}^{1,\gamma}$ set is obtained by relating our problem to a vectorial Alt-Caffarelli problem as in [KL18] or as in the more recent result [MTV21], depending on the sign of the perturbation. We finally conclude through second order shape derivative arguments for small perturbations of the ball.

Section 4.5 follows the steps of Section 4.4 in the case of the vectorial problem, with in addition a careful examination of the dependency of the constants in terms of the multiplicity of the eigenspace, obtained by following the proof of [KL18] on the one hand and through a full proof of a vectorial version of [MTV21] on the other hand.

The last section is devoted to the discussion of the consequences, namely the proof of Theorem 4.1.4 and of the reverse Kohler-Jobin inequality, Corollary 4.1.5.

4.2 Some preliminary estimates

We summarize here some results on eigenvalues and eigenfunctions which we will use throughout the paper. Although these results are not original, for the readability of the paper we give short proofs when possible, or at least give some comments about the proofs.

Eigenvalues of the ball. For any $d \in \mathbb{N}$ we define $\mathbb{H}_{n,d}$ the space of harmonic homogeneous polynomials of degree d in n variables $x_1, \dots, x_n$. For any $\alpha > 0$ we denote by J_α the α-th Bessel function

$$J_\alpha(x) = \sum_{p \geq 0} \frac{(-1)^p}{p!\Gamma(p+\alpha+1)} \left(\frac{x}{2}\right)^{2p+\alpha}$$

where Γ is the standard Gamma function and we call $j_{\alpha,p}$ the p-th positive zero of J_α, which is well-defined for every $p \in \mathbb{N}^*$. Then for every eigenvalue $\lambda_k(B)$, there exists a unique $(d,p) \in \mathbb{N} \times \mathbb{N}^*$, such that

$$\lambda_k(B) = j^2_{d+\frac{n-2}{2},p}$$

and, conversely, for every $(d,p) \in \mathbb{N} \times \mathbb{N}^*$, $j^2_{d+\frac{n-2}{2},p}$ is an eigenvalue of B associated to the eigenspace

$$\left\{ x \mapsto \frac{J_{d+\frac{n-2}{2}}\left(j_{d+\frac{n-2}{2},p}|x|\right)}{|x|^{d+\frac{n-2}{2}}} P(x)\,,\ P \in \mathbb{H}_{n,d} \right\}$$

which has dimension

$$\dim(\mathbb{H}_{n,d}) = \begin{cases} 1 & \text{if } d = 0, \\ 2 & \text{if } d > 0, n = 2, \\ (2d+n-2)\frac{(d+n-3)!}{d!(n-2)!} & \text{if } d \geq 0, n \geq 3. \end{cases}$$

In particular an eigenvalue $\lambda_k(B)$ is either simple with a radial eigenfunction, or multiple with only non-radial eigenfunctions. This particular fact (and more generally the fact that any eigenvalue corresponds

to a unique couple (d,p)) is a result due to Siegel [Sie14]. In the literature, it is also called "Bourget's hypothesis" since it has been mentionned in [Bou66], with an incomplete proof.

Eigenvalues and eigenfunctions estimates on general domains. We start by recalling the following inequalities.

Proposition 4.2.1. *Let $\Omega \in \mathcal{A}$, $k \in \mathbb{N}^*$. Then*

$$\left(\frac{n}{n+2}\right)\frac{4\pi^2}{\omega_n^{4/n}}k^{\frac{2}{n}} \leq \lambda_k(\Omega) \leq \left(1+\frac{4}{n}\right)\lambda_1(\Omega)k^{\frac{2}{n}}, \tag{4.12}$$

$$\lambda_1(\Omega)T(\Omega) \leq \omega_n.$$

The lower bound in the first inequality is due to Li and Yau in [LY83, Corollary 1], while the upper bound was obtained by Chen and Yang in [Che07, Theorem 3.1]. On the other hand, the inequality $\lambda_1(\Omega)T(\Omega) \leq \omega_n$ follows directly from using the torsion function as a competitor in the Rayleigh quotient defining λ_1.

Lemma 4.2.2. *Let $\Omega \in \mathcal{A}$, $k \in \mathbb{N}^*$, and let w be the torsion function of Ω and u_k some L^2-normalized eigenfunction. Then*

$$w \leq \frac{1}{2n},\ |u_k| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{\frac{n}{4}},\ |u_k| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{1+\frac{n}{4}}w \text{ in } \Omega,$$

$$\sup_\Omega |\nabla u_k|^2 \leq \left(\frac{1}{n} + \sup_\Omega |\nabla w|^2\right)e^{\frac{1}{4\pi}}\lambda_k(\Omega)^{2+\frac{n}{2}}.$$

Proof. The first estimate $w \leq \frac{1}{2n}$ is a consequence of Talenti's inequality (see [Tal76, Theorem 1 (iv)]): the supremum of the torsion function is maximal on the unit ball, on which the torsion function has the explicit expression $w(x) = \frac{1-|x|^2}{2n}$. Then classical heat kernel estimates (see for instance [Dav89, Ex. 2.1.8]) give $|u_k| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{\frac{n}{4}}$ so

$$\Delta\left(\pm u_k - e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{1+\frac{n}{4}}w\right) = -(\pm)\lambda_k(\Omega)u_k + e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{1+\frac{n}{4}} \geq 0,$$

therefore $|u_k| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{1+\frac{n}{4}}w$ by maximum principle. For the gradient bound, we suppose that ∇w is bounded. By direct computation we have that $\Delta(|\nabla a|^2) \geq 2\nabla a \cdot \nabla(\Delta a)$ for a smooth function $a : \mathbb{R}^n \to \mathbb{R}$. In the case where u_k is smooth inside Ω the computation yields (using as well the bounds on w and u_k)

$$\Delta(|\nabla u_k|^2 + \lambda_k u_k^2) \geq -2\lambda_k(\Omega)^2 u_k^2 \geq -2e^{\frac{1}{4\pi}}\lambda_k(\Omega)^{2+\frac{n}{2}} \text{ in } \Omega,$$

thus giving

$$\Delta\left(|\nabla u_k|^2 + \lambda_k(\Omega)u_k^2 - 2e^{\frac{1}{4\pi}}\lambda_k(\Omega)^{2+\frac{n}{2}}w\right) \geq 0 \text{ in } \Omega.$$

Suppose first that Ω is a $\mathcal{C}^\infty$ domain, then ∇u_k and ∇w extend continuously to the boundary and the inequality $|u_k| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{1+\frac{n}{4}}w$ ensures

$$\forall x \in \partial\Omega,\ |\nabla u_k(x)| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{1+\frac{n}{4}}|\nabla w(x)|,$$

and so by maximum principle:

$$\begin{aligned}\sup_\Omega |\nabla u_k|^2 &\leq 2e^{\frac{1}{4\pi}}\lambda_k(\Omega)^{2+\frac{n}{2}}w + \sup_{\partial\Omega}|\nabla u_k|^2 \\ &\leq \frac{e^{\frac{1}{4\pi}}}{n}\lambda_k(\Omega)^{2+\frac{n}{2}} + e^{\frac{1}{4\pi}}\lambda_k(\Omega)^{2+\frac{n}{2}}\sup_\Omega|\nabla w|^2.\end{aligned}$$

In the general case, by Sard's Theorem and since w is smooth inside Ω we may find arbitrarily small regular values $\varepsilon > 0$ such that $\{w = \varepsilon\} = \partial\{w > \varepsilon\}$ is a smooth hypersurface. Denote $\Omega^\varepsilon = \{w > \varepsilon\}$ and $w^\varepsilon, (u_k^\varepsilon)_{k\in\mathbb{N}^*}$ the associated torsion function and eigenfunctions. Note that $w^\varepsilon = (w-\varepsilon)_+$, so that Ω^ε γ-converges to Ω, since $\|w_\Omega - w_{\Omega_\varepsilon}\|_{L^2(\mathbb{R}^n)} \leq \omega_n\varepsilon \to 0$ (see for example [BB05] for the definition and properties of γ-convergence). In particular, for all $k \geq 1$, $\lambda_k(\Omega^\varepsilon) \to \lambda_k(\Omega)$ thanks to [DS88, Corollaries 3 and 4, pp. 1089-1090]. Now, because u_k^ε is bounded in $H_0^1(\Omega)$ we can assume (up to extraction) that u_k^ε converges strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ to some limit u_k. Passing to the limit in the sense of distributions in $-\Delta u_k^\varepsilon = \lambda_k(\Omega^\varepsilon)u_k^\varepsilon$ we obtain that $(u_k)_{k\in\mathbb{N}}$ is an orthonormal basis for Ω.

Now, since $w^\varepsilon = (w-\varepsilon)_+$ we have

$$\sup_{\Omega^\varepsilon} |\nabla u_k^\varepsilon|^2 \leq \frac{e^{\frac{1}{4\pi}}}{n}\lambda_k(\Omega^\varepsilon)^{2+\frac{n}{2}} + e^{\frac{1}{4\pi}}\lambda_k(\Omega^\varepsilon)^{2+\frac{n}{2}} \sup_{\Omega} |\nabla w|^2$$

Using the L^∞ bound $|u_k^\varepsilon| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega^\varepsilon)^{\frac{n}{4}} \leq 2e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{\frac{n}{4}}$ for small ε, we get that u_k^ε is bounded in $W^{1,\infty}$ as $\varepsilon \to 0$, so that it converges locally uniformly to some Lipschitz function u_k. The uniform gradient bound on ∇u_k^ε transfers to ∇u_k, thus concluding the proof. □

As the next result shows, one can control the difference of eigenvalues by the difference of torsions for two nested domains $\omega \subset \Omega$.

Lemma 4.2.3. *Let $\omega \subset \Omega \subset \mathbb{R}^n$ be two open sets of finite measure. Then*

$$\frac{1}{\lambda_k(\Omega)} - \frac{1}{\lambda_k(\omega)} \leq e^{\frac{1}{4\pi}} k\lambda_k(\Omega)^{\frac{n}{2}} \left[T(\Omega) - T(\omega)\right]$$

Proof. This result is proved in [Buc03, Theorem 3.4], where one has to follow the proof to keep track of the constants (using for instance the L^∞ bound $|u_k| \leq e^{\frac{1}{8\pi}}\lambda_k(\Omega)^{\frac{n}{4}}$). □

The quantitative Faber-Krahn inequality. The Fraenkel asymmetry $\mathcal{F}$, defined for $\Omega \in \mathcal{A}$ as

$$\mathcal{F}(\Omega) = \inf_{x\in\mathbb{R}^n} |(B+x)\Delta\Omega|,$$

plays a crucial role in the following quantitative Faber-Krahn inequality obtained in [BDPV15].

Theorem 4.2.4. *There exists $c_n > 0$ such that for any $\Omega \in \mathcal{A}$,*

$$T(\Omega)^{-1} \geq T(B)^{-1} + c_n\mathcal{F}(\Omega)^2 \tag{4.13}$$

$$\lambda_1(\Omega) \geq \lambda_1(B) + c_n\mathcal{F}(\Omega)^2 \tag{4.14}$$

4.3 Proof of Theorem 4.1.1: the square root bound

Note that from Proposition 4.2.1 we have $\lambda_1(\Omega) \leq \omega_n T(\Omega)^{-1}$. If $T(\Omega)^{-1}$ is close to $T(B)^{-1}$ one can control $\lambda_1(\Omega)$ and, as a consequence, Theorem 4.1.1 provides that $\lambda_k(\Omega)$ is close to $\lambda_k(B)$ when $T(\Omega)^{-1}$ is close to $T(B)^{-1}$. Its proof is obtained as a consequence of the quantitative Faber-Krahn inequality (4.14), growth estimates over $\lambda_k(\Omega)$ from (4.12) and next proposition, which we believe is of independent interest.

Proposition 4.3.1. *Let $\Omega \in \mathcal{A}$. Then it holds*

$$\left|\frac{1}{\lambda_k(\Omega)} - \frac{1}{\lambda_k(B)}\right| \leq \left(1+\frac{4}{n}\right)^{\frac{n}{2}} e^{\frac{1}{4\pi}} k^2\lambda_1(\Omega)^{\frac{n}{2}} \left[T(B) - T(\Omega) + \left(\frac{1}{n}+\frac{1}{n^2}\right)|\Omega\Delta B|\right].$$

We first prove the following lemma.

Lemma 4.3.2. *For any $\Omega \in \mathcal{A}$ we have*

$$T(\Omega) - T(\Omega \cap B) \leq \left(\frac{1}{n} + \frac{1}{n^2}\right) |\Omega \setminus B|$$

Proof. We write $w := w_\Omega$ and $v = w_B$. Then letting $\tilde{w} = w \wedge v$, we have $\tilde{w} \in H_0^1(\Omega \cap B)$ so

$$T(\Omega \cap B) \geq \int_{\Omega \cap B} \left(2\tilde{w} - |\nabla \tilde{w}|^2\right) = \int_{\Omega \cap B} \left(2(w \wedge v) - |\nabla (w \wedge v)|^2\right)$$

so

$$\begin{aligned} T(\Omega) - T(\Omega \cap B) + \int_{\Omega \setminus B} |\nabla w|^2 &\leq \int_{\Omega \setminus B} 2w + \int_{B \cap \Omega} \left(2(w - \tilde{w}) + |\nabla (w \wedge v)|^2 - |\nabla w|^2\right) \\ &\leq \int_{\Omega \setminus B} 2w + \int_{B \cap \Omega} \left(2(w - v)_+ + 2\nabla (w \wedge v) \cdot \nabla (w \wedge v - w)\right) \\ &= \int_{\Omega \setminus B} 2w + \int_{B \cap \Omega} \left(2(w - v)_+ - 2\nabla (w \wedge v) \cdot \nabla (w - v)_+\right) \end{aligned}$$

Notice that $\nabla (w \wedge v) \cdot \nabla (w - v)_+ = \nabla v \cdot \nabla (w - v)_+ = \nabla \cdot ((w - v)_+ \nabla v) + (w - v)_+$ in $\Omega \cap B$ so by Stokes' formula,

$$T(\Omega) - T(\Omega \cap B) + \int_{\Omega \setminus B} |\nabla w|^2 \leq \int_{\Omega \setminus B} 2w - 2v'(1) \int_{\partial B} w$$

Since $-v'(1) = \frac{1}{n}$ we have

$$2(-v'(1)) \int_{\partial B} w \leq \frac{2}{n} \int_{\Omega \setminus B} |\nabla w| \leq \frac{1}{n^2} |\Omega \setminus B| + \int_{\Omega \setminus B} |\nabla w|^2,$$

so

$$T(\Omega) - T(\Omega \cap B) \leq \int_{\Omega \setminus B} 2w + \frac{1}{n^2} |\Omega \setminus B| \leq \left(\frac{1}{n} + \frac{1}{n^2}\right) |\Omega \setminus B|.$$

□

We may now prove Proposition 4.3.1.

Proof of Proposition 4.3.1. Applying the bound from Lemma 4.2.3 to $(\Omega \cap B, B)$ and $(\Omega \cap B, \Omega)$ we have the two inequalities

$$\begin{aligned} \frac{1}{\lambda_k(\Omega)} - \frac{1}{\lambda_k(\Omega \cap B)} &\leq e^{\frac{1}{4\pi}} k \lambda_k(\Omega)^{\frac{n}{2}} \left[T(\Omega) - T(\Omega \cap B)\right] \\ &\leq \left(1 + \frac{4}{n}\right)^{\frac{n}{2}} e^{\frac{1}{4\pi}} k^2 \lambda_1(\Omega)^{\frac{n}{2}} \left[T(\Omega) - T(\Omega \cap B)\right], \end{aligned}$$

$$\begin{aligned} \frac{1}{\lambda_k(B)} - \frac{1}{\lambda_k(\Omega \cap B)} &\leq e^{\frac{1}{4\pi}} k \lambda_k(B)^{\frac{n}{2}} \left[T(B) - T(\Omega \cap B)\right] \\ &\leq \left(1 + \frac{4}{n}\right)^{\frac{n}{2}} e^{\frac{1}{4\pi}} k^2 \lambda_1(\Omega)^{\frac{n}{2}} \left[T(B) - T(\Omega \cap B)\right]. \end{aligned}$$

Combining them, we get

$$\left|\frac{1}{\lambda_k(\Omega)}-\frac{1}{\lambda_k(B)}\right| \leq \left(1+\frac{4}{n}\right)^{\frac{n}{2}} e^{\frac{1}{4\pi}} k^2 \lambda_1(\Omega)^{\frac{n}{2}} \left[T(\Omega)+T(B)-2T(\Omega\cap B)\right]$$
$$= \left(1+\frac{4}{n}\right)^{\frac{n}{2}} e^{\frac{1}{4\pi}} k^2 \lambda_1(\Omega)^{\frac{n}{2}} \left[(T(B)-T(\Omega))+2(T(\Omega)-T(\Omega\cap B))\right].$$

Using Lemma 4.3.2 we thus obtain

$$\left|\frac{1}{\lambda_k(\Omega)}-\frac{1}{\lambda_k(B)}\right| \leq \left(1+\frac{4}{n}\right)^{\frac{n}{2}} e^{\frac{1}{4\pi}} k^2 \lambda_1(\Omega)^{\frac{n}{2}} \left[(T(B)-T(\Omega))+2\left(\frac{1}{n}+\frac{1}{n^2}\right)|\Omega\setminus B|\right]$$

which is the result, since $|\Omega\setminus B| = \frac{1}{2}|\Omega\Delta B|$. □

We can now prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Using Proposition 4.3.1 and applying, up to a translation of Ω, (4.14) as well as (4.12) we get

$$\begin{aligned}
&|\lambda_k(\Omega)-\lambda_k(B)| \\
&\quad\leq \left(1+\frac{4}{n}\right)^{\frac{n}{2}} e^{\frac{1}{4\pi}} k^2 \lambda_1(\Omega)^{\frac{n}{2}} \lambda_k(B)\lambda_k(\Omega) \left[(T(B)-T(\Omega))+\left(\frac{1}{n}+\frac{1}{n^2}\right)|\Omega\setminus B|\right] \\
&\quad\leq C_n k^{2+\frac{4}{n}} \lambda_1(\Omega)^{1+\frac{n}{2}} \left[(T(B)-T(\Omega))+C_n\sqrt{T(\Omega)^{-1}-T(B)^{-1}}\right].
\end{aligned}$$

This concludes the proof when $T(\Omega)\geq \frac{1}{2}T(B)$ (notice that in this case, thanks to $\lambda_1(\Omega)\leq \omega_n T(\Omega)^{-1}$ from Proposition 4.2.1, we bound $\lambda_1(\Omega)\leq C_n'$ for some $C_n'>0$). When $T(\Omega)\leq \frac{1}{2}T(B)$ we write more direcly

$$\begin{aligned}
|\lambda_k(\Omega)-\lambda_k(B)| &\leq \left(1+\frac{4}{n}\right) k^{\frac{2}{n}} \left(\lambda_1(\Omega)+\lambda_1(B)\right) \leq 2\left(1+\frac{4}{n}\right) k^{\frac{2}{n}} \lambda_1(\Omega) \\
&\leq 2\sqrt{2\omega_n}\left(1+\frac{4}{n}\right) k^{\frac{2}{n}} \lambda_1(\Omega)^{\frac{1}{2}} \left(T(\Omega)^{-1}-T(B)^{-1}\right)^{\frac{1}{2}}
\end{aligned}$$

where we used the estimates from Proposition 4.2.1, thus getting the result also in this case. □

4.4 Proof of Theorem 4.1.2: the linear bound

Let us fix $k\geq 1$ as in Theorem 4.1.2, such that $\lambda_k(B)$ is simple (we also include $k=1$, as it will give non-trivial results). In order to prove Theorem 4.1.2, the goal is to prove that when $\delta\in\mathbb{R}$ is close to 0 (depending on n and k) the ball is the only minimizer of the functional

$$\Omega\in\mathcal{A}\mapsto T(\Omega)^{-1}+\delta\lambda_k(\Omega). \tag{4.15}$$

Proposition 4.4.1. *There exists $c_n>0$ such that for any $\delta\in\mathbb{R}$ with $|\delta|\leq c_n k^{-\left(4+\frac{8}{n}\right)} g_n(k)$ the ball is the unique minimizer of (4.15).*

Remark 4.4.2. Let us remind that $g_n(k)$ has been defined in (4.10). As far as we know, there is no explicit lower bound of $g_n(k)$; it was proved by Siegel (see [Sie14] or [Wat44, 15.28], referred to as "Bourget's hypothesis") that zeroes of different Bessel functions are distincts, but with no quantified separation between successive zeroes. We conjecture that there exists an exponent $\kappa>1$ such that for any $m,p,q\in\mathbb{N}^*$, $\mu\in\mathbb{N}/2$, it holds that

$$|j_{\mu,p}-j_{\mu+m,q}|\geq j_{\mu,p}^{-\kappa}.$$

The validity of this conjecture would improve the quality of our bounds.

In Proposition 4.4.1 we prove minimality of the ball for both positive and negative δ, thus obtaining a bound of $(\lambda_k(\Omega) - \lambda_k(B))$ on both sides and proving Theorem 4.1.2. Thanks to (4.9) this will directly imply the inequality (4.1) with $\alpha = 1$ as a consequence, by distinguishing the regimes $\lambda_1(\Omega) \simeq \lambda_1(B)$ and $\lambda_1(\Omega) \gg \lambda_1(B)$ (using also the growth estimate $\lambda_k(\Omega) \leq C_{n,k}\lambda_1(\Omega)$ from Proposition 4.2.1 in this latter case).

The plan of proof of Proposition 4.4.1 is the following.

- For δ close enough to 0 we prove the existence of a minimizer in (4.15) such that its torsion function w moreover verifies some uniform bounds $|\nabla w| \leq C_n$ and a non-degeneracy condition: for all $x \in \Omega$, $r \in (0, r_n]$
$$\fint_{\partial B_{x,r}} w \geq c_n r.$$
The case $\delta < 0$ raises extra difficulties. We obtain existence through careful concentration-compactness methods, first as a capacity measure and second as a quasi-open set; the uniform estimates are obtained by perturbing Ω and controlling the variation of λ_k by the variations of the torsion T. This is done in Section 4.4.1.

- In Section 4.4.2, we prove that if Ω solves (4.15), then its torsion function w and L^2-normalized k-th eigenfunction u_k verify, in some sense that will be made precise, the equations
$$\begin{cases} -\Delta w = 1, \ -\Delta u_k = \lambda_k(\Omega) u_k & \text{in } \Omega \\ |\nabla w|^2 + T(\Omega)^2 \delta |\nabla u_k|^2 = Q & \text{on } \partial\Omega \end{cases}$$
where $Q > 0$ is a constant which is arbitrarily close to $\frac{1}{n^2}$ when $\delta \to 0$. This part of the proof uses blow-up methods similar to [KL18] and [CS05].

- We show in Section 4.4.3 that, as $\delta \to 0$, $\partial\Omega$ is an arbitrarily small $\mathcal{C}^{2,\gamma}$ graph on ∂B (up to translation). The case $\delta > 0$ relies on the results from [KL18] while the case $\delta < 0$ is obtained by applying the results from [MTV21].

- Finally, in Section 4.4.4 we prove a Fuglede-type result, namely that the ball is optimal for (4.15) among smooth nearly spherical sets, that is to say arbitrarily small $\mathcal{C}^{2,\gamma}$ perturbations of the ball, through a second shape derivative estimate which follows the method of [DL19].

Throughout the proof we extensively use the two following notations:

- $a \lesssim b$ when $a \leq C_n b$ for some (possibly) large $C_n > 0$ which only depends on the dimension n.

- $a \ll b$ when $a \leq c_n b$ for some $c_n > 0$ that can be made as small as we want, and which only depends on the dimension n.

In both cases the notation does not involve a dependence on the order of the eigenvalue k.

4.4.1 Existence of a minimizer

To prove existence we first prove some a priori estimates for sets whose energy $T^{-1} + \delta\lambda_k$ is bounded from above by the one of the ball, which we may suppose to be verified without loss of generality for any element of a minimizing sequence. This is the object of next Lemma.

Lemma 4.4.3. *Let $\Omega \in \mathcal{A}$ be such that*
$$T(\Omega)^{-1} + \delta\lambda_k(\Omega) \leq T(B)^{-1} + \delta\lambda_k(B),$$

and suppose Ω is translated such that $\mathcal{F}(\Omega) = |\Omega \Delta B|$. Then if $|\delta| \ll k^{-\frac{2}{n}}$ the following inequalities hold

$$|\Omega \Delta B| \lesssim k^{\frac{1}{n}} |\delta|^{\frac{1}{2}}, T(\Omega)^{-1} \lesssim 1, \ \lambda_k(\Omega) \lesssim k^{\frac{2}{n}},$$

$$T(\Omega)^{-1} - T(B)^{-1} \lesssim k^{\frac{2}{n}} |\delta| \quad \textit{and for all } i \in \mathbb{N}^*, |\lambda_i(\Omega) - \lambda_i(B)| \lesssim i^{2+\frac{4}{n}} k^{\frac{1}{n}} |\delta|^{\frac{1}{2}},$$

$$\|w_\Omega - w_B\|_{L^1(\mathbb{R}^n)} \lesssim k^{\frac{1}{n}} |\delta|^{\frac{1}{2}}.$$

Proof. Thanks to the upper bound from (4.12), we have

$$T(\Omega)^{-1} - T(B)^{-1} \leq \delta \left(\lambda_k(B) - \lambda_k(\Omega)\right) \lesssim k^{\frac{2}{n}} \lambda_1(\Omega) |\delta| \lesssim k^{\frac{2}{n}} T(\Omega)^{-1} |\delta|$$

so when $|\delta| \ll k^{-\frac{2}{n}}$ we get that $T(\Omega)^{-1} \lesssim 1$, and using the same series of inequalities together with $T(\Omega) \leq T(B)$ we deduce

$$T(B) - T(\Omega) \lesssim k^{\frac{2}{n}} |\delta|.$$

Applying the quantitative Saint-Venant inequality (4.13), we get

$$|\Omega \Delta B| \lesssim k^{\frac{1}{n}} |\delta|^{\frac{1}{2}} \text{ when } |\delta| \ll k^{-\frac{2}{n}},$$

from which we also deduce, using Theorem 4.1.1, that for any $i \in \mathbb{N}^*$ it holds

$$|\lambda_i(\Omega) - \lambda_i(B)| \lesssim i^{2+\frac{4}{n}} k^{\frac{1}{n}} |\delta|^{\frac{1}{2}}.$$

For the third item, we write

$$\begin{aligned}
\|w_\Omega - w_B\|_{L^1(\mathbb{R}^n)} &\leq \|w_\Omega - w_{\Omega \cap B}\|_{L^1(\mathbb{R}^n)} + \|w_B - w_{\Omega \cap B}\|_{L^1(\mathbb{R}^n)} \\
&= T(B) - T(\Omega) + 2(T(\Omega) - T(\Omega \cap B)) \\
&\leq T(B) - T(\Omega) + \left(\frac{1}{n} + \frac{1}{n^2}\right) |\Omega \Delta B|,
\end{aligned}$$

where we used Lemma 4.3.2 and $|\Omega \Delta B| = 2|\Omega \setminus B|$ in the last line. We obtain the last result by recalling that $|\Omega \Delta B| \lesssim k^{\frac{1}{n}} |\delta|^{\frac{1}{2}}$. □

We may use the bound from Theorem 4.1.1 in order to improve the decay of the quantities from the previous lemma, in terms of δ.

Lemma 4.4.4. *Under the same hypotheses as in Lemma 4.4.3 it holds*

$$\begin{aligned}
T(\Omega)^{-1} - T(B)^{-1} &\lesssim k^{4+\frac{8}{n}} |\delta|^2, \\
|\Omega \Delta B| &\lesssim k^{2+\frac{4}{n}} |\delta|, \\
\forall i \in \mathbb{N}^*, \ |\lambda_i(\Omega) - \lambda_i(B)| &\lesssim i^{2+\frac{4}{n}} k^{2+\frac{4}{n}} |\delta|.
\end{aligned}$$

As a consequence, if $|\delta| \ll k^{-(4+\frac{8}{n})} g_n(k)$, $\lambda_k(\Omega)$ is simple.

Proof. Thanks to Theorem 4.1.1 and using minimality we can write

$$T(\Omega)^{-1} - T(B)^{-1} \leq \delta \left(\lambda_k(B) - \lambda_k(\Omega)\right) \lesssim k^{2+\frac{4}{n}} |\delta| \left(T(\Omega)^{-1} - T(B)^{-1}\right)^{\frac{1}{2}}$$

from which the three announced estimates follow. We deduce that $\lambda_k(\Omega)$ is simple by applying separately $|\lambda_i(\Omega) - \lambda_i(B)| \lesssim i^{2+\frac{4}{n}} k^{2+\frac{4}{n}} |\delta|$ for $i = k-1, k, k+1$. □

Proposition 4.4.5. *If $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$, then the functional (4.15) has a minimizer in the class of quasi-open sets of measure ω_n.*

Let us mention that most of our efforts will be concerned with dealing with the case $\delta < 0$, as in the case $\delta > 0$ one can argue more directly. We chose here to treat both cases at the same time in order to simplify the presentation.

To prove this proposition, we will use the setting of capacitary measures (see for instance [BDM93b]). A capacitary measure is a nonnegative Borel measure μ, possibly infinite valued, such that $\mu(E) = 0$ as soon as E has zero capacity. We typically assign to any quasi-open set A the capacitary measure

$$\infty_{\mathbb{R}^n \setminus A}(E) := \begin{cases} +\infty & \text{if } \mathrm{Cap}(E \setminus A) > 0 \\ 0 & \text{else.} \end{cases}$$

Given a capacitary measure μ, we define the regular set of μ, denoted A_μ as the union of all finely open sets of finite μ-measure (the fine topology being the coarsest topology making all super-harmonic functions continuous, see [BB05, p.77] and the references therein). If A_μ has finite Lebesgue measure, we define the torsion $T(\mu)$ and the eigenvalues $\lambda_k(\mu)$ of a capacitary measure as follows ($\mathscr{L}^n$ denotes the n-dimensional Lebesgue measure):

$$\begin{aligned} T(\mu) &:= \sup_{u \in H^1(\mathbb{R}^n) \cap L^2(\mu)} \int_{\mathbb{R}^n} \left(2u - |\nabla u|^2\right) d\mathscr{L}^n - \int_{\mathbb{R}^n} u^2 d\mu \\ &= \sup_{H^1(\mathbb{R}^n) \cap L^2(\mu)} \frac{\left(\int_\Omega u d\mathscr{L}^n\right)^2}{\int_{\mathbb{R}^n} |\nabla u|^2 d\mathscr{L}^n + \int_{\mathbb{R}^n} u^2 d\mu} = \int_{\mathbb{R}^n} w_\mu, \\ \lambda_k(\mu) &:= \inf \left\{ \sup_{v \in V} \frac{\int_{\mathbb{R}^n} |\nabla v|^2 d\mathscr{L}^n + \int_{\mathbb{R}^n} v^2 d\mu}{\int_{\mathbb{R}^n} v^2 d\mathscr{L}^n},\ V \subset H^1(\mathbb{R}^n) \cap L^2(\mu) \text{ of dimension } k \right\} \\ &= \int_{\mathbb{R}^n} |\nabla u_{k,\mu}|^2 d\mathscr{L}^n + \int_{\mathbb{R}^n} u_{k,\mu}^2 d\mu. \end{aligned}$$

Above, w_μ is the torsion function associated to μ and is a variational solution of

$$\begin{cases} -\Delta w_\mu + \mu w_\mu = 1 \text{ in } [H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \mu)]', \\ w_\mu \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \mu), \end{cases} \tag{4.16}$$

and $(u_{k,\mu})_{k \in \mathbb{N}^*}$ is a choice of a L^2-orthonormal basis of eigenfunctions associated to μ, that verify

$$\begin{cases} -\Delta u_{k,\mu} + \mu u_{k,\mu} = \lambda_k(\mu) u_{k,\mu} \text{ in } [H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \mu)]', \\ u_{k,\mu} \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \mu). \end{cases}$$

Note that $\mu \mapsto T(\mu)$ and $\mu \mapsto \lambda_k(\mu)$ are continuous for the $L^1(\mathbb{R}^n)$ distance between the associated torsion functions w_μ (which is called γ-distance, see [BDM93b]). Moreover $A_\mu = \{w_\mu > 0\}$, up to a set of zero capacity.

Proof. (of Proposition 4.4.5) A first remark is that when $|\delta|$ is small enough the measure constraint $|\Omega| = \omega_n$ may be relaxed into $|\Omega| \leq \omega_n$, since any set Ω which does not saturate the constraint $|\Omega| \leq \omega_n$ may be dilated into a set with lower energy. Indeed let $|\Omega| =: (1-t)\omega_n$ for some $t > 0$, then $(1-t)^{-\frac{1}{n}}\Omega$ is still admissible and using $\lambda_k(\Omega) \lesssim k^{\frac{2}{n}} T(\Omega)^{-1}$ (from Lemma 4.2.1) we get

$$\begin{aligned} T((1-t)^{-\frac{1}{n}}\Omega)^{-1} + \delta\lambda_k((1-t)^{-\frac{1}{n}}\Omega) &= (1-t)^{\frac{n+2}{n}} T(\Omega)^{-1} + (1-t)^{\frac{2}{n}} \delta\lambda_k(\Omega) \\ &\leq (1-t)T(\Omega)^{-1} + \delta\lambda_k(\Omega) + |\delta| t \lambda_k(\Omega) \\ &\leq T(\Omega)^{-1} + \delta\lambda_k(\Omega) - t\left(T(\Omega)^{-1} - C_n k^{\frac{2}{n}} |\delta| T(\Omega)^{-1}\right) \\ &< T(\Omega)^{-1} + \delta\lambda_k(\Omega) \text{ when } |\delta| \ll k^{-\frac{2}{n}}. \end{aligned} \tag{4.17}$$

Let $(\Omega_p)_{p\in\mathbb{N}}$ be a minimizing sequence of $T^{-1}+\delta\lambda_k$ in $\mathcal{A}$. By replacing Ω_p by B if needed we can assume without loss of generality that $T^{-1}(\Omega_p)+\delta\lambda_k(\Omega_p)\leq T^{-1}(B)+\delta\lambda_k(B)$ for each p, so that Ω_p satisfies the hypothesis of Lemma 4.4.3. Then by Lemma 4.4.3 we have a bound on the Fraenkel asymmetry $\mathcal{F}(\Omega_p)\lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}$ so, up to translation, we suppose

$$|\Omega_p\Delta B|\lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}.$$

Let us first prove that this sequence γ-converges to a capacitary measure μ, meaning that the associated torsion functions w_{Ω_p} converges weakly in $H^1(\mathbb{R}^n)$ to the function w_μ given by (4.16) and $A_\mu=\{w_\mu>0\}$ verifies $|A_\mu|=\omega_n$.

By concentration-compactness for sequences of open sets of bounded measure (see [Buc00, Th 2.2.]), in order to prove that convergence occurs we must exclude the dichotomy behaviour. We thus assume by contradiction that we are in the latter situation, meaning that one can find $\tilde{\Omega}_p=\Omega_p^1\sqcup\Omega_p^2\subset\Omega_p$ with $\mathrm{dist}(\Omega_p^1,\Omega_p^2)\to\infty$, $\liminf_{p\to\infty}|\Omega_p^i|>0$, and verifying $\|w_{\Omega_p}-w_{\tilde{\Omega}_p}\|_{L^2(\mathbb{R}^n)}\to 0$. As a consequence $|T(\tilde{\Omega}_p)-T(\Omega_p)|\to 0$ and $|\lambda_k(\tilde{\Omega}_p)-\lambda_k(\Omega_p)|\to 0$, and $\tilde{\Omega}_p$ is therefore still a minimizing sequence. By Lemma 4.4.3 we have $T(\Omega_p)^{-1}-T(B)^{-1}\lesssim k^{\frac{2}{n}}|\delta|$, hence we also have $T(\tilde{\Omega}_p)^{-1}-T(B)^{-1}\lesssim k^{\frac{2}{n}}|\delta|$ and one can check that this ensures $|\Omega_p\setminus\tilde{\Omega}_p|\lesssim k^{\frac{2}{n}}|\delta|$ using the Saint-Venant inequality. We therefore have $|\tilde{\Omega}_p\Delta B|\leq|\Omega_p\Delta B|+|\Omega_p\setminus\tilde{\Omega}_p|\lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}$ for $|\delta|\ll k^{-\frac{2}{n}}$. Furthermore, since $d(\Omega_p^1,\Omega_p^2)\to+\infty$ we have (say) that $|\Omega_p^1\Delta B|\lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}$ and $|\Omega_p^2|\lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}$.

We claim that $\lambda_k(\tilde{\Omega}_p)=\lambda_k(\Omega_p^1)$ when $|\delta|\ll k^{-2-\frac{2}{n}}$. Indeed on the one hand by Faber-Krahn inequality we have

$$\lambda_1(\Omega_p^2)\gtrsim|\Omega_p^2|^{-\frac{2}{n}}\gtrsim k^{-\frac{2}{n^2}}|\delta|^{-\frac{1}{n}}$$

and on the other hand thanks to Proposition 4.2.1

$$\lambda_k(\Omega_p^1)\lesssim k^{\frac{2}{n}}T(\Omega_p^1)^{-1}\lesssim k^{\frac{2}{n}}.$$

To justify the last inequality we note that by Saint-Venant it holds $T(\Omega_p^2)\lesssim k^{\frac{n+2}{n^2}}|\delta|^{\frac{n+2}{2n}}$ and thanks to the a priori estimates from Lemma 4.4.3 we have

$$T(\Omega_p^1)=T(\tilde{\Omega}_p)-T(\Omega_p^2)\geq T(B)-C_n\left(k^{\frac{2}{n}}|\delta|+k^{\frac{(n+2)}{n^2}}|\delta|^{\frac{n+2}{2n}}\right)\geq\frac{1}{2}T(B)\text{ for }|\delta|\ll k^{-\frac{2}{n}}.$$

Hence $\lambda_1(\Omega_p^2)\geq\lambda_k(\Omega_p^1)$ for $|\delta|\ll k^{-2-\frac{2}{n}}$ and therefore $\lambda_k(\tilde{\Omega}_p)=\lambda_k(\Omega_p^1)$ for the same δ; as a consequence

$$T(\tilde{\Omega}_p)^{-1}+\delta\lambda_k(\tilde{\Omega}_p)=\left(T(\Omega_p^1)+T(\Omega_p^2)\right)^{-1}+\delta\lambda_k(\Omega_p^1).$$

Set $t_p\in(0,1)$ such that $|\Omega_p^2|=t_p\omega_n$, so $t_p\lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}$ and $\liminf_{p\to\infty}t_p>0$. We now argue that $(1-t_p)^{-\frac{1}{n}}\Omega_p^1$ is a strictly better minimizing sequence. Indeed, since $T(\Omega_p^2)\lesssim t_p^{\frac{n+2}{n}}\lesssim k^{\frac{n+2}{n^2}}|\delta|^{\frac{n+2}{2n}}$ and $T(\Omega_p^1)\gtrsim 1$, one has $\frac{1}{T(\Omega_p^1)}\leq\frac{1}{T(\Omega_p^1)+T(\Omega_p^2)}+C_nt_p^{\frac{n+2}{n}}$ for some $C_n>0$ so that

$$\begin{aligned}
\frac{1}{T\left((1-t_p)^{-\frac{1}{n}}\Omega_p^1\right)}+\delta\lambda_k\left((1-t_p)^{-\frac{1}{n}}\Omega_p^1\right)&=(1-t_p)^{\frac{n+2}{n}}\frac{1}{T\left(\Omega_p^1\right)}+(1-t_p)^{\frac{2}{n}}\delta\lambda_k\left(\Omega_p^1\right)\\
&\leq\frac{1}{T\left(\Omega_p^1\right)}+\delta\lambda_k\left(\Omega_p^1\right)-t_p\left(\frac{1}{T(\Omega_p^1)}-|\delta|\lambda_k(\Omega_p^1)\right)\\
&\leq\frac{1}{T\left(\Omega_p^1\right)+T(\Omega_p^2)}+\delta\lambda_k\left(\Omega_p^1\right)-t_p\left(\frac{1}{T(\Omega_p^1)}-C_nt_p^{\frac{2}{n}}-|\delta|\lambda_k(\Omega_p^1)\right)\\
&\leq\frac{1}{T\left(\Omega_p^1\right)+T(\Omega_p^2)}+\delta\lambda_k\left(\Omega_p^1\right)-t_p\left(\frac{1}{T(B)}-C_nk^{\frac{2}{n^2}}|\delta|^{\frac{1}{n}}-C_nk^{\frac{2}{n}}|\delta|\right).
\end{aligned}$$

Since $\liminf_{p\to\infty} t_p > 0$ this provides a strictly better minimizing sequence when $|\delta| \ll k^{-\frac{2}{n}}$, providing a contradiction and thus proving that dichotomy does not occur.

Thanks to [Buc00, Th 2.2.] we deduce that there exists some capacitary measure μ such that after extraction (and translation of the Ω_p) the sequence $(\Omega_p)_p$ γ-converges to μ. In particular one has convergence of the torsional rigidity and eigenvalues. We let w_μ be the associated torsion function and $A_\mu = \{w_\mu > 0\}$ the (quasi-open) associated domain. Notice that we can assume without loss of generality that the limiting measure verifies $\mu(E) = \infty$ whenever $\mathrm{Cap}(E \setminus A_\mu) > 0$, since this leaves w_μ unchanged.

We have $|A_\mu| \leq \omega_n$ by a.e. pointwise convergence of the torsion functions. We now prove that μ corresponds to some quasi-open domain, precisely that $\mu = \infty_{\mathbb{R}^n \setminus A_\mu}$. The idea is to use optimality of μ to prove that the torsions of μ and A_μ are equal. We denote by w_{A_μ} the torsion function of A_μ. We have (see [BDM93b])

$$\begin{cases} -\Delta w_\mu + \mu w_\mu \leq 1 \text{ in } \mathcal{D}'(\mathbb{R}^n) \\ w_\mu \in H^1(\mathbb{R}^n) \cap L^2(\mu) \end{cases}, \text{ while } \begin{cases} -\Delta w_{A_\mu} = 1 \text{ in } A_\mu, \\ w_{A_\mu} \in H^1_0(A_\mu). \end{cases}$$

By maximum principle, it holds that $w_{A_\mu} \geq w_\mu$, implying in particular $T(A_\mu) \geq T(\mu)$. Thanks to [BBGLB22, Lemma 3.1], the sequence $\Omega_p \cap A_\mu$ still γ-converges to μ. As a consequence, by applying Lemma 4.2.3 to A_μ and $\Omega_p \cap A_\mu$ and passing to the limit one gets

$$0 \leq \frac{1}{\lambda_k(A_\mu)} - \frac{1}{\lambda_k(\mu)} \leq e^{\frac{1}{4\pi}} k \lambda_k(A_\mu)^{\frac{n}{2}} \left[T(A_\mu) - T(\mu)\right],$$

which we rewrite, using that $\lambda_k(A_\mu) \leq \lambda_k(\mu) \lesssim k^{\frac{2}{n}}$ (using monotonicity and Lemma 4.4.3), as

$$0 \leq \lambda_k(\mu) - \lambda_k(A_\mu) \lesssim k^{2+\frac{4}{n}} \left[T(\mu)^{-1} - T(A_\mu)^{-1}\right].$$

Then by minimality of μ,

$$\begin{aligned} T(\mu)^{-1} + \delta\lambda_k(\mu) &\leq T(A_\mu)^{-1} + \delta\lambda_k(A_\mu) \\ &\leq T(\mu)^{-1} + \delta\lambda_k(\mu) + \left(1 - C_n k^{2+\frac{4}{n}}|\delta|\right)\left(T(A_\mu)^{-1} - T(\mu)^{-1}\right). \end{aligned}$$

When $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$ this gives $T(A_\mu) \leq T(\mu)$, hence $T(A_\mu) = T(\mu)$. Now, since $w_\mu \in H^1_0(A_\mu)$ we deduce

$$\int_{\mathbb{R}^n} \left(2w_\mu - |\nabla w_\mu|^2\right) d\mathscr{L}^n \leq T(A_\mu) = T(\mu) \leq \int_{\mathbb{R}^n} \left(2w_\mu - |\nabla w_\mu|^2\right) d\mathscr{L}^n - \int_{\mathbb{R}^n} w_\mu^2 d\mu$$

thus implying $\int_{\mathbb{R}^n} w_\mu^2 d\mu = 0$. As a consequence $\mu = 0$ in A_μ, meaning $\mu = \infty_{\mathbb{R}^n \setminus A_\mu}$. Hence A_μ is a minimizer of the functionnal (4.15) in the class of quasi-open sets of measure ω_n, thus concluding the proof. □

We now prove some first regularity properties of minimizers of (4.15).

Lemma 4.4.6. *Let Ω be a minimizer of (4.15) in the class of quasi-open sets of measure ω_n, and suppose that $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$. Then Ω is bounded and there exists $c_n, C_n, r_n > 0$ such that $\|\nabla w_\Omega\|_{L^\infty(\mathbb{R}^n)} \leq C_n$, $\|\nabla u_k\|_{L^\infty(\mathbb{R}^n)} \leq C_n k^{\frac{2}{n}+\frac{1}{2}}$, diam$(\Omega) \leq C_n$ and for all $x \in \mathbb{R}^n$, $r \in (0, r_n)$,*

$$\fint_{\partial B_{x,r}} w_\Omega < c_n r \text{ implies } w_\Omega|_{B_{x,r/2}} = 0. \tag{4.18}$$

In particular, the open set $\{w_\Omega > 0\}(\underset{q.e.}{=} \Omega)$ is an open minimizer of (4.15).

In the sequel, property (4.18) will be referred to as non-degeneracy, as it accounts in a weak sense for the fact that $|\nabla w_\Omega|$ stays away from 0 near $\partial\Omega$.

Proof. **Lipschitz regularity.** Let us first prove the Lipschitz regularity of w_Ω, which will imply the Lipschitz regularity of the eigenfunctions u_k by the estimates of Lemma 4.2.2. To prove the Lipschitz regularity it is enough to prove the following property on the torsional rigidity: for any open set $\tilde{\Omega}$ that contains Ω such that $|\tilde{\Omega} \setminus \Omega|$ is small enough, we have

$$T(\Omega)^{-1} \leq T(\tilde{\Omega})^{-1} + \Lambda_n |\tilde{\Omega} \setminus \Omega| \tag{4.19}$$

Indeed, consider then some ball $B_{x,r}$ for any small enough r such that this inequality applies to $\tilde{\Omega} = \Omega \cup B_{x,r}$; for any $\tilde{w} \in H^1_0(\Omega \cup B_{x,r})$ coinciding with w outside of $B_{x,r}$, writing $T(\Omega \cup B_{x,r}) \geq \int_{\mathbb{R}^n} 2\tilde{w} - |\nabla \tilde{w}|^2$ we get by rearranging (4.19):

$$\int_{B_{x,r}} \left(|\nabla w|^2 - \frac{1}{2} w \right) \leq \int_{B_{x,r}} \left(|\nabla \tilde{w}|^2 - \frac{1}{2} \tilde{w} \right) + \Lambda'_n r^n$$

for some $\Lambda'_n > 0$. This corresponds to the notion of quasi-minimizer of [BMPV15, Definition 3.1] for $f = 1$, so we may apply [BMPV15, Th 3.3] to get a uniform Lipschitz bound. Let us therefore prove claim (4.19). Let $\tilde{\Omega}$ be an open set that contains Ω with $|\tilde{\Omega}| \leq 2|\Omega|$. We separate the case $\delta > 0$ and $\delta < 0$ for clarity:

- Case $\delta > 0$. By monotonicity of λ_k we have $\lambda_k(\Omega) \geq \lambda_k(\tilde{\Omega})$, so using minimality of Ω against the competitor $\left(\frac{|\Omega|}{|\tilde{\Omega}|}\right)^{\frac{1}{n}} \tilde{\Omega}$ we have

$$T(\Omega)^{-1} \leq \left(\frac{|\tilde{\Omega}|}{|\Omega|} \right)^{\frac{n+2}{n}} T(\tilde{\Omega})^{-1} + \delta \left[\left(\frac{|\tilde{\Omega}|}{|\Omega|} \right)^{\frac{2}{n}} - 1 \right] \lambda_k(\Omega)$$

  which implies

$$T(\Omega)^{-1} \leq T(\tilde{\Omega})^{-1} + C_n \left(1 + |\delta| k^{\frac{2}{n}} \right) |\tilde{\Omega} \setminus \Omega|$$

  thanks to $T(\tilde{\Omega})^{-1} \leq T(\Omega)^{-1} \lesssim 1$ and $\lambda_k(\Omega) \lesssim k^{\frac{2}{n}}$ from Lemma 4.4.3. We thus get (4.19) as soon as $|\delta| k^{\frac{2}{n}} \lesssim 1$.

- Case $\delta < 0$. We deal with this case similarly as is the positive case, using also the estimate from Lemma 4.2.3. Comparing the energy of Ω to the energy of $\left(\frac{|\Omega|}{|\tilde{\Omega}|}\right)^{\frac{1}{n}} \tilde{\Omega}$ gives

$$\begin{aligned} T(\Omega)^{-1} - T(\tilde{\Omega})^{-1} &\leq \delta \left(\lambda_k(\tilde{\Omega}) - \lambda_k(\Omega) \right) + \left(\left(\frac{|\tilde{\Omega}|}{|\Omega|} \right)^{\frac{n+2}{n}} - 1 \right) T(\tilde{\Omega})^{-1} \\ &\leq |\delta| e^{\frac{1}{4\pi}} k \lambda_k(\Omega) \lambda_k(\tilde{\Omega})^{1+\frac{n}{2}} \left[T(\tilde{\Omega}) - T(\Omega) \right] + C_n |\tilde{\Omega} \setminus \Omega| \\ &\leq C'_n |\delta| k^{2+\frac{4}{n}} \left[T(\Omega)^{-1} - T(\tilde{\Omega})^{-1} \right] + C_n |\tilde{\Omega} \setminus \Omega| \end{aligned}$$

  for some $C_n, C'_n > 0$. When $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$ we get (4.19).

Non-degeneracy property. The non-degeneracy is obtained by similar arguments, by choosing sets $\tilde{\Omega}$ such that $\tilde{\Omega} \subset \Omega$. Let us prove that for any $\tilde{\Omega} \subset \Omega$ with $|\tilde{\Omega}] \geq \frac{1}{2}|\Omega|$ it holds

$$T(\Omega)^{-1} + \Lambda_n |\Omega \setminus \tilde{\Omega}| \leq T(\tilde{\Omega})^{-1}. \tag{4.20}$$

This is enough to obtain the non-degeneracy property, thanks to [Buc12, Lemma 1]. This time, it is the case $\delta > 0$ which requires a more careful analysis.

- Case $\delta < 0$. Consider any open set $\tilde{\Omega}$ contained in Ω with $|\tilde{\Omega}| \geq \frac{1}{2}|\Omega|$. By monotonicity $\lambda_k(\Omega) \leq \lambda_k(\tilde{\Omega})$, hence testing minimality of Ω against $\left(\frac{|\Omega|}{|\tilde{\Omega}|}\right)^{\frac{1}{n}} \tilde{\Omega}$ we have

$$T(\Omega)^{-1} - \delta \left[\left(\frac{|\tilde{\Omega}|}{|\Omega|} \right)^{\frac{2}{n}} - 1 \right] \lambda_k(\Omega) \leq \left(\frac{|\tilde{\Omega}|}{|\Omega|} \right)^{\frac{n+2}{n}} T(\tilde{\Omega})^{-1} \leq T(\tilde{\Omega})^{-1}$$

  which provides (4.20) for $|\delta| \ll k^{-\frac{2}{n}}$, using $\lambda_k(\Omega) \lesssim k^{\frac{2}{n}}$ from Lemma 4.4.3.

- Case $\delta > 0$. We proceed as before, using in addition the estimate of Proposition 4.2.3. Using minimality of Ω against the competitor $\left(\frac{|\Omega|}{|\tilde{\Omega}|}\right)^{\frac{1}{n}} \tilde{\Omega}$ we get

$$\begin{aligned} T(\Omega)^{-1} - T(\tilde{\Omega})^{-1} &\leq \delta \left(\lambda_k(\tilde{\Omega}) - \lambda_k(\Omega) \right) + \left(\left(\frac{|\tilde{\Omega}|}{|\Omega|} \right)^{\frac{n+2}{n}} - 1 \right) T(\tilde{\Omega})^{-1} \\ &\leq \delta e^{\frac{1}{4\pi}} k \lambda_k(\Omega)^{1+\frac{n}{2}} \lambda_k(\tilde{\Omega}) \left[T(\Omega) - T(\tilde{\Omega}) \right] - C_n |\tilde{\Omega} \setminus \Omega| \\ &\leq C_n' \delta k^{2+\frac{4}{n}} \left[T(\tilde{\Omega})^{-1} - T(\Omega)^{-1} \right] - C_n |\tilde{\Omega} \setminus \Omega| \end{aligned}$$

  for some $C_n, C_n' > 0$, using $T(\tilde{\Omega})^{-1} \gtrsim T(B)^{-1}$ and also $\lambda_k(\Omega) \lesssim k^{\frac{2}{n}}$ from Lemma 4.4.3. We get (4.20) for $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$.

Bound on diam(Ω). We have shown above that $\|\nabla w_\Omega\|_{L^\infty(\mathbb{R}^n)} \lesssim 1$. Hence by Gagliardo-Nirenberg inequality,

$$\|w_\Omega - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} \lesssim \|\nabla(w_\Omega - w_B)\|_{L^\infty(\mathbb{R}^n)}^{\frac{n}{n+1}} \|w_\Omega - w_B\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n+1}} \lesssim \|w_\Omega - w_B\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n+1}} \lesssim \left(k^{\frac{1}{n}} |\delta|^{\frac{1}{2}} \right)^{\frac{1}{n+1}} \tag{4.21}$$

where we also used that $\|w_\Omega - w_B\|_{L^1(\mathbb{R}^n)} \lesssim k^{\frac{1}{n}} |\delta|^{\frac{1}{2}}$ from Lemma 4.4.3. Let now c_n and r_n denote the non-degeneracy constants found above. Then for any $x \in \mathbb{R}^n \setminus B_{0,1+r_n}$ we have

$$\fint_{\partial B_{x,r_n}} w_\Omega = \fint_{\partial B_{x,r_n}} (w_\Omega - w_B) \lesssim \left(k^{\frac{1}{n}} |\delta|^{\frac{1}{2}} \right)^{\frac{1}{n+1}}$$

and this is strictly less than $c_n r_n$ for $|\delta| \ll k^{-\frac{2}{n}}$. Hence $w_\Omega(x) = 0$ for such x, so that we find $\Omega \subset B_{1+\frac{1}{2}r_n}$. This gives the desired upper bound on diam(Ω), thus concluding the proof.

□

4.4.2 Blow-ups and viscosity solutions

Let us first observe that if Ω is a minimizer of (4.15), then it is also a minimizer (among open sets of any measure) of the associated scale-invariant functional. This is stated in next Lemma.

Lemma 4.4.7. *Let Ω be a quasi-open minimizer of (4.15) in the class of quasi-open sets of measure ω_n. Then it is a minimizer of*

$$A \in \{\textit{quasi open sets}\} \mapsto |A|^{\frac{2}{n}} \left(\frac{|A|}{\omega_n T(A)} + \delta \lambda_k(A) \right).$$

Proof. This follows directly from the scale-invariance of $A \mapsto |A|^{\frac{2}{n}} \left(\frac{|A|}{\omega_n T(A)} + \delta \lambda_k(A) \right)$. □

Consider now a smooth vector field $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Assuming enough regularity on Ω, the shape derivative

$$\frac{d}{dt}\bigg|_{t=0} J((\mathrm{Id} + t\xi)(\Omega))$$

of this functional at Ω in the direction ξ is given by

$$\int_{\partial\Omega} \left[\omega_n^{\frac{2}{n}-1} \left(\frac{n+2}{nT(\Omega)} + \frac{2}{n}\delta\lambda_k(\Omega) \right) - \frac{\omega_n^{\frac{2}{n}}}{T(\Omega)^2} |\nabla w|^2 - \omega_n^{\frac{2}{n}} \delta |\nabla u_k|^2 \right] (\xi \cdot \nu_\Omega) d\mathscr{H}^{n-1},$$

where ν_Ω is the outward unit normal vector of Ω (see for instance [HP18] for the expressions of the shape derivatives of $|\cdot|$, T and λ_k). So letting

$$Q := \frac{T(\Omega)^2}{\omega_n} \left(\frac{n+2}{nT(\Omega)} + \frac{2}{n}\delta\lambda_k(\Omega) \right) \tag{4.22}$$

(which depends not only on the parameter but also on the minimizer Ω), we expect an overdetermined boundary condition $|\nabla w|^2 + T(\Omega)^2\delta|\nabla u_k|^2 = Q$. Note that when $\delta \to 0$, using that $T(\Omega)^{-1} - T(B)^{-1} \lesssim |\delta| k^{\frac{2}{n}}$ and $\lambda_k(\Omega) \lesssim k^{\frac{2}{n}}$ (see Lemma 4.4.3) we find

$$Q \to \frac{1}{n^2},$$

which is expected because it corresponds to the value of $|\nabla w_B|^2_{|\partial B}$. We may estimate its rate of convergence

$$\left| Q - \frac{1}{n^2} \right| \lesssim k^{\frac{2}{n}} |\delta|. \tag{4.23}$$

Let us now prove that these informal considerations hold true for blow-ups around points with contact spheres. This is the object of Lemma 4.4.10 below. We first need the notion of contact point.

Definition 4.4.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $x \in \partial\Omega$. We say that x is a contact point of Ω if there exists $R > 0$ and $\nu \in \mathbb{S}^{n-1}$ verifying*

$$B_{x+R\nu,R} \subset \Omega \text{ or } B_{x-R\nu,R} \subset \mathbb{R}^n \setminus \Omega.$$

The vector ν is called inward "normal" vector of Ω at x.

For a function $w : \mathbb{R}^n \to \mathbb{R}$ and $z \in \mathbb{R}^n$, $r > 0$ we will denote $(w)_{z,r}$ the rescaled function $(w)_{z,r} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(w)_{z,r}(x) := \frac{w(z + rx)}{r}.$$

Let us remind a classical lemma of the one-phase free boundary problem. We say that a function w verifies property (4.24) if

$$|\nabla w| \leq C \text{ and } \forall r \in (0,1), x \in \mathbb{R}^n \ \fint_{\partial B_{x,r}} w < cr \text{ implies } w|_{B(x,r/2)} = 0 \tag{4.24}$$

for some constants $c, C > 0$.

Lemma 4.4.9. *Let $w \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}_+)$ be verifying $w(0) = 0$, property (4.24) for some constants c and C and $|\Delta w| 1_{\{w>0\}} \in L^\infty_{loc}(\mathbb{R}^n)$. Then there exists a subsequence $r_i \to 0$ and a function $\overline{w} \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}_+)$ verifying $\overline{w}(0) = 0$, property (4.24) for the same constants, $\Delta\overline{w} = 0$ on $\{\overline{w} > 0\}$ such that*

$$w_{0,r_i} \xrightarrow[\mathcal{C}^0_{loc}(\mathbb{R}^n)]{} \overline{w}, \ w_{0,r_i} \xrightarrow[\mathcal{C}^1_{loc}(\{\overline{w}>0\})]{} \overline{w},$$
$$\{w_{0,r_i} > 0\} \xrightarrow[loc.Hausd.]{} \{\overline{w} > 0\}, \ \{w_{0,r_i} = 0\} \xrightarrow[loc.Hausd.]{} \{\overline{w} = 0\}.$$

Proof. Functions $(w_{0,r})$ all verify the same Lipschitz bound so there exists a subsequence (w_{0,r_i}) that converges locally uniformly to some $\overline{w} \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}_+)$. Property (4.24) directly transfers to $\overline{w}$ for the same constants. Letting now U be some open set compactly included in $\{\overline{w} > 0\}$, we have that $U \subset \{w_{0,r_i} > 0\}$ for any large enough i with $|\Delta w_{0,r_i}| \leq \|\Delta w\|_{L^\infty(U)} r_i$ on U, giving both that $\Delta \overline{w} = 0$ in $\{\overline{w} > 0\}$ and the local $\mathcal{C}^1$ convergence in the support using $\mathcal{C}^{1,\alpha}$ elliptic estimates. Finally, the local Hausdorff convergence of the supports and their complements is obtained by non-degeneracy and (near-)harmonicity of w and (w_{0,r_i}) (see for instance [Vel19, Section 6]). □

Lemma 4.4.10. *Let $\Omega \subset \mathbb{R}^n$ be a minimizer of (4.15). Let $z \in \partial\Omega$ be a contact point of Ω with inward vector ν. Then provided $|\delta| \ll k^{-\left(4+\frac{8}{n}\right)} g_n(k)$, there exists $\alpha > 0$, $\beta \in \mathbb{R}$ and a positive sequence $s_i \to 0$ such that*

$$\begin{aligned} (w)_{z,s_i} &\underset{\mathcal{C}^0_{loc}(\mathbb{R}^n)}{\longrightarrow} \alpha(x \cdot \nu)_+, \\ (u_k)_{z,s_i} &\underset{\mathcal{C}^0_{loc}(\mathbb{R}^n)}{\longrightarrow} \beta(x \cdot \nu)_+, \end{aligned} \tag{4.25}$$

as $i \to \infty$, and

$$\alpha^2 + T(\Omega)^2 \delta \beta^2 = Q \tag{4.26}$$

where Q is defined in (4.22).

Remark 4.4.11. One could prove that the blow-up is unique, meaning that this holds for every sequence $s_i \to 0$; however this will not be useful to us.

Proof. Up to a displacement we assume $z = 0$, $\nu = e_n$. We also write $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ and

$$w_r(x) := \frac{w(rx)}{r}, \ u_r(x) := \frac{u_k(rx)}{r}$$

We start by proving that (w, u_k) admits $(\alpha x_n^+, \beta x_n^+)$ as a blow-up (for the same subsequence) at 0 for some $\alpha > 0$, $\beta \in \mathbb{R}$.

Blow-up for an exterior contact sphere. Supposing that there is an exterior contact sphere $\mathbb{B} := B_{-Re_n,R}$, we prove that $w(x) = \alpha x_n^+ + o(|x|)$ and $u_k(x) = \beta x_n^+ + o(|x|)$ for some $\alpha > 0$, $\beta \in \mathbb{R}$, thus getting (4.25). We follow the method of [CS05, Lemma 11.17], using the non-degeneracy and Lipschitz bounds on w and u_k from Lemma 4.4.6.

Let us first prove the expansion for w: set

$$G(x) = \begin{cases} \left(R \log\left(\frac{|x+Re_n|}{R}\right)\right)_+ & \text{if } n = 2 \\ \frac{\left(R^{2-n} - |x+Re_n|^{2-n}\right)_+}{(n-2)R^{1-n}} & \text{if } n \geq 3 \end{cases}, \quad \forall m \in \mathbb{N}, \ \alpha_m := \inf\{\alpha \geq 0 : w \leq \alpha G \text{ in } B_{2^{-m}}\}.$$

Above, α_m is well-defined and finite since $w(x) \lesssim d(x, \mathbb{B})$ whereas $G(x) \gtrsim d(x, \mathbb{B})$ on B_1. It is also bounded from below by a positive constant due to the non-degeneracy property. The sequence (α_m) decreases and therefore we can set $\alpha = \lim_{m\to\infty} \alpha_m$. We claim that $w(x) = \alpha G(x) + o(|x|)$, which is sufficient for proving the expansion, considering that $G(x) = x_n^+ + o(|x|)$.

Suppose it is not the case, meaning there is some sequence of points $(x^p)_{p\in\mathbb{N}}$ in $\mathbb{R}^n \setminus \mathbb{B}$ converging to 0 and some $\varepsilon \in (0, 1]$ such that

$$\forall p \in \mathbb{N}, \ w(x^p) < \alpha G(x^p) - \varepsilon |x^p|.$$

We let L be a Lipschitz constant of $w - \alpha G$, and we will suppose without loss of generality that $\varepsilon \ll L$. Letting $y^p = x^p + \frac{\varepsilon}{2L}|x^p| e_n$, then $\frac{1}{2}|x^p| \leq |y^p| \leq 2|x^p|$ and by the Lipschitz bounds we have

$$\forall p, \ w(y^p) - \alpha G(y^p) < -\frac{\varepsilon}{4}|y^p|,$$

as well as

$$y_n^p = \frac{\varepsilon}{2L}|x^p| + x_n \geq \frac{\varepsilon}{4L}|y^p| - \frac{1}{2R}|x^p|^2 \geq \frac{\varepsilon}{8L}|y^p|,$$

where the last inequality holds for any large enough p and we have used that $x_n^p \geq \frac{-1}{2R}|x^p|^2$ (since $x^p \notin \mathbb{B}$). We now let $r^p = |y^p|$ and B^p the ball of center y^p/r^p and of radius $\frac{\varepsilon}{16L}$ on which we have, still by the Lipschitz bound,

$$(w - \alpha G)_{r^p} \leq -\frac{\varepsilon}{8}$$

and for which $\mathrm{dist}(B^p, \partial(B_2 \setminus \mathbb{B})) \geq \frac{\varepsilon}{16L}$. Let now φ^p be the continuous function defined by

$$\begin{cases} \varphi^p = \eta & \text{in } B^p \\ \varphi^p = 0 & \text{on } \partial(B_2 \setminus \mathbb{B}) \\ \Delta\varphi^p = 2r^p & \text{in } (B_2 \setminus \mathbb{B}) \setminus B^p \end{cases}$$

where $\eta > 0$ is fixed small ; if $\eta \leq \frac{\varepsilon}{8}$ is small enough we have $\varphi^p \leq \alpha G_{r^p}$ in B_2 by maximum principle for all p. Then for a large enough p we have $\varphi^p \geq 0$ and $\varphi^p \geq cG_{r_p}$ in $B_{1/2}$ for some $c > 0$ by Hopf's lemma. We claim that for a large enough p in this case

$$\left(w - \left(\alpha + \frac{c}{2}\right) G\right)_{r^p} \leq -\varphi^p$$

by maximum principle in the domain $\omega^p := \{w_{r^p} > 0\} \cap (B_2 \setminus (\mathbb{B} \cup B^p))$. Indeed suppose p is large enough such that $w_{r^p} \leq \left(\alpha + \frac{c}{2}\right) G_{r^p}$ in B_2, then

$$\Delta\left(w - \left(\alpha + \frac{c}{2}\right) G\right)_{r^p} = -r^p > -2r^p = -\Delta\varphi^p \text{ on } \omega^p$$

and the inequality is verified on $\partial\omega^p$:

$$\begin{aligned} &\left(w - \left(\alpha + \frac{c}{2}\right) G\right)_{r^p} + \varphi^p \leq \varphi^p - \alpha G_{r^p} \leq 0 \text{ on } \{w_{r^p} = 0\}, \\ &\left(w - \left(\alpha + \frac{c}{2}\right) G\right)_{r^p} + \varphi^p = w_{r^p} - \left(\alpha + \frac{c}{2}\right) G_{r^p} \leq 0 \text{ on } \partial B_2, \\ &\left(w - \left(\alpha + \frac{c}{2}\right) G\right)_{r^p} + \varphi^p \leq -\frac{\varepsilon}{8} + \varphi^p \leq 0 \text{ on } B^p. \end{aligned}$$

This implies $w(x) \leq \left(\alpha - \frac{c}{2}\right) G(x)$ in some neighbourhood of the origin, which contradicts the definition of $\alpha = \inf_m \alpha_m$. This gives the announced expansion for w and hence (4.25) for w.
The exact same reasoning can then be done for $w + cu_k$ for any c chosen such that $w + cu_k$ is positive in its support (which holds for $c \ll k^{-\left(\frac{1}{2}+\frac{2}{n}\right)}$ by Lemma 4.2.2), thus getting the existence of $\beta \in \mathbb{R}$ such that (4.25) holds true for u_k. This finishes the proof of (4.25) in the case of an exterior contact sphere.

Blow-up for an interior contact sphere. Assume now that there is an interior contact sphere $B_{Re_n,R} \subset \Omega$; in particular for any blow-up $(\overline{w}, \overline{u}_k)$ of (w, u_k) at 0 we have $\mathbb{H}^n \subset \{\overline{w} > 0\}$. We apply [CS05, Lemma 11.17 and Remark 11.18] to $\overline{w}$ (and $\overline{w} + c\overline{u_k}$ for a small enough $c \ll k^{-\left(\frac{1}{2}+\frac{2}{n}\right)}$); this gives $\overline{w}(x) = \alpha x_n^+ + o(|x|)$ and $\overline{u_k}(x) = \beta x_n^+ + o(|x|)$ in $\mathbb{H}^n$ for some $\alpha > 0$, $\beta \in \mathbb{R}$.

We remind that a blow-up (at 0) of a blow-up (at 0) of w is still a blow-up of w: indeed if $w_{0,r_i} \to \overline{w}$ and $\overline{w}_{0,s_i} \to \tilde{w}$ then there is some extraction $\varphi(i)$ such that $w_{0,r_{\varphi(i)}s_i} \to \tilde{w}$.

As a a consequence, there is a blow-up of w, u_k at 0 (that we still denote $\overline{w}, \overline{u_k}$) such that $\overline{w}(x) = \alpha x_n^+$ and $\overline{u_k}(x) = \beta x_n^+$ in $\mathbb{H}^n$.

We now prove that $\tilde{w}(x) = o(|x|)$ on $\mathbb{R}^n \setminus \mathbb{H}^n$, which is enough to conclude since $(\alpha x_n^+, \beta x_n^+)$ is then a blow-up of (w, u_k) at 0. Arguing by contradiction we assume that $\overline{w}(x) = o(|x|)$ is not verified on

$\mathbb{R}^n \setminus \mathbb{H}^n$, so that in particular $\{\overline{w} > 0\} \cap (\mathbb{R}^n \setminus \overline{\mathbb{H}^n})$ is a non-empty open set which accumulates at 0 and since $\overline{w}_{|\partial\mathbb{H}^n} \equiv 0$, then $\overline{w}1_{\mathbb{R}^n\setminus\mathbb{H}^n}$ is continuous and admits $B_{e_n,1}$ as an exterior contact sphere at 0. We can therefore proceed as in the exterior sphere condition case to deduce that there exists $\gamma \geq 0$ such that $\overline{w}(x) = \gamma x_n^- + o(|x|)$ on $\{\overline{w} > 0\} \cap (\mathbb{R}^n \setminus \mathbb{H}^n)$.

Thanks to the contradiction hypothesis we must have $\gamma > 0$. In particular the density of $\{\overline{w} = 0\}$ at the origin is zero. We remind that $\overline{w}$ is a blow-up of w at 0 for some sequence $r_i \to 0$, so

$$\lim_{\tau\to 0}\lim_{i\to\infty} \frac{|B_{\tau r_i} \setminus \Omega|}{(\tau r_i)^n} = \lim_{\tau\to 0} \frac{|B_\tau \setminus \{\overline{w} > 0\}|}{|B_\tau|} = 0. \tag{4.27}$$

Let then $s_i = \tau r_i$ for some $\tau > 0$ to be fixed later. We arrive to a contradiction by proving that the energy of $\tilde{\Omega}_i := \left(\frac{\omega_n}{|\Omega\cup B_{s_i}|}\right)^{\frac{1}{n}} \Omega \cup B_{s_i}$ is strictly lower. We use the following lemma to build a good competitor for $T(\Omega \cup B_{s_i})$.

Lemma 4.4.12. *There exists $\theta_n > 0$ and $\varepsilon_n > 0$ such that the following property holds: for any $u \in H^1(B_1, \mathbb{R}_+)$, such that $\|u - x_n^+\|_{L^\infty(B_1\cap\mathbb{H}^n)} < \varepsilon_n$, let $\mathcal{H}u$ be the harmonic extension of $u_{|\partial B_1}$ in B_1, then*

$$\int_{B_1} |\nabla u|^2 \geq \theta_n + \int_{B_1} |\nabla \mathcal{H}u|^2.$$

Proof. Suppose that $\|u - x_n^+\|_{L^\infty(B_1\cap\mathbb{H}^n)} \leq \varepsilon$ for some $\varepsilon > 0$ and let us adjust ε so that the conclusion holds. We have

$$\begin{aligned}\int_B (|\nabla u|^2 - |\nabla \mathcal{H}u|^2) &= \int_B |\nabla(u - \mathcal{H}u)|^2 \geq \lambda_1(B) \int_B |u - \mathcal{H}u|^2 \\ &\geq \frac{2\lambda_1(B)}{\omega_n} \left(\int_{B\cap\mathbb{H}^n} (\mathcal{H}u - u)_+\right)^2,\end{aligned}$$

where we used Faber-Krahn and Cauchy-Schwarz inequalities. Since

$$\int_{B\cap\mathbb{H}^n} (\mathcal{H}u - u)_+ \geq \int_{B\cap\mathbb{H}^n} (\mathcal{H}x_n^+ - x_n^+) - \omega_n\varepsilon$$

then by taking $\varepsilon = \varepsilon_n := \frac{1}{4} \fint_{B\cap\mathbb{H}^n} (\mathcal{H}x_n^+ - x_n^+)$ and $\theta_n := \frac{1}{2} \int_{B\cap\mathbb{H}^n} (\mathcal{H}x_n^+ - x_n^+)$ the conclusion follows. □

As a consequence, using the harmonic extension of w in B_{s_i} as a test function for $T(\Omega \cup B_{s_i})$, and using the fact that $\frac{1}{\alpha} w_{s_i} \underset{\mathcal{C}^0(B\cap\mathbb{H}^n)}{\longrightarrow} x_n^+$, we find that from any large enough i,

$$T(\Omega \cup B_{s_i}) - T(\Omega) \geq s_i^n \int_{B_1} \left(2(\mathcal{H}u_{s_i} - u_{s_i}) - |\nabla \mathcal{H}u_{s_i}|^2 + |\nabla u_{s_i}|^2\right) \geq \theta_n \alpha s_i^n$$

so $T(\Omega \cup B_{s_i})^{-1} - T(\Omega)^{-1} \gtrsim s_i^n$. At the same time we have by Lemma 4.2.3

$$\lambda_k(\Omega) - \lambda_k(\Omega \cup B_{s_i}) \lesssim k^{2+\frac{4}{n}} (T(\Omega \cup B_{s_i}) - T(\Omega)) \lesssim k^{2+\frac{4}{n}} (T(\Omega)^{-1} - T(\Omega \cup B_{s_i})^{-1})$$

where we also used Lemma 4.4.3 to write $\lambda_k(\Omega \cup B_{s_i}) \leq \lambda_k(\Omega) \lesssim k^{\frac{2}{n}}$. We now compare the energy of Ω and $\tilde{\Omega}_i := \left(\frac{\omega_n}{|\Omega\cup B_{s_i}|}\right)^{\frac{1}{n}} \Omega \cup B_{s_i}$:

$$\begin{aligned}0 &\leq \left(T^{-1} + \delta\lambda_k\right)\left(\tilde{\Omega}_i\right) - \left(T^{-1} + \delta\lambda_k\right)(\Omega) \\ &= \left(\frac{|\Omega \cup B_{s_i}|}{\omega_n}\right)^{\frac{n+2}{n}} T(\Omega \cup B_{s_i})^{-1} - T(\Omega)^{-1} + \delta\left(\left(\frac{|\Omega \cup B_{s_i}|}{\omega_n}\right)^{\frac{2}{n}} \lambda_k(\Omega \cup B_{s_i}) - \lambda_k(\Omega)\right) \\ &\leq \left(1 - C_n|\delta|k^{2+\frac{4}{n}}\right)\left(T(\Omega \cup B_{s_i})^{-1} - T(\Omega)^{-1}\right) + (C_n' + k^{\frac{2}{n}} C_n'')|B_{s_i} \setminus \Omega|\end{aligned}$$

for some constants $C_n, C_n', C_n'' > 0$: as a consequence, when $|\delta| \ll k^{-2-\frac{4}{n}}$ we get

$$|B_{s_i} \setminus \Omega| \gtrsim T(\Omega)^{-1} - T(\Omega \cup B_{s_i})^{-1} \gtrsim s_i^n.$$

Finally, we get $|B_{s_i} \setminus \Omega| \geq c_n s_i^n$ for all $i \in \mathbb{N}$, for some constant $c_n > 0$, which is in contradiction with (4.27) for large $i \in \mathbb{N}$ when τ is chosen small enough. As a consequence we have $\overline{w}(x) = o(|x|)$ on $\mathbb{H}^n$, thus finishing the proof of the interior sphere case.

Relation between α and β. Let $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $\zeta^t = I + t\zeta$, which is a diffeomorphism for any small enough $t \in \mathbb{R}$. Since for $|\delta| \ll k^{-\left(4+\frac{8}{n}\right)} g_n(k)$ we have that $\lambda_k(\Omega)$ is simple by Lemma 4.4.4, we may compute the shape derivatives of T, λ_k and $|\cdot|$ at the bounded open set Ω (see respectively [Lau20, Proposition 6] and [LLdC06, Theorem 2.6 (iii)] for the derivatives of T and λ_k). We have

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \left|\zeta^t(\Omega)\right| &= \int_\Omega \nabla \cdot \zeta, \\
\frac{d}{dt}\bigg|_{t=0} T\left(\zeta^t(\Omega)\right) &= \int_\Omega \left[\left(2w - |\nabla w|^2\right) \nabla \cdot \zeta + 2\nabla w \cdot D\zeta \cdot \nabla w\right], \\
\frac{d}{dt}\bigg|_{t=0} \lambda_k\left(\zeta^t(\Omega)\right) &= \int_\Omega \left[\left(|\nabla u_k|^2 - \lambda_k(\Omega) u_k^2\right) \nabla \cdot \zeta - 2\nabla u_k \cdot D\zeta \cdot \nabla u_k\right].
\end{aligned}$$

Thanks to Lemma 4.4.7, the optimality condition writes

$$\frac{d}{dt}\bigg|_{t=0} \left[\left|\zeta^t(\Omega)\right|^{\frac{2}{n}} \left(\frac{\left|\zeta^t(\Omega)\right|}{\omega_n T\left(\zeta^t(\Omega)\right)} + \delta \lambda_k\left(\zeta^t(\Omega)\right)\right)\right] = 0.$$

It gives, after simplification,

$$\begin{aligned}
\int_\Omega \left[\left(|\nabla w|^2 + T(\Omega)^2 \delta |\nabla u_k|^2 + Q\right) \nabla \cdot \zeta - 2\left(\nabla w \cdot D\zeta \cdot \nabla w + T(\Omega)^2 \delta \nabla u_k \cdot D\zeta \cdot \nabla u_k\right)\right] \\
= \int_\Omega \left(2w + T(\Omega)^2 \delta \lambda_k(\Omega) u_k^2\right) \nabla \cdot \zeta.
\end{aligned}$$

We now replace ζ with $\zeta_i(x) := \zeta(x/s_i)$, where s_i is a positive sequence for which (4.25) holds, and we rescale the previous equality to obtain

$$\begin{aligned}
\int_{s_i^{-1}\Omega} \Big(&\left(|\nabla w_{s_i}|^2 + T(\Omega)^2 \delta |\nabla u_{k,s_i}|^2 + Q\right) \nabla \cdot \zeta \\
&-2\left(\nabla w_{s_i} \cdot D\zeta \cdot \nabla w_{s_i} + T(\Omega)^2 \delta \nabla u_{k,s_i} \cdot D\zeta \cdot \nabla u_{k,s_i}\right)\Big) \\
&= s_i \int_{s_i^{-1}\Omega} \left(2w_{s_i} + s_i T(\Omega)^2 \delta \lambda_k(\Omega) (u_{k,s_i})^2\right) \nabla \cdot \zeta.
\end{aligned}$$

Now, we have shown in the first part that $w_{s_i} \to \alpha x_n^+$ and $u_{k,s_i} \to \beta x_n^+$ in the $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{C}^1_{\mathrm{loc}}(\mathbb{H}^n)$ sense. Using also the L^∞ and Lipschitz bounds on w_{s_i} and u_{k,s_i}, and recalling that $s_i^{-1}\Omega = \{w_{s_i} > 0\}$ converges locally in $\mathbb{R}^n$ in the Hausdorff sense to $\mathbb{H}^n$ thanks to Lemma 4.4.9, every term above passes to the limit and we get

$$\int_{\mathbb{H}^n} \left[\left(\alpha^2 + T(\Omega)^2 \delta \beta^2 + Q\right) \nabla \cdot \zeta - 2\left(\alpha^2 + T(\Omega)^2 \delta \beta^2\right) \partial_n \zeta_n\right] = 0.$$

Applying Stoke's theorem, this gives $\int_{\partial \mathbb{H}^n} \left(\alpha^2 + T(\Omega)^2 \delta \beta^2 - Q\right)(\zeta \cdot e_n) = 0$. Since $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ was chosen arbitrarily, we conclude that $\alpha^2 + T(\Omega)^2 \delta \beta^2 = Q$. $\square$

4.4.3 Minimizers are nearly spherical

In this section we prove that under sufficient smallness of δ, minimizers are nearly spherical sets. In this context, "nearly spherical" means that

$$\Omega = B_h := \{s(1+h(x))x,\ s \in [0,1),\ x \in \partial B\},$$

where $h \in \mathcal{C}^{2,\gamma}\left(\partial B, [-\frac{1}{2}, \frac{1}{2}]\right)$ for some $\gamma > 0$ with a bound $\|h\|_{C^{2,\gamma}} \lesssim 1$. This is achieved in Lemma 4.4.15. It will be useful for us to consider *centered* sets, where we say $\Omega \in \mathcal{A}$ is centered when $\mathrm{bar}(\Omega) := \fint_\Omega x dx$ is well-defined and equals zero. Note that since the functional under study $T^{-1} + \delta\lambda_k$ is translation invariant, there is no loss of generality in assuming that a given minimizer is centered.

Lemma 4.4.13. *Let Ω be a centered minimizer of (4.15) for $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$. Then we have*

$$\|w_\Omega - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} \lesssim \left(k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}},$$
$$|\Omega \Delta B| \lesssim \mathcal{F}(\Omega).$$

Proof. Suppose first that Ω is translated into $\tilde{\Omega}$ so that $\mathcal{F}(\tilde{\Omega}) = |\tilde{\Omega}\Delta B|$ (we will keep denoting it Ω for simplicity). If $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$ then we have shown in (4.21) that $\|w_\Omega - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} \lesssim \left(k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}}$. Now, since Ω is bounded by a dimensional constant thanks to Lemma 4.4.6, we have $|\mathrm{bar}(\Omega)| = |\mathrm{bar}(\Omega) - \mathrm{bar}(B)| \lesssim |\Omega\Delta B| \lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}$ using also Lemma 4.4.3. As a consequence, we deduce

$$\|w_{\Omega-\mathrm{bar}(\Omega)} - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} \leq \|w_\Omega - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} + \|w_{B-\mathrm{bar}(\Omega)} - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} \lesssim \left(k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}}$$

as well as

$$|(\Omega - \mathrm{bar}(\Omega))\Delta B| \leq |\Omega\Delta B| + |(B + \mathrm{bar}(\Omega))\Delta B| \lesssim \mathcal{F}(\Omega).$$

□

Lemma 4.4.14. *Let Ω be a centered minimizer of (4.15). If $|\delta| \ll k^{-\left(4+\frac{8}{n}\right)}g_n(k)$ then we have*

$$\partial\Omega = \{(1+h(x))x,\ x \in \partial B\},$$

where $h \in \mathcal{C}^{1,\gamma}\left(\partial B, [-\frac{1}{2}, \frac{1}{2}]\right)$ for some $\gamma = \gamma_n \in (0,1)$ depending only on n, and $\|h\|_{\mathcal{C}^{1,\gamma}(\partial B)} \lesssim 1$.

Proof. We separate the cases $\delta < 0$ and $\delta > 0$.

Case $\delta < 0$. Set

$$\overline{w} = Q^{-\frac{1}{2}}\left(w + T(\Omega)\sqrt{-\delta}u_k\right) \text{ and } \underline{w} = Q^{-\frac{1}{2}}\left(w - T(\Omega)\sqrt{-\delta}u_k\right).$$

For $|\delta| \ll k^{-1-\frac{4}{n}}$ the functions $\overline{w}$ and $\underline{w}$ are positive on their support (due to the estimates from Lemma 4.2.2). Since $|\delta| \ll k^{-4-\frac{8}{n}}g_n(k)$, Lemma 4.4.10 applies to ensure that the couple $(\overline{w}, \underline{w})$ is a viscosity solution in the sense of [MTV21, Definition 2.4] of the system

$$\begin{cases} -\Delta\overline{w} = Q^{-\frac{1}{2}}\left(1 + T(\Omega)\sqrt{-\delta}\lambda_k(\Omega)u_k\right) & (\Omega), \\ -\Delta\underline{w} = Q^{-\frac{1}{2}}\left(1 - T(\Omega)\sqrt{-\delta}\lambda_k(\Omega)u_k\right) & (\Omega), \\ \overline{w}, \underline{w} > 0 & (\Omega), \\ \overline{w} = \underline{w} = 0 & (\partial\Omega), \\ \partial_\nu\overline{w} \cdot \partial_\nu\underline{w} = 1 & (\partial\Omega). \end{cases}$$

Note that $\overline{w}$ and $\underline{w}$ both converge uniformly to $\left(\frac{1-|x|^2}{2}\right)_+$ as $k^{1+\frac{4}{n}}|\delta| \to 0$. In fact, by respectively Lemma 4.4.13, inequality (4.23) and Lemma 4.2.2 we have

$$\|w_\Omega - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} \lesssim \left(k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}}, \ \left|Q - \frac{1}{n^2}\right| \lesssim k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}, \ \sqrt{-\delta}|u_k| \lesssim k^{\frac{1}{2}}|\delta|^{\frac{1}{2}},$$

so that

$$\left\|\overline{w} - \left(\frac{1-|x|^2}{2}\right)_+\right\|_{\mathcal{C}^0(\mathbb{R}^n)} + \left\|\underline{w} - \left(\frac{1-|x|^2}{2}\right)_+\right\|_{\mathcal{C}^0(\mathbb{R}^n)} \lesssim \left(k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}}.$$

Our goal is now to apply the $\mathcal{C}^{1,\gamma}$ regularity theorem [MTV21, Theorem 3.1] for balls $B_{x,r}$ with $x \in \partial B$ and sufficiently small $r > 0$. To simplify notations we assume that $x = -e_n$ and let $r > 0$ be a radius which will be fixed later. Since $\|(w_B)_{-e_n,r} - x_n\|_{\mathcal{C}^0(B_1)} \lesssim r$ we deduce from the convergence of $\overline{w}$ and $\underline{w}$

$$\|\overline{w}_{-e_n,r} - x_n\|_{\mathcal{C}^0(B_1)} + \|\overline{w}_{-e_n,r} - x_n\|_{\mathcal{C}^0(B_1)} \lesssim r + r^{-1}\left(k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}}$$
$$|\Delta\overline{w}_{-e_n,r}| + |\Delta\underline{w}_{-e_n,r}| \lesssim r \text{ in } r^{-1}\Omega$$

where we also used that $|\Delta\overline{w}_{-e_n,r}|, |\Delta\underline{w}_{-e_n,r}| \lesssim 1 + |\delta|^{\frac{1}{2}}k^{\frac{1}{2}+\frac{4}{n}} \lesssim 1$ in $r^{-1}\Omega$ thanks to $\lambda_k(\Omega) \lesssim k^{\frac{2}{n}}$ (see Lemma 4.4.3) and the choice of δ.

We let $\varepsilon := \sqrt{r}$ and choose r small enough so that the ε-regularity Theorem [MTV21, Theorem 3.1] applies to ε (note that our inequalities are up to a dimensional constant, so the choice of ε may also differ up to a constant). Then when $\left(k^{\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}} \ll r^2$ the couple $(\overline{w}_{-e_n,r}, \underline{w}_{-e_n,r})$ is ε-flat so by [MTV21, Theorem 3.1], $\partial\{\overline{w}_{-e_n,r} > 0\} \cap B_{\frac{1}{2}}$ is a $\mathcal{C}^{1,\gamma}$ graph with controlled $\mathcal{C}^{1,\gamma}$ norm (for some dimensional constant $\gamma = \gamma_n \in (0,1)$), meaning that there exists $g : \left[-\frac{1}{2}, \frac{1}{2}\right] \to [-1,1]$ such that

$$\{\overline{w}_{-e_n,r} > 0\} \cap B_{\frac{1}{2}} = \left\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, \ |x'| \le \frac{1}{2}, \ x_n \ge g(x)\right\}, \ \|g\|_{\mathcal{C}^{1,\gamma}\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)} \lesssim 1.$$

This translates into the fact that $\partial\Omega \cap B_{-e_n,r/2}$ is a $\mathcal{C}^{1,\gamma}$ graph with graph function $\tilde{g} := r(-1+g)$. Covering $\partial\Omega$ with a finite number of such balls of radius $r/2$ we find $h : \partial B \to \mathbb{R}$ such that $\partial\Omega = \{(1+h(x))x, x \in \partial B\}$ with $\|h\|_{\mathcal{C}^{1,\gamma}(\partial B)} \lesssim 1$.

Case $\delta > 0$. We analogously set

$$\overline{w} = Q^{-\frac{1}{2}}\left(w + T(\Omega)\sqrt{\delta}u_k\right) \text{ and } \underline{w} = Q^{-\frac{1}{2}}\left(w - T(\Omega)\sqrt{\delta}u_k\right),$$

This time the couple $(\overline{w}, \underline{w})$ is a viscosity solution with boundary condition $\frac{(\partial_\nu\overline{w})^2 + (\partial_\nu\underline{w})^2}{2} = 1$ in the sense given by [KL18, Definition 4.1]. By the exact same argument as in the previous case we may apply [KL18, Theorem 7.2] to the couple $(\overline{w}_{x,r}, \underline{w}_{x,r})$ for $x \in \partial B$ and some dimensional r, thus providing again the existence of $h : \partial B \to \mathbb{R}$ such that $\partial\Omega = \{(1+h(x))x, x \in \partial B\}$ with $\|h\|_{\mathcal{C}^{1,\gamma}(\partial B)} \lesssim 1$ (with a possibly different dimensional constant γ).

□

Lemma 4.4.15. *Let Ω be a centered minimizer of (4.15). If $|\delta| \ll k^{-\left(4+\frac{8}{n}\right)}g_n(k)$ then*

$$\partial\Omega = \{(1+h(x))x, \ x \in \partial B\}$$

where $h \in \mathcal{C}^{2,\gamma}\left(\partial B, \left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ for some dimensional $\gamma = \gamma_n \in (0,1)$ and $\|h\|_{\mathcal{C}^{2,\gamma}(\partial B)} \lesssim 1$.

In this proof we denote by C_r the cylinder $C_r := \{x = (x', x_n) : |x'| \le r, |x_n| \le r\} = B_r^{n-1} \times [-r, r]$. For a function $a(x')$ of $N-1$ variables we denote by ∇' its gradient.

Proof. We follow the general method of partial hodograph transform, as is detailed for instance in [KL18, CSY18]. Since $|\delta| \ll k^{-4-\frac{8}{n}} g_n(k)$ we can apply Lemma 4.4.14 to Ω, thus giving that Ω is the graph on the sphere of a $\mathcal{C}^{1,\gamma}$ function h with $\|h\|_{C^{1,\gamma}(\partial B)} \lesssim 1$. By classical $\mathcal{C}^{1,\gamma}$ elliptic regularity (see for instance [HL11, Theorem 3.13]) and the $W^{1,\infty}$ bounds from Lemmas 4.2.2 and 4.4.6 we have $\|w_\Omega\|_{\mathcal{C}^{1,\gamma}(\Omega)} \lesssim 1$ and $\|u_k\|_{\mathcal{C}^{1,\gamma}(\Omega)} \lesssim k^{\frac{1}{2}+\frac{4}{n}}$. Let us also remind that $\partial_\nu w_\Omega$ lies between two dimensional positive constants thanks to the non-degeneracy and Lipschitz bound of w_Ω from Lemma 4.4.6.

Consider now any $x \in \partial\Omega$. Up to translation and rotation of Ω we simply suppose that $x = 0$ with inward normal vector $\nu(0) = e_n$, and call $\alpha = \partial_n w_\Omega(0)$. Letting $r > 0$ which we will fix later on (depending only on n), then for any small enough r we have, according to Lemma 4.4.14,

$$\left(r^{-1}\Omega\right) \cap C_2 = \left\{x = (x', x_n) \in C_2 : x_n \geq g(x')\right\},$$

where $\|g\|_{\mathcal{C}^{1,\gamma}(B_2^{n-1})} \lesssim r^\gamma$ (using that $\nabla' g(0) = 0$ and the $\mathcal{C}^{1,\gamma}$ bound) and $\|(w_\Omega)_r - \alpha x_n\|_{\mathcal{C}^{1,\gamma}(r^{-1}\Omega\cap C_2)} \lesssim r^\gamma$. Consider the function

$$\begin{array}{rccc} \phi : & \frac{\Omega}{r} \cap C_2 & \to & \mathbb{H}^n \cap C_3 \\ & x = (x', x_n) & \mapsto & \left(x', \frac{w(rx)}{\alpha r}\right). \end{array}$$

Since $\|D\phi - I_n\|_{\mathcal{C}^{0,\gamma}(r^{-1}\Omega\cap C_2)} \lesssim r^\gamma$ and $\|h\|_{\mathcal{C}^{1,\gamma}(\partial B)} \lesssim 1$, then for a small enough r (only depending on the dimension) thanks to the inverse function theorem applied at $x = 0$ we have that ϕ is a $\mathcal{C}^{1,\gamma}$-diffeomorphism on its image, and its image compactly contains $C_1 \cap \mathbb{H}^n$. We call its inverse ψ, which may also be written as

$$\psi(x) =: (x', v(x))$$

for some function v such that

$$\|v - x_n\|_{\mathcal{C}^{1,\gamma}(C_1\cap\mathbb{H}^n)} \lesssim r^\gamma.$$

Noting that the graph of $v_{|C_1\cap\partial\mathbb{H}^n}$ parametrizes $\partial\Omega \cap C_r$, our goal is now to obtain a $\mathcal{C}^{2,\gamma}$ bound on this function. Let $v_k = (u_k)_r \circ \psi$, and

$$A(p) := \begin{bmatrix} I_{n-1} & -\frac{p'}{p_n} \\ -\frac{(p')^*}{p_n} & \frac{1+|p'|^2}{p_n^2} \end{bmatrix}.$$

Then, using the relations

$$\begin{cases} \partial_i v \circ \phi + \partial_i w_r \partial_n v \circ \phi = 0, \text{ for all } i < n, \\ \partial_n w_r \partial_n v \circ \phi = 1, \end{cases}$$

obtained by differentiating $v \circ \phi(x) = x_n$, we get that (v, v_k) verifies the following elliptic systems:

$$\begin{cases} \operatorname{Tr}\left[A(\nabla v)\nabla^2 v\right] = \frac{\partial_n v}{\alpha} & (C_1 \cap \mathbb{H}^n), \\ \frac{Q}{\alpha^2}(\partial_n v)^2 - \left(1 + \delta|\nabla v_k|^2\right)|\nabla' v|^2 = 1 & (C_1 \cap \partial\mathbb{H}^n), \\ \operatorname{Tr}\left[A(\nabla v)\nabla^2 v_k\right] = \frac{\partial_n v_k}{\alpha} - \lambda_k v_k & (C_1 \cap \mathbb{H}^n), \\ v_k = 0 & (C_1 \cap \partial\mathbb{H}^n). \end{cases} \tag{4.28}$$

Note that $\|A(\nabla v) - I_n\|_{\mathcal{C}^{0,\gamma}(C_1\cap\mathbb{H}^n)} \lesssim r^\gamma$. The Dirichlet condition over $C_1 \cap \partial\mathbb{H}^n$ enables us to extend by odd reflection v_k over $C_1 \cap (\mathbb{R}^n \setminus \mathbb{H}^n)$, so that by interior elliptic estimates in C_1 we get

$$\begin{aligned} \|v_k\|_{\mathcal{C}^{2,\gamma}(C_{9/10}\cap\mathbb{H}^n)} &\lesssim \|v_k\|_{\mathcal{C}^{1,\gamma}(C_1\cap\mathbb{H}^n)} + k^{\frac{2}{n}}\|v_k\|_{\mathcal{C}^{0,\gamma}(C_1\cap\mathbb{H}^n)} \\ &\lesssim \|(u_k)_r\|_{\mathcal{C}^{1,\gamma}(C_2\cap\Omega/r)} + k^{\frac{2}{n}}\|(u_k)_r\|_{\mathcal{C}^{0,\gamma}(C_2\cap\Omega/r)} \\ &\lesssim \|u_k\|_{\mathcal{C}^{1,\gamma}(\Omega)} + k^{\frac{2}{n}}\|u_k\|_{\mathcal{C}^{0,\gamma}(\Omega)} \lesssim k^{\frac{1}{2}+\frac{4}{n}}. \end{aligned}$$

We can then consider the first equation alone, as an elliptic equation with a nonlinear oblique Neumann boundary condition. Defining $q := \frac{\alpha^2(1+\delta|\nabla v_k|^2)}{Q}$ and using the $\mathcal{C}^{2,\gamma}$ estimate on v_k we have

$$\|\nabla q\|_{\mathcal{C}^{0,\gamma}(\partial\mathbb{H}^n\cap C_{9/10})} \lesssim k^{1+\frac{6}{n}}|\delta| \lesssim r^\gamma,$$

where the second inequality is true when $|\delta| \ll k^{-1-\frac{6}{n}}r^\gamma$. We define $\chi : \mathbb{H}^n \to \mathbb{R}^n$ the smooth vector field given by

$$\forall p = (p', p_n) \in \mathbb{R}^{n-1} \times (0,\infty),\ \chi(p', p_n) = \left(p', -\frac{1+|p'|^2}{p_n}\right).$$

Let z be the unique solution to

$$\begin{cases} \Delta z = \partial_n v - 1 & (C_1 \cap \mathbb{H}^n), \\ z = 0 & \partial(C_1 \cap \mathbb{H}^n), \end{cases}$$

and set $\zeta = x_n e_n + \nabla z$. We have

$$\|\nabla z\|_{\mathcal{C}^{1,\gamma}(C_{1/2}\cap\mathbb{H}^n)} \lesssim \|\partial_n v - 1\|_{\mathcal{C}^{0,\gamma}(C_1\cap\mathbb{H}^n)} \lesssim r^\gamma.$$

Then since $\nabla\cdot(\chi(\nabla v)) = \mathrm{Tr}(A(\nabla v)\nabla^2 v)$ and $\nabla\cdot\zeta = \partial_n v$ we may rewrite the equations of v from (4.28) as

$$\begin{cases} \nabla\cdot\chi(\nabla v) = \frac{\nabla\cdot\zeta}{\alpha} & (C_1\cap\mathbb{H}^n), \\ (\partial_n v)^2 - q|\nabla' v|^2 = \frac{\alpha^2}{Q} & (C_1\cap\partial\mathbb{H}^n). \end{cases} \tag{4.29}$$

Let $\tau \in (0, 1/4)$ be small enough so that $\psi(x+\tau e_i)$ is well-defined for $x \in C_{1/4}\cap\mathbb{H}^n$ and $i \in \{1,\dots,n-1\}$. Let $i \in \{1,\dots,n-1\}$ and set

$$\begin{aligned} v_{\tau,i}(x) &= \frac{v(x+\tau e_i) - v(x)}{\tau}, \\ C_{\tau,i}(x) &= \int_0^1 D\chi\left((1-t)\nabla v(x) + t\nabla v(x+\tau e_i)\right)dt, \\ \zeta_{\tau,i}(x) &= \frac{\zeta(x+\tau e_i) - \zeta(x)}{\tau}, \\ a_{\tau,i}(x) &= \frac{\nabla' v(x+\tau e_i) + \nabla' v(x)}{\partial_n v(x+\tau e_i) + \partial_n v(x)} q(x), \\ b_{\tau,i}(x) &= \frac{|\nabla' v(x+\tau e_i)|^2}{\partial_n v(x+\tau e_i) + \partial_n v(x)} \frac{q(x+\tau e_i) - q(x)}{\tau}. \end{aligned}$$

Then the equations (4.29) translate into the following equations for $v_{\tau,i}$

$$\begin{cases} \nabla\cdot(C_{\tau,i}\nabla v_{\tau,i}) = \frac{\nabla\cdot\zeta_{\tau,i}}{\alpha} & (C_{1/4}\cap\mathbb{H}^n), \\ \partial_n v_{\tau,i} = a_{\tau,i}\cdot\nabla' v_{\tau,i} + b_{\tau,i} & (C_{1/4}\cap\partial\mathbb{H}^n). \end{cases} \tag{4.30}$$

We have

$$\begin{aligned} &\|v_{\tau,i}\|_{\mathcal{C}^{0,\gamma}(C_{1/4}\cap\mathbb{H}^n)} + \|C_{\tau,i} - I_n\|_{\mathcal{C}^{0,\gamma}(C_{1/4}\cap\mathbb{H}^n)} \\ &\qquad + \|\zeta_{\tau,i}\|_{\mathcal{C}^{0,\gamma}(C_{1/4}\cap\mathbb{H}^n)} + \|a_{\tau,i}\|_{\mathcal{C}^{0,\gamma}(C_{1/4}\cap\partial\mathbb{H}^n)} + \|b_{\tau,i}\|_{\mathcal{C}^{0,\gamma}(C_{1/4}\cap\partial\mathbb{H}^n)} \lesssim r^\gamma, \end{aligned}$$

where the upper bound is independent of τ. By Schauder estimates on elliptic equations with oblique conditions (see for instance [GT01, Theorem 6.26]) we get that $\|v_{\tau,i}\|_{\mathcal{C}^{1,\gamma}(C_{1/8}\cap\mathbb{H}^n)} \lesssim r^\gamma$ independently of τ, so that letting $\tau \to 0$ we obtain $\|\partial_i v\|_{\mathcal{C}^{1,\gamma}(C_{1/8}\cap\mathbb{H}^n)} \lesssim r^\gamma$ for any $1 \le i < n$. On the other hand thanks to equations (4.28), $\partial_{nn} v$ may be written as

$$\partial_{nn} v = \frac{(\partial_n v)^2}{1+|\nabla' v|^2}\left(2\frac{\nabla' v\cdot\nabla'\partial_n v}{\partial_n v} - \Delta' v + \frac{\partial_n v}{\alpha}\right),$$

so that we also get $\|\partial_{nn} v\|_{\mathcal{C}^{0,\gamma}(C_{1/8}\cap\mathbb{H}^n)} \lesssim r^\gamma$, yielding in turn $\|v\|_{\mathcal{C}^{2,\gamma}(C_{1/8}\cap\mathbb{H}^n)} \lesssim r^\gamma$.

Therefore by covering $\partial\Omega$ with a finite number of such cylinders of radius $r/64$ we find $h : \partial B \to \mathbb{R}$ such that $\partial\Omega = \{(1+h(x))x, x \in \partial B\}$ with $\|h\|_{\mathcal{C}^{2,\gamma}(\partial B)} \lesssim 1$. This concludes the proof. □

4.4.4 Minimality of the ball among nearly spherical sets.

This section is dedicated to the proof of Proposition 4.4.1 in the case where Ω is nearly spherical. Letting $\gamma = \gamma_n \in (0,1)$ be the exponent given by Lemma 4.4.15, throughout this section we fix for once some $\alpha \in (0,\gamma)$.

Definition 4.4.16. *An open set $\Omega \subset \mathbb{R}^n$ is said to be nearly spherical if $|\Omega| = |B|$ and there is a function $h \in \mathcal{C}^{2,\alpha}(\partial B, [-\frac{1}{2}, \frac{1}{2}])$ with $\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)} \leq 1$ such that $\Omega = B_h$, where*

$$B_h := \{s(1+h(x))x, s \in [0,1), x \in \partial B\}.$$

By convention, $h\nu_B$ is extended as a vector field from $\mathbb{R}^n$ to $\mathbb{R}^n$ by the expression

$$\zeta(x) = \varphi(|x|)h\left(\frac{x}{|x|}\right)\frac{x}{|x|}, \tag{4.31}$$

where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*, [0,1])$ is such that $\varphi \equiv 1$ on $[1/2, 3/2]$, $\varphi \equiv 0$ on $[0, 1/4]$ and φ is nondecreasing on $[0, 1/2]$. Thus $\zeta^t(x) = x + t\zeta(x)$ is a $\mathcal{C}^{2,\alpha}$-diffeomorphism from B to B_{th} for all $|t| \leq 1$.
Finally, we remind that Ω is said to be centered when its barycenter is at the origin.

To be consistent with the notation B_r of the centered ball of radius r it would probably be more natural to denote instead B_{1+h} the nearly spherical set, but we will however carry on with the notation B_h through the whole section for sake of simplicity.

The local minimality result is the following.

Proposition 4.4.17. *Let B_h be a nearly spherical centered set such that $\|h\|_{L^1(\partial B)} \ll k^{-1-\frac{4}{n}} g_n(k)$. Then when $|\delta| \ll k^{-\left(1+\frac{8}{n}\right)} g_n(k)$ we have*

$$T(B_h)^{-1} + \delta\lambda_k(B_h) \geq T(B)^{-1} + \delta\lambda_k(B)$$

with equality if and only if $h \equiv 0$.

This will be obtained by performing a second-order Taylor expansion of the functional $T^{-1} + \delta\lambda_k$. The rough idea is the following: on the one hand, the first shape derivative (taken among measure-preserving variation) of T^{-1} and λ_k vanish, while on the other hand the second shape derivative of T^{-1} is coercive in $H^{1/2}(\partial B)$ (in some sense that takes into account the invariance by translation) and the second shape derivative of λ_k is bounded in $H^{1/2}(\partial B)$. This will be enough to get the local minimality of $T^{-1} + \delta\lambda_k$ at the ball.

We begin by a Lemma which states that the eigenvalues and eigenfunctions may be followed smoothly. It includes the case of degenerate eigenvalues, which will also be useful in next section.

Lemma 4.4.18. *Let $h \in \mathcal{C}^{2,\alpha}\left(\partial B, [-\frac{1}{2}, \frac{1}{2}]\right)$ and ζ the corresponding vector field (in accordance with (4.31)). Then there exists real analytic functions*

$$t \in [-1,1] \mapsto \mu_i(t) \in \mathbb{R},\ t \in [-1,1] \mapsto \widehat{u}_i(t) \in H_0^1(B)$$

verifying $\mu_i(0) = \lambda_i(B)$ for every $i \in \mathbb{N}^*$ and such that, denoting $u_i(t) = \widehat{u}_i(t) \circ (\zeta^t)^{-1}$, the functions $(u_i(t))_{i\in\mathbb{N}^*}$ form an orthonormal basis of (non-ordered) eigenfunctions of B_{th} associated to $(\mu_i(t))_{i\in\mathbb{N}^*}$ and

$$t \in [-1,1] \mapsto u_i(t) \in L^2(\mathbb{R}^n)$$

is differentiable with $u_i'(t) \in H^1(B_{th})$. Moreover, we have the expressions

$$\begin{aligned}
\mu_i'(t) &= -\int_{\partial B_{th}} |\nabla u_i(t)|^2 (\zeta \cdot \nu_t), \\
\mu_i''(t) &= \int_{\partial B_{th}} |\nabla u_i(t)|^2 \left(H_t(\zeta \cdot \nu_t)^2 - b_t(\zeta_{\tau_t}, \zeta_{\tau_t}) + 2\zeta_{\tau_t} \cdot \nabla_{|\partial B_{th}}(\zeta \cdot \nu_t) \right) \\
&\qquad + 2\int_{B_{th}} \left(|\nabla u_i'(t)|^2 - \mu_i(t)|u_i'(t)|^2 \right),
\end{aligned}$$

where b_t is the second fundamental form of ∂B_{th}, H_t its (outward) mean curvature, ν_t its (outward) normal vector and $\zeta_{\tau_t} := \zeta - (\zeta \cdot \nu_t)\nu_t$. Finally, $u_i'(t)$ verifies

$$\begin{cases}
-\Delta u_i'(t) - \mu_i(t) u_i'(t) = \mu_i'(t) u_i(t) & (B_{th}), \\
u_i'(t) = -(\zeta \cdot \nu_t)\partial_{\nu_t} u_i(t) & (\partial B_{th}), \\
\forall j \in \mathbb{N}^*, \ \int_{B_{th}} (u_i'(t) u_j(t) + u_i(t) u_j'(t)) = 0.
\end{cases} \tag{4.32}$$

Proof. For each $i \in \mathbb{N}^*$ and $|t| \leq 1$, u_i is an eigenfunction on B_{th} associated to $\lambda_i(B_{th})$ if and only if $\widehat{u_i} := u_i \circ \zeta^t$ verifies

$$\nabla \cdot \left[J_t (D\zeta^t)^{-1} ((D\zeta^t)^{-1})^* \nabla \widehat{u_i} \right] = \lambda_i(B_{th}) J_t \widehat{u_i},$$

where $J_t := \det(D\zeta^t)$. Letting $\widehat{u_i} := \frac{\widehat{v_i}}{\sqrt{J_t}}$, the family $((\widehat{v_i}, \lambda_i(B_{th})))_{i \in \mathbb{N}^*}$ is consisting of the eigenelements of the self-adjoint operator

$$T_t v := -\frac{1}{\sqrt{J_t}} \nabla \cdot \left[J_t (D\zeta^t)^{-1} ((D\zeta^t)^{-1})^* \nabla \frac{v}{\sqrt{J_t}} \right].$$

We apply the result [Kat95, VII.3.5. Theorem 3.9] to the family of self-adjoint operators T_t as defined above, over $L^2(B)$ with fixed domain $D(T_t) = D_0 := H^2(B) \cap H_0^1(B)$. This provides the existence of real analytic rearrangements $t \in [-1,1] \mapsto \mu_i(t)$ and $t \in [-1,1] \mapsto \widehat{u_i}(t) \in L^2(B)$ of respectively eigenvalues and orthonormal eigenfunctions for the operator T_t such that $\mu_i(0) = \lambda_i(B)$ for all i. Writing

$$\widehat{u_i}(t) = R_t \left(\mu_i(t) \widehat{u_i}(t) \right),$$

where R_t is the resolvent of T_t, and using that $t \in [-1,1] \mapsto R_t \in \mathcal{L}(H^{-1}(B), H_0^1(B))$ is real analytic (by implicit function theorem), we improve the analyticity of the eigenfunctions into $t \in [-1,1] \mapsto \widehat{u_i}(t) \in H_0^1(B)$.

Now, by construction we have that

$$(u_i(t), \mu_i(t)) := \left(\frac{\widehat{u_i}(t) \circ (\zeta^t)^{-1}}{\sqrt{J_t \circ (\zeta^t)^{-1}}}, \mu_i(t) \right)$$

are eigenvalues and (orthonormal) eigenfunctions of the Dirichlet Laplacian over B_{th}. Since $t \mapsto \widehat{u_i}(t) \in H_0^1(B)$ is differentiable, one proves as in [HP18, Theorem 5.3.1] that the map $t \in [-1,1] \mapsto u_i(t) \in L^2(\mathbb{R}^n)$ is differentiable. The expressions of the first and second derivative are then classical formulas which we derive as in [HP18, Section 5.9.3]. Let us remind how these expressions are found.

First derivative. The map $t \in [-1,1] \mapsto u_i(t) \in L^2(\mathbb{R}^n)$ is differentiable with derivative $u_i'(t)$ verifying $u_i'(t) + \nabla u_i(t) \cdot \zeta \in H_0^1(B_{th})$. One can therefore differentiate

$$\begin{cases}
-\Delta u_i(t) = \mu_i(t) u_i(t), & (B_{th}) \\
\int_{B_{th}} u_i(t) u_j(t) = \delta_{ij}
\end{cases}$$

to deduce that $u_i'(t)$ verifies the equation and the boundary conditions of (4.32). Integrating by parts (see [HP18, (5.87), (5.88)]) we get the expression

$$\mu_i'(t) = -\int_{\partial B_{th}} (\partial_{\nu_t} u_i(t))^2 (\zeta \cdot \nu_t)$$

Second derivative. We write the first derivative as an integral on the interior

$$\mu_i'(t) = -\int_{B_{th}} \nabla \cdot \left(|\nabla u_i(t)|^2 \zeta\right)$$

and apply the differentiation formula [HP18, Corollary 5.2.8]. The same computations as in [HP18, Section 5.9.6] lead to an analogous expression to the case of a simple eigenvalue (note that in this reference, the authors assume the sets to be $\mathcal{C}^3$; nevertheless, it is classical by using approximations by smooth functions that the result remains for $\mathcal{C}^{2,\alpha}$ sets, see for example [DP00, Section 5]):

$$\begin{aligned}
\mu_i''(t) &= 2\int_{\partial B_{th}} u_i'(t)\partial_{\nu_t} u_i'(t) + \int_{\partial B_{th}} (\partial_{\nu_t} u_i(t))^2 \left[H_t(\zeta\cdot\nu_t)^2 - b_t((\zeta)_{\tau_t}, (\zeta)_{\tau_t}) + 2\nabla_{\tau_t}(\zeta\cdot\nu_t)\cdot(\zeta)_{\tau_t}\right] \\
&= 2\int_{B_{th}} \left(|\nabla u_i'(t)|^2 - \mu_i(t)|u_i'(t)|^2\right) \\
&\qquad + \int_{\partial B_{th}} (\partial_{\nu_t} u_i(t))^2 \left[H_t(\zeta\cdot\nu_t)^2 - b_t((\zeta)_{\tau_t}, (\zeta)_{\tau_t}) + 2\nabla_{\tau_t}(\zeta\cdot\nu_t)\cdot(\zeta)_{\tau_t}\right]
\end{aligned}$$

where $\zeta_{\tau_t} := \zeta_{|\partial B_{th}}$ and ∇_{τ_t} is the gradient over ∂B_{th}.

□

Proposition 4.4.19. *Let B_h be a nearly spherical set such that $\|h\|_{L^1(\partial B)} \ll k^{-1-\frac{4}{n}} g_n(k)$, then*

$$|\lambda_k(B_h) - \lambda_k(B)| \le C_n \frac{k^{1+\frac{8}{n}}}{g_n(k)} \|h\|^2_{H^{1/2}(\partial B)}.$$

Here the $H^{1/2}(\partial B)$ norm is defined as

$$\|h\|^2_{H^{1/2}(\partial B)} = \|h\|^2_{L^2(\partial B)} + \int_B |\nabla \mathcal{H}h|^2$$

where $\mathcal{H}h$ is the harmonic extension of h in B. This is equivalent to the usual Gagliardo-Nirenberg norm $\|h\|^2_{L^2(\partial B)} + \iint_{\partial B^2} \frac{|h(x)-h(y)|^2}{|x-y|^n}$.

Proof of Proposition 4.4.19. Recall that by definition of a nearly spherical set it holds $\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)} \le 1$.

Let $(u_i(t), \mu_i(t))_{i\in\mathbb{N}^*}$ be the eigenelements of B_{th} as defined in Lemma 4.4.18. We claim that for any $|t| \le 1$ and $i \ne k$ we have $|\mu_i(t) - \mu_k(t)| \ge \frac{1}{2} g_n(k)$ when $\|h\|_{L^1(\partial B)} \ll k^{-1-\frac{4}{n}} g_n(k)$. Indeed we have $|u_i(t)| \lesssim \mu_i(t)^{\frac{n}{4}}$ by [Dav89, Example 2.1.8] and Proposition 4.2.1, hence using classical elliptic regularity we deduce

$$\|u_i(t)\|_{\mathcal{C}^{1,\alpha}(B_{th})} \lesssim \|\mu_i(t) u_i(t)\|_{L^\infty(B_{th})} \lesssim \mu_i(t)^{1+\frac{n}{4}}.$$

Using the expression of $\mu_i'(t)$ from Lemma 4.4.18 we get

$$|\mu_i'(t)| \lesssim \|h\|_{L^1(\partial B)} \mu_i(t)^{2+\frac{n}{2}}.$$

Integrating this expression we get that for any $|t| \le 1$ and $i \in \mathbb{N}$,

$$\left|\mu_i(t)^{-1-\frac{n}{2}} - \mu_i(0)^{-1-\frac{n}{2}}\right| \lesssim \|h\|_{L^1(\partial B)}$$

and for any $i \neq k$ we have

$$\left|\mu_i(0)^{-1-\frac{n}{2}} - \mu_k(0)^{-1-\frac{n}{2}}\right| \gtrsim k^{-1-\frac{4}{n}} g_n(k),$$

so when $\|h\|_{L^1(\partial B)} \ll k^{-1-\frac{4}{n}} g_n(k)$ we get the claim. In particular this means that $\mu_k(t) = \lambda_k(B_{th})$ for such h, for any $|t| \leq 1$.

By elliptic regularity we again have

$$\|u_k(t)\|_{\mathcal{C}^{1,\alpha}(B_{th})} \lesssim k^{\frac{1}{2}+\frac{2}{n}}, \ \|u_k(t)\|_{\mathcal{C}^{2,\alpha}(B_{th})} \lesssim k^{\frac{1}{2}+\frac{4}{n}}. \tag{4.33}$$

The eigenvalue $\lambda_k(B)$ being simple, the associated eigenfunction $u_k(0)$ is radial, so that by setting $|\nabla u_k(0)|^2_{|\partial B} =: c_{n,k}(\lesssim k^{1+\frac{4}{n}})$ we have $\frac{d}{ds}\big|_{s=0}(\lambda_k(B_{sh}) + c_{n,k}|B_{sh}|) = 0$. As a consequence, by Taylor expansion and recalling that $|B_h| = |B|$, there exists some $t \in [0,1]$ such that

$$\lambda_k(B_h) - \lambda_k(B) = \frac{1}{2}\left(\frac{d^2}{ds^2}\bigg|_{s=t} \lambda_k(B_{sh}) + c_{n,k}\,\frac{d^2}{ds^2}\bigg|_{s=t} |B_{sh}|\right),$$

which we rewrite

$$\mu_k(B_h) - \mu_k(B) = \frac{1}{2}\left(\mu_k''(t) + c_{n,k}\,\frac{d^2}{ds^2}\bigg|_{s=t} |B_{sh}|\right)$$

To reduce notation we fix t and do not write the dependency in t in the rest of the proof, and set instead $\Omega := B_{th}$, $u_i := u_i(t)$, $v := u_k'(t)$ and write μ_i, μ_i', μ_i'' in place of $\mu_i(t)$, $\mu_i'(t)$, $\mu_i''(t)$. The expression of μ_k'' thus reads

$$\mu_k'' = \int_\Omega 2\left(|\nabla v|^2 - \mu_k v^2\right) + \int_{\partial\Omega} |\nabla u_k|^2 \left[H(\zeta\cdot\nu)^2 - b(\zeta_{|\partial\Omega}, \zeta_{|\partial\Omega}) + 2\zeta\cdot\nabla_{|\partial\Omega}(\zeta\cdot\nu)\right],$$

where v verifies

$$\begin{cases} -\Delta v - \mu_k v = \mu_k' u_k & (\Omega), \\ v = -(\zeta\cdot\nu)\partial_\nu u_k & (\partial\Omega), \\ \int_\Omega v u_k = 0. \end{cases} \tag{4.34}$$

On the other hand we have the following expression for the second derivative of the volume (see [DL19, Theorem 2.1 and Lemma 2.8])

$$\frac{d^2}{ds^2}\bigg|_{s=t} |B_{sh}| = \int_{\partial\Omega} H(\zeta\cdot\nu)^2 + \int_{\partial\Omega} b(\zeta_{|\partial\Omega}, \zeta_{|\partial\Omega}) - 2\zeta_{|\partial\Omega}\cdot\nabla_{|\partial\Omega}(\zeta\cdot\nu),$$

where we recall that $\zeta_{|\partial B} = h\nu_B$ is extended thanks to (4.31). Let us now bound each term independently.

Estimate of v. We write $v = z + w$, where z is harmonic and $w \in H_0^1(\Omega)$. In particular z is the harmonic extension of $-(\zeta\cdot\nu)\partial_\nu u_k$, hence

$$\|z\|_{H^1(\Omega)} \lesssim \|(\zeta\cdot\nu)\partial_\nu u_k\|_{H^{1/2}(\partial\Omega)} \lesssim k^{\frac{1}{2}+\frac{4}{n}}\|h\|_{H^{1/2}(\partial B)},$$
$$\|z\|_{L^2(\Omega)} \lesssim \|z\|_{L^2(\partial\Omega)} \lesssim k^{\frac{1}{2}+\frac{2}{n}}\|h\|_{H^{1/2}(\partial B)}.$$

where we used $\|h\|_{\mathcal{C}^{2,\alpha}} \leq 1$. We also used the general property

$$\|fg\|_{H^{1/2}(\partial B)} \lesssim \|f\|_{L^\infty(\partial B)}\|g\|_{H^{1/2}(\partial B)} + \|\nabla f\|_{L^\infty(\partial B)}\|g\|_{L^2(\partial B)},$$

which is a consequence of the inequality

$$\begin{aligned}\int_B |\nabla\mathcal{H}(fg)|^2 \leq \int_B |\nabla(\mathcal{H}(f)\mathcal{H}(g))|^2 &\leq 2\int_B |\nabla\mathcal{H}f|^2|\mathcal{H}g|^2 + |\mathcal{H}f|^2|\nabla\mathcal{H}g|^2 \\ &\leq 2\|\nabla f\|^2_{L^\infty(\partial B)}\|g\|^2_{L^2(\partial B)} + \|f\|^2_{L^\infty(\partial B)}\int_B |\nabla\mathcal{H}g|^2.\end{aligned}$$

Now w verifies $-\Delta w - \mu_k w = \mu_k z + \mu_k' u_k$, with $\int_\Omega w u_k = \frac{\mu_k'}{\mu_k}$. Let now $i \neq k$, this equation implies

$$(\mu_i - \mu_k) \int_\Omega w u_i = \mu_k \int_\Omega z u_i,$$

whereas for $i = k$ we simply have

$$\int_\Omega w u_k = -\int_\Omega z u_k.$$

Using the spectral decomposition, we have

$$\begin{aligned}
\int_\Omega \left(|\nabla v|^2 - \mu_k v^2\right) &= \int_\Omega \left(|\nabla z|^2 + |\nabla w|^2 - \mu_k v^2\right) \\
&= \int_\Omega |\nabla z|^2 + \sum_{i \in \mathbb{N}^*} \left(\mu_i \left(\int_\Omega w u_i \right)^2 - \mu_k \left(\int_\Omega v u_i \right)^2 \right) \\
&= \int_\Omega |\nabla z|^2 + \mu_k \left(\int_\Omega z u_k \right)^2 + \sum_{i \neq k} \left(\frac{\mu_k^2}{\mu_k - \mu_i} - \mu_k \right) \left(\int_\Omega z u_i \right)^2
\end{aligned}$$

so

$$\begin{aligned}
\left| \int_\Omega \left(|\nabla v|^2 - \mu_k v^2\right) \right| &\lesssim \|\nabla z\|_{L^2(\Omega)}^2 + k^{\frac{2}{n}} \|z\|_{L^2(\Omega)}^2 + g_n(k)^{-1} k^{\frac{4}{n}} \|z\|_{L^2(\Omega)}^2 \\
&\lesssim g_n(k)^{-1} k^{1+\frac{8}{n}} \|h\|_{H^{1/2}(\partial B)}^2.
\end{aligned}$$

Curvature terms. Using again $\|h\|_{\mathcal{C}^{2,\alpha}} \leq 1$ we directly have

$$\left| \int_{\partial\Omega} |\nabla u_k|^2 \left[H(\zeta \cdot \nu)^2 - b(\zeta_{|\partial\Omega}, \zeta_{|\partial\Omega}) \right] \right| \lesssim \|\nabla u_k\|_{L^\infty(\Omega)}^2 \|\zeta\|_{L^2(\partial\Omega)}^2 \lesssim k^{1+\frac{4}{n}} \|h\|_{H^{1/2}(\partial B)}^2,$$

$$\left| \int_{\partial\Omega} \left[H(\zeta \cdot \nu)^2 + b(\zeta_{|\partial\Omega}, \zeta_{|\partial\Omega}) \right] \right| \lesssim \|h\|_{H^{1/2}(\partial B)}^2.$$

Last term. We have $\|\nabla_{|\partial\Omega} \zeta\|_{H^{-1/2}(\partial\Omega)} \lesssim \|\zeta\|_{H^{1/2}(\partial\Omega)}$ and

$$\begin{aligned}
\||\nabla u_k|^2 \zeta\|_{H^{1/2}(\partial\Omega)} &\lesssim \|\nabla_{|\partial\Omega} |\nabla u_k|^2\|_{L^\infty(\partial\Omega)} \|\zeta\|_{H^{1/2}(\partial\Omega)} \\
&\lesssim \|\nabla u_k\|_{L^\infty(\Omega)} \|\nabla^2 u_k\|_{L^\infty(\Omega)} \|\zeta\|_{H^{1/2}(\partial\Omega)} \lesssim k^{1+\frac{6}{n}} \|h\|_{H^{1/2}(\partial B)},
\end{aligned}$$

so

$$\left| \int_{\partial\Omega} |\nabla u_k|^2 \zeta \cdot \nabla_{|\partial\Omega} (\zeta \cdot \nu) \right| \lesssim k^{1+\frac{6}{n}} \|h\|_{H^{1/2}(\partial B)}^2.$$

We also have

$$\left| \int_{\partial\Omega} \zeta_{|\partial\Omega} \cdot \nabla_{|\partial\Omega} (\zeta \cdot \nu) \right| \lesssim \|h\|_{H^{1/2}(\partial B)}^2.$$

Adding all the estimates we get

$$|\mu_k''| \lesssim \frac{k^{1+\frac{8}{n}}}{g_n(k)} \|h\|_{H^{1/2}(\partial B)}^2 \lesssim \frac{k^{1+\frac{8}{n}}}{g_n(k)} \|h\|_{H^{1/2}(\partial B)}^2$$

and

$$\left| c_{n,k} \left. \frac{d^2}{ds^2} \right|_{s=t} |B_{sh}| \right| \lesssim k^{1+\frac{4}{n}} \|h\|^2_{H^{1/2}(\partial B)}$$

so

$$|\mu_k(B_h) - \mu_k(B)| \lesssim \frac{k^{1+\frac{8}{n}}}{g_n(k)} \|h\|^2_{H^{1/2}(\partial B)}.$$

This concludes the proof of the proposition. □

We can now prove minimality of the ball for nearly spherical sets.

Proof of Proposition 4.4.17. It was proven in [BDPV15, Theorem 3.3] that if $\beta \in (0,1)$, then for any B_h with $|B_h| = 1$ and $\mathrm{bar}(B_h) = 0$ and $\|h\|_{\mathcal{C}^{2,\beta}(\partial B)} \ll 1$ it holds

$$T(B_h) \leq T(B) - \frac{1}{32n^2} \|h\|^2_{H^{1/2}(\partial B)} \tag{4.35}$$

By interpolation, using for instance the general inequalities

$$\|h\|_{\mathcal{C}^{2,\beta}(\partial B)} \leq C_{n,\beta} \|h\|^{\frac{\alpha-\beta}{2+\alpha}}_{\mathcal{C}^0(\partial B)} \|h\|^{\frac{2+\beta}{2+\alpha}}_{\mathcal{C}^{2,\alpha}(\partial B)}, \text{ for any } \beta \in (0,\alpha)$$
$$\|h\|_{\mathcal{C}^0(\partial B)} \lesssim \|h\|^{\frac{1}{n+1}}_{L^1(\partial B)} \|h\|^{\frac{n}{n+1}}_{\mathcal{C}^{0,1}(\partial B)},$$

then for any $\beta \in (0,\alpha)$ there exists $\kappa = \kappa_{n,\beta} \in (0,1)$ such that

$$\|h\|_{\mathcal{C}^{2,\beta}(\partial B)} \lesssim \|h\|^{\kappa}_{L^1(\partial B)} \|h\|^{1-\kappa}_{\mathcal{C}^{2,\alpha}(\partial B)}$$

so that $\|h\|_{\mathcal{C}^{2,\beta}(\partial B)} \ll 1$. We can therefore apply (4.35), which together with Proposition 4.4.19 yield

$$\begin{aligned} T(B_h)^{-1} + \delta\lambda_k(B_h) &\geq \left(T(B) - \frac{1}{32n^2} \|h\|^2_{H^{1/2}(\partial B)} \right)^{-1} + \delta\lambda_k(B) - |\delta| \frac{k^{1+\frac{8}{n}}}{g_n(k)} \|h\|^2_{H^{1/2}(\partial B)} \\ &\geq T(B)^{-1} + \delta\lambda_k(B) \end{aligned}$$

where the last line holds provided δ is sufficiently small ($|\delta| \ll k^{-\left(1+\frac{8}{n}\right)} g_n(k)$), with equality if and only if $h = 0$. This finishes the proof. □

4.4.5 Conclusion.

Proof of Proposition 4.4.1. Recall that we have assumed $|\delta| \ll k^{-4-\frac{8}{n}} g_n(k)$. Proposition 4.4.5 applies and there exists a minimizer Ω to the functional (4.15). By Lemma 4.4.15, up to translation Ω is a centered minimizer of the form B_h with $\|h\|_{\mathcal{C}^{2,\gamma}(\partial B)} \lesssim 1$. By Lemma 4.4.13 and Lemma 4.4.4 we have

$$\|h\|_{L^1(\partial B)} \lesssim |\Omega \Delta B| \lesssim \mathcal{F}(\Omega) \lesssim k^{2+\frac{4}{n}} |\delta|.$$

Hence Ω is nearly spherical in the sense of Definition 4.4.16, since $k^{2+\frac{4}{n}}|\delta| \ll 1$ and by interpolation there is some $\kappa = \kappa_n \in (0,1)$ such that

$$\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)} \lesssim \|h\|^{\kappa}_{L^1(\partial B)} \|h\|^{1-\kappa}_{\mathcal{C}^{2,\gamma}(\partial B)} \lesssim \left(k^{2+\frac{4}{n}} |\delta| \right)^{\kappa}.$$

Now, since δ verifies $k^{2+\frac{4}{n}}|\delta| \ll k^{-1-\frac{4}{n}} g_n(k)$ (because $|\delta| \ll k^{-3-\frac{8}{n}} g_n(k)$) we can apply Proposition 4.4.17 to Ω, so that we obtain that Ω is a ball. This finishes the proof. □

4.5 Proof of Theorem 4.1.3: linear bound on clusters of multiple eigenvalues

Consider $1 \leq k \leq l$ such that

$$\lambda_{k-1}(B) < \lambda_k(B) = \cdots = \lambda_l(B) < \lambda_{l+1}(B).$$

We denote the multiplicity of the eigenspace by $m = l - k + 1 \geq 1$. Note that l and k are comparable since (see Lemma 4.2.1)

$$l^{\frac{2}{n}} \lesssim \lambda_l(B) = \lambda_k(B) \lesssim k^{\frac{2}{n}}.$$

In dimension $n = 2$ it is known that the multiplicity of an eigenspace is at most 2. On the other hand, in dimensions $n \geq 3$ the multiplicity of the eigenspace may get arbitrarily large, since the dimension of degree d homogeneous harmonic polynomials in three variables is $2d + 1$.

In order to prove Theorem 4.1.3, we prove that the ball is the unique minimizer of the functional

$$\Omega \in \mathcal{A} \mapsto T(\Omega)^{-1} + \delta \sum_{i=k}^{l} \lambda_i(\Omega), \tag{4.36}$$

for any $\delta \in \mathbb{R}$ sufficiently close to 0.

We prove the following.

Proposition 4.5.1. *There exists $c_n > 0$ such that for any $\delta \in \mathbb{R}$ with $|\delta| \leq c_n k^{-\left(6+\frac{10}{n}\right)} g_n(k)$, the ball is the unique minimizer of (4.36).*

Remark 4.5.2. This result admits the following natural generalization, following the same proof: let $k \leq l$ such that $\lambda_{k-1}(B) < \lambda_k(B)$ and $\lambda_l(B) < \lambda_{l+1}(B)$ (note that we do not ask $\lambda_l(B) = \lambda_k(B)$). Then for any $\Omega \in \mathcal{A}$:

$$\begin{aligned} &\left| \sum_{i=k}^{l} \lambda_i(\Omega) - \lambda_i(B) \right| \\ &\leq C_n l^{6+\frac{10}{n}} \min\left\{\lambda_{l+1}(B) - \lambda_l(B), \lambda_k(B) - \lambda_{k-1}(B)\right\}^{-1} \left(T(\Omega)^{-1} - T(B)^{-1}\right), \end{aligned}$$

where $C_n > 0$ is some dimensional constant.

The proof follows the same plan as in the non-degenerate case. The main difference with the case when $\lambda_k(B)$ is simple concerns the $C^{1,\gamma}$ estimates of a minimizer Ω. When $\delta > 0$ we again apply the results from [KL18] - with the extra effort of obtaining estimates uniform in the multiplicity m. On the other hand, when $\delta < 0$ we cannot directly apply [MTV21]; instead we see (4.36) as a vectorial version of the problem studied in [MTV21], and follow the strategy of [MTV21] (as in [DS11] for the one-phase free boundary problem) by proving first some partial Harnack inequality (see Propositions 4.5.14) and then get by contradiction an improvement of flatness (see Proposition 4.5.15) in order to get $C^{1,\gamma}$ regularity of a minimizer Ω. In this second case we also follow carefully the dependency of the estimates in the multiplicity.

The existence of a solution for (4.36), the first regularity estimates and the existence of blow-ups are proved in the same way than in Section 4.4 (see Proposition 4.4.5, Proposition 4.4.6 and Proposition 4.4.10, respectively). The passage from $\mathcal{C}^{1,\gamma}$ to $\mathcal{C}^{2,\gamma}$ is also similar. We gather the results and emphasize on the slight differences in the proofs.

Lemma 4.5.3. *Let Ω be a domain such that*

$$T(\Omega)^{-1} + \delta \sum_{i=k}^{l} \lambda_i(\Omega) \leq T(B)^{-1} + \delta \sum_{i=k}^{l} \lambda_i(B).$$

Then if $|\delta| \ll k^{-1-\frac{2}{n}}$, we have the following properties (up to a translation of Ω):

- $|\Omega \Delta B| \lesssim k^{\frac{1}{2}+\frac{1}{n}}|\delta|^{\frac{1}{2}}$.
- *It holds*

$$T(\Omega)^{-1} \lesssim 1, \text{ and for } k \leq i \leq l, \ \lambda_i(\Omega) \lesssim k^{\frac{2}{n}},$$
$$T(\Omega)^{-1} - T(B)^{-1} \lesssim k^{1+\frac{2}{n}}|\delta|, \quad \text{and for all } i \in \mathbb{N}^*, |\lambda_i(\Omega) - \lambda_i(B)| \lesssim i^{2+\frac{4}{n}} k^{\frac{1}{2}+\frac{1}{n}}|\delta|^{\frac{1}{2}}$$

- $\|w_\Omega - w_B\|_{L^1(\mathbb{R}^n)} \lesssim k^{\frac{1}{2}+\frac{1}{n}}|\delta|^{\frac{1}{2}}$.

Remark 4.5.4. This may be refined (using the bounds from Theorem 4.1.1) into

$$T(\Omega)^{-1} - T(B)^{-1} \lesssim k^{6+\frac{8}{n}}|\delta|^2,$$
$$\forall i \in \mathbb{N}^*, \ |\lambda_i(\Omega) - \lambda_i(B)| \lesssim i^{2+\frac{4}{n}} k^{3+\frac{4}{n}}|\delta|.$$

Proof. We write

$$T(\Omega)^{-1} - T(B)^{-1} \leq \delta \sum_{i=k}^{l} (\lambda_k(B) - \lambda_k(\Omega)) \lesssim T(\Omega)^{-1} k^{1+\frac{2}{n}}|\delta|$$

where we used Proposition 4.2.1 and the fact that the multiplicity verifies $m \lesssim k$. We then proceed exactly as in Lemma 4.4.3. □

Proposition 4.5.5. *If $|\delta| \ll k^{-\left(3+\frac{4}{n}\right)}$, then the functional (4.36) has a minimizer $\Omega \in \mathcal{A}$, and there exists $c_n, C_n, r_n > 0$ such that $\|\nabla w_\Omega\|_{L^\infty(\mathbb{R}^n)} \leq C_n$, for all $k \leq i \leq l$, $\|\nabla u_i\|_{L^\infty(\mathbb{R}^n)} \leq C_n k^{\frac{1}{2}+\frac{2}{n}}$ and for all $x \in \mathbb{R}^n$, $r \in (0, r_n)$,*

$$\fint_{\partial B_{x,r}} w_\Omega < c_n r \text{ implies } w_\Omega|_{B_{x,r/2}} = 0$$

Moreover, Ω is open and bounded with diam$(\Omega) \lesssim 1$, and up to translating Ω we have

$$\|w_\Omega - w_B\|_{\mathcal{C}^0(\mathbb{R}^n)} \lesssim \left(k^{\frac{1}{2}+\frac{1}{n}}|\delta|^{\frac{1}{2}}\right)^{\frac{1}{n+1}},$$
$$|\Omega \Delta B| \lesssim k^{3+\frac{4}{n}}|\delta|.$$

Proof. This proof is completely similar to the proof of Propositions 4.4.5, 4.4.6 and 4.4.13. The condition $|\delta| \ll k^{-\left(3+\frac{4}{n}\right)}$ in place of $|\delta| \ll k^{-\left(2+\frac{4}{n}\right)}$ in Proposition 4.4.5 again comes from the multiplicity estimate $m \lesssim k$, as well as some of the estimates above.

□

Analogously to the case of a simple eigenvalue, we set

$$Q := \frac{T(\Omega)^2}{\omega_n}\left(\frac{n+2}{nT(\Omega)} + \frac{2}{n}\delta \sum_{i=k}^{l} \lambda_i(\Omega)\right) \tag{4.37}$$

and we have

$$\left|Q - \frac{1}{n^2}\right| \lesssim k^{1+\frac{2}{n}}|\delta|. \tag{4.38}$$

Lemma 4.5.6. *Let $\Omega \in \mathcal{A}$ be a minimizer of (4.36) and suppose $|\delta| \ll k^{-\left(6+\frac{8}{n}\right)} g_n(k)$. Suppose $z \in \partial\Omega$ has a contact sphere on either side with inward vector ν. Then there exists $\beta > 0$, $\beta_i \in \mathbb{R}$, and a sequence $s_j \to 0$ such that*

$$\begin{aligned} (w)_{z,s_j} &\underset{\mathcal{C}^0_{loc}(\mathbb{R}^n)}{\longrightarrow} \beta(x\cdot\nu)_+, \\ (u_i)_{z,s_j} &\underset{\mathcal{C}^0_{loc}(\mathbb{R}^n)}{\longrightarrow} \beta_i(x\cdot\nu)_+ \text{ for any } k \le i \le l \end{aligned} \tag{4.39}$$

as $j \to \infty$, and

$$\beta^2 + T(\Omega)^2\delta\sum_{i=k}^{l}\beta_i^2 = Q, \tag{4.40}$$

where Q is defined in (4.37).

Proof. Since $|\delta| \ll k^{-\left(6+\frac{8}{n}\right)} g_n(k)$ we have $\lambda_{k-1}(\Omega) < \lambda_k(\Omega) \le \lambda_l(\Omega) < \lambda_{l+1}(\Omega)$ thanks to Remark 4.5.4. The proof is then completely analogous to the proof of Proposition 4.4.10, the only difference lying in the computation of the shape derivative: while each λ_i is not necessarily differentiable, the sum $\sum_{i=k}^{l}\lambda_i$ it is, thanks to [LLdC06, Theorem 2.6], and we have

$$\frac{d}{dt}\bigg|_{t=0}\left(\sum_{i=k}^{l}\lambda_i\right)\left(\zeta^t(\Omega)\right) = \sum_{i=k}^{l}\int_\Omega\left[\left(|\nabla u_i|^2 - \lambda_k(\Omega)u_i^2\right)\nabla\cdot\zeta - 2\nabla u_i\cdot D\zeta\cdot\nabla u_i\right],$$

where $(u_i)_{k\le i\le l}$ is an orthonormal basis of the eigenspaces associated to $(\lambda_i(\Omega))_{k\le i\le l}$. □

4.5.1 Harnack inequality

Let us start by introducing the space of viscosity solutions relevant to us, which we will be denoted by $\mathcal{S}_{m,\delta}(L)$, m being the multiplicity of the eigenspace associated to $\lambda_k(B)$.

Definition 4.5.7. *Let $L \ge 1$, $\delta \in \mathbb{R}$, $m \in \mathbb{N}^*$. We define $\mathcal{S}_{m,\delta}(L)$ to be the set of functions $(v, \overline{v}_1, \underline{v}_1, \dots, \overline{v}_m, \underline{v}_m) \in H^1(B_1, \mathbb{R}_+)^{2m+1}$ such that*

$$\begin{aligned} |\nabla v| &\le L, \\ |\nabla\overline{v}_i|, |\nabla\underline{v}_i| &\le L|\delta|^{\frac{1}{4}}, \\ 0 < \frac{\overline{v}_i}{v}, \frac{\underline{v}_i}{v} &\le L|\delta|^{\frac{1}{4}} \quad \text{in } \{v > 0\}, \\ \frac{1}{L} < \frac{\overline{v}_i}{\underline{v}_i} &\le L \quad \text{in } \{v > 0\} \end{aligned}$$

and for every $z \in \partial\{v > 0\}$ with a contact sphere on either side with inward normal vector ν, there exists numbers $\alpha > 0, \overline{\alpha}_i > 0$, $\underline{\alpha}_i > 0$ such that $x \mapsto (\alpha(x\cdot\nu)_+, \overline{\alpha}_1(x\cdot\nu)_+, \dots, \underline{\alpha}_m(x\cdot\nu)_+)$ is a blow-up of $(v, \overline{v}_1, \dots, \underline{v}_m)$ at z in the sense of (4.39) and

$$\begin{aligned} \alpha^2 + \sum_{i=1}^{m}\frac{\overline{\alpha}_i^2 + \underline{\alpha}_i^2}{2} &= 1 && \text{if } \delta > 0, \\ \alpha^2 + \sum_{i=1}^{m}\overline{\alpha}_i\underline{\alpha}_i &= 1 && \text{if } \delta < 0. \end{aligned}$$

Note that this last condition may be written as $(\partial_\nu v)^2 + \sum_{i=1}^{m}\frac{(\partial\overline{v}_i)^2 + (\partial_\nu\underline{v}_i)^2}{2} = 1$ (resp $(\partial_\nu v)^2 + \sum_{i=1}^{m}(\partial\overline{v}_i)(\partial_\nu\underline{v}_i) = 1$ when $\delta < 0$) on $\partial\{v > 0\}$ in the viscosity sense, although the traces of the gradients are not actually well-defined here.

Remark 4.5.8. According to this definition, $\overline{\alpha}_i$ and $\underline{\alpha}_i$ are bounded by $L|\delta|^{\frac{1}{4}}$, so that $\alpha^2 \geq 1 - L^2 m|\delta|^{\frac{1}{2}}$. In particular $\alpha \geq \frac{1}{2}$ when $L^2 m|\delta|^{\frac{1}{2}} \leq \frac{1}{8}$, which is the hypothesis we will make in order to obtain Harnack inequalities.

In next Lemma we link this definition of viscosity solutions to our free boundary problem. Let Ω be a minimizer of (4.36) and let $w, u_k, \dots, u_l$ be its torsion function and eigenfunctions associated to the eigenvalues $(\lambda_i(\Omega))_{k \leq i \leq l}$. Set $m = l - k + 1$, and $B_{x,r}$ a ball of $\mathbb{R}^n$. We let

$$\begin{aligned} v(y) &= \frac{1}{r} Q^{-\frac{1}{2}} \left(1 - mT(\Omega)^2 |\delta|^{\frac{1}{2}}\right)^{\frac{1}{2}} w(x + ry), \\ \overline{v}_i(y) &= \frac{1}{r} Q^{-\frac{1}{2}} T(\Omega) |\delta|^{\frac{1}{4}} \left(w + |\delta|^{\frac{1}{4}} u_{k+i-1}\right)(x + ry), i = 1, \dots, m, \\ \underline{v}_i(y) &= \frac{1}{r} Q^{-\frac{1}{2}} T(\Omega) |\delta|^{\frac{1}{4}} \left(w - |\delta|^{\frac{1}{4}} u_{k+i-1}\right)(x + ry), i = 1, \dots, m. \end{aligned}$$

Lemma 4.5.9. *There exists $C_n, L_n > 0$ such that if $|\delta| \leq C_n k^{-2-\frac{8}{n}}$ then for any $r \leq 1$,*

$$(v, \overline{v}_1, \underline{v}_1, \dots, \overline{v}_m, \underline{v}_m) \in \mathcal{S}_{m,\delta}(L_n).$$

Proof. Since $|u_i| \lesssim k^{\frac{2}{n}+\frac{1}{2}} w$ for $k \leq i \leq l$ (recall Lemma 4.2.2) then for $|\delta| \ll k^{-2-\frac{8}{n}}$ we have $0 < \overline{v}_i, \underline{v}_i \lesssim |\delta|^{\frac{1}{4}} v$ and also $1 \lesssim \overline{v_i}/\underline{v_i} \lesssim 1$. By Proposition 4.5.5 and (4.38) we have $|\nabla w| \lesssim 1$ and $|\delta|^{\frac{1}{4}} |\nabla u_i| \lesssim |\delta|^{\frac{1}{4}} k^{\frac{2}{n}+\frac{1}{2}} \lesssim 1$ since $|\delta| \ll k^{-2-\frac{8}{n}}$, hence there exists $L = L_n$ verifying the properties of Definition 4.5.7. Finally, for any $z \in \partial\{v > 0\}$ which has a contact sphere with inward vector ν, thanks to Lemma 4.5.6 there exists blow-ups $(\beta(x \cdot \nu)_+, \beta_k(x \cdot \nu)_+, \dots, \beta_l(x \cdot \nu)_+)$ of $(w, u_k, \dots, u_l)$ at z such that

$$\beta^2 + T(\Omega)^2 \delta \sum_{i=k}^{l} \beta_i^2 = Q$$

which may be rearranged as

$$\left(1 - mT(\Omega)^2 |\delta|^{\frac{1}{2}}\right) \beta^2 + T(\Omega)^2 |\delta|^{\frac{1}{2}} \sum_{i=k}^{l} \left(\beta^2 + \operatorname{sign}(\delta) |\delta|^{\frac{1}{2}} \beta_i^2\right) = Q.$$

Letting

$$\begin{aligned} \alpha &= Q^{-\frac{1}{2}} \left(1 - mT(\Omega)^2 |\delta|^{\frac{1}{2}}\right)^{\frac{1}{2}} \beta, \\ \overline{\alpha}_i &= Q^{-\frac{1}{2}} T(\Omega) |\delta|^{\frac{1}{4}} \left(\beta + |\delta|^{\frac{1}{4}} \beta_{k+i-1}\right), i = 1, \dots, m, \\ \underline{\alpha}_i &= Q^{-\frac{1}{2}} T(\Omega) |\delta|^{\frac{1}{4}} \left(\beta - |\delta|^{\frac{1}{4}} \beta_{k+i-1}\right), i = 1, \dots, m, \end{aligned}$$

these correspond to the gradients of the blow-ups of $(v, \overline{v}_1, \dots, \underline{v}_m)$ at z, thus concluding the proof. □

As in [MTV21], the key observation consists in noting that if $(v, \overline{v}_1, \underline{v}_1, \dots, \overline{v}_m, \underline{v}_m) \in \mathcal{S}_{m,\delta}(L)$, then

$$(v, \sqrt{\overline{v}_1 \underline{v}_1}, \dots, \sqrt{\overline{v}_m \underline{v}_m})$$

is a supersolution of a vectorial problem of the type of [KL18] because (see [MTV21, Lemma 2.9 and Remark 4.1])

$$\begin{aligned} \Delta \sqrt{\overline{v}_i \underline{v}_i} &\leq \frac{1}{2} \left(\sqrt{\frac{\underline{v}_i}{\overline{v}_i}} \Delta \overline{v}_i + \sqrt{\frac{\overline{v}_i}{\underline{v}_i}} \Delta \underline{v}_i \right) \leq \sqrt{L} \frac{(\Delta \overline{v}_i)_+ + (\Delta \underline{v}_i)_+}{2} \\ \partial_\nu \sqrt{\overline{v}_i \underline{v}_i} &= \sqrt{(\partial_\nu \overline{v}_i)(\partial_\nu \underline{v}_i)} \text{ at blow-ups of contact points} \end{aligned} \tag{4.41}$$

Similarly for any positive $(c_i)_{i=1,\dots,m}$ with $c_i \in [1/(2L), 2L]$ we have that

$$\left(v, \frac{1}{2}(c_1\overline{v}_1 + c_1^{-1}\underline{v}_1), \dots, \frac{1}{2}(c_m\overline{v}_m + c_m^{-1}\underline{v}_m)\right)$$

is a subsolution in the sense that

$$\begin{aligned} \Delta\frac{c_i\overline{v}_i + c_i^{-1}\underline{v}_i}{2} &\geq -\frac{c_i + c_i^{-1}}{2}\varepsilon^2, \\ \partial_\nu\left(\frac{c_i\overline{v}_i + c_i^{-1}\underline{v}_i}{2}\right) &\geq \sqrt{(\partial_\nu\overline{v}_i)(\partial_\nu\underline{v}_i)} \text{ at blow-ups of contact points.} \end{aligned} \tag{4.42}$$

We prove an ε-regularity result, meaning we prove that if $(v, \overline{v}_1, \underline{v}_1, \dots, \overline{v}_m, \underline{v}_m)$ is sufficiently flat in a ball B_1, then it is smooth in a smaller ball $B_{1/2}$.

Definition 4.5.10. *We say $(v, \overline{v}_1, \underline{v}_1, \dots, \overline{v}_m, \underline{v}_m) \in \mathcal{S}_{m,\delta}(L)$ is ε-flat with parameters*

$$a, b, (\alpha, \overline{\alpha}_1, \underline{\alpha}_1, \dots, \overline{\alpha}_m, \underline{\alpha}_m)$$

when $|a|, |b| \leq \varepsilon$ and

$$\begin{aligned} &0 \leq b - a \leq \varepsilon \\ &\alpha^2 + \sum_{i=1}^m \frac{\overline{\alpha}_i^2 + \underline{\alpha}_i^2}{2} = 1 \text{ if } \delta > 0, \alpha^2 + \sum_{i=1}^m \overline{\alpha}_i\underline{\alpha}_i = 1 \text{ if } \delta < 0, \\ &(x_n + a)_+ \leq \frac{v(x)}{\alpha}, \frac{\overline{v}_i(x)}{\overline{\alpha}_i}, \frac{\underline{v}_i(x)}{\underline{\alpha}_i} \leq (x_n + b)_+ \text{ in } B_1, \\ &\frac{|\Delta v|}{\alpha}, \frac{|\Delta\overline{v}_i|}{\overline{\alpha}_i}, \frac{|\Delta\underline{v}_i|}{\underline{\alpha}_i} < \varepsilon^2 \text{ in } B_1 \cap \{v > 0\}. \end{aligned} \tag{4.43}$$

We remark that the second and third equations of (4.43) (evaluated at $x \to e_n$) directly imply, for a small enough ε,

$$\frac{\overline{\alpha}_i}{\underline{\alpha}_i} \in \left[\frac{1}{2L}, 2L\right] \text{ and } \overline{\alpha}_i, \underline{\alpha}_i \leq 2L|\delta|^{\frac{1}{4}}. \tag{4.44}$$

In all the following we let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}, [0,1])$ such that $\eta \equiv 1$ on $[-3/5, 3/5]$ and $\eta = 0$ outside of $[-4/5, 4/5]$. Then for any small enough $t \in \mathbb{R}$ we set

$$H_t := \{(x', x_n) \in \mathbb{R}^n : x_n > -t\eta(|x'|)\}.$$

We define three functions depending on t:

$$\begin{cases} \Delta\varphi_t = 0 & (B \cap H_t) \\ \varphi_t = x_n & (\partial B \cap H_t) \\ \varphi_t = 0 & (B \cap \partial H_t) \end{cases}, \begin{cases} \Delta\psi_t = 0 & (B \cap H_t) \setminus B\left(\frac{1}{2}e_n, \frac{1}{4}\right) \\ \psi_t = 1 & B\left(\frac{1}{2}e_n, \frac{1}{4}\right) \\ \psi_t = 0 & \partial(B \cap H_t) \end{cases}, \begin{cases} \Delta\zeta_t = 2 & (B \cap H_t) \\ \zeta_t = -\chi & \partial(B \cap H_t) \end{cases}$$

where $\chi \in \mathcal{C}^\infty(B_1, [0,1])$ is such that $\chi > 0$ on $B_1 \setminus \overline{B}_{\frac{9}{10}}$ and $\chi \equiv 0$ on $B_{\frac{9}{10}}$. Note that $\zeta_t \leq 0$ and $\psi_t \geq 0$ for each t.

Lemma 4.5.11. *There exists $t_n \in (0,1)$, $c_n, d_n > 0$ such that for any $t \in (-t_n, t_n)$,*

$$\partial_n\psi_t \geq c_n, \ |\nabla\varphi_t - e_n| \leq d_n t, \ |\zeta_t| \leq d_n, \ |\nabla\zeta_t| \leq d_n, \text{ on } \partial H_t \cap \left\{|x'| \leq \frac{9}{10}\right\}$$

and

$$\text{If } t>0,\ \varphi_t \geq \left(x_n + \frac{1}{32}t\right)_+ \quad \text{on } B_{\frac{1}{8\sqrt{n}}} \cap H_t,$$
$$\text{If } t<0,\ \varphi_t \leq \left(x_n + \frac{1}{32}t\right)_+ \quad \text{on } B_{\frac{1}{8\sqrt{n}}} \cap H_t.$$

Proof. For the three first estimates, we only explain how the estimate of $\partial_n \psi_t$ is obtained, as the two others are derived analogously. We have first that $\partial_n \psi_0 \geq c > 0$ for some $c > 0$ by Hopf Lemma, while on the other hand by elliptic estimates $\|\partial_n(\psi_t \circ T_t - \psi_0)\|_{L^\infty(H_0 \cap B_{\frac{9}{10}})} \lesssim t$ where T_t is a diffeomorphism sending $B \cap H_t$ over $B \cap H_0 = B \cap \mathbb{H}^n$, thus giving $\partial_n \psi_t \geq c_n$ over $\partial H_t \cap \left\{|x'| \leq \frac{9}{10}\right\}$ for any $|t| \leq t_n$ for some dimensional $c_n > 0$ and $t_n > 0$.

For the second point, we consider for $0 \leq t \leq \frac{1}{4\sqrt{n}}$

$$P_t(x) := x_n + 4t\left(n\left(x_n - \frac{1}{4\sqrt{n}}\right)^2 - |x'|^2\right).$$

We check that $P_t(x) \leq x_n^+ \leq \phi_t$ on $\partial\left(B^{n-1}_{\frac{1}{2}} \times \left[-t, \frac{1}{4\sqrt{n}}\right]\right)$, while $\Delta P_t = 8t$, so by maximum principe we have $P_t \leq \varphi_t$ on $B^{n-1}_{\frac{1}{2}} \times \left[-t, \frac{1}{4\sqrt{n}}\right]$. Since we have $P_t(x) \geq x_n + \frac{1}{32}t$ on $B_{\frac{1}{8\sqrt{n}}} \cap H_t$ we deduce the claim in this case. The case $t < 0$ is treated similarly.

□

Proposition 4.5.12. *Let $L \geq 1$, $\delta \in \mathbb{R}$ and $m \in \mathbb{N}^*$ be such that $L^2 m|\delta|^{\frac{1}{2}} \leq \frac{1}{8}$. Then there exists $c_n > 0$ such that for any $\varepsilon \ll L^{-\frac{1}{2}}$ and any $(v, \overline{v}_1, \underline{v}_1, \ldots, \overline{v}_m, \underline{v}_m) \in \mathcal{S}_{m,\delta}(L)$ that is ε-flat in the sense of Definition 4.5.10 with parameters*

$$a, b, (\alpha, \overline{\alpha}_1, \underline{\alpha}_1, \ldots, \overline{\alpha}_m, \underline{\alpha}_m),$$

then there exists a', b' such that $a \leq a' \leq b' \leq b$, $b' - a' \leq (1 - c_n)\varepsilon$ and

$$(x_n + a')_+ \leq \frac{v(x)}{\alpha},\ \frac{\overline{v}_i(x)}{\overline{\alpha}_i},\ \frac{\underline{v}_i(x)}{\underline{\alpha}_i} \leq (x_n + b')_+ \text{ on } B_{\frac{1}{8\sqrt{n}}}.$$

Remark 4.5.13. The hypothesis $L^2 m|\delta|^{\frac{1}{2}} \leq \frac{1}{8}$ may be replaced by $L^2 m|\delta|^{\frac{1}{2}} \leq 1 - \eta$ for any $\eta > 0$, but how small ε needs to be would depend on η.

Proof. First note that by the estimates (4.44) and the hypothesis $L^2 m|\delta|^{\frac{1}{2}} \leq \frac{1}{8}$ we get $\alpha^2 \geq \frac{1}{2}$.

We suppose without loss of generality that $b - a \geq \frac{1}{2}\varepsilon$ otherwise we are done. As a consequence we have either $\frac{v(\frac{1}{2}e_n)}{\alpha} \geq \frac{1}{2} + a + \frac{\varepsilon}{4}$ (Case A) or $\frac{v(\frac{1}{2}e_n)}{\alpha} \leq \frac{1}{2} + b - \frac{\varepsilon}{4}$ (Case B).

Case A. Without loss of generality, we can set $a = 0$, meaning $\frac{v(\frac{1}{2}e_n)}{\alpha} \geq \frac{1}{2} + \frac{\varepsilon}{4}$.

Assume first $\delta < 0$. Since $|\Delta v| < \alpha\varepsilon^2$ in $B_1 \cap \{v > 0\}$, then for a small enough ε we get by the usual Harnack inequality applied to the positive function $v - \alpha x_n$ the existence of some $\sigma_n \in (0, 1/4)$ such that

$$\frac{v(x)}{\alpha} \geq x_n + 2\sigma_n\varepsilon \text{ on } B\left(\frac{1}{2}e_n, \frac{1}{4}\right). \tag{4.45}$$

We now consider the set of $t \geq 0$ such that the following $m+1$ inequalities are all verified on $B \cap H_t$:

$$\begin{aligned} \frac{v}{\alpha} &\geq \varphi_t + \varepsilon\sigma_n\psi_t + \varepsilon^2\zeta_t, \\ \sqrt{\frac{\overline{v}_i}{\overline{\alpha}_i} \cdot \frac{\underline{v}_i}{\underline{\alpha}_i}} &\geq \varphi_t + L^{\frac{1}{2}}\varepsilon^2\zeta_t. \end{aligned} \tag{4.46}$$

Note first that for each $t \geq 0$,

$$\begin{aligned} \frac{\Delta v}{\alpha} &< \varepsilon^2 = \Delta(\varphi_t + \varepsilon\sigma_n\psi_t + \varepsilon^2\zeta_t) \text{ in } B \cap H_t \setminus \overline{B}(\frac{1}{2}e_n, \frac{1}{4}), \\ \Delta\sqrt{\frac{\overline{v}_i}{\overline{\alpha}_i} \cdot \frac{\underline{v}_i}{\underline{\alpha}_i}} &\leq L^{\frac{1}{2}}\varepsilon^2 < \Delta\left(\varphi_t + L^{\frac{1}{2}}\varepsilon^2\zeta_t\right) \text{ in } B \cap H_t, \end{aligned}$$

where we used (4.41) in the second series of inequalities.

By (4.45), it holds $v(x)/\alpha \geq x_n + \sigma_n\varepsilon\psi_0(x)$. As a consequence the inequality for v is verified at $t = 0$ by maximum principle inside $B \cap H_0 \setminus B(\frac{1}{2}e_n, \frac{1}{4})$, and for $\sqrt{\frac{\overline{v}_i}{\overline{\alpha}_i} \cdot \frac{\underline{v}_i}{\underline{\alpha}_i}}$ by maximum principle inside $B \cap H_0$, using also $v(x)/\alpha, \overline{v}_i/\overline{\alpha}_i, \underline{v}_i/\underline{\alpha}_i \geq x_n$ in B_1 and noting that $\varphi_0 = x_n$ and $\zeta_t \leq 0$. We can therefore consider the largest $t \geq 0$ such that the inequalities (4.46) are verified in $B \cap H_t$. We want to prove that $t \geq \vartheta_n\varepsilon$ for some dimensional $\vartheta_n > 0$. Note that we lose no generality in supposing that t is at most comparable to ε (meaning $t \ll \varepsilon$), since the claim holds otherwise.

At the maximal t there is a equality in one of the inequalities (4.46) at some point $x \in \overline{B \cap H_t}$. Let us consider the possible cases.

- We cannot have $x \in \partial(H_t \cap B) \setminus \overline{B_{\frac{9}{10}}}$, since $\zeta_t < 0$, $\psi_t = 0$ and $\phi_t = x_n$ over this set.
- Suppose that $x \in B \cap H_t$. Then let us show that in this case we must have $v(x) > 0$. Otherwise, we would have $v(x) = \overline{v}_i(x) = \underline{v}_i(x) = 0$ so that necessarily $x_n \leq 0$. But $\phi_t + L^{\frac{1}{2}}\varepsilon^2\zeta_t > 0$ (and likewise $\varphi_t + \varepsilon\sigma_n\psi_t + \varepsilon^2\zeta_t > 0$) over $B \cap H_t \cap \{x_n \leq 0\}$, which comes from $\phi_t + L^{\frac{1}{2}}\varepsilon^2\zeta_t = \phi_t - L^{\frac{1}{2}}\varepsilon^2\chi = 0$ over $B \cap \partial H_t \cap \{x_n \leq 0\}$ and $|\nabla(\phi_t + L^{\frac{1}{2}}\varepsilon^2\zeta_t) - e_n| \ll 1$ thanks to Lemma 4.5.11. As a consequence $v(x) > 0$ and we can apply the maximum principle inside $B \cap H_t \cap \{v > 0\}$ to get that the equality cannot happen for $\overline{v}_i$. On the other hand, equality cannot happen for v by maximum principle inside $\{v > 0\} \cap B \cap H_t \setminus \overline{B}(\frac{1}{2}e_n, \frac{1}{4})$, and since in $\overline{B}(\frac{1}{2}e_n, \frac{1}{4})$ one has for $t \lesssim \varepsilon$:

$$\varphi_t + \varepsilon\sigma_n\psi_t + \varepsilon^2\zeta_t \leq \varphi_t + \varepsilon\sigma_n \leq x_n + C_n t + \varepsilon\sigma_n < x_n + 2\sigma_n\varepsilon.$$

As a consequence, $x \in \partial H_t \cap \overline{B_{\frac{9}{10}}}$. Therefore $(\varphi_t + \varepsilon\sigma_n\psi_t + \varepsilon^2\zeta_t)(x) = (\varphi_t + L^{\frac{1}{2}}\varepsilon^2\zeta_t)(x) = 0$ so that $v(x) = \overline{v}_i(x) = \underline{v}_i(x) = 0$ and there is equality in all of the inequalities (4.46). Since on the other hand one has $B \cap H_t \subset \{v > 0\}$ by the argument above, hence at any interior contact sphere for $B \cap H_t$ at x there exists a blow-up of $(v, \overline{v}_1, \ldots, \underline{v}_m)$ for the form $z \mapsto (\beta(z-x)\cdot\nu, \overline{\beta}_1(z-x)\cdot\nu, \underline{\beta}_1(z-x)\cdot\nu, \ldots, \underline{\beta}_m(z-x)\cdot\nu)$ as in Definition 4.5.7.As a consequence we have the viscosity condition:

$$
\begin{aligned}
1 = \beta^2 + \sum_{i=1}^m \overline{\beta}_i \underline{\beta}_i &\geq \alpha^2 |\nabla(\varphi_t + \varepsilon\sigma_n\psi_t + \varepsilon^2\zeta_t)(x)|^2 + \sum_{i=1}^m \overline{\alpha}_i\underline{\alpha}_i |\nabla(\varphi_t + L^{\frac{1}{2}}\varepsilon^2\zeta_t)(x)|^2 \\
&\geq \alpha^2 \left[(\partial_n\varphi_t)(x)^2 + 2\partial_n\varphi_t(x)\partial_n\psi_t(x)\sigma_n\varepsilon - C_n\varepsilon^2\right] + \sum_{i=1}^m \overline{\alpha}_i\underline{\alpha}_i \left[(\partial_n\varphi_t(x))^2 - C_n L^{\frac{1}{2}}\varepsilon^2\right] \\
&\text{for some large enough dimensional constant } C_n > 0 \\
&\geq \partial_n\varphi_t(x)^2 + 2\alpha^2\partial_n\varphi_t(x)\partial_n\psi_t(x)\sigma_n\varepsilon - C_n L^{\frac{1}{2}}\varepsilon^2 \\
&\geq 1 - 2d_n t + \frac{1}{2}c_n\sigma_n\varepsilon - C_n L^{\frac{1}{2}}\varepsilon^2
\end{aligned}
$$

where $c_n, d_n > 0$ come from Lemma 4.5.11. So when $\varepsilon \ll L^{-\frac{1}{2}}$ we get

$$
t \geq \frac{c_n\sigma_n}{8d_n}\varepsilon =: \vartheta_n\varepsilon
$$

so that using Lemma 4.5.11 we find for any $y \in B_{\frac{1}{8\sqrt{n}}} \cap H_t$

$$
\frac{v(y)}{\alpha}, \frac{\overline{v}_i(y)}{\overline{\alpha}_i}, \frac{\underline{v}_i(y)}{\underline{\alpha}_i} \geq \varphi_t(y) + \sqrt{L}\varepsilon^2\zeta_t(y) \geq y_n + \frac{1}{32}\vartheta_n\varepsilon - \sqrt{L}\|\zeta_t\|_\infty\varepsilon^2 \geq y_n + \frac{1}{64}\vartheta_n\varepsilon
$$

for $\varepsilon \ll 1$. For $y \in B_{\frac{1}{8\sqrt{n}}} \setminus H_t$ the above inequalities always hold, since $y_n + \frac{1}{64}\vartheta_n\varepsilon \leq 0$ and the functions are non-negative. This finishes the proof when $\delta < 0$.

The case $\delta > 0$ follows the same strategy, and was proven in [KL18, Theorem 5.1] (here in addition we keep track of the constants).

Case B. We suppose without loss of generality that $b = 0$, meaning $\frac{v(\frac{1}{2}e_n)}{\alpha} \leq \frac{1}{2} - \frac{\varepsilon}{4}$. The proof here follows the same outline for which we give rough details.

First, the Harnack inequality applied to $\alpha x_n - v$ gives the existence of some $\sigma_n > 0$ such that $\frac{v(x)}{\alpha} \leq x_n - 2\sigma_n\varepsilon$ on $B\left(\frac{1}{2}e_n, \frac{1}{4}\right)$. We now consider the largest $t > 0$ such that all the following inequalities are verified in $B \cap H_{-t}$:

$$
\begin{aligned}
\frac{v}{\alpha} &\leq \varphi_{-t} - \varepsilon\sigma_n\psi_{-t} - \varepsilon^2\zeta_{-t}, \\
\frac{1}{2}\left(\sqrt{\frac{\underline{\alpha}_i}{\overline{\alpha}_i}}\overline{v}_i + \sqrt{\frac{\overline{\alpha}_i}{\underline{\alpha}_i}}\underline{v}_i\right) &\leq \varphi_{-t} - (2L)^{\frac{1}{2}}\varepsilon^2\zeta_{-t}.
\end{aligned}
$$

It is verified at $t = 0$ by the previous remark and maximum principle. We then identify a contact point x associated to the largest t that we suppose small compared to ε: it is not inside $B \cap H_{-t}$ by maximum principle since

$$
\begin{aligned}
\frac{\Delta v}{\alpha} &> \Delta(\varphi_{-t} - \varepsilon\sigma_n\psi_{-t} + \varepsilon^2\zeta_{-t}) \text{ in } B \cap H_{-t} \setminus \overline{B}(\frac{1}{2}e_n, \frac{1}{4}) \\
\Delta\frac{1}{2}\left(\sqrt{\frac{\underline{\alpha}_i}{\overline{\alpha}_i}}\overline{v}_i + \sqrt{\frac{\overline{\alpha}_i}{\underline{\alpha}_i}}\underline{v}_i\right) &> \Delta\left(\varphi_{-t} - (2L)^{\frac{1}{2}}\varepsilon^2\zeta_{-t}\right) \text{ in } B \cap H_{-t}.
\end{aligned}
$$

where we use the estimates (4.44) on the last line. The contact point is not in $\partial(B_1 \cap H_{-t}) \setminus \overline{B_{\frac{9}{10}}}$ for the same reason as earlier, and so finally it is in $\partial H_{-t} \cap \overline{B_{\frac{9}{10}}}$. We then use the boundary condition (understood in the viscosity sense at the contact point)

$$
1 \leq (\partial_\nu v)^2 + \sum_{i=1}^m \left(\partial_\nu \frac{1}{2}\left(\sqrt{\frac{\underline{\alpha}_i}{\overline{\alpha}_i}}\overline{v}_i + \sqrt{\frac{\overline{\alpha}_i}{\underline{\alpha}_i}}\underline{v}_i\right)\right)^2 \leq 1 + a_n t - b_n\varepsilon
$$

for some $a_n, b_n > 0$. This yields $t \geq \vartheta_n \varepsilon$ for some $\vartheta_n > 0$ and we conclude by the last property of lemma 4.5.11. The case $\delta > 0$ is similar.

□

Proposition 4.5.14. *Let $L \geq 1$, $\delta \in \mathbb{R}$ and $m \in \mathbb{N}^*$ be such that $L^2 m|\delta|^{\frac{1}{2}} \leq \frac{1}{8}$. Then there exists $C_n > 0$ and $\kappa_n \in (0,1)$, such that the following holds: for any $\varepsilon \ll L^{-\frac{1}{2}}$, and for any $(v, \overline{v}_1, \underline{v}_1, \ldots, \overline{v}_m, \underline{v}_m) \in \mathcal{S}_{m,\delta}(L)$ that is ε-flat in the sense of definition 4.5.10 with parameters*

$$a, b, (\alpha, \overline{\alpha}_1, \underline{\alpha}_1, \ldots, \overline{\alpha}_m, \underline{\alpha}_m),$$

then defining

$$V(x) = \frac{v(x) - \alpha x_n}{\alpha \varepsilon}, \ \overline{V}_i(x) = \frac{\overline{v}_i(x) - \overline{\alpha}_i x_n}{\overline{\alpha}_i \varepsilon}, \ \underline{V}_i(x) = \frac{\underline{v}_i(x) - \underline{\alpha}_i x_n}{\underline{\alpha}_i \varepsilon},$$

$$W_i(x) = \frac{\sqrt{\overline{v}_i \underline{v}_i} - \sqrt{\overline{\alpha}_i \underline{\alpha}_i} x_n}{\sqrt{\overline{\alpha}_i \underline{\alpha}_i} \varepsilon}$$

we have that for any $x, y \in B_{\frac{1}{2}} \cap \{v > 0\}$ such that $|x - y| > \varepsilon$,

$$|V(x) - V(y)|, \left|\overline{V}_i(x) - \overline{V}_i(y)\right|, \left|\overline{V}_i(x) - \overline{V}_i(y)\right|, |W_i(x) - W_i(y)| \leq C_n |x - y|^{\kappa_n}. \tag{4.47}$$

Proof. This is obtained by applying successively the previous Lemma, as in [DS11, Corollary 3.2] or [Vel19, Lemma 7.14]. □

4.5.2 Flatness improvement

We let $\tau = \tau_n \in (0,1)$ be a fixed universal constant (depending only on n) such that for any harmonic function $h : B_1 \to [-5, 5]$ and for any $x \in B_\tau$ it holds

$$|h(x) - h(0) - x \cdot \nabla h(0)| \leq \frac{1}{8}\tau \tag{4.48}$$

Proposition 4.5.15. *Let $L \geq 1$, $\delta \in \mathbb{R}$ and $m \in \mathbb{N}^*$ with $L^2 m^2 |\delta|^{\frac{1}{2}} \leq \frac{1}{8}$. Then there exists $\varepsilon_n(L)$ such that we have the following flatness reduction property for any $\varepsilon < \varepsilon_n(L)$. Suppose $(v, \overline{v}_1, \underline{v}_1, \ldots, \overline{v}_m, \underline{v}_m) \in \mathcal{S}_{m,\delta}(L)$ is ε-flat in the sense of Definition 4.5.10 with parameters*

$$a, b, (\alpha, \overline{\alpha}_1, \underline{\alpha}_1, \ldots, \overline{\alpha}_m, \underline{\alpha}_m),$$

Then there exists $a' \leq b'$, $e' \in \mathbb{S}^{n-1}$, and $\alpha', \overline{\alpha}'_i, \underline{\alpha}'_i$ verifying

$$\alpha'^2 + \sum_{i=1}^m \frac{\overline{\alpha}_i'^2 + \underline{\alpha}_i'^2}{2} = 1 \text{ if } \delta > 0, \ \alpha'^2 + \sum_{i=1}^m \overline{\alpha}'_i \underline{\alpha}'_i = 1 \text{ if } \delta < 0$$

such that

$$\left(e' \cdot x + a'\right) \leq \frac{v(\tau x)}{\tau \alpha'}, \ \frac{\overline{v}_i(\tau x)}{\tau \overline{\alpha}'_i}, \ \frac{\underline{v}_i(\tau x)}{\tau \underline{\alpha}'_i} \leq \left(e' \cdot x + b'\right) \text{ on } B_1 \cap \frac{\{v > 0\}}{\tau}.$$

and $b' - a' \leq \frac{1}{2}\varepsilon$, with moreover

$$|e' - e_n|, \ \left|1 - \frac{\alpha'}{\alpha}\right|, \ \left|1 - \frac{\overline{\alpha'}_i}{\overline{\alpha}_i}\right|, \ \left|1 - \frac{\underline{\alpha'}_i}{\underline{\alpha}_i}\right| \lesssim \varepsilon$$

This follows the proof of [KL18, Th 6.1], with a difference when the multiplicity of the eigenspace goes to infinity.

Proof. We proceed by contradiction and compactness. Suppose there exists a sequence $\varepsilon_p \to 0$ (we drop the index p and just write $\varepsilon \to 0$ to lighten the notations) and some sequences

$$(v^\varepsilon, \overline{v}_1^\varepsilon, \underline{v}_1^\varepsilon, \dots, \overline{v}_{m^\varepsilon}^\varepsilon, \underline{v}_{m^\varepsilon}^\varepsilon) \in \mathcal{S}_{m^\varepsilon,\delta^\varepsilon}(L),\ a^\varepsilon, b^\varepsilon, (\alpha^\varepsilon, \overline{\alpha}_1^\varepsilon, \underline{\alpha}_1^\varepsilon, \dots, \overline{\alpha}_{m^\varepsilon}^\varepsilon, \underline{\alpha}_{m^\varepsilon}^\varepsilon)$$

which verify the hypotheses but not the conclusion. This means that one of the functions v^ε, $\overline{v}_1^\varepsilon$, $\underline{v}_1^\varepsilon$, ..., $\overline{v}_{m^\varepsilon}^\varepsilon$, $\underline{v}_{m^\varepsilon}^\varepsilon$ does not verify the flatness improvement on B_τ.

Consider the sequences

$$V^\varepsilon(x) = \frac{v^\varepsilon(x) - \alpha^\varepsilon x_n}{\alpha^\varepsilon \varepsilon},\ \overline{V}_i^\varepsilon(x) = \frac{\overline{v}_i^\varepsilon(x) - \overline{\alpha}_i^\varepsilon x_n}{\overline{\alpha}_i^\varepsilon \varepsilon},\ \underline{V}_i^\varepsilon(x) = \frac{\underline{v}_i^\varepsilon(x) - \underline{\alpha}_i^\varepsilon x_n}{\underline{\alpha}_i^\varepsilon \varepsilon},$$

$$W_i^\varepsilon(x) = \frac{\sqrt{\overline{v}_i^\varepsilon \underline{v}_i^\varepsilon} - \sqrt{\overline{\alpha}_i^\varepsilon \underline{\alpha}_i^\varepsilon} x_n}{\sqrt{\overline{\alpha}_i^\varepsilon \underline{\alpha}_i^\varepsilon}\varepsilon}.$$

We also write $\Omega^\varepsilon = B_1 \cap \{v^\varepsilon > 0\}$ their (common) domain of definition, which converges locally Hausdorff to $B_1 \cap \mathbb{H}^n$ since $\{x \in B_1, x_n > \varepsilon\} \subset \Omega^\varepsilon \subset \{x \in B_1, x_n > -\varepsilon\}$. Each function has values in $[-1,1]$, with laplacian bounded by ε in Ω^ε. Moreover thanks to Proposition 4.5.12, they verify the Hölder-type property (4.47) for some $\kappa_n \in (0,1)$ up to the boundary $\partial\mathbb{H}^n$.

After extraction in ε we have a local Hausdorff convergence of the graphs of V^ε, $\overline{V}_i^\varepsilon$, $\underline{V}_i^\varepsilon$ on Ω^ε to the graphs of functions $V, \overline{V}_i, \underline{V}_i : B \cap \mathbb{H}^n \to [-1,1]$, which are in $\mathcal{C}^{0,\kappa_n}_{\text{loc}}(B \cap \overline{\mathbb{H}^n})$ and harmonic in $B \cap \mathbb{H}^n$ (see for instance [Vel19, Lemma 7.15]). The functions $(W_i^\varepsilon)_{\varepsilon\to 0}$ verify the same ocillation reduction so after extraction their graph converge in the local Hausdorff sense to a limit W_i, which we identify (by taking a limit for any $x \in B \cap \{x_n > \delta\}$) as

$$W_i = \frac{\overline{V}_i + \underline{V}_i}{2}.$$

We now distinguish four cases depending on whether m^ε is stationary at some finite value $m \in \mathbb{N}^*$ or not, and whether $\delta > 0$ or $\delta < 0$. We detail the cases $\delta < 0$ and outline the cases $\delta > 0$, which may be found (without control of the constants in m) in [KL18].

Case $m^\varepsilon \to m$, $\delta < 0$. We lose no generality in assuming $m^\varepsilon = m$ for all ε. Up to extraction there exists $\alpha, \overline{\alpha}_i, \underline{\alpha}_i \geq 0$ such that

$$\alpha^\varepsilon \to \alpha,\ \overline{\alpha}_i^\varepsilon \to \overline{\alpha}_i,\ \underline{\alpha}_i^\varepsilon \to \underline{\alpha}_i,$$

$V - \overline{V}_i$ and $V - \underline{V}_i$ verifies a Dirichlet boundary condition on $B_1 \cap \{x_n = 0\}$ (since $V^\varepsilon - \underline{V}_i^\varepsilon = V^\varepsilon - \overline{V}_i^\varepsilon = 0$ on $\partial\Omega^\varepsilon$). This makes $2m$ Dirichlet boundary conditions for $2m+1$ harmonic functions, and we claim that we have an additional boundary condition

$$\partial_n h = 0 \text{ in } B_1 \cap \{x_n = 0\},\ \text{where } h = \left(\alpha^2 V + \sum_{i=1}^m \overline{\alpha}_i \underline{\alpha}_i \frac{\overline{V}_i + \underline{V}_i}{2}\right), \tag{4.49}$$

holding in the viscosity sense.

Inequality $\partial_n h \leq 0$. To prove this claim set $x^0 \in B_1 \cap \{x_n = 0\}$, and we suppose by contradiction that there are constants $p > 0$, $z \in \mathbb{R}^{n-1} \times \{0\}$, $\sigma > 0$ such that

$$h(x) \geq h(x^0) + p x_n + z \cdot (x - x^0) + \sigma\left(x_n^2 - \frac{1}{n+1}\left|x - x^0\right|^2\right) =: \varphi(x),\ \forall x \in B(x^0, \rho) \cap \mathbb{H}^n \tag{4.50}$$

Note that we can always change p into $p/2$, replace σ by some arbitrarily large $\sigma' \geq \sigma$ and ρ by some small enough $\rho' < \rho$ such that the equality holds only at $x = x^0$. Since the functions $V - h$, $\overline{V}_i - h$ and

$\underline{V}_i - h$ are harmonic and vanish on $B_1 \cap \{x_n = 0\}$, they are smooth over $B_1 \cap \overline{\mathbb{H}^n}$ so that there exists q, $\overline{q}_i$, $\underline{q}_i \in \mathbb{R}$ such that

$$\begin{aligned}
(V-h)(x) &= qx_n + \mathcal{O}\left(|x-x^0|^2\right), \\
(\overline{V}_i - h)(x) &= \overline{q}_i x_n + \mathcal{O}\left(|x-x^0|^2\right), \\
(\underline{V}_i - h)(x) &= \underline{q}_i x_n + \mathcal{O}\left(|x-x^0|^2\right), \\
(W_i - h)(x) &= \frac{\overline{q}_i + \underline{q}_i}{2} x_n + \mathcal{O}\left(|x-x^0|^2\right) =: q_i x_n + \mathcal{O}\left(|x-x^0|^2\right),
\end{aligned}$$

which verify in addition $\alpha^2 q + \sum_{i=1}^m \overline{\alpha}_i \underline{\alpha}_i q_i = 0$. Up to reducing p and increasing σ, we have by uniform interior $\mathcal{C}^2$ estimates on the harmonic functions:

$$V(x) \geq qx_n + \varphi(x), \ W_i(x) \geq q_i x_n + \varphi(x)$$

in a neighbourhood of x^0 in $B \cap \mathbb{H}^n$, which we denote by $B_{x^0,\rho} \cap B \cap \mathbb{H}^n$. Then by the local uniform Hausdorff convergence of the graphs there exists $c^\varepsilon \to 0$ such that

$$V^\varepsilon(x) \geq qx_n + \varphi(x) - c^\varepsilon, \ W_i^\varepsilon \geq q_i x_n + \varphi(x) - c^\varepsilon,$$

for $x \in B_{x^0,\rho} \cap \Omega^\varepsilon$. This may be rewritten as

$$\begin{aligned}
v^\varepsilon(x) &\geq \alpha^\varepsilon \left(x_n + \varepsilon q x_n + \varepsilon\varphi(x) - \varepsilon c^\varepsilon\right), \\
\sqrt{\overline{v}_i^\varepsilon \underline{v}_i^\varepsilon}(x) &\geq \sqrt{\overline{\alpha}_i^\varepsilon \underline{\alpha}_i^\varepsilon} \left(x_n + \varepsilon \overline{q}_i x_n + \varepsilon\varphi(x) - \varepsilon c^\varepsilon\right).
\end{aligned}$$

Hence, up to changing c^ε into $2c^\varepsilon + C\varepsilon$, for some large enough constant C (that does not depend on ε), we have for any $x \in B_{x^0,\rho} \cap \Omega^\varepsilon$:

$$v^\varepsilon(x) \geq \alpha^\varepsilon(1+\varepsilon q)\psi_0(x), \ \sqrt{\overline{v}_i^\varepsilon \underline{v}_i^\varepsilon}(x) \geq \sqrt{\overline{\alpha}_i^\varepsilon \underline{\alpha}_i^\varepsilon}(1+\varepsilon q_i)\psi_0(x),$$

where for any $t \geq 0$ we have defined

$$\psi_t(x) = x_n + \varepsilon\varphi(x) - \varepsilon c^\varepsilon + t.$$

We can therefore consider the maximal $t^\varepsilon \geq 0$ such that the previous set of inequalities still holds for the functions v^ε, $\sqrt{\overline{v}_i^\varepsilon \underline{v}_i^\varepsilon}$. There must be a contact point $x^\varepsilon \in \overline{B_{x^0,\rho} \cap \Omega^\varepsilon}$ for one of the functions, and we may assume without loss of generality (up to changing p into $p/2$, increasing σ and reducing ρ accordingly) that $x^\varepsilon \to x^0$.

Suppose $x^\varepsilon \in \Omega^\varepsilon$ and that it is a contact point for v^ε (the same argument holds for the other functions W_i^ε), and we then have by maximum principle

$$(1+\varepsilon q)\varepsilon \frac{2\sigma n}{n+1} = (1+\varepsilon q)\varepsilon \Delta\varphi(x^\varepsilon) \leq \frac{1}{\alpha^\varepsilon}\Delta v^\varepsilon(x^\varepsilon) \leq \varepsilon^2,$$

which is a contradiction for a small enough ε. As a consequence $x^\varepsilon \in \partial\Omega^\varepsilon$, so that there is equality in all of the above inequalities at x^ε. Hence there exists a blow-up of $(v, \overline{v}_1, \ldots, \underline{v}_m)$ at x^ε, and comparing the derivatives at this blow-up (4.41) we get

$$1 \geq |\alpha^\varepsilon|^2(1+\varepsilon q)^2 |\nabla\psi_{t^\varepsilon}(x^\varepsilon)|^2 + \sum_{i=1}^m \overline{\alpha}_i^\varepsilon \underline{\alpha}_i^\varepsilon (1+\varepsilon q_i)^2 |\nabla\psi_{t^\varepsilon}(x^\varepsilon)|^2$$

which after simplification becomes

$$|\nabla\psi_t(x^\varepsilon)|^2 \leq 1 + o_{\varepsilon\to 0}(\varepsilon).$$

This is a contradiction since $|\nabla\psi_t(x^\varepsilon)|^2 = 1 + 2p\varepsilon + o_{\varepsilon\to 0}(\varepsilon)$. As a consequence we get $\partial_n h \leq 0$ on $B \cap \{x_n = 0\}$ in the viscosity sense.

Inequality $\partial_n h \geq 0$. Suppose indeed that this time $h(x) \leq \varphi(x)$ with $p < 0$ and $\sigma < 0$. Following the previous reasonning, for some sequence $c^\varepsilon \to 0$ and in some neighbourhood $B(x^0, \rho) \cap \mathbb{H}^n$ we have this time

$$v^\varepsilon(x) \leq \alpha^\varepsilon(1+\varepsilon q)\psi_0(x),\ \overline{v}_i^\varepsilon(x) \leq \overline{\alpha}_i^\varepsilon(1+\varepsilon\overline{q}_i)\psi_0(x),\ \underline{v}_i^\varepsilon(x) \leq \underline{\alpha}_i^\varepsilon(1+\varepsilon\underline{q}_i)\psi_0(x),$$

where ψ_t is defined as previously. Consider the largest t such that the inequalities

$$v^\varepsilon(x) \leq \alpha^\varepsilon(1+\varepsilon q)\psi_t(x),\ \frac{1}{2}\left(\sqrt{\frac{\underline{\alpha}_i^\varepsilon}{\overline{\alpha}_i^\varepsilon}}\overline{v}_i^\varepsilon(x) + \sqrt{\frac{\overline{\alpha}_i^\varepsilon}{\underline{\alpha}_i^\varepsilon}}\underline{v}_i^\varepsilon\right) \leq \sqrt{\overline{\alpha}_i^\varepsilon\underline{\alpha}_i^\varepsilon}(1+\varepsilon q_i)\psi_t(x)$$

are verified in $B(x^0, \rho) \cap \mathbb{H}^n$: at the largest t there is some contact point x^ε, and either $x^\varepsilon \notin \Omega^\varepsilon$ by maximum principle as earlier (we use here the estimates 4.44) or $x^\varepsilon \in \partial\Omega^\varepsilon$ and we have the viscosity condition

$$1 \leq |\alpha^\varepsilon|^2(1+\varepsilon q)^2|\nabla\psi_{t^\varepsilon}(x^\varepsilon)|^2 + \sum_{i=1}^m \overline{\alpha}_i^\varepsilon\underline{\alpha}_i^\varepsilon(1+\varepsilon q_i)^2|\nabla\psi_{t^\varepsilon}(x^\varepsilon)|^2$$

which after simplification becomes

$$|\nabla\psi_t(x^\varepsilon)|^2 (= 1 + 2p\varepsilon + o(\varepsilon)) \geq 1 + o_{\varepsilon\to 0}(\varepsilon).$$

Since $p < 0$ this is a contradiction for a small enough ε.

Now that the Neumann boundary condition (4.49) is verified in the viscosity sense, h may be extended a a smooth harmonic function on B_1 by an even reflexion through $\partial\mathbb{H}^n$.

We may now conclude: V, $\overline{V}_i$, $\underline{V}_i$ may be respectively extended as harmonic function V', $\overline{V}_i'$, $\underline{V}_i'$ on B_1 (through reflections), with values in $[-5, 5]$. Indeed we first write

$$V = \left(\alpha^2 V + \sum_{i=1}^m \overline{\alpha}_i\underline{\alpha}_i\frac{\overline{V}_i + \underline{V}_i}{2}\right) + \sum_{i=1}^m \overline{\alpha}_i\underline{\alpha}_i\frac{(V-\overline{V}_i) + (V-\underline{V}_i)}{2}.$$

The first term extends by even reflection thanks to (4.49), and the second by odd reflection (since $V - \overline{V}_i = V - \underline{V}_i = 0$ over $B_1 \cap \{x_n = 0\}$), and as a consequence V extends harmonically with a bound $|V| \leq 3$. We then write for each i, $\overline{V}_i = V - (V - \overline{V}_i)$ and $\underline{V}_i = V - (V - \underline{V}_i)$, so that $\overline{V}_i, \underline{V}_i$ extend harmonically in B_1 into functions bounded by 5. Recalling (4.48) we find $c \in \mathbb{R}$, $z \in \mathbb{R}^{n-1} \times 0$, $q, \overline{q}_i, \underline{q}_i \in \mathbb{R}$ such that for any $x \in B_\tau$,

$$\begin{aligned}
\left|V(x) - c - z\cdot x' - qx_n\right| &\leq \frac{1}{8}\tau,\\
\left|\overline{V}_i(x) - c - z\cdot x' - \overline{q}_i x_n\right| &\leq \frac{1}{8}\tau,\\
\left|\underline{V}_i(x) - c - z\cdot x' - \underline{q}_i x_n\right| &\leq \frac{1}{8}\tau.
\end{aligned}$$

Thus by Haudsdorff convergence of the graphs, for any small enough ε and any $x \in B_\tau$ we have

$$\left|\frac{v^\varepsilon(x) - \alpha^\varepsilon x_n}{\varepsilon\alpha^\varepsilon} - c - z\cdot x' - qx_n\right| \leq \frac{1}{6}\tau \tag{4.51}$$

and the same holds accordingly for $\overline{v}_i^\varepsilon$, $\underline{v}_i^\varepsilon$. Set now

$$S^\varepsilon = (1+\varepsilon q)^2(\alpha^\varepsilon)^2 + \sum_{i=1}^m \overline{\alpha}_i^\varepsilon \underline{\alpha}_i^\varepsilon (1+\varepsilon \overline{q}_i)(1+\varepsilon \underline{q}_i).$$

Since $q\alpha^2 + \sum_{i=1}^m \overline{\alpha}_i^\varepsilon \underline{\alpha}_i^\varepsilon \frac{\overline{q}_i + \underline{q}_i}{2} = 0$, then $|S^\varepsilon - 1| = o(\varepsilon)$. Let now

$$\alpha^{\varepsilon\prime} = \frac{(1+\varepsilon q)\alpha^\varepsilon}{\sqrt{S^\varepsilon}},\ \overline{\alpha}_i^{\varepsilon\prime} = \frac{(1+\varepsilon q_i)\overline{\alpha}_i^\varepsilon}{\sqrt{S^\varepsilon}},\ \underline{\alpha}_i^{\varepsilon\prime} = \frac{(1+\varepsilon q_i)\underline{\alpha}_i^\varepsilon}{\sqrt{S^\varepsilon}},$$

and let

$$e' = \frac{e_n + \varepsilon z}{\sqrt{1+\varepsilon^2|z'|^2}},\ a' = \frac{c}{\tau} - \frac{1}{4}\varepsilon,\ b' = \frac{c}{\tau} + \frac{1}{4}\varepsilon.$$

Then (4.51) may be rewritten as

$$\begin{aligned}
\big[(1+\varepsilon q)e_n + \varepsilon z'\big]\cdot x + \varepsilon c - \frac{1}{6}\tau \le & \frac{v^\varepsilon(x)}{\alpha^\varepsilon} \le \big[(1+\varepsilon q)e_n + \varepsilon z'\big]\cdot x + \varepsilon c + \frac{1}{6}\tau,\\
\big[(1+\varepsilon \overline{q}_i)e_n + \varepsilon z'\big]\cdot x + \varepsilon c - \frac{1}{6}\tau \le & \frac{\overline{v}_i^\varepsilon(x)}{\overline{\alpha}_i^\varepsilon} \le \big[(1+\varepsilon \overline{q}_i)e_n + \varepsilon z'\big]\cdot x + \varepsilon c + \frac{1}{6}\tau,\\
\Big[(1+\varepsilon \underline{q}_i)e_n + \varepsilon z'\Big]\cdot x + \varepsilon c - \frac{1}{6}\tau \le & \frac{\underline{v}_i^\varepsilon(x)}{\underline{\alpha}_i^\varepsilon} \le \Big[(1+\varepsilon \underline{q}_i)e_n + \varepsilon z'\Big]\cdot x + \varepsilon c + \frac{1}{6}\tau,
\end{aligned}$$

for any $x \in B_\tau \cap \Omega^\varepsilon$, which simplifies as $\varepsilon \to 0$ to

$$e'\cdot x + a' \le \frac{v^\varepsilon(x)}{\alpha'^\varepsilon}, \frac{\overline{v}_i^\varepsilon(x)}{\overline{\alpha}_i'^\varepsilon}, \frac{\underline{v}_i^\varepsilon(x)}{\underline{\alpha}_i'^\varepsilon} \le e'\cdot x + b',\ \forall x \in \Omega^\varepsilon \cap B_\tau$$

so that all functions verify the flatness improvement, which is a contradiction for small enough ε. This concludes the proof in this case.

Case $m^\varepsilon \to m$, $\delta > 0$. This case follows more closely [KL18]; the only difference here is that the Neumann boundary condition verified at the limit is

$$\partial_n\left(\alpha^2 V + \sum_{i=1}^m \frac{\overline{\alpha}_i^2 \overline{V}_i + \underline{\alpha}_i^2 \underline{V}_i}{2}\right) = 0 \text{ in } B_1 \cap \{x_n = 0\}.$$

Case $m^\varepsilon \to \infty$. In this case we treat $\delta > 0$ and $\delta < 0$ at once. Define $V^\varepsilon, \overline{V}_i^\varepsilon$ and $\underline{V}_i^\varepsilon$ as previously and thanks to Lemma 4.5.16 below, we may change their order (in i) and assume a convergence to some limits $V, \overline{V}_i$ and $\underline{V}_i$ as $\varepsilon \to 0$ (in the sense of local Hausdorff convergence of the graphs) which is uniform in i. We still have the Dirichlet boundary condition $V - \overline{V}_i = V - \underline{V}_i = 0$ on $B_1 \cap \{x_n = 0\}$. Due to the estimates (4.44) we have

$$\big|1 - (\alpha^\varepsilon)^2\big| \le 4L^2|\delta^\varepsilon|^{\frac{1}{2}} m^\varepsilon \le 2L^2(m^\varepsilon)^{-1} \to 0.$$

We now prove the Neumann boundary condition (in the viscosity sense)

$$\partial_n V = 0 \text{ on } B \cap \{x_n = 0\}. \tag{4.52}$$

We proceed as previously: letting $x^0 \in B_1 \cap \{x_n = 0\}$, we pick a polynomial function defined as in (4.50) with $p > 0$ touching V from below at a point x^0. Writing $\overline{V}_i(x) = V(x) + \overline{q}_i x_n + \mathcal{O}(|x - x^0|^2)$ (where the remainder term only depends on $1 - |x^0|$ and n, by uniform regularity of harmonic functions) and

similarly for $\underline{V}_i$, using the uniform convergence of the graphs we get that in a neighborhood $B_{x^0,\rho} \cap \Omega^\varepsilon$ of x^0 it holds for every i:

$$v^\varepsilon(x) \geq \alpha\psi_0(x),\ \overline{v}_i^\varepsilon(x) \geq \overline{\alpha}_i(1+\varepsilon\overline{q}_i)\psi_0(x),\ \underline{v}_i^\varepsilon(x) \geq \underline{\alpha}_i(1+\varepsilon\underline{q}_i)\psi_0(x) \text{ if } \delta > 0,$$
$$v^\varepsilon(x) \geq \alpha\psi_0(x),\ \sqrt{\overline{v}_i^\varepsilon(x)\underline{v}_i^\varepsilon(x)} \geq \sqrt{\overline{\alpha}_i\underline{\alpha}_i^\varepsilon}\left(1+\varepsilon\frac{\overline{q}_i+\underline{q}_i}{2}\right)\psi_0(x) \text{ if } \delta < 0,$$

where for any $t \geq 0$ we have defined

$$\psi_t(x) = x_n + \varepsilon\varphi(x) - \varepsilon c^\varepsilon + t$$

and $c^\varepsilon \to 0^+$. Note that the $(\overline{q}_i, \underline{q}_i)$ are uniformly bounded by a constant M only depending on n and $1 - |x^0|$.

Taking then the largest $t^\varepsilon \geq 0$ such that these inequalities are verified over $B_{x^0,\rho} \cap \Omega^\varepsilon$ for every i, then there is a contact point x^ε either for the function v^ε or one of the $\overline{v}_i^\varepsilon, \underline{v}_i^\varepsilon$, with $x^\varepsilon \to x^0$. Then as previously by maximum principle the contact point x^ε lies in $\partial\Omega^\varepsilon$ when ε is small enough, and there is equality at x^ε in all of the inequalities. Comparing the derivatives at x^ε we get (with the viscosity condition)

$$\begin{aligned}
1 &\geq |\alpha^\varepsilon|^2|\nabla\psi_{t^\varepsilon}(x^\varepsilon)|^2 + \sum_{i=1}^{m^\varepsilon}\frac{1}{2}\left(|\overline{\alpha}_i^\varepsilon|^2(1+\varepsilon\overline{q}_i)^2 + |\underline{\alpha}_i^\varepsilon|^2(1+\varepsilon\underline{q}_i)^2\right)|\nabla\psi_{t^\varepsilon}(x^\varepsilon)|^2 \\
&\geq \left(1 - 2L^2m^\varepsilon|\delta^\varepsilon|^{\frac{1}{2}}(M\varepsilon + M^2\varepsilon^2)\right)|\nabla\psi_{t^\varepsilon}(x^\varepsilon)|^2 \\
&\geq \left(1 - \frac{M}{m^\varepsilon}\varepsilon + o_{\varepsilon\to 0}(\varepsilon)\right)(1+\varepsilon\partial_n\varphi(x^\varepsilon)),
\end{aligned}$$

which is a contradiction for small enough ε since $\partial_n\varphi(x^\varepsilon) \to p > 0$. This ensures $\partial_n V \leq 0$ over $B_1 \cap \{x_n = 0\}$ in the viscosity sense.

Likewise we get $\partial_n V \geq 0$ on $\{x_n = 0\}$ in the viscosity sense. As a consequence V verifies the Neumann boundary condition (4.52). We may extend V by an even reflexion and the $V - \overline{V}_i$, $V - \underline{V}_i$ by odd reflexions, so that V, $\overline{V}_i, \underline{V}_i$ extend as harmonic function on B_1 with values in $[-3, 3]$, and relying on (4.48) we obtain as previously a contradiction, finishing the proof in the case $m^\varepsilon \to \infty$. □

To deal with the case $m^\varepsilon \to \infty$ we made use of the following Lemma, which is a general statement on sequences in compact metric spaces.

Lemma 4.5.16. *Let (X, d) be a metric compact space. Let m_k be a sequence of integers such that $m_k \to \infty$ and $(x_j^k)_{k\in\mathbb{N}^*, 1\leq j\leq m_k}$ be a sequence in X. Then there exists a sequence of permutations $\sigma^k \in \mathfrak{S}(1, m_k)$ and a sequence $(x_j)_{j\in\mathbb{N}^*}$ such that*

$$\liminf_{k\to\infty} \sup_{1\leq j\leq m_k} d\left(x^k_{\sigma^k(j)}, x_j\right) = 0.$$

We do not claim that this lemma is original, but since we have not found any reference in the litterature we provide hereafter a short proof.

Proof. Note first that it is enough to prove the Lemma for the Cantor set $X = \{0,1\}^{\mathbb{N}^*}$ endowed with the dyadic metric $d(x, y) = 2^{-\inf\{i\geq 1: x(i)\neq y(i)\}}$, as it surjects continuously onto any compact metric space. We write $X_N = \{0,1\}^N$ and $\pi_N : X \to X_N$ the projection onto the first N coordinates. Let $\varphi_1 : \mathbb{N}^* \to \mathbb{N}^*$ be an extraction such that the number of 0's and 1's among

$$\left(\pi_1 x_1^{\varphi_1(k)}, \ldots, \pi_1 x_{m_{\varphi_1(k)}}^{\varphi_1(k)}\right)$$

is nondecreasing in k. Starting from φ_1 we define recursively φ_N in the following way: if φ_{N-1} is given we build φ_N as an extraction of φ_{N-1} verifying that the number of occurrences of each $b \in X_N$ in

$$\left(\pi_N x_1^{\varphi_N(k)}, \dots, \pi_N x_{m_{\varphi_N(k)}}^{\varphi_N(k)}\right)$$

is nondecreasing in k. We finally set $\varphi(k) := \varphi_k(k)$. We now define a sequence of permutations $\sigma^k \in \mathfrak{S}_{m_{\varphi(k)}}$ as follows: let σ^1 be the identity, and provided σ^k is given we define σ^{k+1} such that for each

$$\left(\pi_k x_1^{\varphi(k+1)}, \dots, \pi_k x_{m_{\varphi(k+1)}}^{\varphi(k+1)}\right)$$

there are at least as many occurrences of each element of X_k as in

$$\left(\pi_k x_1^{\varphi(k)}, \dots, \pi_k x_{m_{\varphi(k)}}^{\varphi(k)}\right)$$

This is ensured by finding $\sigma^{k+1} \in \mathfrak{S}_{m_{\varphi(k+1)}}$ such that for each $j \in \{1, \dots, m_{\varphi(k)}\}$, we have

$$\pi_k x_{\sigma^{k+1}(j)}^{\varphi(k+1)} = \pi_k x_{\sigma^k(j)}^{\varphi(k)}.$$

We now define x_j as the unique element of X such that for every $k \in \mathbb{N}^*$, as soon as $m_{\varphi(k)} \geq j$ we have

$$\pi_k x_j = \pi_k x_{\sigma^k(j)}^{\varphi(k)}.$$

By construction this gives for every $k \in \mathbb{N}^*$, $j \in \{1, \dots, m_{\varphi(k)}\}$:

$$d\left(x_{\sigma^k(j)}^{\varphi(k)}, x_j\right) \leq \frac{1}{2^{k+1}}.$$

Hence $(x_{\sigma^k(j)}^{\varphi(k)}) \underset{k\to\infty}{\longrightarrow} (x_j)$ thus concluding the proof. □

Corollary 4.5.17. *Let $L \geq 1$ and $\delta \in \mathbb{R}$. Then there exists $\varepsilon_n(L) > 0$, $\gamma_n \in (0,1)$ verifying the following property. For any $m \in \mathbb{N}^*$, $\delta \in \mathbb{R}$ verifying $L^2 m^2 |\delta|^{\frac{1}{2}} \leq \frac{1}{8}$, and for any $\varepsilon < \varepsilon_n(L)$, $(v, \overline{v}_1, \underline{v}_1, \dots, \overline{v}_m, \underline{v}_m) \in \mathcal{S}_{m,\delta}(L)$ that is ε-flat in the sense of Definition 4.5.10, then there exists $g \in \mathcal{C}^{1,\gamma_n}\left(B_{1/2}^{n-1}, [-\varepsilon, \varepsilon]\right)$ such that $\|g\|_{\mathcal{C}^{1,\gamma_n}\left(B_{1/2}^{n-1}\right)} \lesssim \varepsilon$ and*

$$\{v > 0\} \cap \left(B_{1/2}^{n-1} \times [-1/2, 1/2]\right) = \left\{(x', x_n) \in B_{1/2}^{n-1} \times [-1/2, 1/2] : \ x_n \geq g(x')\right\}.$$

Proof. This step is classical, and comes from iterating the flatness improvement Proposition 4.5.15 as is done in [Vel19, Theorem 8.1]. □

This implies that for small enough δ any minimizer Ω has its boundary which can be written as a $\mathcal{C}^{2,\gamma}$ on the sphere. This is the object of next Lemma.

Lemma 4.5.18. *Suppose that $|\delta| \ll k^{-\left(6+\frac{8}{n}\right)} g_n(k)$ and let Ω be a centered minimizer of (4.36). Then there exists $\gamma = \gamma_n \in (0,1)$ and $g \in \mathcal{C}^{2,\gamma}\left(\partial B, \left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ with $\|h\|_{\mathcal{C}^{2,\gamma}(\partial B)} \lesssim 1$ such that*

$$\partial\Omega = \{(1 + h(x))x, \ x \in \partial B\}$$

Proof. For the $\mathcal{C}^{1,\gamma}$ estimate on h we proceed exactly as in the proof of Lemma 4.4.14. We again use partial hodograph transform as in the proof of Lemma 4.4.15. This time instead of (4.28) we obtain the system

$$\begin{cases} \mathrm{Tr}\left[A(\nabla v)\nabla^2 v\right] = \frac{\partial_n v}{\alpha} & (B\cap\mathbb{H}^n), \\ \frac{Q}{\alpha^2}(\partial_n v)^2 - \left(1+\delta\sum_{i=k}^l |\nabla v_i|^2\right)|\nabla' v|^2 = 1 & (B\cap\partial\mathbb{H}^n), \\ \mathrm{Tr}\left[A(\nabla v)\nabla^2 v_i\right] = \frac{\partial_n v_i}{\alpha} - \lambda_k v_i & (B\cap\mathbb{H}^n), \\ v_i = 0 & (B\cap\partial\mathbb{H}^n). \end{cases}$$

We then proceed similarly as in 4.4.15 to get the announced $\mathcal{C}^{2,\gamma}$ estimate: first the v_is are locally $\mathcal{C}^{2,\gamma}$ up to the boundary $\partial\mathbb{H}^n$ by elliptic regularity up to the boundary with Dirichlet condition. Then v verifies a nonlinear oblique boundary condition, and by considering the discrete derivative $v_{\tau,i} = \frac{v(x+\tau e_i)-v(x)}{\tau}$ we get a $\mathcal{C}^{2,\gamma}$ estimate on v as well. □

4.5.3 Minimality of the ball among nearly spherical sets.

The purpose of this subsection is to show the minimality of the ball for the functional $T^{-1}+\delta\sum_{i=k}^l\lambda_i$ among nearly spherical sets in the sense of Definition 4.4.16. The strategy is the same as in Subsection 4.4.4, with some differences due to the fact that we are considering multiple eigenvalues. We will make use of Lemma 4.4.18 which still applies to multiple eigenvalues. The minimality result is the following.

This time in the definition of a *nearly spherical* (see Definition 4.4.16) we rather fix $\alpha\in(0,\gamma)$ where $\gamma=\gamma_n$ is given by Lemma 4.5.18 instead of Lemma 4.4.15, although for simplicity we do not introduce additional definition and notations.

Proposition 4.5.19. *Let B_h be a nearly spherical set such that $\|h\|_{L^1(\partial B)} \ll k^{-1-\frac{4}{n}}g_n(k)$ and suppose that $|\delta| \ll k^{-\left(2+\frac{8}{n}\right)}g_n(k)$. Then we have*

$$T(B_h)^{-1}+\delta\sum_{i=k}^l\lambda_i(B_h) \geq T(B)^{-1}+\delta\sum_{i=k}^l\lambda_i(B)$$

with equality if and only if $B_h=B$.

As in the case of Proposition 4.4.17 this is achieved by performing a second order Taylor expansion of the functional $T^{-1}+\sum_{i=k}^l\lambda_i$, the main difference being that the eigenvalues $\lambda_k(B)=\ldots=\lambda_l(B)$ are multiple, so that each individual eigenvalue is not differentiable at $\zeta=0$. The key idea is that on the other hand, the sum $\sum_{i=k}^l\lambda_i((\mathrm{Id}+\zeta)(B))$ is smooth in ζ (and even analytic, see [LLdC06, Theorem 2.6]), and has a critical point at the ball.

Proposition 4.5.20. *Let B_h be a nearly spherical set such that $\|h\|_{L^1(\partial B)} \ll k^{-1-\frac{4}{n}}g_n(k)$. Then it holds*

$$\left|\sum_{i=k}^l(\lambda_i(B_h)-\lambda_i(B))\right| \leq C_n\frac{k^{2+\frac{8}{n}}}{g_n(k)}\|h\|^2_{H^{1/2}(\partial B)}.$$

Proof. Recall that by definition of a nearly spherical set it holds $\|h\|_{\mathcal{C}^{2,\alpha}(\partial B)}\leq 1$.

We proceed as in the proof of Proposition 4.4.19: letting $(\mu_i(t),u_i(t))$ be given by Lemma 4.4.18, since $\|h\|_{L^1(\partial B)} \ll k^{-1-\frac{4}{n}}g_n(k)$ we have by the same argument that $|\mu_i(t)-\mu_j(t)|\geq\frac{1}{2}g_n(k)$ for any $|t|\leq 1$ and i,j such that $i\in\{k,\ldots,l\}$, $j\notin\{k,\ldots,l\}$. As a consequence, $u_k(t),\ldots,u_l(t)$ is an orthonormal basis of eigenfunctions of the sum of eigenspaces corresponding to $(\lambda_i(B_{th}))_{i=k,\ldots,l}$, and $(\mu_i(t))_{i=k,\ldots,l}$ is a permutation of $(\lambda_i(B_{th}))_{i=k,\ldots,l}$.

We have

$$\frac{d}{dt}\Big|_{t=0}\sum_{i=k}^{l}\lambda_i(B_{th}) = \sum_{i=k}^{l}\mu_i'(0) = -\int_{\partial B}\left(\sum_{i=k}^{l}|\nabla u_i(0)|^2\right)\zeta\cdot\nu.$$

Because of the structure of the eigenfunctions of the ball, we know that $\sum_{i=k}^{l}|\nabla u_i(0)|^2$ is a constant, which we denote by $c_{n,k}$, and we thus have

$$\frac{d}{dt}\Big|_{t=0}\left(\sum_{i=k}^{l}\lambda_i(B_{th}) + c_{n,k}|B_{th}|\right) = 0.$$

Also by elliptic regularity we have $c_{n,k} \lesssim k^{1+\frac{6}{n}}$. Using a Taylor formula and since $|B| = |B_h|$, there exists some $t \in [0,1]$ such that

$$\sum_{i=k}^{l}(\lambda_i(B_h) - \lambda_i(B)) = \frac{1}{2}\left(\sum_{i=k}^{l}\mu_i''(t) + c_{n,k}\,\frac{d^2}{ds^2}\Big|_{s=t}|B_{sh}|\right).$$

To reduce notation we fix t and we do not write the dependency in t in the rest of the proof. We thus set $\Omega = B_{th}$, $u_i = u_i(t)$, $v_i := u_i'(t)$ and write μ_i,μ_i',μ_i'' in place of $\mu_i(t)$,$\mu_i'(t)$,$\mu_i''(t)$. The expression of μ_i'' takes the form

$$\mu_i'' = \int_\Omega 2\left(|\nabla v_i|^2 - \mu_i v_i^2\right) + \int_{\partial\Omega}|\nabla u_i|^2\left[H(\zeta\cdot\nu)^2 - b(\zeta_\tau,\zeta_\tau) + 2\zeta_\tau\cdot\nabla_{|\partial\Omega}(\zeta\cdot\nu)\right] \tag{4.53}$$

and each v_i verifies

$$\begin{cases} -\Delta v_i - \mu_i v_i = \mu_i' u_i & (\Omega),\\ v_i = -(\zeta\cdot\nu)\partial_\nu u_i & (\partial\Omega),\\ \int_\Omega(v_i u_j + v_j u_i) = 0, & \forall j\in\mathbb{N}^*,\end{cases} \tag{4.54}$$

where the last line is a consequence of $\int_{B_{th}} u_i(t)u_j(t) = \delta_{ij}$ for all t.

The "geometric" terms in (4.53) are estimated exactly as in the proof of Proposition 4.4.19. We thus have

$$\left|\int_{\partial\Omega}|\nabla u_i|^2\left[H(\zeta\cdot\nu)^2 - b(\zeta_{|\partial\Omega},\zeta_{|\partial\Omega}) + 2\zeta\cdot\nabla_{|\partial\Omega}(\zeta\cdot\nu)\right]\right| \lesssim k^{1+\frac{6}{n}}\|h\|^2_{H^{1/2}(\partial B)}.$$

For estimating the first term of (4.53), there is a difference with the case of a simple eigenvalue lying in the fact that we do not have a good control of $\int_\Omega v_i u_j$ when $i,j\in\{k,\dots,l\}$. Refining the analysis we will see that these terms in fact cancel in the sum $\sum_{i\in I}\mu_i''$.

Set $I = \{k,\dots,l\}$ and for each $i\in I$ write $v_i = z_i + w_i$ with z_i the harmonic extension of $-(\partial_\nu u_i)\zeta\cdot\nu$ and $w_i \in H_0^1(\Omega)$. The functions z_i verify the same estimates as in the proof of Proposition 4.4.19:

$$\|z_i\|_{H^1(\Omega)} \lesssim k^{\frac{1}{2}+\frac{4}{n}}\|h\|_{H^{1/2}(\partial B)},\ \|z_i\|_{L^2(\Omega)} \lesssim k^{\frac{1}{2}+\frac{2}{n}}\|h\|_{H^{1/2}(\partial B)}.$$

The function w_i verifies $-(\Delta+\mu_i)w_i = \mu_i z_i + \mu_i' u_i$ which ensures

$$\forall j\neq i,\ (\mu_j-\mu_i)\int_\Omega w_i u_j = \mu_i\int_\Omega z_i u_j \tag{4.55}$$

or, written differently,

$$\forall j\neq i,\ (\mu_j-\mu_i)\int_\Omega v_i u_j = \mu_j\int_\Omega z_i u_j. \tag{4.56}$$

We have

$$\begin{aligned}\sum_{i\in I}\int_\Omega \left(|\nabla v_i|^2-\mu_i|v_i|^2\right) &= \sum_{i\in I}\int_\Omega \left(|\nabla z_i|^2+|\nabla w_i|^2-\mu_i v_i^2\right)\\ &= \sum_{i\in I}\int_\Omega |\nabla z_i|^2+\sum_{i\in I, j\in\mathbb{N}^*}\mu_j\left(\int_\Omega w_iu_j\right)^2-\mu_i\left(\int_\Omega v_iu_j\right)^2\\ &= \sum_{i\in I}\int_\Omega |\nabla z_i|^2+\sum_{i\in I, j\notin I}\left(\frac{\mu_i^2}{\mu_j-\mu_i}-\mu_i\right)\left(\int_\Omega z_iu_j\right)^2\\ &+\sum_{i\in I, j\in I}\mu_j\left(\int_\Omega z_iu_j\right)^2-(\mu_j-\mu_i)\left(\int_\Omega v_iu_j\right)^2,\end{aligned}$$

where we used (4.56) in the last line. Thanks to the orthogonality conditions from (4.54), we have $\sum_{i,j\in I}(\mu_i-\mu_j)\left(\int_\Omega v_iu_j\right)^2=0$, hence we conclude

$$\begin{aligned}\left|\sum_{i\in I}\int_\Omega \left(|\nabla v_i|^2-\mu_i|v_i|^2\right)\right| &\lesssim \sum_{i\in I}\|\nabla z_i\|^2_{L^2(\Omega)}+g_n(k)^{-1}k^{\frac{4}{n}}\sum_{i\in I}\|z_i\|^2_{L^2(\Omega)}\\ &\lesssim g_n(k)^{-1}k^{1+\frac{8}{n}}|I|\|h\|^2_{H^{1/2}(\partial B)}\\ &\lesssim g_n(k)^{-1}k^{2+\frac{8}{n}}\|h\|^2_{H^{1/2}(\partial B)}.\end{aligned}$$

As a consequence $\left|\sum_{i=k}^l \mu_i''(t)\right|\lesssim k^{2+\frac{8}{n}}g_n(k)^{-1}\|h\|^2_{H^{1/2}(\partial B)}$, thus finishing the proof.

□

Proof of Proposition 4.5.19. This is done exactly as in the proof of Proposition 4.4.17.

□

4.5.4 Conclusion

Proof of Proposition 4.5.1. We proceed exactly as in the proof of Proposition 4.4.1. Since $|\delta|\ll k^{-\left(3+\frac{4}{n}\right)}$ then by application of Proposition 4.5.5 there exists a minimizer Ω (which we can suppose to be centered) with $|\Omega\Delta B|\lesssim k^{3+\frac{4}{n}}|\delta|$. By Lemma 4.5.18, since $|\delta|\ll k^{-6-\frac{8}{n}}g_n(k)$ then $\Omega=B_h$ where $\|h\|_{\mathcal{C}^{2,\gamma}(\partial B)}\lesssim 1$ and

$$\|h\|_{L^1(\partial B)}\lesssim|\Omega\Delta B|\lesssim k^{3+\frac{4}{n}}|\delta|.$$

From this we get by interpolation between L^1 and $\mathcal{C}^{2,\gamma}$ norms that when $k^{3+\frac{4}{n}}|\delta|\ll 1$, then Ω is nearly spherical in the sense of Definition 4.4.16. When $k^{3+\frac{4}{n}}|\delta|\ll k^{-2-\frac{8}{n}}g_n(k)$ (which is verified for $|\delta|\ll k^{-6-\frac{10}{n}}g_n(k)$) we can therefore apply Proposition 4.5.19 to conclude that Ω is a ball.

□

4.6 Discussion and consequences

4.6.1 About the sharpness of the results

We prove in the proposition below that the exponents 1/2 and 1 on the right-hand side of (4.1) given by Theorems 4.1.1, 4.1.2 and 4.1.3 are sharp for every k in dimension $n=2$. Proving it for any dimension would require a full second order analysis of the spectrum of smooth deformations of the ball in every dimension, in the spirit of the two dimensional work of [Ber15].

Proposition 4.6.1. *Let $n = 2$ and $k \geq 2$. There exists a constant $c_k > 0$ and $\Omega^\varepsilon = \phi^\varepsilon(B)$ a sequence of domains (in $\mathcal{A}$) with $\|\phi^\varepsilon - Id\|_{\mathcal{C}^1} \leq \varepsilon$ such that*

$$\text{if } \lambda_k(B) \text{ is simple } |\lambda_k(\Omega^\varepsilon) - \lambda_k(B)| \geq c_k(\lambda_1(\Omega) - \lambda_1(B)), \tag{4.57}$$

$$\text{if } \lambda_k(B) \text{ is double } |\lambda_k(\Omega^\varepsilon) - \lambda_k(B)| \geq c_k\left(\lambda_1(\Omega) - \lambda_1(B)\right)^{\frac{1}{2}}, \tag{4.58}$$

with each side going to 0 as $\varepsilon \to 0$.

Proof. Suppose first $\lambda_k(B)$ is simple. Then, following [Ber15, Lemmas 1 and 3] we get an explicit perturbation ϕ^ε of the identity, expressed as a Fourier series, which is preserving the area at the second order and for which the second order term in the asymptotic developments of both $\lambda_1(\phi^\varepsilon(B))$ and $\lambda_k(\phi^\varepsilon(B))$ are non vanishing, proving (4.57).

If $\lambda_k(B)$ is double, (for instance $\lambda_k(B) = \lambda_{k+1}(B)$), then for any vector field $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that $\int_{\partial B} \zeta \cdot x = 0$ the directional derivative of λ_k, λ_{k+1} in the direction ζ are respectively the first and second eigenvalues of the symmetric matrix

$$\begin{pmatrix} -\int_{\partial B} |\nabla u_k|^2 \zeta \cdot x & -\int_{\partial B} \nabla u_k \cdot \nabla u_{k+1} \zeta \cdot x \\ -\int_{\partial B} \nabla u_k \cdot \nabla u_{k+1} \zeta \cdot x & -\int_{\partial B} |\nabla u_{k+1}|^2 \zeta \cdot x \end{pmatrix}.$$

Moreover, since the functions (u_k, u_{k+1}) are not radial, we may choose a field ζ such that $\int_{\partial B} |\nabla u_k|^2 \zeta \cdot x \neq 0$, which gives a non-zero matrix with two nonzero (opposite) eigenvalues. Letting $\Omega^\varepsilon = \frac{(I+\varepsilon\zeta)(\Omega)}{|(I+\varepsilon\zeta)(\Omega)|^{1/2}}$ we have $|\lambda_k(\Omega^\varepsilon) - \lambda_k(B)| > c\varepsilon$ for some $c > 0$ and small enough ε, whereas $\lambda_1(\Omega^\varepsilon) - \lambda_1(B) < C\varepsilon^2$. □

Remark 4.6.2. In the same spirit an application of [Ber15, Lemmas 5] proves that for any double eigenvalue $\lambda_k(B) = \lambda_{k-1}(B)$ for $k \neq 3$, there exists a sequence of domains (Ω^ε) as in Proposition 4.6.1 such that

$$|\lambda_{k-1}(\Omega^\varepsilon) + \lambda_k(\Omega^\varepsilon) - 2\lambda_k(B)| \geq c_k\left(\lambda_1(\Omega) - \lambda_1(B)\right)$$

About the linear control of a multiple eigenvalue. A consequence of Theorem 4.1.3 is the following one side linear control: for any $k \geq 2$ with $\lambda_k(B)$ multiple and $\Omega \in \mathcal{A}$ such that $\lambda_1(\Omega) \leq 2\lambda_1(B)$, we have

$$\begin{aligned} &\text{if } \lambda_k(B) < \lambda_{k+1}(B), \text{ then } \lambda_k(\Omega) - \lambda_k(B) \geq -C_{n,k}(\lambda_1(\Omega) - \lambda_1(B)), \\ &\text{if } \lambda_{k-1}(B) < \lambda_k(B), \text{ then } \lambda_k(\Omega) - \lambda_k(B) \leq C_{n,k}(\lambda_1(\Omega) - \lambda_1(B)). \end{aligned}$$

The second one generalizes (4.5) which was observed for $k = 2$.

4.6.2 Proof of the reverse Kohler-Jobin inequality: Corollary 4.1.5

The linear bound from Theorem 4.1.2 (or equivalently Proposition 4.4.1) gives us a reverse form of the Kohler-Jobin inequality. This answers, in full generality, the question raised in [vdBBP21].

Proof of Corollary 4.1.5. By Proposition 4.4.1 there exists some $\delta_n > 0$ such that $\mathcal{A} \ni \Omega \mapsto T^{-1}(\Omega) - \delta_n \lambda_1(\Omega)$ is minimal on the ball. Let $p \geq 1$ and $\Omega \in \mathcal{A}$ be such that $T(\Omega)\lambda_1(\Omega)^{\frac{1}{p}} > T(B)\lambda_1(B)^{\frac{1}{p}}$, then

$$\frac{\lambda_1(\Omega)}{\lambda_1(B)} > \left(\frac{T(B)}{T(\Omega)}\right)^p \geq 1 + p\left(\frac{T(B)}{T(\Omega)} - 1\right) \geq 1 + \delta_n p T(B)\left(\lambda_1(\Omega) - \lambda_1(B)\right)$$

which implies $p < p_n := (\delta_n T(B)\lambda_1(B))^{-1}$. □

Note that we do not have explicit information on the value of p_n even in low dimension, as the proof of Proposition 4.4.1 relies on a contradiction and compactness argument at several points.

4.6.3 Proof of Theorem 4.1.4

Proof of Theorem 4.1.4. It will be useful to us to partition $\{1,\dots,k\}$ by intervals $(I_l)_{l=1,\dots,p}$ consisting of the clusters of eigenvalues. Let c_l be the common value of $\frac{\partial F}{\partial \lambda_i}(\lambda_1(B),\dots,\lambda_k(B))$ for $i \in I_l$. Then there exists some $C>0$ such that for any $\lambda \in \mathbb{R}_+^{k*}$:

$$\left| F(\lambda_1,\dots,\lambda_k) - F(\lambda_1(B),\dots,\lambda_k(B)) - \sum_{s=1}^{p} c_s \sum_{i\in I_s} [\lambda_i - \lambda_i(B)] \right| \leq C \sum_{i=k}^{l} (\lambda_i - \lambda_i(B))^2.$$

Applying Theorem 4.1.3 to each $\sum_{i\in I_s}[\lambda_i(\Omega)-\lambda_i(B)]$ and Theorem 4.1.1 to each $(\lambda_i(\Omega)-\lambda_i(B))^2$, we thus get the existence of some $D>0$ such that

$$|F(\lambda_1(\Omega),\dots,\lambda_k(\Omega)) - F(\lambda_1(B),\dots,\lambda_k(B))| \leq D\left(T(\Omega)^{-1} - T(B)^{-1}\right) T(\Omega)^{-1}. \tag{4.59}$$

Consider $\Omega \in \mathcal{A}$ a domain such that

$$T^{-1}(\Omega) + \delta F(\lambda_1(\Omega),\dots,\lambda_k(\Omega)) \leq T^{-1}(B) + \delta F(\lambda_1(B),\dots,\lambda_k(B))$$

for some $\delta \in \mathbb{R}$. Due to Proposition (4.2.1), this gives

$$\begin{aligned} T^{-1}(\Omega) &\leq T^{-1}(B) + \delta F((\lambda_i(B))_{i=1,\dots,k}) + C|\delta|(1+|(\lambda_i(\Omega))_{i=1,\dots,k}|) \\ &\leq T^{-1}(B) + \delta F((\lambda_i(B))_{i=1,\dots,k}) + CC_{n,k}|\delta|(1+T^{-1}(\Omega)). \end{aligned}$$

for some $C_{n,k}>0$. As a consequence, if $|\delta|$ is small enough then $T^{-1}(\Omega) \leq 2T^{-1}(B)$. Equation (4.59) then provides

$$|F(\lambda_1(\Omega),\dots,\lambda_k(\Omega)) - F(\lambda_1(B),\dots,\lambda_k(B))| \leq 2DT(B)^{-1}\left(T(\Omega)^{-1} - T(B)^{-1}\right)$$

which gives the result when $|\delta| \leq (2D)^{-1}T(B)$. □

Acknowledgements: The authors would like to thank G. Buttazzo, L. Briani and S. Guarino Lo Bianco for providing an early version of [BBGLB22] and for fruitful discussions. This work was partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR).

Bibliography

[AC81] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.

[ACKS01] I. Athanasopoulos, L.A. Caffarelli, C. Kenig, and S. Salsa. An area-Dirichlet integral minimization problem. *Comm. Pure Appl. Math.*, 54(4):479–499, 2001.

[AFM13] E. Acerbi, N. Fusco, and M. Morini. Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.*, 322(2):515–557, 2013.

[AFN11] A. Alvino, V. Ferone, and C. Nitsch. A sharp isoperimetric inequality in the plane. *Journal of the European Mathematical Society*, 013(1):185–206, 2011.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[AKN21] M. Allen, D. Kriventsov, and R. Neumayer. Sharp quantitative faber-krahn inequalities and the alt-caffarelli-friedman monotonicity formula. *arXiv preprint arXiv:2107.03505*, 2021.

[AKN22] M. Allen, D. Krivenstov, and R. Neumayer. Linear stability implies nonlinear stability for faber-krahn type inequalities. *arXiv: 2107.03495*, 2022.

[Amb97] L. Ambrosio. *Corso introduttivo alla teoria geometrica della misura e alle superfici minime*. Appunti dei corsi tenuti da docenti della scuola. Scuola normale superiore, Pisa, 1997.

[Bag67] T. Bagby. The Modulus of a Plane Condenser. *Journal of Mathematics and Mechanics*, 17(4):315–329, 1967.

[BB05] D. Bucur and G. Buttazzo. *Variational Methods in Shape Optimization Problems*, volume 65 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser Basel, 2005.

[BB22] B. Bogosel and D. Bucur. On the Polygonal Faber-Krahn Inequality. *arXiv*, 2022.

[BBGLB22] L. Briani, G. Buttazzo, and S. Guarino L. B. On a reverse kohler-jobin inequality. *arXiv preprint arXiv:2212.05375*, 2022.

[BBH09] D. Bucur, G. Buttazzo, and A. Henrot. Minimization of $\lambda_2(\omega)$ with a perimeter constraint. *Indiana University Mathematics Journal*, 58:2709–2728, 2009.

[BC06] J. Bertrand and B. Colbois. Capacité et inégalité de faber–krahn dans rn. *Journal of Functional Analysis*, 232(1):1–28, 2006.

[BCH17] C. Bianchini, G. Croce, and A. Henrot. On the quantitative isoperimetric inequality in the plane. *ESAIM: Control, Optimisation and Calculus of Variations*, 23(2):517–549, 2017.

[BDM93a] G. Buttazzo and G. Dal Maso. An existence result for a class of shape optimization problems. *Arch. Rational Mech. Anal.*, 122(2):183–195, 1993.

[BDM93b] G. Buttazzo and G. Dal Maso. An existence result for a class of shape optimization problems. *Arch. Rational Mech. Anal.*, 122(2):183–195, 1993.

[BDPV15] L. Brasco, G. De Philippis, and B. Velichkov. Faber-Krahn inequalities in sharp quantitative form. *Duke Mathematical Journal*, 164(9):1777–1831, 2015.

[Ber15] A. Berger. The eigenvalues of the Laplacian with Dirichlet boundary condition in $\mathbb{R}^2$ are almost never minimized by disks. *Ann. Global Anal. Geom.*, 47(3):285–304, 2015.

[BFK95] G. Buttazzo, V. Ferone, and B. Kawohl. Minimum problems over sets of concave functions and related questions. *Math. Nachr.*, 173:71–89, 1995.

[BH21] A. Behzadan and M. Holst. Multiplication in Sobolev spaces, revisited. *Ark. Mat.*, 59(2):275–306, 2021.

[Bha12] P. K. Bhattacharyya. *Distributions*. De Gruyter, 2022-10-05 2012.

[BHP05] T. Briançon, M. Hayouni, and M. Pierre. Lipschitz continuity of state functions in some optimal shaping. *Calc. Var. Partial Differential Equations*, 23(1):13–32, 2005.

[BL07] V. I. Burenkov and P. D. Lamberti. Spectral stability of general non-negative self-adjoint operators with applications to Neumann-type operators. *J. Differential Equations*, 233(2):345–379, 2007.

[BL08] V. I. Burenkov and P. D. Lamberti. Spectral stability of Dirichlet second order uniformly elliptic operators. *J. Differential Equations*, 244(7):1712–1740, 2008.

[BL09] T. Briançon and J. Lamboley. Regularity of the optimal shape for the first eigenvalue of the laplacian with volume and inclusion constraints. *Annales de l'I.H.P. Analyse non linéaire*, 26(4):1149–1163, 2009.

[BMPV15] D. Bucur, D. Mazzoleni, D. Pratelli, and B. Velichkov. Lipschitz regularity of the eigenfunctions on optimal domains. *Archive for Rational Mechanics and Analysis*, 216(1):117–151, 2015.

[BNT10] B. Brandolini, C. Nitsch, and C. Trombetti. An upper bound for nonlinear eigenvalues on convex domains by means of the isoperimetric deficit. *Archiv der Mathematik*, 94(4):391–400, 2010.

[BO16] B. Bogosel and E. Oudet. Qualitative and numerical analysis of a spectral problem with perimeter constraint. *SIAM J. Control Optim.*, 54(1):317–340, 2016.

[Bog19] B. Bogosel. Regularity result for a shape optimization problem under perimeter constraint. *Comm. Anal. Geom.*, 27(7):1523–1547, 2019.

[Bon24] T. Bonnesen. Über das isoperimetrische Defizit ebener Figuren. *Math. Ann.*, 91(3-4):252–268, 1924.

[Bou66] J. Bourget. Mémoire sur le mouvement vibratoire des membranes circulaires. *Ann. Sci. École Norm. Sup.*, 3:55–95, 1866.

[Buc00] D. Bucur. Uniform concentration-compactness for sobolev spaces on variable domains. *Journal of Differential Equations*, 162(2):427–450, 2000.

[Buc03] D. Bucur. Regularity of optimal convex shapes. *J. Convex Anal.*, 10(2):501–516, 2003.

[Buc12] D. Bucur. Minimization of the k-th eigenvalue of the Dirichlet Laplacian. *Arch. Ration. Mech. Anal.*, 206(3):1073–1083, 2012.

[Car02] G. Carlier. Calculus of variations with convexity constraint. *J. Nonlinear Convex Anal.*, 3(2):125–143, 2002.

[CCL13] L. A. Caffarelli, G. Carlier, and P.-L. Lions. $\mathcal{C}^{1,\alpha}$-regularity for variational problems with a convexity constraint and related issues. *Unpublished*, 2013.

[CF10] A. Colesanti and M. Fimiani. The Minkowski problem for torsional rigidity. *Indiana Univ. Math. J.*, 59(3):1013–1039, 2010.

[Che75] D. Chenais. On the existence of a solution in a domain identification problem. *Journal of Mathematical Analysis and Applications*, 52(2):189–219, 1975.

[Che77] D. Chenais. Sur une famille de variétés à bord lipschitziennes. Application à un problème d'identification de domaines. *Annales de l'Institut Fourier*, 27(4):201–231, 1977.

[Che07] H. Cheng, Q.-M.and Yang. Bounds on eigenvalues of dirichlet laplacian. *Mathematische Annalen*, 337(1):159–175, 2007.

[CL12] M. Cicalese and G.-P. Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Archive for Rational Mechanics and Analysis*, 206(2):617–643, 2012.

[CL13] M. Cicalese and G.P. Leonardi. Best constants for the isoperimetric inequality in quantitative form. *Journal of the European Mathematical Society*, 15:1101–1129, 01 2013.

[CLR01] G. Carlier and T. Lachand-Robert. Regularity of solutions for some variational problems subject to a convexity constraint. *Comm. Pure Appl. Math.*, 54(5):583–594, 2001.

[CS03] A. Colesanti and P. Salani. The Brunn-Minkowski inequality for p-capacity of convex bodies. *Mathematische Annalen*, 327(3):459–479, 2003.

[CS05] L. A. Caffarelli and S. Salsa. *A geometric approach to free boundary problems*, volume 68. American Mathematical Soc., 2005.

[CSY18] L. A. Caffarelli, H. Shahgholian, and K. Yeressian. A minimization problem with free boundary related to a cooperative system. *Duke Mathematical Journal*, 167(10):1825–1882, 2018.

[Dam02] M. Dambrine. On variations of the shape Hessian and sufficient conditions for the stability of critical shapes. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 96(1):95–121, 2002.

[Dav89] E. B. Davies. *Heat Kernels and Spectral Theory.* Cambridge University Press, Cambridge, 1989.

[DL19] M. Dambrine and J. Lamboley. Stability in shape optimization with second variation. *Journal of Differential Equations*, 267(5):3009–3045, 2019.

[DP00] M. Dambrine and M. Pierre. About stability of equilibrium shapes. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*, 34(4):811–834, 2000.

[DPF15] G. De Philippis and A. Figalli. Optimal regularity of the convex envelope. *Transactions of the American Mathematical Society*, 367(6):4407–4422, June 2015.

[DPLPV18] G. De Philippis, J. Lamboley, M. Pierre, and B. Velichkov. Regularity of minimizers of shape optimization problems involving perimeter. *J. Math. Pures Appl.*, 109:147–181, 2018.

[DPMM21] G. De Philippis, M. Marini, and E. Mukoseeva. The sharp quantitative isocapacitary inequality. *Rev. Mat. Iberoam.*, 37(6):2191–2228, 2021.

[DPV14] G. De Philippis and B. Velichkov. Existence and regularity of minimizers for some spectral functionals with perimeter constraint. *Appl. Math. Optim.*, 69(2):199–231, 2014.

[DS88] N. Dunford and J. T. Schwartz. *Linear operators. Part II.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication.

[DS11] D. De Silva. Free boundary regularity for a problem with right hand side. *Interfaces and free boundaries*, 13(2):223–238, 2011.

[EFT05] L. Esposito, N. Fusco, and C. Trombetti. A quantitative version of the isoperimetric inequality : the anisotropic case. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 4(4):619–651, 2005.

[FFM$^+$15] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini. Isoperimetry and stability properties of balls with respect to nonlocal energies. *Comm. Math. Phys.*, 336(1):441–507, 2015.

[FG08] Ilaria Fragalà and Filippo Gazzola. Partially overdetermined elliptic boundary value problems. *J. Differential Equations*, 245(5):1299–1322, 2008.

[FL21] I. Ftouhi and J. Lamboley. Blaschke-santaló diagram for volume, perimeter, and first dirichlet eigenvalue. *SIAM Journal on Mathematical Analysis*, 53(2):1670–1710, 2021.

[FM11] A. Figalli and F. Maggi. On the shape of liquid drops and crystals in the small mass regime. *Archive for Rational Mechanics and Analysis*, 201(1):143–207, 2011.

[FMP08] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.

[FMP09] N. Fusco, F. Maggi, and A. Pratelli. Stability estimates for certain Faber-Krahn, isocapacitary and Cheeger inequalities. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 8(1):51–71, 2009.

[FMP10] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.

[Fra21] Frank, R. L. and Nam, P. T. Existence and nonexistence in the liquid drop model. *Calc. Var. Partial Differential Equations*, 60(6):Paper No. 223, 12, 2021.

[Fto21] I. Ftouhi. *Diagrammes de Blaschke-Santaló et autres problèmes en optimisation de forme.* Thèse Sorbonne université, 2021.

[Fug89] B. Fuglede. Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbb{R}^n$. *Transactions of the American Mathematical Society*, 314(2):619–638, 1989.

[Fus15] N. Fusco. The quantitative isoperimetric inequality and related topics. *Bull. Math. Sci.*, 5(3):517–607, 2015.

[GNR15] M. Goldman, M. Novaga, and B. Ruffini. Existence and stability for a non-local isoperimetric model of charged liquid drops. *Arch. Rat. Mech. Anal.*, 217(1):1–36, 2015.

[GNR18] M. Goldman, M. Novaga, and B. Ruffini. On minimizers of an isoperimetric problem with long-range interactions and convexity constraint. *Analysis & PDE*, 11(5):1113–1142, 2018.

[GT01] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[Hal92] R. R. Hall. A quantitative isoperimetric inequality in n-dimensional space. *J. Reine Angew. Math.*, 428:161–176, 1992.

[Hen06] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[Hen20] A. Henrot. *Shape optimization and spectral theory*. De Gruyter, Berlin, Boston, 2020.

[HL11] Q. Han and F. Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[HO03] A. Henrot and E. Oudet. Minimizing the second eigenvalue of the Laplace operator with Dirichlet boundary conditions. *Arch. Ration. Mech. Anal.*, 169(1):73–87, 2003.

[HP18] A. Henrot and M. Pierre. *Shape Variation and Optimization, A Geometrical Analysis*, volume 28. EMS Tracts in Mathematics, 2018.

[Jul14] V. Julin. Isoperimetric problem with a coulomb repulsive term. *Indiana University Mathematics Journal*, 63(1):77–89, 2014.

[Kat95] T. Kato. *Perturbation Theory for Linear Operators*. Springer Berlin, Heidelberg, 1995.

[Ken85] A. U. Kennington. Power concavity and boundary value problems. *Indiana Univ. Math. J.*, 34(3):687–704, 1985.

[KJ78] M.-T. Kohler-Jobin. Une méthode de comparaison isopérimétrique de fonctionnelles de domaines de la physique mathématique. II. Cas inhomogène: une inégalité isopérimétrique entre la fréquence fondamentale d'une membrane et l'énergie d'équilibre d'un problème de Poisson. *Z. Angew. Math. Phys.*, 29(5):767–776, 1978.

[KL18] D. Kriventsov and F. Lin. Regularity for shape optimizers: the nondegenerate case. *Communications on Pure and Applied Mathematics*, 71(8):1535–1596, 2018.

[KL19] D. Kriventsov and F. Lin. Regularity for shape optimizers: the degenerate case. *Comm. Pure Appl. Math.*, 72(8):1678–1721, 2019.

[KM13] H. Knüpfer and C. B. Muratov. On an isoperimetric problem with a competing nonlocal term I: The planar case. *Comm. Pure Appl. Math.*, 66(7):1129–1162, 2013.

[KM14] H. Knüpfer and C. B. Muratov. On an isoperimetric problem with a competing nonlocal term ii: The general case. *Communications on Pure and Applied Mathematics*, 67(12):1974–1994, 2014.

[KN77] D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):373–391, 1977.

[Lam11] J. Lamboley. About Hölder-regularity of the convex shape minimizing λ_2. *Appl. Anal.*, 90(2):263–278, 2011.

[Lan07] N. Landais. A regularity result in a shape optimization problem with perimeter. *J. Convex Anal.*, 14(4):785–806, 2007.

[Lau20] A. Laurain. Distributed and boundary expressions of first and second order shape derivatives in nonsmooth domains. *J. Math. Pures Appl. (9)*, 134:328–368, 2020.

[LLdC06] P. D. Lamberti and M. Lanza de Cristoforis. Critical points of the symmetric functions of the eigenvalues of the laplace operator and overdetermined problems. *Journal of the Mathematical Society of Japan*, 58(1):231–245, 1 2006.

[LN10] J. Lamboley and A. Novruzi. Polygons as optimal shapes with convexity constraint. *SIAM J. Control Optim.*, 48(5):3003–3025, 2009/10.

[LNP12] J. Lamboley, A. Novruzi, and M. Pierre. Regularity and singularities of optimal convex shapes in the plane. *Arch. Ration. Mech. Anal.*, 205(1):311–343, 2012.

[LNP22] J. Lamboley, A. Novruzi, and M. Pierre. Polygons as maximizers of dirichlet energy or first eigenvalue of dirichlet-laplacian among convex planar domains. *Advances in Calculus of Variations*, 2022.

[LO14] J. Lu and F. Otto. Nonexistence of a minimizer for thomas–fermi–dirac–von weizsäcker model. *Communications on Pure and Applied Mathematics*, 67:1605–1617, 2014.

[LP23] J. Lamboley and R. Prunier. Regularity in shape optimization under convexity constraint. *Calc. Var. Partial Differential Equations*, 62(3):Paper No. 101, 44, 2023.

[LRP01a] T. Lachand-Robert and M. A. Peletier. An example of non-convex minimization and an application to newton's problem of the body of least resistance. *Annales de l'I.H.P. Analyse non linéaire*, 18(2):179–198, 2001.

[LRP01b] T. Lachand-Robert and M. A. Peletier. Newton's problem of the body of minimal resistance in the class of convex developable functions. *Math. Nachr.*, 226:153–176, 2001.

[LWZ22] D. Lokutsievskiy, G. Wachsmuth, and M. Zelikin. Non-optimality of conical parts for newton's problem of minimal resistance in the class of convex bodies and the limiting case of infinite height. *Calculus of Variations and Partial Differential Equations*, 61(31), 2022.

[LY83] P. Li and S.-T. Yau. On the schrödinger equation and the eigenvalue problem. *Communications in Mathematical Physics*, 88(3):309–318, 1983.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[Maz85] V.G. Maz'ya. *Sobolev Spaces*. Springer Series in Soviet Mathematics. Springer-Verlag Berlin Heidelberg, 1985.

[Maz09] V.G. Maz'ya. Boundedness of the gradient of a solution to the Neumann-Laplace problem in a convex domain. *Comptes Rendus Mathematique*, 347(9):517–520, 2009.

[MP19] D. Mazzoleni and A. Pratelli. Some estimates for the higher eigenvalues of sets close to the ball. *Journal of Spectral Theory*, 9(4):1385–1403, 2019.

[MR21] D. Mazzoleni and B. Ruffini. A spectral shape optimization problem with a nonlocal competing term. *Calc. Var. Partial Differential Equations*, 60(3):Paper No. 114, 46, 2021.

[MTV17] D. Mazzoleni, S. Terracini, and B. Velichkov. Regularity of the optimal sets for some spectral functionals. *Geom. Funct. Anal.*, 27(2):373–426, 2017.

[MTV21] F. P. Maiale, G. Tortone, and B. Velichkov. Epsilon-regularity for the solutions of a free boundary system. *arXiv preprint arXiv:2108.03606*, 2021.

[Nit14] C. Nitsch. An isoperimetric result for the fundamental frequency via domain derivative. *Calculus of Variations and Partial Differential Equations*, 49(1):323–335, 2014.

[OK13] B. Osting and C.-Y. Kao. Minimal convex combinations of sequential Laplace-Dirichlet eigenvalues. *SIAM J. Sci. Comput.*, 35(3):B731–B750, 2013.

[OK14] B. Osting and C.-Y. Kao. inimal convex combinations of three sequential Laplace-Dirichlet eigenvalues. *Appl. Math. Optim.*, 69(1):123–139, 2014.

[Oss78] R. Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84(6):1182–1238, 1978.

[Peg21] M. Pegon. Large mass minimizers for isoperimetric problems with integrable nonlocal potentials. *Nonlinear Anal.*, 211:Paper No. 112395, 48, 2021.

[Pet22] R. Petit. *Reconstruction of piecewise constant images via total variation regularization: exact support recovery and grid-free numerical methods.* Thèse, Université Paris Dauphine-PSL, December 2022.

[PW61] L. E Payne and H. F Weinberger. Some isoperimetric inequalities for membrane frequencies and torsional rigidity. *Journal of Mathematical Analysis and Applications*, 2(2):210–216, 1961.

[RC98] J-C Rochet and P. Choné. Ironing, sweeping and multidimensional screening. *Econometrica*, 66(4), 1998.

[Ros04] M. Ross. The Lipschitz continuity of Neumann eigenvalues on convex domains. *Hokkaido Mathematical Journal*, 33(2):369 – 381, 2004.

[RS11] T. Runst and W. Sickel. *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations.* De Gruyter, 2022-10-04 2011.

[Sch12] F. Schulze. Partial differential equations ii. *Lecture notes available at https://www.felixschulze.eu/index.php/publications*, 2012.

[Sch14] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, expanded edition, 2014.

[Ser71] J. Serrin. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.*, 43:304–318, 1971.

[Sie14] Carl L. Siegel. Über einige Anwendungen diophantischer Approximationen [reprint of Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1]. In *On some applications of Diophantine approximations*, volume 2 of *Quad./Monogr.*, pages 81–138. Ed. Norm., Pisa, 2014.

[SZ97] E. Stredulinsky and W. P. Ziemer. Area minimizing sets subject to a volume constraint in a convex set. *J. Geom. Anal.*, 7(4):653–677, 1997.

[Tal76] G. Talenti. Elliptic equations and rearrangements. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 3(4):697–718, 1976.

[Tam84] I. Tamanini. *Regularity results for almost minimal oriented hypersurfaces.* Quaderni del Dipartimento di Matematica dell'Universit'a di Lecce, http://cvgmt.sns.it/paper/1807/, 1984.

[Tam88] I. Tamanini. Variational problems of least area type with constraints. *Ann. Univ. Ferrara Sez. VII (N.S.)*, 34:183–217, 1988.

[Tho15] M. Thomas. Uniform Poincaré-Sobolev and isoperimetric inequalities for classes of domains. *Discrete & Continuous Dynamical Systems*, 35(6):2741–2761, 2015.

[vdBBP21] M. van den Berg, G. Buttazzo, and A. Pratelli. On relations between principal eigenvalue and torsional rigidity. *Commun. Contemp. Math.*, 23(8):Paper No. 2050093, 28, 2021.

[Vel19] B. Velichkov. Regularity of the one-phase free boundaries. *Lecture notes available at http://cvgmt. sns. it/paper/4367*, 2019.

[Wat44] G. N. Watson. *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944.

[Wei99] G. S. Weiss. Partial regularity for a minimum problem with free boundary. *J. Geom. Anal.*, 9(2):317–326, 1999.